Tensor Networks for Operator-Algebraic Gauge Theories

by

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This thesis is dedicated to my beautiful fiancée Julia. Everything I do is a love letter to you; this one happens to be written in math.

Abstract

Quantum field theory (QFT) is a conceptual framework for understanding the behaviour of subatomic particles—the most successful, mathematically rigorous formulation of QFT is in the language of operator algebras. In this thesis, we describe the construction of specific kinds of QFTs, called gauge theories, using operator-algebraic methods. Once we have described their construction in detail, we use tensor network methods (which are at the centre of modern quantum physics) to build approximations of these QFTs. We finish with a discussion on the relationship between our tensor networks and those used in toy models of the AdS/CFT correspondence.

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Above all, thanks be to God, who holds all things together.

Contents

Chapter 1

Introduction

Motivated by the study of quantum mechanics and its mathematical formalism, John von Neumann introduced the notion of *rings of operators* in 1930 [31]. Throughout the 1930s and 1940s, von Neumann and Francis Murray developed the basic theory of these rings, which are known today as *von Neumann algebras*. In 1943, Israel Gelfand and Mark Naimark considered a broader class of von Neumann's rings of operators [16]. In doing so, they initiated the theory of *C*-algebras*.

While we can rightly think of von Neumann algebras as a special class of C*-algebras, this is not always useful. Instead, we often think of C*-algebras as being a non-commutative generalization of topology and von Neumann algebras being a non-commutative generalization of measure theory. More broadly, the theory of *operator algebras* can be thought of as the study of non-commutative generalizations of classical theory.

Today, the study of operator algebras is diverse and mathematically interesting in its own right. Operator algebraists have made (and continue to make) connections from this field to knot theory, mathematical logic, and beyond. This thesis, however, focuses on operator algebras primarily in the setting in which they were borne: mathematical physics. In particular, we are interested in studying the construction of a certain kind of *quantum field theory* (QFT).

Broadly speaking, QFT refers to a number of physical theories, all of which aim to combine special relativity and quantum mechanics. Algebraic QFT offers a rigorous mathematical formulation of quantum systems with infinitely many degrees of freedom[1]. In this framework, a QFT is a net of von Neumann algebras, indexed by open regions of space-time, satisfying certain properties called the Haag-Kastler axioms [19]. Each element of such a net is called a *local observable*.

In what follows, we endeavour to explain the construction of certain QFTs called *gauge theories* using operator-algebraic methods due to Arnaud Brothier and Alexander Stottmeister [6]. Moreover, we wish to connect their construction to toy mod-

[1] Cf. the canonical quantization of classical mechanics, which applies to systems of n degrees of freedom [18, Section I, Chapter I.1].

els ([33], [34]) of the *anti-de Sitter / conformal field theory* (AdS/CFT) correspondence. The AdS/CFT correspondence is a conjectured relationship between anti-de Sitter (AdS) spaces (used in theories of quantum gravity) and conformal field theories (CFTs), i.e., QFTs which are invariant under conformal transformations (i.e., angle-preserving maps).

The bulk of this thesis is focused on understanding and presenting the construction in [6]. In Chapter 2, we begin with a preliminary overview of the relevant parts of the theory of operator algebras, all of which is standard material. In Chapter 3, we look at the background material specific to the construction in [6]. We begin our presentation of their construction in Chapter 4; here, we construct a net of C*-algebras and its limit $\mathscr{M}_0$. In Chapter 5, we explain how Brothier and Stottmeister introduce a state $\overline{\omega}$ on $\mathscr{M}_0$ in order to obtain an associated von Neumann algebra $(\mathscr{M}, \Omega_{\overline{\omega}})$. We conclude by making preliminary connections between this construction and toy models of the AdS/CFT correspondence via *tensor networks* in Chapter 6.

We make no claim to the originality of the main results in Chapters 2-5 (inclusive). In particular, if a result is indicated as being pulled from the main paper of interest [6], then the proof provided here will be substantially the same as the source. We have added extra details and included computations, as necessary, to clarify matters for the reader. The original content of this thesis is mostly relegated to Chapter 6, in which we make novel connections between the aforementioned constructions.

Chapter 2

Operator Algebraic Preliminaries

We begin with the basics needed for a graduate student who has taken a senior undergraduate / first-year graduate sequence of analysis courses (i.e., measure theory and functional analysis) but has not necessarily studied operator algebras. Familiarity with the basic theory of operators on Hilbert spaces (e.g., the material covered in [4, Chapter 1]) will be extremely useful. We endeavour to provide the necessary foundational material here and some fundamental results to give the reader a taste of the theory of operator algebras. This section is by no means comprehensive—helpful, standard references include Blackadar's *Operator Algebras*, [4] and all three volumes of Takesaki's *Theory of Operator Algebras*, [37], [38], [39].

The reader who is already familiar with this topic may wish to skip to Chapter 3 for the preliminaries specific to the main operator-algebraic construction of the paper.

2.1 C*-Algebras

As Rodger and Hammerstein wrote, and Julie Andrews famously sang: let's start at the very beginning, a very good place to start.

Definition 2.1.1. A *Banach algebra* is an algebra $\mathfrak{A}$ over $\mathbb{C}$ endowed with a sub-multiplicative norm $\|\cdot\|$ such that $(\mathfrak{A}, \|\cdot\|)$ is a Banach space. This means that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathfrak{A}$ and $\mathfrak{A}$ is complete with respect to $\|\cdot\|$. We say that $\mathfrak{A}$ is *unital* if it contains a multiplicative identity, which we will denote by e (or by 1 when this makes sense contextually). In this case, we require $\|e\| = 1$.

Fact 2.1.2. Any non-unital Banach algebra can be embedded in a unital Banach algebra via a process called *unitization* or *unitalization*—we refer the reader to [4, Section II.1.2] for details. We denote the unitization of $\mathfrak{A}$ by $\tilde{\mathfrak{A}}$; if $\mathfrak{A}$ is unital, then $\mathfrak{A} = \tilde{\mathfrak{A}}$.

Definition 2.1.3. Let $\mathfrak{A}$ be a Banach algebra. The spectrum of an element $x \in \mathfrak{A}$ is given by

$$\sigma_{\mathfrak{A}}(x) = \left\{ \lambda \in \mathbb{C} : x - \lambda 1 \text{ is not invertible in } \tilde{\mathfrak{A}} \right\}.$$

If $\mathfrak{A}$ is non-unital, then $0 \in \sigma_{\mathfrak{A}}(x)$ for every $x \in \mathfrak{A}$.

Definition 2.1.4. An *involution* or a $*$-*operation* on an algebra $\mathfrak{A}$ is a map $^* : \mathfrak{A} \to \mathfrak{A}$ satisfying

$$(a+b)^* = a^* + b^*,\ (\lambda a)^* = \overline{\lambda} a^*,\ (ab)^* = b^* a^*,\ a^{**} = a$$

for all $a, b \in \mathfrak{A}, \lambda \in \mathbb{C}$. An algebra equipped with an involution is called an *involutive algebra* or a $*$-*algebra*. We adopt the convention of always calling an involutive Banach algebra a *Banach* $*$-*algebra*.[1]

Definition 2.1.5. Let $\mathfrak{A}$ be a Banach $*$-algebra. An element $a \in \mathfrak{A}$ is said to be

(i) *normal* if $a^* a = a a^*$.

(ii) *self-adjoint* (or *Hermitian*) if $a^* = a$.

If $\mathfrak{A}$ is unital, then an element $a \in \mathfrak{A}$ is said to be

(iii) *unitary* if $a^* a = a a^* = e$.

Definition 2.1.6. A *C*-algebra* $\mathfrak{A}$ is a Banach $*$-algebra such that

$$\|a^* a\| = \|a\|^2 \tag{2.1}$$

for all $a \in \mathfrak{A}$. We call (2.1) the *C*-condition*.

Proposition 2.1.7. *Let $\mathfrak{A}$ be a C*-algebra. Then $\|a^*\| = \|a\|$ for all $a \in \mathfrak{A}$.*

Proof. This is trivial if $a = 0$, so let us assume $a \in \mathfrak{A} \setminus \{0\}$. By the C*-condition and submultiplicativity of the norm, we have

$$\begin{aligned} \|a\|^2 &= \|a^* a\| \\ &\leq \|a^*\| \|a\| \end{aligned}$$

Hence, $\|a^*\| \leq \|a\|$. Applying the same argument to a^* yields $\|a^*\| \leq \|a^{**}\| = \|a\|$, and so the $*$-operation is isometric. □

Example 2.1.8. The space of complex numbers, along with the usual norm and complex conjugation, is a C*-algebra.

The canonical example of a C*-algebra is $\mathcal{B}(H)$, the space of bounded linear operators on a Hilbert space, endowed with the operator norm and $*$-operation given by the adjoint. More generally, any subalgebra $\mathfrak{A}$ of $\mathcal{B}(H)$ which is both norm-closed and self-adjoint is a C*-algebra. If $\mathfrak{A}$ is a self-adjoint unital subalgebra of $\mathcal{B}(H)$, we will call it a *concrete C*-algebra*. There are other C*-algebras which are not obviously of this form, such as the one described in the following example.

[1] Some authors define a Banach $*$-algebra to be an involutive Banach algebra with *isometric* $*$-operation. Most natural examples of Banach $*$-algebras will have this property, but it is not strictly necessary.

Example 2.1.9. Let X be a locally compact Hausdorff space. Recall that a continuous function f on X is said to *vanish at infinity* if the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact for all $\epsilon > 0$. The space of all continuous functions on X vanishing at infinity, denoted $\mathscr{C}_0(X)$, forms a commutative C*-algebra with the sup-norm and complex conjugation as the $*$-operation.

Recall that, in general, a *homomorphism* is a type of map which preserves the structure between two algebraic structures of the same type. In the case of C*-algebras, this role is played by maps which preserve both the involution and the algebra structure.

Definition 2.1.10. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras with $*$-operations denoted by $(\cdot)^*$ and $(\cdot)^\dagger$, respectively. A $*$-*homomorphism* $\psi : \mathfrak{A} \to \mathfrak{B}$ is a map such that

(i) $\psi(xy) = \psi(x)\psi(y)$ for any $x, y \in \mathfrak{A}$, and

(ii) $\psi(x^*) = (\psi(x))^\dagger$ for any $x \in \mathfrak{A}$.

A $*$-*isomorphism* is a bijective $*$-homomorphism.

One can show that any $*$-homomorphism $\psi : \mathfrak{A} \to \mathfrak{B}$ is contractive, i.e., $\|\psi(a)\| \leq \|a\|$ for any $a \in \mathfrak{A}$ (see [4, Corollary II.1.6.6]). Moreover, we can show that an injective $*$-homomorphism is isometric (see, e.g. [4, Corollary II.2.2.9]). This implies that every $*$-isomorphism is isometric; we will sometimes still refer to a map as being an "isometric $*$-isomorphism" for emphasis.

Theorem 2.1.11. *[10, Theorem I.3.1] Let $\mathfrak{A}$ be an abelian C*-algebra. Then there exists a locally compact Hausdorff space X such that $\mathfrak{A}$ is (isometrically) $*$-isomorphic to $\mathscr{C}_0(X)$.*

With the preceding example and theorem in mind, it becomes clear why we can think of general C*-algebras as "noncommutative topological spaces".

Definition 2.1.12. Let $\mathfrak{A}, \mathfrak{B}$ be C*-algebras.

(i) An element a in $\mathfrak{A}$ is said to be *positive* if $a = b^*b$ for some $b \in \mathfrak{A}$; equivalently, if $a = a^*$ and $\sigma_{\mathfrak{A}}(x) \subseteq [0, \infty)$. If a is positive, we write $a \geq 0$. We denote the set of all positive elements in $\mathfrak{A}$ by $\mathfrak{A}^+$.

(ii) A linear map $f : \mathfrak{A} \to \mathfrak{B}$ is said to be *positive* if $f(a) \in \mathfrak{B}^+$ whenever $a \in \mathfrak{A}^+$. If $\mathfrak{B} = \mathbb{C}$, then f is called a *positive linear functional.*

(iii) A *state* is a positive linear function of norm 1.

  (a) We say that a state is *tracial* if $f(ab) = f(ba)$ for all $a, b \in \mathfrak{A}$.

  (b) If $\mathfrak{A}$ is a C*-subalgebra of $\mathcal{B}(H)$ then we say that a state f is a *vector state* of $\mathfrak{A}$ if $f(a) = \langle ax|x\rangle$ for some unit vector $x \in H$.

(c) Let $\mathcal{S}(\mathfrak{A})$ denote the set of all states on $\mathfrak{A}$. This is known as the *state space* of $\mathfrak{A}$.

Proposition 2.1.13. *[4, Proposition II.6.2.5] Let $\mathfrak{A}$ be a unital C*-algebra and let f be a linear functional. Then f is positive if and only if f is bounded and $\|f\| = f(e)$.*

Corollary 2.1.14. *Given a unital C*-algebra $\mathfrak{A}$, a positive linear functional f is a state if and only if $f(e) = 1$.*

Definition 2.1.15. Let $\mathfrak{A}$ be a C*-algebra and let H be a Hilbert space. A $*$-*representation* π of $\mathfrak{A}$ on H is a $*$-homomorphism of $\mathfrak{A}$ into $\mathcal{B}(H)$. We say that $\pi : \mathfrak{A} \to \mathcal{B}(H)$ is

(i) *cyclic* if there is a vector $\xi \in H$ such that $\pi(\mathfrak{A})\xi$ is dense in H; in this case, we call ξ a *cyclic vector* for π,

(ii) *faithful* if π is injective,

(iii) *non-degenerate* if $\pi(\mathfrak{A})H$ is dense in H,

(iv) *algebraically irreducible* if $\pi(\mathfrak{A})$ has no proper invariant subspaces, and

(v) *topologically irreducible* if $\pi(\mathfrak{A})$ has no proper closed invariant subspaces.

Given a $*$-representation $\pi : \mathfrak{A} \to \mathcal{B}(H)$ and a vector $\xi \in H$, the function $\phi(x) = \langle \pi(x)\xi | \xi \rangle$ for all $x \in A$ defines a positive linear functional on $\mathfrak{A}$.

Example 2.1.16. Let $\mathfrak{A}$ is a C*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. Let $\iota : \mathfrak{A} \to \mathcal{B}(H)$ be the canonical inclusion map. Then ι is an injective $*$-homomorphism, hence it is a faithful representation of $\mathfrak{A}$ on H.

It is clear from the definitions above that if a representation is algebraically irreducible then it is also topologically irreducible—if $\pi(\mathfrak{A})$ has no proper invariant subspaces at all, it certainly has no closed ones. Interestingly, one can show that the opposite claim also holds.

Theorem 2.1.17 (Kadison Transitivity)**.** *[4, Theorem II.6.12] Let π be a topologically irreducible representation of a C*-algebra $\mathfrak{A}$ on a Hilbert space H and let $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ be vectors in H with $\{\xi_1, \ldots, \xi_n\}$ linearly independent. Then there is an $a \in \mathfrak{A}$ such that $\pi(a)\xi_k = \eta_k$ for $1 \leq k \leq n$. In particular, π is algebraically irreducible.*

Because the definitions of *algebraically irreducible* and *topologically irreducible* representations coincide for C*-algebras, we can simply say that a representation is *irreducible* if either (and hence both) of the properties hold.

The astute reader may have noticed that we previously said an arbitrary C*-algebra $\mathfrak{A}$ is "not obviously" of the form $\mathfrak{A} \subseteq \mathcal{B}(H)$ for some Hilbert space H. In fact, we can use states to represent any C*-algebra on a Hilbert space. This process is called the Gelfand-Naimark-Segal (GNS) construction.

Theorem 2.1.18 (GNS Construction)**.** *[10, Theorem I.9.6] Let f be a positive linear functional on a C*-algebra $\mathfrak{A}$. Then there is a representation π_f of $\mathfrak{A}$ on a Hilbert space H and a vector $x_f \in H$ which is a cyclic vector for $\pi_f(\mathfrak{A})$ such that $\|x_f\|^2 = \|f\|$ and $f(a) = \langle \pi_f(a)x_f | x_f \rangle$ for all $a \in \mathfrak{A}$.*

Corollary 2.1.19 (Gelfand-Naimark Theorem)**.** *[10, Theorem I.9.12] Every C*-algebra is isometrically $*$-isomorphic to a C*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.*

The proof of the Gelfand-Naimark Theorem makes use of a representation called the *universal representation* of $\mathfrak{A}$, defined by $\pi := \bigoplus_{f \in \mathcal{S}(\mathfrak{A})} \pi_f$. Observe that each state $f \in \pi(\mathfrak{A})$ is a vector state.

It turns out that certain classes of C*-algebras can be completely characterized by their irreducible representations.

Example 2.1.20. A C*-algebra is commutative if and only if every irreducible representation is one-dimensional (see [25, Section 10.4] for details).

At the heart of the main paper of study [6], the authors are interested in taking limits of systems of C*-algebras. Before moving on to the next section, we would like to make this notion explicit.

Definition 2.1.21. An *inductive system* (or *directed system*) of C*-algebras is a collection $\{(A_i, \phi_{ij}) : i, j \in I, i \leq j\}$ where I is a directed set, each A_i is a C*-algebra, and ϕ_{ij} is a $*$-homomorphism from A_i to A_j such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ for $i \leq j \leq k$.

As each ϕ_{ij} is norm-decreasing, there is a naturally induced C*-seminorm on the (algebraic) direct limit of the system $\varinjlim_{\text{alg}} A_i$, which is defined by

$$\|a\| = \lim_{j>i} \|\phi_{ij}(a)\| = \inf_{j>i} \|\phi_{ij}(a)\|$$

for each $a \in A_i$. The *inductive limit* is the C*-algebra obtained by modding out elements of seminorm 0 in $\varinjlim_{\text{alg}} A_i$ and taking its completion. We will denote this algebra as $\mathfrak{A} := \varinjlim_i (A_i, \phi_{ij})$ or simply $\mathfrak{A} := \varinjlim_i (A_i)$ if the $*$-homomorphisms are clear from context.

Remark 2.1.22. Let (A_i), $i \in I$ be a net of C*-algebras.

(i) If $\mathfrak{A}$ is the inductive limit of the system (A_i, ϕ_{ij}), then for each i there is a natural $*$-homomorphism $\phi_i : A_i \to \mathfrak{A}$.

(ii) If all the connecting maps are injective (and hence isometric), we can think of the (algebraic) direct limit as $\cup A_i/_\sim$ where $\sim$ is the equivalence relation defined by identifying $a \in A_i$ with $\phi_{ij}(a) \in A_j$ for any $i < j$; the inductive limit is the completion of this $*$-algebra.

Example 2.1.23. [4, Example II.8.2.2 (i)] An inductive system (A_i, ϕ_{ij}) where each $A_i = \mathscr{C}(X_i)$ and the connecting maps ϕ_{ij} are unital corresponds exactly to to an inverse system of compact Hausdorff spaces (X_i, f_{ij}). Moreover $\varinjlim(A_i, \phi_{ij}) \cong \mathscr{C}(\varprojlim(X_i, f_{ij}))$.

Definition 2.1.24. Let $(A_n, \phi_{n,n+1})$ be a sequence of finite dimensional C*-algebras with appropriately defined connecting maps. If $\mathfrak{A} = \varinjlim_n A_n$, then we say that $\mathfrak{A}$ is an *approximately finite dimensional C*-algebra*. These are often simply called *AF algebras*.

2.2 von Neumann Algebras

There is a special class of C*-algebras known as *von Neumann algebras*, which are interesting in their own right. As every von Neumann algebra is a C*-algebra, one can certainly make use of all the theory and techniques explored above to study them. However, it is often more useful to study von Neumann algebras in their own context. In some ways, von Neumann algebras are more well-behaved than general C*-algebras and in other ways, they are pathological. We will not explore the depth or breadth of von Neumann algebra theory here—the interested reader can consult [4, Chapter 3] for a more comprehensive overview.

In order to make sense of the theory of von Neumann algebras (and indeed, in order to define what a von Neumann algebra is), we will need to work with several different topologies on $\mathcal{B}(H)$. In this section, and further throughout this thesis, H will always denote a Hilbert space.

Definition 2.2.1. In addition to the standard operator norm topology, we define two more topologies on $\mathcal{B}(H)$.

(i) The *weak operator topology* (WOT) is the weakest topology on $\mathcal{B}(H)$ such that the sets $\mathcal{W}(T, x, y) = \{A \in \mathcal{B}(H) : |\langle (T - A)x|y\rangle| < 1\}$ are open. A net $(T_\alpha)_{\alpha\in\Lambda}$ converges weakly to T, denoted $T_\alpha \xrightarrow{WOT} T$, if and only if $(\langle T_\alpha x|y\rangle - \langle Tx|y\rangle) \to 0$ for every $x, y \in H$.

(ii) Similarly, the *strong operator topology* (SOT) on $\mathcal{B}(H)$ is the topology defined by the open sets $\mathcal{S}(T, x) = \{A \in \mathcal{B}(H) : \|(T - A)x\| < 1\}$, $T \in \mathcal{B}(H)$, $x \in H$. A net $(T_\alpha)_{\alpha\in\Lambda}$ converges strongly to T, denoted $T_\alpha \xrightarrow{SOT} T$, if and only if $\|T_\alpha x - Tx\| \to 0$ for every $x \in H$.

It is not hard to see that the WOT is weaker than the SOT and SOT is weaker than the (operator) norm topology. If a subset $\mathcal{S}$ of $\mathcal{B}(H)$ is closed under the weak operator (resp. strong operator) topology, we say that $\mathcal{S}$ is WOT-closed (resp. SOT-closed). We write $\overline{\mathcal{S}}^{\text{WOT}}$ (resp. $\overline{\mathcal{S}}^{\text{SOT}}$) for the WOT (resp. SOT) closures of a subset $\mathcal{S}$.

Proposition 2.2.2.

(i) The adjoint is WOT-continuous.

(ii) Both left and right multiplication by a fixed operator are WOT-continuous.

(iii) Both left and right multiplication by a fixed operator are SOT-continous.

We include the proof of the first two parts of this proposition here as these are often relegated to exercises in introductory texts.

Proof. (i) Let let (T_α) be a net of operators in $\mathcal{B}(H)$ such that $T_\alpha \xrightarrow{WOT} T$. That is, $\langle T_\alpha x|y\rangle \to \langle Tx|y\rangle$ for any $x, y \in H$, so

$$\begin{aligned}\langle T_\alpha^* y|x\rangle = \overline{\langle T_\alpha x|y\rangle} &\to \overline{\langle Tx|y\rangle} \\ &= \langle T^* y|x\rangle.\end{aligned}$$

Hence $T_\alpha^* \xrightarrow{WOT} T^*$.

(ii) Let T, S be operators in $\mathcal{B}(H)$ and let (T_α) be a net such that $T_\alpha \xrightarrow{WOT} T$. By definition, $\langle T_\alpha x|y\rangle \to \langle Tx|y\rangle$ for any $x, y \in H$. First, let us consider right multiplication. We have

$$\begin{aligned}\langle (T_\alpha S)x|y\rangle = \langle T_\alpha(Sx)|y\rangle &= \langle T_\alpha z|y\rangle \qquad [\text{denoting } Sx := z] \\ &\to \langle Tz|y\rangle = \langle (TS)x|y\rangle.\end{aligned}$$

Hence $T_\alpha S \xrightarrow{WOT} TS$. Now, let us multiply by S on the left. We have

$$\begin{aligned}\langle (ST_\alpha)x|y\rangle = \langle T_\alpha x|S^* y\rangle &= \langle T_\alpha x|w\rangle \qquad [\text{denoting } S^* y := w] \\ &\to \langle Tx|w\rangle = \langle Tx|S^* y\rangle = \langle (ST)x|y\rangle.\end{aligned}$$

Hence $ST_\alpha \xrightarrow{WOT} ST$

(iii) The proof is similar.

□

Before we introduce von Neumann algebras, we will introduce a class of operators in $\mathcal{B}(H)$ which we use to define one more topology of interest.

Definition 2.2.3. Let H be a Hilbert space with an orthonormal basis (ONB) $\{e_i\}_{i\in I}$ and $A \in \mathcal{B}(H)^+$. The *trace* of A, denoted by $\mathrm{Tr}(A)$, is given by

$$\mathrm{Tr}(A) := \sum_{i\in I} \langle Ae_i|e_i\rangle.$$

A bounded linear operator $T \in \mathcal{B}(H)$ is said to be *trace-class* if $\mathrm{Tr}(|T|) < \infty$ where $|T| := \sqrt{T^*T}$. The space of all trace-class operators on H is denoted by $\mathcal{T}(H)$.

Definition 2.2.4. The *weak* topology* on $\mathcal{B}(H)$ is the topology generated by the seminorms $\mathcal{B}(H) \ni S \mapsto |\mathrm{Tr}(ST)|$ where $T \in \mathcal{T}(H)$. A net $(S_\alpha)_{\alpha\in\Lambda}$ converges to S weak*, denoted $S_\alpha \xrightarrow{w^*} S$, if and only if $\mathrm{Tr}(S_\alpha T) \to \mathrm{Tr}(ST)$ for every $T \in \mathcal{T}(H)$.

Definition 2.2.5. A *von Neumann algebra* on a Hilbert space H is a unital C*-subalgebra of $\mathcal{B}(H)$ which is WOT-closed.

Definition 2.2.6. A von Neumann algebra $\mathfrak{M}$ is said to be *approximately finite dimensional* or *hyperfinite* if there is a directed collection $\{M_i\}$ of finite dimensional $*$-subalgebras with $\overline{\cup M_i}^{\mathrm{WOT}} = \mathfrak{M}$.

Definition 2.2.7. The *commutant* of a subset $\mathcal{S}$ of $\mathcal{B}(H)$ is the set of operators T which commute with every operator in $\mathcal{S}$. That is,

$$\mathcal{S}' := \{T \in \mathcal{B}(H) : ST = TS \text{ for all } S \in \mathcal{S}\}.$$

The *double commutant* of $\mathcal{S}$ is defined by $\mathcal{S}'' = (\mathcal{S}')'$.

Clearly, taking the commutant is a purely algebraic operation. However, the following results show that it also contains information about the topology.

Proposition 2.2.8. *Let $\mathcal{S}$ be a subset of $\mathcal{B}(H)$.*

(i) The commutant $\mathcal{S}'$ is WOT-closed.

(ii) If $\mathcal{S}$ is self-adjoint, then $\mathcal{S}'$ is a self-adjoint unital algebra.

Proof. This follows immediately from the fact that both left and right multiplication (by fixed operators) are WOT-continuous. □

Corollary 2.2.9. *The commutant of a concrete C*-algebra is always a von Neumann algebra.*

Theorem 2.2.10 (von Neumann Double Commutant)**.** *[10, Theorem I.7.1] Let $\mathfrak{M}$ be a concrete C*-algebra. Then*

$$\mathfrak{M}'' = \overline{\mathfrak{M}}^{WOT} = \overline{\mathfrak{M}}^{SOT}.$$

Definition 2.2.11. A positive linear map $\phi : \mathfrak{M} \to \mathcal{N}$ between von Neumann algebras is *normal* if $\phi(x_\alpha) \xrightarrow{SOT} \phi(x)$ for any increasing net (x_α) in $\mathfrak{M}$ such that $(x_\alpha) \xrightarrow{SOT} x$. If we take $\mathcal{N} := \mathbb{C}$, then ϕ is called a *normal linear functional*.

Example 2.2.12. Let (X, Ω, μ) be a σ-finite measure space and denote the Hilbert space $L^2(X, \mu)$ by H. Endow $L^\infty(X, \mu)$, the space of (complex-valued) essentially bounded μ-measurable functions, with pointwise multiplication. We can view $L^\infty(X, \mu)$ as an algebra of multiplication operators on H, making it a commutative von Neumann algebra.

Theorem 2.2.13. *[4, Corollary III.1.5.18] For any commutative von Neumann algebra $\mathfrak{M}$, there exists a measure space (X, Ω, μ) such that $\mathfrak{M} \cong L^\infty(X, \mu)$.*

With the preceding example and theorem in mind, it becomes clear why we can think of general von Neumann algebras as "noncommutative measure spaces".

Definition 2.2.14. The *center of a von Neumann algebra* $\mathfrak{M}$ is the von Neumann subalgebra $\mathcal{Z}(\mathfrak{M}) := \mathfrak{M} \cap \mathfrak{M}'$. If $\mathcal{Z}(\mathfrak{M}) = \mathbb{C}I$, we call $\mathfrak{M}$ a *factor*. If $\mathcal{Z}(\mathfrak{M}) = \mathfrak{M}$, we say that $\mathfrak{M}$ is *abelian*.

As we said above, the theory of von Neumann algebras is often studied in a much different way than the theory of C*-algebras. A major reason for this is that, as a consequence of the definition, every von Neumann algebra contains an abundance of *projections*.

Definition 2.2.15. A *projection* $p \in \mathcal{B}(H)$ is a self-adjoint, idempotent operator, i.e., $p = p^* = p^2$. If $\mathcal{X} := \{p(v) : v \in H\}$, i.e., if $\mathcal{X}$ is the range of p, we say that p projects onto the closed subspace $\mathcal{X} \subseteq H$.

On the other hand, recall that if $\mathcal{X}$ is any closed subspace of H, then every $v \in H$ can be written as $x + y$ where $x \in \mathcal{X}$ and $y \in \mathcal{X}^\perp$.[2] The map $p_\mathcal{X} : v \mapsto x$ is called the (orthogonal) *projection onto* $\mathcal{X}$. This map is a projection in the sense of the preceding definition, meaning there is a one-to-one correspondence between projections in $\mathcal{B}(H)$ and closed subspaces of H.

Definition 2.2.16. Let $\mathfrak{A}$ be a C*-algebra and let $x \in \mathfrak{A}$. We say that x is

(i) a *projection* if $x = x^* = x^2$,

(ii) a *partial isometry* if $x = xx^*x$.

Definition 2.2.17. Let p, q be projections in a C*-algebra $\mathfrak{A}$. We say that

(i) p is a *subprojection* of q if $p \leq q$ (as elements of $\mathfrak{A}^+$),

(ii) p and q are (Murray von Neumann) *equivalent*, written $p \sim q$, if there is a partial isometry $u \in \mathfrak{A}$ with $u^*u = p$ and $uu^* = q$.

Fact 2.2.18. If $\mathfrak{M}$ is a von Neumann algebra and $\{p_i\}$ is a set of mutually orthogonal projections in $\mathfrak{M}$, then $\bigwedge p_i$ and $\bigvee p_i$ are in $\mathfrak{M}$. In other words, the projections in $\mathfrak{M}$ form a complete lattice [4, Property I.9.2.1(ii)].

Proposition 2.2.19 (Proposition III.1.1.1, [4])**.** *Let $\mathfrak{M}$ be a von Neumann algebra and let $\{p_i\}_{i \in I}$ be a set of mutually orthogonal projections in $\mathfrak{M}$. Then the net of finite sums of the p_i converges strongly to $\bigwedge_{i \in I} p_i$ which we denote $\sum_{i \in I} p_i$.*

[2] For clarity, $\mathcal{X}^\perp$ denotes the orthogonal complement of $\mathcal{X}$, i.e.,

$$\mathcal{X}^\perp := \{v \in H : \langle x|v \rangle = 0, \text{ for all } x \in \mathcal{X}\}.$$

Definition 2.2.20. A projection p in von Neumann algebra $\mathfrak{M}$ is said to be

(i) *minimal* if $p \neq 0$ and $q \leq p$ implies $q = 0$ or $q = p$; equivalently, $\dim(p\mathfrak{M}p) = 1$,

(ii) *abelian* if $p\mathfrak{M}p := \{pzp : z \in \mathfrak{M}\}$ is commutative,

(iii) *finite* if $p \sim q \leq p$ implies $q = p$,

(iv) *semifinite* if there exists a family $\{p_i\}_{i \in I}$ of pairwise orthogonal, finite projections such that $p = \sum_{i \in I} p_i$,

(v) *purely infinite* if $p \neq 0$ and if q is a finite projection such that $q \leq p$ then $q = 0$,

(vi) *properly infinite* if $p \neq 0$ and for all non-zero projections $z \in \mathcal{Z}(\mathfrak{M})$, the projection zp is not finite,

(vii) *central* if $p \in \mathcal{Z}(\mathfrak{M})$.

Furthermore, $\mathfrak{M}$ itself is said to be *finite, semifinite, purely infinite,* or *properly infinite* if $1 \in \mathfrak{M}$ has the corresponding property.

One can show that, for any von Neumann algebra $\mathfrak{M}$, there exists a unique central projection $z_f \in \mathfrak{M}$ such that z_f is finite and $1 - z_f$ is properly infinite. We call $\mathfrak{M}z_f$ the *finite part* and $\mathfrak{M}(1 - z_f)$ the *properly infinite part* of $\mathfrak{M}$. Similarly, we can show that there is a central projection z_{sf} such that z_{sf} is semifinite and $1 - z_{sf}$ is purely infinite. Then $\mathfrak{M}z_{sf}$ is called the *semifinite part* and $\mathfrak{M}(1 - z_{sf})$ is called the *purely infinite part* of $\mathfrak{M}$ (see [4, III.1.4.1, III.1.4.3] for details).

Definition 2.2.21. A von Neumann algebra $\mathfrak{M}$ is said to be of

(i) *Type I* if every non-zero projection $p \in \mathfrak{M}$ has a non-zero, abelian subprojection,

(ii) *Type II* if $1 \in \mathfrak{M}$ is semifinite and $\mathfrak{M}$ has no non-zero, abelian projections; equivalently, if $\mathfrak{M}$ is semifinite and has no non-zero, abelian projections,

(iii) *Type III* if $1 \in \mathfrak{M}$ is purely infinite; equivalently, if $\mathfrak{M}$ is purely infinite.

Theorem 2.2.22. *[30, Theorem 5.3.2] Let $\mathfrak{M}$ be a von Neumann algebra. Then there exist unique central, pairwise orthogonal projections $z_I, z_{II}, z_{III} \in \mathcal{Z}(\mathfrak{M})$ such that $z_I + z_{II} + z_{III} = 1$ and $\mathfrak{M}z_T$ is Type T for $T \in \{I, II, III\}$.*

Corollary 2.2.23. *[4, III.1.4.7] A von Neumann algebra $\mathfrak{M}$ can be canonically written as*

$$\mathfrak{M} = \mathfrak{M}z_I \oplus \mathfrak{M}z_{II} \oplus \mathfrak{M}z_{III}.$$

Moreover, a factor is either Type I, Type II, or Type III.

We can describe the structure of Type I von Neumann algebras in further detail. If n is a cardinal number, then we say that a Type I von Neumann algebra $\mathfrak{M}$ is of *Type* I_n f $1 \in \mathfrak{M}$ is a sum of n equivalent abelian projections. It turns out that if $\mathfrak{M}$ is Type I_n, then $\mathfrak{M} \cong \mathcal{B}(H_n)\bar{\otimes}Z_n$, where H_n is an n-dimensional Hilbert space and Z_n is a commutative von Neumann algebra [4, III.1.5.12].

Similarly, we can further decompose the Type II part of a von Neumann algebra $\mathfrak{M}$. Recalling that z_f is the finite part of $\mathfrak{M}$ and z_{II} is the central summand for the Type II part of $\mathfrak{M}$, set $z_{II_1} := z_{II}z_f$ and $z_{II_\infty} := z_{II}(1 - z_f)$. Then $\mathfrak{M}z_{II_1}$ is called the *Type* II_1 part $\mathfrak{M}z_{II_\infty}$ is called the *Type* II_∞ part of $\mathfrak{M}$.

There is also a further decomposition of the Type III part, However, in order to define this decomposition, we would need more background material that is outside the scope of this thesis, so we refer the interested reader to [4, Section III.4]. Indeed (spoiler alert), the von Neumann algebras in the main construction(s) coming from [6] do not have any Type III component.

2.3 Group Operator Algebras

As the name suggests, a *topological group* is a set G endowed with both the structure of a group and of a topological space. The topology on G is such that the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous (where $G \times G$ is endowed with the product topology). Further, we require that the topology on G be separable.

We refer to a topological group G in terms of its topology. For example, if the topology on G is locally compact, then G is called a *locally compact group*. We are interested in relating (topological) groups to the kinds of algebras studied above.

Definition 2.3.1. A (unitary) *character* of a topological group G is a continuous homomorphism $G \to \mathbb{T}$, where

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\},$$

denotes the circle group.

Definition 2.3.2. Let G be a group. The *complex group algebra* for G is the set of formal sums, $\mathbb{C}[G] := \{\sum_g a_g g \mid a_g \in \mathbb{C},\ \mathrm{supp}(a_g) \text{ finite }\}$. Here, $\mathrm{supp}(a_g)$ denotes the support of a_g, i.e., the set of $g \in G$ for which $a_g \neq 0$. Multiplication in $\mathbb{C}[G]$ is defined by setting

$$a_g g \cdot a_h h = (a_g a_h)(gh),$$

and extending linearly. Moreover, defining $g^* = g^{-1}$ and extending conjugate linearly, $\mathbb{C}[G]$ becomes a *complex group $*$-algebra*.

In the sequel, we assume that G is a compact group unless otherwise specified. Much of the general theory outlined below can be generalized to the case where G is locally compact, but this is not necessary for our purposes.

Definition 2.3.3. A *left* (respectively, *right*) *Haar measure* on G is a nonzero regular Borel measure μ that is left (resp., right) translation invariant. That is, for every Borel set $A \subseteq G$ and every $g \in G$, $\mu(gA) = \mu(A)$ (resp., $\mu(Ag) = \mu(A)$).

Example 2.3.4. Consider the space of complex-valued square integrable functions on G with respect to the Haar measure m_G

$$L^2(G) := \{f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dm_G < \infty\}.$$

This is a Hilbert space under the inner product $\langle f_1 | f_2 \rangle = \int_{g \in G} f_1(g)\overline{f_2(g)}$. The von Neumann algebra of bounded linear operators on $L^2(G)$ is denoted by $\mathcal{B}(L^2(G))$.

