Seeking a Hypergeometric Closed Form for Map Enumeration

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ABSTRACT

This thesis starts by introducing the literature review of “Map Enumeration” and the definition of “Generating functions” and “Hypergeometric closed form”. We examine several basic algorithms for finding hypergeometric solutions to linear recurrence relations with polynomial coefficients. We introduce GFUN Maple package and develop a Maple program to analyze generating functions of four important families of rooted maps on the plane, the torus, and the projective plane. It is proved that the coefficients of their generating functions cannot be expressed in closed form.

KEY WORDS

Map Enumeration, Generating functions, Hypergeometric closed form, 3-connected planar maps, 4-connected planar triangulations, Torus, Projective plane

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LIST OF SYMBOLS

IN	nonnegative integers
Z	all integers
P	positive integers
C	complex numbers

CHAPTER 1 MAP ENUMERATION

1.1 MOTIVATION AND OBJECTIVE

Map enumeration is widely used in many research areas. However, most enumeration solutions are complicated forms that may cause inconveniences in further applications. Several days' work of try to simplify these mammoth expressions manually may be fruitless. This might be caused by computation errors. Or no desired simple expression exists at all. Therefore, we want to use Maple programs to find hypergeometric closed form algorithmically or prove that no such closed form exists at all.

In this thesis we develop a Maple function `Hyperhomo( )` based on the Algorithm Hyper. It is designed to find hypergeometric solutions for a linear homogeneous recurrence with polynomial coefficients. We use GFUN package to derive the linear recursions for the coefficients of four important families of maps. Then we use `Hyperhomo( )` to prove that none of them have hypergeometric closed form.

1.2 ORGANIZATION

This thesis is organized as follows: Chapter 1 starts with basic concepts of maps and map enumeration. Chapter 2 discusses algebraic and D-finite generating functions and their relation to linear recursions that have polynomial coefficients. Then, four algorithms that are used to find hypergeometric closed form are introduced in Chapter 3. Chapter 4 deals with four map enumeration problems.

1.3 ENUMERATIVE COMBINATORICS

Enumerative combinatorics is the science of counting, whereby questions such as “Given a description of a finite set, what is the cardinality?” can be solved.

Some map enumeration problems, solved by either asymptotic enumeration methods or algebraic enumeration methods, will be discussed later. Asymptotic enumeration focuses on an approximate and simple formula, which gives the order of growth of the cardinalities as a function of their parameters. Algebraic enumeration considers the exact result that either can be computed from explicit formulas, or from generating functions and recurrence relations. Details of generating functions in enumeration problems will be discussed in Chapter 2.

1.4 HISTORY OF THE FOUR COLOR THEOREM [27]

“The Four Color Map Conjecture”, introduced by DeMorgan, “is central in the branch of mathematics known as combinatorics.” (Tutte [25]). Francis Guthrie, a mathematician, lawyer, and botanist, is the actual initiator. Dating back to 1852, he tried to color a map of counties of England, and noticed that four colors sufficed. He asked his brother Frederick if it was true that any map could be colored using four colors in such a way that adjacent regions (i.e., those sharing a common boundary segment, not just a point.) receive different colors. Frederick then communicated the conjecture to DeMorgan, who proposed this question to his colleague Hamilton. But this question created no stir at that time.

Thirty years later, the first printed reference by Arthur Cayley appeared in 1878. Kempe tried to give the first complete ‘proof’ in 1878, but Heawood pointed out a mistake in

Kempe's proof in 1890. Another failed proof was held out by Guthrie Peter Tait in 1880. But Petersen pointed out a gap in Tait's argument in 1891.

In the years after Heawood's discovery that five colors are sufficient to color any map drawn on either the sphere or the plane, many papers studied the Four-Color problem at great length but without any big success. Finally, Kenneth Appel and Wolfgang Haken provided a computer-assisted proof of the Four Color Theorem in 1976. However, even the Appel-Haken proof is not completely satisfactory.

Tutte [25] categorized the above methods as two types: "qualitative" and "quantitative". The latter one studied the exact number of n -colorings of a map. For example, there are 24 ways to color a map M in four given colors if M has only 4 mutually contiguous countries. Tutte considered these two methods and preferred the quantitative one. Tutte began his study of map enumeration in the 1960s when he noticed that map enumeration method could lead to a positive proof of the Four Color conjecture.

The study of Map Enumeration has blossomed and established itself as an important mathematical tool in many research fields. Arthur Cayley first applied it to the study of enumerating alkanes C_nH_{2n+2} with certain number of carbon atoms. Nowadays in Biology, it is used to study the structure of genes and proteins. In Chemistry, it is used to study the arrangements of atoms in molecules and crystals. In Statistical Mechanics, Baxter [1] studied the q -state Potts model randomized over a non-separable rooted map. Different from other research works that use field-theoretic techniques and matrix models, he took combinatorial approaches based on the equivalence of dichromatic polynomials in two variables of a graph G with the partition function of a q -state Potts

model on a graph G , with $q = (x-1)(y-1)$. In Quantum Field Theory, Hooft [12] studied the critical effects for large g 's (where g is the gauge coupling constant.) of Quantum Chromodynamics in a zero-dimensional model. Instead of using matrix theory, he started from the functional equations, and counted the number of corresponding planar maps with various types of restrictions. It is easier to calculate in the diagrammatical way, which also provides more transparent relations of functional equations.

1.5 MAPS

1.5.1 DEFINITIONS

Several definitions and notations are being introduced here to make this thesis more self-contained.

Surface:

Kinsey [13] defines a surface from the topological point of view: A surface can be a 2-dimensional manifold. Every point in this topological space has a neighborhood topologically equivalent to a 2-dimensional open disc with center x and radius r ,

$$D^2(x, r) = \{y \in R^2 : \|x - y\| < r\}.$$

Map:

A map M is a non-null connected graph G embedded in a surface. Surface here could be a plane or a 2-sphere. Hence, a planar map is a connected graph that is embedded in the plane. Graph G separates the surface into a number of connected and disjoint regions, which are called the faces of map M . The vertices and edges of the defining graph are referred to as the vertices and edges of the resulting map. The external face is an

unbounded face and unique for every map. The valency of a vertex is the number of incident edges. The valency of a face is also the number of incident edges.

Note that the graphs discussed here are allowed to have loops and multiple edges, and must be finite. A graph is connected if and only if it has paths between any pair of vertices. (see Figure 1)

Figure 1. Connected graph and disconnected graph

The connectivity $\kappa(G)$ of a graph G is the minimal number of vertices whose removal will make G a disconnected graph or a vertex-map (the simplest map has a single vertex only and no edges). A graph is k -connected if $\kappa(G) \geq k$. A map is k -connected if the underlying graph is k -connected.

A map is rooted when an edge e is selected as the root-edge, one of the vertices incident with e is selected as the root-vertex, and one of the faces incident with e is selected as root-face respectively. Normally the exterior face is the root-face.

Some particular maps

A triangulation is a planar map that each face is a triangle. In other words, each face has a valency 3.

A torus is a surface having genus 1, and therefore possessing a single “hole”. A torus in three-dimensional space has a shape much like that of a donut.

Figure 2. Assembling a planar map to a torus

A projective plane is a 2-dimensional manifold. Intuitively, a projective plane is topologically equivalent to taking a disc and sewing each point on the boundary to the one diametrically opposite it.

Figure 3. A projective plane

Two rooted maps M_1 and M_2 are considered combinatorially equivalent if there is a homeomorphism¹ f of the surface onto itself that transforms M_1 into M_2 with preservation of the natural cyclic sequence of the approaches² to each vertex, the natural cyclic

¹ Homeomorphism is an equivalence relation and one-to-one correspondence between points in two geometric figures or topological spaces which is continuous in both directions. It is also called a continuous transformation. It means that deformations like stretching, bending, or shrinking are allowed. But no tearing or gluing is allowed.

² Each vertex of a map M is incident with at least one edge if M is not a vertex-map. For any vertex v , the adjacent portions of the incident edges can be regarded as short segments radiating from v . These adjacent portions are called the approaches to v . The approaches have a natural cyclic order. Any vertex v has a natural cyclic sequence of edges by substituting each approach by corresponding edge.

sequence of edges round each face, and the root. In the study of map enumeration, we do not distinguish combinatorially equivalent maps.

1.5.2 EXAMPLE OF MAP ENUMERATION

The following example ([21]) helps us understand what an enumeration problem looks like.

For $n=0$, $a_0=1$, the corresponding map is called a vertex-map.

For $n=1$, $a_1=2$,

Figure 4. Rooted maps with one edge

For $n=2$, $a_2=9$,

Figure 5. Rooted maps with two edges

Tutte [21] answered the question “ How many (combinatorially distinct) rooted planar

maps are there with n edges? ” That is $a_n = \frac{2(2n)!3^n}{n!(n+2)!}$.

This is an example of enumeration with respect to the number of edges. Other specifications can be the number of faces, vertices, and connectivity.

1.6 A SURVEY OF MAP ENUMERATION

It was Tutte who founded the enumeration theory of planar maps in the sixties to deal with questions such as how many combinatorially distinct maps there are with n edges, or with k vertices and l faces, or some other parameters. Following his original paper [23] on the enumeration theory on planar maps, a series of papers on enumeration of planar triangulations brought up some new results. Most of the results are for rooted maps because rooting changes maps to asymmetrical maps, making enumeration easier.

In this section, references and a very brief description of some results in the map enumeration will be covered.

1.6.1 ROOTED PLANAR MAPS

1.6.1.1 Rooted trees

Map enumeration began long ago with the enumeration of trees. However, extensive work on the enumeration of maps did not begin until Tutte's work in the 1960's. There exist many species of trees and elaborated details of the enumeration of trees since they are widely used in computer science. Tutte [21] solved the general case of a rooted plane tree. A rooted tree is a tree in which one vertex is picked as the root. A planar tree is a special map that the underlying graph is a tree. Such a map has only one face, the outer

face. The number of rooted plane trees with n edges is $t_n = \frac{(2n)!}{n!(n+1)!}$.

1.6.1.2 Rooted planar triangulations

So far, all the possible n -connected planar triangulations have already been enumerated since no planar map is 6-connected.

- Mullin [17] showed that the number of 2-connected (allowing multiple edges)

triangulations with n vertices is $\frac{3(3n)!2^n}{n!(2n+2)!}$.

- Tutte [23] provided the number of 3-connected triangulations with $n+3$ vertices,

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

- Tutte [23] studied the number of 4-connected planar triangulations, which are enumerated with respect to vertices. He obtained the algebraic generating function:

$$f = \frac{t(t-1)}{(1+t)^2} \text{ and } x = t(1-t)^2.$$

- Gao, Wanless, and Wormald [9] found the generating function whose coefficients denote the number of 5-connected rooted planar triangulations. It is

$$T(w,0) = 4w(s+1)^3 \frac{w(3s-1) + (3s-2)(s+1)^3}{(w-s^3+3s+2)^3} - \frac{3w^3}{1+w} + w + 3w^3 - w^4, \quad \text{where}$$

$w^2 - 2(4s^2 + 2s + 1)(s+1)^2 w - s(s+2)(s+1)^4 = 0$. Furthermore, the asymptotic formula

was derived. That is $T_n = \frac{3c_3}{4\sqrt{\pi}} n^{-5/2} w_0^{-n} (1 + o(1/n))$, where $w_0 = 0.24775354$ and

$$c_3 = 0.00067543010.$$

1.6.1.3 Other rooted planar maps

- The number of rooted connected maps with n edges was determined by Tutte [22].

The result is $\frac{3^n (2n)! 2}{n! (n+2)!}$.

- Tutte [22] determined the number of rooted 2-connected maps with n edges as

$$\frac{(3n-3)! 2}{n! (2n-1)!}.$$

- Tutte [22] provided an explicit generating function for the number of rooted 3-connected planar maps that are enumerated with respect to edges. That is

$$C(x) = x^2 \left(\frac{2}{1+x} + 2x \right) - \frac{u}{(1+2u)^3}, \text{ where } u = x(1+u)^2.$$

1.6.2 SOME NON-PLANAR ROOTED MAPS

There are various results of non-planar map enumeration, such as non-planar maps with arbitrary genus, quadrangulations, etc. Details can be found in Bender and Richmond [2].

- Bender, Canfield and Robinson [3] derived the explicit expression of the generating

$$\text{function: } T = \frac{(R-1)^2}{12R^2(R+2)}, \text{ where } R = \sqrt{1-12x}.$$

- Bender, Canfield and Robinson [3] studied the number of rooted maps, enumerated with respect to edges, on the projective plane. The derived explicit generating

$$\text{function is } P = \frac{\frac{2R+1}{3} - \sqrt{R(R+2)/3}}{2x}, \text{ where } R = \sqrt{1-12x}.$$

So far, we have seen some marvelous expressions for the results of map enumeration. The properties and algorithms used to find a hypergeometric closed form will be discussed in the following chapters.

CHAPTER 2 GENERATING FUNCTIONS

When dealing with enumerative combinatorics, there are several standard ways to express the counting function¹:

1. An explicit closed formula involving only well-known functions and no summation symbols. It is the simplest and most satisfactory form, however it only exists in rare cases.
2. A recurrence expression for the counting function. It is given in terms of previously calculated counting functions.
3. An estimated expression of the counting function $f(i)$. This estimation is obtained by an asymptotic formula $f(n) \sim g(n)$, where $g(n)$ is a “familiar function”. For many reasons, we prefer the estimate to the explicit “formula”.
4. A generating function, an “object” that represents the counting function.

2.1 DEFINITIONS AND NOTATIONS

Generating functions are a powerful tool in Enumerative Combinatorics. The generating function for a set S is $G(s) = \sum_{\alpha \in S} w(\alpha)$, where $w(\alpha)$ is the assigned weight of each element of S . Thus, the concept of a generating function for a set is a generalization of the concept of cardinality. Usually the weights are monomials in a ring of polynomials or power series, and they may belong to an abelian group². In a typical application each α has a ‘length’ $l(\alpha)$, and the weight of α is $x^{l(\alpha)}$, where x is an indeterminate. If the

¹ Richard P. Stanley, *Enumerative Combinatorics*, pp1-3

² If all the elements of a group commute (i.e., $AB = BA$ for all elements A and B), this group is called an Abelian group.

generating function $\sum_{\alpha \in S} x^{l(\alpha)}$ is known, the number of elements of S of each length is known.

Why bother with a generating function? It is not just an alternative way to represent a sequence. The analytical properties of a generating function also reflect many properties of the corresponding sequence. Particularly, it is easy to compute asymptotic properties of a sequence from its generating function because we can perform algebraic manipulations on the sequence. (there is an obvious correspondence between operations performed on the sequence and operations performed on the generating function.) Also, the generating function is often more compact than the “closed form” of the sequence, and it is available even when no closed form exists.

Rational, Algebraic and D-finite are three classes of generating functions. Stanley [20] says “*D-finite* functions are a natural generalization of algebraic functions, while the algebraic functions are a natural generalization of rational functions”. Figure 6 (see Stanley [20]) shows the hierarchy of these three generating functions.

Figure 6. Hierarchy of generating functions

Let K be a commutative ring. The set of all polynomials with coefficients in K is denoted as $K[x]$. While a polynomial represents a finite sequence, there is a power series for an

infinite sequence $\{a_n\}$. The set of all power series with coefficients in K is a commutative ring itself, denoted by $K[[x]]$. $K(x)$ is the quotient field of $K[[x]]$ and it stands for the field of rational functions over K . The quotient field of $K[[x]]$ is $K((x)) = K[[x]][1/x]$.

2.1.1 RATIONAL GENERATING FUNCTIONS

Let $\alpha_1, \alpha_2, \dots, \alpha_d$ be a fixed sequence of complex numbers, $d \geq 1$ and $\alpha_d \neq 0$. The following conditions on a function $f: \mathbb{N} \rightarrow C$ are equivalent:

$$1. \sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)}, \text{ where } Q(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_d x^d \text{ and } P(x) \text{ is a}$$

polynomial in x of degree less than d .

$$2. \text{ For all } n \geq 0, f(n+d) + \alpha_1 f(n+d-1) + \alpha_2 f(n+d-2) + \dots + \alpha_d f(n) = 0 \quad (2.1)$$

$$3. \text{ For all } n \geq 0, f(n) = \sum_{i=1}^k P_i(n) \gamma_i^n, \text{ where } 1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_d x^d = \prod_{i=1}^k (1 - \gamma_i x)^{d_i},$$

the γ_i 's are distinct and $P_i(n)$ is a polynomial in n of degree less than d_i .

2.1.2 ALGEBRAIC GENERATING FUNCTIONS

Let K be a field. A formal power series $\eta \in K[[x]]$ is *algebraic* if there exist polynomials

$P_0(x), \dots, P_d(x) \in K[x]$, not all zero, such that

$$P_0(x) + P_1(x)\eta + \dots + P_d(x)\eta^d = 0. \quad (2.2)$$

The degree of η is the smallest positive integer d that holds for (2.2). An algebraic series

η has degree one if and only if η is rational.

2.1.2.1 Example of Algebraic series

Figure 7. Polygon triangulations

The Catalan numbers are an integer sequence $\{C_n\}$ which appears in many enumeration problems, such as “In how many ways can a regular n -gon be divided into $n-2$ triangles if different orientations are counted separately?”. Figure 7 (see [29]) graphically illustrates the above question. The Catalan numbers take the explicit formula:

$$C_n \equiv \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} \frac{(2n)!}{n!^2} = \frac{(2n)!}{(n+1)!n!}. \text{ The generating function for this sequence is}$$

$$\frac{1 - \sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + \dots.$$

Let $y = \frac{1 - \sqrt{1-4x}}{2x}$, then $1 - 2xy = \sqrt{1-4x}$, $4x^2y^2 - 4xy + 4x = 0$. We obtain $y^2x - y + 1 = 0$,

which satisfies (2.2). Therefore $\frac{1 - \sqrt{1-4x}}{2x}$ is an algebraic generating function.

