Equilibrium Behaviour of Double-Sided Queueing System with Dependent Matching Time

by

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Abstract

In this thesis, we consider the equilibrium behaviour of a double-ended queueing system with dependent matching time in the context of taxi-passenger systems at airport terminal pickup. We extend the standard taxi-passenger model by considering random matching time between taxis and passengers in an airport terminal pickup setting. For two types of matching time distribution, we examine this model through analysis of equilibrium behaviour and optimal strategies. We demonstrate in detail how to derive the equilibrium joining strategies for passengers arriving at the terminal and the existence of a socially optimal strategies for partially observable and fully observable cases. Numerical experiments are used to examine the behaviour of social welfare and compare cases.

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Contents

Table of Notation

$N(t)$	$\triangleq$	state of the double-ended queue at time t
$\mathcal{F}$	$\triangleq$	state space
λ_1	$\triangleq$	passenger arrival rate
λ_0	$\triangleq$	taxi arrival rate when there is no passenger queue
λ_2	$\triangleq$	taxi arrival rate when there is a non-zero passenger queue
q	$\triangleq$	passenger joining probability
q_e	$\triangleq$	equilibrium passenger joining probability
q_o	$\triangleq$	socially optimal passenger joining probability
n^s	$\triangleq$	passenger joining threshold
n_e	$\triangleq$	equilibrium passenger joining threshold
n^*	$\triangleq$	socially optimal passenger joining threshold
λ_P^*	$\triangleq$	effective passenger arrival rate
λ_T^*	$\triangleq$	effective taxi arrival rate
ρ_0	$=$	$\frac{\lambda_1}{\lambda_0}$
ρ_1	$=$	$\frac{\lambda_1}{\lambda_2}$
ρ_2	$=$	$\frac{\lambda_1 q}{\lambda_2}$
π_n	$\triangleq$	stationary probability of system in state n
C_P	$\triangleq$	per unit waiting cost for passengers
C_T	$\triangleq$	per unit waiting cost for taxis
$C_{M,P}$	$\triangleq$	per unit matching time cost for passengers
$C_{M,T}$	$\triangleq$	per unit matching time cost for T
R	$\triangleq$	value of reward for passenger of taxi service
p	$\triangleq$	taxi fare
N	$\triangleq$	centralized taxi holding capacity
M	$\triangleq$	matching time
L_P	$\triangleq$	passenger queue length
L_T	$\triangleq$	taxi queue length
W_P	$\triangleq$	passenger waiting time
W_T	$\triangleq$	taxi waiting time
$U(\cdot)$	$\triangleq$	passenger utility of joining the passenger queue (assuming the passenger queue is non-zero) given passenger joining probability or threshold
$S(\cdot)$	$\triangleq$	social welfare of joining the passenger queue (assuming the passenger queue is non-zero) given passenger joining probability or threshold

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Chapter 1

Introduction

1.1 Motivation

Queueing theory is the study of systems where objects wait in a line to be served. It is used to study customers waiting to be served, telecommunication networks, delivery of goods, traffic systems, changes in a population, and many more real world scenarios when the arrival and service of customers is stochastic. It is important to be able to find properties of these types of systems in order to build valid models to examine and predict properties such as queue lengths, wait times, and optimal strategies for customers and servers. Queueing theory provides valuable tools in the analysis of transportation models. In particular, the taxi-passenger model has been studied previously in the queueing context, both as traditional one-sided models [5] and as double-ended queueing models, first studied by Kendall in 1951 [11]. In the era of technological innovation, we have seen the rapid rise of peer-to-peer transportation applications and other disruptor of traditional transportation structures dominating the transportation sector. With the skyrocketing popularity of these alternatives to traditional transportation and even more paradigm shifting technology on the horizon, such as autonomous vehicles, it is more important than ever to study and optimize the operation of taxi models. Not only would further knowledge in this area benefit taxi operators in competing with newer transportation models, but a deep understanding of how to choose strategies to maximize a taxi operator's own utility or social welfare would be greatly beneficial in the current environment of technological innovation which often creates extremely rapid shifts in a industry's operation. Taxi operators need to become adaptable to current and future technologies and co-exist with many competitors. Governments, systems designers, and other organizations that depend heavily on local transportation services or have some level of control over them would also benefit from understanding how to optimize taxi operations. These models can even be extended to the study of new transportation technologies such as ridehailing apps and autonomous vehicles and even food and goods delivery as they deal with the matching of servers and customers in a similar setting. In an age where a passenger's options for local transportation are ever growing, it is useful to study the system that would afford the passenger the highest utility and the passenger's own strategies when choosing among transportation options. It is also important to consider the social welfare of all involved parties: the government, taxi drivers and the taxi company, and passengers.

Much of the literature in taxi-passenger queueing models presents them in a double-ended queue with a taxi queue in one direction and a passenger queue in the other, where taxis and passengers at the front of their respective queues are matched and leave the system together. Often, the matching time of taxis and passengers is considered to be negligible. This assumption allows for a simplification of the double-ended queue into a more traditional one-sided queue, and thus allows for the application of known properties and methods for queueing models to obtain model results. However, non-zero matching time models have not been studied as thoroughly due to the difficulty in obtaining explicit analytical results [17]. Often, non-zero matching time models are examined only using simulation and numerical experiments [12] [14]. While the zero matching time assumption is useful in obtaining model results, this assumption may limit how well these models can represent real world situations. It is often not the case in real world situations that a passenger and taxi will be able to match and leave the system immediately, especially when the matching happens at areas of high demand where there may be a congestion of passengers and/or taxis looking to be matched. Longer and varying matching times can arise from taxis and passengers looking for an appropriate match, loading passengers onto taxis, waiting for pickup locations to become free, taxis traveling a short distance to pick up a passenger, and various other common scenarios. While more complex than the zero matching time model, it is important to consider a random matching time in a taxi-passenger setting.

While the study of double-ended models in queueing theory has largely been focused in the taxi-passenger context, it has broader implications such as in organ transplants [7], financial trading systems [1], new technology based peer-to-peer transportation and delivery systems, and more.

1.2 Organization

The thesis is organized as follows:

Chapter 2 provides a detailed literature review of double-ended queueing systems and taxi-passenger models. Preliminaries for understanding the use of equilibrium behaviour in queueing systems and the standard setup for a double-ended queue with zero matching time are provided. Details for a taxi-passenger model proposed by Wang and Liu in a 2019 paper [18] are provided and important literature results are shown and further simplified when appropriate. The extended matching time model this thesis proposes is introduced.

Chapter 3 discusses the taxi-passenger model when matching time follows a two mass point distribution. Two levels of information are considered: the partially observable case and fully observable case. The passenger utility function is found and used to find the passenger's equilibrium strategies. The system's overall social welfare function is found and the socially optimal strategy is discussed. The social welfare is studied through numerical experiments.

Chapter 4 uses a piecewise uniform distribution for matching time and again analyzes both the partially observable and fully observable cases. The passenger utility function is derived to find the equilibrium strategy. The social welfare function is found and numerical experiments are used to further analyze the social welfare and compare its behavior to the cases in chapter 3.

Chapter 5 gives concluding remarks and suggestions for further study.

Chapter 2

Double-Ended Taxi-Passenger Queueing System

2.1 Literature review

Double-ended queueing systems were first discussed by Kendall in 1951 in the taxi-passenger context [11]. Kendall presented a simple model which had taxis and passengers both arriving according to Poisson processes, matching, then leaving the system instantaneously. The Poisson arrivals and zero matching time allowed for the system to be studied as a one-dimensional queue which could be modeled by the Skellam distribution which represents the difference of two independent Poisson random variables. This model was further explored by Dobbie in 1961 with time dependent arrival rates using the Irwin-Skellam distribution [6]. Another solution for the time dependent taxi-passenger model and further properties, including the conditions for the existence of the limiting distribution, was proposed by Giveen in 1963 [8]. These earlier taxi-passenger queueing systems assumed Possion arrivals of both taxis and passengers and instantaneous matching times.

Further research attempted to analyze these systems in more realistic settings, taking into account different properties that might affect the taxi-passenger model in the real world. In 1965, Kashyap considered general arrival of taxis while passengers still arrived according to Poisson process, both with limited waiting space [10]. Impatient behaviour, where passengers and taxis may reneg due to impatience, was first analyzed in a model with time dependent Poisson arrivals by Conolly, Parthasarathy, and Selvaraju in 2002 [3].

A major motivation for understanding taxi-passenger systems is to understand how best to construct these systems for optimizing profit and reward for both taxi companies and individual passengers and optimizing social welfare. This understanding will advise taxi companies, governments, and other transportation planners on best policies and strategies. Optimization strategies and equilibrium behaviour has been studied in queueing theory extensively. Most notably, Hassin and Haviv's *To Queue or Not To Queue* (2003) is a comprehensive survey of the application of equilibrium behaviour to queueing theory [9]. Optimization of a queueing system was first studied by Noar (1969) by analyzing the optimal join/balk strategy in an M/M/1 queue and levying tolls as a way to optimize for social welfare [13]. Yang and Yang (2011) considered the equilibrium behaviour of the aggregate market, where taxis and passengers meet in a network according to a general meeting function and general bilateral searching, with market friction [20]. Shi and Lian looked at strategic behaviour of passengers and social welfare optimization strategies for the taxi-passenger system under different conditions [15] [16]. These conditions included the level of information passengers had when making the decision to join or balk (whether the system was fully observable, partially observable, or fully unobservable) and whether the system was centralized or decentralized (whether the behaviour was controlled by a centralized body that valued social welfare maximization or the passengers and taxis behaved independently to maximize their own utility). Wang, Wang, and Zhang (2017) studied the equilibrium behaviour of double-ended taxi-passenger queues under a gated policy, again in settings with differing levels of information [19]. Chai et al. (2019) considered equilibrium behaviour in a batch matching model with impatient servers and boundedly rational passengers [2]. Wang and Liu (2019) studied passenger equilibrium strategies in the taxi-passenger model under differing levels of information with dynamic taxi control (where the taxi behaviour may change depending on the system state) [18].

The double-ended queueing models cited thus far have dealt with systems where taxis and passengers match and leave the system instantaneously. With non-zero matching times, double-ended queueing systems are substantially more complicated as they cannot be simplified to be the difference of two random variables. Simulation methods have been used to study double-ended queues with non-zero matching time by Kim et al. (2010) [12]. A simple model with level dependent matching time has been studied analytically and algorithmically by Shi, Lian, and Shang (2015) using matrix analytic methods [17]. This thesis will focus on the airport pickup taxi-passenger scenario to study the equilibrium behaviour of a double-ended queue with non-zero matching time.

Airport taxi pickup has been studied though never as a double-ended queue to the best of the author's knowledge. Curry, Vany, and Feldman (1977) has studied airport pickup by taxis following a Poisson arrival process (single-ended queue) with competition from public transportation (random clearing) [5]. Airport taxi pickup models are compared through simulation by Passos et al. (2009) [14]. Conway et al. (2012) provides an overview of taxi structures at multiple airports with a focus on the centralized taxi holding area and dispatching procedures at John F. Kennedy International Airport (JFK) in New York City [4]. Yazici, Kamga, and Singhal (2016) goes into further detail specifying the structure of JFK and making policy suggestions based using the equilibrium strategy approach the analyze a logistic regression model for taxi decisions [21].

2.2 Preliminaries

2.2.1 Equilibrium behaviour in queueing systems

The game theory approach is often applied to queueing theory with the goal of understanding how the choices of different participants in a system impacts system outcomes and how they can be optimized for individuals or overall social welfare.

When analyzing equilibrium behaviour, the system under study is considered a non-cooperative game where every individual/organization has their own goals and own measures of reward and cost for entering the system and completing service. The metric researches wish to maximize is either the individual utility of a customer/player/organization or the overall social welfare of every player in the system [9]. These are represented by the utility function and social welfare function:

$$\text{Utility} = \text{Reward for service} - (\text{Payment for service} + \text{Cost of waiting}),$$
$$\text{Social Welfare} = \text{Rewards for service} - \text{Costs of waiting}.$$

The utility function represents an individual player's net gain from the game. In this thesis, the player we consider is the passenger and the game we consider is taking a taxi and completing service. The social welfare function represents the overall net gain for every player in the system, in this case both the passenger and taxi. Such games consider the Nash equilibrium, which represents the threshold at which a player has increased their own utility as much as possible. Socially optimal strategies are also considered as the strategy such that the social welfare among all players is maximized. This would be utilized in a cooperative case where one or more parties have incentive to maximize the overall net gain in the game. For example, in the airport taxi-passenger case, the airport or government might want to have a socially optimal system in place.

The behaviour and strategy of players refers to their choice to balk, choose not to join the system upon arrival, or reneg, leaving the system once they have joined. Depending on the information available to a customer when they enter into a system, different types of strategies are used. Let R represent reward for service, p represent payment for service, C_P, represent per unit waiting cost for a customer, and C_T represent per unit waiting cost for a server.

For unobservable or partially observable cases where arriving customers cannot see the entire queue length, their strategy is based on their probability of joining the system rather than balking, $q = P(\text{join})$. The equilibrium joining probability q_e is determined based on the potential customer arrival rate, Λ, the effective customer arrival rate, $\lambda^* = \Lambda q$, and $W(\lambda)$, the expected waiting time for a joining customer when customer arrival rate is λ. The utility and social welfare functions can be written:

$$\text{Customer Utility} = R - p - C_P W(\lambda),$$
$$\text{Social Welfare} = R - (C_P + C_T) W(\lambda).$$

The equilibrium probability represents the probability at which the customer will have made a net gain by joining the system, and therefore should join, rather than balking. By Hassin and Haviv, the equilibrium probability is given:

$$q_e = \begin{cases} 0, & p + CW(0) \geq R \\ q_e^*, & p + CW(0) < R < p + CW(\Lambda) \\ 1, & p + CW(\Lambda) \leq R \end{cases}$$

where q_e^* is such that $R - p - C_P W(\Lambda q_e) = 0$. A customer's socially optimal joining probability is defined $q^* = \arg\max_{0 \leq q \leq 1} \ R - (C_P + C_T) W(\Lambda q^*)$ [9].

For the observable case, customers are able to see the state of the system upon arrival and use this information to decide whether they join or balk. Rather than a probability, the customer makes their decision based on a threshold n_s such that if the state $N(t)$ is less than n_s, then they join, otherwise they balk. Letting W be the average waiting time for one customer, we can write the utility and social welfare functions with respect to $N(t) = n$ as:

$$\text{Customer Utility} = U_n = R - p - (n+1)C_P W,$$
$$\text{Social Welfare} = S_n = R - (n+1)(C_P + C_T)W.$$

The equilibrium threshold is given $n_s = \{n \ |U_{n-1} \geq 0, U_n < 0\}$. The socially optimal threshold is given $n^* = \arg\max_{n \in \mathbb{Z}^+} \ S_n$ [13].

2.2.2 Double-ended queues with zero matching time

The double-ended queueing model was first considered by Kendall briefly in a 1951 paper where he discussed many types of queueing models. In this model, there is a taxi queue, N_1, where taxis arrive according to a Poisson process with rate α and a passenger queue, N_2, where passengers arrive according to a Poisson process with arrival rate β. Since this model has instantaneous matching between passengers and taxis, at least one queue must be empty at any given moment. Kendall proposes the queue $Q = N_t - N_2$ which follows the Skellam distribution, transforming a two-dimensional queue system $\{N_1, N_2\}$ to a one-dimensional queue Q. This one-dimensional queue is positive when there is a taxi queue and negative when there is a passenger queue. Kendall shows that the mean queue length is finite only when the mean arrival rates between taxis and queues are equal [11].

The zero matching time models discussed in the Literature Review extend from Kendall's original model and take advantage of the simplification from a two-dimensional queue to a one-dimensional queue to analyze double-ended systems. This way, the system can be modeled as a modified birth and death process where the state may be negative. In some cases, such as in Wang and Liu [18], the system is similar to a truncated birth and death process which is shifted so that the state space begins at $-N$, the taxi waiting capacity, rather than 0.

2.3 Taxi-passenger queueing model with dynamic controls (Wang and Liu) [18]

The study developed in this thesis is an extension of the taxi-passenger double-ended queue model presented by Wang and Liu [18] which used dynamic controls for taxi arrival times depending on the state of the passenger queue. This model will be extended in this thesis by including non-zero matching time to the analysis.

The model considered in this paper is a double-ended queue with state space $\mathcal{F} = \{-N, -N+1, -N+2, \cdots, -1, 0, 1, 2, \cdots\}$ such that when $N(t) < 0$, there is a taxi queue of length $-N(t)$ and no passenger queue, when $N(t) = 0$, there is no taxi queue nor passenger queue, and when $N(t) > 0$, there is a passenger queue of length $N(t)$ and no taxi queue. The taxi waiting area has a capacity of N.

The dynamic control in this model describes the taxi arrival rate dynamically changing depending on the state of the system. Passengers arrive to the system following a Poisson process with rate λ_1 which represents the potential rate of customers. Upon arrival, if there is a non-zero taxi queue and no passenger queue (i.e. $N(t) < 0$), they automatically join. If there is no taxi queue (i.e. $N(t) \geq 0$), then passenger may join or balk given the information the passenger has about the system. This model is unique to other double-ended models for the taxi-passenger system as it employs dynamic controls for taxis which allows the taxi arrival rate to change depending on the state of the passenger queue. Taxis arrive according to Poisson process with rate λ_0 if there are no passengers waiting (i.e. $N(t) \leq 0$) and rate λ_2 if there is a non-zero passenger queue (i.e. $N(t) > 0$).

The per unit waiting cost for taxis and passengers are C_T and C_P, respectively. A passenger's reward at the end of service is R. The taxi fare paid by the passenger at the end of service is p. While in reality, fares are not always constant and can also be represented with a random variable, for simplicity we use a constant p which can be thought of as the mean fare for travelers taking the taxi from the airport. Some taxi companies, such as New York's Medallion taxis, charge a flat fare price for rides from John F. Kennedy international airport to any Manhattan destination [21].

This paper focused on studying the equilibrium and social behaviour of this model under two levels of visibility. Passengers upon arrival may have knowledge of the taxi queue state (partially observable case), both taxi and passenger queue states (observable case), or only the taxi and passenger arrival rates (unobservable case). Wang and Liu [18] analyzed the utility function for a passenger and the social welfare function for both passengers and taxis given passenger joining probability

(q) or joining threshold (n_s). They found the stationary probabilities for the model depending on the level of visibility and, in some cases, the equilibrium and socially optimal passenger joining probabilities or thresholds.