Let $\lambda : G \to L^2(G)$ denote the left regular representation of G on $L^2(G)$. That is, let λ denote the representation of G on the Hilbert space $L^2(G)$ defined by $\lambda_g \xi(h) = \xi(g^{-1}h)$ for all $g, h \in G$ and $\xi \in L^2(G)$. In a slight abuse of notation, we will now use $\mathbb{C}[G]$ to refer to the set of formal sums $\{\sum_g a_g \lambda_g : a_g \in \mathbb{C}, \mathrm{supp}(a_g) \text{ finite}\} \subseteq \mathcal{B}(L^2(G))$.

The C*-algebra generated by $\{\lambda_g : g \in G\}$ is the norm-completion of $\mathbb{C}[G]$ with respect to the norm of $\mathcal{B}(L^2(G))$, i.e.,

$$C^*(\{\lambda_g : g \in G\}) := \overline{\mathbb{C}[G]}^{\|\cdot\|_{\mathcal{B}(L^2(G))}},$$

and so $C^*(\{\lambda_g : g \in G\})$ is a C*-subalgebra of $\mathcal{B}(L^2(G))$.

We can also define von Neumann algebras in a similar way. The von Neumann algebra generated by the left regular representation of G is given by

$$LG := \lambda(G)'' \subseteq \mathcal{B}(L^2(G)).$$

Alternatively, let $\rho : G \to L^2(G)$ be the right regular representation of G on $L^2(G)$ given by $\rho_g \xi(h) = \xi(hg)$. Then the von Neumann algebra generated by the right regular representation of G is given by

$$RG := \rho(G)'' \subseteq \mathcal{B}(L^2(G)).$$

Definition 2.3.5. Let $\mathscr{C}(G)$ be the space of continuous functions $G \to \mathbb{C}$ and let μ be the Haar measure. We can endow this space with a multiplication (convolution), an inner product, and a norm, by setting

$$\begin{aligned}
(f * g)(s) &= \int_G f(t) g(t^{-1}s) d\mu(t) \\
f^*(s) &= \overline{f(s^{-1})} \\
\|f\|_1 &= \int_G |f| d\mu.
\end{aligned}$$

These operations and norm make $\mathscr{C}(G)$ into a normed $*$-algebra and we define $L^1(G)$ to be its completion.

Definition 2.3.6. Suppose π is a (strongly continuous) unitary representation of the compact group G on a Hilbert space H, i.e., $\pi : G \to \mathcal{U}(H)$ is a group homomorphism such that $g \mapsto \pi(g)\xi$ is a norm continuous function for every $\xi \in H$. If $f \in L^1(G)$, the operator $\pi(f) = \int_G f(t)\pi(t)d\mu(t)$ is well-defined (in the weak sense) and bounded. In fact, $\|\pi(f)\| \leq \|f\|_1$. This representation of $L^1(G)$ is called the *integrated form* of π.

Example 2.3.7. If λ is the left-regular representation of G and $f \in L^1(G)$, then the operator $\lambda(f)$ is convolution with f on the left. In other words, for $g \in L^1(G)$ and $s \in G$,

$$\lambda(f) : g(s) \mapsto \int_G f(t)\lambda(t)g(s)dt = \int_G f(t)g(t^{-1}s)dt = (f * g)(s).$$

Theorem 2.3.8. *[4, Theorem II.10.2.2] Every non-degenerate $*$-representation of $L^1(G)$ as a Banach $*$-algebra on a Hilbert space arises from a strongly continuous unitary representation of G. Thus, there is a one-to-one correspondence between the representation theories of G and of $L^1(G)$.*

Definition 2.3.9. Let G be a compact group. The *(full) group C*-algebra* $C^*(G)$ is the universal enveloping C*-algebra of the Banach $*$-algebra $L^1(G)$. In other words, if $f \in L^1(G)$, define a norm by

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ a representation of } L^1(G)\}^3.$$

Then take $C^*(G)$ be the completion of $L^1(G)$ with respect to this norm.

It follows immediately from the definition and Theorem 2.3.8 that there is a correspondence between (strongly continuous unitary) representations of G and (non-degenerate) representations of $C^*(G)$. Indeed, if π is a representation of G (and hence of $C^*(G)$), then $\pi(G)'' = \pi(C^*(G))''$. In particular, π is an irreducible representation of G if and only if it is an irreducible representation of $C^*(G)$.

Definition 2.3.10. The completion of $L^1(G)$ with respect to the norm $\|f\|_r = \|\lambda(f)\|$, where λ is the left regular representation, is called the *reduced group C*-algebra* $C^*_r(G)$.

One may show that if G is compact (which will often be the case), then $C^*(G) \cong C^*_r(G)$. The two definitions need not coincide when G is locally compact (see [39, Chapter XII, Section 4] and [2, Section 8.B.]). In particular, they coincide if and only if the group G is *amenable* (see [4, Section II.10.2.6]).

[3] To see that $\|\cdot\|$ is indeed a norm on $L^1(G)$ (rather than a seminorm), we make use of the left regular representation λ of G; if $0 \neq f \in L^1(G)$ then $\lambda(f) \neq 0$.

2.4 Crossed Products

Groups are the mathematical structures which encode symmetries of physical systems. We would like to incorporate symmetries of physical systems into the context of operator algebras—one way this can be done is by taking the *crossed product* of an algebra by a group.

The general idea is that given a C*-algebra $\mathfrak{A}$, a compact topological group G, and a (continuous) action α of G on $\mathfrak{A}$, we would like to construct a larger C*-algebra $\mathfrak{A} \rtimes_\alpha G$ containing both $\mathfrak{A}$ and a group of unitaries which are isomorphic to G and which implement the action α.[4]

Definition 2.4.1. Let G be a group and let $\mathfrak{A}$ be a C*-algebra. An *action of G on $\mathfrak{A}$* is a homomorphism $G \to \mathrm{Aut}(\mathfrak{A})$ denoted $g \mapsto a_g$. In particular, for each $g \in G$, there exists an automorphism $a_g : \mathfrak{A} \to \mathfrak{A}$ such that $a_e = \mathrm{id}_{\mathfrak{A}}$ and $a_g \circ a_h = a_{gh}$ where e is the group identity element and $g, h \in G$.

When we are working with topological groups (which is the case throughout this thesis), we require that the function $G \to \mathfrak{A}$, $g \mapsto a_g(x)$ be continuous for all $x \in \mathfrak{A}$.

Definition 2.4.2. A *C*-dynamical system* is a triple $(\mathfrak{A}, G, \alpha)$ consisting of a C*-algebra $\mathfrak{A}$, a compact group G, and a homomorphism α of G into the automorphism group of $\mathfrak{A}$[5], which is norm-continuous (α is called a continuous action of G on $\mathfrak{A}$ and denoted $\alpha : G \curvearrowright \mathfrak{A}$).

Definition 2.4.3. Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system. A *covariant representation* of $(\mathfrak{A}, G, \alpha)$ is a pair of representations (π, ρ) of $\mathfrak{A}$ and G respectively on the same Hilbert space, such that the following relation holds for all $a \in A$ and $t \in G$,

$$\rho(t)\pi(a)\rho(t)^* = \pi(\alpha_t(a)).$$

We require that π be non-degenerate and ρ be strongly continuous.

Definition 2.4.4. Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system and let $\mathscr{C}(G, \mathfrak{A})$ be the set of continuous functions $G \to \mathfrak{A}$. Define a multiplication (convolution) and involution on $\mathscr{C}(G, \mathfrak{A})$ by,

$$[f * g](t) = \int_G f(s)\alpha_s(g(s^{-1}t))d\mu(s)$$
$$f^*(t) = \alpha_t(f(t^{-1})^*)$$

Then $\mathscr{C}(G, \mathfrak{A})$ becomes a normed $*$-algebra with $\|f\|_1 = \int_G \|f(t)\| d\mu(t)$. We define the *covariance algebra* $L^1(G, \mathfrak{A})$ to be the completion.

[4] This idea does not work perfectly if $\mathfrak{A}$ is not unital and/or G is not discrete; this description is simply meant to motivate the definition.

[5] We denote the automorphism group of X by $\mathrm{Aut}(X)$.

Definition 2.4.5. If (π, ρ) is a covariant representation of $(\mathfrak{A}, G, \alpha)$ on H, then there is an associated representation $\pi \rtimes \rho$ of $L^1(G, \mathfrak{A})$ given by

$$[\pi \rtimes \rho](f) = \int_G \pi(f(t))\rho(t)d\mu(t).$$

We have that $\|[\pi \rtimes \rho](f)\| \leq \|f\|_1$. The representation $\pi \rtimes \rho$ of $L^1(G, \mathfrak{A})$ is called the *integrated form* of (π, ρ).

Definition 2.4.6. The *(full) crossed-product* of $(\mathfrak{A}, G, \alpha)$, denoted $\mathfrak{A} \rtimes_\alpha G$, is the completion of $L^1(G, \mathfrak{A})$ with respect to the norm given by

$$\|f\| = \sup\{\|[\pi \rtimes \rho](f)\| : (\pi, \rho) \text{ a covariant representation of } (\mathfrak{A}, G, \alpha)\}.$$

Now, suppose $(\mathfrak{A}, G, \alpha)$ is a C*-dynamical system. Let π be a non-degenerate, faithful representation of $\mathfrak{A}$ on a Hilbert space H. Define representations π_α and λ of $\mathfrak{A}$ and G, respectively, on $L^2(G, H)$ by

$$\begin{aligned} ([\pi_\alpha(x)]\xi)(t) &= \alpha_{t^{-1}}(x)(\xi(t)) \\ [\lambda(t)\xi](s) &= \xi(t^{-1}s). \end{aligned}$$

Then (π_α, λ) is a covariant representation of $(\mathfrak{A}, G, \alpha)$ and the corresponding representation $\pi_\alpha \rtimes \lambda$ of $L^1(G, \mathfrak{A})$ is faithful. Then for $f \in L^1(G, \mathfrak{A})$ we can define a norm

$$\|f\|_r = \|[\pi_\alpha \rtimes \lambda](f)\| \leq \|f\|_1$$

Definition 2.4.7. The completion of $L^1(G, \mathfrak{A})$ with respect to $\|\cdot\|_r$ is called the *reduced crossed-product* of $(\mathfrak{A}, G, \alpha)$, denoted $\mathfrak{A} \rtimes_{\alpha,r} G$.

One may show that when G is compact then $\mathfrak{A} \rtimes_{\alpha,r} G \cong \mathfrak{A} \rtimes_\alpha G$. However, the two crossed-products need not coincide when G is locally compact.

We would like to have some sort of analogue of dynamical systems and crossed-products specifically for von Neumann algebras. In the definition of a C*-dynamical system, we took a homomorphism $\alpha : G \to \mathrm{Aut}(\mathfrak{A})$ which was norm-continuous. As we know, the norm topology is too strong for von Neumann algebras, so we adjust our definition accordingly.

Definition 2.4.8. Let G be a compact group, $\mathfrak{M}$ a von Neumann algebra, and α an action of G on $\mathfrak{M}$ which is point-WOT-continuous (or equivalently, point-SOT-continuous). We call $(\mathfrak{M}, G, \alpha)$ a *W*-dynamical system.*

In general, if α is a WOT-continuous action of G on $\mathfrak{M}$, it is not necessarily norm-continuous. This means that a W*-dynamical system $(\mathfrak{M}, G, \alpha)$ need not be a C*-dynamical system—however, the two definitions coincide if G is discrete.

Given a W*-dynamical system $(\mathfrak{M}, G, \alpha)$ and the left Haar measure μ on G, then we can realize $\mathfrak{M}$ and G on the Hilbert space $L^2((G, \mu), H) \cong L^2(G, \mu) \otimes H$ in the

same way as in the reduced crossed-product above. Namely, $[\pi(x)f](s) = \alpha_{s^{-1}}(x)f(s)$ for $x \in \mathfrak{M}$ and $\lambda(t)f(s) = f(t^{-1}s)$ for $t \in G$.

Definition 2.4.9. In the above setting, (π, λ) is a covariant representation of $(\mathfrak{M}, G, \alpha)$ and the *von Neumann crossed-product* of $\mathfrak{M}$ by G is given by

$$\mathfrak{M} \bar{\rtimes}_\alpha G := \{\pi(\mathfrak{M}), \lambda(G)\}''.$$

2.5 Tensor Products

Given two (or more) C*-algebras, we would like to be able to create a larger C*-algebra. In Section 2.1, we saw how to take limits of infinite directed systems of C*-algebras, but this is not our only option.

If $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras, we can form their *algebraic tensor product* $\mathfrak{A} \odot \mathfrak{B}$ over $\mathbb{C}$. This space has a natural structure as a $*$-algebra with multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$ and involution $(a \otimes b)^* = a^* \otimes b^*$. One may show that C*-norms on $\mathfrak{A} \odot \mathfrak{B}$ exist; if $\|\cdot\|_\gamma$ is a C*-norm on $\mathfrak{A} \odot \mathfrak{B}$, we write $\mathfrak{A} \otimes_\gamma \mathfrak{B}$ for the C*-algebra given by the completion of $\mathfrak{A} \odot \mathfrak{B}$ with respect to $\|\cdot\|_\gamma$. As an algebra, $\mathfrak{A} \odot \mathfrak{B}$ has the following universal property.

Proposition 2.5.1. *[4, Paragraph II.9.12] Let $\mathfrak{C}$ be a complex $*$-algebra. If π_A and π_B are $*$-homomorphisms such that $\pi_A(A)$ and $\pi_B(B)$ commute, then there is a unique $*$-homomorphism $\pi : \mathfrak{A} \odot \mathfrak{B} \to \mathfrak{C}$, such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$ for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.*

Taking $\mathfrak{C} = \mathcal{B}(H)$ above, we get $*$-representations of $\mathfrak{A} \odot \mathfrak{B}$ and hence induced C*-seminorms. A standard way to generate such representations is via tensor products of Hilbert spaces.

If π_A and π_B are representations of $\mathfrak{A}$ and $\mathfrak{B}$ on H_1 and H_2 respectively, we can form the representation $\pi = \pi_A \otimes \pi_B$ of $\mathfrak{A} \odot \mathfrak{B}$ on $H_1 \otimes H_2$ by $\pi(a \otimes b) = \pi_A(a) \otimes \pi_B(b)$. If π_A and π_B are faithful, then so is $\pi_A \otimes \pi_B$, so $\mathfrak{A} \odot \mathfrak{B}$ has at least one C*-norm. Also, for any π_A and π_B, we have

$$\left\| (\pi_A \otimes \pi_B) \left(\sum_{i=1}^n a_i \otimes b_i \right) \right\| \leq \sum_{i=1}^n \|a_i\| \|b_i\|.$$

This means that

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\min} = \sup \left\{ \left\| (\pi_A \otimes \pi_B) \left(\sum_{i=1}^n a_i \otimes b_i \right) \right\| : \pi_A \text{ rep. of } \mathfrak{A},\ \pi_B \text{ rep. of } \mathfrak{B} \right\}$$

is finite and hence a C*-norm.

Definition 2.5.2. We call $\|\cdot\|_{\min}$ the *minimal (or spatial) C*-norm* on $\mathfrak{A} \odot \mathfrak{B}$. The completion of $\mathfrak{A} \odot \mathfrak{B}$ with respect to this norm is written $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ and it is called the *minimal (or spatial) tensor product* of $\mathfrak{A}$ and $\mathfrak{B}$. There can be other C*-norms on $\mathfrak{A} \odot \mathfrak{B}$. However, throughout this thesis, we are only interested in the minimal C*-norm defined above.

Fact 2.5.3. There are several interesting properties of the minimal tensor product, which we will use. They are presented here without proof. The interested reader is encouraged to check out [4, Section II.9] for more detail.

(i) If $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ and $\psi : \mathfrak{B}_1 \to \mathfrak{B}_2$ are $*$-homomorphisms, then there is a natural induced $*$-homomorphism

$$\phi \otimes \psi : \mathfrak{A}_1 \odot \mathfrak{B}_1 \to \mathfrak{A}_2 \otimes \mathfrak{B}_2$$

which induces a $*$-homomorphism

$$\phi \otimes_{\min} \psi : \mathfrak{A}_1 \otimes_{\min} \mathfrak{B}_1 \to \mathfrak{A}_2 \otimes_{\min} \mathfrak{B}_2.$$

(ii) If $\hat{\mathfrak{A}}$ is a C*-subalgebra of $\mathfrak{A}$ and $\mathfrak{B}$ is any C*-algebra, then the natural inclusion of $\hat{\mathfrak{A}} \odot \mathfrak{B}$ into $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ extends to an isometric embedding of $\hat{\mathfrak{A}} \otimes_{\min} \mathfrak{B}$ into $\mathfrak{A} \otimes_{\min} \mathfrak{B}$.

(iii) Minimal tensor products commute with inductive limits when the connecting maps are injective. If $\mathfrak{A} = \varinjlim_i (A_i, \phi_{ij})$ is an inductive system of C*-algebras (with injective connecting maps) and $\mathfrak{B}$ is any C*-algebra then the inductive system $(A_i \otimes_{\min} \mathfrak{B}, \phi_{ij} \otimes \mathrm{id}_B)$ is isomorphic to $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. Moreover, the maps $\phi_{ij} \otimes \mathrm{id}_B$ are injective.

It is clear that the definition of the minimal tensor product can be extended to finite sets of C*-algebras, i.e., there is an obvious way to define $A_1 \otimes_{\min} \cdots \otimes_{\min} A_n$ for C*-algebras $A_1, \ldots, A_n$. In fact, in the case of unital C*-algebras, we can actually define the *infinite tensor product*.

Definition 2.5.4. If $\{A_i\}_{i \in I}$ is a set of unital C*-algebras, then for every finite set $\mathcal{F} = \{i_1, \ldots, i_n\} \subseteq I$ set $A_{\mathcal{F}} = A_{i_1} \otimes_{\min} \cdots \otimes_{\min} A_{i_n}$. If $\mathcal{F} \subseteq \mathcal{G}$, then there is a natural isomorphism $A_{\mathcal{G}} \cong A_{\mathcal{F}} \otimes_{\min} A_{\mathcal{G} \setminus \mathcal{F}}$. Moreover, since each A_i is unital, there is a natural inclusion of $A_{\mathcal{F}}$ into $A_{\mathcal{G}}$ given by $x \mapsto x \otimes 1_{\mathcal{G} \setminus \mathcal{F}}$. Thus, the $A_{\mathcal{F}}$ form an inductive system and we define the *infinite minimal tensor product* $\otimes_{i \in I}^{\min} A_i$ to be the inductive limit of this system.

The notion of the spatial tensor product can also be applied to von Neumann algebras.

Definition 2.5.5. If $\mathfrak{M}_i$ is a von Neumann algebra on H_i (for $i = 1, 2$), then $\mathfrak{M}_1 \odot \mathfrak{M}_2 \subseteq \mathcal{B}(H_1) \odot \mathcal{B}(H_2)$ acts naturally on $H_1 \otimes H_2$. The *(von Neumann algebra) tensor*

product of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is the WOT-closure of the algebraic tensor product inside $\mathcal{B}(H_1 \otimes H_2)$. That is,

$$\mathfrak{M}_1 \bar{\otimes} \mathfrak{M}_2 := \overline{\mathfrak{M}_1 \odot \mathfrak{M}_2}^{WOT} \subseteq \mathcal{B}(H_1 \otimes H_2).$$

By the von Neumann Double Commutant Theorem 2.2.10, this means $\mathfrak{M}_1 \bar{\otimes} \mathfrak{M}_2 = (\mathfrak{M}_1 \odot \mathfrak{M}_2)''$.

Remark 2.5.6.

(i) If $\mathfrak{M}_1, \mathfrak{M}_2$ are infinite dimensional von Neumann algebras, then $\mathfrak{M}_1 \bar{\otimes} \mathfrak{M}_2$ is larger than the norm closure $\mathfrak{M}_1 \otimes_{\min} \mathfrak{M}_2$.

(ii) For any Hilbert spaces H_1, H_2, we have $\mathcal{B}(H_1) \bar{\otimes} \mathcal{B}(H_2) \cong \mathcal{B}(H \otimes K)$.

Example 2.5.7. Let each $L^\infty(X_i, \mu_i)$ act on $L^2(X_i, \mu_i)$ by multiplication (for $i = 1, 2$). Then

$$L^\infty(X_1, \mu_1) \bar{\otimes} L^\infty(X_2, \mu_2) \cong L^\infty(X_1 \times X_2, \mu_1 \times \mu_2)$$

acting on

$$L^2(X_1, \mu_1) \otimes L^2(X_2, \mu_2) \cong L^2(X_1 \times X_2, \mu_1 \times \mu_2).$$

Moreover, we can define the notion of infinite tensor products for von Neumann algebras.

First, let $\{H_i : i \in I\}$ be a collection of Hilbert spaces and ξ_i a unit vector in H_i. Then we can construct the *infinite tensor product of the H_i with respect to ξ_i*, which we denote $\otimes_i(H_i, \xi_i)$. For each finite $F \subseteq I$, let $H_F = \odot_{i \in F} H_i$ with the natural inner product. If $F \subseteq G$, identify H_F with the closed subspace $H_F \otimes (\bigotimes_{i \in G \setminus F} \xi_i) \subseteq H_G$. Let H_0 be the inductive limit of the H_F; elements of H_0 are linear combinations of elementary tensors $\otimes \eta_i$, where $\eta_i = \xi_i$ for all but finitely many i. We set $\otimes_i(H_i, \xi_i)$ to be the completion of H_0 and note that it has a distinguished unit vector $\xi = \otimes \xi_i$. If M_i is a unital C*-subalgebra of $\mathcal{B}(H_i)$, then the infinite, minimal C*-tensor product $\otimes_{\min} M_i$ acts naturally as a C*-algebra of operators on $\otimes_i(H_i, \xi_i)$.

Definition 2.5.8. If each M_i in the above situation is a von Neumann algebra, then we can define the *infinite von Neumann algebraic tensor product* of the M_i with respect to ξ_i to be $\overline{\otimes_i}(M_i, \xi_i)$, the von Neumann algebra generated by the algebraic infinite tensor product.

Example 2.5.9. An infinite tensor product of finite dimensional (or more generally, Type I) von Neumann algebras is approximately finite dimensional (hyperfinite).

Example 2.5.10. For each $n \in \mathbb{N}$, let M_n denote a copy of $M_2(\mathbb{C})$—the C*-algebra of 2×2 matrices with complex entries—and let τ be its unique tracial state. Then $\overline{\otimes_{n \in \mathbb{N}}}(M_n, \tau)$ is a hyperfinite Type II_1 factor, denoted $\mathcal{R}$.

Theorem 2.5.11. *[4, Section III.3.4] Let $\mathfrak{M}$ be a hyperfinite II_1 factor with separable predual*[6]*. Then $\mathfrak{M} \cong \mathcal{R}$.*

[6]The predual of a von Neumann algebra $\mathfrak{M}$ is the unique Banach space X such that $X^* \cong \mathfrak{M}$.

Chapter 3

Setting the Stage

This chapter focuses on the basic elements and formalism which are used by Brothier and Stottmeister in [6], but which are not (necessarily) standard material for an operator algebraist.

3.1 Category Theory

Following the conventions of [6], we will use a category theoretical approach throughout this thesis. This approach is often used in the literature, but rarely discussed in the classroom setting. As such, this chapter will be used to establish some basic definitions and facts. This section is based heavily on [11].

Definition 3.1.1. A *category* $\mathcal{C}$ consists of

- a class of *objects*—often denoted $\mathrm{Obj}(\mathcal{C})$;
- for any pair of objects X, Y a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms*—to denote a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we will often write $f : X \to Y$;
- a *composition law* for morphisms—for any three objects X, Y, Z, the composition law is a function $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, which we denote $(g, f) \mapsto g \circ f$ (or simply gf);
- for every object X, a distinguished morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$—we refer to this as the "identity of X".

Further, there are two axioms which must hold:

(i) *Associativity*: If $f : X \to Y$, $g : Y \to Z$, and $h : Z \to X$ are morphisms, then $(h \circ g) \circ f = h \circ (g \circ f)$.

(ii) *Identity*: For any morphism $f : X \to Y$, $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.

Definition 3.1.2. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is called an *isomorphism* if there is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In this case, we also say that f is invertible.

Example 3.1.3. The class of all sets and all functions between sets forms a category, denoted by Set. In this category, composition is the usual composition of functions and identities are the identity maps; the isomorphisms are bijective functions.

Example 3.1.4. There are several interesting categories of C*-algebras, e.g., the category where objects are (unital) C*-algebras and morphisms are (unital) $*$-homomorphisms. The composition law for morphisms is the usual function composition and the identity morphism is the usual identity map. We will discuss the unital case in greater detail in Section 3.5.

Definition 3.1.5. Let $\mathcal{C}, \mathcal{D}$ be two categories. A *(covariant) functor* F from $\mathcal{C}$ to $\mathcal{D}$, written $F : \mathcal{C} \to \mathcal{D}$ is a function which assigns every object X of $\mathcal{C}$ to an object $F(X)$ of $\mathcal{D}$ and every morphism $f : X \to Y$ in $\mathcal{C}$ to a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$, such that all identities and the composition law are preserved. That is,

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \text{ and } F(g \circ f) = F(g) \circ F(f).$$

A *contravariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ differs from a covariant functor in that it associates each morphism $f : X \to Y$ in $\mathcal{C}$ to a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$ such that identities are preserved and the direction of the composition is reversed. That is,

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \text{ and } F(g \circ f) = F(f) \circ F(g).$$

Fact 3.1.6. If $F : \mathcal{C} \to \mathcal{D}$ is a functor and f is an isomorphism in $\mathcal{C}$ with inverse f^{-1}, then $F(f)$ is an isomorphism with inverse $F(f^{-1})$ [11, Remark 5].

Definition 3.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. For every pair of objects $X, Y \in \mathcal{C}$, the functor F induces a function

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)).$$

We say that F is *faithful* if $F_{X,Y}$ is injective.

Definition 3.1.8. A *concrete category* is a pair $(\mathcal{C}, U)$ such that $\mathcal{C}$ is a category and $U : \mathcal{C} \to \mathrm{Set}$ is a faithful functor.

In a concrete category, we think of the faithful functor U as assigning every object of $\mathcal{C}$ to its "underlying set" and every morphism in $\mathcal{C}$ to its "underlying function".

Definition 3.1.9. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* or *morphism of functors* between F and G is a family of morphisms $\alpha = \{\alpha_X : F(X) \to G(X)\}_{X \in \mathrm{Obj}(\mathcal{C})}$

in $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ such that the following diagram commutes.[1]

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

We write $\alpha : F \to G$ to denote that α is a morphism from F to G. We say that α is a *natural isomorphism* if each component morphism α_X is an isomorphism.

Definition 3.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence* if there exists some functor $G : \mathcal{D} \to \mathcal{C}$ and two isomorphisms of functors $\alpha : G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $\beta : F \circ G \cong \mathrm{id}_{\mathcal{D}}$. We call an equivalence $F : \mathcal{C} \to \mathcal{C}$ an *autoequivalence*.

3.1.1 Tensor Categories

It will be convenient to make use of various categories with an additional tensor product structure throughout this thesis. Because the definition of a *tensor category* varies from source to source, we will take a brief detour to introduce the notion we are using here. In particular, the definition we are using coincides with the definition of a *monoidal category* given in [13].

Definition 3.1.11. A *tensor category* is a sextuple $(\mathcal{C}, \otimes, a, \mathbf{1}, \ell, r)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor called the *tensor product* bifunctor,
$a : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$ is a natural isomorphism

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \qquad X, Y, Z \in \mathrm{Obj}(\mathcal{C})$$

called the *associativity constraint*, $\mathbf{1}$ is an object of $\mathcal{C}$ called the *identity object*, and $\ell_X : \mathbf{1} \otimes X \to X$ and $r_X : X \otimes \mathbf{1} \to X$ are natural isomorphisms called the left and right *unit constraints*, respectively. We require that $(\mathcal{C}, \otimes, a, \mathbf{1}, \ell, r)$ satisfies the following two axioms, which we describe using commutative diagrams.

(i) **The triangle axiom.** The following commutes for all $X, Y \in \mathrm{Obj}(\mathcal{C})$.

$$\begin{tikzcd} (X \otimes \mathbf{1}) \otimes Y \arrow[rr, "a_{X,\mathbf{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \ell_Y"] \\ & X \otimes Y & \end{tikzcd}$$

[1] By commutes, we mean that $G(f) \circ \alpha_X = \alpha_Y \circ F(f)$.

(ii) **The pentagon axiom.** The following commutes for all $W, X, Y, Z \in \mathrm{Obj}(\mathcal{C})$.

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_Z \otimes a_{X,Y,Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

We often refer to a tensor category by a triple $(\mathcal{C}, \otimes, \mathbf{1})$, if the remaining data are clear from context.

Example 3.1.12. Let Gr be the category whose objects are groups and morphisms are group homomorphisms. Then $(\mathrm{Gr}, \times, \{e\})$ is a tensor category, where $\times$ is the direct product of groups.

3.2 Binary Forests

We are almost ready to describe the category of binary forests. First, we briefly review some standard terminology from graph theory, using the relevant chapters of [3] as our reference.

Recall that a *graph* is a pair $G := (V, E)$ consisting of a set V, whose members are called *vertices*, and a set E consisting of unordered pairs of distinct elements of V[2], whose members are called *edges*. We call V the *vertex set* and E the *edge set*.

Definition 3.2.1. Two vertices v, w in a graph $G = (V, E)$ are said to be *adjacent* if the set $\{v, w\} \in E$. The *degree* of a vertex v is the number of edges of G which contain v. More formally, the degree of $v \in V$ is the cardinality of the set D_v where

$$D_v := \{e \in E : v \in e\}.$$

Definition 3.2.2. A *walk* in a graph $G = (V, E)$ is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that v_i and v_{i+1} are adjacent for all $1 \leq i \leq k - 1$. If all of the vertices in a walk are distinct, then it is called a *path*. If all of the vertices in the walk are distinct

[2]In general, one may choose to allow edges of the form $e = \{v, v\}$ which are called *loops* or *self-edges*. However, we intentionally choose here to remove this possibility, as we neither want nor need these in the sequel.

except $v_1 = v_k$, is is called a *cycle*. If v, w are distinct vertices which can be joined by a path, we write $v \sim w$.

The interested reader can verify that $\sim$ is an equivalence relation on the vertex set V of G. This means that V can be partitioned into disjoint equivalence classes where two vertices are in the same class if they can be joined by a path.

Definition 3.2.3. Suppose $G = (V, E)$ is a graph and the partition of V corresponding to the equivalence relation $\sim$ is $V = \bigcup_{i \in I} V_i$. For each $i \in I$, let E_i denote the subset of E consisting of edges whose ends are both in V_i. The graphs $G_i = (V_i, E_i)$ are called the *components* of G. If G has only one component, it is said to be *connected*.

Definition 3.2.4. A *tree* is a connected graph which contains no cycles. A *forest* is a graph whose components are all trees, i.e., it is a disjoint union of trees. A *rooted tree* is a tree with distinguished vertex, called the *root*, which has degree 1.

Because a tree is connected, there is a path between any two vertices. In particular, in a rooted tree there is a path between the root and every other vertex. On the path from the root to a vertex w, if v immediately precedes w then v is called the *parent* of w and w is called a *child* of v. A *leaf* of a rooted tree is a vertex with no children.

Definition 3.2.5. A *binary (rooted) tree* is a rooted tree such that each vertex (apart from the root) has either zero or two children.

Throughout this thesis, the only trees we are interested in are binary rooted trees. For convenience's sake, we will typically refer to them as "binary trees" or simply "trees".

Example 3.2.6. A binary tree with 3 leaves:

$t =$  .

Again, as a matter of convenience, we will refer to a forest of binary rooted trees as a "binary forest" or simply as a "forest".

Example 3.2.7. A binary forest composed of 3 trees:

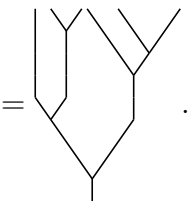
$f =$  .

It is natural to put an order on the roots and the leaves of a forest—we will always count them from left to right. In the previous example, we could label the roots $1, 2, 3$ and the leaves $1, 2, \ldots, 6$.

Now, we want to endow the collection of binary forests with the structure of a tensor category. Let $\mathcal{F}$ be the category whose objects are the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and morphisms between n and m are the set of binary forests with n roots and m leaves, which we think of as mapping n to m and denote as $\mathcal{F}(n, m)$.

Consider two forests g, f such that f has n leaves and g has n roots. We think of the composition $g \circ f$ – which we will denote gf – as the vertical concatenation of g on top of f. Namely, the ith leaf of f lines up with the ith root of g.

Example 3.2.8. Using the tree t from Example 3.2.6 and the forest f from Example 3.2.7, we get the composition:

$f \circ t =$  .

We denote the set of all trees by $\mathfrak{T}$. We will often work with the rooted binary trees with one and two leaves, which we will denote as I and Y respectively. The letters are chosen both for the obvious fact that they look like the trees they are representing (see below) and for the fact that I is the identity morphism id_1 in $\mathcal{F}$. In the literature, Y is sometimes referred to as the *caret*. We will refer to these two trees as the "generating trees", which we justify in the following proposition.

$I :=$ $Y :=$  .

The tensor product of objects is simply addition, $n \otimes m = n + m$ for each $n, m \in \mathbb{N}$. The tensor product of morphisms is horizontal concatenation (of forests). It is not hard to see that this fully describes a tensor category as in Definition 3.1.11.[3]

Proposition 3.2.9. *If $n \geq 1$ and $1 \leq j \leq n$, let $f_{j,n}$ be the forest with n roots and $n+1$ leaves, whose jth tree is Y and all others are I, i.e., $f_{j,n} := I^{\otimes j-1} \otimes Y \otimes I^{\otimes n-j}$. Then any morphism is the composition of some $f_{j,n}$, hence the tensor category $\mathcal{F}$ is generated by I and Y.*

Proof. We proceed by induction. If $f \in \mathcal{F}(n, n+1)$, then clearly $f = f_{j,n} = I^{\otimes j-1} \otimes Y \otimes I^{\otimes n-j}$ where $1 \leq j \leq n$. If $f \in \mathcal{F}(n, n+2)$ then $f = g \circ h$ where $h \in \mathcal{F}(n, n+1)$ and $g \in \mathcal{F}(n+1, n+2) = \mathcal{F}(n+1, (n+1)+1)$. Thus, $h = f_{j,n}$ for some $1 \leq j \leq n$ and $g = f_{k,n+1}$ for some $1 \leq k \leq n+1$. So the morphism f is composed of morphisms of the correct form. Now suppose the hypothesis holds for any $f \in \mathcal{F}(n, n+k)$. If $f \in \mathcal{F}(n, n+k+1)$ then $f = g \circ h$ where $h \in \mathcal{F}(n, n+k)$ and $g \in \mathcal{F}(n+k, (n+k)+1)$. By the preceding work, this means that f is the composition of the basic morphisms. Hence, the tensor category is generated by the trees I and Y. □

Definition 3.2.10. We call a forest of the form $f_{j,n} = I^{\otimes j-1} \otimes Y \otimes I^{\otimes n-j}$ an *elementary forest*.

[3] Actually, to be rigorous, we also need an identity to satisfy the definition. The identity object is $0 \in \mathbb{N}$ and the morphism from 0 to 0 is the empty forest, i.e., the forest with no vertices and no edges. We will never use these objects in our construction(s).

By the preceding proposition, any binary rooted forest f is the composition of elementary forests.

Now, let $((\mathcal{D}, U), \otimes, 1)$ be a concrete tensor category and let $\Phi : \mathcal{F} \to \mathcal{D}$ be a tensor functor. Since $\Phi(n) = \otimes_{k=1}^{n} \Phi(1)$, then $\Phi(n)$ is completely characterized by $\Phi(1)$. Further, for any elementary forest $f_{j,n}$, we have $\Phi(f_{j,n}) = \mathrm{id}^{n-j} \otimes \Phi(Y) \otimes \mathrm{id}^{\otimes n-j}$, where id is the identity of $\Phi(1)$. Since any forest f is a composition of elementary forests, we see that $\Phi(f)$ is characterized by the morphism $R := \Phi(Y) \in \mathrm{Hom}_{\mathcal{D}}(\Phi(1), \Phi(1) \otimes \Phi(1))$. Conversely, given an object A in a concrete tensor category $((\mathcal{D}, U), \otimes, 1)$ together with a morphism $R : A \to A \otimes A$, we can define a tensor functor $\Psi : \mathcal{F} \to \mathcal{D}$.

3.3 Dyadic rationals and partitions

As mentioned previously, the direct limit of C*-algebras is at the very heart of the main construction in [6]. For reasons that will become clear later, we would like to be able to index our collection by the *dyadic rationals*.

Definition 3.3.1. The set of all *dyadic rationals* inside the half-open interval $[0, 1)$ is the set $\mathbb{D} = \left\{ \frac{a}{2^n} : n \geq 1, 0 \leq a \leq 2^n - 1, a, n \in \mathbb{N} \right\}$.

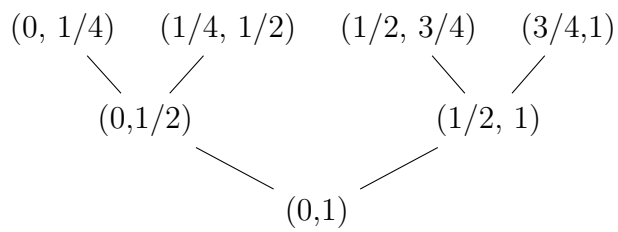
We do not include the point 1 because we identify $[0, 1)$ with the circle $\mathbb{R}/\mathbb{Z}$ and $\mathbb{D}$ with the set of dyadic rationals inside of it.

Definition 3.3.2. A *standard dyadic interval* (s.d.i.) is any subinterval of $[0, 1)$ of the form $\left(\frac{a}{2^n}, \frac{a+1}{2^n}\right)$ with $a, n \in \mathbb{N}$. Because an s.d.i is a subinterval of $[0, 1)$, it necessarily follows that $\frac{a+1}{2^n} \leq 1$.

Definition 3.3.3. A *standard dyadic partition* (s.d.p.) of $[0, 1)$ is a finite sequence of dyadic rationals $0 = d_1 < d_2 < \ldots < d_n < d_{n+1} = 1$ such that (d_j, d_{j+1}) is an s.d.i. for any $1 \leq j \leq n$.

