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**LA THÈSE A ÉTÉ
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SOME DISTRIBUTIONAL TRANSFORMATIONS
AND ABELIAN THEOREMS

by

UMANATH TIWARI

A thesis submitted to the Faculty of Graduate Studies
in partial fulfillment of the requirements for the
degree of DOCTOR OF PHILOSOPHY.

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The undersigned recommend to the Faculty
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ABSTRACT

The present dissertation consists of three chapters, each of which deals with a specific problem of a distributional transformation.

In Chapter I, two complex inversion formulas of Byrne and Love for generalized Stieltjes transformation of functions are shown to be valid for a class of distributions. This is accomplished by transferring the complex inversion formulas on the testing function space of a class of distributions and then showing that the limiting process in the resulting formula converges in the topology of the testing function space. The main results of this chapter are contained in Theorems 4.1, 4.2, 5.1, 5.4 and 5.5. Theorems 5.1 and 5.5 are the extensions of the complex inversion theorems of Byrne and Love.

In Chapter II, initial value and final value theorems of distributional Stieltjes transformation defined in Chapter I, are proved. To prove the initial value theorem Lojasiewicz's definition of the value of a distribution at a point is used while final value theorem is proved by using the definition of the value of a distribution at infinity. This chapter is concluded by proving an initial value and a final value Abelian theorems for the

generalized Stieltjes transformation of functions considered by Byrne and Love. Theorems 2.1, 3.3, 4.1 and 4.2 contain the main results of this chapter.

Due to the wide applicability of various integral transformations in solving boundary value problems, a variety of integral transforms have been extended to the various classes of distributions. However, still there are some transformations which have not yielded to such generalizations. The Mehler-Fock transform is one of these. In Chapter III, a successful attempt is made to change this situation. This is achieved by defining a testing function space containing the kernel $P_{-\frac{1}{2}+i\tau}(chx)$ for each τ and then defining the transformation $f^*(\tau)$ of a distribution f by

$$f^*(\tau) = \langle f(x), P_{-\frac{1}{2}+i\tau}(chx) \rangle$$

The corresponding inversion formula is proved by establishing

$$\lim_{N \rightarrow \infty} \int_0^N \tau \tanh \pi \tau f^*(\tau) P_{-\frac{1}{2}+i\tau}(chx) \sinh \tau d\tau, \phi(\tau) \rangle = \langle f, \phi \rangle$$

In fact this is the main result of Chapter III. A big portion of this chapter is devoted to proving this theorem. Also, Theorems 3.2 and 3.3 contain results of some significance.

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INTRODUCTION

In the past twenty-five years, a variety of integral transformations have been extended to the various classes of distributions. The first successful attempt in this direction was made by L. Schwartz [18]. Schwartz extended the classical Fourier and Laplace transformations to certain classes of distributions. Later, in mid-1960's, A.H. Zemanian [24] extended Laplace, Mellin, Hankel, K, Weierstrass and convolution transformations to distributions. Several other names, in this area of research, are worth mentioning. Amongst these are Pandey, Koh, Lavoine, Benedetto, Cooper, Pathak and others.

The present dissertation consists of three chapters, each of which deals with a specific problem of a distributional transformation.

In Chapter I, two complex inversion theorems for Stieltjes transforms of functions established by Byrne and Love [2], have been extended to distributions. The work of Pandey and Zemanian [11], Pandey [14] and Pathak [15] on distributional Stieltjes transformation is worth mentioning. Pandey and Zemanian extended Widder's complex inversion theory to distributions while Pathak extended Sumner's [20] inversion formula to distributions. Chapter II deals with

the initial value and final value Abelian theorems for the distributional Stieltjes transformation defined in Chapter I. The initial value theorem for the distributional Stieltjes transformation has been proved in the sense of Lojasiewicz [10]. Here, I may point out that Misra [12] has attempted to extend this theorem but his proof suffers from a drawback, [12, last two lines of the page 595]. I have not only removed this deficiency but also have extended his result to a more general case of generalized Stieltjes transformation with a complex parameter p . The final value Abelian theorem for the distributional transformation has been proved in the sense of Lavoine and Misra [7]. Finally, this chapter has been concluded with the extension of initial value and final value Abelian theorems of Carmichael [3] to the generalized Stieltjes transformation considered by Byrne and Love [2].

In Chapter III I have extended Mehler-Fock transformation to distributions and have proved the inversion theorem in the distributional sense. Historically, Mehler-Fock transformation and its inversion formula appeared in a paper by Mehler [11] for the first time. The derivation used by Mehler is purely formal. Fock [5], in early forties made investigations as to the class of functions for which Mehler's inversion formula is valid while Lebedev [8] in

the late forties established Parseval's formula for this transformation. Vilenkin [22] and Rosenthal [17] gave a generalization of Mehler-Fock transformation independent of each other. In recent years, considerable interest has been displayed in the use of this transformation to the solution of boundary value problems in the mathematical theory of elasticity. The interested readers are referred to the work of Uflyand [21] and Sneddon [19].

The terminology and notation used in this dissertation is that of Zemanian [24]. I denotes the open interval $(0, \infty)$ and all testing functions herein are defined on I . The letters, x, y, t, θ have been used to denote the real variables in I while z denotes a complex variable. In Chapter III, the letters τ, ξ and η denote non-negative real parameters. In Chapters I and II, s and p denote complex numbers whereas σ and ω denote real numbers. The symbol $\langle f, \phi \rangle$ denotes a unique complex number assigned by the functional f to some element ϕ of its testing function space. Whenever I wish to emphasize the fact that a certain equation is a definition, I have used the symbol $\triangleq$ for equality. Sometimes, a distribution f has been denoted by $f(x)$. This is merely to indicate that the corresponding testing function space has x as its independent variable. The k th derivative of a distribution or a function is

denoted by $D^k f(x)$. As usual, Γx denotes the gamma function. Finally, $\mathcal{D}(I)$ is the space of infinitely differentiable complex valued functions with compact supports on I . The topology of $\mathcal{D}(I)$ is that which makes it dual $\mathcal{D}'(I)$, the space of Schwartz distributions [18].

A triple numbering system has been used for all definitions, theorems, lemmas, corollaries and equations. The first number in Roman numeral indicates the Chapter, second the section in which they appear while the third refers to the theorem, lemma or definition number. If the Roman numeral is omitted, then either the reference to the chapter is obvious or else it refers to the chapter in which it appears. The equation numbers are enclosed in the round brackets. For example, II-(4.2) refers to equation number (4.2) of Chapter II whereas Lemma III-1.5 refers to the Lemma 1.5 of Chapter III.

CHAPTER I

THE STIELTJES TRANSFORM OF DISTRIBUTIONS

1. INTRODUCTION

Let p be any complex number except zero and the negative integers.

Then for all s in the "cut plane", that is all complex numbers except those which are negative real or zero, the Stieltjes Transform in its general form is defined by:

$$F(s) = \int_0^{\infty} \frac{f(t) dt}{(s+t)^p} \quad (1.1)$$

The following inversion theorems for particular values of p and s are well known.

THEOREM 1.1 (Widder). If $f(t)$ belongs to $L(0, R)$ for every positive R and is such that the integral

$$F(x) = \int_0^{\infty} \frac{f(t) dt}{t+x}$$

converges for $x > 0$, then $F(s)$ exists for complex s in the cut plane and

$$\lim_{\eta \rightarrow 0+} \frac{F(-\xi - i\eta) - F(-\xi + i\eta)}{2\pi i} = \frac{f(\xi+) + f(\xi-)}{2}$$

for any positive ξ at which $f(\xi+)$ and $f(\xi-)$ both exist.

THEOREM 1.2 (Sumner). If $p > 0$, $f(t)$ is locally integrable in $[0, \infty]$, the improper Lebesgue integral.

$$F(s) = \int_0^{\infty} \frac{f(t) dt}{(s+t)^p}$$

converges (for a certain value of s in the cut plane and so for all), $t > 0$ and the limits $f(t \pm 0)$ exist, then:

$$\frac{1}{2} [f(t+0) + f(t-0)] = \lim_{\eta \rightarrow 0+} \frac{-1}{2\pi i} \int_0^t dx \int_{C_{\eta x}} (x+z)^{p-1} F_1(z) dz,$$

where $C_{\eta x}$ is a contour in the cut plane from $-x-i\eta$ to $-x+i\eta$.

THEOREM 1.3 (Byrne and Love). If $\text{Re } p > 1$, f is locally integrable in $[0, \infty)$, improper Lebesgue integral (1.1) converges, and $\lambda > 0$; then, for each positive x for which the Lebesgue limits $f(x \pm 0)$ exists,

$$\frac{1}{2} \{f(x+0) + f(x-0)\} = \lim_{\eta \rightarrow 0+} \frac{p-1}{2\pi i} \int_{-x}^{\lambda} (x+t)^{p-2} \{F(t-i\eta) - F(t+i\eta)\} dt. \quad [2, \text{p. 349}]$$

THEOREM 1.4 (Byrne and Love). If $\text{Re } p > 1$, $\frac{f(t)}{1+t^2} \in L(0, \infty)$ and the improper Lebesgue integral

$$F(s) = \int_0^{\infty} \frac{f(t)}{(s+t)^p} dt$$

converges, then for each positive x for which the Lebesgue limit $f(x \pm 0)$ exists,

$$\frac{1}{2} \{f(x+0) + f(x-0)\} = \lim_{\eta \rightarrow 0+} \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} \{F(t-i\eta) - F(t+i\eta)\} dt \quad [2, \text{p. 352}]$$

Theorem 1.1 has been extended to distributions by Pandey and Zemanian [13] and Pandey [14]. Theorem 1.2 was extended to distribution by Pathak [15]. Our object is to extend theorems 1.3 and 1.4 of Byrne and Love to generalized functions (distributions).

2. THE TESTING FUNCTION SPACE, $S_\alpha(I)$ AND ITS DUAL

An infinitely differentiable complex valued function $\phi(x)$ defined over $I=(0,\infty)$ belongs to the testing function space $S_\alpha(I)$ if,

$$\gamma_k(\phi) = \sup_{0 < x < \infty} \left| (1+x)^\alpha x^k \left(\frac{d}{dx} \right)^k \phi(x) \right| < \infty$$

for all $k=0,1,2,\dots$, where α is a fixed real number. Clearly, $S_\alpha(I)$ is a vector space with respect to the field of complex numbers. The zero element of the vector space $S_\alpha(I)$ is the function defined over I which is identically zero. The topology over $S_\alpha(I)$ is generated by the collection of seminorms $\{\gamma_k\}_{k=0}^\infty$. [24;p.8]. We say that a sequence $\{\phi_v\}$ where ϕ_v belongs to $S_\alpha(I)$ converges in $S_\alpha(I)$ to $\phi(x)$ if for each fixed k , $\gamma_k(\phi_v - \phi)$ tends to zero as v tends to ∞ . The space $S_\alpha(I)$ is a locally convex Hausdorff topological vector space. The space $\mathcal{D}(I)$ is a vector subspace of $S_\alpha(I)$ and the topology of $\mathcal{D}(I)$ is stronger than the topology induced on $\mathcal{D}(I)$ by $S_\alpha(I)$ and as such the restriction of any member of $S'_\alpha(I)$ to $\mathcal{D}(I)$ is in $\mathcal{D}'(I)$, where $S'_\alpha(I)$ and $\mathcal{D}'(I)$ denote the dual spaces of $S_\alpha(I)$ and $\mathcal{D}(I)$ respectively. We say that a sequence $\{\phi_v\}_{v=1}^\infty$ where $\phi_v(x)$ belongs to $S_\alpha(I)$ is a Cauchy sequence in $S_\alpha(I)$ if $\gamma_k(\phi_u - \phi_v)$ goes to zero for any non-negative integer k as u and v both tend to infinity independently of each other. It can be readily seen that $S_\alpha(I)$ is sequentially complete.

3. PRELIMINARY LEMMAS

We will now prove some lemmas which will be used in sequel.

LEMMA 3.1 [2, p. 331]. If p and z are complex numbers, $\mu = \operatorname{Re} p$ and $\nu = \operatorname{Im} p$, then

$$|z|^{\mu} e^{-\pi|\nu|} \leq |z^p| \leq |z|^{\mu} e^{\pi|\nu|} \quad (3.1)$$

Proof:

$$\begin{aligned} |z^p| &= |e^{p \ln z}| \\ &= e^{\mu \ln |z| - \nu \arg z} \\ &= |z|^{\mu} e^{-\nu \arg z} \\ &= |z|^{\mu} e^{-\nu \arg z} \end{aligned}$$

Hence, the lemma follows.

Notice that if p is real or z is positive, the exponential factor in (3.1) may be omitted and the equality holds.

LEMMA 3.2 [2, p. 333]. If p , a and b are complex, c and x are real, $c < x$ and $a+c$ and $b+c$ are neither zero nor negative real, then

$$\int_c^x \frac{(b+t)^{p-1}}{(a+t)^{p+1}} dt = \frac{1}{p(a-b)} \left(\frac{(b+x)^p}{(a+x)^p} - \frac{(b+c)^p}{(a+c)^p} \right)$$

Proof: We can write:

$$\frac{(b+t)^{p-1}}{(a+t)^{p+1}} = \frac{1}{p(a-b)} \frac{d}{dt} \frac{(b+t)^p}{(a+t)^p} \quad (3.2)$$

Observe that the left-hand side is continuous in $c \leq t \leq x$ because $a+t$ and $b+t$ are neither zero nor negative real there. Therefore, by integrating (3.2), we obtain the desired result.

LEMMA 3.3. If $\mu = \operatorname{Re} p > 0$, $v = \operatorname{Im} p$ and n, x and y are positive, then:

$$\begin{aligned} \left| \frac{x^p}{(y+in)^p} - 1 \right| &\leq |p| e^{2\pi|v|} \left| \frac{x}{y+in} - 1 \right|, \quad \text{if } \mu \geq 1 \text{ and } x \leq y \\ &\leq |p| e^{2\pi|v|} \left| \frac{x+in}{y+in} \right|^{\mu-1} \left| \frac{x}{y+in} - 1 \right|, \quad \text{if } \mu \geq 1 \text{ and } x \geq y \\ &\leq |p| e^{2\pi|v|} \left| \frac{x}{y+in} \right|^{\mu-1} \left| \frac{x}{y+in} - 1 \right|, \quad \text{if } \mu \leq 1 \text{ and } x \leq y \\ &\leq |p| e^{2\pi|v|} \left| \frac{y}{y+in} \right|^{\mu-1} \left| \frac{x}{y+in} - 1 \right|, \quad \text{if } \mu \leq 1 \text{ and } x \geq y. \end{aligned}$$

$$\begin{aligned} \text{Proof: } |x^p - (y+in)^p| &= \left| \int_{y+in}^x p z^{p-1} dz \right| \\ &\leq |x-y-in| |p| \max |z^{p-1}| \end{aligned}$$

where $z=(1-t)x+t(y+in)$ for $0 \leq t \leq 1$. Thus,

$$|x+t(y-x)| \leq |z| \leq \max(|x|, |y+in|)$$

so that

$$\min(x, y) \leq |z| \leq \max(|x|, |y+in|)$$

Let $\mu \geq 1$ and $y \geq x$. Then, $|z| \leq |y+in|$ and
 $|x^p - (y+in)^p| \leq |x-y-in| |p| |y+in|^{\mu-1} e^{\pi|v|}$

$$\left| \frac{x^p - (y+in)^p}{(y+in)^p} \right| \leq \frac{|x-y-in| |p|}{|y+in|^{\mu} e^{-\pi|v|}} |p| |y+in|^{\mu-1} e^{\pi|v|}$$

(ii) Let $\mu \geq 1$ and $y \leq x$. Then, $|z| \leq |x+in|$ and

$$\left| \frac{x^p - (y+in)^p}{(y+in)^p} \right| \leq \frac{|x-y-in|}{|y+in|^{\mu} e^{-\pi|v|}} |p| |x+in|^{\mu-1} e^{\pi|v|}$$

(iii) Let $\mu \leq 1$ and $y \geq x$. Then, $|z| \geq x$ and

$$\left| \frac{x^p - (y+in)^p}{(y+in)^p} \right| \leq \frac{|x-y-in|}{|y+in|^{\mu} e^{-\pi|v|}} |p| x^{\mu-1} e^{\pi|v|}$$

(iv) Let $\mu \leq 1$ and $y \leq x$. Then, $|z| \leq y$ and the result follows as in (iii).

LEMMA 3.4. If $0 < \operatorname{Re} p < \alpha < 1$, there is $H=H(p, \alpha)$ independent of x and ζ such that

$$\left| \frac{x^p}{\zeta^p} - 1 \right| \leq H \left| \frac{x}{\zeta} \right|^{\alpha-1} \left| \frac{x}{\zeta} - 1 \right|$$

for all real $x > 0$ and complex ζ not negative real or zero.

Proof: Let $z = \frac{x}{\zeta}$; then z varies in the complex plane cut along the closed negative real axis. The function

$$f(z) = \frac{z^p - 1}{z - 1} \cdot z^{1-\alpha}$$

is continuous in the cut plane except possibly at $z=1$; and it becomes continuous there also if we define $f(1)=p$.

Let $\mu = \operatorname{Re} p$ and $\nu = \operatorname{Im} p$. As $z \rightarrow 0$ in the cut plane,

$$|z^{p+1-\alpha} - z^{1-\alpha}| \leq |z|^{\mu+1-\alpha} e^{\pi|\nu|} + |z|^{1-\alpha} \rightarrow 0$$

and so $f(z) \rightarrow 0$. Thus, $f(z)$ is bounded in the neighbourhood of zero. And

$$f\left(\frac{1}{z}\right) = \frac{1-z^p}{1-z} \cdot z^{\alpha-p}$$

is similarly bounded in the neighbourhood of zero, so that $f(z)$ is bounded in the neighbourhood of infinity. Thus, $f(z)$ is bounded in the cut plane outside an annulus.

If $r > 0$ and $z \rightarrow -r$ in the upper half plane,

$$f(z) \rightarrow \frac{r^p e^{i\pi p} - 1}{r + 1} \cdot r^{1-\alpha} e^{-i\pi\alpha}; \text{ call this } f(-r).$$

Then, $f(z)$ so extended is continuous in the closed upper semi-annulus, and since this is compact, $f(z)$ is bounded on it.

Similarly, $f(z)$, with a different extension to the negative real axis is bounded on the closed lower semi-annulus.

Thus, $f(z)$ as defined without these extensions is bounded on the whole cut plane. This proves the lemma, H being any such bound to $|f(z)|$.

The hypothesis could be relaxed to $0 \leq \operatorname{Re} p \leq 1$ with very little change in the proof.

LEMMA 3.5. If $\alpha < 1$, $\operatorname{Re} p$, x and η are positive constants,

$$\omega_\lambda(y) = \frac{(x+\lambda)^{p-1}}{(y-x+i\eta)(y+\lambda+i\eta)^{p-1}}$$

and

$$\omega(y) = \frac{1}{y-x+i\eta},$$

then, $\omega_\lambda \rightarrow \omega$ in S_α as $\lambda \rightarrow \infty$.

Proof:

(i) Show that $\omega \in S_\alpha$.

For each non-negative integer k and $y > 0$,

$$|(1+y)^\alpha y^k \omega^{(k)}(y)| = \frac{k! (1+y)^\alpha y^k}{|y-x+i\eta|^{k+1}} = \frac{k! (1+y)^\alpha y^k}{\{(y-x)^2 + \eta^2\}^{\frac{1}{2}(k+1)}}$$

where $\omega^{(k)}(y) \equiv \frac{\partial^k}{\partial y^k} \omega(y)$. If $0 < y \leq 2x$, this is at most $k! (1+2x)^\alpha (2x)^k / \eta^{k+1}$, independent of y . If $y \geq 2x$, it is at most:

$$\frac{k! (1+y)^\alpha y^k}{\{(y-x)^2 + \eta^2\}^{\frac{1}{2}(k+1)}} = \frac{k! (1+y)^\alpha (2y)^{k+1}}{y (y^2 + 4\eta^2)^{\frac{1}{2}(k+1)}} \leq 2^{k+1} k! \frac{1+2x}{2x}$$

(ii) Have to show that $J_{\lambda,k}(y) \rightarrow 0$, uniformly on $y \geq 0$ as $\lambda \rightarrow \infty$ where

$$J_{\lambda,k}(y) = (1+y)^\alpha y^k \frac{d^k}{dy^k} \left\{ \frac{1}{y-x+in} \left(\frac{(x+\lambda)^{p-1}}{(y+\lambda+in)^{p-1}} - 1 \right) \right\}$$

It will be enough to suppose that $\alpha > 0$ since otherwise $(1+y)^\alpha \leq 1$. We may also suppose that $\lambda \geq 1$. By Leibnitz's rule,

$$\begin{aligned} J_{\lambda,k}(y) &= (1+y)^\alpha (-y)^k \frac{k!}{(y-x+in)^{k+1}} \left(\frac{x+\lambda}{(y+\lambda+in)^{p-1}} - 1 \right) + \\ &\quad (1+y)^\alpha (-y)^k \sum_{r=1}^k \binom{k}{r} \frac{(k-r)!}{(y-x+in)^{k-r+1}} \cdot \frac{(p-1)_r (x+\lambda)^{p-1}}{(y+\lambda+in)^{p-1+r}} \\ &= (1+y)^\alpha (-y)^k \sum_{r=0}^k \frac{k!}{r!} \frac{1}{(y-x+in)^{k-r+1}} \cdot \frac{(p-1)_r}{(y+\lambda+in)} \left(\frac{(x+\lambda)^{p-1}}{(y+\lambda+in)^{p-1}} - 1 \right) \\ &\quad + (1+y)^\alpha (-y)^k \sum_{r=1}^k \frac{k!}{r!} \frac{1}{(y-x+in)^{k-r+1}} \cdot \frac{(p-1)_r}{(y+\lambda+in)^r} \\ &= \sum_{r=0}^k (-1)^k k! \frac{(p-1)_r}{r!} L_{\lambda,k,r}(y) + \sum_{r=1}^k (-1)^k k! \frac{(p-1)_r}{r!} M_{\lambda,k,r}(y) \end{aligned}$$

We show that each of these $2k+1$ terms tend to 0, uniformly on $y \geq 0$, as $\lambda \rightarrow \infty$. We use Lemma 3.3 with $|p|e^{2\pi|v|}$ replaced by $K=K(p)$; and with p, x, y replaced by $p-1, x+\lambda, y+\lambda$. We repeatedly use $0 \leq r \leq k$.