2.3.1 Literature results (Wang and Liu) [18]

In this section, key results from Wang and Liu [18] are given. More detailed proofs are provided and results are simplified.

Proposition 2.1 Let $\rho_0 = \frac{\lambda_1}{\lambda_0}$, $\rho_1 = \frac{\lambda_1}{\lambda_2}$, and $\rho_2 = \frac{\lambda_1 q}{\lambda_2}$. From Wang and Liu [18], we have the stationary probabilities of the double-ended system with dynamic control for the partially observable case given as:

$$\pi_{-N} = \frac{(1-\rho_0)(1-\rho_2)}{1-\rho_2-\rho_0^{N+1}+\rho_0^N\rho_2}.$$

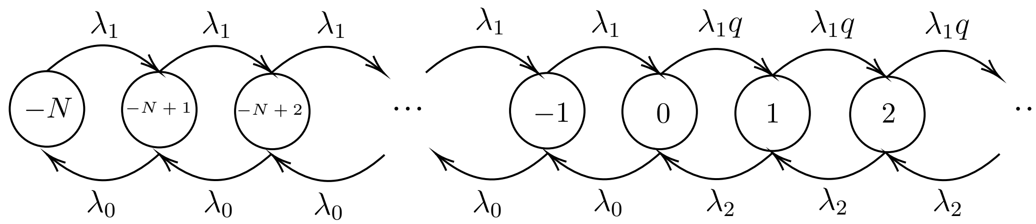
For $n = -N+1, -N+2, -N+3, \cdots,$

$$\pi_n = \begin{cases} \rho_0^{N+n}\pi_{-N}, & -N < n \leq 0 \\ \rho_0^N\rho_2^n\pi_{-N}, & \text{otherwise} \end{cases}.$$

The stability condition is $\rho_2 = \frac{\lambda_1 q}{\lambda_2} < 1$.

PROOF. Wang and Liu [18], presented a rough proof for the stationary distribution of the partially observable case. Here we will provide more detail. With the consideration of joining probability, the system's transition diagram is shown in figure 2.1.

Figure 2.1: State transition diagram for Wang and Liu [18] double-ended queueing model for a passenger-taxi system with dynamic controls for the partially observable case.



Note that the transition diagram looks similar to a birth and death process where the lowest value of the state space is $-N$ rather than 0.

From the transition diagram, we find the balance equations:

$$\begin{aligned}
\lambda_1\pi_{-N} &= \lambda_0\pi_{-N+1}, \\
(\lambda_0+\lambda_1)\pi_n &= \lambda_0\pi_{n+1}+\lambda_1\pi_{n-1} \quad , -N+1 \le n \le -1, \\
(\lambda_0+\lambda_1 q)\pi_0 &= \lambda_1\pi_{-1}+\lambda_2\pi_1, \\
(\lambda_0+\lambda_1 q)\pi_n &= \lambda_1 q\pi_{n-1}+\lambda_2\pi_{n+1} \quad , n \ge 1.
\end{aligned}$$

By adding each equation to the preceding, we have:

$$\begin{aligned}
\lambda_1\pi_n &= \lambda_0\pi_{n+1} \quad , -N \le n \le -1, \\
\lambda_1 q\pi_n &= \lambda_2\pi_{n+1} \quad , n \ge 0.
\end{aligned}$$

By rearranging we have:

$$\pi_{n+1} = \begin{cases} \frac{\lambda_1}{\lambda_0}\pi_n, & -N \leq n \leq -1 \\ \frac{\lambda_1 q}{\lambda_2}\pi_n, & n \geq 0 \end{cases} = \begin{cases} \rho_0\pi_n, & -N \leq n \leq -1 \\ \rho_2\pi_n, & n \geq 0 \end{cases}.$$

Now, we find that:

$$\begin{aligned} \pi_{-N+1} &= \rho_0\pi_{-N}, \\ \pi_{-N+2} &= \rho_0^2\pi_{-N}, \\ &\vdots \\ \pi_{-1} &= \rho_0^{N-1}\pi_{-N}, \\ \pi_0 &= \rho_0^N\pi_{-N}, \\ \pi_1 &= \rho_0^N\rho_2\pi_{-N}, \\ \pi_2 &= \rho_0^N\rho_2^2\pi_{-N}, \\ \pi_3 &= \rho_0^N\rho_2^3\pi_{-N}, \\ &\vdots \end{aligned}$$

Therefore, we have that:

$$\pi_n = \begin{cases} \rho_0^{N+n}\pi_{-N}, & -N \leq n \leq 0 \\ \rho_0^N\rho_2^n\pi_{-N}, & n > 0 \end{cases}.$$

Now, by normalizing property of stationary distribution:

$$
\begin{aligned}
1 &= \sum_{n=-N}^{\infty} \pi_n \\
1 &= \pi_{-N}\rho_0^N \left[\sum_{i=0}^{N} \left(\frac{1}{\rho_0}\right)^i + \sum_{i=1}^{\infty} \rho_2^i \right] \\
1 &= \pi_{-N}\rho_0^N \left[\frac{1-\rho_0^{-N-1}}{1-\frac{1}{\rho_0}} + \frac{\rho_2}{1-\rho_2} \right] \\
1 &= \pi_{-N} \left[\frac{\rho_0^{N+1}-1}{\rho_0 - 1} + \frac{\rho_0^N \rho_2}{1-\rho_2} \right] \\
1 &= \pi_{-N} \left[\frac{(\rho_0^{N+1}-1)(1-\rho_2) + \rho_0^N \rho_2(\rho_0 - 1)}{(\rho_0-1)(1-\rho_2)} \right] \\
1 &= \pi_{-N} \left[\frac{\rho_0^{N+1} - \rho_0^{N+1}\rho_2 - 1 + \rho_2 + \rho_0^{N+1}\rho_2 - \rho_2\rho_0^N}{(\rho_0-1)(1-\rho_2)} \right] \\
1 &= \pi_{-N} \left[\frac{1-\rho_2-\rho_0^{N+1}+\rho_2\rho_0^N}{(1-\rho_0)(1-\rho_2)} \right] \\
\pi_{-N} &= \frac{(1-\rho_0)(1-\rho_2)}{1-\rho_2-\rho_0^{N+1}+\rho_2\rho_0^N}
\end{aligned}
$$

The geometric series used above require the conditions $\left|\frac{1}{\rho_0}\right| \neq 1$, or equivalently $\lambda_0 \neq \lambda_1$, and $|\rho_2| < 1$, or equivalently $\lambda_1 q < \lambda_2$. Thus, we have the final stationary probabilities:

$$
\pi_n = \begin{cases} \frac{\rho_0^{N+n}(1-\rho_0)(1-\rho_2)}{1-\rho_2-\rho_0^{N+1}+\rho_2\rho_0^N}, & -N \leq n \leq 0 \\ \frac{\rho_0^N \rho_2^n (1-\rho_0)(1-\rho_2)}{1-\rho_2-\rho_0^{N+1}+\rho_2\rho_0^N}, & n > 0 \end{cases}.
$$

□

Lemma 2.1 The effective arrival rate for taxis and passengers in the partially observable case are equal and is given:

$$\lambda^* = \pi_{-N}\lambda_1\left(\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \frac{\lambda_2\rho_0^N q}{\lambda_2 - \lambda_1 q}\right)$$

PROOF. We will show the effective arrival rates for passengers and taxis are equal and arrive at a more simplified result than the one provided in Wang and Liu [18]. The effective arrival rate of process refers to the rate of arrival adjusted to reflect the arrival process without passengers who have balked. We will individually examine passenger effective arrival rate, λ_P^*, and taxi effective arrival rate, λ_T^*.

$$\begin{aligned}
\lambda_P^* &= \lambda_1 \sum_{n=-N}^{-1} \pi_n + \lambda_1 q \sum_{n=0}^{\infty} \pi_n \\
&= \pi_{-N}\rho_0^N\lambda_1\left[\sum_{n=1}^{N}\left(\frac{1}{\rho_0}\right)^n + q\sum_{n=0}^{\infty}\rho_2^n\right] \\
&= \pi_{-N}\rho_0^N\lambda_1\left[\frac{1-\rho_0^{-N}}{\rho_0(1-\rho_0^{-1})} + \frac{q}{1-\rho_2}\right] \\
&= \pi_{-N}\lambda_1\left[\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \frac{\rho_0^N q}{1 - \frac{\lambda_1 q}{\lambda_2}}\right] \\
&= \pi_{-N}\lambda_1\left[\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \frac{\lambda_2\rho_0^N q}{\lambda_2 - \lambda_1 q}\right]
\end{aligned}$$

$$
\begin{aligned}
\lambda_T^* &= \lambda_0 \sum_{n=-N+1}^{0} \pi_n + \lambda_2 \sum_{n=1}^{\infty} \pi_n \\
&= \pi_{-N}\rho_0^N \left[\lambda_0 \sum_{n=0}^{N-1} \left(\frac{1}{\rho_0}\right)^n + \lambda_2 \sum_{n=1}^{\infty} \rho_2^n \right] \\
&= \pi_{-N}\rho_0^N \left[\frac{\lambda_0\lambda_1(1-\rho_0^{-N})}{\lambda_1-\lambda_0} + \frac{\lambda_1\lambda_2 q}{\lambda_2-\lambda_1 q} \right] \\
&= \pi_{-N}\lambda_1 \left[\frac{\lambda_0(\rho_0^N-1)}{\lambda_1-\lambda_0} + \frac{\lambda_2\rho_0^N q}{\lambda_2-\lambda_1 q} \right]
\end{aligned}
$$

Therefore, $\lambda_P^* = \lambda_T^* = \lambda^*$ as needed. □

Lemma 2.2 The expected queue lengths for passenger and taxi queues in the partially observable case are given, respectively:

$$E(L_P) = \frac{\pi_{-N}\rho_0^N\rho_2}{(1-\rho_2)^2},$$

$$E(L_T) = \frac{\pi_{-N}\rho_0^{N+1}\left(1-(N+1)\rho_0^{-N} + N\rho_0^{-N-1}\right)}{(\rho_0-1)^2}.$$

[18]

PROOF. Again, we provide proof for a more simplified result for expected queue length of passengers, $E(L_P)$, and taxis, $E(L_T)$.

$$
\begin{aligned}
E(L_P) &= \sum_{n=0}^{\infty} n\pi_n = \pi_{-N}\rho_0^N \sum_{n=0}^{\infty} n\rho_2^n = \frac{\pi_{-N}\rho_0^N\rho_2}{(1-\rho_2)^2} \\
E(L_T) &= \sum_{n=-N}^{0} -n\pi_n = \pi_{-N}\rho_0^N \sum_{n=0}^{N} n\left(\frac{1}{\rho_0}\right)^n = \frac{\pi_{-N}(\rho_0^N - N + 1 + N\rho_0^{-1})}{\rho_0(1-\frac{1}{\rho_0})^2}
\end{aligned}
$$

□

Lemma 2.3 The expected waiting times for passenger and taxi queues for the partially observable case are given:

$$E(W_P) = \frac{\pi_{-N}\rho_0^N\rho_2}{\lambda^*(1-\rho_2)^2},$$

$$E(W_T) = \frac{\pi_{-N}\rho_0^{N+1}\left(1-(N+1)\rho_0^{-N}+N\rho_0^{-N-1}\right)}{\lambda^*\left(\rho_0-1\right)^2}.$$

[18]

PROOF. This is a direct result of Little's law. □

Proposition 2.2 The expected waiting time for a passenger who arrives to a system with no taxi queue (i.e. $N(t) \geq 0$) in the partially observable case is given:

$$E(W) := E(W_P|N(t) \geq 0) = \frac{1}{\lambda_2 - \lambda_1 q}.$$

PROOF. Again, this proof is based off a proof sketch given in Wang and Liu [18] and filled in with more detail.

Note that by Little's law:

$$\begin{aligned}
E(W_P|N(t)\geq 0) &= \frac{E(W_P, N(t)\geq 0)}{P(N(t)\geq 0)}\\
&= \frac{\frac{E(L_P)}{\lambda_1 q}}{P(N(t)\geq 0)}\\
&= \frac{\frac{\pi_{-N}\rho_0^N\rho_2}{(1-\rho_2)^2\lambda_1 q}}{\sum_{n=0}^{\infty}\pi_n}\\
&= \frac{\rho_2}{(1-\rho_2)^2\lambda_1 q\sum_{n=0}^{\infty}\rho_2^n}\\
&= \frac{\lambda_1 q}{(1-\rho_2)\lambda_1 q\lambda_2}\\
&= \frac{1}{\lambda_2-\lambda_1 q}
\end{aligned}$$

□

Proposition 2.3 Let $\rho_0 = \frac{\lambda_1}{\lambda_0}$, $\rho_1 = \frac{\lambda_1}{\lambda_2}$. From Wang and Liu [18], we have the stationary probabilities of the double-ended system with dynamic control in the observable case given as:

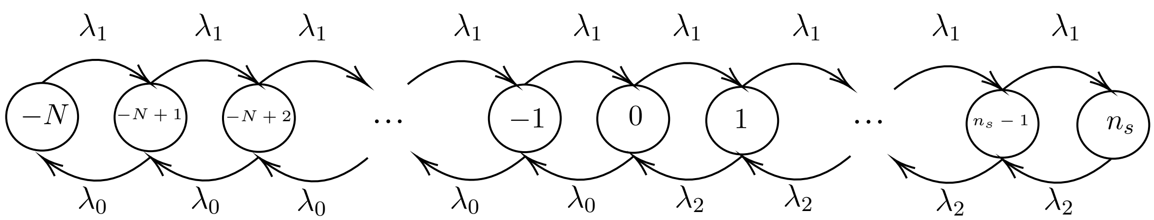
$$\pi_{-N} = \frac{(1-\rho_0)(1-\rho_1)}{1-\rho_0^{N+1}+\rho_1+\rho_0^N\rho_1-\rho_0^N\rho_1^{n_s+1}+\rho_0^{N+1}\rho_1^{n_s+1}}.$$

For $n = -N+1, -N+2, -N+3, \cdots,$

$$\pi_n = \begin{cases}\rho_0^{N+n}\pi_{-N}, & -N < n \leq 0\\ \rho_0^N\rho_1^n\pi_{-N}, & \text{otherwise}\end{cases}.$$

Proof. Wang and Liu [18], presented a rough proof for the stationary distribution of the observable case. Here we will provide more detail. In the observable case, there is a threshold, n_s such that if a passenger arrives to the system and sees less than n_s passengers they join and otherwise balk. Therefore, the queue has a capacity of n_s since no new passengers will join the queue when it is of length n_s. With the consideration of joining probability, the system's transition diagram is shown in figure 2.2.

Figure 2.2: State transition diagram for Wang and Liu [18] double-ended queueing model for a passenger-taxi system with dynamic controls in the observable case with threshold n_s.



Note that the transition diagram looks similar to a birth and death process where the lowest value of the state space is $-N$ rather than 0 and the capacity of the queue is n_s.

From the transition diagram, we find the balance equations:

$$\lambda_1\pi_{-N} = \lambda_0\pi_{-N+1},$$

$$(\lambda_0 + \lambda_1)\pi_n = \lambda_0\pi_{n+1} + \lambda_1\pi_{n-1} \quad , -N+1 \leq n \leq -1,$$

$$(\lambda_0 + \lambda_1)\pi_0 = \lambda_1\pi_{-1} + \lambda_2\pi_1,$$

$$(\lambda_0 + \lambda_1)\pi_n = \lambda_1\pi_{n-1} + \lambda_2\pi_{n+1} \quad , 1 \leq n \leq n_s.$$

By adding each equation to the preceding, we have:

$$\lambda_1\pi_n = \lambda_0\pi_{n+1} \quad , -N \le n \le -1,$$

$$\lambda_1\pi_n = \lambda_2\pi_{n+1} \quad , 0 \le n < n_s.$$

By rearranging we have:

$$\pi_{n+1} = \begin{cases} \frac{\lambda_1}{\lambda_0}\pi_n, & -N \le n \le -1 \\ \frac{\lambda_1}{\lambda_2}\pi_n, & n \ge 0 \end{cases} = \begin{cases} \rho_0\pi_n, & -N \le n \le -1 \\ \rho_1\pi_n, & 0 \le n < n_s \end{cases}.$$

Now, we find that:

$$\begin{aligned} \pi_{-N+1} &= \rho_0\pi_{-N}, \\ \pi_{-N+2} &= \rho_0^2\pi_{-N}, \\ &\vdots \\ \pi_0 &= \rho_0^N\pi_{-N}, \\ \pi_1 &= \rho_0^N\rho_1\pi_{-N}, \\ \pi_2 &= \rho_0^N\rho_1^2\pi_{-N}, \\ &\vdots \\ \pi_{n_s-1} &= \rho_0^N\rho_1^{n_s-1}\pi_{-N}, \\ \pi_{n_s} &= \rho_0^N\rho_1^{n_s}\pi_{-N}. \end{aligned}$$

Therefore, we have that:

$$\pi_n = \begin{cases} \rho_0^{N+n}\pi_{-N}, & -N \leq n \leq 0 \\ \rho_0^N \rho_1^n \pi_{-N}, & 0 \leq n < n_s \end{cases}.$$

Now, by normalizing property of stationary distribution:

$$\begin{aligned} 1 &= \sum_{n=-N}^{n_s} \pi_n \\ 1 &= \pi_{-N}\rho_0^N \left[\sum_{i=0}^{N} \left(\frac{1}{\rho_0}\right)^i + \sum_{i=1}^{n_s} \rho_1^i \right] \\ 1 &= \pi_{-N}\rho_0^N \left[\frac{1-\rho_0^{-N-1}}{1-\frac{1}{\rho_0}} + \frac{\rho_1(1-\rho_1^{n_s})}{1-\rho_1} \right] \\ 1 &= \pi_{-N} \left[\frac{(\rho_0^{N+1}-1)(1-\rho_1) + \rho_0^N\rho_1(\rho_0-1)(1-\rho_1^{n_s})}{(\rho_0-1)(1-\rho_1)} \right] \\ \pi_{-N} &= \frac{(1-\rho_0)(1-\rho_1)}{1-\rho_0^{N+1}+\rho_1+\rho_0^N\rho_1-\rho_0^N\rho_1^{n_s+1}+\rho_0^{N+1}\rho_1^{n_s+1}} \end{aligned}$$

The geometric series used above require that $\left|\frac{1}{\rho_0}\right| \neq 1$, or equivalently $\lambda_0 \neq \lambda_1$, and $|\rho_1| \neq 1$, or equivalently $\lambda_1 \neq \lambda_2$. □