Strictly speaking, this is not a partition of $[0, 1)$ since the union of the s.d.i.'s does not contain the right boundary points $d_j, 1 \leq j \leq n + 1$. However, it is just as well for us to consider half-open intervals with left boundaries included, i.e., we can take s.d.i's of the form $[d_j, d_{j+1})$ instead. When we refer to s.d.p.'s, typically we will be referring to these half-open intervals. We will call the union of (the sets of) open s.d.i.'s an *open s.d.p.* of the unit interval.

Now we would like to relate dyadic rationals to binary trees. Let t_∞ be the infinite binary rooted tree. We label its root by $(0, 1)$ and the successor of a vertex labelled (d, d') is labelled by $(d, \frac{d+d'}{2})$ to the left and $(\frac{d+d'}{2}, d')$ to the right. Immediately, we see that the collection of all the labels is given by the collection of (open) s.d.i.'s. The first two levels of this tree are given as follows:



We can view a finite rooted binary tree $t \in \mathfrak{T}$ with n leaves as a rooted sub-tree of t_∞. We label the jth leaf of t by the label (e_j, e'_j) corresponding to the associated vertex in t_∞. We have that $e_j = e'_j =: d_j$ for $2 \leq j \leq n+1$, $e_1 = 0 =: d_1$, and $e'_n = 1 =: d_{n+1}$.

Example 3.3.4. The complete tree t_m is the rooted binary tree with 2^m leaves all at a distance m from its root. When $m = 2$, we have the beginning of t_∞ as given above.

Via the labelling of a finite rooted binary tree $t \in \mathfrak{T}$, we obtain a sequence of dyadic rationals $0 = d_1 < d_2 < \ldots < d_n < d_{n+1} = 1$ which gives an open s.d.p. of $(0, 1)$. This process yields a bijection from $\mathfrak{T}$ to the set of open s.d.p.'s of the unit interval.

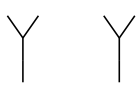
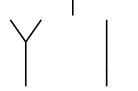
Example 3.3.5. The tree

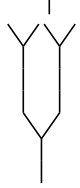
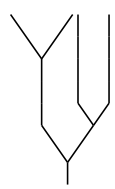
$t =$ 

has the associated open s.d.p. $\{(0, 1/4), (1/4, 1/2), (1/2, 1)\}$. In general, we will denote to the open s.d.p. of a tree t by $\mathcal{I}(t)$.

It will be useful to consider the set of trees $\mathfrak{T}$ as a partially ordered set under the following relation: $s \leq t$ if and only if there exists a forest f such that $t = fs$. This partial order corresponds to the binary relation between pairs of partitions where one is finer than the other. Further, this relation makes $\mathfrak{T}$ a directed set, since any two trees $s, t \in \mathfrak{T}$ are both smaller than the complete tree t_m, for m sufficiently large.

Example 3.3.6. If $s =$  and $t =$  , then we can show $s, t \leq t_2$.

Let $f_1 :=$  and let $f_2 :=$  .

Then, $f_1 \circ s =$  $\cong$  $= t_2$ and $f_2 \circ t =$  $\cong$  $= t_2$.

3.4 Thompson's groups and Jones' actions

Thompson's groups $F \subseteq T \subseteq V$ are three groups which were defined by Richard

Thompson in a collection of unpublished notes from 1965. The groups T and V were the first examples of finitely-presented groups that were also infinite and simple. These groups are interesting in their own right and open questions about them remain (including whether or not F is amenable) [8].

Here, we are not particularly interested in the details about the groups. However, we provide a definition of F and briefly explain the machinery due to Vaughan F.R. Jones which allows us to produce actions of Thompson's groups on the C*-algebras we construct in the sequel. All the details presented here are included in Sections 1.3 and 1.4 of [6]. The canonical reference for Thompson's groups is [8]; for further details on the method of constructing actions, we refer the reader to [22].

Definition 3.4.1. Thompson's group F is the set of piecewise linear homeomorphisms from the closed unit interval $[0, 1]$ to itself such that

(i) they are differentiable except at finitely many dyadic rationals, and

(ii) their derivatives are powers of 2 (on the intervals where they are differentiable).

It turns out that F can actually be described as the *group of fractions* associated to the category of binary forests $\mathcal{F}$. It is constructed in the following way.

Let $\{(t, s)\}$ be the set of pairs of trees such that both t and s have the same number of leaves. We define a relation between two pairs of trees and write $(p, q) \simeq (t, s)$ if $p = ft$ and $q = fs$ where n is the number of leaves of s (and hence t) and n is also the number of roots of the forest f. We quotient $\{(t, s)\}$ by the equivalence relation generated by $\simeq$ and denote the equivalence class of a pair (t, s) by $\frac{t}{s}$.

Definition 3.4.2. The *group of fractions* $G_{\mathcal{F}}$ associated to the category $\mathcal{F}$ (with fixed object $1 \in \mathbb{N}$) is the set of equivalence classes with multiplication

$$\frac{t}{s}\frac{v}{u} = \frac{pt}{qu},$$

where $ps = qv$, identity $e = \frac{s}{s}$, and inverse $\frac{t}{s}^{-1} = \frac{s}{t}$.

It turns out that $G_{\mathcal{F}}$ is isomorphic to Thompson's group F. Moreover, we can describe Thompson's group V as the group of fractions of the category of *symmetric forests* $\mathcal{SF}$ and Thompson's group T is isomorphic to the group of fractions of the category of *affine forests* $\mathcal{AF}$ (for the relevant definitions and results, we refer the reader back to the materials cited above).

We can generalize the notion of groups of fractions for a category with a distinguished object, provided the category satisfies certain axioms. Suppose $(\mathcal{C}, x)$ is a category and distinguished object for which we can construct a group of fractions $G_{\mathcal{C}}$. Jones showed that, if we have a functor $\Psi : \mathcal{C} \to \mathcal{D}$, we can construct an action of $G_{\mathcal{C}}$ which depends on Ψ. We call such an action a *Jones action*.

We do not present the details here, as we do not focus on the Jones actions in our presentation of the material from [6]. From time to time, we will mention that there

are suitable Jones actions for our constructions, but we will refer the reader to the original paper for the details.

3.5 The category of unital C*-algebras

We would like to exploit the tidiness of the category theoretic approach when it comes to taking limits of C*-algebras. In particular, we will work with the category of unital C*-algebras with injective unital $*$-homomorphisms, which we denote C*-alg.

Let $\Psi : \mathcal{F} \to$ C*-alg be a functor, let $(B_t, t \in \mathfrak{T})$ denote the associated directed system with connecting maps $\iota_t^{ft} : B_t \to B_{ft}$, and let B_Ψ denote the system's algebraic inductive limit. Since each ι_t^{ft} is an injective $*$-homomorphism between C*-algebras, it must necessarily be an isometry. Thus, there exists a C*-norm $\|\cdot\|$ on the inductive limit $*$-algebra B_Ψ. By taking the completion of B_Ψ with respect to this norm, we get a C*-algebra $\mathscr{B}_0$. This completion $\mathscr{B}_0$ is the unique C*-limit of the directed system $(B_t, t \in \mathfrak{T})$, so we are justified in also choosing to denote it by $\varinjlim_{t\in\mathfrak{T}} B_t$ and referring to it as the direct limit of C*-algebras.

Moreover, it is not hard to see that (C*-alg, $\otimes_{\min}, \mathbb{C}$) describes a tensor category. Thus, as explained in Section 3.2, we can define a tensor functor $\Psi : \mathcal{F} \to$ (C*-alg, $\otimes_{\min}$) via a C*-algebra B and an injective unital $*$-homomorphism $R : B \to B \otimes_{\min} B$.

In the sequel, it will be useful to consider the inductive limit of C*-algebras $\otimes_{d\in E}^{\min} B$, where E runs over the finite subsets of the dyadic rationals $\mathbb{D}$. We will denote the unique C*-completion of this limit by $\otimes_{d\in\mathbb{D}}^{\min} B$.

With all this in mind, we will prove the following proposition, which is at the heart of the construction(s) used throughout this thesis.

Proposition 3.5.1. *[6, Proposition 1.3] Consider a unital C*-algebra B and the map $R : B \to B \otimes_{\min} B$, $b \mapsto b \otimes 1$. Let $\Psi : \mathcal{F} \to$ (C*-alg, $\otimes_{\min}$) be the associated tensor functor. Then there exists an isomorphism of C*-algebras $J : \mathscr{B}_0 \to \otimes_{d\in\mathbb{D}}^{\min} B$. Moreover, there is a Jones action $\alpha_\Psi : V \to Aut(\mathscr{B}_0)$.*

Proof. Consider a tree t with n leaves and the associated s.d.p. $0 = d_1 < d_2 < \ldots < d_n < d_{n+1} = 1$. Let $\mathbb{D}(t) := \{d_1, \ldots, d_n\}$ and let $J_t : B_t \to \otimes_{d\in\mathbb{D}}^{\min} B$ be the unital embedding induced by the inclusion $\mathbb{D}(t) \subseteq \mathbb{D}$. As above, let $f := f_{j,n}$ be the forest with n roots, $n+1$ leaves, and jth tree equal to Y.

For an elementary tensor $x = x_1 \otimes \ldots \otimes x_n$, we have that

$$\iota_t^{ft}(x) = \iota_t^{ft}(x_1 \otimes \ldots \otimes x_n) = x_1 \otimes \ldots \otimes x_j \otimes 1 \otimes x_{j+1} \otimes x_n.$$

Therefore, for any elementary tensor x we have that $J_{ft} \circ \iota_t^{ft}(x) = J_t(x)$. Extending by linearity and density, we get that $J_{ft} \circ \iota_t^{ft} = J_t$.

By Proposition 3.2.9, we know that any forest is a finite composition of forests $f_{j,n}$. As such, the family of maps $(J_t, t \in \mathfrak{T})$ is compatible with respect to the directed system $(B_t, t \in \mathfrak{T})$ and so it induces a densely defined map $J : \varinjlim_{\text{alg}} B_t \to \otimes_{d\in\mathbb{D}}^{\min} B$.

This map is isometric because each J_t is an injective $*$-homomorphism (and so each J_t is isometric). Thus J extends to the C*-algebra $\mathscr{B}_0$ as an injective $*$-homomorphism.

For any dyadic rational $d \in \mathbb{D}$, we can find a tree containing d in its associated s.d.p. Therefore, any elementary tensor appears in the range of J, meaning that the range of J is dense and hence surjective (as it is a morphism between C*-algebras). This shows that $\mathscr{B}_0 \cong \otimes_{d\in\mathbb{D}}^{\min} B$, as required.

The details for the Jones action are available in the cited proposition. □

3.5.1 States and von Neumann algebras

As above, assume that we have a functor $\Psi : \mathcal{F} \to$ C*-alg inducing a directed system of C*-algebras $(B_t, t \in \mathfrak{T})$ with inclusion maps $\iota_t^{ft} : B_t \to B_{ft}$ and the C*-direct limit $\mathscr{B}_0$. In order to (ultimately) prove the main theorem, we will need a state on $\mathscr{B}_0$ so that we can work with the von Neumann algebra induced from it (via the GNS construction). With this in mind, let us also assume that we have a family of states $(\omega_t : B_t \to \mathbb{C}, t \in \mathfrak{T})$ such that ι_t^{ft} is state-preserving for any tree t and any forest f.

Proposition 3.5.2. *In the setting described above, there exists a unique state $\overline{\omega}$ on $\mathscr{B}_0$ such that $\overline{\omega}(b) = \omega_t(b)$ for any $b \in B_t$.*

Proof. We begin by defining a linear functional $\overline{\omega}$ on $\mathscr{B}_0$. By construction, $\cup_{t\in\mathfrak{T}} B_t$ is a dense $*$-subalgebra of $\mathscr{B}_0$. Thus, we can set $\overline{\omega}(b) = \omega_t(b)$ for each $b \in B_t$ and extend by linearity. By uniform boundedness of the family of ω_t's, the linear functional $\overline{\omega}$ is bounded. To show that $\overline{\omega}$ is a state, it remains to show that it is positive and that $\|\overline{\omega}\| = 1$.

Let e be the unit in $\mathscr{B}_0$. By the definition of $\overline{\omega}$, we have that $\|\overline{\omega}\| = \overline{\omega}(e) = 1$. By Proposition 2.1.13 and Corollary 2.1.14, we get that $\overline{\omega}$ is a state.

The uniqueness of $\overline{\omega}$ follows from the fact that it is required to agree with the family of ω_t's on a dense $*$-subalgebra. □

Given the pair $(\mathscr{B}_0, \overline{\omega})$, we perform the GNS construction (Theorem 2.1.16 above) in order to obtain a triple (π, H, Ω), where (π, H) is a $*$-representation of $\mathscr{B}_0$ and Ω is a cyclic vector such that $\langle \pi(b)\Omega|\Omega\rangle = \overline{\omega}(b)$ for any $b \in \mathscr{B}_0$. We let $\mathscr{B}$ denote the von Neumann algebra given by the WOT-completion $\pi(\mathscr{B}_0)''$ and, as it is convenient, we abuse notation by using $\overline{\omega}$ to also denote the unique normal extension of the limit state to $\mathscr{B}$ (via the cyclic vector obtained from the GNS construction).

In order to prove a statement about the family of states above, we will need to make use of the following lemma.

Lemma 3.5.3. *[4, Proposition II.8.2.4] Let $\mathfrak{A}$ be a C*-algebra and let $\{A_t\}$ be a collection of C*-subalgebras such that $A_0 = \cup A_t$ is a dense $*$-subalgebra of $\mathfrak{A}$. If J is any closed ideal of $\mathfrak{A}$ then $J \cap A_0 = \cup_t (J \cap A_t)$ is dense in J.*

Proposition 3.5.4. *If each state ω_t in the construction above is faithful, then the GNS representation π of $\mathscr{B}_0$ is also faithful.*

Proof. At each level, we can perform the GNS construction on (B_t, ω_t). Each representation π_t of B_t obtained in this way is faithful and hence isometric.

The GNS representation π of $\mathscr{B}_0$ contains π_t (in the sense that $\pi = \pi_t$ when the former is restricted to B_t), so π is isometric when restricted to B_t. By density, we get that π is isometric on the whole algebra $\mathscr{B}_0$.

To show faithfulness, we use Lemma 3.5.3. We have a C*-algebra $\mathscr{B}_0$, a family $(B_t, t \in \mathfrak{T})$ of C*-subalgebras of $\mathscr{B}_0$, and we denote a dense $*$-algebra of $\mathscr{B}_0$ by $B_0 := \cup_t B_t$. For the closed ideal of $\mathscr{B}_0$, we take $J := \ker(\pi)$.

Fix $t \in \mathfrak{T}$, then pick some $b_t \in J \cap B_t$. Since $b_t \in J$, we get $\pi(b_t) = 0$ which implies $\pi(b_t^* b_t) = 0$.

By construction, we have $0 = \overline{\omega}(b_t^* b_t) = \langle \pi(b_t^* b_t)\Omega | \Omega \rangle = \langle \pi_t(b_t^* b_t)\Omega_t | \Omega_t \rangle = \omega_t(b_t^* b_t)$. By assumption, each ω_t is faithful, so we have $b_t^* b_t = 0$, which implies $b_t = 0$. In other words, the representation π is faithful. □

Remark 3.5.5. The preceding proposition shows that given a family of faithful states ω_t in our construction, we have that π is a faithful representation of the C*-algebra $\mathscr{B}_0$. However, in general, we cannot guarantee that $\overline{\omega}$ is a faithful state on the von Neumann algebra $\mathscr{B}$.

Chapter 4

Construction of a C*-algebra

Our first goal is to study and describe the limit of a net of C*-algebras indexed by binary forests defined in [6].

4.1 Directed system of C*-algebras

Let G be any separable compact group with Haar measure m_G. Let $\mathcal{B}(L^2(G))$ be the von Neumann algebra of all bounded linear operators on the Hilbert space $L^2(G)$ of the complex-valued square integrable operators on G with respect to m_G. We define a C*-subalgebra $M \subseteq \mathcal{B}(L^2(G))$ which we will use to build the directed system of interest.

As in Section 2.3, we consider the norm-completion of the complex group algebra $\mathbb{C}G$ inside $\mathcal{B}(L^2(G))$. Using notation consistent with [6] and above, we take $N := \overline{\mathbb{C}[G]}$. We also define another C*-algebra, namely $Q := \mathscr{C}(G)$, the commutative C*-algebra of continuous functions on G.

Proposition 4.1.1. *$Q := \mathscr{C}(G)$ is (isometrically isomorphic to) a C*-subalgebra of $\mathcal{B}(L^2(G))$.*

Proof. Define a map $\mathcal{M} : \mathscr{C}(G) \to \mathcal{B}(L^2(G))$ by $\mathcal{M}(f) = M_f$ where M_f is the left multiplication operator $M_f(\xi(g)) = f(g)\xi(g)$ for all $\xi \in L^2(G)$, $g \in G$. It is clear that $\mathcal{M}$ is a unital $*$-homomorphism. Moreover, as G is compact, the constant function 1 is an element of $L^2(G)$. Therefore, we have $\mathcal{M}(f)(1) = M_f(1) = f \in \mathcal{B}(L^2(G))$ and so $\mathcal{M}$ is injective and thus isometric. $\square$

Brothier and Stottmeister use these two C*-algebras to build M. Following their lead, we let $M := C^*(N, Q)$, i.e., we define M to be the C*-subalgebra of $\mathcal{B}(L^2(G))$ generated by $\overline{\mathbb{C}[G]}$ and $\mathscr{C}(G)$.

Proposition 4.1.2. *M is the norm completion inside $\mathcal{B}(L^2(G))$ of the algebraic*

crossed-product, defined by

$$\mathscr{C}(G) \rtimes_{alg} G := \left\{ \sum_{g \in G} a_g \lambda_g : a_g \in \mathscr{C}(G),\ supp(g \mapsto a_g) \text{ finite} \right\}, \tag{4.1}$$

where supp($g \mapsto a_g$) being finite means that $a_g = 0$ for all but finitely many $g \in G$.

Proof. We would like to show that $\overline{\mathscr{C}(G) \rtimes_{\text{alg}} G} = C^*(\overline{\mathbb{C}[G]}, \mathscr{C}(G))$. For any $a_g \in \mathscr{C}(G)$ (with finite support), it is clear that $a_g \lambda_g \in C^*(\overline{\mathbb{C}[G]}, \mathscr{C}(G))$. Taking sums of elements of this form, we see that

$$\begin{aligned} &\left\{ \sum_{g \in G} a_g \lambda_g : a_g \in \mathscr{C}(G),\ \text{supp}(g \mapsto a_g) \text{ finite} \right\} \subseteq C^*(\overline{\mathbb{C}[G]}, \mathscr{C}(G)) \\ \implies &\mathscr{C}(G) \rtimes_{\text{alg}} G \subseteq C^*(\overline{\mathbb{C}[G]}, \mathscr{C}(G)). \end{aligned}$$

Now we want to show that $\mathscr{C}(G) \rtimes_{\text{alg}} G$ is a $*$-subalgebra of $C^*(\overline{\mathbb{C}[G]}, \mathscr{C}(G))$. First, we want to show that it is closed under multiplication. Let a_g, b_h denote finitely supported elements of $\mathscr{C}(G)$ and let λ be the left regular representation. We have

$$\begin{aligned} \left(\sum_{g \in G} a_g \lambda_g \right) \left(\sum_{h \in G} b_h \lambda_h \right) &= \sum_{g,h \in G} a_g \lambda_g b_h \lambda_h \\ &= \sum_{g,h} a_g \lambda_g b_h \lambda_g^{-1} \lambda_g \lambda_h \\ &= \sum_{g,h} a_g (\lambda_g b_h \lambda_g^{-1}) \lambda_{gh}. \end{aligned} \tag{4.2}$$

By the previous proposition, we can view functions as multiplication operators. In particular, for any $f \in \mathscr{C}(G)$ we have $\lambda_g f \lambda_g^{-1} \mapsto \lambda_g (M_f) \lambda_g^{-1}$. Thus, for $\xi \in L^2(G)$, $k \in G$,

$$\begin{aligned} (\lambda_g (M_f) \lambda_g^{-1})(\xi)(k) = \lambda_g((M_f)\lambda_g^{-1})\xi)(k) &= M_f(\lambda_g^{-1}\xi)(g^{-1}k) \\ &= f(g^{-1}k)\lambda_g^{-1}\xi(g^{-1}k) \\ &= f(g^{-1}k)\xi(k) \\ &= (\lambda_g f)(k)\xi(k) \\ &= M_{\lambda_g f}\xi(k) \\ \implies \lambda_g M_f \lambda_g^{-1} &= M_{\lambda_g f}, \end{aligned}$$

where the latter element can be mapped back to $\lambda_g f \in \mathscr{C}(G)$. In particular, then

$\lambda_g b_h \lambda_g^{-1} \in \mathscr{C}(G)$ and so returning to equation (4.2) we see that

$$\left(\sum_{g\in G} a_g\lambda_g\right)\left(\sum_{h\in G} b_h\lambda_h\right) = \sum_{g,h} a_g(\lambda_g b_h \lambda_g^{-1})\lambda_{gh} \in \mathscr{C}(G) \rtimes_{\text{alg}} G.$$

Moreover, $\mathscr{C}(G) \rtimes_{\text{alg}} G$ is closed under the $*$-operation. We have

$$\begin{aligned}\left(\sum_{g\in G} a_g\lambda_g\right)^* &= \sum_g \lambda_g^{-1} a_g^* \\ &= \sum_g (\lambda_g^{-1} a_g^* \lambda_g)\lambda_g^{-1} \in \mathscr{C}(G) \rtimes_{\text{alg}} G.\end{aligned}$$

Thus $\mathscr{C}(G) \rtimes_{\text{alg}} G$ is a $*$-subalgebra of M (and so its norm-closure is a C*-subalgebra of M). It remains to show that $\overline{\mathscr{C}(G) \rtimes_{\text{alg}} G}$ contains both $\overline{\mathbb{C}[G]}$ and $\mathscr{C}(G)$. First, let us fix an arbitrary $g \in G$. Define a function $a : G \to \mathscr{C}(G)$ by $a(h) = a_h = \delta_{h,g}1$ where $1 \in \mathscr{C}(G)$ is the constant function equal to 1. Then $\sum_{h\in G} a_h\lambda_h = \lambda_g \in \mathscr{C}(G) \rtimes_{\text{alg}} G$. We can define functions in this way for every $g \in G$ and so

$$\begin{aligned}\text{span}\{\lambda_g\} &\subseteq \mathscr{C}(G) \rtimes_{\text{alg}} G \\ \implies \overline{\mathbb{C}[G]} = \overline{\text{span}\{\lambda_g\}} &\subseteq \overline{\mathscr{C}(G) \rtimes_{\text{alg}} G}.\end{aligned}$$

Finally, for any $f \in \mathscr{C}(G)$, we can define $a : G \to \mathscr{C}(G)$ by $a(g) = a_g = \delta_{g,e}f$. Then $\sum_{h\in G} a_g\lambda_g = f\lambda_e = f \in \mathscr{C}(G) \rtimes_{\text{alg}} G$. Since f was arbitrary, this means we have $\mathscr{C}(G) \subseteq \mathscr{C}(G) \rtimes_{\text{alg}} G$. Putting everything together, we get that

$$M = C^*(\overline{\mathbb{C}[G]}, \mathscr{C}(G)) = \overline{\mathscr{C}(G) \rtimes_{\text{alg}} G},$$

as desired. □

It is worth noting M is isomorphic to the reduced crossed-product exactly when G is discrete (and hence finite, as G is also compact). Later on, we will restrict to this case; however, we chose to keep the results general for as long as possible.

In view of our goals and Proposition 3.5.1 above, we define the morphism

$$R : M \to M \otimes M, R(b) = \text{Ad}(u)(b \otimes 1), \tag{4.3}$$

where $u\xi(g,h) = \xi(gh,h)$ for all $\xi \in L^2(G \times G)$. This defines a tensor functor

$$\Psi : \mathcal{F} \to (\text{C*-alg}, \otimes_{\min}), \Psi(1) := M, \Psi(Y) = R.$$

By the discussion in the preceding section, this will provide us with the direct limit of C*-algebras $\mathscr{M}_0 := \varinjlim_{t\in\mathfrak{T}} M_t$ (as well as a Jones action $\alpha_\Psi : V \curvearrowright \mathscr{M}_0$).

Lemma 4.1.3. *[6, Lemma 2.1] We have the following equalities:*

$$R\left(\sum_{g\in G} a_g\lambda_g\right) = \sum_{g\in G}(a_g\circ\mu_G)\lambda_{g,e}, \tag{4.4}$$

where $\mu_G : G\times G\to G$ is the group multiplication, $g\mapsto a_g$ has finite support with values in Q, and $\lambda_{g,e} := \lambda_g\otimes\lambda_e = \lambda_g\otimes id$. In particular, we have that $R(a) = a\circ\mu_G$ for any $a\in Q$ and $R(b) = b\otimes 1$ for any $b\in N$.

For completeness, we include the proof of this lemma, which was left as an exercise in [6].

Proof. Let $b := \lambda_g$ for some $g\in G$. By definition of R, $R(\lambda_g) = u(\lambda_g\otimes\mathrm{id})u^*$. We want to show that $u(\lambda_g\otimes\mathrm{id})u^* = (\lambda_g\otimes\mathrm{id})$. Recalling that $u\xi(g,h) = \xi(gh,h)$ and $u^*\xi(g,h) = \xi(gh^{-1},h)$, we have:

$$\begin{aligned} u(\lambda_g\otimes\mathrm{id})u^*\xi(s,t) = (\lambda_g\otimes\mathrm{id})u^*(\xi(st,t)) &= u^*(\xi)(g^{-1}st,t)\\ &= \xi(g^{-1}s,t)\\ &= (\lambda_g\otimes\mathrm{id})\xi(s,t). \end{aligned}$$

Thus, $R(\lambda_g) = \lambda_g\otimes\mathrm{id}$ for all $g\in G$. So by linearity and density, $R(b) = b\otimes 1$ for $b\in\overline{\mathrm{span}\{\lambda_g\}} = N$, hence $R(N)\subseteq N\otimes N$.

Now we want to understand what happens to the other "part" of M under R. For each $f\in\mathscr{C}(G)$, we can consider the associated multiplication operator M_f, allowing us to view $\mathscr{C}(G)\subseteq\mathcal{B}(L^2(G))$. Following the definition, we get:

$$\begin{aligned} R(M_f)\xi(s,t) = u(M_f\otimes\mathrm{id})u^*\xi(s,t) &= (M_f\otimes\mathrm{id})u^*\xi(st,t)\\ &= f(st)u^*\xi(st,t)\\ &= f(st)\xi(s,t)\\ &= f(\mu_G(s,t))\xi(s,t). \end{aligned}$$

Thus, $R(M_f) = M_{f\circ\mu_G}$, where $M_{f\circ\mu_G}\in\mathscr{C}(G\times G) = \mathscr{C}(G)\otimes_{\min}\mathscr{C}(G)$ (see [4, Theorem II.9.4.4]). That is, up to identification of $f\in\mathscr{C}(G) = Q$ with the appropriate multiplication operator, we get $R(f) = f\circ\mu_G$, hence $R(Q)\subseteq Q\otimes Q$.

Finally, in general, we get

$$\begin{aligned} R\left(\sum_g a_g\lambda_g\right) = \sum_g R(a_g)R(\lambda_g) &= \sum_g (a_g\circ\mu_G)(\lambda_g\otimes\mathrm{id})\\ &= \sum_g (a_g\circ\mu_G)\lambda_{g,e}, \end{aligned}$$

as desired. □

In conclusion, we can think of R as acting "canonically" on the two pieces $N = \overline{\mathbb{C}[G]}$ and $Q = \mathscr{C}(G)$. Hence, it acts canonically on $C^*(N, Q)$ and $W^*(N, Q)$, that is, it acts canonically on both the C*-algebra and the von Neumann algebra generated by N and Q.[1] We also note that $M \otimes M$ embeds WOT-densely inside $\mathcal{B}(L^2(G \times G))$.

Remark 4.1.4. The lemma above can be generalized slightly.

(i) If $a \in L^\infty(G)$, the von Neumann algebra of bounded measurable maps on G, then the identity $R(a) = a \circ \mu_G$ is still valid.

(ii) If $b \in LG$, the von Neumann algebra generated by the left regular representation of G, then $R(b) = b \otimes 1$ is also still valid.

Fact 4.1.5. For a separable, compact group G, any $n \in \mathbb{N}$, and the minimal tensor product, we have that $\otimes_{k=1}^n \mathscr{C}(G) \cong \mathscr{C}(G^n)$ and $\otimes_{k=1}^n \overline{\mathbb{C}[G]} \cong \overline{\mathbb{C}[G^n]}$.

By the previous lemma, we can use the map R to define two tensor functors $\Psi_Q, \Psi_N : \mathcal{F} \to (\text{C*-alg}, \otimes_{\min})$. In particular, we have

$$\Psi_Q(1) = Q, \Psi_Q(n) = \otimes_{k=1}^n Q = \otimes_{k=1}^n \mathscr{C}(G), \text{ and } \Psi_Q(Y) = R|_Q$$

and

$$\Psi_N(1) = N, \Psi_N(n) = \otimes_{k=1}^n N = \otimes_{k=1}^n \overline{\mathbb{C}[G]}, \text{ and } \Psi_N(Y) = R|_N.$$

Notation 4.1.6. Let t be a tree with n leaves. We will let G_t denote a copy of $G^n = G \times \cdots \times G$ indexed by the leaves of t. Moreover, we write:

$$\begin{aligned} Q_t &= \mathscr{C}(G_t), \\ N_t &= \overline{\mathbb{C}[G_t]}, \text{ and} \\ M_t &= \mathscr{C}(G_t) \rtimes G_{t,d}, \end{aligned}$$

where the subscript d refers to the fact that we think of G_t as being endowed with the discrete topology.

Using the notation and the two tensor functors described above, we obtain three directed systems of C*-algebras $(Q_t, t \in \mathfrak{T})$, $(N_t, t \in \mathfrak{T})$ and $(M_t, t \in \mathfrak{T})$. We will study both the C*-subalgebras of $\mathscr{M}_0 := \varinjlim_{t \in \mathfrak{T}} M_t$ given by the limits $\mathscr{Q}_0 = \varinjlim_{t \in \mathfrak{T}} Q_t$ and $\mathscr{N}_0 = \varinjlim_{t \in \mathfrak{T}} N_t$.

4.1.1 The commutative part $\mathscr{Q}_0$

At the group level, there are (at least) two different ways that we can describe the limit algebra Q_0.

[1] For any compact group G, we know that the algebra of compact operators $\mathcal{K}(L^2(G))$ is isomorphic to $\mathscr{C}(G) \rtimes_\lambda G$ and further, that $\overline{\mathcal{K}(L^2(G))}^{WOT} = \mathcal{B}(L^2(G))$, [4, Theorem II.10.4.3]. Therefore, $W^*(N, Q) = \mathcal{B}(L^2(G))$ and so the claim follows.

First, let us consider the contravariant tensor functor $\Xi : \mathcal{F} \to \text{Set}$ given by $\Xi(1) = G$ and $\Xi(Y) = \mu_G$, where μ_G is the group multiplication. Here, Set denotes the tensor category of sets equipped with the Cartesian product. This functor gives rise to an inverse system of groups $(G_t, t \in \mathfrak{T})$ with the maps $p_s^t : G_t \to G_s$ if $s \leq t$ (i.e., if $t = fs$ for some forest f) defined by the group multiplication. Hence, the inverse limit is

$$\varprojlim_{t \in \mathfrak{T}} G_t := \left\{ x = (x_t)_{t \in \mathfrak{T}} \in \prod_{t \in \mathfrak{T}} G_t : \Xi(f)(x_{ft}) = x_t, t \in \mathfrak{T}, f \in \mathcal{F} \right\}.$$

Whereas the limit above is indexed by the (number of) leaves of a tree, we can instead index by the s.d.i.'s associated to a given tree.

Notation 4.1.7. Recall that for a tree $t \in \mathfrak{T}$, we denote its set of s.d.i.'s by $\mathcal{I}(t)$. We will denote the set of all s.d.i.'s by $\mathcal{I}$. For an s.d.i. $I \in \mathcal{I}$, let I_1 and I_2 denote its first and second half, respectively.

Now, let us consider the product of copies of G indexed by s.d.i.'s. In other words, we consider the product space $\prod_{I \in \mathcal{I}} G$. Define a subset of this space by

$$\overline{\mathscr{A}} := \left\{ x \in \prod_{I \in \mathcal{I}} G : x(I) = x(I_1)x(I_2) \right\}.$$

Here, it is important to notice that there is a bijection between $\overline{\mathscr{A}}$ and $\lim_{\overleftarrow{t \in \mathfrak{T}}} G_t$. In particular, we can define a function $j : \overline{\mathscr{A}} \to \varprojlim_{t \in \mathfrak{T}} G_t$ by $(j(x))_t := x|_{\mathcal{I}(t)}$. Its inverse is given by $j^{-1}(y)(I) = y_t(I)$ if $I \in \mathcal{I}(t)$.

Notation 4.1.8. Let $p_s : \varprojlim_{t \in \mathfrak{T}} G_t \to G_s$ denote the projection built from all p_s^r with $s \leq r$.

We equip the compact topological space G_t with its Borel σ-algebra and its Haar measure m_{G_t}. Since group multiplication $\mu_G : G \times G \to G$ is continuous and probability measure preserving (p.m.p.), each $\Xi(f)$ is also continuous and p.m.p. Thus, $\lim_{\overleftarrow{t \in \mathfrak{T}}} G_t$ is a compact topological space.

In order to continue with our construction, we will need to make use of the following standard theorem.

Theorem 4.1.9 (Riesz-Markov-Kakutani[2]). *[36, Theorem 2.14] Let X be a locally compact Hausdorff space. For any positive linear functional ψ on $\mathscr{C}_c(X)$, there exists a unique Radon measure μ on X such that for all $f \in \mathscr{C}_c(X)$*

$$\psi(f) = \int_X f(x)d\mu(x).$$

Proposition 4.1.10. *There is a unique probability measure $\overline{m}$ on $\varprojlim_{t\in\mathfrak{T}} G_t$ such that each projection $p_s : (\varprojlim_{t\in\mathfrak{T}} G, \overline{m}) \to (G_s, m_{G_s})$ is p.m.p.*

Proof. We have a compact topological space $\overline{\mathscr{A}} = \varprojlim_{t\in\mathfrak{T}} G_t$ and the set of continuous functions on $\overline{\mathscr{A}}$ denoted $\mathscr{C}(\overline{\mathscr{A}}) = \varinjlim_{t\in\mathfrak{T}} \mathscr{C}(G_t)$. Each Haar measure m_{G_t} induces the Haar integral $\phi_t : \mathscr{C}(G_t) \to \mathbb{C}$ given by

$$\phi_t(f) = \int_{G_t} f(x)dm_{G_t}(x).$$

Note that the family $(\phi_t)_{t\in\mathfrak{T}}$ is compatible with the direct limit structure. By an argument similar to the one used in Proposition 3.5.2, this family induces a unique linear functional $\phi \in \overline{\mathscr{A}}$. The map $\phi : \overline{\mathscr{A}} \to \mathbb{C}$ satisfies the hypotheses of the Riesz-Markov-Kakutani Theorem and so there exists a unique probability measure $\overline{m}$ on $\varprojlim_{t\in\mathfrak{T}} G_t$. Further, each projection $p_s : (\varprojlim_{t\in\mathfrak{T}} G, \overline{m}) \to (G_s, m_{G_s})$ is p.m.p. because each map $p_s^r, s \leq r$ is p.m.p. □

Using the map j defined above, we endow $\overline{\mathscr{A}}$ with the restriction of the product topology of $\prod_{I\in\mathcal{I}} G$ to $\overline{\mathscr{A}}$ and a probability measure $m_{\overline{\mathscr{A}}}$.

Proposition 4.1.11. *[6, Proposition 2.4] There is an isomorphism of C*-algebras between the direct limit $\mathscr{Q}_0$ and $\mathscr{C}(\overline{\mathscr{A}})$.*

Proof. Recall that $\Xi : \mathcal{F} \to \text{Set}$ is the contravariant tensor functor given by $\Xi(1) = G$ and $\Xi(Y) = \mu_G$, where μ_G is the group multiplication. Let $\mathscr{C} : X \to \mathscr{C}(X)$ denote the contravariant functor which sends a compact topological space X to the C*-algebra of continuous functions on X. It is easy to see that the functor $\Psi_Q : \mathcal{F} \to$ C*-alg is obtained by the composition $\mathscr{C} \circ \Xi$. For each $t \in \mathfrak{T}$, define a map $J_t : \mathscr{C}(G_t) \to \mathscr{C}(\overline{\mathscr{A}})$ by $J_t(f) = f \circ p_t$. By an argument similar to the one in Proposition 3.5.1, we can extend this family of maps to an isomorphism $J : \mathscr{Q}_0 \to \mathscr{C}(\overline{\mathscr{A}})$. □

As it is convenient, we will abuse the notation slightly by directly identifying $J(\mathscr{C}(G_t))$ with $\mathscr{C}(G_t)$.

[2]Here we follow the lead of Brothier and Stottmeister in referring to this theorem by all three author names, rather than calling it the (generalized) Riesz Representation theorem. Riesz (1909) originally introduced this result for continuous functions on the unit interval [35], Markov (1938) extended the result to certain non-compact spaces [29], and Kakutani (1941) extended the result to compact Hausdorff spaces [26].

Remark 4.1.12. It is worth noting that, although we think of $\overline{\mathscr{A}}$ as being the limit of groups, it does not have a natural group structure unless the group G is abelian. This is because group multiplication μ_G is a morphism in the category of groups if and only if G is abelian.

4.1.2 The convolution part $\mathscr{N}_0$

Consider the group $(\prod_{\mathbb{D}} G, \cdot)$ where $\prod_{\mathbb{D}} G$ is the set of all maps from the dyadic rationals $\mathbb{D}$ to our group G, and $(\cdot)$ is the pointwise multiplication. As before, let "supp" denote the support of a map; that is, for $f \in \prod_{\mathbb{D}} G$, $\mathrm{supp}(f) := \{d \in \mathbb{D} : f(d) \neq e\}$. We are interested in a particular subgroup, namely

$$\oplus_{\mathbb{D}} G = \left\{ g \in \prod_{\mathbb{D}} G : \mathrm{supp}(g) \text{ finite} \right\},$$

and its group ring $\mathbb{C}[\oplus_{\mathbb{D}} G]$.

