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Non-Parametric, Non-Sequential Change-Point Analysis

by

William J. Pouliot

A thesis submitted to

the Faculty of Graduate Studies and Research

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics

Carleton University

Ottawa, Ontario, Canada

December, 2001

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Graduate Studies and Research acceptance of the thesis
Non-Parametric, Non-Sequential Change-point Analysis

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in partial fulfillment of the requirements
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Abstract

This dissertation is concerned with extending the non-parametric at most one change in the variance problem studied by E. Gombay, L. Horváth and M. Hušková [17] for the at most one change in variance case. The extension parallels the geometric method that was developed by M. R. Orasch in [23], [24] for constructing U-statistics based tests which can be used to detect multiple changes in the distribution of independent observations. The extension in Chapters 4 and 6 is achieved via results concerning weighted approximations of partial sum processes due to B. Szyszkowicz [31] - [34]. In particular a special case of B. Szyszkowicz [34, Theorem 2.1] is extended in the context of [17].

Chapter 1 provides many well-known results which are detailed in some definitions, lemmas and theorems. They figure prominently in the proofs of many of the theorems, lemmas and propositions to come.

Chapter 2 begins with a description of the generic change-point problem, and details some practical situations where change-point analysis can be used. This then leads to the statistical parameterization of change-point analysis and the construction of partial sum and U-statistic processes.

Chapter 3 introduces the at most two change-point partial sum process together with a detailed study of the asymptotic properties of certain functionals of this process under different assumptions. The asymptotic behaviour of these processes is studied under both the null $H_O^{(2)}$ and the alternative $H_A^{(2)}$.

Chapter 4 repeats the results established in chapter 3 but for weighted partial sum processes. In this case, asymptotic results are established only under $H_0^{(2)}$ and not $H_A^{(2)}$. It is expected that the test statistics constructed, as functionals of this process, should have greater power for detecting change-points on the tails than

their unweighted counterparts.

Chapter 5 extends results developed in chapter 3 to the at most s change-points in variance setting. Again, the limiting behaviour of certain functionals of the developed partial sum process is studied under both $H_O^{(s)}$ and $H_A^{(s)}$.

Chapter 6 extends the results developed in Chapter 4 regarding weights to the at most s changes in variance setting. Again, the behaviour is analyzed through a number of theorems but only under $H_O^{(s)}$.

Chapter 7 tabulates the limiting distribution of a number of change in mean statistics developed therein that are based on M. Csörgő and L. Horváth [5] via B. Szyszkowicz [34, Theorem 2.1], and can be used in testing the AMOC in mean hypothesis. Then a detailed Monte Carlo study is conducted where the power of many competing test statistics for the at most one change in mean hypothesis is explored. This chapter is founded on [25], a manuscript in preperation with Markus R. Orasch.

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Chapter 1

Definitions and Theorems

Since most of change-point analysis involves the study of the asymptotic behaviour of functionals of partial sums, it is important to provide some basic definitions.

Definition 1.0.1 A family of real valued random variables $\{X(t) := X(t, \omega); t \in T \subset \mathbb{R}\}$, defined on a probability space $\{\Omega, \mathcal{A}, \mathbb{P}\}$ will be called a stochastic process.

T , in most cases, is either a countable set or an interval and accordingly we talk about a discrete time or a continuous time stochastic process. For any fixed t , $X(t) = X(t, \omega)$ is a random variable. Throughout, we follow convention and suppress ω in notation and refer to a stochastic process as $\{X(t)\}$.

Following Csörgő and Révész [10, pages 21-22], we give their definition of a standard Wiener process. Here, as well as throughout, $\stackrel{D}{=}$ stands for equality in distribution.

Definition 1.0.2 A stochastic process $\{W(t) := W(t, \omega); 0 \leq t < \infty\}$, where $\omega \in \Omega$, and $\{\Omega, \mathcal{A}, \mathbb{P}\}$ is a probability space, is called a standard Wiener process if

1. $W(t) - W(s) \stackrel{D}{=} N(0, t - s)$ for all $0 \leq s < t < \infty$ and $W(0) = 0$,
2. $W(t)$ is an independent increment process, that is $W(t_2) - W(t_1)$, $W(t_4) - W(t_3), \dots, W(t_{2l}) - W(t_{2l-1})$ are independent random variables for all $0 \leq$

$$t_1 < t_2 \leq t_3 < t_4 \leq \cdots \leq t_{2l-1} < t_{2l} < \infty \ (l = 1, 2, \dots),$$

3. the sample path function $W(t, \omega)$ is continuous in t with probability one.

Remark 1.0.1 The assumption that $W(0) = 0$ is of convenience only. If a process $\{\bar{W}(t); 0 \leq t < \infty\}$ satisfies the other assumptions but not $\{\bar{W}(0) = 0\}$, then the Wiener process defined as $\{W(t) := \bar{W}(t) - \bar{W}(0); 0 \leq t < \infty\}$ will be a standard Wiener process.

Lemma 1.0.1 *The covariance function of a standard Wiener process is*

$$R(s, t) = \mathbf{E}(W(s)W(t)) = t \wedge s. \quad (1.1)$$

We define now a s -time parameter Gaussian process $\Gamma(t_1, \dots, t_s)$ as a linear combination of a standard Wiener process.

Definition 1.0.3 The s -time parameter Gaussian process $\Gamma(t_1, \dots, t_s)$ is defined by

$$\Gamma(t_1, \dots, t_s) := \sum_{i=1}^s (t_{i+1} - t_{i-1})W(t_i) - t_s W(1) \quad 0 \leq t_1 \leq t_2 \leq \cdots \leq t_s \leq 1, \quad (1.2)$$

where $t_0 := 0$ and $t_{s+1} := 1$.

Lemma 1.0.2 *For $\Gamma(t_1, \dots, t_s)$ defined in Definition 1.0.3, we have*

$$\begin{aligned} \mathbf{E}\Gamma(t_1, \dots, t_s) &= 0, \\ \mathbf{E}(\Gamma(t_1, \dots, t_s))^2 &= \sum_{i=1}^s (1 - t_{i+1} - t_i)^2 (t_{i+1} - t_i), \\ \mathbf{E}\Gamma(t_1, \dots, t_s)\Gamma(r_1, \dots, r_s) &= \sum_{i=1}^s \sum_{j=1}^s (t_{i+1} - t_{i-1})(r_{j+1} - r_{j-1})(t_i \wedge r_j) - t_s r_s, \end{aligned}$$

where $t_0 = r_0 := 0$ and $t_{s+1} = r_{s+1} := 1$.

Proof. The first is obvious. The covariance is a result of $\sum_{i=1}^s (t_{i+1} - t_{i-1})t_i = t_s$.

If, in Definition 1.0.3, $s = 1$, we have the following representation of a *Brownian bridge* $\{B(t); 0 \leq t \leq 1\}$.

Lemma 1.0.3 *Let $\{W(t); 0 \leq t < \infty\}$ be a standard Wiener process. Then we have the following representation of a Brownian bridge $B(t)$,*

$$B(t) := W(t) - tW(1) \quad 0 \leq t \leq 1. \quad (1.3)$$

If, in Definition 1.0.3, $s = 2$, we have the following representation of the two-time parameter Gaussian process $\Gamma(t_1, t_2)$.

Lemma 1.0.4 *Let $\{W(t); 0 \leq t < \infty\}$ be a standard Wiener process. Then*

$$\Gamma(t_1, t_2) := t_2W(t_1) + (1 - t_1)W(t_2) - t_2W(1). \quad (1.4)$$

Lemma 1.0.5 *Let $\{W(t); 0 \leq t < \infty\}$ be a standard Wiener process and define*

$$\Gamma_n(t_1, \dots, t_s) := \frac{\sum_{i=1}^s (t_{i+1} - t_{i-1})W(nt_i)}{\sqrt{n}} - \frac{t_s W(n)}{\sqrt{n}} \quad 0 \leq t_1 \leq t_2 \cdots \leq t_s \leq 1,$$

then, for each $n \geq 1$,

$$\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq t_2 \cdots \leq t_s \leq 1\} \stackrel{\mathcal{D}}{=} \{\Gamma(t_1, \dots, t_s); 0 \leq t_1 \leq t_2 \cdots \leq t_s \leq 1\},$$

where $t_0 := 0$ and $t_{s+1} := 1$.

Proof. Using Lemma 1.0.2, we see, for each $n \geq 1$, that the variance and covariance of the two Gaussian processes are the same or we can simply say that for each $n \geq 1$

$$\{W(t); 0 \leq t \leq 1\} \stackrel{\mathcal{D}}{=} \frac{1}{\sqrt{n}}\{W(nt); 0 \leq t \leq 1\}.$$

If, in Lemma 1.0.5, $s = 2$, we have the following representation of the two-time parameter Gaussian process $\Gamma_n(t_1, t_2)$.

Lemma 1.0.6 *Let $\{W(t); 0 \leq t < \infty\}$ be a standard Wiener process. Then*

$$\Gamma_n(t_1, t_2) := \frac{t_2 W(nt_1) + (1 - t_1)W(nt_2) - t_2 W(n)}{\sqrt{n}} \quad (1.5)$$

The direct, sufficient part of Kolmogorov's Strong Law of Large Numbers (SLLN) will be used frequently in the sequel. Here, and throughout, $\xrightarrow{a.s.}$ stands for almost sure convergence.

Theorem 1.0.1 *(Strong Law of Large Numbers SLLN) If $X_1, X_2, \dots$ are independent and identically distributed random variables with $\mathbb{E}|X_1| < \infty$, then, as $n \rightarrow \infty$,*

$$\frac{\sum_{i=1}^n X_i}{n} \xrightarrow{a.s.} \mathbb{E}X_1.$$

We make use also of two laws of the iterated logarithm (LIL), which are provided in Theorems 1.0.2 and 1.0.3.

Theorem 1.0.2 *If $X_1, X_2, \dots$ are independent and identically distributed random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$, then we have*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \log \log n}} \max_{1 \leq k \leq n} \frac{1}{k^{1/2}} \left| \sum_{i=1}^k X_i \right| \xrightarrow{a.s.} 1.$$

Proof. For a proof of Theorem 1.0.1 we refer to M. Csörgö and L. Horváth [3, page 368].

Theorem 1.0.3 *(Hartman-Wintner LIL) Let $X_1, X_2, \dots$ be independent and identically distributed random variables with $\mathbb{E}X_1 = \mu$ and finite variance σ^2 . Then we have*

$$\limsup_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} \left| \frac{\sum_{i=1}^{[nt]} (X_i - \mu)}{\sqrt{2n\sigma^2 \log \log n}} \right| \xrightarrow{a.s.} 1.$$

We provide now the following inequality due to Kolmogorov.

Theorem 1.0.4 (*Kolmogorov*) Suppose that $X_1, \dots, X_n$ are independent random variables with mean 0 and finite variances. For $\alpha > 0$,

$$P \left\{ \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\lfloor (n+1)t \rfloor} X_i \right| \geq \alpha \right\} \leq \frac{\mathbb{E}[\sum_{i=1}^n X_i]^2}{\alpha^2}. \quad (1.6)$$

Proof. Refer to P. Billingsley [2].

Theorem 1.0.5 (*Hájek-Rényi*) If X_i , $1 \leq i \leq n$, are independent random variables with $\mathbb{E}X_i = 0$ and finite variances $\sigma_i^2 := \mathbb{E}X_i^2$. Let $S_k := \sum_{i=1}^k X_i$, and if $\{c_i\}_{i=1}^n$ is a non-increasing sequence of positive constants, then, for any $\varepsilon > 0$, we have

$$P \left\{ \max_{1 \leq k \leq n} c_k |S_k| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon^2} \left\{ \sum_{k=1}^n c_k^2 \mathbb{E}(X_k^2) \right\}. \quad (1.7)$$

The last three theorems provided are the Lindeberg-Lévy and Lindeberg-Feller versions of the Central Limit Theorem (CLT) and Donsker's functional CLT.

Theorem 1.0.6 (*The Lindeberg-Lévy CLT*). Suppose $\{X_i\}_{i=1}^\infty$ is an independent and identically distributed sequence of random variables with finite $\mathbb{E}X_1 = \mu$ and $\mathbb{E}(X_1 - \mu)^2 = \sigma^2 > 0$ and put $S_n := X_1 + \dots + X_n$. Then, as $n \rightarrow \infty$,

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0, 1).$$

Theorem 1.0.7 (*The Lindeberg-Feller CLT*). Suppose for each n that the sequence of random variables $X_{n,1}, \dots, X_{n,r_n}$ is independent with mean $\mathbb{E}(X_{nk}) = 0$ and finite variance $\sigma_{nk}^2 = \mathbb{E}(X_{nk}^2)$, and set $s_n^2 := \sum_{k=1}^{r_n} \sigma_{nk}^2$. If

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{|X_{nk}| \geq \varepsilon s_n} X_{nk}^2 dP = 0$$

holds for all positive ε , then, $n \rightarrow \infty$,

$$\frac{S_n}{s_n} \xrightarrow{\mathcal{D}} N(0, 1).$$

Theorem 1.0.8 (Donsker 1951). Suppose $\{X_n\}_{n=1}^\infty$ is an independent and identically distributed sequence of random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = \sigma^2 > 0$.

Define

$$S_n(t) := \frac{S_{[nt]} + X_{[nt]+1}(nt - [nt])}{\sigma\sqrt{n}}, \quad 0 \leq t \leq 1,$$

let $\{W(t); 0 \leq t \leq 1\}$ be a standard Wiener process. Then, as $n \rightarrow \infty$,

$$h(S_n(t)) \xrightarrow{\mathcal{D}} h(W(t)),$$

for all $h : C[0, 1] \rightarrow \mathbb{R}$ that are continuous in sup-norm on C , or sup-norm continuous except at points forming a set of Wiener measure 0.

Chapter 2

Topics on Non-parametric, Non-sequential Change-point Analysis

2.1 The Generic Change-point Problem

Let us consider a random process that generates independent observations indexed by a non-random factor that will be referred to as time. These observations are collected over varying values of time with the purpose of monitoring the process for uniformity in distribution. One might suspect that there has been some changes in the random process, in this regard during data collection and, consequently, considers the following questions.

1. Has there in fact been at least one change in the underlying distribution of the random process?
2. If there have been changes in our process, for what value of time did it occur?
3. If there has been one change, has there been two or more changes as well?

The above description of a generic change-point problem is sufficiently general that it can adequately describe many statistical problems. This may include:

- (a) Industrial production involving output produced to a required quality such as the brix content of orange juice.
- (b) Variation of share prices traded on major stock exchanges.
- (c) Times between aircraft arrivals at large airports.
- (d) Annual flow of water in the Nile River.

Statistical analysis of these change-point problems depends on the mode of data collection. If data collection is to be continued until some random time that is to be determined via analysing the process as time goes by, then the appropriate statistical procedure is called sequential. If, however, a large finite collection of data was collected with the purpose of determining if at least one change-point occurred, then this is referred to as non-sequential. Both sequential and non-sequential statistical modes of data collection can use either parametric or non-parametric methods. Parametric methods generally assume the random observations follow some parametric distribution such as $N(\mu, \sigma^2)$, whereas non-parametric methods make no such assumption.

Change-point analysis has been an active area of research over the past 45 years. Some of this research dates as far back as 1954 with the work of E. S. Page [26], [27]. Nevertheless, many questions still remain unanswered. For example, D. Wolfe and E. Schectman [35], list many questions and make reference to the following list.

