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**Dynamic Modeling and Stability Analysis of
a “Smart” Hingeless Helicopter Blade**

by

Solly Solaiman, B.Eng.

Carleton University

A thesis submitted to
The Faculty of Graduate Study and Research
in partial fulfillment of
the requirements for the degree of
Master of Engineering

Department of Mechanical and Aerospace Engineering

Ottawa-Carleton Institute

For Mechanical and Aerospace Engineering

Carleton University
Ottawa, Ontario
September 10, 1999
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acceptance of the thesis

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Thesis Co-Supervisor

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Chairman,

Department of Mechanical and Aerospace Engineering

Carleton University

September 24, 1999

Abstract

This study was carried out as part of research contract for Structures, Materials and Propulsion Laboratory (SMPL), Institute for Aerospace Research (IAR) of National Research Council (NRC) under contract No. 31184-6-0296/001/ST: "Helicopter Individual Blade Control Strategies Using Smart Structures Technology".

The main objective of this project is to investigate the use of a "smart" spring, developed at the National Research Council of Canada, to actively control the dynamics of an individual hingeless blade. The conceptual design of the "smart" spring has been granted several patents.

A set of dynamic equations of motion of the "smart" blade, which describe the unsteady blade motion near the equilibrium operating condition, are derived based on the work done by Hodges and Dowell (Ref. 11). In order to present a realistic case study, a model of a "smart" blade is developed based on a reduced scale EUROCOPTER BO105 helicopter blade.

Harmonic parametric actuation is then applied to the "smart" blade using a square wave (on-off) control law. To evaluate the stability of the periodic-coefficient equations of motion associated with the Individual Blade Control, Floquet method is used. Finally, the results are compared with those associated with the uncontrolled blade.

Significant improvements in the damping are observed. Thus, the application of the Individual Blade Control using the "smart" spring technology can be suggested to improve aeroelastic phenomena such as flutter and divergence on a hingeless helicopter blade.

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To the glory of God, I dedicate this work.

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List of Symbols

a	Lift-curve slope
a_1, b_1, e_1, f_1	Proportionality constants of the cross-section dimensions of the “smart” blade
A	Section area, $\iint_A d\eta d\zeta$
$A_i - O_{ij}$	Integrals of the products of basis functions
b	Number of blades
$B(t)$	Solution matrix to a homogeneous periodic system
$b(T), b(U)$	Kinetic and strain energy boundary terms
B_1^*	Blade cross-section integral
B_2^*	Blade cross-section integral
c	Blade chord
c_1, d_1	Cross-section dimensions of the “smart” blade
c_{d_0}	Drag coefficient
C_k	Coefficient associated with the solution to a homogeneous periodic system
$[C]$	Modal damping matrix
$[\Delta C]$	Modal damping matrix due to “smart” spring
C_1	Blade cross-section integral
C_1^*	Blade cross-section integral

$C(k)$	Theodorsen's function
C_T	Rotor thrust coefficient
D	Airfoil profile drag per unit length
e	Mass centroid offset from elastic axis, positive when in front of the elastic axis
e_A	Tensile axis (area centroid) offset from elastic axis, positive when in front of the elastic axis
E	Young's modulus
G	Shear modulus
$[G]$	Modal damping matrix without the aerodynamic terms
h	Vertical displacement of two dimensional airfoil section normal to free-stream velocity component V
$\bar{i}, \bar{j}, \bar{k}$	Unit vectors associated with undeformed blade coordinate systems
$\bar{i}', \bar{j}', \bar{k}'$	Unit vectors associated with deformed blade coordinate systems
$\bar{I}, \bar{J}, \bar{K}$	Fixed unit vectors associated with undeformed blade coordinate systems, before undergoing a pre-cone displacement
$[I]$	Identity matrix
I_y, I_z	Area moments of inertia, $\iint_A \zeta^2 d\eta d\zeta, \iint_A \eta^2 d\eta d\zeta$
J	Area polar moment of inertia corrected for warping, $\iint_A (\hat{\eta}^2 + \hat{\zeta}^2) d\eta d\zeta$
k	Reduced frequency (Greenberg aerodynamic formulation), $\omega c / 2V$
k_A^2	Squared area radius of gyration, $\iint_A (\eta^2 + \zeta^2) d\eta d\zeta / A$
k_m^2	Squared mass radius of gyration, $k_{m_1}^2 + k_{m_2}^2$
$k_{m_1}^2, k_{m_2}^2$	$\iint_A \rho \zeta^2 d\eta d\zeta / m,$ $\iint_A \rho \eta^2 d\eta d\zeta / m$

K	Non-dimensional parameter
$[K]$	Modal stiffness matrix
$[K_s]$	Modal stiffness matrix involving the structural terms only
$[K_a]$	Modal stiffness matrix involving the aerodynamic terms only
$[\Delta K]$	Modal stiffness matrix due to “smart” spring
L	Aerodynamic lift per unit length, also dimension of matrix
L_u, L_v, L_w	Aerodynamic forces in the u , v - and w -directions per unit of length
m	Mass per unit of length, $\iint_A \rho d\eta d\zeta$
M	Aerodynamic pitching moment per unit length
M_1	Number of rotating, coupled modes
$M_x, M_{x'}$	Twisting moment about x and x'
$M_y, M_{y'}$	Bending moment about y and y'
$M_z, M_{z'}$	Bending moment about z and z'
M_ϕ	Aerodynamic moment in the ϕ -direction per unit of length
$[M]$	Modal mass matrix
$[M_m]$	Diagonalized modal mass matrix
$[\Delta M]$	Modal mass matrix due to “smart” spring
n	Frequency of harmonic parametric actuation of the “smart” spring
N	Number of non-rotating, uncoupled modes in the Galerkin formulation
O	A fixed point on the axis of rotation, at the root of the blade
$[P], [P^*]$	State space matrices of the perturbation equations
p_k	Exponential constant associated with the solution to a harmonic periodic system
$P_{x'}$	Warp term in the strain energy expression

Q	Inertial frame of reference
r	Distance along the deformed elastic axis
R	Rotor radius
$\Re$	Flap-lag structural coupling parameter
S	Aerodynamic force per unit length tangent to the airfoil
$S_{x'}$	Twisting moment due to shear stress
t	Time and blade thickness
T	Period of actuation, kinetic energy, and also blade tension
u, v, w	Blade extensional, lead-lag and flap bending displacements (along x, y, z)
U	Strain energy; blade airfoil velocity with respect to the fluid
$[U]$	Matrix of free vibration, non-rotating eigenvectors
v_i	Rotor induced velocity
V	Free stream velocity component of two dimensional airfoil
$\bar{V}$	Velocity of a point on the blade
$V_x, V_{x'}$	Axial force (tension) in the x and x' direction
$V_y, V_{y'}$	Shear force in the y and y' direction
$V_z, V_{z'}$	Shear force in the z and z' direction
V_j, W_j, V_0, W_0	Lead-lag and flap generalized coordinates and trim deflections
x, y, z	Undeformed blade coordinate system (x is the elastic axis)
$\bar{x}_1, \bar{x}_2$	Non-dimensional starting and ending positions of the “smart” spring in the longitudinal direction of the blade
x_p, y_p	Orthogonal projections on the x and y coordinate system after the blade has been rotated through β_{pc}

x', y', z'	Deformed blade coordinate system (x' is the elastic axis)
$\{X\}$	Vector of generalized perturbations
$\bar{Y}_u, \bar{Y}_v, \bar{Y}_w, \bar{Y}_\phi$	Notation for writing the energy expressions in a concise form
$Z_u, \bar{Z}_u$	Notation for writing the energy expressions in a concise form
$Z_v, \bar{Z}_v, Z_w, \bar{Z}_w$	Notation for writing the energy expressions in a concise form
$Z_\phi, \bar{Z}_\phi, Z_\psi, \bar{Z}_\psi$	Notation for writing the energy expressions in a concise form
$Z_\phi, \bar{Z}_\phi$	Notation for writing the energy expressions in a concise form
α	Airfoil section angle of attack
$\alpha_j, \beta_j, \gamma_j$	Constants for assumed mode shapes
β_{pc}	Pre-cone angle
γ	Lock number, $3\rho_\infty acR/m$
ε	Small parameter
$\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13},$ $\varepsilon_{22}, \varepsilon_{23}, \varepsilon_{33}$	Tensor strain components
$\varepsilon_{xx}, \varepsilon_{x\eta}, \varepsilon_{x\zeta}$	Engineering strain components
η, ζ	Principle coordinate system of the blade cross section
$\hat{\eta}, \hat{\zeta}$	$\eta - \frac{\partial \lambda}{\partial \zeta}, \zeta + \frac{\partial \lambda}{\partial \eta}$
θ	Blade pitch angle
$\Theta(\bar{x})$	Non-rotating torsional mode shape
κ	Non-dimensional torsional rigidity
λ	Eigenvalues of the transition matrix
$\lambda(\eta, \zeta)$	Warp function
Λ_1, Λ_2	Non-dimensional bending stiffnesses

μ, μ_1, μ_2	Non-dimensional radii of gyration
ν	Poisson's ratio
ρ, ρ_∞	Structural mass density, air density
σ	Rotor solidity, $bc/\pi R$
$\sigma_{11}, \sigma_{12}, \sigma_{13},$ $\sigma_{22}, \sigma_{23}, \sigma_{33}$	Tensor stress components
$\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$	Engineering stress components
ϕ, ϕ_0, ϕ_j	Torsional displacement, steady state torsional displacement, and torsional generalized coordinate
ψ	Azimuth angle
$\Psi_j(\bar{x})$	Lead-lag and flap assumed mode shape
ω, ξ	Frequency and damping of the aeroelastic mode
$\omega_v, \omega_w, \omega_\phi$	Fundamental lead-lag, flap, and torsion natural frequencies
Ω	Rotor spin velocity
$()'$	Derivative with respect to x
$(\dot{})$	Derivative with respect to t or ψ
$(\bar{})$	Non-dimensionalized quantity
$\Delta()$	Refers to perturbation quantities
Superscript T	Refers to matrix transpose operation
Subscript $xj(k)(1)(2)$	Refers to modal integrals due to smart spring
Subscript C	Refers to circulatory aerodynamic term
Subscript $_{NC}$	Refers to non-circulatory aerodynamic term
Subscript $_{NR}$	Refers to non-rotating non-dimensional natural frequency
Subscript $_{ss}$	Refers to “smart” spring

Chapter One

Introduction

1.1. Overview of Helicopters

A helicopter is an aircraft that uses large diameter, low disk loading rotary wings to provide lift, propulsion, and control. Aerodynamic forces on a helicopter are generated by the relative velocity of the rotating wings with respect to the air. In contrast, for a fixed wing aircraft, the lift, propulsion, and control forces are provided largely by separate aerodynamic surfaces.

The fundamental characteristic of a helicopter is the capability of vertical take-off and landing, which is in contrast to a fixed wing aircraft that requires a horizontal motion to take-off and land. This characteristic makes a helicopter the appropriate choice for different applications, such as transportation in urban and mountainous areas where good maneuvering ability in relatively tight spots is needed. A specific helicopter operation mode in which there is no relative velocity with respect to air, either in vertical or horizontal direction, is called hover.

Several criteria must be considered in designing a helicopter due to the nature of the operation. Some of the design criteria are as follows:

1. The vertical flight capability demands a higher power requirement than a fixed wing aircraft of the same weight and size, a factor that influences the operation costs.
2. A large transmission is required to deliver the power to the rotor at high torque and low speed.

3. The blade, being a long, slender beam, introduces vibration to the rotor shaft and the airframe of the helicopter. As a result, pilot fatigue, passenger discomfort, and maintenance costs are of important considerations.
4. Alternating loads on the mechanically complex rotor decrease the life of the components thus resulting in high maintenance costs.
5. The stability and control characteristics are often marginal, especially in hover, unless a reliable control system is used. This particular design criterion is the basis of this study.
6. Aircraft noise is an important factor in helicopter operation since it often involves operations close to urban areas. In addition, pilot and passenger comfort due to the aircraft noise is an important criterion.

1.2. Helicopter Configurations

The rotor arrangement plays an important role in the stability and the control characteristics of the helicopter. When power is delivered to the rotor through the shaft, there is a torque associated with it. Since in a steady flight no net forces and moments are allowed, the torque must be balanced in some manner. Two methods are generally used to balance the torque, which lead to the following helicopter configurations:

1. **The single main rotor and tail configuration** uses a small auxiliary tail rotor to provide the torque balance and yaw control (Fig. 1.1). The tail rotor is usually placed on a shaft that is parallel to the helicopter lateral axis, and is usually placed vertically. In this configuration, the thrust produced by the tail rotor acts on a moment arm about the main rotor shaft, balancing the torque of the main rotor.

2. **The twin main rotor configuration** uses two counter rotating rotors, of equal size and loading, to produce equal and opposing torque. This configuration has the advantage of having no yaw moment. The rotor-to-rotor aerodynamics interference, however, absorbs approximately an equal amount of power. The most common twin rotor arrangement is the tandem helicopter configuration, where the rotors are placed fore and aft of the fuselage with the aft rotor placed at a slightly higher position with significant rotor disk overlap (Fig. 1.2). The less common twin rotor configuration is the side by side arrangement.

1.3. Helicopter Rotors

To achieve an efficient vertical flight a large rotor is required. A high aspect ratio is also needed to achieve a good aerodynamic efficiency. As a result, the blade is designed as a long, slender, and flexible structure. Consequently, large motions of the rotor blade are produced by the aerodynamic loads, which can lead to large stresses in the blade and large moments at the root, which are transmitted to the fuselage. Therefore, emphasis must be given to achieve the optimum design of the rotor blade and the hub.

A frequently adopted design solution in the early development of helicopter has been to use hinges at the root to allow free motion of the blade, which eliminate bending moments at the root (Fig. 1.3). Out of plane blade motion about the hinge lying in the rotor disk is called the flap motion. Motion about the vertical hinge produces in-plane deflection called the lead-lag motion. For a hingeless blade, the flap and lead-lag motions are the motions in the out-of-plane and in-plane directions, respectively. The pitch of the blade is often changed to control the aerodynamic loads of the rotor. This

motion is accomplished by a rotation about a pitch bearing at the root. With the advancement of material and design technology, some current blades have been designed by eliminating some or all of the hinges. Since the loads become higher for the hingeless arrangement, the blade and the hub must be able to accommodate substantial structural bending associated with the motions. Hence the blade motions become a dominant factor in the stability of the rotor.

The mechanical arrangement associated with the flap and lag hinges at the hub provides the fundamental classification of rotor types as follows:

1. **Articulated rotor:** The blades are attached to the hub with flap and lag hinges (Fig. 1.3).
2. **Teetering rotor:** Two blades forming a continuous structure are attached to the rotor shaft with a single flap hinge in a teetering arrangement without lag hinges. When the rotor has three or more blades attached to the hub, the hinges are replaced by a gimbal or universal joint arrangement.
3. **Hingeless rotor:** The blades are attached to the hub without flap or lag hinges, but often with a pitch bearing. In this configuration, each blade is installed as a cantilevered structure to the hub so that blade motion occurs through bending at the root (Fig. 1.4).

1.4. Aeroelastic Stability of a Hingeless Helicopter Blade

Helicopter rotor aeroelasticity is a complex problem involving structural, inertial and aerodynamic loads. In general, there are two sources contributing to aeroelastic instability problems in helicopter rotors. The first is the coupling between the dynamics

of the rotor and that of the airframe. The second source is the dynamic coupling involving the modes of the individual blades. Therefore, availability of a sound dynamic model is instrumental in a thorough understanding of rotor blade dynamics in order to investigate and analyze the stability problem associated with helicopter rotor aeroelasticity.

Presently, two types of helicopter rotor configurations are predominant in the rotorcraft industry. They are the hingeless blade, in which the flap and lead-lag hinges are absent, and the more mechanically complex configuration of the articulated blade with the flap and lead-lag hinges. The hingeless configuration is of particular interest due to the fact that it affords an increased controllability. However, due to the nature of its configuration, a hingeless rotor can be easily susceptible to instability problems. This problem is caused by the non-linear coupling that occurs between the flap, lead-lag and torsional motions of the cantilever blade. The strong coupling of these motions makes the analysis of the hingeless rotor a complicated and important subject.

Although a substantial amount of research in the past has focused on the hingeless rotor dynamics, there still remain many aspects of the problem that need to be investigated. The main objectives of this study are: 1) to investigate and 2) to improve the stability of the system by incorporating a so-called “smart” spring into the design of the blade. As an initial attempt in this regard, only hover flight conditions are considered in this work. Moreover, as suggested in Ref. 12, the rotor is modeled as a long, straight, slender, homogeneous, isotropic beam with zero pre-twist for which the offsets between the axial tension, elastic center, mass and aerodynamic axes are considered to be negligible.

1.5. Dynamic Modeling of Rotor Blades

Early investigation of flap-lag-torsion interaction in elastic cantilever rotor blades were pioneered by simpler analyses of a torsionally rigid blade with spring restrained hinges at the root that modeled the flap and lag degrees of freedom. These analyses revealed complicated flap-lag coupling in the aerodynamic and inertial terms of the resulting equations of motion. At the time, Houbolt and Brooks (Ref. 15) developed the most comprehensive equations of motion to describe the behaviour of a rotating flexible beam.

The investigation by Hodges and Ormiston (Ref. 14) using the rigid blade approximation in hover, showed that the flexibility distribution in the outboard and inboard of the pitch bearing (the degree of the elastic structural coupling between flap-lag motions) played an important role in the aeroelastic stability. The flap-lag stability of an elastic cantilever blade with uniform properties was investigated by Hodges and Ormiston (Refs. 22, 14) who obtained non-linear partial differential equations of motion. They concluded that the rigid blade approximation held true to accurately predict lead-lag damping when the flap-lag structural coupling was modeled properly. Friedmann and Tong (Ref. 6, 7) studied similar equations without the flap-lag structural coupling to determine the stability characteristics of the helicopter blade. An asymptotic expansion procedure was employed in a simple linearized stability analysis with multiple time scales and the results indicated certain configurations to be unstable. When a more elaborate analysis was used the results indicated large amplitude limit cycle behaviour.

Realizing the restrictions of the rigid blade model, efforts to investigate the complete problem of a torsionally elastic cantilever blade were made subsequently. Mil et al (Ref.

19) found that torsion deflections of an elastic hingeless rotor blade are strongly influenced by non-linear moments caused by flap and lag bending (these are discussed in Section 2.4.5). The bending-torsion structural coupling was found to be proportional to the product of the flap and lead-lag displacements and the difference between the two bending stiffnesses.

One of the first efforts to understand the non-linear flap-lag-torsion coupling of a flexible cantilever blade was undertaken by Arcidiacono (Ref. 1). In this work, the equations of motion developed were derived in a reasonably detailed and complete manner with some non-linear inertial terms being neglected.

Friedmann and Tong (Ref. 6) also developed flap-lag-torsion equations of motion by considering an elastic flap-lag uniform blade with torsion deflections modeled as rigid body pitching motion (root torsion). The results presented in Ref. 6 showed that torsion motion was important and the stability was strongly influenced by the number and types of assumed mode shapes. The structural coupling was not considered in these reports.

Using two complementary methods: 1) the variational method based on the Hamilton's principle, and 2) the Newtonian method based on the equilibrium of a differential blade element, Hodges and Dowell (Ref. 11) corrected and extended Hodges' derivation of non-linear flap-lag-torsion equations of motion of a twisted non-uniform rotor blade (Ref. 10). Hodges and Ormiston (Ref. 12) used these equations for a solid uniform rotor blade cross-section and found that most hingeless configurations without a pre-cone angle were stable. The main destabilizing factors were found to be the structural coupling between the flap and lag bending as well as low torsion stiffness values.

In the present investigation, the equations of motion developed by Hodges and Dowell (Ref. 11) are used to evaluate the stability of an elastic cantilever rotor blade with zero pre-twist and negligible offsets between the elastic, mass, aerodynamics, and tension axes. A comprehensive, yet succinct, derivation of equations of motion developed by Hodges and Dowell (Ref. 11) is presented. This is deemed to be crucial for achieving the objectives of the investigation as outlined earlier. In fact, the incorporation of the “smart” spring at any desired location along the blade to improve its aeroelastic stability necessitates a further elaboration and expansion of these equations of motion. This is accomplished by defining flap, lag, and torsion stiffnesses as well as radii of gyrations of the cross-section as functions of the distance from the hub along the blade. These functions are then incorporated in the application of the Galerkin method to obtain the root loci of the perturbed equations, which in turn are used to investigate the stability of the system. In this process, the non-linear terms in the perturbed equations of motion are linearized about the trim (steady state) displacements. This allows employing the conventional eigenvalue approach in the stability analysis without neglecting the non-linear terms.

1.6. General Solution Methodology

The non-linear, integro-partial differential equations of motion with variable coefficients are solved by Galerkin’s method. To this end, once these equations are appropriately non-dimensionalized, in order to apply Galerkin’s method, the flap, lead-lag and torsional displacements are prescribed as series in generalized coordinates and mode shape functions.

This leads to two sets of equations of motion. By substituting the steady state quantities of the generalized displacements into the non-linear modal equations of motion a first set of non-linear algebraic equilibrium equations is obtained. A second set of equations is available by substituting both the steady state and perturbation quantities of the generalized displacements into the modal equations, subtracting the equilibrium equations, and neglecting all the non-linear products of the perturbation quantities. This results in the linearized equations of perturbation in which the coefficients are functions of the equilibrium solution. These equations of perturbation define the unsteady blade motion about the equilibrium displacements, the stability of which is investigated by analyzing the eigenvalues of the system.

Chapter Two

Non-Linear Equations of Motion

2.1. General Parameters of a Hingeless Rotor Configuration

In the stability analysis of a hingeless helicopter rotor, the structural configuration parameters associated with a cantilevered rotor blade become of significant importance. The present study is concerned with primary configuration parameters that affect the aeroelastic stability of a hingeless rotor blade. These parameters are briefly discussed in this section.

In general, a hingeless helicopter rotor consists of two main flexible segments, namely: the inboard and the outboard segments as outlined in Fig. 1.4. The inboard segment is attached to the hub by the root end. The outboard segment is connected to the inboard segment by a pitch bearing. The pitch bearing can be rotated through a pitch link that is modeled as a spring element attached to a swash plate. This arrangement allows a rigid body pitching motion of the outboard segment. Generally, the inboard and the outboard segments have non-uniformity in mass, bending, and torsion rigidity. This non-uniformity is the cause of complicated structural couplings between flap, lead-lag, and torsion deflections. Other important parameters are pre-cone, droop, sweep, and torque offset, which is also shown in Fig. 2.1. These parameters minimize the steady state stresses and improve the flying quality of the helicopter.

The above features may be different in some hingeless configurations. It is common, for example, for the inboard segment to be replaced entirely by a pitch bearing attached to the hub, thus, leaving only the single blade outboard segment. Also common is to

eliminate the pitch bearing by having a flexible inboard segment to accommodate the pitching motion by twisting the inboard segment.

The blade structure by itself also contributes to the aeroelastic stability of the rotor. A blade structure may be composed of spar, skin, stiffeners, and balance weight components or it may be constructed from many layers of composite materials. Various arrangements are made to meet the desired aerodynamic, strength, and stiffness properties.

Tailoring structural coupling becomes an important consideration in designing a good helicopter rotor blade. Well-designed pitch-flap and pitch-lead-lag couplings, for example, will improve the flying quality and stability of the helicopter operation. Similar results can be achieved by designing a good kinematics of the pitch link connection between the outboard segment and the swash plate. This is specially true if the inboard segment is long and flexible, resulting in increasing pitching motion due to the increase in bending.

The analysis outlined in this chapter, which is a review of the work by Hodges and Dowell (Ref. 11), treats a flexible, uniform rotor blade with pre-cone and in the absence of sweep, droop, and torque offset. The blade is also assumed to have a zero axial pre-twist and no offsets between the mass, elastic, tension, and aerodynamic axes. The equations of motion are derived for a single outboard segment without the flexible inboard segment. In this case, the structural coupling between the inboard and the outboard segments is represented by introducing a special parameter, β , whose value ranges from 0 (uncoupled case) to 1 (completely coupled case). A further discussion

regarding the structural coupling between the inboard and outboard segments will be covered in Section 2.4.5.

2.2. Coordinate Systems

The undeformed blade is identified by the coordinate system x, y, z with the x -axis coinciding with the elastic axis, with respect to inertial frame Q , that is represented by X, Y, Z attached to the hub at the root of the blade Fig. 2.2. This undeformed coordinate system rotates with a constant angular velocity Ω . The y -axis lies in the plane of rotation which is identified by the plane X, x_p, Y . x_p is the orthogonal projection of the x -axis on the plane of rotation through a small pre-cone angle, β_{pc} , Fig. 2.2. The blade is subjected to extension, flap bending, lead-lag bending, and torsional rotation ϕ , and pitch angle rotation θ . The components of deformation along the x, y, z -axes are represented by u, v, w , respectively.

A second coordinate system, x', y', z' is associated with the deformation of the blade. This coordinate system is fixed to the deformed blade with y' and z' axes parallel to the principal coordinate system η, ζ of the blade cross-section. The projection of the blade cross-section, before and after deformation on the y, z plane is shown in Fig. 2.3. Before deformation, the blade is rotated through a pitch angle of θ (Fig. 2.3). After deformation, a point on the elastic axis is displaced by u, v, w and the cross-section is subjected to an elastic angle of twist ϕ . Note that the cross-section of the blade after deformation does not lie on the y, z plane, the projection of it is shown in Fig. 2.3.

2.3. Blade Equations of Motion

As mentioned earlier, the main objective of this project is to investigate the use of a “smart” spring to actively control the dynamics of an individual hingeless blade. An important and fundamental aspect of this work depends on employing a non-linear 3-D dynamic model for a single helicopter rotor blade. Hodges and Dowell (Ref. 11) developed non-linear equations of motion for a rotating non-uniform cantilever rotor blade using both the variational method based on Hamilton’s principle as well as the Newtonian method based on equilibrium. The variational method that is reviewed in this chapter is of particular interest due to its rigorous and systematic approach. In general, the resulting equations of motion are valid to second order for long, straight, slender, isotropic, homogeneous beams with mass and area centroid axes offsets from the elastic axis. Also varying mass and cross-sectional stiffness properties are allowed along the beam axis as well as a varying pre-twist with small pre-cone angle undergoing moderate displacements.

The displacements considered are the axial displacement u , two transverse displacements: v , for lead-lag bending and w , for flap and one torsional displacement, ϕ . The ordering scheme, that is discussed in detail in Section 2.4.4, is based on the restriction that the squares of the bending slopes, the torsion deformation, the chord/radius, and the thickness/radius ratios are small compared to unity. The strain-displacement relations are obtained using non-linear exact transformation between deformed and undeformed coordinate systems and they play a significant role in the development of the strain energy component of the Hamilton’s principle.

Initially, this dynamic model is used to determine the stability of an uncontrolled hingeless blade. Next, a “smart” spring is incorporated at the blade section near the root. A new set of equations of motion is developed to accommodate the inclusion of the “smart” spring in the helicopter rotor (Section 3.2). Finally, the dynamic behavior of the controlled rotor is investigated using this latter set of equations of motion to compare the stability of a controlled rotor with that of the uncontrolled one.

2.4. Derivation of Equations by Hamilton Principle

The derivation of equations of motion for a rotating cantilever blade using Hamilton’s principle is presented in this section. It includes a brief review of the work achieved by Hodges and Dowell (Ref. 11) which is of fundamental importance to the work developed later on during the present research. The blade is modeled as a long, straight, homogeneous, isotropic beam with varying mass and stiffness properties along its axial direction. In this formulation, the mass, tension, and aerodynamic axes of the blade are not coincident with the elastic axis and a small pre-cone angle is allowed.

Hamilton’s principle may be expressed as:

$$\int_{t_1}^{t_2} [\delta(U - T) - \delta W] dt = 0 \quad (2.1)$$

where U is the strain energy, T is the kinetic energy, and W is the work due to the external loads. Moreover, δ denotes a variation on the quantity, t represents time, with t_1 and t_2 corresponding to initial and final arbitrary time respectively.

2.4.1. Strain Energy Contributions

The expression for the first variation of strain energy δU , in terms of engineering stress-strain relation is given by:

$$\delta U = \int_0^R \int_A (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{x\eta} \delta \varepsilon_{x\eta} + \sigma_{x\zeta} \delta \varepsilon_{x\zeta}) d\eta d\zeta dx \quad (2.2.1)$$

where R and A represent the radius of the blade and the cross-section area, respectively. Moreover, σ, ε denote engineering stress and engineering strain, respectively. In their salient work, Hodges and Dowell (Ref. 11) have obtained the following strain displacement equations:

$$\begin{aligned} \delta \varepsilon_{xx} &= \delta u' + v' \delta v' + w' \delta w' + (\eta^2 + \zeta^2)(\theta + \phi)' \delta \phi' - \lambda \delta \phi'' \\ &\quad - [\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)](\delta v'' + w'' \delta \phi) \\ &\quad - [\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)](\delta w'' - v'' \delta \phi) \\ \delta \varepsilon_{x\eta} &= -\hat{\zeta} \delta \phi' \\ \delta \varepsilon_{x\zeta} &= \hat{\eta} \delta \phi' \end{aligned} \quad (2.2.2)$$

where ' denotes the first derivative with respect to x . Moreover, the following notation is used:

$$\begin{aligned} \hat{\eta} &= \eta - \frac{\partial \lambda}{\partial \zeta} \\ \hat{\zeta} &= \zeta + \frac{\partial \lambda}{\partial \eta} \end{aligned} \quad (2.2.3)$$

with λ representing the warp function.

Noting that:

$$\begin{aligned} \sigma_{xx} &= E \varepsilon_{xx} \\ \sigma_{x\eta} &= G \varepsilon_{x\eta} \\ \sigma_{x\zeta} &= G \varepsilon_{x\zeta} \end{aligned} \quad (2.2.4)$$

where E and G represent Young's modulus and shear modulus, and using (2.2.2), it can be shown that (Ref. 11):

$$\begin{aligned} \delta U = \int_0^R \{ & V_{x'} (\delta u' + v' \delta v' + w' \delta w') + (S_{x'} + T_{x'}) \delta \phi' + P_{x'} \delta \phi'' \\ & + [M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)] (\delta v'' + w'' \delta \phi) \\ & + [M_{z'} \sin(\theta + \phi) - M_{y'} \cos(\theta + \phi)] (\delta w'' - v'' \delta \phi) \} dx \end{aligned} \quad (2.3)$$

where the following internal forces and moments are defined:

$$\begin{aligned} V_{x'} &\equiv \iint_A \sigma_{xx} d\eta d\zeta \\ &= EA \left\{ u' + \frac{v'^2}{2} + \frac{w'^2}{2} + k_A^2 \theta' \phi' - e_A [v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)] \right\} \end{aligned} \quad (2.4.1)$$

$$S_{x'} \equiv \iint_A (\hat{\eta} \sigma_{x\zeta} - \hat{\zeta} \sigma_{x\eta}) d\eta d\zeta = GJ \phi' \quad (2.4.2)$$

$$\begin{aligned} T_{x'} &\equiv \iint_A (\theta + \phi)' \sigma_{xx} (\eta^2 + \zeta^2) d\eta d\zeta = EA k_A^2 (\theta + \phi)' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \\ &\quad + EB_1^* \theta'^2 \phi' - EB_2^* \theta' (v'' \cos \theta + w'' \sin \theta) \end{aligned} \quad (2.4.3)$$

$$P_{x'} \equiv - \iint_A \lambda \sigma_{xx} d\eta d\zeta = EC_1 \phi'' + EC_1^* (w'' \cos \theta - v'' \sin \theta) \quad (2.4.4)$$

$$M_{y'} \equiv \iint_A \zeta \sigma_{xx} d\eta d\zeta = EI_{y'} [v'' \sin(\theta + \phi) - w'' \cos(\theta + \phi)] - EC_1^* \phi'' \quad (2.4.5)$$

$$\begin{aligned} M_{z'} &\equiv - \iint_A \eta \sigma_{xx} d\eta d\zeta = EI_{z'} [v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)] \\ &\quad - EA e_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) - EB_2^* \theta' \phi' \end{aligned} \quad (2.4.6)$$

In equations (2.4), the following section integrals are used:

$$\begin{aligned}
A &= \iint_A d\eta d\zeta \\
Ae_A &= \iint_A \eta d\eta d\zeta \\
I_{y'} &= \iint_A \zeta^2 d\eta d\zeta \\
I_{z'} &= \iint_A \eta^2 d\eta d\zeta \\
Ak_A^2 &= \iint_A (\eta^2 + \zeta^2) d\eta d\zeta \\
J &= \iint_A (\hat{\eta}^2 + \hat{\zeta}^2) d\eta d\zeta \\
B_1^* &= \iint_A (\eta^2 + \zeta^2)^2 d\eta d\zeta \\
B_2^* &= \iint_A \eta(\eta^2 + \zeta^2) d\eta d\zeta \\
C_1 &= \iint_A \lambda^2 d\eta d\zeta \\
C_1^* &= \iint_A \zeta \lambda d\eta d\zeta
\end{aligned} \tag{2.5}$$

These integrals are to be evaluated over the structurally effective portion of the cross-section. The effective cross-sectional area carrying the tension is A ; the chord-wise and flap-wise moments of inertia are $I_{z'}$ and $I_{y'}$. J is the torsional constant, which includes cross-section warping. Without warping, J is the torsional rigidity. The tension axis offset from the elastic axis is e_A . λ is the warp function and C_1 is the warping rigidity. B_1^* , B_2^* , and C_1^* are sectional constants, the first two of which are equivalent to those used by Houbolt and Brooks (Ref. 15). Ak_A^2 is the polar moment of inertia.