For a tree $t \in \mathfrak{T}$ with associated s.d.p $0 = d_1 < d_2 < \ldots < d_n < d_{n+1} = 1$, we have an embedding of groups $\iota_t : G_t \to \oplus_{\mathbb{D}} G$ given by the canonical inclusion $\mathbb{D}(t) := \{d_1, \ldots, d_n\} \subseteq \mathbb{D}$. Thus, we think of G_t as the subgroup of maps supported in $\mathbb{D}(t)$, i.e.,

$$G_t = \{f \in \oplus_{\mathbb{D}} G : \mathrm{supp}(f) = \mathbb{D}(t)\}.$$

Further, let $\otimes_{\mathbb{D}} N$ be the infinite minimal tensor product of the C*-algebra $N = \overline{\mathbb{C}[G]}$ over the set $\mathbb{D}$.

Proposition 4.1.13. *[6, Proposition 2.6] The direct limit $\mathscr{N}_0$ is isomorphic to $\otimes_{\mathbb{D}} N$.*

Proof. By Lemma 4.1.3, we know that for any $a \in N$, $R(a) = a \otimes \mathrm{id}$ and $R(N) \subseteq N \otimes N$. Therefore, by Proposition 3.5.1, we can construct an isomorphism $J : \mathscr{N}_0 \to \otimes_{\mathbb{D}} N$. □

We can think of $\oplus_{\mathbb{D}} G$ as the limit of an inductive system in the tensor category $(\mathrm{Gr}, \times, \{e\})$ described in Section 3.1.1. In this perspective, we can see that there is a well-defined tensor functor $\Upsilon : \mathcal{F} \to \mathrm{Gr}$ such that

$$\Upsilon(1) = G \text{ and } \Upsilon(Y)(g) = (g, e), g \in G.$$

In this way, the usual topology on $\oplus_{\mathbb{D}} G$ is the inductive one corresponding to the restriction of the box topology of $\prod_{\mathbb{D}} G$.

Now, let $\mathbb{C}[\cdot] : H \to \mathbb{C}[H]$ be the functor that associates a group to its group ring $*$-algebra. Then $\Upsilon \circ \mathbb{C}[\cdot]$ defines a functor similar to Ψ_N; in this construction, however, we do not take the closure. Indeed, the group ring $\mathbb{C}[\oplus_{\mathbb{D}} G]$ is a dense $*$-subalgebra of $\otimes_{\mathbb{D}} \overline{\mathbb{C}[G]}$ and $J(\lambda_{G_t}(g)) = u(\iota_t(g))$, where λ_{G_t} is the left regular representation of G_t on $L^2(G_t)$, $g \in G$, $t \in \mathfrak{T}$, and $u : \oplus_{\mathbb{D}} G \to \otimes_{\mathbb{D}} \overline{\mathbb{C}[G]}$ is the canonical embedding.

4.1.3 The inductive limit $\mathscr{M}_0$

As above, we identify the commutative part $\mathscr{Q}_0$ with $\mathscr{C}(\overline{\mathscr{A}})$ and the convolution part $\mathscr{N}_0$ with $\otimes^{\min}_{d\in\mathbb{D}}\overline{\mathbb{C}[G]}$. Now we are ready to study the interplay between these two algebras and the construction of $\mathscr{M}_0$.

The group $\oplus_{\mathbb{D}}G$ admits an action on $\overline{\mathscr{A}}$, which is given by

$$g \cdot x(d, d') = g(d)x(d, d'), \tag{4.5}$$

where (d, d') is an s.d.i, $g \in \oplus_{\mathbb{D}}G$, and $x \in \overline{\mathscr{A}}$. This action does not restrict to an action of $\oplus_{\mathbb{D}}G$ on the spaces (G_t, m_{G_t}), $t \in \mathfrak{T}$. However, we do have the following result.

Lemma 4.1.14. *For any $g \in \oplus_{\mathbb{D}}G$, we can find a tree $t_g \in \mathfrak{T}$ such that the map $g\cdot : (G_s, m_{G_s}) \to (G_s, m_{G_s})$ is well-defined for each $s \geq t_g$. Furthermore, this map is continuous and p.m.p.*

Proof. Fix some g in $\oplus_{\mathbb{D}}G$. Let $\mathcal{P}$ denote the coarsest s.d.p. containing $\mathrm{supp}(g)$ and let t_g be the tree whose leaves can be decorated in the appropriate way (as in Section 3.3) by $\mathcal{P}$. Choose any $s \in \mathfrak{T}$ such that $s \geq t_g$. Then $\mathcal{I}(s)$, the s.d.p. associated to s, is finer than $\mathcal{I}(t_g) = \mathcal{P}$. Furthermore, if $f \in G_s$ then we have

$$\mathrm{supp}(f) = \mathbb{D}(s) \supseteq \mathbb{D}(t_g) \supseteq \mathrm{supp}(g).$$

It follows immediately that the map $g\cdot : f \mapsto g \cdot f$, where $(g \cdot f)(d) = g(d)f(d)$ for any $d \in \mathbb{D}$, is well-defined—the preceding work shows that that $\mathrm{supp}(g \cdot f) \subseteq \mathrm{supp}(f) = \mathcal{I}(s)$. That this map is continuous and p.m.p. follows from the fact that the group multiplication is continuous and p.m.p. □

These maps give us a family of continuous actions on $\overline{\mathscr{A}}$ by the subgroups $\oplus_{\mathbb{D}(t)}G$, where $\mathbb{D}(t) = \{d_1, \ldots, d_n\}$ is the set of boundary points associated to the s.d.p. of the tree t. These actions are compatible with the inverse system (G_t, m_{G_t}), $t \in \mathfrak{T}$ which is given by the contravariant functor Ξ described above. We conclude that $\oplus_{\mathbb{D}}G$ acts by p.m.p. homeomorphism on $(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}})$.

Remark 4.1.15. We have that each element $g \in \oplus_{\mathbb{D}}G$ acts continuously and p.m.p. on $\overline{\mathscr{A}}$; we do not need any statement regarding the continuity of the morphism $\oplus_{\mathbb{D}}G \to \mathrm{Aut}(\overline{\mathscr{A}})$. As such, we have not (yet) defined any topology on $\mathrm{Aut}(\overline{\mathscr{A}})$.

Let $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\mathrm{alg}} \oplus_{\mathbb{D}}G$ denote the $*$-algebra generated by the group ring $\mathbb{C}[\oplus_{\mathbb{D}}G]$ and $\mathscr{C}(\overline{\mathscr{A}})$.

Theorem 4.1.16. *[6, Theorem 2.7] The $*$-algebra $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{alg} \oplus_{\mathbb{D}}G$ embeds as a dense $*$-subalgebra of $\mathscr{M}_0$. Furthermore, there exists a unique faithful $*$-representation*

$$\pi : \mathscr{M}_0 \to \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}))$$

such that

$$\pi(bu_g)\xi(x) = b(x)\xi(g^{-1}x), b \in \mathscr{C}(\overline{\mathscr{A}}), g \in \oplus_{\mathbb{D}}G, \xi \in L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}), x \in \overline{\mathscr{A}}.$$

Proof. Let us denote the embedding of the group $\oplus_{\mathbb{D}}G$ inside the group of unitaries of $\mathscr{N}_0$ by $u : g \mapsto u_g$. Consider a tree t with associated s.d.p $0 = d_1 < d_2 < \cdots < d_n < d_{n+1} = 1$, let $\mathbb{D}(t) = d_1, \ldots, d_n$, and let $\mathcal{I}(t)$ be the set of s.d.i. (d_j, d_{j+1}) with $1 \leq j \leq n$. Recall that $M_t := \mathscr{C}(G_t) \rtimes G_{t,d}$ and that G_t can be identified with either $\oplus_{\mathcal{I}(t)}G$ and/or $\oplus_{\mathbb{D}(t)}G$. Thus, we have $M_t \cong \mathscr{C}(\oplus_{\mathcal{I}(t)}G) \rtimes (\oplus_{\mathbb{D}(t)}G)_d$, where we think of $\oplus_{\mathbb{D}(t)}G$ being endowed with the discrete topology. Moreover, observe that $\oplus_{\mathbb{D}(t)}G$ acts on $\oplus_{\mathcal{I}(t)}G$ via the formula (4.5). Therefore, we can see $\mathscr{C}(\oplus_{\mathcal{I}(t)}G) \rtimes_{\text{alg}} \oplus_{\mathbb{D}(t)}G$ (which is a dense $*$-subalgebra of M_t) as a $*$-subalgebra of $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\text{alg}} \oplus_{\mathbb{D}}G$. By the preceding work, we know that the embeddings inside $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\text{alg}} \oplus_{\mathbb{D}}G$ respect both directed systems and so it embeds inside $\mathscr{M}_0$. Moreover, we have

$$\bigcup_{t \in \mathfrak{T}} \left(\mathscr{C}(\oplus_{\mathcal{I}(t)}G) \rtimes_{\text{alg}} \oplus_{\mathbb{D}(t)}G\right) \subseteq \mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\text{alg}} \oplus_{\mathbb{D}}G,$$

where the union on the left-hand side is known to be dense inside $\mathscr{M}_0$. Thus, $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\text{alg}} \oplus_{\mathbb{D}}G$ must also be a dense $*$-subalgebra of $\mathscr{M}_0$.

Now we want to show that we have a directed system of faithful representations in our construction. Recall from Example 2.1.16 that, in general, if $\mathfrak{A}$ is a C*-subalgebra of some $\mathcal{B}(H)$ then the inclusion map $\mathfrak{A} \to \mathcal{B}(H)$ is a faithful representation of $\mathfrak{A}$ on H. By virtue of M being a C*-subalgebra of $\mathcal{B}(L^2(G))$, it is clear that each M_t is a C*-subalgebra of $\mathcal{B}(L^2(G_t))$. Thus we have a faithful representation $\pi_t : M_t \to \mathcal{B}(L^2(G_t))$ at every level of the system. Moreover, our directed system of faithfully represented C*-algebras induces a directed system of Hilbert spaces. We have $(L^2(G_t, m_{G_t}), t \in \mathfrak{T})$ with inclusion maps $w_t^{ft} : (L^2(G_t, m_{G_t})) \to (L^2(G_{ft}, m_{G_{ft}}))$ which are given by

$$w_t^{ft}(\xi)(x) := \xi(p_t^{ft}(x)),$$

for $\xi \in L^2(G_t, m_{G_t}), x \in G_{ft}$. These inclusion maps are isometries because the projections p_t^{ft} are p.m.p. So, by construction, the limit of the directed system of Hilbert spaces is $L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}})$.

It remains to show that there is a faithful $*$-representation on the direct limit.

Recall that $\iota_t^{ft} : M_t \to M_{ft}$ is the embedding induced by the functor $\Psi : \mathcal{F} \to$ C*-alg. These embeddings are compatible with our system of Hilbert spaces in the sense that

$$w_t^{ft}(\pi_t(a_t)\xi_t) = \pi_{ft}(\iota_t^{ft}(a_t))w_t^{ft}(\xi_t),$$

for any $a_t \in M_t$, $\xi_t \in L^2(G_t)$. As such we can define a $*$-representation

$$\pi : \varinjlim_{\text{alg}} M_t \to \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}))$$

such that $\pi_t(a_t)\xi_t = \pi(a_t)\xi_t$ for any $a_t \in M_t, \xi_t \in L^2(G_t)$. Since each π_t is injective, π is an isometry which extends to a faithful representation on the C*-algebraic completion $\mathscr{M}_0$ of the algebraic direct limit $\varinjlim_{\text{alg}} M_t$.

Finally, for any $t \in \mathfrak{T}$, $b \in \mathscr{C}(\oplus_{\mathcal{I}(t)}G)$, $g \in \oplus_{\mathbb{D}(t)}G$ and $\xi_t \in L^2(G_t)$, we have that $\pi(bu_g)\xi_t = b\xi_t(g^{-1}\cdot)$. By density, we get the desired formula. □

We have omitted the discussion of the Jones action here—that there is a suitable action of V on the commutative part $\mathscr{Q}_0$ and on the convolution part $\mathscr{N}_0$ is proved in Sections 2.2 and 2.3 of [6], respectively. Moreover, the authors show in [6, Section 2.4] that the action $V \curvearrowright \mathscr{M}_0$ is unitarily implemented.

4.2 Action of the gauge group

As the title suggests, the particular kinds of QFT that Brothier and Stottmeister have constructed [6] are *gauge theories*. Roughly speaking, a gauge theory is one which is invariant under transformations of the gauge group, which represents the "internal symmetries" of the system. The physics and general theory of this are outside of the scope of this thesis, but we present the general idea here.

In our setting, the *gauge group* is defined as $\Gamma := \prod_{\mathbb{D}} G$, i.e., the infinite product of G over the set $\mathbb{D}$ equipped with the Tychonoff topology. In other words, the group of all maps $g : \mathbb{D} \to G$ with the pointwise convergence topology. As G is separable and compact, so is Γ.

Consider a tree t with n leaves and associated s.d.p. given by $0 = d_1 < d_2 < \cdots < d_n < d_{n+1} = 1$. Recall that we can identify the group G_t with $\oplus_{\mathcal{I}(t)}G$ where $\mathcal{I}(t) = \{(d_i, d_{i+1}) : 1 \leq i \leq n\}$. At the level t, we define the gauge action:

$$Z_t : \Gamma \curvearrowright G_t, \quad Z_t(s)(g)(d_i, d_{i+1}) = s(d_i)g(d_i, d_{i+1})s(d_{i+1})^{-1}.$$

By virtue of the fact that we are working with the circle $\mathbb{R}/\mathbb{Z}$, we identify 0 with 1 and so we have the periodicity condition $s(d_{n+1}) = s(1) = s(0) = s(d_1)$. This action is p.m.p. with respect to the Haar measure m_{G_t} because G_t is compact (and hence unimodular, see, e.g. [14, Corollary 2.28]). Thus, the action provides a unitary representation

$$W_t : \Gamma \to \mathcal{U}(L^2(G_t)), \quad W_t(s)\xi(x) := \xi(Z_t(s^{-1})x),$$

for $s \in \Gamma$, $x \in G_t$, and $\xi \in L^2(G_t)$. Further, let $\gamma_t : \Gamma \curvearrowright \mathcal{B}(L^2(G_t))$ denote the adjoint action

$$\gamma_t(s)(a) := W_t(s)aW_t(s)^*,$$

for $s \in \Gamma$ and $a \in \mathcal{B}(L^2(G_t))$.

Brothier and Stottmeister prove [6, Proposition 2.9] that the C*-algebra $M_t = \mathscr{C}(G_t) \rtimes G_{t,d} \subseteq \mathcal{B}(L^2(G_t))$ is stable under the action $\gamma_t : \Gamma \curvearrowright \mathcal{B}(L^2(G_t))$.

Further, recall that the C*-algebraic limit $\mathscr{M}_0 := \varinjlim_{t\in\mathfrak{T}} M_t$ is isomorphic to the norm completion of $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\text{alg}} \oplus_{\mathbb{D}} G \subseteq \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}))$. It turns out that there exists a unique action $Z : \Gamma \curvearrowright (\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}})$ such that if we project $\overline{\mathscr{A}}$ onto G_t, we obtain the action Z_t which we defined above. In particular, we have

$$Z(s)(x)(d, d') := s(d)x(d, d')s(d')^{-1},$$

for $s \in \Gamma$, $x \in \overline{\mathscr{A}}$, and (d, d') an s.d.i. Moreover, there is a corresponding unitary representation given by

$$W : \Gamma \to \mathcal{U}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}})), \quad W(s)\xi(x) := \xi(Z(s)^{-1}x),$$

for $s \in \Gamma$, $x \in \overline{\mathscr{A}}$, and $\xi \in L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}})$. The associated adjoint action is given by

$$\gamma : \Gamma \curvearrowright \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}})), \quad \gamma(s)(a) := W(s)aW(s)^*,$$

for $s \in \Gamma$ and $a \in \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}))$.

In [6, Proposition 2.10] the authors show that the C*-algebra $\mathscr{M}_0 \subseteq \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}))$ is stable under the action $\gamma : \Gamma \curvearrowright \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}))$. Moreover, they show that the Jones action $V \curvearrowright \mathscr{M}_0$ is compatible with the gauge group action in a suitable sense.

4.3 Local algebras of fields

Let O be a connected, open subset of the circle. Let $t \in \mathfrak{T}$ be a tree and let $\mathcal{I}(t)$ be the associated collection of open s.d.i.'s. In the preceding section, we showed that

$$M_t \cong \otimes_{I\in\mathcal{I}(t)} M \cong \otimes_{I\in\mathcal{I}(t)} (\mathscr{C}(G) \rtimes G_d) \cong \mathscr{C}(G_t) \rtimes G_{t,d}.$$

It is not hard to see that we can also identify the group G_t with $\oplus_{\mathcal{I}(t)} G$.

Given a tree t, let $\mathcal{I}(t, O)$ denote the subset of s.d.i.'s $I \in \mathcal{I}(t)$ with $I \subseteq O$. With this notation in mind, we can define a C*-subalgebra of M_t by

$$M_t(O) := \big(\otimes_{I\in\mathcal{I}(t,O)} M \big) \otimes \big(\otimes_{J\in\mathcal{I}(t)\setminus\mathcal{I}(t,O)} \mathbb{C}1_t \big).$$

Moreover, we can define a subgroup of G_t by $G_{t,O} := \oplus_{\mathcal{I}(t,O)} G$ via the identification of G_t with $\oplus_{\mathcal{I}(t)} G$ above. Both the coordinate projection

$$p_{t,O} : G_t \to G_{t,O}, (g_I)_{I\in\mathcal{I}(t)} \mapsto (g_I)_{I\in\mathcal{I}(t,O)}$$

and the group embedding

$$\iota_{t,O} : G_{t,O} \to G_t, (g_I)_{I\in\mathcal{I}(t,O)} \mapsto (\tilde{g}_I)_{I\in\mathcal{I}(t)},$$

where

$$\tilde{g}_I = \begin{cases} g_I & \text{if } I \in I(t,O) \\ e & \text{otherwise} \end{cases},$$

are well-defined.

If $t \in \mathfrak{T}$ is a tree with n leaves and $f \in \mathcal{F}$ is a forest with n roots, then we can define the projections

$$p_t^{ft} : G_{ft,O} \to G_{t,O}, x \mapsto p_{t,O} \circ p_t^{ft} \circ \iota_{ft,O}(x),$$

where $p_t^{ft} : G_{ft} \to G_t$ are the projections given by group multiplication described in Section 4.1.1.

Proposition 4.3.1. *The inclusion $M_t(O) \subseteq M_t$ corresponds to the embedding*

$$\begin{aligned} j_{t,O} : \mathscr{C}(G_{t,O}) \rtimes G_{t,O,d} &\to \mathscr{C}(G_t) \rtimes G_{t,d} \\ \sum_{g \in G_{t,O}} a_g \lambda_g &\mapsto \sum_{g \in G_{t,O}} (a_g \circ p_{t,O}) \lambda_{\iota_{t,O}(g)}, \end{aligned}$$

where $a_g \in \mathscr{C}(G_{t,O})$ and $g \mapsto a_g$ has finite support.

Proof. This follows immediately Lemma 4.1.3 and the relevant local definitions. □

Recall that $\overline{\mathscr{A}} = \{x \in \prod_I G : x(I) = x(I_1)x(I_2)\}$, where I_1, I_2 are the first and second halves of the s.d.i. I, respectively. This is a compact space, equipped with the product (subspace) topology. We define a subspace

$$\begin{aligned} \overline{\mathscr{A}}(O) &:= \overline{\mathscr{A}} \cap \prod_{I \subseteq O} G \\ &= \left\{ x \in \prod_{I \subseteq O} G : x(I) = x(I_1)x(I_2) \right\} \end{aligned}$$

and denote the associated coordinate projection $p_O : \overline{\mathscr{A}} \to \overline{\mathscr{A}}(O)$.

Let us consider the system given by the collection of spaces $(G_{t,O}, t \in \mathfrak{T})$ along with the projections $p_t^{ft}(O) : G_{ft,O} \to G_{t,O}$. By an argument similar to the one in Section 4.1.1, we can see that $\overline{\mathscr{A}}(O)$ is the inverse limit of this system.

Finally, let $\mathbb{D}(O)$ be the set of dyadic rationals such that $d \in O$ or d is the left boundary point of the connected open set O; we define an inclusion map

$$\iota_O : \oplus_{\mathbb{D}(O)} G \to \oplus_{\mathbb{D}} G, g(d) \mapsto \hat{g}(d),$$

where

$$\hat{g}(d) = \begin{cases} g(d) & \text{if } d \in \mathbb{D}(O) \\ e & \text{otherwise} \end{cases}.$$

The goal of the main theorem of this section is to show that the net of C*-algebras $\mathscr{M}_0(O)$ satisfies properties very similar to those in the definition of a *conformal net* (which is what a net of local observable algebras is called in the case when the QFT of interest is a CFT). For explicit formulations of these nets, see [15, Definition 2.3] for a definition in the context of an in-depth study of 2-dimensional (algebraic) CFT; the introduction of [21] offers a definition which is particularly relevant for our construction.

Theorem 4.3.2. *[6, Proposition 2.12] Let O, O_1, O_2 be connected, open subsets of the circle and let $t \in \mathfrak{T}$ be a fixed tree.*

(1) If $O_1 \subseteq O_2$, then $M_t(O_1) \subseteq M_t(O_2)$.

(2) If $O_1 \cap O_2 = \emptyset$, then $M_t(O_1)$ and $M_t(O_2)$ mutually commute.

(3) If $s \in \mathfrak{T}$ and $s \geq t$, then $M_t(O)$ is a unital C-subalgebra of $M_s(O)$ when M_t is identified as a subalgebra of M_s (via the functor Ψ defined in Section 3.5).*

(4) The norm closure of the union of the C-algebras $(M_s(O), s \in \mathfrak{T})$ is a unital C*-subalgebra $\mathscr{M}_0(O) \subseteq \mathscr{M}_0$. The algebraic crossed-product $\mathscr{C}(\overline{\mathscr{A}})(O) \rtimes_{alg} \oplus_{\mathbb{D}(O)} G$ (suitably defined) embeds as a dense $*$-subalgebra in $\mathscr{M}_0(O)$.*

(5) The inclusion $\mathscr{M}_0(O) \subseteq \mathscr{M}_0$ restricts to the embedding

$$j_O : \mathscr{C}(\overline{\mathscr{A}}(O)) \rtimes_{alg} \oplus_{\mathbb{D}(O)} G \to \mathscr{C}(\overline{\mathscr{A}}) \rtimes_{alg} \oplus_{\mathbb{D}} G$$
$$\sum_{g \in \oplus_{\mathbb{D}(O)} G} a_g u_g \mapsto \sum_{g \in \oplus_{\mathbb{D}} G} (a_g \circ p_O) u_{\iota_O(g)},$$

where $g \mapsto a_g \in \mathscr{C}(\overline{\mathscr{A}})$ has finite support.

(6) If $O_1 \subseteq O_2$, then $\mathscr{M}_0(O_1) \subseteq \mathscr{M}_0(O_2)$.

(7) If $O_1 \cap O_2 = \emptyset$, then $\mathscr{M}_0(O_1)$ and $\mathscr{M}_0(O_2)$ mutually commute.

(8) If v is an element of Thompson's group T and $\alpha : V \curvearrowright \mathscr{M}_0$ is the Jones action, then

$$\alpha(v)\mathscr{M}_0(O) = \mathscr{M}_0(vO).$$

Proof. Parts (1) and (2) follow immediately from the fact that $M_t(O)$ is defined in terms of tensor products over the set $\mathcal{I}(t)$. The statement in (3) comes from the fact that the directed system $(M_t, \iota_t^s, t \leq s \in \mathfrak{T})$ has unital $*$-homomorphic embeddings. By (3), we have that $(M_t(O), t \in \mathfrak{T})$ is nested and by definition, $M_t(O) \subseteq M_t$ so the first statement of (4) is obvious. By identifying M_t with $\mathscr{C}(G_t) \rtimes G_{t,d}$ and $\mathscr{M}_0$ with the completion of $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{alg} \oplus_{\mathbb{D}} G$, the second statement of (4) and (5) follow from the same idea as the first part of the proof of Theorem 4.1.3.

We included the final statement about the Jones action for completeness. As we have omitted the previous details about it, we refer the reader back to the cited theorem of [6] for a proof. □

Moreover, recall that we had previously defined the gauge group $\Gamma := \prod_{\mathbb{D}} G$ and referenced the fact that $\mathscr{M}_0 \subseteq \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}))$ is stable under the gauge group action $\gamma : \Gamma \curvearrowright \mathcal{B}(L^2(\overline{\mathscr{A}}, m_{\overline{\mathscr{A}}}))$. It turns out that we can actually define the *localized gauge group* $\Gamma(O) := \prod_{\mathbb{D}\cap\overline{O}} G$. Letting $q_O : \Gamma \to \Gamma(O)$ be the coordinate projection, we define the action

$$Z_O : \Gamma(O) \curvearrowright \overline{\mathscr{A}}(O), \quad Z_O(s)(x)(d, d') := s(d)x(d, d')s(d')^{-1},$$

for $s \in \Gamma(O)$, $x \in \overline{\mathscr{A}}(O)$, and (d, d') a s.d.i. which is contained in O. Observe that we took the product over all dyadic rationals which are in the closure of O—we did this because because the formula of Z_O given above requires both the values $s(d)$ and $s(d')^{-1}$.

The authors show that there is suitable localized gauge group action $\gamma_O : \Gamma(O) \curvearrowright \mathscr{M}_0(O)$ corresponding to the restriction of the gauge group action on $\mathscr{M}_0(O)$. Furthermore, the family of localized gauge group actions is equivariant with respect to the Jones action, so that for any $v \in T$, $\mathrm{Ad}(\alpha(v)) \circ \gamma_O = \gamma_{vO}$.

Chapter 5

Construction of a von Neumann algebra with a state

In the previous chapter, we described the limit C*-algebra $\mathscr{M}_0$ in detail. Now, we want to find a state $\overline{\omega}$ on $\mathscr{M}_0$ and study the associated von Neumann algebra $(\mathscr{M}, \Omega_{\overline{\omega}})$ obtained via the GNS construction.

5.1 General construction

First, we want to consider a state on the "building block" C*-algebra, which we chose to be $M := \overline{\mathscr{C}(G) \rtimes_{\text{alg}} G} \subseteq \mathcal{B}(L^2(G))$. The choice of state used here was physically motivated by the family of states associated to the heat kernel of a compact Lie group. The details of this choice are summarized in Remark 3.1, [6] and further detailed in the companion article [7].

5.1.1 Construction of a state

Recall that a separable, compact group G has a distinguished unitary representation, the regular representation λ_G on $L^2(G) := L^2(G, m_G)$, where m_G is the normalized Haar measure. The *unitary dual* of G is the set $\hat{G}$ of (equivalence classes of) irreducible unitary representations of G. Since G is compact and separable, this set is countable and all the irreducible representations are finite dimensional (see [2, Introduction] for a summary of the fundamentals of the representation theory of compact and abelian groups).

Notation 5.1.1. Each unitary representation of G is denoted by (π, H_π). Let $\{e_i^\pi\}_{i=1}^{d_\pi}$ be an ONB for H_π. The dimension of H_π is denoted d_π and Tr_{H_π} is the non-normalized trace. Additionally, let us write $\chi_\pi(g) := \text{Tr}_{H_\pi}(\pi(g))$, $g \in G$ for the character associated to π.

Definition 5.1.2. The functions $\phi_{v,u}(g) = \langle \pi(g)u|v\rangle$ with $u, v \in H_\pi$ are called matrix elements of π. When u, v are elements of an ONB for H_π, then $\phi_{v,u}(g)$ is an entry in

the matrix for $\pi(g)$ with respect to that ONB. In particular, write

$$\pi_{ij}(g) := \phi_{e_i^\pi, e_j^\pi}(g) = \langle \pi(g) e_j^\pi | e_i^\pi \rangle.$$

Using the notation above, note that $\chi_\pi = \sum_i \pi_{ii}$.

We will need to make use of some facts from the theory of representations of compact groups. For our purposes, these results are a means to an end. The interested reader is encouraged to check out either the cited portions of [14] or the lecture notes [40] for additional context and detailed proofs.

Notation 5.1.3. We are interested in certain subspaces of $\mathscr{C}(G)$, which we will denote by

$$\begin{aligned} \mathcal{E}_\pi &:= \operatorname{span}\big(\{\pi_{ij} : 1 \le i, j \le d_\pi\}\big), \text{ and} \\ \mathcal{E} &:= \operatorname{span}\Big(\bigcup_{\pi \in \hat{G}} \mathcal{E}_\pi\Big). \end{aligned}$$

Theorem 5.1.4 (Peter-Weyl)**.** *[14, Theorem 5.12] Let G be a compact group. Then $\mathcal{E}$ is uniformly dense in $\mathscr{C}(G)$, $L^2(G) = \bigoplus_{\pi \in \hat{G}} \mathcal{E}_\pi$, and $\{\sqrt{d_\pi}\pi_{ij} : 1 \le i, j \le d_\pi, \pi \in \hat{G}\}$ is an ONB for $L^2(G)$.*

By Peter-Weyl, we have that if $f \in L^2(G)$ then $f = \sum_{\pi \in \hat{G}} d_\pi f * \chi_\pi$. In other words, $d_\pi f * \chi_\pi$ is the orthogonal projection of f onto $\mathcal{E}_\pi$.

In the case where π, π' are unitary representations and $f = \chi_\pi$ is an irreducible character, we have

$$\chi_\pi * \chi_{\pi'} = \begin{cases} d_\pi^{-1} \chi_\pi & \text{if } \pi = \pi' \\ 0 & \text{otherwise} \end{cases}$$

Definition 5.1.5. A function ϕ on G is called *central* if $\phi(gh) = \phi(hg)$ for all $g, h \in G$. The set of all central functions in $L^p(G)$ (resp. $\mathscr{C}(G)$) is denoted by $\mathcal{Z}(L^p(G))$ (resp. $\mathcal{Z}(\mathscr{C}(G))$).

Example 5.1.6. By virtue of the trace being cyclic and π being a homomorphism, observe that each character χ_π defines a central function:

$$\begin{aligned} \chi_\pi(gh) = \operatorname{Tr}_{H_\pi}(\pi(gh)) &= \operatorname{Tr}_{H_\pi}(\pi(g)\pi(h)) \\ &= \operatorname{Tr}_{H_\pi}(\pi(h)\pi(g)) = \operatorname{Tr}_{H_\pi}(\pi(hg)) = \chi_\pi(hg). \end{aligned}$$

Fact 5.1.7. Considering $L^p(G)$ and $\mathscr{C}(G)$ as Banach algebras under convolution, $\mathcal{Z}(L^p(G))$ and $\mathcal{Z}(\mathscr{C}(G))$ coincide exactly with their respective centers. Moreover, one can show that $\{\chi_\pi : \pi \in \hat{G}\}$ is an ONB of $\mathcal{Z}(L^2(G))$ (see [14, Theorem 5.21 and Theorem 5.23]).

We begin this construction by choosing a strictly positive probability measure

$m \in \mathrm{Prob}(\hat{G})$ on $\hat{G}$ and we define a function $h_0 : \hat{G} \to [0,1]$ so that $h_0(\pi) = m(\{\pi\})$[1]. We use this map to define a continuous function $h \in L^1(G) := L^1(G, m_G)$ by

$$h := \sum_{\pi \in \hat{G}} d_\pi^{-1} h_0(\pi) \chi_\pi.$$

This allows us to define a linear functional ω on $\mathcal{B}(L^2(G))$. In particular, we define $\omega(b) := \mathrm{Tr}(b\lambda(h))$ for $b \in \mathcal{B}(L^2(G))$. An argument (analogous to the ones used in [14], referenced above) shows that each character χ defines a central projection $\lambda(\chi)$ of the group von Neumann algebra LG such that the family $\{\lambda(\chi)\}_\chi$ partitions the identity. It follows immediately that $\lambda(h) \in \mathcal{Z}(LG)$.

Lemma 5.1.8. *[6, Lemma 3.2]The functional ω defined above is a faithful normal state on $\mathcal{B}(L^2(G))$ and*

$$\omega\left(\sum_{g \in G} b_g \lambda_g\right) = \sum_{g \in G} \left(\int_G b_g dm_G\right) h(g^{-1}), \tag{5.1}$$

where $b_g \in L^\infty(G)$ is equal to zero for all but finitely many $g \in G$.

Proof. For each Hilbert space H_π, let $\{\delta_k^\pi : 1 \leq k \leq d_\pi\}$ be an ONB of H_π. Take $\pi \in \hat{G}$ and define the function $\pi_{n,m}(g) := \langle \pi(g)\delta_m^\pi | \delta_n^\pi \rangle$ for $g \in G$. It follows from Peter-Weyl 5.1.4 that $\{\sqrt{d_\pi}\pi_{n,m} : \pi \in \hat{G}, 1 \leq n, m \leq d_\pi\}$ is an ONB of $L^2(G)$.

Moreover, we can show that

$$\begin{aligned}
\chi_\pi * \pi'_{n,m}(g) &= \int_G \chi_\pi(h) \pi'_{n,m}(h^{-1}g) dm_G \\
&= \int_G \mathrm{Tr}_{H_\pi}(\pi(h)) \pi'_{n,m}(h^{-1}g) dm_G \\
&= \int_G \Big(\sum_{k=1}^{d_\pi} \langle \pi(h)\delta_k^\pi | \delta_k^\pi \rangle \Big) \langle \pi'(h^{-1}g)\delta_m^\pi | \delta_n^\pi \rangle dm_G \\
&= \begin{cases} \pi_{n,n}(g) & \text{if } \pi = \pi' \text{ and } n = m \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Now, consider a set of scalars $\{a_\pi\}_{\pi \in \hat{G}}$ such that $\sum_{\pi \in \hat{G}} |a_\pi| d_\pi < \infty$. Since $\hat{G}$ is countable, we can define some listing of the elements $\pi_1, \pi_2, \ldots \in \hat{G}$ and, in turn, define a series of functions whose n-th partial sum is of the form $\sum_{k=1}^n a_{\pi_k} \chi_{\pi_k}$. This series converges uniformly on G to a continuous function $a := \sum_{k=1}^\infty a_{\pi_k} \chi_{\pi_k} = \sum_{\pi \in \hat{G}} a_\pi \chi_\pi$.

[1]Such a probability measure m exists if and only if G is separable.

By the formula above, we have that $\mathrm{Tr}(\lambda(\chi_\pi)) = d_\pi$ for any $\pi \in \hat{G}$ and so,

$$\mathrm{Tr}(\lambda(a)) = a(e) = \sum_{\pi \in \hat{G}} a_\pi d_\pi.$$

Moreover, if $b = \pi_{n,m}$ (and we identify b with the associated pointwise multiplication operator $b \in L^\infty(G)$) then $\mathrm{Tr}(b\lambda(a)) = 0$ unless π is the trivial representation for any a, as above. So, for arbitrary $b \in L^\infty(G)$, we have

$$\mathrm{Tr}(b\lambda(a)) = \langle b|1\rangle \mathrm{Tr}(\lambda(a)) = \Big(\int_G b\, dm_G\Big)\mathrm{Tr}(\lambda(a)).$$

Further, we have that

$$h(e) = \sum_{\pi \in \hat{G}} d_\pi^{-1} h_0(\pi) d_\pi = \sum_{\pi \in \hat{G}} h_0(\pi) = 1$$

and

$$\lambda_g \lambda(h) = \lambda(h^g)$$

where $h^g(x) = h(g^{-1}x)$. Thus

$$\mathrm{Tr}(\lambda_g \lambda(h)) = h^g(e) = h(g^{-1}),$$

which gives us formula 5.1, as required.

It remains to show that this state is faithful. As described above, the set of characters forms a set of orthogonal projections inside LG which sum up to the identity operator. Further, $h_0(\pi) = m(\{\pi\}) \neq 0$ for any $\pi \in \hat{G}$. So, as the linear combination of characters and h_0, the operator $\lambda(h)$ is positive and has strictly positive spectrum. Now, suppose that $\omega(a^*a) = 0$ for some $a \in \mathcal{B}(L^2(G))$. Then, by definition $\omega(a^*a) = \mathrm{Tr}(a^*a\lambda(h)) = \mathrm{Tr}(ba^*ab) = 0$, where b is the positive square root of $\lambda(h)$, i.e., $b^2 = \lambda(h)$. Since $\lambda(h)$ has strictly positive specturm, it must be invertible and therefore so is b. Since Tr is faithful, this means $ba^*ab = 0$ and by invertibility of b, we get that $a^*a = 0$ which implies that $\omega := \mathrm{Tr}(\cdot\lambda(h))$ is faithful. □

Now, let us choose a map $m : \mathbb{D} \to \mathrm{Prob}(\hat{G}), d \mapsto m_d$ such that m_d is strictly positive for any $d \in \mathbb{D}$; this generates a family of measures. Now, for each $d \in \mathbb{D}$, let $h_{d,0}$ be the positive map satisfying

$$m_d(A) = \sum_{\pi \in A} h_{d,0}(\pi)$$

for any $A \subseteq \hat{G}$.

This allows us to define an element of LG associated to each $d \in \mathbb{D}$,

$$h_d := \sum_{\pi \in \hat{G}} \frac{h_{d,0}(\pi)}{d_\pi} \chi_\pi.$$

Each element $\lambda(h_d)$ belongs to the center of LG and defines a normal faithful state $\omega_d : b \mapsto \mathrm{Tr}(b\lambda(h_d))$ on $\mathcal{B}(L^2(G))$, which we will restrict to M.

At this point, the notation is getting rather heavy handed, so we will take the liberty of identifying h_d with $\lambda(h_d)$. This identification is justified by the injectivity of λ.

Recall that, given a tree $t \in \mathfrak{T}$ with n leaves and associated s.d.p. $0 = d_1 < d_2 < \ldots < d_n < d_{n+1} = 1$, we have $M_t = \mathscr{C}(G_t) \rtimes G_{t,d} \subseteq \mathcal{B}(L^2(G_t))$. We freely identify $\mathcal{B}(L^2(G_t))$ with $\mathcal{B}(L^2(G))^{\bar{\otimes} n}$, the nth von Neumann tensor power of $\mathcal{B}(L^2(G))$. At each level, we want to use a similar construction as above, so we define an element $h_t := h_{d_1} \otimes \cdots \otimes h_{d_n} \in \mathcal{Z}(LG_t)$, a probability measure $m_t := m_G \otimes \cdots \otimes m_G$ on G_t[2] and a state $\omega_t : b \mapsto \mathrm{Tr}(bh_t)$ on M_t.