- (a) Confidence intervals and bounds for the change-point.
- (b) Relative properties of naturally competing non-parametric point estimators for the change-point.

- (c) Non-parametric inference procedures for data involving possibly more than one change-point.

We shall see that item (c) figures prominently in what is to come.

2.2 From a Non-parametric Two Sample Problem to a Change-point Problem

Let $\{X_1, \dots, X_{k-1}, X_k, X_{k+1}, \dots, X_n\}$ be independent random variables, where X_i for $i = 1, \dots, k$ have cumulative distribution function (CDF) $F(x)$, and X_i for $i = k+1, \dots, n$ has CDF $F(x - \Delta)$, $-\infty < \Delta < \infty$. Δ is referred to as a location or shift parameter. This parametrization of the non-parametric change-point problem will be used in Chapter 7. As specified, this is considered to be a two sample problem with unknown location parameter Δ . However, when the integer k is unknown, this is no longer a standard two sample problem but becomes what is now referred to as a *change-point problem*.

Page [26] and [27] was the first to consider a non-parametric parametrization of the at most one change-point (AMOC) problem. In particular, he considered testing

$$H_O: \Delta = 0$$

versus

$$H_A: \Delta \neq 0,$$

under the assumption that the initial mean $\mathbf{E}X_1 = \theta_o$ is known.

Page introduced $S_k^* = \sum_{i=1}^k V_j$, where $S_0^* = 0$,

$$V_j = \begin{cases} a & \text{if } X_i > \theta_o \\ -b & \text{if } X_i \leq \theta_o, \end{cases}$$

and $a > 0$ and $b > 0$ are constants chosen so that $\mathbf{E}(V_j) = 0$. For example, his

decision rule rejects H_O in favour of the alternative of one change and $\Delta > 0$ if

$$T_n^* = \max_{0 \leq k \leq n} \left\{ S_k^* - \min_{0 \leq j \leq k} S_j^* \right\}$$

is too large. Gombay [18] has derived the exact finite sample distribution of T_n^* for the special case when the scores are $a = 1$ and $b = -1$.

Further along these lines, let $k = [nt]$ $0 \leq t \leq 1$ and set $S_{[nt]} := \sum_{i=1}^{[nt]} X_i$. Then if θ_o is known, H_O could be rejected in favour of H_A with $\Delta \neq 0$ if

$$T_n = \sup_{0 \leq t \leq 1} |S_{[nt]} - [nt]\theta_o|$$

is too large. We note that, under H_O

$$\frac{T_n}{n^{1/2}\sigma} \xrightarrow{D} \sup_{0 \leq t \leq 1} |W(t)|,$$

as $n \rightarrow \infty$, where $W(t)$ is a standard Wiener process and $\sigma^2 := \mathbb{E}(X_1 - \theta)^2$. This result follows from Donsker's theorem restated on $D[0, 1]$.

If, as is more frequently the case, θ_o is unknown, then set $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}$, which results in the following statistic

$$\hat{T}_n = \sup_{0 \leq t \leq 1} |S_{[nt]} - [nt]\bar{X}_n|,$$

whose large values would reject $H_O : \Delta = 0$ in favour of H_A with $\Delta \neq 0$. We note that under H_A

$$\frac{\hat{T}_n}{n^{1/2}\sigma} \xrightarrow{D} \sup_{0 \leq t \leq 1} |B(t)|, \quad (2.1)$$

as $n \rightarrow \infty$, where $B(t)$ is a Brownian bridge. Again, this follows from Donsker's theorem on $D[0, 1]$, and the same is true with σ replaced by any consistent sequence of estimators $\{\hat{\sigma}_n\}$.

Another way of arriving at the statistic $\hat{T}_n$ as above is to consider the family of statistics

$$\begin{aligned} & \frac{k(n-k)}{n} \left(\frac{S_k}{k} - \frac{S_n - S_k}{n-k} \right) \\ &= \left(S_k - \frac{k}{n} S_n \right), \quad 1 \leq k \leq n, \end{aligned} \quad (2.2)$$

where in the first line one compares the sample mean at time k to the sample mean after time k , i.e., the sample mean before to that after the unknown time of change. Thus, on taking the maximum in k in (2.2), we arrive at $\hat{T}_n$ again.

We note also that the above rationalisation of (2.2) and $\hat{T}_n$ is also valid for testing

$$\begin{aligned} H_O^{(1)} : X_1, \dots, X_n \text{ are i.i.d. r.v's with } \mathbb{E}X_1 = \mu \text{ and} \\ \sigma^2 = \text{Var}(X_1) = \mathbb{E}(X_1 - \mu)^2 < \infty \end{aligned} \quad (2.3)$$

against the alternative of having AMOC in the mean

$$\begin{aligned} H_A^{(1)} : \text{there is an integer } k^*, \quad 1 \leq k^* < \infty, \text{ such that } \mathbb{E}X_1 = \mathbb{E}X_2 = \dots \\ \dots = \mathbb{E}X_{k^*} \neq \mathbb{E}X_{k^*+1} = \dots \mathbb{E}X_n. \end{aligned} \quad (2.4)$$

In what follows, we will see that functionals of the family of statistics in (2.2) play an important role in some recent developments of change-point analysis.

It is well-known that appropriate functionals of (2.2) are more sensitive to detecting changes in the mean for values of k near $\frac{n}{2}$ than for values of k near 0 or n . As a possible remedy, Csörgő and Horváth [5], [6], [7] introduce classes of weight functions which should improve the power of test statistics that are weighted sup functionals of (2.2). Two chapters of this thesis, one theoretical the other a simulation, will consider the use of weight functions and their role in improving the power of such tests.

The family of statistics in (2.2), as specified, is only appropriate for testing hypotheses regarding changes in the mean and cannot directly accommodate more general changes such as changes in the cumulative distribution function that are unrelated to changes in the mean. To test more general hypotheses, Csörgő and Horváth consider statistics based on processes of U-statistics. Again, they consider both unweighted and weighted versions of such processes and the types of hypotheses that can be handled. For more information regarding these matters we refer to M. Csörgő and L. Horváth [5] [6] and [7].

Given the large class of hypotheses that can be considered within a U-statistics methodology, they will play an important role in developing statistics suited to testing hypotheses of at most two change-points (AMTC).

2.3 U-Statistics

With the above as motivation, we provide now a more detailed account of U-statistics. We follow the route that is developed by Randles and Wolfe [28]. U-statistics were first studied by Hoeffding in his famous article [19] in 1948. We begin with the definition of one sample U-statistics which leads naturally to the two sample version.

Definition 2.3.1 A parameter γ is said to be estimable of degree r for the family of distributions $\mathcal{F}$ if r is the smallest sample size for which there exists a function $h^*(x_1, \dots, x_r)$ such that

$$\mathbb{E}_F[h^*(X_1, \dots, X_r)] = \gamma$$

for every distribution $F(\cdot) \in \mathcal{F}$, where $X_1, \dots, X_r$ denotes a random sample from $F(x)$ and $h^*(X_1, \dots, X_r)$ is a statistic and thus does not depend on $F(\cdot)$.

Let $X_1, \dots, X_n$ be a random sample (independent and identically distributed random variables) with $n \geq r$ and CDF $F(\cdot) \in \mathcal{F}$.

Definition 2.3.2 A U-statistic for the estimable parameter γ of degree r is created with kernel $h^*(\cdot)$ by forming

$$U(X_1, \dots, X_n) = \frac{1}{\binom{n}{r}} \sum_{\beta \in B} h^*(x_{\beta_1}, \dots, x_{\beta_r}), \quad (2.5)$$

where $B = \left\{ \beta \mid \beta \text{ is one of the } \binom{n}{r} \text{ unordered subsets of } r \text{ integers chosen without replacement from } \{1, \dots, n\} \right\}$.

To derive a U-statistic for the variance, we set $h^*(x_1, x_2) := \frac{1}{2}(x_1 - x_2)^2$ and substitute this into (2.5) to obtain the following.

$$\begin{aligned} U(X_1, \dots, X_n) &= \frac{1}{\binom{n}{2}} \sum_{\beta \in B} \frac{1}{2} (X_{\beta_1} - X_{\beta_2})^2 \\ &= \frac{1}{n(n-1)} \left[(n-1) \sum_{i=1}^n X_i^2 - 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n X_i X_j \right] \\ &= \frac{1}{n(n-1)} \left[n \sum_{i=1}^n X_i^2 - \left(\sum_{i=1}^n X_i \right)^2 \right] \\ &=: s^2. \end{aligned}$$

One sample U-statistics can be extended to two samples and to s samples as well. Since our interest here concerns statistics that are capable of detecting at most two change-points, this can be best done within a two sample U-statistic framework. As we will see, two sample U-statistics provide an important method for constructing statistics suited to testing for two or more changes.

Definition 2.3.3 Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be independent random samples from distributions with CDF's $F(x)$ and $G(x)$, respectively. A parameter γ is said

to be estimable of degree (r, s) , for the distributions (F, G) in the family $\mathcal{F}$, if r and s are the smallest sample sizes for which there exists an estimator of γ that is unbiased for every $(F, G) \in \mathcal{F}$.

Definition 2.3.4 For an estimable parameter γ of degree (r, s) with kernel $h^*(\cdot)$, a two-sample U-statistic has, for $m \geq r$ and $n \geq s$, the form

$$U(X_1, \dots, X_m; Y_1, \dots, Y_n) = \frac{1}{\binom{m}{r} \binom{n}{s}} \sum_{\alpha \in A} \sum_{\beta \in B} h^*(X_1, \dots, X_r, Y_1, \dots, Y_s), \quad (2.6)$$

where $A[B]$ is the collection of all subsets of $r[s]$ integers chosen without replacement from the integers $\{1, \dots, m\}[\{1, \dots, n\}]$.

In testing for a change in the mean, one might consider $h^*(x_i, y_j) := x_i - y_j$, as $E(X_i - Y_j) = -\Delta$. This follows as a consequence of the specification of the non-parametric two sample problem given in Section 2.2 (cf. H_O versus H_A). As specified, we have $X \stackrel{D}{=} Y - \Delta$ which implies $\mathbb{E}X = \mathbb{E}Y - \Delta$ (cf. Randles and Wolfe [28]). If this kernel is substituted into equation (2.6), we have

$$\begin{aligned} U(X_1, \dots, X_r; Y_1, \dots, Y_{n-r}) &= \frac{1}{\binom{r}{1} \binom{n-r}{1}} \sum_{i=1}^r \sum_{j=1}^{n-r} (X_i - Y_j) \\ &= (n-r) \frac{\sum_{i=1}^r X_i}{r(n-r)} - r \frac{\sum_{j=1}^{n-r} Y_j}{r(n-r)} \\ &= \frac{\sum_{i=1}^r X_i}{r} - \frac{\sum_{j=1}^{n-r} Y_j}{n-r} \end{aligned}$$

We see that this family of two sample statistics coincides with that given in (2.2). This confirms the possibility and importance of U-statistics for developing statistics in change-point analysis.

As has been mentioned previously, current research has extended non-parametric change-point analysis by developing methods more suited to entertaining more general hypotheses and also by stressing the important role weight functions can have

on improving the power of such tests. Recently, E. Gombay, L. Horváth and M. Hušková [17, 1996] have used the change in the mean approach to test for at most one change (AMOC) in variance. More specifically, they use a variant of (2.2), created by replacing the summands in S_k and S_n with $(X_i - \mu)^2$, to test the hypothesis of AMOC in variance. Moreover, they studied the asymptotics of weighted functionals of this variant.

For the purpose of this thesis, we are interested in extending their results regarding AMOC in the variance to AMTC in variance. Moreover, when the AMTC statistic is developed, we shall see that it extends naturally to at most s changes in variance (AMSC).

To construct the AMTC in variance test statistic, we use the two sample U-statistic with $h^*(Y_i, Y_j) = Y_i - Y_j$ where $Y_i = (X_i - \mu)^2$. We incorporate the possibility of two change-points by separating the random sample into three separate blocks. Accordingly, let $\{X_1, X_2, \dots, X_{k_1-1}, X_{k_1}, X_{k_1+1}, \dots, X_{k_2-1}, X_{k_2}, X_{k_2+1}, \dots, X_n\}$, where k_1 and k_2 are integers with the property $1 \leq k_1 \leq k_2 < n$. Viewing $k_1 \leq k_2$ as the unknown times of possible change in the variance suggests that the statistic should compare $\{X_1, X_2, \dots, X_{k_1-1}, X_{k_1}\}$ with $\{X_{k_1+1}, X_{k_1+2}, \dots, X_{k_2-1}, X_{k_2}\}$ and $\{X_{k_2+1}, \dots, X_n\}$, while $\{X_{k_1+1}, X_{k_1+2}, \dots, X_{k_2-1}, X_{k_2}\}$ should be compared with $\{X_{k_2+1}, X_{k_2+2}, \dots, X_{n-1}, X_n\}$. This results in the two parameter partial sum process.

$$\begin{aligned}
 Z_{k_1, k_2} &= \sum_{i=1}^{k_1} \sum_{j=k_1+1}^{k_2} (Y_i - Y_j) + \sum_{i=1}^{k_1} \sum_{j=k_2+1}^n (Y_i - Y_j) \\
 &\quad + \sum_{i=k_1+1}^{k_2} \sum_{j=k_2+1}^n (Y_i - Y_j) \\
 &= (k_2 - k_1) \sum_{i=1}^{k_1} Y_i - k_1 \sum_{j=k_1+1}^{k_2} Y_j + (n - k_2) \sum_{j=1}^{k_1} Y_j
 \end{aligned}$$

$$\begin{aligned}
& -k_1 \sum_{j=k_2+1}^n Y_j + (n-k_2) \sum_{i=k_1+1}^{k_2} Y_i - (k_2-k_1) \sum_{j=k_2+1}^n Y_j \\
& = -k_1 \sum_{i=1}^{k_2} Y_i - k_2 \sum_{i=k_1+1}^n Y_i + n \sum_{i=1}^{k_1} Y_i + n \sum_{i=k_1+1}^{k_2} Y_i \\
& = -k_1 \sum_{i=1}^{k_2} Y_i - k_2 \sum_{j=k_1+1}^n Y_j + n \sum_{i=1}^{k_2} Y_i \\
& = n \sum_{i=1}^{k_2} Y_i - k_1 \sum_{i=1}^{k_2} Y_i + k_2 \sum_{i=1}^{k_1} Y_i - k_2 \sum_{i=1}^n Y_i \\
& = k_2 S_{k_1} + (n-k_1) S_{k_2} - k_2 S_n \quad 1 \leq k_1 \leq k_2 < n. \tag{2.7}
\end{aligned}$$

Equation (2.7) is a two sample U-statistic type process in k_1 and k_2 for each n . We note that in case of $k_1 = k_2$, then the family of statistics in (2.7) reduces to that of (2.2).