(iii) Let $\mu = \operatorname{Re} p \geq 2$. Keeping everything fixed except y and λ ,

$$|L_{\lambda,k,r}(y)| \leq \frac{K(1+y)^\alpha y^k}{n^{k-r} |\lambda+in|^{r+1}} \max \left(1, \left| \frac{x+\lambda+in}{y+\lambda+in} \right|^{\mu-2} \right)$$

$$\text{So, } \sup_{0 \leq y \leq x} |L_{\lambda,k,r}(y)| \leq \frac{K(1+x)^\alpha x^k}{n^{k-r} |\lambda+in|^{r+1}} \left| \frac{x+\lambda+in}{\lambda+in} \right|^{\mu-2} \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

Again, for $y \geq x$

$$\begin{aligned} |L_{\lambda,k,r}(y)| &\leq K \left| \frac{y}{y-x+in} \right|^{k-r} \left| \frac{y}{y+\lambda+in} \right|^r \left| \frac{1+y}{y+\lambda+in} \right|^\alpha \frac{1}{|y+\lambda+in|^{1-\alpha}} \\ &\leq K \left(2 + \frac{x^2}{n^2} \right)^{k-r} \left| \frac{y+\lambda}{y+\lambda+in} \right|^{r+\alpha} \frac{1}{|x+\lambda+in|^{1-\alpha}} \end{aligned}$$

So that

$$\sup_{y \geq x} |L_{\lambda,k,r}(y)| \leq K \left(2 + \frac{x^2}{n^2} \right)^{k-r} \frac{1}{(\lambda+x)^{1-\alpha}} \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

These two results give convergence of $L_{\lambda,k,r}(y)$ to 0 uniformly on $y \geq 0$.

(iv) Let $1 < \mu < 2$. Again, by Lemma 3.3,

$$|L_{\lambda,k,r}(y)| \leq \frac{K(1+y)^\alpha y^k}{|y-x+in|^{k-r} |y+\lambda+in|^{r+1}} \cdot \frac{\max[(x+\lambda)^{\mu-2}, (y+\lambda)^{\mu-2}]}{|y+\lambda+in|^{\mu-2}} \quad (3.3)$$

For $y \leq x$,

$$|L_{\lambda,k,r}(y)| \leq \frac{K(1+x)^\alpha x^k}{n^{k-r}} \cdot \frac{(y+\lambda)^{\mu-2}}{|y+\lambda+in|^{r+\mu-1}}, \text{ since } \mu-2 < 0$$

$$\leq \frac{K(1+x)^{\alpha} x^k}{n^{k-r}} \frac{1}{(y+\lambda)^{r+1}}, \text{ since } r+\mu-1 > 0$$

$$\sup_{0 \leq y \leq x} |L_{\lambda, k, r}(y)| \leq \frac{K(1+x)^{\alpha} x^k}{n^{k-r} \lambda^{r+1}} \rightarrow 0 \text{ as } \lambda \rightarrow \infty, \text{ even if } r=0.$$

For $y \geq x$, we have as in (iii) but using the formula (3.3),

$$\begin{aligned} |L_{\lambda, k, r}(y)| &\leq K \left| \frac{y}{y-x+i\eta} \right|^{k-r} \left| \frac{y}{y+\lambda+i\eta} \right|^r \frac{(\lambda+y)^{\alpha} (\lambda+x)^{u-2}}{|y+\lambda+i\eta|^{\mu-1}} \\ &\leq K \left(2 + \frac{x^2}{\eta^2} \right)^{k-r} \left| \frac{y+\lambda}{y+\lambda+i\eta} \right|^r (\lambda+y)^{\alpha-\mu+1} (\lambda+x)^{u-2} \\ &\leq K \left(2 + \frac{x^2}{\eta^2} \right)^{k-r} (\lambda+x)^{\alpha-\mu+1+\mu-2} \text{ if } \mu-1 \geq \alpha, \end{aligned}$$

and so,

$$\sup_{y \geq x} |L_{\lambda, k, r}(y)| \leq K \left(2 + \frac{x^2}{\eta^2} \right)^{(k-r)} (\lambda+x)^{\alpha-1} \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

(v) If $0 < \mu-1 < \alpha < 1$, on the other hand, we replace the latter paragraph of (iv) by an application of Lemma 3.3, with p, x, ζ replaced by $p-1, x+\lambda$ and $y+\lambda+i\eta$. This gives us using $\lambda \geq 1$,

$$\begin{aligned} |L_{\lambda, k, r}(y)| &\leq \frac{(1+y)^{\alpha} y^k}{|y-x+i\eta|^{k-r+1} |y+\lambda+i\eta|^r} \cdot H \cdot \frac{(x+\lambda)^{\alpha-1} |x-y-i\eta|}{|y+\lambda+i\eta|^{\alpha}} \\ &\leq H \left| \frac{y}{y-x+i\eta} \right|^{k-r} \left| \frac{y}{y+\lambda+i\eta} \right|^r \left| \frac{y+\lambda}{y+\lambda+i\eta} \right|^{\alpha} (x+\lambda)^{\alpha-1} \end{aligned}$$

so that

$$\sup_{y \geq x} |L_{\lambda, k, r}(y)| \leq H \left(2 + \frac{x^2}{\eta^2}\right)^{k-r} \lambda^{\alpha-1} \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

(vi) Having shown in (iii), (iv) and (v) that $L_{\lambda, k, r}(y) \rightarrow 0$ uniformly on $y \geq 0$, it remains to show the same thing for $M_{\lambda, k, r}(y)$. But, now r is restricted to $1 \leq r \leq k$; the case $r=0$ does not occur, and this plays a vital part.

$$\begin{aligned} |M_{\lambda, k, r}(y)| &= \frac{(1+y)^\alpha y^k}{|y-x+i\eta|^{k-r+1} |y+\lambda+i\eta|^r} \\ &\leq \left| \frac{y}{y-x+i\eta} \right|^{k-r+1} \left| \frac{y}{y+\lambda+i\eta} \right|^{r-1} \frac{(y+\lambda)^\alpha}{|y+\lambda+i\eta|} \\ &\leq \left| \frac{y}{y-x+i\eta} \right|^{k-r+1} (y+\lambda)^{\alpha-1} \text{ for all } y \geq 0 \end{aligned}$$

and so

$$\sup_{y \geq 0} |M_{\lambda, k, r}(y)| \leq \left(2 + \frac{x^2}{\eta^2}\right)^{k-r+1} \lambda^{\alpha-1} \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

(vii) In (ii) and (v), we have shown in particular that $\omega_\lambda - \omega \in S_\alpha$. and in (i) we showed that $\omega \in S'_\alpha$. Consequently, $\omega_\lambda \in S'_\alpha$. These, together with the uniform convergence shown in (ii) to (vi), establish that $\omega_\lambda \rightarrow \omega$ in S_α completing the proof of Lemma 3.4.

LEMMA 3.6. For a complex s not negative real or zero, $\frac{1}{(s+x)^p}$ belongs to S_α where $\alpha \leq \operatorname{Re} p$.

Proof: Let $s = \sigma + i\omega$.

$$\left| (1+x)^\alpha x^k \left(\frac{d}{dx} \right)^k \left[\frac{1}{(s+x)^p} \right] \right| = \frac{(1+x)^\alpha x^k |^{(p)}k|}{|(s+x)^{p+k}|}$$

$$\leq \frac{(1+x)^\alpha x^k e^{\pi |\operatorname{Im} p|}}{[(\sigma+x)^2 + \omega^2]^{1/2(\operatorname{Re} p + k)}} \quad \text{by Lemma 3.1}$$

(i) If $\sigma > 0$, then for $0 < x \leq \sigma$

$$\frac{(1+x)^\alpha x^k}{[(\sigma+x)^2 + \omega^2]^{1/2(\operatorname{Re} p + k)}} \text{ is at most } \frac{(1+\sigma)^\alpha}{\sigma^{\operatorname{Re} p}} \text{ and}$$

for $x > \sigma$, this is bounded by $(1+\sigma)^{\alpha - \operatorname{Re} p} (1 + \frac{1}{\sigma})^{\operatorname{Re} p}$.

(ii) If $\sigma = 0$ then ω cannot be 0. For $0 < x < |\omega|$

$$\frac{(1+x)^\alpha x^k}{[x^2 + \omega^2]^{1/2(\operatorname{Re} p + k)}} \text{ is less than } \frac{(1+|\omega|)^\alpha}{|\omega|^{\operatorname{Re} p}} \text{ and for } x > |\omega|$$

it is bounded by $(1+|\omega|)^{\alpha - \operatorname{Re} p} (1 + 1/|\omega|)^{\operatorname{Re} p}$.

(iii) If $\sigma < 0$, write $\sigma = -\rho$ where ρ is a positive real number. For $0 < x \leq \rho$; $\rho < M$,

$$\frac{(1+x)^\alpha x^k}{[(x-\rho)^2 + \omega^2]^{1/2(\operatorname{Re} p + k)}} \text{ is at most } \frac{(1+\rho)^\alpha}{|\omega|^{\operatorname{Re} p}} \text{ and}$$

for $x > M > p$, this is bounded by $((1+M)^\alpha) / (M^{\operatorname{Re} p} (1 - \frac{\rho}{M})^{\operatorname{Re} p + K})$.

In (i) to (iii), we have shown that $\gamma_k[1/(s+x)^p]$ is finite.

This completes the proof of lemma.

COROLLARY 3.7. If m and k both assume non-negative integral values and Ω is a compact set of the complex plane not meeting the negative real axis, then for fixed non-negative integers m and k , there exists a constant B_Ω satisfying

$$\gamma_m \left[\frac{1}{(s+x)^{p+k}} \right] \leq B_\Omega < \infty$$

uniformly for all s lying in the compact set Ω of the complex plane not meeting the negative real axis on the origin.

Proof: Using the compactness of the set Ω , the corollary is immediate from the previous lemma.

4. THE DISTRIBUTIONAL STIELTJES TRANSFORMATION

Let $\alpha < \operatorname{Re} p$. Then, the distributional Stieltjes transformation $F(s)$ of an arbitrary element $f \in S'_\alpha$ is defined by

$$F(s) = \mathbb{A} \left\langle f(x), \frac{1}{(s+x)^p} \right\rangle \quad (4.1)$$

where s belongs to the complex plane cut along the negative real axis including the origin.

THEOREM 4.1. For an arbitrary $f \in S'_\alpha$ and $\alpha < \operatorname{Re} p$ let $F(s)$ be defined by equation (4.1). Then, for $m=1, 2, \dots$

$$\left(\frac{d}{ds}\right)^m F(s) = \langle f(x), \frac{(-1)^m (p)_m}{(s+x)^{p+m}} \rangle \quad (4.2)$$

where $(p)_m = p(p+1)(p+2) \dots (p+m-1)$.

Proof: If p is such that s^p does not have a branch cut in the complex plane, the proof can be given in a way similar to that given in [13, Lemma 2a]. If p is such that s^p has a branch cut (along the negative real axis for the sake of definiteness), then choose the contour of integration as shown in the complex-plane cut along the negative real axis below.

Here, C_1 and C_2 are arcs of two concentric circles with centre at origin and C is the contour of integration as shown. The radii of C_1 and C_2 and paths L_1 and L_2 are so chosen that the point s is contained in the region bounded by the contour. Let $d = \inf_{\zeta \in C} |\zeta - s|$ and choose $|\Delta s| < \frac{d}{2}$.

Now

$$\begin{aligned} \frac{F(s+\Delta s) - F(s)}{\Delta s} &= \langle f(t), \frac{-p}{(s+t)^{p+1}} \rangle \\ &= \langle f(t), \frac{1}{\Delta s} \left\{ \frac{1}{(s+\Delta s+t)^p} - \frac{1}{(s+t)^p} \right\} + \frac{p}{(s+t)^{p+1}} \rangle \\ &= \langle f(t), \frac{1}{\Delta s} \cdot \frac{1}{2\pi i} \cdot \int_C \frac{1}{(z+t)^p} \left[\frac{1}{z-s-\Delta s} - \frac{1}{z-s} - \frac{\Delta s}{(z-s)^2} \right] dz \rangle \end{aligned}$$

where C is the contour as shown in the diagram

$$\begin{aligned} &= \langle f(t), \frac{\Delta s}{2\pi i} \int_C \frac{1}{(z+t)^p} \frac{1}{(z-s)^2(z-s-\Delta s)} dz \rangle \\ &= \langle f(t), \theta_{\Delta s} \rangle \end{aligned} \tag{4.3}$$

where

$$\theta_{\Delta s} = \frac{\Delta s}{2\pi i} \int_C \frac{1}{(z+t)^p} \frac{1}{(z-s)^2(z-s-\Delta s)} dz$$

We now wish to show that $\theta_{\Delta s} \rightarrow 0$ in $S_a(I)$ as $\Delta s \rightarrow 0$.

$$(1+t)^{\alpha} t^k \left(\frac{d}{dt}\right)^k \theta_{\Delta s} = \frac{\Delta s}{2\pi i} (-1)^k \int_C \frac{(1+t)^{\alpha} t^k (p)_k}{(z+t)^{p+k}} \cdot \frac{1}{(z-s)^2(z-s-\Delta s)} dz$$

Using Corollary 3.6, we have

$$r_k(\theta_{\Delta s}) \leq B_c \frac{|\Delta s|}{2\pi} \cdot \frac{L}{d^3}, \quad (4.4)$$

where L is the length of the contour C and B_c is the uniform bound of

$$\left| \frac{(1+t)^{\alpha} t^k}{(z+t)^{p+k}} \right| \text{ for all } z \text{ lying on the closed contour } C \text{ and } t > 0.$$

Letting $\Delta s \rightarrow 0$ in (4.3) and using (4.4), we get

$$F'(s) = \langle f(t), \frac{-p}{(s+t)^{p+1}} \rangle$$

We now use induction on the order of derivative of $F(s)$. Assume that the equation (4.2) holds for any m , say $m=n$. Then, as before, we can write:

$$\frac{1}{\Delta s} [F^{(n)}(s+\Delta s) - F^{(n)}(s)] = \langle f(t), \frac{(-1)^{n+1} (p)_{n+1}}{(s+t)^{p+n+1}} \rangle$$

$$= \langle f(t), \theta_{n, \Delta(s)}(t) \rangle$$

where

$$\theta_{n,\Delta s}(t) = \frac{(-1)^n (p)_n \Delta s}{2\pi i} \int_C \frac{1}{(z+t)^{p+n}} \frac{1}{(z-s)^2 (z-s-\Delta s)} dz$$

Now, using Corollary 3.7, it can be easily seen that

$$\gamma_k(\theta_{n,\Delta s}^{(t)}) \leq B'_C \cdot \frac{\Delta s}{\pi} \frac{L}{d^3} \rightarrow 0 \text{ as } \Delta s \rightarrow 0$$

where B'_C is the uniform bound of

$$\left| \frac{(p)_n (1+t)^{\alpha_k} t^k}{(z+t)^{p+n+k}} \right| \text{ for all } z \text{ on the closed contour } C.$$

This proves the theorem.

THEOREM 4.2. The function $F^{(m)}(x)$ for real x where $F(s)$ is the Stieltjes transform of $f \in S'_\alpha$, satisfies the following relation:

$$F^{(m)}(x) = \begin{cases} O[x^{-m}] \text{ as } x \rightarrow \infty & \text{if } \alpha < \operatorname{Re} p \\ O[x^{-k}] \text{ as } x \rightarrow \infty & \text{if } \alpha = \operatorname{Re} p \\ O[x^{-k-\operatorname{Re} p}] \text{ as } x \rightarrow 0 & \text{if } \alpha \leq \operatorname{Re} p \end{cases}$$

Proof: By the boundedness property of distributions [24, p. 18], there exists a constant C and a non-negative integer n such that:

$$\begin{aligned} \left| \langle f(t), \frac{(-1)^m (p)_m}{(s+t)^{p+m}} \rangle \right| &\leq C \max_{0 \leq k \leq n} \gamma_k \left[\frac{(-1)^m (p)_m}{(s+t)^{p+m}} \right] \\ &\leq C \max_{0 \leq k \leq n} \sup_{0 < t < \infty} \frac{|(1+t)^{\alpha_k} t^k (p)_{m+k} e^{\pi |\operatorname{Im} p|}|}{|s+t|^{\operatorname{Re} p + m + k}} \end{aligned}$$

Therefore, for x real,

$$|F^{(m)}(x)| \leq A \max_{0 \leq k \leq n} \sup_{0 < t < \infty} \frac{(1+t)^{\alpha} t^k}{|x+t|^{\operatorname{Re} p + k + m}}$$

where A is positive constant. As $x \rightarrow \infty$, we can write:

$$\begin{aligned} |F^{(m)}(x)| &\leq A \max_{0 \leq k \leq n} \frac{(1+x)^{\alpha} x^k}{x^{\operatorname{Re} p + k + m}} \\ &\leq A (1+x)^{\alpha - \operatorname{Re} p} \left(1 + \frac{1}{x}\right)^{\operatorname{Re} p} x^{-m} \end{aligned}$$

Therefore,

$$|F^{(m)}(x)| = O[x^{-m}] \text{ as } x \rightarrow \infty \text{ for } \alpha \leq \operatorname{Re} p.$$

Similarly, as $x \rightarrow 0+$, we obtain:

$$|F^{(m)}(x)| = O[x^{-\operatorname{Re} p - m}] \text{ for } \alpha \leq \operatorname{Re} p.$$

5. COMPLEX INVERSION THEOREMS

We are now ready to prove our first inversion theorem.

THEOREM 5.1. For a fixed $\alpha < 1$ and $\operatorname{Re} p > 1$, let $f \in S_{\alpha}^1(I)$ and let $F(s)$ be the Stieltjes transform of $f(t)$ as defined by (4.1). Then,

$$\begin{aligned} \lim_{\eta \rightarrow 0} &< \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} [F(t-i\eta) - F(t+i\eta)] dt, \phi(x) > \\ &= \langle f, \phi \rangle \quad \text{for all } \phi \in \mathcal{D}(I) \end{aligned}$$

Proof: First consider

$$\int_{-x}^{\lambda} (x+t)^{p-2} F(t-in) dt = \int_{-x}^{\lambda} (x+t)^{p-2} \langle f(y), \frac{1}{(y+t-in)^p} \rangle dt \quad (5.1)$$

For fixed x and t ,

$$\frac{(x+t)^{p-2}}{(y+t-in)^p} \in S_{\alpha}(I).$$

Indeed,

$$\left| (1+y)^{\alpha} y^k \left(\frac{d}{dy} \right)^k \left(\frac{(x+t)^{p-2}}{(y+t-in)^p} \right) \right| = \left| (x+t)^{p-2} (p)_k \frac{(1+y)^{\alpha} y^k}{(y+t-in)^{p+k}} \right|$$

For $y \geq 1$, this is at most $2^{\alpha} (x+t)^{\operatorname{Re} p-2} |(p)_k| e^{2\pi |\operatorname{Im} p|}$ and for $0 < y \leq 1$ it is bounded by

$$|(p)_k| (x+t)^{\operatorname{Re} p-2} \cdot \frac{e^{2\pi |\operatorname{Im} p|}}{(t^2+n^2)^{\frac{1}{2}} (\operatorname{Re} p+k)}$$

Since, in view of Theorem 4.1, $F(s)$ is analytic in the cut plane, the left-hand side integral in (5.1) is meaningful. By using the technique of Riemann sums it can be shown that for $\epsilon > 0$,

$$\begin{aligned} & \int_{-x+\epsilon}^{\lambda} (x+t)^{p-2} F(t-in) dt \\ &= \langle f(y), \int_{-x+\epsilon}^{\lambda} \frac{(x+t)^{p-2}}{(y+t-in)^p} dt \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle f(y), \frac{1}{p-1} \cdot \frac{1}{(x-y+i\eta)} \left[\frac{(x+\lambda)^{p-1}}{(y+\lambda-i\eta)^{p-1}} - \frac{\epsilon^{p-1}}{(y-x+\epsilon-i\eta)^{p-1}} \right] \rangle \\
&\quad \text{(by Lemma 3.2)} \\
&= I \text{ (say).}
\end{aligned}$$

One can easily check that as $\epsilon \rightarrow 0+$, $\frac{\epsilon^{p-1}}{(y-x+\epsilon-i\eta)^{p-1}} \rightarrow 0$ in $S_\alpha(I)$ for $\operatorname{Re} p > 1 > \alpha$ and for fixed λ , η and x . Therefore, letting $\epsilon \rightarrow 0$, we get:

$$\int_{-x}^{\lambda} (x+t)^{p-2} F(t-i\eta) dt = \langle f(y), \frac{(\lambda+x)^{p-1}}{(p-1)(y-x-i\eta)(y+\lambda-i\eta)^{p-1}} \rangle \quad (5.2)$$

In view of Lemma 3.5, it follows that as $\lambda \rightarrow \infty$

$$\frac{1}{(y-x-i\eta)} \left(\frac{\lambda+x}{(y+\lambda-i\eta)} \right)^{p-1} \rightarrow \frac{1}{y-x-i\eta} \text{ in } S_\alpha(I) \quad (5.3)$$

for fixed x and η .