Proposition 5.1.9. *[6, Proposition 3.3] Let $t \in \mathfrak{T}$ be a fixed tree of n leaves with associated s.d.p. $0 = d_1 < d_2 < \ldots < d_n < d_{n+1} = 1$ and identify M_t with the nth tensor power of M. The state ω_t on M_t is (the restriction of) a normal faithful state and satisfies the equality*

$$\omega_t\left(\sum_{g \in G_t} a_g \lambda_g\right) = \sum_{g \in G_t} \left(\int_{G_t} a_g dm_t\right) \prod_{j=1}^{n} h_{d_j}(g_j^{-1}), \tag{5.2}$$

where $g \mapsto a_g \in \mathscr{C}(G)$ has finite support. Moreover, the embedding $\Psi(f) : M_t \to M_{ft}$ is state-preserving for any forest f with n roots.

Proof. Because $h_t := h_{d_1} \otimes \ldots \otimes h_{d_n}$, we have that $\mathrm{Tr}(\cdot h_t) = \otimes_{j=1}^{n} \mathrm{Tr}(\cdot h_{d_j})$ and so we identify ω_t with $\omega_{d_1} \otimes \ldots \otimes \omega_{d_n}$. By Lemma 5.1.7, ω_d is (the restriction of) a normal faithful state for any $d \in \mathbb{D}$ and therefore, so is ω_t.

In order to show that ω_t satisfies the equality in the statement, we show what it does to each part of $M_t = \mathscr{C}(G_t) \rtimes G_{t,d}$. First, take $a = a_1 \otimes \cdots \otimes a_n$ be an elementary tensor in $\mathscr{C}(G_t)$. By applying ω_t to a, we obtain

$$\omega_t(a) = \prod_{j=1}^{n} \omega_{d_j}(a_j) = \prod_{j=1}^{n} \int_G a_j \, dm_G = \int_{G_t} a \, dm_{G_t}.$$

Next, consider an element $g = (g_1, \ldots, g_n) \in G_t$. Applying ω_t to λ_g yields

$$\omega_t(\lambda_g) = \prod_{j=1}^{n} \omega_{d_j}(\lambda_{g_j}) = \prod_{j=1}^{n} \mathrm{Tr}(\lambda_{g_j} h_{d_j}) = \prod_{j=1}^{n} h_{d_j}(g_j^{-1}).$$

[2]Recall that m_G is the normalized Haar measure on G.

We claim that $h_t \in \mathcal{Z}(LG_t)$ is a trace-class operator with trace equal to one. To prove this, we briefly revert back to using the (strictly speaking, more correct) notation $\lambda(h_t) = \lambda(h_{d_1}) \otimes \cdots \otimes \lambda(h_{d_n}) \in \mathcal{Z}(G_t)$.

It follows from Peter-Weyl theory that for a compact group G and its left-regular representation $\lambda : G \to L^2(G)$,

$$\lambda \cong \bigoplus_{\pi \in \hat{G}} d_\pi \cdot \pi.$$

Therefore, $LG \cong \bigoplus_\pi (I_{d_\pi} \otimes M_{d_\pi}(\mathbb{C}))$ where $M_{d_\pi}(\mathbb{C})$ is the space of $d_\pi \times d_\pi$ complex-valued matrices, I_{d_π} is the $d_\pi \times d_\pi$ identity matrix, and $\otimes$ is the Kronecker product. In other words, LG is block diagonal in $\mathcal{B}(L^2(G))$. Since the trace of block diagonal matrices is equal to the sum of the trace of each block, and for any $d \in \mathbb{D}$, we have $\lambda(h_d) = \sum_{\pi \in \hat{G}} \frac{h_{d,0}(\pi)}{d_\pi} \lambda(\chi_\pi)$, we have:

$$\begin{aligned}
\|\lambda(h_d)\|_{\mathcal{T}(L^2(G))} &= \sum_{\pi \in \hat{G}} \left\| \frac{h_{d,0}(\pi)}{d_\pi} \lambda(\chi_\pi) \right\|_1 \\
&= \sum_{\pi \in \hat{G}} h_{d,0}(\pi) \\
&= m_d(\hat{G}) = 1.
\end{aligned}$$

That is, each $\lambda(h_d)$ is a trace-class operator with trace equal to 1. The claim follows from the definition of $\lambda(h_t)$ as the tensor product of trace-class operators, each with trace equal to 1.

Identifying h_t with $\lambda(h_t)$, we have that $\mathrm{Tr}(a\lambda_g h_t) = \mathrm{Tr}(ah_t)\mathrm{Tr}(\lambda_g h_t)$. Putting this together with the equations above, we get

$$\begin{aligned}
\omega_t \left(\sum_{g \in G_t} a_g \lambda_g \right) = \mathrm{Tr} \left(\left(\sum_{g \in G_t} a_g \lambda_g \right) h_t \right) &= \sum_{g \in G_t} \mathrm{Tr}(a_g \lambda_g h_t) \\
&= \sum_{g \in G_t} \mathrm{Tr}(a_g h_t) \mathrm{Tr}(\lambda_g h_t) \\
&= \sum_{g \in G_t} \left(\int_{G_t} a_g dm_t \right) \prod_{j=1}^{n} h_{d_j}(g_j^{-1}).
\end{aligned}$$

In other words, we have proved that equation (5.2) holds.

Recall that any forest is a composition of elementary forests $f_{j,n} = I^{\otimes j-1} \otimes Y \otimes I^{\otimes n-j}$. This means that it is sufficient to show that the embedding $\Psi(f_{j,n}) : M_t \to M_{f_{j,n}t}$ is state-preserving.

Now, let $f := f_{j,n}$ and let $e_j := \frac{d_j + d_{j+1}}{2}$. We want to show that $\omega_{ft} \circ \Psi(f) = \omega_t$. By density, it suffices to show that $\omega_{ft} \circ \Psi(f)(x) = \omega_t(x)$ for elementary tensors $x = x_1 \otimes \cdots \otimes x_n \in M_t$.

We have

$$\begin{aligned}\omega_{ft} \circ \Psi(f)(x) &= \omega_{ft}(x_1 \otimes \cdots \otimes x_{j-1} \otimes R(x_j) \otimes x_{j+1} \otimes \cdots \otimes x_n) \\ &= \omega_{d_1}(x_1) \otimes \cdots \otimes \omega_{d_{j-1}}(x_{j-1})[\omega_{d_j} \otimes \omega_{e_j}](R(x_j)) \otimes \omega_{d_{j+1}}(x_{j+1}) \otimes \cdots \otimes \omega_{d_n}(x_n) \\ &= [\omega_{d_j} \otimes \omega_{e_j}](u(x_j \otimes \mathrm{id})u^*) \prod_{i \neq j} \omega_{d_i}(x_i) \\ &= \mathrm{Tr}(u(x_j \otimes \mathrm{id})u^*(h_{d_j} \otimes h_{e_j})) \prod_{i \neq j} \omega_{d_i}(x_i)\end{aligned}$$

The operator u is in the von Neumann algebra $R(G)\bar{\otimes}L^\infty(G)$ (see, e.g. [23]). For any probability measure μ on G, define a normal, unital, positive [3] map on $\mathcal{B}(L^2(G))$ given by

$$\Theta(\mu)(x) = (\mathrm{id} \otimes \mu)u(x \otimes 1)u^* = \int_G \rho(s)x\rho(s)^* d\mu(s),$$

where ρ is the right-regular representation. This map is trace-preserving in the sense that for any trace-class operator x, $\mathrm{Tr}(\Theta(\mu)(x)) = \mathrm{Tr}(x)$ (see [9, Proposition 3.1]). Since h_{e_j} is a positive trace-class operator with trace 1, its restriction (as a positive linear functional) to $L^\infty(G)$ defines a normal state in $L^1(G)$, namely the normalized Haar measure m_G. Since the first leg of u lies in $R(G)$, it follows that for $h_{d_j} \otimes h_{e_j} \in L(G \times G)$, we have

$$\begin{aligned}\mathrm{Tr}(u(x_j \otimes \mathrm{id})u^*(h_{d_j} \otimes h_{e_j})) &= (\mathrm{Tr} \otimes \mathrm{Tr})(u(x_j h_{d_j} \otimes \mathrm{id})u^*(1 \otimes h_{e_j})) \\ &= \mathrm{Tr}((\mathrm{id} \otimes m_G)u(x_j h_{d_j} \otimes 1)u^*) \\ &= \mathrm{Tr}(\Theta(m_G)(x_j h_{d_j})) \\ &= \omega_{d_j}(x_j),\end{aligned}$$

where the third equality uses the fact that the second leg of u lies in $L^\infty(G)$ and the restriction of $\mathrm{Tr}(\cdot h_{e_j})$ to $L^\infty(G)$ is just integration with respect to m_G. In particular, this means

$$\begin{aligned}\omega_{ft} \circ \Psi(f)(x) &= \omega_{d_j}(x_j) \prod_{i \neq j}^{n} \omega_{d_i}(x_i) \\ &= \prod_{i \neq j}^{n} \omega_{d_i}(x_i) \\ &= \omega_t(x),\end{aligned}$$

as required. □

[3] In fact this map is normal, unital, and *completely positive*, but we did not define this notion in our preliminaries. See [4, Section II.6.9] for details.

5.1.2 The limit state and the GNS completion

By construction, we now find ourselves in the setting of Section 3.5.1. In particular, we can define a unique state $\overline{\omega}$ on $\mathscr{M}_0$ by setting $\overline{\omega} = \omega_t(b)$ for any $b \in M_t,\ t \in \mathfrak{T}$. We define $\mathscr{M}$ to be the GNS completion of $\mathscr{M}_0$ with respect to $\overline{\omega}$. For convenience, we use $\overline{\omega}$ to denote the unique normal extension of the state on the von Neumann algebra $\mathscr{M}$.

Recall from Section 4.1.2 that:

(i) the group ring $\mathbb{C}[\oplus_{\mathbb{D}} G]$ embeds as a dense $*$-subalgebra inside $\mathscr{N}_0$,

(ii) the group $\oplus_{\mathbb{D}} G$ acts on $\overline{\mathscr{A}}$ by the formula $gx(I) := g(d)x(I)$ where I is an s.d.i. starting at $d \in \mathbb{D}$.

Moreover, we had previously define a $*$-algebra

$$\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\text{alg}} \oplus_{\mathbb{D}} G := \left\{ \sum_{g \in \oplus_{\mathbb{D}} G} b_g u_g : b_g \in \mathscr{C}(\overline{\mathscr{A}}), \text{supp}(g \mapsto b_g) \text{ finite} \right\},$$

where $\text{Ad}(u_g)b(x) = b(g^{-1}x)$ for any $g \in \oplus_{\mathbb{D}} G$, $b \in \mathscr{C}(\overline{\mathscr{A}})$, and $x \in \overline{\mathscr{A}}$.

Theorem 5.1.10. *[6, Theorem 3.4] There is an injective $*$-morphism $J : \mathscr{C}(\overline{\mathscr{A}}) \rtimes_{alg} \oplus_{\mathbb{D}} G \to \mathscr{M}$ with WOT-dense image such that*

$$\overline{\omega} \circ J \left(\sum_{g \in \oplus_{\mathbb{D}} G} b_g u_g \right) = \sum_{g \in \oplus_{\mathbb{D}} G} \left(\int_{\overline{\mathscr{A}}} b_g(t) dm_{\overline{\mathscr{A}}}(t) \prod_{d \in supp(g)} h_d(g(d)^{-1}) \right). \tag{5.3}$$

Proof. Recall that in Theorem 4.1.3, we showed that $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\text{alg}} \oplus_{\mathbb{D}} G$ embeds as a dense $*$-subalgebra in $\mathscr{M}_0$. Since $\mathscr{M}$ is the WOT-completion $\pi(\mathscr{M}_0)''$, we get that the image of $\mathscr{C}(\overline{\mathscr{A}}) \rtimes_{\text{alg}} \oplus_{\mathbb{D}} G$ under J is WOT-dense in $\mathscr{M}$. By Proposition 3.5.4, the GNS representation associated to $\overline{\omega}$ is faithful, which implies that J is injective. At each level t, we have the formula given in equation (5.2) and so equation (5.3) holds for the algebraic crossed-product. □

Additionally, one may show that the gauge group action $\gamma : \Gamma \curvearrowright \mathscr{M}_0$ defined in Section 4.2 is state-preserving and so it extends to an action on the von Neumann algebra $\mathscr{M}$. If we require all the measures m_d to be equal, then the Jones action $\alpha_\Psi : V \curvearrowright \mathscr{M}_0$ is also state-preserving and extends to an action on $\mathscr{M}$.

Remark 5.1.11. We conclude this section with a brief discussion about the choice of the "building block" M used in our construction.

One may show (see the discussion at the end of Section 3.2 and [6, Remark 3.5]) that R can be extended to an algebra morphism $\mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G)) \otimes \mathcal{B}(L^2(G))$ and the state ω can be extended to a normal state on all of $\mathcal{B}(L^2(G))$. Further, given

a tree $t \in \mathfrak{T}$ with n leaves, we can extend the state ω_t to a normal state on $\mathcal{B}(L^2(G_t))$. So, for *any* WOT-dense $*$-subalgebra $\tilde{M}$ of $\mathcal{B}(L^2(G))$ with $R(\tilde{M}) \subseteq \tilde{M} \otimes \tilde{M}$, we can define a directed system $((\tilde{M}, \omega_t|_{\tilde{M}_t}), t \in \mathfrak{T})$ of $*$-algebras which are equipped with states and state-preserving isometric maps. From this system, we can construct a limit $*$-algebra $\tilde{\mathscr{M}}_0$ with a state $\tilde{\overline{\omega}}$, from which we can build a von Neumann algebra $(\tilde{\mathscr{M}}, \tilde{\overline{\omega}})$ via the GNS construction. By using a map similar to the one in the preceding theorem, we could show that there is a state-preserving isomorphism from $(\tilde{\mathscr{M}}, \tilde{\overline{\omega}})$ onto the von Neumann algebra $(\mathscr{M}, \overline{\omega})$ that we (actually) built above.

As such, we can say that the pair $(\mathscr{M}, \overline{\omega})$ does not really depend on M. The choice of M is the minimal C*-algebra which includes all unitaries λ_g, $g \in G$ and all multiplication operators $a \in \mathscr{C}(G)$.

5.2 An abelian group with a single measure

By placing further restrictions on the topological group G and the family of measures used in the construction, we are able to describe the von Neumann algebra $(\mathscr{M}, \overline{\omega})$ in greater detail. In this section, we consider the case where the compact, separable group G is abelian and all the measures m_d are equal to a single strictly positive measure $m \in \mathrm{Prob}(\hat{G})$. In order to do this construction, we want to make use of some general theory.

First, recall that if a compact group G is abelian, then its unitary dual $\hat{G}$ coincides exactly with its *Pontrjagin dual*, the countable discrete abelian group of characters of G (see [14, Chapter 4] for details).

Second, from harmonic analysis (see, e.g. [14, Chapter 4]), we know that when G is compact and abelian, the usual Fourier transform $F : L^1(G) \to \mathscr{C}_0(\hat{G})$ can be extended to a unitary transformation $U_F : L^2(G) \to \ell^2(\hat{G})$. The extended map takes characters of G and maps them to delta functions of $\ell^2(\hat{G})$; that is, $U_F : \chi \mapsto \delta_\chi$.

Because $\hat{G}$, the dual of a compact, separable, abelian group G, is "nice" and it is easy to move from $L^2(G)$ to $\ell^2(\hat{G})$ via the unitary Fourier transformation, the construction (and its analysis) in this section will be done using the dual.

Notation 5.2.1. We denote the left regular representation by $\lambda : \hat{G} \to \ell^2(\hat{G})$ and we identify $\ell^\infty(\hat{G})$ with the pointwise multiplication operators acting on $\ell^2(\hat{G})$.

Recall that the building block of our construction was the C*-algebra $M = C^*(Q, N) = C^*(\mathscr{C}(G), \overline{\mathbb{C}[G]})$. In order to make sense of the dual version of our basic building block, let us investigate what happens to both parts of M.

Proposition 5.2.2. *If G is a compact, separable, abelian group, then*

(i) $Ad(U_F)(\mathscr{C}(G)) = C_r^*(\hat{G})$.

(ii) $Ad(U_F)(\overline{\mathbb{C}[G]}) = Char(\hat{G})$, *the C*-algebra generated by characters of* $\hat{G}$.

Proof. For any $f \in L^\infty(G)$, we can view elements as "being" their associated multiplication operator in $M_f \in \mathcal{B}(L^2(G))$.

(i) Let $\gamma \in \hat{G}$. Taking arbitrary $\chi \in \hat{G}$, we have

$$\begin{aligned} U_F M_\gamma U_F^* \delta_\chi = U_F(M_\gamma \cdot \chi) &= U_F(\gamma \cdot \chi) \\ &= \delta_{\gamma \cdot \chi} \\ &= \lambda_\gamma \delta_\chi. \end{aligned}$$

Since $\overline{\operatorname{span}(\hat{G})} = \mathscr{C}(G)$ and $C_r^*(\hat{G})$ is generated by λ_γ, $\gamma \in \hat{G}$, we are done.

(ii) Let us denote the evaluation function on $\hat{G}$ by $\mathrm{ev}_g(\chi) = \chi(g)$ for all $\chi \in \hat{G}$. Clearly, we have $C^*(\{\mathrm{ev}_g : g \in G\}) = \mathrm{Char}(\hat{G})$. Now for arbitrary $g \in G$, $\chi \in \hat{G}$, we have

$$\begin{aligned} U_F \lambda_g U_F^* \delta_\chi = U_F \lambda_g \chi &= U_F(\chi(g^{-1})\chi) \\ &= \chi(g^{-1}) U_F \chi \\ &= \chi(g^{-1}) \delta_\chi \\ &= \mathrm{ev}_{g^{-1}}(\chi) \delta_\chi \\ &= M_{\mathrm{ev}_{g^{-1}}} \delta_\chi. \end{aligned}$$

Notice that for any $f \in \ell^\infty(\hat{G})$ and $\gamma \in \hat{G}$, we have $M_f \delta_\chi(\gamma) = f(\gamma)\delta_\chi(\gamma) = f(\chi)\delta_\chi(\gamma)$. So by the work above,

$$U_F \lambda_g U_F^* \delta_\chi = M_{\mathrm{ev}_{g^{-1}}} \delta_\chi = \mathrm{ev}_{g^{-1}}(\chi) \delta_\chi.$$

Since $\overline{\mathbb{C}[G]} = \overline{\operatorname{span}(\{\lambda_g : g \in G\})}$, we get the desired equality.

□

Since $\hat{G}$ is abelian, the full and reduced group C*-algebras coincide, so we can drop the "reduced" notation without any confusion and simply write $\mathrm{Ad}(U_F)(\mathscr{C}(G)) = C^*(\hat{G})$. Moreover, we will identify $\mathcal{B}(L^2(G))$ with $\mathcal{B}(\ell^2(\hat{G}))$ via the Fourier transform.

Notation 5.2.3. With the goal of using similar notation as before, we will write $\hat{Q} := \mathrm{Ad}(U_F)(Q) = C^*(\hat{G})$ and $\hat{N} := \mathrm{Ad}(U_F)(N) = \mathrm{Char}(\hat{G})$. Our new building block will be $\hat{M} := C^*(\hat{Q}, \hat{N}) = \mathrm{Char}(\hat{G}) \subseteq \mathcal{B}(\ell^2(\hat{G}))$.

Note that $\hat{G}$ acts on itself via the left regular representation and we can lift this action to $\mathrm{Char}(\hat{G})$. So by an argument similar to the one in Proposition 4.1, we get that $\hat{M} = \mathrm{Char}(\hat{G}) \rtimes \hat{G}$. Note that here we have the *actual* crossed-product (not just algebraic) since $\hat{G}$ is discrete.

Notation 5.2.4. By doing the same Fourier transform identification at each level, we get $\hat{Q}_t := C^*(\hat{G}_t)$, $\hat{N}_t := \mathrm{Char}(\hat{G}_t)$, and $\hat{M}_t := \mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t$. If t is a tree with n leaves then, as before, $\hat{Q}_t = \otimes_{k=1}^n \hat{Q}$, $\hat{N}_t = \otimes_{k=1}^n \hat{N}$, and $\hat{M}_t = \otimes_{k=1}^n \hat{M}$ (where $\otimes$ is the minimal tensor product).

Remark 5.2.5. In Chapter 4, Q was the commutative part of M and N was the convolution part. The Fourier transform swaps these: $\hat{Q}$ is now the convolution part of $\hat{M}$ and $\hat{N}$ is the commutative part.

In the original construction, the inclusion map $R : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G)) \otimes \mathcal{B}(L^2(G))$ was given by $R(b) = \mathrm{Ad}(u)(b \otimes 1)$ where $u\xi(g,h) := \xi(gh,h)$ for all $\xi \in L^2(G \times G)$. Now, we want our inclusion map to take $\mathcal{B}(\ell^2(\hat{G}))$ to $\mathcal{B}(\ell^2(\hat{G})) \otimes \mathcal{B}(\ell^2(\hat{G}))$. In particular, we define

$$\hat{R} := \mathrm{Ad}(U_F \otimes U_F) \circ R \circ \mathrm{Ad}(U_F^*).$$

In the previous section, we constructed a state given by $\omega(b) := \mathrm{Tr}(b\lambda(h))$ for $b \in \mathcal{B}(L^2(G))$. In order to have a state which is suitable to our new context, we will use $\hat{\omega} := \omega \circ \mathrm{Ad}(U_F^*)$.

Recall that, since G is abelian, the dimension of any irreducible unitary representation π is $d_\pi = 1$ (see [14, Section 4.1]). So, instead of matrix coefficients $\pi_{n,m}$, we just have characters χ. Therefore, the map h is given by $h = \sum_{\chi \in \hat{G}} h_0(\chi)\chi$ and the associated measure $m \in \mathrm{Prob}(\hat{G})$ satisfies $m(A) = \sum_{\chi \in A} h_0(\chi)$ whenever $A \subseteq \hat{G}$.

Lemma 5.2.6. *[6, Lemma 3.6] The inclusion map yields the equality*

$$\hat{R}\Big(\sum_{\chi \in \hat{G}} b_\chi \lambda_\chi\Big) = \sum_{\chi \in \hat{G}} b_\chi \lambda_\chi \otimes \lambda_\chi,$$

where b_χ is a character on $\hat{G}$ with finite support. In particular, $\hat{R}(\hat{Q}) \subseteq \hat{Q} \otimes \hat{Q}$ and $\hat{R}(\hat{N}) \subseteq \hat{N} \otimes \hat{N}$. The state $\hat{\omega}$ satisfies the equalities

$$\begin{aligned}\hat{\omega}\Big(\sum_{\chi \in \hat{G}} b_\chi \lambda_\chi\Big) &= \sum_{g \in \hat{G}} b_e(g) h_0(g) \\ &= \int_{\hat{G}} b_e(g) dm(g),\end{aligned}$$

where e is the identity in $\hat{G}$.

In [6], the authors leave the proof as a simple consequence of their Lemma 3.2 (our Lemma 5.1.7). For the sake of completeness, however, we will spell out the details below.

Proof. First, we show the desired equality for the inclusion map.

Using the definition, we have

$$\begin{aligned}\hat{R}\Big(\sum_{\chi\in\hat{G}} b_\chi\lambda_\chi\Big) &= \mathrm{Ad}(U_F\otimes U_F)\circ R\Big(\sum_{\chi\in\hat{G}} U_F^* b_\chi U_F\cdot U_F^*\lambda_\chi U_F\Big)\\ &= \mathrm{Ad}(U_F\otimes U_F)\Big(\sum_{\chi\in\hat{G}} R(U_F^* b_\chi U_F)R(U_F^*\lambda_\chi U_F)\Big)\\ &= \mathrm{Ad}(U_F\otimes U_F)\Big(\sum_{\chi\in\hat{G}} (U_F^* b_\chi U_F\otimes \mathrm{id})\cdot(U_F^*\lambda_\chi U_F\circ\mu_G)\Big),\end{aligned}$$

where the last line follows from the equalities computed in Lemma 4.1.3 (and μ_G is the group multiplication). Notice that for $\chi,\gamma\in\hat{G}$, we have

$$\begin{aligned}U_F^*\lambda_\chi U_F\gamma &= U_F^*\lambda_\chi\delta_\gamma\\ &= U_F^*\delta_{\chi\cdot\gamma}\\ &= \chi\gamma.\end{aligned}$$

In other words, $U_F^*\lambda_\chi U_F = M_\chi$[4] as operators. Returning to our computation of $\hat{R}$, we have $U_F^*\lambda_\chi U_F\circ\mu_G = M_\chi\circ\mu_G$, which we see as an element of $\mathscr{C}(G\times G)$ by writing $\chi\circ\mu_G$. Now, for $g,h\in G$, we have

$$\begin{aligned}\chi\circ\mu_G(g,h) = \chi(gh) &= \chi(g)\chi(h)\\ &= (\chi\otimes\chi)(g,h).\end{aligned}$$

In other words, $\chi\circ\mu_G = \chi\otimes\chi$, which means

$$\begin{aligned}M_\chi\circ\mu_G = M_{\chi\otimes\chi} &= M_\chi\otimes M_\chi\\ &= U_F^*\lambda_\chi U_F\otimes U_F^*\lambda_\chi U_F.\end{aligned}$$

Thus,

$$\begin{aligned}\hat{R}\Big(\sum_{\chi\in\hat{G}} b_\chi\lambda_\chi\Big) &= \mathrm{Ad}(U_F\otimes U_F)\Big(\sum_{\chi\in\hat{G}}(U_F^* b_\chi U_F\otimes\mathrm{id})\cdot(U_F^*\lambda_\chi U_F\otimes U_F^*\lambda_\chi U_F)\Big)\\ &= \sum_{\chi\in\hat{G}}(b_\chi\otimes\mathrm{id})(\lambda_\chi\otimes\lambda_\chi)\\ &= \sum_{\chi\in\hat{G}} b_\chi\lambda_\chi\otimes\lambda_\chi,\end{aligned}$$

as required.

[4]Recall that M_χ refers to left multiplication by χ.

Now, let $b \in \mathcal{B}(\ell^2(\hat{G}))$. Following the definition of the state, we have

$$\begin{aligned}\hat{\omega}(B) = \omega(U_F^* b U_F) &= \mathrm{Tr}_{L^2(G)}(U_F^* b U_F \lambda(h)) \\ &= \mathrm{Tr}_{\ell^2(\hat{G})}(b U_F \lambda(h) U_F^*).\end{aligned}$$

Observe that $U_F \lambda(h) U_F^* = \sum_{\chi \in \hat{G}} h_0(\chi) U_F \lambda(\chi) U_F^*$, where $\lambda(\chi)$ is the projection onto $\mathbb{C}\chi$. That is, for $\gamma \in \hat{G}$,

$$\lambda(\chi)\gamma = \delta_{\chi,\gamma}\chi = \big(|\chi \rangle\!\langle \chi|\big)\gamma$$

where $|\chi \rangle\!\langle \chi|$ is the Dirac outer product notation. Thus,

$$\begin{aligned}U_F \lambda(h) U_F^* &= \sum_{\chi \in \hat{G}} h_0(\chi) U_F |\chi \rangle\!\langle \chi| U_F^* \\ &= \sum_{\chi \in \hat{G}} h_0(\chi) |\delta_\chi \rangle\!\langle \delta_\chi|.\end{aligned}$$

So we have

$$\begin{aligned}\hat{\omega}\left(\sum_{\chi \in \hat{G}} b_\chi \lambda_\chi\right) &= \sum_{\chi \in \hat{G}} \mathrm{Tr}\left(b_\chi \lambda_\chi U_F \lambda(h) U_F^*\right) \\ &= \sum_{\chi \in \hat{G}} \mathrm{Tr}\Big(b_\chi \lambda_\chi \Big(\sum_{g \in \hat{G}} h_0(g) |\delta_g \rangle\!\langle \delta_g|\Big)\Big) \\ &= \sum_{\chi \in \hat{G}} \sum_{g \in \hat{G}} h_0(g) \mathrm{Tr}(b_\chi \lambda_\chi |\delta_g \rangle\!\langle \delta_g|) \\ &= \sum_{\chi \in \hat{G}} \sum_{g \in \hat{G}} h_0(g) \langle b_\chi \lambda_\chi \delta_g | \delta_g \rangle \\ &= \sum_{\chi \in \hat{G}} \sum_{g \in \hat{G}} h_0(g) \langle b_\chi \delta_{\chi \cdot g} | \delta_g \rangle.\end{aligned}$$

Note that $b_\chi \in \mathrm{Char}(\hat{G}) \subseteq \ell^\infty(\hat{G})$ is diagonal in the standard basis $\{\delta_g : g \in \hat{G}\}$. This means that $\langle b_\chi \delta_{\chi \cdot g} | \delta_g \rangle = 0$ unless $\chi \cdot g = g$ or in other words, $\langle b_\chi \delta_{\chi \cdot g} | \delta_g \rangle$ is non-zero exactly when $\chi = e$.

Therefore,

$$\begin{aligned}\hat{\omega}\left(\sum_{\chi\in\hat{G}} b_\chi \lambda_\chi\right) &= \sum_{g\in\hat{G}} h_0(g)\langle b_e\delta_g|\delta_g\rangle \\ &= \sum_{g\in\hat{G}} b_e(g)h_0(g) \\ &= \sum_{g\in\hat{G}} b_e(g)m(\{g\}) \\ &= \int_{\hat{G}} b_e(g)dm(g),\end{aligned}$$

as claimed. □

As before, we obtain two systems of C*-algebras $(\hat{Q}_t, t \in \mathfrak{T})$ and $(\hat{N}_t, t \in \mathfrak{T})$. For notational simplicity, we drop the "hat" notation in our C*-algebras moving forward, giving us systems $(Q_t, t \in \mathfrak{T})$ and $(N_t, t \in \mathfrak{T})$ with C*-completion of the algebraic direct limits and denoted by $\mathscr{Q}_0$ and $\mathscr{N}_0$, respectively. At each level, we have a state ω_t on M_t and we also have a limit state $\overline{\omega}$.

Before describing $\mathscr{Q}_0$ and $\mathscr{N}_0$ in more detail, we introduce some groups we would like to work with.

Notation 5.2.7.

(i) Let $\hat{G}_{fr}$ denote the group of all maps $g : \mathbb{D} \to \hat{G}$ for which there exists a s.d.p. $0 = d_1 < d_2 < \cdots < d_n < d_{n+1} = 1$ such that g is constant on each half-open interval $[d_j, d_{j+1}) \cap \mathbb{D}$ for $1 \leq j \leq n$.

(ii) Let $\hat{G}^{\mathbb{D}}$ denote the infinite product of groups $\prod_{d\in\mathbb{D}} \hat{G}$ endowed with the product topology, where each copy of $\hat{G}$ is equipped with the discrete topology.

The group $\hat{G}_{fr}$ acts on $\hat{G}^{\mathbb{D}}$ by pointwise multiplication. That is, the action $\hat{G}_{fr} \curvearrowright \hat{G}^{\mathbb{D}}$ is given by the formula

$$(g \cdot x)(d) = g(d)x(d) \tag{5.4}$$

for all $g \in \hat{G}_{fr}, x \in \hat{G}^{\mathbb{D}}, d \in \mathbb{D}$.

Let $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$ be the infinite (minimal) tensor product of the C*-algebra $\mathrm{Char}(\hat{G})$ over $\mathbb{D}$, which is (isomorphic to) the unique C*-completion of the algebraic direct limit of $\mathrm{Char}(\hat{G}^E)$ where E runs over the collection of finite subsets of $\mathbb{D}$. In this way, we see that the action (5.4) provides us with an action of $\hat{G}_{fr}$ on $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$, where we view the latter as an algebra of functions over $\hat{G}^{\mathbb{D}}$.

Notation 5.2.8.

(i) Let u_χ, $\chi \in \hat{G}_{fr}$ denote the classical embedding of $\hat{G}_{fr}$ inside $C^*(\hat{G}_{fr})$.

(ii) Let $m^{\mathbb{D}}$ denote the infinite tensor product of the measure $m \in \mathrm{Prob}(\hat{G})$, making it a probability measure on $\hat{G}^{\mathbb{D}}$.

We are now ready to describe $\mathscr{Q}_0$ and $\mathscr{N}_0$.

Lemma 5.2.9. *The C*-limit $\mathscr{Q}_0 := \varinjlim_{t\in\mathfrak{T}} Q_t$ is isomorphic to the (reduced) C*-algebra $C^*(\hat{G}_{fr})$. The C*-limit $\mathscr{N}_0 := \varinjlim_{t\in\mathfrak{T}} N_t$ is the infinite (minimal) tensor product $\otimes_{\mathbb{D}} Char(\hat{G})$.*

Proof. Let $\Psi_Q, \Psi_N : \mathcal{F} \to$ C*-alg be the tensor functors induced by $\hat{R}$ such that $\Psi_Q(1) = Q$ and $\Psi_N(1) = N$. By Lemma 5.2.6, those functors are defined by the maps $\hat{R}_Q(\lambda_\chi) = \lambda_\chi \otimes \lambda_\chi$ for $\chi \in \hat{G}$ and $\hat{R}_N(a) = a \otimes \mathrm{id}$ for $a \in N$, respectively. That $\mathscr{N}_0 = \otimes_{\mathbb{D}} N = \otimes_{\mathbb{D}} \mathrm{Char}(\hat{G})$ is immediate from Proposition 3.5.1. There is a bit more detail required to prove the statement about $\mathscr{Q}_0$.

Let $(\Gamma, \times, e)$ be the category of countable, discrete groups with direct products as the tensor structure. Consider the tensor functor $\Upsilon : \mathcal{F} \to \Gamma$ given by $\Upsilon(1) = \hat{G}$ and $\Upsilon(Y)(\chi) = (\chi, \chi) \in \hat{G} \times \hat{G}$. This gives us a directed system of groups $(\hat{G}_t, t \in \mathfrak{T})$. Fix a tree $t \in \mathfrak{T}$ with s.d.p. $0 = d_1 < d_2 < \cdots < d_n < d_{n+1} = 1$ and embed $\hat{G}_t$ into $\hat{G}_{fr}$ via

$$j_t(g)(d) = g_i,$$

for $d_i \leq d < d_{i+1}$. The family of maps $(j_t, t \in \mathfrak{T})$ is compatible with $(\hat{G}_t, t \in \mathfrak{T})$, which gives us an embedding of $\varprojlim_t \hat{G}_t$ into $\hat{G}_{fr}$. Using the definition of $\hat{G}_{fr}$, it is clear that this map is surjective.

Now, consider the tensor functor $C_r^* : \Gamma \to$ C*-alg, which takes a countable, discrete group to its reduced group C*-algebra. As functors, $\Psi_Q = C_r^* \circ \Upsilon$ (cf. the analogous claim at the end of Section 4.1.2). Therefore $\mathscr{Q}_0 \cong C_r^*(\hat{G}_{fr})$. □

We would like to describe a WOT-dense $*$-subalgebra of $\mathscr{M}$. In [6], this algebra is chosen to be the algebraic crossed-product $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes_{\mathrm{alg}} \hat{G}_{fr}$ with the action of $\hat{G}_{fr}$ on $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$ coming from (5.4). However, we will show that the argument holds for the reduced crossed-product (and we will make use of the fact that this coincides with the full crossed-product as $\hat{G}_{fr}$ is a discrete, abelian group).

Proposition 5.2.10. *[6, Proposition 3.8] Consider the crossed-product $\otimes_{\mathbb{D}} Char(\hat{G}) \rtimes \hat{G}_{fr}$ for the action* (5.4)*. This embeds as a WOT-dense $*$-subalgebra of $\mathscr{M}$. Moreover,*

$$\overline{\omega}\Big(\sum_{\chi\in\hat{G}_{fr}} b_\chi u_\chi\Big) = \int_{\hat{G}^{\mathbb{D}}} b_e(t) dm^{\mathbb{D}}(t),$$

where $b_\chi \in \otimes_{\mathbb{D}} Char(\hat{G})$ is finitely supported.

Proof. Let $\iota_t : \mathrm{Char}(\hat{G}_t) \to \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$ and $j_t : \hat{G}_t \to \hat{G}_{fr}$ be the embeddings given by the directed system of C*-algebras $\mathrm{Char}(\hat{G}_t)$ and groups $\hat{G}_t$, as constructed in the previous lemma. To get the desired embedding requires a bit of work. We begin with making some notation explicit, in order to keep everything as clear as possible.

Let α denote the action of $\hat{G}_{fr}$ on $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$ and let π be a non-degenerate, faithful $*$-representation of the C*-algebra $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$ on some Hilbert space H. We view $(\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})) \rtimes_r \hat{G}_{fr} \subseteq \mathcal{B}(\ell^2(\hat{G}_{fr}) \otimes H)$. Let $u : \hat{G}_{fr} \to \mathcal{U}(\ell^2(\hat{G}_{fr}) \otimes H)$ be the embedding arising from the left regular representation, i.e.,

$$u_g = \lambda(g) \otimes 1_H, \quad g \in \hat{G}_{fr}.$$

We define a $*$-representation $\pi_\alpha : \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \to \mathcal{B}(\ell^2(\hat{G}_{fr}) \otimes H)$ such that

$$\pi_\alpha(b)(\delta_g \otimes \xi) = \delta_g \otimes \pi(\alpha_{g^{-1}}(b))\xi,$$ [5]

for $b \in \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$, $g \in \hat{G}_{fr}$ and $\xi \in H$.

For a fixed $t \in \mathfrak{T}$, we want to induce a $*$-representation

$$\kappa_t : \mathrm{Char}(\hat{G}_t) \rtimes_r \hat{G}_t \to \mathcal{B}(\ell^2(\hat{G}_{fr}) \otimes H)$$

via the canonical embeddings ι_t and j_t. To do this, we need to show the covariance condition between the appropriate representations, namely $\pi_\alpha \circ \iota_t$ and $\lambda \circ j_t$, on $\ell^2(\hat{G}_{fr}) \otimes H$.