Orasch [23] and [24] has developed a geometric method for constructing a test statistics capable of testing the AMTC in the mean null hypothesis. As our statistic is based on the change in mean framework, we can also use this geometric method to construct a test statistics for testing our AMTC in variance null hypothesis. Let X_i , for $i = 1, \dots, n$, $n \geq 1$, be specified in (3.1). Set $Y_i = (X_i - \mu)^2$ and $S_k := \sum_{i=1}^k Y_i$, and consider testing the following

$$H_O^{(2)} : k_1^* \geq n$$

against the AMTC changes in variance alternative

$$H_A^{(2)} : 1 \leq k_1^* \leq k_2^* < n.$$

We use now the geometric argument that was developed by Orasch [24, page 110-112], to illustrate testing for AMTC in variance. Let $m(t) = t$, $t \in \mathbb{R}$, which joins under $H_O^{(2)}$ all points $(k, \frac{1}{\sigma^2} \mathbf{E}(S_k))$, $0 \leq k < n$. Without loss of generality, we may assume that $\sigma^2 = 1$. Then in Figure 2.1¹, we join all the points $(k, \mathbf{E}\{S_k\})$,

¹Figure 2.1 was reproduced from [24, page 111] by permission of Markus Orasch.

via the line $m(t) = t$. We pick $k_1 \in \{1, \dots, n-2\}$ and $k_2 \in \{k_1 + 1, \dots, n-1\}$. We draw a horizontal line starting from $B := (0, \mathbb{E}\{S_{k_1}\})$, containing the point $(k_1, \mathbb{E}\{S_{k_1}\})$, and with terminus $C := (k_2, \mathbb{E}\{S_{k_1}\})$. We then draw a vertical line from the terminus that intersects the t -axis. We denote the point of intersection by $D := (k_2, \mathbb{E}\{S_0\})$, where we define $S_0 := 0$. In this way we construct a rectangle, denoted by $ABCD$ as depicted in Figure 2.1 where $A := (0, \mathbb{E}\{S_0\})$ with length k_2 and height $\mathbb{E}\{S_{k_1}\}$. We also draw a horizontal line starting from $F := (k_1, \mathbb{E}\{S_{k_2}\})$, containing the points $(k_2, \mathbb{E}\{S_{k_2}\})$ and terminus $G := (n, \mathbb{E}\{S_{k_2}\})$. We then draw two vertical lines, one from the starting point and the other from the terminus, and both intersecting the t -axis at $E := (k_1, \mathbb{E}\{S_0\})$ and $H := (n, \mathbb{E}\{S_0\})$ respectively. Similarly, we construct another rectangle, denoted by $EFGH$ (refer to Figure 2.1) with length $(n - k_1)$ and height $\mathbb{E}\{S_{k_2}\}$.

Figure 2.1: A geometrical interpretation of $\mathbb{E}\{k_2 S_{k_1} + (n - k_1) S_{k_2} - k_2 S_n\} = 0$ under $H_0^{(2)}$.

Reflecting each point of the rectangle $EFGH$ around the 45 degree line $m = t$, we get the new rectangle $BIJC$, where $B := (0, \mathbb{E}\{S_{k_1}\})$ is the reflection of $E := (k_1, \mathbb{E}\{S_0\})$, $I := (0, \mathbb{E}\{S_n\})$ is the reflection point of $H := (n, \mathbb{E}\{S_0\})$, $J := (k_2, \mathbb{E}\{S_n\})$ is the reflection of $G := (n, \mathbb{E}\{S_{k_2}\})$ and $C := (k_2, \mathbb{E}\{S_{k_1}\})$ is the reflection of $F := (k_1, \mathbb{E}\{S_{k_2}\})$. This new rectangle has length k_2 and height $\mathbb{E}\{S_n - S_{k_1}\}$.

Moreover, combining rectangles $ABCD$ with $BIJC$, we have the resultant rectangle $AIJD$, with width k_2 and height $\mathbb{E}\{S_n\}$. Rectangle $AIJD$, because of this reflective property under $H_O^{(2)}$, has area equal to the sum of the areas of rectangles $ABCD$ and $EFGH$. Due to the equality in areas of these rectangles, we have

$$\begin{aligned}
 0 &= k_2 \mathbb{E}\{S_{k_1}\} + (n - k_1) \mathbb{E}\{S_{k_2}\} - k_2 \mathbb{E}\{S_n\} \\
 &= \mathbb{E} \left(\sum_{i=1}^{k_1} \sum_{j=k_1+1}^n ((X_i - \mu)^2 - (X_j - \mu)^2) \right. \\
 &\quad \left. + \sum_{i=k_1+1}^{k_2} \sum_{j=k_2+1}^n ((X_i - \mu)^2 - (X_j - \mu)^2) \right) \\
 &= 0.
 \end{aligned} \tag{2.8}$$

For algebraic justification, we refer the reader to (2.7). We see that the family of statistics constructed by way of geometry (cf. (2.8)) corresponds directly with that family of statistics developed via U-statistics (cf. (2.7)). Moreover, we see that, for each k_1 and k_2 , $1 \leq k_1 \leq k_2 < n$, this family of statistics (cf. (2.8)) is an unbiased estimator of zero. And as such, we conclude directly that testing for AMTC in variance corresponds to comparing the areas of three different rectangles.

Hence, by comparing the areas of particular rectangles, one is able to construct a two parameter family of partial sum statistics as given in (2.7). Furthermore, one can, by comparing areas of $s+1$ rectangles, construct a s parameter family of partial sum statistics whose functionals can be used in testing more general hypotheses.

More will said regarding this matter in the following sections and, indeed, in chapters to come.

2.4 A U-Statistic Type Stochastic Process

Orasch [24] has developed an s parameter family of partial sum statistics which can be used to entertain very general hypotheses regarding parameters of interest. The family of processes he develops in equation 5.1.1 [24, page 132] for testing his $H_O^{(s)}$ versus $H_A^{(s)}$ [24, page 132] is as follows.

$$Z_{k_1, \dots, k_s} := \sum_{m=1}^s \sum_{i=k_{m-1}+1}^{k_m} \sum_{l=m}^s \sum_{j=k_l+1}^{k_{l+1}} h(X_i, X_j), \quad 1 \leq k_1 \leq k_2 \leq \dots \leq k_s < n, \quad (2.9)$$

where $k_0 := 0$ and $k_{s+1} := n$.

With this s parameter family of partial sums, we are now able to construct a process from which statistics can be developed to test $H_O^{(2)}$ against $H_A^{(s)}$ which reads as follows: $H_A^{(s)} : 1 \leq k_1^* \leq k_2^* \leq \dots \leq k_s^* < n$ is the at most s change-points (AMSC) in variance hypothesis. Given (2.9) and $H_A^{(s)}$, we set $h(Y_i, Y_j) := Y_i - Y_j$ and then, after lengthy algebra, the version of the Gombay, Horváth and Hušková's [17] family of statistics is produced and is given in (2.15)- (2.17). Since the algebra required to produce (2.15) is non-trivial, all the algebraic details are provided below.

$$\begin{aligned} Z_{k_1, \dots, k_s} &= \sum_{i=1}^s \sum_{i=k_{m-1}+1}^{k_m} \sum_{l=m}^s \sum_{j=k_l+1}^{k_{l+1}} (Y_i - Y_j) \\ &= \sum_{i=1}^{k_1} \left(\sum_{k_1+1}^{k_2} (Y_i - Y_j) + \dots + \sum_{k_s+1}^n (Y_i - Y_j) \right) \\ &\quad + \sum_{k_1+1}^{k_2} \left(\sum_{k_2+1}^{k_3} (Y_i - Y_j) + \dots + \sum_{k_s+1}^n (Y_i - Y_j) \right) \end{aligned}$$

$$\begin{aligned}
& + \cdots + \sum_{i=k_{s-1}+1}^{k_s} \sum_{j=k_s+1}^n (Y_i - Y_j) \\
& = \sum_{i=1}^{k_1} \left(\sum_{j=k_1+1}^{k_2} Y_i + \cdots + \sum_{j=k_s+1}^n Y_i - \sum_{j=k_1+1}^{k_s} Y_j - \cdots - \sum_{j=k_s+1}^n Y_j \right) \\
& \quad + \sum_{i=k_1+1}^{k_2} \left(\sum_{j=k_2+1}^{k_3} Y_i + \cdots + \sum_{j=k_s+1}^n Y_i - \sum_{j=k_2+1}^{k_3} Y_j - \cdots - \sum_{j=k_s+1}^n Y_j \right) \\
& \quad + \cdots + \sum_{i=k_{s-1}}^{k_s} \sum_{j=k_s+1}^n Y_i - \sum_{i=k_{s-1}+1}^{k_s} \sum_{j=k_s+1}^n Y_j. \\
& = \sum_{i=1}^{k_1} \left((k_2 - k_1)Y_i + \cdots + (n - k_s)Y_i - \sum_{j=k_1+1}^{k_2} Y_j - \cdots - \sum_{j=k_s+1}^n Y_j \right) \\
& \quad + \sum_{i=k_1+1}^{k_2} \left((k_3 - k_2)Y_i + \cdots + (n - k_s)Y_i - \sum_{j=k_2+1}^{k_3} Y_j - \cdots - \sum_{j=k_s+1}^n Y_j \right) \\
& \quad + \cdots + (n - k_s) \sum_{i=k_{s-1}+1}^{k_s} Y_i - (k_s - k_{s-1}) \sum_{j=k_s+1}^n Y_j \\
& = \sum_{l=1}^s \left((n - k_l) \sum_{i=k_{l-1}+1}^{k_l} Y_i - (k_l - k_{l-1}) \sum_{i=k_l+1}^n Y_i \right) \\
& = \sum_{l=1}^s \left(n \sum_{i=k_{l-1}+1}^{k_l} Y_i - k_l \sum_{i=k_{l-1}+1}^n Y_i + k_{l-1} \sum_{i=k_l+1}^n Y_i \right) \\
& = \sum_{l=1}^s \left(n \sum_{i=k_{l-1}+1}^{k_l} Y_i + k_l \sum_{i=1}^{k_{l-1}} Y_i - k_{l-1} \sum_{i=1}^{k_l} Y_i - k_l \sum_{i=1}^n Y_i + k_{l-1} \sum_{i=1}^n Y_i \right) \\
& = \sum_{l=1}^s \left(n \sum_{i=k_{l-1}+1}^{k_l} Y_i + k_l \sum_{i=1}^{k_{l-1}} Y_i - k_{l-1} \sum_{i=1}^{k_l} Y_i - (k_l - k_{l-1}) \sum_{i=1}^n Y_i \right) \\
& = n \sum_{l=1}^s \sum_{i=k_{l-1}+1}^{k_l} Y_i + \sum_{l=1}^s k_l \sum_{i=1}^{k_{l-1}} Y_i - \sum_{l=1}^s k_{l-1} \sum_{i=1}^{k_l} Y_i - \sum_{l=1}^s (k_l - k_{l-1}) \sum_{i=1}^n Y_i \quad (2.10)
\end{aligned}$$

First define $k_0 := 0$ and $k_{s+1} := n$. Then take the first, second and fourth pair of sums in equation (2.10) and expand them as shown below. The first two pair of sums can be rewritten as is shown in (2.11) and (2.12).

$$n \sum_{l=1}^s \sum_{i=k_{l-1}+1}^{k_l} Y_i = n \left(\sum_{i=1}^{k_1} Y_i + \sum_{i=k_1+1}^{k_2} Y_i + \cdots + \sum_{i=k_{s-1}}^{k_s} Y_i \right)$$

$$= n \sum_{i=1}^{k_s} Y_i \quad (2.11)$$

and

$$\begin{aligned} \sum_{l=1}^s k_l \sum_{i=1}^{k_{l-1}} Y_i &= k_1 \sum_{i=1}^{k_0} Y_i + k_2 \sum_{i=1}^{k_1} Y_i + \cdots + k_s \sum_{i=1}^{k_{s-1}} Y_i \\ &= k_2 \sum_{i=1}^{k_1} Y_i + \cdots + k_s \sum_{i=1}^{k_{s-1}} Y_i \end{aligned} \quad (2.12)$$

Combining sums in equations (2.11) and (2.12), results in the following sum

$$\begin{aligned} k_2 \sum_{i=1}^{k_1} Y_i + \cdots + k_s \sum_{i=1}^{k_{s-1}} Y_i + n \sum_{i=1}^{k_s} Y_i \\ = \sum_{l=1}^s k_{l+1} \sum_{i=1}^{k_l} Y_i. \end{aligned} \quad (2.13)$$

The fourth sum in equation (2.10), when expanded, telescopes which simplifies the sum as shown in (2.14).

$$\begin{aligned} (k_1 - k_0) \sum_{i=1}^n Y_i + \cdots + (k_s - k_{s-1}) \sum_{i=1}^n Y_i \\ = k_s \sum_{i=1}^n Y_i \end{aligned} \quad (2.14)$$

Using the results from equations (2.13) and (2.14), the sum in equation (2.10) can be expressed as

$$\begin{aligned} Z_{k_1, \dots, k_s} &= \sum_{l=1}^s k_{l+1} \sum_{i=1}^{k_l} Y_i - \sum_{l=1}^s k_{l-1} \sum_{i=1}^{k_l} Y_i - k_s \sum_{i=1}^n Y_i \\ &= \sum_{l=1}^s (k_{l+1} - k_{l-1}) \sum_{i=1}^{k_l} Y_i - k_s \sum_{i=1}^n Y_i. \end{aligned} \quad (2.15)$$

If, in (2.15), $k_l := [(n+1)t_l]$, where $t_0 := 0$ and $t_{s+1} := \frac{n}{n+1}$ and $t_1, \dots, t_s$ satisfy $0 \leq t_1 \leq \cdots \leq t_s \leq t_{s+1} < 1$, results in (2.16).

$$Z_{k_1, \dots, k_s} = \sum_{l=1}^s ([(n+1)t_{l+1}] - [(n+1)t_{l-1}]) \sum_{i=1}^{k_l} Y_i - [(n+1)t_s] \sum_{i=1}^n Y_i. \quad (2.16)$$

Dividing the sum in (2.16) by $n^{3/2}$ yields an s -time parameter stochastic processes, for each fixed $n \geq 1$, as shown in (2.17).

$$M_n^{(1)}(t_1, \dots, t_s) := n^{-3/2} \left\{ \sum_{l=1}^s \left(([(n+1)t_{l+1}] - [(n+1)t_{l-1}]) \sum_{i=1}^{[(n+1)t_l]} Y_i \right) - [(n+1)t_s] \sum_{i=1}^n Y_i \right\}, \quad 0 \leq t_1 \leq t_2 \leq \dots \leq t_s < 1. \quad (2.17)$$

The partial sum process defined by (2.17) is the process from which statistics will be constructed, either as specified or with some modification, to test $H_O^{(s)}$ the AMSC in variance null hypothesis versus $H_A^{(s)}$.

If we set $s = 2$ in (2.15), one would expect to reproduce the two parameter partial sum process derived in (2.7). To this end we set $s = 2$.

$$\begin{aligned} Z_{k_1, k_2} &= \sum_{l=1}^2 \left((k_{l+1} - k_{l-1}) \sum_{i=1}^{k_l} Y_i \right) - k_2 \sum_{i=1}^n Y_i \\ &= k_2 \sum_{i=1}^{k_1} Y_i + (n - k_1) \sum_{i=1}^{k_2} Y_i - k_2 \sum_{i=1}^n Y_i, \quad 1 \leq k_1 \leq k_2 < n. \end{aligned} \quad (2.18)$$

We also have the 2-time parameter version of (2.17), as shown below.

$$\begin{aligned} M_n^{(1)}(t_1, t_2) &= n^{-3/2} \left\{ \sum_{l=1}^2 \left(([(n+1)t_{l+1}] - [(n+1)t_{l-1}]) \sum_{i=1}^{[(n+1)t_l]} Y_i \right) - [(n+1)t_2] \sum_{i=1}^n Y_i \right\} \\ &\quad 0 \leq t_1 \leq t_2 < 1. \end{aligned} \quad (2.19)$$

We also have an equivalent representation of (2.19) which views it as a sum of four partial sums, where each partial sum is paired with a second partial sum.