Rational generating functions $\sum f(n)x^n$ have an important property that coefficients $f(n)$ satisfy a simple recurrence (2.1). Algebraic generating functions have the similar

property, and the type of recurrence satisfied by the coefficients of an algebraic generating function also holds for the coefficients of more general series, *D-finite* series.

2.1.3 D-FINITE GENERATING FUNCTIONS

2.1.3.1 Definition

A function $f \in K[[x]]$ is ***D-finite*** if and only if f satisfies an ordinary linear differential equation with polynomial coefficients (a holonomic differential equation), i.e., there exist $d \in \mathbb{N}$ and polynomials $p_0(x), p_1(x), \dots, p_d(x) \in K[x]$ with $p_d(x) \neq 0$, such that

$$p_d(x)f^{(d)}(x) + p_{d-1}(x)f^{(d-1)}(x) + \dots + p_1(x)f'(x) + p_0(x)f(x) = 0 \quad (2.3)$$

The nonnegative integer d denotes the order of the D-finite (short for differentiably finite) function. A D-finite function is also called a holonomic function.

The following theorem gives alternative ways for defining a D-finite function.

2.1.3.2 Theorem (Stanley [20])

Let $u \in K[[x]]$. The following three conditions are equivalent:

1. The vector space over $K(x)$ spanned by u and all its derivatives $u', u'', \dots$ is finite-dimensional. In symbols,

$$\dim_{K(x)} [K(x)u + K(x)u' + K(x)u'' + \dots] < \infty \quad (2.4)$$

2. There exist polynomials $p_0(x), p_1(x), \dots, p_d(x) \in K[x]$ with $p_d(x) \neq 0$, such that

$$p_d(x)u^{(d)} + p_{d-1}(x)u^{(d-1)} + \dots + p_1(x)u' + p_0(x)u = 0 \quad (2.5)$$

where $u^{(j)} = d^j u / dx^j$.

3. There exist polynomials $q_0(x), \dots, q_m(x), q(x) \in K[x]$, with $q_m(x) \neq 0$, such that

$$q_m(x)u^{(m)} + q_{m-1}(x)u^{(m-1)} + \dots + q_1(x)u' + q_0(x)u = q(x) \quad (2.6)$$

If u satisfies any of the above three conditions, then u is a **D-finite power series**.

2.1.3.3 Examples of D-finite Series

- $u = e^x$ is D-finite, since $u' = u$.
- $y = \frac{1 - \sqrt{1 - 4x}}{2x}$ (the generating function of Catalan numbers), is D-Finite. Since

it satisfies a linear differential equation $1 + (-1 + 2x)y(x) + (-x + 4x^2)y'(x) = 0$.

2.1.3.4 P-recursive

A function $f: \mathbb{N} \rightarrow K$ is called **P-recursive** (polynomially recursive) if there exist polynomials $P_0, \dots, P_e \in K[n]$ with $P_e \neq 0$, and for all $n \in \mathbb{N}$, f satisfies a homogeneous linear recurrence of finite degree with polynomial coefficients, such that

$$p_e(n)f(n+e) + p_{e-1}(n)f(n+e-1) + \dots + p_0(n)f(n) = 0 \quad (2.7)$$

2.1.3.5 Proposition

Let $u = \sum_{n \geq 0} f(n)x^n \in K[[x]]$. Then u is D-Finite if and only if f is P-recursive.

2.1.3.6 Example

The n -th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$ is a P-recursive function, since

$$\frac{C_{n+1}}{C_n} = \frac{(2n+2)!}{(n+2)[(n+1)!]^2} \frac{(n+1)(n!)^2}{(2n)!} = \frac{2(2n+1)}{n+2}. \text{ Therefore, } y = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n \text{ is D-}$$

Finite.

2.1.3.7 Theorem ([20], Theorem 6.4.6)

Let $u \in K[[x]]$ be algebraic of degree d . Then u is *D-Finite*. In other words, u satisfies an equation (2.5) of order d , or satisfies an equation (2.6) of order $m=d-1$.

Proposition 2.1.3.5 and Theorem 2.1.3.7 together show that the coefficients of an algebraic power series satisfy the recurrence relation (2.7)

2.1.3.8 Example

The following example shows how to get a recurrence relation for the coefficients of a

generating function $y = \sum_{n \geq 0} f(n)x^n = \frac{1}{\sqrt{1-6x+x^2}}$

- Let $F(x) \in K(x)$ with $\text{char}K=0$ and $F(0)=1$. Suppose $F(x)=y^2$, i.e.,

$F(x)^{1/2} = y$. Then $F'(x) = (y^2)' = 2yy'$, multiplying by y to both sides yields

$$F'(x)y = 2F(x)y' \quad (2.8)$$

- Suppose $F(x) = A(x)/B(x)$, where $A, B \in K[x]$. For this example, $A(x) = 1$,

$B(x) = b_0 + b_1x + \dots + b_r x^r = 1 - 6x + x^2$. Then equation (2.8) becomes

$$\frac{(A'B - B'A)}{B^2} y = 2F(x)y' = \frac{2A}{B} y', \text{ i.e., } 2AB y' = (A'B - B'A)y. \quad (2.9)$$

- Equating coefficients of x^{n+r-1} in (2.9) yields $\sum_{j=0}^r b_{r-j}(2n+2j+r-j)f(n+j) = 0$,

where $r=2$, $b_0=1$, $b_1=-6$, and $b_2=1$. Then the linear recurrence for the

coefficients of $y = \sum_{n \geq 0} f(n)x^n$ is determined as

$$(n+1)f(n) - 3(2n+3)f(n+1) + (n+2)f(n+2) = 0.$$

In the next section, a Maple package called GFUN, used to manipulate generating functions, will be discussed. We will use GFUN to solve the previous example again.

Up to now, we have noticed that all algebraic series are D-finite. But not all D-finite series are algebraic. For example, e^x is D-finite, but not algebraic.

2.2 GFUN PACKAGE

GFUN package was designed by B. Salvy, P. Zimmermann and E. Murray as a tool to manipulate and discover generating functions. GFUN is a part of the Maple share library. It can be regarded as the forefather of the Mathematica package GeneratingFunctions.

This package has two parts. The first part is for computing with generating functions. For instance, given two generating functions defined by linear differential equations with polynomial coefficients, there is a procedure to compute the differential equation satisfied by their product. Given the first few terms of the sequence, the second part of GFUN contains functions that will help to determine the generating function. In some cases, the answer is “explicit”. In most cases, when such an explicit expression does not exist, a differential or algebraic equation satisfied by the generating function will be given.

The functions used in this paper to deal with generating functions are: `algfuntoalgeq`, `algeqtodiffeq`, `algeqtoseries` and `diffeqtorec`. Details of these four functions will be introduced later. The details of the GFUN package can be found in the Maple help file.

To load this package, type the following Maple commands in a Maple session.

```
> with( share ):
> with( gfun ):
```

2.2.1 GFUN[ALGFUNTOALGEQ] ([26])

Objective: find a polynomial equation satisfied by an algebraic function

Calling Sequence: `algfuntoalgeq(expr, y(z), ini, typ)`

Parameters:

`expr` - an algebraic or radical function in `z`

`y, z` - the variables to use in the resulting polynomial

`ini` - (optional) to be filled with a set of initial conditions to specify a solution of the resulting polynomial equation

`typ` - (optional) the type of coefficients to use in the polynomial equation. Must be either the string 'rational' or 'algebraic'. The default is 'rational'.

This function returns a polynomial in `y` and `z` that has **expr** as a root. If the parameter **ini** is used, a set of initial conditions will be assigned for the polynomial.

2.2.1.1 Example

```
>algfuntoalgeq(1/12*(sqrt(1-12*x)-1)^2/((1-12*x)*(sqrt(1-12*x)+2)),y(x));
(3-60x + 144 x^2 + 1728 x^3 )y^2 + (-1 - 96 x^2 + 20 x)y + x^2
```

2.2.2 GFUN[ALGEQTODIFFEQ] ([26])

Objective: compute a differential equation for an algebraic function

Calling Sequence: `algeqtodiffeq(p, y(z), inits)`

Parameters:

`p` - a polynom in `y` and `z` (or a polynomial equation)

`y` - the name of the algebraic function

`z` - the generic variable

`inits` - (optional) a set of initial conditions

The polynomial p defines an algebraic function, $\text{RootOf}(p, y)$ in Maple terms. This function computes a linear differential equation with polynomial coefficients verified by the function $y(z)$. This equation is of order at most $\text{degree}(p, y)-1$.

The output contains initial conditions in zero ($y(0)$, $D(y)(0)$, etc.) if **`inits`** is specified.

Comtet³ pointed out the algorithm used in `algeqtodiffeq[ ]`. Given an irreducible polynomial P with two variables, this function computes a holonomic equation satisfied by any function $y(z)$, solution of $P(z, y(z))=0$. Salvy and Zimmermann [19] shows it works in the following steps:

- Differentiating $P(z, y(z))$ with respect to z yields $y'(z) = A(y, z)/B(y, z)$, with A and B two polynomials.
- Since P is irreducible, P and B are relatively prime. The extended gcd algorithm produces three polynomials $u(y, z)$, $v(y, z)$ and $g(z)$, where $uB + vP = g$. C is

$$\text{defined as } C = Au \bmod P \text{ and deduce } y'(z) = \frac{C(y, z)}{g(z)} \quad (2.10)$$

- Differentiating equation (2.10) repetitively and reducing the right-hand side by means of (2.10) to suppress y' and modulo P to eliminate large powers of y . After several steps of repetitions (the number of steps is bounded by $\text{degree}(p, y)$), we obtain many linearly equations to eliminate the powers of y by Gaussian elimination. This yields the desired equation.

³ Comtet, L. Calcul pratique des coefficients de Taylor d'une fonction algebrique. *L'Enseignement Mathematique* 10 (1964), 267-270

Note that if P is not irreducible, a recursive splitting can be used to separate it to several irreducible components, then **algeqtodiffeq** is applied to each of them.

2.2.2.1 Example

```
>eq:=(3-60 x + 144 x^2 + 1728 x^3)y^2+(-1 - 96 x^2 + 20 x)y + x^2:
>algeqtodiffeq(eq,y(x),{y(0)=0});
{D(y)(0)= 0,y(0)= 0,16 x^2 + 2x +(576 x^3 -1056 x^2 + 84x)y(x)
+ (-1 + 22 x - 88 x^2 - 480 x^3 + 1152 x^4 )y'(x) }
```

2.2.3 GFUN[DIFFEQTOREC] ([26])

Objective: convert a linear differential equation into a recurrence

Calling Sequence: `diffeqtorec(deqns, y(z), u(n))`

Parameters:

- `deqns` - a linear differential equation with polynomial coefficients or a set of one differential equation and initial conditions
- `y, z` - the name and the variable of the function
- `u, n` - the name and the index of the recurrence

Let f be a power series solution of the **deqns**. If $u(n)$ is the n -th Taylor coefficient of f around zero, this procedure converts a holonomic equation $c_k(z) y^{(k)}(z) + \dots + c_1(z) y'(z) + c_0(z)y(z) + b(z) = 0$ for the function $y(z)$ into a linear recurrence for the numbers $u(n)$, with rational coefficients in n . The algorithm of this procedure works by translating each $z^k y^{(j)}(z)$ into $\frac{(n-k+j)!}{(n-k)!} u_{n+j-k}$, and the inhomogeneous term $b(z)$ yields the initial conditions of the recurrence.

2.2.3.1 Example ([19]): a “real” problem in the news group sci.math:

```

>eq:=(335*t^2+1290)*diff(f(t),t,t)+1540*t*diff(f(t),t)+4687
    20*f(t)=544;
eq:=(335 t^2 +1290)f''(t)+ 1540t f'(t)+ 468720f(t)= 544
> diffeqtorec(eq,f(t),u(n));

{(93744+241n+67n^2)u(n)+(258n^2+774n+516)u(n+2), u(0) = _C[0],
u(1) = _C[1], u(2) = -7812/43_C[0]+136/645}

```

Therefore the Taylor coefficients u_n of $f(t)$ satisfy the recurrence as follows:

$$u_{n+2} = \frac{241n + 67n^2 + 93744}{516 + 258n^2 + 774n} u_n, \text{ for } n \geq 1,$$

2.2.4 GFUN[ALGEQTOSERIES] ([26])

Objective: Puiseux expansions of algebraic functions

Calling Sequence: `algeqtoseries(pol, x, y, order, pos_slopes)`

Parameters: `pol` - a polynomial equation in two variables x and y

`x, y` - variable names

`order` - positive integer: order of the expansions

`pos_slopes` - (optional) only branches tending to 0 are computed

2.2.4.1 Example

```

> algeqtoseries((3-60*x+144*x^2+1728*x^3)*y^2+(-96*x^2-
1+20*x)*y+x^2,x,y,8);
[1/3+15x^2 + 108x^3 + 1485x^4 + 16200x^5 + 192942x^6 + 2245320x^7
+ O(x^8), x^2 + 20 x^3 + 307 x^4 + 4280 x^5 + 56914x^6 + 736568 x^7
+ 9370183x^8 + 117822512x^9 + O(x^10)]

```

2.2.5 EXAMPLE SOLVED BY GFUN

The generating function $y = \frac{1}{\sqrt{1-6x+x^2}}$ (same as the example 2.1.3.8) is used to illustrate the idea of how to find a linear recurrence for the coefficients of a generating function by GFUN.

- This is a simple generating function. Then we can obtain the linear differential equation directly by differentiating y with respect to x , which yields

$$y' = \frac{-(\sqrt{1-6x+x^2})'}{1-6x+x^2} = \frac{\frac{-1}{2\sqrt{1-6x+x^2}}(-6+2x)}{1-6x+x^2} = \frac{-y(-3+x)}{1-6x+x^2}.$$

So the resulting linear differential equation is $y'(1-6x+x^2)+y(-3+x)=0$ i.e.,

$$x^2y' - 6xy' + y' - 3y + yx = 0. \quad (2.11)$$

- Translating each $x^k y^{(j)}(x)$ of (2.11) into $\frac{(n-k+j)!}{(n-k)!} u_{n+j-k}$ yields

$$\frac{(n-1)!}{(n-2)!} u_{n-1} - 6 \frac{(n)!}{(n-1)!} u_n + \frac{(n+1)!}{(n)!} u_{n+1} - 3 \frac{n!}{n!} u_n + \frac{(n-1)!}{(n-1)!} u_{n-1} = nu_{n-1} - 3(2n+1)u_n + (n+1)u_{n+1}$$

Substituting $n-1$ with n , we have $(n+1)u_n - 3(2n+3)u_{n+1} + (n+2)u_{n+2} = 0$, the same

linear recurrence relation as found previously in section 2.1.3.8.

According to Salvy and Zimmermann [19], “The fastest known method to compute coefficients of an algebraic function is to use successively *algeqtodiffeq*, *diffeqtorec* and *rectoproc*.” (the procedure outputs a Maple procedure that, given a nonnegative integer n as the input, gives the n -th term $u(n)$ of the linear recurrence).

CHAPTER 3 BASIC ALGORITHMS

In the study of combinatorial mathematics, it is common to see a sum of expression involving binomial coefficients, factorials, rational functions and power series. Usually people want to evaluate the complex expression mechanically and try to represent it in a simple “**hypergeometric closed form**”, which only involves a fixed number of hypergeometric terms. This is the main objective of this thesis. The followings are two marvelous hypergeometric identities.

$$\sum_{j=0}^n \binom{n}{j}^2 = \binom{2n}{n}$$

$$\sum_{i+j+k=n} \binom{i+j}{i} \binom{j+k}{j} \binom{k+i}{k} = \sum_{0 \leq j \leq n} \binom{2j}{j}$$

Thanks to the works of Sister Celine, Gosper, Zeilberger, and Petkovšek, the task of determining whether a given hypergeometric sum is expressible in a simple closed form and, if so, finding the form, is now completely solved. Furthermore, it can be solved by mathematical programs. The underlying mathematical theories and Maple programs of these methods will be discussed in this chapter. Algorithms are accompanied by theorems that describe circumstances under which these algorithms are guaranteed to work.

The first algorithm is Sister Celine’s method, which was generalized by Zeilberger; the second is Gosper’s algorithm; Zeilberger’s algorithm is the third; and Algorithm Hyper is the last. The main theme that runs through these methods is *recurrence*, which is derived by using functions of the GFUN Maple package.

3.1 HYPERGEOMETRIC SERIES AND CLOSED FORM

Hypergeometric series:

A geometric series $\sum_{k \geq 0} t_k$ is a series for which the ratio of every two consecutive terms

t_{k+1}/t_k is a constant function of the summation index k . A more general case is that the

ratio t_{k+1}/t_k is a rational function of the summation index k . It produces a series called a

hypergeometric series, with $t_0 = 1$ and $\frac{t_{k+1}}{t_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\dots(k+a_p)}{(k+b_1)(k+b_2)\dots(k+b_q)(k+1)} x$,

where x is an arbitrary constant and P, Q are polynomials in k . t_k of a given

hypergeometric series $\sum_{k \geq 0} t_k x^k$ is called a hypergeometric term.

Hypergeometric closed form([18], pp151):

A function $f(n)$ is said to be of closed form if it can be expressed as a linear combination of a fixed number, say r , of hypergeometric terms. The number r must be an absolute constant, i.e., it must be independent of all variables and parameters of the problem.

3.2 SISTER CELINE'S METHOD

The idea of algorithmically finding a recurrence relation for hypergeometric terms and solving the recurrence explicitly first came from the Ph.D. thesis of Sister Mary Celine Fasemyer in 1945. The method is effective and easy to implement. Although it has been superseded by faster ones, Sister Celine's method contains the original idea on which later algorithms are built.

We start by illustrating this method with a simple example. Please note that we follow the summation range of [18]. Throughout this thesis, if we do not specify the range of

summation, the sums are considered to extend over all integers, including positive and negative. However, for equations like $\sum_k k \binom{n}{k}$, the binomial coefficient $\binom{n}{k}$ vanishes if $k < 0$ or $k > n \geq 0$; which means only finitely many terms contribute.