Because of the anti-symmetric nature of the warping function, λ , together with the symmetry of the cross-section about the η axis, the following integrals vanish and, thus, do not appear in the equation (2.4):

$$\begin{aligned}
\iint_A (\lambda) d\eta d\zeta &= 0 \\
\iint_A (\eta\lambda) d\eta d\zeta &= 0 \\
\iint_A (\eta\zeta) d\eta d\zeta &= 0 \\
\iint_A (\zeta) d\eta d\zeta &= 0 \\
\iint_A (\zeta\{\eta^2 + \zeta^2\}) d\eta d\zeta &= 0 \\
\iint_A (\lambda\{\eta^2 + \zeta^2\}) d\eta d\zeta &= 0
\end{aligned} \tag{2.6}$$

Integration by parts of equation (2.3) yields:

$$\delta U = \int_0^R (\bar{Y}_u \delta u + \bar{Y}_v \delta v + \bar{Y}_w \delta w + \bar{Y}_\phi \delta \phi) dx + b(U) \tag{2.7}$$

where

$$\bar{Y}_u = -(V_x)' \tag{2.8.1}$$

$$\bar{Y}_v = [M_z \cos(\theta + \phi) + M_y \sin(\theta + \phi)]'' - (V_x v')' \tag{2.8.2}$$

$$\bar{Y}_w = [M_z \sin(\theta + \phi) - M_y \cos(\theta + \phi)]'' - (V_x w')' \tag{2.8.3}$$

$$\begin{aligned}
\bar{Y}_\phi &= (P_x)'' - (S_x + T_x)' - v''[M_z \sin(\theta + \phi) - M_y \cos(\theta + \phi)] \\
&\quad + w''[M_z \cos(\theta + \phi) + M_y \sin(\theta + \phi)]
\end{aligned} \tag{2.8.4}$$

$$\begin{aligned}
b(U) &= V_x \delta u \Big|_0^R + \{V_x v' - [M_z \cos(\theta + \phi) + M_y \sin(\theta + \phi)]'\} \delta v \Big|_0^R \\
&\quad + [M_z \cos(\theta + \phi) + M_y \sin(\theta + \phi)] \delta v' \Big|_0^R \\
&\quad + [M_z \sin(\theta + \phi) - M_y \cos(\theta + \phi)] \delta w' \Big|_0^R \\
&\quad + \{V_x w' - [M_z \sin(\theta + \phi) - M_y \cos(\theta + \phi)]'\} \delta w \Big|_0^R \\
&\quad + [S_x + T_x - (P_x)'] \delta \phi \Big|_0^R + P_x \delta \phi' \Big|_0^R
\end{aligned} \tag{2.8.5}$$

2.4.2. Kinetic Energy Contributions

The position of an arbitrary point after the deformation of the beam, as derived in Ref. 11, is given by (x_1, y_1, z_1) where:

$$\begin{aligned} x_1 &= x + u - \lambda\phi' - v'[\eta\cos(\theta + \phi) - \zeta\sin(\theta + \phi)] - w'[\eta\sin(\theta + \phi) + \zeta\cos(\theta + \phi)] \\ y_1 &= v + \eta\cos(\theta + \phi) - \zeta\sin(\theta + \phi) \\ z_1 &= w + \eta\sin(\theta + \phi) + \zeta\cos(\theta + \phi) \end{aligned} \quad (2.9)$$

or

$$\begin{aligned} x_1 &= x + u - \lambda\phi' - v'(y_1 - v) - w'(z_1 - w) \\ y_1 &= v + (y_1 - v) \\ z_1 &= w + (z_1 - w) \end{aligned} \quad (2.10)$$

where $-\lambda\phi'$ represents the axial displacement due to warp.

The velocity of this point with respect to the inertial frame of reference $\underline{Q}$, is

$$\vec{V} = \frac{\partial \vec{r}}{\partial t} + \Omega \vec{K} \times \vec{r} \quad (2.11)$$

where:

$$\begin{aligned} \vec{r} &= x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k} \\ \frac{\partial \vec{r}}{\partial t} &= \dot{x}_1 \vec{i} + \dot{y}_1 \vec{j} + \dot{z}_1 \vec{k} \\ \Omega \vec{K} \times \vec{r} &= -\Omega y_1 \cos \beta_{pc} \vec{i} + (\Omega x_1 \cos \beta_{pc} - \Omega z_1 \sin \beta_{pc}) \vec{j} + \Omega y_1 \sin \beta_{pc} \vec{k} \end{aligned} \quad (2.12)$$

The first term of the velocity equation is the time derivative of the position vector with respect to the rotating coordinate system. The second term corresponds to the contribution of the rotating coordinate system.

For a material with a mass density, ρ , the kinetic energy is given by

$$T = \frac{1}{2} \int_0^R \iint_A \rho \vec{V} \cdot \vec{V} d\eta d\zeta dx \quad (2.13)$$

and the variation of the kinetic energy is

$$\delta T = \int_0^R \iint_A \rho \vec{V} \cdot \delta \vec{V} d\eta d\zeta dx \quad (2.14)$$

where

$$\begin{aligned} \delta \vec{V} = & (\delta \dot{x}_1 - \Omega \cos \beta_{pc} \delta y_1) \vec{i} + (\delta \dot{y}_1 + \Omega \cos \beta_{pc} \delta x_1 - \Omega \sin \beta_{pc} \delta z_1) \vec{j} \\ & + (\delta \dot{z}_1 + \Omega \sin \beta_{pc} \delta y_1) \vec{k} \end{aligned} \quad (2.15)$$

Using (2.14) in (2.1) and performing integration by parts, and truncating the resulting expressions consistent with the ordering scheme that was employed by Hodges and Dowell (Ref. 11) we may combine the various terms to obtain:

$$\delta T = \int_0^R \iint_A (Z_u \delta u + Z_v \delta v + Z_w \delta w + Z_\phi \delta \phi + Z_{v'} \delta v' + Z_{w'} \delta w' + Z_{\phi'} \delta \phi') \rho d\eta d\zeta dx \quad (2.16)$$

where

$$\begin{aligned} Z_u &= \Omega^2 x + 2\Omega \dot{v} \\ Z_v &= \Omega^2 [v + (y_1 - v)] - \ddot{v} + \ddot{\phi}(z_1 - w) + 2\Omega \beta_{pc} \dot{w} - 2\Omega [\dot{u} - \dot{v}'(y_1 - v) - \dot{w}'(z_1 - w)] \\ Z_w &= -\Omega^2 \beta_{pc} x - 2\Omega \beta_{pc} \dot{v} - \ddot{w} - \ddot{\phi}(y_1 - v) \\ Z_\phi &= -\Omega^2 x(y_1 - v)w' + \Omega^2 x(z_1 - w)v' - \Omega^2 [v + (y_1 - v)](z_1 - w) \\ &\quad + \ddot{v}(z_1 - w) - \ddot{w}(y_1 - v) - \Omega^2 \beta_{pc} x(y_1 - v) - \ddot{\phi}[(y_1 - v)^2 + (z_1 - w)^2] \\ Z_{v'} &= -(\Omega^2 x + 2\Omega \dot{v})(y_1 - v) \\ Z_{w'} &= -(\Omega^2 x + 2\Omega \dot{v})(z_1 - w) \\ Z_{\phi'} &= -\Omega^2 \lambda x \end{aligned} \quad (2.17)$$

where ($\dot{}$) and ($\ddot{}$) denote the first and second derivatives with respect to time, respectively. Integrating over the blade cross-section, the variation of kinetic energy becomes

$$\delta T = \int_0^R (\bar{Z}_u \delta u + \bar{Z}_v \delta v + \bar{Z}_w \delta w + \bar{Z}_\phi \delta \phi + \bar{Z}_{v'} \delta v' + \bar{Z}_{w'} \delta w') dx \quad (2.18)$$

where

$$\begin{aligned}
\bar{Z}_u &= m(\Omega^2 x + 2\Omega \dot{v}) \\
\bar{Z}_v &= m\Omega^2 [v + e \cos(\theta + \phi)] + 2m\Omega(\beta_{pc} \dot{w} - \dot{u}) + 2me\Omega(\dot{v}' \cos \theta + \dot{w}' \sin \theta) \\
&\quad - m\ddot{v} + me\ddot{\phi} \sin \theta \\
\bar{Z}_w &= -m\beta_{pc}(\Omega^2 x + 2\Omega \dot{v}) - m\ddot{w} - me\ddot{\phi} \cos \theta \\
\bar{Z}_\phi &= -mk_m^2 \ddot{\phi} - m\Omega^2 (k_{m2}^2 - k_{m1}^2) \cos(\theta + \phi) \sin(\theta + \phi) \\
&\quad - me\Omega^2 x (w' \cos \theta - v' \sin \theta) - me\Omega^2 v \sin \theta \\
&\quad - me\Omega^2 \beta_{pc} x \cos \theta + me(\ddot{v} \sin \theta - \ddot{w} \cos \theta) \\
\bar{Z}_{v'} &= -me[\Omega^2 x \cos(\theta + \phi) + 2\Omega \dot{v} \cos \theta] \\
\bar{Z}_{w'} &= -me[\Omega^2 x \sin(\theta + \phi) + 2\Omega \dot{v} \sin \theta]
\end{aligned} \tag{2.19}$$

Assuming cross-section symmetry about the η axis and an anti-symmetric warp function

λ , the cross-sectional integrals are given by:

$$\begin{aligned}
m &\equiv \iint_A \rho d\eta d\zeta; \\
me &\equiv \iint_A \rho \eta d\eta d\zeta; \\
mk_{m1}^2 &\equiv \iint_A \rho \zeta^2 d\eta d\zeta; \\
mk_{m2}^2 &\equiv \iint_A \rho \eta^2 d\eta d\zeta; \\
k_{m1}^2 + k_{m2}^2 &= k_m^2; \\
\iint_A \rho \zeta d\eta d\zeta &= \iint_A \rho \eta \zeta d\eta d\zeta; = \iint_A \rho \lambda d\eta d\zeta = 0; \\
me \cos(\theta + \phi) &= \iint_A \rho (y_1 - v) d\eta d\zeta; \\
me \sin(\theta + \phi) &= \iint_A \rho (z_1 - w) d\eta d\zeta; \\
m(k_{m2}^2 - k_{m1}^2) \cos(\theta + \phi) \sin(\theta + \phi) &= \iint_A \rho (y_1 - v)(z_1 - w) d\eta d\zeta; \\
mk_m^2 &= \iint_A \rho [(y_1 - v)^2 + (z_1 - w)^2] d\eta d\zeta
\end{aligned} \tag{2.20}$$

In the above expressions, m , e , mk_m^2 , mk_{m1}^2 , and mk_{m2}^2 represent mass per unit length, the offset of the mass center from the elastic center, mass polar moment of inertia, mass moment of inertia in the flap-wise and chord-wise directions, respectively.

The differential kinetic energy given by equation (2.19) is then integrated by parts to obtain:

$$\delta T = \int_0^R \left[\bar{Z}_u \delta u + \left(\bar{Z}_v - \bar{Z}_v' \right) \delta v + \left(\bar{Z}_w - \bar{Z}_w' \right) \delta w + \bar{Z}_\phi \delta \phi \right] dx + b(T) \quad (2.21)$$

where the boundary condition associated with the integration by parts is given by:

$$b(T) = \bar{Z}_v \delta v|_0^R + \bar{Z}_w \delta w|_0^R \quad (2.22)$$

2.4.3. Generalized Non-Conservative Forces Contributions

The last term of the Hamilton's equation (2.1) is the virtual work by non-conservative loads as follows

$$\delta W = \int_0^R \left(L_u \delta u + L_v \delta v + L_w \delta w + M_\phi \delta \phi \right) dx \quad (2.23)$$

where L_u , L_v , L_w , are the applied distributed loads in the x , y , z directions, respectively, and M_ϕ is the applied distributed torque. These non-conservative forces include all loads other than elastic and inertial forces (aerodynamic and non-conservative mechanical forces). In this study, the non-conservative loads are due to only the aerodynamic loads that are discussed later in Section 2.4.6.

2.4.4. Governing Differential Equations of Motion

By substituting the 1st variation of kinetic and strain energy terms and the virtual work terms into Hamilton's equation and collecting the terms accordingly, one obtains the variational equations as well as the corresponding boundary conditions for the beam. For arbitrary, admissible displacements $\delta u, \delta v, \delta w, \delta \phi$, the coefficients of these variations in the integrand must vanish for all x from 0 to R . Thus, the application of Hamilton's principle results in the following equations (Ref. 11):

1. From the radial displacement, δu :

$$-T' - m\Omega^2 x - 2m\Omega\dot{v} = L_u \quad (2.24)$$

2. From the lead-lag displacement, δv :

$$\begin{aligned} & -(Tv')' + \left\{ -E Ae_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \cos(\theta + \phi) - EB_2^* \theta' \phi' \cos \theta - EC_1^* \phi'' \sin \theta \right. \\ & + [EI_{z'} \cos^2(\theta + \phi) + EI_{y'} \sin^2(\theta + \phi)] v'' + (EI_{z'} - EI_{y'}) \cos(\theta + \phi) \sin(\theta + \phi) w'' \}'' \\ & + 2m\Omega \dot{u} + m\ddot{v} - me\ddot{\phi} \sin \theta - 2me\Omega(\dot{v}' \cos \theta + \dot{w}' \sin \theta) - m\Omega^2[v + e \cos(\theta + \phi)] \\ & - 2m\Omega\beta_{pc} \dot{w} - \{ me[\Omega^2 x \cos(\theta + \phi) + 2\Omega\dot{v} \cos \theta] \}' = L_v \end{aligned} \quad (2.25)$$

3. From the flap displacement, δw :

$$\begin{aligned} & -(Tw')' + \left\{ -E Ae_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \sin(\theta + \phi) - EB_2^* \theta' \phi' \sin \theta + EC_1^* \phi'' \cos \theta \right. \\ & + (EI_{z'} - EI_{y'}) \cos(\theta + \phi) \sin(\theta + \phi) v'' + [EI_{z'} \sin^2(\theta + \phi) + EI_{y'} \cos^2(\theta + \phi)] w'' \}'' \\ & + m\ddot{w} + me\ddot{\phi} \cos \theta + 2m\Omega\beta_{pc} \dot{v} - \{ me[\Omega^2 x \sin(\theta + \phi) + 2\Omega\dot{v} \sin \theta] \}' = \\ & L_w - m\Omega^2 \beta_{pc} x \end{aligned} \quad (2.26)$$

4. From the torsional displacement, $\delta\phi$:

$$\begin{aligned} & -[E A k_A^2 (\theta + \phi)' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) + EB_1^* \theta'^2 \phi' - EB_2^* \theta' (v'' \cos \theta + w'' \sin \theta)]' \\ & - E A e_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) (w'' \cos \theta - v'' \sin \theta) - (GJ\phi')' + [EC_1 \phi'' + EC_1^* (w'' \cos \theta \\ & - v'' \sin \theta)]'' + (EI_{z'} - EI_{y'}) [(w''^2 - v''^2) \cos \theta \sin \theta + v'' w'' \cos 2\theta] + m k_m^2 \ddot{\phi} \\ & + m\Omega^2 \phi (k_{m2}^2 - k_{m1}^2) \cos 2\theta + me[\Omega^2 x (w' \cos \theta - v' \sin \theta) - (\ddot{v} - \Omega^2 v) \sin \theta \\ & + \ddot{w} \cos \theta] = M_\phi - m\Omega^2 (k_{m2}^2 - k_{m1}^2) \cos \theta \sin \theta - me\Omega^2 \beta_{pc} x \cos \theta \end{aligned} \quad (2.27)$$

where the tension is given by:

$$T \equiv V_{x'} = EA \left\{ u' + \frac{v'^2}{2} + \frac{w'^2}{2} + k_A^2 \theta' \phi' - e_A [v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)] \right\} \quad (2.28)$$

The boundary conditions that are obtained from the terms associated with the strain energy and the kinetic energy become:

$$\begin{aligned}
 b(U) - b(T) = & \\
 & T\delta u_0^R + (Tv' - \{[EI_{\dot{z}} \cos^2(\theta + \phi) + EI_{y'} \sin^2(\theta + \phi)]v'' \\
 & + (EI_{\dot{z}} - EI_{y'}) \cos(\theta + \phi) \sin(\theta + \phi)w'' - EAe_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \cos(\theta + \phi) - EB_2^* \theta' \phi' \cos \theta \\
 & - EC_1^* \phi'' \sin \theta\}' + me[\Omega^2 x \cos(\theta + \phi) + 2\Omega \dot{v} \cos \theta])\delta v_0^R + \{[EI_{\dot{z}} \cos^2(\theta + \phi) + EI_{y'} \sin^2(\theta + \phi)]v'' \\
 & + (EI_{\dot{z}} - EI_{y'}) \cos(\theta + \phi) \sin(\theta + \phi)w'' - EAe_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \cos(\theta + \phi) - EB_2^* \theta' \phi' \cos \theta \\
 & - EC_1^* \phi'' \sin \theta\}' \delta v_0^R + (Tw' - \{(EI_{\dot{z}} - EI_{y'}) \cos(\theta + \phi) \sin(\theta + \phi)v'' + [EI_{\dot{z}} \sin^2(\theta + \phi) \\
 & + EI_{y'} \cos^2(\theta + \phi)]w'' - EAe_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \sin(\theta + \phi) - EB_2^* \theta' \phi' \sin \theta + EC_1^* \phi'' \cos \theta\}' \\
 & + me[\Omega^2 x \sin(\theta + \phi) + 2\Omega \dot{v} \sin \theta])\delta w_0^R + \{(EI_{\dot{z}} - EI_{y'}) \cos(\theta + \phi) \sin(\theta + \phi)v'' + [EI_{\dot{z}} \sin^2(\theta + \phi) \\
 & + EI_{y'} \cos^2(\theta + \phi)]w'' - EAe_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \sin(\theta + \phi) - EB_2^* \theta' \phi' \sin \theta + EC_1^* \phi'' \cos \theta\}' \delta w_0^R \\
 & + \{GJ\phi' + EAk_A^2(\theta + \phi)' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) + EB_1^* \theta'^2 \phi' - EB_2^* \theta'(v'' \cos \theta + w'' \sin \theta) - [EC_1 \phi'' \\
 & + EC_1^*(w'' \cos \theta - v'' \sin \theta)]'\} \delta \phi_0^R + [EC_1 \phi'' + EC_1^*(w'' \cos \theta - v'' \sin \theta)] \delta \phi_0^R \quad (2.29)
 \end{aligned}$$

The more detailed derivation of Equations 2.24 - 2.29 and a discussion of various terms in these equations is presented in the salient work by Hodges and Dowell (Ref. 11). It should also be noted that an important feature of the derivation of these equations has been the introduction of an ordering scheme in which the order of magnitude of the bending slopes is represented by a small parameter ε . Then, the assumed order of magnitude of all quantities used in the derivation is as shown below. Within any equation, the terms of order of magnitude of $O(\varepsilon^3)$ were dropped, unless they contribute to the damping.

$$\begin{aligned}
\frac{c}{R}, \frac{v}{R}, \frac{w}{R} &= O(\varepsilon) \\
\frac{v_i}{\Omega R} &= O(\varepsilon) \\
\beta_{pc} &= O(\varepsilon) \\
\phi &= O(\varepsilon) \\
\theta(\text{aerodynamic forces only}) &= O(\varepsilon) \\
\theta &= O(1) \\
\frac{c_{d0}}{a} &= O(\varepsilon^2) \\
\frac{d}{\Omega dt}, R \frac{d}{dx} &= O(1) \\
\frac{EA}{m\Omega^2 R^2} &= O(\varepsilon^{-2}) \\
\frac{EI_y}{m\Omega^2 R^4}, \frac{EI_z}{m\Omega^2 R^4}, \frac{GJ}{m\Omega^2 R^4} &= O(1) \\
\frac{k_A}{R}, \frac{k_{m1}}{R}, \frac{k_{m2}}{R}, \frac{k_m}{R} &= O(\varepsilon)
\end{aligned} \tag{2.30}$$

2.4.5. Simplified Blade Equations of Motion

For the investigation at hand, some simplifications of the blade equations of motion (2.24 to 2.29) need to be made. These simplifications reflect the fact that no chord-wise offsets are assumed to be present between the elastic, mass, tension, and aerodynamic centers on the blade cross-section. Moreover, the unloaded blade is assumed to be free of any pre-twist along the axial direction. This simplified model, which is a better representation of a real rotor blade, leads to the following partial differential equations of motion with the radial aerodynamic force component L_u being neglected as a higher order term (Ref. 11):

1. From the radial displacement, δu :

$$-T' - m\Omega^2 x - 2m\Omega\dot{v} = 0 \quad (2.31)$$

2. From the lead-lag displacement, δv :

$$\begin{aligned} & -(Tv')' + [EI_{z'} - (EI_{z'} - EI_{y'}) \sin^2 \theta] v'''' + (EI_{z'} - EI_{y'}) \\ & \left[\frac{\sin 2\theta}{2} w'''' - \sin 2\theta (\phi v'')'' + \cos 2\theta (\phi w'')'' \right] + 2m\Omega(\dot{u} - \beta_{pc} \dot{w}) \\ & + m(\ddot{v} - \Omega^2 v) = L_v \end{aligned} \quad (2.32)$$

3. From the flap displacement, δw :

$$\begin{aligned} & -(Tw')' + [EI_{y'} + (EI_{z'} - EI_{y'}) \sin^2 \theta] w'''' + (EI_{z'} - EI_{y'}) \\ & \left[\frac{\sin 2\theta}{2} v'''' + \sin 2\theta (\phi w'')'' + \cos 2\theta (\phi v'')'' \right] \\ & + m\ddot{w} + 2m\Omega\beta_{pc} \dot{v} = L_w - m\Omega^2 \beta_{pc} x \end{aligned} \quad (2.33)$$

4. From the torsional displacement, $\delta\phi$:

$$\begin{aligned} & (EI_{z'} - EI_{y'})[(w''^2 - v''^2) \frac{\sin 2\theta}{2} + v''w'' \cos 2\theta] - k_A^2 (T\phi')' - (GJ\phi'') \\ & + mk_m^2 \ddot{\phi} + m\Omega^2 \phi (k_{m2}^2 - k_{m1}^2) \cos 2\theta = M_\phi - m\Omega^2 (k_{m2}^2 - k_{m1}^2) \frac{\sin 2\theta}{2} \end{aligned} \quad (2.34)$$

where the tension is given by:

$$T = EA(u' + \frac{v'^2}{2} + \frac{w'^2}{2}) \quad (2.35)$$

The mass and structural properties of the rotor blade cross-section are represented by the mass per unit length m , the flap-wise and chord-wise principal mass moments of inertia mk_{m1}^2 and mk_{m2}^2 , the torsional mass moment of inertia mk_m^2 , the radial (axial) stiffness EA , the flap-wise and chord-wise bending stiffnesses $EI_{y'}$ and $EI_{z'}$, the polar radius of gyration k_A , and the torsional rigidity GJ . The different terms appearing in the equations of motion shall be briefly described. The inertia terms will be discussed first.

In the radial displacement equation δu , the term $2m\Omega\dot{v}$ represents centrifugal force due to lead-lag velocity. When the tension in the equation is expressed as a function of the lead-lag velocity and substituted in $(Tv')'$ and $(Tw')'$ in lead-lag and flap equations respectively, one will obtain non-linear terms. Similarly, in the lead-lag equation δv , the term $2m\Omega\dot{u}$ represents Coriolis force due to radial velocity. This term will also produce non-linearity when the tension is expressed as a function of u . The two terms above must be included carefully since they represent Coriolis and centrifugal coupling between radial extension, flap and lead-lag displacements.

Structural terms play a significant role in the stability characteristics of a hingeless helicopter rotor blade operation. One can observe that there are two important structural terms in the torsion equation $\delta\phi$, namely $GJ\phi''$ and the term associated with $(EI_z - EI_y)$. The first term is the familiar torsion rigidity term. The second term is proportional to the difference of the lead-lag and flap bending stiffnesses and the products of the bending curvatures. This term is associated with non-linear bending-torsion coupling that is quite strong and sensitive to many configuration parameters. In the lead-lag δv and flap δw equations, two important terms associated with $(EI_z - EI_y)$ exist. The first term is due to linear flap-lag structural coupling while the second term is due to non-linear bending-torsion coupling.

As it can be detected from equations (2.25 - 2.26), in the general case of non-uniform blade configurations, the flap-lag structural coupling terms also depend on blade twist ϕ as well as EI_z and EI_y distributions. One of the most significant non-uniform features that strongly influences the stability of the helicopter blade is the difference between the

stiffnesses of the blade segments inboard and outboard of the pitch bearing. Because of this non-uniformity, as the pitch link is rotated, the principal elastic axes of the outboard segment rotate through an angle θ while those of the inboard segment do not. Therefore, the resultant effective orientation of the principal axes depends on blade geometry and the relative magnitude of the bending stiffnesses of the two blade segments. Although the stability of the system is heavily influenced by the variations in these coupling effects, nevertheless, based on the assumption of a simple single segment uniform beam, they have not been included in modeling the dynamics of the rotor up to this stage. In order to avoid further complexity in the governing equations of motion of the blade, rather than restoring to a more general blade configuration and a more sophisticated analysis, an approximate representation of these effects has been suggested by Hodges and Ormiston (Ref. 12). In this approach, the average inclination of the principal elastic axes of a non-uniform rotor is assumed to be equal to some fraction of the inclination of the principal axes of a uniform single segment beam. With this approach, the pitch angle, θ , is replaced by $\Re\theta$ for the terms associated with the flap and lag stiffness. The parameter $\Re$ can be assigned any arbitrary value between 0 to 1, where 1 represents a completely coupled case and 0 represents separated segments.

So far, four unknown displacements u , v , w , and ϕ are considered in the mathematical model that is presented. This model can be simplified by eliminating u and T from the equations of motion. To this end, the radial extension equation δu is integrated to obtain:

$$T = \frac{m\Omega^2}{2}(R^2 - x^2) + 2m\Omega \int_x^R \dot{v} \, dx \quad (2.36)$$

This expression for tension can then be used in δv , δw , and $\delta \phi$ equations. When it is substituted in the $\delta \phi$ equation, the second term is neglected since it is of a higher order, $O(\varepsilon^3)$. Moreover, using equations (2.35) and (2.36), one can obtain:

$$u' = \frac{m\Omega^2 R^2}{EA} \left(\frac{1-x^2/R^2}{2} \right) + \frac{2m\Omega^2 R^2}{EA} \int_x^R \dot{v} \frac{dx}{\Omega R^2} - \frac{v'^2}{2} - \frac{w'^2}{2} \quad (2.37)$$

This expression is then integrated with respect to x and differentiated with respect to time resulting in:

$$\dot{u} = \frac{2m\Omega^2 R^2}{EA} \int_0^x \int_x^R \frac{\ddot{v}}{\Omega^2 R^2} dx dx - \int_0^x (\dot{v}'v' + \dot{w}'w') dx \quad (2.38)$$

The first term in the above equation can be neglected for convenience since it is of a higher order. Essentially, this is equivalent to assuming that the rotor blade is in-extensional due to perturbation bending deflections, i.e., radial displacement of the rotor blade is a purely geometric consequence of the transverse bending deflections of the blade (Ref. 12). It should be noted that such an assumption is valid only for sufficiently large values of dimensionless radial stiffness $EA/m\Omega^2 R^2$. Finally, $\dot{u}$ given by equation (2.38) may be used in the Coriolis term, $2m\Omega\dot{u}$, in the δv equation. Thus, the following three non-linear, integro-partial differential equations for v , w , and ϕ with variable coefficients in the x -direction are obtained:

1. Lead-lag equation, δv :

$$\begin{aligned} & -\frac{m\Omega^2}{2} \left[v'(R^2 - x^2) \right]' - 2m\Omega \left(v' \int_x^R \dot{v} dx \right)' + [EI_{z'} - (EI_{z'} - EI_{y'}) \sin^2(\Re \theta)] v'''' \\ & + (EI_{z'} - EI_{y'}) \frac{\sin(2\Re \theta)}{2} w'''' + (EI_{z'} - EI_{y'}) \left[-\sin(2\Re \theta) (\phi v'')'' + \cos(2\Re \theta) (\phi w'')'' \right] \quad (2.39) \\ & - 2m\Omega \beta_{pc} \dot{w} - 2m\Omega \int_0^x (v' \dot{v}' + w' \dot{w}') dx + m(\ddot{v} - \Omega^2 v) = L_v \end{aligned}$$

2. Flap equation, δw :

$$\begin{aligned}
 & -\frac{m\Omega^2}{2} \left[w'(R^2 - x^2) \right]' - 2m\Omega \left(w' \int_x^R \dot{v} dx \right)' + [EI_{y'} + (EI_{z'} - EI_{y'}) \sin^2(\Re \theta)] w'''' \\
 & + (EI_{z'} - EI_{y'}) \frac{\sin(2\Re \theta)}{2} v'''' + (EI_{z'} - EI_{y'}) \left[\cos(2\Re \theta) (\phi v'')'' + \sin(2\Re \theta) (\phi w'')'' \right] \\
 & + 2m\Omega \beta_{pc} \dot{v} + m\ddot{w} = L_w - m\Omega^2 \beta_{pc} x
 \end{aligned} \tag{2.40}$$

3. Torsion equation, $\delta \phi$:

$$\begin{aligned}
 & -\frac{m\Omega^2}{2} k_{\phi}^2 \left[\phi'(R^2 - x^2) \right]' - GJ\phi'' + mk_m^2 \ddot{\phi} + m\Omega^2 (k_{m2}^2 - k_{m1}^2) \phi \cos(2\theta) \\
 & + (EI_{z'} - EI_{y'}) \left[(w''^2 - v''^2) \frac{\sin(2\Re \theta)}{2} + v''w'' \cos(2\Re \theta) \right] \\
 & = M_\phi - m\Omega^2 (k_{m2}^2 - k_{m1}^2) \frac{\sin(2\theta)}{2}
 \end{aligned} \tag{2.41}$$

2.4.6. Aerodynamic Loads

The hover aerodynamic loads on the rotor blade used in this study are based on Greenberg's extension of the Theodorsen's theory (Ref. 8). This model is valid for two-dimensional airfoil undergoing a sinusoidal motion in pulsating incompressible flow. The basis of this formulation is the strip theory in which only velocity components in the directions perpendicular to the span-wise axis of the blade influences the aerodynamic loads. A quasi-steady approximation of the unsteady theory for low reduced frequency k is employed in which the Theodorsen function $C(k)$ is taken to be unity. The classical blade element momentum theory is used to calculate the steady inflow velocity for the rotor. Since in the stability analysis of a hovering rotor one is mainly concerned with low

frequencies due to blade bending, the simplifying approximations made in determining aerodynamic loads are considered to be satisfactory (Ref. 12).

Theodorsen's theory assumes that the airfoil is pivoted about a certain axis, which in general, may be distinct from the aerodynamic axis. The airfoil is pitched at an angle $\varepsilon(t)$ with respect to the free stream flowing at constant velocity V . The airfoil is displaced vertically at the same time with velocity $\dot{h}$ positive downwards (Fig. 2.4). Greenberg extended this theory to incorporate a pulsating free stream velocity $V(t)$. The lift and the pitching moment per unit length are composed of circulatory and non-circulatory components (denoted by the subscripts c and $_{NC}$, respectively) as follows:

$$\begin{aligned} L &= L_c + L_{NC} \\ M &= M_c + M_{NC} \end{aligned} \quad (2.42)$$

where L and M denote the lift and the pitching moment respectively.

When the airfoil pivot axis, i.e. the elastic axis, coincides with the aerodynamic axis, these components are:

$$\begin{aligned} L_{NC} &= \frac{\rho_{\infty} a c}{2} \frac{c}{4} \left(\ddot{h} + V \dot{\varepsilon} + V \dot{\varepsilon} + \frac{c}{4} \ddot{\varepsilon} \right) \\ L_c &= \frac{\rho_{\infty} a c}{2} V \left(\dot{h} + V \varepsilon + \frac{c}{2} \dot{\varepsilon} \right) \\ M_{NC} &= -\frac{c}{4} L_{NC} - \frac{\rho_{\infty} a c}{2} \left(\frac{c}{4} \right)^3 \frac{\ddot{\varepsilon}}{2} \\ M_c &= -\frac{\rho_{\infty} a c}{2} \left(\frac{c}{4} \right)^2 V \dot{\varepsilon} \end{aligned} \quad (2.43)$$

where ρ_{∞} is the air density, c is the length of the blade cross-section's chord, and a is the airfoil lift curve slope.

It is important to note that ε is the angular position of the airfoil in space while the angle between the airfoil chord line and the resultant fluid velocity U is the instantaneous

angle of attack of the airfoil α . For convenience, it is desirable to express the aerodynamic loads as the velocity components in the principal axes system y' and z' , namely U_T and U_P . For small values of ε and α , it can be seen that:

$$\begin{aligned} U_P &\cong -\dot{h} - V\varepsilon \\ U &= \sqrt{U_T^2 + U_P^2} \cong V \end{aligned} \quad (2.44)$$

Substitution of the above approximation into the lift components results in:

$$\begin{aligned} L_{NC} &= \frac{\rho_\infty a c}{2} \frac{c}{4} \left(-\dot{U}_P + \frac{c}{4} \ddot{\varepsilon} \right) \\ L_C &= \frac{\rho_\infty a c}{2} U \left(-U_P + \frac{c}{2} \dot{\varepsilon} \right) \end{aligned} \quad (2.45)$$

Next the aerodynamic forces on the airfoil in the directions parallel and perpendicular to the airfoil chord are considered. The non-circulatory lift acts normal to the airfoil while the circulatory lift acts normal to the resultant blade velocity U (Fig. 2.5). An aerodynamic profile drag force per unit length, acting parallel to the blade resultant velocity is given by:

$$D = \frac{\rho_\infty a c}{2} \frac{c_{d0}}{a} (U_T^2 + U_P^2) \quad (2.46)$$

where c_{d0} represents a constant profile drag coefficient.

Combining the aerodynamic forces in the principal axes system, one gets:

$$\begin{aligned} T &= L_C \cos \alpha + L_{NC} + D \sin \alpha \\ S &= L_C \sin \alpha - D \cos \alpha \end{aligned} \quad (2.47)$$

where T and S correspond to the aerodynamic forces in the z' -direction and y' -direction respectively. As can be seen in (Fig. 2.4) :

$$\begin{aligned}\cos \alpha &= \frac{U_T}{U} = \frac{U_T}{\sqrt{U_T^2 + U_P^2}} \\ \sin \alpha &= \frac{-U_P}{U} = \frac{U_P}{\sqrt{U_T^2 + U_P^2}}\end{aligned}\quad (2.48)$$

By substituting equations (2.45), (2.46), and (2.48) into (2.47) and noting that $c_{d0} \alpha$ is negligible with respect to unity, one gets:

$$\begin{aligned}T &= \frac{\rho_\infty a c}{2} \left[-U_P U_T + \frac{c}{2} U_T \dot{\epsilon} - \frac{c}{4} \dot{U}_P + \left(\frac{c}{4} \right)^2 \ddot{\epsilon} \right] \\ S &= \frac{\rho_\infty a c}{2} \left[U_P^2 - \frac{c}{2} U_P \dot{\epsilon} - \frac{c_{d0}}{a} U_T^2 \right]\end{aligned}\quad (2.49)$$

Similarly, from (2.43) and (2.44) the expressions for aerodynamic pitching moments can be written as:

$$\begin{aligned}M_{NC} &= -\frac{\rho_\infty a c}{2} \left(\frac{c}{4} \right)^2 \left(-\dot{U}_P + \frac{3c}{8} \ddot{\epsilon} \right) \\ M_C &\equiv -\frac{\rho_\infty a c}{2} \left(\frac{c}{4} \right)^2 U_T \dot{\epsilon} \\ M_\phi = M &\equiv -\frac{\rho_\infty a c}{2} \left(\frac{c}{4} \right)^2 \left(U_T \dot{\epsilon} - \dot{U}_P + \frac{3c}{8} \ddot{\epsilon} \right)\end{aligned}\quad (2.50)$$

where, $U \equiv U_T$ has been used in M_C and M_ϕ is the total twisting moment.