Through pairing, the partial sum process can be interpreted as comparing "mean before" with "mean after" á la (2.2) (cf. (2.20)). Again, due to the non-trivial algebra required to express $M_n^{(1)}(t_1, t_2)$ in such a form, all algebraic details will be provided.

$$\begin{aligned}
M_n^{(1)}(t_1, t_2) &= n^{-3/2} \left\{ \sum_{l=1}^2 \left(([(n+1)t_{l+1}] - [(n+1)t_{l-1}]) \sum_{i=1}^{[(n+1)t_l]} Y_i \right) \right. \\
&\quad \left. - [(n+1)t_2] \sum_{i=1}^n Y_i \right\} \\
&= n^{-3/2} \left\{ [(n+1)t_2] \sum_{i=1}^{[(n+1)t_1]} Y_i + (n - [(n+1)t_1]) \sum_{i=1}^{[(n+1)t_2]} Y_i - [(n+1)t_2] \sum_{i=1}^n Y_i \right\} \\
&= n^{-1/2} \left\{ \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} Y_i + \left(1 - \frac{[(n+1)t_1]}{n} \right) \sum_{i=1}^{[(n+1)t_2]} Y_i - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n Y_i \right\} \\
&= n^{-1/2} \left\{ \sum_{i=1}^{[(n+1)t_1]} Y_i + \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} Y_i \right. \\
&\quad \left. - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n Y_i + \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} Y_i \right\} \\
&= n^{-1/2} \left\{ \left(1 - \frac{[(n+1)t_1]}{n} \right) \sum_{i=1}^{[(n+1)t_1]} Y_i + \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i - \frac{[(n+1)t_1]}{n} \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i \right. \\
&\quad \left. - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n Y_i + \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} Y_i \right\} \\
&= n^{-1/2} \left\{ \left(1 - \frac{[(n+1)t_1]}{n} \right) \sum_{i=1}^{[(n+1)t_1]} Y_i - \frac{[(n+1)t_1]}{n} \sum_{i=[(n+1)t_1]}^{[(n+1)t_2]} Y_i \right. \\
&\quad \left. - \frac{[(n+1)t_1]}{n} \sum_{i=[(n+1)t_1]}^n Y_i + \frac{[(n+1)t_1]}{n} \sum_{i=[(n+1)t_1]}^n Y_i - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n Y_i \right. \\
&\quad \left. + \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} Y_i + \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i \right\} \\
&= n^{-1/2} \left\{ \left(1 - \frac{[(n+1)t_1]}{n} \right) \sum_{i=1}^{[(n+1)t_1]} Y_i - \frac{[(n+1)t_1]}{n} \sum_{i=[(n+1)t_1]}^n Y_i - \frac{[(n+1)t_1]}{n} \sum_{i=[(n+1)t_1]}^n Y_i \right. \\
&\quad \left. + \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i \right\}
\end{aligned}$$

$$\begin{aligned}
& - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n Y_i + \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} Y_i + \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i \Big\} \\
= & n^{-1/2} \left\{ \frac{[(n+1)t_1]}{n} \frac{(1 - [(n+1)t_1])}{n} \left(\frac{n}{[(n+1)t_1]} \sum_{i=1}^{[(n+1)t_1]} Y_i \right. \right. \\
& \left. \left. - \frac{n}{n - [(n+1)t_1]} \sum_{i=[(n+1)t_1]+1}^n Y_i \right) + \frac{[(n+1)t_1]}{n} \sum_{i=[(n+1)t_2]+1}^n Y_i \right. \\
& \left. - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_2]+1}^n Y_i + \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_2]} Y_i \right. \\
& \left. + \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} Y_i + \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i \right\} \\
= & n^{-1/2} \left\{ \frac{[(n+1)t_1]}{n} \frac{(n - [(n+1)t_1])}{n} \left(\frac{n}{[(n+1)t_1]} \sum_{i=1}^{[(n+1)t_1]} Y_i \right. \right. \\
& \left. \left. - \frac{n}{n - [(n+1)t_1]} \sum_{i=[(n+1)t_1]+1}^n Y_i \right) - \frac{([(n+1)t_2] - [(n+1)t_1])}{n} \sum_{i=[(n+1)t_2]+1}^n Y_i \right. \\
& \left. + \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i \right\} \\
= & n^{-1/2} \left\{ \frac{[(n+1)t_1]}{n} \frac{(n - [(n+1)t_1])}{n} \left(\frac{n}{[(n+1)t_1]} \sum_{i=1}^{[(n+1)t_1]} Y_i \right. \right. \\
& \left. \left. - \frac{n}{(n - [(n+1)t_1])} \sum_{i=[(n+1)t_2]+1}^n Y_i \right) \right. \\
& \left. + \frac{(n - [(n+1)t_2])}{n} \frac{([(n+1)t_2] - [(n+1)t_1])}{n} \left(\frac{n}{[(n+1)t_2] - [(n+1)t_1]} \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i \right. \right. \\
& \left. \left. - \frac{n}{n - [(n+1)t_2]} \sum_{i=[(n+1)t_2]+1}^n Y_i \right) \right\} \\
= & n^{1/2} \left\{ \frac{[(n+1)t_1]}{n} \frac{(n - [(n+1)t_1])}{n} \left(\frac{1}{[(n+1)t_1]} \sum_{i=1}^{[(n+1)t_1]} Y_i \right. \right. \\
& \left. \left. - \frac{1}{(n - [(n+1)t_1])} \sum_{i=[(n+1)t_2]+1}^n Y_i \right) \right. \\
& \left. + \frac{(n - [(n+1)t_2])}{n} \frac{([(n+1)t_2] - [(n+1)t_1])}{n} \left(\frac{1}{[(n+1)t_2] - [(n+1)t_1]} \sum_{i=[(n+1)t_1]+1}^{[(n+1)t_2]} Y_i \right. \right.
\end{aligned}$$

$$- \frac{1}{n - [(n+1)t_2]} \sum_{i=[(n+1)t_2]+1}^n Y_i \Bigg) \Bigg\}. \quad (2.20)$$

As a last note, we provide the following observations as lemmas, as they will be used repeatedly in chapters to come.

Lemma 2.4.1 *We have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t \leq 1} \left| \frac{[(n+1)t]}{n} - t \right| = o(1).$$

Proof.

$$\begin{aligned} \sup_{0 \leq t \leq 1} \left| \frac{[(n+1)t]}{n} - t \right| &= \sup_{0 \leq t \leq 1} \left| \frac{[(n+1)t] - t(n+1) + t}{n} \right| \\ &\leq \sup_{0 \leq t \leq 1} \left| \frac{(n+1)t - t(n+1) + t}{n} \right| \\ &= \frac{1}{n} \\ &= o(1), \text{ as } n \rightarrow \infty. \end{aligned}$$

Lemma 2.4.2 *Let $s \geq 1$, and $0 \leq t_1 \leq \dots \leq t_s \leq t_{s+1} = 1$. Then*

$$\sum_{l=1}^s (t_{l+1} - t_{l-1})t_l - t_s = 0,$$

where $t_0 := 0$ and $t_{s+1} := 1$.

Proof. Expanding the sum, we have

$$\begin{aligned} (t_2 - t_0)t_1 + (t_3 - t_1)t_2 + (t_4 - t_2)t_3 + \dots + (t_s - t_{s-2})t_{s-1} + (t_{s+1} - t_{s-1})t_s - t_s \\ = t_{s+1}t_s - t_s \\ = t_s - t_s = 0, \end{aligned}$$

as $t_{s+1} := 1$.

Lemma 2.4.3 *For $n \geq 1$, we have*

$$\begin{aligned}
 1 - \frac{[(n+1)t]}{n} &\leq \frac{n - (n+1)t + 1}{n} \\
 &= \frac{(n+1) - (n+1)t}{n} \\
 &= \frac{n+1}{n}(1-t).
 \end{aligned}$$

Chapter 3

Two Change-points

3.1 Asymptotics Under the Null Hypothesis $H_O^{(2)}$

Statistical inference has hypothesis testing as one of its major components, and by way of the Neyman-Pearson Lemma provides an often used method by which a researcher tests a hypothesis. Given the probability of a type I error, the probability of rejecting the null assumption when it is true, the Lemma enables one to find that rejection region with highest power. Also, the possibility to construct uniformly most powerful tests is one of the achievements of this Lemma.

However, one hypothesis test often leads to another. With reference to this research, if one were to find a change in variance at time k^* , one might test for the presence of a second change in variance by employing the test a second time. While the Neyman-Pearson Lemma permits control of test-by-test error, it does not allow control of experiment-wide error. In particular, if one were to test for the presence of a second change in variance, an additional test would be necessary. While this has no effect on test-by-test error, it does affect the experiment-wide error rate. If, for example, the error rate was set at the frequently used 0.05 percent, then the probability of jointly making no error would be 0.9025, assuming of course that the two tests are independent. Since the probability of no error is 0.9025, the probability

of making an error on at least one of the tests is $1 - 0.9025 = 0.0975$, which is larger than the 0.05 test-by-test error rate. Because of the increase in the error rates, one would like to develop a test that controls not only test-by-test but experiment-wide error rates as well. Much of what follows deals with the construction of one such test statistic.

3.1.1 Population Mean Known

Given a set of observations $\{X_1, \dots, X_n\}$ for $n = 1, 2, \dots$, one might be interested in testing for the presence of at most two changes in their variance at two distinct, yet unknown times. With positive constants σ, τ and ρ , let

$$X_i = \begin{cases} \mu + \sigma\epsilon_i, & 1 \leq i \leq k_1^*, \\ \mu + \tau\epsilon_i, & k_1^* < i \leq k_2^*, \\ \mu + \rho\epsilon_i, & k_2^* < i \leq n, \end{cases} \quad (3.1)$$

where

$$\begin{aligned} &\epsilon_i \text{ are independent and identically} \\ &\text{distributed random variables with } \mathbb{E}\epsilon_i = 0, \mathbb{E}\epsilon_i^2 = 1 \\ &\text{and } \mathbb{E}|\epsilon_i|^4 < \infty, \quad i = 1, \dots, n. \end{aligned} \quad (3.2)$$

The values of the parameters μ, σ, τ, ρ and k_1^* and k_2^* are unknown. Assuming that at least one of the inequalities hold: $\sigma \neq \tau$, $\tau \neq \rho$ or $\sigma \neq \rho$, the no change in variance null hypothesis can be formulated as

$$H_O^{(2)} : k_1^* \geq n$$

versus the AMTC changes in variance alternative

$$H_A^{(2)} : 1 \leq k_1^* \leq k_2^* < n.$$

E. Gombay, L. Horváth and M. Hušková [17] use the change in mean framework as in (2.2) to develop a statistic suited to testing the hypothesis of AMOC in variance (cf. $H_O^{(2)}$ versus $H_A^{(2)}$ with $k_1 = k_2$). In what follows, we use the change in mean approach to extend their statistic so that it can test $H_O^{(2)}$ against the AMTC change in variance hypothesis of $H_A^{(2)}$. As can be seen, the only difference between the AMTC statistic and Gombay, Horváth and Hušková's AMOC statistic is that the AMTC compares three sets of random variables, one with the other two, to detect for the presence of at most two changes, rather than comparing only two sets (cf. Sections 2.3 and 2.4 and equation (3.5)).

Let $Y_i := (X_i - \mu)^2$ and $\gamma^2 := \mathbb{E}Y_i^2 - (\sigma^2)^2$, for $i = 1, \dots, n$, and $n \geq 1$. We require

$$0 < \mathbb{E}Y_i^2 < \infty \quad (3.3)$$

which implies

$$\mathbb{E}X_i^4 < \infty \quad i = 1, \dots, n. \quad (3.4)$$

Hence the requirement of (3.2).

The test statistic for at most two changes in variance can be based on the following two-time parameter stochastic process. As developed in Chapter 2, via equation (2.19), we arrive at the following partial sum process.

$$\begin{aligned} M_n^{(1)}(t_1, t_2) &:= n^{-1/2} \left\{ \left(1 - \frac{[(n+1)t_1]}{n} \right) \sum_{i=1}^{[(n+1)t_2]} Y_i \right. \\ &\quad \left. + \frac{[(n+1)t_2]}{n} \left(\sum_{i=1}^{[(n+1)t_1]} Y_i - \sum_{i=1}^n Y_i \right) \right\}, \quad (3.5) \\ &\quad 0 \leq t_1 \leq t_2 < 1. \end{aligned}$$

Theorem 3.1.1 Assume $H_O^{(2)}$, (3.1), (3.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}_{n \in \mathbb{N}}$ such that, as $n \rightarrow \infty$,

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| = o_p(1), \quad (3.6)$$

and, for each $n \geq 1$,

$$\{\Gamma_n(t_1, t_2) : 0 \leq t_1 \leq t_2 \leq 1\} \stackrel{\mathcal{D}}{=} \{\Gamma(t_1, t_2) : 0 \leq t_1 \leq t_2 \leq 1\},$$

where $\Gamma(t_1, t_2)$ and $\Gamma_n(t_1, t_2)$ are defined in Lemmas 1.0.4 and 1.0.6, respectively.

Proof. Using the definitions of $M_n^{(1)}(t_1, t_2)$ and $\Gamma_n(t_1, t_2)$ as given in (3.5) and Lemma 1.0.6 respectively, via (3.6), we have, where now the random variables Y_i $i = 1, \dots, n$ and the standard Wiener process $\{W(nt); 0 \leq t \leq 1\}$ are viewed to be on the same probability space which is that of Major (1979),

$$\begin{aligned} I(n) = \sup_{0 < t_1 \leq t_2 < 1} & \left| \frac{n}{n^{3/2}} \sum_{i=1}^{[(n+1)t_2]} \frac{(Y_i - \sigma^2)}{\gamma} - \frac{[(n+1)t_1]}{n^{3/2}} \sum_{i=1}^{[(n+1)t_2]} \frac{(Y_i - \sigma^2)}{\gamma} \right. \\ & - \frac{[(n+1)t_2]}{n^{3/2}} \sum_{i=1}^n \frac{(Y_i - \sigma^2)}{\gamma} + \frac{[(n+1)t_2]}{n^{3/2}} \sum_{i=1}^{[(n+1)t_1]} \frac{(Y_i - \sigma^2)}{\gamma} \\ & \left. - \frac{1}{n^{1/2}} \{t_2 W(nt_1) + (1 - t_1)W(nt_2) - t_2 W(n)\} \right|. \end{aligned} \quad (3.7)$$

In (3.7), we added and subtracted the term

$$\sigma^2([(n+1)t_2]n - [(n+1)t_2][(n+1)t_1] - n[(n+1)t_2] + [(n+1)t_2][(n+1)t_1])$$

which, by Lemma 2.4.2, equals 0 or can be confirmed by inspection as well.