Fix $b_t \in \mathrm{Char}(\hat{G}_t)$, $g \in \hat{G}_t$. Then for all $h \in \hat{G}_{fr}$, $\xi \in H$, we have,

$$\pi_\alpha \circ \iota_t(\alpha_g^t(b_t))(\delta_h \otimes \xi) = \delta_h \otimes \pi(\alpha_{h^{-1}}(\iota_t(\alpha_g^t(b_t))))\xi.$$

Notice that

$$\begin{aligned}\iota_t(\alpha_g^t(b_t)) &= \iota_t(g \cdot b_t)\\ &= j_t(g) \cdot \iota_t(b_t)\\ &= \alpha_{j_t(g)}(\iota_t(b_t)).\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_{h^{-1}}(\iota_t(\alpha_g^t(b_t))) &= \alpha_{h^{-1}}(\alpha_{j_t(g)}(\iota_t(b_t)))\\ &= \alpha_{h^{-1}\cdot j_t(g)}(\iota_t(b_t)).\end{aligned}$$

[5] This coincides with the definition of π_α in the reduced crossed-product, Definition 2.4.7

This implies that

$$\begin{aligned}\pi_\alpha(\iota_t(\alpha_{g^t}(b_t)))(\delta_h \otimes \xi) &= \delta_h \otimes \pi(\alpha_{h^{-1}\cdot j_t(g)}(\iota_t(b_t)))\xi \\ &= \delta_h \otimes \pi(\alpha_{(j_t(g)^{-1}\cdot h)^{-1}}(\iota_t(b_t)))\xi \\ &= (\lambda(j_t(g)) \otimes 1)[\delta_{j_t(g)^{-1}\cdot h} \otimes \pi(\alpha_{(j_t(g)^{-1}\cdot h)^{-1}}(\iota_t(b_t)))\xi] \\ &= (\lambda(j_t(g)) \otimes 1)\pi_\alpha(\iota_t(b_t))[\delta_{j_t(g)^{-1}\cdot h} \otimes \xi] \\ &= (\lambda(j_t(g)) \otimes 1)\pi_\alpha(\iota_t(b_t))(\lambda(j_t(g)^{-1}) \otimes 1)[\delta_h \otimes \xi].\end{aligned}$$

Thus, $(\pi_\alpha \circ \iota_t, \lambda \circ j_t)$ is a covariant representation of $\hat{G}_t \curvearrowright \mathrm{Char}(\hat{G}_t)$ on $\ell^2(\hat{G}_{fr}) \otimes H$. This induces the appropriate $*$-representation κ_t where

$$\kappa_t\left(\sum_g b_g u_g\right) = \sum_g \iota_t(b_g) j_t(u_g),$$

for $b : \hat{G}_t \to \mathrm{Char}(\hat{G}_t)$ with finite support. Further, since the action $\hat{G}_t \curvearrowright \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$ restricts to the canonical action $\hat{G}_t \curvearrowright \mathrm{Char}(\hat{G}_t)$, we have that $\mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t$ embeds into $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes \hat{G}_t$ canonically via the map

$$\sum_g b_g u_g \mapsto \sum_g \iota_t(b_g) u_g.$$

Similarly, as $j_t(\hat{G}_t)$ is an open subgroup of $\hat{G}_{fr}$, we have an embedding of $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes \hat{G}_t$ into $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes \hat{G}_{fr}$ (see [4, II.10.3.17] for details on the functoriality of the crossed-products).

As κ_t is the composition of these two canonical embeddings, it is isometric. Moreover, given trees $s \leq t$, the following diagram commutes.

$$\begin{tikzcd} \mathrm{Char}(\hat{G}_s) \rtimes \hat{G}_s \arrow[r, "\kappa_s"] \arrow[d, "\iota_t^s"] & \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes \hat{G}_{fr} \\ \mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t \arrow[ur, "\kappa_t"] & \end{tikzcd}$$

Therefore, we obtain a $*$-homomorphism

$$\varinjlim_{t \in \mathfrak{T}} \kappa_t =: \kappa : \varinjlim_{t \in \mathfrak{T}} \mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t \to \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes \hat{G}_{fr}.$$

Clearly κ is isometric on the dense subset $\cup_{t \in \mathfrak{T}}\mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t$ inside the direct limit, so κ is isometric on all of $\varinjlim_{t \in \mathfrak{T}} \mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t$. Moreover the range is dense, since

$$\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) = \overline{\cup_t \iota_t(\mathrm{Char}(\hat{G}_t))}$$

and

$$\hat{G}_{fr} = \varprojlim_{t\in\mathfrak{T}} \hat{G}_t.$$

Thus $\mathscr{M}_0 = \varinjlim_{t\in\mathfrak{T}} \mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t \cong \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes \hat{G}_{fr}$.

Let K denote the embedding obtained by taking the inverse of κ and extending $\cup_{t\in\mathfrak{T}}\mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t$ to $\mathscr{M}_0$. Further, let $\pi : \mathscr{M}_0 \to \mathcal{B}(L^2(\mathscr{M}_0, \overline{\omega}))$ be the representation associated with the GNS construction used to obtain $\mathscr{M}$. That is,

$$\begin{aligned}\mathscr{M} = \pi(\mathscr{M}_0)'' &= \overline{\pi(\mathscr{M}_0)}^{\mathrm{SOT}} \\ &= \overline{\pi(\mathscr{M}_0)}^{\mathrm{WOT}} \\ &= \overline{\pi(\mathscr{M}_0)}^{w^*}.\end{aligned}$$

In other words, $\mathscr{M}_0$ is WOT-dense in $\mathscr{M}$. So K is the desired WOT-dense embedding of $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes \hat{G}_{fr}$ inside $\mathscr{M}$.

Now, given $\chi \in \hat{G}_t$ and $b_\chi \in \mathrm{Char}(\hat{G}_t)$, it is clear from the definition of ω_t and the computation done in Lemma 5.2.6 that we have $\omega_t(b_\chi\lambda_\chi) = \delta_{\chi,e}\int_{\hat{G}_t} b_e(g)dm_t(g)$. Now, suppose we have $\sum_{\chi\in\hat{G}_{fr}} b_\chi u_\chi \in \mathscr{M}_0$ with b_χ finitely supported. Then

$$\overline{\omega}\left(\sum_{\chi\in\hat{G}_{fr}} b_\chi u_\chi\right) = \int_{\hat{G}^{\mathbb{D}}} b_e(t)dm^{\mathbb{D}}(t),$$

as required. □

Recall that at each level t, the algebraic-crossed-product $\mathrm{Char}(\hat{G}_t) \rtimes_{\mathrm{alg}} \hat{G}_t$ is a norm-dense subset of the crossed-product C*-algebra $\mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t$ living inside $\mathcal{B}(\ell^2(\hat{G}_t))$. Therefore, the algebraic limit of the $*$-subalgebras $\mathrm{Char}(\hat{G}_t) \rtimes_{\mathrm{alg}} \hat{G}_t$ (which are norm-dense in $\mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t$) is also WOT-dense in $\mathscr{M}$, so we are justified in using the algebraic crossed-product when it suffices to show a given statement for a WOT-dense subset of $\mathscr{M}$.

Again, we have omitted certain physically motivated details; we state them here for the sake of completeness. First, the authors show in [6, Proposition 3.8] that there is indeed a restriction of the Jones action to an action $V \curvearrowright \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes_{\mathrm{alg}} \hat{G}_{fr}$. Second, in [6, Proposition 3.9], they show that there is a well-defined group action $\gamma : \Gamma \curvearrowright \mathscr{M}$ in our dual picture.

Now, let $(\pi, \mathfrak{H}, \Omega)$ be the triple associated to the GNS completion of $(\mathscr{M}_0, \overline{\omega})$. In order to more fully understand $\mathscr{M}$, we first want to describe the action of the $*$-algebra $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes_{\mathrm{alg}} \hat{G}_{fr}$ on $\mathfrak{H}$.

Definition 5.2.11. Given $g \in \hat{G}_{fr}$, let $g_*m^{\mathbb{D}}$ be the *pushforward measure* defined by the formula

$$g_*m^{\mathbb{D}}(C) = m^{\mathbb{D}}(g^{-1}\cdot C) = \prod_{d\in\mathbb{D}} m^{\mathbb{D}}(g(d)^{-1}C(d)),$$

for any measurable cylinder $C = \prod_{d\in\mathbb{D}} C(d)$.

Lemma 5.2.12. *[6, Lemma 3.10] There exists a unitary transformation*

$$W : \mathfrak{H} \to \bigoplus_{k\in\hat{G}_{fr}} L^2(\hat{G}^{\mathbb{D}}, k_* m^{\mathbb{D}})$$

such that

$$Ad(W)\pi(a)(\xi_k)_{k\in\hat{G}_{fr}} = (a\xi_k)_{k\in\hat{G}_{fr}}$$

and

$$Ad(W)\pi(u_g)(\xi_k)_{k\in\hat{G}_{fr}} = (\xi^g_{g^{-1}k})_{k\in\hat{G}_{fr}}$$

for all $a \in \otimes_{\mathbb{D}} Char(\hat{G}), g \in \hat{G}_{fr}$, *and vectors* $\xi = (\xi_k)_{k\in\hat{G}_{fr}}$, *where* $\xi^g_{g^{-1}k}(x) := \xi_{g^{-1}k}(g^{-1}x)$.

Proof. For simplicity, let us set $B := \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$. We can think of B as a space of bounded, measurable functions $\hat{G}^{\mathbb{D}} \to \mathbb{C}$, so that $B \subseteq L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})$. By the definition of the GNS construction and the previous proposition, we have that $\mathfrak{H}$ is the closure of $B \rtimes_{\mathrm{alg}} \hat{G}_{fr}$ with respect to the inner product $\langle\xi|\eta\rangle := \overline{\omega}(\eta^*\xi)$. As a vector space, we can view $B \rtimes_{\mathrm{alg}} \hat{G}_{fr}$ as the direct sum $\oplus_{k\in\hat{G}_{fr}} Bu_k$, where $u_k, k \in \hat{G}_{fr}$ is the usual embedding of $\hat{G}_{fr}$ in $C^*(\hat{G}_{fr})$.

Note that for $\xi, \eta \in B$ and $g, k \in \hat{G}_{fr}$ we have

$$\langle \xi u_g | \eta u_k\rangle = \begin{cases} \int_{\hat{G}^{\mathbb{D}}} \overline{\eta}\xi d(g_* m^{\mathbb{D}}) & \text{if } g = k \\ 0 & \text{otherwise} \end{cases}.$$

By orthogonality, we have a direct sum at a the Hilbert space level, $\mathfrak{H} \cong \oplus_{k\in\hat{G}_{fr}} \mathfrak{H}_k$, where $\mathfrak{H}_k = L^2(\hat{G}^{\mathbb{D}}, k_* m^{\mathbb{D}})$. Let W be the unitary transformation induced by this isomorphism, namely

$$W : \sum_k (\xi_k u_k) \mapsto \sum_k \xi_k.$$

□

In order to achieve our main goal of completely describing $(\mathscr{M}, \overline{\omega})$, we will need to make use of a standard (albeit non-trivial) result from measure theory. We do not prove it here as it is fairly involved and not illuminating for our purposes.

Before stating the theorem, let us recall some useful definitions. Two measures on a measure space are called

(i) *singular* if their supports are disjoint,

(ii) *equivalent* if they have the same null sets.

Theorem 5.2.13 (Kakutani's Theorem[6]). *[27] Let X be a countable measurable space with two infinite families of probability measures $\mu_d, \nu_d, d \in \mathbb{D}$. Assume that μ_d is equivalent to ν_d for any $d \in \mathbb{D}$ and put $\mu := \otimes_{d\in\mathbb{D}}\mu_d$ and $\nu := \otimes_{d\in\mathbb{D}}\nu_d$ the infinite tensor products of measures defined on the product σ-algebra of the product space $X^{\mathbb{D}}$. Denote by $\rho(\mu_d, \nu_d)$ the quantity $\sum_{x\in X}\sqrt{\mu_d(x)\nu_d(x)} \in (0, 1]$. Then the measures μ and ν are either equivalent or singular with each other. Moreover, they are equivalent if and only if*

$$-\sum_{d\in\mathbb{D}} \log(\rho(\mu_d, \nu_d)) < +\infty.$$

We can deduce the following lemma, whose detailed proof is available in [6].

Lemma 5.2.14. *[6, Lemma 3.12] The measures $g_*m^{\mathbb{D}}$ and $g'_*m^{\mathbb{D}}$ are mutually singular if and only if there exists $d \in \mathbb{D}$ such that $g(d)_*m \neq g'(d)_*m$ where $g, g' \in \hat{G}_{fr}$. Otherwise the measures are equal. If the group $\hat{G}$ is torsion free, then the family of measures $(g_*m^{\mathbb{D}}, g \in \hat{G}_{fr})$ are mutually singular.*

Definition 5.2.15. We define a subgroup N of $\hat{G}$ by

$$N := \{g \in \hat{G} : g_*m = m\}.$$

We define a subgroup N_{fr} of $\hat{G}_{fr}$ by

$$\{g \in \hat{G}_{fr} : g(d) \in N \text{ for all } d \in \mathbb{D}\}.$$

Let $\sigma : \hat{G}_{fr}/N_{fr} \to \hat{G}_{fr}$ be a section and consider the cocycle $\kappa : \hat{G}_{fr} \times \hat{G}_{fr}/N_{fr}$ defined by $(g, \gamma) \mapsto \sigma(g\gamma)^{-1}g\sigma(\gamma)$. Define a group action $\hat{G}_{fr} \curvearrowright (\hat{G}^{\mathbb{D}} \times \hat{G}_{fr}/N_{fr})$ by

$$g \cdot (z, \gamma) := (\kappa(g, \gamma)z, g\gamma). \tag{5.5}$$

Equip the space $\hat{G}^{\mathbb{D}} \times \hat{G}_{fr}/N_{fr}$ with the measure $m^{\mathbb{D}} \otimes \mu_c$ where μ_c is the counting measure (i.e., the discrete Haar measure). Since the action $N_{fr} \curvearrowright (\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})$ is p.m.p., the action $\hat{G}_{fr} \curvearrowright (\hat{G}^{\mathbb{D}} \times \hat{G}_{fr}/N_{fr}, m^{\mathbb{D}} \otimes \mu_c)$ is measure preserving.

We are ready to define the crossed-product von Neumann algebra of interest, $\mathscr{B} := L^\infty(\hat{G}^{\mathbb{D}} \times \hat{G}_{fr}/N_{fr}, m^{\mathbb{D}} \otimes \mu_c)\bar{\rtimes}\hat{G}_{fr}$. Denote the unitaries of $\mathscr{B}$ implementing the action by v_s for $s \in \hat{G}_{fr}$. There is a unique normal state on $\mathscr{B}$ such that

$$\overline{\omega}_{\mathscr{B}}\Big(\sum_{g\in\hat{G}_{fr}} A_g v_g\Big) = \int_{\hat{G}^{\mathbb{D}}} A_e(z, \bar{e})dm^{\mathbb{D}}(z),$$

where $A_g \in L^\infty(\hat{G}^{\mathbb{D}} \times \hat{G}_{fr}/N_{fr}, m^{\mathbb{D}} \otimes \mu_c)$ for $g \in \hat{G}_{fr}$.

[6]This particular formulation of Kakutani's theorem comes from [6, Theorem 3.11]

Additionally, there is an action of Thompson's group, $\alpha_{\mathscr{B}} : V \curvearrowright \mathscr{B}$ which leaves the state $\overline{\omega}_{\mathscr{B}}$ invariant (the details are available in the paragraph preceding the statement of [6, Theorem 3.13]).

Lemma 5.2.16. *The von Neumann algebra $\mathscr{B}$ does not have any Type III direct summand.*

Proof. First, note that the group N_{fr} acts in a probability measure preserving way on $(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})$ and so the action $\alpha : \hat{G}_{fr} \curvearrowright (\hat{G}^{\mathbb{D}} \times \hat{G}_{fr}/N_{fr}, m^{\mathbb{D}} \otimes \mu_c)$ is measure preserving. Moreover, we can define a corresponding action on

$$L^\infty(X, \mu) := L^\infty(\hat{G}^{\mathbb{D}} \times \hat{G}_{fr}/N_{fr}, m^{\mathbb{D}} \otimes \mu_c)$$

by $g \cdot x(s,t) = x(\alpha_{g^{-1}}(s,t))$.

Now, let ϕ denote integration with respect to $m^{\mathbb{D}} \otimes \mu_c$. This is a normal, semifinite trace on $L^\infty(X, \mu)$ and it is $\hat{G}_{fr}$-invariant (by invariance of $m^{\mathbb{D}} \otimes \mu_c$). Let $\Phi : L^\infty(X, \mu) \bar{\rtimes} \hat{G}_{fr} \to L^\infty(X, \mu)$ be the canonical conditional expectation (see [38, Chapter X, Theorem 1.17]). Then $\phi \circ \Phi$ is a normal, semifinite trace on $L^\infty(X, \mu) \bar{\rtimes} \hat{G}_{fr}$, which means $L^\infty(X, \mu) \bar{\rtimes} \hat{G}_{fr}$ is semifinite, i.e., it has no Type III component. □

We are ready to describe the von Neumann algebra $(\mathscr{M}, \overline{\omega})$ as a crossed-product von Neumann algebra when G is a compact, abelian, separable group and $m \in \mathrm{Prob}(\hat{G})$. Furthermore, we can be even more explicit in our description in two "opposite" cases. In the first case, we will further specify that $\hat{G}$ is *torsion free* meaning that the only element of $\hat{G}$ with finite order is the identity. In the second case, we will take G to be a finite group.

Theorem 5.2.17. *[6, Theorem 3.13] Let G be a compact, abelian, separable group and $m \in Prob(\hat{G})$ a strictly positive probability measure. There is a state-preserving von Neumann algebra isomorphism $\psi : (\mathscr{M}, \overline{\omega}) \to (\mathscr{B}, \overline{\omega}_{\mathscr{B}})$. In particular, $\mathscr{M}$ does not have any Type III component. Moreover, the Jones action $\alpha_\Psi : V \curvearrowright \mathscr{M}$ preserves the state $\overline{\omega}$ and is conjugate to the action $\alpha_{\mathscr{B}}$ via ψ. That is, $Ad(\psi) \circ \alpha_\Psi = \alpha_{\mathscr{B}}$.*

(i) If $\hat{G}$ is torsion free, then

$$(\mathscr{M}, \overline{\omega}) \cong (L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}}) \bar{\otimes} \mathcal{B}(\ell^2(\hat{G}_{fr})), m^{\mathbb{D}} \otimes \langle \cdot \delta_e | \delta_e \rangle),$$

which is a Type I_∞ von Neumann algebra with a diffuse center and equipped with a non-faithful state.

(ii) If G is a finite group and m is $\hat{G}$-invariant, then

$$(\mathscr{M}, \overline{\omega}) \cong \mathcal{R},$$

where $\mathcal{R}$ is the hyperfinite Type II_1 factor equipped with its trace.

Proof. Let $(X,\mu) := (\hat{G}^{\mathbb{D}} \times \hat{G}_{fr}/N_{fr}, m^{\mathbb{D}} \otimes \mu_c)$. Recall that we have a measure-preserving action $\hat{G}_{fr} \curvearrowright (X,\mu)$ given by

$$g \cdot (z,\gamma) = (\sigma(g\gamma)^{-1} g\sigma(\gamma)z, g\gamma),$$

for $g \in \hat{G}_{fr}, z \in \hat{G}^{\mathbb{D}}, \gamma \in \hat{G}_{fr}/N_{fr}$. If $\xi : X \to \mathbb{C}$ is any map and $g \in \hat{G}_{fr}$, we write $\xi^g := \xi(g^{-1}x)$, $x \in X$ for this action.

We can write elements of $\mathscr{B}$ as formal sums $\sum_{g\in\hat{G}_{fr}} A_g v_g$ with $A_g \in L^\infty(X,\mu)$. Moreover, we can assume that $\mathscr{B}$ is represented on the Hilbert space $K := L^2(X,\mu) \otimes \ell^2(\hat{G}_{fr})$ and acts in the following way[7] :

$$v_g(\xi \otimes \delta_k) := \xi^g \otimes \delta_{gk},$$

for $g, k \in \hat{G}_{fr}$, $\xi \in L^2(X,\mu)$ and

$$A(\xi \otimes \delta_k) = (A\xi) \otimes \delta_k,$$

where $(A\xi)(y) := A(y)\xi(y)$ and $A \in L^\infty(X,\mu)$.

Given $a \in \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$, define the map $\psi(a) \in L^\infty(X,\mu)$ by $\psi(a)(z,\gamma) := a(\sigma(\gamma)z)$ for any $z \in \hat{G}^{\mathbb{D}}$, $\gamma \in \hat{G}_{fr}/N_{fr}$.

We have that

$$\begin{aligned}\psi(\overline{a})(z,\gamma) = \overline{a}(\sigma(\gamma)z) &= \overline{a(\sigma(\gamma)z)} \\ &= \overline{\psi(a)(z,\gamma)} \\ &= \overline{\psi(a)}(z,\gamma).\end{aligned}$$

Moreover, setting $a^g(y) := a(g^{-1}y)$ for $a \in \mathrm{Char}(\hat{G})$, $g \in \hat{G}_{fr}$, and $y \in \hat{G}^{\mathbb{D}}$, we have that

$$\begin{aligned}\psi(a^g)(z,\gamma) = a^g(\sigma(\gamma)z) &= a(g^{-1}(\sigma(\gamma)z)) \\ &= \psi(a)(g^{-1}(\sigma(\gamma)z)) \\ &= (\psi(a))^g(z,\gamma).\end{aligned}$$

In other words, we have $\overline{\psi(a)} = \psi(\overline{a})$ and $\psi(a^g) = (\psi(a))^g$ for any $a \in \mathrm{Char}(\hat{G})$ and $g \in \hat{G}_{fr}$.

In turn, we define the $*$-morphism $\psi : \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes_{\mathrm{alg}} \hat{G}_{fr} \to \mathscr{B}$ by

$$\sum_{g\in\hat{G}_{fr}} a_g u_g \mapsto \sum_{g\in\hat{G}_{fr}} \psi(a_g) v_g.$$

[7]This is called the "implemented crossed-product" in [25, Chapter 13]. This action may appear to be non-standard to some operator algebraists, who are more familiar with what those authors call the "abstract crossed-product".

So we have

$$\begin{aligned}
\overline{\omega}_{\mathscr{B}} \circ \psi\Big(\sum_g a_g u_g\Big) &= \overline{\omega}_{\mathscr{B}}\Big(\sum_g \psi(a_g) v_g\Big) \\
&= \int_{\hat{G}^{\mathbb{D}}} \psi(a_e)(z, \bar{e}) dm^{\mathbb{D}}(z) \\
&= \int_{\hat{G}^{\mathbb{D}}} a_e(\sigma(\bar{e})z) dm^{\mathbb{D}}(z) \\
&= \int_{\hat{G}^{D}} a_e(z)\sigma(\bar{e})_* m^{\mathbb{D}}(z) \\
&= \int_{\hat{G}^{\mathbb{D}}} a_e(z) dm^{\mathbb{D}}(z) \\
&= \overline{\omega}\Big(\sum_g a_g u_g\Big),
\end{aligned}$$

where the second last line follows from the fact that $\sigma(\bar{e}) \in N_{fr}$. Therefore, ψ is a densely defined, state-preserving $*$-morphism from $(\mathscr{M}, \overline{\omega})$ to $(\mathscr{B}, \overline{\omega}_{\mathscr{B}})$. We want to show that ψ extends to a normal isomorphism from $\mathscr{M}$ onto $\mathscr{B}$.

Recall that $K = L^2(X, \mu) \otimes \ell^2(\hat{G}_{fr})$, which we can canonically identify as $L^2(\hat{G}, m^{\mathbb{D}}) \otimes \ell^2(\hat{G}_{fr}/N_{fr}) \otimes \ell^2(\hat{G}_{fr})$. Under this identification, we have

$$v_g(\eta \otimes \delta_\gamma \otimes \delta_k) = \eta^{\kappa(g,\gamma)} \otimes \delta_{g\gamma} \otimes \delta_{gk}, \text{ and}$$

$$A(\eta \otimes \delta_\gamma \otimes \delta_k) = (A(\cdot, \gamma)\eta) \otimes \delta_\gamma \otimes \delta_k,$$

for $g, k \in \hat{G}_{fr}$, $\eta \in L^2(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})$, $\gamma \in \hat{G}_{fr}/N_{fr}$, and $A \in L^\infty(X, \mu)$. Thus, if $a \in \otimes_{\mathbb{D}} \mathrm{Char}(\hat{G})$, then

$$\psi(a)(\eta \otimes \delta_\gamma \otimes \delta_k) = (a(\sigma(\gamma)\cdot)\eta) \otimes \delta_\gamma \otimes \delta_k.$$

Now, for any $\gamma \in \hat{G}_{fr}/N_{fr}$, define the subspace

$$K_\gamma := \left\{ \sum_{k \in \hat{G}_{fr}} \eta_k \otimes \delta_{k\gamma} \otimes \delta_k : \eta_k \in L^2(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}}), \sum_k \|\eta_k\|^2 < \infty \right\} \subseteq K.$$

This Hilbert space is closed under the action of $\mathscr{B}$, so it is a $\mathscr{B}$-module. We can decompose K into $\mathscr{B}$-modules as follows:

$$K = \oplus_{\gamma \in \hat{G}_{fr}/N_{fr}} K_\gamma.$$

Now let us consider the transformation $U_\gamma : K_\gamma \to \oplus_{k\in\hat{G}_{fr}} L^2(\hat{G}^{\mathbb{D}}, k_* m^{\mathbb{D}})$ given by

$$\sum_{k\in\hat{G}_{fr}} \eta_k \otimes \delta_{k\gamma} \otimes \delta_k \mapsto (\eta_k^{\sigma(k\gamma)})_{k\cdot\sigma(\gamma)\in\hat{G}_{fr}}.$$

This is a unitary transformation:

$$\begin{aligned}
\|U_\gamma(\eta \otimes \delta_{k\gamma} \otimes \delta_k)\|^2 &= \|\eta^{\sigma(k\gamma)}\|^2_{L^2(\hat{G}^{\mathbb{D}},(k\cdot\sigma(\gamma))_* m^{\mathbb{D}})} \\
&= \int_{\hat{G}^{\mathbb{D}}} |\eta^{\sigma(k\gamma)}(x)|^2 d(k\cdot\sigma(\gamma))_* m^{\mathbb{D}} \\
&= \int_{\hat{G}^{\mathbb{D}}} |\eta^{\sigma(k\gamma)}(k\cdot\sigma(\gamma)\cdot x)|^2 dm^{\mathbb{D}} \\
&= \int_{\hat{G}^{\mathbb{D}}} |\eta^{\sigma(k\gamma)}(\sigma(k\gamma)^{-1}\cdot k\cdot\sigma(\gamma)\cdot x)|^2 dm^{\mathbb{D}} \\
&= \int_{\hat{G}^{\mathbb{D}}} |\eta(x)|^2 dm^{\mathbb{D}} \\
&= \|\eta \otimes \delta_{k\gamma} \otimes \delta_k\|^2_{K_\gamma}.
\end{aligned}$$

Moreover, the map satisfies an "intertwining" relationship. Let $(\pi, \mathcal{H})$ be the GNS representation associated to $(\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes_{\mathrm{alg}} \hat{G}_{fr}, \overline{\omega})$ described in Lemma 5.2.12. Then for any $\xi \in K_\gamma$ and $a \in \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$, $g \in \hat{G}_{fr}$,

$$\begin{aligned}
U_\gamma\psi(au_g)(\eta \otimes \delta_{k\gamma} \otimes \delta_k) &= U_\gamma((a(\sigma(gk\gamma)\cdot)\eta^{\kappa(g,k\gamma)}) \otimes \delta_{gk\gamma} \otimes \delta_{gk}) \\
&= (0, 0, \ldots, (a(\sigma(gk\gamma)\cdot)\eta^{\kappa(g,k\gamma)})^{\sigma(gk\gamma)}, 0, \ldots),
\end{aligned}$$

where the non-zero entry is in the $gk\cdot\sigma(\gamma)$th slot. In particular, this entry is

$$\begin{aligned}
(a(\sigma(gk\gamma)\cdot)\eta^{\kappa(g,k\gamma)})^{\sigma(gk\gamma)}(x) &= (a(\sigma(gk\gamma)\cdot)\eta^{\kappa(g,k\gamma)})(\sigma(gk\gamma)^{-1}x) \\
&= a(x)\eta^{\kappa(g,k\gamma)}(\sigma(gk\gamma)^{-1}x) \\
&= a(x)\eta(\sigma(k\gamma)^{-1}\cdot g^{-1}\cdot x) \\
&= a(\eta^{g\sigma(k\gamma)})(x).
\end{aligned}$$

So $U_\gamma\psi(au_g)(\eta \otimes \delta_{k\gamma} \otimes \delta_k) = (0, 0, \ldots, a\eta^{g\sigma(k\gamma)}, 0, \ldots)$.

Similarly, we have

$$\begin{aligned}
\pi(au_g)U_\gamma\xi &= \pi(au_g)U_\gamma(\eta \otimes \delta_{k\gamma} \otimes \delta_k) \\
&= \pi(au_g)(0, 0, \ldots, \eta^{\sigma(k\gamma)}, 0, \ldots) \\
&= \pi(a)(\pi(u_g)(0, 0, \ldots, \eta^{\sigma(k\gamma)}, 0, \ldots)) \\
&= \pi(a)(0, 0, \ldots, (\eta^{\sigma(k\gamma)})^g, 0, \ldots) \\
&= (0, 0, \ldots, a(\eta^{\sigma(k\gamma)})^g, 0, \ldots).
\end{aligned}$$

So, $U_\gamma \psi(au_g)\xi = \pi(au_g)U_\gamma\xi$. Therefore, we see that U_γ provides an isomorphism between the $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes_{\mathrm{alg}} \hat{G}_{fr}$-modules K_γ and the GNS module $\mathfrak{H}$. Therefore, the $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \rtimes_{\mathrm{alg}} \hat{G}_{fr}$-module K is isomorphic to the direct sum $\oplus_{\hat{G}_{fr}/N_{fr}}\mathfrak{H}$. Therefore, ψ extends to an isomorphism from $\mathscr{M}$ onto the WOT-closure of the range of ψ.

It remains to show that the range of ψ is WOT-dense in $\mathscr{B}$. The set of unitary operators v_g, $g \in \hat{G}_{fr}$ is necessarily in the range of ψ by the definition given above. Therefore, we only need to prove that the WOT-closure of $\psi(\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}))$ is equal to $L^\infty(X,\mu)$.

For each $g \in \hat{G}_{fr}$, define the representation $\pi_g : \otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}) \to \mathcal{B}(L^2(\hat{G}^{\mathbb{D}}, g_* m^{\mathbb{D}}))$ where the maps act by pointwise multiplication. Since $\mathrm{Char}(\hat{G})$ is WOT-dense in $\ell^\infty(\hat{G})$ and $L^\infty(\hat{G}^{\mathbb{D}}, g_* m^{\mathbb{D}})$ is obtained via the infinite tensor product $\otimes_{d\in\mathbb{D}}(\ell^\infty(\hat{G}), g(d)_* m)$, then the range of π_g must be WOT-dense inside $L^\infty(\hat{G}^{\mathbb{D}}, g_* m^{\mathbb{D}})$. By Lemma 5.2.14, the family of measures $(\sigma(\gamma)_* m^{\mathbb{D}}, \gamma \in \hat{G}_{fr}/N_{fr})$ are mutually singular. This implies that the representations $(\pi_{\sigma(\gamma)}, \gamma \in \hat{G}_{fr}/N_{fr})$ are mutually disjoint (see [1, Theorem 2.2.2]). Moreover, the WOT-completion of $\oplus_{\gamma\in\hat{G}_{fr}/N_{fr}}\pi_{\sigma(\gamma)}(\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}))$ is equal to the direct sum $\oplus_{\gamma\in\hat{G}_{fr}/N_{fr}} L^\infty(\hat{G}^{\mathbb{D}}, \sigma(\gamma)_* m^{\mathbb{D}})$, as the former is the WOT-completion of a sum of disjoint representations and the latter is the sum of the WOT-completions, (again, we refer the reader to [1, Section 2]). Identifying $L^\infty(X,\mu)$ as the direct sum $\oplus_{\gamma\in\hat{G}_{fr}/N_{fr}} L^\infty(\hat{G}^{\mathbb{D}}, \sigma(\gamma)_* m^{\mathbb{D}})$, we can see that, as a $L^\infty(X,\mu)$-module, K is isomorphic to a direct sum of the canonical module $\oplus_{\gamma\in\hat{G}_{fr}/N_{fr}} L^2(\hat{G}^{\mathbb{D}}, \sigma(\gamma)_* m^{\mathbb{D}})$. Under this identification, the range of $\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G})$ under ψ is the diagonal subalgebra $\oplus_{\gamma\in\hat{G}_{fr}/N_{fr}}\pi_{\sigma(\gamma)}(\otimes_{\mathbb{D}}\mathrm{Char}(\hat{G}))$. Therefore, $\psi(\otimes_{\mathbb{D}}(\hat{G}))$ is WOT-dense in $L^\infty(X,\mu)$, as required. This implies that the range of ψ is WOT-dense in $\mathscr{B}$.

We showed above that $\mathscr{B}$ has no Type III direct summands. As $*$-isomorphisms preserve the type structure of von Neumann algebras, $\mathscr{M}$ must have direct summands of Type 1 and/or Type II, only. The proof of the statement about the Jones action is detailed in the cited proof.

Finally, let us consider the two "opposite" cases.

(i) Suppose that $\hat{G}$ is torsion free. Then by Lemma 5.2.14, N and N_{fr} are trivial. Therefore the cocycle κ is also trivial and so the action $\hat{G}_{fr} \curvearrowright (\hat{G}^{\mathbb{D}} \times \hat{G}_{fr})$ is given by the formula $g \cdot (x,k) := (x, gk)$. Therefore

$$\mathscr{B} = L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})\bar{\otimes}(\ell^\infty(\hat{G}_{fr})\bar{\rtimes}\hat{G}_{fr}) \cong L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})\bar{\otimes}\mathcal{B}(\ell^2(\hat{G}_{fr})).$$

As $\mathscr{B}$ is the von Neumann tensor product of bounded operators on an infinite dimensional Hilbert space (in this case, $H = \ell^2(\hat{G}_{fr})$) and a commutative von Neumann algebra (namely $L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})$), a standard result (e.g. [4, III.5.12.])

tells us that it is Type I_∞. Moreover, we have that

$$\begin{aligned}\mathcal{Z}(\mathscr{B}) &= \mathcal{Z}(L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})\bar{\otimes}\mathcal{B}(\ell^2(\hat{G}_{fr}))) \\ &= \mathcal{Z}(L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}}))\bar{\otimes}\mathcal{Z}(\mathcal{B}(\ell^2(\hat{G}_{fr}))) \\ &= L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})\bar{\otimes}\mathbb{C}1 \\ &\cong L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}}).\end{aligned}$$

This is a rather canonical example of an abelian von Neumann algebra with no minimal projections (i.e., diffuse), so the claim about the center holds. Finally, for some $g \neq e \in \hat{G}_{fr}$, we have $\delta_g \perp \delta_e$ and so

$$\langle |\delta_g \rangle\!\langle \delta_g| \delta_e | \delta_e \rangle = ||\delta_g \rangle\!\langle \delta_e||^2 = 0,$$

implying that the state $m^{\mathbb{D}} \otimes \langle \cdot \delta_e | \delta_e \rangle$ is not faithful.

(ii) Now, suppose G is a finite group and m is the Haar measure of $\hat{G}$. Then $N = \hat{G}$ and $N_{fr} = \hat{G}_{fr}$. Therefore

$$\mathscr{B} = L^\infty(\hat{G}^{\mathbb{D}}, m^{\mathbb{D}})\bar{\rtimes}\hat{G}_{fr},$$

where the group action is given by $g \cdot x = gx$. Moreover, we have that

$$\overline{\omega}_{\mathscr{B}}(\sum_{g \in \hat{G}_{fr}} a_g v_g) = m^{\mathbb{D}}(a_e).$$

This is a normal, faithful, tracial state, meaning that $\mathscr{B}$ is finite by a straightforward argument (see, e.g. [20, Remark 10.4]). Moreover, since $\mathscr{B}$ is built from a net of finite factors $\mathcal{B}(\ell^2(\hat{G}_t))$, $t \in \mathfrak{T}$, then $\mathscr{B}$ is a factor by [39, XIV, Lemma 2.13] [8]); it is hyperfinite as each of the $\mathcal{B}(\ell^2(\hat{G}_t)), t \in \mathfrak{T}$ are finite dimensional. Finally, since $\mathscr{B}$ is infinite dimensional, we get that $(\mathscr{B}, \overline{\omega}_{\mathscr{B}})$ is necessarily the hyperfinite II_1 factor $\mathcal{R}$ equipped with its unique trace.

□

Remark 5.2.18. We have reached the end of the portion of the thesis which deals directly with the construction of operator-algebraic gauge theories. Before we move on, we have a few remarks to make.

(i) Notice that at each level, we have a faithful state ω_t on M_t. However, at the limit, the state $\overline{\omega}$ need not be faithful (as in the case when $\hat{G}$ is torsion free). Brothier and Stottmeister point out in [6, Remark 3.14] that this affects our ability to place a non-trivial time evolution on the system—the details of this are beyond the scope of this thesis.

[8] The referenced statement assumes that one has an increasing *sequence* of finite factors. However, the proof does not rely on countability, so the result holds for the more general case here.

(ii) We can recover the von Neumann field algebras $\mathscr{M}(O)$, where O is a connected, open subset of the circle as we did in Section 4.3. In particular, the localized von Neumann algebra $\mathscr{M}(O)$ is the WOT-completion of $\mathscr{M}_0(O)$ inside $\mathscr{M}$. With a little work, we can extend the result of Theorem 4.3.2 to see that $\{\mathscr{M}(O)\}$ describes something like a conformal net of von Neumann algebras. We also have a "nice" localized gauge group action.