Lemma 2.4 The effective arrival rate for taxis and passengers in the observable case are equal and written as:

$$\lambda^* = \pi_{-N}\lambda_1 \left(\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \frac{\lambda_2\rho_0^N(1-\rho_1^{n_s})}{\lambda_2 - \lambda_1} \right).$$

Proof. The effective arrival rate of process refers to the rate of arrival adjusted to reflect the arrival process without passengers who have balked. We will individually examine passenger effective arrival rate, λ_P^*, and taxi effective arrival rate, λ_T^*.

$$\begin{aligned}
\lambda_P^* &= \lambda_1 \sum_{n=-N}^{-1} \pi_n + \lambda_1 q \sum_{n=0}^{n_s} \pi_n \\
&= \pi_{-N} \rho_0^N \lambda_1 \left[\sum_{n=1}^{N} \left(\frac{1}{\rho_0} \right)^n + q \sum_{n=0}^{n_s} \rho_1^n \right] \\
&= \pi_{-N} \rho_0^N \lambda_1 \left[\frac{1-\rho_0^{-N}}{\rho_0(1-\rho_0^{-1})} + \frac{1-\rho_1^{n_s}}{1-\rho_1} \right] \\
&= \pi_{-N} \lambda_1 \left(\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \frac{\lambda_2 \rho_0^N (1-\rho_1^{n_s})}{\lambda_2 - \lambda_1} \right)
\end{aligned}$$

$$\begin{aligned}
\lambda_T^* &= \lambda_0 \sum_{n=-N+1}^{0} \pi_n + \lambda_2 \sum_{n=1}^{n_s} \pi_n \\
&= \pi_{-N} \rho_0^N \left[\lambda_0 \sum_{n=0}^{N-1} \left(\frac{1}{\rho_0} \right)^n + \lambda_2 \sum_{n=1}^{n_s} \rho_1^n \right] \\
&= \pi_{-N} \rho_0^N \left[\frac{\lambda_0 \lambda_1 (1-\rho_0^{-N})}{\lambda_1 - \lambda_0} + \frac{\lambda_2 \rho_1 (\rho_1^{n_s})}{1-\rho_1} \right] \\
&= \pi_{-N} \lambda_1 \left(\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \frac{\lambda_2 \rho_0^N (1-\rho_1^{n_s})}{\lambda_2 - \lambda_1} \right)
\end{aligned}$$

Therefore, $\lambda_P^* = \lambda_T^* = \lambda^*$ as needed. □

Lemma 2.5 The expected queue lengths for passenger and taxi queues in the observable case are given:

$$E(L_P) = \frac{\pi_{-N} \rho_0^N \rho_1 \left[1 - (n_s+1)\rho_1^{n_s} + n_s \rho_1^{n_s+1} \right]}{(1-\rho_1)^2},$$

$$E(L_T) = \frac{\pi_{-N}\rho_0^{N+1}\left(1-(N+1)\rho_0^{-N}+N\rho_0^{-N-1}\right)}{(\rho_0-1)^2}.$$

[18]

PROOF. Again, we provide proof for a more simplified result for expected queue length of passengers, $E(L_P)$, and taxis, $E(L_T)$.

$$E(L_P) = \sum_{n=0}^{n_s} n\pi_n = \pi_{-N}\rho_0^N \sum_{n=0}^{n_s} n\rho_1^n = \frac{\pi_{-N}\rho_0^N\rho_1\left[1-(n_s+1)\rho_1^{n_s}+n_s\rho_1^{n_s+1}\right]}{(1-\rho_1)^2},$$

$$E(L_T) = \sum_{n=-N}^{0} -n\pi_n = \pi_{-N}\rho_0^N \sum_{n=0}^{N} n\left(\frac{1}{\rho_0}\right)^n = \frac{\pi_{-N}(\rho_0^N - N + 1 + N\rho_0^{-1})}{\rho_0(1-\frac{1}{\rho_0})^2}.$$

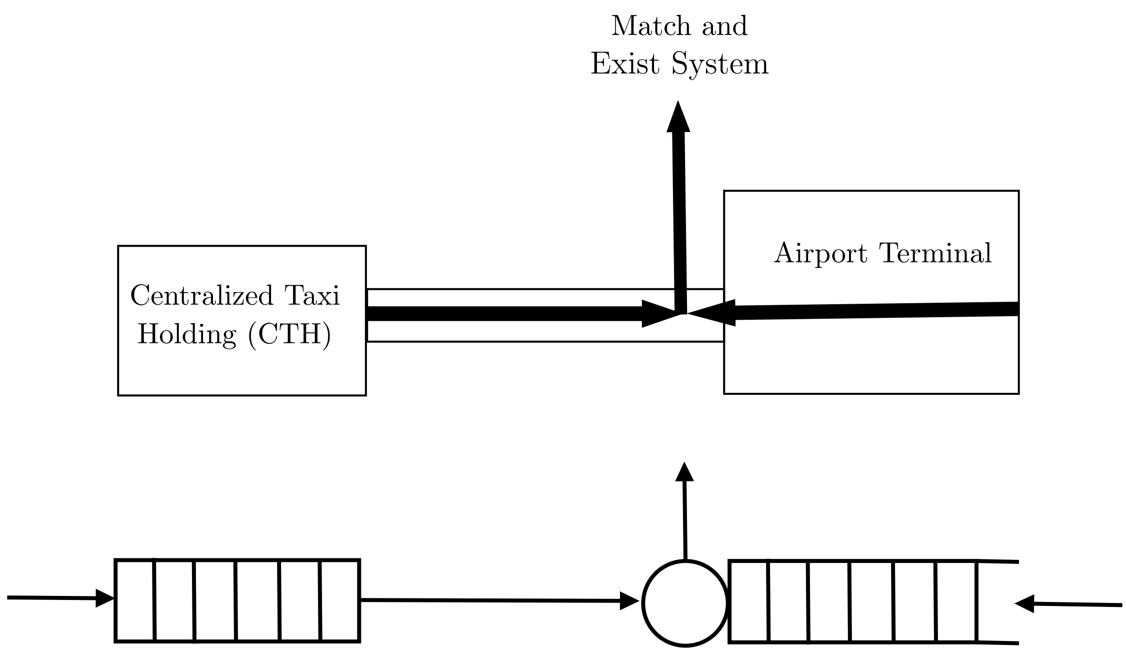
□

2.4 Extended matching time model

The double-ended taxi-passenger queueing model with dynamic controls presented in Wang and Liu [18] is extended it by considering a non-zero matching time between taxis and passengers described in the previous section. The Wang and Liu [18] model is particularly relevant in the airport taxi pickup scenario. Taxis will wait in a finite capacity waiting area, the centralized taxi holding (CTH), and a dispatcher will direct them to proceed to the terminal pickup area. It can be assumed that the dispatcher will send taxis to the terminal pickup according to the dynamic control Poisson rates described in Wang and Liu [18].

The matching time, M, describes the time it takes for taxis to travel from the CTH to the terminal pickup area, park, connect with the passenger, and load that passenger and their luggage before departure. Passengers who arrive at the terminal have many choices for transportation: taxi, airport shuttle, app-based ridehailing services, and more. Therefore, the joining and balking passenger probability or threshold is important to consider. Groups of passengers traveling together can be considered as one single passenger.

Figure 2.3: Diagram of passenger-taxi double-ended queue in the airport pickup scenario showing the taxi queue with finite capacity and a taxi's path from the CTH to terminal where the taxi matches with a passenger from the passenger queue and exits the system.



In previously studied double-ended queueing models with non-zero matching time, the system was represented as a two-dimensional queueing process such as $\{N_P(t), N_T(t)\}$ where $N_P(t)$ and $N_T(t)$ represent the queues for passengers and taxis respectively [18]. Due to non-zero matching time, these two queues could both be non-zero at once, which is a much more complex system to study than double-ended queues with zero matching time which could be represented as a one dimensional queue $N(t) = N_P(t) - N_T(t)$ since zero matching time means at least one queue is always empty. In this extended matching time model, a taxi in transit from the CTH to terminal pickup can be thought of as artificially designated to a passenger. Therefore, this taxi and passenger pair can be thought of as no longer available in the system, effectively following the Wang and Liu [18] model. While the taxi and passenger have already been artificially "matched", there is still a random matching time which is dependent on the system state. This is similar to the operation of ridehailing apps where a passenger and driver are matched, and therefore unavailable to the rest of the system, but there is still a period of waiting time while the driver travels to pick up the passenger from their current location. While both passenger and taxi can be thought of as removed from the system during this matching time, the matching time is significant to both parties when considering strategic behaviour. Therefore, in this model we may take advantage of the simplicity of a one-dimensional model while still examining the equilibrium behaviour effects of matching time.

Matching time, M, between taxis and passengers will be represented by a random variable which is dependent on the state of the taxi-passenger system, $N(t)$. We consider two distributions for this conditional matching time random variable $(M|N(t) = n)$, a simple two mass point distribution and a piecewise continuous uniform distribution.

Chapter 3

Two Mass Point Matching Time

First consider the simple two mass point matching time random variable where $M \in \{k_1, k_2\}$ for $k_1 < k_2 \in \mathbb{Z}^+$ and:

$$P(M = k_1 | N(t) = n) = \begin{cases} 1, & n \leq 0 \\ 0, & \text{otherwise} \end{cases},$$

$$P(M = k_2 | N(t) = n) = \begin{cases} 0, & n \leq 0 \\ 1, & \text{otherwise} \end{cases}.$$

It is reasonable to assume this distribution as it considers matching time in two cases of the system: when there is no passenger queue versus when there is no taxi queue at the terminal and passengers are waiting. It stands to reason that when there is no passenger queue, the internal airport roads between the CTH and terminal will have less traffic and parking and loading of passengers and luggage will take less time.

For the two mass point distribution, we have:

$$E(M|N(t) = n) = \begin{cases} k_1, & n \leq 0 \\ k_2, & \text{otherwise} \end{cases}.$$

3.1 Partially observable case for two mass matching time

In the partially observable case, the passenger upon arrival can see the state of the taxi queue and has knowledge of the passenger arrival rate but cannot see the state of the passenger queue. This would represent the scenario where the airport terminal might display the number of taxis currently in CTH. If the taxi queue is non-zero, then there is no passenger queue so the arriving passenger joins. However, if the taxi queue is zero and passengers are waiting, then the arriving passenger joins with probability q.

In this section, we look at how to find the equilibrium and socially optimal joining probabilities.

3.1.1 Passenger utility for the partially observable case with two point matching time

Since the passenger's choice to join or balk when arriving at a system is the most important choice being made in this system, the joining probability q is the quantity of interest. We also focus on the scenario when a passenger arrives to a system with no taxi queue as this is the scenario where this decision takes place. We begin by looking at the utility function to find the equilibrium joining probability.

Recall:

- R represents the passenger's intrinsic reward for taxi service
- p represents taxi fare
- C_P represents the passenger's cost per unit time of waiting associated with service
- $C_{M,P}$ represents the passenger's cost per unit time of waiting associated with matching time
- λ_1 represents passenger arrival rate
- λ_0 represents taxi arrival rate when there is no passenger queue
- λ_2 represnts the taxi arrival rate when there is a non-zero passenger queue
- N represents the central taxi holding capacity
- q representing the passenger joining probability

Lemma 3.1 The utility function for a passenger arriving to a partially observable system with two point matching time where there is no taxi queue is given:

$$U(q) = R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P}\pi_{-N}\rho_0^N \left[k_1 \left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1 - \lambda_0} \right) + k_2 \left(\frac{\lambda_1 q}{\lambda_2 - \lambda_1 q} \right) \right].$$

PROOF. The utility function for this scenario is:

$$U(q) = R - p - C_P E(W) - C_{M,P} E(M).$$

The last term, $C_{M,P}E(M)$, represents the expected cost incurred due to the matching time. Since the matching time is a random variable which depends on the state of the system but is unknown by arriving passengers, it does not affect the value of $E(W)$. Additionally, there will be non-zero matching time for any case of a passenger and taxi matching and leaving the system, regardless of whether there is a queue for taxis or passengers. Thus, it is reasonable to include the expected cost in the utility function this way.

Now, examine the utility function:

$$\begin{aligned}
U(q) &= R - p - C_P E(W) - C_{M,P} E(M) \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \left[\sum_{n=-N}^{\infty} E(M|N(t) = n)\pi_n \right] \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \left[k_1 \pi_{-N} + k_1 \sum_{n=-N+1}^{0} \rho_0^{N+n} \pi_{-N} + k_2 \sum_{n=1}^{\infty} \rho_0^N \rho_2^n \pi_{-N} \right] \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \pi_{-N} \rho_0^N \left[k_1 \sum_{n=-N}^{0} \rho_0^n + k_2 \sum_{n=1}^{\infty} \rho_2^n \right] \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \pi_{-N} \rho_0^N \left[k_1 \sum_{n=0}^{N} \rho_0^{-n} + k_2 \sum_{n=1}^{\infty} \rho_2^n \right] \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \pi_{-N} \rho_0^N \left[k_1 \left(\frac{1 - \rho_0^{-N-1}}{1 - \frac{1}{\rho_0}} \right) + k_2 \left(\frac{\rho_2}{1 - \rho_2} \right) \right] \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \pi_{-N} \rho_0^N \left[k_1 \left(\frac{\lambda_1 (1 - \rho_0^{-N-1})}{\lambda_1 - \lambda_0} \right) + k_2 \left(\frac{\lambda_1 q}{\lambda_2 - \lambda_1 q} \right) \right]
\end{aligned}$$

□

Theorem 3.1 The equilibrium passenger joining probability, q_e, upon entering a partially observable system with two point matching time is given:

$$q_e = \begin{cases} 0, & p + \frac{C_p}{\lambda_2} + C_{M,P}\pi_{-N}\rho_0^N\left[k_1\left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right)\right] \geq R \\ q_e^*, & \frac{C_P}{\lambda_2} < R - p - \frac{C_{M,P}\pi_{-N}\rho_0^N k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0} < \frac{C_P + C_{M,P}\pi_{-N}\rho_0^N k_2\lambda_1}{\lambda_2-\lambda_1} \\ 1, & p + \frac{C_P}{\lambda_2-\lambda_1} + C_{M,P}\pi_{-N}\rho_0^N\left[k_1\left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right) + k_2\left(\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\right] \leq R \end{cases}$$

where:

$$q_e^* = \frac{R - p - C_{M,P}\pi_{-N}\rho_0^N\left[\frac{k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right]\lambda_2 - C_P}{\left(R - p - C_{M,P}\pi_{-N}\rho_0^N\left[\frac{k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0} + k_2\right]\right)\lambda_1}.$$

Proof.

The equilibrium probability is piecewise on three intervals.

The first interval is when the utility of a system with no new customers joining is less than or equal to 0 ($U(q) \leq 0$). Here, the equilibrium probability is zero as a joining passenger would not have any gain even if no other customer enters the queue. When there are no customers entering the queue, the queue waiting time for a passenger is $\frac{1}{\lambda_2}$ and the matching time is:

$$\sum_{n=-N}^{0} E(M|N(t)=n)\pi_n = \frac{\pi_{-N}\rho_0^N k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}.$$

Therefore, the utility of this case is:

$$0 \geq R - p - \frac{C_p}{\lambda_2} - C_{M,P}\pi_{-N}\rho_0^N \left[k_1 \left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1 - \lambda_0} \right) \right].$$

Thus, when $p + \frac{C_p}{\lambda_2} + C_{M,P}\pi_{-N}\rho_0^N \left[k_1 \left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0} \right) \right] \geq R$, the equilibrium probability is $q_e = 0$.

The second interval is when the utility under the full potential arrival rate, λ_1, is non-negative. Here, the equilibrium probability is 1. Since the utility is always non-negative, even when the maximum amount of other passengers are arriving to the system, the passenger should always join. When the effective passenger arrival rate is λ_1, the queue waiting time per passenger is $\frac{1}{\lambda_2 - \lambda_1}$ and the matching time is:

$$\sum_{n=-N}^{\infty} E(M|N(t) = n)\pi_n = \pi_{-N}\rho_0^N \left[k_1 \left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1 - \lambda_0} \right) + k_2 \left(\frac{\lambda_1}{\lambda_2 - \lambda_1} \right) \right].$$

Therefore, when $p + \frac{C_P}{\lambda_2 - \lambda_1} + C_{M,P}\pi_{-N}\rho_0^N \left[k_1 \left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0} \right) + k_2 \left(\frac{\lambda_1}{\lambda_2-\lambda_1} \right) \right] \leq R$, the equilibrium probability is given $q_e = 1$.

The final interval is given by

$$p + \frac{C_p}{\lambda_2} + C_{M,P}\pi_{-N}\rho_0^N \left(\frac{k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1 - \lambda_0} \right) < R$$

and

$$R < p + \frac{C_P}{\lambda_2 - \lambda_1} + C_{M,P}\pi_{-N}\rho_0^N \cdot \left[\frac{k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1 - \lambda_0} + \frac{k_2\lambda_1}{\lambda_2 - \lambda_1} \right].$$

By combining the inequalities and rearranging, we find:

$$\frac{C_P}{\lambda_2} < R - p - \frac{C_{M,P}\pi_{-N}\rho_0^N k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0} < \frac{C_P + C_{M,P}\pi_{-N}\rho_0^N k_2\lambda_1}{\lambda_2-\lambda_1}.$$

To find the equilibrium joining probability in the final interval, q_e^*, set the utility function to zero such that $U(q_e^*) = 0$ and solve.

$$0 = R - p - \frac{C_P}{\lambda_2 - \lambda_1 q_e^*} - C_{M,P}\pi_{-N}\rho_0^N\left[k_1\left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right) + k_2\left(\frac{\lambda_1 q_e^*}{\lambda_2-\lambda_1 q_e^*}\right)\right]$$

$$\frac{C_P + C_{M,P}\pi_{-N}\rho_0^N k_2\lambda_1 q_e^*}{\lambda_2 - \lambda_1 q_e^*} = R - p - C_{M,P}\pi_{-N}\rho_0^N\left[\frac{k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right]$$

Let:

$$A_1 = R - p - C_{M,P}\pi_{-N}\rho_0^N\left[\frac{k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right],$$

$$A_2 = C_{M,P}\pi_{-N}\rho_0^N k_2.$$

Then, we have:

$$\begin{aligned}
\frac{C_p + A_2\lambda_1 q_e^*}{\lambda_2 - \lambda_1 q_e^*} &= A_1 \\
C_P + A_2\lambda_1 q_e^* &= A_1\lambda_2 - A_1\lambda_1 q_e^* \\
(A_1 + A_2)\lambda_1 q_e^* &= A_1\lambda_2 - C_P \\
q_e^* &= \frac{A_1\lambda_2 - C_P}{(A_1 + A_2)\lambda_1} \\
q_e^* &= \frac{R - p - C_{M,P}\pi_{-N}\rho_0^N\left[\frac{k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right]\lambda_2 - C_P}{\left(R - p - C_{M,P}\pi_{-N}\rho_0^N\left[\frac{k_1\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0} + k_2\right]\right)\lambda_1}
\end{aligned}$$

□

3.1.2 Social welfare for the partially observable case for two mass matching time

The social welfare function takes into account both the utility of the passenger and taxi. This section will look at how an arriving passenger's joining probability would affect social welfare.