3.2.1 EXAMPLE¹

Let $f(n) = \sum_k k \binom{n}{k}$ ($n = 0, 1, 2, \dots$), we try to find a “closed form” of $f(n)$.

- First step is to look for a recurrence relation for summand $F(n, k) = k \binom{n}{k}$, which is a function of two variables (n, k) . Assume the recurrence takes the form as

$$a(n)F(n, k) + b(n)F(n+1, k) + c(n)F(n, k+1) + d(n)F(n+1, k+1) = 0. \quad (3.1)$$

- Divide (3.1) by $F(n, k)$ and get the following equation (3.2)

$$a + b \frac{F(n+1, k)}{F(n, k)} + c \frac{F(n, k+1)}{F(n, k)} + d \frac{F(n+1, k+1)}{F(n, k)} = a + b \frac{n+1}{n+1-k} + c \frac{n-k}{k} + d \frac{n+1}{k} = 0.$$

Putting (3.2) in a common denominator $k(n+1-k)$ and then collecting by the power of k yields the numerator

$$(d + (c+2d)n + (c+d)n^2) + (a + b - c - d + (a + b - 2c - d)n)k + (c - a)k^2. \quad (3.3)$$

- The assumed recurrence (3.1) will exist if this numerator identically vanishes, for all n and k . Since the coefficients a, b, c, d of (3.1) are permitted to depend on n only, the coefficient of (3.3) for each power of k must vanish. This gives a system of three equations in four unknowns.

¹ Marko Petkovšek, Herbert S. Wilf, Doron Zeilberger A=B pp55

$$\begin{bmatrix} 0 & 0 & n(n+1) & (n+1)^2 \\ n+1 & n+1 & -2n-1 & -(n+1) \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.4)$$

The solution of (3.4) gives the values of the coefficients and ensures that the assumed “ k -free” recurrence that formed as (3.1) exists, namely

$$-\left(1+\frac{1}{n}\right)F(n,k) - \left(1+\frac{1}{n}\right)F(n,k+1) + F(n+1,k+1) = 0. \quad (3.5)$$

- The final step is to sum (3.5) over all integers k . Since the coefficients are free of k 's, it is easy to get the recurrence of $f(n)$ from the recurrence of the summand. Then we can solve the recurrence of the sum and get the evaluation of the sum in closed form or prove that no such closed form exists. Here, we have

$$\begin{aligned} -\left(1+\frac{1}{n}\right)f(n) - \left(1+\frac{1}{n}\right)f(n) + f(n+1) &= 0 \\ f(n+1) &= 2\frac{n+1}{n}f(n) = 2^2\frac{n+1}{n}\frac{n}{n-1}f(n-1) = \dots = 2^n(n+1)f(1), \end{aligned}$$

($n=1, 2, \dots$; $f(1)=1$). So $f(n) = n2^{n-1}$, for all $n \geq 0$.

3.2.2 SISTER CELINE'S GENERAL ALGORITHM

Given a sum $f(n) = \sum_k F(n,k)$, Sister Celine's method operates by finding a recurrence

$$\text{for } F(n,k) \text{ of the form } \sum_{i=0}^I \sum_{j=0}^J a_{i,j}(n) F(n-j, k-i) = 0. \quad (3.6)$$

It proceeds as follows ²:

1. Fix trial values of I and J .

² Marko Petkovšek, Herbert S. Wilf, Doron Zeilberger A=B pp59

2. Assume a recurrence formula in the form of (3.6), where the coefficients $a_{i,j}(n)$ are to be solved, if possible.
3. Divide each term of (3.6) by $F(n,k)$ and reduce every ratio $F(n-j, k-i)/F(n,k)$ by simplifying the ratios of its constituent factorials so that only rational functions in n and k remain.
4. Put the resulting expression over a common denominator, then collect the numerator as a polynomial in k .
5. Solve the system of linear equations that results after setting the coefficients of each power of k in the numerator to 0 for the unknown coefficients $a_{i,j}$.
6. If no solution results, repeat step 1 to step 5 with larger I and (or) J .

The Fundamental theorem guarantees that this algorithm always succeeds when a certain hypothesis is satisfied for large enough I and J (which can be estimated in advance).

3.2.3 THE FUNDAMENTAL THEOREM

Definition: A function $F(n, k)$ is said to be a *proper hypergeometric term* if it can be written in the form

$$F(n, k) = P(n, k) \frac{\prod_{i=1}^{uu} (a_i n + b_i k + c_i)!}{\prod_{i=1}^{vv} (u_i n + v_i k + w_i)!} x^k \quad (3.7)$$

where x is an indeterminate over the complex numbers, and

1. $P(n, k)$ is a polynomial in n and k ,
2. the a 's, b 's, u 's, v 's are specific integers,
3. the quantities uu , vv are finite, nonnegative, and specific integers.

A function $F(n, k)$ of the form as (3.7) is well defined at a point (n, k) when none of the numbers $\{a_i n + b_i k + c_i\}_{i=1}^m$ is negative.

3.2.3.1 Theorem

Let $F(n, k)$ be a proper hypergeometric term. Then F satisfies a k -free recurrence relation.

In other words, the recurrence $\sum_{i=0}^I \sum_{j=0}^J a_{i,j}(n) F(n-j, k-i) = 0$ exists and holds for every point (n, k) , at which $F(n, k) \neq 0$. In this recurrence, I and J are positive integers, and $a_{i,j}(n)$, for $i=0, \dots, I$, $j=0, \dots, J$ are not all zero polynomials. All the values of $F(n, k)$ in this recurrence are well defined. Furthermore, there is a recurrence with $(I, J) = (I^*, J^*)$, where $J^* = \sum_s |b_s| + \sum_s |v_s|$, and

$$I^* = 1 + \deg(P) + J^* \left(\left\{ \sum_s |a_s| + \sum_s |u_s| \right\} - 1 \right). \quad (I^* \text{ and } J^* \text{ are the estimation of } I \text{ and } J.)$$

3.2.4 MAPLE PROGRAM

The Maple program for Sister Celine's method, `celine()`, is contained in Maple Package **EKHAD** that is available at <http://www.cis.upenn.edu/~wilf/AeqB.html>. The calling sequence of Sister Celine's method is `celine(f, ii, jj)`, where f is the summand, and ii , jj are the sizes of the recurrence that we assumed.

Here is the output for the same question as example (3.2.1) by using procedure `celine()`.

```
>Celine((n,k)->k*n!/(k!*(n-k)!),1,1);
b[3] n F(n-1,k) - (n-1) b[3] F(n,k) + b[3] F(n-1,k-1) == 0
```

In this output, $b[3]$ can be ignored since $b[3]$ is an arbitrary constant. And the result by Maple is identical to (3.5) of example (3.2.1) after replacing n and k by $n-1$ and $k-1$ in (3.5)

3.3 GOSPER'S ALGORITHM

Gosper's algorithm is one of the landmarks in the history of computerization of solving closed form summation. It not only completely solves the problem of indefinite hypergeometric summation (called indefinite because the upper limit of summation is not a fixed number), but also is vital in the procedures of the creative telescoping algorithm.

Given a sum $s_n = \sum_{k=0}^{n-1} t_k$, where t_k is a hypergeometric term that depends on k only (hence the term ratio $r(k) = t_{k+1}/t_k$ is a rational function of k), Gosper's algorithm goes through a decision procedure of finding a given hypergeometric term t_k has a hypergeometric antidifference, i.e., whether there is a hypergeometric term z_n such that

$$z_{n+1} - z_n = t_n. \quad (3.8)$$

If we can find such a z_n successfully, t_n is called *Gosper-summable*, and we can deduce s_n from $z_n = z_{n-1} + t_{n-1} = z_{n-2} + t_{n-2} + t_{n-1} = \dots = z_0 + \sum_{k=0}^{n-1} t_k = s_n + c$, where $c = z_0$ is a constant. Otherwise no hypergeometric term antidifference exists. According to [18, Theorem 5.6.3], if Gosper's algorithm does not return a solution successfully, it has proved that no such closed form exists for the given sum.

3.3.1 HYPERGEOMETRICS TO RATIONALS TO POLYNOMIALS

For a hypothetical hypergeometric term z_n that satisfies equation (3.8), the ratio of

$$\frac{z_n}{t_n} = \frac{z_n}{z_{n+1} - z_n} = \frac{1}{\frac{z_{n+1}}{z_n} - 1} \quad (3.9)$$

is a rational function of n .

Assume $y(n)$ is an unknown rational function of n , and let $z_n = y(n)t_n$. Then $z_{n+1} - z_n = t_n$ can be written as

$$r(n)y(n+1) - y(n) = 1. \quad (3.10)$$

The problem of finding hypergeometric solutions of (3.8) is now reduced to finding rational solutions of $y(n)$ for a first-order linear recurrence (3.10). Gosper reduced this problem further by finding polynomial solutions to another first-order recurrence.

$$\text{Assume } r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}, \quad (3.11)$$

where $a(n)$, $b(n)$, $c(n)$ are polynomials and

$$\gcd(a(n), b(n+h)) = 1, \text{ for all } h \in \mathbb{N}. \quad (3.12)$$

Gosper suggested to look for a nonzero solution taking the form as

$$y(n) = \frac{b(n-1)x(n)}{c(n)}, \quad (3.13)$$

where $x(n)$ is an unknown rational function of n . Substituting (3.11) and (3.13) into (3.10), then we get

$$a(n)x(n+1) - b(n-1)x(n) = c(n). \quad (3.14)$$

Since $x(n)$ turns out to be a polynomial solution³, it is easy to solve coefficients $a(n)$, $b(n)$ and $c(n)$. If $x(n)$ is nonzero, then there is a hypergeometric term antidifference

$$z_n = \frac{b(n-1)x(n)}{c(n)} t_n. \text{ Thus the procedure of finding hypergeometric solutions of}$$

$z_{n+1} - z_n = t_n$ has become a procedure of finding polynomial solutions of (3.14).

³ Marko Petkovšek, Herbert S. Wilf, Doron Zeilberger A=B pp76, Theorem 5.2.1

3.3.2 THE FULL ALGORITHM

3.3.2.1 Theorem

Let K be a field of characteristic zero and $r \in K[n]$ be a nonzero rational function. Then r

can always be written as a unique factorization $r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}$, with polynomials $a, b, c \in K[n]$, where

- (1) $\gcd(a(n), b(n+h)) = 1$, for every nonnegative integer h ,
- (2) $\gcd(a(n), c(n)) = 1$,
- (3) $\gcd(b(n), c(n+1)) = 1$.

3.3.2.2 Gosper's Algorithm:

INPUT: A hypergeometric term t_n

1. Make the consecutive term ratio $r(n) = t_{n+1}/t_n$, which is a rational function.
2. Try to factorize $r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}$ by the following two steps, where $a(n), b(n), c(n)$ are polynomials satisfying $\gcd(a(n), b(n+h)) = 1$, for all $h \in \mathbb{N}$.

2.1 Let $r(n) = Z \frac{f(n)}{g(n)}$, where f, g are monic relatively prime polynomials and Z is a

constant.

Let $R(h) := \text{Resultant}_n(f(n), g(n+h))$; $S = \{h_1, h_2, \dots, h_N\}$ be the set of nonnegative integer zeros of $R(h)$ ($N \geq 0, 0 \leq h_1 \leq h_2 \leq \dots \leq h_N$).

2.2 $p_0(n) := f(n); q_0(n) := g(n)$;

for $j=1, 2, \dots, N$ do: $s_j(n) := \gcd(p_{j-1}(n), q_{j-1}(n+h_j))$;

$$p_j(n) := p_{j-1}(n) / s_j(n);$$

$$q_j(n) := q_{j-1}(n) / s_j(n - h_j).$$

$$a(n) := Zp_N(n); b(n) := q_N(n); c(n) := \prod_{i=1}^N \prod_{j=1}^{h_i} s_i(n - j).$$

3. Try to get nonzero polynomial solutions $x(n)$ of $a(n)x(n+1) - b(n-1)x(n) = c(n)$.

3.1 If $\deg a(n) \neq \deg b(n)$ or $\text{lc}^4 a(n) \neq \text{lc} b(n)$ then

$$D := \{\deg c(n) - \max\{\deg a(n), \deg b(n)\}\}$$

else let A and B be as in the following equations respectively;

$$a(n) = \lambda n^k + A n^{k-1} + O(n^{k-2}), \quad b(n-1) = \lambda n^k + B n^{k-1} + O(n^{k-2})$$

$$D := \{\deg c(n) - \deg a(n) + 1, (B - A) / \text{lca}(n)\}.$$

Let $D := D \cap \mathbb{N}$, If $D = \emptyset$ then return “no nonzero polynomial solution exists”

and stop, else return $d := \max D$.

3.2 Using the method of undetermined coefficients, get a nonzero polynomial solution $x(n)$ of (3.14), of degree d or less. If no such solution exists, return “no nonzero polynomial solution” and stop.

4. Return $\frac{b(n-1)x(n)}{c(n)} t_n$ and stop if a nonzero $x(n)$ exists.

3.3.3 Example and Maple program [18]

3.3.3.1 Example

Let $S_n = \sum_{k=0}^n (4k+1) \frac{k!}{(2k+1)!}$. Can this sum be expressed in closed form?

⁴ $\text{lc}(p)$ is the leading coefficient of polynomial p

- Let $t_n = (4n+1) \frac{n!}{(2n+1)!}$, then the rational ratio is $r(n) = \frac{t_{n+1}}{t_n} = \frac{(4n+5)}{2(4n+1)(2n+3)}$. It is

easy to get the satisfying $a(n)$, $b(n)$, $c(n)$ that $a(n)=1$, $b(n)=2(2n+3)$, $c(n)=4n+1$.

Thus, $y(n) = \frac{2(2n+1)}{4n+1} x(n)$.

- Then we have $z_n = y(n)t_n = \frac{2(2n+1)}{4n+1} (4n+1) \frac{n!}{(2n+1)!} x(n)$, where $x(n)$ is a

polynomial in n . There is a recurrence for $x(n)$: $x(n+1) - 2(2n+1)x(n) = 4n+1$.

We use the method of undertermined coefficients⁵ to solve the recurrence. This is a

simple example, we get $x(n) = -1$ and $z_n = -2 \frac{n!}{(2n)!}$ immediately.

- Now, we know that $S_n = \sum_{k=0}^n (4k+1) \frac{k!}{(2k+1)!}$ has a closed form, $S_n = 2 - \frac{n!}{(2n+1)!}$.

3.3.3.2 Maple program

Gosper's Algorithm uses the built-in Maple function **sum()** to handle summation.

```
> sum (4^k / binomial(2*k, k), k=0..n);
> (2/3*n+1/3)*4^(n+1)/binomial(2*n+2, n+1)+1/3
```

3.4 ZEILBERGER'S ALGORITHM

Zeilberger's algorithm, also called the method of *creative telescoping* by van der Poorten, finds a recurrence relation for a given hypergeometric summand $F(n, k)$. Basically, it solves the same problem that Sister Celine's algorithm does. However the form of recurrence derived from Zeilberger's algorithm is different, and Zeilberger's algorithm

⁵ For a degree bound d , make a generic polynomial of degree d , substitute it into the recurrence. Then equate the coefficients of same power of n , and solve the resulting linear algebraic equations of $d+1$ unknown coefficients.

works much faster. Furthermore, this method, which plays an important role in the study of definite sums, expresses the sum as a function of a variable upper limit.

3.4.1 HOW THE ALGORITHM WORKS

A notation has to be mentioned first. Let N, K denote the forward shift operator in arguments n and k respectively. For example, $N g(n, k) = g(n+1, k)$, $K g(n, k) = g(n, k+1)$.

In order to get a recurrence relation for a given a sum $f(n) = \sum_k F(n, k)$, where $F(n, k)$ is a hypergeometric term in n and k , we first try to find a recurrence relation for the summand $F(n, k)$. A theorem, which will be discussed later, guarantees that we can “always” find a recurrence such as

$$p(n, N)F(n, k) = (K - 1)G(n, k). \quad (3.15)$$

where $G(n, k)$ is a hypergeometric term and the **difference operator**

$p(n, N) = a_0(n) + a_1(n)N + a_2(n)N^2 + \dots + a_J(n)N^J$ and the coefficients $\{a_i(n)\}_0^J$ are polynomials in n . Equation(3.15) can be expressed as

$$\sum_{j=0}^J a_j(n)F(n+j, k) = G(n, k+1) - G(n, k). \quad (3.16)$$

Zeilberger’s Algorithm gives us not only a recurrence for the summand but also a rational function $R(n, k)$, where $G(n, k) = R(n, k)F(n, k)$. Since the coefficients $\{a_i(n)\}_0^J$ are free of k , summing both sides of (3.16) over all integer values of k brings up

$$\sum_{j=0}^J a_j(n)f(n+j) = 0. \quad (3.17)$$

There are several possible scenarios for equation (3.17).

- $J=1$

$a_0(n)f(n) + a_1(n)f(n+1) = 0$ is a first-order recurrence with polynomial coefficients.

Then our desired sum $f(n) = f(0) \prod_{j=0}^{n-1} (-a_0(j)/a_1(j))$ is a hypergeometric term.

- $J > 1$

Even in this case, we may be lucky to find the coefficients $\{a_i(n)\}_0^J$ to be constant.

The hypergeometric solution of $f(n)$ is easy to obtained. If it is not the case, we can still handle the complex situation by using the Algorithm Hyper of Petovšek, which may give us a solution as a linear combination of a fixed number of hypergeometric terms in n or return “No simple evaluation exists”.

3.4.2 EXISTENCE OF THE RECURRENCE AND IMPLEMENTATION

Zeilberger’s algorithm makes strong use of the assumed recurrence of the form

$$\sum_{j=0}^J a_j(n)F(n+j, k) = G(n, k+1) - G(n, k),$$

which is guaranteed to exist when the following theorem is satisfied.

3.4.2.1 Theorem

If $F(n, k)$ is a proper hypergeometric term. Then F satisfies a nontrivial recurrence of the form $\sum_{j=0}^J a_j(n)F(n+j, k) = G(n, k+1) - G(n, k)$, and $R(n, k) = G(n, k)/F(n, k)$ is a rational function of n and k .