U_P , U_T and $\dot{\epsilon}$ can be expressed in terms of v , w , and ϕ as follows (Ref. 12):

$$\begin{aligned}U_P &\equiv -\Omega x \left(\theta + \phi + \int_0^x v' w'' dx \right) - (\theta + \phi) \dot{v} + v_i + \dot{w} + \Omega v (\beta_{pc} + w') \\ U_T &\equiv \Omega x + \dot{v} \\ \dot{\epsilon} &\equiv \dot{\phi} + \Omega (\beta_{pc} + w')\end{aligned}\quad (2.51)$$

where v_i is the induced inflow velocity that is taken to be steady and uniform along the blade radius and is approximated by the value of the non-uniform inflow given by blade

momentum theory at the blade angle which is equal to the pitch angle plus the 75% of the equilibrium elastic twist ϕ_0 at the tip. Thus:

$$v_i = \text{sgn}\left[\theta + 0.75 * \phi_0|_{np}\right] \Omega R \frac{\pi \sigma}{8} \left(\sqrt{1 + \frac{12}{\pi \sigma} |\theta + 0.75 * \phi_0|_{np}|} - 1 \right) \quad (2.52)$$

where $\sigma = bc / \pi R$ is the blade solidity and b is the number of blades. It must be noted, however, that the expression in (2.52) differs from that of Ref. 12 in choosing the blade angle contributing to the induced velocity. In Ref. 12, the blade angle is taken to be equal to the pitch angle plus the equilibrium elastic twist ϕ_0 at 0.75 of the radius.

Assuming a small collective pitch angle $(\theta + \phi)$, and using the transformation matrix between (x, y, z) and (x', y', z') coordinate systems (Ref. 11), the aerodynamic loads S and T can be resolved into L_v and L_w parallel to the y and z directions, respectively, as:

$$\begin{aligned} L_v &= S - T(\theta + \phi) \\ L_w &= T + S(\theta + \phi) \end{aligned} \quad (2.53)$$

By substituting equations (2.51) into the third of equations (2.50) and using equations (2.49), (2.51), and (2.53), one gets:

$$\begin{aligned} L_v &\equiv \frac{\rho_\infty a c}{2} \left\{ v_i^2 - \Omega^2 x^2 \frac{c_{d0}}{a} - \Omega x v_i (\theta + \phi) - \left[2\Omega x \frac{c_{d0}}{a} + (\theta + \phi) v_i \right] \dot{v} \right. \\ &\quad \left. + [2v_i - \Omega x (\theta + \phi)] \dot{w} \right\} \end{aligned} \quad (2.54)$$

$$\begin{aligned} L_w &\equiv \frac{\rho_\infty a c}{2} \left\{ -\Omega x v_i + \Omega^2 x^2 (\theta + \phi + \int_0^x v' w'' dx) - \Omega^2 x v (\beta_{pc} + w') \right. \\ &\quad \left. + \Omega^2 \frac{x c}{2} (\beta_{pc} + w') + [2\Omega x (\theta + \phi) - v_i] \dot{v} - \Omega x \dot{w} + \frac{3}{4} c \Omega x \dot{\phi} - \frac{c}{4} \ddot{w} \right\} \end{aligned} \quad (2.55)$$

$$M_\phi \equiv -\frac{\rho_\infty a c}{2} \left(\frac{c^2}{8} \Omega x \dot{\phi} \right) \quad (2.56)$$

In deriving these expressions, the non-linear product terms and higher order terms are neglected because they do not contribute to the linearized stability analysis.

2.4.7. Non-Dimensional Form of Blade Equations of Motion

The non-linear, variable coefficient, integro-partial differential equations of motion with variable coefficients that are given in (2.39) – (2.41) can be combined with equations (2.54) - (2.56) and expressed in non-dimensional form by dividing the lead-lag and flap equations by $m\Omega^2 R$ and the torsion equation by $m\Omega^2 R^2$. The resulting equations can be used to obtain a corresponding set of non-linear ordinary differential equations by applying Galerkin's method. The non-dimensional equations read as:

1. Lead-lag displacement, $\delta\bar{v}$:

$$\begin{aligned}
 & -(\bar{v}') \int_{\bar{x}}^1 (\bar{x} + 2\bar{v}) d\bar{x})' + \Lambda_2 \bar{v}'''' - (\Lambda_2 - \Lambda_1) \sin^2(\Re\theta) \bar{v}'''' + (\Lambda_2 - \Lambda_1) \\
 & \left[\frac{\sin(2\Re\theta)}{2} \bar{w}'''' - \sin(2\Re\theta)(\phi\bar{w}'')'' + \cos(2\Re\theta)(\phi\bar{v}'')'' \right] - 2\beta_{pc} \dot{\bar{w}} - 2 \int_0^{\bar{x}} \bar{v}' \dot{\bar{v}} d\bar{x} \\
 & - 2 \int_0^{\bar{x}} \bar{w}' \dot{\bar{w}} d\bar{x} + \ddot{\bar{v}} - \bar{v} + \frac{\gamma}{6} \{ \bar{x} \bar{v}_i \phi + [2 \frac{c_{d0}}{a} \bar{x} + (\theta + \phi) \bar{v}_i] \dot{\bar{v}} + [2 \bar{v}_i - \bar{x}(\theta + \phi)] \dot{\bar{w}} \} \\
 & = \frac{\gamma}{6} \{ \bar{v}_i^2 - \frac{c_{d0}}{a} \bar{x}^2 - \bar{x} \bar{v}_i \theta \}
 \end{aligned} \tag{2.57}$$

2. Flap displacement, $\delta\bar{w}$:

$$\begin{aligned}
 & -(\bar{w}') \int_{\bar{x}}^1 (\bar{x} + 2\bar{v}) d\bar{x})' + \Lambda_1 \bar{w}'''' + (\Lambda_2 - \Lambda_1) \sin^2(\Re\theta) \bar{w}'''' + (\Lambda_2 - \Lambda_1) \\
 & \left[\frac{\sin(2\Re\theta)}{2} \bar{v}'''' + \sin(2\Re\theta)(\phi\bar{w}'')'' + \cos(2\Re\theta)(\phi\bar{v}'')'' \right] + 2\beta_{pc} \dot{\bar{v}} + \ddot{\bar{w}} (1 + \frac{\gamma\bar{c}}{24}) \\
 & + \frac{\gamma}{6} \{ -\bar{x}^2 (\phi + \int_0^{\bar{x}} \bar{v}' \bar{w}'' d\bar{x}) + \bar{x} \bar{v} (\beta_{pc} + \bar{w}') - \frac{\bar{c}}{2} \bar{x} \bar{w}' - [2\bar{x}(\theta + \phi) - \bar{v}_i] \dot{\bar{v}} + \bar{x} \dot{\bar{w}} - \frac{3}{4} \bar{c} \bar{x} \dot{\phi} \} \\
 & = -\beta_{pc} \bar{x} + \frac{\gamma}{6} (-\bar{x} \bar{v}_i + \bar{x}^2 \theta + \frac{\bar{c}}{2} \bar{x} \beta_{pc})
 \end{aligned} \tag{2.58}$$

3. Torsional displacement, $\delta\phi$:

$$\begin{aligned}
 & -\frac{\mu^2 K}{2} \{ (1 - \bar{x}^2) \phi' \}' - \kappa \phi'' + (\Lambda_2 - \Lambda_1) \left[\frac{\bar{w}''^2 - \bar{v}''^2}{2} \sin(2\Re\theta) + \bar{v}'' \bar{w}'' \cos(2\Re\theta) \right] \\
 & + \mu^2 \ddot{\phi} + (\mu_2^2 - \mu_1^2) \phi \cos(2\theta) + \frac{\gamma \bar{c}^2}{48} \bar{x} \dot{\phi} = -(\mu_2^2 - \mu_1^2) \frac{\sin(2\theta)}{2}
 \end{aligned} \tag{2.59}$$

where the dimensionless parameters are:

$$\begin{aligned}
 K &= \frac{k_A^2}{k_m^2} & \bar{x} &= \frac{x}{R} & \bar{v} &= \frac{v}{R} \\
 \mu_1 &= \frac{k_{m1}}{R} & \Lambda_1 &= \frac{EI_{y'}}{m\Omega^2 R^4} & \bar{w} &= \frac{w}{R} \\
 \mu_2 &= \frac{k_{m2}}{R} & \Lambda_2 &= \frac{EI_{z'}}{m\Omega^2 R^4} & \bar{v}_i &= \frac{v_i}{\Omega R} \\
 \mu &= \frac{k_m}{R} & \kappa &= \frac{GJ}{m\Omega^2 R^4} & ()' &= \frac{\partial}{\partial \bar{x}} () \\
 \bar{c} &= \frac{c}{R} & \gamma &= \frac{3\rho_\infty acR}{m} & () &= \frac{\partial}{\partial \psi} = \frac{\partial}{\Omega \bar{c} t} ()
 \end{aligned} \tag{2.60}$$

2.5. Solving the Equations of Motion

These equations of motion are then solved by Galerkin's method using four coupled, rotating mode shapes, i.e., the 1st lead-lag, 1st and 2nd flap, and 1st torsion modes, to evaluate the stability of small perturbation motions about the equilibrium operating condition. Initially, these coupled, rotating mode shapes are determined by first calculating the equilibrium position and formulating the stability analysis using the standard, uncoupled, non-rotating cantilever beam mode shapes. The coupled mode shapes are then determined from an approximate free vibration analysis of the rotating beam about the deformed shape of the equilibrium operating condition.

In order to apply Galerkin's method to the non-dimensional equations of motion, the flap, lead-lag and torsional displacements each is prescribed as a series of generalized coordinates and mode shape functions as follows:

$$\begin{aligned}\bar{v} &= \sum_{j=1}^N V_j(\psi) \Psi_j(\bar{x}) \\ \bar{w} &= \sum_{j=1}^N W_j(\psi) \Psi_j(\bar{x}) \\ \phi &= \sum_{j=1}^N \phi_j(\psi) \Theta_j(\bar{x})\end{aligned}\tag{2.61}$$

where:

$$\begin{aligned}\psi &= \Omega t \\ \bar{x} &= \frac{x}{R} \\ \Psi_j(\bar{x}) &= \cosh(\beta_j \bar{x}) - \cos(\beta_j \bar{x}) - \alpha_j [\sinh(\beta_j \bar{x}) - \sin(\beta_j \bar{x})] \\ \Theta_j(\bar{x}) &= \sqrt{2} \sin(\gamma_j \bar{x})\end{aligned}\tag{2.62}$$

The assumed mode shapes, $\Psi_j(\bar{x})$ and $\Theta_j(\bar{x})$, are the standard non-rotating, uncoupled mode shapes of a uniform straight cantilevered beam. The constants α_j and β_j are tabulated in (Ref. 4) and $\gamma_j = \pi(j - 0.5)$. By substituting the generalized lead-lag, flap, and torsion coordinates and mode shape functions in the equations of motion (2.57 – 2.59) and applying Galerkin's method, one will obtain the following set of $3N$ non-linear ordinary modal differential equations of motion in terms of generalized coordinates V_j , W_j , ϕ_j :

1. Lead-lag equations, δV_i :

$$\begin{aligned}
 & \sum_{j=1}^N \left\langle V_j \{D_y + [\Lambda_2 - (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^4 \delta_y - \delta_y\} \right. \\
 & + W_j \frac{\sin(2\Re \theta)}{2} (\Lambda_2 - \Lambda_1) \beta_j^4 \delta_y + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jk} \phi_j [-V_k \sin(2\Re \theta) + W_k \cos(2\Re \theta)] \\
 & - 2\beta_{pc} \delta_y \dot{W}_j + 2 \sum_{k=1}^N (F_{ikj} - F_{jkn}) V_k \dot{V}_j - 2 \sum_{k=1}^N F_{jkn} W_k \dot{W}_j + \delta_y \ddot{V}_j \\
 & + \frac{\gamma}{6} [\bar{v}_i I_{y\phi_j} + \left(2 \frac{c_{d0}}{a} E_y + \theta \bar{v}_i \delta_y + \bar{v}_i \sum_{k=1}^N \phi_k H_{yk} \right) \dot{V}_j \\
 & \left. + \left(2 \bar{v}_i \delta_y - \theta E_y - \sum_{k=1}^N \phi_k G_{yk} \right) \dot{W}_j \right] = \frac{\gamma}{6} [\bar{v}_i^2 A_i - \frac{c_{d0}}{a} C_i - \bar{v}_i \theta B_i], \quad i = 1, 2, \dots, N
 \end{aligned} \tag{2.63}$$

2. Flap equations, δW_i :

$$\begin{aligned}
 & \sum_{j=1}^N \left\langle V_j (\Lambda_2 - \Lambda_1) \frac{\sin(2\Re \theta)}{2} \beta_j^4 \delta_y + W_j \{D_y + [\Lambda_1 + (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^4 \delta_y\} \right. \\
 & + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jk} \phi_j [W_k \sin(2\Re \theta) + V_k \cos(2\Re \theta)] + 2\beta_{pc} \delta_y \dot{V}_j + 2 \sum_{k=1}^N F_{ikj} W_k \dot{V}_j \\
 & + \delta_y \left(1 + \frac{\gamma \bar{c}}{24} \right) \ddot{W}_j + \frac{\gamma}{6} [-J_{yy} \phi_j + \sum_{k=1}^N L_{yk} V_j W_k + \beta_{pc} E_y V_j - \frac{\bar{c}}{2} O_y W_j - (2\theta E_y + 2 \sum_{k=1}^N \phi_k G_{yk} \\
 & - \bar{v}_i \delta_y) \dot{V}_j + E_y \dot{W}_j - \frac{3\bar{c}}{4} I_{y\phi_j} \dot{\phi}_j] = -\beta_{pc} B_i + \frac{\gamma}{6} [-\bar{v}_i B_i + \theta C_i + \frac{\bar{c}}{2} \beta_{pc} B_i], \quad i = 1, 2, \dots, N
 \end{aligned} \tag{2.64}$$

3. Torsion equations, $\delta \phi_i$:

$$\begin{aligned}
 & \sum_{j=1}^N \left\langle \phi_j \left\{ \mu^2 K N_y + [\kappa \gamma_j^2 + (\mu_2^2 - \mu_1^2) \cos(2\theta)] \delta_y \right\} + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{yk} [\cos(2\Re \theta) V_j W_k \right. \\
 & + \frac{\sin(2\Re \theta)}{2} (W_j W_k - V_j V_k)] + \mu^2 \delta_y \ddot{\phi}_j + \frac{\gamma \bar{c}^2}{48} M_y \dot{\phi}_j \Bigg\rangle = \\
 & - (\mu_2^2 - \mu_1^2) \frac{\sin(2\theta)}{\sqrt{2} \gamma_i}; \quad i = 1, 2, \dots, N
 \end{aligned} \tag{2.65}$$

where

$$\begin{aligned}
A_i &= \int_0^1 \Psi_i d\bar{x} & J_{ij} &= \int_0^1 \bar{x}^2 \Psi_i \Theta_j d\bar{x} \\
B_i &= \int_0^1 \bar{x} \Psi_i d\bar{x} & K_{ijk} &= \int_0^1 \Theta_i \Psi_j'' \Psi_k'' d\bar{x} \\
C_i &= \int_0^1 \bar{x}^2 \Psi_i d\bar{x} & L_{ijk} &= \int_0^1 \bar{x} \Psi_i \Psi_j \Psi_k' d\bar{x} \\
D_{ij} &= \int_0^1 \frac{1-\bar{x}^2}{2} \Psi_i' \Psi_j' d\bar{x} & & - \int_0^1 \bar{x}^2 \Psi_i \int_0^{\bar{x}} \Psi_j'(\bar{x}_1) \Psi_k''(\bar{x}_1) d\bar{x}_1 d\bar{x} \\
E_{ij} &= \int_0^1 \bar{x} \Psi_i \Psi_j d\bar{x} & M_{ij} &= \int_0^1 \bar{x} \Theta_i \Theta_j d\bar{x} \\
F_{ijk} &= -\frac{1}{\beta_k^4} \int_0^1 \Psi_i' \Psi_j' \Psi_k''' d\bar{x} & N_{ij} &= \int_0^1 \frac{1-\bar{x}^2}{2} \Theta_i' \Theta_j' d\bar{x} \\
G_{ijk} &= \int_0^1 \bar{x} \Psi_i \Psi_j \Theta_k d\bar{x} & O_{ij} &= \int_0^1 \bar{x} \Psi_i \Psi_j' d\bar{x} \\
H_{ijk} &= \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} & \delta_{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} = \int_0^1 \Theta_i \Theta_j d\bar{x} \\
I_{ij} &= \int_0^1 \bar{x} \Psi_i \Theta_j d\bar{x} & &= \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases}
\end{aligned} \tag{2.66}$$

The non-linear, non-homogeneous, constant-coefficient ordinary differential equations of motion (2.63 – 2.65) are to be linearized for small perturbation quantities about the equilibrium operating condition. To achieve this, the time dependent generalized coordinates are decomposed into steady equilibrium quantities and small unsteady perturbation quantities as follows:

$$\begin{aligned}
V_j(\psi) &= V_{0j} + \Delta V_j(\psi) \\
W_j(\psi) &= W_{0j} + \Delta W_j(\psi) \\
\phi_j(\psi) &= \phi_{0j} + \Delta \phi_j(\psi)
\end{aligned} \tag{2.67}$$

Thus, $3N$ equilibrium deflection equations can be obtained by substituting the steady quantities into equations (2.63 - 2.65). This results in:

1. Lead-lag equations, δV_{0i} :

$$\begin{aligned} & \sum_{j=1}^N \left\langle V_{0j} \{D_{0j} + [\Lambda_2 - (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^{-4} \delta_{0j} - \delta_{0j}\} + W_{0j} \frac{\sin(2\Re \theta)}{2} (\Lambda_2 - \Lambda_1) \beta_j^{-4} \delta_{0j} \right. \\ & \left. + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jk} \phi_{0j} [-V_{0k} \sin(2\Re \theta) + W_{0k} \cos(2\Re \theta)] + \frac{\gamma}{6} [\bar{v}_i I_{ij} \phi_{0j}] \right\rangle = \\ & \frac{\gamma}{6} [\bar{v}_i^2 A_i - \frac{c_{d0}}{a} C_i - \bar{v}_i \theta B_i]; \quad i = 1, 2, \dots, N \end{aligned} \quad (2.68)$$

2. Flap equations, δW_{0i} :

$$\begin{aligned} & \sum_{j=1}^N \left\langle V_{0j} (\Lambda_2 - \Lambda_1) \frac{\sin(2\Re \theta)}{2} \beta_j^{-4} \delta_{0j} + W_{0j} \{D_{0j} + [\Lambda_1 + (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^{-4} \delta_{0j}\} \right. \\ & \left. + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jk} \phi_{0j} [W_{0k} \sin(2\Re \theta) + V_{0k} \cos(2\Re \theta)] + \frac{\gamma}{6} [-J_{ij} \phi_{0j} \right. \\ & \left. + \sum_{k=1}^N L_{ijk} V_{0j} W_{0k} + \beta_{pc} E_{ij} V_{0j} - \frac{\bar{c}}{2} O_{ij} W_{0j}] \right\rangle \\ & = -\beta_{pc} B_i + \frac{\gamma}{6} [-\bar{v}_i B_i + \theta C_i + \frac{\bar{c}}{2} \beta_{pc} B_i]; \quad i = 1, 2, \dots, N \end{aligned} \quad (2.69)$$

3. Torsion equations, $\delta \phi_{0i}$:

$$\begin{aligned} & \sum_{j=1}^N \left\langle \phi_{0j} \left\{ \mu^2 K N_{0j} + [\kappa \gamma_j^2 + (\mu_2^2 - \mu_1^2) \cos(2\theta)] \delta_{0j} \right\} + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jk} [\cos(2\Re \theta) I_{0j}^* W_{0k} \right. \\ & \left. + \frac{\sin(2\Re \theta)}{2} (W_{0j} W_{0k} - V_{0j} V_{0k}) \right\rangle = -(\mu_2^2 - \mu_1^2) \frac{\sin(2\theta)}{\sqrt{2} \gamma_i}; \quad i = 1, 2, \dots, N \end{aligned} \quad (2.70)$$

Next, perturbation equations are obtained. This is achieved by first substituting equations (2.67) into the non-dimensional equations (2.63 - 2.65), subtracting the equilibrium equations (2.68 - 2.70), and discarding all non-linear products of the perturbation quantities. These equations represent perturbation motion near the equilibrium operating condition and it is the stability of this perturbation that would be of concern. These

perturbation equations are linear, homogeneous, constant-coefficient ordinary differential equations in the familiar form of:

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{0\} \quad (2.71)$$

the coefficients of which are functions of the equilibrium solution, such that:

- the mass matrix $[M]$ is given by:

$$[M] = \begin{bmatrix} \delta_{\eta} & 0 & 0 \\ 0 & \delta_{\eta} \left(1 + \frac{\gamma \bar{c}}{24} \right) & 0 \\ 0 & 0 & \mu^2 \delta_{\eta} \end{bmatrix}, \quad (2.72)$$

- the damping matrix $[C]$ is given by:

$$[C] = 2 \begin{bmatrix} \sum_{k=1}^N (F_{ikj} - F_{jki}) V_{0k} & -\beta_{pc} \delta_{\eta} - \sum_{k=1}^N F_{jki} W_{0k} & 0 \\ \beta_{pc} \delta_{\eta} + \sum_{k=1}^N F_{ikj} W_{0k} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{\gamma}{6} \begin{bmatrix} 2 \frac{c_{d0}}{a} E_{\eta} + \theta \bar{v}_t \delta_{\eta} - \bar{v}_t \sum_{k=1}^N H_{\eta k} \phi_{0k} & 2 \bar{v}_t \delta_{\eta} - \theta E_{\eta} - \sum_{k=1}^N G_{\eta k} \phi_{0k} & 0 \\ -2 \theta E_{\eta} + \bar{v}_t \delta_{\eta} - 2 \sum_{k=1}^N G_{\eta k} \phi_{0k} & E_{\eta} & -\frac{3 \bar{c}}{4} I_{\eta} \\ 0 & 0 & \frac{\bar{c}^2}{8} M_{\eta} \end{bmatrix}, \quad (2.73)$$

- the stiffness matrix $[K]$ is given by:

$$[K] = [K_s] + [K_a]$$

$$= \begin{bmatrix} K_{s11} & K_{s12} & K_{s13} \\ K_{s21} & K_{s22} & K_{s23} \\ K_{s31} & K_{s32} & K_{s33} \end{bmatrix} + \begin{bmatrix} K_{a11} & K_{a12} & K_{a13} \\ K_{a21} & K_{a22} & K_{a23} \\ K_{a31} & K_{a32} & K_{a33} \end{bmatrix}, \quad (2.74)$$

where $[K_a]$ is the part of the stiffness matrix due to the aerodynamic contribution and $[K_s]$

is the one due to the structural contribution, the elements of which are:

$$K_{a11} = K_{a12} = K_{a31} = K_{a32} = K_{a33} = 0$$

$$K_{a13} = \frac{\gamma}{6} \bar{v}_i I_{ij}$$

$$K_{a21} = \frac{\gamma}{6} (\beta_{pc} E_{ij} + \sum_{k=1}^N L_{ijk} W_{0k})$$

$$K_{a22} = \frac{\gamma}{6} (-\frac{\bar{c}}{2} O_{ij} + \sum_{k=1}^N L_{ikj} V_{0k})$$

$$K_{a23} = -\frac{\gamma}{6} J_{ij}$$

$$K_{s11} = D_{ij} + [\Lambda_2 - (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^4 \delta_{ij} - \delta_{ij} - (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{kji} \phi_{0k} \sin(2\Re \theta) \quad (2.75)$$

$$K_{s12} = \frac{\sin(2\Re \theta)}{2} (\Lambda_2 - \Lambda_1) \beta_j^4 \delta_{ij} + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{kji} \phi_{0k} \cos(2\Re \theta)$$

$$K_{s13} = (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} [W_{0k} \cos(2\Re \theta) - V_{0k} \sin(2\Re \theta)]$$

$$K_{s21} = \frac{\sin(2\Re \theta)}{2} (\Lambda_2 - \Lambda_1) \beta_j^4 \delta_{ij} + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{kji} \phi_{0k} \cos(2\Re \theta)$$

$$K_{s22} = D_{ij} + [\Lambda_1 + (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^4 \delta_{ij} + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{kji} \phi_{0k} \sin(2\Re \theta)$$

$$K_{s23} = (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} [V_{0k} \cos(2\Re \theta) + W_{0k} \sin(2\Re \theta)]$$

$$K_{s31} = (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{ijk} [W_{0k} \cos(2\Re \theta) - V_{0k} \sin(2\Re \theta)]$$

$$K_{s32} = (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{ijk} [V_{0k} \cos(2\Re \theta) + W_{0k} \sin(2\Re \theta)]$$

$$K_{s33} = \mu^2 K N_{ij} + [\kappa \gamma_j^2 + (\mu_2^2 - \mu_1^2) \cos(2\theta)] \delta_{ij}$$

- and the perturbation vector is given by:

$$\{X\} = \begin{Bmatrix} \Delta V_j \\ \Delta W_j \\ \Delta \phi_j \end{Bmatrix} \quad (2.76)$$

where, as a symmetric matrix $[M]$, and two asymmetric matrices $[C]$ and $[K]$, each matrix is $3N \times 3N$. The $[M]$ matrix contains the contribution of both aerodynamic and inertial terms; the $[C]$ matrix has contributions due to both gyroscopic and aerodynamic damping terms and the $[K]$ matrix is banded, containing stiffness and aerodynamic terms. The aerodynamic term in $[M]$, namely $\gamma \bar{c}/24$, is known as the apparent mass. The apparent mass is usually of a smaller order of magnitude and thus is neglected in this study.

The stability of the perturbation motion about the equilibrium operating condition is determined by the eigenvalues of the $6N \times 6N$ matrix $[P]$ given by:

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} = [P] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \quad (2.77)$$

Since the primary concern is the stability at lower frequencies of operation (first lead-lag, first and second flap, and first torsion frequencies), there is a value of N for which the eigenvalues can be considered to converge. The existing literature shows that the typical value of N for a hingeless helicopter blade to give a suitable convergence of lower eigenvalues is 5 (Ref. 12). In the present analysis $N=6$ was used, thus, the matrix $[P]$ is 36×36 . For computational purpose, it is desirable to reduce the size of matrix $[P]$ without affecting the eigenvalues (Ref. 12). This is made possible by transformation of the modal coordinates. To accomplish this, we must first consider the free vibration of the hingeless

helicopter blade about the equilibrium deflection state. The governing equation of the free vibration case is given by:

$$[M]\{\ddot{X}\} + [G]\{\dot{X}\} + [K_s]\{X\} = \{0\} \quad (2.78)$$

where the subscript s corresponds to the stiffness matrix without the aerodynamic contribution. Both $[M]$ and $[K_s]$ are symmetric; and $[G]$ is the damping matrix without the aerodynamic contribution (gyroscopic terms only). The presence of $[G]$ results in complex eigenvectors, which is not desirable computationally. Thus, the free vibration equation is approximated by:

$$[M]\{\ddot{X}\} + [K_s]\{X\} = \{0\} \quad (2.79)$$

The resulting eigenvectors of the approximated free vibration case are orthogonal with respect to $[M]$:

$$[U]^T [M] [U] = [M_m] \quad (2.80)$$

where $[M_m]$ is a diagonal mass matrix, $[U]$ denotes the matrix of free vibration eigenvectors, and the superscript T corresponds to matrix transpose operation. A so-called principal coordinates transformation of coordinate systems can be achieved by replacing $\{X\}$ by $[U] \{X\}$ (Ref. 18). By pre-multiplying equations (2.71) by $[U]^T$, one may take advantage of diagonal matrix $[M_m]$ to obtain:

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M_m]^{-1}[U]^T[K][U] & -[M_m]^{-1}[U]^T[C][U] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} = [P^*] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \quad (2.81)$$

Because of the nature of the transformation, both $[P]$ and $[P^*]$ have the same eigenvalues. Another consequence of this principal coordinate transformation is that the eigenvectors of $[U]$ that correspond to the higher frequencies can be removed without affecting the eigenvectors of the low frequency modes of interest. Thus, the $6N \times 6N$ matrix $[P]$ will be

reduced to $2M_1 \times 2M_1$ matrix $[P^*]$ where M_1 is the number of low frequency modes that are retained. For this study, $N = 6$ and $M_1 = 4$ (corresponding to the first lead-lag, the first and the second flap, and the first torsion frequencies) were used. The new equations of motion (2.81) represent the M_1 coupled stiffness and damping matrices of rotating modes. The old equations of motion (2.79) represent uncoupled, non-rotating mode shapes.

Chapter Three

Introduction of a “Smart” Spring

3.1. Hingeless Helicopter Rotors as “Smart” Structures

This chapter presents the development and simulation of the behavior of a “smart” hingeless helicopter blade. The spectrum of loads associated with the non-homogeneous forcing loads is tuned by superimposing a harmonic system on the rotating frame. The Individual Blade Control (IBC) concept pioneered by Kretz (Ref. 17) and Ham (Ref. 9) is used in this approach. The IBC concept allows the control of a broad range of frequencies by using actuators installed either on the helicopter swash-plate or individually attached on the blades. In this way, using a number of control subsystems, one may control a number of modes of vibration introduced during the operation of the aircraft. The IBC method is very appropriate for situations in which each blade has its own characteristics. By synthesizing a suitable control system on each individual blade, the modes of vibration experienced by that particular blade can be minimized leading to reduced loads transferred to the helicopter airframe.

Two strategies are available in connection with the use of the IBC approach. The first one involves the use of embedded adaptive materials in the helicopter blade’s composite construction. This is intended to induce strain deformations, which are controlled externally. The strategy allows the possibility of achieving special features of the application of distributed control theory, such as superior robustness. In addition, it has the advantage of being smoother from the aerodynamic point of view. However, several studies have shown that this strategy overestimates the capability of the adaptive

materials to induce the necessary strain deformations under typical helicopter operational loads.

This leads to the utilization of a second strategy, namely the lumped parameter control approach which was initiated by Spangler and Hall (Ref. 25). This approach is based on the idea of controlling some important part of a helicopter blade using “smart” actuators. To date, the most accepted solution using this approach has been the use of a “smart” flap by installing adaptive actuators at the flaps on the trailing edge of the helicopter blade. Another idea has been to use adaptive actuators to deform the airfoil shape to actively control rotary wing characteristics such as: dynamic stall, blade vortex interaction, and shock. However, an interesting alternative, on which the present work is based on, was introduced by Nitzsche, Lammering, and Breitbach (Ref. 21). According to this concept, by actuating the “smart” spring implemented at the blade root, one may control the aeroelastic stability of the rotors. The actuation of the “smart” spring affects the lead-lag, flap, and torsional stiffness of the blade over a certain section of its length.

3.2. Governing Dynamic Equations of “Smart” Blades

The governing equations of motion for a uniform, untwisted, cantilever rotor blade without chord-wise offsets between the elastic, mass, tension, and aerodynamic center axes were obtained in Chapter 2. However, as it was noted earlier, due to the nature of its configuration, the hingeless blade can be easily susceptible to instability problems. This is caused by non-linear coupling that occurs between the flap, lead-lag, and torsional motions of the cantilever blade. To improve on the aeroelastic stability of the blade it is suggested that “smart” materials be introduced at desired locations along the blade.

When actuated such a “smart” material will act as a “smart” spring affecting the cross-sectional characteristics of the blade. The introduction and actuation of the “smart” materials necessitates a proper modification of the equations of motion developed by Hodges and Dowell (Ref. 11). This is accomplished by defining the flap, lag, and torsion stiffnesses as well as the radii of gyrations of the blade cross-section as functions of the distance from the hub along the blade. Next, these functions are incorporated in the application of the Galerkin method to obtain the root loci of the perturbation equations, which are used to investigate the stability of the system.

To model the “smart” spring one may start with the general equations of motion (2.24 - 2.28). As before, we hold the assumptions of concentric mass, elastic, aerodynamic, and tension axes, as well as the absence of the pre-twist along the blade to give a better representation of the real rotor blade. The blade cross-section is also assumed to be symmetric about η (Fig. 2.3) with anti-symmetric warp function λ about η . In comparison to the equations of motion obtained earlier, in the derivation of the governing equations of motion for the “smart” blade, all the terms related to mass and cross-sectional properties of the blade are now considered to be functions of x (Figs. 3.1), and thus they vary along the blade axis rather than being constant quantities, i.e.,

$$\begin{aligned} Z_t &= Z ; & 0 \leq \bar{x} < \bar{x}_1 ; \bar{x}_2 < \bar{x} \leq 1 \\ &= Z + Z_{ss} ; & \bar{x}_1 \leq \bar{x} \leq \bar{x}_2 \end{aligned} \quad (3.1)$$

where Z represents any of the following properties: m , mk_m , mk_{m1} , mk_{m2} , EA , EI_z , EI_y , GJ . The subscript t means total, i.e., the rotor blade plus the “smart” spring, whereas the subscript ss refers to the “smart” spring only.