Recall:

- C_T represents the taxi's cost per unit time of waiting associated with service
- $C_{M,T}$ represents the taxi's cost per unit time of waiting associated with matching time

Theorem 3.2 The social welfare function, $S(q)$, upon entering a partially observable system with two point matching time is given:

$$S(q) = A_3 + A_4\left[\frac{R - C_P q}{(\lambda_2 - \lambda_1 q)^2}\right] - A_5\Bigg[A_6 + \frac{\lambda_0 k_2(\rho_0^N - 1)q + \lambda_2\rho_0^N(1-\rho_0^{-N-1})}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_1 q)} + \frac{\lambda_2 \rho_0^N k_2 q^2}{(\lambda_2 - \lambda_1 q)^2}\Bigg]$$

where:

$$A_3 = \frac{R\pi_{-N}\lambda_0\lambda_1(\rho_0^N - 1)}{\lambda_1 - \lambda_0} - \frac{C_T\pi_{-N}\rho_0^{N-1}(1-\rho_0^{-N}(1+N) + N\rho_0^{-(N+1)})}{(1-\frac{1}{\rho_0})^2},$$

$$A_4 = \pi_{-N}\rho_0^N\lambda_1\lambda_2,$$

$$A_5 = -(C_{M,P} + C_{M,T})\pi_{-N}^2\lambda_1^2\rho_0^N,$$

$$A_6 = \frac{\lambda_0 k_1(\rho_0^N - 1)(1-\rho_0^{-N-1})}{(\lambda_1 - \lambda_0)^2}.$$

Proof. The social welfare function adds the utility for both passengers and taxis, scaled by their respective effective arrival rates.

$$\begin{aligned} S(q) &= \lambda^*\left(R - p - C_P E(W_P) - C_{M,P}E(M)\right) + \lambda^*\left(p - C_T E(W_T) - C_{M,T}E(M)\right) \\ &= \lambda^*\left(R - C_P E(W_P) - C_T E(W_T)\right) + \lambda^* E(M)\left(-C_{M,P} - C_{M,T}\right). \end{aligned}$$

Let $S_0(q) = \lambda^*\left(R - C_P E(W_P) - C_T E(W_T)\right)$ represent the social welfare of the system without consideration of matching time and $S_M(q) = \lambda^* E(M)\left(-C_{M,P} - C_{M,T}\right)$ represent the social welfare associated with matching time. Then, we have: $S(q) = S_0(q) + S_M(q)$

By Wang and Liu [18], $S_0(q)$ is concave down and has a maximum when $q =$

$\min\left\{\{q \mid \frac{d}{dq}S_0(q)=0\},1\right\}$.

$$
\begin{aligned}
S_0(q) &= \lambda^*\left(R-p-C_PE(W_P)\right)+\lambda^*\left(p-C_TE(W_T)\right)\\
&= \lambda^*R - C_PE(L_P) - C_TE(L_T)\\
&= \lambda^*R - \frac{C_P\pi_{-N}\rho_0^N\rho_2}{(1-\rho_2)^2} - \frac{C_T\pi_{-N}\rho_0^{N-1}(1-\rho_0^{-N}(1+N)+N\rho_0^{-(N+1)})}{(1-\frac{1}{\rho_0})^2}\\
&= R\pi_{-N}\lambda_1\left[\frac{\lambda_0(\rho_0^N-1)}{\lambda_1-\lambda_0}+\frac{\lambda_2\rho_0^Nq}{\lambda_2-\lambda_1q}\right] - \frac{C_P\pi_{-N}\rho_0^N\lambda_1\lambda_2q}{(\lambda_2-\lambda_1q)^2}\\
&\quad - \frac{C_T\pi_{-N}\rho_0^{N-1}(1-\rho_0^{-N}(1+N)+N\rho_0^{-(N+1)})}{(1-\frac{1}{\rho_0})^2}
\end{aligned}
$$

Let:

$$
\begin{aligned}
A_3 &= \frac{R\pi_{-N}\lambda_0\lambda_1(\rho_0^N-1)}{\lambda_1-\lambda_0} - \frac{C_T\pi_{-N}\rho_0^{N-1}(1-\rho_0^{-N}(1+N)+N\rho_0^{-(N+1)})}{(1-\frac{1}{\rho_0})^2},\\
A_4 &= \pi_{-N}\rho_0^N\lambda_1\lambda_2.
\end{aligned}
$$

Then, we have $S_0(q)=A_3+A_4\left[\frac{R-C_Pq}{(\lambda_2-\lambda_1q)^2}\right]$ where A_4 is clearly always positive.

Now, examine $S_M(q)$:

$$
\begin{aligned}
S_M(q) &= \lambda^*E(M)\left(-C_{M,P}-C_{M,T}\right)\\
&= -\lambda^*\left(C_{M,P}+C_{M,T}\right)\pi_{-N}\rho_0^N\left[k_1\left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right)+k_2\left(\frac{\lambda_1q}{\lambda_2-\lambda_1q}\right)\right]\\
&= -\left(C_{M,P}+C_{M,T}\right)\pi_{-N}\rho_0^N\left[\pi_{-N}\lambda_1\left(\frac{\lambda_0(\rho_0^N-1)}{\lambda_1-\lambda_0}+\frac{\lambda_2\rho_0^Nq}{\lambda_2-\lambda_1q}\right)\right]\times\\
&\quad \left[k_1\left(\frac{\lambda_1(1-\rho_0^{-N-1})}{\lambda_1-\lambda_0}\right)+k_2\left(\frac{\lambda_1q}{\lambda_2-\lambda_1q}\right)\right]
\end{aligned}
$$

Let

$$A_5 = -(C_{M,P} + C_{M,T})\pi_{-N}^2 \lambda_1^2 \rho_0^N,$$

$$A_6 = \frac{\lambda_0 k_1(\rho_0^N - 1)(1 - \rho_0^{-N-1})}{(\lambda_1 - \lambda_0)^2}.$$

Since costs of waiting, probabilities, and arrival rates are always positive, A_5 is always negative.

Note that when $\lambda_0 < \lambda_1$, then $\rho_0 > 1$ and therefore $(\rho_0^N - 1)$ and $(1 - \rho_0^{-N-1})$ are both positive. Similarly, when $\lambda_0 > \lambda_1$, $(\rho_0^N - 1)$ and $(1 - \rho_0^{-N-1})$ are both negative. Since λ_0 and k_1 are both positive, A_6 is always positive.

Then we have:

$$S_M(q) = A_5 \left[A_6 + \frac{\lambda_0 k_2(\rho_0^N - 1)q + \lambda_2 \rho_0^N (1 - \rho_0^{-N-1})}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_1 q)} + \frac{\lambda_2 \rho_0^N k_2 q^2}{(\lambda_2 - \lambda_1 q)^2} \right].$$

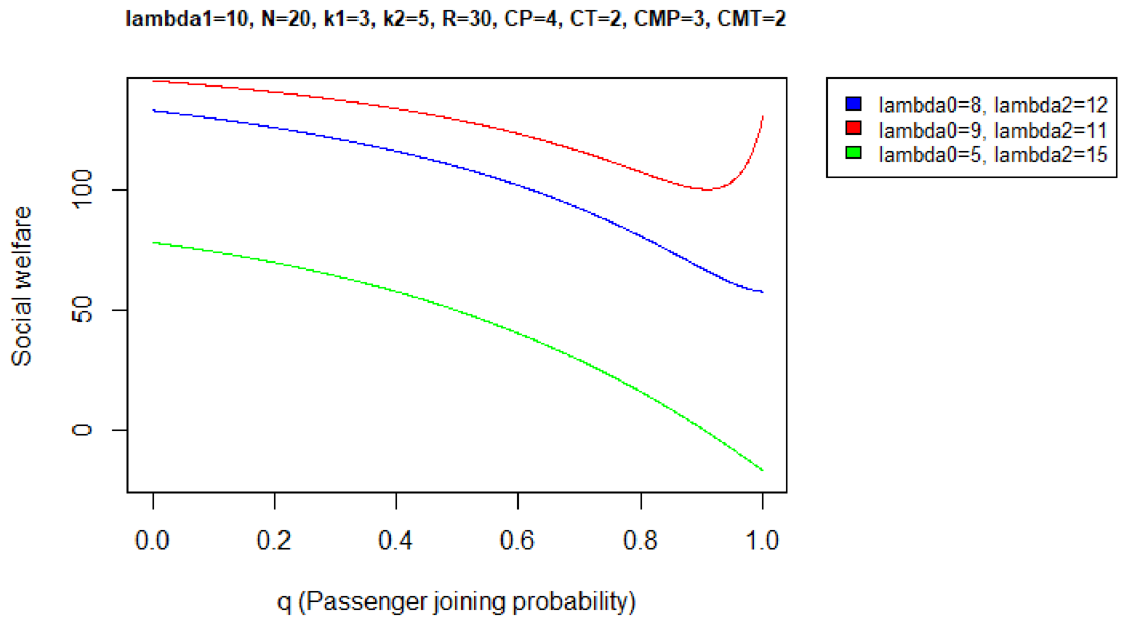
□

Numerical experiments for social welfare of partially observable two mass point matching time case

As the social welfare function is too complicated for direct analysis, we conduct some numerical experiments to illustrate the behaviour of social welfare under different conditions.

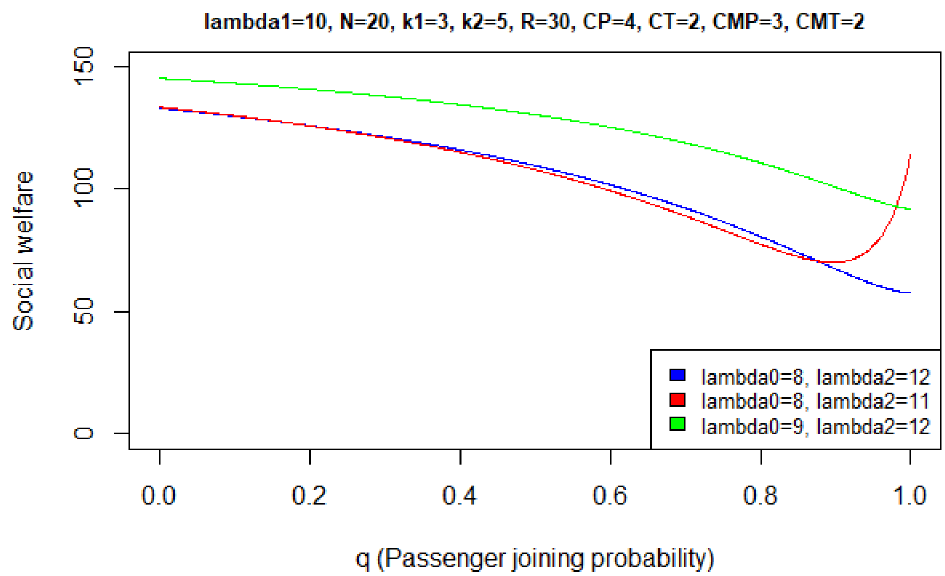
Figure 3.1 shows the social welfare by passenger joining probability for three different levels of arrival rates, λ_0, λ_1, and λ_2. We see that in general, the social welfare decreases as passenger joining probability increases. When the taxi arrival

Figure 3.1: Social welfare by the passenger joining probability (partially observable case with two mass point matching time) when $\lambda_1 = 10, N = 20, k_1 = 3, k_2 = 5, R = 30, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$.



rates are closer to the passenger arrival rate, we see the social welfare rises as passenger joining probability approaches 1 and social welfare is higher in general. When the taxi arrival rates are further from the passenger arrival rate, the social welfare is lower for all q and there is a more steep decline of social welfare when passenger joining probability approaches 1. When the taxi arrival rates are closer to the passenger arrival rate, the socially optimal strategy for the passenger would to either always balk ($q = 0$) or always join ($q = 1$). When taxi arrival rates are further from the passenger arrival rate, the passenger's socially optimal strategy is to always balk. This indicates that it is in the taxi operators' interest to have taxi arrival rates close to passenger arrival rates as it will increase social welfare as well as being the only case where passengers joining the queue leads to higher social welfare.

Figure 3.2: Social welfare by the passenger joining probability (partially observable case with two mass point matching time) when $\lambda_1 = 10, N = 20, k_1 = 3, k_2 = 5, R = 30, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$, varying λ_0 and λ_1 individually.

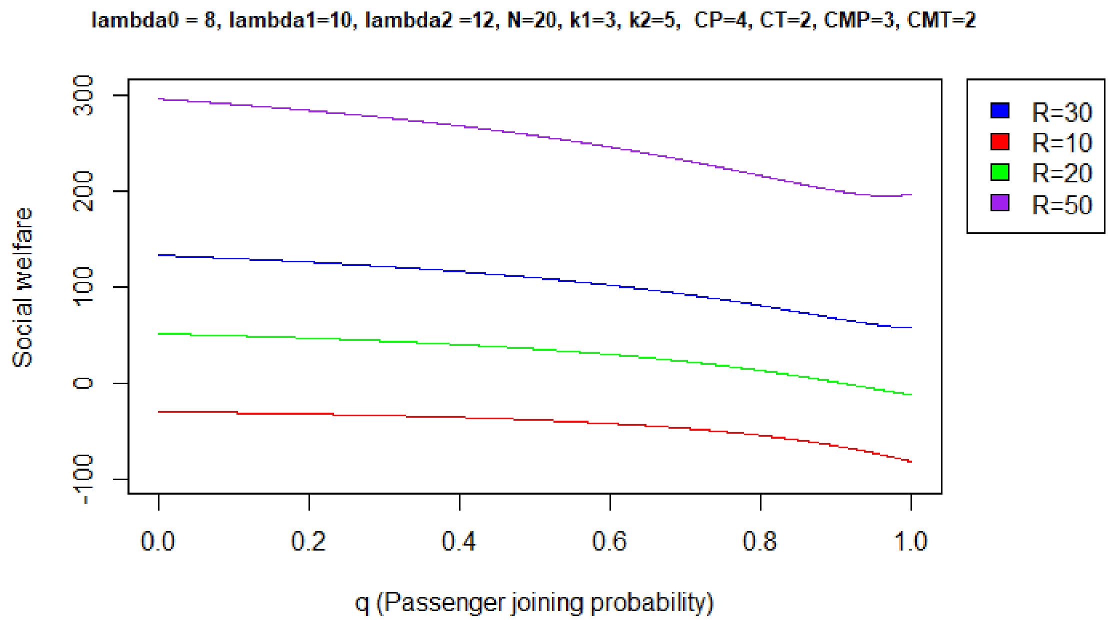


In figure 3.2, we compare the social welfare function where $\lambda_0 = 8$, $\lambda_1 = 10$, and $\lambda_2 = 12$ to cases where λ_0 and λ_2 change separately. We see that change in λ_2, the taxi arrival rate when the taxi queue (CTH) is empty and passenger queue is not, changes the shape of the social welfare function. Change in λ_0, the taxi arrival rate when the passenger queue is empty, seems to change the function's shape slightly but causes a vertical shift in the social welfare function. Therefore, controlling both taxi arrival rates is important for the taxi operator to maximize social welfare though controlling λ_2 would be a priority as it leads to the passenger's socially optimal strategy of always joining, resulting in more customers for the taxi company.

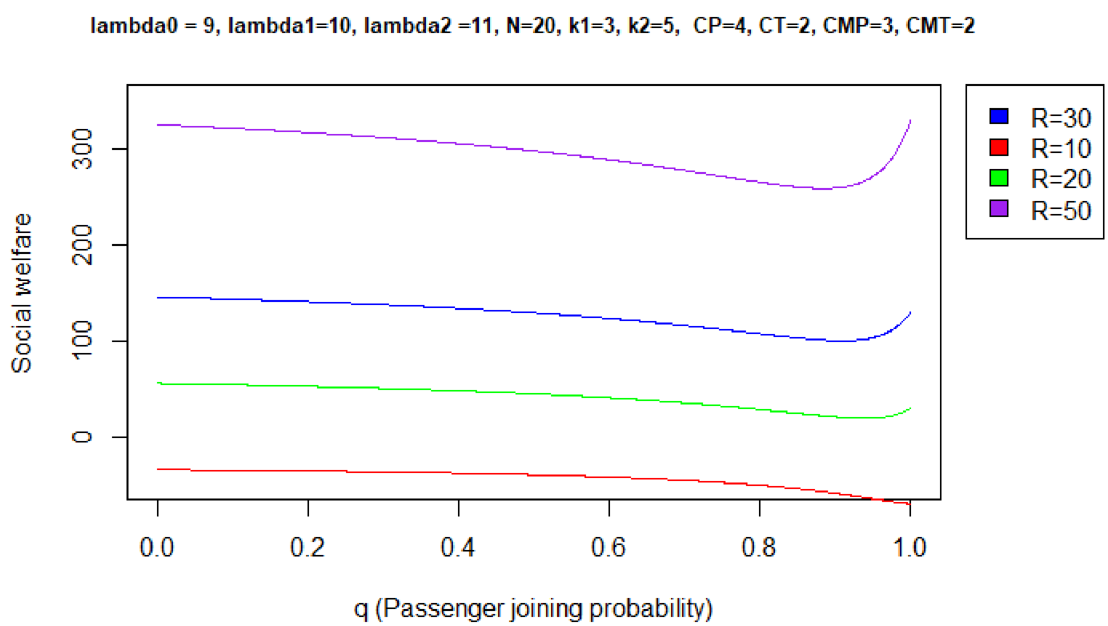
Figure 3.3 shows the social welfare function at differing levels of passenger reward, R, for both the $\lambda_0 = 8,\ \lambda_1 = 10,\ \lambda_2 = 12$ case and the $\lambda_0 = 9,\ \lambda_1 = 10,\ \lambda_2 = 11$ case. Recall that R represents the intrinsic value of reward the passenger would receive at the completion of service. As expected, increasing reward causes a vertical increase in the social welfare function. Figure 3.3(b) shows that when the taxi arrival rates are closer to the passenger arrival rate, a higher R leads to a steeper rise in social welfare when q approaches 1 whereas for very low R, this rise in social welfare does not occur. A passenger may value a taxi ride higher if other means of transportation out of the airport is more expensive, inconvenient, or unavailable. For taxi companies, it is important to understand how passengers' reward from a taxi ride would influence the social welfare and adjust taxi arrival rates and fare prices in response.