To simplify notation set $Z_i = \frac{Y_i - \sigma^2}{\gamma}$. Equation (3.7) can now be majorized by the following sums.

$$\begin{aligned}
I(n) \leq & \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{1}{n^{1/2}} \sum_{i=1}^{[(n+1)t_2]} Z_i - \frac{W(nt_2)}{n^{1/2}} \right| + \sup_{0 \leq t_1 \leq t_2 \leq 1} t_1 \left| \frac{1}{n^{1/2}} \sum_{i=1}^{[(n+1)t_2]} Z_i - \frac{W(nt_2)}{n^{1/2}} \right| \\
& + \sup_{0 \leq t_1 \leq t_2 \leq 1} t_2 \left| \frac{1}{n^{1/2}} \sum_{i=1}^{[(n+1)t_1]} Z_i - \frac{W(nt_1)}{n^{1/2}} \right| + \sup_{0 < t_1 \leq t_2 < 1} t_2 \left| \frac{1}{n^{1/2}} \sum_{i=1}^n Z_i - \frac{W(n)}{n^{1/2}} \right| \\
& + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{[(n+1)t_1]}{n} - t_1 \right| \left| \frac{1}{n^{1/2}} \sum_{i=1}^{[(n+1)t_2]} Z_i \right| + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{[(n+1)t_2]}{n} - t_2 \right| \left| \frac{1}{n^{1/2}} \sum_{i=1}^{[(n+1)t_1]} Z_i \right| \\
& + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{[(n+1)t_2]}{n} - t_2 \right| \left| \frac{1}{n^{1/2}} \sum_{i=1}^n Z_i \right|. \tag{3.8}
\end{aligned}$$

We establish now the following result.

$$\begin{aligned}
I(n) &= o_P(1) + o_P(1) + o_P(1) + o_P(1) \\
&\quad + O_P(1) \cdot o(1) + O_P(1) \cdot o(1) + O_P(1) \cdot o(1) \\
&= o_P(1), \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

As the first four terms in equation (3.8) are similar, we show the result for only one of them. Using Csörgő and Révész (1981, Theorem S.2.2.1 by Major [22, 1979] combined with (S.2.2.2)) which asserts the existence of a Wiener process $W(nt)$ such that, as $n \rightarrow \infty$,

$$\frac{1}{n^{1/2}} \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{[(n+1)t]} Z_i - W(nt) \right| = o_P(1).$$

Hence, we have

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{1}{n^{1/2}} \sum_{i=1}^{[(n+1)t_2]} Z_i - \frac{W(nt_2)}{n^{1/2}} \right| = o_P(1), \quad \text{as } n \rightarrow \infty.$$

For the last three terms in equation (3.8), again only one will be shown to be of order $o_P(1)$ in sup-norm, as $n \rightarrow \infty$, as the remaining terms share the same property.

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{[(n+1)t_2]}{n} - t_2 \right| \left| \frac{1}{n^{1/2}} \sum_{i=1}^{[(n+1)t_1]} Z_i \right|$$

$$\begin{aligned}
&\leq \sup_{0 \leq t_2 \leq 1} \left| \frac{[(n+1)t_2]}{n} - t_2 \right| \sup_{0 \leq t_1 \leq 1} \left| \frac{1}{n^{1/2}} \sum_{i=1}^{[(n+1)t_1]} Z_i \right| \\
&= o(1) \cdot O_P(1) \\
&= o_P(1), \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

which follows from Lemma 2.4.1 and Donsker's theorem on $D[0, 1]$.

An immediate consequence of the above result is the following corollary.

Corollary 3.1.1 *Under the conditions given in Theorem 3.1.1, we have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 \leq t_1 \leq t_2 \leq 1} |\Gamma(t_1, t_2)|.$$

3.1.2 Population Mean Unknown

The partial sum process developed in (3.5) depends upon the population mean and therefore cannot be used to construct a statistic when it is unknown. If we replace the population mean with a consistent estimator, it can then be used to construct a statistic. In what follows, we explore the effect this may have on the limiting distribution of certain functionals of the thus altered partial sum process.

First define $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Under $H_O^{(2)}$, this is a consistent estimator of μ , the population mean. Let

$$\bar{Y}_i := (X_i - \bar{X}_n)^2, \quad i = 1, \dots, n, \quad n \geq 1. \quad (3.9)$$

In view of (3.5), we introduce the following two-time parameter partial sum process.

$$\bar{M}_n^{(1)}(t_1, t_2) := n^{-1/2} \left\{ \left(1 - \frac{[(n+1)t_1]}{n} \right) \sum_{i=1}^{[(n+1)t_2]} \bar{Y}_i \right\}$$

$$+ \frac{[(n+1)t_2]}{n} \left(\sum_{i=1}^{[(n+1)t_1]} \bar{Y}_i - \sum_{i=1}^n \bar{Y}_i \right) \Bigg\}, \quad (3.10)$$

$$0 \leq t_1 \leq t_2 < 1,$$

and compare it to that of (3.5). In this regard we have

Lemma 3.1.1 *Assume $H_O^{(2)}$, (3.1), (3.2) and $\infty > \gamma > 0$. Then, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1)}(t_1, t_2) \right| = o_P(1).$$

Proof.

$$\begin{aligned} n^{1/2} \bar{M}_n^{(1)}(t_1, t_2) &= \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu + \mu - \bar{X}_n)^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu + \mu - \bar{X}_n)^2 \\ &\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_i - \mu + \mu - \bar{X}_n)^2 \\ &= \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu)^2 - 2(\bar{X}_n - \mu) \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu) \\ &\quad + [(n+1)t_2](\bar{X}_n - \mu)^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu)^2 \\ &\quad + 2 \frac{[(n+1)t_1]}{n} (\bar{X}_n - \mu) \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu) - \frac{[(n+1)t_1]}{n} [(n+1)t_2](\bar{X}_n - \mu)^2 \\ &\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_i - \mu)^2 + 2 \frac{[(n+1)t_2]}{n} (\bar{X}_n - \mu) \sum_{i=[(n+1)t_1]+1}^n (X_i - \mu) \\ &\quad - \frac{[(n+1)t_2]}{n} (n - [(n+1)t_1])(\bar{X}_n - \mu)^2. \end{aligned}$$

Grouping terms, we arrive at

$$\begin{aligned} n^{1/2} \bar{M}_n^{(1)}(t_1, t_2) &= \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu)^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu)^2 \\ &\quad + \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} (X_i - \mu)^2 - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n (X_i - \mu)^2 \end{aligned}$$

$$\begin{aligned}
& -2(\bar{X}_n - \mu) \left\{ \frac{(n - [(n+1)t_1])}{n} \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu) \right. \\
& + \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} (X_i - \mu) - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n (X_i - \mu) \left. \right\} \\
& + (\bar{X}_n - \mu)^2 \left\{ [(n+1)t_2] - \frac{[(n+1)t_1]}{n} [(n+1)t_2] \right. \\
& \left. - \frac{[(n+1)t_2]}{n} (n - [(n+1)t_1]) \right\}.
\end{aligned}$$

Since

$$\frac{(\bar{X}_n - \mu)^2}{n^{1/2}} \left\{ [(n+1)t_2] - \frac{[(n+1)t_1]}{n} [(n+1)t_2] - \frac{[(n+1)t_2]}{n} (n - [(n+1)t_1]) \right\} = 0,$$

we conclude that $\bar{M}_n^{(1)}(t_1, t_2)$ can be expressed as $M_n^{(1)}(t_1, t_2)$ plus some cross-product terms and write

$$\begin{aligned}
\bar{M}_n^{(1)}(t_1, t_2) &= M_n^{(1)}(t_1, t_2) - 2 \frac{(\bar{X}_n - \mu)}{n^{1/2}} \left\{ \frac{(n - [(n+1)t_1])}{n} \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu) \right. \\
&+ \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} (X_i - \mu) - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n (X_i - \mu) \left. \right\}.
\end{aligned} \tag{3.11}$$

Using (3.11), the difference between $\bar{M}_n^{(1)}(t_1, t_2)$ and $M_n^{(1)}(t_1, t_2)$ can be majorized to yield the following.

$$\begin{aligned}
\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \frac{M_n^{(1)}(t_1, t_2)}{\gamma} \right| &\leq 2|\bar{X}_n - \mu| \left\{ \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\sum_{i=1}^{[(n+1)t_2]} (X_i - \mu)}{n^{1/2}\gamma} \right| \right. \\
&+ 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{[(n+1)t_1]}{n} \frac{\sum_{i=1}^{[(n+1)t_2]} (X_i - \mu)}{n^{1/2}\gamma} \right| \\
&+ 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{[(n+1)t_2]}{n} \frac{\sum_{i=1}^n (X_i - \mu)}{n^{1/2}\gamma} \right| \\
&\left. + 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{[(n+1)t_2]}{n} \frac{\sum_{i=1}^{[(n+1)t_1]} (X_i - \mu)}{n^{1/2}\gamma} \right| \right\}. \tag{3.12}
\end{aligned}$$

Each of the first four terms on the right hand side of the inequality can be majorized further. Only one will be shown, as the remaining are majorized in a similar fashion.

We have

$$|\bar{X}_n - \mu| \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} (X_i - \mu)}{n^{1/2}\gamma} \right| \leq |\bar{X}_n - \mu| \sup_{0 \leq t_2 \leq 1} \left| \frac{\sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} (X_i - \mu)}{n^{1/2}\gamma} \right|.$$

Moreover,

$$|\bar{X}_n - \mu| \sup_{0 \leq t_2 \leq 1} \left| \frac{\sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} (X_i - \mu)}{n^{1/2}\gamma} \right| = o_P(1) \cdot O_P(1) = o_P(1), \quad \text{as } n \rightarrow \infty.$$

The first term is of order $o_P(1)$ due to the SLLN, while the second term is of order $O_P(1)$ by Donsker's theorem on $D[0, 1]$. One can use the same argument to show that the other terms are also of order $o_P(1)$, as $n \rightarrow \infty$.

Theorem 3.1.2 Assume $H_O^{(2)}$, (3.1), (3.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}_{n \in \mathbb{N}}$ such that, as $n \rightarrow \infty$,

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| = o_P(1),$$

and, for each $n \geq 1$,

$$\{\Gamma_n(t_1, t_2) : 0 \leq t_1 \leq t_2 \leq 1\} \stackrel{D}{=} \{\Gamma(t_1, t_2) : 0 \leq t_1 \leq t_2 \leq 1\},$$

where $\Gamma(t_1, t_2)$ and $\Gamma_n(t_1, t_2)$ are defined in Lemmas 1.0.4 and 1.0.6 respectively.

Proof. We have that

$$\begin{aligned} \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| &\leq \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \frac{M_n^{(1)}(t_1, t_2)}{\gamma} \right| \\ &\quad + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| \\ &= o_P(1) + o_P(1) = o_P(1), \quad \text{as } n \rightarrow \infty, \end{aligned}$$

by Lemma 3.1.1 and Theorem 3.1.1.

Corollary 3.1.2 *Under the conditions given in Theorem 3.1.2, we have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 \leq t_1 \leq t_2 \leq 1} |\Gamma(t_1, t_2)|.$$

3.1.3 Small Changes in the Population Mean

Along the lines of equation (2.6) in E. Gombay, L. Horváth and M. Hušková [17], we consider now a generalisation of their model. Assume that

$$X_{i,n} = \begin{cases} \mu_{in} + \sigma \epsilon_i, & 1 \leq i \leq k_1^*, \\ \mu_{in} + \tau \epsilon_i, & k_1^* < i \leq k_2^*, \\ \mu_{in} + \rho \epsilon_i, & k_2^* < i \leq n, \end{cases} \quad (3.13)$$

where $\mu_{in}, \dots, \mu_{nn}$ are unknown constants, the ϵ_i 's, $i = 1, \dots, n$, $n \geq 1$, are i.i.d random variables that satisfy the conditions given in (3.2). As in (2.7) of [17], let

$$\max_{1 \leq i \leq n} (\mu_{in} - \bar{\mu})^2 = o(n^{-1/2}), \quad (3.14)$$

where $\bar{\mu} = \frac{1}{n} \sum_{1 \leq i \leq n} \mu_{in}$.

In view of (3.13), let

$$Y_{i,n}^* := (X_{i,n} - \mu_{in})^2, \quad i = 1, \dots, n, \quad n \geq 1,$$

and define

$$\begin{aligned} M_n^{(1*)}(t_1, t_2) &:= n^{-1/2} \left\{ \left(1 - \frac{[(n+1)t_1]}{n} \right) \sum_{i=1}^{[(n+1)t_2]} Y_{i,n}^* \right. \\ &\quad \left. + \frac{[(n+1)t_2]}{n} \left(\sum_{i=1}^{[(n+1)t_1]} Y_{i,n}^* - \sum_{i=1}^n Y_{i,n}^* \right) \right\}, \end{aligned} \quad (3.15)$$

$$0 \leq t_1 \leq t_2 < 1.$$

$M_n^{(1*)}(t_1, t_2)$ has the same form as $M_n^{(1)}(t_1, t_2)$ (cf. (3.5)), only in this case μ , in the latter, is replaced by μ_{in} in $M_n^{(1)}(t_1, t_2)$. Recall (3.10) for the sake of the next lemma.