3.4.2.2 Implementation

The creative telescoping algorithm gets an enormous speed up from its complicated implementation. At the very beginning, we only know a recurrence relation (3.16) exists.

The coefficients $\{a_i(n)\}_0^J$ on the left side are unknown, the order J of the recurrence is

unknown (the Fundamental Theorem only gives an upper bound), and $G(n, k)$ of the right side of the (3.16) is unknown.

- First fix an assumed order J , and then look for a recurrence of that order. If not successful, try a higher order until a solution is found or the upper bound is reached.

Let t_k denote $a_0 F(n, k) + a_1 F(n+1, k) + \dots + a_J F(n+J, k)$, then the term ratio is

$$\frac{t_{k+1}}{t_k} = \frac{\sum_{j=0}^J a_j F(n+j, k+1) / F(n, k+1)}{\sum_{j=0}^J a_j F(n+j, k) / F(n, k)} \frac{F(n, k+1)}{F(n, k)}. \quad (3.18)$$

Assume $\frac{F(n, k+1)}{F(n, k)} = \frac{r_1(n, k)}{r_2(n, k)}$, $\frac{F(n, k)}{F(n-1, k)} = \frac{s_1(n, k)}{s_2(n, k)}$, r 's and s 's are polynomials.

$$\bullet \quad \frac{F(n+j, k)}{F(n, k)} = \prod_{i=0}^{j-1} \frac{F(n+j-i, k)}{F(n+j-i-1, k)} = \prod_{i=0}^{j-1} \frac{s_1(n+j-i, k)}{s_2(n+j-i, k)} \quad (3.19)$$

$$\begin{aligned} \frac{t_{k+1}}{t_k} &= \frac{\sum_{j=0}^J a_j \left\{ \prod_{i=0}^{j-1} \frac{s_1(n+j-i, k+1)}{s_2(n+j-i, k+1)} \right\} r_1(n, k)}{\sum_{j=0}^J a_j \left\{ \prod_{i=0}^{j-1} \frac{s_1(n+j-i, k)}{s_2(n+j-i, k)} \right\} r_2(n, k)} = \\ &= \frac{\sum_{j=0}^J a_j \left\{ \prod_{i=0}^{j-1} s_1(n+j-i, k+1) \prod_{r=j+1}^J s_2(n+r, k+1) \right\}}{\sum_{j=0}^J a_j \left\{ \prod_{i=0}^{j-1} s_1(n+j-i, k) \prod_{r=j+1}^J s_2(n+r, k) \right\}} \times \frac{r_1(n, k) \prod_{r=1}^J s_2(n+r, k)}{r_2(n, k) \prod_{r=1}^J s_2(n+r, k+1)} \end{aligned}$$

$$\text{Then we have } \frac{t_{k+1}}{t_k} = \frac{p_0(k+1) r(k)}{p_0(k) s(k)}, \quad (3.20)$$

$$\text{where } p_0(k) = \sum_{j=0}^J a_j \left\{ \prod_{i=0}^{j-1} s_1(n+j-i, k) \prod_{r=j+1}^J s_2(n+r, k) \right\} \quad (3.21)$$

$$\text{and } r(k) = r_1(n, k) \prod_{r=1}^J s_2(n+r, k), \quad s(k) = r_2(n, k) \prod_{r=1}^J s_2(n+r, k+1).$$

Note that only $p_0(k)$ has the assumed coefficients a_j .

- By the theorem (3.3.2.1), $\frac{r(k)}{s(k)} = \frac{p_1(k+1)}{p_1(k)} \frac{p_2(k)}{p_3(k)}$, $p_2(k)$ and $p_3(k)$ are co-prime.

$\gcd(p_2(k), p_3(k+j)) = 1$, for all nonnegative integers j . Thus, we can factorize

$$\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \frac{p_2(k)}{p_3(k)}, \text{ where } p(k) = p_0(k)p_1(k). \quad (3.22)$$

- Now the original question becomes a standard procedure that Gosper's Algorithm can handle. t_k turns out to be an indefinitely summable hypergeometric term if and only if the recurrence

$$p_2(k)b(k+1) - p_3(k-1)b(k) = p(k) \quad (3.23)$$

has a polynomial solution $b(k)$.

With the help of Gosper's algorithm and method of undetermined coefficients, we obtain a polynomial solution $b(k)$ of (3.23) if it does exist. After solving all

coefficients $\{a_i(n)\}_0^J$, we obtain $G(n, k) = \frac{p_3(k-1)}{p(k)} b(k) t_k$. If there are no solutions of

$b(k)$ for this assumed J , and J has not reached the bound, we have to repeat the procedure of solving $b(k)$ for a higher order J .

3.4.3 EXAMPLE AND MAPLE PROGRAM

Here we show an example handled by the Maple program `ct()` in the EKHAD package.

Given a sum $f(n) = \sum_k (-1)^k \binom{n}{k} \left/ \binom{x+k}{k} \right.$,

```
>ct((-1)^k*(binomial(n,k)/binomial(x+k,k)),1,k,n,N);
(n+x+(-n-1-x)N)F(n,k)=G(n,k+1)-G(n,k)
```

where $G=RF$ and $R(n, k) = k(x+k) / (n+1-k)$.

Summing over k , we get $f(n+1) = \frac{n+x}{n+x+1} f(n)$, which together with $f(0) = 1$, gives

the desired closed form as $f(n) = \frac{x}{(x+n)}$.

3.5 ALGORITHM HYPER

After applying Zeilberger's algorithm to the given sum $f(n) = \sum_k F(n, k)$, we get a linear recurrence with polynomial coefficients that the given sum $f(n)$ satisfies. If this recurrence is of order $J = 0$ or $J = 1$, or of order $J > 1$ and coefficients are constant, the closed form solution will pop out immediately. For other complicated cases where there is no apparent solution, no answer could be found until Petkovšek's algorithm Hyper came out. Algorithm Hyper is a decision procedure to determine all hypergeometric solutions of a given recurrence relation with polynomial coefficients that the given sum satisfies. Algorithm Hyper completes the job of computing a "closed form" together with Zeilberger's algorithm when such a closed form does exist. It also can prove that there is no such "closed form" if algorithm Hyper cannot get one.

3.5.1 HOW HYPER WORKS

Let K be a field of characteristic of zero and $K^{\mathbb{N}}$ denotes the set of all sequences $(a(n)_{n=0}^{\infty})$ whose terms belong to K .

3.5.1.1 Linear recurrence operator

First, we use the following examples to illustrate Shift operator $N: K^{\mathbb{N}} \rightarrow K^{\mathbb{N}}$.

- $N(a(0), a(1), \dots) = (a(1), a(2), \dots)$

- $(Na)(n)=a(n+1)$

The linear recurrence operator on K^{IN} takes the form as $L = \sum_{k=0}^r a_k N^k$, where $a_k \in K^{IN}$.

A linear recurrence relation in K^{IN} can be represented as $Ly = f$. If $a_r \neq 0$ and $a_0 \neq 0$, order of L is r . If $f = 0$, $Ly = f$ is homogeneous, otherwise it is inhomogeneous. It has to be converted to be homogeneous before algorithm Hyper is applied to it.

3.5.1.2 Seeking hypergeometric solutions

- Let F be a field of characteristic zero and K be an extension field of F . Try to solve $Ly = 0$, where L is a linear recurrence operator with polynomial coefficients over F .

First consider a simple case, a second order homogeneous recurrence

$$p(n)y(n+2) + q(n)y(n+1) + r(n)y(n) = 0. \quad (3.24)$$

Assume that $y(n)$ is a hypergeometric solution of equation (3.24), then the term ratio

$S(n) = y(n+1)/y(n)$ is a rational function. Substituting it into (3.24), we have

$$p(n)S(n+1)S(n) + q(n)S(n) + r(n) = 0. \quad (3.25)$$

According to Theorem (3.3.2.1), $S(n)$ can be factorized as $S(n) = z \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}$,

where $z \in K \setminus \{0\}$ and a, b, c are monic polynomials⁶ satisfying the three properties of

Theorem (3.3.2.1). Then $p(n)S(n+1)S(n) + q(n)S(n) + r(n) = 0$ is converted to

$$z^2 p(n)a(n+1)a(n)c(n+2) + zq(n)b(n+1)a(n)c(n+1) + r(n)b(n+1)b(n)c(n) = 0. \quad (3.26)$$

Cancel $a(n)b(n+1)$ from two sides of equation (3.26) to obtain

⁶ A polynomial $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, in which the coefficient of the highest order term is 1.

$$z^2 \frac{p(n)}{b(n+1)} a(n+1)c(n+2) + zq(n)c(n+1) + \frac{r(n)}{a(n)} b(n)c(n) = 0. \quad (3.27)$$

The leading coefficients of the left side of (3.27) and the choice of $a(n)$ and $b(n)$ (Theorem 3.3.2.1 leave us a finite set of candidates) help to determine the values of z .

- For each possible triple of fixed $a(n)$, $b(n)$ and z , Algorithm Poly helps to find nonzero polynomial solutions $c(n)$ of (3.27). If there is a nonzero solution of $c(n)$, which means (3.24) has a hypergeometric solution. If the algorithm cannot find a nonzero solution, the given sum cannot be expressed in closed form.
- Check the solutions we obtained to see if they agree with the initial conditions of sum $f(n)$ for enough consecutive values of n .

3.5.1.3 Algorithm Hyper ([18]-page 152)

INPUT: Polynomials $p_i(n)$ over F , for $i = 0, 1, \dots, d$; an extension field K of F .

OUTPUT: A hypergeometric solution or a proof that no such solution exists.

1. For all monic factors $a(n)$ of $p_0(n)$ and $b(n)$ of $p_d(n-d+1)$ over K do:

$$P_i(n) := p_i(n) \prod_{j=0}^{i-1} a(n+j) \prod_{j=i}^{d-1} b(n+j), \text{ for } i = 0, 1, \dots, d;$$

$$m := \max_{0 \leq i \leq d} \deg P_i(n); \text{ let } \alpha_i \text{ be the coefficient of } n^m \text{ in } P_i(n), \text{ for } i = 0, 1, \dots, d;$$

$$\text{for all nonzero } z \in K \text{ such that } \sum_{i=0}^d \alpha_i z^i = 0$$

do:

- 1.1 Make an auxiliary recurrence $\sum_{i=0}^d z^i P_i(n)c(n+i) = 0$, with a polynomial $c(n)$

$$\text{in } n. \text{ Let } g_i(n) = z^i P_i(n), \text{ for } i = 0, 1, \dots, d \text{ and } q_j = \sum_{i=j}^d \binom{i}{j} g_i, \text{ for } 0 \leq j \leq r.$$

1.2 Compute the degree upper bound for $c(n)$ by $d = \max \{ \deg f - b, -b - 1, d_1 \}$,

Where

$$b := \max_{0 \leq j \leq r} (\deg q_j - j), \quad d_1 = \max \{ x \in \mathbb{N}; \alpha(x) = 0 \} \text{ and } \alpha(x) = \sum_{\substack{0 \leq j \leq r \\ \deg q_j - j = b}} \text{lc}(q_j) x^j.^7$$

1.3 Using the method of undetermined coefficients, find all nonzero polynomial solutions $c(n)$ over K for $\sum_{i=0}^d z^i P_i(n) c(n+i) = 0$. For each $c(n)$ return a nonzero hypergeometric solution $y(n)$ that satisfies $y(n+1) = S(n)y(n)$ and stop.

2. Return 0 and stop.

3.5.1.4 Efficiency considerations

We may face a complicated case with lots of pairs of monic factors $a(n)$ and $b(n)$. Petkovšek, Wilf and Zeilberger [18] provides some suggestions about how to speed up the implementation.

- a) First of all, omit all pairs $a(n), b(n)$ which do not satisfy the first property of Theroem(3.3.2.1), i.e, $\gcd(a(n), b(n+h)) \neq 1$ for some nonnegative integers h .
- b) Reduce the degree of recurrence $\sum_{i=0}^d z^i P_i(n) c(n+i) = 0$ by canceling $a(n)b(n+1)$ as we did in (3.27) for the case $d = 2$.

⁷ $a^{\downarrow}$ denotes falling factorial function $a(a-1)(a-j+1) \dots$. $\text{lc}(p)$ is the leading coefficient of polynomial p

- c) The coefficients of $\sum_{i=0}^d \alpha_i z^i = 0$ make z only depend on the difference $d(a, b) = \deg b(n) - \deg a(n)$ and have nothing to do with $a(n)$ $b(n)$ themselves. It would be good to test pairs of factors $a(n)$, $b(n)$ in order of the value of $d(a, b)$.
- d) Omit the $a(n)$, $b(n)$ that $d(a, b)$ makes equation $\sum_{i=0}^d \alpha_i z^i = 0$ have no nonzero solution. For example, it happens when $\deg p_d(n) = \deg p_0(n) \geq \deg p_i(n)$ for $0 \leq i \leq d$, and $d(a, b) \neq 0$. In this case we only need to work with $a(n)$ and $b(n)$ of the same degree.

3.5.2 EXAMPLE([18]-PAGE 153)

Consider a given recurrence $(n-1)y(n+2) - (n^2 + 3n - 2)y(n+1) + 2n(n+1)y(n) = 0$, where $p_2(n) = n-1$, $p_1(n) = -(n^2 + 3n - 2)$, $p_0(n) = 2n(n+1)$. The monic factors $a(n)$ of $p_0(n)$ are 1, n , $n+1$, and $n(n+1)$, and $b(n)$ of $p_2(n-2+1)$ are 1 and $n-2$.

Case 1:

Taking $a(n) = b(n) = 1$, yields $z = 2$ and an easily solvable auxiliary recurrence $2(n-1)c(n+2) - (n^2 + 3n - 2)c(n+1) + n(n+1)c(n) = 0$.

The polynomial solution $c(n)$ is 1. It gives $S(n) = 2$ and $y(n) = 2^n$.

Case 2:

Taking $a(n) = n+1$, $b(n) = 1$ yields $z^2 - z = 0$, i.e., $z = 1$ and an auxiliary recurrence $(n-1)(n+2)c(n+2) - (n^2 + 3n - 2)c(n+1) + 2nc(n) = 0$. This recurrence has the same polynomial solution $c(n) = 1$. Hence, $S(n) = n+1$ and $y(n) = n!$.

The details of checking the remaining cases of $a(n)$ and $b(n)$ won't be given here since we have found the general solution formed as $y(n) = C2^n + Dn!$, where C, D are arbitrary constants.

3.5.3 MY OWN HYPER PROGRAM

No existing Maple program can fulfill Algorithm Hyper, which is implemented in the Mathematica function **Hyper[eqn, y(n)]**, where **eqn** is the recurrence relation and **y(n)** is the unknown sum. The output here is a list of the term ratio $S(n) = y(n+1)/y(n)$.

Based on the idea of Petkovšek, we made our own program to solve the linear recurrence with polynomial coefficients. Details of the full program are disclosed in Appendix 2. It deals with a homogeneous recurrence $Ly = 0$. The inhomogeneous recurrences have to be transformed to homogeneous one first.

3.5.3.1 Testing of Hyperhomo()

In order to check the correctness of Hyperhomo(), let's apply it to the recurrence of example (3.5.2). It is $(n-1)y(n+2) - (n^2 + 3n - 2)y(n+1) + 2n(n+1)y(n) = 0$.

There are five pairs of $a(n)$ and $b(n)$ that satisfying the $\gcd(a(n), b(n+h)) = 1$, for every nonnegative integer h .

Output:

- Case 1: $a(n) = b(n) = 1$
 $Z=2$, and degree upper bound is 0. $y(n) = 2^n$
- Case 2: $a(n)=1, b(n)=n-2$

There is no solution of Z .

- Case 3: $a(n)=n, b(n)=1$

$Z=1$, and degree upper bound is 1. $y(n) = n!$

- Case 4: $a(n)=n+1, b(n)=1$

$Z=1$, and degree upper bound is 0. $y(n) = n!$

- Case 5: $a(n)=n(n+1), b(n)=1$

Z has solution of 0.

The closed form solution derived from Hyperhomo() is $y(n) = C2^n + Dn!$, where C, D are arbitrary constants. This is same as example (3.5.2). Hence, it is proved that this program works properly.

CHAPTER 4 HYPERGEOMETRIC CLOSED FORM

In previous chapters, we have studied some basic concepts of map enumeration and generating functions. Some marvelous expressions for the results of map enumeration were discussed. Furthermore, algorithms of finding a hypergeometric closed form have been introduced. This chapter tries to represent four sum expressions, the results of the enumeration for some specific families of maps, in closed form algorithmically.

4.1 ROOTED MAPS ON THE TORUS

$T = \frac{(R-1)^2}{12R^2(R+2)}$, where $R = \sqrt{1-12x}$, is the ordinary generating function by the

number of edges for rooted maps on the torus.

Step 1: Find the recurrence relation

```
>R:=(1-12*x)^(1/2):
>T:=(R-1)^2/(12*R^2*(R+2)):
>eq:=algfuntoalgeq(T, y(x));
eq:=(3-60x + 144 x^2 + 1728 x^3 )y^2 + (-1 - 96 x^2 + 20 x)y +x^2
>eq1:=algeqtodiffeq(eq,y(x),{y(0)=0});
eq1:= {D(y)(0) = 0, y(0) = 0, 16 x^2 + 2 x + (576 x^3- 1056 x^2
+ 84x)y(x) +(-1 + 22 x - 88 x^2 - 480 x^3 + 1152x^4)y'(x)
> diffeqtorec(eq1,y(x),u(n));
{u(1) = 0, u(0) = 0, (576 + 1152 n) u(n) + (-1536 - 480 n)
u(n + 1)+ (-92 - 88 n) u(n + 2) + (22 n + 66) u(n + 3)
+ (-n - 4) u(n + 4), u(2) = 1, u(3) = 20}
```

(4.1)

Step 2: Find all hypergeometric solutions for (4.1)

First step is to find all monic factors $a(n)$ of $p_0(n)$ and $b(n)$ of $p_d(n-d+1)$, which satisfy the condition $\gcd(a(n), b(n+h)) = 1$, for every $h \in \mathbb{N}$. The possible factors are the follows: $a(n) = \{1, n + 1/2\}$, $b(n) = \{1, n + 1\}$. We also observed that all the coefficients of recurrence (4.1) are of the same degree, therefore the equation for z will have no nonzero solution unless $a(n)$, $b(n)$ are of the same degree.