The derivation of the equations of motion of the “smart” blade starts with the simplification of the general equations (2.24 – 2.28) by eliminating the pre-twist and the offsets between the mass, tension, elastic, and aerodynamic axes to obtain:

1. Axial displacement equation, δu :

$$-T' - m\Omega^2 x - 2m\Omega \dot{v} = L_u \quad (3.2)$$

2. Lead-lag displacement equation, δv :

$$\begin{aligned} & -(Tv')' + \{[EI_{z'} \cos^2(\theta + \phi) + EI_{y'} \sin^2(\theta + \phi)]v'' + (EI_{z'} - EI_{y'}) \frac{\sin 2(\theta + \phi)}{2} w'''\}'' \\ & + 2m\Omega \dot{u} + m\ddot{v} - m\Omega^2 v - 2m\Omega \beta_{pc} \dot{w} = L_v \end{aligned} \quad (3.3)$$

3. Flap displacement equation, δw :

$$\begin{aligned} & -(Tw')' + \{(EI_{z'} - EI_{y'}) \frac{\sin 2(\theta + \phi)}{2} v'' + [EI_{z'} \sin^2(\theta + \phi) + EI_{y'} \cos^2(\theta + \phi)]w'''\}'' \\ & + m\ddot{w} + 2m\Omega \beta_{pc} \dot{v} = L_w - m\Omega^2 \beta_{pc} x \end{aligned} \quad (3.4)$$

4. Torsional displacement equation, $\delta \phi$:

$$\begin{aligned} & -(k_{A'}^2 T\phi')' - (GJ\phi')' + (EI_{z'} - EI_{y'})[(w''^2 - v''^2) \frac{\sin 2\theta}{2} + v''w'' \cos 2\theta] + mk_m^2 \ddot{\phi} \\ & + m\Omega^2 \phi(k_{m2}^2 - k_{m1}^2) \cos 2\theta = M_\phi - m\Omega^2 (k_{m2}^2 - k_{m1}^2) \frac{\sin 2\theta}{2} \end{aligned} \quad (3.5)$$

where

$$T \equiv EA \left\{ u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right\} \quad (3.6)$$

Next, Equation (3.6) is substituted in Equations (3.2 – 3.5) eliminating expressions for the axial displacement u and tension T . The details of this manipulation are outlined in

Section 2.4.5. As before, the non-conservative loads are represented by the Greenberg's aerodynamic forces and moments (Eqs. 2.54 – 2.56). The resulting non-linear, variable coefficient, integro-partial differential equations that govern the motion of a non-uniform, untwisted, cantilevered “smart” blade may be reduced to ordinary differential equations by Galerkin's method. In so doing, one may again use the same basis functions as given in equation (2.62). The resulting ordinary differential equations of motion are then normalized with respect to the inertia term $m\Omega^2 R$ for the lead-lag and flap equations; and with respect to $m\Omega^2 R^2$ for the torsion equation. Finally the following equations of motion, expressed in the modal generalized coordinates are obtained:

1. Lead-lag equations, δV_i :

$$\begin{aligned}
& \sum_{j=1}^N \left\{ V_j \{ D_{ij} + \bar{m}_{ss} D_{xij} + [\Lambda_2 - (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^{-4} \delta_{ij} \right. \\
& + [\Lambda_{2ss} - (\Lambda_{2ss} - \Lambda_{1ss}) \sin^2(\Re \theta)] \Delta_{xij1} - (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) \} \\
& + W_j \frac{\sin(2\Re \theta)}{2} \{ (\Lambda_2 - \Lambda_1) \beta_j^{-4} \delta_{ij} + (\Lambda_{2ss} - \Lambda_{1ss}) \Delta_{xij1} \} \\
& + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} \phi_j [-V_k \sin(2\Re \theta) + W_k \cos(2\Re \theta)] \\
& + (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xjki} \phi_j [-V_k \sin(2\Re \theta) + W_k \cos(2\Re \theta)] \\
& - 2\beta_{pc} \dot{W}_j (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) + 2 \sum_{k=1}^N (F_{ikj} - F_{jki} + \bar{m}_{ss} (F_{xikj} - F_{xjki})) V_k \dot{V}_j \\
& - 2 \sum_{k=1}^N (F_{jki} + \bar{m}_{ss} F_{xjki}) W_k \dot{W}_j + (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) \ddot{V}_j \\
& + \frac{\gamma}{6} [\bar{v}_i I_{ij} \phi_j + \left(2 \frac{c_{d0}}{a} E_{ij} + \theta \bar{v}_i \delta_{ij} + \bar{v}_i \sum_{k=1}^N \phi_k H_{ijk} \right) \dot{V}_j \\
& \left(2 \bar{v}_i \delta_{ij} - \theta E_{ij} - \sum_{k=1}^N \phi_k G_{ijk} \right) \dot{W}_j] = \frac{\gamma}{6} [\bar{v}_i^2 A_i - \frac{c_{d0}}{a} C_i - \bar{v}_i \theta B_i]; \quad i = 1, 2, \dots, N
\end{aligned} \tag{3.7}$$

2. Flap equations, δW_i :

$$\begin{aligned}
& \sum_{j=1}^N \left\langle V_j \frac{\sin(2\Re\theta)}{2} \{(\Lambda_2 - \Lambda_1)\beta_j^4 \delta_{ij} + (\Lambda_{2ss} - \Lambda_{1ss})\Delta_{xj1}\} + W_j \{(D_{ij} + \bar{m}_{ss}D_{xj}) \right. \\
& + [\Lambda_1 + (\Lambda_2 - \Lambda_1)\sin^2(\Re\theta)]\beta_j^4 \delta_{ij} + [\Lambda_{1ss} + (\Lambda_{2ss} - \Lambda_{1ss})\sin^2(\Re\theta)]\Delta_{xj1}\} \\
& + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jkt} \phi_j [W_k \sin(2\Re\theta) + V_k \cos(2\Re\theta)] \\
& + (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xjk} \phi_j [W_k \sin(2\Re\theta) + V_k \cos(2\Re\theta)] \\
& + 2\beta_{pc} \dot{V}_j (\delta_{ij} + \bar{m}_{ss}\delta_{xj1}) + 2 \sum_{k=1}^N (F_{ikj} + \bar{m}_{ss}F_{xikj}) W_k \dot{V}_j + (\delta_{ij} + \bar{m}_{ss}\delta_{xj1}) \ddot{W}_j \\
& + \frac{\gamma}{6} [-J_{ij} \phi_j + \sum_{k=1}^N L_{ijk} V_j W_k + \beta_{pc} E_{ij} V_j - \frac{\bar{c}}{2} O_{ij} W_j - (2\theta E_{ij} + 2 \sum_{k=1}^N \phi_k G_{ijk} \\
& - \bar{v}_i \delta_{ij}) \dot{V}_j + E_{ij} \dot{W}_j - \frac{3\bar{c}}{4} I_{ij} \dot{\phi}_j] \rangle = -\beta_{pc} B_i + \frac{\gamma}{6} [-\bar{v}_i B_i + \theta C_i + \frac{\bar{c}}{2} \beta_{pc} B_i]; i = 1, 2, \dots, N
\end{aligned} \tag{3.8}$$

3. Torsion equations, $\delta\phi_i$:

$$\begin{aligned}
& \sum_{j=1}^N \left\langle \phi_j \{ \mu^2 K N_{ij} + \mu_{ss}^2 K_{ss} N_{xj} + [\kappa \gamma_j^2 + (\mu_2^2 - \mu_1^2) \cos(2\theta)] \delta_{ij} \right. \\
& + \kappa_{ss} \Delta_{xj2} + (\mu_{2ss}^2 - \mu_{1ss}^2) \cos(2\theta) \delta_{xj2} \} \\
& + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{ijk} [\cos(2\Re\theta) V_j W_k + \frac{\sin(2\Re\theta)}{2} (W_j W_k - V_j V_k)] \\
& + (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xjk} [\cos(2\Re\theta) V_j W_k + \frac{\sin(2\Re\theta)}{2} (W_j W_k - V_j V_k)] \\
& + (\mu^2 \delta_{ij} + \mu_{ss}^2 \delta_{xj2}) \ddot{\phi}_j + \frac{\gamma \bar{c}}{48} M_{ij} \dot{\phi}_j \rangle = \\
& - (\mu_2^2 - \mu_1^2) \frac{\sin(2\theta)}{\sqrt{2}\gamma_i} - (\mu_{2ss}^2 - \mu_{1ss}^2) \frac{\sin(2\theta)}{\sqrt{2}\gamma_i}; \quad i = 1, 2, \dots, N
\end{aligned} \tag{3.9}$$

The non-dimensional parameters and the integrals of the products of the basis functions appearing in (3.7 – 3.9) are as follows:

$$\begin{aligned}
K_{ss} &= \frac{k_{As}^2}{k_m^2} & \mu_{1ss} &= \frac{k_{m1ss}}{R} \\
\mu_{2ss} &= \frac{k_{m2ss}}{R} & \mu_{ss} &= \frac{k_{mss}}{R} \\
\bar{m}_{ss} &= \frac{m_{ss}}{m} & \Lambda_{1ss} &= \frac{(EI_{y'})_{ss}}{m\Omega^2 R^4} \\
\Lambda_{2ss} &= \frac{(EI_{z'})_{ss}}{m\Omega^2 R^4} & \kappa_{ss} &= \frac{(GJ)_{ss}}{m\Omega^2 R^4} \\
\delta_{xy1} &= \int_{\bar{x}_1}^{\bar{x}_2} \Psi_i \Psi_j d\bar{x} & \delta_{xy2} &= \int_{\bar{x}_1}^{\bar{x}_2} \Theta_i \Theta_j d\bar{x} \\
\Delta_{xy1} &= \int_{\bar{x}_1}^{\bar{x}_2} \Psi_i'' \Psi_j'' d\bar{x} & \Delta_{xy2} &= \int_{\bar{x}_1}^{\bar{x}_2} \Theta_i' \Theta_j' d\bar{x} \\
D_{xy} &= \int_{\bar{x}_1}^{\bar{x}_2} \frac{1-\bar{x}^2}{2} \Psi_i' \Psi_j' d\bar{x} & N_{xy} &= \int_{\bar{x}_1}^{\bar{x}_2} \frac{1-\bar{x}^2}{2} \Theta_i' \Theta_j' d\bar{x} \\
K_{xyk} &= \int_{\bar{x}_1}^{\bar{x}_2} \Theta_i \Psi_j'' \Psi_k'' d\bar{x} & F_{xyk} &= -\frac{1}{\beta_k^4} \int_{\bar{x}_1}^{\bar{x}_2} \Psi_i' \Psi_j' \Psi_k''' d\bar{x}
\end{aligned} \tag{3.10}$$

All other parameters in Equations (3.7 – 3.9) are the same as defined in Chapter 2.

One can obtain the perturbation equations by firstly decomposing the generalized coordinates into steady equilibrium quantities and small unsteady perturbation quantities (Equation 2.67). Following the procedure in Section 2.5, the perturbation equations read as follows:

$$\langle [M] + [\Delta M] \rangle \{ \ddot{X} \} + \langle [C] + [\Delta C] \rangle \{ \dot{X} \} + \langle [K] + [\Delta K] \rangle \{ X \} = \{ 0 \} \tag{3.11}$$

where $[M]$, $[C]$, and $[K]$ are the same as defined earlier and:

- the mass matrix due to the “smart” spring is given by:

$$[\Delta M] = \begin{bmatrix} \bar{m}_{ss} \delta_{xy1} & 0 & 0 \\ 0 & \bar{m}_{ss} \delta_{xy1} & 0 \\ 0 & 0 & \mu_{ss}^2 \delta_{xy2} \end{bmatrix}, \quad (3.12)$$

- the damping matrix due to the “smart” spring is given by:

$$[\Delta C] = 2 \begin{bmatrix} \bar{m}_{ss} \sum_{k=1}^N (F_{xkj} - F_{xjk1}) V_{0k} & -\bar{m}_{ss} \beta_{pc} \delta_{xy1} - \bar{m}_{ss} \sum_{k=1}^N F_{xjk1} W_{0k} & 0 \\ \bar{m}_{ss} \beta_{pc} \delta_{xy1} + \bar{m}_{ss} \sum_{k=1}^N F_{xkj} W_{0k} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (3.13)$$

- the stiffness matrix due to the “smart” spring is given by:

$$[\Delta K] = \begin{bmatrix} \Delta K_{11} & \Delta K_{12} & \Delta K_{13} \\ \Delta K_{21} & \Delta K_{22} & \Delta K_{23} \\ \Delta K_{31} & \Delta K_{32} & \Delta K_{33} \end{bmatrix}, \quad (3.14)$$

with the elements of $[\Delta K]$ as follows:

$$\begin{aligned}
\Delta K_{11} &= [\Lambda_{2ss} - (\Lambda_{2ss} - \Lambda_{1ss}) \sin^2(\Re \theta)] \Delta_{xy1} + \bar{m}_{ss} (D_{xy} - \delta_{xy1}) \\
&\quad - (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xkj} \phi_{0k} \sin(2\Re \theta) \\
\Delta K_{12} &= (\Lambda_{2ss} - \Lambda_{1ss}) \frac{\sin(2\Re \theta)}{2} \Delta_{xy1} + (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xkj} \phi_{0k} \cos(2\Re \theta) \\
\Delta K_{13} &= (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xkj} [W_{0k} \cos(2\Re \theta) - V_{0k} \sin(2\Re \theta)] \\
\Delta K_{21} &= (\Lambda_{2ss} - \Lambda_{1ss}) \frac{\sin(2\Re \theta)}{2} \Delta_{xy1} + (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xkj} \phi_{0k} \cos(2\Re \theta) \\
\Delta K_{22} &= \bar{m}_{ss} D_{xy} + [\Lambda_{1ss} + (\Lambda_{2ss} - \Lambda_{1ss}) \sin^2(\Re \theta)] \Delta_{xy1} + (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xkj} \phi_{0k} \sin(2\Re \theta) \\
\Delta K_{23} &= (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xkj} [V_{0k} \cos(2\Re \theta) + W_{0k} \sin(2\Re \theta)] \\
\Delta K_{31} &= (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xjk} [W_{0k} \cos(2\Re \theta) - V_{0k} \sin(2\Re \theta)] \\
\Delta K_{32} &= (\Lambda_{2ss} - \Lambda_{1ss}) \sum_{k=1}^N K_{xjk} [V_{0k} \cos(2\Re \theta) + W_{0k} \sin(2\Re \theta)] \\
\Delta K_{33} &= \mu_{ss}^2 K_{ss} N_{xy} + \kappa_{ss} \Delta_{xy2} + (\mu_{2ss}^2 - \mu_{1ss}^2) \cos(2\theta) \delta_{xy2}
\end{aligned} \tag{3.15}$$

and as before:

$$\{X\} = \begin{Bmatrix} \Delta V_j \\ \Delta W_j \\ \Delta \phi_j \end{Bmatrix} \tag{3.16}$$

3.3. Stability Analysis of the “Smart” Rotor

The presence of the “smart” spring alters the nature of the stability analysis of the hingeless helicopter blade. Since the “smart” spring is activated only at certain time period intervals, the equations of motion will have periodic coefficients. The mechanism of the activation of the “smart” spring is described in detail in Section 3.4. This new system differs from the original system, which employed constant coefficients. As a consequence, this periodic system requires a more complex approach for stability analysis. Some well-known methods that can be used to solve equations with periodic coefficients are the Floquet method, the multi-blade coordinate, and the harmonic balance. The Floquet method is used in the present study because of its simplicity and ease of application. For a detailed explanation of the Floquet method one may refer to Dugundji and Wendell (Ref. 5).

The analysis starts with rewriting the blade equations of motion in the following form:

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M(t)]^{-1} [K(t)] & -[M(t)]^{-1} [C(t)] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} = [P(t)] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \quad (3.17)$$

where $[M]$, $[K]$, and $[C]$ are now functions of time t representing actuation and de-actuation of the “smart” spring. Over one blade rotation of 2π , assume that the “smart” spring is actuated n times and de-actuated the same number of times. Thus, there are n periodic intervals of magnitude $T = 2\pi / n$, each one of which is divided into two equal subintervals of π / n magnitude (Fig. 3.2). Thus:

$$\begin{aligned}
[M(t)] &= [M] + [\Delta M]; & 2j \frac{\pi}{n} \leq \Omega \leq (2j+1) \frac{\pi}{n}, j = 0, 1, 2, \dots, n-1 \\
&= [M]; & (2j+1) \frac{\pi}{n} < \Omega < (2j+2) \frac{\pi}{n}, j = 0, 1, 2, \dots, n-1 \\
[C(t)] &= [C] + [\Delta C]; & 2j \frac{\pi}{n} \leq \Omega \leq (2j+1) \frac{\pi}{n}, j = 0, 1, 2, \dots, n-1 \\
&= [C]; & (2j+1) \frac{\pi}{n} < \Omega < (2j+2) \frac{\pi}{n}, j = 0, 1, 2, \dots, n-1 \\
[K(t)] &= [K] + [\Delta K]; & 2j \frac{\pi}{n} \leq \Omega \leq (2j+1) \frac{\pi}{n}, j = 0, 1, 2, \dots, n-1 \\
&= [K]; & (2j+1) \frac{\pi}{n} < \Omega < (2j+2) \frac{\pi}{n}, j = 0, 1, 2, \dots, n-1
\end{aligned} \tag{3.18}$$

According to the Floquet theorem, a periodic system with no forcing terms has an exponential solution of the form:

$$\{X(t)\} = B(t) \{C_k e^{p_k t}\} \tag{3.19}$$

where $B(t)$ is the solution matrix to the periodic system, C_k is a constant, p_k is the exponential constant, and t is time. Moreover,

$$\begin{aligned}
\{X(t)\} &\in L^{2M_1 \times 1} \\
B(t) &\in L^{2M_1 \times 2M_1} \\
\{C_k e^{p_k t}\} &\in L^{2M_1 \times 1}
\end{aligned} \tag{3.20}$$

and L represents the size of the matrix, . Since $B(t)$ is periodic over period T , then: $B(0) = B(T)$. It follows from Equations (3.10) and (3.11):

$$\begin{aligned}
\{X(0)\} &= B(0) \{C_k\} \\
\{X(T)\} &= B(T) \{C_k e^{p_k T}\} = B(0) \{C_k e^{p_k T}\}
\end{aligned} \tag{3.21}$$

One can also express the solution of the periodic equations of motion as a linear combination of independent vectors as follows:

$$\{X(T)\} = \begin{bmatrix} X^{(1)} & X^{(2)} & \dots \end{bmatrix} \begin{Bmatrix} X_1(0) \\ X_2(0) \\ \vdots \end{Bmatrix} = [Q(T,0)] \begin{Bmatrix} X_1(0) \\ X_2(0) \\ \vdots \end{Bmatrix} \quad (3.22)$$

where $X^{(1)}$ is the solution at $t = T$ of the periodic equation of motion (3.17) for the initial conditions $X_1(0) = 1$ and all the remaining $X_i(0) = 0$, $X^{(2)}$ is the solution for the initial conditions $X_2(0) = 1$ and all the remaining $X_i(0) = 0$, etc. The $2M_1 \times 2M_1$ matrix, $[Q(T,0)]$, is called the state transition matrix between $t = 0$ to $t = T$. The transition matrix, $[Q(T,0)]$, can be obtained using either:

1. “multiple pass approach” where the equations of motion are integrated $2M_1$ times to cover all initial conditions, or
2. “a single pass approach” which is integrating once, but keeping all initial conditions simultaneously.

In this study, the first approach was used to obtain the state transition matrix. Using Equations (3.21), (3.22) gives:

$$[Q(T,0)] [\{B(0)\}_1 C_1 + \{B(0)\}_2 C_2 + \dots] = \{B(0)\}_1 C_1 e^{p_1 T} + \{B(0)\}_2 C_2 e^{p_2 T} + \dots \quad (3.23)$$

Since the coefficients C_k are independent, one must have:

$$[Q(T,0)] \{B(0)\}_k = \lambda_k \{B(0)\}_k \quad (3.24)$$

One can observe that λ_k are the eigenvalues of the transition matrix, and it can be shown that:

$$\begin{aligned}
\lambda_k &= e^{p_k T} \\
p_k &= \left(\frac{1}{T} \right) \ln(\lambda_k) = \xi_k + i \omega_k \\
\xi_k &= \left(\frac{1}{T} \right) \ln |\lambda_k| = \left(\frac{1}{2T} \right) \ln [(\lambda_k)_R^2 + (\lambda_k)_I^2] \\
\omega_k &= \left(\frac{1}{T} \right) \tan^{-1} [(\lambda_k)_I / (\lambda_k)_R]
\end{aligned} \tag{3.16}$$

where the subscripts R and I of λ_k correspond to its real and imaginary parts respectively.

The real part of p_k , which is ξ_k , represents the growth /decay rate of the response to the system. A positive value indicates instability. The imaginary part ω_k , is the frequency. Because of the function $\tan^{-1}$, ω_k is multi-valued but the numerical execution of the code calculates its principal value, therefore, ω_k can be obtained as the basic frequency plus or minus an integer multiple of $2\pi / T$. The value of ω_k can be chosen from the principal value of ω_k or the frequency expected from the physical considerations, (Refs. 16 and 23). For this reason, the Static Regime, in which the “smart” spring is actuated at all time, is included in the numerical analysis to indicate the physical region of ω_k . A more detailed explanation of the Static and different regimes is provided in Section 4.1.

3.4. Selection of Parameters for the “Smart” Blade

In order to study the stability of a hingeless helicopter rotor blade in hover flight, several configuration parameters must be specified. In general, there are six types of configuration parameters, namely: structural, inertial, geometric, aerodynamic, coupling, and control parameters. Each type of configuration parameter will be discussed and the summary of the numerical values of each parameter is presented in Table 3.1.

The specified aerodynamic, geometric, control, and coupling parameters for the blade model used in this study represent the typical values for a hingeless rotor blade. For the geometric pre-cone angle β_{pc} a baseline value of zero radian is chosen. Next, the solidity ratio $\sigma = (bc)/(\pi R)$, is specified. For this study, the number of blades b is taken to be 4 and the ratio of the chord length to the radius c/R is 6%. Therefore, the solidity ratio is 0.077. The aerodynamic parameters selected in this study are: the Lock number $\gamma = 5$, the drag coefficient $c_{d0} = 0.01$, and the lift curve slope $a = 2\pi/\text{rad}$. The only control parameter being considered, which defines the rotor thrust, is the pitch angle θ , the baseline value of which is taken to be zero. The structural coupling parameter $\mathfrak{R}$, which represents the stiffness distribution characteristics of the rotor, can be assigned any value between 0 (uncoupled case) to 1 (completely coupled case). The baseline value for the coupling parameter is taken to be 1. Further discussions on the coupling parameter $\mathfrak{R}$ can be found in Section 2.4.5.

The structural and inertial parameters play very important roles in the stability of the system because they contribute to the natural frequencies of the rotor blade. In order to present a realistic study case, a model of a helicopter blade is developed. The structure of the “smart” blade model consists of (Figs. 3.3, 3.4):

1. A uniform main structure covered with a skin to form an airfoil cross-section
2. “Smart” materials at the corners of the main structure located at any desired locations along the blade.
3. Non-structural filler materials (foam) between the blade structure and the skin.

The first step in the design of the “smart” blade is to configure the main structure of the blade. For this purpose, the design was based on the parameters of a reduced scale of

the EUROCOPTER BO105 helicopter blade. Using the reduced scale BO105 data (obtained through private communication), the designed rotor is assigned non-dimensional rotating frequencies (normalized with respect to the operating speed of 110 rad/sec) of 0.7, 1.14, 3.9 corresponding to the first lead-lag, first flap, and first torsion modes, respectively, with a radius span of 2 meters.

In general, there is no simple relationship to correlate the structural and inertial parameters to the rotating blade natural frequencies. For this study, the following approach was used. First, the uncoupled, non-rotating natural frequencies of the reduced blade were determined using the appropriate data from experimental calibration of a BO105 that correlate the natural frequencies of the non-rotating blade to the rotating one. Next, employing the functions that relate the structural and inertial parameters to the uncoupled, non-rotating natural frequencies of the blade, the structural and inertial parameters for the intended design were calculated. The relationships used are:

$$\begin{aligned}\frac{EI_{z'}}{m\Omega^2 R^4} &= \frac{(\omega_{l, NR} / \Omega)^2}{\beta_1^4} \\ \frac{EI_{y'}}{m\Omega^2 R^4} &= \frac{(\omega_{w, NR} / \Omega)^2}{\beta_1^4} \\ \frac{GJ}{m\Omega^2 k_m^2 R^2} &= \frac{(\omega_{\phi, NR} / \Omega)^2}{\gamma_1^2}\end{aligned}\tag{3.17}$$

where the subscript NR refers to the non-rotating case. By utilizing different material choices, the optimal moments of inertia were found, and thus, the dimensions of the main structure were specified and the radii of gyration of the cross-section were also calculated. For this study, the material chosen for the optimal dimensions of the main structure is graphite – epoxy composite.

The “smart” spring may be installed along the blade as shown in (Fig. 3.4). When actuated, the “smart” spring will expand to close the gap that exists between it and the main structure. As the result of the actuation, the stiffness characteristics of the blade along the length of the spring will increase. In this study, piezoelectric material is chosen as the “smart” spring because of its strength and ability to undergo a considerable amount of deformation upon the application of a reasonable electric field (the material properties are tabulated in Table 3.2). Depending on its location and the length of the “smart” spring, the stability characteristics of the system will vary. Two design cases were considered in the present study. For the first design case the “smart” spring is installed between 0.05 to 0.2, and for the second design case between 0.05 to 0.40 in the normalized direction of the blade (cf. Section 4.3).

Chapter Four

Aeroelasticity Stability Analysis

4.1. Overview

The stability analysis of the hingeless “smart” blade was carried out by determining and comparing the eigenvalues of the controlled blade versus the uncontrolled rotor. The dynamic equations of motion (2.77) with the constant-coefficients were used for the uncontrolled blade, whereas for the controlled blade, equation (3.17) with periodic coefficients was used to obtain the system eigenvalues. In general, these eigenvalues are complex conjugate pairs, which consist of both real and imaginary parts. The real part of an eigenvalue represents the growth /decay of the system’s response, while the imaginary part is the oscillation frequency of this response.

In general, two types of instability can occur in the operation of a helicopter rotor blade, i.e., flutter and divergence. An accurate definition of each phenomena is given by Bisplinghoff, Ashley, and Halfman (Ref. 2) as follows:

1. Flutter is a **dynamic** instability occurring in an aircraft in flight, at a speed called the flutter speed, where the elasticity of the structure plays an essential part in the instability.
2. Divergence is a **static** instability of a lifting surface of an aircraft in flight, at a speed called the divergence speed, where the elasticity of the lifting surface plays an essential role in the instability.

Flutter and divergence instability of a system is related to the root loci, i.e., the eigenvalues of the system. Flutter is characterized by the presence of a root in the right-

half-plane of the s -plane, which is the half-plane representing the instability region of the Laplace domain, with a non-zero imaginary part. On the other hand, divergence is recognized by the presence of a root on the real axis of right-half-plane of the Laplace domain.

Certain parameters were chosen as the basis for an aeroelastic parametric study of instability in this investigation. These parameters are:

1. The span-wise location and the length of the “smart” spring, i.e. the starting and ending position of the “smart” spring, $\bar{x}_1$ and $\bar{x}_2$, in Equations (3.1).
2. The structural coupling $\mathfrak{R}$.
3. The pitch angle θ .

By varying the values of the parameters above, different system responses are obtained for the 1st lead-lag, 1st and 2nd flap, and 1st torsion modes and the aeroelastic stability of the system can thus be analyzed. Additional parameters that are associated with the “smart” blade are tabulated in Table 3.1.

Another important aspect that must be noted in the stability analysis of the hingeless “smart” blade is the activation regime of the “smart” spring that results in various systems as outlined below:

1. Baseline Regime:

The system consisting of only the main structure without the “smart” spring.

2. Static Regime:

The system consisting of the main structure and the “smart” spring that is activated all the time.

3. IBC 3 Regime:

The system consisting of the main structure and the “smart” spring that is activated and deactivated 3 times per rotor revolution.

4. IBC 4 Regime:

The system consisting of the main structure and the “smart” spring that is activated and deactivated 4 times per rotor revolution.

5. IBC 5 Regime:

The system consisting of the main structure and the “smart” spring that is activated and deactivated 5 times per rotor revolution.

As mentioned earlier, for the Baseline and Static regimes the equations of motion should have constant coefficients. Hence, for the Baseline, i.e., in the absence of the “smart” spring, the system’s perturbation is defined by equations (2.71). For the Static Regime, i.e., continuously activated spring, where the system’s perturbations are defined by equations (3.11), $[M]+[\Delta M]$, $[C]+[\Delta C]$, and $[K]+[\Delta K]$ are constants for all time t . In both cases, the corresponding eigenvalues can be determined using the conventional eigenvalue-eigenvector technique. In contrast, since IBC 3, IBC 4, and IBC 5 are periodic systems (due to the harmonic activation of the “smart” spring), the eigenvalues are determined using Floquet’s method as outlined in Section 3.3. The Static Regime is included for two reasons. First, it provides an insight into the behavior of the hingeless blade without periodic activation and deactivation of the “smart” spring. In this case, the “smart” blade is modeled and analyzed as a “composite” structure. Second, the imaginary parts of this regime are used to normalize the multi-valued oscillation frequencies of the IBC regimes as explained in Section 3.3.

4.2. Steady State Deflections

It is important to note that the elements of the state-space matrices consist of the steady state modal quantities of the lead-lag, flap, and torsional deflections as well as the integration of the products of the basis functions. Therefore, the steady state deflections, also known as trim deflections, of the hingeless blade must first be determined for two reasons, namely:

1. The trim deflections contribute to the coefficients of the linearized perturbation equations.
2. The trim deflections define the operating condition of the rotor, which when added to the commanded pitch angle, determine the rotor thrust in the hover.

The trim deflections are given in terms of dimensionless tip deflections that are related to the equilibrium generalized coordinates by the following relations (Ref. 12):

$$\begin{aligned}\bar{v}_0|_{np} &= 2 \sum_{j=1}^N V_{0j} (-1)^{j-1} \\ \bar{w}_0|_{np} &= 2 \sum_{j=1}^N W_{0j} (-1)^{j-1} \\ \phi_0|_{np} &= \sqrt{2} \sum_{j=1}^N \phi_{0j} (-1)^{j-1}\end{aligned}\tag{4.1}$$

Fig. 4.1 shows the trim deflections of the hingeless blade for the Baseline Regime as functions of pitch angle θ , for both the coupled and the uncoupled case, i.e., $\mathfrak{R} = 1$ and $\mathfrak{R} = 0$, respectively. The blade pitch angle θ , is varied from zero to a high nominal operation value of 0.3. One may expect that increasing the pitch angle will increase the aerodynamic loads on the blade thus generally, increasing the magnitude of the equilibrium deflections. It is observed that the structural coupling parameter $\mathfrak{R}$ has very little effect for all deflections at low pitch angles, i.e., $\theta < 0.15$.

The equilibrium torsional deflection depends on both the inertial as well as the structural moments in a more complex manner. At low pitch angles, the inertial terms, which are not influenced by the structural coupling, dominate the torsional moments causing a positive deflection, which is evident in Fig. 4.1. The influence of the structural moments on the torsional deflection is not as simple. One may see from the equation of torsional equilibrium, Eq. 2.70, that there are two terms in the equation that result from the contribution of the difference between the lead-lag and flap stiffnesses, i.e.,

$$(\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{\eta k} [\cos(2\Re\theta) V_{0j} W_{0k} + \frac{\sin(2\Re\theta)}{2} (W_{0j} W_{0k} - V_{0j} V_{0k})] \quad (4.2)$$

Let us first consider the uncoupled case, $\Re = 0$, where the second term vanishes. The first term is a positive torsional moment, which acts to add to the deflection caused by the inertial moment. For the coupled case, $\Re = 1$, the contribution from the second term to the moment counteracts the inertial moment, which tends to reduce the deflection as clearly demonstrated in Fig. 4.1.

4.3. Aeroelasticity Stability Analysis

Aeroelastic stability of a hingeless “smart” blade depends on several parameters. As already mentioned in Section 4.1, some of these parameters have been chosen to analyze the sensitive operational regions towards instability behavior of the blade. Additional parameters associated with the “smart” blade are tabulated in Table 3.1. Various combinations of the chosen parameters result in 4 design cases as follows:

1. **Design Case 1:** $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 1.0$.
2. **Design Case 2:** $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$.

3. **Design Case 3:** $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 1.0$.

4. **Design Case 4:** $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$.

The aeroelastic flap-lag-torsion stability of the hingeless “smart” blade for the above four design cases is presented in the following. A MATLAB computer code named Rutlo45.m (Appendix C), which prompts the user to input the parameters $\bar{x}_1, \bar{x}_2, \Re, \theta$, and IBC activation frequency, was written to obtain the eigenvalues of the dynamic equations of motion of the hingeless “smart” blade. The user manual for this code is also included at the beginning of Appendix C. This code includes several subroutines that perform various tasks such as:

1. Performing the numerical integration of the products of the basis functions, which is supported by a MATLAB subroutine called Newton.m
2. Calculating the steady state deflections
3. Assembling the matrices, and
4. Solving for the eigenvalues of the equations of motion.

The results corresponding to each of the above four design cases are presented in four sets of figures, each comprising four figures, i.e., Figures (4.2.1 – 4.2.4), (4.3.1 – 4.3.4), (4.4.1 – 4.4.4), and (4.5.1 – 4.5.4). The variation of damping and frequency as functions of pitch angle for each mode, i.e., the 1st lead-lag, 1st and 2nd flap, and 1st torsion, are plotted respectively in:

1. Figures 4.2.1, 4.2.2, 4.2.3, and 4.2.4 for Design Case 1.
2. Figures 4.3.1, 4.3.2, 4.3.3, and 4.3.4 for Design Case 2.
3. Figures 4.4.1, 4.4.2, 4.4.3, and 4.4.4 for Design Case 3.
4. Figures 4.5.1, 4.5.2, 4.5.3, and 4.5.4 for Design Case 4.

With reference to these plots, it should be noted that a pitch angle of 0.3 is a considerably larger value than a typical operational pitch angle. Thus, this investigation covers the stability analysis of a hingeless blade for complete practical applications. The data that were obtained and used to produce the above mentioned plots are tabulated in Tables (4.1.1 – 4.1.8) for Design Case 1, Tables (4.2.1 – 4.2.10) for Design Case 2, Tables (4.3.1 – 4.3.8) for Design Case 3, Tables (4.4.1 – 4.4.14) for Design Case 4. It should be noted that the individual data points from these tables were used by the MATLAB built-in function SPLINE to plot the corresponding graphs. This results in a best-fit graph for each design case.

For a given configuration of “smart” spring at zero pitch angle, i.e., $\theta = 0$, the structural coupling $\Re$ becomes ineffective since $\Re\theta = 0$. At this operating condition, both first and second flap modes are highly damped due to the large aerodynamic damping produced by the flapping velocity (Tables 4.1.1, 4.2.1, 4.3.1, and 4.4.1). However, the lead-lag damping due to the aerodynamic forces is low. According to Ormiston and Hodges (Ref. 22), structural damping acts like aerodynamic profile drag damping, which is also low for zero pitch operation. Therefore, the 1st lead-lag mode starts from a very close proximity from the zero damping line as can be observed in Figures 4.2.1, 4.3.1, 4.4.1, and 4.5.1.

4.3.1. Comparative Study

In conducting the aeroelastic analysis of the “smart” blade, the correctness and accuracy of the calculated eigenvalues are of primary importance.. To ensure the accuracy of the computer codes (rutlo9.m, rutlo10.m, and rutlo11.m) that were developed

to calculate the eigenvalues, the results obtained in the present work were compared to corresponding results published in the existing literature. The work reported by Hodges and Ormiston (Ref. 12) is used as for this purpose. These researchers conducted a stability analysis of a uniform helicopter blade in the absence of pre-twist and chord-wise offsets between the elastic, aerodynamic, tension, and mass axis. The properties of this blade, which are given in Table 4.5, were used to calculate eigenvalues using the codes developed in the present work. Three modes were considered, i.e., 1st lead-lag, 1st flap, and 1st torsion. The eigenvalues produced by both Hodges and Ormiston and the author are tabulated in Tables 4.6.1 – 4.6.3 for pitch angles of 0.0, 0.3, and 0.5, respectively. Considering that a slightly different expression was used in the present work for v_i as well as the fact that the values for Hodges and Ormiston's work were interpolated from graphs presented in their work, the reasonable agreement of the two results is self evident. Thus, the reliability of the computer codes (rutlo9.m, rutlo10.m, and rutlo11.m), which were the basis of the more sophisticated subsequent set of codes as outlined in Section 4.3, was established.