Lemma 3.1.2 *Assume $H_O^{(2)}$, (3.13), (3.14) all hold and $\infty > \gamma > 0$. Then, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1*)}(t_1, t_2) \right| = o_P(1).$$

Proof. Consider (cf. (3.10))

$$\begin{aligned} n^{1/2} \bar{M}_n^{(1)}(t_1, t_2) &= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{X}_n)^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{X}_n)^2 \\ &\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \bar{X}_n)^2 \\ &=: I_1(n, t_1, t_2) + I_2(n, t_1, t_2) + I_3(n, t_1, t_2), \end{aligned}$$

$$\begin{aligned} I_1(n, t_1, t_2) &= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{\mu} + \bar{\mu} - \bar{X}_n)^2 \\ &= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{\mu})^2 - 2(\bar{X}_n - \bar{\mu}) \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{\mu}) + [(n+1)t_2](\bar{X}_n - \bar{\mu})^2 \\ &=: I_1^a(n, t_1, t_2) + I_1^b(n, t_1, t_2) + I_1^c(n, t_1, t_2), \end{aligned}$$

$$\begin{aligned} I_2(n, t_1, t_2) &= - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{\mu} + \bar{\mu} - \bar{X}_n)^2 \\ &= - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{\mu})^2 + 2 \frac{[(n+1)t_1]}{n} (\bar{X}_n - \bar{\mu}) \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{\mu}) \\ &\quad - \frac{[(n+1)t_1]}{n} [(n+1)t_2](\bar{X}_n - \bar{\mu})^2 \\ &=: I_2^a(n, t_1, t_2) + I_2^b(n, t_1, t_2) + I_2^c(n, t_1, t_2), \end{aligned}$$

$$\begin{aligned}
I_3(n, t_1, t_2) &= - \frac{[(n+1)t_2]}{n} \sum_{[(n+1)t_1]+1}^n (X_{i,n} - \bar{\mu} + \bar{\mu} - \bar{X}_n)^2 \\
&= - \frac{[(n+1)t_2]}{n} \sum_{[(n+1)t_1]+1}^n (X_{i,n} - \bar{\mu})^2 + 2 \frac{[(n+1)t_2]}{n} (\bar{X}_n - \bar{\mu}) \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \bar{\mu}) \\
&\quad - \frac{[(n+1)t_2]}{n} (n - [(n+1)t_1]) (\bar{X}_n - \bar{\mu})^2 \\
&=: I_3^a(n, t_1, t_2) + I_3^b(n, t_1, t_2) + I_3^c(n, t_1, t_2).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_1^a(n, t_1, t_2) &= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})^2 \\
&= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})^2 + 2 \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \\
&\quad + \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2, \\
I_1^b(n, t_1, t_2) &= -2(\bar{X}_n - \bar{\mu}) \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu}) \\
&= -2(\bar{X}_n - \bar{\mu}) \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in}) - 2(\bar{X}_n - \bar{\mu}) \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu}),
\end{aligned}$$

$$\begin{aligned}
I_2^a(n, t_1, t_2) &= - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})^2 \\
&= - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})^2 \\
&\quad - 2 \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \bar{\mu}_{in})(\bar{\mu}_{in} - \bar{\mu}) \\
&\quad - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2, \\
I_2^b(n, t_1, t_2) &= 2 \frac{[(n+1)t_1]}{n} (\bar{X}_n - \bar{\mu}) \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})
\end{aligned}$$

$$\begin{aligned}
&= 2 \frac{[(n+1)t_1]}{n} (\bar{X}_n - \bar{\mu}) \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in}) \\
&\quad + 2 \frac{[(n+1)t_1]}{n} (\bar{X}_n - \bar{\mu}) \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu}),
\end{aligned}$$

$$\begin{aligned}
I_3^a(n, t_1, t_2) &= - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})^2 \\
&= - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in})^2 \\
&\quad - 2 \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \\
&\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (\mu_{in} - \bar{\mu})^2, \\
I_3^b(n, t_1, t_2) &= 2 \frac{[(n+1)t_2]}{n} (\bar{X}_n - \bar{\mu}) \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in} + \mu_{in} + \bar{\mu}) \\
&= 2 \frac{[(n+1)t_2]}{n} (\bar{X}_n - \bar{\mu}) \left\{ \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in}) + \sum_{i=[(n+1)t_1]+1}^n (\mu_{in} - \bar{\mu}) \right\}.
\end{aligned}$$

Reorganizing the above terms, we define

$$\begin{aligned}
D_n^{(1)}(t_1, t_2) &:= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})^2 \\
&\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in})^2, \\
D_n^{(2)}(t_1, t_2) &:= 2 \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \\
&\quad - 2 \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \\
&\quad - 2 \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}), \\
D_n^{(3)}(t_1, t_2) &:= \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2,
\end{aligned} \tag{3.16}$$

$$- \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (\mu_{in} - \bar{\mu})^2 \quad (3.17)$$

$$D_n^{(4)}(t_1, t_2) := -2(\bar{X}_n - \bar{\mu}) \left\{ \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in}) - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in}) \right. \\ \left. - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in}) \right\}, \quad (3.18)$$

$$D_n^{(5)}(t_1, t_2) := -2(\bar{X}_n - \bar{\mu}) \left\{ \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu}) - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu}) \right. \\ \left. - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (\mu_{in} - \bar{\mu}) \right\}, \quad (3.19)$$

$$D_n^{(6)}(t_1, t_2) := (\bar{X}_n - \bar{\mu})^2 \left\{ [(n+1)t_2] - \frac{[(n+1)t_1]}{n} [(n+1)t_2] \right. \\ \left. - \frac{[(n+1)t_2]}{n} (n - [(n+1)t_1]) \right\}.$$

Using the fact that $D_n^{(6)}(t_1, t_2) = 0$, which is easily seen by inspection, along with the above expressions, equation (3.10) can be rewritten as

$$n^{1/2} \bar{M}_n^{(1)}(t_1, t_2) = n^{1/2} M_n^{(1*)}(t_1, t_2) + D_n^{(2)}(t_1, t_2) + D_n^{(3)}(t_1, t_2) + D_n^{(4)}(t_1, t_2) \\ + D_n^{(5)}(t_1, t_2), \quad (3.20)$$

where $M_n^{(1*)}(t_1, t_2)$ is defined in (3.15). From (3.16) - (3.19), we now have the following majorization concerning the main line of Lemma 3.1.2.

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} \right| \\ \leq \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{D_n^{(2)}(t_1, t_2)}{n^{1/2}\gamma} \right| + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{D_n^{(3)}(t_1, t_2)}{n^{1/2}\gamma} \right| \\ + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{D_n^{(4)}(t_1, t_2)}{n^{1/2}\gamma} \right| + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{D_n^{(5)}(t_1, t_2)}{n^{1/2}\gamma} \right| \quad (3.21)$$

Each term on the right side bound of (3.21) can be majorized. We begin with the first term in (3.21).

$$\begin{aligned}
\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{D_n^{(2)}(t_1, t_2)}{n^{1/2}\gamma} \right| &\leq 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}\gamma} \right| \\
&\quad + 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{[(n+1)t_1]}{n} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}\gamma} \right| \\
&\quad + 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{[(n+1)t_2]}{n} \left| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}\gamma} \right| \\
&\quad + 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{[(n+1)t_2]}{n} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}\gamma} \right| \\
&=: J_1(n) + J_2(n) + J_3(n) + J_4(n) \\
&= o_p(1), \tag{3.22}
\end{aligned}$$

as $n \rightarrow \infty$.

We show, for $n = 1, \dots, 4$, that

$$J_i(n) = o_p(1), \text{ as } n \rightarrow \infty. \tag{3.23}$$

To this end, we make use of Theorem 1.0.4 and arrive, as $n \rightarrow \infty$, at

$$\begin{aligned}
\mathbb{P} \left\{ \sup_{1 \leq [(n+1)t_2] \leq n} \left| \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right| \geq \varepsilon \sqrt{n} \right\} \\
\leq \frac{1}{n\varepsilon^2} \sum_{i=1}^n \mathbb{E}[(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})^2] \\
\leq \frac{1}{n\varepsilon^2} \max_{1 \leq i \leq n} (\mu_{in} - \bar{\mu})^2 \sum_{i=1}^n \mathbb{E}(X_{i,n} - \mu_{in})^2 \\
\leq \frac{\sigma^2}{\varepsilon^2} \max_{1 \leq i \leq n} (\mu_{in} - \bar{\mu})^2 \\
= o(n^{-1/2}) \\
= o(1), \tag{3.24}
\end{aligned}$$

and (3.22) then follows

Continuing, we also have

$$\begin{aligned} \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{D_n^{(3)}(t_1, t_2)}{n^{1/2}\gamma} \right| &\leq 3 \max_{1 \leq i \leq n} \frac{(\mu_{in} - \bar{\mu})^2}{n^{1/2}\gamma} = n^{1/2} o(n^{-1/2}) \\ &= o(1), \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.25)$$

Moreover,

$$\begin{aligned} \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{D_n^{(4)}(t_1, t_2)}{n^{1/2}\gamma} \right| &\leq 2|(\bar{X}_n - \bar{\mu})| \left\{ \sup_{0 \leq t_2 \leq 1} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})}{n^{1/2}} \right| \right. \\ &\quad + \sup_{0 \leq t_2 \leq 1} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})}{n^{1/2}} \right| + \sup_{0 \leq t_1 \leq 1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{(X_{i,n} - \mu_{in})}{n^{1/2}} \right| \\ &\quad \left. + \left| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})}{n^{1/2}} \right| \right\} \\ &= o_P(1), \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (3.26)$$

which follows by consistency of $\bar{X}_n$ and (3.27) below.

$$\begin{aligned} \mathbb{P} \left\{ \sup_{1 \leq [(n+1)t] \leq n} \left| \sum_{i=1}^{[(n+1)t]} (X_{i,n} - \mu_{in}) \right| \geq A\sqrt{n} \right\} &\leq \frac{1}{A^2 \cdot n} \sum_{i=1}^n \sigma^2 \\ &= \frac{1}{A^2} \sigma^2 \\ &= 0, \end{aligned} \quad (3.27)$$

as $A \rightarrow \infty$ for each n . As the remaining terms also follow by (3.27), we have (3.26).

To show the next result, we make use of both the Cauchy-Schwarz inequality and the Lindeberg-Feller CLT.

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{D_n^{(5)}(t_1, t_2)}{n^{1/2}\gamma} \right| = o_P(1), \quad \text{as } n \rightarrow \infty, \quad (3.28)$$

Referring to (3.19), we have

$$\begin{aligned}
|n^{1/2}(\bar{X}_n - \bar{\mu})| & \left| \sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} \frac{1}{n^{1/2}} \frac{(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
& \leq |n^{1/2}(\bar{X}_n - \bar{\mu})| \left(\sum_{i=1}^n \frac{(\mu_{in} - \bar{\mu})^2}{n^{1/2}} \right)^{1/2} \\
& = O_P(1) \cdot o(1) = o_P(1). \tag{3.29}
\end{aligned}$$

as $n \rightarrow \infty$, by the Lindeberg-Feller CLT and (3.25). We must check that the random variables in the term $(\bar{X}_n - \bar{\mu})$ satisfy the Lindeberg condition. Let $A_n := \{\omega; |\epsilon_i(\omega)| > \varepsilon \cdot n^{1/2}\}$, then A_n is a non-increasing sequence of events in n . Moreover, $\bigcap_{n=1}^{\infty} A_n = \mathbf{A}$.

$$\begin{aligned}
\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n\sigma^2} \int_{A_n} \epsilon_i^2 dF & \leq \lim_{n \rightarrow \infty} \frac{1}{\sigma^2} \sqrt{\int \mathbf{1}_{A_n} dF} \sqrt{\int |\epsilon_i|^4 dF} \\
& = \frac{\mathbf{E}(|\epsilon_i|^4)}{\sigma^2} \lim_{n \rightarrow \infty} \sqrt{\int \mathbf{1}_{A_n} dF} \\
& = \frac{\mathbf{E}(|\epsilon_i|^4)}{\sigma^2} \sqrt{\int \mathbf{1}_{\mathbf{A}} dF} \\
& = 0, \tag{3.30}
\end{aligned}$$

as $n \rightarrow \infty$, by the Bounded Convergence Theorem. So the condition is satisfied.

We also note that

$$n^{1/2}(\bar{X}_n - \bar{\mu}) = \frac{n^{1/2}\sigma \sum_{i=1}^n (X_{i,n} - \mu_{in})}{n^{1/2}} = \sigma \frac{\sum_{i=1}^n (X_{i,n} - \mu_{in})}{n^{1/2}\sigma}. \tag{3.31}$$

Since the remaining terms in $D_n^{(5)}(t_1, t_2)$ (cf. (3.19)) are similar, we appeal to (3.29) to show that these terms are also $o_P(1)$ in sup-norm and (3.28) then follows.

Finally, in terms of Gaussian processes, we also conclude

Theorem 3.1.3 *Assume $H_O^{(2)}$, (3.2), (3.13), (3.14) all hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}_{n \in \mathbb{N}}$ such that, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| = o_P(1).$$

and, for each $n \geq 1$,

$$\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\} \stackrel{\mathcal{D}}{=} \{\Gamma(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}.$$

where $\Gamma(t_1, t_2)$ and $\Gamma_n(t_1, t_2)$ are defined in Lemmas 1.0.4 and 1.0.6, respectively.

Proof. In the probability space of Major [22, 1979] (cf. Theorem S.2.2.1 in [10]) we have

$$\begin{aligned} \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| &\leq \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} \right| \\ &\quad + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| \\ &= o_P(1) + o_P(1) \\ &= o_P(1), \quad \text{as } n \rightarrow \infty, \end{aligned}$$

by Lemma 3.1.2 and Theorem 3.1.2.

Corollary 3.1.3 *Under the conditions given in Theorem 3.1.3, we have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 \leq t_1 \leq t_2 \leq 1} |\Gamma(t_1, t_2)|.$$

In view of (3.5), let $Y_{i,n} := (X_{i,n} - \bar{\mu})^2$, $i = 1, \dots, n$, $n \geq 1$, and in terms of the newly defined $Y_{i,n}$, define

$$\begin{aligned} M_n^{(3)}(t_1, t_2) &:= n^{-1/2} \left\{ \left(1 - \frac{[(n+1)t_1]}{n} \right) \sum_{i=1}^{[(n+1)t_2]} Y_{i,n} \right. \\ &\quad \left. + \frac{[(n+1)t_2]}{n} \left(\sum_{i=1}^{[(n+1)t_1]} Y_{i,n} - \sum_{i=1}^n Y_{i,n} \right) \right\}, \quad (3.32) \\ &\quad 0 \leq t_1 \leq t_2 < 1. \end{aligned}$$

Lemma 3.1.3 Assume $H_O^{(2)}$, (3.2), (3.13), (3.14) all hold and $\infty > \gamma > 0$. Then,

as $n \rightarrow \infty$,

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{1}{\gamma} \left| M_n^{(3)}(t_1, t_2) - M_n^{(1*)}(t_1, t_2) \right| = o_P(1).$$

Proof.

$$\begin{aligned} n^{1/2} M_n^{(3)}(t_1, t_2) &= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})^2 \\ &\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})^2 \\ &= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})^2 + 2 \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})(X_{i,n} - \bar{\mu}) \\ &\quad + \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})^2 \\ &\quad - 2 \frac{[(n+1)t_1]}{n} \sum_{i=[(n+1)t_1]+1}^n (\mu_{in} - \bar{\mu})(X_{i,n} - \mu_{in}) - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2 \\ &\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in})^2 - 2 \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \\ &\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (\mu_{in} - \bar{\mu})^2. \end{aligned}$$

Grouping terms, we obtain

$$\begin{aligned} n^{1/2} \bar{M}_n^{(3)}(t_1, t_2) &= \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})^2 - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_1]} (X_{i,n} - \mu_{in})^2 \\ &\quad - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (X_{i,n} - \mu_{in})^2 + 2 \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})(X_{i,n} - \mu_{in}) \\ &\quad - 2 \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})(X_{i,n} - \mu_{in}) \\ &\quad - 2 \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (\mu_{in} - \bar{\mu})(X_{i,n} - \mu_{in}) + \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2 \\ &\quad - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2 - \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} (\mu_{in} - \bar{\mu})^2. \end{aligned}$$

We see from above that $M_n^{(3)}(t_1, t_2)$ has the following representation in terms of

$$M_n^{(1*)}(t_1, t_2).$$

$$\begin{aligned}
M_n^{(3)}(t_1, t_2) &= M_n^{(1*)}(t_1, t_2) \\
&+ 2n^{-1/2} \left\{ \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})(X_{i,n} - \mu_{in}) - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_1]} (\mu_{in} - \bar{\mu})(X_{i,n} - \mu_{in}) \right. \\
&- \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n (\mu_{in} - \bar{\mu})(X_{i,n} - \mu_{in}) \left. \right\} + \sum_{i=1}^{[(n+1)t_2]} \frac{(\mu_{in} - \bar{\mu})^2}{n^{1/2}} \\
&- \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} \frac{(\mu_{in} - \bar{\mu})^2}{n^{1/2}} - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_1]+1}^n \frac{(\mu_{in} - \bar{\mu})^2}{n^{1/2}}. \tag{3.33}
\end{aligned}$$

Using (3.33), the difference between $M_n^{(3)}(t_1, t_2)$ and $M_n^{(1*)}(t_1, t_2)$ can be majorized as follows.

$$\begin{aligned}
\sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(3)}(t_1, t_2)}{\gamma} - \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} \right| &\leq 2 \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}\gamma} \right| \\
&+ 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{[(n+1)t_1]}{n} \left| \frac{\sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}\gamma} \right| \\
&+ 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{[(n+1)t_2]}{n} \left| \frac{\sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}\gamma} \right| \\
&+ 2 \sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{[(n+1)t_2]}{n} \left| \frac{\sum_{i=1}^{[(n+1)t_1]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}\gamma} \right| \\
&+ 3 \left| \sum_{i=1}^n \frac{(\mu_{in} - \bar{\mu})^2}{n^{1/2}\gamma} \right| \\
&= o_P(1), \tag{3.34}
\end{aligned}$$

as $n \rightarrow \infty$, by (3.22) and (3.25).