Therefore, letting $\lambda \rightarrow \infty$ in (5.2), we obtain

$$\int_{-x}^{\infty} (x+t)^{p-2} F(t-i\eta) dt = \langle f(y), \frac{1}{p-1} \cdot \frac{1}{y-x-i\eta} \rangle \quad (5.4)$$

Using a similar argument, we can show that

$$\int_{-x}^{\infty} (x+t)^{p-2} F(t+i\eta) dt = \langle f(y), \frac{1}{p-1} \cdot \frac{1}{y-x+i\eta} \rangle \quad (5.5)$$

Combining Equations (5.4) and (5.5), we get

$$\frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} [F(t-i\eta) - F(t+i\eta)] dt$$

$$\begin{aligned}
&= \langle f(y), \frac{1}{2\pi i} \left(\frac{1}{y-x-in} - \frac{1}{y-x+i\eta} \right) \rangle \\
&= \langle f(y), \frac{\eta}{\pi[(y-x)^2 + \eta^2]} \rangle
\end{aligned} \tag{5.6}$$

Now using the technique of Riemann sums, we obtain

$$\begin{aligned}
&\langle \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} [F(t-in) - F(t+i\eta)] dt, \phi(x) \rangle \\
&= \langle f(y), \int_a^b \frac{\eta \phi(x) dx}{\pi[(y-x)^2 + \eta^2]} \rangle
\end{aligned} \tag{5.7}$$

where the support of $\phi(x) \in \mathcal{D}(I)$ is contained in (a, b) , $b > a > 0$. Using the same technique as followed in proving Theorem 2 of [14] one can show that

$$\frac{1}{\pi} \int_a^b \frac{\eta \phi(x) dx}{(y-x)^2 + \eta^2} \rightarrow \phi(y) \tag{5.8}$$

in the topology of $S'_\alpha(I)$ as $\eta \rightarrow 0+$. Therefore, letting $\eta \rightarrow 0+$ in (5.7), we have

$$\lim_{\eta \rightarrow 0+} \langle \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} [F(t-in) - F(t+i\eta)] dt, \phi(x) \rangle = \langle f, \phi \rangle$$

This completes the proof of the theorem.

To prove our other inversion theorems, we require a couple of Lemmas.

LEMMA 5.2. Let $t, x, \eta > 0$. Then, for finite $b > a > 0$ and $\alpha < 1$,

$$\lim_{\eta \rightarrow 0+} (1+x)^\alpha \eta \int_a^b \frac{|t-x|}{(t-x)^2 + \eta^2} dt = 0$$

uniformly for all $x > 0$.

Proof: Let

$$I = (1+x)^\alpha \eta \int_a^b \frac{|t-x|}{(t-x)^2 + \eta^2} dt$$

Since $\sup_{\substack{x > N > b \\ a \leq t \leq b}} \left| (1+x)^\alpha \frac{(t-x)}{(t-x)^2 + \eta^2} \right|$ is bounded, for $\epsilon > 0$, there exists a

positive $N > b$ and $0 < q < 1$ such that

$$|I| < \epsilon \tag{5.9}$$

uniformly for all $\eta \in (0, q)$ and $x > N$.

Now assume that δ is a positive number $\leq \min(1, \frac{a}{2})$ and for $\delta \leq x \leq N$ let us write.

$$I = (1+x)^\alpha \eta \left(\int_a^{a+x-\delta} + \int_{a+x-\delta}^{a+x+\delta} + \int_{a+x+\delta}^b \right) \frac{|t-x|}{(t-x)^2 + \eta^2} dt. \tag{5.10}$$

Denote the three expressions on the right-hand side of Eqn. (5.10) by I_1 , I_2 and I_3 respectively.

Now

$$I_2 = n \int_{a+x-\delta}^{a+x+\delta} \frac{|t-x| (1+x)^\alpha}{(t-x)^2 + \eta^2} dt$$

$$\leq n \cdot 2\delta \frac{1}{2\eta} (1+N)^\alpha = \delta (1+N)^\alpha$$

Now choose δ such that $\delta (1+N)^\alpha < \varepsilon$ and fix δ this way. Therefore

$$|I_2| < \varepsilon \quad (5.11)$$

uniformly for all $x \in [\delta, N]$ and $\eta \in (0, q]$.

$$I_3 = n \int_{a+x+\delta}^b \frac{(1+x)^\alpha |t-x|}{(t-x)^2 + \eta^2} dt = (1+x)^\alpha \eta \left[\int_{a+x+\delta}^x + \int_x^b \right] \frac{|t-x|}{(t-x)^2 + \eta^2} dt$$

$$= \frac{n}{2} \ln \frac{(b-x)^2 + \eta^2}{(a+\delta)^2 + \eta^2} (1+x)^\alpha, \quad \text{if } \delta < x < b$$

$$= (1+x)^\alpha \frac{n}{2} \ln \frac{\eta^4}{[(a+\delta)^2 + \eta^2][(b-x)^2 + \eta^2]} \quad \text{if } x > b.$$

Therefore, $I_3 \rightarrow 0$ as $\eta \rightarrow 0+$

(5.12)

uniformly for all $x \in [\delta, N]$.

Next,

$$I_1 = (1+x)^\alpha \eta \int_a^{a+x-\delta} \frac{|t-x| dt}{(t-x)^2 + \eta^2}$$

Now for $\delta \leq x \leq a$

$$I_1 = (1+x)^\alpha \frac{n}{2} \ln \left(\frac{(a-\delta)^2 + \eta^2}{(a-x)^2 + \eta^2} \right) \rightarrow 0 \quad \text{as } \eta \rightarrow 0+ \quad (5.13)$$

uniformly for all x lying in $[\delta, a]$.

For $a \leq x \leq N$,

$$\begin{aligned}
 I_1 &= - \left[\int_a^x \frac{t-x}{(t-x)^2 + \eta^2} dt + \int_x^{a+x-\delta} \eta \frac{t-x}{(t-x)^2 + \eta^2} dt \right] (1+x)^\alpha \\
 &= \left[-\frac{\eta}{2} \ln \frac{\eta^2}{(a-x)^2 + \eta^2} + \frac{\eta}{2} \ln \frac{(a-\delta)^2 + \eta^2}{\eta^2} \right] (1+x)^\alpha \\
 &\rightarrow 0 \quad \text{as } \eta \rightarrow 0+ \text{ uniformly for all } x \in [a, N]
 \end{aligned} \tag{5.14}$$

Combining results (5.11) through (5.14) we have

$$\lim_{\eta \rightarrow 0+} |I| \leq \epsilon \tag{5.15}$$

Now, for $0 < x \leq \delta$, we can see that

$$\begin{aligned}
 |I| &\leq \eta \int_a^b \frac{(t-x)(1+\delta)^\alpha}{(t-x)^2 + \eta^2} dt \\
 &\leq (1+\delta)^\alpha \eta \ln \left[\frac{(b-x)^2 + \eta^2}{(a-x)^2 + \eta^2} \right] \rightarrow 0
 \end{aligned} \tag{5.16}$$

as $\eta \rightarrow 0+$ uniformly for all $x \in [0, \delta]$.

Combining results (5.15) and (5.16), we get

$$\lim_{\eta \rightarrow 0+} |I| \leq \epsilon \quad \text{uniformly for all } x > 0.$$

Thus, the proof of the lemma is complete.

LEMMA 5.3. Let $\operatorname{Re} p > 1 > \alpha$. Assume that t, x, λ and η are all positive numbers and $\phi(t) \in \mathcal{D}(I)$. Then,

$$I(x, t) \triangleq \frac{1}{2\pi i} \int_a^b \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] - \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \right\} \phi(t) dt \rightarrow 0$$

as $\eta \rightarrow 0+$ in the topology of S_α where the support of $\phi(t) \in \mathcal{D}(I)$ is contained in (a, b) ; $b > a > 0$.

Proof: We have to show that for each $m=0, 1, 2, \dots$ $(1+x)^\alpha x^m D_x^m I(x, \eta) \rightarrow 0$ as $\eta \rightarrow 0$ uniformly.

It can be easily shown that

$$D_x^m I(x, \eta) = O\left(\frac{1}{x^{m+1}}\right) \quad \text{as } x \rightarrow \infty$$

uniformly for all η satisfying $0 < \eta < 1$. So,

$$(1+x)^\alpha x^m D_x^m I(x, \eta) \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

uniformly for all $\eta \in (0, 1]$. Therefore, for $\epsilon > 0$ there exists $N > 0$ such that

$$|(1+x)^{\alpha} x^m D_x^m(x, \eta)| < \epsilon \quad (5.18)$$

uniformly for all $x \geq N$, and $0 < \eta < 1$.

Now consider the case $0 < x \leq N$. We will first give the proof for $m=0$ and complete the proof for $m=1, 2, 3, \dots$ by using the result for the case $m=0$.

I. The case $m=0$. Write

$$\begin{aligned} (1+x)^{\alpha} I(x, \eta) &= \frac{(1+x)^{\alpha}}{2\pi i} \int_a^b \frac{x-t}{(x-t)^2 + \eta^2} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - \left(\frac{\lambda+t}{\lambda+x-i\eta} \right)^{p-1} \right] \phi(t) dt \\ &\quad - (1+x)^{\alpha} \cdot \frac{\eta}{2\pi} \int_a^b \frac{\phi(t)}{(x-t)^2 + \eta^2} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 + \left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \phi(t) dt \\ &= I_1(x, \eta) - I_2(x, \eta) \quad (\text{say}) \end{aligned}$$

First, consider $I_1(x, \eta)$. Since

$$\begin{aligned} \left| \frac{1}{(x+\lambda+i\eta)^{p-1}} - \frac{1}{(x+\lambda-i\eta)^{p-1}} \right| &= \left| -(p-1) \int_{x+\lambda-i\eta}^{x+\lambda+i\eta} z^{-p} dz \right| \\ &\leq 2|p-1| \eta \frac{e^{\pi |\operatorname{Im} p|}}{(x+\lambda)^{\operatorname{Re} p}} \\ &\leq 2|p-1| \eta e^{\pi |\operatorname{Im} p| / \lambda} \lambda^{\operatorname{Re} p}. \end{aligned}$$

Therefore, we can find a constant $B(p)$ independent of x and η such that

$$|I_1(x, \eta)| \leq B (1+x)^\alpha \eta \int_a^b \frac{|x-t| dt}{(x+t)^{2+\eta^2}}$$

In view of Lemma 5.2, the right-hand side converges to 0 uniformly for all $x > 0$ as $\eta \rightarrow 0+$.

We now consider $I_2(x, \eta)$. For $0 < \eta < 1$ and $a \leq t \leq b$, using Lemma 3.3, we get

(i) For $\operatorname{Re} p \geq 2$, $t \leq x$

$$\begin{aligned} \left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| &\leq |p-1| e^{2\pi |\operatorname{Im} p|} \frac{|t-x|+\eta}{\sqrt{(x+\lambda)^2+\eta^2}}, \quad t \leq x \\ &\leq |p-1| e^{2\pi |\operatorname{Im} p|} \frac{|t-x|+\eta}{\lambda}, \end{aligned} \quad (5.18)$$

(ii) For $\operatorname{Re} p \geq 2$, $t \geq x$

$$\begin{aligned} \left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| &\leq |p-1| e^{2\pi |\operatorname{Im} p|} \left| \frac{t+\lambda+i\eta}{x+\lambda+i\eta} \right|^{\operatorname{Re} p-2} \frac{|t-x|+\eta}{\sqrt{(\lambda+x)^2+\eta^2}} \\ &\leq \frac{|p-1| e^{2\pi |\operatorname{Im} p|}}{\lambda^{\operatorname{Re} p-1}} (b+\lambda+1)^{\operatorname{Re} p-2} (|t-x|+\eta); \quad 0 < \eta < 1 \end{aligned} \quad (5.19)$$

(iii) For $\operatorname{Re} p \leq 2$, $t \leq x$ and $t \in [a, b]$, $b > a > 0$, $0 < \eta < 1$.

$$\begin{aligned} \left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| &\leq |p-1| e^{2\pi |\operatorname{Im} p|} \left| \frac{t+\lambda}{x+\lambda+i\eta} \right|^{\operatorname{Re} p-2} \frac{|t-x|+\eta}{\sqrt{(x+\lambda)^2+\eta^2}} \\ &\leq \frac{|p-1| e^{2\pi |\operatorname{Im} p|} (a+\lambda)^{\operatorname{Re} p-2}}{\lambda^{\operatorname{Re} p-1} \left(1 + \frac{1}{\lambda^2} \right)^{\operatorname{Re} (p-2)/2}} \cdot \frac{(|t-x|+\eta)}{\left(1 + \frac{b}{\lambda} \right)^{\operatorname{Re} p-2}} \end{aligned} \quad (5.20)$$

(iv) For $\operatorname{Re} p \leq 2$, $t \geq x$, $t \in [a, b]$, $b > a > 0$, $0 < \eta < 1$

$$\begin{aligned} \left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| &\leq |p-1| e^{2\pi |\operatorname{Im} p|} \left| \frac{x+\lambda}{x+\lambda+i\eta} \right|^{\operatorname{Re} p-2} \frac{|t-x|+\eta}{\sqrt{(x+\lambda)^2+\eta^2}} \\ &\leq \frac{|p-1| e^{2\pi |\operatorname{Im} p|}}{\lambda} \left[1 + \frac{1}{\lambda^2} \right]^{2-\operatorname{Re} p} (|t-x|+\eta) \quad (5.22) \end{aligned}$$

Therefore, from inequalities (5.19) through (5.22), it is evident that for $\operatorname{Re} p > 1$ and $0 < \eta < 1$, there exists a positive constant $K(\lambda, p)$ independent of t and x satisfying

$$\left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| \leq K(\lambda, p) [(t-x)+\eta] \quad t \in (a, b)$$

Similarly, under the same set of conditions

$$\left| \left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right| \leq K(\lambda, p) [(t-x)+\eta] \quad t \in (a, b)$$

In view of these inequalities we, therefore, have established that

$$|I_2(x, \eta)| \leq \frac{K(\lambda, p) M(1+x)^{\alpha}}{\pi} \int_a^b \frac{\eta (|t-x|+\eta) dt}{(t-x)^2+\eta^2}$$

where $M = \sup_{a \leq t \leq b} |\phi(t)|$

That is

$$|I_2(x, \eta)| \leq \frac{K(\lambda, p)M}{\pi} (1+x)^\alpha \left\{ \eta \int_a^b \frac{|t-x| dt}{(t-x)^2 + \eta^2} + \eta^2 \int_a^b \frac{dt}{(t-x)^2 + \eta^2} \right\}$$

It is a very simple exercise to show that

$$(1+x)^\alpha \eta^2 \int_a^b \frac{dt}{(t-x)^2 + \eta^2} \rightarrow 0$$

as $\eta \rightarrow 0+$ uniformly for all $x > 0$. Therefore, in view of Lemma 5.2, it follows that

$$I_2(x, \eta) \rightarrow 0 \text{ as } \eta \rightarrow 0+ \text{ uniformly for all } x > 0.$$

Thus, we have completed the proof of the fact that

$$(1+x)^\alpha I(x, \eta) \rightarrow 0 \text{ as } \eta \rightarrow 0+ \text{ uniformly for all } x > 0.$$

II) The case $m=1, 2, 3, \dots$

A careful computation along with integration by parts will show that

$$\begin{aligned} (1+x)^\alpha x D_x I(x, \eta) &= \frac{(1+x)^\alpha}{2\pi i} x \int_a^b \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] \right. \\ &\quad \left. - \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \right\} \phi(t) dt \\ &\quad + \frac{(p-1)}{2\pi i} (1+x)^\alpha x \int_a^b [(x+\lambda+i\eta)^{-p} - (x+\lambda-i\eta)^{-p}] (t+\lambda)^{p-2} \phi(t) dt \end{aligned}$$

Using the technique of induction, we obtain:

$$\begin{aligned}
(1+x)^{\alpha} x^m D_x^m I(x, \eta) &= \frac{(1+x)^{\alpha} x^m}{2\pi i} \int_a^b \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] \right. \\
&\quad \left. - \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \right\} \phi^{(m)}(t) dt \\
&\quad + \frac{(1+x)^{\alpha} x^m}{2\pi i} \int_a^b [(x+\lambda+i\eta)^{-p} - (x+\lambda-i\eta)^{-p}] (t+\lambda)^{p-2} \phi^{(m-1)}(t) dt \\
&\quad - \frac{p(p-1)}{2\pi i} (1+x)^{\alpha} x^m \int_a^b [(x+\lambda+i\eta)^{-p-1} - (x+\lambda-i\eta)^{-p-1}] (t+\lambda)^{p-2} \\
&\quad \phi^{(m-2)}(t) dt \\
&\quad + \dots \dots \dots \\
&\quad (-1)^{m-1} \frac{p(p-1) \dots (p+m-2)}{2\pi i} (1+x)^{\alpha} x^m \\
&\quad \int_a^b [(x+\lambda+i\eta)^{-p-m+1} - (x+\lambda-i\eta)^{-p-m+1}] (t+\lambda)^{p-2} \phi(t) dt \quad (5.22)
\end{aligned}$$

Denote the integrals on the right-hand side of (5.22) by $J_1, J_2, \dots, J_{m+1}$ in that order. In view of case $m=0$, $J_1 \rightarrow 0$ as $\eta \rightarrow 0+$ uniformly for all $x \in (0, N]$. To show that other integrals converge to 0 as $\eta \rightarrow 0+$ uniformly for $x \in (0, N]$, we consider the most general integral J_{m+1} . As before,

$$\begin{aligned}
|(x+\lambda+i\eta)^{-p-m+1} - (x+\lambda-i\eta)^{-p-m+1}| &= |(-p-m+1) \int_{x+\lambda-i\eta}^{x+\lambda+i\eta} z^{-p-m} dz| \\
&\leq 2\eta |p+m-1| \frac{e^{\pi |\operatorname{Im} p|}}{(x+\lambda)^{\operatorname{Re} p+m}}
\end{aligned}$$

Therefore, we can find a positive constant C independent of x and η such that

$$|J_{m+1}| \leq \eta C \frac{(1+x)^{\alpha} x^m}{(x+\lambda)^{\operatorname{Re} p+m}} \rightarrow 0 \text{ as } \eta \rightarrow 0+$$

uniformly for all $x \in (0, N]$ and each fixed $m=1, 2, \dots$

Thus, we have proved that

$$(1+x)^{\alpha} x^m D_x^m I(x, \eta) \rightarrow 0 \text{ as } \eta \rightarrow 0+ \text{ uniformly for all } x \in (0, N] \text{ and}$$

each fixed $m=0, 1, 2, 3, \dots$

Combining this fact with inequality (5.17), we have

$$\lim_{\eta \rightarrow 0+} |(1+x)^{\alpha} x^m D_x^m I(x, \eta)| < \epsilon$$

uniformly for all $x > 0$ and each fixed $m=0, 1, 2, \dots$; since ϵ is arbitrary our claim is established.

THEOREM 5.4. Let $\operatorname{Re} p > 1$ and $f(t) \in S'_{\alpha}$. If $F(s)$ is the Stieltjes transform of $f(t)$ defined by (4.1) then for $\lambda > 0$ and each $\phi \in \mathcal{D}(I)$

$$\lim_{\eta \rightarrow 0+} \left\langle \frac{p-1}{2\pi i} \int_{\lambda}^{\infty} [F(y-i\eta) - F(y+i\eta)] (y+t)^{p-2} dy, \phi(t) \right\rangle = 0$$

Proof; By using the same technique as used in proving Theorem 5.1, it can be shown that:

$$\begin{aligned} & \left\langle \frac{p-1}{2\pi i} \int_{\lambda}^{\infty} [F(y-i\eta) - F(y+i\eta)] (y+t)^{p-2} dy, \phi(t) \right\rangle \\ &= \int_a^b \left\langle f(x), \frac{1}{2\pi i} \left\{ \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] \right. \right. \\ & \quad \left. \left. - \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{\lambda+x+i\eta} \right)^{p-1} - 1 \right] \right\} \right\rangle \phi(t) dt \end{aligned}$$

where the support of $\phi(t)$ is contained in (a, b) , $b > a > 0$,

$$= \langle f(x), \frac{1}{2\pi i} \int_a^b \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] \right. \\ \left. - \frac{1}{(x-t-i\eta)} \left[\left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \right\} \phi(t) dt.$$

(By using Riemann's sum technique)

Letting $\eta \rightarrow 0+$, the result follows in view of Lemma 5.3.

THEOREM 5.5. For a fixed $\alpha < 1 < \operatorname{Re} p$, let $f(t) \in S'_\alpha(I)$ and let $F(s)$ be the Stieltjes transform of $f(t)$ defined by (4.1). Then,

$$\lim_{\eta \rightarrow 0+} \langle \frac{p-1}{2\pi i} \int_{-x}^{\lambda} (x+t)^{p-2} [F(t-i\eta) - F(t+i\eta)] dt, \phi(x) \rangle = \langle f, \phi \rangle$$

for all $\phi \in \mathcal{D}(I)$ and $\lambda > 0$.