4.3.2. Design Case 1: $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\mathfrak{R} = 1.0$.

1st lead-lag mode: Figure 4.2.1 and Table 4.1.5 show the eigenvalues corresponding to the 1st lead-lag mode of the hingeless “smart” blade. As the pitch angle is increased from 0 to 0.3, an increase of damping is observed for all systems, i.e., Baseline, Static and IBC 3, 4, and 5 regimes. The oscillation frequency decreases to zero at 0.3 pitch angle for the Baseline, Static, IBC 3, and IBC 4 regimes. Divergence is experienced at 0.3 pitch angle by the Baseline and Static regimes as each has one root on the real axis at

the right-half- plane of the Laplace domain, i.e., a positive damping value and a zero oscillation frequency. At pitch angles greater than 0.3, IBC 4 becomes prone to divergence as one of its roots on the real axis is approaching the stability boundary. Thus, the presence of the “smart” spring as IBC 3, IBC 4, and IBC 5 regimes indeed stabilize the hingeless blade.

1st flap mode: All systems experience decreasing damping for the 1st flap mode when the pitch angle is increased from 0 to 0.3 as can be seen Figure 4.2.2 and Table 4.1.6. A sharp decrease in damping of the Static and IBC 5 regimes make them sensitive to flutter if the pitch angle is increased beyond 0.3.

2nd flap mode: A similar trend as the 1st flap mode is observed in the 2nd flap mode as can be seen in Figure 4.2.3 and Table 4.1.7. As the pitch angle is increased from 0 to 0.3, all systems experience a decrease in damping, i.e., moving towards the unstable region. A very rapid decrease of damping is observed for both IBC 3 and IBC 4 when the pitch angle is increased from 0.2 to 0.3. If the pitch angle was increased beyond 0.3, both IBC 3 and IBC 4 will have experienced flutter.

1st torsion mode: Figure 4.2.4 and Table 4.1.8 show an increase of damping for the 1st torsion mode of all systems as the pitch angle is increased from 0 to 0.3. At 0.3 pitch angle, all systems are highly damped and thus the amplitude of response will quickly decrease. A particularly interesting situation is observed for IBC 3 Regime where the system responds in a damped, non-oscillatory manner at zero pitch angle, while its damping increases as the pitch angle increases. A negative pitch angle on IBC 3 Regime might result in divergence due to the separation of the roots on the real axis of the Laplace domain.

4.3.3. Design Case 2: $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\mathfrak{R} = 0.0$.

This design case is similar to Design Case 1 except for the structural coupling. This design case represents a non-coupled structure, while Design Case 1 corresponds to a completely coupled structure. The results for this Design Case are presented in Figures 4.3.1 - 4.3.4 and Tables 4.2.1 - 4.2.10.

1st lead lag mode: An increase in damping of the 1st lead-lag mode is observed for all systems as the pitch angle is increased from 0 to 0.3 except for the Baseline Regime, as shown in Figure 4.3.1 and Table 4.2.7. For both the Baseline and Static regimes the oscillation frequency decreases to zero, at $\theta = 0.25$ for the Baseline Regime and at $\theta = 0.3$ for the Static Regime. Divergence takes place for the Baseline Regime starting at 0.25 pitch angle while at $\theta = 0.3$ the Static Regime is damped without oscillation. At this point, the Static Regime becomes prone to divergence since the roots are split on the real axis of the Laplace domain, i.e., at the zero frequency. Again, for the 1st lead-lag mode IBC 3, IBC 4, and IBC 5 promote stability to the Baseline Regime.

1st flap mode: The 1st flap mode for this design case is depicted in Figure 4.3.2 and Table 4.2.8. As the pitch angle is increased from 0 to 0.3, the Static, IBC 3, IBC 4, and IBC 5 regimes experience a sharp drop in damping associated with the 1st flap mode leading to flutter for the Static Regime at approximately 0.23 pitch angle. At 0.3 pitch angle, one will also observe that IBC 3, IBC 4, and IBC 5 each has a root near the stability boundary, i.e., at the zero damping line. Therefore, should the pitch angle be increased beyond 0.3, the systems might experience flutter. An interesting trend occurs for the Baseline Regime where initially from 0 to 0.2 pitch angle the damping decreases while it increases from 0.2 to 0.3 pitch angle. This implies that the Baseline Regime has

a self-stabilizing behavior associated with $\Re = 0$, which is a desirable feature from aeroelasticity point of view.

2nd flap mode: Figure 4.3.3 and Table 4.2.9 show the responses of the systems in terms of damping and oscillation frequency for the 2nd flap mode as the pitch angle is increased from 0 to 0.3. The Baseline and Static regimes exhibit a decrease in both damping and oscillation frequency. The IBC 3 Regime is of particular interest because of its response to the increase of the pitch angle. Between 0 to 0.15 pitch angle, both damping and the oscillation frequency decrease until the oscillation frequency reaches zero at 0.15 pitch angle. At this point, the system experiences damped non-oscillatory response as the roots split on the left-half-plane of the Laplace domain, i.e., the negative damping region. When the pitch angle is increased again to 0.3, both damping and oscillation frequency increase to attain a better system stability. The IBC 4 and IBC 5 regimes exhibit an oscillating trend for both damping and oscillation frequency as the pitch angle is increased from 0 to 0.3.

1st torsion mode: An increase in damping of the 1st torsion mode is observed in Figure 4.3.4 and Table 4.2.10 for the Baseline, Static, IBC 4, and IBC 5 regimes. Similar to its corresponding mode in Design Case 1, the IBC 3 shows a different response from the rest of the systems in which the system is damped without oscillation at 0, 0.2, as well as 0.3 pitch angle. Between 0.05 to 0.15 pitch angle, the damping and the oscillation frequency increase to non-zero values to exhibit a damped oscillatory response. Thus, it can be concluded that IBC 3 response for $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$ is sensitive to the pitch angle in the 1st torsion mode.

4.3.4. Design Case 3: $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\mathfrak{R} = 1.0$.

The results for this Design Case are presented in Figures 4.4.1 - 4.4.4 and Tables 4.3.1 - 4.3.8.

1st lead-lag mode: Figure 4.4.1 and Table 4.3.5 show the eigenvalues corresponding to the 1st lead-lag mode of the hingeless “smart” blade where trends similar to Design Case 1 are observed. As the pitch angle is increased from 0 to 0.3, an increase of damping is observed for all systems, i.e., Baseline, Static, IBC 3, IBC 4, and IBC 5 regimes. For all systems, the oscillation frequency decreases to zero at 0.3 pitch angle and separation of roots on the real axis takes place. As a result, divergence is experienced at 0.3 pitch angle by Baseline, IBC 3, IBC 4 and IBC 5 regimes as each has one root on the real axis at the right-half-plane of the Laplace domain, i.e., at the positive damping region. The Static Regime, however, is damped without oscillation and becomes prone to divergence as one of its roots on the real axis is approaching the stability boundary at 0.3 pitch angle. At this condition, with the activation of “smart” spring, the blade still experiences divergence.

1st flap mode: For the 1st flap mode, all systems experience a rapid decrease in damping towards the stability boundary when the pitch angle is increased from 0 to 0.3 as can be seen in Figure 4.4.2 and Table 4.3.6. Flutter occurs for the Static Regime at 0.3 pitch angle while IBC 3 and IBC 4 are sensitive to flutter if the pitch angle is increased beyond 0.3.

2nd flap mode: The Baseline and Static regimes start with a high damping and a non-zero oscillation frequency at 0 pitch angle and continue with a decrease in both the damping and oscillation frequency for the 2nd flap mode (Figure 4.4.3 and Table 4.3.7).

An interesting behavior is exhibited by IBC 3 and IBC 5. A decrease in damping occurs in each system when the pitch angle is increased from 0 to 0.2. This is followed by a reversed effect of increasing damping as the pitch angle is increased from 0.2 to 0.3. Thus, for both IBC 3 and IBC 5, the decay rate of the response of this mode is increased at higher pitch angles rather than promoting unstable response, which is a desirable output. IBC 4 is damped without oscillation at lower pitch angles of 0 to 0.1 and as the pitch angle increases, its response follows the same trends described above.

1st torsion mode: Figure 4.4.4 and Table 4.3.8 show an increase of damping and oscillation frequency for the 1st torsion mode of the Baseline and Static regimes as the pitch angle is increased from 0 to 0.3. At 0.3 pitch angle, the Baseline Regime becomes highly damped and thus the amplitude of response will quickly decrease. Just like the corresponding system in Design Case 1, IBC 3 presents a particularly interesting situation. As the pitch angle is increased from 0 to 0.1, the oscillation frequency drops to zero at 0.1 pitch angle, which results in a damped, non-oscillatory response. As the pitch angle varies from 0.1 to 0.3, damping and frequency increase to non-zero values. IBC 4 responds with an increasing damping as the pitch angle varies from 0 to 0.3. IBC 5 also responds in an interesting manner in which the damping increases as the pitch angle is increased from 0 to 0.2. This is followed by a sharp decline in damping as the pitch angle is further increased to 0.3 thus making the system prone to flutter at higher pitch angle.

4.3.5. Design Case 4: $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\mathfrak{R} = 0.0$.

The results for this design case are presented in Figures 4.5.1 - 4.5.4 and Tables 4.4.1 - 4.4.14.

1st lead-lag mode: In this design case, the IBC 3 Regime experiences an increase in damping up to 0.2 pitch angle and a damping decrease towards 0.3 pitch angle for the 1st lead-lag mode, Figure 4.5.1 and Table 4.4.11. This decrease in damping makes the IBC 3 Regime prone to flutter for 0.3 pitch angle. For IBC 4 as well as IBC 5, damping increases up to 0.25 pitch angle. This is followed by a decrease until $\theta = 0.28$ and an increase again at 0.3 pitch angle. As discussed in Design Case 2, the Baseline system experiences divergence at a pitch angle greater or equal to 0.25. In fact, for this mode the Static Regime exhibits the most favorable situation in which an increasing damping is found as the pitch angle is increased from 0 to 0.3.

1st flap mode: Figure 4.5.2 and Table 4.4.12 indicate that in this design case for the 1st flap mode, as the pitch angle is increased from 0 to 0.3, the Static, IBC 3, IBC 4, and IBC 5 regimes experience a sharp decrease in damping leading to flutter for the Static Regime at approximately 0.25 pitch angle. Flutter also occurs at 0.3 pitch angle for both IBC 3 and IBC 4 regimes. At 0.3 pitch angle, one will also observe that the IBC 5 has a root very close to the stability boundary, i.e., the zero damping line. Therefore, should the pitch angle be increased beyond 0.3, the system might experience flutter. As already discussed in Design Case 2, an interesting trend occurs for the Baseline Regime where the damping initially decreases as θ varies from 0 to 0.2 pitch angle followed by an increase between 0.2 to 0.3. This implies that the Baseline Regime has a self-stabilizing behavior associated with $\mathfrak{R} = 0$, which is a desirable feature from aeroelasticity point of

view. Thus, for this mode, the harmonic excitation of the “smart” spring, in fact, causes an onset of instability in the blade.

2nd flap mode: Figure 4.5.3 and Table 4.4.13 show the responses of the systems in terms of damping and oscillation frequency for the 2nd flap mode as the pitch angle is increased from 0 to 0.3. The Baseline Regime exhibits a decrease in both damping and oscillation frequency. The Static and IBC 3 regimes have almost constant damping throughout the pitch angle range of 0 to 0.28. For all values of pitch angle, the IBC 4 is damped without oscillation. This suggests that this regime can exert an effective control of the 2nd flap mode throughout the pitch angle range. When the pitch angle is increased from 0 to 0.28, the damping in the IBC 5 Regime remains almost constant. An interesting trend happens when the pitch angle approaches 0.3, at which point the oscillation frequency suddenly drops to zero and one of the roots crosses the stability boundary resulting in divergence. Thus, it can be concluded that a parametric resonance (as explained in Ref. 3) takes place on the 2nd flap mode when the “smart” spring extending from 5% to 40% stations, under uncoupled structure condition, is excited 5 times per revolution with 0.3 pitch angle.

1st torsion mode: Similar to Design Case 2, an increase in damping of the 1st torsion mode is observed in Figure 4.5.4 and Table 4.4.14 for the Baseline, Static, IBC 4 and IBC 5 regimes. The sharp increase in damping between 0.25 to 0.3 pitch angle results in highly damped systems. The IBC 3 at this mode shows a different response from the rest of the regimes. At this regime, the damping of the system increases as θ varies from 0 to 0.28. This is followed by a sudden drop in damping and oscillation frequency, which results in damped non-oscillatory response at 0.3 pitch angle. At this point, if the pitch

angle is increased further, the IBC 3 may be subjected to divergence. Considering all design cases, it is noted that there are regions of pitch angle which the IBC 3 Regime induces a damped non-oscillatory response for the 1st torsion mode.

Chapter Five

Summary, Conclusions, and Recommendations

5.1. Summary

This thesis deals with the aeroelastic stability analysis of a hingeless “smart” helicopter rotor blade in which flap, lead-lag, and torsional motions are considered. An important contribution of this work has been the theoretical development and dynamic modeling of a “smart” rotor blade using a “smart” structure technology. In this regard, a realistic helicopter blade cross-section was studied and modeled by considering a reduced-scaled model of EUROCOPTER BO105 blade as the starting point of the design. Based on EUROCOPTER BO105 dimensions, mass, and stiffness properties, the materials and the dimensions of the blade cross-section were modeled.

The main structure of the blade, also addressed as the Baseline Regime, is modeled as a uniform, untwisted, cantilevered beam without chord-wise offsets between the elastic, mass, tension, and aerodynamic axes. The governing equations of perturbation of the Baseline Regime that are used to determine the eigenvalues of the hingeless blade were originally proposed by Hodges and Ormiston (Ref. 12). The inclusion of the “smart” spring necessitates one to derive a new set of dynamic equations of motion. These were obtained based on a set of general equations of motion that were proposed by Hodges and Dowell (Ref. 11) for a twisted, non-uniform, long, straight, slender, homogeneous, isotropic beam, with chord-wise offsets between the elastic, mass, tension, and aerodynamic axes, undergoing moderate displacements (c.f. Section 2.3). Since the “smart” spring is assumed to be harmonically actuated using an on-off control scheme,

the eigenvalues of the controlled blade were determined by the Floquet method (c.f. Section 3.3). Finally, the aeroelastic stability analysis of the rotor blade is carried out by considering the damping and oscillation frequencies extracted from the eigenvalues of the perturbations of the 1st lead-lag, 1st and 2nd flap, as well as the 1st torsion modes of the Baseline, Static, and various IBC regimes.

5.2. Conclusions

In general, the results of the analysis indicate that harmonic parametric excitation of the “smart” spring affects the aeroelastic stability of the hingeless helicopter blade. Some unstable conditions encountered in the Baseline hingeless blade at high pitch angles can be stabilized using IBC. However, due to the complex manner in which various parameters such as the length or position of the “smart” spring, the pitch angle of the rotor, or the frequency of the harmonic actuation of the “smart” spring affect the behaviour of the rotor, one may expect that IBC will not always be beneficial to the overall stability of the system. As noted by Bolotin (Ref. 3), stability problems may arise when the harmonic parametric excitation approaches the frequencies associated with the structural modes. The present study also reveals that the aeroelastic behaviour of the system changes significantly with the presence of the “smart” spring. Considering the improvement in the 1st lead-lag stability of the Baseline Regime for Design Cases 1, 2, and 4 in terms of increasing modal damping at a high pitch angle, i.e., above 0.25 radians (14 degrees), one may suggest the application of “smart” spring to control aeromechanical phenomena such as ground and air resonance. Thus, a carefully

engineered “smart” spring technology, applied on a hingeless helicopter rotor blade, will improve the quality of the helicopter operation.

The Stability of the 1st Lead-Lag Mode: One important finding about the harmonically excited “smart” blade is that it improves the general stability of the 1st lead-lag mode response of the Baseline Regime for both the coupled ($\mathfrak{R} = 1$) and the uncoupled scenarios ($\mathfrak{R} = 0$). The application of the Individual Blade Control (IBC) concept using the “smart” spring technology in IBC 3, IBC 4, and IBC 5 regimes results in the aeroelastic stability for Design Cases 1, 2, and 4 when the Baseline Regime experiences divergence at a high pitch angle, i.e., above 0.25 radians (14 degrees). For Design Case 3, however, at 0.3 pitch angle, the IBC fails to stabilize the system at all regimes.

The Stability of the 1st Flap Mode: The IBC concept is less beneficial in terms of improving the stability of the uncoupled ($\mathfrak{R} = 0$) Baseline Regime for the 1st flap mode. The Baseline Regime was found to have the most desirable response between 0.2 and 0.3 where damping increases after a reduction in its magnitude from 0 to 0.2 pitch angle. Unlike the self-stabilizing response of the Baseline Regime, the harmonic actuation of the “smart” spring in IBC 3, IBC 4, and IBC 5 regimes results in decreased damping of the blade as the pitch angle is increased. In Design Case 2, all of the IBC regimes become prone to flutter at 0.3 pitch angle while in the Design Case 4, only the IBC 5 Regime experiences flutter at 0.3 pitch angle.

The Stability of the 2nd Flap Mode: The 2nd flap mode presents an interesting study case where, depending on the length of the “smart” spring and the frequency of its harmonic actuation as well as the structural coupling, the application of the IBC concept

can be either favourable or unfavourable towards improving the stability of the blade. As one can observe in Design Case 3, the IBC systems significantly improve the stability of the Baseline Regime at high pitch angles by increasing the damping as θ varies from 0.2 to 0.3. For Design Case 4, the harmonic excitation of the “smart” spring at IBC 3 and IBC 4 improves the stability of the Baseline Regime while the excitation of “smart” spring at IBC 5 results in divergence at 0.3 pitch angle. In this particular case, the damping of the IBC 3 stays approximately the same throughout the range of the pitch angle.

The Stability of the 1st Torsion Mode: In general, it is observed that for this mode, damping and oscillation frequency increase as the pitch angle is increased. For Design Case 3, IBC 5 initially increases the damping up to 0.25 pitch angle. This is followed by a sharp drop in damping towards the stability boundary at 0.3 pitch angle. This indicates that for this design, i.e., the “smart” spring located between 5% to 40% in the axial direction, its actuation at IBC 5 will result in undesirable conditions of stability.

The evaluation of the behaviour of the harmonic excitation of the “smart” spring also reveals that under certain conditions, the IBC regimes result in damped non-oscillatory response. At these regions, the eigenvalues of the systems lie on the real axis of the Laplace domain, i.e., zero frequency line. For the 2nd flap mode, IBC 4 behaves in this manner at low pitch angles in Design Case 3 as well as throughout the whole pitch angle range in Design Case 4 with relatively high damping. This implies that for Design Case 4 an IBC 4 helps the stability of the 2nd flap mode of the blade throughout an operation range of 0 up to 0.3 of pitch angle. For each design case, IBC 3 can result in a damped, non- oscillatory response for the 1st torsion mode at certain values of pitch angle.

5.3. Recommendations

The results of the study demonstrate that the harmonic actuation of the “smart” spring changes the dynamics of the system significantly. Thus, a carefully engineered “smart” spring technology, incorporated in the design of a hingeless helicopter rotor blade, will improve the quality of the helicopter operation. Several recommendations can be proposed to move the project towards the realization of the “smart” hingeless helicopter blade.

Hodges and Ormiston (Ref. 12) found that the pre-cone angle of a hingeless helicopter blade has a major effect on the dynamics and stability of the blade. Since the pre-cone angle was set to zero in this study, future stability analysis should consider various pre-cone angles. In order to better understand the aeroelastic stability of the “smart” blade, further alternatives of design cases should be considered with various lengths and locations of the “smart” spring at different values of structural coupling parameter. It is also recommended to consider higher modes, in addition to the 1st lead-lag, 1st and 2nd flap, and the 1st torsion modes as have been studied in this project.

A more sophisticated dynamics equations of motion can be developed to accommodate other physical features of a typical helicopter blade such as: droop and sweep (Fig. 2.1), as well as non-conservative loads due to the mechanical and frictional damping of the control system and the pitch bearing which may promote aeroelastic instability to the blade. A study on the effectiveness of the IBC concept can then be conducted to evaluate the performance of the “smart” spring to improve the stability of the system.

Further modifications of the general equations of motion by Hodges and Dowell (Ref. 11) can be made to accommodate the application of the “smart” spring technology on a thin-wall blade structure to investigate the stability of the system application to improve the stability of the blade.

The Galerkin method is used in this study to form the modal equations of motion by specifying the displacements as series of modal generalized coordinates and continuous mode shape functions. The application of the Galerkin method is justified since the structure dealt with in this study is relatively simple and thus can be modeled as a continuous system quite precisely. If, however, the structure being considered has a complex arrangement, one may recommend the use of the Finite Element Method to model the structure as a system of lumped parameters. One must note a cautious observation that the lumped parameter method does not always represent torsional interactions between adjoining segments as well as a continuous system does.

The most important recommendation that the author would like to propose is the extension of the operation of the “smart” blade from hover to forward flight, which introduces the reverse flow region to the rotor. Because of the azimuth dependant nature of the reverse flow region, periodic aerodynamic loads will appear in the perturbation equations of motion. As a consequence of these periodic loads, a frequency based forced response can be evaluated to analyze the feasibility of the IBC concept in the forward flight. One may also incorporate a more sophisticated aerodynamic model, which includes a free wake produced by the previous blade as well as impulse loads such as gust.

Obviously, from operational and practical points of view, much further work remains to be done in designing an optimal feedback control system that should be incorporated to provide the optimal harmonic actuation of the “smart” blade to improve the response of the system. Finally, all this work should be substantiated and validated by building and testing a properly designed reduced scale hingeless “smart” blade as a model.

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Appendix A

Figures

Fig. 1.1. The single main and tail rotor helicopter configuration (Ref. 16)

Fig. 1.2. The twin main rotor helicopter configuration (Ref. 16)

Fig. 1.3. Schematic of an articulated rotor hub and root (Ref. 16)

Fig. 1.4. Major components of a hingeless helicopter rotor blade (Ref. 12)

Fig. 2.1. Configuration parameters of a hingeless rotor blade (Ref. 12)

Fig. 2.2. Undeformed coordinate systems (Ref. 11)

Fig. 2.3. Rotor blade cross-section, before and after deformation (Ref. 12). Note that y' and z' are projections on the y and z plane.

Fig. 2.4. Airfoil cross-section under general unsteady motion (Ref. 12)

Fig. 2.5. Aerodynamic loads on an airfoil section (Ref. 12)

Fig. 3.1. Cross-sectional mass and stiffness distribution along the span wise direction of the rotor blade

Fig. 3.2. Harmonic parametric actuation of the "smart" spring

Fig. 3.3. Structural cross-section of the "smart" blade

Fig. 3.4. Side view of the "smart" blade structure. The gray area represents the "smart" materials and the triangles represent points of attachment of the "smart" materials. The schematic is not to scale.

Fig. 4.1. Steady state lead-lag, flap, and torsion at the tip as functions of pitch angle for Baseline Regime

Fig. 4.2.1. Damping and frequency of the 1st lead-lag mode of Design Case 1, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\mathcal{R} = 1.0$

Fig. 4.2.2. Damping and frequency of the 1st flap mode of Design Case 1, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 1.0$

Fig. 4.2.3. Damping and frequency of the 2nd flap mode of Design Case 1, i.e.,
 $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 1.0$

Fig. 4.2.4. Damping and frequency of the 1st torsion mode of Design Case 1, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 1.0$

Fig. 4.3.1. Damping and frequency of the 1st lead-lag mode of Design Case 2, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$

Fig. 4.3.2. Damping and frequency of the 1st flap mode of Design Case 2, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$

Fig. 4.3.3. Damping and frequency of the 2nd flap mode of Design Case 2. i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$

Fig. 4.3.4. Damping and frequency of the 1st torsion mode of Design Case 2, i.e.,
 $\bar{x}_1 = 0.05, \bar{x}_2 = 0.20, \Re = 0.0$

Fig. 4.4.1. Damping and frequency of the 1st lead-lag mode of Design Case 3, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\mathfrak{R} = 1.0$

Fig. 4.4.2. Damping and frequency of the 1st flap mode of Design Case 3. i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 1.0$

Fig. 4.4.3. Damping and frequency of the 2nd flap mode of Design Case 3. i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\mathcal{R} = 1.0$

Fig. 4.4.4. Damping and frequency of the 1st torsion mode of Design Case 3, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\mathfrak{R} = 1.0$

Fig. 4.5.1. Damping and frequency of the 1st lead-lag mode of Design Case 4, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\mathfrak{R} = 0.0$

Fig 4.5.2. Damping and frequency of the 1st flap mode of Design Case 4, i.e..
 $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$

Fig. 4.5.3. Damping and frequency of the 2nd flap mode of Design Case 4, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$

Fig 4.5.4. Damping and frequency of the 1st torsion mode of Design Case 4, i.e., $\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$

Appendix B

Tables

Table 3.1. Configuration parameters of the “smart” blade

Parameter	Baseline Value	Additional Values
$EI_z / (m\Omega^2 R^4)$	0.0222 ¹	--
$EI_y / (m\Omega^2 R^4)$	0.0017 ²	--
$GJ / (m\Omega^2 R^4)$	0.0010 ³	--
$\bar{x}_1$	0.05	--
$\bar{x}_2$	0.20	0.40
θ	0.0 rad	Variable up to 0.3 rad
β_{pc}	0.0	--
$\Re$	1	0
$(k_A / k_m)^2$	0.25	--
k_m / R	0.013	--
k_{m1} / k_{m2}	0.2778	--
b	4	--
c_1	48 mm	--
d_1	15 mm	--
a_1	1/3	--
b_1	1/3	--
e_1	1/5	--
f_1	3/5	--
c / R	0.06	--
m	2.9 kg/m	--
Ω	110 rad/sec	--
R	2 m	--
σ	0.077	--
γ	5	--
c_{do}	0.01	--
a	2π /rad	--

¹ Corresponds to $\omega_v / \Omega = 0.7$, $(\omega_v)_{NR} / \Omega = 0.514$ ² Corresponds to $\omega_w / \Omega = 1.14$, $(\omega_w)_{NR} / \Omega = 0.143$ ³ Corresponds to $\omega_\phi / \Omega = 3.9$, $(\omega_\phi)_{NR} / \Omega = 3.770$

Table 3.2. Material properties

Material	E [G Pa]	G [G Pa]	ρ [kg/m³]
Graphite / epoxy composite	142	6	1800
Piezoelectric	71.4	27.5	7500

Table 4.1.1. Eigenvalues of Design Case 1 ($\bar{x}_1 = 0.05, \bar{x}_2 = 0.20, \Re = 1.0$) at $\theta = 0.0$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0011 +0.6560 <i>i</i>	-0.0011 +0.7747 <i>i</i>	-0.0011 +0.7184 <i>i</i>	-0.0011 +0.7181 <i>i</i>	-0.0011 +0.7180 <i>i</i>
1 st flap	-0.3236 +0.9714 <i>i</i>	-0.3287 +1.0721 <i>i</i>	-0.3260 +0.4756 <i>i</i>	-0.3385 +0.9706 <i>i</i>	-0.3342 +0.4810 <i>i</i>
2 nd flap	-0.7581 +1.9077 <i>i</i>	-0.8163 +1.9770 <i>i</i>	-0.7852 +3.4433 <i>i</i>	-0.7855 +4.0448 <i>i</i>	-0.7865 +5.5547 <i>i</i>
1 st torsion	-0.3036 +2.7609 <i>i</i>	-0.3618 +3.2344 <i>i</i>	-0.3175 -0.3524	-0.3221 +4.9974 <i>i</i>	-0.3253 +6.0234 <i>i</i>

Table 4.1.2. Eigenvalues of Design Case 1 ($\bar{x}_1 = 0.05, \bar{x}_2 = 0.20, \Re = 1.0$) at $\theta = 0.1$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0567 +0.6052 <i>i</i>	-0.0466 +0.7661 <i>i</i>	-0.0182 +0.6682 <i>i</i>	-0.0203 +0.6639 <i>i</i>	-0.0201 +0.6576 <i>i</i>
1 st flap	-0.2688 +0.9584 <i>i</i>	-0.2740 +1.0509 <i>i</i>	-0.3382 +0.4766 <i>i</i>	-0.3093 +0.9582 <i>i</i>	-0.3526 +1.0688 <i>i</i>
2 nd flap	-0.7203 +1.8853 <i>i</i>	-0.7868 +1.9669 <i>i</i>	-0.7283 +3.4003 <i>i</i>	-0.7426 +4.0908 <i>i</i>	-0.7473 +5.5961 <i>i</i>
1 st torsion	-0.3855 +3.0182 <i>i</i>	-0.6193 +4.0954 <i>i</i>	-0.4944 +3.5921 <i>i</i>	-0.5069 +4.4248 <i>i</i>	-0.4592 +6.0669 <i>i</i>

Table 4.1.3. Eigenvalues of Design Case 1 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 1.0$) at $\theta = 0.2$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.1583 +0.4224 <i>i</i>	-0.1974 +0.6955 <i>i</i>	-0.1913 +0.5858 <i>i</i>	-0.1859 +0.5825 <i>i</i>	-0.2047 +0.5687 <i>i</i>
1 st flap	-0.1747 +0.9891 <i>i</i>	-0.1300 +1.0460 <i>i</i>	-0.1444 +0.4817 <i>i</i>	-0.1382 +0.9826 <i>i</i>	-0.1327 +1.0132 <i>i</i>
2 nd flap	-0.6816 +1.7241 <i>i</i>	-0.7537 +1.8171 <i>i</i>	-0.7083 +3.2696 <i>i</i>	-0.7114 +4.2243 <i>i</i>	-0.7115 +5.7273 <i>i</i>
1 st torsion	-0.5361 +3.5433 <i>i</i>	-1.0645 +4.9020 <i>i</i>	-0.8042 +3.2650 <i>i</i>	-0.8127 +4.1982 <i>i</i>	-0.7992 +5.7879 <i>i</i>

Table 4.1.4. Eigenvalues of Design Case 1 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 1.0$) at $\theta = 0.3$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	0.4560; -1.4111	0.3193; -1.4709	-0.0823; -1.0907	-0.0006; -1.1435	-0.3403 +0.2051 <i>i</i>
1 st flap	-0.0955 +0.9261 <i>i</i>	-0.0377 +0.9256 <i>i</i>	-0.1839 +0.1736 <i>i</i>	-0.2467 +0.8532 <i>i</i>	-0.0374 +1.0111 <i>i</i>
2 nd flap	-0.4336 +1.1563 <i>i</i>	-0.4399 +1.3356 <i>i</i>	-0.1734 +3.6503 <i>i</i>	-0.1079 +4.9106 <i>i</i>	-0.5011 +5.9574 <i>i</i>
1 st torsion	-0.7835 +4.0090 <i>i</i>	-1.0237 +4.7094 <i>i</i>	-0.9898 +3.1572 <i>i</i>	-1.0069 +4.5645 <i>i</i>	-1.0548 +5.4438 <i>i</i>

Table 4.1.5. Eigenvalues of the 1st lead-lag mode of Design Case 1
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 1.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.0011 +0.6560 <i>i</i>	-0.0011 +0.7747 <i>i</i>	-0.0011 +0.7184 <i>i</i>	-0.0011 +0.7181 <i>i</i>	-0.0011 +0.7180 <i>i</i>
0.1	-0.0567 +0.6052 <i>i</i>	-0.0466 +0.7661 <i>i</i>	-0.0182 +0.6682 <i>i</i>	-0.0203 +0.6639 <i>i</i>	-0.0201 +0.6576 <i>i</i>
0.2	-0.1583 +0.4224 <i>i</i>	-0.1974 +0.6955 <i>i</i>	-0.1913 +0.5858 <i>i</i>	-0.1859 +0.5825 <i>i</i>	-0.2047 +0.5687 <i>i</i>
0.3	0.4560; -1.4111	0.3193; -1.4709	-0.0823; -1.0907	-0.0006; -1.1435	-0.3403 +0.2051 <i>i</i>

Table 4.1.6. Eigenvalues of the 1st flap mode of Design Case 1
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 1.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.3236 +0.9714 <i>i</i>	-0.3287 +1.0721 <i>i</i>	-0.3260 +0.4756 <i>i</i>	-0.3385 +0.9706 <i>i</i>	-0.3342 +0.4810 <i>i</i>
0.1	-0.2688 +0.9584 <i>i</i>	-0.2740 +1.0509 <i>i</i>	-0.3382 +0.4766 <i>i</i>	-0.3093 +0.9582 <i>i</i>	-0.3526 +1.0688 <i>i</i>
0.2	-0.1747 +0.9891 <i>i</i>	-0.1300 +1.0460 <i>i</i>	-0.1444 +0.4817 <i>i</i>	-0.1382 +0.9826 <i>i</i>	-0.1327 +1.0132 <i>i</i>
0.3	-0.0955 +0.9261 <i>i</i>	-0.0377 +0.9256 <i>i</i>	-0.1839 +0.1736 <i>i</i>	-0.2467 +0.8532 <i>i</i>	-0.0374 +1.0111 <i>i</i>

Table 4.1.7. Eigenvalues of the 2nd flap mode of Design Case 1
 ($\bar{x}_1 = 0.05, \bar{x}_2 = 0.20, \Re = 1.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.7581 +1.9077i	-0.8163 +1.9770i	-0.7852 +3.4433i	-0.7855 +4.0448i	-0.7865 +5.5547i
0.1	-0.7203 +1.8853i	-0.7868 +1.9669i	-0.7283 +3.4003i	-0.7426 +4.0908i	-0.7473 +5.5961i
0.2	-0.6816 +1.7241i	-0.7537 +1.8171i	-0.7083 +3.2696i	-0.7114 +4.2243i	-0.7115 +5.7273i
0.3	-0.4336 +1.1563i	-0.4399 +1.3356i	-0.1734 +3.6503i	-0.1079 +4.9106i	-0.5011 +5.9574i

Table 4.1.8. Eigenvalues of the 1st torsion mode of Design Case 1
 ($\bar{x}_1 = 0.05, \bar{x}_2 = 0.20, \Re = 1.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.3036 +2.7609i	-0.3618 +3.2344i	-0.3175; -0.3524	-0.3221 +4.9974i	-0.3253 +6.0234i
0.1	-0.3855 +3.0182i	-0.6193 +4.0954i	-0.4944 +3.5921i	-0.5069 +4.4248i	-0.4592 +6.0669i
0.2	-0.5361 +3.5433i	-1.0645 +4.9020i	-0.8042 +3.2650i	-0.8127 +4.1982i	-0.7992 +5.7879i
0.3	-0.7835 +4.0090i	-1.0237 +4.7094i	-0.9898 +3.1572i	-1.0069 +4.5645i	-1.0548 +5.4438i