Theorem 3.1.4 Assume $H_O^{(2)}$, (3.2), (3.13), (3.14) all hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2) : 0 \leq t_1 \leq t_2 \leq 1\}_{n \in \mathbb{N}}$ such that as, $n \rightarrow \infty$,

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(3)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| = o_P(1)$$

and, for each $n \geq 1$,

$$\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 < 1\} \stackrel{\mathcal{D}}{=} \{\Gamma(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}.$$

where $\Gamma(t_1, t_2)$ and $\Gamma_n(t_1, t_2)$ are defined in Lemmas 1.0.4 and 1.0.6, respectively.

Proof.

We have

$$\begin{aligned} \left| \frac{M_n^{(3)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| &\leq \frac{1}{\gamma} \sup_{0 \leq t_1 \leq t_2 \leq 1} |M_n^{(3)}(t_1, t_2) - M_n^{(1*)}(t_2, t_2)| \\ &\quad + \sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| \\ &= o_P(1), \quad \text{as } n \rightarrow \infty, \end{aligned}$$

by Lemma 3.1.3 and Theorem 3.1.3.

Theorem 3.1.4 implies the following corollary.

Corollary 3.1.4 *Under the conditions given in Theorem 3.1.4, we have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(3)}(t_1, t_2)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 \leq t_1 \leq t_2 \leq 1} |\Gamma(t_1, t_2)|.$$

3.2 Estimating γ^2 under the Null Hypothesis $H_O^{(2)}$

The functionals developed in Section 3.1 under $H_O^{(2)}$ depend on the unknown parameter γ^2 and therefore are not yet statistics. Transforming them into statistics requires estimating γ^2 under $H_O^{(2)}$. To this end, we have under $H_O^{(2)}$ that

$$\gamma^2 = \mathbb{E}[(X_1 - \mu)^4] - (\sigma^2)^2.$$

This form suggests a rather simple estimator of γ^2 , as follows.

$$\hat{\gamma}_{1,n}^2 = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^4}{n} - \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n} \right)^2. \quad (3.35)$$

In view of $H_A^{(2)}$, but still under $H_O^{(2)}$, we extend the estimator of γ^2 suggested by M. Csörgő and L. Horváth [3, page 190] for the AMOC in variance to our setting, the AMTC in variance. Namely, we propose

$$\begin{aligned} \hat{\gamma}_{2,n,k_1,k_2}^2 &= n^{-1} \left\{ \sum_{i=1}^n (X_i - \bar{X}_n)^4 - \frac{1}{k_1} \left(\sum_{i=1}^{k_1} (X_i - \bar{X}_n)^2 \right)^2 \right. \\ &\quad \left. - \frac{1}{k_2 - k_1} \left(\sum_{i=1}^{k_2 - k_1} (X_{k_1+i} - \bar{X}_n)^2 \right)^2 - \frac{1}{n - k_2} \left(\sum_{i=1}^{n - k_2} (X_{k_2+i} - \bar{X}_n)^2 \right)^2 \right\}, \\ &\quad \text{if } 1 \leq k_1 \leq k_2 < n, \\ &= \hat{\gamma}_{1,n}^2, \quad \text{if } k_1 = k_2 = n. \end{aligned} \quad (3.36)$$

Theorem 3.2.1 *If $H_O^{(2)}$ and (3.1), (3.2) hold, then, as $n \rightarrow \infty$, we have*

$$|\hat{\gamma}_{1,n}^2 - \gamma^2| = o_P(1)$$

and

$$\max_{1 \leq k_1 \leq k_2 \leq n} |\hat{\gamma}_{2,n,k_1,k_2}^2 - \gamma^2| = o_P(1).$$

Proof.

After some simple calculations, we have

$$\begin{aligned} \sum_{i=1}^n ((X_i - \mu) - (\bar{X}_n - \mu))^4 &= \sum_{i=1}^n (X_i - \mu)^4 - 4(\bar{X}_n - \mu)^3 \sum_{i=1}^n (X_i - \mu) \\ &\quad - 6(\bar{X}_n - \mu)^2 \sum_{i=1}^n (X_i - \mu)^2 - 4(\bar{X}_n - \mu) \sum_{i=1}^n (X_i - \mu)^3 \\ &\quad + n(\bar{X}_n - \mu)^4 \end{aligned} \quad (3.37)$$

and

$$\begin{aligned} \left(\sum_{i=1}^n ((X_i - \mu) - (\bar{X}_n - \mu))^2 \right)^2 &= \left(\sum_{i=1}^n (X_i - \mu)^2 \right)^2 - 2n(\bar{X}_n - \mu)^2 \sum_{i=1}^n (X_i - \mu)^2 \\ &\quad + n^2(\bar{X}_n - \mu)^4. \end{aligned} \quad (3.38)$$

As a consequence of (3.37) and (3.38), we have the following.

$$\begin{aligned} &\left| \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^4}{n} - \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n} \right)^2 - \gamma^2 \right| \\ &\leq \left| \frac{\sum_{i=1}^n ((X_i - \bar{X}_n)^4 - \mathbb{E}[(X_1 - \mu)^4])}{n} \right| \\ &\quad + \left| \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n} \right)^2 - (\sigma^2)^2 \right| \\ &\leq \left| \frac{\sum_{i=1}^n ((X_i - \mu)^4 - \mathbb{E}[(X_1 - \mu)^4])}{n} \right| \\ &\quad + 4|\bar{X}_n - \mu|^3 \left| \frac{\sum_{i=1}^n (X_i - \mu)}{n} \right| \\ &\quad + 6(\bar{X}_n - \mu)^2 \frac{\sum_{i=1}^n (X_i - \mu)^2}{n} \\ &\quad + 4|\bar{X}_n - \mu| \left| \frac{\sum_{i=1}^n (X_i - \mu)^3}{n} \right| \\ &\quad + (\bar{X}_n - \mu)^4 + \left| \left(\frac{\sum_{i=1}^n (X_i - \mu)^2}{n} \right)^2 - (\sigma^2)^2 \right| \\ &\quad + (\bar{X}_n - \mu)^4 + 2(\bar{X}_n - \mu)^2 \frac{\sum_{i=1}^n (X_i - \mu)^2}{n} \\ &= o_P(1), \end{aligned} \quad (3.39)$$

as $n \rightarrow \infty$, which follows by the SLLN and the CLT.

In order to prove the second statement, we note first that, as $n \rightarrow \infty$,

$$\max_{1 \leq k_1 \leq n} \left| \frac{\sum_{i=1}^{k_1} ((X_i - \mu)^2 - \sigma^2)}{\sqrt{k_1} \log \log n} \right| \sqrt{\frac{\log \log n}{n}} \stackrel{a.s.}{=} O(1) \cdot o(1), \quad (3.40)$$

$$\max_{1 \leq k_2 - k_1 \leq n} \left| \frac{\sum_{i=1}^{k_2 - k_1} ((X_{k_1+i} - \mu)^2 - \sigma^2)}{\sqrt{(k_2 - k_1) \log \log n}} \right| \sqrt{\frac{\log \log n}{n}} \stackrel{a.s.}{=} O(1) \cdot o(1), \quad (3.41)$$

and

$$\max_{1 \leq n-k_2 \leq n} \left| \frac{\sum_{i=1}^{n-k_2} ((X_{k_2+i} - \mu)^2 - \sigma^2)}{\sqrt{(n-k_2) \log \log n}} \right| \sqrt{\frac{\log \log n}{n}} \stackrel{a.s.}{=} O(1) \cdot o(1), \quad (3.42)$$

with all three following from Theorem 1.0.3.

Simple calculations yield

$$\begin{aligned} \frac{1}{nk_1} \left(\sum_{i=1}^{k_1} (X_i - \bar{X}_n)^2 \right)^2 - \frac{k_1}{n} (\sigma^2)^2 &= \frac{1}{nk_1} \left(\sum_{i=1}^{k_1} (X_i - \mu)^2 \right)^2 \\ &+ 4(\bar{X}_n - \mu)^2 \left(\sum_{i=1}^{k_1} \frac{(X_i - \mu)}{(nk_1)^{1/2}} \right)^2 + \frac{k_1}{n} (\bar{X}_n - \mu)^4 \\ &- 4 \frac{k_1}{nk_1} (\bar{X}_n - \mu)^3 \sum_{i=1}^{k_1} (X_i - \mu) \\ &- 4n^{1/2} (\bar{X}_n - \mu) \sum_{i=1}^{k_1} \frac{(X_i - \mu)}{(nk_1)^{1/2}} \sum_{i=1}^{k_1} \frac{(X_i - \mu)^2}{n} \\ &+ 2 \frac{k_1}{nk_1} (\bar{X}_n - \mu)^2 \sum_{i=1}^{k_1} (X_i - \mu)^2 - \frac{k_1}{n} (\sigma^2)^2 \end{aligned} \quad (3.43)$$

and, in the light of this, we obtain

$$\begin{aligned} \max_{1 \leq k_1 \leq n} \left| \frac{1}{nk_1} \left(\sum_{i=1}^{k_1} (X_i - \bar{X}_n)^2 \right)^2 - \frac{k_1}{n} (\sigma^2)^2 \right| \\ \leq \max_{1 \leq k_1 \leq n} \left| \frac{1}{nk_1} \left(\sum_{i=1}^{k_1} (X_i - \mu)^2 \right)^2 - \frac{k_1}{n} (\sigma^2)^2 \right| + (\bar{X}_n - \mu)^4 \\ + 4(\bar{X}_n - \mu)^2 \left(\max_{1 \leq k_1 \leq n} \frac{|\sum_{i=1}^{k_1} (X_i - \mu)|}{(nk_1)^{1/2}} \right)^2 \\ + 4 \left| (\bar{X}_n - \mu)^3 \right| \max_{1 \leq k_1 \leq n} \frac{|\sum_{i=1}^{k_1} (X_i - \mu)|}{n} \\ + 4 \left| n^{1/2} (\bar{X}_n - \mu) \right| \max_{1 \leq k_1 \leq n} \frac{|\sum_{i=1}^{k_1} (X_i - \mu)| \sum_{i=1}^n (X_i - \mu)^2}{(nk_1)^{1/2} n} \\ + 2(\bar{X}_n - \mu)^2 \frac{\sum_{i=1}^n (X_i - \mu)^2}{n} \\ = o_P(1), \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (3.44)$$

which follows by equation (3.40), the CLT and the SLLN.

Using results (3.41) and (3.42), we have the following two results, both similar to (3.44), and proven in a similar fashion.

$$\max_{1 \leq k_2 - k_1 \leq n} \left| \frac{1}{n(k_2 - k_1)} \left(\sum_{i=1}^{k_2 - k_1} (X_{k_1+i} - \bar{X}_n)^2 \right)^2 - \frac{k_2 - k_1}{n} (\sigma^2)^2 \right| = o_P(1) \quad (3.45)$$

and

$$\max_{1 \leq k_2 \leq n} \left| \frac{1}{n(n - k_2)} \left(\sum_{i=1}^{n - k_2} (X_{k_2+i} - \bar{X}_n)^2 \right)^2 - \frac{n - k_2}{n} (\sigma^2)^2 \right| = o_P(1), \quad (3.46)$$

as $n \rightarrow \infty$.

We are now able to conclude that

$$\begin{aligned} \max_{1 \leq k_1 \leq k_2 \leq n} |\hat{\gamma}_{2,n,k_1,k_2}^2 - \gamma^2| &\leq \left| \sum_{i=1}^n \frac{(X_i - \bar{X}_n)^4}{n} - \mathbb{E}[(X_1 - \mu)^4] \right| \\ &+ \max_{1 \leq k_1 \leq n} \left| \frac{1}{nk_1} \left(\sum_{i=1}^{k_1} (X_i - \bar{X}_n)^2 \right)^2 - \frac{k_1}{n} (\sigma^2)^2 \right| \\ &+ \max_{1 \leq k_1 \leq k_2 \leq n} \left| \frac{1}{n(k_2 - k_1)} \left(\sum_{i=1}^{k_2 - k_1} (X_{k_1+i} - \bar{X}_n)^2 \right)^2 - \frac{k_2 - k_1}{n} (\sigma^2)^2 \right| \\ &+ \max_{1 \leq k_2 \leq n} \left| \frac{1}{n(n - k_2)} \left(\sum_{i=1}^{n - k_2} (X_{k_2+i} - \bar{X}_n)^2 \right)^2 - \frac{n - k_2}{n} (\sigma^2)^2 \right| \\ &= o_P(1) + o_P(1) + o_P(1) + o_P(1) \end{aligned} \quad (3.47)$$

$$= o_P(1), \quad \text{as } n \rightarrow \infty. \quad (3.48)$$

In verifying (3.48), we note that the first term in (3.47) is $o_P(1)$, as $n \rightarrow \infty$, by (3.39), while the remaining three terms follow by equations (3.44), (3.45) and (3.46).

The estimator $\hat{\gamma}_{2,n,k_1,k_2}^2$, as specified, still depends on the unknown parameters k_1 and k_2 , and as such, it is not a statistic. To resolve this problem, we propose

setting k_1 and k_2 in equation (3.36) to those values $\hat{k}_1$ and $\hat{k}_2$ that maximize $|M^{(1)}(k_1, k_2)|$, i.e., à la Orasch [24, (4.6.1)],

$$\begin{aligned} (\hat{k}_1, \hat{k}_2) &:= \min \left\{ (k_1, k_2); \left| M_n^{(1)} \left(\frac{k_1}{n+1}, \frac{k_2}{n+1} \right) \right| \right. \\ &\quad \left. = \max_{1 \leq m_1 \leq m_2 < n} \left| M_n^{(1)} \left(\frac{m_1}{n+1}, \frac{m_2}{n+1} \right) \right| \right\}, \end{aligned} \quad (3.49)$$

where $\min \{(k_1, k_2); \dots\}$ is used to denote the point (k_1, k_2) where the maximum occurs which has the smallest distance using Euclidean norm from the origin. Once the estimates $\hat{k}_1$ and $\hat{k}_2$ of parameters k_1 and k_2 have been determined, they could then be substituted for the corresponding parameters in formula (3.36) to arrive at

$$\begin{aligned} \tilde{\gamma}_{2,n,\hat{k}_1,\hat{k}_2}^2 &= n^{-1} \left\{ \sum_{i=1}^n (X_i - \bar{X}_n)^4 - \frac{1}{\hat{k}_1} \left(\sum_{i=1}^{\hat{k}_1} (X_i - \bar{X}_n)^2 \right)^2 \right. \\ &\quad \left. - \frac{1}{\hat{k}_2 - \hat{k}_1} \left(\sum_{i=1}^{\hat{k}_2 - \hat{k}_1} (X_{\hat{k}_1+i} - \bar{X}_n)^2 \right)^2 - \frac{1}{n - \hat{k}_2} \left(\sum_{i=1}^{n - \hat{k}_2} (X_{\hat{k}_2+i} - \bar{X}_n)^2 \right)^2 \right\}. \end{aligned} \quad (3.50)$$

Since the estimator in (3.50) incorporates three different estimates of the variance, if $H_A^{(2)}$ were true, it should be more sensitive to changes in the variance than the estimator given in (3.35). As a consequence, it is expected that this estimator will make functionals of (3.5), along with its derivatives, more sensitive to detecting changes in the variance as well.