Proof: The result follows quite easily in view of Theorems 5.1 and 5.4.

CHAPTER II

ABELIAN THEOREMS

1. DEFINITIONS AND PRELIMINARY LEMMAS

Throughout this Chapter, $S_\alpha(I)$ will denote the testing function space as defined in Chapter I where I is the open interval $(0, \infty)$ and $S'_\alpha(I)$ will denote the dual space of $S_\alpha(I)$.

DEFINITION 1.1 (Lojasiewicz). Let T be a distribution defined in the neighbourhood of the point x_0 . Then we say that T has value c at x_0 if the distributional limit of $T(x_0 + \lambda x)$ as $\lambda \rightarrow 0+$ exists in a neighbourhood of zero and is a constant function c .

DEFINITION 1.2. Let T be a distribution defined on the interval $(0, \infty)$.

Then we say that

$$\lim_{x \rightarrow 0+} T(x) = c$$

if

$$\lim_{\lambda \rightarrow 0+} \langle T(\lambda x), \phi(x) \rangle = \langle c, \phi \rangle$$

for all $\phi \in \mathcal{D}(I)$.

DEFINITION 1.3. The wedge P_k in a complex s -plane for a fixed $k \geq 0$ will be defined by

$$P_k = \{s: s = \sigma + i\omega, \sigma > 0, |\omega| \leq k\sigma\}.$$

LEMMA 1.4. Let $\alpha > \eta + 1 > 0$. Then the regular distribution generated by t^η is an element of $S'_\alpha(I)$ where $\alpha < 1$.

Proof: For $\phi(t) \in S_\alpha(I)$, we have:

$$\begin{aligned} \int_0^\infty t^\eta \phi(t) dt &\leq \sup_{0 < t < \infty} |(1+t)^\alpha \phi(t)| \int_0^\infty \frac{t^\eta}{(1+t)^\alpha} dt \\ &= \gamma_0(\phi) \frac{\Gamma(n+1)\Gamma(\alpha-\eta-1)}{\Gamma(\alpha)} \end{aligned}$$

[9,p13]

since $\phi(t) \in S_\alpha(I)$.

Therefore, $\int_0^\infty t^\eta \phi(t) dt$ exists for all $\phi(t) \in S_\alpha(I)$.

Define

$$\langle t^\eta, \phi(t) \rangle \triangleq \int_0^\infty t^\eta \phi(t) dt \quad (1.1)$$

We shall show that this defines a linear continuous functional on $S'_\alpha(I)$. The linearity is obvious. To prove continuity, let the sequence $\{\phi_\nu\}_{\nu=1}^\infty$ converge to zero in $S_\alpha(I)$. Then, from the same

consideration as above we have

$$|\langle t, \phi_v(t) \rangle| \leq \frac{\gamma_0(\phi_v) \Gamma(\eta+1) \Gamma(\alpha-\eta-1)}{\Gamma(\alpha)} \rightarrow 0$$

as $v \rightarrow \infty$.

Thus the proof of Lemma 1.4 is complete.

LEMMA 1.5. Let p be a complex number satisfying $\operatorname{Re} p > \eta + 1 > 0$ and let s be a non-zero complex number not lying on the negative real axis. Then,

$$s^{p-\eta-1} \int_0^\infty \frac{t^\eta}{(s+t)^p} dt = \frac{\Gamma(p-\eta-1) \Gamma(\eta+1)}{\Gamma(p)} \quad (1.2)$$

Proof: Write $s = \sigma + i\omega$, $\sigma > 0$. First assume that $\omega \neq 0$. Substitute $t = sz$

where s is an arbitrary but fixed complex number. Let Δ denote a half line beginning at origin and passing through $1/s$. Thus,

$$s^{p-n-1} \int_0^\infty t^n (s+t)^{-p} dt = \int_\Delta z^n (1+z)^{-p} dz$$

Now, let A be a point on Δ such that $OA = Re^{i\theta'}$, $\theta' = \arg(\frac{1}{s})$. Denote by AR the arc of the circumference $z = Re^{i\theta}$, $\theta' \leq \theta \leq 0$.

$$\int_\Delta z^n (1+z)^{-p} dz = \lim_{R \rightarrow \infty} \int_{OA} z^n (1+z)^{-p} dz$$

$$\int_{ORA} z^n (1+z)^{-p} dz = 0 \text{ by Cauchy theorem.}$$

Therefore,

$$\int_{OR} x^n (1+x)^{-p} dx + \int_{ArcRA} z^n (1+z)^{-p} dz + \int_{AO} z^n (1+z)^{-p} dz = 0$$

or

$$\int_{OA} z^n (1+z)^{-p} dz = \int_{OR} z^n (1+z)^{-p} dx - \int_{ArcAR} z^n (1+z)^{-p} dz$$

Now, substituting $z = Re^{i\theta}$ in the second integral and using Lemma 3.1 of Chapter I, we have

$$\left| \int_{ArcRA} z^n (1+z)^p dz \right| \leq R^{n+1} \int_0^{|\theta'|} \frac{d\theta}{|(1+z)^p|}$$

$$\leq \frac{\pi}{2} R^{n+1-\operatorname{Re} p} e^{\pi/2 |\operatorname{Im} p|} \rightarrow 0 \text{ as } R \rightarrow \infty$$

since $\operatorname{Re} p > n+1 > 0$.

In the theory of gamma functions, it is well known that for $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$, the integral

$$\int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad [9, p. 13]$$

Therefore, we get

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{OA} z^n (1+z)^{-p} dz &= \lim_{R \rightarrow \infty} \int_0^R x^n (1+x)^{-p} dx \\ &= \frac{\Gamma(p-n-1)\Gamma(n+1)}{\Gamma p} \end{aligned}$$

If $\operatorname{Im} s < 0$, our argument will be similar to the one used above. Thus, the proof of the lemma is complete.

We now quote a result which is merely a consequence of a result proved by Lojasiewicz [16, p. 90].

THEOREM 1.6 (Lojasiewicz). A necessary and sufficient condition that

$\lim_{x \rightarrow 0+} T(x) = c$ where $T \in D'(I)$ is that there exists a non-negative integer n and a continuous function F satisfying $T = C + D^n F$ in an interval $(0, a)$, $a > 0$ and $\lim_{x \rightarrow 0+} \frac{F(x)}{x^n} = 0$.

2. AN INITIAL VALUE THEOREM FOR DISTRIBUTIONAL STIELTJES TRANSFORM

THEOREM 2.1. Let $\operatorname{Re} p > n+1$ and $\alpha < 1$. Assume that s is a non-zero complex number not lying on the negative real axis. If for

$f(t) \in S'_a$ there exists a constant c such that

$$\lim_{t \rightarrow 0^+} \frac{f(t)}{t^\eta} = c \text{ in the sense of } {}^{\text{Lojasiewicz}} \text{ and } F(s) \text{ is the}$$

Stieltjes transform of f defined by

$$F(s) = \langle f(t), \frac{1}{(s+t)^p} \rangle.$$

Then,

$$\lim_{\substack{|s| \rightarrow 0 \\ s \in P_k}} \frac{s^{p-\eta-1} \Gamma(p) F(s)}{\Gamma(p-\eta-1) \Gamma(\eta+1)} = c$$

Proof: Using Lemma 1.4 and 1.5, we have

$$\langle t^\eta, \frac{1}{(s+t)^p} \rangle = \frac{s^{\eta+1-p} \Gamma(p-\eta-1) \Gamma(\eta+1)}{\Gamma(p)}$$

Therefore,

$$\frac{F(s) \Gamma(p) s^{p-\eta-1}}{\Gamma(p-\eta-1) \Gamma(\eta+1)} - c = \frac{\Gamma(p) s^{p-\eta-1}}{\Gamma(p-\eta-1) \Gamma(\eta+1)} \langle f(t) - ct^\eta, \frac{1}{(s+t)^p} \rangle$$

Note that $\frac{f(t)}{t^\eta} \in D'(I)$ and therefore using Theorem 1.6, we can find a non-negative integer m and a finite constant $A > 0$ such that

$$\frac{f(t)}{t^\eta} - c = D^m g$$

over $(0, a)$, $0 < a < A$ where $g(t)$ is a continuous function defined over $(0, a)$ with the property

$$\lim_{t \rightarrow 0^+} \frac{g(t)}{t^m} = 0$$

Now, let $\eta, \delta > 0$ satisfying $\delta < \delta', \delta + \delta' < a, 2\delta < a$ and $\lambda(t)$ be infinitely differentiable non-negative real valued function defined over $(0, \infty)$ such that

$$\lambda(t) = \begin{cases} 1 & \text{for } t > a - \delta \\ 0 & \text{for } t < a - \delta - \delta' \end{cases}$$

Thus, we can write

$$\langle f(t) - ct^\eta, \frac{1}{(s+t)^p} \rangle = \langle f(t) - ct^\eta, \frac{1-\lambda(t)}{(s+t)^p} \rangle + \langle f(t) - ct^\eta, \frac{\lambda(t)}{(s+t)^p} \rangle$$

The support of $\frac{1-\lambda(t)}{(s+t)^p}$ is contained in $(0, a-\delta] \subset (0, a)$. Therefore, we are justified in writing

$$\begin{aligned} \langle f(t) - ct^\eta, \frac{1-\lambda(t)}{(s+t)^p} \rangle &= \langle t^{\eta m} g, \frac{1-\lambda(t)}{(s+t)^p} \rangle \\ &= \langle g(t), (-1)^m D^m \left\{ \frac{t^\eta (1-\lambda(t))}{(s+t)^p} \right\} \rangle \end{aligned}$$

Hence,

$$\frac{\Gamma(s)\Gamma(p)s^{p-\eta-1}}{\Gamma(p-\eta-1)\Gamma(\eta+1)} - c = I_1 + I_2$$

where

$$I_1 = \frac{\Gamma(p)s^{p-\eta-1}}{\Gamma(p-\eta-1)\Gamma(\eta+1)} \langle g(t), (-1)^m D^m \left\{ \frac{t^\eta (1-\lambda(t))}{(s+t)^p} \right\} \rangle$$

and

$$I_2 = \frac{\Gamma(p)s^{p-\eta-1}}{\Gamma(p-\eta-1)\Gamma(\eta+1)} \langle f(t) - ct^\eta, \frac{\lambda(t)}{(s+t)^p} \rangle$$

Now,

$$t^m D^m \left[\frac{t^\eta (1-\lambda(t))}{(s+t)^p} \right] = \sum_{0 \leq j \leq i \leq m} \binom{m}{i} \binom{i}{j} [n]_j (p)_{m-i} (-1)^{m-i} q_{ij}(t) \cdot \frac{t^{m+\eta-j}}{(s+t)^{p+m-1}}$$

where

$$[n]_j = n(n-1) \dots (\eta-j+1);$$

$$(p)_{m-j} = p(p+1) \dots (p+m-j+1)$$

and

$$q_{ij}(t) = D^{i-j} [1-\lambda(t)]$$

So,

$$\left| t^m D^m \left[\frac{t^\eta (1-\lambda(t))}{(s+t)^p} \right] \right| \leq \sum_{0 \leq j \leq i \leq m} a_{ij} \left| \frac{t^{m+\eta-j}}{(s+t)^{p+m-1}} \right|$$

where a_{ij} are obvious bounded positive constants depending upon i, j, m, p and η .

$$|I_1| \leq \left| \frac{\Gamma(p) s^{p-\eta-1}}{\Gamma(p-\eta-1) \Gamma(\eta+1)} \right| \int_0^{a-\delta} \left| \frac{g(t)}{t^\eta} \right| \left| t^m D^m \left[\frac{t^\eta (1-\lambda(t))}{(s+t)^p} \right] \right| dt$$

$$|I_1| \leq \left| \frac{\Gamma(p) s^{p-\eta-1}}{\Gamma(p-\eta-1) \Gamma(\eta+1)} \right| \sum_{0 \leq j \leq i \leq n} a_{ij} M_a^\delta \int_0^{a-\delta} \left| \frac{t^{m+\eta-j}}{(s+t)^{p+m-1}} \right| dt$$

where

$$M_a^\delta = \sup_{0 < t < a-\delta} \left| \frac{g(t)}{t^m} \right|.$$

The supremum exists since $\frac{g(t)}{t^m} \rightarrow 0$ as $t \rightarrow 0+$.

Since

$$|s|^{\operatorname{Re} z} e^{-\pi |\operatorname{Im} z|} \leq |s^z| \leq |s|^{\operatorname{Re} z} e^{\pi |\operatorname{Im} z|}$$

we have

$$|I_1| \leq \left| \frac{\Gamma(p)}{\Gamma(p-\eta-1)\Gamma(\eta+1)} \right| |s|^{\operatorname{Re} p-\eta-1} e^{2\pi |\operatorname{Im} p|} M_a^\delta \times$$

$$\sum_{0 \leq j \leq i \leq m} a_{ij} \int_0^{a-\delta} \frac{t^{m+\eta-j}}{(1+t)^{\operatorname{Re} p+m-i}} dt$$

$$\leq \left| \frac{\Gamma(p)}{\Gamma(p-\eta-1)\Gamma(\eta+1)} \right| (1+k^2)^{\operatorname{Re} p-\eta-1} e^{2\pi |\operatorname{Im} p|} \sum_{0 \leq j \leq i \leq m} |a_{ij}| M_a^\delta \int_0^{a-\delta} \frac{t^{m+\eta-j}}{(1+t)^{\operatorname{Re} p+m-i}} dt$$

$$\leq \left| \frac{\Gamma(p) M_a^\delta}{\Gamma(p-\eta-1)\Gamma(\eta+1)} \right| (1+k^2)^{\frac{1}{2}(\operatorname{Re} p-\eta-1)} e^{2\pi |\operatorname{Im} p|} \sum_{0 \leq j \leq i \leq m} |a_{ij}| \frac{\Gamma(\operatorname{Re} p-\eta-1)\Gamma(m+\eta+1-i)}{\Gamma(\operatorname{Re} p+m-i)}$$

Since $\frac{F(t)}{t^n} \rightarrow 0$ as $t \rightarrow 0^+$ we can select a, δ and δ' small enough such that

$$|I_1| < \epsilon \quad (2.1)$$

where ϵ is an arbitrarily chosen small positive number. Fix a and s this way. Now, $f(t) - ct^n \in S'_\alpha$ as $f(t)$ and t^n both belong to S'_α . Therefore, using boundedness property of generalized functions, we get

$$\begin{aligned} |I_2| &\leq \left| \frac{\Gamma(p)}{\Gamma(p-n-1)\Gamma(n+1)} \right| |s|^{\operatorname{Re} p - n - 1} e^{|\operatorname{Im} p|} \times \\ &\quad \max_{s \in P_k} \sup_{\substack{0 \leq m \leq r \\ t \geq \delta \\ |s| < 1}} \left| (1+t)^{\alpha} t^{m_D m} \left[\frac{\lambda(t)}{(s+t)^p} \right] \right| \\ &\leq \left| \frac{\Gamma(p)}{\Gamma(p-n-1)\Gamma(n+1)} \right| (1+K^2)^{\frac{1}{2}} (\operatorname{Re} p - n - 1)_+^{\operatorname{Re} p - n - 1} e^{|\operatorname{Im} p|} L \end{aligned}$$

where

$$L = \max_{0 \leq m \leq r} \sup_{\substack{t \geq \delta \\ s \in P_k \\ |s| < 1}} \left| (1+t)^{\alpha} t^{m_D m} \left\{ \frac{\lambda(t)}{(s+t)^p} \right\} \right|$$

Therefore, $|I_2| \rightarrow 0$ as $s \rightarrow 0$ in P_k .

(2.2)

Combining (2.1) and (2.2) together, we have proved that

$$\lim_{\substack{|s| \rightarrow 0 \\ s \in P_k}} |I_1 + I_2| \leq \epsilon$$

Since ϵ is arbitrary, our claim is proved.

3. FINAL VALUE THEOREM FOR DISTRIBUTIONAL STIELTJES TRANSFORM

3.1. Value of a Distribution at Infinity (Motivation)

Suppose that f is a distribution satisfying $f = D^k h(x)$ in the interval (N, ∞) where $N > 1$ and $h(x)$ is a continuous function. Then, for a $\phi \in \mathcal{D}(N, \infty)$ we have,

$$\begin{aligned} \langle f, \phi \rangle &= \langle D^k h(x), \phi(x) \rangle = \langle h(x), (-1)^k D^k \phi(x) \rangle \\ &= \langle k! x^{-k} h(x), \frac{x^k}{k!} D^k \phi(x) \rangle \end{aligned}$$

Suppose $k! x^{-k} h(x) \rightarrow C$ as $x \rightarrow \infty$, then for sufficiently large N , the difference $|\langle f, \phi \rangle - \langle C, \phi \rangle|$ can be made less than an arbitrarily small quantity ϵ .

Therefore, the value of a distribution at infinity can be defined as follows.

DEFINITION 3.2. The value of a distribution f at infinity is a constant C if there exists a number $N > 1$ and a non-negative integer k satisfying $D^k h(x) = f$ over (N, ∞) and

$$k! x^{-k} h(x) \rightarrow C \text{ as } x \rightarrow \infty.$$

This definition has been given by Lavoine and Misra [7]. Now, we prove the following theorem which is a minor extension of the result proved by Lavoine and Misra [7].

THEOREM 3.3. $\operatorname{Re} p > n+1 > 0$. If, for $f \in S'_\alpha(I)$, there exists a constant

C such that

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t^n} = C \text{ in the sense of Lavoine and Misra}$$

then

$$\lim_{\substack{|s| \rightarrow \infty \\ s \in P_k}} \frac{s^{p-n-1} \Gamma(p) \Gamma(s)}{\Gamma(p-n-1) \Gamma(n+1)} = C$$

where $F(s)$ is the Stieltjes transform of $f \in S'_\alpha(I)$.

Proof: This theorem can be proved by following the technique of the previous theorem. We, therefore, give an outline of the proof.

Since $\frac{f(t)}{t^n} \in \mathcal{D}'(I)$ and $\frac{f(t)}{t^n} \rightarrow C$, as $t \rightarrow \infty$, we can find a non-negative integer m and a finite number $N > 1$ such that

$$\frac{f(t)}{t^n} - C = D^m h$$

over (N, ∞) where $h(t)$ is a continuous function defined over (N, ∞) with the property

$$\lim_{t \rightarrow \infty} \frac{m! h(t)}{t^m} = 0$$

For $0 < \delta < 1$, let $\lambda(t)$ be an infinitely differentiable function such that

$$\begin{aligned} \lambda(t) &= 1 && \text{for } t < N + \delta \\ &= 0 && \text{for } t > N + 2\delta \end{aligned}$$

where $N > 1$.

As in Theorem 2.1, we can write

$$\frac{F(s)\Gamma(p)s^{p-\eta-1}}{\Gamma(p-\eta-1)\Gamma(\eta+1)} - C = I_1 + I_2$$

where

$$I_1 = \frac{\Gamma(p)s^{p-\eta-1}}{\Gamma(p-\eta-1)\Gamma(\eta+1)} \langle h(t), (-1)^m D^m \left(\frac{t^\eta}{(s+t)^p} (1-\lambda(t)) \right) \rangle$$

and

$$I_2 = \frac{\Gamma(p)s^{p-\eta-1}}{\Gamma(p-\eta-1)\Gamma(\eta+1)} \langle f(t) - ct^\eta, \frac{\lambda(t)}{(s+t)^p} \rangle$$

Now, following the pattern of the proof of the previous theorem, we obtain

$$|I_1| \leq \left| \frac{\Gamma(p)s^{p-\eta-1}}{\Gamma(p-\eta-1)\Gamma(\eta+1)} \right| \int_{N+\delta}^{\infty} \left| \frac{h(t)}{t^m} \right| \left| t^m D^m \left[\frac{t^\eta}{(s+t)^p} (1-\lambda(t)) \right] \right| dt$$

Since $h(t)/t^m \rightarrow 0$ as $t \rightarrow \infty$, we can choose $N > 0$ such that

$$\left| \frac{h(t)}{t^m} \right| < \varepsilon \quad \text{for } t \geq N + \delta$$

Now, denoting $D^\ell(1-\lambda(t))$ by $g_\ell(t)$, and the expression

$$\left| \frac{\Gamma(p)}{\Gamma(p-\eta-1)\Gamma(\eta+1)} (1+K)^{\operatorname{Re} p - \eta - 1} e^{\pi |\operatorname{Im} p|} \right|$$

by $A(K, p, \eta)$ we have

$$|I_1| \leq A(K, p, \eta) \in \sigma^{\operatorname{Re} p - \eta - 1} \int_{N+\delta}^{\infty} \left| t^m D^m \left\{ \frac{t^\eta [1 - \lambda(t)]}{(s+t)^p} \right\} \right| dt$$

Let

$$\sup_{N+\delta < t < \infty} |D^i [1 - \lambda(t)]| = a_i \quad \text{for } i=0, 1, 2, \dots, m$$

Therefore,

$$\begin{aligned} |I_1| &\leq a_0 A(K, p, \eta) \in \sigma^{\operatorname{Re} p - \eta - 1} \int_{N+\delta}^{\infty} \left| t^m D^m \left\{ \frac{t^\eta}{(s+t)^p} \right\} \right| dt \\ &\quad + A(K, p, \eta) \in \sigma^{\operatorname{Re} p - \eta - 1} \sum_{i=1}^m a_i \binom{m}{i} \int_{N+\delta}^{N+2\delta} \left| t^m D^{m-i} \left\{ \frac{t^\eta}{(s+t)^p} \right\} \right| dt \end{aligned}$$

Clearly the expression

$$\sum_{i=1}^m a_i \binom{m}{i} \int_{N+\delta}^{N+2\delta} \left| t^m D^{m-i} \left\{ \frac{t^\eta}{(s+t)^p} \right\} \right| dt = O(|s|^{-\operatorname{Re} p}) \quad \text{as } |s| \rightarrow \infty \text{ in } P_K.$$

Therefore

$$A(K, p, \eta) \in \sigma^{\operatorname{Re} p - \eta - 1} \sum_{i=1}^m a_i \binom{m}{i} \int_{N+\delta}^{N+2\delta} \left| t^m D^{m-i} \left\{ \frac{t^\eta}{(s+t)^p} \right\} \right| dt \rightarrow 0$$

as $|s| \rightarrow \infty$ in P_K .