Table 4.2.1. Eigenvalues of Design Case 2 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$) at $\theta = 0.0$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0011 +0.6560 <i>i</i>	-0.0011 +0.7747 <i>i</i>	-0.0011 +0.7184 <i>i</i>	-0.0011 +0.7181 <i>i</i>	-0.0011 +0.7180 <i>i</i>
1 st flap	-0.3236 +0.9714 <i>i</i>	-0.3287 +1.0721 <i>i</i>	-0.3260 +0.4756 <i>i</i>	-0.3385 +0.9706 <i>i</i>	-0.3342 +0.4810 <i>i</i>
2 nd flap	-0.7581 +1.9077 <i>i</i>	-0.8163 +1.9770 <i>i</i>	-0.7852 +3.4433 <i>i</i>	-0.7855 +4.0448 <i>i</i>	-0.7865 +5.5547 <i>i</i>
1 st torsion	-0.3036 +2.7609 <i>i</i>	-0.3618 +3.2344 <i>i</i>	-0.3175 -0.3524	-0.3221 +4.9974 <i>i</i>	-0.3253 +6.0234 <i>i</i>

Table 4.2.2 Eigenvalues of Design Case 2 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$) at $\theta = 0.05$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0090 +0.6481 <i>i</i>	-0.0077 +0.7728 <i>i</i>	-0.0088 +0.7136 <i>i</i>	-0.0065 +0.7127 <i>i</i>	-0.0077 +0.7144 <i>i</i>
1 st flap	-0.3203 +0.9570 <i>i</i>	-0.3209 +1.0598 <i>i</i>	-0.3303 +0.3122 <i>i</i>	-0.3056 + 0.9621 <i>i</i>	-0.3116 +0.8337 <i>i</i>
2 nd flap	-0.7358 +1.9042 <i>i</i>	-0.7988 +1.9793 <i>i</i>	-0.8136 +3.3828 <i>i</i>	-0.8172 +4.0971 <i>i</i>	-0.8134 +5.5518 <i>i</i>
1 st torsion	-0.3429 +2.8843 <i>i</i>	-0.4706 +3.6294 <i>i</i>	-0.3504 +3.4802 <i>i</i>	-0.3738 +4.6608 <i>i</i>	-0.3703 +5.9946 <i>i</i>

Table 4.2.3. Eigenvalues of Design Case 2 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$) at $\theta = 0.1$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0522 +0.6265 <i>i</i>	-0.0400 +0.7822 <i>i</i>	-0.0106 +0.6705 <i>i</i>	-0.0066 +0.6707 <i>i</i>	-0.0020 +0.6743 <i>i</i>
1 st flap	-0.2831 +0.8969 <i>i</i>	-0.2895 +1.0035 <i>i</i>	-0.3010 +0.4822 <i>i</i>	-0.3077 +0.9953 <i>i</i>	-0.3034 +0.9942 <i>i</i>
2 nd flap	-0.7033 +1.8967 <i>i</i>	-0.7777 +1.9819 <i>i</i>	-0.7460 +3.4369 <i>i</i>	-0.7516 +4.0439 <i>i</i>	-0.7527 +5.5593 <i>i</i>
1 st torsion	-0.4323 +3.2007 <i>i</i>	-0.7449 +4.4179 <i>i</i>	-0.6038 +3.6995 <i>i</i>	-0.5956 +4.1763 <i>i</i>	-0.6033 +6.2216 <i>i</i>

Table 4.2.4. Eigenvalues of Design Case 2 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$) at $\theta = 0.15$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.1704 +0.5227 <i>i</i>	-0.2288 +0.8040 <i>i</i>	-0.1537 +0.5877 <i>i</i>	-0.1307 +0.5901 <i>i</i>	-0.1805 +0.5799 <i>i</i>
1 st flap	-0.1674 +0.8566 <i>i</i>	-0.1074 +0.9039 <i>i</i>	-0.1169 +0.5205 <i>i</i>	-0.1179 +0.9911 <i>i</i>	-0.1182 +0.9688 <i>i</i>
2 nd flap	-0.6841 +1.8786 <i>i</i>	-0.7654 +1.9767 <i>i</i>	-0.6945; -0.3335	-0.7117 +4.4320 <i>i</i>	-0.6729 +5.5072 <i>i</i>
1 st torsion	-0.5467 +3.6001 <i>i</i>	-0.9885 +4.8403 <i>i</i>	-1.0447 +3.0353 <i>i</i>	-0.8690 +4.2556 <i>i</i>	-0.8576 +5.6769 <i>i</i>

Table 4.2.5. Eigenvalues of Design Case 2 ($x_1 = 0.05$; $x_2 = 0.2$; $\Re = 0$) at $\theta = 0.2$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.2326 +0.2272 <i>i</i>	-0.3331 +0.6173 <i>i</i>	-0.2368 +0.3587 <i>i</i>	-0.2529 +0.3410 <i>i</i>	-0.2205 +0.4261 <i>i</i>
1 st flap	-0.1083 +0.8946 <i>i</i>	-0.0128 +0.9811 <i>i</i>	-0.0205 +0.5171 <i>i</i>	-0.0498 +0.9715 <i>i</i>	-0.0727 +1.0008 <i>i</i>
2 nd flap	-0.6691 +1.8269 <i>i</i>	-0.7582 +1.9593 <i>i</i>	-0.7602 +3.3483 <i>i</i>	-0.7773 +4.1163 <i>i</i>	-0.7727 +5.6238 <i>i</i>
1 st torsion	-0.6747 +3.9694 <i>i</i>	-1.1005 +4.8761 <i>i</i>	-1.0724; -0.7817	-0.8646 +4.5067 <i>i</i>	-0.8787 +5.5045 <i>i</i>

Table 4.2.6. Eigenvalues of Design Case 2 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$) at $\theta = 0.3$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	0.4330; -0.6843	-0.3973; -0.5547	-0.3417 +0.5899 <i>i</i>	-0.4006 +0.9299 <i>i</i>	-0.3512 +1.0499 <i>i</i>
1 st flap	-0.2021 +1.1102 <i>i</i>	0.1074 +1.0356 <i>i</i>	-0.0192 +0.4021 <i>i</i>	-0.0118 +0.3755 <i>i</i>	-0.0246 +0.3107 <i>i</i>
2 nd flap	-0.6351 +1.3960 <i>i</i>	-0.7538 +1.8950 <i>i</i>	-0.8863 +3.1524 <i>i</i>	-0.6359 +4.3996 <i>i</i>	-0.6553 +5.9035 <i>i</i>
1 st torsion	-0.7185 +3.8173 <i>i</i>	-1.1293 +4.5897 <i>i</i>	-0.3929; -1.0457	-0.9182 +4.3061 <i>i</i>	-0.9353 +5.6604 <i>i</i>

Table 4.2.7. Eigenvalues of the 1st lead-lag mode of Design Case 2
 ($\bar{x}_1 = 0.05, \bar{x}_2 = 0.20, \Re = 0.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.0011 +0.6560 <i>i</i>	-0.0011 +0.7747 <i>i</i>	-0.0011 +0.7184 <i>i</i>	-0.0011 +0.7181 <i>i</i>	-0.0011 +0.7180 <i>i</i>
0.05	-0.0090 +0.6481 <i>i</i>	-0.0077 +0.7728 <i>i</i>	-0.0088 +0.7136 <i>i</i>	-0.0065 +0.7127 <i>i</i>	-0.0077 +0.7144 <i>i</i>
0.1	-0.0522 +0.6265 <i>i</i>	-0.0400 +0.7822 <i>i</i>	-0.0106 +0.6705 <i>i</i>	-0.0066 +0.6707 <i>i</i>	-0.0020 +0.6743 <i>i</i>
0.15	-0.1704 +0.5227 <i>i</i>	-0.2288 +0.8040 <i>i</i>	-0.1537 +0.5877 <i>i</i>	-0.1307 +0.5901 <i>i</i>	-0.1805 +0.5799 <i>i</i>
0.2	-0.2326 +0.2272 <i>i</i>	-0.3331 +0.6173 <i>i</i>	-0.2368 +0.3587 <i>i</i>	-0.2529 +0.3410 <i>i</i>	-0.2205 +0.4261 <i>i</i>
0.3	0.4330; -0.6843	-0.3973; -0.5547	-0.3417 +0.5899 <i>i</i>	-0.4006 +0.9299 <i>i</i>	-0.3512 +1.0499 <i>i</i>

Table 4.2.8. Eigenvalues of the 1st flap mode of Design Case 2
 ($\bar{x}_1 = 0.05, \bar{x}_2 = 0.20, \Re = 0.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.3236 +0.9714 <i>i</i>	-0.3287 +1.0721 <i>i</i>	-0.3260 +0.4756 <i>i</i>	-0.3385 +0.9706 <i>i</i>	-0.3342 +0.4810 <i>i</i>
0.05	-0.3203 +0.9570 <i>i</i>	-0.3209 +1.0598 <i>i</i>	-0.3303 +0.3122 <i>i</i>	-0.3056 +0.9621 <i>i</i>	-0.3116 +0.8337 <i>i</i>
0.1	-0.2831 +0.8969 <i>i</i>	-0.2895 +1.0035 <i>i</i>	-0.3010 +0.4822 <i>i</i>	-0.3077 +0.9953 <i>i</i>	-0.3034 +0.9942 <i>i</i>
0.15	-0.1674 +0.8566 <i>i</i>	-0.1074 +0.9039 <i>i</i>	-0.1169 +0.5205 <i>i</i>	-0.1179 +0.9911 <i>i</i>	-0.1182 +0.9688 <i>i</i>
0.2	-0.1083 +0.8946 <i>i</i>	-0.0128 +0.9811 <i>i</i>	-0.0205 +0.5171 <i>i</i>	-0.0498 +0.9715 <i>i</i>	-0.0727 +1.0008 <i>i</i>
0.3	-0.2021 +1.1102 <i>i</i>	0.1074 +1.0356 <i>i</i>	-0.0192 +0.4021 <i>i</i>	-0.0118 +0.3755 <i>i</i>	-0.0246 +0.3107 <i>i</i>

Table 4.2.9. Eigenvalues of the 2nd flap mode of Design Case 2
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.7581 +1.9077 <i>i</i>	-0.8163 +1.9770 <i>i</i>	-0.7852 +3.4433 <i>i</i>	-0.7855 +4.0448 <i>i</i>	-0.7865 +5.5547 <i>i</i>
0.05	-0.7358 +1.9042 <i>i</i>	-0.7988 +1.9793 <i>i</i>	-0.8136 +3.3828 <i>i</i>	-0.8172 +4.0971 <i>i</i>	-0.8134 +5.5518 <i>i</i>
0.1	-0.7033 +1.8967 <i>i</i>	-0.7777 +1.9819 <i>i</i>	-0.7460 +3.4369 <i>i</i>	-0.7516 +4.0439 <i>i</i>	-0.7527 +5.5593 <i>i</i>
0.15	-0.6841 +1.8786 <i>i</i>	-0.7654 +1.9767 <i>i</i>	-0.6945; -0.3335	-0.7117 +4.4320 <i>i</i>	-0.6729 +5.5072 <i>i</i>
0.2	-0.6691 +1.8269 <i>i</i>	-0.7582 +1.9593 <i>i</i>	-0.7602 +3.3483 <i>i</i>	-0.7773 +4.1163 <i>i</i>	-0.7727 +5.6238 <i>i</i>
0.3	-0.6351 +1.3960 <i>i</i>	-0.7538 +1.8950 <i>i</i>	-0.8863 +3.1524 <i>i</i>	-0.6359 +4.3996 <i>i</i>	-0.6553 +5.9035 <i>i</i>

Table 4.2.10. Eigenvalues of the 1st torsion mode of Design Case 2
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.20$, $\Re = 0.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.3036 +2.7609 <i>i</i>	-0.3618 +3.2344 <i>i</i>	-0.3175; -0.3524	-0.3221 +4.9974 <i>i</i>	-0.3253 +6.0234 <i>i</i>
0.05	-0.3429 +2.8843 <i>i</i>	-0.4706 +3.6294 <i>i</i>	-0.3504 +3.4802 <i>i</i>	-0.3738 +4.6608 <i>i</i>	-0.3703 +5.9946 <i>i</i>
0.1	-0.4323 +3.2007 <i>i</i>	-0.7449 +4.4179 <i>i</i>	-0.6038 +3.6995 <i>i</i>	-0.5956 +4.1763 <i>i</i>	-0.6033 +6.2216 <i>i</i>
0.15	-0.5467 +3.6001 <i>i</i>	-0.9885 +4.8403 <i>i</i>	-1.0447 +3.0353 <i>i</i>	-0.8690 +4.2556 <i>i</i>	-0.8576 +5.6769 <i>i</i>
0.2	-0.6747 +3.9694 <i>i</i>	-1.1005 +4.8761 <i>i</i>	-1.0724; -0.7817	-0.8646 +4.5067 <i>i</i>	-0.8787 +5.5045 <i>i</i>
0.3	-0.7185 +3.8173 <i>i</i>	-1.1293 +4.5897 <i>i</i>	-0.3929; -1.0457	-0.9182 +4.3061 <i>i</i>	-0.9353 +5.6604 <i>i</i>

Table 4.3.1. Eigenvalues of Design Case 3 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 1.0$) at $\theta = 0.0$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0011 +0.6560 <i>i</i>	-0.0010 +0.9962 <i>i</i>	-0.0010 +0.6503 <i>i</i>	-0.0010 +0.8463 <i>i</i>	-0.0010 +0.8452 <i>i</i>
1 st flap	-0.3236 +0.9714 <i>i</i>	-0.3366 +1.2059 <i>i</i>	-0.3308 +0.4025 <i>i</i>	-0.3481 +0.9165 <i>i</i>	-0.3364 +1.0941 <i>i</i>
2 nd flap	-0.7581 +1.9077 <i>i</i>	-0.9080 +2.1267 <i>i</i>	-0.8457 +3.4952 <i>i</i>	-0.9306; -0.7663	-0.8382 +5.4777 <i>i</i>
1 st torsion	-0.3036 +2.7609 <i>i</i>	-0.1844 +2.7677 <i>i</i>	-0.2307 +3.2133 <i>i</i>	-0.2107 +4.7791 <i>i</i>	-0.2326 +5.2995 <i>i</i>

Table 4.3.2. Eigenvalues of Design Case 3 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 1.0$) at $\theta = 0.1$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0567 +0.6052 <i>i</i>	-0.0343 +1.0091 <i>i</i>	-0.0462 +0.6514 <i>i</i>	-0.0353 +0.8484 <i>i</i>	-0.0446 +0.8474 <i>i</i>
1 st flap	-0.2688 +0.9584 <i>i</i>	-0.3033 +1.1773 <i>i</i>	-0.3036 +0.4243 <i>i</i>	-0.3363 +0.9025 <i>i</i>	-0.3300 +1.0452 <i>i</i>
2 nd flap	-0.7203 +1.8853 <i>i</i>	-0.8866 +2.1079 <i>i</i>	-0.8399 +3.4285 <i>i</i>	-0.9058; -0.7864	-0.8169 +5.4995 <i>i</i>
1 st torsion	-0.3855 +3.0182 <i>i</i>	-0.2168 +2.8494 <i>i</i>	-0.2820; -0.2108	-0.2185 +4.8523 <i>i</i>	-0.2447 +5.5406 <i>i</i>

Table 4.3.3. Eigenvalues of Design Case 3 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 1.0$) at $\theta = 0.2$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.1583 +0.4224 <i>i</i>	-0.2575 +0.9592 <i>i</i>	-0.3677 +0.2081 <i>i</i>	-0.3103 +0.7955 <i>i</i>	-0.3520 +0.8172 <i>i</i>
1 st flap	-0.1747 +0.9891 <i>i</i>	-0.0732 +1.1903 <i>i</i>	-0.0768 +0.6939 <i>i</i>	-0.0680 +0.6769 <i>i</i>	-0.0626 +0.6731 <i>i</i>
2 nd flap	-0.6816 +1.7241 <i>i</i>	-0.8477 +1.9594 <i>i</i>	-0.6733 +3.3118 <i>i</i>	-0.7115 +4.2252 <i>i</i>	-0.7147 +5.7219 <i>i</i>
1 st torsion	-0.5361 +3.5433 <i>i</i>	-0.2661 +3.0229 <i>i</i>	-0.3798 +3.3042 <i>i</i>	-0.4078 +4.7164 <i>i</i>	-0.3683 +6.2446 <i>i</i>

Table 4.3.4. Eigenvalues of Design Case 3 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 1.0$) at $\theta = 0.3$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	0.4560; -1.4111	-0.2823; -0.9139	-1.2626; 0.3361	-0.6376; 0.3702	-1.3485; 0.2978
1 st flap	-0.0955 +0.9261 <i>i</i>	0.0610 +1.2431 <i>i</i>	-0.0413 +0.0892 <i>i</i>	-0.0099 +0.7361 <i>i</i>	-0.0724 +1.1388 <i>i</i>
2 nd flap	-0.4336 +1.1563 <i>i</i>	-0.5206 +1.2149 <i>i</i>	-0.8310 +0.3850 <i>i</i>	-1.0523 +0.5318 <i>i</i>	-0.9505 +1.1497 <i>i</i>
1 st torsion	-0.7835 +4.0090 <i>i</i>	-0.4527 +3.5856 <i>i</i>	-0.3147 +3.5905 <i>i</i>	-0.4545 +4.9619 <i>i</i>	-0.1021 +6.1730 <i>i</i>

Table 4.3.5. Eigenvalues of the 1st lead-lag mode of Design Case 3
 $(\bar{x}_1 = 0.05, \bar{x}_2 = 0.40, \mathfrak{R} = 1.0)$ at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.0011 +0.6560 <i>i</i>	-0.0010 +0.9962 <i>i</i>	-0.0010 +0.6503 <i>i</i>	-0.0010 +0.8463 <i>i</i>	-0.0010 +0.8452 <i>i</i>
0.1	-0.0567 +0.6052 <i>i</i>	-0.0343 +1.0091 <i>i</i>	-0.0462 +0.6514 <i>i</i>	-0.0353 +0.8484 <i>i</i>	-0.0446 +0.8474 <i>i</i>
0.2	-0.1583 +0.4224 <i>i</i>	-0.2575 +0.9592 <i>i</i>	-0.3677 +0.2081 <i>i</i>	-0.3103 +0.7955 <i>i</i>	-0.3520 +0.8172 <i>i</i>
0.3	0.4560; -1.4111	-0.2823; -0.9139	-1.2626; 0.3361	-0.6376; 0.3702	-1.3485; 0.2978

Table 4.3.6. Eigenvalues of the 1st flap mode of Design Case 3
 $(\bar{x}_1 = 0.05, \bar{x}_2 = 0.40, \mathfrak{R} = 1.0)$ at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.3236 +0.9714 <i>i</i>	-0.3366 +1.2059 <i>i</i>	-0.3308 +0.4025 <i>i</i>	-0.3481 +0.9165 <i>i</i>	-0.3364 +1.0941 <i>i</i>
0.1	-0.2688 +0.9584 <i>i</i>	-0.3033 +1.1773 <i>i</i>	-0.3036 +0.4243 <i>i</i>	-0.3363 +0.9025 <i>i</i>	-0.3300 +1.0452 <i>i</i>
0.2	-0.1747 +0.9891 <i>i</i>	-0.0732 +1.1903 <i>i</i>	-0.0768 +0.6939 <i>i</i>	-0.0680 +0.6769 <i>i</i>	-0.0626 +0.6731 <i>i</i>
0.3	-0.0955 +0.9261 <i>i</i>	0.0610 +1.2431 <i>i</i>	-0.0413 +0.0892 <i>i</i>	-0.0099 +0.7361 <i>i</i>	-0.0724 +1.1388 <i>i</i>

Table 4.3.7. Eigenvalues of the 2nd flap mode of Design Case 3
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 1.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.7581 +1.9077 <i>i</i>	-0.9080 +2.1267 <i>i</i>	-0.8457 +3.4952 <i>i</i>	-0.9306; -0.7663	-0.8382 +5.4777 <i>i</i>
0.1	-0.7203 +1.8853 <i>i</i>	-0.8866 +2.1079 <i>i</i>	-0.8399 +3.4285 <i>i</i>	-0.9058; -0.7864	-0.8169 +5.4995 <i>i</i>
0.2	-0.6816 +1.7241 <i>i</i>	-0.8477 +1.9594 <i>i</i>	-0.6733 +3.3118 <i>i</i>	-0.7115 +4.2252 <i>i</i>	-0.7147 +5.7219 <i>i</i>
0.3	-0.4336 +1.1563 <i>i</i>	-0.5206 +1.2149 <i>i</i>	-0.8310 +3.3850 <i>i</i>	-1.0523 +4.5318 <i>i</i>	-0.9505 +6.1497 <i>i</i>

Table 4.3.8. Eigenvalues of the 1st torsion mode of Design Case 3
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 1.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.3036 +2.7609 <i>i</i>	-0.1844 +2.7677 <i>i</i>	-0.2307 +3.2133 <i>i</i>	-0.2107 +4.7791 <i>i</i>	-0.2326 +5.2995 <i>i</i>
0.1	-0.3855 +3.0182 <i>i</i>	-0.2168 +2.8494 <i>i</i>	-0.2820; -0.2108	-0.2185 +4.8523 <i>i</i>	-0.2447 +5.5406 <i>i</i>
0.2	-0.5361 +3.5433 <i>i</i>	-0.2661 +3.0229 <i>i</i>	-0.3798 +3.3042 <i>i</i>	-0.4078 +4.7164 <i>i</i>	-0.3683 +6.2446 <i>i</i>
0.3	-0.7835 +4.0090 <i>i</i>	-0.4527 +3.5856 <i>i</i>	-0.3147 +3.5905 <i>i</i>	-0.4545 +4.9619 <i>i</i>	-0.1021 +6.1730 <i>i</i>

Table 4.4.1. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.0$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0011 +0.6560 <i>i</i>	-0.0010 +0.9962 <i>i</i>	-0.0010 +0.6503 <i>i</i>	-0.0010 +0.8463 <i>i</i>	-0.0010 +0.8452 <i>i</i>
1 st flap	-0.3236 +0.9714 <i>i</i>	-0.3366 +1.2059 <i>i</i>	-0.3308 +0.4025 <i>i</i>	-0.3481 +0.9165 <i>i</i>	-0.3364 +1.0941 <i>i</i>
2 nd flap	-0.7581 +1.9077 <i>i</i>	-0.9080 +2.1267 <i>i</i>	-0.8457 +3.4952 <i>i</i>	-0.9306; -0.7663	-0.8382 +5.4777 <i>i</i>
1 st torsion	-0.3036 +2.7609 <i>i</i>	-0.1844 +2.7677 <i>i</i>	-0.2307 +3.2133 <i>i</i>	-0.2107 +4.7791 <i>i</i>	-0.2326 +5.2995 <i>i</i>

Table 4.4.2. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.05$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0090 +0.6481 <i>i</i>	-0.0055 +0.9978 <i>i</i>	-0.0048 +0.6565 <i>i</i>	-0.0045 +0.8396 <i>i</i>	-0.0045 +0.8382 <i>i</i>
1 st flap	-0.3203 +0.9570 <i>i</i>	-0.3368 +1.1955 <i>i</i>	-0.3253 +0.4201 <i>i</i>	-0.3697 +0.9179 <i>i</i>	-0.3259 +1.0910 <i>i</i>
2 nd flap	-0.7358 +1.9042 <i>i</i>	-0.8947 +2.1174 <i>i</i>	-0.8201 +3.4981 <i>i</i>	-0.9042; -0.7503	-0.8268 +5.4836 <i>i</i>
1 st torsion	-0.3429 +2.8843 <i>i</i>	-0.1979 +2.8014 <i>i</i>	-0.2714 +3.1514 <i>i</i>	-0.2200 +4.8480 <i>i</i>	-0.2643 +5.3537 <i>i</i>

Table 4.4.3. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.08$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0282 +0.6375 <i>i</i>	-0.0134 +1.0054 <i>i</i>	-0.0103 +0.6659 <i>i</i>	-0.0097 +0.8285 <i>i</i>	-0.0093 +0.8269 <i>i</i>
1 st flap	-0.3051 +0.9260 <i>i</i>	-0.3342 +1.1746 <i>i</i>	-0.3271 +0.4298 <i>i</i>	-0.3864 +0.9808 <i>i</i>	-0.3279 +1.0835 <i>i</i>
2 nd flap	-0.7150 +1.9003 <i>i</i>	-0.8799 +2.1041 <i>i</i>	-0.8015 +3.4830 <i>i</i>	-0.8860; -0.7416	-0.8134 +5.4963 <i>i</i>
1 st torsion	-0.3928 +3.0575 <i>i</i>	-0.2147 +2.8542 <i>i</i>	-0.3028 +3.0365 <i>i</i>	-0.2317 +4.9068 <i>i</i>	-0.2911 +5.4660 <i>i</i>

Table 4.4.4. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.1$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.0522 +0.6265 <i>i</i>	-0.0207 +1.0149 <i>i</i>	-0.0148 +0.6744 <i>i</i>	-0.0143 +0.8182 <i>i</i>	-0.0132 +0.8163 <i>i</i>
1 st flap	-0.2831 +0.8969 <i>i</i>	-0.3305 +1.1535 <i>i</i>	-0.3299 +0.4365 <i>i</i>	-0.3866 +0.9447 <i>i</i>	-0.3302 +1.0775 <i>i</i>
2 nd flap	-0.7033 +1.8967 <i>i</i>	-0.8706 +2.0935 <i>i</i>	-0.7889 +3.4709 <i>i</i>	-0.8733; -0.7372	-0.8050 +5.5065 <i>i</i>
1 st torsion	-0.4323 +3.2007 <i>i</i>	-0.2271 +2.9020 <i>i</i>	-0.3264 +3.0605 <i>i</i>	-0.2537 +4.9341 <i>i</i>	-0.3116 +5.5604 <i>i</i>

Table 4.4.5. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.15$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.1704 +0.5227 <i>i</i>	-0.0397 +1.0619 <i>i</i>	-0.0833 +0.5729 <i>i</i>	-0.0895 +0.9173 <i>i</i>	-0.0825 +0.9123 <i>i</i>
1 st flap	-0.1674 +0.8566 <i>i</i>	-0.3181 +1.0712 <i>i</i>	-0.4046 +0.3230 <i>i</i>	-0.4030 +0.6797 <i>i</i>	-0.4017 +0.8121 <i>i</i>
2 nd flap	-0.6841 +1.8786 <i>i</i>	-0.8588 +2.0696 <i>i</i>	-0.7597 +3.4796 <i>i</i>	-0.8293; -0.6912	-0.7633 +5.5071 <i>i</i>
1 st torsion	-0.5467 +3.6001 <i>i</i>	-0.2560 +3.0583 <i>i</i>	-0.2731 +3.6020 <i>i</i>	-0.2679 +4.8996 <i>i</i>	-0.2731 +5.8977 <i>i</i>

Table 4.4.6. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.18$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.2173 +0.3722 <i>i</i>	-0.3187 +1.0057 <i>i</i>	-0.4527 +0.5598 <i>i</i>	-0.4658 +0.4638 <i>i</i>	-0.3995 +1.0090 <i>i</i>
1 st flap	-0.1219 +0.8789 <i>i</i>	-0.0392 +1.1048 <i>i</i>	-0.0523 +0.4453 <i>i</i>	-0.0753 +0.9200 <i>i</i>	-0.1073 +1.0594 <i>i</i>
2 nd flap	-0.6757 +1.8536 <i>i</i>	-0.8621 +2.0628 <i>i</i>	-0.8233 +3.4170 <i>i</i>	-0.9325; -0.7223	-0.8228 +5.5210 <i>i</i>
1 st torsion	-0.6230 +3.8324 <i>i</i>	-0.2723 +3.1726 <i>i</i>	-0.2368 +3.6147 <i>i</i>	-0.1966 +4.6726 <i>i</i>	-0.2355 +5.6833 <i>i</i>

Table 4.4.7. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.2$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	-0.2326 +0.2272 <i>i</i>	-0.3235 +0.9626 <i>i</i>	-0.5159 +0.6567 <i>i</i>	-0.4921 +0.3459 <i>i</i>	-0.4235 +1.1218 <i>i</i>
1 st flap	-0.1083 +0.8946 <i>i</i>	-0.0321 +1.1332 <i>i</i>	-0.0241 +0.4248 <i>i</i>	-0.0474 +0.9001 <i>i</i>	-0.0815 +1.0873 <i>i</i>
2 nd flap	-0.6691 +1.8269 <i>i</i>	-0.8684 +2.0626 <i>i</i>	-0.8250 +3.3996 <i>i</i>	-0.9485; -0.7147	-0.8263 +5.5276 <i>i</i>
1 st torsion	-0.6747 +3.9694 <i>i</i>	-0.2844 +3.2561 <i>i</i>	-0.2315 +3.5528 <i>i</i>	-0.2254 +4.5986 <i>i</i>	-0.2652 +5.6031 <i>i</i>

Table 4.4.8. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.25$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	0.1592; -0.6125	-0.3349 +0.8623 <i>i</i>	-0.5749 +0.6795 <i>i</i>	-0.5747 +0.1767 <i>i</i>	-0.5641 +1.1659 <i>i</i>
1 st flap	-0.1238 +0.9442 <i>i</i>	-0.0053 +1.1993 <i>i</i>	-0.0832 +0.5869 <i>i</i>	-0.0260 +0.5302 <i>i</i>	-0.0260 +0.5242 <i>i</i>
2 nd flap	-0.6419 +1.7016 <i>i</i>	-0.8932 +2.0761 <i>i</i>	-0.8199 +3.3897 <i>i</i>	-0.9691; -0.7392	-0.8637 +5.5533 <i>i</i>
1 st torsion	-0.7807 +4.1679 <i>i</i>	-0.3279 +3.4892 <i>i</i>	-0.1892 +3.3813 <i>i</i>	-0.2123 +4.8874 <i>i</i>	-0.2134 +6.1155 <i>i</i>

Table 4.4.9. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.28$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	0.3211; -0.6826	-0.3399 +0.8030 <i>i</i>	-0.1553 +0.3412 <i>i</i>	-0.2131 +0.9219 <i>i</i>	-0.2540 +1.1703 <i>i</i>
1 st flap	-0.1724 +1.0114 <i>i</i>	0.0147 +1.2382 <i>i</i>	-0.0814 +0.7297 <i>i</i>	-0.1127 +0.7932 <i>i</i>	-0.0751 +0.7990 <i>i</i>
2 nd flap	-0.6221 +1.5559 <i>i</i>	-0.9099 +2.0893 <i>i</i>	-0.8349 +3.2833 <i>i</i>	-0.8134; -0.7082	-0.7743 +5.6597 <i>i</i>
1 st torsion	-0.7779 +4.0599 <i>i</i>	-0.3693 +3.6455 <i>i</i>	-0.6072 +3.6653 <i>i</i>	-0.5921 +4.1594 <i>i</i>	-0.5754 +6.0323 <i>i</i>

Table 4.4.10. Eigenvalues of Design Case 4 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.3$

Mode	Baseline	Static	IBC 3	IBC 4	IBC 5
1 st lead-lag	0.4330; -0.6843	-0.3436 +0.7623 <i>i</i>	-0.1334 +0.3405 <i>i</i>	-0.2664 +0.6814 <i>i</i>	-0.3104 +0.3132 <i>i</i>
1 st flap	-0.2021 +1.1102 <i>i</i>	0.0302 +1.2638 <i>i</i>	0.0020 +0.5556 <i>i</i>	0.0149 +0.2569 <i>i</i>	-0.0617 +1.2058 <i>i</i>
2 nd flap	-0.6351 +1.3960 <i>i</i>	-0.9202 +2.0977 <i>i</i>	-1.3511 +3.1930 <i>i</i>	-0.8300; -0.4095	0.3249; -1.3039
1 st torsion	-0.7185 +3.8173 <i>i</i>	-0.4056 +3.7559 <i>i</i>	-0.2076; -0.1478	-0.7889 +4.7326 <i>i</i>	-0.7986 +5.7403 <i>i</i>

Table 4.4.11. Eigenvalues of the 1st lead-lag mode of Design Case 4
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.0011 +0.6560 <i>i</i>	-0.0010 +0.9962 <i>i</i>	-0.0010 +0.6503 <i>i</i>	-0.0010 +0.8463 <i>i</i>	-0.0010 +0.8452 <i>i</i>
0.05	-0.0090 +0.6481 <i>i</i>	-0.0055 +0.9978 <i>i</i>	-0.0048 +0.6565 <i>i</i>	-0.0045 +0.8396 <i>i</i>	-0.0045 +0.8382 <i>i</i>
0.08	-0.0282 +0.6375 <i>i</i>	-0.0134 +1.0054 <i>i</i>	-0.0103 +0.6659 <i>i</i>	-0.0097 +0.8285 <i>i</i>	-0.0093 +0.8269 <i>i</i>
0.1	-0.0522 +0.6265 <i>i</i>	-0.0207 +1.0149 <i>i</i>	-0.0148 +0.6744 <i>i</i>	-0.0143 +0.8182 <i>i</i>	-0.0132 +0.8163 <i>i</i>
0.15	-0.1704 +0.5227 <i>i</i>	-0.0397 +1.0619 <i>i</i>	-0.0833 +0.5729 <i>i</i>	-0.0895 +0.9173 <i>i</i>	-0.0825 +0.9123 <i>i</i>
0.18	-0.2173 +0.3722 <i>i</i>	-0.3187 +1.0057 <i>i</i>	-0.4527 +0.5598 <i>i</i>	-0.4658 +0.4638 <i>i</i>	-0.3995 +1.0090 <i>i</i>
0.2	-0.2326 +0.2272 <i>i</i>	-0.3235 +0.9626 <i>i</i>	-0.5159 +0.6567 <i>i</i>	-0.4921 +0.3459 <i>i</i>	-0.4235 +1.1218 <i>i</i>
0.25	0.1592; -0.6125	-0.3349 +0.8623 <i>i</i>	-0.5749 +0.6795 <i>i</i>	-0.5747 +0.1767 <i>i</i>	-0.5641 +1.1659 <i>i</i>
0.28	0.3211; -0.6826	-0.3399 +0.8030 <i>i</i>	-0.1553 +0.3412 <i>i</i>	-0.2131 +0.9219 <i>i</i>	-0.2540 +1.1703 <i>i</i>
0.3	0.4330; -0.6843	-0.3436 +0.7623 <i>i</i>	-0.1334 +0.3405 <i>i</i>	-0.2664 +0.6814 <i>i</i>	-0.3104 +0.3132 <i>i</i>