Figure 3.3: Social welfare by the passenger joining probability (partially observable case with two mass point matching time) when $N = 20, k_1 = 3, k_2 = 5, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$, varying by arrival rates and passenger reward, R.

(a) $\lambda_0 = 8, \lambda_1 = 10, \lambda_2 = 12$



(b) $\lambda_0 = 9, \lambda_1 = 10, \lambda_2 = 11$



3.2 Observable case for two mass matching time

In the observable case, arriving passengers can see both the taxi and passenger queue lengths. As before, if the passenger sees a taxi queue upon arrival and no passenger queue (i.e. $N(t) \leq 0$), they will join the system. If the passenger sees a passenger queue of length $n \in \mathbb{Z}^+$, then the system is in state $N(t) = n$. The threshold strategy is a $n_s \in \mathbb{Z}^+$ such that when $N(t) < n_s$, the arriving passenger joins the queue and when $N(t) = n_s$, the arriving passenger balks.

Note that for the model discussed in this thesis, the observable case would have identical results as a partially observable case where the only the passenger queue is seen and the taxi queue is unknown.

When the system is observable, let W_P represent an individual passenger's expected wait time for service and M represent an individual passenger's wait time for matching, assuming they arrive to an empty queue. Then we have:

$$W_P = \frac{1}{\lambda_2}$$
$$M = k_2$$

In this section, we look at the equilibrium and socially optimal threshold strategies.

3.2.1 Passenger utility for the observable case for two mass matching time

Let $U(n)$ denote the utility function for the observable case where an arriving passenger sees that there is a passenger queue of length n, i.e. $N(t) = n$.

Recall:

- R represents the passenger's intrinsic reward for taxi service
- p represents taxi fare
- C_P represents the passenger's cost per unit time of waiting associated with service
- $C_{M,P}$ represents the passenger's cost per unit time of waiting associated with matching time
- C_T represents the taxi's cost per unit time of waiting associated with service
- $C_{M,T}$ represents the taxi's cost per unit time of waiting associated with matching time
- λ_1 represents passenger arrival rate
- λ_0 represents taxi arrival rate when there is no passenger queue
- λ_2 represnts the taxi arrival rate when there is a non-zero passenger queue
- N represents the central taxi holding capacity

Lemma 3.2 The utility function for a passenger arriving to an observable system with two point matching time where there is no taxi queue is given:

$$U(n) = R - p - \frac{(n+1)C_P}{\lambda_2} - (n+1)C_{M,P}k_2.$$

Proof. When an arriving passenger sees there is a passenger queue of length n, they must wait for the n passengers in front of them to be matched and served, then wait for their own service and matching time. As before, per unit waiting cost is C_P and per unit matching time cost is $C_{M,P}$ for passengers. We have:

$$\begin{aligned} U &= R - p - (n+1)C_P W_P - (n+1)C_{M,P}M \\ &= R - p - \frac{(n+1)C_P}{\lambda_2} - (n+1)C_{M,P}k_2 \end{aligned}$$

□

Now that we have the utility function, we can find the equilibrium threshold strategy.

Theorem 3.3 The equilibrium passenger threshold strategy, n_e, upon entering an observable system with two point matching time is $n_e = \left\lfloor \frac{R-p}{\frac{C_P}{\lambda_2}+C_{M,P}k_2} \right\rfloor$.

PROOF. The equilibrium passenger strategy is $n_e \in \mathbb{Z}^+$ such that $U(n_e - 1) \geq 0$ and $U(n_e) < 0$.

Examine the first inequality:

$$\begin{aligned} 0 &\leq U(n_e - 1) \\ 0 &\leq R - p - \frac{n_e C_P}{\lambda_2} - n_e C_{M,P}k_2 \\ n_e \left(\frac{C_P}{\lambda_2} + C_{M,P}k_2 \right) &\leq R - p \\ n_e &\leq \frac{R-p}{\frac{C_P}{\lambda_2} + C_{M,P}k_2} \end{aligned}$$

Now, examine the second inequality:

$$\begin{aligned} 0 &> U(n_e) \\ 0 &> R - p - \frac{(n_e+1)C_P}{\lambda_2} - (n_e+1)C_{M,P}k_2 \\ (n_e+1)\left(\frac{C_P}{\lambda_2} + C_{M,P}k_2\right) &> R - p \\ n_e &\leq \frac{R-p}{\frac{C_P}{\lambda_2} + C_{M,P}k_2} - 1 \end{aligned}$$

Therefore, the positive integer which satisfies both inequalities is $n_e = \left\lfloor \frac{R-p}{\frac{C_P}{\lambda_2}+C_{M,P}k_2} \right\rfloor$.

□

3.2.2 Social welfare for the observable case for two mass matching time

In this section we look at the social welfare function for the observable system when matching time follows a two mass point distribution.

Lemma 3.3 The expected matching time in the observable case when matching time follows the two mass point distribution is always positive and is given as $E(M) = \pi_{-N}\left[D_1 + k_2\lambda_1 D_2(1-\rho_1^{n_s})\right]$ where:

$$D_1 = k_1\left(1 + \frac{\lambda_1(\rho_0^N - 1)}{\lambda_0 - \lambda_1}\right),$$

$$D_2 = \frac{\rho_0^N}{\lambda_1 - \lambda_2}.$$

PROOF.

$$
\begin{aligned}
E(M) &= \sum_{n=-N}^{n_s} E(M|N(t)=n)\pi_n \\
&= \pi_{-N}\left[k_1 + k_1\rho_0^N \sum_{n=-N+1}^{0} \rho_0^n + k_2\rho_0^N \sum_{n=1}^{n_s} \rho_1^n\right] \\
&= \pi_{-N}\left[k_1 + k_1\rho_0^N \left(\frac{\rho_0^N - 1}{1-\frac{1}{\rho_0}}\right) + k_2\rho_0^N \left(\frac{\rho_1(1-\rho_1^{n_s})}{1-\rho_1}\right)\right] \\
&= \pi_{-N}\left[k_1 + k_1\rho_0^N \left(\frac{\lambda_1(\rho_0^N - 1)}{\lambda_1-\lambda_0}\right) + k_2\rho_0^N \left(\frac{\lambda_1(1-\rho_1^{n_s})}{\lambda_2-\lambda_1}\right)\right]
\end{aligned}
$$

Let:

$$D_1 = k_1\left(1 + \frac{\lambda_1(\rho_0^N - 1)}{\lambda_0-\lambda_1}\right),$$

$$D_2 = \frac{\rho_0^N}{\lambda_2-\lambda_1}.$$

Thus, we have:

$$E(M) = \pi_{-N}\left[D_1 + k_2\lambda_1 D_2(1-\rho_1^{n_s})\right].$$

As shown previously, $\frac{\rho_0^{-N}-1}{\lambda_0-\lambda_1}$ is always positive, therefore D_1 is clearly positive.

$D_2(1-\rho_1^{n_s})$ can be shown to be always positive. When $\lambda_1 > \lambda_2$, then $\lambda_2 - \lambda_1$ and therefore D_2 are negative and $\rho_1 > 1$ so $(1-\rho_1^{n_s})$ is negative. When $\lambda_1 < \lambda_2$, then $\lambda_2 - \lambda_1$ and thus D_2 are positive and $\rho_1 < 1$ so $(1-\rho_1^{n_s})$ is positive.

Thus, $E(M)$ is always positive as expected.

□

Theorem 3.4 The social welfare function upon entering an observable system with respect to threshold strategy n with two point matching time is given:

$$S(n) = D_3 + D_4\rho_1^n + D_5 n_s \rho_1^n - \pi_{-N}^2 \lambda_1 (C_{M,P} + C_{M,T}) \left(\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \lambda_2 D_2 (1 - \rho_1^{n_s}) \right) [D_1 + k_2 \lambda_1 D_2 (1 - \rho_1^n)]$$

where:

$$D_3 = \pi_{-N} \left[\frac{\lambda_0\lambda_1(\rho_0^N - 1)R}{\lambda_1 - \lambda_0} + \frac{\lambda_1\lambda_2\rho_0^N R}{\lambda_2 - \lambda_1} - \frac{C_P\rho_0^N\rho_1}{(1-\rho_1)^2} - \frac{C_T\rho_0^{N+1}\left(1 - (N+1)\rho_0^{-N} + N\rho_0^{-N-1}\right)}{(\rho_0 - 1)^2} \right],$$

$$D_4 = \pi_{-N}\rho_0^N \left(\frac{C_P\rho_1}{(1-\rho_1)^2} - \frac{\lambda_1\lambda_2 R \rho_1^{n_s}}{\lambda_2 - \lambda_1} \right),$$

$$D_5 = \frac{\pi_{-N}\rho_0^N C_P \rho_1 (1-\rho_1)}{(1-\rho_1)^2}.$$

PROOF.

$$\begin{aligned} S(n) &= \lambda^*(R - p - C_p E(W_p) - C_{M,P}E(M)) + \lambda^*(p - C_T E(W_T) - C_{M,T}E(M)) \\ &= \lambda^* R - C_P E(L_P) - C_T E(L_T) - \lambda^*(C_{M,P} + C_{M,T})E(M). \end{aligned}$$

We can see the social welfare function can be split into social welfare derived from matching time, $S_M(n)$, and from the double-ended queue with no matching time,

$S_0(n)$. Thus, $S(n) = S_0(n) - S_M(n)$ where:

$$S_0(n) = \lambda^* R - C_P E(L_P) - C_T E(L_T),$$
$$S_M(n) = \lambda^*(C_{M,P} + C_{M,T})E(M).$$

First, consider $S_0(n)$:

$$\begin{aligned}
S_0(n) &= \left[\pi_{-N}\lambda_1\left(\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \frac{\lambda_2\rho_0^N(1-\rho_1^{n_s})}{\lambda_2 - \lambda_1}\right)\right] R \\
&\quad - C_P\left[\frac{\pi_{-N}\rho_0^N\rho_1\left[1-(n_s+1)\rho_1^{n_s} + n_s\rho_1^{n_s+1}\right]}{(1-\rho_1)^2}\right] \\
&\quad - C_T\left[\frac{\pi_{-N}\rho_0^{N+1}\left(1-(N+1)\rho_0^{-N} + N\rho_0^{-N-1}\right)}{(\rho_0-1)^2}\right] \\
&= \pi_{-N}\left[\frac{\lambda_0\lambda_1(\rho_0^N-1)R}{\lambda_1-\lambda_0} + \frac{\lambda_1\lambda_2\rho_0^N R}{\lambda_2-\lambda_1} - \frac{C_P\rho_0^N\rho_1}{(1-\rho_1)^2}\right. \\
&\quad \left. - \frac{C_T\rho_0^{N+1}\left(1-(N+1)\rho_0^{-N}+N\rho_0^{-N-1}\right)}{(\rho_0-1)^2}\right] + \pi_{-N}\rho_0^N\left[-\frac{\lambda_1\lambda_2R\rho_1^{n_s}}{\lambda_2-\lambda_1}\right. \\
&\quad \left. + \frac{C_P\rho_1(1-\rho_1)n_s\rho_1^{n_s}}{(1-\rho_1)^2} + \frac{C_P\rho_1\rho_1^{n_s}}{(1-\rho_1)^2}\right] \\
&= \pi_{-N}\left[\frac{\lambda_0\lambda_1(\rho_0^N-1)R}{\lambda_1-\lambda_0} + \frac{\lambda_1\lambda_2\rho_0^N R}{\lambda_2-\lambda_1} - \frac{C_P\rho_0^N\rho_1}{(1-\rho_1)^2}\right. \\
&\quad \left. - \frac{C_T\rho_0^{N+1}\left(1-(N+1)\rho_0^{-N}+N\rho_0^{-N-1}\right)}{(\rho_0-1)^2}\right] \\
&\quad + \pi_{-N}\rho_0^N\left[\left(\frac{C_P\rho_1}{(1-\rho_1)^2} - \frac{\lambda_1\lambda_2R\rho_1^{n_s}}{\lambda_2-\lambda_1}\right)\rho_1^{n_s} + \left(\frac{C_P\rho_1(1-\rho_1)}{(1-\rho_1)^2}\right)n_s\rho_1^{n_s}\right]
\end{aligned}$$

Let:

$$
\begin{aligned}
D_3 &= \pi_{-N}\left[\frac{\lambda_0\lambda_1(\rho_0^N-1)R}{\lambda_1-\lambda_0}+\frac{\lambda_1\lambda_2\rho_0^N R}{\lambda_2-\lambda_1}-\frac{C_P\rho_0^N\rho_1}{(1-\rho_1)^2}\right. \\
&\qquad \left. -\frac{C_T\rho_0^{N+1}\left(1-(N+1)\rho_0^{-N}+N\rho_0^{-N-1}\right)}{(\rho_0-1)^2}\right], \\
D_4 &= \pi_{-N}\rho_0^N\left(\frac{C_P\rho_1}{(1-\rho_1)^2}-\frac{\lambda_1\lambda_2 R\rho_1^{n_s}}{\lambda_2-\lambda_1}\right), \\
D_5 &= \frac{\pi_{-N}\rho_0^N C_P\rho_1(1-\rho_1)}{(1-\rho_1)^2}.
\end{aligned}
$$

Thus we have $S_0(n) = D_3 + D_4\rho_1^{n_s} + D_5 n_s\rho_1^{n_s}$.

Now, examine $S_M(n)$:

$$
\begin{aligned}
S_M(n) &= \lambda^*(C_{M,P}+C_{M,T})E(M) \\
&= \pi_{-N}^2\lambda_1(C_{M,P}+C_{M,T})\left(\frac{\lambda_0(\rho_0^N-1)}{\lambda_1-\lambda_0}+\lambda_2 D_2(1-\rho_1^{n_s})\right)\left[D_1+k_2\lambda_1 D_2(1-\rho_1^{n_s})\right].
\end{aligned}
$$

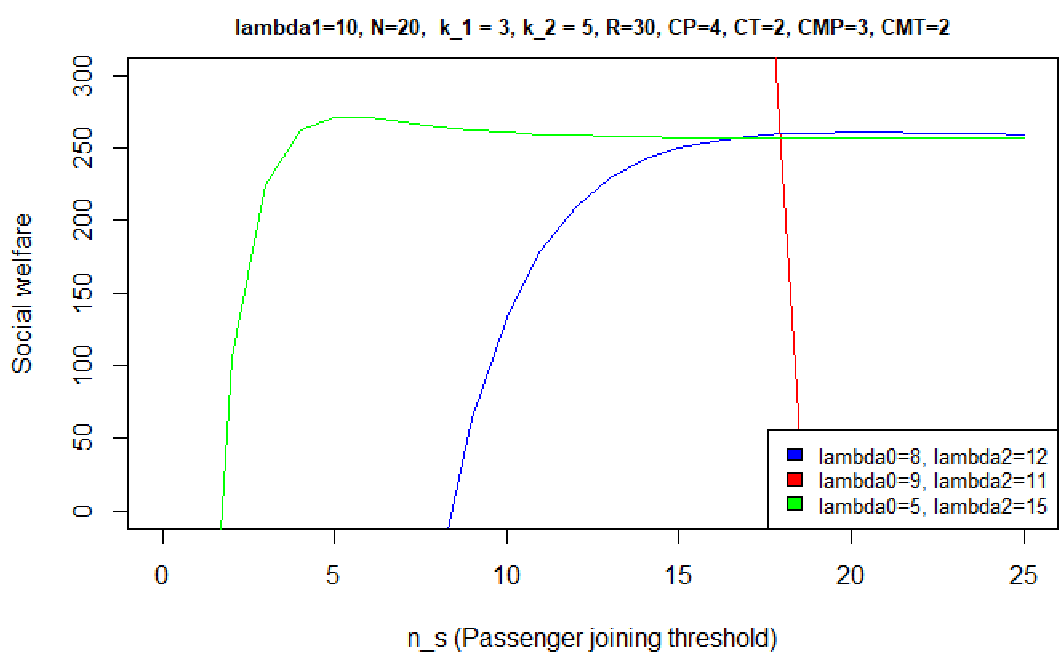
□

Numerical experiments for social welfare of observable two mass point matching time case

Again, we use some numerical experiments to show the characteristics of the social welfare function.

In figure 3.4, we see the social welfare function by the passenger joining threshold for different levels of taxi and passenger arrival rates. When taxi arrival rates are too close to the passenger arrival rate, we see the social welfare function is a steeply decreasing function, meaning social welfare function is maximized when the passenger joining threshold is 0. In other words, the passenger's socially optimal strategy in this case is to never join. When taxi arrival rates are further from the passenger arrival rate, the social welfare function has a passenger joining threshold which maximizes social welfare. As the taxi arrival rates get further from the passenger arrival rate, the socially optimal passenger threshold becomes lower. Taxi companies would prefer a passenger's threshold to be higher so that more customers will join the queue for service. Therefore, this indicates that in the observable case, the taxi operator must balance the taxi arrival rates such that the socially optimal strategy also maximizes number of customers.

Figure 3.4: Social welfare by the passenger joining threshold (observable case with two mass point matching time) when $\lambda_1 = 10, N = 20, k_1 = 3, k_2 = 5, R = 30, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$.



In figure 3.5, we individually change the taxi arrival rates λ_0 and λ_2 to see their effects on the social welfare function. We see that increasing the taxi arrival rate when the taxi queue (CTH) is empty and passenger queue is not, λ_2, shifts the social welfare function upward and to the left, meaning there is overall higher social welfare, but the passenger's socially optimal joining threshold is lower which goes against the goals of the taxi company. Decreasing the taxi arrival rate when the passenger queue is empty, λ_0, causes the shape of the social welfare function to change as well as a shift to the left. This is not optimal for taxi companies as it causes the socially optimal passenger threshold to be lower, and a steeper decrease of overall social welfare as the passenger joining threshold increases.