A careful computation shows that

$$\sigma^{\operatorname{Re} p - \eta - 1} \int_{N+\delta}^{\infty} \left| t^m D^m \left\{ \frac{t^\eta}{(s+t)^p} \right\} \right| dt$$

remains bounded $|s| \rightarrow \infty$ in P_K . Let this bound be M . Therefore

$$\lim_{\substack{|s| \rightarrow \infty \\ s \in P_K}} |I_1| \leq M \cdot A(K, p, \eta) \in \quad (3.1)$$

Next, consider I_2 . Now, using the boundedness property of generalized functions, we get

$$|I_2| \leq A_0^{\operatorname{Re} p - \eta - 1} \max_{0 \leq k \leq r} \sup_{0 \leq t \leq N + 2\delta} |(1+t)^\alpha \sum_{i=0}^k \binom{k}{i} (p)_i D^{k-i}[\lambda(t)] \times$$

$$\frac{t^k}{(s+t)^{\operatorname{Re} p + i}}|$$

$$\leq A(N+2\delta+1)^{k+\alpha} e^{\pi |\operatorname{Im} p|} \sum_{i=0}^k B_i \sigma^{-i-\eta-1} \rightarrow 0 \quad (3.2)$$

as $|s| \rightarrow \infty$ in P_k .

Combining (3.1) and (3.2), we have proved that

$$\lim_{\substack{s \in P_k \\ |s| \rightarrow \infty}} |I_1 + I_2| \leq M \cdot A(K, p, \eta) \cdot \epsilon$$

This proves Theorem 3.3.

4. ABELIAN THEOREMS FOR GENERALIZED STIELTJES TRANSFORMATION

In this section, we shall extend the initial and final value theorems of Carmichael [3] for complex p .

THEOREM 4.1. Let p be a complex number satisfying $\operatorname{Re} p > \eta + 1 > 0$. Let $f(t)$ be a complex valued function such that the Stieltjes transform

$$F(s) = \int_0^\infty \frac{f(t) dt}{(s+t)^p} \quad (4.1)$$

exists for $s = \sigma + i\omega$, $\sigma > 0$ and such that $\frac{f(t)}{t^\eta}$ is bounded on $y < t < \infty$ for all

$y > 0$. Let α be a complex number such that

$$\frac{f(t)}{t^\eta} \rightarrow \alpha \text{ as } t \rightarrow 0+.$$

Then, for each fixed $K \geq 0$,

$$\lim_{\substack{|s| \rightarrow 0 \\ s \in P_K}} \frac{s^{p-\eta-1} \Gamma(p) F(s)}{\Gamma(p-\eta-1) \Gamma(\eta+1)} = \alpha$$

Proof: By Lemma 1.5, for a fixed positive number y we have

$$\begin{aligned} & \left| s^{p-\eta-1} F(s) - \alpha \frac{\Gamma(p-\eta-1) \Gamma(\eta+1)}{\Gamma(p)} \right| \\ &= \left| s^{p-\eta-1} \int_0^\infty \frac{f(t) dt}{(s+t)^p} - \alpha s^{p-\eta-1} \int_0^\infty \frac{t^\eta dt}{(s+t)^p} \right| \\ &\leq |s^{p-\eta-1}| \left| \int_0^y \frac{f(t)-\alpha t^\eta}{(s+t)^p} dt + \int_y^\infty \frac{f(t)-\alpha t^\eta}{(s+t)^p} dt \right| \\ &= |I_1| + |I_2| \quad (\text{say}). \end{aligned} \tag{4.2}$$

At first, consider I_2 .

$$\begin{aligned} |I_2| &= |s^{p-\eta-1}| \left| \int_y^\infty \frac{f(t)-\alpha t^\eta}{(s+t)^p} dt \right| \\ &= |s^{p-\eta-1}| \left| \int_y^\infty \frac{t^\eta \left(\frac{f(t)}{t^\eta} - \alpha \right) dt}{(s+t)^p} \right| \end{aligned}$$

$$\leq \sup_{y \leq t < \infty} \left| \frac{f(t)}{t^\eta} - \alpha \right| |s|^{p-\eta-1} \int_y^\infty \frac{t^\eta}{|(s+t)^p|} dt$$

$$\leq \sup_{y \leq t < \infty} \left| \frac{f(t)}{t^\eta} - \alpha \right| |e^{2\pi i \operatorname{Im} p}| |s|^{\operatorname{Re} p - \eta - 1} \int_y^\infty \frac{t^\eta dt}{(\sigma+t)^{\operatorname{Re} p}}$$

(By Lemma 3.1, Chapter I).

Since $\left| \frac{f(t)}{t^\eta} - \alpha \right|$ is bounded on $y \leq t < \infty$, there exists a constant A such that

$$|I_2| \leq A |s|^{\operatorname{Re} p - \eta - 1} \int_y^\infty t^{\eta - \operatorname{Re} p} dt$$

$$\leq A |s|^{\operatorname{Re} p - \eta - 1} \frac{y^{\eta+1 - \operatorname{Re} p}}{\operatorname{Re} p - \eta - 1}$$

By the hypothesis, $\operatorname{Re} p > \eta + 1$, therefore

$$|I_2| \rightarrow 0 \text{ as } |s| \rightarrow 0 \quad (4.3)$$

in any manner, $s = \sigma + i\omega$, $\sigma > 0$.

Now, consider I_1 . Since $\frac{f(t)}{t^\eta} \rightarrow \alpha$ as $t \rightarrow 0+$, for $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(t)}{t^\eta} - \alpha \right| < \epsilon \quad \text{for } 0 < t < \delta.$$

Therefore, for a fixed $y < \delta$, we can write

$$|I_1| \leq \epsilon |s|^{\operatorname{Re} p - \eta - 1} e^{2\pi |\operatorname{Im} p|} \int_0^y \frac{t^\eta dt}{(\sigma + t)^{\operatorname{Re} p}}$$

$$< \epsilon \sigma^{\operatorname{Re} p - \eta - 1} (1 + K^2)^{\frac{1}{2}} (\operatorname{Re} p - \eta - 1) e^{2\pi |\operatorname{Im} p|} \sigma^{\eta + 1 - \operatorname{Re} p}$$

$$\frac{\Gamma(\operatorname{Re} p - \eta - 1) \Gamma(\eta + 1)}{\Gamma(\operatorname{Re} p)}$$

(By Lemma 1.5)

(4.4)

Combining (4.2), (4.3) and (4.4), we obtain

$$\lim_{\substack{|s| \rightarrow 0 \\ s \in P_k}} |I_1 + I_2| \leq (1 + K^2)^{\frac{1}{2}} (\operatorname{Re} p - \eta - 1) e^{2\pi |\operatorname{Im} p|} \frac{\Gamma(\operatorname{Re} p - \eta - 1) \Gamma(\eta + 1)}{\Gamma(\operatorname{Re} p)}$$

Since ϵ is arbitrary, the result follows.

THEOREM 4.2. Let p be a complex number satisfying $\operatorname{Re} p > \eta + 1 > 0$. Let $f(t)$

be a complex valued function such that the Stieltjes transform

$F(s)$ defined by (4.1) exists for $s = \sigma + i\omega$, $\sigma > 0$ and such that $\frac{f(t)}{t^\eta}$ is

bounded on $0 < t \leq y$ for all $y > 0$. Let α be a complex number such that $\frac{f(t)}{t^\eta} \rightarrow \alpha$

as $t \rightarrow \infty$. Then, for each fixed $K \geq 0$,

$$\lim_{\substack{|s| \rightarrow \infty \\ s \in P_k}} \frac{s^{p - \eta - 1} \Gamma(p) F(s)}{\Gamma(p - \eta - 1) \Gamma(\eta + 1)} = \alpha$$

Proof: For $y > 0$, using Lemma 1.5, we can write:

$$\begin{aligned}
& |s^{p-n-1} F(s) - \frac{\alpha \Gamma(p-n-1) \Gamma(n+1)}{\Gamma(p)}| \\
&= |s^{p-n-1}| \left| \int_0^\infty \frac{(f(t) - \alpha t^n) dt}{(s+t)^p} \right| \\
&= |s^{p-n-1}| \left| \int_0^y \frac{f(t) - \alpha t^n}{(s+t)^p} dt + \int_y^\infty \frac{f(t) - \alpha t^n}{(s+t)^p} dt \right| \\
&= |J_1 + J_2| \quad (\text{say}) \quad (4.5)
\end{aligned}$$

Since $\frac{f(t)}{t^n} \rightarrow \alpha$ as $t \rightarrow \infty$, choosing y sufficiently large, we can make

$$\left| \frac{f(t)}{t^n} - \alpha \right| < \epsilon \quad \text{for all } t > y.$$

Therefore, in view of Lemma 3.1 of Chapter 1, we get

$$\begin{aligned}
|I_2| &\leq e^{2\pi |\operatorname{Im} p|} |s|^{\operatorname{Re} p - n - 1} \sup_{y \leq t < \infty} \left| \frac{f(t)}{t^n} - \alpha \right| \int_y^\infty \frac{t^n dt}{(\sigma + t)^{\operatorname{Re} p}} \\
&\leq \epsilon e^{2\pi |\operatorname{Im} p|} |s|^{\operatorname{Re} p - n - 1} \int_0^\infty \frac{t^n dt}{(\sigma + t)^{\operatorname{Re} p}} \\
&\leq \epsilon e^{2\pi |\operatorname{Im} p|} (1 + K^2)^{\frac{1}{2}(\operatorname{Re} p - n - 1)} \frac{\Gamma(\operatorname{Re} p - n - 1) \Gamma(n+1)}{\Gamma(\operatorname{Re} p)} \quad (4.6)
\end{aligned}$$

Let $\left| \frac{f(t)}{t^n} - \alpha \right|$ be bounded by M on $0 < t \leq y$. Then, by Lemma 3.1 of Chapter I, we get

$$\begin{aligned}
|I_1| &\leq M|s|^{\operatorname{Re} p - n - 1} e^{2\pi |\operatorname{Im} p|} \int_0^y \frac{t^n dt}{(\sigma + t)^{\operatorname{Re} p}} \\
&\leq M \sigma^{\operatorname{Re} p - n - 1} (1 + K^2)^{\frac{1}{2}} (\operatorname{Re} p - n - 1) e^{2\pi |\operatorname{Im} p|} \int_0^y \frac{t^n dt}{\sigma^{\operatorname{Re} p}} \\
&= M \frac{1}{\sigma^{n+1}} (1 + K^2)^{\frac{1}{2}} (\operatorname{Re} p - n - 1) e^{2\pi |\operatorname{Im} p|} \frac{y^{n+1}}{n+1}
\end{aligned}$$

Observe that as $|s| \rightarrow \infty$ in P_k for a fixed $K \geq 0$, σ the real part of s must approach ∞ . Therefore,

$$|I_1| \rightarrow 0 \text{ as } |s| \rightarrow \infty \text{ in } P_k \quad (4.7).$$

Now, combining (4.5), (4.6) and (4.7) together we obtain the desired result.

CHAPTER III

THE MEHLER-FOCK TRANSFORM OF DISTRIBUTIONS

1. INTRODUCTION

The classical Mehler-Fock transformation of a function $f(x)$ is defined by

$$f^*(\tau) = \int_0^\infty f(x) P_{-\frac{1}{2}+i\tau}(\operatorname{ch}x) \operatorname{sh}x dx \quad (1.1)$$

where $P_{-\frac{1}{2}+i\tau}(\operatorname{ch}x)$ is the Legendre function of first kind. The corresponding inversion formula

$$f(x) = \int_0^\infty \tau \operatorname{th} \pi \tau P_{-\frac{1}{2}+i\tau}(\operatorname{ch}x) f^*(\tau) d\tau \quad (1.2)$$

was given by Mehler [11] in a purely formal way. Fock [5] in the early forties established these formulae for a class of functions. In this Chapter, we will extend this transformation to a class of distributions and shall prove the inversion formula in the distributional sense.

The Legendre function used in the definition of the above transformation possesses the following well known properties to be used in the sequel.

$$P_{-\frac{1}{2}+i\tau}(\operatorname{ch}x) = \frac{2}{\pi} \operatorname{ch} \pi \tau \int_0^\infty \frac{\cos \tau \theta d\theta}{\sqrt{2 \operatorname{ch} \theta + 2 \operatorname{ch} t}}, \quad \theta > 0 \quad (1.3)$$

[9, p. 173]

$$\left| P_{-\frac{1}{2}+i\tau}(\operatorname{ch}x) \right| < \frac{x}{2 \operatorname{Sh} \frac{x}{2}} \leq 1, \quad x \geq 0 \quad (1.4)$$

[5, p. 254]

$$P_{-\frac{1}{2}+i\tau}(\text{ch}\theta) = \sqrt{\frac{\theta}{\text{Sh}\theta}} \left\{ J_0(\tau\theta) + \frac{1}{8\tau} (\coth\theta - \frac{1}{\theta}) J_1(\tau\theta) + \dots \right\} \quad (1.5)$$

Here, the general term in brackets has the shape

$$A_n = a_n(\theta) \frac{J_n(\tau\theta)}{(\tau\theta)^n}$$

where $J_n(\tau\theta)$ is the Bessel function of first kind and $a_n(\theta)$ is a polynomial in θ and $\coth\theta$ which with $\theta \rightarrow 0$ behaves like θ^{2n} and with $\theta \rightarrow \infty$ behaves like θ^n . The series (1.5), if τ is great, may be regarded as an asymptotic series fit to be used for all values of θ [5, p. 253].

Using the asymptotic expansions of Bessel functions for large $\tau > 0$ and fixed $\theta > 0$ in (1.5) it follows that

$$P_{-\frac{1}{2}+i\tau}(\text{ch}\theta) = \sqrt{\frac{2}{\pi\tau\text{sh}\theta}} \left[\sin(\tau\theta + \pi/4) + \frac{\coth\theta}{8\tau} \sin(\tau\theta - \pi/4) + O\left(\frac{1}{\tau^2}\right) \right] \quad (1.6)$$

This asymptotic expansion is uniform with respect to θ over a compact subset of $(0, \infty)$.

2. THE TESTING FUNCTION SPACE $M_\alpha(I)$ AND ITS DUAL

Let I denote the open interval $(0, \infty)$. For $\alpha > 0$, let $\zeta(t)$ be the function defined on I by

$$\begin{aligned} \zeta(t) &= t^\alpha, & 0 < t < 1 \\ &= 1, & 1 \leq t \leq \infty \end{aligned}$$

Let $M_\alpha(I)$ be the collection of all infinitely differentiable, complex valued functions $\phi(t)$ defined on I such that for every non-negative

integer k

$$\gamma_k(\phi) \triangleq \sup_{0 < t < \infty} |\zeta(t) \Delta_t^k \phi(t)| < \infty$$

where

$$\Delta_t^k = [D^2 + \coth t D]^k, \quad D \equiv \frac{d}{dt}.$$

One can readily see that $M_\alpha(I)$ is a linear space and each γ_k , $k=0,1,2,\dots$ is a seminorm while γ_0 is a norm. Therefore, the collection of seminorms $\{\gamma_k\}$, $k=0,1,2,\dots$ is separating [24, p. 8]. We equip $M_\alpha(I)$ with the topology generated by the seminorms $\{\gamma_k\}_{k=0}^\infty$. A sequence $\{\phi_\nu\}_{\nu=1}^\infty$ in $M_\alpha(I)$ converges to ϕ in $M_\alpha(I)$ if and only if for each k , $\gamma_k(\phi_\nu - \phi) \rightarrow 0$ as $\nu \rightarrow \infty$. A sequence $\{\phi_\nu\}_{\nu=1}^\infty$ is said to be a Cauchy sequence if for each k , $\gamma_k(\phi_\nu - \phi_\mu) \rightarrow 0$ as μ and ν both tend to infinity independently of each other. It can be shown that $M_\alpha(I)$ is sequentially complete, locally convex topological vector space. $\mathcal{D}(I)$, the space of infinitely differentiable functions of compact support with the usual topology, [18, Vol. I, p.65] is a linear subspace of $M_\alpha(I)$. The topology of $\mathcal{D}(I)$ is stronger than the topology induced by $M_\alpha(I)$. Hence, the restriction of any $f \in M'_\alpha(I)$ to $\mathcal{D}(I)$ is in $\mathcal{D}'(I)$, the dual space of $\mathcal{D}(I)$.

For $\tau \geq 0$, the Legendre function $P_{-\frac{1}{2}+i\tau}(cht)$ is an element of $M_\alpha(I)$. For, $P_{-\frac{1}{2}+i\tau}(cht)$ satisfies the differential equation

$$D^2 y + \coth t \cdot D y + \left(\frac{1}{4} + \tau^2\right)y = 0, \quad D \equiv \frac{d}{dt}$$

Therefore,

$$\Delta_t [P_{-\frac{1}{2}+i\tau} (cht)] = - (1/4+\tau^2) P_{-\frac{1}{2}+i\tau} (cht) \quad (2.1)$$

Using Equation (2.1) and inequality (1.4), we get

$$\begin{aligned} \gamma_k [P_{-\frac{1}{2}+i\tau} (cht)] &= \sup_{0 < t < \infty} |\zeta(t) \Delta_t^k (P_{-\frac{1}{2}+i\tau} (cht))| \\ &\leq \sup_{0 < t < \infty} \zeta(t) (1+\tau^2)^k \frac{t^{\frac{k}{2}}}{\text{Sh } \frac{t}{2}} < \infty \\ &k=0,1,2,\dots \end{aligned}$$

3. THE DISTRIBUTIONAL MEHLER-FOCK TRANSFORMATION

For $f \in M'_0(I)$ and $\tau \geq 0$, define the distributional Mehler-Fock transformation $f^*(\tau)$ of f by

$$f^*(\tau) = \langle f(t), P_{-\frac{1}{2}+i\tau} (cht) \rangle \quad (3.1)$$

LEMMA 3.1. $\frac{\partial}{\partial \xi} P_{-\frac{1}{2}+i\xi} (cht)$ and $\frac{\partial^2}{\partial \xi^2} P_{-\frac{1}{2}+i\xi} (cht)$ are uniformly bounded

over $0 < t < \infty$.

Proof. In view of the integral representation (1.3), viz.,

$$P_{-\frac{1}{2}+i\xi} (cht) = \frac{2}{\pi} \text{ch} \pi \xi \int_0^\infty \frac{\cos \xi \theta d\theta}{\sqrt{(2\text{ch} \theta + 2cht)}}$$

we have

$$\begin{aligned}
\frac{\partial}{\partial \xi} P_{-\frac{1}{2}+i\xi}(\text{cht}) &= 2 \operatorname{sh} \pi \xi \int_0^\infty \frac{\cos \xi \theta d\theta}{\sqrt{(2\operatorname{ch} \theta + 2\text{cht})}} - \frac{2}{\pi} \operatorname{ch} \pi \xi \int_0^\infty \frac{\theta \sin \xi \theta d\theta}{\sqrt{(2\operatorname{ch} \theta + 2\text{cht})}} \\
&= \pi t h \pi \xi P_{-\frac{1}{2}+i\xi}(\text{cht}) - \frac{2}{\pi} \operatorname{ch} \pi \xi \int_0^\infty \frac{\theta \sin \xi \theta d\theta}{\sqrt{(2\operatorname{ch} \theta + 2\text{cht})}}
\end{aligned}$$

The first term of the right-hand side is obviously bounded for all $t \in (0, \infty)$ and a fixed ξ , hence consider the second term. Now,

$$\begin{aligned}
J &= \int_0^\infty \frac{\theta \sin \xi \theta d\theta}{\sqrt{(2\operatorname{ch} \theta + 2\text{cht})}} \leq \int_0^\infty \frac{\theta d\theta}{\sqrt{2(\operatorname{ch} \theta + 1)}} \\
&= \frac{1}{2} \int_0^\infty \frac{\theta d\theta}{\operatorname{ch} \frac{\theta}{2}} \\
&= \int_0^\infty \frac{\theta d\theta}{e^{\theta/2} + e^{-\theta/2}} \\
&\leq \int_0^\infty \theta e^{-\theta/2} d\theta \\
&= 4
\end{aligned}$$

This proves that $\frac{\partial}{\partial \xi} P_{-\frac{1}{2}+i\xi}(\text{cht})$ is bounded for all $t \in (0, \infty)$.

Similarly, it can be proved that $\frac{\partial^2}{\partial \xi^2} P_{-\frac{1}{2}+i\xi}(\text{cht})$ is also bounded for all $t \in (0, \infty)$.

THEOREM 3.2. For $f \in M'_\alpha(I)$ and $\tau \geq 0$ let $f^*(\tau)$ be defined by (3.1).