Table 4.4.12. Eigenvalues of the 1st flap mode of Design Case 4
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.3236 +0.9714 <i>i</i>	-0.3366 +1.2059 <i>i</i>	-0.3308 +0.4025 <i>i</i>	-0.3481 +0.9165 <i>i</i>	-0.3364 +1.0941 <i>i</i>
0.05	-0.3203 +0.9570 <i>i</i>	-0.3368 +1.1955 <i>i</i>	-0.3253 +0.4201 <i>i</i>	-0.3697 +0.9179 <i>i</i>	-0.3259 +1.0910 <i>i</i>
0.08	-0.3051 +0.9260 <i>i</i>	-0.3342 +1.1746 <i>i</i>	-0.3271 +0.4298 <i>i</i>	-0.3864 +0.9808 <i>i</i>	-0.3279 +1.0835 <i>i</i>
0.1	-0.2831 +0.8969 <i>i</i>	-0.3305 +1.1535 <i>i</i>	-0.3299 +0.4365 <i>i</i>	-0.3866 +0.9447 <i>i</i>	-0.3302 +1.0775 <i>i</i>
0.15	-0.1674 +0.8566 <i>i</i>	-0.3181 +1.0712 <i>i</i>	-0.4046 +0.3230 <i>i</i>	-0.4030 +0.6797 <i>i</i>	-0.4017 +0.8121 <i>i</i>
0.18	-0.1219 +0.8789 <i>i</i>	-0.0392 +1.1048 <i>i</i>	-0.0523 +0.4453 <i>i</i>	-0.0753 +0.9200 <i>i</i>	-0.1073 +1.0594 <i>i</i>
0.2	-0.1083 +0.8946 <i>i</i>	-0.0321 +1.1332 <i>i</i>	-0.0241 +0.4248 <i>i</i>	-0.0474 +0.9001 <i>i</i>	-0.0815 +1.0873 <i>i</i>
0.25	-0.1238 +0.9442 <i>i</i>	-0.0053 +1.1993 <i>i</i>	-0.0832 +0.5869 <i>i</i>	-0.0260 +0.5302 <i>i</i>	-0.0260 +0.5242 <i>i</i>
0.28	-0.1724 +1.0114 <i>i</i>	0.0147 +1.2382 <i>i</i>	-0.0814 +0.7297 <i>i</i>	-0.1127 +0.7932 <i>i</i>	-0.0751 +0.7990 <i>i</i>
0.3	-0.2021 +1.1102 <i>i</i>	0.0302 +1.2638 <i>i</i>	0.0020 +0.5556 <i>i</i>	0.0149 +0.2569 <i>i</i>	-0.0617 +1.2058 <i>i</i>

Table 4.4.13. Eigenvalues of the 2nd flap mode of Design Case 4
($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\Re = 0.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.7581 +1.9077 <i>i</i>	-0.9080 +2.1267 <i>i</i>	-0.8457 +3.4952 <i>i</i>	-0.9306; -0.7663	-0.8382 +5.4777 <i>i</i>
0.05	-0.7358 +1.9042 <i>i</i>	-0.8947 +2.1174 <i>i</i>	-0.8201 +3.4981 <i>i</i>	-0.9042; -0.7503	-0.8268 +5.4836 <i>i</i>
0.08	-0.7150 +1.9003 <i>i</i>	-0.8799 +2.1041 <i>i</i>	-0.8015 +3.4830 <i>i</i>	-0.8860; -0.7416	-0.8134 +5.4963 <i>i</i>
0.1	-0.7033 +1.8967 <i>i</i>	-0.8706 +2.0935 <i>i</i>	-0.7889 +3.4709 <i>i</i>	-0.8733; -0.7372	-0.8050 +5.5065 <i>i</i>
0.15	-0.6841 +1.8786 <i>i</i>	-0.8588 +2.0696 <i>i</i>	-0.7597 +3.4796 <i>i</i>	-0.8293; -0.6912	-0.7633 +5.5071 <i>i</i>
0.18	-0.6757 +1.8536 <i>i</i>	-0.8621 +2.0628 <i>i</i>	-0.8233 +3.4170 <i>i</i>	-0.9325; -0.7223	-0.8228 +5.5210 <i>i</i>
0.2	-0.6691 +1.8269 <i>i</i>	-0.8684 +2.0626 <i>i</i>	-0.8250 +3.3996 <i>i</i>	-0.9485; -0.7147	-0.8263 +5.5276 <i>i</i>
0.25	-0.6419 +1.7016 <i>i</i>	-0.8932 +2.0761 <i>i</i>	-0.8199 +3.3897 <i>i</i>	-0.9691; -0.7392	-0.8637 +5.5533 <i>i</i>
0.28	-0.6221 +1.5559 <i>i</i>	-0.9099 +2.0893 <i>i</i>	-0.8349 +3.2833 <i>i</i>	-0.8134; -0.7082	-0.7743 +5.6597 <i>i</i>
0.3	-0.6351 +1.3960 <i>i</i>	-0.9202 +2.0977 <i>i</i>	-1.3511 +3.1930 <i>i</i>	-0.8300; -0.4095	0.3249; -1.3039

Table 4.4.14. Eigenvalues of the 1st torsion mode of Design Case 4
 ($\bar{x}_1 = 0.05$, $\bar{x}_2 = 0.40$, $\mathfrak{R} = 0.0$) at $\theta = 0.0$ to 0.3

θ [rad]	Baseline	Static	IBC 3	IBC 4	IBC 5
0.0	-0.3036 +2.7609 <i>i</i>	-0.1844 +2.7677 <i>i</i>	-0.2307 +3.2133 <i>i</i>	-0.2107 +4.7791 <i>i</i>	-0.2326 +5.2995 <i>i</i>
0.05	-0.3429 +2.8843 <i>i</i>	-0.1979 +2.8014 <i>i</i>	-0.2714 +3.1514 <i>i</i>	-0.2200 +4.8480 <i>i</i>	-0.2643 +5.3537 <i>i</i>
0.08	-0.3928 +3.0575 <i>i</i>	-0.2147 +2.8542 <i>i</i>	-0.3028 +3.0365 <i>i</i>	-0.2317 +4.9068 <i>i</i>	-0.2911 +5.4660 <i>i</i>
0.1	-0.4323 +3.2007 <i>i</i>	-0.2271 +2.9020 <i>i</i>	-0.3264 +3.0605 <i>i</i>	-0.2537 +4.9341 <i>i</i>	-0.3116 +5.5604 <i>i</i>
0.15	-0.5467 +3.6001 <i>i</i>	-0.2560 +3.0583 <i>i</i>	-0.2731 +3.6020 <i>i</i>	-0.2679 +4.8996 <i>i</i>	-0.2731 +5.8977 <i>i</i>
0.18	-0.6230 +3.8324 <i>i</i>	-0.2723 +3.1726 <i>i</i>	-0.2368 +3.6147 <i>i</i>	-0.1966 +4.6726 <i>i</i>	-0.2355 +5.6833 <i>i</i>
0.2	-0.6747 +3.9694 <i>i</i>	-0.2844 +3.2561 <i>i</i>	-0.2315 +3.5528 <i>i</i>	-0.2254 +4.5986 <i>i</i>	-0.2652 +5.6031 <i>i</i>
0.25	-0.7807 +4.1679 <i>i</i>	-0.3279 +3.4892 <i>i</i>	-0.1892 +3.3813 <i>i</i>	-0.2123 +4.8874 <i>i</i>	-0.2134 +6.1155 <i>i</i>
0.28	-0.7779 +4.0599 <i>i</i>	-0.3693 +3.6455 <i>i</i>	-0.6072 +3.6653 <i>i</i>	-0.5921 +4.1594 <i>i</i>	-0.5754 +6.0323 <i>i</i>
0.3	-0.7185 +3.8173 <i>i</i>	-0.4056 +3.7559 <i>i</i>	-0.2076; -0.1478	-0.7889 +4.7326 <i>i</i>	-0.7986 +5.7403 <i>i</i>

Table 4.5. Configuration Parameters of of an elastic hingeless rotor blade in Ref. 12 for comparison study

Parameter	Baseline Value	Additional Values
ω_v / Ω	1.5	--
ω_w / Ω	1.15	--
ω_ϕ / Ω	2.5	--
θ	0.0 rad	Variable up to 0.3 rad
β_{pc}	0.0	--
$\Re$	1	--
k_m / R	0.025	--
k_{m1} / k_{m2}	0.0	--
γ	5	--
σ	0.10	--
c/R	$\pi / 40$	--
b	4	--
c_{d0} / a	$0.01 / 2\pi$	--

Table 4.6.1. Comparison of eigenvalues of an elastic hingeless rotor blade dynamic response as calculated by Hodges (Ref. 12) to those obtained by the author. Completely coupled case: $\mathcal{R} = 1, \omega_v / \Omega = 1.5, \omega_w / \Omega = 1.15, \omega_\phi / \Omega = 2.5, \theta = 0.0$

Mode	Hodges	Author
1 st lead-lag	$-0.001 + 1.50i$	$-0.0010 + 1.5024i$
1 st flap	$-0.33 + 1.08i$	$-0.3212 + 1.0651i$
1 st torsion	$-0.36 + 2.48i$	$-0.3655 + 2.3496i$

Table 4.6.2. Comparison of eigenvalues of an elastic hingeless rotor blade dynamic response as calculated by Hodges (Ref. 12) to those obtained by the author. Completely coupled case: $\mathcal{R} = 1, \omega_v / \Omega = 1.5, \omega_w / \Omega = 1.15, \omega_\phi / \Omega = 2.5, \theta = 0.3$

Mode	Hodges	Author
1 st lead-lag	$-0.015 + 1.94i$	$-0.0196 + 1.9063i$
1 st flap	$-0.2 + 0.86i$	$-0.2095 + 0.9551i$
1 st torsion	$-0.48 + 2.14i$	$-0.5338 + 1.9929i$

Table 4.6.3. Comparison of eigenvalues of an elastic hingeless rotor blade dynamic response as calculated by Hodges (Ref. 12) to those obtained by the author. Completely coupled case: $\mathcal{R} = 1, \omega_v / \Omega = 1.5, \omega_w / \Omega = 1.15, \omega_\phi / \Omega = 2.5, \theta = 0.5$

Mode	Hodges	Author
1 st lead-lag	$0.45 + 2.12i$	$0.3823 + 1.9905i$
1 st flap	$-0.14 + 0.73i$	$-0.0478 + 0.9273i$
1 st torsion	$-1.02 + 2.0i$	$-1.1936 + 1.8257i$

Appendix C

Computer Codes

C1. User Manual:

1. Open a MATLAB window and change to the appropriate directory containing the files described in C2.
2. Specify the non-dimensional starting **x1i** and ending **x2i** locations of the “smart” spring, the pitch angle **Tei** in radians, structural coupling parameter **Rci**, and activation frequencies of the “smart” spring **Nmi**.
3. Run the code to obtain various information about the system (including the eigenvalues) using the following command:
[X, vi, Tip, Va2, Va2_f, p3, p4, p5, p6] = rutlo45(x1i, x2i, Tei, Rci, Nmi)
4. The parameters obtained are:

X	Modal steady state quantities
vi	Induced in-flow velocity
Tip	Matrix containing lead-lag, flap, and torsion Baseline trim deflections at the tip
Va2	Eigenvalues of Baseline Regime
Va2_f	Eigenvalues of Static Regime
p3	Unsorted eigenvalues of the IBC regimes obtained from the transition matrix
p4	Sorted eigenvalues of the IBC regimes obtained by sorting p3 according to the relative magnitude of the damping (real part) of Va2_f
p5	Sorted eigenvalues of the IBC regimes obtained by normalizing the frequency (imaginary part) of p4 with respect to that of Va2_f
p6	Matrix consisting of Va2 , Va2_f , and p5 as column vectors
5. The following must be noted in determining the eigenvalues of IBC regimes:
 - a. After obtaining **p4**, **p5**, and **p6** from **rutlo45.m**, it is necessary to evaluate (using one own rational judgement) whether the eigenvalues are matched to the appropriate modes since the criteria of sorting was only according to the relative magnitudes of the damping of the Static Regime.
 - b. Once the modes are decided for the eigenvalues, it might be necessary to normalize the imaginary parts of the IBC regimes by adding or subtracting by integer multiples of $2\pi / T$, i.e., 3 for IBC 3, 4 for IBC 4, etc.

C2. Computer Code Flow Chart for Eigenvalues Calculation

Code / Routine / Subroutine	Purpose
Rutlo45.m	Calculates modal and trim steady state deflections, induced in-flow velocity, eigenvalues of Baseline, Static, and IBC regimes.
Big1.m Newton3.m	Calculates the integrals of the products of the basis function associated with the Baseline Regime. Performs numerical integration.
Stry33.m Sted38.m	Calculates modal and trim steady state deflections, and induced in-flow velocity. Calculates modal steady state deflections.
Mck25.m	Calculates mass, damping, and stiffness matrices of the Baseline Regime.
Smart5.m Bigx4.m Newton2.m	Calculates mass, damping, and stiffness matrices associated with the activation of the "smart" spring. Calculates the integrals of the products of the basis function associated with the "smart" spring. Performs numerical integration.
XprimeA.m	Sets up the state space matrix describing the activation of the "smart" spring.
XprimeB.m	Sets up the state space matrix describing the de-activation of the "smart" spring.
Mysort1.m	Sorts the eigenvalues of the IBC regimes p3 according to the relative magnitude of the damping (real part) of Va2_f and returns it as p4 .
Mul2.m	Normalizes the imaginary parts of the IBC regimes p4 with respect to Va2_f by adding or subtracting by integer multiples of $2\pi / T$, i.e., 3 for IBC 3, 4 for IBC 4, etc, and returns it as p5 .

[illegible]

```

% Helicopter blade parameter specifications
Wv = 0.5141;% Non-dimensional non-rotating lead-lag frequency,
%associated rotating value 0.7
Ww = 0.1428;% Non-dimensional non-rotating flap frequency, associated
%rotating value 1.14
Wt = 3.7699;% Non-dimensional non-rotating torsion frequency,
%associated rotating value 3.9

Om= 110;      % Rotational speed in [rad/sec]
R = 2;        % Radius in [m]

E1 = 142e3;    % Young's Modulus of Graphite /Epoxy in [N/mm^2]=[M Pa]
G1=6e3;% Shear Modulus of Graphite /Epoxy in [N/mm^2]=[M Pa]
Ro1 = 1.8e3;% Graphite /Epoxy mass density in [kg/m^3]; *1e-6[gr/mm^3]
E2 = 71.4e3;   % Young's Modulus of piezoelectric in [N/mm^2]=[M Pa]
nu2=0.3;% Poisson ratio of piezoelectric
G2=E2/(2*(1+nu2));% Shear Modulus of piezoelectric in [N/mm^2]=[M Pa]
Ro2 = 7.5e3;   % Piezoelectric mass density in [kg/m^3]; *1e-6[gr/mm^3]

c= 48.2809;% Dimension of the structure in the blade chord direction
in [mm]
d= 15.2201;% Dimension of the structure in perpendicular to the blade
chord direction in [mm]
a=1/3;b=1-2*a;% Proportionality constants in the blade chord direction
ee=1/5;f=1-2*ee;% Proportionality constants in the perpendicular
%direction to blade chord
km=26.4896;% Mass radius of gyration in [mm]
n=4;% Ratio of (km/ka)^2
m=2.8995;
% mass/length of the whole structure without piezoelectric in [kg/m]

Rc = Rci;% Structural coupling
Num_b = 4;% Number of blade
ce = 0.121/R;% Normalized blade chord
Si = ce*(Num_b/pi); % Solidity ratio
Ga = 5;% Lock number
Cd = 1.59e-3;% Normalized drag coeff
Bp = 0;% Precone angle in rad; no precone

~~~~~

% Cross section properties of the rotor structure without the smart
materials
Iz=d*c^3/12*(2*ee*b^3+f);% Moment of Inertia in [mm^4]
Stz=E1*Iz*1e-6/(m*Om^2*R^4);% Non-dimensional Moment of Inertia

Iy=c*d^3/12*(2*b*ee^3+6*b*ee*(ee+f)^2+f^3);% Moment of Inertia in
[mm^4]
Sty=E1*Iy*1e-6/(m*Om^2*R^4);% Non-dimensional Moment of Inertia

J=Iz+Iy;% Polar Moment of Inertia in [mm^4]
Stj=G1*J*1e-6/(m*Om^2*R^4);% Non-dimensional Moment of Inertia

A=c*d*(2*b*ee+f);% Cross section area in [mm^2]

```

$Ro = km^2 \cdot m / (I_z + I_y) \cdot 1e6$; % mass/length of the whole structure without piezoelectric in $[kg/m^3]$

$km1 = \sqrt{Ro \cdot I_y / m \cdot 1e-6}$; % first mass radius of gyration in $[mm]$
 $Mu1 = km1 \cdot 1e-3 / R$; % non-dimensional first mass radius of gyration

$km2 = \sqrt{Ro \cdot I_z / m \cdot 1e-6}$; % second mass radius of gyration in $[mm]$
 $Mu2 = km2 \cdot 1e-3 / R$; % non-dimensional second mass radius of gyration

$Mu = \sqrt{Mu1^2 + Mu2^2}$; % non-dimensional mass radius of gyration

$ka = km / \sqrt{n}$; % area radius of gyration in $[mm]$
 $kl = 1/n$; % Ratio of $(ka/km)^2$

~~~~~

% Cross section properties of the rotor structure with the smart materials

$I_{z\_f} = d \cdot c^3 / 12$; % Moment of Inertia in $[mm^4]$
 $I_{z\_p} = I_{z\_f} - I_z$; % Moment of Inertia due to smart materials in $[mm^4]$
 $Stz\_p = E2 \cdot I_{z\_p} \cdot 1e-6 / (m \cdot \Omega^2 \cdot R^4)$; % Non-dimensional Moment of Inertia due to smart materials

$I_{y\_f} = c \cdot d^3 / 12$; % Moment of Inertia in $[mm^4]$
 $I_{y\_p} = I_{y\_f} - I_y$; % Moment of Inertia due to smart materials in $[mm^4]$
 $Sty\_p = E2 \cdot I_{y\_p} \cdot 1e-6 / (m \cdot \Omega^2 \cdot R^4)$; % Non-dimensional Moment of Inertia due to smart materials

$J\_f = I_{z\_f} + I_{y\_f}$; % Polar Moment of Inertia in $[mm^4]$
 $J\_p = J\_f - J$; % Polar Moment of Inertia due to smart materials in $[mm^4]$
 $Stj\_p = G2 \cdot J\_p \cdot 1e-6 / (m \cdot \Omega^2 \cdot R^4)$; % Non-dimensional Moment of Inertia due to smart materials

$A\_f = c \cdot d$; % Cross section area in $[mm^2]$
 $A\_p = A\_f - A$; % Cross section area due to smart materials in $[mm^2]$

$ka\_f = \sqrt{(I_{z\_f} + I_{y\_f}) / A\_f}$; % Area radius of gyration in $[mm]$
 $ka\_p = \sqrt{ka\_f^2 - ka^2}$; % Area radius of gyration due to smart materials in $[mm]$
 $kl\_p = (ka\_p / km)^2$; % Non-dimensional ratio of $(ka\_p / km)^2$

$m\_f = m + 4 \cdot Ro^2 \cdot A\_p \cdot 1e-6$; % Mass/length in $[kg/m]$
 $m\_p = m\_f - m$; % Mass/length due to smart materials in $[kg/m]$

$km1\_f = \sqrt{(Ro \cdot I_y + 4 \cdot Ro^2 \cdot I_{y\_p}) \cdot 1e-6 / m\_f}$; % first mass radius of gyration in $[mm]$
 $km1\_p = \sqrt{km1\_f^2 - km1^2}$; % first mass radius of gyration due to smart materials in $[mm]$
 $Mu1\_p = km1\_p \cdot 1e-3 / R$; % Non-dimensional first mass radius of gyration due to smart materials

$km2\_f = \sqrt{(Ro \cdot I_z + 4 \cdot Ro^2 \cdot I_{z\_p}) \cdot 1e-6 / m\_f}$; % second mass radius of gyration in $[mm]$
 $km2\_p = \sqrt{km2\_f^2 - km2^2}$; % second mass radius of gyration due to smart materials in $[mm]$
 $Mu2\_p = km2\_p \cdot 1e-3 / R$

```

% Non-dimensional second mass radius of gyration due to smart materials

Mu_p=sqrt(Mu1_p^2+Mu2_p^2);
% Non-dimensional mass radius of gyration due to smart materials

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Mode shapes eigenvalues & non-dimensional integrals of products of
mode shapes
[bej,gaj,Ai,Bi,Ci,Dij,Eij,Iij,Jij,Mij,deij,Nij,Hij,Fijk,Fikj,Fjki,Gijk,
Hijk,Kijk,Kjki,Kkji,Lijk,Likj] = big1;
[sb,sbl] = size(bej); % sb is number of modes; sbl is dummy variable

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% This section calculates the steady state solution of the blade
motion
[X,vi,T,V,W]=stry33;
Tip=[V W T];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Obtain the mass matrix without the aerodynamics terms;
% Obtain the damping matrix;
% Obtain the stiffness matrix;
% Obtain the structural stiffness matrix without the aerodynamics
term;
[M,C,Ks,Kt]=mck25(X,vi); % Steady state quantities and in flow velocity
[M_p,C_p,Ks_p] = smart5(X);
% M_f = M ; C_f=C; Kt_f=Kt; Ks_f=Ks;
M_f = M + M_p; C_f=C+C_p; Kt_f=Kt+Ks_p; Ks_f=Ks+Ks_p;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Solve for the eigenvalues and eigenvectors of the free vibration
% case without damping
P0 =inv(M)*Ks;
P0_f =inv(M_f)*Ks_f;
[Vc, Va] = eig(P0); % Va is the squared of the eigenvalues
[Vc_f, Va_f] = eig(P0_f); % Va_f is the squared of the eigenvalues
Va=diag(Va); Va_f=diag(Va_f);
[Val,j]=sort(Va); % Sort the eigenvalues in ascending order
[Val_f,j_f]=sort(Va_f); % Sort the eigenvalues in ascending order
% Sort the eigenvectors in ascending order
J=3*sb;
for k=1:J
    Vc1(:,k) = Vc(:,j(k));
    Vc1_f(:,k) = Vc_f(:,j_f(k));
end
% Select the first four eigenvectors: 1st lead-lag mode, 1st and 2nd
% flap modes, 1st torsion

```

```

for k=1:4
    U(:,k) = Vc1(:,k);
    U_f(:,k) = Vc1_f(:,k);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Obtain P1 = P* matrix to describe the state space model
Mm=U'*M*U;
P11 = zeros(4);
P12 = eye(4);
P21 = -inv(Mm)*U'*Kt*U;
P22 = -inv(Mm)*U'*C*U;
P1 = [P11 P12;P21 P22];

% Draw root locus
[Vc2,Va2]=eig(P1);
Va2=diag(Va2);
[Va2,j]=sort(Va2); % Sort the eigenvalues in ascending order
% Sort the eigenvectors in ascending order
J=8;
for k=1:J
    Vc2(:,k) = Vc2(:,j(k));
end

% Obtain P1_f = P* matrix to describe the state space model
Mm_f=U_f'*M_f*U_f;
P11_f = zeros(4);
P12_f = eye(4);
P21_f = -inv(Mm_f)*U_f'*Kt_f*U_f;
P22_f = -inv(Mm_f)*U_f'*C_f*U_f;
P1_f = [P11_f P12_f;P21_f P22_f];

% Draw root locus
[Vc2_f,Va2_f]=eig(P1_f);
Va2_f=diag(Va2_f);
[Va2_f,j_f]=sort(Va2_f); % Sort the eigenvalues in ascending order
% Sort the eigenvectors in ascending order
J=8;
for k=1:J
    Vc2_f(:,k) = Vc2_f(:,j_f(k));
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% This section calculates the eigenvalues and eigenvectors
%corresponding to the periodic
% activation of the smart spring

for l=1:length(Nmi)

Nb=1;% Number of interval being investigated
Nm=Nmi(1,1);% Number of periodic intervals
t0=0;tf=(2*pi)/Nm;% Initial and final values of the first period

```

```

int=(tf-t0);% Period's length
% Divide time range into time intervals of one half period length
for k=0:(2*Nb)
    T1(k+1,1)=t0+k*int/2;
end

% Obtain the transition matrix
for i=1:8
    x0=zeros(8,1);
    x0(i,1)=1;
    for j=1:(2*Nb)
        tspan = [T1(j,1) T1(j+1,1)];
        % Determine the equations of motion corresponding to the time
        % interval
        if mod(j,2)==0
            [t,x]=ode45('xprimeB',tspan,x0);
        else
            [t,x]=ode45('xprimeA',tspan,x0);
        end
        [a1,b1]=size(x);
        x0=x(a1,:);
    end % end of j loop
    Q2(:,i)=x0;% Transition matrix
end % end of i loop

[Vc22, Va22] = eig(Q2);
% Eigenvalues and eigenvectors of the transition matrix
Va22=diag(Va22); [Va22,j]=sort(Va22);
% Sort the eigenvalues in ascending order
for i=1:8
    p_re2(i,1) = 1/tf*log(abs(Va22(i,1)));
    % Growth/decay rate of the motion
    p_im2(i,1) = 1/tf*atan(imag(Va22(i,1))/real(Va22(i,1)));
    p2(i,1)=p_re2(i,1)+p_im2(i,1)*sqrt(-1);
end
[p3(:,1)]=p2;% Calculates the eigen values of the system activated
%at Nmi /rev accordingly
end% end of 1 loop

% The purpose of this section is to match the arrangement of the
%elements of each
% column matrix p3 (which contains the eigen values of the system
%activated at Nmi /rev
% as its 1st, 2nd, 3rd columns) according to its relative magnitude
%such that it resembles
% that of Va2_f (non-activated system). This is done because the
%modes of the eigenvalues
% of Va2_f is known while those of p3 aren't.

for i=1:length(Nmi)
    p4(:,i)=mysort1(Va2_f,p3(:,i));
    p4r(:,i)=real(p4(:,i));
    p4i(:,i)=imag(p4(:,i));
end

```

```

%*****

```

```

~ The purpose of this section is to normalize the frequencies in p3
~ (which contains the eigen
~ values of the system activated at Nmi /rev as its columns) with
~ respect to
~ those of Va2_f (non-activated system)

```

```

function
[bej,gaj,Ai,Bi,Ci,Dij,Eij,Iij,Jij,Mij,deij,Nij,Hij,Fijk,Fikj,Fjki,Gijk,
Hijk,Kijk,Kjki,Kkji,Lijk,Likj] = bigl
% function
%[bej,gaj,Ai,Bi,Ci,Dij,Eij,Iij,Jij,Mij,deij,Nij,Hij,Fijk,Fikj,Fjki,Gijk
%,Hijk,Kijk,Kjki,Kkji,Lijk,Likj] = bigl;
% calculates integration of products of basis functions
% This program also takes advantage of the numerical integration
%program, newton3.m,
% written by Fred Nitzsche and modified by Solaiman A

% n = degree of polynomial interpolator
n= 5;
% N = number of integrating points
N=21;

% Get the integrating matrix
[Ln,x,bn] = newton3(n,N);

% Variables Beta and Alfa (associated with the natural frequencies) for
% fixed - free beam configuration
Beta=[1.875104068712 4.694091132974 7.854757438238 10.99554073488
14.13716839105 17.27875953209].';
Alfa= -[0.7340955137589 1.018467318759 0.9992244965174 1.000033553252
0.9999985501087 1.000000062656].';

bej = Beta; gaj = Alfa;

% N1= Number of modes, N2= Dummy variable
[N1,N2]=size(Beta);

% Construct the basis function matrix for the integration terms
for i=1:N1
    for j=1:N
        Y(j,i)= cosh(Beta(i,1)*x(j,1))-
cos(Beta(i,1)*x(j,1))+Alfa(i,1)*(sinh(Beta(i,1)*x(j,1))-
sin(Beta(i,1)*x(j,1)));
        Yp(j,i)= Alfa(i,1)*Beta(i,1)*(cosh(Beta(i,1)*x(j,1))-
cos(Beta(i,1)*x(j,1))) +
Beta(i,1)*(sinh(Beta(i,1)*x(j,1))+sin(Beta(i,1)*x(j,1)));
        Ypp(j,i)=
(Beta(i,1)^2)*(cosh(Beta(i,1)*x(j,1))+cos(Beta(i,1)*x(j,1)))+Alfa(i,1)*
(Beta(i,1)^2)*(sinh(Beta(i,1)*x(j,1))+sin(Beta(i,1)*x(j,1)));
        Yppp(j,i)=
Alfa(i,1)*(Beta(i,1)^3)*(cosh(Beta(i,1)*x(j,1))+cos(Beta(i,1)*x(j,1)))
+ (Beta(i,1)^3)*(sinh(Beta(i,1)*x(j,1))-sin(Beta(i,1)*x(j,1)));
        Gamma = pi * (i - 0.5);
        W(j,i)= sqrt(2) *sin(Gamma * x(j,1));
        Wp(j,i)= sqrt(2)* Gamma *cos(Gamma * x(j,1));
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    Ai(i,1) = bn * Ln * diag(Y(:,i))*ones(N,1);
end

```



```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    Bi(i,1) = bn * Ln * diag(x.* Y(:,i))*ones(N,1);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

deij = eye(N1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    Ci(i,1) = bn * Ln * diag(x.* x.* Y(:,i))*ones(N,1);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        Dij(i,j) = 0.5 * bn * Ln * diag((ones(N,1)-(x.*x)).* Yp(:,i)
.* Yp(:,j))*ones(N,1);
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        Eij(i,j) = bn * Ln * diag(x.* Y(:,i) .* Y(:,j))*ones(N,1);
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        Iij(i,j) = bn * Ln * diag(x.* Y(:,i) .* W(:,j))*ones(N,1);
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        Jij(i,j) = bn * Ln * diag(x.* x.* Y(:,i) .* W(:,j))*ones(N,1);
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        Mij(i,j) = bn * Ln * diag(x.* W(:,i) .* W(:,j))*ones(N,1);
    end
end

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        Nij(i,j) = 0.5 * bn * Ln * diag((ones(N,1)-(x.*x)).* Wp(:,i)
.* Wp(:,j))*ones(N,1);
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        Hij(i,j) = bn * Ln * diag(x.* Y(:,i) .* Yp(:,j))*ones(N,1);
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Fijk(i,j,k) = -(1/(Beta(k,1)^4))* bn * Ln * diag(Yp(:,i) .*
Yp(:,j) .* Yppp(:,k))*ones(N,1);
        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Fikj(i,j,k) = -(1/(Beta(j,1)^4))* bn * Ln * diag(Yp(:,i) .*
Yp(:,k) .* Yppp(:,j))*ones(N,1);
        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Fjki(i,j,k) = -(1/(Beta(i,1)^4))* bn * Ln * diag(Yp(:,j) .*
Yp(:,k) .* Yppp(:,i))*ones(N,1);
        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Gijk(i,j,k) = bn * Ln * diag(x.* Y(:,i) .* Y(:,j)).*
W(:,k))*ones(N,1);
        end
    end
end

```

```

        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Hij(k,i,j) = bn * Ln * diag(Y(:,i) .* Y(:,j) .* W(:,k)) * ones(N,1);
        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Kij(k,i,j) = bn * Ln * diag(W(:,i) .* Ypp(:,j) .*
Ypp(:,k)) * ones(N,1);
        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Kj(k,i,j) = bn * Ln * diag(W(:,j) .* Ypp(:,k) .*
Ypp(:,i)) * ones(N,1);
        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Kk(k,i,j) = bn * Ln * diag(W(:,k) .* Ypp(:,j) .*
Ypp(:,i)) * ones(N,1);
        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            Lijk(i,j,k) = bn * Ln * diag(x.* x.* Y(:,i) .* Y(:,j) .*
Yp(:,k)) * ones(N,1) ...
- bn*Ln* diag(x.*x.*Y(:,i) .*
(Ln*Yp(:,j) .* Ypp(:,k))) * ones(N,1);
        end
    end
end

```



```

function [Ln,rn,bn] = newton3(n,N)
% function [Ln,rn,bn] = newton3(n,N)
%
% Newton3 integrating and weighting matrices of nth. order (up to 7)
%
% Fred Nitzsche
% 31.5.92
% modified by Solaiman A
if n<1 | n>7,
    n=input('Order of integrating matrix must be btw. 1 & 7. Input
n ')
    end
if (N-n) <=0,
    N=input('too few discretizing points for this matrix. Input new
N ')
    end
if n==1,
    wn=      [1,1];
    ncheck=2;
    end
if n==2,
    wn=      [5,8,-1];
    ncheck=12;
    end
if n==3,
    wn=      [9,19,-5,1;
              -1,13,13,-1];
    ncheck=24;
    end
if n==4,
    wn=      [251,646,-264,106,-19;
              -19,346,456,-74,11];
    ncheck=720;
    end
if n==5,
    wn=      [475,1427,-798,482,-173,27;
              -27,637,1022,-258,77,-11;
              11,-93,802,802,-93,11];
    ncheck=1440;
    end
if n==6,
    wn=      [19087,65112,-46461,37504,-20211,6312,-863;
              -863,25128,46989,-16256,7299,-2088,271;
              271,-2760,30819,37504,-6771,1608,-191];
    ncheck=60480;
    end
if n==7,
    wn=      [36799,139849,-121797,123133,-88547,41499,-11351,1375;
              -1375,47799,101349,-44797,26883,-11547,2999,-351;
              351,-4183,57627,81693,-20227,7227,-1719,191;
              -191,1879,-9531,68323,68323,-9531,1879,-191];
    ncheck=120960;
    end
[nl,nc]=size(wn);
w=[wn(1:nl-1,1:nc),zeros(nl-1,N-nc)];
i=1;
while (N-nc-i+1)>=0,
    % fill with repeated rows

```

```

w=[w;
  zeros(1,i-1),wn(nl,1:nc),zeros(1,N-nc-i+1)];
i=i+1;
end
[nl1,nc1]=size(w);
if (N-nl1-1)==nl, % order is even
    w=[w;w(nl:-1:1,N:-1:1)];
else % order is odd
    w=[w;w(nl-1:-1:1,N:-1:1)];
end
w=[zeros(1,N);w];
for i=2:N,
    sum=0;
    for j=1:N,
        sum=sum+w(i,j);
    end
    if(sum~=ncheck)
        disp('line sum does not match ncheck on line:')
        i
        sum
        ncheck
        return
    end
end
h=1/(N-1);
w=h/ncheck*w;
s=tril(ones(N));
Ln=s*w;
xc=linspace(0,2*pi,N)';
if (nargout>1),
    rn=linspace(0,1,N)';
    bn=[zeros(1,N-1),1];
end
return

```

```

function [X,vi,T,V,W]=stry33
% function [X,vi,T,V,W]=stry33 calculates:
% X = Modal steady state quantities
% vi = induced in-flow velocity
% T = Steady state torsion deflection at the tip
% V = Steady state lead-lag deflection at the tip
% W = Steady state flap deflection at the tip

global sb

% Initial guess
x0 = zeros(3*sb+2,1);

% Option to show results
options(1)=1;

% Options to set up tolerance values
options(2) = 1e-4;
options(3) = 1e-4;

% Set the number of iterations
options(14) = 1000;

% Invoke optimizer
X = fsolve('Sted38',x0,options);

% Steady state lead-lag deflection at the tip
V=0;
for i=1:sb
    V=V+X(i)*(-1)^(i+1);
end
V=2*V;

% Steady state flap deflection at the tip
W=0;
for i=1:sb
    W=W+X(i+sb)*(-1)^(i+1);
end
W=2*W;

% Steady state torsion deflection at the tip
T = 4/3*X(3*sb+1);

% Induced in-flow velocity
vi = X(3*sb+2);

% Modal steady state quantities
X = X(1:3*sb);