Figure 3.5: Social welfare by the passenger joining threshold (observable case with two mass point matching time) when $\lambda_1 = 10, N = 20, k_1 = 3, k_2 = 5, R = 30, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$, varying λ_0 and λ_1 individually.

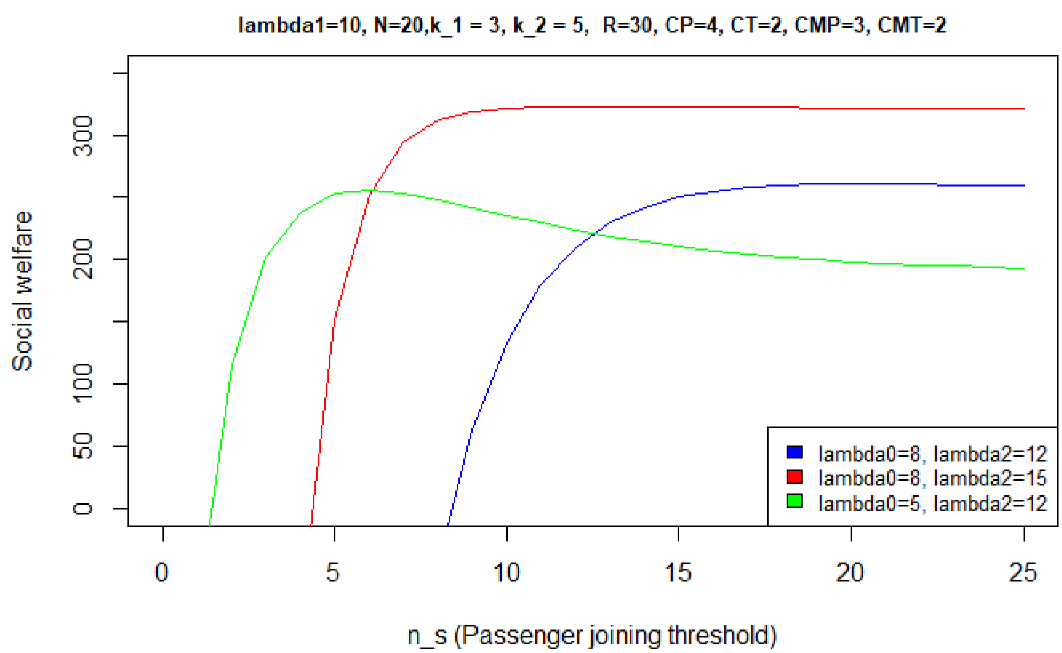
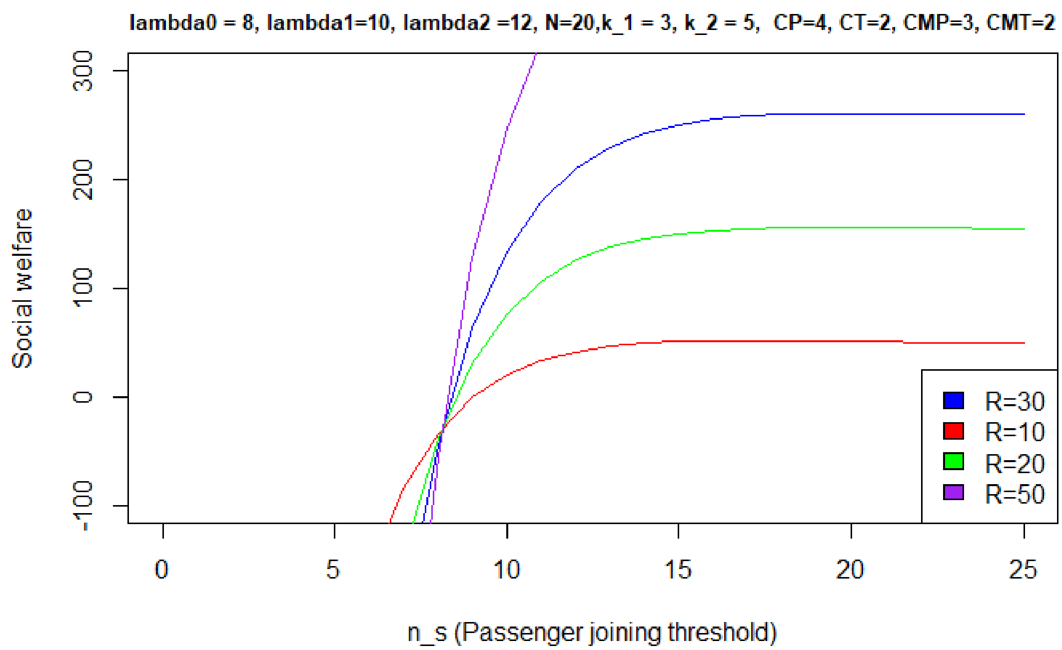


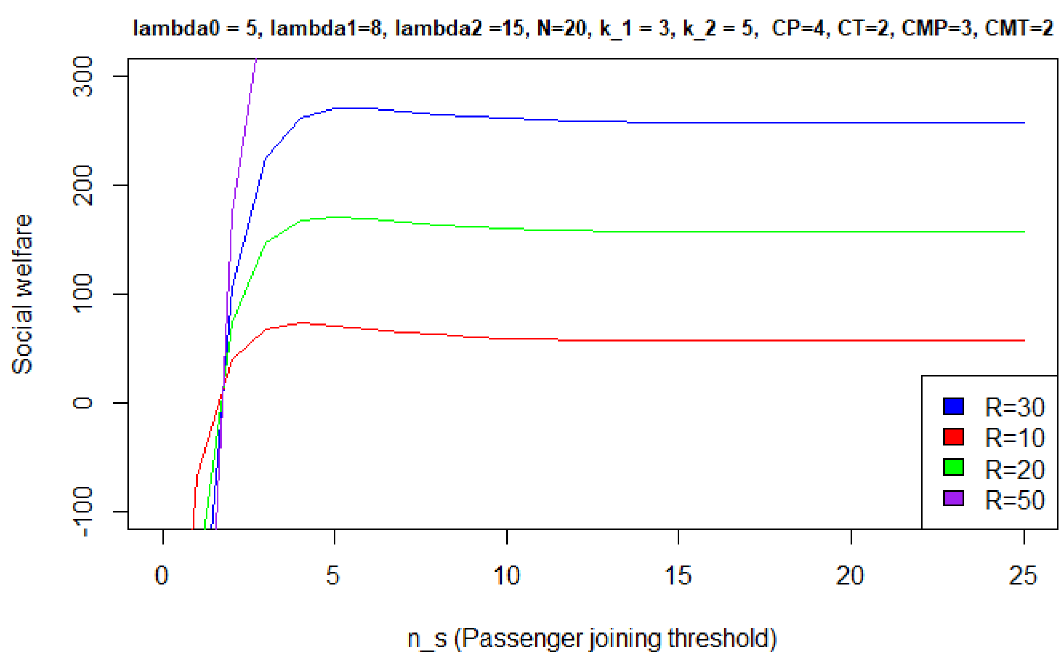
Figure 3.6 looks at how the social welfare function is impacted by change in the passenger reward, R, in both the case when $\lambda_0 = 8, \lambda_1 = 10, \lambda_2 = 12$ and when $\lambda_0 = 5, \lambda_1 = 10, \lambda_2 = 15$. In both cases, as passenger reward increases, not only is there a vertical increase in social welfare, but the steepness of the incline increases as well, showing that increasing R increase the sensitivity of social welfare to the passenger joining threshold. While the socially optimal passenger threshold does also increase with higher R, this increase if very small.

Figure 3.6: Social welfare by the passenger joining threshold (observable case with two mass point matching time) when $N = 20, k_1 = 3, k_2 = 5, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$.

(a) $\lambda_0 = 8, \lambda_1 = 10, \lambda_2 = 12$



(b) $\lambda_0 = 5, \lambda_1 = 1, \lambda_2 = 15$



Chapter 4

Piecewise Uniform Matching Time

Another reasonable distribution is piecewise continuous uniform distribution of matching time given system state. For intervals $\Delta_1 = b_1 - a_1$, $\Delta_2 = b_2 - a_2$, and $\Delta_3 = b_3 - a_3$ such that $a \leq a_1 < b_1 \leq b$, $a_2 < b_2$, and $a_3 < b_3$. Assume $(M|N(t) = n) \sim U\left(a(n), b(n)\right)$ where:

$$a(n) = \begin{cases} a_1, & n = -N \\ a_2, & -N < n \leq 0 \\ a_3, & \text{otherwise} \end{cases},$$

$$b(n) = \begin{cases} b_1, & n = -N \\ b_2, & -N < n \leq 0 \\ b_3, & \text{otherwise} \end{cases} \cdot$$

This distribution for matching time is reasonable as transit from the CTH to the terminal pickup location should be uniform under certain conditions, specifically different levels of demand which can be represented by the state of the system. The bounds of this continuous uniform random variable will vary depending on if the system is at full CTH capacity with no waiting passengers ($N(t) = -N$), has taxis waiting in CTH but not at capacity and no waiting passengers ($-N < N(t) \leq 0$), or has no waiting taxis but passengers are waiting at the terminal ($N(t) > 0$). We can further make assumptions about how much variability there would be (i.e. how wide the intervals are) for these three scenarios. It is reasonable to assume that when no taxis may enter the system and there are no waiting passengers, traffic will be low so $a_1 < a_i$, $b_1 < b_i$, and $\Delta_1 < \Delta_i$ for $i = 2, 3$. Similarly, it can be assumed that traffic between the CTH and terminal is highest and has greatest range when there are taxis waiting CTH but it is not full since these taxis will be waiting to be sent to the terminal as passengers arrive and already matched taxis and other vehicles will be departing terminal once they have picked up their passengers. Therefore, we have $\Delta_1 < \Delta_3 < \Delta_2$.

Let $d_i = a_i + b_i$ for $i = 1, 2, 3$. Therefore we have:

$$E(M|N(t) = n) = \begin{cases} \frac{d_1}{2}, & n = -N \\ \frac{d_2}{2}, & -N < n \leq 0 \\ \frac{d_3}{2}, & \text{otherwise} \end{cases} \cdot$$

Based on the previous assumptions, we can say $d_1 < d_2$ and $d_1 < d_3$ since when the CTH is full, the matching time is shorter and has less variability so both bounds of matching time are lower.

In the following results, we will see how the more complex matching time following a piecewise uniform distribution is similar to and differs from the simplistic two mass point matching time distribution case.

4.1 Partially observable case for piecewise uniform matching time

4.1.1 Passenger utility for the partially observable case for piecewise uniform matching time

Recall:

- R represents the passenger's intrinsic reward for taxi service
- p represents taxi fare
- C_P represents the passenger's cost per unit time of waiting associated with

service

- $C_{M,P}$ represents the passenger's cost per unit time of waiting associated with matching time
- C_T represents the taxi's cost per unit time of waiting associated with service
- $C_{M,T}$ represents the taxi's cost per unit time of waiting associated with matching time
- λ_1 represents passenger arrival rate
- λ_0 represents taxi arrival rate when there is no passenger queue
- λ_2 represnts the taxi arrival rate when there is a non-zero passenger queue
- N represents the central taxi holding capacity
- q represents passenger joining probability

Lemma 4.1 The utility function for a passenger arriving to a partially observable system with piecewise uniform matching time where there is no taxi queue is given:

$$U(q) = R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - \frac{C_{M,P}\pi_{-N}}{2}\left[d_1 + \frac{d_2\lambda_0(1-\rho_0)}{\lambda_1 - \lambda_0} + \frac{d_3\rho_0^N\lambda_1 q}{\lambda_2 - \lambda_1 q}\right].$$

Proof. We use the same results from Wang and Liu [18] which are used in the two mass point matching time distribution model's partially observable case and derive the utility function in a similar manner as in the previous section:

$$
\begin{aligned}
U(q) &= R - p - C_P E(W) - C_{M,P} E(M) \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \sum_{n=-N}^{\infty} E(M|N(t) = n)\pi_n \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \left[d_1 + d_2 \sum_{n=-N+1}^{0} \rho_0^n + d_3 \sum_{1}^{\infty} \rho_2^n \right] \pi_{-N} \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} \left[d_1 + d_2 \left(\frac{1 - \rho_0^{-N-1}}{1 - \frac{1}{\rho_0}} \right) + d_3 \left(\frac{\rho_2}{1 - \rho_2} \right) \right] \pi_{-N} \\
&= R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - \frac{C_{M,P}\pi_{-N}}{2} \left[d_1 + \frac{d_2 \lambda_0 (1 - \rho_0)}{\lambda_1 - \lambda_0} + \frac{d_3 \rho_0^N \lambda_1 q}{\lambda_2 - \lambda_1 q} \right]
\end{aligned}
$$

Let:

$$
\begin{aligned}
A_7 &= \frac{\pi_{-N}}{2}, \\
A_8 &= d_1 + \frac{d_2 \lambda_0 (\rho_0^N - 1)}{\lambda_1 - \lambda_0}.
\end{aligned}
$$

It is clear that A_7 is always positive as π_{-N} is positive. We have shown previously that $(\rho_0^N - 1)$ and $(\lambda_1 - \lambda_0)$ are always the same sign, so A_8 is always positive as well. Thus, $U(q) = R - p - \frac{C_P}{\lambda_2 - \lambda_1 q} - C_{M,P} A_7 \left[A_8 + \frac{d_3 \rho_0^N \lambda_1 q}{\lambda_2 - \lambda_1 q} \right]$. □

Theorem 4.1 The equilibrium passenger joining probability for a passenger arriving to a partially observable system with piecewise uniform matching time where there is no taxi queue is given:

$$q_e = \begin{cases} 0, & p + \frac{C_P}{\lambda_2} + C_{M,P}A_7A_8 \geq R \\ q_e^*, & \frac{C_P}{\lambda_2} < R - p - C_{M,P}A_7A_8 < \frac{C_P + C_{M,P}A_7 d_3 \rho_0^N \lambda_1}{\lambda_2 - \lambda_1} \\ 1, & p + \frac{C_P}{\lambda_2 - \lambda_1} + C_{M,P}A_7\left[A_8 + \frac{\rho_0^N d_3 \lambda_1}{\lambda_2 - \lambda_1}\right] \leq R \end{cases}.$$

where:

$$q_e^* = \frac{\lambda_2(R - p - C_{M,P}A_7A_8) - C_P}{\lambda_1\left[C_{M,P}A_7d_3\rho_0^N + R - p - C_{M,P}A_7A_8\right]}.$$

PROOF.

Similar to chapter 3, the equilibrium probability here is piecewise on three intervals.

The first interval is when the utility of a system with no customers joining is zero or lower. Here, the equilibrium probability is zero as a joining passenger would not have any gain even if no other customer enters the queue. When there are no customers entering the queue, the queue waiting time for a passenger is $\frac{1}{\lambda_2}$ and the matching time is:

$$\sum_{n=-N}^{0} E(M|N(t) = n)\pi_n = A_7A_8.$$

Therefore, when $p + \frac{C_P}{\lambda_2} + C_{M,P}A_7A_8 \geq R$, the equilibrium probability is $q_e = 0$.

The second interval is when the utility under the full potential arrival rate λ_1 is non-negative. Here, the equilibrium probability is 1. Since the utility is always non-negative, even when the maximum amount of other passengers are arriving to the system, the passenger should always join. When the effective passenger arrival rate is λ_1, the queue waiting time per passenger is $\frac{1}{\lambda_2-\lambda_1}$ and the matching time is:

$$\sum_{n=-N}^{\infty} E(M|N(t)=n)\pi_n = A_7\left[A_8 + \frac{d_3\rho_0^N\lambda_1}{\lambda_2-\lambda_1}\right].$$

Therefore, when $p+\frac{C_P}{\lambda_2-\lambda_1}+C_{M,P}A_7\left[A_8+\frac{d_3\rho_0^N\lambda_1}{\lambda_2-\lambda_1}\right] \leq R$, the equilibrium probability is given $q_e = 1$

The final interval is given by

$$p+\frac{C_P}{\lambda_2}+C_{M,P}A_7A_8 < R < p+\frac{C_P}{\lambda_2-\lambda_1}+C_{M,P}A_7\left[A_8+\frac{d_3\rho_0^N\lambda_1}{\lambda_2-\lambda_1}\right]$$
$$\frac{C_P}{\lambda_2} < R-p-C_{M,P}A_7A_8 < \frac{C_P+C_{M,P}A_7d_1\rho_0^N\lambda_1}{\lambda_2-\lambda_1}$$

To find the equilibrium joining probability in the final interval, q_e^*, set the utility function to zero such that $U(q_e^*)=0$ and solve.

$$\frac{C_P+C_{M,P}A_7d_3\rho_0^N\lambda_1q_e}{\lambda_2-\lambda_1q_e} = R-p-C_{M,P}A_7A_8$$
$$C_P+C_{M,P}A_7d_3\rho_0^N\lambda_1q_e = (\lambda_2-\lambda_1q_e)(R-p-C_{M,P}A_7A_8)$$
$$(C_{M,P}A_7d_3\rho_0^N+(R-p-C_{M,P}A_7A_8))\lambda_1q_e = (\lambda_2(R-p-C_{M,P}A_7A_8)-C_P)$$
$$q_e = \frac{\lambda_2(R-p-C_{M,P}A_7A_8)-C_P}{\lambda_1\left[C_{M,P}A_7d_3\rho_0^N+R-p-C_{M,P}A_7A_8\right]}$$

□

4.1.2 Social welfare for the partially observable case for piecewise uniform matching time

Theorem 4.2 In the partially observable case for piecewise uniform matching time, the social welfare function is given:

$$S(q) = A_3 + A_4\left[\frac{R - C_P q}{(\lambda_2 - \lambda_1 q)^2}\right] - A_9\left[\frac{A_8\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + A_{10}\left(\frac{q}{\lambda_2 - \lambda_1 q}\right)\right.$$
$$\left. + A_{11}\left(\frac{q^2}{(\lambda_2 - \lambda_1 q)^2}\right)\right]$$

where:

$$A_3 = \frac{R\pi_{-N}\lambda_0\lambda_1(\rho_0^N - 1)}{\lambda_1 - \lambda_0} - \frac{C_T\pi_{-N}\rho_0^{N-1}(1 - \rho_0^{-N}(1+N) + N\rho_0^{-(N+1)})}{(1 - \frac{1}{\rho_0})^2},$$

$$A_4 = \pi_{-N}\rho_0^N\lambda_1\lambda_2,$$

$$A_7 = \frac{\pi_{-N}}{2},$$

$$A_8 = d_1 + \frac{d_2\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0},$$

$$A_9 = \pi_{-N}\lambda_1 A_7(C_{M,P} + C_{M,T}),$$

$$A_{10} = A_8\lambda_2\rho_0^N + \frac{d_3\lambda_0\rho_0^N(\rho_0^N - 1)}{\lambda_1 - \lambda_0},$$

$$A_{11} = \lambda_1\lambda_2 d_3\rho_0^{2N}.$$

Proof. By the proof of theorem 2.4, we know that:

$$E(M) = A_7\left[A_8 + \frac{d_3\rho_0^N\lambda_1 q}{\lambda_2 - \lambda_1 q}\right].$$