Case 1. $\{a(n), b(n)\}$ is $\{1, 1\}$

$1152 - 480z - 88z^2 + 22z^3 - z^4 = 0$ has three nonzero solutions $\{-4, 2, 12\}$.

Case 1.1 $\{a(n), b(n), z\} = \{1, 1, 2\}$

The auxiliary recurrence does not have nonzero polynomial solution $c(n)$.

Case 1.2 $\{a(n), b(n), z\} = \{1, 1, -4\}$

The auxiliary recurrence has a degree upper bound 0. The method of undetermined coefficients proved that $c(n)$ is an arbitrary nonzero constant. Hence,

$$\frac{u(n+1)}{u(n)} = z \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)} = -4. \text{ But this solution for } u(n) \text{ does not satisfy the initial}$$

conditions ($u(1)=0$, $u(0)=0$, $u(2)=1$, $u(3)=20$, etc.). Therefore, no hypergeometric solution of $u(n)$ exists.

Case 1.3 $\{a(n), b(n), z\} = \{1, 1, 12\}$

The polynomial solution $c(n)$ for this case is an arbitrary nonzero constant. Hence

$$\frac{u(n+1)}{u(n)} = 12. \text{ But this } u(n) \text{ does not satisfy the initial conditions, which means that}$$

no hypergeometric solution of $u(n)$ exists.

Case 2 $\{a(n), b(n)\}$ is $\{n + 1/2, n + 1\}$

$1152 - 480z - 88z^2 + 22z^3 - z^4 = 0$ gives same nonzero solutions $\{-4, 2, 12\}$.

Case 2.1 $\{a(n), b(n), z\} = \{n + 1/2, n + 1, 2\}$

There is no nonzero polynomial solution since the degree upper bound is $-3/2$. So no hypergeometric solution exists.

Case 2.2 $\{a(n), b(n), z\} = \{n + 1/2, n + 1, -4\}$

The degree upper bound of $c(n)$ is $1/2$, which means there is no nonzero polynomial solution of $c(n)$. So $u(n)$ does not have hypergeometric solution in this case.

Case 2.3 $\{a(n), b(n), z\} = \{n + 1/2, n + 1, 12\}$

No hypergeometric solution exists since 0 is the only solution for the auxiliary recurrence.

Conclusion:

After checking all possible triples of $\{a(n), b(n), z\}$, Algorithm Hyper proved that the hypergeometric solution we got do not agree with the initial conditions. Therefore, the

coefficients of $T = \frac{(R-1)^2}{12R^2(R+2)}$, where $R = \sqrt{1-12x}$, do not have a closed form.

4.2 ROOTED MAPS ON THE PROJECTIVE PLANE

$P = \frac{\frac{2R+1}{3} - \sqrt{R(R+2)/3}}{2x}$, where $R = \sqrt{1-12x}$, is the ordinary generating function for

the number of combinatorially distinct rooted maps which are enumerated with respect to edges on the projective plane.

Step 1: Find a recurrence relation

$>R := (1-12*x)^(1/2) :$

```

>G:=((2*R+1)/3-((R*(R+2))/3)^(1/2))/(2*x):
>deq:=algfuntoalgeq(G,y(x));
  deq:= 9y^4x^3 - 6 y^3x^2 + (42 x^2 - 2 x)y^2 + (-1 + 10 x)y + x
>deq1:=algeqtodiffeq(deq,y(x));
  deq1:={y(0) = 0, -36+(-5184x^2 + 648x)y(x)+(-24 -46656x^3
+3888x^2 + 288x)y'(x) +(3 - 41472x^4 +1728x^3 +1008x^2
-108x) y''(x)+ (-6912x^5 + 288x^3 -32 x^2 +x)y'''(x)}
>diffeqtorec(deq1,y(x),u(n));
{(-5184 -19008 n - 20736 n^2 - 6912 n^3 )u(n)+ (4536 + 5616 n
+ 1728 n^2)u(n + 1)+(3888 n + 2592+ 1872n^2 + 288 n^3)u(n + 2)
+(-916 n -912 -300 n^2 -32 n^3)u(n + 3)+(12n^2 + 47 n + 60+ n^3)
u(n + 4),u(3) = 32 _C[0] + 66, u(1) = _C[0], u(2) = 4 _C[0]
+ 6, u(0) = 0}
(4.2)

```

Step 2: Find the hypergeometric solutions for (4.2)

First, find all monic factors $a(n)$ of $p_0(n)$ and $b(n)$ of $p_d(n-d+1)$, which satisfy $\gcd(a(n), b(n+h)) = 1$, for all nonnegative integers h . There are 40 pairs of $\{a(n), b(n)\}$ that satisfy the above condition. The followings are all candidates:

$$a(n) = \{1, n+1, n+3/2, (n+1)(n+3/2), n+1/2, (n+1)(n+1/2), (n+3/2)(n+1/2), (n+1)(n+3/2)(n+1/2)\}$$

$$b(n) = \{1, n+1, n, (n+1)n, n+2, (n+1)(n+2), n(n+2), (n+2)(n+1)n\}.$$

Then refer to the fourth scenario of efficiency consideration for Algorithm Hyper, some pairs of $a(n), b(n)$ that $d(a,b) \neq 0$ can be skipped.

Hence, there are only 11 pairs of satisfying $a(n), b(n)$.

Case 1 $\{a(n), b(n)\}$ is $\{1, 1\}$

$-6912 + 288z^2 - 32z^3 + z^4 = 0$ has two solutions $\{-4, 12\}$.

The auxiliary recurrence relations corresponding to both z 's do not have nonzero polynomial solutions for $c(n)$. Therefore, there is no hypergeometric solution.

Case 2 $\{a(n), b(n)\}$ is $\{n+3/2, n+2\}$

$-6912 + 288z^2 - 32z^3 + z^4 = 0$ has nonzero solutions $\{-4, 12\}$.

The auxiliary recurrences for both z 's do not own a nonzero polynomial solution.

Case 3 $\{a(n), b(n)\}$ is $\{n+3/2, n+1\}$

This case has same nonzero solutions of z , $\{-4, 12\}$. But none of these z 's has a nonzero polynomial solution $c(n)$.

Case 4 $\{a(n), b(n)\}$ is $\{n+3/2, n\}$

The nonzero solutions of z , $\{-4, 12\}$, do not bring up a nonzero polynomial solution $c(n)$.

Case 5 $\{a(n), b(n)\}$ is $\{n+1/2, n+2\}$

The solutions of z are $\{-4, 12\}$.

Case 5.1 $\{a(n), b(n), z\} = \{n+1/2, n+2, -4\}$

For $z = -4$, the auxiliary recurrence has one solution, zero.

Case 5.2 $\{a(n), b(n), z\} = \{n+1/2, n+2, 12\}$

This z making the auxiliary recurrence has one solution, an arbitrary constant. i.e.,

$\frac{u(n+1)}{u(n)} = \frac{12n+6}{n+2}$. But this solution is in conflict with the initial conditions.

Case 6 $\{a(n), b(n)\}$ is $\{n+1/2, n+1\}$

Case 7 $\{a(n), b(n)\}$ is $\{n+1/2, n\}$

There are no nonzero solutions of z for case 6 and case 7, which means these two pairs of $a(n), b(n)$ do not have hypergeometric solutions for the recurrence (4.2).

Case 8 $\{a(n), b(n)\}$ is $\{(n + 3/2)(n + 1/2), (n+2)(n + 1)\}$

For this case, $z = \{-4, 12\}$. But there are no nonzero polynomial solutions for auxiliary recurrences.

Case 9 $\{a(n), b(n)\}$ is $\{(n + 3/2)(n + 1/2), (n+2)n\}$

Case 10 $\{a(n), b(n)\}$ is $\{(n + 3/2)(n + 1/2), (n+1)n\}$

The above two cases have same nonzero solutions $\{-4, 12\}$ for z . The corresponding auxiliary recurrence for each case does not have a polynomial solution.

Case 11 $\{a(n), b(n)\}$ is $\{(n + 1), (n + 2)\}$

For this case, $z = \{-4, 12\}$. But the auxiliary recurrences do not have polynomial solutions. Hence, there is no hypergeometric solution of $u(n)$.

Conclusion:

After checking all these satisfying monic factors, we have one nonzero polynomial solution for $c(n)$. The corresponding term ratio is $\frac{u(n+1)}{u(n)} = z \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)} = \frac{12n+6}{n+2}$.

This solution does not satisfy the initial conditions. ($u(0) = 0$, $u(1) = 1$, $u(2) = 10$, $u(3) = 98$, $u(4) = 982$, $u(5) = 10062$, etc.,) Therefore, it is proved that the number of rooted maps, which are enumerated with respect to edges on the projective plane, does not have a closed form.

4.3 3-CONNECTED PLANAR ROOTED MAPS

The generating function for the 3-connected planar maps, enumerated with respect to

edges, is $C(x) = x^2 \left(\frac{2}{1+x} + 2x \right) - \frac{u}{(1+2u)^3}$, where $u = x(1+u)^2$.

Step 1: Find the recurrence

```
> eq0:=x*(1+u)^2=u:
> eq1:=x^2*(2/(1+x)+2*x)-u^2/(1+2*u)^3=y:
> G:=eliminate( {eq0,eq1}, u);
```

$$G = \left\{ u = -\frac{x(3yx + 8y - 6x^4 + 10x^5 - 5yx^2 - 15x^2 + x - 6x^3)}{-28x^5 + 14yx^2 + 12x^6 - 6yx^3 - x + 17x^2 - 22x^3 - 12x^4 - 8y + 12yx}, \right. \\ \left. \{108x^9 + 32x^{10} + 248x^7 + 182x^6 + 212x^8 + 4x^{11} - 4yx^8 + y^2x^5 + 8y^2 - 20yx^2 \right. \\ \left. + 25y^2x^3 + 38y^2x^2 - 104yx^6 - 32yx^7 + 8y^2x^4 - 178yx^4 - 93yx^3 - 182yx^5 \right. \\ \left. + 28y^2x - x^3 + 10x^4 + yx + 73x^5\} \right\}$$

The **eliminate**[26] procedure eliminates some given variables from a set of equations. It returns an expression sequence of lists. Each list represents a possible elimination of the given variables. The first component of every solution is a set of equations, where the left hand side of each equation is the variable that has been eliminated and the right hand side is its corresponding value. The second component of each solution is the set of remaining equations once the given variables have been eliminated.

```
> deq:=algeqtodiffeq(op(1,op(2,G)), y(x), {y(0)=0});
deq:={y(0)=0, -28x^7-54x^6 -28x^5 +26x^4 +20x^3 + 2x^2 +(2x^4-18x^3
-40x^2 -18 x + 2)y(x)+(4x^5 + 15x^4 + 16x^3 + 3 x^2- 2x)y'(x)}
> diffeqtorec(deq,y(x),u(n));
{(2 + 4 n)u(n)+(-3+15 n)u(n + 1)+(-8+16n)u(n + 2)+(-9 +3 n)
u(n + 3)+(-6-2n)u(n + 4), u(7)=24 _C[0], u(4) = -61/4 _C[0],
u(5)= 279/16_C[0], u(6)= 1 - 153/32_C[0], u(3)= 2+33/2 _C[0]
u(1) = _C[0], u(2) = 1 - 15/2 _C[0], u(0) = 0} (4.3)
```

Step 2: Find the hypergeometric solutions for (4.3)

First, find all monic factors $a(n)$ of $p_0(n)$ and $b(n)$ of $p_d(n-d+1)$, which satisfy $\gcd(a(n), b(n+h))=1$, for all nonnegative integers h . In order to have nonzero solutions of z , $\deg a(n)$ must equal to $\deg b(n)$ since the degree of all the coefficients of recurrence (4.3) are same.

The candidates of monic factors are: $a(n)=\{1, n+1/2\}$, $b(n)=\{1, n\}$.

Case1 $\{a(n), b(n)\}$ is $\{1, 1\}$

The solutions of $4+15z+16z^2+3z^3-2z^4=0$ are $\{-1, 4, -1/2\}$.

Case1.1 $\{a(n), b(n), z\} = \{1, 1, -1/2\}$

For this case, the degree upper bound of auxiliary recurrence is 2. The method of

undetermined coefficients gives us the solution $c(n) = \frac{27}{16}c_3n^2 - \frac{45}{16}c_3n + c_3$, where

c_3 is an arbitrary constant. But the corresponding solution of $u(n)$ does not satisfy the initial conditions.

Case 1.2 $\{a(n), b(n), z\} = \{1, 1, -1\}$

Degree upper bound of $c(n)$ is zero. The method of undetermined coefficients shows

$c(n)$ is an arbitrary constant. So $\frac{u(n+1)}{u(n)} = -1$. But this $u(n)$ does not satisfy the initial

conditions, which means there is no hypergeometric solution of $u(n)$.

Case 1.3 $\{a(n), b(n), z\} = \{1, 1, 4\}$

This auxiliary recurrence does not have polynomial solutions.

Case 2 $\{a(n), b(n)\}$ is $\{n+1/2, n\}$

The solutions of $4+15z+16z^2+3z^3-2z^4=0$ are $\{-1, 4, -1/2\}$.

Case 2.1 $\{a(n), b(n), z\} = \{n+1/2, n, -1\}$

Case 2.2 $\{a(n), b(n), z\} = \{n+1/2, n, 4\}$

Case 2.3 $\{a(n), b(n), z\} = \{n+1/2, n, -1/2\}$

For the above three cases, none of auxiliary recurrences have polynomial solutions for $c(n)$. Therefore, there is no hypergeometric solution of $u(n)$ for case 2.

Conclusion:

Algorithm Hyper proved that the number of 3-connected rooted planar maps, which are enumerated with respect to edges, cannot be expressed in hypergeometric closed form.

4.4 4-CONNECTED PLANAR TRIANGULATIONS

The generating function for the 4-connected planar triangulations (enumerated with

respect to vertices) is $f = \frac{t(t-1)}{(1+t)^2}$, where $x = t(1-t)^2$.

Step 1: Find the recurrence relation

```
>f=t*(t-1)/(1+t)^2:
```

```
>x=t*(1-t)^2:
```

```
>E:=eliminate({f=t*(t-1)/(1+t)^2,x=t*(1-t)^2},t);
```

```
      x f^2 - 2 x f + x - 4 f^2 + f
E:=[{t = -----}, {20 x f^2 - x f + 16 f^3
      f(8 f + 1)
+ 8 x f^3 + x^2 f^3 - 3 x^2 f^2 + 3 x^2 f - x^2}]
```

```
>deq:=algeqtodiffeq(op(1,op(2,E)), f(x), {f(0)=0});
```

```
deq := {f(0) = 0, 6 x^2+(-6 x^2 + 8-88x)f(x)+(-8x-54x^3 +118 x^2)
diff(f(x),x)+(16 x^2-27 x^4-104 x^3)diff(f(x),`$`(x,2))}
```

```
>diffeqtoec(deq,f(x),u(n));
```

$$\{u(0) = 0, (-6-27n - 27n^2)u(n) + (30+14n-104n^2)u(n+1) + (24+40n+16n^2)u(n+2), u(1) = -C[0], u(2) = -1/4-5/4C[0]\}$$

where $C[0]$ is an arbitrary constant (4.4)

Step 2: Find the hypergeometric solutions for (4.4)

First, we find all monic factors $a(n)$ of $p_0(n)$ and $b(n)$ of $p_d(n-d+1)$ that satisfy $\gcd(a(n), b(n+h))=1$, for all nonnegative integers h . Also $a(n)$, $b(n)$ must be of the same degree in order to have the nonzero solution. Hence, the following monic factors bring up 6 pairs of satisfying $a(n)$ and $b(n)$.

$$a(n)=\{1, n+2/3, n+1/3, (n+2/3)(n+1/3)\}, b(n)=\{1, n, n+1/2, n(n+1/2)\}.$$

Case1 $\{a(n), b(n)\}$ is $\{1, 1\}$

$$-27-104z+16z^2=0 \text{ has two nonzero solutions } \{-1/4, 27/4\}.$$

$$\textbf{Case1.1} \{a(n), b(n), z\} = \{1, 1, -1/4\}$$

For this case, the degree upper bound of auxiliary recurrence is one. The method of undetermined coefficients proved $c(n)=0$, which means that there is no hypergeometric solution.

$$\textbf{Case 1.2} \{a(n), b(n), z\} = \{1, 1, 27/4\}$$

There is no polynomial solution of $c(n)$ for this auxiliary recurrence.

Case 2 $\{a(n), b(n)\}$ is $\{n+1/3, n\}$

$$-27-104z+16z^2=0 \text{ has two nonzero solutions } \{-1/4, 27/4\}.$$

Algorithm Poly checked the two recurrences and proved that there is no polynomial solution of $c(n)$ for both z 's. Therefore, case 2 has no hypergeometric solution.

Case 3 $\{a(n), b(n)\}$ is $\{n+1/3, n+1/2\}$

$-27 - 104z + 16z^2 = 0$ has two nonzero solutions $\{-1/4, 27/4\}$.

The recurrences of these two sub cases do not have nonzero polynomial solutions.

Case 4 $\{a(n), b(n)\}$ is $\{n+2/3, n\}$

$-27 - 104z + 16z^2 = 0$ has two nonzero solutions $\{-1/4, 27/4\}$.

The two sub cases do not have nonzero polynomial solutions of $c(n)$. So there is no hypergeometric solution for (4.4).

Case 5 $\{a(n), b(n)\}$ is $\{n+2/3, n+1/2\}$

$-27 - 104z + 16z^2 = 0$ has two nonzero solutions $\{-1/4, 27/4\}$.