```

```

function S = Sted38(X)
% Sted38(x) defines the non-linear steady state deflections equations
% of a helicopter rotor blade as derived by Hodges and Ormiston.
%
% The equations consist of: generalized coordinates (lead-lag, flap,
% torsion)
%           integrals of the shape functions (Dij, Kijk, etc)
%           adjustable parameters (precone angle, structural
% coupling, etc)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Defining the adjustable parameters
global Stz Sty Stj Rc Te Bp ce Si Mu Mul Mu2 k1
% Structural and geometric parameters w.o. piezo
global Ga Cd % aerodynamics parameters
% Non dimensional integrals of the products of the basis functions
global bej gaj deij Dij Ai Bi Ci Hij Jij Nij Lijk Likj Kkji Kijk Kjki
global Iij Eij Mij Gijk Hijk Fikj Fjki
global sb

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Steady state deflections equations of motion

% 75% of total torsion deflection at the tip of the blade
T75=0;
for i=1:sb
    T75=T75+X(i+2*sb)*(-1)^(i+1);
end
T75=0.75*sqrt(2)*T75-X(3*sb+1);

% Inflow equation
vi = sign(Te+X(3*sb+1))*pi*Si/8*(sqrt(1+12/(pi*Si)*abs(Te+X(3*sb+1))))-
1)-X(3*sb+2);

% Lead-lag, V, equation
% Initialize counter
P = zeros(sb,1);
Q = zeros(sb,1);
% Loop
for i=1:sb
    for j=1:sb

        P(i) = P(i) + X(j)*(Dij(i,j)+(Stz-(Stz-
Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j)-deij(i,j))+ ...
            X(j+sb)*0.5*sin(2*Rc*Te)*(Stz-
Sty)*bej(j)^4*deij(i,j)+Ga/6*X(3*sb+2)*Iij(i,j)*X(j+2*sb);

        for k=1:sb
            Q(i) = Q(i) + (Stz-Sty)*Kjki(j,i,k)*X(j+2*sb)*(-
X(k)*sin(2*Rc*Te)+X(k+sb)*cos(2*Rc*Te));
        end % k loop
    end % j loop

    P(i) = P(i) + Q(i) - Ga/6*(X(3*sb+2)^2*Ai(i)-Cd*Ci(i)-
X(3*sb+2)*Te*Bi(i));

```



```

end; % i loop

% Flap, W, equation
% Initialize counter
Z = zeros(sb,1);
U = zeros(sb,1);
% Loop
for i=1:sb
    for j=1:sb

        Z(i) = Z(i) + X(j)*0.5*sin(2*Rc*Te)*(Stz-Sty)*bej(j)^4*deij(i,j)+
        ...
        X(j+sb)*(Dij(i,j)+(Sty+(Stz-
        Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j))+ ...
        Ga/6*(Bp*Eij(i,j)*X(j)-ce/2*Hij(i,j)*X(j+sb)-Jij(i,j)*X(j+2*sb));

        for k=1:sb
            U(i) = U(i) + (Stz-
            Sty)*(Kjki(j,i,k)*X(j+2*sb)*(X(k)*cos(2*Rc*Te)+X(k+sb)*sin(2*Rc*Te)))+
            ...
            Ga/6*Lijk(i,j,k)*X(j)*X(k+sb);
        end % k loop
    end % j loop

    Z(i) = Z(i) + U(i) + Bp*Bi(i)-Ga/6*(Te*Ci(i)-
    X(3*sb+2)*Bi(i)+ce/2*Bp*Bi(i));
end; % i loop

% Torsion, T, equation
% Initialize counter
D = zeros(sb,1);
E = zeros(sb,1);
% Loop
for i=1:sb
    for j=1:sb

        D(i) = D(i) + X(j+2*sb)*(Mu^2*k1*Nij(i,j)+(Stj*gaj(j)^2+(Mu2^2-
        Mu1^2)*cos(2*Te))*deij(i,j));

        for k=1:sb
            E(i) = E(i) +(Stz-
            Sty)*Kijk(i,j,k)*(sin(2*Rc*Te)/2*(X(j+sb)*X(k+sb)-X(j)*X(k))+ ...
            cos(2*Rc*Te)*X(j)*X(k));
        end % k loop

    end % j loop

    D(i) = D(i) + E(i) + (Mu2^2-Mu1^2)*sin(2*Te)/(sqrt(2)*gaj(i));
end; % i loop

% Arrange equations to be optimized
S = [ T75; vi; P; Z; D];

%+-----+

```

```

function [M,C,Ks,Kt]=mck25(X,vi)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [M,C,Ks,Kt]=mck25 creates damping matrix of helicopter
%blade equations of motion
% as derived by Hodges and Ormiston.
%
% This matrix involved some non-dimensional integrals of products of
% mode shapes
% which are produced by some other .m files created by Solaiman and
% supported by
% newton.m programmed by Nitzsche.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Defining the adjustable parameters
global Stz Sty Stj Rc Te Bp ce Si Mu Mu1 Mu2 k1
% Structural and geometric parameters w.o. piezo
global Stz_p Sty_p Stj_p Mu_p Mu1_p Mu2_p k1_p
% Structural and geometric parameters due to piezo
global Ga Cd % aerodynamics parameters
% Non dimensional integrals of the products of the basis functions
global bej gaj deij Dij Ai Bi Ci Hij Jij Nij Lij Likj Kkji Kijk Kjki
global Iij Eij Mij Gijk Hijk Fikj Fjki
global sb

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Mass matrix construction.
dl=sb;
M11 = deij;
M22 = deij;
M33 = deij*Mu^2;
M12 = zeros(dl);
M13 = zeros(dl);
M21 = zeros(dl);
M23 = zeros(dl);
M31 = zeros(dl);
M32 = zeros(dl);
M = [M11 M12 M13;M21 M22 M23; M31 M32 M33];

% Damping matrix construction
% Structural damping
% Element (1,1)
% Initialize counter
Cv111 = zeros(sb);
% Loop
for k=1:sb
    Cv111 = Cv111 + 2*(Fikj(:, :,k)-Fjki(:, :,k))*X(k);
end % k loop
Cv11=Cv111;
Cv{1,1}=Cv11;

% Element (1,2)

```

```

Cv121 = -2*Bp*deij;
% Initialize counter
Cv122 = zeros(sb);
% Loop
for k=1:sb
    Cv122 = Cv122 - 2*Fjki(:, :, k)*X(k+sb);
end % k loop
Cv12 = Cv121 + Cv122;
Cv{1,2} = Cv12;

% Element (1,3)
Cv131 = zeros(sb);
Cv13 = Cv131;
Cv{1,3} = Cv13;

% Element (2,1)
Cv211 = 2*Bp*deij;
% Initialize counter
Cv212 = zeros(sb);
% Loop
for k=1:sb
    Cv212 = Cv212 + 2*Fikj(:, :, k)*X(k+sb);
end % k loop
Cv21 = Cv211 + Cv212;
Cv{2,1} = Cv21;

% Element (2,2)
Cv221 = zeros(sb);
Cv22 = Cv221;
Cv{2,2} = Cv22;

% Element (2,3)
Cv231 = zeros(sb);
Cv23 = Cv231;
Cv{2,3} = Cv23;

% Element (3,1)
Cv311 = zeros(sb);
Cv31 = Cv311;
Cv{3,1} = Cv31;

% Element (3,2)
Cv321 = zeros(sb);
Cv32 = Cv321;
Cv{3,2} = Cv32;

% Element (3,3)
Cv331 = zeros(sb);
Cv33 = Cv331;
Cv{3,3} = Cv33;

% Aerodynamic damping matrix
% Element (1,1)
Cal11 = Ga/6*2*Cd*Eij;
Cal12 = Ga/6*Te*vi*deij;
% Initialize counter
Cal13 = zeros(sb);

```

```

% Loop
for k=1:sb
    Ca113 = Ca113 + Ga/6*vi*Hijk(:, :, k)*X(k+2*sb);
end % k loop
Ca11 = Ca111+Ca112+Ca113;
Ca{1,1}=Ca11;

% Element (1,2)
Ca121 = -Ga/6*Te*Eij;
Ca122 = Ga/6*2*vi*deij;
% Initialize counter
Ca123 = zeros(sb);
% Loop
for k=1:sb
    Ca123 = Ca123 - Ga/6*Gijk(:, :, k)*X(k+2*sb);
end % k loop
Ca12 = Ca121+Ca122+Ca123;
Ca{1,2}=Ca12;

% Element (1,3)
Ca131 = zeros(sb);
Ca13 = Ca131;
Ca{1,3} = Ca13;

% Element (2,1)
Ca211 = -Ga/6*2*Te*Eij;
Ca212 = Ga/6*vi*deij;
% Initialize counter
Ca213 = zeros(sb);
% Loop
for k=1:sb
    Ca213 = Ca213 - 2*Ga/6*Gijk(:, :, k)*X(k+2*sb);
end % k loop
Ca21 = Ca211+Ca212+Ca213;
Ca{2,1}=Ca21;

% Element (2,2)
Ca221 = Ga/6*Eij;
Ca22 = Ca221;
Ca{2,2} = Ca22;

% Element (2,3)
Ca231 = -0.75*Ga/6*ce*Iij;
Ca23 = Ca231;
Ca{2,3} = Ca23;

% Element (3,1)
Ca311 = zeros(sb);
Ca31 = Ca311;
Ca{3,1} = Ca31;

% Element (3,2)
Ca321 = zeros(sb);
Ca32 = Ca321;
Ca{3,2} = Ca32;

```

```

% Element (3,3)
Ca331 = Ga/6*ce^2/8*Mij;
Ca33 = Ca331;
Ca{3,3} = Ca33;

% Display aerodynamics damping matrix, Ca
% celldisp(Ca)

% Total damping matrix
C{1,1} = Cv11 + Ca11;
C{1,2} = Cv12 + Ca12;
C{1,3} = Cv13 + Ca13;
C{2,1} = Cv21 + Ca21;
C{2,2} = Cv22 + Ca22;
C{2,3} = Cv23 + Ca23;
C{3,1} = Cv31 + Ca31;
C{3,2} = Cv32 + Ca32;
C{3,3} = Cv33 + Ca33;
% celldisp(C);

C = [C{1,1} C{1,2} C{1,3}; C{2,1} C{2,2} C{2,3}; C{3,1} C{3,2} C{3,3}];

% Stiffness matrix construction
% Structural stiffness matrix
% Element (1,1)
for i=1:sb
    for j=1:sb
        Ks111(i,j) = (Stz-(Stz-Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j);
    end % j loop
end % i loop
% Initialize counter
Ks112 = zeros(sb);
% Loop
for k=1:sb
    Ks112 = Ks112 + (Stz-Sty)*sin(2*Rc*Te)*Kkji(:, :, k)*X(k+2*sb);
end % k loop
Ks11 = Dij+Ks111-deij-Ks112;

% Element (1,2)
for i=1:sb
    for j=1:sb
        Ks121(i,j) = (Stz-Sty)*(sin(2*Rc*Te))/2*bej(j)^4*deij(i,j);
    end % j loop
end % i loop
% Initialize counter
Ks122 = zeros(sb);
% Loop
for k=1:sb
    Ks122 = Ks122 + (Stz-Sty)*cos(2*Rc*Te)*Kkji(:, :, k)*X(k+2*sb);
end % k loop
Ks12 = Ks121+Ks122;

% Element (1,3)
% Initialize counter

```

```

Ks131 = zeros(sb);
% Loop
for k=1:sb
    Ks131 = Ks131 + (Stz-Sty)*Kjki(:, :, k)*(X(k+sb)*cos(2*Rc*Te)-
X(k)*sin(2*Rc*Te));
end % k loop
Ks13 = Ks131;

% Element (2,1)
Ks21 = Ks12;

% Element (2,2)
for i=1:sb
    for j=1:sb
        Ks221(i,j) = (Sty+(Stz-Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j);
    end % j loop
end % i loop
Ks222 = Ks112;
Ks22 = Dij+Ks221+Ks222;

% Element (2,3)
% Initialize counter
Ks231 = zeros(sb);
% Loop
for k=1:sb
    Ks231 = Ks231 + (Stz-
Sty)*Kjki(:, :, k)*(X(k+sb)*sin(2*Rc*Te)+X(k)*cos(2*Rc*Te));
end % k loop
Ks23 = Ks231;

% Element (3,1)
% Initialize counter
Ks311 = zeros(sb);
% Loop
for k=1:sb
    Ks311 = Ks311 + (Stz-Sty)*Kijk(:, :, k)*(X(k+sb)*cos(2*Rc*Te)-
X(k)*sin(2*Rc*Te));
end % k loop
Ks31 = Ks311;

% Element (3,2)
% Initialize counter
Ks321 = zeros(sb);
% Loop
for k=1:sb
    Ks321 = Ks321 + (Stz-
Sty)*Kijk(:, :, k)*(X(k+sb)*sin(2*Rc*Te)+X(k)*cos(2*Rc*Te));
end % k loop
Ks32 = Ks321;

% Element (3,3)

```

```

Ks331=Mu^2*k1*Nij+(Mu2^2-Mu1^2)*cos(2*Te)*deij;
for i=1:sb
    for j=1:sb
        Ks332(i,j) = Stj*gaj(j)^2*deij(i,j);
    end % j loop
end % i loop
Ks33=Ks331+Ks332;

Ks = [Ks11 Ks12 Ks13;Ks21 Ks22 Ks23; Ks31 Ks32 Ks33];

% Construct aerodynamic stiffness matrix, Kar
% Element (1,1)
Kal1 = zeros(sb);

% Element (1,2)
Kal2 = zeros(sb);

% Element (1,3)
Kal3 = Ga/6*vi*Iij;

% Element (2,1)
Ka211 = Ga/6*Bp*Eij;
% Initialize counter
Ka212 = zeros(sb);
% Loop
for k=1:sb
    Ka212 = Ka212 + Ga/6*Lijk(:, :, k)*X(k+sb);
end % k loop
Ka21 = Ka211+Ka212;

% Element (2,2)
Ka221 = Ga/6*(-ce/2*Hij);
% Initialize counter
Ka222 = zeros(sb);
% Loop
for k=1:sb
    Ka222 = Ka222 + Ga/6*Likj(:, :, k)*X(k);
end % k loop
Ka22 = Ka221+Ka222;

% Element (2,3)
Ka23 = Ga/6*(-Jij);

% Element (3,1)
Ka31 = zeros(sb);

% Element (3,2)
Ka32 = zeros(sb);

% Element (3,3)
Ka33 = zeros(sb);

% Aerodynamics stiffness matrix, Ka
Ka = [Kal1 Kal2 Kal3; Ka21 Ka22 Ka23; Ka31 Ka32 Ka33];

% Total stiffness matrix
Kt = Ks+Ka;

```

```

function [M_p,C_p,Ks_p] = smart5(X)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% function [M_p,C_p,Ks_p] = smart5(X) calculates the additional terms
% on
% the M, C, K matrices due to the inclusion of the smart springs.
% The input are: X=steady state deflection
%               x1=starting position of the smart springs
%               x2=ending position of the smart springs
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Defining the adjustable parameters
global Stz Sty Stj Rc Te Bp ce Si Mu Mul Mu2 kl
% Structural and geometric parameters w.o. piezo
global m m_p ka
% mass parameters w. and w.o. piezo and radius of gyration without
% piezo
global Stz_p Sty_p Stj_p Mu_p Mul_p Mu2_p kl_p
% Structural and geometric parameters due to piezo
global Ga Cd % aerodynamics parameters
global x1 x2
% Non dimensional integrals of the products of the basis functions
global bej gaj deij Dij Ai Bi Ci Hij Jij Nij Lijk Likj Kkji Kijk Kjki
global Iij Eij Mij Gijk Hijk Fikj Fjki
global sb

% Integrals of the products of the basis functions
[dlxij,dlxij2,Dlxij1,Dlxij2,dxij,nxij,fxijk,fxjki,fxikj,kxijk,kxjki,kxk
ji] = bigx4(x1,x2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% This section calculates the delta M matrix
d1 = sb;% Number of modes
M_p11 = m_p/m*dlxij;
M_p22 = m_p/m*dlxij;
M_p33 = real(dlxij2*Mu_p^2);
% Because Mu_p^2 is negative, must use real( ) to avoid +0.00i
M_p12 = zeros(d1);
M_p13 = zeros(d1);
M_p21 = zeros(d1);
M_p23 = zeros(d1);
M_p31 = zeros(d1);
M_p32 = zeros(d1);
M_p = [M_p11 M_p12 M_p13;M_p21 M_p22 M_p23; M_p31 M_p32 M_p33];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% This section calculates the delta C matrix
% Element (1,1)
Cv111 = zeros(d1);% Initialize counter
% Loop
for k=1:d1
    Cv111 = Cv111 + (fxikj(:, :,k)-fxjki(:, :,k))*X(k);

```



```

end % k loop
Cv11=2*m_p/m*Cv111;
Cv{1,1}=Cv11;

% Element (1,2)
Cv121 = -2*Bp*m_p/m*dlxij;
% Initialize counter
Cv122 = zeros(sb);
% Loop
for k=1:sb
    Cv122 = Cv122 - 2*m_p/m*fxjki(:, :, k)*X(k+sb);
end % k loop
Cv12 = Cv121 + Cv122;
Cv{1,2} = Cv12;

% Element (1,3)
Cv131 = zeros(d1);
Cv13 = Cv131;
Cv{1,3} = Cv13;

% Element (2,1)
Cv211 = 2*m_p/m*Bp*dlxij;
% Initialize counter
Cv212 = zeros(d1);
% Loop
for k=1:sb
    Cv212 = Cv212 + 2*m_p/m*fxikj(:, :, k)*X(k+sb);
end % k loop
Cv21 = Cv211 + Cv212;
Cv{2,1} = Cv21;

% Element (2,2)
Cv221 = zeros(d1);
Cv22 = Cv221;
Cv{2,2} = Cv22;

% Element (2,3)
Cv231 = zeros(d1);
Cv23 = Cv231;
Cv{2,3} = Cv23;

% Element (3,1)
Cv311 = zeros(d1);
Cv31 = Cv311;
Cv{3,1} = Cv31;

% Element (3,2)
Cv321 = zeros(d1);
Cv32 = Cv321;
Cv{3,2} = Cv32;

% Element (3,3)
Cv331 = zeros(d1);
Cv33 = Cv331;
Cv{3,3} = Cv33;

% Delta C matrix

```

```
C_p = [Cv{1,1} Cv{1,2} Cv{1,3};Cv{2,1} Cv{2,2} Cv{2,3}; Cv{3,1} Cv{3,2}
Cv{3,3}];
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Structural stiffness matrix
```

```
% Element (1,1)
```

```
Ks111 = (Stz_p-(Stz_p-Sty_p)*(sin(Rc*Te))^2)*Dlxij1;
```

```
% Initialize counter
```

```
Ks112 = zeros(d1);
```

```
% Loop
```

```
for k=1:d1
```

```
    Ks112 = Ks112 + (Stz_p-Sty_p)*sin(2*Rc*Te)*kxkji(:, :, k)*X(k+2*sb);
```

```
end % k loop
```

```
Ks113= m_p/m*(dxij-dlxij);
```

```
Ks11 = Ks111-Ks112+Ks113;
```

```
% Element (1,2)
```

```
Ks121 = (Stz_p-Sty_p)*(sin(2*Rc*Te))/2*Dlxij1;
```

```
% Initialize counter
```

```
Ks122 = zeros(d1);
```

```
% Loop
```

```
for k=1:d1
```

```
    Ks122 = Ks122 + (Stz_p-Sty_p)*cos(2*Rc*Te)*kxkji(:, :, k)*X(k+2*sb);
```

```
end % k loop
```

```
Ks12 = Ks121+Ks122;
```

```
% Element (1,3)
```

```
% Initialize counter
```

```
Ks131 = zeros(d1);
```

```
% Loop
```

```
for k=1:d1
```

```
    Ks131 = Ks131 + (Stz_p-Sty_p)*kxjki(:, :, k)*(X(k+sb)*cos(2*Rc*Te)-
X(k)*sin(2*Rc*Te));
```

```
end % k loop
```

```
Ks13 = Ks131;
```

```
% Element (2,1)
```

```
Ks21 = Ks12;
```

```
% Element (2,2)
```

```
Ks221 = (Sty_p+(Stz_p-Sty_p)*(sin(Rc*Te))^2)*Dlxij1;
```

```
Ks222 = Ks112;
```

```
Ks223= m_p/m*(dxij);
```

```
Ks22 = Ks221+Ks222+Ks223;
```

```
% Element (2,3)
```

```
% Initialize counter
```

```
Ks231 = zeros(d1);
```

```
% Loop
```

```
for k=1:d1
```

```

    Ks231 = Ks231 + (Stz_p-
Sty_p)*kxjki(:, :, k) * (X(k+sb)*sin(2*Rc*Te)+X(k)*cos(2*Rc*Te));
end % k loop
Ks23 = Ks231;

% Element (3,1)
% Initialize counter
Ks311 = zeros(d1);
% Loop
for k=1:d1
    Ks311 = Ks311 + (Stz_p-Sty_p)*kxijk(:, :, k) * (X(k+sb)*cos(2*Rc*Te) -
X(k)*sin(2*Rc*Te));
end % k loop
Ks31 = Ks311;

% Element (3,2)
% Initialize counter
Ks321 = zeros(d1);
% Loop
for k=1:d1
    Ks321 = Ks321 + (Stz_p-
Sty_p)*kxijk(:, :, k) * (X(k+sb)*sin(2*Rc*Te)+X(k)*cos(2*Rc*Te));
end % k loop
Ks32 = Ks321;

% Element (3,3)
Ks331=real(Mu_p^2*k1_p*nxij+(Mu2_p^2-Mu1_p^2)*cos(2*Te)*dlxij);
Ks332 = Stj_p*Dlxij2;
Ks33=Ks331+Ks332;

Ks_p = [Ks11 Ks12 Ks13;Ks21 Ks22 Ks23; Ks31 Ks32 Ks33];

```

```

function
[dlxij,dlxij2,Dlxij1,Dlxij2,dxij,nxij,fxijk,fxjki,fxikj,kxijk,kxjki,kxk
ji] = bigx4(x1,x2)
% function
%[dlxij,dlxij2,Dlxij1,Dlxij2,dxij,nxij,fxijk,fxjki,fxikj,kxijk,kxjki,kx
%kji] = bigx4(x1,x2)
% calculates integration of products of basis functions
% associated with "smart" springs from x1 and x2
% The input are:
% x1 = non-dimensional starting position of the "smart" torsional
%spring
% x2 = non-dimensional ending position of the "smart" torsional spring
% x2 must be greater or equal to x1, both are between 0 and 1 and of
%multiplication of 0.01
% The integrals of the products of the basis functions from x1 and x2
%to 1 respectively
% are calculated
% This program also takes advantage of the numerical integration
%program, newton2.m,
% written by Fred Nitzsche and modified by Solaiman A

% n = degree of polynomial interpolator
n= 5;
% N = number of integrating points
N=21;

% Check if x1 is less or equal to x2
if x1 >= x2
    disp('x2 must be greater or equal to x1. Input x1 and x2')
end

% Check if x1 and x2 falls in the range corresponding to N
x=linspace(0,1,N)'; % Create the discretizing location matrix
i=find(x<=x1);
il=length(i);
x1=x(il);
j=find(x<=x2);
jl=length(j);
x2=x(jl);
xi=[x1 x2].';
[Ni,Nn]=size(xi);

% Get the integrating matrix
[Ln,bn1,bn2,bn] = newton2(n,N,il,jl);

% Variables Beta and Alfa (associated with the natural frequencies) for
% fixed - free beam configuration
Beta=[1.875104068712 4.694091132974 7.854757438238 10.99554073488
14.13716839105 17.27875953209].';
Alfa= -[0.7340955137589 1.018467318759 0.9992244965174 1.000033553252
0.9999985501087 1.000000062656].';

% N1= Number of modes, N2= Dummy variable
[N1,N2]=size(Beta);

```

```

% Construct the basis function matrix for the integration terms
for i=1:N1
    for j=1:N
        Y(j,i)= cosh(Beta(i,1)*x(j,1))-
        cos(Beta(i,1)*x(j,1))+Alfa(i,1)*(sinh(Beta(i,1)*x(j,1))-
        sin(Beta(i,1)*x(j,1)));
        Yp(j,i)= Alfa(i,1)*Beta(i,1)*(cosh(Beta(i,1)*x(j,1))-
        cos(Beta(i,1)*x(j,1))) +
        Beta(i,1)*(sinh(Beta(i,1)*x(j,1))+sin(Beta(i,1)*x(j,1)));
        Ypp(j,i)=
        (Beta(i,1)^2)*(cosh(Beta(i,1)*x(j,1))+cos(Beta(i,1)*x(j,1)))+Alfa(i,1)*
        (Beta(i,1)^2)*(sinh(Beta(i,1)*x(j,1))+sin(Beta(i,1)*x(j,1)));
        Yppp(j,i)=
        Alfa(i,1)*(Beta(i,1)^3)*(cosh(Beta(i,1)*x(j,1))+cos(Beta(i,1)*x(j,1)))
        + (Beta(i,1)^3)*(sinh(Beta(i,1)*x(j,1))-sin(Beta(i,1)*x(j,1)));
        Gamma = pi * (i - 0.5);
        W(j,i)= sqrt(2) *sin(Gamma * x(j,1));
        Wp(j,i)= sqrt(2) * Gamma *cos(Gamma * x(j,1));
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        % Integrate from 0 to x1
        I1(i,j) = bn1 * Ln * diag(Y(:,i) .* Y(:,j))*ones(N,1);
        % Integrate from 0 to x2
        I2(i,j) = bn2 * Ln * diag(Y(:,i) .* Y(:,j))*ones(N,1);
    end
end

dlxij = I2-I1; % Integration from x1 to x2

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Integrate the products of the basis functions from the x1 and x2 to
% 1 respectively
for i=1:N1
    for j=1:N1
        % Integrate from 0 to x1
        I1(i,j) = bn1 * Ln * diag(W(:,i) .* W(:,j))*ones(N,1);
        % Integrate from 0 to x2
        I2(i,j) = bn2 * Ln * diag(W(:,i) .* W(:,j))*ones(N,1);
    end
end

dlxij2 = I2-I1; % Integration from x1 to x2

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:N1
    for j=1:N1
        % Integrate from 0 to x1
        I1(i,j) = bn1 * Ln * diag(Ypp(:,i) .* Ypp(:,j))*ones(N,1);
        % Integrate from 0 to x2
        I2(i,j) = bn2 * Ln * diag(Ypp(:,i) .* Ypp(:,j))*ones(N,1);
    end
end

```



```

        for j=1:N1
            % Integrate from 0 to x1
            Fijk_1(i,j,k) = -(1/(Beta(k,1)^4))* bn1 * Ln * diag(Yp(:,i) .*
            Yp(:,j) .* Yppp(:,k))*ones(N,1);
            % Integrate from 0 to x2
            Fijk_2(i,j,k) = -(1/(Beta(k,1)^4))* bn2 * Ln * diag(Yp(:,i) .*
            Yp(:,j) .* Yppp(:,k))*ones(N,1);
        end
    end
end

% Integration from x1 to x2
fxijk= Fijk_2-Fijk_1;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            % Integrate from 0 to x1
            Fjki_1(i,j,k) = -(1/(Beta(i,1)^4))* bn1 * Ln * diag(Yp(:,j) .*
            Yp(:,k) .* Yppp(:,i))*ones(N,1);
            % Integrate from 0 to x2
            Fjki_2(i,j,k) = -(1/(Beta(i,1)^4))* bn2 * Ln * diag(Yp(:,j) .*
            Yp(:,k) .* Yppp(:,i))*ones(N,1);
        end
    end
end

% Integration from x1 to x2
fxjki= Fjki_2-Fjki_1;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for k=1:N1
    for i=1:N1
        for j=1:N1
            % Integrate from 0 to x1
            Fikj_1(i,j,k) = -(1/(Beta(j,1)^4))* bn1 * Ln * diag(Yp(:,i) .*
            Yp(:,k) .* Yppp(:,j))*ones(N,1);
            % Integrate from 0 to x2
            Fikj_2(i,j,k) = -(1/(Beta(j,1)^4))* bn2 * Ln * diag(Yp(:,i) .*
            Yp(:,k) .* Yppp(:,j))*ones(N,1);
        end
    end
end

% Integration from x1 to x2
fxikj= Fikj_2-Fikj_1;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Integrate the products of the basis functions from the x1 and x2 to
% 1 respectively
for k=1:N1
    for i=1:N1
        for j=1:N1

```



```

function [Ln,bn1,bn2,bn] = newton2(n,N,il,jl)
% [Ln,bn1,bn2,bn] = newton2(n,N,il,jl)
%
% Newton integrating and weighting matrices of nth. order (up to 7)
%
% Fred Nitzsche
% 31.5.92
% Solaiman A
% 17.5.98

if n<1 | n>7,
    n=input('Order of integrating matrix must be btw. 1 & 7. Input
n ')
    end
if (N-n) <=0,
    N=input('too few discretizing points for this matrix. Input new
N ')
    end
if n==1,
    wn=      [1,1];
    ncheck=2;
    end
if n==2,
    wn=      [5,8,-1];
    ncheck=12;
    end
if n==3,
    wn=      [9,19,-5,1;
              -1,13,13,-1];
    ncheck=24;
    end
if n==4,
    wn=      [251,646,-264,106,-19;
              -19,346,456,-74,11];
    ncheck=720;
    end
if n==5,
    wn=      [475,1427,-798,482,-173,27;
              -27,637,1022,-258,77,-11;
              11,-93,802,802,-93,11];
    ncheck=1440;
    end
if n==6,
    wn=      [19087,65112,-46461,37504,-20211,6312,-863;
              -863,25128,46989,-16256,7299,-2088,271;
              271,-2760,30819,37504,-6771,1608,-191];
    ncheck=60480;
    end
if n==7,
    wn=      [36799,139849,-121797,123133,-88547,41499,-11351,1375;
              -1375,47799,101349,-44797,26883,-11547,2999,-351;
              351,-4183,57627,81693,-20227,7227,-1719,191;
              -191,1879,-9531,68323,68323,-9531,1879,-191];
    ncheck=120960;
    end
[nl,nc]=size(wn);

```

```

w=[wn(1:nl-1,1:nc),zeros(nl-1,N-nc)];
i=1;
while (N-nc-i+1)>=0, % fill with repeated rows
    w=[w;
        zeros(1,i-1),wn(nl,1:nc),zeros(1,N-nc-i+1)];
    i=i+1;
end
[nl1,nc1]=size(w);
if (N-nl1-1)==nl, % order is even
    w=[w;w(nl:-1:1,N:-1:1)];
else % order is odd
    w=[w;w(nl-1:-1:1,N:-1:1)];
end
w=[zeros(1,N);w];
for i=2:N,
    sum=0;
    for j=1:N,
        sum=sum+w(i,j);
    end
    if(sum~=ncheck)
        disp('line sum does not match ncheck on line:')
        i
        sum
        ncheck
        return
    end
end
h=1/(N-1);
w=h/ncheck*w;
s=tril(ones(N));
Ln=s*w;
xc=linspace(0,2*pi,N)';
%plot(xc,2*pi*Ln*cos(xc),xc,sin(xc),'o')
%title('Check for Integrating Matrix'),xlabel('x - rad'),ylabel('sin(x)
[c = EXACT]')
if (nargout>1),
    bn1=[zeros(1,i1-1),1,zeros(1,N-i1)];
    bn2=[zeros(1,j1-1),1,zeros(1,N-j1)];
    bn=[zeros(1,N-1),1];
end
return

```

[illegible]

[illegible]

```

function B3=mysort1(A,B)
% function B3=mysort(A,B) is written to match the arrangement of the
% elements of column
% matrix B according to its relative magnitude such that it resembles
% that of the real part
% elements of column matrix A
% for example: A = [3+1i;1+2i;2+3i]; B= [5+6i;6+7i;7+5i]; then B3=
% [7+5i;5+6i;6+7i]

[A1,A2]=sort(real(A));
[B1,B2]=sort(real(B));
[A3,A4]=sort(A2);

for i=1:length(A)
    B3r(i,1)=B1(A4(i,1),1);
    B3i(i,1)=imag(B(A4(i,1),1));
end

B3=B3r+B3i*sqrt(-1);

```

```

function B1 = mul2(A,B,a)
% function B1 = mul2(A,B) calculates the normalized frequency column
% matrix B1 of the "smart" blade
% The input of the program are:
% A = the physical frequency column matrix
% B = the principal frequency column matrix
% a = activation frequency of the "smart" spring

[a1,a2]=size(A);
[b1,b2]=size(B);

for i=1:a1
    if A(i,1)>= 0 & B(i,1)>= 0
        dB = abs(A(i,1)-B(i,1));
        if dB < a/2
            B1(i,1) = B(i,1);
        elseif dB >= a/2
            Br = dB/(a);
            Br = round (Br);
            if B(i,1) < 0
                B1(i,1) = B(i,1) - Br*a;
            elseif B(i,1) >= 0
                B1(i,1) = B(i,1) + Br*a;
            end
        end
    end

elseif A(i,1)< 0 & B(i,1)>= 0
    A(i,1) = abs(A(i,1));
    dB = abs(A(i,1)-B(i,1));
    if dB < a/2
        B1(i,1) = B(i,1);
    elseif dB >= a/2
        Br = dB/(a);
        Br = round (Br);
        if B(i,1) < 0
            B1(i,1) = B(i,1) - Br*a;
        elseif B(i,1) >= 0
            B1(i,1) = B(i,1) + Br*a;
        end
    end
end

elseif A(i,1)< 0 & B(i,1)< 0
    dB = abs(A(i,1)-B(i,1));
    if dB < a/2
        B1(i,1) = B(i,1);
    elseif dB >= a/2
        Br = dB/(a);
        Br = round (Br);
        if B(i,1) < 0
            B1(i,1) = B(i,1) - Br*a;
        elseif B(i,1) >= 0
            B1(i,1) = B(i,1) + Br*a;
        end
    end
end

elseif A(i,1)>= 0 & B(i,1)< 0
    A(i,1) = -A(i,1);

```

```
        dB = abs(A(i,1)-B(i,1));  
    if dB < a/2  
        B1(i,1) = B(i,1);  
        elseif dB >= a/2  
            Br = dB/(a);  
            Br = round (Br);  
            if B(i,1) < 0  
                B1(i,1) = B(i,1) - Br*a;  
            elseif B(i,1) >= 0  
                B1(i,1) = B(i,1) + Br*a;  
            end  
        end  
    end  
  
end  
end
```