As before,

$$
\begin{aligned}
S(q) &= \lambda_P^*(R - p - C_P E(W_P) - C_{M,P} E(M)) + \lambda_T^*(p - C_T E(W_T) - C_{M,T} E(M)) \\
&= \lambda^* \left(R - C_P E(W_P) - C_T E(W_T)\right) + \lambda^* E(M) \left(-C_{M,P} - C_{M,T}\right) \\
&= S_0(q) + \lambda^* A_7 \left[A_8 + \frac{d_3 \rho_0^N \lambda_1 q}{\lambda_2 - \lambda_1 q}\right] \left(-C_{M,P} - C_{M,T}\right)
\end{aligned}
$$

Separate $S(q)$ into social welfare as a result of waiting time, $S_0(q)$, and social welfare as a result of matching time, $S_M(q)$. Since $S_0(q)$ as been examined before, we will focus on $S_M(q)$:

$$
\begin{aligned}
S_M(q) &= \lambda^* E(M) \left(-C_{M,P} - C_{M,T}\right) \\
&= \pi_{-N} \lambda_1 \left[\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \frac{\lambda_2 \rho_0^N q}{\lambda_2 - \lambda_1 q}\right] A_7 \left[A_8 + \frac{d_3 \rho_0^N \lambda_1 q}{\lambda_2 - \lambda_1 q}\right] \left(-C_{M,P} - C_{M,T}\right) \\
&= -\pi_{-N} \lambda_1 A_7 (C_{M,P} + C_{M,T}) \left[\frac{A_8 \lambda_0 (\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \left(A_8 \lambda_2 \rho_0^N + \frac{d_3 \lambda_0 \rho_0^N (\rho_0^N - 1)}{\lambda_1 - \lambda_0}\right)\right. \\
&\quad \left. \cdot \left(\frac{q}{\lambda_2 - \lambda_1 q}\right) + \lambda_1 \lambda_2 d_3 \rho_0^{2N} \left(\frac{q^2}{(\lambda_2 - \lambda_1 q)^2}\right)\right]
\end{aligned}
$$

Let:

$$
\begin{aligned}
A_9 &= \pi_{-N} \lambda_1 A_7 (C_{M,P} + C_{M,T}), \\
A_{10} &= A_8 \lambda_2 \rho_0^N + \frac{d_3 \lambda_0 \rho_0^N (\rho_0^N - 1)}{\lambda_1 - \lambda_0}, \\
A_{11} &= \lambda_1 \lambda_2 d_3 \rho_0^{2N}.
\end{aligned}
$$

These three constants, A_9, A_{10}, and A_{11} are all always positive following the same reasoning as in previous proofs. Thus, the social welfare as a result of matching time can be expressed:

$$S_M(q) = -A_9 \left[\frac{A_8 \lambda_0 (\rho_0^N - 1)}{\lambda_1 - \lambda_0} + A_{10} \left(\frac{q}{\lambda_2 - \lambda_1 q} \right) + A_{11} \left(\frac{q^2}{(\lambda_2 - \lambda_1 q)^2} \right) \right].$$

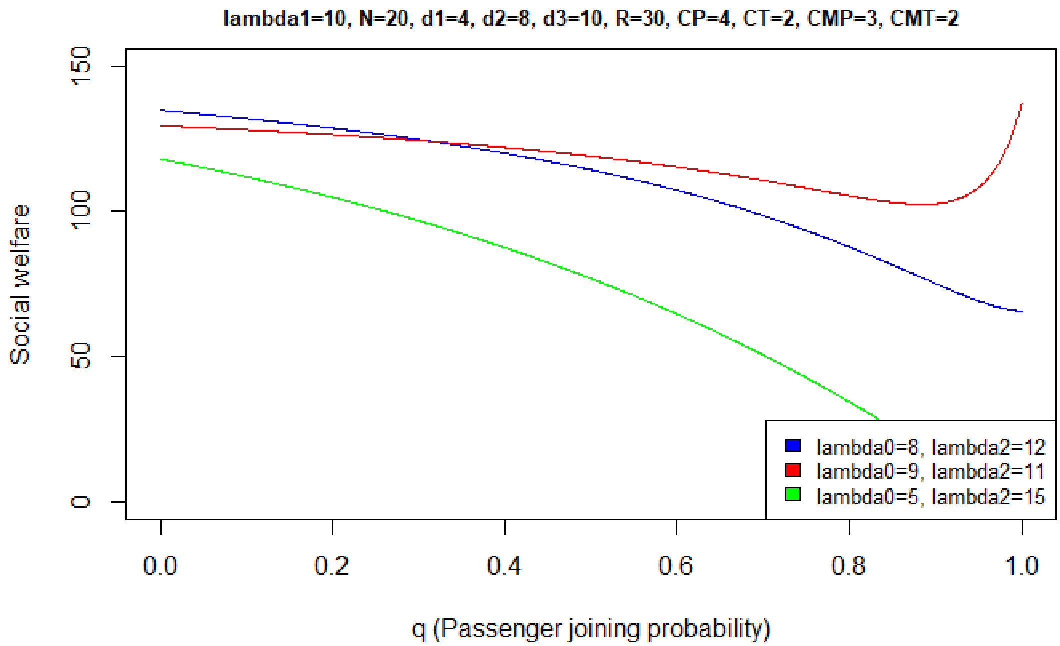
□

Numerical experiments for social welfare of partially observable piecewise uniform matching time case

Numerical experiments are done to interpret the characteristics of the social welfare function and how changing values affect the social welfare.

In figure 4.1, we can see that as the taxi arrival rates get closer to the passenger arrival rate, social welfare in general increases. Similar to the two mass point matching time case, when the taxi arrival rates are sufficiently close to the passenger arrival rate, the social welfare function increases as passenger joining probability approaches 1. When the taxi arrival rates are further from the passenger arrival rate, the social welfare decreases as passenger joining probability increases.

Figure 4.1: Social welfare by the passenger joining probability (partially observable case with two mass point matching time) when $\lambda_1 = 10, N = 20, d_1 = 2, d_2 = 5, d_3 = 4, R = 30, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$.



In figure 4.2, we individually change the taxi arrival rate when the passenger queue is empty, λ_0, and when the passenger queue is non-empty, λ_2. Like the two mass point matching time case, change in λ_2 results in a rise in social welfare as passenger joining probability approaches 1. Therefore, the taxi company would want the taxi arrival rate when there is a non-empty passenger queue to be close to the passenger arrival rate. Change in λ_0 results in the social welfare curve being flatter, meaning social welfare becomes less sensitive to passenger joining probability when passenger queue is empty though it does still decrease.

Figure 4.2: Social welfare by the passenger joining probability (partially observable case with piecewise uniform matching time) when $\lambda_1 = 10, N = 20, d_1 = 4, d_2 = 8, d_3 = 10, R = 30, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$, varying λ_0 and λ_1 individually.

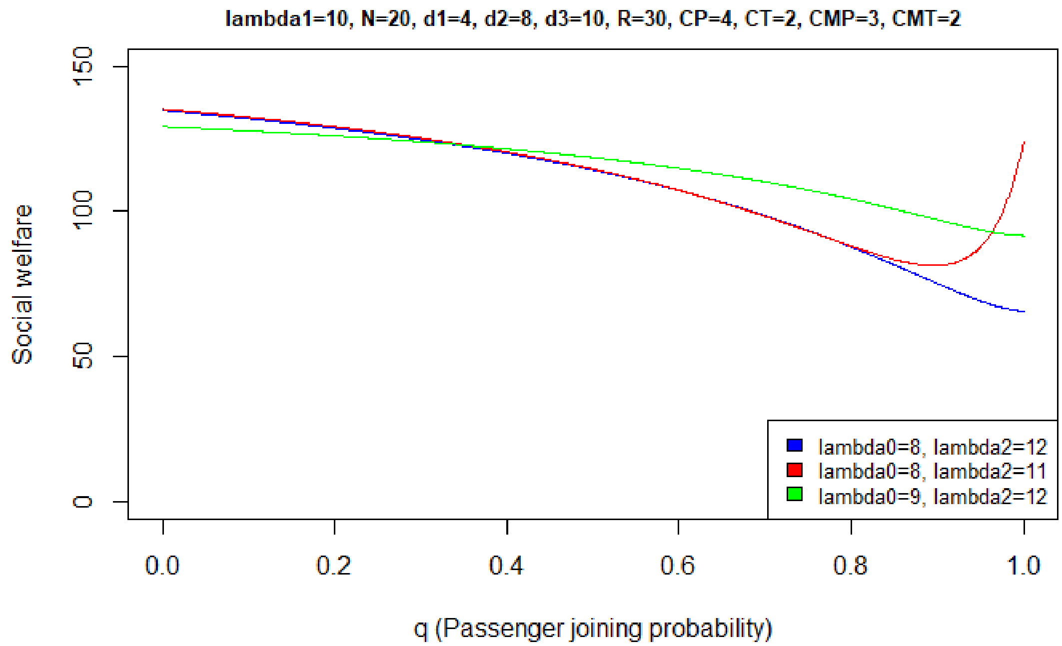
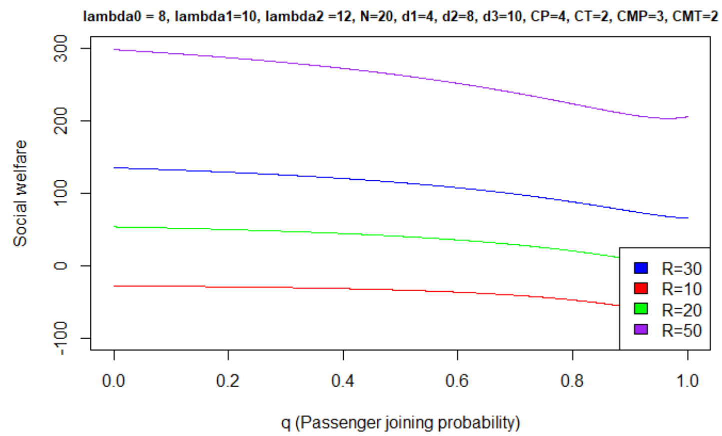


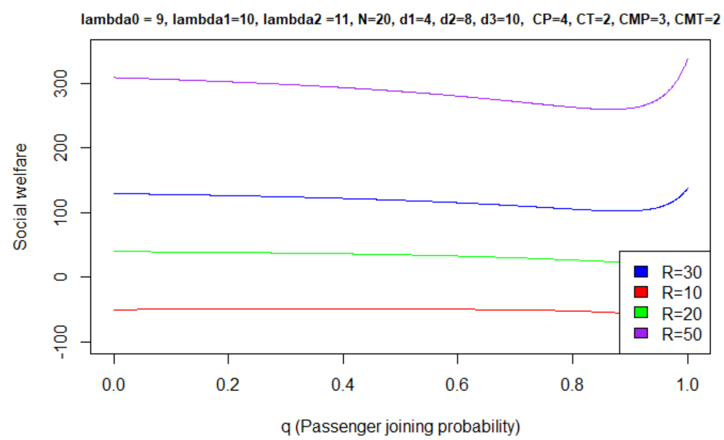
Figure 4.3 shows how change in the passenger reward from service, R, affects the social welfare function. Similar to previous cases, when R increases, there is a vertical increase in the social welfare function. When the taxi arrival rates are sufficiently close to the passenger arrival rate, higher R also leads to a steeper increase of social welfare when passenger joining probability approaches 1. The relationship between the taxi arrival rates and social welfare here is very similar to that in the corresponding two mass point matching time case.

Figure 4.3: Social welfare by the passenger joining probability (partially observable case with piecewise uniform matching time) when $N = 20, d_1 = 2, d_2 = 5, d_3 = 4, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$.

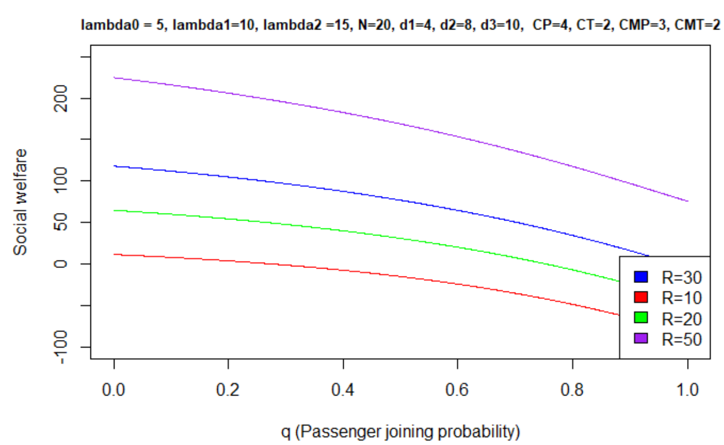
(a) $\lambda_0 = 8, \lambda_1 = 10, \lambda_2 = 12$



(b) $\lambda_0 = 9, \lambda_1 = 10, \lambda_2 = 111$



(c) $\lambda_0 = 5, \lambda_1 = 10, \lambda_2 = 15$



4.2 Observable case for piecewise uniform matching time

In the observable case, arriving passengers can see both the taxi and passenger queue lengths. As before, if the passenger sees a taxi queue upon arrival and no passenger queue (i.e. $N(t) \leq 0$), they will join the system. The threshold strategy is a $n_s \in \mathbb{Z}^+$ such that when $N(t) < n_s$, the arriving passenger joins the queue and when $N(t) = n_s$, the arriving passenger balks.

When the system is observable, an individual passenger's expected wait time for matching, assuming they arrive to an empty queue and matching time is piecewise uniform, is given $M = \frac{a_3+b_3}{2}$.

In this section, we look at the equilibrium and socially optimal threshold strategies.

4.2.1 Passenger utility for the observable case for piecewise uniform matching time

Let $U(n)$ denote the utility function for the observable case where an arriving passenger sees that there is a passenger queue of length n, i.e. $N(t) = n$.

Recall:

- R represents the passenger's intrinsic reward for taxi service
- p represents taxi fare
- C_P represents the passenger's cost per unit time of waiting associated with service

- $C_{M,P}$ represents the passenger's cost per unit time of waiting associated with matching time
- C_T represents the taxi's cost per unit time of waiting associated with service
- $C_{M,T}$ represents the taxi's cost per unit time of waiting associated with matching time
- λ_1 represents passenger arrival rate
- λ_0 represents taxi arrival rate when there is no passenger queue
- λ_2 represnts the taxi arrival rate when there is a non-zero passenger queue
- N represents the central taxi holding capacity

Lemma 4.2 The utility function for a passenger arriving to an observable system with piecewise uniform matching time where there is no taxi queue is given:

$$U(n) = R - p - \frac{(n+1)C_P}{\lambda_2} - \frac{(n+1)C_{M,P}(a_3+b_3)}{2}.$$

Proof. When an arriving passenger sees there is a passenger queue of length n, they must wait for the n passengers in front of them to be matched and served, then wait for their own service and matching time. As before, per unit waiting cost is C_P and per unit matching time cost is $C_{M,P}$ for passengers. We have:

$$\begin{aligned} U &= R - p - (n+1)C_P W_P - (n+1)C_{M,P} M \\ &= R - p - \frac{(n+1)C_P}{\lambda_2} - \frac{(n+1)C_{M,P}(a_3+b_3)}{2}. \end{aligned}$$

□

Now that we have the utility function, we can find the equilibrium threshold strategy.

Theorem 4.3 The equilibrium passenger threshold strategy, n_e, upon entering an observable system with piecewise uniform matching time is $n_e = \left\lfloor \frac{R-p}{\frac{C_P}{\lambda_2}+\frac{C_{M,P}(a_3+b_3)}{2}} \right\rfloor$.

Proof. The equilibrium passenger strategy is $n_e \in \mathbb{Z}^+$ such that $U(n_e - 1) \geq 0$ and $U(n_e) < 0$. This proof is similar to the analogous proof given in chapter 3.

Examine the first inequality:

$$0 \leq R - p - \frac{n_e C_P}{\lambda_2} - n_e C_{M,P} k_2$$
$$n_e \leq \frac{R-p}{\frac{C_P}{\lambda_2} + \frac{C_{M,P}(a_3+b_3)}{2}}$$

Now, examine the second inequality:

$$0 > R - p - \frac{(n_e+1) C_P}{\lambda_2} - (n_e+1) C_{M,P} k_2$$
$$n_e \leq \frac{R-p}{\frac{C_P}{\lambda_2} + \frac{C_{M,P}(a_3+b_3)}{2}} - 1$$

Therefore, the positive integer which satisfies both inequalities is $n_e = \left\lfloor \frac{R-p}{\frac{C_P}{\lambda_2}+\frac{C_{M,P}(a_3+b_3)}{2}} \right\rfloor$. The threshold is rounded down as it is the highest integer which would satisfy the above inequality. □

4.2.2 Social welfare for the observable case for piecewise uniform matching time

In this section, we look at the social welfare function for the observable system when matching time follows a piecewise uniform distribution.

Lemma 4.3 The expected matching time in the observable case when matching time follows a piecewise uniform is positive and given as $E(M) = A_7 \left[D_6 + d_3 D_2 \lambda_1 (1 - \rho_1^{n_s})\right]$ where:

$$D_6 = d_1 + d_2 \rho_0^N \left(\frac{\lambda_1(\rho_0^N - 1)}{\lambda_1 - \lambda_0} \right).$$

Proof.

$$\begin{aligned} E(M) &= \sum_{n=-N}^{n_s} E(M|N(t) = n)\pi_n \\ &= \frac{\pi_{-N}}{2} \left[d_1 + d_2 \rho_0^N \sum_{n=-N+1}^{0} \rho_0^n + d_3 \rho_0^N \sum_{n=1}^{n_s} \rho_1^n \right] \\ &= \pi_{-N} \left[d_1 + d_2 \rho_0^N \left(\frac{\rho_0^N - 1}{1 - \frac{1}{\rho_0}} \right) + d_3 \rho_0^N \left(\frac{\rho_1(1 - \rho_1^{n_s})}{1 - \rho_1} \right) \right] \\ &= \frac{\pi_{-N}}{2} \left[d_1 + d_2 \rho_0^N \left(\frac{\lambda_1(\rho_0^N - 1)}{\lambda_1 - \lambda_0} \right) + d_3 \rho_0^N \left(\frac{\lambda_1(1 - \rho_1^{n_s})}{\lambda_2 - \lambda_1} \right) \right] \end{aligned}$$

Let:

$$D_6 = d_1 + d_2 \rho_0^N \left(\frac{\lambda_1(\rho_0^N - 1)}{\lambda_1 - \lambda_0} \right).$$

Since $\frac{\rho_0^N - 1}{\lambda_1 - \lambda_0}$ is always positive, D_6 is always positive.