Furthermore, in the definition of $\mathscr{B}$, we can replace each $\mathbb{D}$ with $\mathbb{D}(O)$ and use the appropriate embeddings and projections in order to describe $\mathscr{M}(O) \subseteq \mathscr{M}$ in terms of crossed-products of von Neumann algebras. That is, for any connected, open $O \subset S^1$, we can define:

(1) The group $\hat{G}_{fr}(O)$, the restriction of maps in $g \in \hat{G}_{fr}$ to $\mathbb{D}(O)$,

(2) The group $\hat{G}^{\mathbb{D}(O)}$, the infinite product of groups $\prod_{d\in\mathbb{D}(O)} \hat{G}$ endowed with the product topology (and each copy of $\hat{G}$ has the discrete topology), and

(3) The group $N_{fr}(O) := \{g \in \hat{G}_{fr}(O) : g(d) \in N \text{ for all } d \in \mathbb{D}(O)\}$.

From this, we can define local crossed-product von Neumann algebras

$$\mathscr{B}(O) := L^\infty(\hat{G}^{\mathbb{D}(O)} \times \hat{G}_{fr}(O)/N_{fr}(O), m^{\mathbb{D}(O)} \otimes \mu_c) \rtimes \hat{G}_{fr}(O),$$

which are isomorphic to the local algebras $\mathscr{M}(O)$.

Chapter 6

Tensor Networks

Over the past two decades, substantial progress has been made in putting aspects of QFT into a rigorous mathematical framework. In large part, this progress has followed from establishing connections between QFT and quantum information. It is within this context that tensor networks have emerged as a powerful tool.

In this chapter, we will cover the basics of tensor networks and explore how to build a tensor network approximation for the simplest case of the operator algebraic gauge theories in the previous chapters. We use [5] as our main source for the basics of tensors and tensor networks and both [34] and [33] for our discussion of perfect tensors. Some familiarity with the basics of quantum mechanics, (e.g., at the level of [32, Chapter 2]) will be useful for the reader.

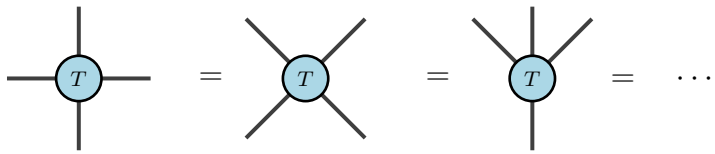
6.1 Tensor Network Basics

We begin this chapter with a brief introduction to the basics of tensors and tensor networks. This section is not comprehensive; we refer the reader to [5] for an in-depth collection of introductory notes.

In physics, *tensors* are generalizations of matrices (which can be thought of as generalizations of vectors). Just as a d-dimensional (complex) vector is an element of $\mathbb{C}^d$ and an $n \times m$ matrix is an element of $\mathbb{C}^{n\times m}$, an *n-index* tensor of dimensions $d_1 \times \cdots \times d_n$ is an element of $\mathbb{C}^{d_1\times\cdots\times d_n}$. Conversely, this means we can think of scalars, vectors, and matrices as tensors with $0, 1$, and 2 indices, respectively.

If T is an n-index tensor such that each index j_k has dimension d_k, it is often useful to explicitly denote it as $T_{j_1,\ldots,j_n} \in \mathbb{C}^{d_1\times\cdots\times d_n}$. In tensor network notation (TNN), we draw an n-index tensor T as a vertex (in this thesis, we will use (labelled) circles for vertices) with n edges (which we call *legs*) sticking out of it. Generally speaking, the location of the legs does not matter in TNN; however, later on we would like to fix an orientation of the graph in order to "interpret" the graphs as maps between Hilbert spaces. If we have a particular interpretation in mind, it is helpful to draw the legs in a way that is conducive to this understanding.

Example 6.1.1. A 4-index tensor T:



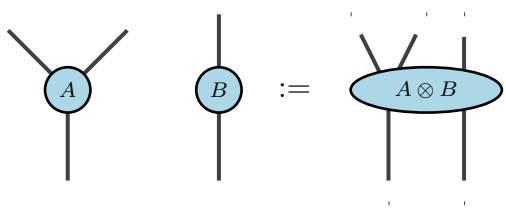
TNN allows us to build larger tensors by composing tensors together. There are many useful operations that can be represented graphically using TNN. We are interested in building networks of tensors via *contraction* of two (or more) tensors – the contraction operation is built from the *tensor product* followed by a *trace* between indices of the two tensors.

Mathematically, the tensor product is a generalization of the outer product of vectors. Namely, given an r-index tensor $A_{i_1,\ldots,i_r} \in \mathbb{C}^{d_{i_1}\times\cdots\times d_{i_r}}$ and an s-index tensor $B_{j_1,\ldots,j_s} \in \mathbb{C}^{d_{j_1}\times\cdots\times d_{j_s}}$, their tensor product is an $(r+s)$-index tensor $(A\otimes B)_{i_1,\ldots,i_r,j_1,\ldots,j_s} \in \mathbb{C}^{d_{i_1}\times\cdots\times d_{i_r}\times d_{j_1}\times\cdots\times d_{j_s}}$. In particular, the value of their tensor product on a given set of indices is the element-wise product of the values of each of the two tensors; we can write this as

$$[A\otimes B]_{i_1,\ldots,i_r,j_1,\ldots,j_s} := A_{i_1,\ldots,i_r}\cdot B_{j_1,\ldots,j_s}.$$

Diagrammatically, the tensor product is simply given by placing two tensors next to each other.

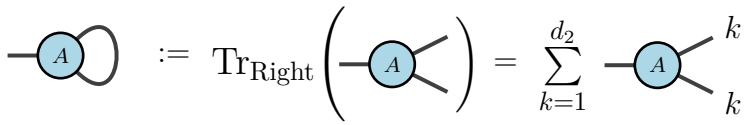
Example 6.1.2. Given the 3-index tensor A and the 2-index tensor B (below), we can denote the 5-index tensor $A\otimes B$ as



The next important operation is the (partial) trace. Given a tensor $A_{i_1,\ldots,i_r} \in \mathbb{C}^{d_1\times\cdots\times d_r}$ for which the nth and mth indices have the same dimension (i.e., $d_n = d_m$), the partial trace over these two dimensions is a joint summation over that index. We can write this as

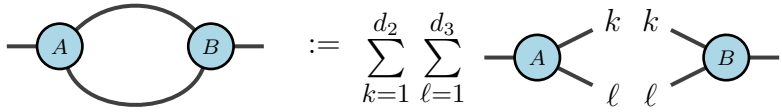
$$[\mathrm{Tr}_{n,m}A]_{i_1,\ldots,i_{n-1},i_{n+1},\ldots,i_{m-1},i_{m+1}\ldots,i_r} := \sum_{k=1}^{d_n} A_{i_1,\ldots,i_{n-1},k,i_{n+1},\ldots,i_{m-1},k,i_{m+1}\ldots,i_r}.$$

Example 6.1.3. Graphically, tracing over the two right-hand indices (each with dimension d_2) of a 3-index tensor A would look like:



Finally, contraction between two tensors corresponds to first taking their tensor product and then taking the trace between indices of the two tensors.

Example 6.1.4. Graphically, contracting two 3-index tensors $A_{i_1,i_2,i_3} \in \mathbb{C}^{d_{i_1} \times d_{i_2} \times d_{i_3}}$ and $B_{j_1,j_2,j_3} \in \mathbb{C}^{d_{j_1} \times d_{j_2} \times d_{j_3}}$ where both i_2, j_2 and i_3, j_3 have common dimensions d_2 and d_3, respectively:



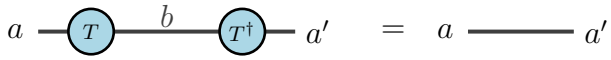
We can think of maps between Hilbert spaces as tensors (and vice versa). If H_A and H_B are Hilbert spaces of finite-dimension with $\{|a\rangle\}$ being an ONB of H_A and $\{|b\rangle\}$ and ONB of H_B then a map $T : H_A \to H_B$ can be represented as a 2-index tensor $T_{a,b}$ where

$$T : |a\rangle \mapsto \sum_b T_{ba}|b\rangle.$$

Moreover, $T : H_A \to H_B$ is an isometry if and only if

$$\sum_b T^\dagger_{a'b} T_{ba} = \delta_{a',a},$$

where $(\cdot)^\dagger$ denotes the conjugate transpose. In this case, we call T an *isometric tensor*. In TNN, the line segment without any nodes denotes the identity map, so an isometric tensor can be described in the following way:



Given an n-index tensor $T_{j_1,\ldots,j_n}$, where each index j_i has dimension d_i, we can bipartition the set of indices into two complementary sets, say $A = \{j_1, \ldots, j_k\}$ and $A^c = \{j_{k+1}, \ldots, j_n\}$. Moreover, we can define the Hilbert space associated to A by $H_A := \bigotimes_{i=1}^k \mathbb{C}^{d_i}$ and similarly, we can define the Hilbert space associated to A^c by $H_{A^c} := \bigotimes_{i=j+1}^n \mathbb{C}^{d_i}$. In this way, we can think of $T_{j_1,\ldots,j_n}$ as a map $T : H_A \to H_{A^c}$.

A useful technique which can be applied to isometric tensors is "pushing an operator through a tensor". The idea is that any operator O acting on the incoming

(left-most) leg of an isometric tensor can be replaced by an operator O' of the same norm applied to the outgoing (right-most) leg. This follows from the fact that if T is an isometry, then $TO = TOT^\dagger T = (TOT^\dagger)T$ and $O' := TOT^\dagger$ is an operator with the same norm as O. Conversely, if a 2-index tensor T has the property that any unitary operator U acting on the incoming leg can be replaced by a unitary operator U' applied to the outgoing leg, then T is proportional to an isometric tensor.

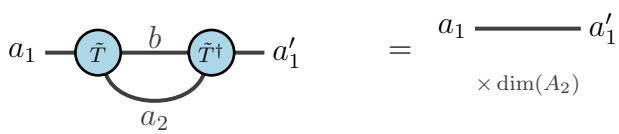
Further, if $T : H_A \to H_B$ is an isometry and $H_A = H_{A_2} \otimes H_{A_1}$, then we can "move" one of the input Hilbert space factors to the output and obtain a map which is proportional to an isometry. That is, given an isometric map $T : H_{A_2} \otimes H_{A_1} \to H_B$, acting on ONBs by

$$T : |a_2 a_1\rangle \mapsto \sum_b T_{ba_2a_1} |b\rangle,$$

we can obtain a new map $\tilde{T} : H_{A_1} \to B \otimes H_{A_2}$ with

$$\tilde{T} : |a_1\rangle \mapsto |ba_2\rangle T_{ba_2a_1},$$

such that $\tilde{T}^\dagger \tilde{T} = \dim(A_2) I_{A_1}$. In diagrammatic TNN, we have the following:



6.1.1 Perfect Tensors

We are interested in using a special class of isometric tensors, which were originally introduced in [34].

Definition 6.1.5. An n-index tensor $T_{j_1,j_2,\ldots,j_n}$ is a *perfect tensor* if, for any bipartition of its indices into a pair of complementary sets $\{j_1, j_2, \ldots, j_n\} = A \cup A^c$ such that, without loss of generality, $|A| \leq |A^c|$, T is proportional to an isometry from H_A to H_{A^c}.

In order to check that a $2n$-index tensor $T_{j_1,\ldots,j_{2n}}$ is perfect, it suffices to check that $T : H_A \to H_{A^c}$ is a unitary transformation when $|A| = |A^c| = n$, i.e., it suffices to check that both $T^\dagger T = I$ and $TT^\dagger = I$. This follows from the property illustrated above—we can factor the input Hilbert space and "move" one of these factors to the output and obtain a map which is proportional to an isometry. In particular, this means that we can still identify $T_{j_1,\ldots,j_{2n}}$ with a map which is proportional to an isometry for $|A| < n$.

Fact 6.1.6. We can relate (perfect) tensors to other ideas in quantum information theory. In general, any tensor T with m indices, each ranging over d values, describes a pure quantum state $|\psi_T\rangle$ of m d-dimensional spins where

$$|\psi_T\rangle = T_{a_1,a_2,\ldots,a_m} |a_1, a_2, \ldots, a_m\rangle,$$

up to some scalar multiple (or normalization factor, as it is referred to in the literature). A perfect tensor with $2n$ indices describes a pure state of $2n$ spins such that any set of n spins is maximally entangled with the complementary set of n spins (see [34, Section 2]). We call such states *absolutely maximally entangled* (AME). Conversely, any AME state induces a perfect tensor.

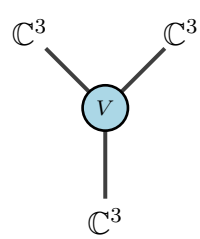
We will only consider the case of tensors whose legs each have the same dimension d. In this case, an n-index tensor T is associated to a vector $|\psi_T\rangle \in (\mathbb{C}^d)^{\otimes n}$.

Example 6.1.7. [33, Example 2.4] The map $V : \mathbb{C}^3 \to \mathbb{C}^3 \otimes \mathbb{C}^3$ given by

$$\langle jk|V|l\rangle = \begin{cases} 0 & \text{if } j = k, k = l, \text{ or } j = l \\ 1 & \text{otherwise} \end{cases}$$

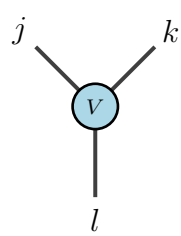
describes a perfect tensor with $n = 3$ indices and each leg having dimension $d = 3$.

Proof. We can view V diagrammatically as



.

In order to study this tensor, we fix a direction on V, label the legs, and read the following figure from bottom to top:



.

We can denote this map as $V_l^{j,k}$ to make the direction clear. For notational simplicity, however, we will simply refer to it as V for the time being. We have

$$\begin{aligned} V|0\rangle &= \sum_{j,k=0}^{2} \langle jk|V|0\rangle |jk\rangle \\ &= \sum_{j=0}^{2} \sum_{k=0}^{2} \langle jk|V|0\rangle |jk\rangle \\ &= |21\rangle + |12\rangle. \end{aligned}$$

Similarly,

$$V|1\rangle = |02\rangle + |20\rangle$$
$$V|2\rangle = |01\rangle + |10\rangle.$$

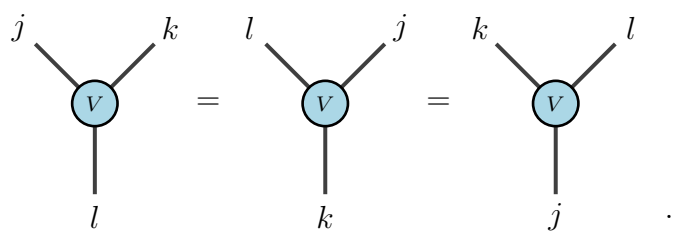
There is an associated state $|\psi_V\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ such that

$$\psi_V = \sum_{j,k,l=0}^{2} V_{jkl}|jkl\rangle,$$

where the coefficients are $V_{jkl} = \langle jk|V|l\rangle$. After normalizing each of the $V|l\rangle$ by a factor of $\frac{1}{\sqrt{2}}$, we get

$$\begin{aligned}\psi_V &= \sum_{j,k,l=0}^{2} V_{jkl}|jkl\rangle \\ &= \sum_{l=0}^{2}\Big(\sum_{j,k=0}^{2}\langle jk|V|l\rangle|jkl\rangle\Big) \\ &= \frac{1}{\sqrt{6}}\Big(|210\rangle + |120\rangle + |021\rangle + |201\rangle + |012\rangle + |102\rangle\Big)\end{aligned}$$

Recall that we previously fixed a direction so that we were actually working with the map $V_l^{j,k}$. However, from the previous equation we see that the associated state will be the same regardless of which way we read the diagram. A tensor with this property is said to be *rotationally invariant*. In particular, we have



By rotational variance and the fact that $V_l^{j,k}$ is proportional to an isometry, we get that V is a perfect tensor. □

Example 6.1.8. [33, Example 2.3] Let $n = 3$ and $d = 4$ and define the map $V : \mathbb{C}^2 \otimes \mathbb{C}^2 \to (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)$ by

$$V|j\rangle|k\rangle = \frac{1}{2}|j\rangle|\Phi^-\rangle|k\rangle,$$

where $|\Phi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is one of the maximally entangled Bell states [32,

Example 1.3.6]. This example also describes a perfect tensor.

Proof. Let $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ denote the standard basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$. Then by definition of V

$$\begin{aligned} V|00\rangle &= \frac{1}{2}|0\rangle|\Phi^-\rangle|0\rangle \\ V|01\rangle &= \frac{1}{2}|0\rangle|\Phi^-\rangle|1\rangle \\ V|10\rangle &= \frac{1}{2}|1\rangle|\Phi^-\rangle|0\rangle \\ V|11\rangle &= \frac{1}{2}|1\rangle|\Phi^-\rangle|1\rangle. \end{aligned}$$

There is an associated state $|\psi_V\rangle \in (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)$ such that

$$\psi_V = \sum_{j,k,l=0}^{1} V_{(ij),(kl)(mn)}|ijklmn\rangle,$$

where the coefficients are given by

$$\begin{aligned} V_{(ij),(kl)(mn)} &= \langle V(ij)|klmn\rangle \\ &= \langle i|\Phi^-\rangle j|klmn\rangle \\ &= \frac{1}{2\sqrt{2}}\langle i(|01\rangle - |10\rangle)j|klmn\rangle \\ &= \frac{1}{2\sqrt{2}}\big(\langle i(01)j|klmn\rangle - \langle i(10)j|klmn\rangle\big) \\ &= \frac{1}{2\sqrt{2}}\big(\delta_{i,k}\cdot\delta_{0,l}\cdot\delta_{1,m}\cdot\delta_{j,n} - \delta_{i,k}\cdot\delta_{1,l}\cdot\delta_{0,m}\cdot\delta_{j,n}\big). \end{aligned}$$

In order to show that this is a perfect tensor, we need to show that it is (proportional to) an isometry in each direction. In other words, if we denote the indices of V by $1, 2, 3$ then we need to show that each of the maps $V_1^{2,3}$, $V_2^{1,3}$, and $V_3^{1,2}$ are proportional to isometries.

We have

$$\begin{aligned} V_1^{2,3}|ij\rangle &= \sum_{k,l,m,n=0}^{1} V_{(ij),(kl)(mn)}|k\rangle|l\rangle \otimes |m\rangle|n\rangle \\ &= \sum_{l,m=0}^{1} V_{(ij),(kl)(mn)}|i\rangle|l\rangle \otimes |m\rangle|j\rangle \\ &= \frac{1}{2\sqrt{2}}|i\rangle(|01\rangle - |10\rangle)|j\rangle \\ &= \frac{1}{2}|i\rangle|\Phi^-\rangle|j\rangle. \end{aligned}$$

As expected, this is just the map V, which is clearly (proportional to) an isometry. Moreover, we have

$$\begin{aligned} V_2^{1,3}|kl\rangle &= \sum_{i,j,m,n=0}^{1} V_{(ij),(kl)(mn)}|i\rangle|j\rangle \otimes |m\rangle|n\rangle \\ &= \sum_{m,n=0}^{1} V_{(ij),(kl)(mn)}|k\rangle|n\rangle \otimes |m\rangle|n\rangle \\ &= \frac{1}{2\sqrt{2}} \sum_{n=0}^{1} \delta_{0,l}|k\rangle|n\rangle \otimes |1\rangle|n\rangle - \delta_{1,l}|k\rangle|n\rangle \otimes |0\rangle|n\rangle. \end{aligned}$$

There are a few extra steps required to show that this is (proportional to) an isometry. First, recall (see, e.g. [32, Section I.3]) that the Pauli X matrix is the Hermitian, unitary matrix acting as a quantum NOT gate, and is written as

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with respect to the standard basis in $\mathbb{C}^2$. Further, the SWAP gate, i.e., the Hermitian, unitary matrix which swaps two qubits, can be written as

$$\text{SWAP} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with respect to the standard basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

We have that

$$\begin{aligned}(I_2 \otimes \mathrm{SWAP} \otimes I_2) \circ V_2^{1,3}|kl\rangle &= \frac{1}{2\sqrt{2}} \sum_{n=0}^{1} \delta_{0,l}|k\rangle \mathrm{SWAP}(|n\rangle \otimes |1\rangle)|n\rangle - \delta_{1,l}|k\rangle \mathrm{SWAP}(|n\rangle \otimes |0\rangle)|n\rangle \\ &= \frac{1}{2\sqrt{2}} \sum_{n=0}^{1} \delta_{0,l}|k\rangle|1\rangle \otimes |nn\rangle - \delta_{1,l}|k\rangle|0\rangle \otimes |nn\rangle \\ &= \left(\frac{-1}{2}\right)^l (|k\rangle \otimes X|l\rangle \otimes \frac{1}{\sqrt{2}} \sum_{n=0}^{1} |nn\rangle) \\ &= \left(\frac{-1}{2}\right)^l (|k\rangle \otimes X|l\rangle \otimes |\Phi^+\rangle),\end{aligned}$$

where $|\Phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (another maximally entangled Bell state [32, Example 1.3.6]).

This composition gives a map which applies a unitary to $|kl\rangle$ (conditioned on l) and adjoins a maximally entangled state, which is clearly proportional to an isometry. As $V_2^{1,3}$ is equal to the composition (up to a unitary swap in positions 2 and 3), then $V_2^{1,3}$ is also proportional to an isometry.

Similarly, we have

$$\begin{aligned}V_3^{1,2}|mn\rangle &= \sum_{i,j,k,l=0}^{1} V_{(ij),(kl)(mn)}|i\rangle|j\rangle \otimes |k\rangle|l\rangle \\ &= \sum_{k,l=0}^{1} V_{(ij),(kl)(mn)}|k\rangle|n\rangle \otimes |k\rangle|l\rangle \\ &= \frac{1}{2\sqrt{2}} \sum_{k=0}^{1} \delta_{1,m}|k\rangle|n\rangle \otimes |k\rangle|0\rangle - \delta_{0,m}|k\rangle|n\rangle \otimes |k\rangle|1\rangle.\end{aligned}$$

Moreover, we have

$$(I_2 \otimes \mathrm{SWAP} \otimes I_2) \circ V_3^{1,2}|mn\rangle = \left(\frac{-1}{2}\right)^{m+1} (|\Phi^+\rangle \otimes |n\rangle \otimes X|m\rangle).$$

So, by the same type of argument as above, $V_3^{1,2}$ is proportional to an isometry, as required. □

6.2 From operator algebras to tensor networks

So far, this chapter likely feels completely disconnected from the previous work. However, we can connect the abstract operator-algebraic constructions from previous chapters to the realm of Hilbert spaces (and from there, to tensor networks) via the

GNS construction with respect to coherent families of states.

First, let us recall the setting from our construction in Section 5.2. Namely, we take our compact separable group G to be abelian and we choose a single strictly positive measure $m \in \mathrm{Prob}(\hat{G})$. We have a directed system of C*-algebras $M_t := \mathrm{Char}(\hat{G}_t) \rtimes \hat{G}_t$, each with a normal, faithful state ω_t, and the C*-algebraic limit $\varinjlim_{t\in\mathfrak{T}} M_t := \mathscr{M}_0$ with a unique normal, faithful (limit) state $\overline{\omega}$. The maps $k_t^s : M_t \to M_s$ for $t \leq s \in \mathfrak{T}$ are state-preserving injective unital homomorphisms, so $\omega_t = \omega_s \circ k_t^s$.

Recall also that the GNS construction for (M_t, ω_t) gives a triple (π_t, H_t, Ω_t) where $\pi_t : M_t \to \mathcal{B}(H_t)$ is a $*$-homomorphism and $\langle \pi_t(a)\Omega_t | \Omega_t \rangle_{H_t} = \omega_t(a)$ for $a \in M_t$.

The following is a fairly straightforward result, but the reference given in [7] (the companion article to the source of our main construction) leads to something which is left as a series of exercises. As a matter of completeness, we prove it here.

Proposition 6.2.1. *[7, Section 2.3.2] A family of C*-algebras and states satisfying the conditions above induces an inductive system of Hilbert spaces with connecting maps $R_t^s : H_t \to H_s$ such that*

$$\begin{aligned} (R_t^s)^*(R_t^s) &= 1_{H_t}, & t \leq s \\ (R_s^v)(R_t^s) &= R_t^v, & t \leq s \leq v. \end{aligned}$$

and

$$\pi_s(k_t^s(m))R_t^s = R_t^s \pi_t(m) \tag{1}$$

$$R_t^s \Omega_t = \Omega_s \tag{2}$$

$$\omega_t(a^* b) = \langle \pi_t(b)\Omega_t | \pi_t(a)\Omega_t \rangle_{H_t}, \tag{3}$$

where $a, b, m \in M_t$ and $t \leq s$.

Proof. Let (π_t, H_t, Ω_t) be the GNS triple associated to M_t for a given $t \in \mathfrak{T}$. Since Ω_t is cyclic by the GNS construction, then $H_t := \overline{\{\pi_t(a)\Omega_t : a \in M_t\}}^{\|\cdot\|}$. In other words, any $x \in H_t$ can be norm-approximated by $\pi_t(a)\Omega_t$ for some $a \in M_t$. Denote $D_t := \{\pi_t(a)\Omega_t : a \in M_t\}$. For a tree $s \geq t$, define a map:

$$\begin{aligned} r_t^s : & D_t \to H_s \\ & \pi_t(a)\Omega_t \mapsto \pi_s(k_t^s(a))\Omega_s. \end{aligned}$$

We extend this map to $\overline{D_t} = H_t$ to obtain $R_t^s : H_t \to H_s$. Given trees $t \leq s$ and $a \in M_t$, we have $(R_t^s)^*(R_t^s)(\pi_t(a)\Omega_t) = (R_t^s)^*(\pi_s(k_t^s(a))\Omega_s)$.

So

$$\begin{aligned}
\langle \pi_t(b)\Omega_t | (R_t^s)^*(\pi_s(k_t^s(a))\Omega_s)\rangle &= \langle (R_t^s)(\pi_t(b)\Omega_t) | \pi_s(k_t^s(a))\Omega_s\rangle \\
&= \langle \pi_s(k_t^s(b))\Omega_s) | \pi_s(k_t^s(a))\Omega_s\rangle \\
&= \langle (\pi_s(k_t^s(a)))^*\pi_s(k_t^s(b))\Omega_s) | \Omega_s\rangle \\
&= \langle \pi_s(k_t^s(a^*))\pi_s(k_t^s(b))\Omega_s) | \Omega_s\rangle \\
&= \langle \pi_s(k_t^s(a^*b))\Omega_s | \Omega_s\rangle \\
&= \omega_s(k_t^s(a^*b)) \\
&= \omega_t(a^*b).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \pi_t(b)\Omega_t | 1_{H_t}(\pi_t(a)\Omega_t)\rangle &= \langle \pi_t(b)\Omega_t | \pi_t(a)\Omega_t\rangle \\
&= \omega_t(a^*b) \\
\implies (R_t^s)^*(R_t^s) &= 1_{H_t}.
\end{aligned}$$

Now, for trees $t \leq s \leq v$, we have

$$\begin{aligned}
(R_s^v)(R_t^s)(\pi_t(a)\Omega_t) &= R_s^v(\pi_s(k_t^s(a))\Omega_s) \\
&= \pi_v(k_s^v(k_s^t(a)))\Omega_v \\
&= \pi_v(k_t^v(a))\Omega_v \\
&= R_t^v(\pi_t(a)\Omega_t).
\end{aligned}$$

So we have shown that the connecting maps $R_t^s : H_t \to H_s$ satisfy the conditions in the statement of the proposition.

Moreover, we can show the remaining three conditions, as follows.

(1) For $t \leq s$ and $a, b \in M_t$:

$$\begin{aligned}
\pi_s(k_t^s(a))R_t^s(\pi_t(b)\Omega_t) &= \pi_s(k_t^s(a))\pi_s(k_t^s(b))\Omega_s \\
&= \pi_s(k_t^s(ab))\Omega_s \\
&= R_t^s(\pi_t(ab))\Omega_t \\
&= R_t^s\pi_t(a)(\pi_t(b)\Omega_t) \\
\implies \pi_s(k_t^s(a))R_t^s &= R_t^s\pi_t(a).
\end{aligned}$$

(2) For any $t \in \mathfrak{T}$, $\pi_t(e)\Omega_t = \Omega_t$, so for $t \leq s$:

$$\begin{aligned}
R_t^s\pi_t(e)\Omega_t &= \pi_s(k_t^s(e))\Omega_s \\
&= \Omega_s \\
\implies R_t^s\Omega_t &= \Omega_s.
\end{aligned}$$

(3) For $a, b \in M_t$:

$$\begin{aligned}\omega_t(a^*b) &= \langle \pi_t(a^*b)\Omega_t|\Omega_t\rangle \\ &= \langle \pi_t(a^*)\pi_t(b)\Omega_t|\Omega_t\rangle \\ &= \langle \pi_t(b)\Omega_t|\pi_t(a)\Omega_t\rangle.\end{aligned}$$

Thus, we have an inductive system of Hilbert spaces, as required. □

Proposition 6.2.2. *[7, Proposition 2.31] Given a coherent family of states $\{\omega_t\}_{t\in\mathfrak{T}}$ as above, there is a unique (projective) limit state $\overline{\omega} = \varprojlim_{t\in\mathfrak{T}} \omega_t$ together with a representation $\pi : \mathscr{M}_0 \to \mathcal{B}(\mathfrak{H})$ on the inductive limit Hilbert space $\mathfrak{H} = \varinjlim_{t\in\mathfrak{T}} H_t$ with cyclic vector $\Omega \in \mathfrak{H}$. Moreover, we have:*

$$\begin{aligned}\pi(k_t)(m)R_t &= R_t\pi_t(m) \\ R_t\Omega_t &= \Omega \\ \overline{\omega}(a^*b) &= \langle \pi(b)\Omega|\pi(a)\Omega\rangle_{\mathfrak{H}},\end{aligned}$$

for $m \in M_t$ and $a, b \in \mathscr{M}_0$. Here, $k_t : M_t \to \mathscr{M}_0$ and $R_t : H_t \to \mathfrak{H}$ are the natural maps associated with the inductive limit construction.

The proof of this proposition is substantially the same as the proof of Proposition 6.2.1, so we omit it here; further details can be gleaned from the hints accompanying [25, Exercises 11.5.26–11.5.28].

Definition 6.2.3. We call the WOT-closure $\mathscr{M} = \pi(\mathscr{M}_0)''$ the *semi-continuum* von Neumann field algebra.

We want to use the two preceding propositions to take the operator algebraic construction described in the previous chapter and build a tensor network from the inductive system of Hilbert spaces.

We will only consider the simplest case where G is a finite, abelian group. Recall that, in this specialized case, the unitary dual $\hat{G}$ coincides with the Pontrjagin dual, and so $\hat{G}$ is a finite, discrete, abelian group. The building block of our construction is $M = \mathrm{Char}(\hat{G}) \rtimes \hat{G} \cong \mathcal{B}(\ell^2(\hat{G}))$ and if t is a tree with n_t leaves, $M_t = \otimes_{k=1}^{n_t} M$. Moreover, let $\tau = \frac{1}{|\hat{G}|}\sum_g (\cdot)_{gg}$ be the unique tracial state on $\mathcal{B}(\ell^2(\hat{G}))$ and use this to define a trace $\omega_t = \otimes_{k=1}^{n_t}\tau$.

Consider the two generating trees, $t_0 := I$ and $t_1 := Y$. Then $t_0 \leq t_1$ since $t_1 = f \circ t_0$ where $f := Y$. Then $M_{t_0} = \mathcal{B}(\ell^2(\hat{G}))$ and $M_{t_1} = \mathcal{B}(\ell^2(\hat{G})) \otimes \mathcal{B}(\ell^2(\hat{G}))$. Moreover, we have the inclusion map $\hat{R} : \mathcal{B}(\ell^2(\hat{G})) \to \mathcal{B}(\ell^2(\hat{G})) \otimes \mathcal{B}(\ell^2(\hat{G}))$ given by

$$\hat{R}(x) = \mathrm{Ad}(\hat{u})(x \otimes 1),$$

where $\hat{u} = (U_F \otimes U_F)u(U_F^* \otimes 1)$ and $u\xi(g, h) = \xi(gh, h)$ for all $\xi \in \ell^2(G \times G)$.

First, let us describe a suitable GNS construction for $(M_{t_0}, \omega_{t_0}) = (\mathcal{B}(\ell^2(\hat{G})), \tau)$. We take $H_{t_0} := \ell^2(\hat{G}) \otimes \ell^2(\hat{G})$ with

$$\begin{aligned}\langle \xi \otimes \bar{\eta} | \alpha \otimes \bar{\beta} \rangle_{H_{t_0}} &= \tau(\xi \eta^* \alpha \beta^*) \\ &= \frac{1}{|\hat{G}|} \mathrm{Tr}(\xi \eta^* \alpha \beta^*) \\ &= \frac{1}{|\hat{G}|} \langle \xi \eta^* | \alpha \beta^* \rangle_{HS},\end{aligned}$$

where $\langle \cdot | \cdot \rangle_{HS}$ refers to the Hilbert Schmidt inner product. Now, fix the standard basis of $\mathcal{B}(\ell^2(\hat{G}))$ (given by matrix units labelled by elements of $\hat{G}$) and define the GNS map $\Lambda_{t_0} : \mathcal{B}(\ell^2(\hat{G})) \to H_{t_0}$ relative to this. In particular, we define

$$\Lambda_{t_0} : [a_{\chi\gamma}] \mapsto \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi\gamma} a_{\chi\gamma} |\chi\gamma\rangle.$$

The $*$-representation of interest is

$$\pi_{t_0} : \mathcal{B}(\ell^2(\hat{G})) \ni x \mapsto x \otimes 1 \in \mathcal{B}(\ell^2(\hat{G}) \otimes \ell^2(\hat{G})).$$

Since $\mathcal{B}(\ell^2(\hat{G}))$ is unital, we have that the cyclic vector for π_{t_0} is

$$\Omega_{t_0} = \Lambda_{t_0}(1) = \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi} |\chi\chi\rangle,$$

and it is clear that $\tau(x) = \langle (x \otimes 1)\Omega_{t_0} | \Omega_{t_0} \rangle$, as required. So π_{t_0} is the GNS representation of $\mathcal{B}(\ell^2(\hat{G}))$ associated to τ.

We have a similar GNS construction for $(M_{t_1}, \omega_{t_1}) = (\mathcal{B}(\ell^2(\hat{G})) \otimes \mathcal{B}(\ell^2(\hat{G})), \tau \otimes \tau)$. Namely, we take the Hilbert space $H_{t_1} = \ell^2(\hat{G}) \otimes \ell^2(\hat{G}) \otimes \ell^2(\hat{G}) \otimes \ell^2(\hat{G})$, the $*$-representation $\pi_{t_1} : x \otimes y \mapsto (x \otimes 1) \otimes (y \otimes 1)$, and cyclic vector $\Omega_{t_1} = \Omega_{t_0} \otimes \Omega_{t_0}$.

Proposition 6.2.4. *The isometry $R_{t_0}^{t_1} : \ell^2(\hat{G})^{\otimes 2} \to \ell^2(\hat{G})^{\otimes 4}$ connecting the Hilbert spaces induced by the GNS construction is given by*

$$R_{t_0}^{t_1}(|\chi\rangle \otimes |\gamma\rangle) = |\chi\rangle \otimes |\gamma\rangle \otimes (\lambda_{\chi\gamma^{-1}} \otimes 1)\Omega_{t_0}.$$

Proof. By Proposition 6.2.1, we have $R_{t_0}^{t_1}(\pi_{t_0}(a)\Omega_{t_0}) = \pi_{t_1}(\hat{R}(a))\Omega_{t_1}$ for $a \in \mathcal{B}(\ell^2(\hat{G}))$. That is,

$$\begin{aligned}R_{t_0}^{t_1}((a \otimes 1)\Omega_{t_0}) &= \pi_{t_1}(\hat{u}(a \otimes 1)\hat{u}^*)\Omega_{t_1} \\ &= \pi_{t_1}(\hat{u}(a \otimes 1)\hat{u}^*)(\Omega_{t_0} \otimes \Omega_{t_0}).\end{aligned}$$

Any $a \in \mathcal{B}(\ell^2(\hat{G}))$ is some linear combination of $b_\chi \lambda_\chi$, $\chi \in \hat{G}$. By linearity, we only

need to focus on $a = b_\chi \lambda_\chi$; by Lemma 5.2.6, we already know that $\hat{R}(a) = b_\chi \lambda_\chi \otimes \lambda_\chi$ in this case. So we have

$$\begin{aligned} R_{t_0}^{t_1}((a \otimes 1)\Omega_{t_0}) = R_{t_0}^{t_1}((b_\chi \lambda_\chi \otimes 1)\Omega_{t_0}) &= \pi_{t_1}((b_\chi \lambda_\chi \otimes \lambda_\chi))(\Omega_{t_0}. \otimes \Omega_{t_0}) \\ &= (b_\chi \lambda_\chi \otimes 1)\Omega_{t_0} \otimes (\lambda_\chi \otimes 1)\Omega_{t_0}. \end{aligned}$$

We want to describe what $R_{t_0}^{t_1}$ does to a standard basis element $|\chi\rangle \otimes |\gamma\rangle$ of $\ell^2(\hat{G})^{\otimes 2}$. Observe that

$$\begin{aligned} \sqrt{|\hat{G}|}\,(|\chi \times \gamma| \otimes 1)\,\frac{1}{\sqrt{|\hat{G}|}} \sum_\xi |\xi\rangle \otimes |\xi\rangle &= \sum_\xi |\chi\rangle\langle\gamma|\xi\rangle \otimes |\xi\rangle \\ &= \sum_\xi |\chi\rangle \delta_{\gamma,\xi} \otimes |\xi\rangle \\ &= |\chi\rangle \otimes |\gamma\rangle. \end{aligned}$$

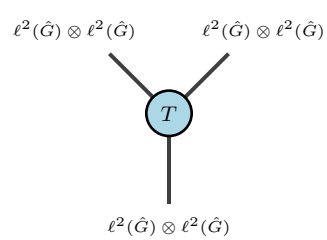
In order to complete the proof, notice that $|\chi \times \gamma| = |\chi \times \chi| \cdot \lambda_{\chi\gamma^{-1}}$ and $|\chi \times \chi|$ is the same as δ_χ, when we view it as a function on $\hat{G}$.