Then,

$$\frac{d}{d\tau} f^*(\tau) = \langle f(t), \frac{\partial}{\partial \tau} P_{-\frac{1}{2}+i\tau}(cht) \rangle$$

Proof: For $\Delta\tau \neq 0$, satisfying $|\Delta\tau| < \frac{1}{2}\tau$, we can write

$$\frac{F(\tau+\Delta\tau) - F(\tau)}{\Delta\tau} = \langle f(t), \frac{\partial}{\partial \tau} P_{-\frac{1}{2}+i\tau}(cht) \rangle = \langle f(t), \psi_{\Delta\tau}(cht) \rangle$$

where

$$\psi_{\Delta\tau}(cht) = \frac{1}{\Delta\tau} \left\{ P_{-\frac{1}{2}+i(\tau+\Delta\tau)}(cht) - P_{-\frac{1}{2}+i\tau}(cht) \right\} = \frac{\partial}{\partial \tau} P_{-\frac{1}{2}+i\tau}(cht)$$

We will prove that $\psi_{\Delta\tau}(cht) \rightarrow 0$ in $M'_\alpha(I)$ as $\Delta\tau \rightarrow 0$. For a non-negative integer k , we have:

$$\begin{aligned} \Delta_t^k [\psi_{\Delta\tau}(cht)] &= \frac{(-1)^k}{\Delta\tau} \left\{ \left[\frac{1}{4} + (\tau+\Delta\tau)^2 \right]^k P_{-\frac{1}{2}+i(\tau+\Delta\tau)}(cht) - \left[\frac{1}{4} + \tau^2 \right]^k P_{-\frac{1}{2}+i\tau}(cht) \right\} \\ &= \Delta_t^k \left(\frac{\partial}{\partial \tau} P_{-\frac{1}{2}+i\tau}(cht) \right) \\ &= \frac{(-1)^k}{\Delta\tau} \left\{ \left[\frac{1}{4} + (\tau+\Delta\tau)^2 \right]^k P_{-\frac{1}{2}+i(\tau+\Delta\tau)}(cht) - \left[\frac{1}{4} + \tau^2 \right]^k P_{-\frac{1}{2}+i\tau}(cht) \right. \\ &\quad \left. - \frac{\partial}{\partial \tau} \left[\frac{1}{4} + \tau^2 \right]^k P_{-\frac{1}{2}+i\tau}(cht) \right\} \cdot \Delta\tau \\ &= \frac{(-1)^k}{\Delta\tau} \left\{ \int_{\tau}^{\tau+\Delta\tau} \frac{\partial}{\partial \eta} \left[\left(\frac{1}{4} + \eta^2 \right)^k P_{-\frac{1}{2}+i\eta}(cht) \right] d\eta \right. \\ &\quad \left. - \frac{\partial}{\partial \tau} \left[\left(\frac{1}{4} + \tau^2 \right)^k P_{-\frac{1}{2}+i\tau}(cht) \right] \Delta\tau \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^k}{\Delta\tau} \left\{ \int_{\tau}^{\tau+\Delta\tau} \frac{\partial}{\partial n} \left[\left(\frac{1}{4} + n^2 \right)^k P_{-\frac{1}{2}+in}(\text{cht}) \right] dn - \right. \\
&\quad \left. \frac{\partial}{\partial \tau} \left[\left(\frac{1}{4} + \tau^2 \right)^k P_{-\frac{1}{2}+i\tau}(\text{cht}) \right] \int_{\tau}^{\tau+\Delta\tau} dn \right\} \\
&= \frac{(-1)^k}{\Delta\tau} \int_{\tau}^{\tau+\Delta\tau} \left\{ \frac{\partial}{\partial n} \left[\left(\frac{1}{4} + n^2 \right)^k P_{-\frac{1}{2}+in}(\text{cht}) \right] - \frac{\partial}{\partial \tau} \left[\left(\frac{1}{4} + \tau^2 \right)^k P_{-\frac{1}{2}+i\tau}(\text{cht}) \right] \right\} dn \\
&= \frac{(-1)^k}{\Delta\tau} \int_{\tau}^{\tau+\Delta\tau} \int_{\tau}^n \frac{\partial^2}{\partial \xi^2} \left[\left(\frac{1}{4} + \tau^2 \right)^k P_{-\frac{1}{2}+i\xi}(\text{cht}) \right] d\xi
\end{aligned}$$

$$|\zeta(t) \Delta_{\tau}^k [\psi_{\Delta\tau}(\text{cht})]| \leq |\Delta\tau| \sup_{\substack{\tau-\Delta\tau \leq \xi \leq \tau+\Delta\tau \\ 0 < t < \infty}} \left| \zeta(\tau) \frac{\partial^2}{\partial \xi^2} \left[\left(\frac{1}{4} + \xi^2 \right)^k P_{-\frac{1}{2}+i\xi}(\text{cht}) \right] \right|$$

$$= |\Delta\tau| \sup_{\substack{\tau-\Delta\tau \leq \xi \leq \tau+\Delta\tau \\ 0 < t < \infty}} \left| 2k(\xi^2 + 2(k-1)\xi + \frac{1}{4}) \left(\frac{1}{4} + \xi \right)^{k-2} \zeta(t) P_{-\frac{1}{2}+i\xi}(\text{cht}) \right.$$

$$\left. + 4k\xi \left(\frac{1}{4} + \xi^2 \right)^{k-1} \zeta(t) \frac{\partial}{\partial \xi} [P_{-\frac{1}{2}+i\xi}(\text{cht})] \right|$$

$$+ \left(\frac{1}{4} + \xi^2 \right)^k \zeta(t) \frac{\partial^2}{\partial \xi^2} [P_{-\frac{1}{2}+i\xi}(\text{cht})] \Big|$$

In view of the integral representation (1.3) and Lemma 3.1, the Legendre function $P_{-\frac{1}{2}+i\xi}(\text{cht})$ and its partial derivatives $\frac{\partial}{\partial \xi} P_{-\frac{1}{2}+i\xi}(\text{cht})$ and $\frac{\partial^2}{\partial \xi^2} P_{-\frac{1}{2}+i\xi}(\text{cht})$ are uniformly bounded on I. Therefore, we can find a positive constant Q such that:

$$(i) \sup_{\substack{0 < t < \infty \\ \tau-\Delta\tau \leq \xi \leq \tau+\Delta\tau}} \left| 2k(\xi^2 + 2(k-1)\xi + \frac{1}{4}) \left(\frac{1}{4} + \xi \right)^{k-2} \zeta(t) P_{-\frac{1}{2}+i\xi}(\text{cht}) \right| < \frac{Q}{3}$$

$$(ii) \sup_{\substack{0 < t < \infty \\ \tau - |\Delta\tau| < \xi < \tau + |\Delta\tau|}} |4k\xi \left(\frac{1}{4} + \xi^2\right)^{k-1} \zeta(t) \frac{\partial}{\partial \xi} [P_{-\frac{1}{2}+i\xi}(\text{cht})]| < \frac{Q}{3}$$

$$\text{and } (iii) \sup_{\substack{0 < t < \infty \\ \tau - |\Delta\tau| < \xi < \tau + |\Delta\tau|}} \left| \left(\frac{1}{4} + \xi^2\right)^k \zeta(t) \frac{\partial^2}{\partial \xi^2} [P_{-\frac{1}{2}+i\xi}(\text{cht})] \right| < \frac{Q}{3}$$

Consequently,

$$|\zeta(t) \Delta_t^k \psi_{\Delta\tau}(\text{cht})| < |\Delta\tau| Q,$$

and therefore,

$$\gamma_k(\psi_{\Delta\tau}) \rightarrow 0 \text{ as } \Delta\tau \rightarrow 0 \text{ for all } k.$$

THEOREM 3.3. For $f \in M'_\alpha(I)$, let $f^*(\tau)$ be defined by (3.1).

Then,

$$\begin{aligned} f^*(\tau) &= O(1) & \text{as } \tau \rightarrow 0 \\ &= O(\tau^{2n}) & \text{as } \tau \rightarrow \infty \end{aligned}$$

where n is a non-negative integer depending upon f .

Proof: If $f \in M'_\alpha(I)$ then by boundedness property of distributions

[24, p. 18] there exists a constant C and a non-negative integer n such that

$$\begin{aligned} |\langle f(t), P_{-\frac{1}{2}+i\tau}(\text{cht}) \rangle| &\leq C \max_{0 \leq k \leq n} \gamma_k(P_{-\frac{1}{2}+i\tau}(\text{cht})) \\ &= C \max_{0 \leq k \leq n} \sup_{0 < t < \infty} |\zeta(t) \Delta_t^k P_{-\frac{1}{2}+i\tau}(\text{cht})| \\ &= C \max_{0 \leq k \leq n} \sup_{0 < t < \infty} \left(\frac{1}{4} + \tau^2\right)^k \zeta(t) |P_{-\frac{1}{2}+i\tau}(\text{cht})| \end{aligned}$$

$$\leq C \max(1+\tau^2)^n,$$

since $|P_{-\frac{1}{2}+i\tau}(cht)| < 1$. Therefore,

$$|f^*(\tau)| \leq C(1+\tau^2)^n$$

and this implies the theorem.

LEMMA 3.4. Let $f \in M'_2(I)$ then for a positive integer N

$$\begin{aligned} & \int_0^N \tau \theta \pi \tau P_{-\frac{1}{2}+i\tau}(cht) \langle f(x), P_{-\frac{1}{2}+i\tau}(chx) \rangle d\tau \\ &= \langle f(x), \int_0^N \tau \theta \pi \tau P_{-\frac{1}{2}+i\tau}(chx) P_{-\frac{1}{2}+i\tau}(cht) d\tau \rangle \end{aligned} \quad (3.2)$$

Proof: The expression (3.2) is meaningful since the integral on the right-hand side belongs to $M_2(I)$. Indeed, for fixed $t > 0$, let

$$\theta(x, t) = \int_0^N \tau \theta \pi \tau P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) d\tau.$$

Then,

$$\begin{aligned} \Delta_x^k[\theta(x, t)] &= \int_0^N \tau \theta \pi \tau P_{-\frac{1}{2}+i\tau}(cht) \Delta_x^k(P_{-\frac{1}{2}+i\tau}(chx)) d\tau \\ &= (-1)^k \int_0^N \tau \theta \pi \tau \left(\frac{1+\tau^2}{4}\right)^k P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) d\tau \end{aligned}$$

Therefore,

$$\sup_{0 < x < \infty} |\zeta(x) \Delta_x^k [\theta(x, t)]| = \sup_{0 < x < \infty} \zeta(x) \times$$

$$\left| \int_0^N \tau \operatorname{th} \pi \tau \left(\frac{1}{4} + \tau^2 \right)^k P_{-\frac{1}{2} + i\tau}(\operatorname{cht}) P_{-\frac{1}{2} + i\tau}(\operatorname{ch} x) d\tau \right|$$

$$\leq \int_0^N \tau \operatorname{th} \pi \tau \left(\frac{1}{4} + \tau^2 \right)^k d\tau < \infty$$

since $|P_{-\frac{1}{2} + i\tau}(\operatorname{ch} x)| \leq 1$ for all x and all τ . The fact that both sides of (3.2) are equal can be proved by Riemann sum technique.

4. INVERSION OF THE DISTRIBUTIONAL MEHLER-FOCK TRANSFORMATION

For x, t and y in $I = (0, \infty)$ and $N > 0$, define

$$(a) \quad G_N(t, x) \triangleq \int_0^N \tau \operatorname{th} \pi \tau P_{-\frac{1}{2} + i\tau}(\operatorname{ch} x) P_{-\frac{1}{2} + i\tau}(\operatorname{cht}) \operatorname{sht} d\tau$$

$$(b) \quad \psi_N(y, x) \triangleq \int_0^N \tau \operatorname{th} \pi \tau P_{-\frac{1}{2} + i\tau}(\operatorname{ch} x) R_{-\frac{1}{2} + i\tau}(y) d\tau$$

$$\text{where } R_{-\frac{1}{2} + i\tau}(y) \triangleq \int_0^y \operatorname{sht} P_{-\frac{1}{2} + i\tau}(\operatorname{cht}) dt$$

$$(c) \quad \theta_N(y, x) \triangleq \int_0^N \tau \operatorname{th} \pi \tau P_{-\frac{1}{2}+i\tau}(chx) T_{-\frac{1}{2}+i\tau}(y) d\tau$$

$$\text{where } T_{-\frac{1}{2}+i\tau}(y) \equiv \int_0^y t s h t P_{-\frac{1}{2}+i\tau}(cht) dt$$

In the proof of Lemma 4.2, the application of Fock's theorem [19, p. 390] is required. Therefore, we state this theorem without proof.

THEOREM 4.1 (Fock). Let the function $g(t)$ be defined on $[1, \infty)$ such that the integral

$$\int_1^{\infty} \frac{|g(t)| dt}{\sqrt{t+1}}$$

exists, then at every point t in whose neighbourhood $g(t)$ has a bounded variation,

$$\frac{1}{2} [g(t+0) + g(t-0)] = \int_0^{\infty} P_{-\frac{1}{2}+i\tau}(t) g^*(\tau) d\tau$$

where

$$g^*(\tau) = \tau \operatorname{th} \pi \tau \int_1^{\infty} P_{-\frac{1}{2}+i\tau}(t) g(t) dt.$$

We shall now prove the following eight lemmas required to prove

Theorem 4.9.

LEMMA 4.2. For a fixed $y > 0$,

$$(a) \lim_{N \rightarrow \infty} \psi_N(y, x) = \begin{cases} 1 & \text{for } 0 < x < y \\ \frac{1}{2} & \text{for } x = y \\ 0 & \text{for } x > y \end{cases}$$

$$(b) \lim_{N \rightarrow \infty} \theta_N(y, x) = \begin{cases} x & \text{for } 0 < x < y \\ x/2 & \text{for } x = y \\ 0 & \text{for } x > y \end{cases}$$

Proof:

(a) In Fock's theorem, write cht for t and set

$$\begin{aligned} g(cht) &= 1 && \text{for } 0 < t < y \\ &= 0 && \text{for } t > y \end{aligned}$$

It is readily seen that the conditions of Fock's theorem are satisfied by $g(cht)$. Therefore,

$$\int_0^{\infty} P_{-\frac{1}{2}+i\tau}(cht) g(cht) sht dt = \int_0^y P_{-\frac{1}{2}+i\tau}(cht) sht dt \equiv R_{-\frac{1}{2}+i\tau}(y)$$

Now, using Fock's inversion theorem, we get:

$$\int_0^{\infty} \tau \tanh \pi \tau P_{-\frac{1}{2}+i\tau}(chx) R_{-\frac{1}{2}+i\tau}(cht) d\tau = \begin{cases} 1 & \text{for } 0 < x < y \\ \frac{1}{2} & \text{for } x = y \\ 0 & \text{for } x > y \end{cases}$$

The left-hand side of the above equation is nothing but the limit of $\psi_N(x, y)$ as $N \rightarrow \infty$.

(b) This part of the lemma can be proved in a similar way by defining $g(\text{cht})$ in Fock's theorem by

$$\begin{aligned} g(\text{cht}) &= t && \text{for } 0 < t < y \\ &= 0 && \text{for } t > y. \end{aligned}$$

The function $g(\text{cht})$ satisfies the conditions of Fock's inversion theorem. Therefore,

$$\int_0^{\infty} t \text{th} \tau P_{-\frac{1}{2}+i\tau}(\text{ch}x) T_{-\frac{1}{2}+i\tau}(y) d\tau = \begin{cases} x & \text{if } 0 < x < y \\ x/2 & \text{if } x=y \\ 0 & \text{if } x > y \end{cases}$$

That is,

$$\lim_{N \rightarrow \infty} \theta_N(x, y) = \begin{cases} x, & 0 < x < y \\ x/2, & x=y \\ 0, & x > y \end{cases}$$

LEMMA 4.3. Let a and b be any two real numbers satisfying $0 < a < b$.

Then,

$$\begin{aligned} \text{(i)} \quad \int_a^b G_N(t, x) dt &= 1 && \text{for } x \in (a, b) \\ &= 0 && \text{for } x \notin [a, b] \\ &= \frac{1}{2} && \text{at } x=a, b \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \lim_{N \rightarrow \infty} \int_a^b t G_N(t, x) dt &= x && \text{for } x \in (a, b) \\ &= 0 && \text{for } x \notin [a, b] \\ &= \frac{1}{2} && \text{for } x=a, b \end{aligned}$$

Proof: Part (i) can be proved as follows.

$$\begin{aligned}
 \int_a^b G_N(t, x) dt &= \int_0^N \tau \theta \pi \tau P_{-\frac{1}{2}+i\tau}(chx) d\tau \int_a^b P_{-\frac{1}{2}+i\tau}(cht) sht dt \\
 &= \int_0^N \tau \theta \pi \tau P_{-\frac{1}{2}+i\tau}(chx) [R_{-\frac{1}{2}+i\tau}(b) - R_{-\frac{1}{2}+i\tau}(a)] d\tau \\
 &= \psi_N(b, x) - \psi_N(a, x)
 \end{aligned}$$

Therefore, in view of Lemma 4.2(a), we obtain

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \int_a^b G_N(t, x) dt &= 1 \quad \text{for } x \in (a, b) \\
 &= 0 \quad \text{for } x \notin [a, b] \\
 &= \frac{1}{2} \quad \text{for } x = a, b.
 \end{aligned}$$

Part (ii) of this lemma can be proved in a similar way. In fact,

$$\begin{aligned}
 \int_a^b t G_N(t, x) dt &= \int_0^N \tau \theta \pi \tau P_{-\frac{1}{2}+i\tau}(chx) d\tau \int_a^b \tau sht P_{-\frac{1}{2}+i\tau}(cht) dt \\
 &= \theta_N(b, x) - \theta_N(a, x)
 \end{aligned}$$

The conclusion is immediate from Lemma 4.2(b).

CONVENTION: In the rest of this Chapter, L will denote an arbitrarily large but fixed positive number.

COROLLARY 4.4. Let $b > a > 0$. Then, for any $x \in (0, L]$

$$\lim_{N \rightarrow \infty} \int_a^b (t-x) G_N(t, x) dt = 0$$

Proof: Since

$$\int_a^b (t-x) G_N(t, x) dt = \int_a^b t G_N(t, x) dt - x \int_a^b G_N(t, x) dt$$

In view of the previous lemma,

$$\lim_{N \rightarrow \infty} \int_a^b (t-x) G_N(t, x) dt = 0 \quad \text{for all } x \text{ in } (0, L].$$

LEMMA 4.5. Let $b > a > 0$. Then, for a fixed δ satisfying $0 < \delta < \min(\frac{1}{2}, \frac{a}{2})$

$$(i) \quad \int_a^{x-\delta} (t-x) G_N(t, x) dt \rightarrow 0$$

uniformly for all $x \in (a, L]$ as $N \rightarrow \infty$.

$$(ii) \quad \int_{x+\delta}^b (t-x) G_N(t, x) dt \rightarrow 0$$

uniformly for all $x \in [\delta, b-\delta]$ as $N \rightarrow \infty$.

Proof:

(i) Changing the order of integration, we can write

$$\int_a^{x-\delta} (x-t) G_N(t, x) dt = \int_0^N \tau \operatorname{th} \pi \tau d\tau \int_a^{x-\delta} (x-t) \operatorname{sht} P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) dt$$

$$= \int_0^\infty \tau \operatorname{th} \pi \tau d\tau \int_a^{x-\delta} (x-t) \operatorname{sht} P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) dt,$$

$$= \int_N^\infty \tau \operatorname{th} \pi \tau d\tau \int_a^{x-\delta} (x-t) \operatorname{sht} P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) dt$$

Now, by Corollary 4.4,

$$I_1 = \int_0^\infty \tau \operatorname{th} \pi \tau d\tau \int_a^{x-\delta} (x-t) \operatorname{sht} P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) dt = 0,$$

and for sufficiently large N , using Equation (1.5), we have

$$I_2 = \int_N^\infty \tau \operatorname{th} \pi \tau d\tau \int_a^{x-\delta} (x-t) \operatorname{sht} P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) dt$$

$$= \frac{1}{\pi \sqrt{\operatorname{sh} x}} \int_N^\infty \tau \operatorname{th} \pi \tau [1 + O(\frac{1}{\tau})] d\tau \int_a^{x-\delta} \sqrt{\operatorname{sht}}(x-t) \cdot 2 \sin(x\tau + \frac{\pi}{4}) \sin(\tau\tau + \frac{\pi}{4}) dt.$$

$$+ \frac{1}{8\pi \sqrt{\operatorname{sh} x}} \int_N^\infty \frac{\tau \operatorname{th} \pi \tau}{\tau} [1 + O(\frac{1}{\tau})] d\tau \int_a^{x-\delta} (x-t) \operatorname{coth} t \sqrt{\operatorname{sht}} \cdot 2 \sin(x\tau + \frac{\pi}{4}) \sin(\tau\tau - \frac{\pi}{4})$$

$$+ \frac{2}{\pi \sqrt{\operatorname{sh} x}} \int_N^\infty \tau \operatorname{th} \pi \tau [1 + O(\frac{1}{\tau})] d\tau \int_a^{x-\delta} (x-t) \sqrt{\operatorname{sht}} \sin(x\tau + \frac{\pi}{4}) O(\frac{1}{\tau^2}) dt.$$

$$= I_1 + L_2 + L_3 \quad (\text{say}),$$

the orders being uniform with respect to t and x lying in compact subsets of $(0, \infty)$. At first consider L_1 .

$$L_1 = \frac{1}{\pi \sqrt{\text{sh} x}} \int_N^\infty \text{th} \pi \tau \left[1 + O\left(\frac{1}{\tau}\right) \right] d\tau \int_a^{x-\delta} (x-t) \sqrt{\text{sh} t} [\cos \tau(x-t) + \sin \tau(x+t)] dt$$

Now, integrating by parts twice, we get

$$\begin{aligned} L_1 = & \frac{1}{\pi \sqrt{\text{sh} x}} \int_N^\infty \left[1 + O\left(\frac{1}{\tau}\right) \right] \frac{\text{th} \pi \tau}{\tau} d\tau \left\{ (x-a) \sqrt{\text{sh} a} (\sin \tau(x-a) + \cos \tau(x+a)) \right. \\ & - \delta \sqrt{\text{sh}(x-\delta)} (\sin \delta \tau + \cos \tau(2x-\delta)) + \frac{1}{\tau} \left[\left(\frac{\delta \text{ch}(x-\delta)}{2 \sqrt{\text{sh}(x-\delta)}} - \sqrt{\text{sh}(x-\delta)} \right) \right. \\ & \times (\cos \tau \delta + \sin \tau(2x-\delta)) - \left(\frac{\frac{1}{2}(x-a) \text{cha}}{\sqrt{\text{sh} a}} - \sqrt{\text{sh} a} \right) \cdot (\cos \tau(x-a) + \sin \tau(x+a)) \\ & \left. \left. - \int_a^{x-\delta} \left(\frac{1}{2} \sqrt{\text{sh} t} (x-t) - \frac{1}{4} \frac{(x-t) \text{ch}^2 t}{(\text{sh} t)^{3/2}} - \frac{\text{cht}}{\sqrt{\text{sh} t}} \right) (\cos \tau(x-t) + \sin \tau(x+t)) dt \right] \right\} \end{aligned}$$

Therefore, a careful computation shows that

$$\begin{aligned} |L_1| \leq & \frac{\sqrt{\text{sh} a}}{\pi \sqrt{\text{sh} x}} \left| \int_N^\infty \frac{\text{th} \pi \tau}{\tau} [(x-a) \sin \tau(x-a) + (x-a) \cos \tau(x+a)] d\tau \right| \\ & + \frac{\delta}{\pi} \sqrt{\frac{\text{sh}(x-\delta)}{\text{sh} x}} \left| \int_N^\infty \frac{\text{th} \pi \tau}{\tau} [\sin \pi \delta + \cos \tau(2x-\delta)] d\tau \right| \\ & + A \int_N^\infty \frac{d\tau}{\tau^2} + B \int_N^\infty \frac{d\tau}{\tau^3} \end{aligned}$$

where A and B are positive constants independent of x . Let Q_i , $i=1,2,3,4,5$ represent the integrals on the right-hand side in that order. It can be easily seen that Q_3 and Q_4 approach zero uniformly for all x as $N \rightarrow \infty$.