Therefore:

$$E(M) = A_7 \left[D_6 + d_3 D_2 \lambda_1 (1 - \rho_1^{n_s}) \right].$$

By the proof of lemma 3.3, $D_2(1-\rho_1^{n_s})$ is always positive and thus $E(M)$ is always positive as expected.

□

Theorem 4.4 The social welfare function upon entering an observable system with respect to threshold strategy n with piecewise uniform matching time is given:

$$S(n) = D_3 + D_4\rho_1^{n_s} + D_5 n_s \rho_1^{n_s} - \frac{1}{2}\pi_{-N}^2 \lambda_1 (C_{M,P} + C_{M,T}) \left(\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} \right.$$
$$\left. + \lambda_2 D_2 (1 - \rho_1^{n_s}) \right) \left[D_6 + d_3 D_2 \lambda_1 (1 - \rho_1^{n_s}) \right]$$

where:

$$D_3 = \pi_{-N} \left[\frac{\lambda_0\lambda_1(\rho_0^N - 1)R}{\lambda_1 - \lambda_0} + \frac{\lambda_1\lambda_2\rho_0^N R}{\lambda_2 - \lambda_1} - \frac{C_P\rho_0^N\rho_1}{(1-\rho_1)^2} \right.$$
$$\left. - \frac{C_T\rho_0^{N+1}\left(1 - (N+1)\rho_0^{-N} + N\rho_0^{-N-1}\right)}{(\rho_0 - 1)^2} \right],$$
$$D_4 = \pi_{-N}\rho_0^N \left(\frac{C_P\rho_1}{(1-\rho_1)^2} - \frac{\lambda_1\lambda_2 R\rho_1^{n_s}}{\lambda_2 - \lambda_1} \right),$$
$$D_5 = \frac{\pi_{-N}\rho_0^N C_P \rho_1 (1-\rho_1)}{(1-\rho_1)^2},$$
$$D_6 = d_1 + d_2\rho_0^N \left(\frac{\lambda_1(\rho_0^N - 1)}{\lambda_1 - \lambda_0} \right).$$

PROOF.

$$\begin{aligned} S(n) &= \lambda^*(R - p - C_p E(W_p) - C_{M,P}E(M)) + \lambda^*(p - C_T E(W_T) - C_{M,T}E(M)) \\ &= \lambda^* R - C_P E(L_P) - C_T E(L_T) - \lambda^*(C_{M,P} + C_{M,T})E(M) \end{aligned}$$

We can see the social welfare function can be split into social welfare derived from matching time, $S_M(n)$, and from the double-ended queue with no matching time, $S_0(n)$. Thus, $S(n) = S_0(n) - S_M(n)$ where:

$$\begin{aligned} S_0(n) &= \lambda^* R - C_P E(L_P) - C_T E(L_T) \\ S_M(n) &= \lambda^*(C_{M,P} + C_{M,T})E(M) \end{aligned}$$

By the proof of theorem 3.4, we have $S_0(n) = D_3 + D_4\rho_1^{n_s} + D_5 n_s \rho_1^{n_s}$.

Now, examine $S_M(n)$:

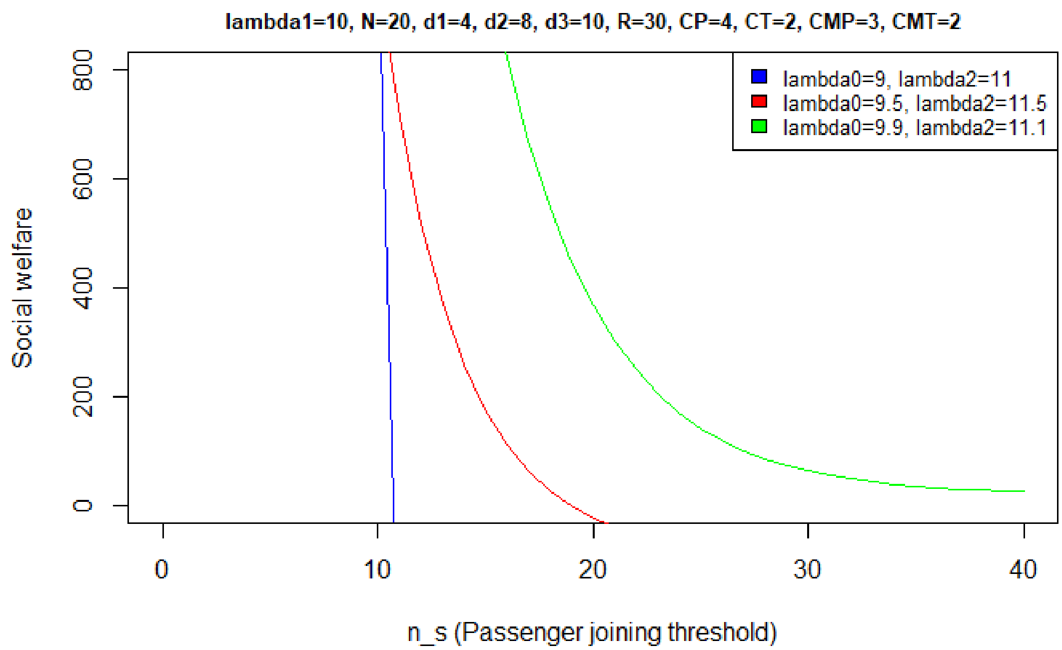
$$\begin{aligned} S_M(n) &= \lambda^*(C_{M,P} + C_{M,T})E(M) \\ &= \frac{1}{2}\pi_{-N}^2 \lambda_1 (C_{M,P} + C_{M,T}) \left(\frac{\lambda_0(\rho_0^N - 1)}{\lambda_1 - \lambda_0} + \lambda_2 D_2(1 - \rho_1^{n_s}) \right) [D_6 + d_3 D_2 \lambda_1 (1 - \rho_1^{n_s})] \end{aligned}$$

□

Numerical experiments for social welfare of observable piecewise uniform matching time case

Numerical experiments are performed to analyze the characteristics of the social welfare function.

Figure 4.4: Social welfare by the passenger joining threshold (observable case with piecewise uniform matching time) when $\lambda_1 = 10, N = 20, d_1 = 2, d_2 = 4, d_3 = 5, R = 30, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$.



In figure 4.4, we look at the social welfare function by passenger joining threshold for different values of the taxi and passenger arrival rates. We see that social welfare is always a decreasing function of passenger joining threshold. As taxi arrival rates approaches the passenger arrival rate, the social welfare increases and the steep decline levels off as threshold increases. Therefore, it would be important for taxi operators to have taxi arrival rates sufficiently close to the passenger arrival rate to balance social welfare with having a high enough passenger joining threshold to gain customers. Here, the behaviour of the social welfare function is very different from the corresponding two mass point matching time case.

In figure 4.5, we look at how the taxi arrival rates individually affect social welfare.

We see that change in λ_0, the taxi arrival rate when the passenger queue is empty, causes a horizontal shift in the social welfare function, also slightly changing the shape of the function. The social welfare function is much more sensitive to change in λ_2, the taxi arrival rate when taxi queue is empty and passenger queue non-empty, and as λ_2 approaches the passenger arrival rate, the social welfare increases.

Figure 4.5: Social welfare by the passenger joining threshold (observable case with piecewise uniform matching time) when $\lambda_1 = 10, N = 20, d_1 = 2, d_2 = 4, d_3 = 5, R = 30, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$, varying λ_0 and λ_2 individually.

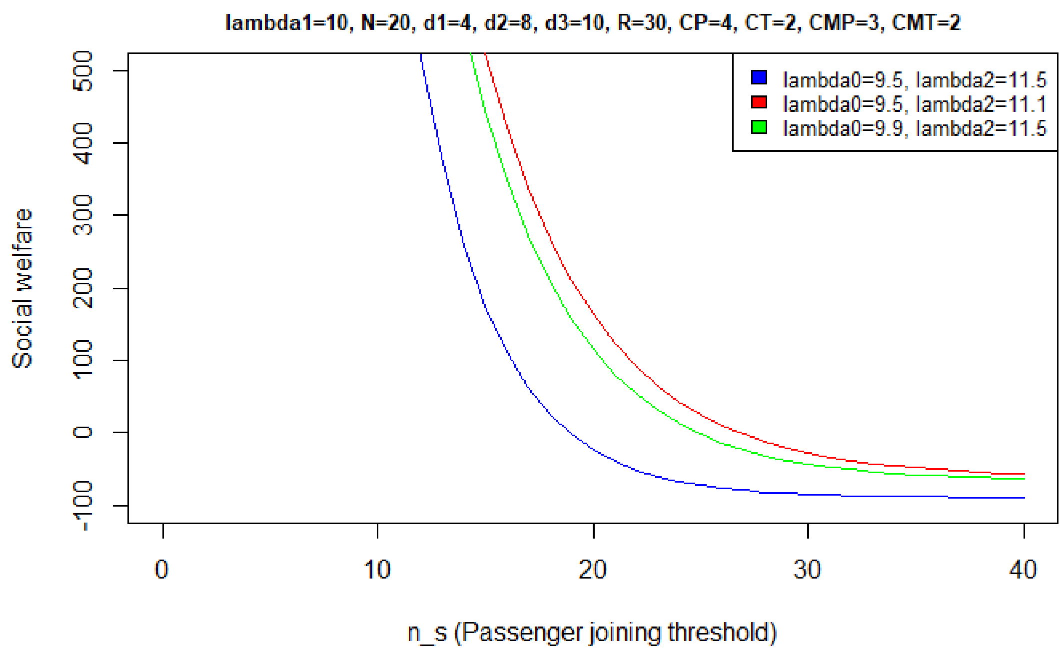
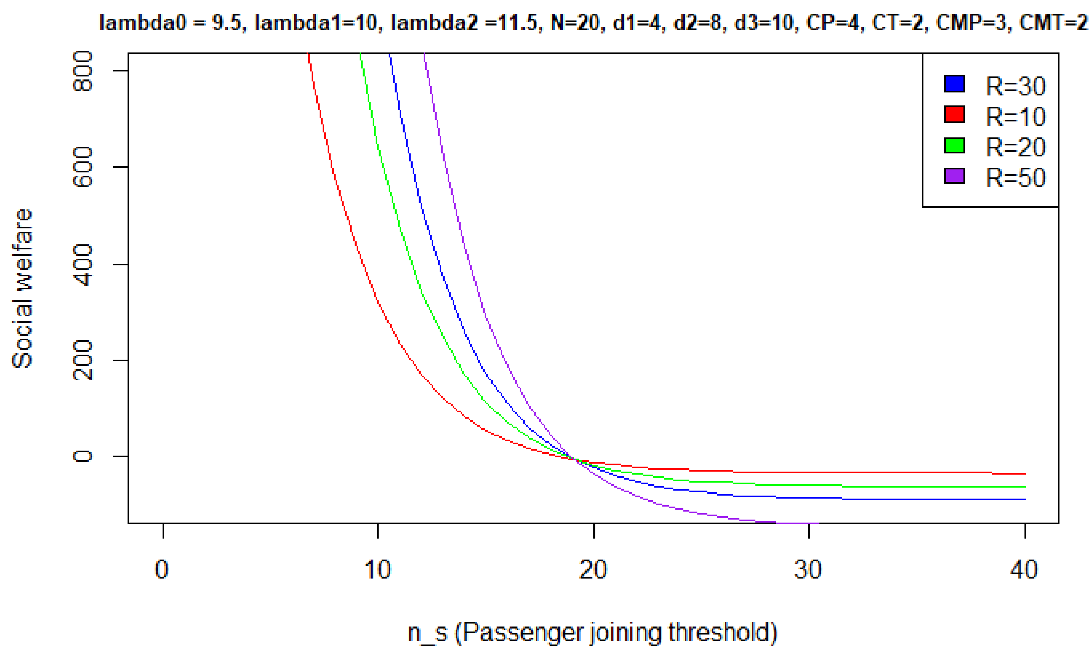


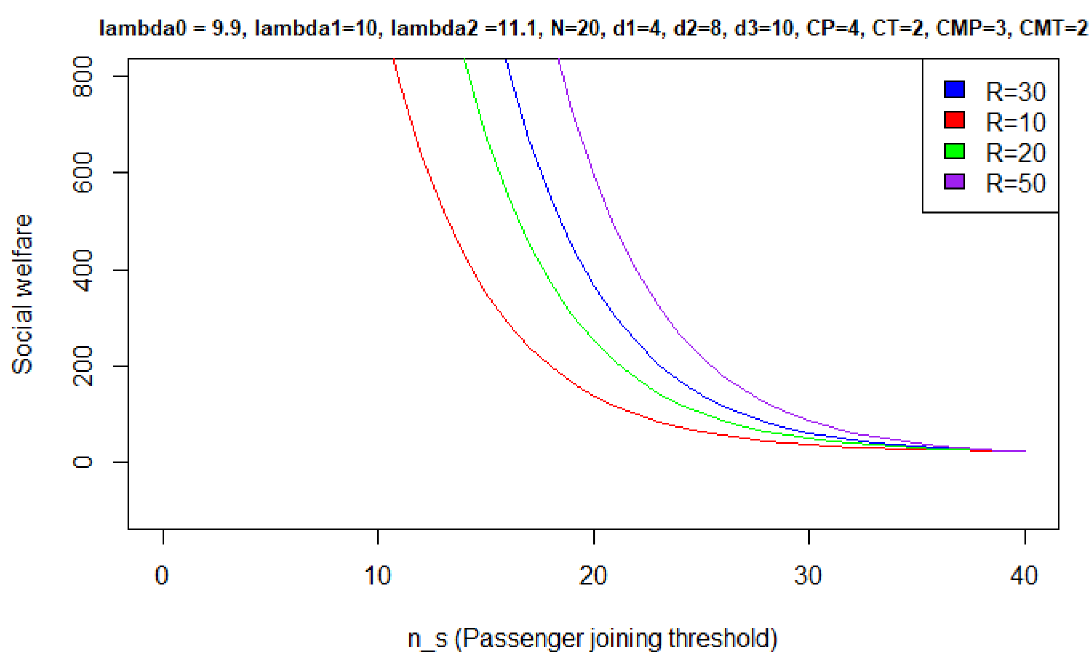
Figure 4.6 shows how the passenger reward, R, affects the social welfare function for two different levels of taxi and passenger arrival rates. We see that for lower passenger joining threshold, a higher R increases social welfare. However, after a certain point, higher R leads to a decreased social welfare function. In gen-

Figure 4.6: Social welfare by the passenger joining threshold (observable case with piecewise uniform matching time) when $N = 20, d_1 = 2, d_2 = 4, d_3 = 5, C_P = 4, C_T = 2, C_{M,P} = 3, C_{M,T} = 2$.

(a) $\lambda_0 = 9.5, \lambda_1 = 10, \lambda_2 = 11.5$



(b) $\lambda_0 = 9.9, \lambda_1 = 10, \lambda_2 = 11.1$



eral, higher passenger reward leads to social welfare to decrease more steeply as the passenger joining threshold increases. This relationship seems inverse to the relationship between R and social welfare in the corresponding two mass point matching time case where for lower values of the passenger joining threshold, social welfare was lower with higher R, and for higher values of the threshold, social welfare was higher for higher R.

Chapter 5

Conclusion and Discussion

5.1 Conclusion

The study of double-ended queues is important, not only in the airport taxi-passenger context, but also in its application to other areas such as organ transplants, app based transportation and delivery services, and more. While there is extensive literature in this subject, double-ended queues with non-zero matching time has been difficult to study due to the complexity matching time adds. In this thesis, we constructed a taxi-passenger double-ended queueing model by adding a random matching time to a taxi-passenger model with dynamic control and studied its equilibrium and social behaviour. We studied two types of matching time distributions: a two mass point distribution and piecewise uniform distribution, each under partially observable and observable conditions. The passenger utility functions and social welfare functions were derived. For the partially observable cases, equilibrium passenger joining probabilities were found and in the observable case, equilibrium passenger joining thresholds were found. This would inform passengers of their equilibrium strategy: to join or balk. The behaviour of social welfare for each case was studied using numerical experiments, showing how changing arrival rates and other factors affect social welfare. The social welfare between cases was also compared and found similarities between the partially observable cases of social welfare. By taking an equilibrium behaviour and social welfare approach, we were able to study how matching time affects a double-ended queueing process while circumventing some of the complications typical of this subject.

5.2 Future work

Future work in this area may consider more complex matching time distributions such as normal distribution dependent on the double-ended queue's system state. In this thesis, the matching time was simplified by construing the distribution based on intervals of the system state space. A normal distribution where the mean is a function of the system state $N(t) = n$ would be a more complex but a more realistic model.

Different arrival processes and queueing disciplines can also be considered for the passengers and taxis. While this thesis considered passenger balking behaviour, it may be more realistic to consider reneging in taxi-passenger contexts where the passenger does nt incur a loss (such as a rideshare cancellation fee) if they choose to leave the queue after joining. Though first-come-first-serve discipline is realistic for taxis, it may not be appropriate for certain airport terminals where the pick-up location is more spread out.

Another valuable direction is to consider the taxi's utility and equilibrium and optimal strategies. This can be compared to the passenger strategies to determine decisions taxi operators may make in response to the expected passenger behaviour. Further research can also consider how policies (such as adjusting fare or adding additional incentives or determents for taxis and passengers) can be implemented to balance the system such that the optimal strategies for both taxis and passengers are similar.

While this thesis focuses on the airport taxi-passenger context, this model can be easily extended to a regular taxi-passenger context. Additionally, it can be used to study other applications of double-ended queues. This may be valuable as matching times are very relevant to peer-to-peer app based transportation and delivery services such as Uber and Foodora and, as these services become more popular, more specific study into these models would be worthwhile.

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