There are no polynomial solutions of $c(n)$ for the two sub cases. Therefore, no hypergeometric solution exists.

Case 6 $\{a(n), b(n)\}$ is $\{(n+1/3)(n+2/3), n(n+1/2)\}$

$-27 - 104z + 16z^2 = 0$ has two nonzero solutions $\{-1/4, 27/4\}$.

For these two z 's, there are no polynomial solutions for the auxiliary recurrences. Hence, case 6 does not own a hypergeometric solution.

Conclusion:

Algorithm Hyper proved that the number of 4-connected planar triangulations, which are enumerated respect to vertices can not be expressed in closed form.

CHAPTER 5 SUMMARY AND CONCLUSIONS

We started with introducing the basics of map and made a survey of some solved problems of map enumeration. Then generating functions and some functions of GFUN Maple package are introduced. After defining “hypergeometric closed form”, we discussed four basic algorithms that are used to find a “closed form”. Furthermore, we developed a Maple program Hyperhomo() that fulfills Algorithm Hyper. In other words, it is designed to solve a linear recurrence with polynomial coefficients.

In the last part of this thesis, Chapter 4, we dealt with four questions: seeking a hypergeometric closed form for the number of rooted maps on the torus, rooted maps on the projective plane, 3-connected planar maps, and 4-connected planar triangulations. Our conclusion is that none of the four map enumeration problems has a hypergeometric closed form.

BIBLIOGRAPHY

- [1] R. J. Baxter, *Dichromatic polynomials and potts models summed over rooted maps*, Annals of Combinatorics 5 (2001) pp.17-36.
- [2] E.A. Bender and L.B. Richmond, *A Survey of the asymptotic behaviour of maps*, Journal of Combinatorial Theory, Series B, V40, No 3, June (1986).
- [3] E.A. Bender, E.R. Canfield and R.W. Robinson, *The enumeration of maps on the torus and the projective plane*, Canadian Math. Bull, V31 (3), 1988.
- [4] E.A. Bender, Z.C. Gao, L.B. Richmond and N.C. Wormald, *Asymptotic properties of rooted 3-connected maps on surfaces*, J. Austral Math. Soc (series A) 60 (1996).
- [5] E.A. Bender, *Some unsolved problems in map enumeration*, Bull. Inst. Combin. Appl. 3 (1991) pp.51-56.
- [6] W. G. Brown and W. T. Tutte, *On the enumeration of rooted non-separable planar maps*, Canad. J. Math.16 (1964), pp.572-577.
- [7] W. G. Brown, *Enumeration of triangulations of the disk*, Proc. London Math.Soc(3) 14 (1964), pp.746-768.
- [8] Rudolf Fritsch and Gerda Fritsch, *The Four-color theorem*, Springer 1998.
- [9] Z. C. Gao, I. M. Wanless and N.C. Wormald, *Counting 5-connected planar triangulations*, John Wiley&Sons, Journal of Graph Theory 38: pp18-35, 2001.
- [10] I. M. Gessel and R. P. Stanley, *Algebraic enumeration*, Handbook of Combinatorics, 1995, pp1021-1062.
- [11] J. M. Harris, J. L. Hirst and M. J. Mossingh, *Combinatorics and graph theory*, Springer 2000.
- [12] Gerard' t Hooft, *Counting Planar Diagrams with various restrictions*, Nucl. Phys. B538 (1999) pp.389-410.
- [13] L. Christine Kinsey, *Topology of surfaces*, Springer-Verlag, 1993.

- [14] Yanpei Liu, *Counting rooted planar maps*, C&O Res. Rep. 83-26, University of Waterloo, 1983.
- [15] R. C. Mullin and P.J. Schellenberg, *The enumeration of c-Nets via quadrangulations*, Journal of Combinatorial Theory 4, pp.259-276 (1968).
- [16] R. C. Mullin, *Enumeration of rooted triangular maps*, American Mathematics Monthly, vol. 71 (1964), pp.1007—1010.
- [17] R. C. Mullin, *On counting rooted triangular maps*. Canadian Journal of Mathematics, vol. 17 (1965), pp. 373—382.
- [18] M. Petkovšek, H.S. Wilf, and D. Zeilberger, *A=B*, D Wellesley, MA: A. K. Peters, 1996.
- [19] B. Salvy and P. Zimmermann, *GFUN: A maple package for the manipulation of generating and holonomic functions in one variable*, ACM Transactions on Mathematical Software, Volume 20, 1992.
- [20] Richard P. Stanley, *Enumerative combinatorics*, Cambridge University Press, 1997.
- [21] W.T. Tutte, *The enumeration theory of planar maps*, in “A Survey of Combinatorial Theory”(J.N.Srivastava et al., Eds), pp437-448, North-Holland, Amsterdam, 1973.
- [22] W.T. Tutte, *A census of planar maps*, Canad. Journal of Math.15(1963), pp249-271.
- [23] W.T. Tutte, *A census of planar triangulations*, Canad. Journal of Math.14 (1962), pp.21-38.
- [24] W.T. Tutte, *On the enumeration of planar maps*, Bull. Amer. Math. Soc. 74(1968), pp.64-74.
- [25] W.T. Tutte, *Map-coloring problems and chromatic polynomials*, America Scientist (1974) pp.702-705.
- [26] Maple help file.
- [27] *The Four color Theorem*, <http://www.math.gatech.edu/~thomas/FC/fourcolor.html>.

Appendix

[A.1.1---Rooted maps on the torus

```
[ > restart;
> with(share):
See ?share and ?share,contents for information about the share library
> with(gfun):
Share Library: gfun
Authors: Salvy, Bruno and Zimmermann, Paul.
Description: package for the manipulation and discovery of generating
functions
```

Generating function

```
[ > R:= (1-12*x)^(1/2);
                                     R := sqrt(1 - 12 x)
> T:=(R-1)^2/(12*R^2*(R+2));
                                     T := 1 / (12 (1 - 12 x) (sqrt(1 - 12 x) + 2))
> deq1:=algfuntoalgeq(T,y(x));
deq1 := (3 - 60 x + 144 x^2 + 1728 x^3) y^2 + (-1 + 20 x - 96 x^2) y + x^2
> deq2:=algeqtodiffeq(deq1,y(x),{y(0)=0});
deq2 := { 16 x^2 + 2 x + (576 x^3 - 1056 x^2 + 84 x) y(x)
          + (-1 + 22 x - 88 x^2 - 480 x^3 + 1152 x^4) (d/dx y(x)) , D(y)(0) = 0,
          y(0) = 0 }
> rec:=diffeqtorec(deq2,y(x),u(n));
rec := { (576 + 1152 n) u(n) + (-1536 - 480 n) u(n + 1)
          + (-92 - 88 n) u(n + 2) + (22 n + 66) u(n + 3) + (-n - 4) u(n + 4),
          u(2) = 1, u(3) = 20, u(1) = 0, u(0) = 0 }
> algeqtoseries(deq1,x,y,8);
[ 1/3 + 15 x^2 + 108 x^3 + 1485 x^4 + 16200 x^5 + 192942 x^6 + 2245320 x^7 +
  O(x^8), x^2 + 20 x^3 + 307 x^4 + 4280 x^5 + 56914 x^6 + 736568 x^7 + 9370183
  x^8 + 117822512 x^9 + O(x^10) ]
```

```

> receq:=(576+1152*n)*u(n)+(-1536-480*n)*u(n+1)+(-
  92-88*n)*u(n+2)+(22*n+66)*u(n+3)+(-n-4)*u(n+4);
receq:=(576+1152 n) u(n)+(-1536-480 n) u(n+1)
  +(-92-88 n) u(n+2)+(22 n+66) u(n+3)+(-n-4) u(n+4)
> p[4]:=n->(-n-4): p[3]:=n->22*n+66:
  p[2]:=n->-92-88*n: p[1]:=n->-1536-480*n:
  p[0]:=n->576+1152*n:
> d:=4: fp0:=factors(p[0](n));
  fp1:=factors(p[d](n-d+1));


$$fp0 := \left[ 1152, \left[ \left[ \frac{1}{2} + n, 1 \right] \right] \right]$$



$$fp1 := [-1, [[n+1, 1]]]$$

# m, the number of factors determines the number of levels of next for loop
> m:=nops(op(2,fp0));


$$m := 1$$

> for i1 from 0 to 1 do
>   a[i1]:= op(1,op(1,op(2,fp0)))^i1;
>   b[i1]:= op(1,op(1,op(2,fp1)))^i1;
> od:
# The following part computes the number of a(n), b(n) that satisfy the
gcd(a,b) condition, k denote the number of pairs contains (a(n), b(n))
> k:=0: unassign('h'): hs:=[]: poly:=0:
> for i from 0 to 50 do A[i]:=0: B[i]:=0:
  polyh[i]:=0: od:
> for i from 0 to 2^m-1 do
>   for j from 0 to 2^m-1 do flag:=1:
>     if gcd(a[i],b[j])=1 and
       degree(a[i])=degree(b[j]) then
       poly:=resultant(a[i],subs(n=n+h,b[j]),n);
       hs:=[isolve(poly=0)];
>     if hs=[] then A[k]:=a[i]: B[k]:=b[j]:
       polyh[k]:=poly: k:=k+1:
>     elif nops(hs)>1 then for s from 1 to nops(hs) do
       if type(op(2,op(1,op(s,hs))),negint) then
       flag:=flag*1:
       else flag:=flag*(-1); break: fi:

```

```

od:
  if flag=1 then A[k]:=a[i]: B[k]:=b[j]:
    polyh[k]:=poly: k:=k+1: fi:
>   elif type(op(2,op(1,op(1,hs))), negint)
    then A[k]:=a[i]: B[k]:=b[j]:
    polyh[k]:=poly: k:=k+1:
>   fi;
>   fi;
>   od;
>   od;
>   k;
2
[ # pls change the directory "c:/hyperal.txt" to your local directory
[ > read "c:/hyperal.txt";
[ > for i from 0 to k-1 do "a"=A[i]; "b"=B[i]; od;
    "a" = 1
    "b" = 1
    "a" =  $\frac{1}{2} + n$ 
    "b" =  $n + 1$ 
[ > for i from 0 to k-1 do a:=A[i]: b:=B[i]:
  print(case[i+1]); Hyperhomo(p,a,b,d): od:
    case1
    "no degree upbound for polynomial solution of c(n)"
    "degree upbound" = 0
    "a" = 1
    "b" = 1
    "z" = -4
    "auxirec" =  $(576 + 1152 n) c(n) - 4 (-1536 - 480 n) c(n + 1)$ 
    +  $16 (-92 - 88 n) c(n + 2) - 64 (22 n + 66) c(n + 3)$ 
    +  $256 (-n - 4) c(n + 4)$ 
    "degree upbound" = 0

```

$$"a" = 1$$

$$"b" = 1$$

$$"z" = 12$$

$$\begin{aligned} "auxirec" = & (576 + 1152 n) c(n) + 12 (-1536 - 480 n) c(n+1) \\ & + 144 (-92 - 88 n) c(n+2) + 1728 (22 n + 66) c(n+3) \\ & + 20736 (-n - 4) c(n+4) \end{aligned}$$

case₂

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

$$"degree\ upbound" = 0$$

$$"a" = \frac{1}{2} + n$$

$$"b" = n + 1$$

$$"z" = 12$$

$$\begin{aligned} "auxirec" = & (576 + 1152 n) (n+1) (n+2) (n+3) (n+4) c(n) \\ & + 12 (-1536 - 480 n) \left(\frac{1}{2} + n\right) (n+2) (n+3) (n+4) c(n+1) \\ & + 144 (-92 - 88 n) \left(\frac{1}{2} + n\right) \left(\frac{3}{2} + n\right) (n+3) (n+4) c(n+2) \\ & + 1728 (22 n + 66) \left(\frac{1}{2} + n\right) \left(\frac{3}{2} + n\right) \left(\frac{5}{2} + n\right) (n+4) c(n+3) \\ & + 20736 (-n - 4) \left(\frac{1}{2} + n\right) \left(\frac{3}{2} + n\right) \left(\frac{5}{2} + n\right) \left(\frac{7}{2} + n\right) c(n+4) \end{aligned}$$

[>

solve the above three auxiliary recurrences that have valid degree up bound for a polynomial

auxirec1: degree up bound is zero, a=1,b=1,z=12

```
> auxirec1 := (576+1152*n)*c(n)+12*(-1536-480*n)*c(n+1)+144*(-92-88*n)*c(n+2)+1728*(22*n+66)*c(n+3)+20736*(-n-4)*c(n+4);
```

```
auxirec1 := (576 + 1152 n) c(n) + 12 (-1536 - 480 n) c(n+1)
+ 144 (-92 - 88 n) c(n+2) + 1728 (22 n + 66) c(n+3)
```



```

+ 20736 (-n - 4) c(n + 4)
> for i from 0 to 4 do auxirec1:=subs(c(n+i)=c,
auxirec1); od:
> auxirec1;
(576 + 1152 n) c + 12 (-1536 - 480 n) c + 144 (-92 - 88 n) c
+ 1728 (22 n + 66) c + 20736 (-n - 4) c
> solve(auxirec1, c);
c
# c(n) can be any nonzero constant. Hence, u(n+1)/u(n)=12. But this
solution of u(n) doesn't satisfy the initial condition, (u(1) = 0, u(0) = 0, u(2)
= 1, u(3) = 20), which is given by diffeqtoec.

```

auxirec2: degree up bound is zero, a=1, b=1, z=-4

```

> auxirec2:=(576+1152*n)*c(n)-4*(-1536-480*n)*c(n+
1)+16*(-92-88*n)*c(n+2)-64*(22*n+66)*c(n+3)+256*
(-n-4)*c(n+4);
auxirec2:=(576 + 1152 n) c(n) - 4 (-1536 - 480 n) c(n + 1)
+ 16 (-92 - 88 n) c(n + 2) - 64 (22 n + 66) c(n + 3)
+ 256 (-n - 4) c(n + 4)
> for i from 0 to 4 do auxirec2:=subs(c(n+i)=c,
auxirec2); od:
> auxirec2;
(576 + 1152 n) c - 4 (-1536 - 480 n) c + 16 (-92 - 88 n) c
- 64 (22 n + 66) c + 256 (-n - 4) c
> solve(auxirec2, c);
c
# c(n) can be any nonzero constant. Hence, u(n+1)/u(n)=-4. But this
hypergeometric solution of u(n) doesn't satisfy the initial condition too. (u(1)
= 0, u(0) = 0, u(2) = 1, u(3) = 20).

```

auxirec3: degree up bound is zero, a=n+1/2, b=n+1, z=12

```

> auxirec3:=(576+1152*n)*(n+1)*(n+2)*(n+3)*(n+4)*c
(n)+12*(-1536-480*n)*(1/2+n)*(n+2)*(n+3)*(n+4)*c
(n+1)+144*(-92-88*n)*(1/2+n)*(3/2+n)*(n+3)*(n+4)
*c(n+2)+1728*(22*n+66)*(1/2+n)*(3/2+n)*(5/2+n)*(
n+4)*c(n+3)+20736*(-n-4)*(1/2+n)*(3/2+n)*(5/2+n)
*(7/2+n)*c(n+4);

```

```

auxirec3 := (576 + 1152 n) (n + 1) (n + 2) (n + 3) (n + 4) c(n)
+ 12 (-1536 - 480 n)  $\left(\frac{1}{2} + n\right)$  (n + 2) (n + 3) (n + 4) c(n + 1)
+ 144 (-92 - 88 n)  $\left(\frac{1}{2} + n\right)$   $\left(\frac{3}{2} + n\right)$  (n + 3) (n + 4) c(n + 2)
+ 1728 (22 n + 66)  $\left(\frac{1}{2} + n\right)$   $\left(\frac{3}{2} + n\right)$   $\left(\frac{5}{2} + n\right)$  (n + 4) c(n + 3)
+ 20736 (-n - 4)  $\left(\frac{1}{2} + n\right)$   $\left(\frac{3}{2} + n\right)$   $\left(\frac{5}{2} + n\right)$   $\left(\frac{7}{2} + n\right)$  c(n + 4)
[ > for i from 0 to 4 do auxirec3:=subs(c(n+i)=c,
  auxirec3); od:
[ > expand(auxirec3);
  -15552 c - 7776 c n2 - 34992 c n
[ > solve(auxirec3,c);
  0
[ # c(n) is zero. So hypergeometric solution of u(n) for this case.
[ # Conclusion: There is no hypergeometric closed form for the number of rooted
  maps which are enumerated w.r.t.edges on the torus .