Now, integrating by parts, we have

$$\begin{aligned}
 Q_1 &= \frac{\sqrt{\text{sha}}}{\pi \sqrt{\text{sh}x}} \left| \int_N^\infty \frac{\text{th}\pi\tau}{\tau} [(x-a)\sin\tau(x-a) + (x-a)\cos\tau(x+a)] d\tau \right| \\
 &= \frac{1}{\pi} \sqrt{\frac{\text{sha}}{\text{sh}x}} \left| \frac{\text{th}\pi N}{N} \left[\cos N(x-a) - \left(\frac{x-a}{x+a}\right) \sin\tau(x+a) \right] \right. \\
 &\quad \left. - \int_N^\infty \left[\frac{\pi \text{sech}^2 \pi\tau}{\tau} - \frac{\text{th}\pi\tau}{\tau^2} \right] \left[\cos\tau(x-a) - \left(\frac{x-a}{x+a}\right) \sin\tau(x+a) \right] d\tau \right| \\
 &= \frac{1}{\pi} \sqrt{\frac{\text{sha}}{\text{sh}x}} O\left(\frac{1}{N}\right) \quad \text{as } N \rightarrow \infty.
 \end{aligned}$$

Thus, $Q_1 \rightarrow 0$ uniformly for all $x \in (a, L]$ as $N \rightarrow \infty$.

Next consider

$$\begin{aligned}
 Q_2 &= \frac{\delta}{\pi} \sqrt{\frac{\text{sh}(x-\delta)}{\text{sh}x}} \left| \int_N^\infty \frac{\text{th}\pi\tau}{\tau} [\sin\delta\tau + \cos\tau(2x-\delta)] d\tau \right| \\
 &= \frac{\delta}{\pi} \sqrt{\frac{\text{sh}(x-\delta)}{\text{sh}x}} \left| \frac{\text{th}\pi N}{N} \left(\frac{\cos\delta N}{\delta} - \frac{\sin N(2x-\delta)}{2x-\delta} \right) \right. \\
 &\quad \left. - \int_N^\infty \left(\frac{\pi \text{sech}^2 \pi\tau}{\tau} - \frac{\text{th}\pi\tau}{\tau^2} \right) \left(-\frac{\cos\tau\delta}{\delta} + \frac{\sin\tau(2x-\delta)}{2x-\delta} \right) d\tau \right|
 \end{aligned}$$

(integrating by parts)

$$\leq O\left(\frac{1}{N}\right) \quad \text{for all } x \in (a, L]$$

Hence, $Q_2 \rightarrow 0$ uniformly as $N \rightarrow \infty$. Similarly, it can be shown that L_2 and $L_3 \rightarrow 0$ uniformly for all $x \in (a, L]$ as $N \rightarrow \infty$. Thus $I_2 \rightarrow 0$ as $N \rightarrow \infty$ uniformly for all $x \in (a, L]$.

Altogether we have, therefore, proved that

$$\int_a^{x-\delta} (t-x)G_N(t,x)dt \rightarrow 0$$

uniformly for all $x \in (a, L]$ as $N \rightarrow \infty$.

(ii) Proceeding as in part (i), for sufficiently large N , one can write

$$\int_{x+\delta}^b (x-t)G_N(t,x)dt = J_1 - J_2$$

where

$$J_1 = \int_0^\infty \tau \operatorname{th} \pi \tau d\tau \int_{x+\delta}^b (x-t) \operatorname{sht} P_{-\frac{1}{2}+i\tau}(\operatorname{ch} x) P_{-\frac{1}{2}+i\tau}(\operatorname{ch} t) dt = 0$$

(By Corollary 4.4)

and

$$\begin{aligned} J_2 &= \int_N^\infty \tau \operatorname{th} \pi \tau d\tau \int_{x+\delta}^b (x-t) \operatorname{sht} P_{-\frac{1}{2}+i\tau}(\operatorname{ch} x) P_{-\frac{1}{2}+i\tau}(\operatorname{ch} t) dt \\ &= \frac{1}{\pi \sqrt{\operatorname{sh} x}} \int_N^\infty \operatorname{th} \pi \tau \left[1 + O\left(\frac{1}{\tau}\right)\right] d\tau \int_{x+\delta}^b \sqrt{\operatorname{sht}}(x-t) [\cos \tau(x-t) + \sin \tau(x+t)] dt \\ &\quad + \frac{1}{\pi \sqrt{\operatorname{sh} x}} \int_N^\infty \operatorname{th} \pi \tau \left[1 + O\left(\frac{1}{\tau}\right)\right] d\tau \int_{x+\delta}^b \sqrt{\operatorname{sht}}(x-t) \coth t \cdot 2 \sin\left(x\tau + \frac{\pi}{4}\right) \sin\left(t\tau - \frac{\pi}{4}\right) dt \\ &\quad + \frac{2}{\pi \sqrt{\operatorname{sh} x}} \int_N^\infty \operatorname{th} \pi \tau \left[1 + O\left(\frac{1}{\tau}\right)\right] d\tau \int_{x+\delta}^b (x-t) \sqrt{\operatorname{sht}} \sin\left(x\tau + \frac{\pi}{4}\right) O\left(\frac{1}{\tau^2}\right) dt \\ &= L'_1 + L'_2 + L'_3 \quad (\text{say}). \end{aligned}$$

Consider L'_1 at first. Integrating the inner integral by parts twice, we have

$$\begin{aligned}
L_1' = & \frac{1}{\pi \sqrt{\text{sh} x}} \int_N^\infty \frac{\text{th} \pi \tau}{\tau} \left[1 + O\left(\frac{1}{\tau}\right) \right] \left\{ \sqrt{\text{sh} b} (x-b) (\sin \tau (b-x) - \cos \tau (x+b)) \right. \\
& + \delta \sqrt{\text{sh} (x+\delta)} (\sin \delta \tau - \cos \tau (2x+\delta)) + \frac{1}{\tau} \left[\left(\frac{1}{2} \frac{(x-b) \text{ch} b}{\sqrt{\text{sh} b}} - \sqrt{\text{sh} b} \right) \right. \\
& \times (\cos \tau (x-b) + \sin \tau (x+b)) + \left(\frac{\delta}{2} \frac{\text{ch} (x+\delta)}{\sqrt{\text{sh} (x+\delta)}} + \sqrt{\text{sh} (x+\delta)} \right) \\
& \times (\text{ch} \tau \delta + \cos \tau (2x+\delta)) - \int_{x+\delta}^b \left(\frac{1}{2} \sqrt{\text{sh} t} (x-t) - \frac{\text{cht}}{\sqrt{\text{sh} t}} - \frac{(x-t) \text{ch}^2 t}{4 (\text{sh} t)^{3/2}} \right) \\
& \left. \left. \times (\cos \tau (x-t) + \sin \tau (x+t)) dt \right\} d\tau
\end{aligned}$$

Therefore, we can find positive constants D, E and F independent of x such that

$$\begin{aligned}
|L_1| \leq & \frac{\sqrt{\text{sh} b}}{\pi \sqrt{\text{sh} x}} \left| \int_N^\infty \frac{\text{th} \pi \tau}{\tau} [(x-b) \sin \tau (x-b) - (x-b) \cos \tau (x+b)] d\tau \right| \\
& + \frac{\delta}{\pi} \sqrt{\frac{\text{sh} (x+\delta)}{\text{sh} x}} \left| \int_N^\infty \frac{\text{th} \pi \tau}{\tau} [\sin \tau \delta - \cos \tau (2x+\delta)] d\tau \right| \\
& + D \int_N^\infty \frac{d\tau}{\tau^2} + E \int_N^\infty \frac{d\tau}{\tau^3}
\end{aligned}$$

Clearly, the last two integrals on the right-hand side converge to zero uniformly for all x as $N \rightarrow \infty$. Let P_1 and P_2 denote the first two integrals on the right-hand side in that order. Now,

$$\begin{aligned}
P_1 &= \frac{1}{\pi} \sqrt{\frac{\sinh b}{\sinh x}} \left| \int_N^\infty \frac{\tanh \pi \tau}{\tau} [(x-b) \sin \tau(x-b) - (x-b) \cos \tau(x+b)] d\tau \right| \\
&= \frac{1}{\pi} \sqrt{\frac{\sinh b}{\sinh x}} \left| \frac{\tanh \pi N}{N} \left(\cos N(x-b) + \frac{x-b}{x+b} \sin N(x+b) \right) \right. \\
&\quad \left. + \int_N^\infty \left(\frac{\pi \operatorname{sech}^2 \pi \tau}{\tau} - \frac{\tanh \pi \tau}{\tau^2} \right) \left(\cos \tau(x-b) + \frac{x-b}{x+b} \sin \tau(x+b) \right) d\tau \right| \\
&\quad \text{(Integrating by parts)}
\end{aligned}$$

Thus, we can find a constant A independent of x such that

$$P_1 < \frac{A}{N}$$

Therefore, $P_1 \rightarrow 0$ as $N \rightarrow \infty$ uniformly for all $x \in (\delta, b-\delta)$. Similarly, it can be shown that $P_2 \rightarrow 0$ as $N \rightarrow \infty$ uniformly for all $x \in (\delta, b-\delta)$. Thus $L_1' \rightarrow 0$ as $N \rightarrow \infty$ uniformly for $x \in (\delta, b-\delta)$. Similarly, it can be shown that L_2' and $L_3' \rightarrow 0$ as $N \rightarrow \infty$ uniformly for all $x \in (\delta, b-\delta)$. This completes the proof of the lemma.

LEMMA 4.6. For $b > a > 0$, let $t \in [a, b]$. Then, for $x \in [\delta, L]$ where

$0 < \delta < \min(\frac{a}{2}, \frac{1}{2})$, $L > \delta$ is a fixed number, $(x-t)G_N(t, x)$ is uniformly bounded for all x , t and $N > 0$.

Proof: $(x-t)G_N(t, x) \equiv 0$ whenever $t=x$. Therefore, assume that $t \neq x$. Now we consider two cases.

Case I. Let N be any fixed positive number, say N_0 . Then, using inequality (1.4) which holds for all x and τ , we have

$$(x-t)G_{N_0}(t, x) \leq |(x-t)| \cdot \frac{t/2}{\sinh t/2} \cdot \frac{x/2}{\sinh x/2} \int_0^{N_0} \tau \tanh \pi \tau d\tau$$

$$|(x-t)G_N(t,x)| \leq \sup_{x,t} |x-t| \frac{t/2}{\text{sh } t/2} \cdot \frac{x/2}{\text{sh } x/2} \int_0^{N_0} \tau \text{th} \pi \tau < \infty \quad (4.1)$$

Case II. Let N_0 be a large fixed number. Then, for all $N > N_0$

$$(x-t)G_N(t,x) = (x-t)G_{N_0}(t,x) + I \quad (4.2)$$

where

$$\begin{aligned} I &= (x-t) \text{sht} \int_{N_0}^N \tau \text{th} \pi \tau P_{-\frac{1}{2}+i\tau}(chx) P_{-\frac{1}{2}+i\tau}(cht) d\tau \\ &\quad \text{(Using Equation (1.5))} \\ &= \frac{(x-t)}{\pi} \sqrt{\frac{\text{sht}}{\text{sh}x}} \int_{N_0}^N [1+O(\frac{1}{\tau})]^2 \text{th} \pi \tau \cdot 2 \sin(x\tau + \frac{\pi}{4}) \sin(t\tau + \frac{\pi}{4}) d\tau \\ &= \frac{(x-t)}{\pi} \sqrt{\frac{\text{sht}}{\text{sh}x}} \int_{N_0}^N [1+O(\frac{1}{\tau})]^2 \text{th} \pi \tau (\cos \tau(x-t) + \sin \tau(x+t)) d\tau \end{aligned}$$

Therefore, we can find positive constants A and B independent of x such that,

$$\begin{aligned} |I| &\leq \frac{1}{\pi} \sqrt{\frac{\text{sht}}{\text{sh}x}} \left| \int_{N_0}^N (x-t) \text{th} \pi \tau g_\tau(t,x) d\tau \right| + A \sqrt{\frac{\text{sht}}{\text{sh}x}} \left| \int_{N_0}^N (x-t) \times \right. \\ &\quad \left. \frac{\text{th} \pi \tau}{\tau} g_\tau(t,x) d\tau \right| + B \frac{1}{\pi} \sqrt{\frac{\text{sht}^2}{\text{sh}x}} \left| \int_{N_0}^N (x-t) \frac{\text{th} \pi \tau}{\tau^2} g_\tau(t,x) d\tau \right| \quad (4.3) \end{aligned}$$

where

$$g_\tau(t,x) = \cos \tau(x-t) + \sin \tau(x+t).$$

Let I_1 , I_2 and I_3 denote the corresponding integrals on the right-hand side of (4.3) in that order. First consider I_1 . Integration by parts yields

$$\begin{aligned} I_1 = & \frac{1}{\pi} \sqrt{\frac{sh t}{sh x}} \left[\operatorname{th} \pi N_0 \left[\sin N_0 (x-t) + \left(\frac{x-t}{x+t} \right) \cos N_0 (x+t) \right] \right. \\ & - \operatorname{th} \pi N \left[\sin N (x-t) + \left(\frac{x-t}{x+t} \right) \cos N (x+t) \right] \\ & \left. + \int_{N_0}^N \pi \operatorname{sech}^2 \pi \tau \left[\sin \tau (x-t) + \left(\frac{x-t}{x+t} \right) \cos \tau (x+t) \right] d\tau \right] \end{aligned}$$

So,

$$I_1 \leq C \sup_{x,t} \sqrt{\frac{sh t}{sh x}} \left(1 + \frac{|x-t|}{x+t} \right) \left(1 + \int_{N_0}^N \pi \operatorname{sech}^2 \pi \tau d\tau \right) \quad (4.4)$$

where C is a positive constant.

Clearly, I_1 is bounded by a constant independent of x, t for all $N > N_0$. Similarly, it can be shown that I_2 and I_3 are uniformly bounded for all $x \in [\delta, L]$, $t \in [a, b]$ and $N > 0$. Thus, combining Equations (4.1) to (4.4), we have shown that $(x-t)G_N(t, x)$ is uniformly bounded for all x, t and $N > 0$.

LEMMA 4.7. Let $\phi(t) \in \mathcal{D}(I)$ with its support contained in $[a, b]$. Then, for $0 < \delta < \min(\frac{1}{2}, \frac{\pi}{2})$,

$$(i) \int_a^{x-\delta} G_N(t,x) [\phi(t) - \phi(x)] dt \rightarrow 0 \text{ as } N \rightarrow \infty$$

uniformly for all $x \in [\delta, L]$.

$$(ii) \int_{x+\delta}^b G_N(t,x) [\phi(t) - \phi(x)] dt \rightarrow 0 \text{ as } N \rightarrow \infty$$

uniformly for all $x \geq 2\delta$.

Proof:

(i) For a fixed $x > a + \delta$, $t \neq x$. Therefore, we can write

$$\begin{aligned} \int_a^{x-\delta} G_N(t,x) [\phi(t) - \phi(x)] dt &= \int_a^{x-\delta} G_N(t,x) (t-x) \cdot \frac{\phi(t) - \phi(x)}{t-x} dt \\ &= \int_a^{x-\delta} G_N(t,x) (t-x) \theta(x,t) dt \end{aligned} \quad (4.5)$$

where $\theta(t,x)$ is defined by

$$\theta(t,x) = \frac{\phi(t) - \phi(x)}{t-x}, \quad \text{for } t \neq x$$

$$= \phi'(x), \quad \text{for } t = x$$

Clearly, $\theta(t, x)$ is a continuous function of t and x . For $x \leq a$, the integral (4.5) vanishes. Therefore, it is enough to prove the lemma for all $x \in (a, L]$.

Consider at first the case $0 \leq t \leq x - \delta$. In view of Lemma 4.6, we can find a constant K such that

$$|G_N(t, x)(t - x)| < K$$

for all $x \in (a, L]$, $t \in (a, L - \delta]$ and $N > 0$.

Now, let a set of lines parallel to the coordinate axes divide the region bounded by $a < x \leq L$ and $a \leq t \leq x - \delta$ into n subdivisions $D_1, D_2, D_3, \dots, D_n$, so small that the variation of $\theta(t, x)$ in each subdivision is less than $\epsilon/2K(L - \delta - a)$. Let $\psi(t, x)$ be the function which in the interior of each subdivision D_i , $i = 1, 2, \dots, n$ has a constant value $C_i \triangleq \theta(t_i, x_i)$; where (t_i, x_i) is an arbitrary point in the interior of the i th subdivision D_i , $i = 1, 2, \dots, n$. At the boundary of each subdivision, we can take $\psi(t, x)$ to be zero. Therefore, $\psi(t, x)$ has a finite set of values $C_1, C_2, \dots, C_n, 0$. Thus, for a fixed x the line segment from a to $x - \delta$ is divided into n subintervals $(x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n)$ where $x_0 = a$ and $x - \delta = x_n$, in the interior of each of which $\psi(t, x)$ is a constant. Therefore, for a fixed x ,

$$\psi(t, x) = \sum_{i=1}^n C_i \chi_i(t, x)$$

where $\chi_i(t, x)$ is the characteristic function of the i th subinterval.

Since

$$|\theta(t, x) - \psi(t, x)| < \frac{\epsilon}{2K(L - \delta - a)}$$

therefore

$$\int_a^{x-\delta} |\theta(t, x) - \psi(t, x)| dt < \frac{\epsilon}{2K}$$

Now consider

$$v. \left| \int_a^{x-\delta} G_N(t, x) (t-x) \theta(t, x) dt \right|$$

$$\leq \left| \int_a^{x-\delta} G_N(t, x) (t-x) [\theta(t, x) - \psi(t, x)] dt \right|$$

$$+ \left| \int_a^{x-\delta} \psi(t, x) G_N(t, x) (t-x) dt \right|$$

$$\leq \int_a^{x-\delta} |G_N(t, x) (t-x)| |\theta(t, x) - \psi(t, x)| dt$$

$$+ \left| \sum_{i=1}^n C_i \int_{x_{i-1}}^{x_i} G_N(t, x) (t-x) dt \right|$$

$$\leq K \int_a^{x-\delta} |\theta(t,x) - \psi(t,x)| dt + \sum_{i=1}^n |C_i| \left| \int_{x_{i-1}}^x G_N(t,x)(x-t) dt \right|$$

$$< K \frac{\varepsilon}{2K} + \sum_{i=1}^n |C_i| \left| \int_{x_{i-1}}^{x_i} G_N(t,x)(x-t) dt \right|$$

Since x lies outside the interval

$$[x_i, x_{i+1}]$$

for each $i=1,2,3,\dots,r$ where $r \leq n$, in view of Lemma 4.5.

$$\left| \int_{x_i}^{x_{i+1}} G_N(t,x)(x-t) dt \right| \rightarrow 0$$

as $N \rightarrow \infty$ uniformly for all $x \in [a, L]$. A positive number N_ε , independent of x , can be so chosen that

$$\left| \int_{x_{i-1}}^{x_i} G_N(t,x)(x-t) dt \right| < \frac{\varepsilon/2}{\sum_{i=1}^n |C_i|}$$

for $i=1,2,\dots,n$ and for all x and all $N \geq N_\varepsilon$.