Therefore, we have

$$\begin{aligned} R_{t_0}^{t_1}(|\chi\rangle \otimes |\gamma\rangle) &= R_{t_0}^{t_1}\left(\sqrt{|\hat{G}|}(|\chi \times \chi|\lambda_{\chi\gamma^{-1}} \otimes 1)\Omega_{t_0}\right) \\ &= \sqrt{|\hat{G}|}(|\chi \times \chi|\lambda_{\chi\gamma^{-1}} \otimes 1)\Omega_{t_0} \otimes (\lambda_{\chi\gamma^{-1}} \otimes 1)\Omega_{t_0} \\ &= |\chi\rangle \otimes |\gamma\rangle \otimes (\lambda_{\chi\gamma^{-1}} \otimes 1)\Omega_{t_0}, \end{aligned}$$

where the last line follows from the fact that we already showed $R_{t_0}^{t_1}((b_\chi \lambda_\chi \otimes 1)\Omega_{t_0}) = (b_\chi \lambda_\chi \otimes 1)\Omega_{t_0} \otimes (\lambda_\chi \otimes 1)\Omega_{t_0}$. □

We want to define a tensor T based on $R_{t_0}^{t_1}$. We can view the map as a tensor $T : (\ell^2(\hat{G}) \otimes \ell^2(\hat{G})) \to (\ell^2(\hat{G}) \otimes \ell^2(\hat{G}))^{\otimes 2}$. Diagramatically, we have the following, which we read from the bottom upwards:



We label the legs $1, 2$ and 3 starting at the bottom leg and moving in a counterclockwise direction. Moreover, let (χ_i, γ_i) refer to basis vectors of the copy of $\ell^2(\hat{G}) \otimes \ell^2(\hat{G})$

at the ith leg. The coefficients of T are:

$$\begin{aligned}
T_{(\chi_1,\gamma_1),(\chi_2,\gamma_2),(\chi_3\gamma_3)} &= \langle T(|\chi_1 \otimes \gamma_1\rangle)||\chi_2 \otimes \gamma_2\rangle \otimes |\chi_3 \otimes \gamma_3\rangle\rangle \\
&= \langle |\chi_1 \otimes \gamma_1\rangle \otimes (\lambda_{\chi_1\gamma_1^{-1}} \otimes 1)\Omega_{t_0}||\chi_2 \otimes \gamma_2\rangle \otimes |\chi_3 \otimes \gamma_3\rangle\rangle \\
&= \delta_{\chi_1,\chi_2}\delta_{\chi_1,\chi_2}\langle(\lambda_{\chi_1\gamma_1^{-1}} \otimes 1)\Omega_{t_0}||\chi_3 \otimes \gamma_3\rangle\rangle \\
&= \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_2}}{\sqrt{|\hat{G}|}} \sum_{\chi} \langle(\lambda_{\chi_1\gamma_1^{-1}} \otimes 1)|\chi\rangle \otimes |\chi\rangle||\chi_3\rangle \otimes |\gamma_3\rangle\rangle \\
&= \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_2}}{\sqrt{|\hat{G}|}} \sum_{\chi} \langle|\chi_1\gamma_1^{-1} \cdot \chi\rangle \otimes |\chi\rangle||\chi_3\rangle \otimes |\gamma_3\rangle\rangle \\
&= \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_2}}{\sqrt{|\hat{G}|}} \langle|\chi_1\gamma_1^{-1} \cdot \gamma_3\rangle||\chi_3\rangle\rangle \\
&= \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_2}\delta_{\chi_1\gamma_1^{-1}\cdot\gamma_3,\chi_3}}{\sqrt{|\hat{G}|}}.
\end{aligned}$$

Since $R_{t_0}^{t_1}$ was the map coming from our GNS construction, we know that it is an isometry. By construction, $T_1^{2,3} = R_{t_0}^{t_1}$, so the tensor is necessarily an isometry in this direction. Moreover, we can see from the coefficients that T is symmetric in legs 1 and 2, so $T_2^{1,3}$ is also an isometry. In order for us to connect this construction to the work done in [33], we need T to be a perfect tensor, i.e., we want it to be (proportional to) an isometry in the third direction.

We have

$$\begin{aligned}
T_3^{1,2}(|\chi_3 \otimes \gamma_3\rangle) &= \sum_{\chi_1,\gamma_1,\chi_2,\gamma_2} T_{(\chi_1,\gamma_1),(\chi_2,\gamma_2),(\chi_3\gamma_3)}(|\chi_1 \otimes \gamma_1\rangle \otimes |\chi_2 \otimes \gamma_2\rangle) \\
&= \sum_{\chi_1,\gamma_1,\chi_2,\gamma_2} \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_2}\delta_{\chi_1\gamma_1^{-1}\cdot\gamma_3,\chi_3}}{\sqrt{|\hat{G}|}}(|\chi_1 \otimes \gamma_1\rangle \otimes |\chi_2 \otimes \gamma_2\rangle) \\
&= \sum_{\chi,\gamma} \frac{\delta_{\chi\gamma^{-1}\cdot\gamma_3,\chi_3}}{\sqrt{|\hat{G}|}}|\chi \otimes \gamma\rangle \otimes |\chi \otimes \gamma\rangle \\
&= \sum_{\chi,\gamma} \frac{\langle\lambda_{\chi\gamma^{-1}}|\gamma_3\rangle||\chi_3\rangle\rangle}{\sqrt{|\hat{G}|}}|\chi \otimes \gamma\rangle \otimes |\chi \otimes \gamma\rangle.
\end{aligned}$$

From this, one may show that

$$\begin{aligned}\left\| T_3^{1,2}\Big(\sum_{\chi_3,\gamma_3} c_{\chi_3\gamma_3}|\chi_3\otimes\gamma_3\rangle\Big)\right\|^2 &= \sum_{\chi,\gamma}\left|\sum_{\chi_3,\gamma_3}\frac{c_{\chi_3\gamma_3}}{\sqrt{|\hat{G}|}}\mathrm{Tr}(\lambda_{\chi\gamma^{-1}}|\gamma_3\rangle\langle\chi_3|)\right|^2\\ &= \frac{1}{\sqrt{|\hat{G}|}}\sum_{\chi,\gamma}\left|\mathrm{Tr}\left(\lambda_{\chi\gamma^{-1}}\left(\sum_{\chi_3,\gamma_3}c_{\chi_3\gamma_3}|\gamma_3\rangle\langle\chi_3|\right)\right)\right|^2\\ &= \sum_{\gamma}\left|\sum_{\chi_3,\gamma_3}\langle c_{\chi_3\gamma_3}|\gamma_3\rangle\langle\chi_3||\lambda_\gamma\rangle_{HS}\right|^2.\end{aligned}$$

This shows that $T_3^{1,2}$ is *not* (proportional to) an isometry and hence, T is not a perfect tensor.

In order to understand why this example did not work, we will look at a concrete example of the imperfect case above and compare it with an example which we already know yields a perfect tensor.

Example 6.2.5. We take $G = \mathbb{Z}_2$ and make the details of the construction completely explicit. In particular, $\hat{G} = \mathbb{Z}_2$, $Q = C^*(\hat{G}) = C^*(\mathbb{Z}_2) = \mathbb{C}^2$ and $N = \mathrm{Char}(\hat{G}) = \mathrm{Char}(\mathbb{Z}_2) = \ell^\infty(\mathbb{Z}_2)$, where the last equality follows from the fact that $\mathrm{Char}(K) = \ell^\infty(K)$ for any finite group K.

In this case, our building block is $M = \mathrm{Char}(\hat{G}) \rtimes \hat{G} = \ell^\infty(\mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong \mathcal{B}(\ell^2(\mathbb{Z}_2))$. Notice that

$$\ell^2(\mathbb{Z}_2) = \{f : \mathbb{Z}_2 \to \mathbb{C} : |f(0)|^2 + |f(1)|^2 < \infty\} \cong \mathbb{C}^2.$$

So, $M \cong M_2(\mathbb{C})$, the C*-algebra of 2×2 matrices with complex entries. We endow $M_2(\mathbb{C})$ with the usual normalized trace $\tau := \frac{1}{2}\sum_{i=1}^2 T_{ii}$, where T_{ii} is the ith diagonal entry of a matrix $T \in M_2(\mathbb{C})$.

Moreover, for $t_0 := I$ and $t_1 := Y$, we have $M_{t_0} = M_2(\mathbb{C})$ and $M_{t_1} = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, which are connected via $\hat{R} : M_2(\mathbb{C}) \to (M_2(\mathbb{C}) \otimes M_2(\mathbb{C}))$.

By the preceding work, we have that the GNS construction for $(M_2(\mathbb{C}), \tau)$ yields the triple $(\pi_{t_0}, H_{t_0}, \Omega_{t_0})$ with the $*$-representation $\pi_{t_0} : A \mapsto A \otimes I_2$, the Hilbert space $H_{t_0} = \mathbb{C}^2 \otimes \mathbb{C}^2$, and cyclic vector $\Omega_{t_0} = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle =: |\Phi^+\rangle$ (one of the maximally entangled Bell states [32, Example 1.3.6]). Continuing further from the example above, the GNS construction for $(M_2(\mathbb{C}) \otimes M_2(\mathbb{C}), \tau \otimes \tau)$ yields the triple $(\pi_{t_1}, H_{t_1}, \Omega_{t_1})$ where the $*$-representation $\pi_{t_1} : A\otimes B \mapsto (A\otimes I_2)\otimes(B\otimes I_2)$, the Hilbert space $H_{t_1} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, and cyclic vector $\Omega_{t_1} = |\Phi^+\rangle \otimes |\Phi^+\rangle$. By Proposition 6.2.4, this construction gives us an isometry connecting the Hilbert spaces. Namely, we have

$$T : \mathbb{C}^2 \otimes \mathbb{C}^2 \to (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)$$

given by

$$T|j\rangle|k\rangle = |j\rangle|k\rangle \otimes (\lambda_{jk^{-1}} \otimes 1)|\Phi^+\rangle.$$

Recall that in Example 6.1.8, we showed that the (very similiar) map

$$V : \mathbb{C}^2 \otimes \mathbb{C}^2 \to (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)$$

defined by

$$V|j\rangle|k\rangle = \frac{1}{2}|j\rangle|\Phi^-\rangle|k\rangle,$$

describes a perfect tensor. So what's the problem?

The major difference in these two maps is the location of the entanglement. In the example of the perfect tensor, the entanglement is *between* the second and third legs, whereas in the example arising from our operator algebraic construction, the entanglement is relegated to the third leg alone.

With this example in mind, we modify the GNS construction slightly to get a connecting map between Hilbert spaces with the entanglement in the "middle".

We leave the construction for $(M_{t_0}, \omega_{t_0}) = (\mathcal{B}(\ell^2(\hat{G})), \tau)$ the same. Namely, we have the Hilbert space $H_{t_0} := \ell^2(\hat{G}) \otimes \ell^2(\hat{G})$, the GNS map

$$\Lambda_{t_0} : \mathcal{B}(\ell^2(\hat{G})) \ni [a_{\chi\gamma}] \mapsto \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi\gamma} a_{\chi\gamma}|\chi\gamma\rangle \in \ell^2(\hat{G}) \otimes \ell^2(\hat{G}),$$

the $*$-representation

$$\pi_{t_0} : \mathcal{B}(\ell^2(\hat{G})) \ni x \mapsto x \otimes 1 \in \mathcal{B}(\ell^2(\hat{G}) \otimes \ell^2(\hat{G})),$$

and the cyclic vector

$$\Omega_{t_0} = \Lambda_{t_0}(1) = \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi} |\chi\chi\rangle.$$

We adjust our construction for $(M_{t_1}, \omega_{t_1}) = (\mathcal{B}(\ell^2(\hat{G})) \otimes \mathcal{B}(\ell^2(\hat{G})), \tau \otimes \tau)$. We leave the Hilbert space as $H_{t_1} = \ell^2(\hat{G}) \otimes \ell^2(\hat{G}) \otimes \ell^2(\hat{G}) \otimes \ell^2(\hat{G})$, however, we take the GNS map to be $\Lambda_{t_1} : \mathcal{B}(\ell^2(\hat{G})) \otimes \mathcal{B}(\ell^2(\hat{G})) \to \ell^2(\hat{G}) \otimes \ell^2(\hat{G}) \otimes \ell^2(\hat{G}) \otimes \ell^2(\hat{G})$ such that

$$\Lambda_{t_1} : [a_{\chi_1\gamma_1\chi_2\gamma_2}] \mapsto \frac{1}{|\hat{G}|} \sum_{\chi_1,\gamma_1,\chi_2,\gamma_2} a_{\chi_1\gamma_1\chi_2\gamma_2}|\chi_1\gamma_1\rangle \otimes |\gamma_2\chi_2\rangle,$$

the $*$-representation to be $\pi_{t_1} : \mathcal{B}(\ell^2(\hat{G})) \otimes \mathcal{B}(\ell^2(\hat{G})) \to \mathcal{B}(\ell^2(\hat{G}) \otimes \ell^2(\hat{G}) \otimes \ell^2(\hat{G}) \otimes \ell^2(\hat{G}))$ given by

$$\pi_{t_1} : x \otimes y \mapsto x \otimes y \otimes 1 \otimes 1,$$

and the cyclic vector $\Omega_{t_1} = \Lambda_{t_1}(1) = \frac{1}{|\hat{G}|} \sum_{\chi,\gamma} |\chi\gamma\rangle \otimes |\gamma\chi\rangle$.

Proposition 6.2.6. *There is an isometry* $\tilde{R}^{t_1}_{t_0} : \ell^2(\hat{G})^{\otimes 2} \to \ell^2(\hat{G})^{\otimes 4}$ *connecting the Hilbert spaces induced by the modified GNS construction given by*

$$\tilde{R}^{t_1}_{t_0}(|\chi\rangle \otimes |\gamma\rangle) = \frac{1}{|\hat{G}|}\mathcal{U}(|\chi\rangle \otimes (\lambda_{\chi\gamma^{-1}} \otimes 1)\Omega_{t_0} \otimes \gamma),$$

where $\mathcal{U}$ *is the unitary map which acts as the identity on the first two legs and swaps the entries in the third and fourth legs.*

Proof. This result follows from a simple computation, which is almost identical to the one in Proposition 6.2.4. □

Now we can construct a perfect tensor based on the modified GNS construction. In particular, we define

$$\tilde{T}(|\chi \otimes \gamma\rangle) = |\chi\rangle \otimes (\lambda_{\chi\gamma^{-1}} \otimes 1)\Omega_{t_0} \otimes |\gamma\rangle.$$

The coefficients of the tensor are given by

$$\begin{aligned}
\tilde{T}_{(\chi_1,\gamma_1),(\chi_2,\gamma_2),(\chi_3\gamma_3)} &= \langle \tilde{T}(|\chi_1 \otimes \gamma_1\rangle)||\chi_2 \otimes \gamma_2\rangle \otimes |\chi_3 \otimes \gamma_3\rangle\rangle \\
&= \langle |\chi_1\rangle \otimes (\lambda_{\chi_1\gamma_1^{-1}} \otimes 1)\Omega_{t_0} \otimes |\gamma_1\rangle||\chi_2 \otimes \gamma_2\rangle \otimes |\chi_3 \otimes \gamma_3\rangle\rangle \\
&= \delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_3}\langle (\lambda_{\chi_1\gamma_1^{-1}} \otimes 1)\Omega_{t_0}||\gamma_2\rangle \otimes |\chi_3\rangle\rangle \\
&= \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_3}}{\sqrt{|\hat{G}|}} \sum_{\chi} \langle (\lambda_{\chi_1\gamma_1^{-1}} \otimes 1)|\chi\rangle \otimes |\chi\rangle||\gamma_2\rangle \otimes |\chi_3\rangle\rangle \\
&= \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_3}}{\sqrt{|\hat{G}|}} \sum_{\chi} \langle \chi_1\gamma_1^{-1} \cdot \chi|\gamma_2\rangle\langle\chi|\chi_3\rangle \\
&= \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_3}\delta_{\chi_1\gamma_1^{-1}\cdot\chi_3,\gamma_2}}{\sqrt{|\hat{G}|}}.
\end{aligned}$$

Moreover, a simple computation shows that

$$\tilde{T}^{2,3}_1(|\chi_1 \otimes \gamma_1\rangle) = |\chi_1\rangle \otimes (\lambda_{\chi_1\gamma_1^{-1}} \otimes 1)\Omega_{t_0} \otimes |\gamma_1\rangle = \tilde{T}(|\chi_1 \otimes \gamma_1\rangle).$$

This is an isometry as $\tilde{T}$ is equal to the GNS isometry $\tilde{R}^{t_1}_{t_0}$ (up to a unitary).

If we interpret the tensor as starting at the second leg, we have:

$$\begin{aligned}
\tilde{T}_2^{1,3}(|\chi_2 \otimes \gamma_2\rangle) &= \sum_{\chi_1,\gamma_1,\chi_3,\gamma_3} \tilde{T}_{(\chi_1,\gamma_1),(\chi_2,\gamma_2),(\chi_3\gamma_3)} |\chi_1 \otimes \gamma_1\rangle \otimes |\chi_3\gamma_3\rangle \\
&= \sum_{\chi_1,\gamma_1,\chi_3,\gamma_3} \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_3}\delta_{\chi_1\gamma_1^{-1}\cdot\chi_3,\gamma_2}}{\sqrt{|\hat{G}|}} |\chi_1 \otimes \gamma_1\rangle \otimes |\chi_3 \otimes \gamma_3\rangle \\
&= \sum_{\chi_3,\gamma_3} \frac{\delta_{\chi_2\gamma_3^{-1}\cdot\chi_3,\gamma_2}}{\sqrt{|\hat{G}|}} |\chi_2 \otimes \gamma_3\rangle \otimes |\chi_3 \otimes \gamma_3\rangle \\
&= |\chi_2\rangle \otimes \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi} (|\chi_2\gamma_2^{-1}\cdot\chi\rangle \otimes |\chi\rangle \otimes |\chi_2\gamma_2^{-1}\cdot\chi\rangle) \\
&= |\chi_2\rangle \otimes (\lambda_{\chi_2\gamma_2^{-1}} \otimes 1 \otimes \lambda_{\chi_2\gamma_2^{-1}}) \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi} |\chi \otimes \chi \otimes \chi\rangle \\
&= |\chi_2\rangle \otimes (\lambda_{\chi_2\gamma_2^{-1}} \otimes 1 \otimes \lambda_{\chi_2\gamma_2^{-1}}) |\Theta\rangle,
\end{aligned}$$

where $|\Theta\rangle := \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi} |\chi \otimes \chi \otimes \chi\rangle$ is a GHZ-type state ([17]).
Additionally,

$$\begin{aligned}
\langle \tilde{T}_2^{1,3}(\chi_2\gamma_2) | \tilde{T}_2^{1,3}(\chi_2'\gamma_2')\rangle &= \langle |\chi_2\rangle \otimes (\lambda_{\chi_2\gamma_2^{-1}} \otimes 1 \otimes \lambda_{\chi_2\gamma_2^{-1}})|\Theta\rangle || \chi_2'\rangle \otimes (\lambda_{\chi_2'\gamma_2'^{-1}} \otimes 1 \otimes \lambda_{\chi_2'\gamma_2'^{-1}})|\Theta\rangle\rangle \\
&= \delta_{\chi_2,\chi_2'} \langle (\lambda_{\gamma_2^{-1}\gamma_2'} \otimes 1 \otimes \lambda_{\gamma_2^{-1}\gamma_2'})\Theta | \Theta\rangle \\
&= \delta_{\chi_2,\chi_2'} \frac{1}{|\hat{G}|} \sum_{\chi} \langle \gamma_2^{-1}\gamma_2' \cdot \chi | \chi\rangle \cdot \langle \chi | \chi\rangle \cdot \langle \gamma_2^{-1}\gamma_2' \cdot \chi | \chi\rangle \\
&= \delta_{\chi_2,\chi_2'} \delta_{\gamma_2,\gamma_2'}.
\end{aligned}$$

That is, $\tilde{T}_2^{1,3}$ preserves inner products on basis vectors and so it is an isometry.

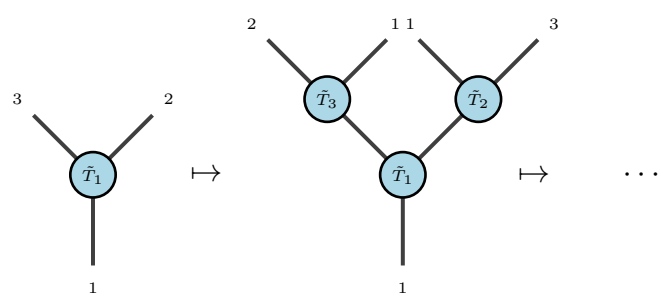
In the final direction, we have:

$$\begin{aligned}
\tilde{T}_3^{1,2}(|\chi_3 \otimes \gamma_3\rangle) &= \sum_{\chi_1,\gamma_1,\chi_2,\gamma_2} \tilde{T}_{(\chi_1,\gamma_1),(\chi_2,\gamma_2),(\chi_3\gamma_3)} |\chi_1 \otimes \gamma_1\rangle \otimes |\chi_2\gamma_2\rangle \\
&= \sum_{\chi_1,\gamma_1,\chi_2,\gamma_2} \frac{\delta_{\chi_1,\chi_2}\delta_{\gamma_1,\gamma_3}\delta_{\chi_1\gamma_1^{-1}\cdot\chi_3,\gamma_2}}{\sqrt{|\hat{G}|}} |\chi_1 \otimes \gamma_1\rangle \otimes |\chi_2\gamma_2\rangle \\
&= \sum_{\chi_2,\gamma_2} \frac{\delta_{\chi_2\gamma_3^{-1}\cdot\chi_3,\gamma_2}}{\sqrt{|\hat{G}|}} |\chi_2 \otimes \gamma_3\rangle \otimes |\chi_2 \otimes \gamma_2\rangle \\
&= \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi} (|\chi\rangle \otimes |\gamma_3\rangle \otimes |\chi\rangle \otimes |\chi \cdot \chi_3\gamma_3^{-1}\rangle) \\
&= (\mathrm{SWAP} \otimes \mathrm{id} \otimes \lambda_{\chi_3\gamma_3^{-1}})(|\gamma_3\rangle \otimes |\Theta\rangle).
\end{aligned}$$

We can use a similar argument as before to show that $\tilde{T}_3^{1,2}$ also preserves inner products on basis vectors and so it is an isometry. Therefore $\tilde{T}$ is a perfect tensor.

Remark 6.2.7. Although $\tilde{T}$ is a perfect tensor, it is *not* rotationally invariant. Clearly the maps $\tilde{T}_1^{2,3}, \tilde{T}_2^{1,3}$ and $\tilde{T}_3^{1,2}$ are not the same.

Now, fixing an orientation of $\tilde{T}$, the perfect tensor we described above, observe that we can generate a tensor network by contracting the appropriately oriented copy of T at each leg (other than the "root"). We denote $\tilde{T}_1 := \tilde{T}_1^{2,3}$; $\tilde{T}_2 := \tilde{T}_2^{1,3}$; and $\tilde{T}_3 := \tilde{T}_3^{1,2}$, in order to keep things tidy.



(6.1)

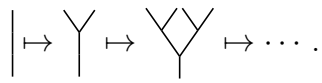
Recall that we defined the complete tree t_m to be the rooted binary tree with 2^m leaves all at a distance m from the root. Notice that the morphism $f_m := Y^{\otimes 2^m}$ takes the complete tree t_m to t_{m+1} (see, e.g., Example 3.3.6 above, in which we composed the forest $Y \otimes Y$ with the tree Y). Then $(t_m, f_m)_{m\in\mathbb{N}}$ is an inductive sequence in the category of binary forests.

In the case where the compact, separable group G is both finite and abelian and we pick the positive probability measure m to be the Haar measure on $\hat{G}$, we can describe an associated sequence of C*-algebras, as in [6]. In particular, we have a sequence of C*-algebras of the form $(M_{t_m}) \cong (\mathcal{B}(\ell^2(\hat{G}_{t_m})))$, where $\hat{G}_{t_m}$ is the product of 2^m copies of $\hat{G}$ and the connecting map $k_{t_m}^{t_{m+1}} : M_{t_m} \to M_{t_{m+1}}$ is given by applying

$\hat{R}$ to each copy of $\mathcal{B}(\ell^2(\hat{G}))$, i.e., applying $\hat{R}$ at the C*-algebra level whenever Y is applied at the binary tree level, as in Proposition 3.5.1.

Now, recall from Section 3.3 that for any tree $t \in \mathfrak{T}$, there exists some $n \in \mathbb{N}$ such that $t \leq t_n$. Therefore, we can identify any $b \in M_t$ with $k_t^{t_n}(b) \in M_{t_n}$, where $k_t^{t_n} : M_t \to M_{t_n}$ is the connecting map from our operator-algebraic construction. In this way, we *should* have that the C*-direct limit of the sequence of C*-algebras $\mathscr{M}_1 := \varinjlim_{m\in\mathbb{N}} M_{t_m}$ is equal to $\varinjlim_{t\in\mathfrak{T}} M_t =: \mathscr{M}_0$[1]. In other words, our first claim is that taking the limit of C*-algebras over the directed set of complete binary trees is "enough" to obtain the limit of C*-algebras over the directed set of *all* binary trees.

Graphically, our sequence of complete trees (below) looks strikingly like the tensor network above:



It turns out that this resemblance is not just superficial. First, consider the tensor network built from contracting appropriately oriented copies of the original (imperfect) tensor T obtained via the GNS construction. It is clear that this describes exactly the Hilbert space setting for the sequence of C*-algebras $(M_{t_m})_{m\in\mathbb{N}}$. Taking the limit Hilbert space $\mathfrak{H}$ generated by this tensor network, we obtain a unique state $\overline{\omega}$, a representation $\rho : \mathscr{M}_1 \to \mathcal{B}(\mathfrak{H})$, and a cyclic vector $\Omega \in \mathfrak{H}$, as in Proposition 6.2.2. Moreover, if our claim is correct that $\mathscr{M}_1 \cong \mathscr{M}_0$, then we should also get that $\rho(\mathscr{M}_1)'' \cong \mathscr{M} \cong \mathcal{R}$. In this way, we will think of the tensor network obtained from copies of T as a *tensor network approximation* of the operator-algebraic construction given by Brothier and Stottmeister [6].

We can do something similar in the case of the perfect tensor network. The representation of M_{t_m} at each level of the tensor network is different because of the modifications made to the GNS construction and so the representation $\tilde{\rho}$ on the limit Hilbert space will also be different. However, because our GNS modifications really only involved composition with a unitary, we will still obtain something that could reasonably be called a *unitarily equivalent tensor network approximation.*

6.3 Connections and Extensions

In 2015, a group of physicists constructed a family of toy models for the AdS/CFT correspondence based on quantum error-correcting codes with a tensor network structure [34]. These models are "exactly solvable" (an exact mapping of bulk operators to boundary operators can be constructed) and capture key features of the holographic principle, which underlies the AdS/CFT correspondence. Subsequently, a method for introducing dynamics on these type of models was described [33]. Recently, an infinite-dimensional analogue of the model was introduced, [28].

[1]This argument requires more detail to be made completely rigorous. Due to time limitations inherent to a master's thesis, we hope that the reader is convinced by this sketch of a proof.

At the outset, one goal of this thesis was to completely understand how the tensor networks associated to the rigorous operator algebraic construction of the gauge theories in [6] are related to the dynamical toy model(s) of the AdS/CFT correspondence in [33]. It seems that this goal was overly ambitious given the time constraints of a master's degree; as such, this problem remains open. However, in this final section, we will discuss the results of Osborne and Stiegemann and endeavour to give a non-rigorous explanation of their relationship. We borrow generously from [33] and use modified versions of their figures in several places.

As a matter of situating their results, we will take the following for granted—the interested reader is directed to consult [33, Section 2.2] and references therein for additional details.

(i) n-dimensional anti-de Sitter spaces (AdS_n) are a certain kind of manifold used in theories of quantum gravity.

(ii) The Poincaré disk model is a well-known model of 2-dimensional hyperbolic geometry[2]; we will denote the Poincaré disk by $\mathcal{D}$.

(iii) The $(2+1)$-dimensional anti-de Sitter space (AdS_3) can be visualized as a cylinder whose equal-time slices are copies of $\mathcal{D}$ (see Figure 6.1).

(iv) The AdS/CFT correspondence is a conjectured relationship between theories of quantum gravity in AdS_n and conformal field theories in $(n-1)$-dimensions.

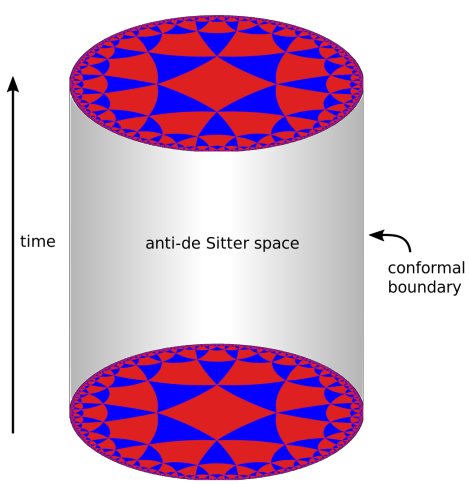
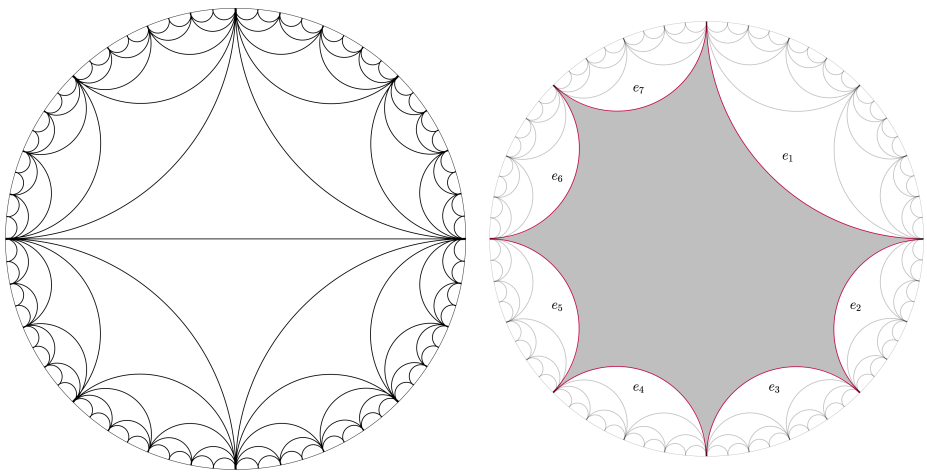


Figure 6.1: AdS_3 is a stack of hyperbolic disks, each representing the state of the universe at a given time. Source: [12].

[2]That is, a geometry satisfying all the postulates of Euclidean geometry *except* for the "parallel postulate". In hyperbolic geometry, this postulate becomes: For any straight line L and any point P not on it, there are at least two other infinitely extending straight lines which pass through P and never intersect L.

In [33], the authors build discretized models of $\mathcal{D}$ via special coverings of polygons called *tessellations*. Specifically, they focus on the *dyadic tessellation*. As the name suggests, there is some sense in which this tessellation is related to the dyadic rationals—the points of the tessellation on the boundary $\partial\mathcal{D}$ are dyadic rationals.

A *cutoff* of a given tessellation is a convex finite area of $\mathcal{D}$ bounded by a closed curve of finitely many geodesics (i.e., circles meeting the boundary $\partial\mathcal{D}$ at right angles). An example of a cutoff with boundary $\gamma = (e_1, e_2, \ldots, e_7)$ is shown in Figure 6.2 (b). The set of all cutoffs forms a directed set.



(a) The dyadic tessellation. (b) A cutoff γ.

Figure 6.2

The construction in [33] involves finding certain states and Hilbert spaces H_γ associated to a given cutoff γ coming from the dyadic tessellation. Then, they are interested in taking the direct limit of these H_γ (under some appropriate connecting maps). In particular, Osborne and Stiegemann build states associated to a cutoff γ of the dyadic tessellation by first fixing a (rotationally invariant) perfect tensor V with each leg Hilbert space being $\mathbb{C}^d$. Then, they place V in every triangle of the cutoff and contract tensor indices whenever two triangles share a common edge.

The associated Hilbert space is the tensor product of the leg Hilbert space $\mathbb{C}^d$ over each edge of the cutoff γ. In particular, for our example, we have $H_\gamma := \otimes_{j=1}^{7}\mathbb{C}^d$. Recall that a tensor is thought of as a map between Hilbert spaces via a bipartitioning of the legs $\{j_1, j_2, \ldots, j_n\}$ into sets A, A^c. Taking $A = \emptyset$, this yields a *holographic state* $|\psi_\gamma\rangle$ on H_γ. Moreover, given two cutoffs $\gamma \leq \gamma'$, we have an isometry $T_\gamma^{\gamma'} : H_\gamma \to H_{\gamma'}$ coming from the tensor networks associated to the two cutoffs; now we can take the direct limit of the system $(H_\gamma, T_\gamma^{\gamma'})_\gamma$. Osborne and Stiegemann call the Hilbert space $\mathfrak{H} = \lim_\gamma H_\gamma$ the *semi-continuous limit*, which they view as a discrete model of the CFT Hilbert space in the AdS/CFT correspondence.

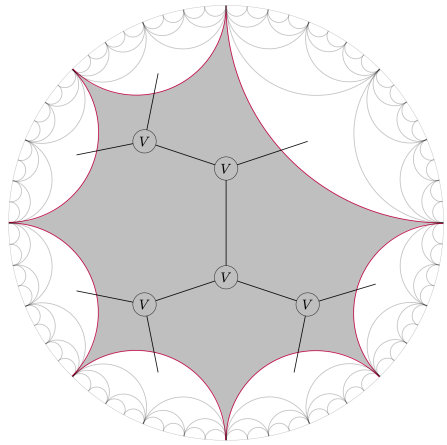
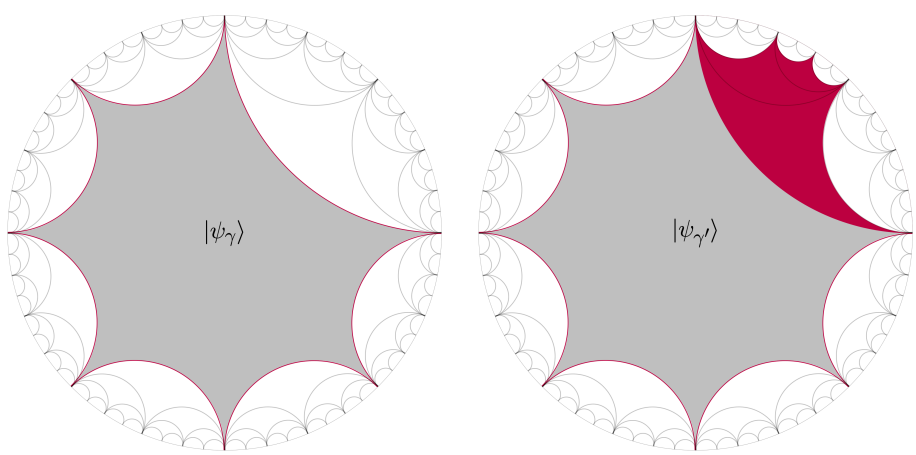


Figure 6.3: Building a holographic state.

So what does this have to do with our operator-algebraic construction(s)? Well, there is a one-to-one correspondence between cutoffs coming from a particular tessellation and partitions of the circle S^1 via the intervals determined by the endpoints of the geodesics coming from the same tessellation. In particular, that means that there is a bijection between cutoffs of the dyadic tessellation and standard dyadic partitions (s.d.p.'s).



(a) Holographic state $|\psi_\gamma\rangle$. (b) Holographic state on $\gamma' \geq \gamma$.

Figure 6.4

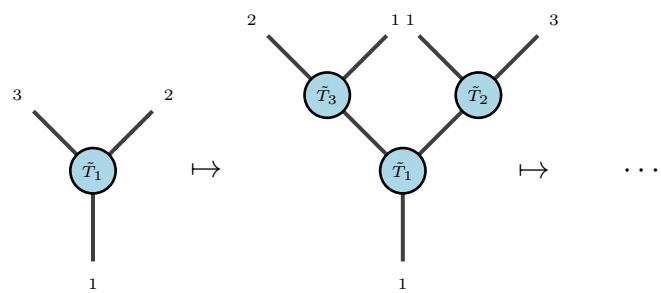
Example 6.3.1. Looking at the cutoff γ shown in Figure 6.2(b) (as well as subsequent figures), we see that it is naturally associated to the s.d.p.

$$\{[0, 1/4), [1/4, 3/8), [3/8, 1/2), [1/2, 5/8), [5/8, 3/4), [3/4, 7/8), [7/8, 1)\},$$

via the labelling of the "north pole" of the boundary circle as $0 = 1 \in \mathbb{R}/\mathbb{Z} \cong S^1$.

Moreover, as we saw in Section 3.3, there is a bijection between s.d.p's and the collection of all binary trees $\mathfrak{T}$. Therefore, there is necessarily a bijection between cutoffs of the dyadic tessellation and binary trees. In particular, our C*-algebraic directed system indexed by trees becomes a Hilbert space directed system indexed by cutoffs of the dyadic tessellation (via the GNS construction and the aforementioned bijection).

Now, recall that we previously defined a (perfect) tensor network approximation for the sequence of C*-algebras indexed by defined the complete binary trees. Namely, we described this tensor network via contractions of appropriately oriented copies of $\tilde{T} : \ell^2(\hat{G})^{\otimes 2} \to \ell^2(\hat{G})^{\otimes 4}$.



Placing this tensor network inside the unit disk (with the appropriate geometry), we get something rather like the construction of holographic states in [33].

Moreover, Osborne and Stiegemann introduce dynamics on the boundary of their toy model via Thompson's group T. We know that there is a well-defined action of Thompson's group V (which contains T as a subgroup) on the von Neumann algebra $\mathscr{M}$ in our operator-algebraic construction and it preserves the limit state $\overline{\omega}$. As such, it seems that we should be able to interpret the net of local field algebras $\mathscr{M}(O)$ from [6] as the local observable algebras on the semi-continuous limit Hilbert space $\mathfrak{H}$ coming from the tensor network construction.

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