```

A.1.2--Rooted maps on the projective plane

```

> restart:
> with(share): with(gfun):
# The following part first finds all the monic factors that satisfy gcd( a,
# b(n+h))=1 for all nonnegative integers h. Then only the pairs with
# degree(a(n))= degree (b(n)) are considered.
> R:=(1-12*x)^(1/2);

$$R := \sqrt{1 - 12x}$$

> G:=((2*R+1)/3-((R*(R+2))/3)^(1/2))/(2*x);

$$G := \frac{\frac{1}{3} \sqrt{1 - 12x} + \frac{1}{3} - \frac{1}{3} \sqrt{3} \sqrt{\sqrt{1 - 12x} (\sqrt{1 - 12x} + 2)}}{2x}$$

> deq:=algfuntoalgeq(G,y(x));

$$deq := 9y^4x^3 - 6y^3x^2 + (42x^2 - 2x)y^2 + (-1 + 10x)y + x$$

> deq1:=algeqtodiffeq(deq,y(x));

$$deq1 := \{y(0) = 0, -36 + (-5184x^2 + 648x)y(x) \\ + (-24 - 46656x^3 + 3888x^2 + 288x) \left( \frac{\partial}{\partial x} y(x) \right) \\ + (3 - 41472x^4 + 1728x^3 + 1008x^2 - 108x) \left( \frac{\partial^2}{\partial x^2} y(x) \right) \\ + (-6912x^5 + 288x^3 - 32x^2 + x) \left( \frac{\partial^3}{\partial x^3} y(x) \right) \}$$

> diffeqtorec(deq1,y(x),u(n));

$$\{(-5184 - 19008n - 20736n^2 - 6912n^3)u(n) \\ + (4536 + 5616n + 1728n^2)u(n+1) \\ + (3888n + 2592 + 1872n^2 + 288n^3)u(n+2) \\ + (-916n - 912 - 300n^2 - 32n^3)u(n+3) \\ + (12n^2 + 47n + 60 + n^3)u(n+4), u(2) = 4\_C_0 + 6, u(3) = 32\_C_0 + 66, \\ u(1) = \_C_0, u(0) = 0 \}$$

> algeqtoseries(deq,x,y,7);

$$[x^{-1} - 4 - 13x - 82x^2 - 638x^3 - 5518x^4 - 50886x^5 + O(x^6),$$


```

```

RootOf(9 _Z^2 + 3 _Z + 1) x^-1 + 2 + (-18 RootOf(9 _Z^2 + 3 _Z + 1) + 3) x
+ (18 - 108 RootOf(9 _Z^2 + 3 _Z + 1)) x^2 +
(108 - 972 RootOf(9 _Z^2 + 3 _Z + 1)) x^3 +
(810 - 8748 RootOf(9 _Z^2 + 3 _Z + 1)) x^4 +
(-84564 RootOf(9 _Z^2 + 3 _Z + 1) + 6318) x^5 + O(x^6),
x + 10 x^2 + 98 x^3 + 982 x^4 + 10062 x^5 + 105024 x^6 + 1112757 x^7 + O(x^8)]
> d:=4:
> p[4]:=n->12*n^2+47*n+60+n^3:
p[3]:=n->-916*n-912-300*n^2-32*n^3:
p[2]:=n->3888*n+2592+1872*n^2+288*n^3:
p[1]:=n->4536+5616*n+1728*n^2:
p[0]:=n->-5184-19008*n-20736*n^2-6912*n^3:
> fp0:=factors(p[0](n)); fp1:=factors(p[d](n-d+1));
fp0 := [ -6912, [ [n+1, 1], [n+3/2, 1], [n+1/2, 1] ] ]
fp1 := [ 1, [ [n+2, 1], [n, 1], [n+1, 1] ] ]
> m:=nops(op(2,fp0));
# m , the number of factors means the number of levels of next for loop
m := 3
> for i1 from 0 to 1 do
>   for i2 from 0 to 1 do
>     for i3 from 0 to 1 do
>       a[i1+2*i2+2^2*i3]:= op(1,op(1,op(2,fp0)))^i1*
op(1,op(2,op(2,fp0)))^i2 *
op(1,op(3,op(2,fp0)))^i3;
>       b[i1+2*i2+2^2*i3]:= op(1,op(1,op(2,fp1)))^i1*
op(1,op(2,op(2,fp1)))^i2 *
op(1,op(3,op(2,fp1)))^i3;
> od; od; od;
> for i from 0 to 2^m-1 do a[i]: od:
> for i from 0 to 2^m-1 do b[i]; od:
# The following part computes the number of a(n), b(n) that satisfy the gcd(a,b)
condition , k denote the number of pairs of a(n) and b(n)
> k:=0: unassign('h'): hs:=[]: poly:=0:

```

```

> for i from 0 to 50 do A[i]:=0: B[i]:=0:
    polyh[i]:=0: od:
> for i from 0 to 2^m-1 do
>   for j from 0 to 2^m-1 do   flag:=1:
>     if gcd(a[i],b[j])=1 then
>       poly:=resultant(a[i],subs(n=n+h,b[j]),n);
>       hs:=[isolve(poly=0)];
>       if hs=[] then A[k]:=a[i]: B[k]:=b[j]:
>         polyh[k]:=poly: k:=k+1:
>       elif nops(hs)>1 then for s from 1 to nops(hs) do
>         if type(op(2,op(1,op(s,hs))), negint)
>         then flag:=flag*1: else flag:=flag*(-1); break:
>         fi: od:
>         if flag=1 then A[k]:=a[i]: B[k]:=b[j]:
>           polyh[k]:=poly: k:=k+1: fi:
>       elif type(op(2,op(1,op(1,hs))), negint)
>         then A[k]:=a[i]: B[k]:=b[j]:
>           polyh[k]:=poly: k:=k+1:
>       fi;
>     fi;
>   od;
> od;
> k;

```

40

```

# further reduce some pairs of a(n) b(n) that degree(a) is not equal to degree(b).
> ind:=0:
> for i from 0 to k-1 do if degree(A[i])=degree(B[i])
>   then AA[ind]:=A[i]; BB[ind]:=B[i]; ind:=ind+1: fi;
>   od;
> for i from 0 to ind-1 do "a"=AA[i]; "b"=BB[i]; od;
    "a" = 1
    "b" = 1
    "a" = n + 1
    "b" = n + 2
    "a" = n +  $\frac{3}{2}$ 

```

$$"b" = n + 2$$

$$"a" = n + \frac{3}{2}$$

$$"b" = n$$

$$"a" = n + \frac{3}{2}$$

$$"b" = n + 1$$

$$"a" = n + \frac{1}{2}$$

$$"b" = n + 2$$

$$"a" = n + \frac{1}{2}$$

$$"b" = n$$

$$"a" = n + \frac{1}{2}$$

$$"b" = n + 1$$

$$"a" = \left(n + \frac{3}{2}\right) \left(n + \frac{1}{2}\right)$$

$$"b" = (n + 2) n$$

$$"a" = \left(n + \frac{3}{2}\right) \left(n + \frac{1}{2}\right)$$

$$"b" = (n + 2) (n + 1)$$

$$"a" = \left(n + \frac{3}{2}\right) \left(n + \frac{1}{2}\right)$$

$$"b" = n (n + 1)$$

```
[ > "ind"=ind;
```

```
        "ind" = 11
```

```
[ # please change the directory "c:/hyperal.txt" to your local directory
```

```
[ > read "c:/hyperal.txt":
```

```
  > for i from 0 to ind-1 do a:=AA[i]: b:=BB[i]: print  
    (case[i+1]); Hyperhomo(p,a,b,d): od:
```

```
        casei
```

"no degree upbound for polynomial solution of $c(n)$ "

"no degree upbound for polynomial solution of $c(n)$ "

$case_2$

"no degree upbound for polynomial solution of $c(n)$ "

"no degree upbound for polynomial solution of $c(n)$ "

$case_3$

"no degree upbound for polynomial solution of $c(n)$ "

"no degree upbound for polynomial solution of $c(n)$ "

$case_4$

"no degree upbound for polynomial solution of $c(n)$ "

"no degree upbound for polynomial solution of $c(n)$ "

$case_5$

"no degree upbound for polynomial solution of $c(n)$ "

"no degree upbound for polynomial solution of $c(n)$ "

$case_6$

"degree upbound" = 0

$$"a" = n + \frac{1}{2}$$

$$"b" = n + 2$$

$$"z" = -4$$

$$"auxirec" = (-5184 - 19008n - 20736n^2 - 6912n^3)(n+2)(n+3)(n+4)(n+5)c(n)$$

$$-4(4536 + 5616n + 1728n^2)\left(n + \frac{1}{2}\right)(n+3)(n+4)(n+5)c(n+1) +$$

$$16(3888n + 2592 + 1872n^2 + 288n^3)\left(n + \frac{3}{2}\right)\left(n + \frac{1}{2}\right)(n+4)(n+5)$$

$$c(n+2) - 64$$

$$(-916n - 912 - 300n^2 - 32n^3)\left(n + \frac{3}{2}\right)\left(n + \frac{1}{2}\right)\left(n + \frac{5}{2}\right)(n+5)c(n+3)$$

$$+ 256 (12 n^2 + 47 n + 60 + n^3) \left(n + \frac{3}{2}\right) \left(n + \frac{1}{2}\right) \left(n + \frac{5}{2}\right) \left(n + \frac{7}{2}\right) c(n+4)$$

"degree upbound" = 0

$$"a" = n + \frac{1}{2}$$

$$"b" = n + 2$$

$$"z" = 12$$

$$"auxirec" = (-5184 - 19008 n - 20736 n^2 - 6912 n^3) (n+2) (n+3) (n+4) (n+5) c(n)$$

$$+ 12 (4536 + 5616 n + 1728 n^2) \left(n + \frac{1}{2}\right) (n+3) (n+4) (n+5) c(n+1)$$

$$+ 144 (3888 n + 2592 + 1872 n^2 + 288 n^3) \left(n + \frac{3}{2}\right) \left(n + \frac{1}{2}\right) (n+4) (n+5)$$

$$c(n+2) + 1728$$

$$(-916 n - 912 - 300 n^2 - 32 n^3) \left(n + \frac{3}{2}\right) \left(n + \frac{1}{2}\right) \left(n + \frac{5}{2}\right) (n+5) c(n+3)$$

$$+ 20736 (12 n^2 + 47 n + 60 + n^3) \left(n + \frac{3}{2}\right) \left(n + \frac{1}{2}\right) \left(n + \frac{5}{2}\right) \left(n + \frac{7}{2}\right) c(n+4)$$

case₇

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

case₈

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

case₉

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

case₁₀

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

case₁₁

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

auxirec1: degree up bound is zero and a(n)=n+1/2, b(n)=n+2, z=12

```
> auxirec1:=(-5184-19008*n-20736*n^2-6912*n^3)*(n+
2)*(n+3)*(n+4)*(n+5)*c(n)+12*(4536+5616*n+1728*n
^2)*(n+1/2)*(n+4)*(n+3)*(n+5)*c(n+1)+144*(3888*n
+2592+1872*n^2+288*n^3)*(n+1/2)*(n+3/2)*(n+4)*(n
+5)*c(n+2)+1728*(-916*n-912-300*n^2-32*n^3)*(n+1
/2)*(n+3/2)*(n+5/2)*(n+5)*c(n+3)+20736*(12*n^2+4
7*n+60+n^3)*(n+1/2)*(n+3/2)*(n+5/2)*(n+7/2)*c(n+
4):
```

```
> for i from 0 to 4 do auxirec1:=subs(c(n+i)=c,
auxirec1); od:
```

```
> auxirec1;
```

$$(-5184 - 19008n - 20736n^2 - 6912n^3)(n+2)(n+3)(n+4)(n+5)c$$

$$+ 12(4536 + 5616n + 1728n^2)\left(\frac{1}{2} + n\right)(n+4)(n+3)(n+5)c +$$

$$144$$

$$(3888n + 2592 + 1872n^2 + 288n^3)\left(\frac{1}{2} + n\right)\left(\frac{3}{2} + n\right)(n+4)(n+5)c$$

$$+ 1728$$

$$(-916n - 912 - 300n^2 - 32n^3)\left(\frac{1}{2} + n\right)\left(\frac{3}{2} + n\right)\left(\frac{5}{2} + n\right)(n+5)c$$

$$+ 20736(12n^2 + 47n + 60 + n^3)\left(\frac{1}{2} + n\right)\left(\frac{3}{2} + n\right)\left(\frac{5}{2} + n\right)\left(\frac{7}{2} + n\right)c$$

```
> solve(auxirec1=0,c);
```

c

c(n) can be any nonzero constant. Hence, $u(n+1)/u(n)=(12n+6)/(n+2)$. But this solution of u(n) doesn't satisfy the initial condition, $(u(0) = 0,$

$u(1) = -C_0, u(2) = 4 - C_0 + 6, u(3) = 32 - C_0 + 66, ?$), which is given by diffeqtorec.

auxirec2: degree up bound is zero and $a(n)=n+1/2$, $b(n)=n+2$, $z=-4$

```
> auxirec2:=(-5184-19008*n-20736*n^2-6912*n^3)*(n+
2)*(n+3)*(n+4)*(n+5)*c(n)-4*(4536+5616*n+1728*n^
2)*(n+1/2)*(n+4)*(n+3)*(n+5)*c(n+1)+16*(3888*n+2
592+1872*n^2+288*n^3)*(n+1/2)*(n+3/2)*(n+4)*(n+5
)*c(n+2)-64*(-916*n-912-300*n^2-32*n^3)*(n+1/2)*
(n+3/2)*(n+5/2)*(n+5)*c(n+3)+256*(12*n^2+47*n+60
+n^3)*(n+1/2)*(n+3/2)*(n+5/2)*(n+7/2)*c(n+4);
```

```
> for i from 0 to 4 do auxirec2:=subs(c(n+i)=c,
auxirec2); od:
```

```
> auxirec2;
```

$$\begin{aligned}
& (-5184 - 19008n - 20736n^2 - 6912n^3)(n+2)(n+3)(n+4) \\
& \quad (n+5)c \\
& \quad - 4(4536 + 5616n + 1728n^2) \left(\frac{1}{2} + n \right) (n+4)(n+3)(n+5)c + 16 \\
& \quad (3888n + 2592 + 1872n^2 + 288n^3) \left(\frac{1}{2} + n \right) \left(\frac{3}{2} + n \right) (n+4)(n+5)c \\
& \quad - 64 \\
& \quad (-916n - 912 - 300n^2 - 32n^3) \left(\frac{1}{2} + n \right) \left(\frac{3}{2} + n \right) \left(\frac{5}{2} + n \right) (n+5)c \\
& \quad + 256(12n^2 + 47n + 60 + n^3) \left(\frac{1}{2} + n \right) \left(\frac{3}{2} + n \right) \left(\frac{5}{2} + n \right) \left(\frac{7}{2} + n \right) c
\end{aligned}$$

```
> solve(auxirec2=0,c);
```

0

[c(n) is zero. u(n) doesn't have a hyper solution for this case.

[Conclusion: The number of rooted maps on the projective, enumerated with respect to edges, does not have a hypergeometric closed form.

[A.1.3---3 connected planar maps

[> restart;

[> eq0:=x*(1+u)^2=u;

$$eq0 := x(1+u)^2 = u$$

[> eq1:=x^2*(2/(1+x)+2*x)-u^2/(1+2*u)^3=y;

$$eq1 := x^2 \left(2 \frac{1}{1+x} + 2x \right) - \frac{u^2}{(1+2u)^3} = y$$

[> G:=eliminate({eq0,eq1}, u);

$$G := \left[\begin{aligned} & \left\{ u = \right. \\ & - \frac{x(-5yx^2 - 6x^4 + 8y + x + 3yx - 15x^2 - 6x^3 + 10x^5)}{12x^6 + 17x^2 - 28x^5 - 8y + 12yx - 12x^4 - 22x^3 - x - 6yx^3 + 14yx^2} \\ & , \{ 108x^9 + 182x^6 + 212x^8 + 32x^{10} + 4x^{11} - 182yx^5 + 248x^7 + yx \\ & - 20yx^2 - 93yx^3 + 38y^2x^2 + 28y^2x - x^3 + 10x^4 + 73x^5 + 8y^2 - 104yx^6 \\ & \left. - 4yx^8 + 8y^2x^4 - 32yx^7 + 25y^2x^3 - 178yx^4 + y^2x^5 \} \right\} \end{aligned} \right]$$

[> with(share): with(gfun):

[> deq:=algeqtodiffeq(op(1,op(2,G)), y(x), {y(0)=0});

$$\begin{aligned} deq := & \{ -28x^7 - 54x^6 - 28x^5 + 26x^4 + 20x^3 + 2x^2 \\ & + (2x^4 - 18x^3 - 40x^2 - 18x + 2)y(x) \\ & + (4x^5 + 15x^4 + 16x^3 + 3x^2 - 2x) \left(\frac{\partial}{\partial x} y(x) \right), y(0) = 0 \} \end{aligned}$$

[> diffeqtorec(deq,y(x),u(n));

$$\begin{aligned} & \{ u(0) = 0, u(3) = 2 + \frac{33}{2} _C_0, u(6) = 1 - \frac{153}{32} _C_0, u(7) = 24 _C_0, \\ & u(4) = -\frac{61}{4} _C_0, u(5) = \frac{279}{16} _C_0, u(1) = _C_0, u(2) = 1 - \frac{15}{2} _C_0, \\ & (2 + 4n)u(n) + (-3 + 15n)u(n+1) + (-8 + 16n)u(n+2) \\ & + (-9 + 3n)u(n+3) + (-6 - 2n)u(n+4) \} \end{aligned}$$

[> algeqtoseries(op(1,op(2,G)),x,y,8);

$$\left[-\frac{1}{8}x + \frac{31}{16}x^2 - \frac{1}{16}x^3 + \frac{61}{32}x^4 - \frac{279}{128}x^5 + \frac{409}{256}x^6 - 3x^7 - \frac{173}{256}x^8 + O(x^9), \right. \\ \left. x^2 + 2x^3 + x^6 + 4x^8 + 6x^9 + O(x^{10}) \right]$$

```

> receq:=(2+4*n)*u(n)+(-3+15*n)*u(n+1)+(-8+16*n)*u(n+
2)+(-9+3*n)*u(n+3)+(-6-2*n)*u(n+4): d:=4:
> p[4]:=n->(-6-2*n): p[3]:=n->(-9+3*n):
p[2]:=n->(-8+16*n): p[1]:=n->(-3+15*n):
p[0]:=n->(2+4*n):
> fp0:=factors(p[0](n));
fp0:=4, [[1/2+n, 1]]
> fp1:=factors(p[4](n-4+1));
fp1:=-2, [[n, 1]]
> m:=nops(op(2,fp0));
# m, the number of factors means the number of levels of next for loop
m:=1
> for i1 from 0 to 1 do
>   a[i1]:= op(1,op(1,op(2,fp0)))^i1 ;
>   b[i1]:= op(1,op(1,op(2,fp1)))^i1 ;
> od:
# The following part computes the number of a(n), b(n) that satisfy the
gcd(a,b+h)=1 (h are all nonnegative integers, k denote the number of sets
contains (a(n), b(n))
> k:=0: unassign('h'): hs:=[]: poly:=0:
> for i from 0 to 50 do A[i]:=0: B[i]:=0:
polyh[i]:=0: od:
> for i from 0 to 2^m-1 do
>   for j from 0 to 2^m-1 do flag:=1:
>     if gcd(a[i],b[j])=1 and
degree(a[i])=degree(b[j]) then
poly:=resultant(a[i],subs(n=n+h,b[j]),n);
hs:=[isolve(poly=0)];
>     if hs=[] then A[k]:=a[i]: B[k]:=b[j]:
polyh[k]:=poly: k:=k+1:

```

```

>   elif nops(hs)>1 then for s from 1 to nops(hs) do
      if type(op(2,op(1,op(s,hs))), negint) then
        flag:=flag*1: else flag:=flag*(-1): break:
      fi: od:
      if flag=1 then A[k]:=a[i]: B[k]:=b[j]:
        polyh[k]:=poly: k:=k+1: fi:
>   elif type(op(2,op(1,op(1,hs))), negint)
      then A[k]:=a[i]: B[k]:=b[j]: polyh[k]:=poly:
        k:=k+1:
>   fi;
>   fi;
>   od;
>   od;
>   k;

2
> for i from 0 to k-1 do "a"=A[i]; "b"=B[i] od;
      "a" = 1
      "b" = 1
      "a" =  $\frac{1}{2} + n$ 
      "b" = n

# pls change the directory "c:/hyperal.txt" to your local directory.
#read in the hypera procedure to find the polynomial solution of c(n),
> read "c:/hyperal.txt";
> for i from 0 to k-1 do a:=A[i]: b:=B[i]:
  print(case[i+1]); Hyperhomo(p,a,b,d); od:
      case1
      "no degree upbound for polynomial solution of c(n)"
      "degree upbound" = 0
      "a" = 1
      "b" = 1
      "z" = -1

"auxirec" = (2 + 4 n) c(n) - (-3 + 15 n) c(n+1) + (-8 + 16 n) c(n+2)
            - (-9 + 3 n) c(n+3) + (-6 - 2 n) c(n+4)

```

"degree upbound" = 2

"a" = 1

"b" = 1

"z" = $-\frac{1}{2}$

$$\begin{aligned} \text{"auxirec"} = & (2 + 4n) c(n) - \frac{1}{2} (-3 + 15n) c(n+1) \\ & + \frac{1}{4} (-8 + 16n) c(n+2) - \frac{1}{8} (-9 + 3n) c(n+3) + \frac{1}{16} (-6 - 2n) c(n+4) \end{aligned}$$

case₂

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

Next step is to solve the auxiliary recurrences that have valid degree bound of polynomial solution

auxirec1: degree up bound is zero, a(n)=1, b(n)=1, z=-1

```
> auxirec1:=(2+4*n)*c(n)-(-3+15*n)*c(n+1)+(-8+16*n)
  )*c(n+2)-(-9+3*n)*c(n+3)+(-6-2*n)*c(n+4):
> for i from 0 to 4 do auxirec1:=subs(c(n+i)=c,
  auxirec1); od:
> auxirec1;
(2+4 n) c - (-3+15 n) c + (-8+16 n) c - (-9+3 n) c
  + (-6-2 n) c
> solve(auxirec1=0,c);
```

c

The solution of c(n) is an arbitrary constant . Hence, u(n+1)/u(n)=-1, this hyper solution of u(n) doesn't satisfy the initial condition

auxirec2: degree up bound is two, a(n)=1, b(n)=1, z=-1/2

```
> auxirec2:=(2+4*n)*c(n)-1/2*(-3+15*n)*c(n+1)+1/4*
  (-8+16*n)*c(n+2)-1/8*(-9+3*n)*c(n+3)+1/16*(-6-2*
  n)*c(n+4):
> cc:=n->c1*n^2+c2*n+c3;
```

```

cc := n → c1 n2 + c2 n + c3
> for i from 0 to 4 do
  auxirec2 := subs(c(n+i)=cc(n+i), auxirec2); od:
> auxirec2;

```

$$\begin{aligned}
& (2+4n)(c1 n^2 + c2 n + c3) \\
& - \frac{1}{2}(-3+15n)(c1(n+1)^2 + c2(n+1) + c3) \\
& + \frac{1}{4}(-8+16n)(c1(n+2)^2 + c2(n+2) + c3) \\
& - \frac{1}{8}(-9+3n)(c1(n+3)^2 + c2(n+3) + c3) \\
& + \frac{1}{16}(-6-2n)(c1(n+4)^2 + c2(n+4) + c3)
\end{aligned}$$

```

> expand(auxirec2);

```

$$-\frac{19}{8}c1 + \frac{9}{4}c3 + \frac{9}{8}c2n + \frac{15}{8}c1n - \frac{5}{8}c2$$

```

> solve({-5/8*c2-19/8*c1+9/4*c3=0,
  9/8*c2+15/8*c1=0},{c3,c2,c1});

```

$$\{c2 = -\frac{45}{16}c3, c1 = \frac{27}{16}c3, c3 = c3\}$$

```

> S(n) := (-1/2)*(c1*n^2+c2*n+c3);

```

$$S(n) := -\frac{1}{2}c1 n^2 - \frac{1}{2}c2 n - \frac{1}{2}c3$$

```

> subs(c2 = -45/16*c3, c1 = 27/16*c3, S(n));

```

$$-\frac{27}{32}c3 n^2 + \frac{45}{32}n c3 - \frac{1}{2}c3$$

this polynomial solution of c(n) => u(n+1)/u(n)=

$-\frac{27 c3 n^2}{32} + \frac{45 c3 n}{32} - \frac{1 c3}{2}$. But this solution of u(n) doesn't agree with the two series generated by `algeqtoseries()`. Hence, hyper solution doesn't exist in this case.

Conclusion: There is no hypergeometric closed form for the number of 3-connected rooted planar maps which are enumerated w.r.t. edges.

A.1.4---4connected planar triangulations

```
> restart;
```

```
> f:=t*(t-1)/(1+t)^2;
```

```
> x:=t*(1-t)^2;
```

$$f = \frac{t(t-1)}{(1+t)^2}$$

$$x = t(1-t)^2$$

```
> E:=eliminate({f=t*(t-1)/(1+t)^2,x=t*(1-t)^2},t);
```

$$E := \left[\left\{ t = \frac{x f^2 - 2 x f + x + f - 4 f^2}{f(8 f + 1)} \right\}, \right.$$

$$\left. \left\{ -x f + 20 x f^2 + 16 f^3 + 8 x f^3 + x^2 f^3 - 3 x^2 f^2 + 3 x^2 f - x^2 \right\} \right]$$

```
> with(share):
```

See ?share and ?share,contents for information about the share library

```
> with(gfun):
```

Share Library: gfun

Authors: Salvy, Bruno and Zimmermann, Paul.

Description: package for the manipulation and discovery of generating functions

```
> deq:=algeqtodiffeq(op(1,op(2,E)),f(x),{f(0)=0});
```

$$deq := \{ f(0) = 0, 6x^2 + (-6x^2 + 8 - 88x)f(x)$$

$$+ (-8x - 54x^3 + 118x^2) \left(\frac{\partial}{\partial x} f(x) \right) + (16x^2 - 27x^4 - 104x^3) \left(\frac{\partial^2}{\partial x^2} f(x) \right) \}$$

```
> algeqtoseries(op(1,op(2,E)),x,f,8);
```

$$\left[-\frac{1}{4}\sqrt{x} - \frac{1}{8}x - \frac{3}{32}x^{(3/2)} - \frac{3}{32}x^2 - \frac{59}{512}x^{(5/2)} - \frac{21}{128}x^3 - \frac{1063}{4096}x^{(7/2)} \right.$$

$$- \frac{227}{512}x^4 + O(x^{(9/2)}), \frac{1}{4}\sqrt{x} - \frac{1}{8}x + \frac{3}{32}x^{(3/2)} - \frac{3}{32}x^2 + \frac{59}{512}x^{(5/2)} - \frac{21}{128}x^3$$

$$+ \frac{1063}{4096}x^{(7/2)} - \frac{227}{512}x^4 + O(x^{(9/2)}),$$

$$\left. -x + x^2 + x^4 + 3x^5 + 12x^6 + 52x^7 + 241x^8 + O(x^9) \right]$$

```
> diffeqtorec(deq,f(x),u(n));
```



```

{ u(1) = -C0, u(2) = - $\frac{1}{4}$  -  $\frac{5}{4}$  C0, u(0) = 0, (-6 - 27 n - 27 n2) u(n)
  + (30 + 14 n - 104 n2) u(n+1) + (24 + 40 n + 16 n2) u(n+2) }
> receq := (-6 - 27*n - 27*n^2)*u(n) + (30 + 14*n - 104*n^2)*u(n+
  1) + (24 + 40*n + 16*n^2)*u(n+2):
> d:=2:
> for i from 0 to d do p[i]:=n->0 od:
> p[0]:=n->op(1,op(1,receq)):
  p[1]:=n->op(1,op(2,receq)):
  p[2]:=n->op(1,op(3,receq)):
> fp0:=factors(p[0](n));
  fp0 :=  $\left[ -27, \left[ \left[ n + \frac{2}{3}, 1 \right], \left[ n + \frac{1}{3}, 1 \right] \right] \right]$ 
> fp1:=factors(subs(n=n-1,p[2](n)));
  fp1 :=  $\left[ 16, \left[ \left[ \frac{1}{2} + n, 1 \right], [n, 1] \right] \right]$ 
# m, the number of factors means the number of levels of next for loop
> m:=nops(op(2,fp0));
  m := 2
> for i1 from 0 to 1 do
>   for i2 from 0 to 1 do
>     a[i1+2*i2] := op(1,op(1,op(2,fp0)))^i1 *
      op(1,op(2,op(2,fp0)))^i2 :
>     b[i1+2*i2] := op(1,op(1,op(2,fp1)))^i1 *
      op(1,op(2,op(2,fp1)))^i2 :
>   od; od;
>
> k:=0: unassign('h'): hs:=[]: poly:=0:
> for i from 0 to 50 do A[i]:=0: B[i]:=0:
  polyh[i]:=0: od:
> for i from 0 to 2^m-1 do
>   for j from 0 to 2^m-1 do flag:=1:
>     if gcd(a[i],b[j])=1 and
      degree(a[i])=degree(b[j]) then
      poly:=resultant(a[i],subs(n=n+h,b[j]),n);

```

```

    hs:=[isolve(poly=0)];
>   if hs=[] then A[k]:=a[i]: B[k]:=b[j]:
        polyh[k]:=poly: k:=k+1:
>   elif nops(hs)>1 then for s from 1 to nops(hs) do
        if type(op(2,op(1,op(s,hs))), negint) then
            flag:=flag*1: else flag:=flag*(-1): break:
        fi: od:
        if flag=1 then A[k]:=a[i]: B[k]:=b[j]:
            polyh[k]:=poly: k:=k+1: fi:
>   elif type(op(2,op(1,op(1,hs))), negint) then
        A[k]:=a[i]: B[k]:=b[j]: polyh[k]:=poly:
        k:=k+1:
>       fi;
>       fi;
>   od;
> od;
> k;

```

6

```

> for i from 0 to k-1 do "a"=A[i]: "b"=B[i] od;

```

$$"a" = 1$$

$$"b" = 1$$

$$"a" = n + \frac{2}{3}$$

$$"b" = \frac{1}{2} + n$$

$$"a" = n + \frac{2}{3}$$

$$"b" = n$$

$$"a" = n + \frac{1}{3}$$

$$"b" = \frac{1}{2} + n$$

$$"a" = n + \frac{1}{3}$$

$$\begin{aligned} \text{"b"} &= n \\ \text{"a"} &= \left(n + \frac{2}{3}\right) \left(n + \frac{1}{3}\right) \\ \text{"b"} &= \left(\frac{1}{2} + n\right) n \end{aligned}$$

```
# pls change the directory "c:/hypera1.txt" to your local directory.
#read in the hypera procedure to find the polynomial solution of c(n),
> read "c:/hypera1.txt":
> for i from 0 to k-1 do a:=A[i]; b:=B[i];
  print(case[i+1]); Hyperhomo(p,a,b,d); od:
```

*case*₁

"degree upbound" = 1

"a" = 1

"b" = 1

"z" = $-\frac{1}{4}$

$$\begin{aligned} \text{"auxirec"} &= (-6 - 27n - 27n^2) c(n) - \frac{1}{4} (30 + 14n - 104n^2) c(n+1) \\ &+ \frac{1}{16} (24 + 40n + 16n^2) c(n+2) \end{aligned}$$

"no degree upbound for polynomial solution of c(n)"

*case*₂

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

*case*₃

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

*case*₄

"no degree upbound for polynomial solution of c(n)"

"no degree upbound for polynomial solution of c(n)"

*case*₅

"no degree upbound for polynomial solution of $c(n)$ "

"no degree upbound for polynomial solution of $c(n)$ "

$case_6$

"no degree upbound for polynomial solution of $c(n)$ "

"no degree upbound for polynomial solution of $c(n)$ "

solve the above auxiliary recurrences that have valid degree up bound for polynomial solution of $c(n)$

auxirec1: degree up bound is 1, $a(n)=1$, $b(n)=1$, $z=-1/4$

```
> auxirec1:=(-6-27*n-27*n^2)*c(n)-1/4*(30+14*n-104
  *n^2)*c(n+1)+1/16*(24+40*n+16*n^2)*c(n+2):
```

```
> cc:=n->c1*n+c2;
```

$cc := n \rightarrow c1 n + c2$

```
> for i from 0 to 2 do
```

```
  auxirec1:=subs(c(n+i)=cc(n+i), auxirec1); od:
```

```
> auxirec1;
```

$(-6 - 27 n - 27 n^2) (c1 n + c2)$

$-\frac{1}{4} (30 + 14 n - 104 n^2) (c1 (n + 1) + c2)$

$+\frac{1}{16} (24 + 40 n + 16 n^2) (c1 (n + 2) + c2)$

```
> solve(auxirec1=0, {c1, c2});
```

$\{c2 = -\frac{3}{8} c1, c1 = c1\}$

```
> S(n):=(-1/4)*((-8/3)*n*c2+c2);
```

$S(n) := \frac{2}{3} n c2 - \frac{1}{4} c2$

The hyper solution of $u(n)$ based this polynomial solution of $c(n)$ doesn't satisfy the initial condition $\{u(0)=0, u(1)=C0, u(2)=-(1/4)-(5/4)*c0,$

$series(-1 x + 1 x^2 + 1 x^4 + 3 x^5 + 12 x^6 + 52 x^7 + 241 x^8 + O(x^9), x, 9),)$

which is given by `diffeqtoec()` and `algeqtoseries()`.

Conclusion: There is no hypergeometric closed form for the number of

4-connected rooted planar triangulations which are enumerated w.r.t. vertices.

A.2 Hyperhomo[p,a,b,d]

#Objective: find hypergeometric solutions for a homogeneous recurrence $Ly=0$, where L is a linear recurrence operator.

#Calling sequence: p - coefficients of a given linear recursion

a,b - a pair of monic factors satisfy certain requirements

d - the order of the given recurrence

```
> Hyperhomo:=proc(p,a,b,d)
> local
  t1,t2,i,j,k,P,m,de,polyz,zs,newzs,inz,indexz,lp,z,
  auxirec,q,deg,polyd,qj,dx,ubd,debd,temp,a1,b1,bp:
  # compute nonzero z's for the given a and b
> a1:=n->a; b1:=n->b; t1:=1: t2:=1: 'i': 'j': 'k':
  'inz':'inzs':
> for i from 0 to d do
>   t1:=1: t2:=1:
>   for j from 0 to i-1 do t1:=t1*subs(n=n+j,a1(n)):
    od:
>   for k from i to d-1 do t2:=t2*subs(n=n+k,b1(n)):
    od:
>   P[i]:=p[i](n)*t1*t2;
> od;
> for i from 0 to d do de[i]:=0: od:
> for i from 0 to d do de[i]:= degree(expand(P[i]));
  od:
> 'm': m:=max(seq(de[i],i=0..d));
> polyz:=0;
> for i from 0 to d do polyz:=polyz+coeff(P[i],n,m)*z^i;
  od;
> 'zs': newzs:=(): zs:={solve(polyz=0,z)};
> if zs={} then print(" z has no solution! ");
> elif zs={0} then print("z has solution of 0
                        only"):
> else for inz from 1 to nops(zs minus {0}) do
>   if type(op(inz,zs),rational) then
>     newzs:=newzs union {op(inz,zs)}:
```

```

>      fi:      od:
> fi:
# for every nonzero z , make a auxiliary recurrence and find polynomial solution
of c(n)
> for indexz from 1 to nops(newzs) do
>   z:=newzs[indexz]: auxirec:=0:
>   for i from 0 to d do
>     lp[i]:=z^i*P[i]:
>     auxirec:=auxirec + z^i*P[i]*c(n+i):
>   od:
  # the auxiliary recurrence is here, applying algorithm POLY to solve c(n)
>   for i from 0 to d do q[i]:=0: od:
>   for j from 0 to d do
>     for i from j to d do
>       q[j]:=q[j]+binomial(i,j)*lp[i]: od:
>   od:
>   for i from 0 to d do deq[i]:=0 od:
>   for i from 0 to d do deq[i]:=degree(expand(q[i]))-i: od:
>   unassign('bp'): bp:= max(seq(deq[i],i=0..d));
>   polyd:=0: qj:={}: unassign('temp'): dx:={}:
#solve the polynomial for degree upbound of c(n)
>   unassign('x'):
>   for i from 0 to d do
>     if deq[i]=bp then
>       polyd:=polyd+lcoeff(expand(q[i]))*x^i; fi:
>     od:
>   dx:={solve(polyd,x)}:
>   temp:=max(seq(dx[i],i=1..nops(dx))):
>   ubd:=max(degree(0)-bp,-bp-1,temp):
>   if type(ubd,nonnegint) then print("degree upbound" =ubd);
  # the degree upbound is denoted as ubd
>   print("a"=a1(n)); print("b"=b1(n));
>   print("z"=z); print("auxirec"= auxirec);
>   else print("no degree upbound for polynomial
>     solution of c(n)");
>   fi:
> od:
> end:

```