Therefore,

$$\left| \int_a^{x-\delta} G_N(t,x)(x-t)\theta(t,x) dt \right| < \varepsilon$$

provided $N \geq N_\epsilon$, for all $x \in [X, L]$.

The case $x - \delta \leq t \leq a$, $x \in [a, L]$ can similarly be disposed off.

(ii) Next consider

$$\int_{x+\delta}^b G_N(t, x) [\phi(t) - \phi(x)] dt$$

Since $x + \delta \leq t \leq b$, t cannot be equal to x . Therefore, define $\theta(t, x)$ as in part (i) of this lemma and write

$$I \equiv \int_{x+\delta}^b G_N(t, x) [\phi(t) - \phi(x)] dt = \int_{x+\delta}^b G_N(t, x) (t-x) \theta(t, x) dt$$

Since $I \equiv 0$, for $x \geq b$, it is enough to prove the lemma for $2\delta \leq x \leq b$. Now following the pattern of the proof for part (i) of this lemma, we can easily show that

$$\int_{x+\delta}^b G_N(t, x) [\phi(t) - \phi(x)] dt \rightarrow 0$$

as $N \rightarrow \infty$ uniformly for all $x \geq 2\delta$.

LEMMA 4.8. Let $\phi(t) \in \mathcal{D}(I)$ with its support contained in the interval $[a, b]$. Then,

$$\int_a^b G_N(t, x) \phi(t) dt \rightarrow \phi(x),$$

in $M_\alpha(I)$ as $N \rightarrow \infty$.

Proof: It can be easily seen that

$$\Delta_x G_N(t, x) = \text{sht } \Delta_t \hat{G}_N(t, x)$$

where

$$\hat{G}_N(t, x) = \int_0^N \tau i h \pi \tau P_{-\frac{1}{2} + i\tau}(chx) \Delta_t [P_{-\frac{1}{2} + i\tau}(cht)] d\tau$$

Now,

$$\begin{aligned} \Delta_x \int_a^b G_N(t, x) \phi(t) dt &= \int_a^b \text{sht } \Delta_t \hat{G}_N(t, x) \phi(t) dt \\ &= \int_a^b G_N(t, x) \Delta_t [\phi(t)] dt \end{aligned}$$

(Integration by parts).

Therefore, operating by Δ_x successively k -times, we get

$$\Delta_x^k \int_a^b G_N(t, x) \phi(t) dt = \int_a^b G_N(t, x) \phi_k(t) dt$$

where

$$\phi_k(t) \equiv \Delta_t^k \phi(t)$$

Now, using Lemma 4.3, we can write

$$\begin{aligned} \lim_{N \rightarrow \infty} \zeta(x) \Delta_x^k \left[\int_a^b G_N(t, x) \phi(t) dt - \phi(x) \right] \\ = \lim_{N \rightarrow \infty} \zeta(x) \int_a^b G_N(t, x) [\phi_k(t) - \phi_k(x)] dt. \end{aligned}$$

It is therefore reduced to proving that

$$\zeta(x) \int_a^b G_N(t, x) [\psi(t) - \psi(x)] dt \rightarrow 0$$

uniformly for all x as $N \rightarrow \infty$ where $\psi(t) \in D(I)$ with its support contained in $[a, b]$.

For a fixed $x \geq 2\delta$, where $0 < \delta < \min(\frac{a}{2}, \frac{1}{2})$, we can write

$$\zeta(x) \int_a^b G_N(t, x) [\psi(t) - \psi(x)] dt = \zeta(x) \left(\int_a^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^b \right) G_N(t, x) [\psi(t) - \psi(x)] dt.$$

$$= I_1 + I_2 + I_3$$

where I_i , $i=1,2,3$ denotes the corresponding integral.

At first consider I_2 . For $x \geq b+\delta$ or $x \leq a-\delta$, I_2 is clearly zero.

Therefore, we consider I_2 for the case when $a-\delta < x < b+\delta$. We can write

$$I_2 = \zeta(x) \int_{x-\delta}^{x+\delta} G_N(t, x) dt \int_x^t \psi'(\eta) d\eta$$

Therefore,

$$|I_2| \leq \sup_{a \leq \eta \leq b} |\psi'(\eta)| \int_{x-\delta}^{x+\delta} |G_N(t, x)(t-x)| dt$$

$$\leq 2\delta \sup_{a \leq \eta \leq b} |\psi'(\eta)| \cdot \sup_{\substack{a-\frac{1}{2} \leq x \leq b+\frac{1}{2} \\ a-\frac{1}{2} \leq t \leq b+\frac{1}{2}}} |G_N(t, x)(t-x)|$$

Now, using Lemma 4.6, we can find a constant $D > 0$ and independent of δ such that

$$|I_2| \leq D\delta$$

For a given $\epsilon > 0$ we can choose $\delta < \min(\frac{a}{2}, \frac{1}{2}, \frac{\epsilon}{D})$ to obtain

$$|I_2| < \frac{\epsilon}{2} \quad (4.6)$$

Next, consider I_1 . Since $I_1 = 0$ if $x \leq a$. Therefore, we have to consider $x > a$. In view of Lemma 4.7, $I_1 \rightarrow 0$ uniformly for all $x \in (a, L]$ as $N \rightarrow \infty$. Hence, assume that $x > L$, where L is arbitrarily large number greater than b . Since $\psi(t)$ is of compact support contained in $[a, b]$

$$\begin{aligned} I_1 &= \zeta(x) \int_a^{x-\delta} G_N(t, x) \psi(t) dt \\ &= \zeta(x) \int_a^b G_N(t, x) \psi(t) dt \\ &= \zeta(x) \int_a^b \psi(t) sh^{\delta} dt \int_0^N \tau th \tau P_{-\frac{1}{2}+i\tau}(chx) P_{-\frac{1}{2}+i\tau}(cht) d\tau \end{aligned}$$

Let N_0 be a sufficiently large but fixed number greater than one. Then, using (1.4), we have

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$$|I_1| \leq \zeta(x) \int_a^b |\psi(t)| dt \int_0^N \tau \operatorname{th} \pi \tau \frac{x/2}{\operatorname{sh} x/2} \frac{t/2}{\operatorname{sh} t/2} \operatorname{sh} t d\tau$$

$$+ \zeta(x) \left| \int_a^b \psi(t) dt \int_{N_0}^N \tau \operatorname{th} \pi \tau P_{-\frac{1}{2}+i\tau}(\operatorname{ch} x) P_{-\frac{1}{2}+i\tau}(\operatorname{ch} t) \operatorname{sh} t d\tau \right|$$

$$= O(xe^{-x/2}) + \frac{1}{\pi \sqrt{\operatorname{sh} x}} \int_a^b \psi(t) \sqrt{\operatorname{sh} t} dt \int_{N_0}^N \tau \operatorname{th} \pi \tau [1+O(\frac{1}{\tau})] d\tau$$

$$2 \sin(x\tau + \frac{\pi}{4}) \sin(t\tau + \frac{\pi}{4}) d\tau + \frac{1}{8} \int_{N_0}^N \tau \operatorname{th} \pi \tau [1+O(\frac{1}{\tau})] d\tau$$

$$\coth t \cdot 2 \sin(x\tau + \frac{\pi}{4}) \sin(t\tau - \frac{\pi}{4}) d\tau + 2 \int_{N_0}^N \tau \operatorname{th} \pi \tau [1+O(\frac{1}{\tau})] \sin(x\tau + \frac{\pi}{4}) O(\frac{1}{\tau}) d\tau$$

$$= O(xe^{-x/2}) + Q_1 + Q_2 + Q_3 \quad (\text{say});$$

Now, integrating by parts, we have

$$|Q_1| \leq \frac{1}{\pi \sqrt{\operatorname{sh} x}} \left| \int_a^b \psi(t) \sqrt{\operatorname{sh} t} \left\{ \operatorname{th} \pi N \left(\frac{\sin(x-t)N}{x-t} - \frac{\cos N(x+t)}{x+t} \right) \right. \right.$$

$$\left. - \operatorname{th} \pi N_0 \left(\frac{\sin N_0(x-t)}{x-t} - \frac{\cos N_0(x+t)}{x+t} \right) \right.$$

$$\left. - \int_{N_0}^N \pi \operatorname{sech}^2 \pi \tau \left(\frac{\sin \tau(x-t)}{x-t} - \frac{\cos \tau(x+t)}{x+t} \right) d\tau \right\} dt \Big|$$

$$\leq O(e^{-x/2}), \quad x \rightarrow \infty.$$

Similarly, it can be shown that Q_2 and Q_3 are $O(x^\alpha e^{-x/2})$, where $\alpha \geq 0$, as

$x \rightarrow \infty$.

Therefore, I_1 can be made $< \epsilon/2$

(4.7)

for all $N > 0$ by choosing $x > L$, for sufficiently large L .

Finally, consider I_3 . By Lemma 4.7, $I_3 \rightarrow 0$ as $N \rightarrow \infty$

(4.8)

uniformly for all $x \geq 2\delta$.

Combining (4.6), (4.7) and (4.8) together, we have

$$\lim I \leq \epsilon \text{ uniformly for all } x \geq 2\delta$$

(4.9)

For $0 < x \leq 2\delta$, write

$$\begin{aligned} I &= x^\alpha \left(\int_a^{x+\delta} + \int_{x+\delta}^b \right) G_N(t, x) [\psi(t) - \psi(x)] dt \\ &= J_1 + J_2 \text{ (say)} \end{aligned}$$

Now,

$$J_1 = x^\alpha \int_a^{x+\delta} G_N(t, x) [\psi(t) - \psi(x)] dt$$

vanishes for $x+\delta \leq a$. Therefore, we take $x > a-\delta$. Since $a > 2\delta$, $x > \delta$.

Hence, we have to consider the case $\delta < x < 2\delta$.

$$|J_1| \leq x^\alpha \int_a^{x+\delta} |G_N(t, x)| |\psi(t) - \psi(x)| dt$$

$$\leq x^\alpha \int_a^{x+\delta} |G_N(t, x)| dt \left| \int_x^t \psi'(\eta) d\eta \right|$$

$$\begin{aligned}
&\leq x^\alpha \int_a^{x+\delta} |G_N(t, x)| dt \int_x^{x+\delta} |\psi'(n)| dn \\
&\leq \delta \sup_{a < n < b} |\psi'(n)| \int_a^{x+\delta} |G_N(t, x)| dt \\
&\leq \delta \sup_{a < n < b} |\psi'(n)| \int_a^{3/2} |G_N(tx)| dt
\end{aligned}$$

Since the last integral is bounded uniformly for all $x \in (\delta, 2\delta)$, we can find a positive constant C independent of ϵ satisfying

$$|J_1| \leq \delta C = \delta D \frac{C}{D} < \frac{\epsilon}{2} \frac{C}{D} \quad (4.10)$$

Next, consider J_2 . Since $0 < x < 2\delta$ and $2\delta < \min(a, 1)$

$$\begin{aligned}
J_2 &= x^\alpha \int_{x+\delta}^b G_N(t, x) \psi(t) dt \\
&= x^\alpha \int_a^b G_N(t, x) \psi(t) dt
\end{aligned}$$

Now,

$$\begin{aligned}
|J_2| &\leq x^\alpha \int_a^b |\psi(t)| sht dt \int_0^{N_0} \tau \theta \pi \tau |P_{-\frac{1}{2}+i\tau}(chx) P_{-\frac{1}{2}+i\tau}(cht)| d\tau \\
&\quad + x^\alpha \left| \int_a^b \psi(t) sht dt \int_{N_0}^N \tau \theta \pi \tau P_{-\frac{1}{2}+i\tau}(chx) P_{-\frac{1}{2}+i\tau}(cht) d\tau \right|
\end{aligned}$$

$$\begin{aligned}
&\leq x^\alpha \int_a^b |\psi(t)| \operatorname{sh} t dt \int_0^{N_0} t \operatorname{th} \pi t \frac{x/2}{\operatorname{sh} x/2} \frac{t/2}{\operatorname{sh} t/2} d\tau \\
&\quad + \frac{2}{\pi} x^\alpha \int_{N_0}^N \sqrt{\tau} \operatorname{th} \pi \tau \left| \int_a^b \psi(t) \sqrt{\operatorname{sh} t} \sin(\tau t + \frac{\pi}{4}) dt \right. \\
&\quad + \frac{1}{2\pi} \int_a^b \psi(t) \sqrt{\operatorname{sh} t} \operatorname{coth} t \sin(\tau t - \frac{\pi}{4}) d\tau + \frac{A}{\tau^2} \left| \int_a^b \psi(t) \sqrt{\operatorname{sh} t} dt \right| d\tau \\
&\leq O(x^\alpha) + Q_1 + Q_2 + Q_3 \quad (\text{say}).
\end{aligned}$$

Here, we have used the result (1.4) and the uniform asymptotic expansion (1.6) for $P_{-\frac{1}{2}+it}(x)$ where τ is large and x is small. First consider Q_1 . Integration by parts twice yields,

$$|I_1| \leq \sqrt{\frac{2}{\pi}} x^\alpha \int_{N_0}^N \frac{\operatorname{th} \pi \tau}{\tau^{3/2}} d\tau \int_a^b \sin(\tau t + \frac{\pi}{4}) dt$$

$$\begin{aligned}
&\left(\psi''(t) \sqrt{\operatorname{sh} t} + \frac{\psi'(t) \operatorname{ch} t}{\sqrt{\operatorname{sh} t}} + \frac{\psi(t)}{2} \sqrt{\operatorname{sh} t} - \frac{\psi(t)}{4} \frac{\operatorname{ch}^2 t}{(\operatorname{sh} t)^{3/2}} \right) dt \\
&= O(x^\alpha) \quad \text{as } x \rightarrow 0+
\end{aligned}$$

Similarly, it can be shown that Q_2 and Q_3 are $O(x^\alpha)$ as $x \rightarrow 0$.

Therefore,

$$\begin{aligned}
|J_2| &= O(x^\alpha) \quad \text{as } x \rightarrow 0 \\
&\leq Mx^\alpha \leq M2^\alpha \delta^\alpha \leq \frac{M2^\alpha \epsilon^\alpha}{D^\alpha} \quad (4.11)
\end{aligned}$$

Combining (4.10) and (4.11) together, we have

$$\lim_{N \rightarrow \infty} |I_1| \leq \frac{\epsilon}{2} \frac{C}{D} + \frac{M2^\alpha \epsilon^\alpha}{D^\alpha} \quad \text{for } 0 < x \leq 2\delta \quad (4.12)$$

Combining (4.9) and (4.12) together, we have

$$\lim_{N \rightarrow \infty} I = 0 \quad \text{uniformly for all } x > 0$$

This completes the proof of the lemma.

Now we state and prove our inversion theorem.

THEOREM 4.9. Let $f \in M'_\alpha(I)$ and let $f^*(\tau)$ be the distributional Mehler-Fock transformation of f defined by (3.1). Then, for each $\phi \in D(I)$

$$\left\langle \int_0^N t^{\frac{1}{2} + i\tau} f^*(\tau) P_{-\frac{1}{2} + i\tau}(cht) sht d\tau, \phi(t) \right\rangle \rightarrow \langle f, \phi \rangle$$

as $N \rightarrow \infty$.

Proof: Assume that the support of $\phi(t)$ is contained in the interval $[a, b] \subset (0, \infty)$. In view of Theorem 3.2, $f^*(\tau)$ is a continuous function of τ . The integral

$$\int_0^N t^{\frac{1}{2} + i\tau} f^*(\tau) P_{-\frac{1}{2} + i\tau}(cht) sht d\tau,$$

therefore is a continuous function of t and as a consequence it generates a regular distribution. Hence we can write

$$\left\langle \int_0^N t^{\frac{1}{2} + i\tau} f^*(\tau) P_{-\frac{1}{2} + i\tau}(cht) sht d\tau, \phi(t) \right\rangle$$

$$= \int_a^b \phi(t) sht dt \int_0^N \tau th \pi \tau P_{-\frac{1}{2}+i\tau}(cht) \langle f(x), P_{-\frac{1}{2}+i\tau}(chx) \rangle d\tau$$

$$= I \text{ (say)}$$

By Lemma 3.4, we have

$$I = \int_a^b \phi(t) sht \langle f(x), \int_0^N \tau th \pi \tau P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) d\tau \rangle dt \quad (4.13)$$

Once again, using Riemann sums technique of Lemma 3.4, we can write (4.13) as

$$I = \langle f(x), \int_a^b \phi(t) sht dt \int_0^N \tau th \pi \tau P_{-\frac{1}{2}+i\tau}(cht) P_{-\frac{1}{2}+i\tau}(chx) d\tau \rangle$$

$$= \langle f(x), \int_a^b \phi(t) G_N(t, x) dt \rangle$$

In view of Lemma 4.8, the integral

$$\int_a^b \phi(t) G_N(t, x) dt \rightarrow \phi(x) \text{ in } M_\alpha(I) \text{ as } N \rightarrow \infty.$$

Therefore, as $N \rightarrow \infty$

$$\langle f(x), \int_a^b \phi(t) G_N(t, x) dt \rangle \rightarrow \langle f, \phi \rangle.$$

Q.E.D.

CONCLUSION

Now I would like to sum-up some of the achievements of this dissertation.

In Chapter I, two complex inversion formulas of Byrne and Love for generalized Stieltjes transformation of functions were shown to be valid for a class of distributions. This was accomplished by transferring the complex inversion formulas on the testing function space of a class of distributions and then showing that the limiting process in the resulting formula converges in the topology of the testing function space. The main results of this chapter are contained in Theorems 4.1, 4.2, 5.1, 5.4 and 5.5. Theorems 5.1 and 5.5 are the extensions of the complex inversion theorems of Byrne and Love.

In chapter II, initial value and final value theorems of distributional Stieltjes transformation defined in Chapter I, were proved. To prove the initial value theorem I used Lojasiewicz's definition of the value of a distribution at a point while final value theorem was proved by using the definition of the value of a distribution at infinity. This chapter was concluded by proving an initial value and a final value Abelian theorems for the generalized Stieltjes transformation of functions considered by Byrne and Love. Theorems 2.1, 3.3, 4.1 and 4.2 contain the main result of this chapter.

Due to the wide applicability of various integral transformations in solving boundary value problems, a variety of integral

transforms have been extended to the various classes of distributions. However, still there are some transformation which have not yielded to such generalizations. The Mehler-Fock transform is one of these. In Chapter III, a successful attempt was made to change this situation. This was achieved by defining a testing function space containing the kernel $P_{-\frac{1}{2}+i\tau}(chx)$ for each τ and then defining the transformation $f^*(\tau)$ of a distribution f by

$$f^*(\tau) = \langle f(x), P_{-\frac{1}{2}+i\tau}(chx) \rangle$$

The corresponding inversion formula was proved by establishing

$$\lim_{N \rightarrow \infty} \langle \int_0^N \tau \tanh \pi \tau f^*(\tau) P_{-\frac{1}{2}+i\tau}(chx) \sinh \tau d\tau, \phi(\tau) \rangle = \langle f, \phi \rangle$$

In fact this is the main result of Chapter III. A big portion of this chapter was devoted to proving this theorem. Also, Theorems 3.2 and 3.3 contain results of some significance.

Finally, I would like to mention some of the things I have left out or have been forced to consider the way I did.

The treatment in Chapter I requires that $\operatorname{Re} p > 1$. At several places in this chapter I have made use of this fact. To

mention one' the convergence of $\frac{\epsilon^{p-1}}{(y-x-1\eta+\epsilon)^{p-1}}$ to 0 in $S_\alpha(I)$ at page 25 requires that $\text{Re } p > 1$. For if we write $p = \mu + i\nu$ and estimate the supremum by taking $y = x$

$$\sup_{y \in I} \left| \frac{(1+y)^{p-1}}{(y-x-1+\epsilon)^{p-1}} \right| > \frac{(1+x)^{\alpha \mu - 1} \epsilon}{|\epsilon - i\eta|^{1/\mu} e^{\pi |\nu|}} \rightarrow 0$$

for $\mu \leq 1$. So if $\text{Re } p \leq 1$ the inversion theorems of Byrne and Love are not valid for distributions in $S'_\alpha(I)$ defined in Chapter I. This is the reason I proved inversion theorems for $\text{Re } p > 1$. The inversion of distributional Stieltjes transformation for complex p which are not zero or negative integral is still an open question.

For the sake of simplicity I considered the Mehler-Fock transformation in its simpler form. In other words, I did not consider the generalized Mehler-Fock transformation which is defined by

$$f_m^*(\tau) = \int_0^\infty f(x) P_{-\frac{1}{2}+1\tau}^{-m}(chx) shx dx$$

The corresponding inversion formula is given by

$$f(x) = (-1)^m \int_0^\infty \tau th \pi \tau P_{-\frac{1}{2}+1\tau}^m(chx) f_m^*(\tau) d\tau$$

where $p_{-\frac{1}{2}+i\tau}^{\pm m}(\chi x)$ are the associated Legendre functions of first kind. It may be pointed out that the class of functions for which this result is valid is not yet clearly defined. Vilenkin [22] and Rosenthal [17] impose conditions on $f^*(\tau)$ but not on $f(t)$. For $m = 0$ this reduces to the case considered in Chapter III. It seems that with some restrictions on m , this inversion formula may be extended to distributions in a way similar to case $m = 0$. Extension of generalized Mehler-Fock formula to distributions is still an open problem.

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