As Theorem 3.2.1 establishes the uniform consistency of $\hat{\gamma}_{2,n,k_1,k_2}^2$, in k_1 and k_2 , $1 \leq k_1 \leq k_2 \leq n$, we are then able to conclude that $\tilde{\gamma}_{2,n,\hat{k}_1,\hat{k}_2}^2$ is a consistent estimator of γ^2 as well. With this in mind, we make the following remark.

Remark 3.2.1 Since $\hat{\gamma}_{1,n}^2$ and $\tilde{\gamma}_{2,n,\hat{k}_1,\hat{k}_2}^2$ are both consistent estimators of γ^2 under $H_O^{(2)}$ (cf. Theorem 3.2.1), we can replace γ with either one of $\hat{\gamma}_{1,n}$ or $\tilde{\gamma}_{2,n,\hat{k}_1,\hat{k}_2}$ in Corollaries 3.1.1 and 3.1.2, and they will continue to hold true by Slutsky's theorem.

3.3 Asymptotic Results Under $H_A^{(2)}$

3.3.1 Population Mean Known

This section deals with the limiting behaviour of $M_n^{(1)}(t_1, t_2)$ and $\bar{M}_n^{(1)}(t_1, t_2)$, defined in (3.5) and (3.10), respectively, under $H_A^{(2)}$. Using their limiting behaviour under $H_A^{(2)}$, we will establish the consistency of tests developed in Corollaries 3.1.1, 3.1.2 and 3.1.3 under the three cases: when the population mean is known, unknown and then under small changes in the population mean.

Remark 3.3.1 Assume $H_A^{(2)}$, (3.2), put $Y_i := (X_i - \mu)^2$, and define (cf. (3.1))

$$\gamma_A^2 := \begin{cases} \gamma_1^2 := \mathbf{E}Y_i^2 - (\sigma^2)^2, & 1 \leq i \leq k_1^* \\ \gamma_2^2 := \mathbf{E}Y_i^2 - (\tau^2)^2, & k_1^* < i \leq k_2^* \\ \gamma_3^2 := \mathbf{E}Y_i^2 - (\rho^2)^2, & k_2^* < i \leq n. \end{cases}$$

Then, we write

$$\gamma_A = \gamma_1 I\{1 \leq i \leq k_1^*\} + \gamma_2 I\{k_1^* < i \leq k_2^*\} + \gamma_3 I\{k_2^* < i \leq n, \}$$

and note that, given $H_A^{(2)}$ with k_1^* and k_2^* as in Theorem 3.3.1,

$$\text{Var}(Y_i/\gamma_A) = (\mathbf{E}Y_i^2 - (\mathbf{E}Y_i)^2)/\gamma_A^2 = 1, \quad i = 1, \dots, n,$$

and Theorem 1.0.2 is applicable in the indicated three regions for i in the definition of γ_A^2 above.

Under $H_A^{(2)}$, the limiting functions will depend on the unknown location of the two change-points $k_1^* = [n\lambda_1]$ and $k_2^* = [n\lambda_2]$, where $0 < \lambda_1 \leq \lambda_2 < 1$ are fixed.

Theorem 3.3.1 Assume $H_A^{(2)}$ and (3.2) hold, and at least one of $0 < \sigma^2 \neq \tau^2 < \infty$ or $0 < \sigma^2 \neq \rho^2 < \infty$ or $0 < \tau^2 \neq \rho^2 < \infty$ holds. Let $\gamma_A^2 := \mathbf{E}Y_i^2 - (\mathbf{E}Y_i)^2$ where $Y_i := (X_i - \mu)^2$, X_i be as defined in (3.1), and let γ_A be as defined in

Remark 3.3.1. Let $k_1^* = [n\lambda_1]$, and $k_2^* = [n\lambda_2]$, with $0 < \lambda_1 \leq \lambda_2 < 1$ fixed. Then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 < t_2 < 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{n^{1/2}} - m_{\lambda_1, \lambda_2}(t_1, t_2) \right| \stackrel{a.s.}{=} o(1),$$

where

$$m_{\lambda_1, \lambda_2}(t_1, t_2) := \begin{cases} t_2[(\sigma^2 - \rho^2) - \lambda_1(\sigma^2 - \tau^2) - \lambda_2(\tau^2 - \rho^2)] & \text{if } 0 < t_1 < t_2 \leq \lambda_1 \leq \lambda_2 < 1 \\ t_1 t_2(\sigma^2 - \tau^2) - t_1 \lambda_1(\sigma^2 - \tau^2) - t_2[\lambda_1(\sigma^2 - \tau^2) + (1 - \lambda_2)(\tau^2 - \rho^2)] + \lambda_1(\sigma^2 - \tau^2) & \text{if } 0 < t_1 \leq \lambda_1 < t_2 \leq \lambda_2 < 1 \\ t_1 t_2(\sigma^2 - \tau^2) + t_1(\lambda_2 - \lambda_1)(\sigma^2 - \tau^2) + \lambda_2(\sigma^2 - \tau^2) & \text{if } 0 < t_1 \leq \lambda_1 \leq \lambda_2 < t_2 < 1 \\ -t_1 \lambda_1(\sigma^2 - \tau^2) + t_2[(1 - \lambda_2)(\tau^2 - \rho^2)] + \lambda_1(\sigma^2 - \tau^2) & \text{if } 0 < \lambda_1 \leq t_1 < t_2 \leq \lambda_2 < 1 \\ t_1 t_2(\tau^2 - \rho^2) - t_1(\lambda_1(\sigma^2 - \tau^2) + \lambda_2(\tau^2 - \rho^2)) - t_2(\lambda_2(\tau^2 - \rho^2) + \lambda_1(\sigma^2 - \tau^2)) - \lambda_2(\rho^2 - \tau^2) - \lambda_1(\tau^2 - \sigma^2) & \text{if } 0 < \lambda_1 \leq t_1 \leq \lambda_2 < t_2 < 1 \\ -t_1 \lambda_1(\sigma^2 - \tau^2) + \lambda_1(\sigma^2 - \tau^2) + \lambda_2(\sigma^2 - \tau^2) & \text{if } 0 < \lambda_1 \leq \lambda_2 \leq t_1 < t_2 < 1. \end{cases} \quad (3.51)$$

Proof.

The result will be shown for only the region $0 < t_1 < t_2 \leq \lambda_1 \leq \lambda_2 < 1$, as the other regions follow using similar arguments.

Let $0 < t_1 < t_2 \leq \lambda_1 \leq \lambda_2 < 1$ and $M_n^{(1)}(t_1, t_2)$ be as defined in (3.5). Each of the partial sums that figure in the definition of the latter will be shown, for fixed t_1 and t_2 , to converge a.s., as $n \rightarrow \infty$, to a function of the population parameters. By the SLLN, we have the following three results.

$$\sum_{i=1}^{[(n+1)t_1]} \frac{Y_i}{n} = \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_1]} \frac{Y_i}{[(n+1)t_1]} \xrightarrow{a.s.} t_1 \sigma^2, \quad \text{as } n \rightarrow \infty, \quad (3.52)$$

$$\sum_{i=1}^{[(n+1)t_2]} \frac{Y_i}{n} = \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_2]} \frac{Y_i}{[(n+1)t_2]} \xrightarrow{a.s.} t_2 \sigma^2, \quad \text{as } n \rightarrow \infty \quad (3.53)$$

and

$$\begin{aligned} \sum_{i=1}^n \frac{Y_i}{n} &= \frac{[n\lambda_1]}{n} \sum_{i=1}^{[n\lambda_1]} \frac{Y_i}{[n\lambda_1]} + \frac{[n(\lambda_2 - \lambda_1)]}{n} \sum_{i=1}^{[n(\lambda_2 - \lambda_1)]} \frac{Y_{[n\lambda_1]+i}}{[n(\lambda_2 - \lambda_1)]} \\ &\quad + \frac{[n(1 - \lambda_2)]}{n} \sum_{i=1}^{[n(1 - \lambda_2)]} \frac{Y_{[n\lambda_2]+i}}{[n(1 - \lambda_2)]} \\ &\xrightarrow{a.s.} \lambda_1 \sigma^2 + (\lambda_2 - \lambda_1) \tau^2 + (1 - \lambda_2) \rho^2, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.54)$$

Consequently, via (3.52) - (3.54), the definition of $M^{(1)}(t_1, t_2)$, and as $n \rightarrow \infty$, we arrive at

$$\frac{M_n^{(1)}(t_1, t_2)}{n^{1/2}} \xrightarrow{a.s.} t_2((\sigma^2 - \rho^2) - \lambda_1(\sigma^2 - \tau^2) - \lambda_2(\tau^2 - \rho^2)), \quad (3.55)$$

for t_1 and t_2 fixed in the region $0 \leq t_1 < t_2 \leq \lambda_1 \leq \lambda_2 < 1$.

Next it is shown that convergence of the latter is uniform in t_1 and t_2 in this region, and note that one can apply similar arguments to the remaining regions as well. To establish uniform convergence in t_1 and t_2 of (3.55), we need to establish uniform convergence, under $H_A^{(2)}$, for each of the partial sums that constitute $M_n^{(1)}(t_1, t_2)$. We have

$$\begin{aligned} \sup_{0 < t_1 < t_2 \leq \lambda_1 \leq \lambda_2 < 1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{(Y_i - \sigma^2)}{n} \right| &\leq \sup_{0 < t_1 < \lambda_1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{(Y_i - \sigma^2)}{n \cdot \gamma_A \sqrt{2 \log \log n}} \frac{\gamma_A \sqrt{2 \log \log n}}{n} \right| \\ &= \sup_{0 < t_1 < \lambda_1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{(Y_i - \sigma^2)}{[(n+1)t_1]^{1/2} \gamma_A \sqrt{2 \log \log n}} \frac{[(n+1)t_1]^{1/2}}{n^{1/2}} \right| \frac{\gamma_A \sqrt{2 \log \log n}}{n^{1/2}} \\ &\stackrel{a.s.}{=} O(1) \cdot o(1) = o(1), \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (3.56)$$

which follows by Theorem 1.0.2 of Chapter 1 (cf. Remark 3.3.1).

In the light of (3.56), we are able now to establish convergence of $M_n^{(1)}(t_1, t_2)$ in sup-norm.

$$\begin{aligned}
& \sup_{0 < t_1 < t_2 \leq \lambda_1 \leq \lambda_2 < 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{n^{1/2}} - m_{\lambda_1, \lambda_2}(t_1, t_2) \right| \\
& \leq \sup_{0 < t_1 < t_2 \leq \lambda_1 < 1} \left| \frac{[(n+1)t_2]}{n} \sum_{i=1}^{[(n+1)t_1]} \frac{Y_i}{n} - t_2 t_1 \sigma^2 \right| \\
& \quad + \sup_{0 < t_1 < t_2 \leq \lambda_1 < 1} \left| \frac{(n - [(n+1)t_1])}{n} \sum_{i=1}^{[(n+1)t_2]} \frac{Y_i}{n} \right. \\
& \quad \quad \left. - (1 - t_1) t_2 \sigma^2 \right| \\
& \quad + \sup_{0 < t_2 \leq \lambda_1 < 1} \left| \frac{[(n+1)t_2]}{n} \sum_{i=1}^n \frac{Y_i}{n} - t_2 (\lambda_1 \sigma^2 \right. \\
& \quad \quad \left. + (\lambda_2 - \lambda_1) \tau^2 + (1 - \lambda_2) \rho^2) \right| \\
& =: I_1(n) + I_2(n) + I_3(n) \\
& \stackrel{\text{a.s.}}{=} o(1), \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

since each term, $I_1(n)$, $I_2(n)$ and $I_3(n)$, can be majorized by terms that are a.s.

$o(1)$ in sup-norm, as $n \rightarrow \infty$. Indeed, we have

$$\begin{aligned}
I_1(n) & \leq \sup_{0 < t_1 < t_2 \leq \lambda_1 < 1} \left| t_2 \sum_{i=1}^{[(n+1)t_1]} \frac{Y_i}{n} - t_1 t_2 \sigma^2 \right| \\
& \quad + \sup_{0 < t_1 < t_2 \leq \lambda_1 < 1} \left| \frac{[(n+1)t_2]}{n} - t_2 \right| \sup_{0 < t_1 < t_2 \leq \lambda_1 < 1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{Y_i}{n} \right| \\
& \leq \sup_{0 < t_1 < t_2 \leq \lambda_1 < 1} t_2 \left| \sum_{i=1}^{[(n+1)t_1]} \frac{Y_i}{n} - \frac{[(n+1)t_1]}{n} \sigma^2 \right| + \sup_{0 < t_1 < t_2 < 1} \sigma^2 t_2 \left| \frac{[(n+1)t_1]}{n} - t_1 \right| \\
& \quad + \sup_{0 < t_1 < \lambda_1 < 1} \left| \frac{[(n+1)t_2]}{n} - t_2 \right| \sup_{0 < t_1 < \lambda_1 < 1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{Y_i}{n} \right| \\
& \leq \sup_{0 < t_1 < \lambda_1 < 1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{Y_i}{n} - \frac{[(n+1)t_1]}{n} \sigma^2 \right| + \sup_{0 < t_1 < 1} \sigma^2 \left| \frac{[(n+1)t_1]}{n} - t_1 \right|
\end{aligned}$$

$$\begin{aligned}
& + \sup_{0 < t_1 < 1} \left| \frac{[(n+1)t_2]}{n} - t_2 \right| \left| \sum_{i=1}^n \frac{Y_i}{n} \right| \\
& \stackrel{\text{a.s.}}{=} o(1) + o(1) + o(1) \cdot O(1) = o(1), \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

as a consequence of (3.54), (3.56), Lemma 2.4.1. Moreover, $I_2(n)$ is also of order $o(1)$ a.s. in sup-norm, as $n \rightarrow \infty$, by the same argument used to establish this property for $I_1(n)$.

As to $I_3(n)$, we have

$$\begin{aligned}
I_3(n) & \leq \sup_{0 < t_2 < 1} \left| \frac{[(n+1)t_2]}{n} \left| \sum_{i=1}^n \frac{Y_i}{n} - (\lambda_1 \sigma^2 + (\lambda_2 - \lambda_1) \tau^2 + (1 - \lambda_2) \rho^2) \right| \right. \\
& \quad \left. + \sup_{0 < t_2 < 1} \left| \frac{[(n+1)t_2]}{n} - t_2 \right| \left| \lambda_1 \sigma^2 + (\lambda_2 - \lambda_1) \tau^2 + (1 - \lambda_2) \rho^2 \right| \right| \\
& \stackrel{\text{a.s.}}{=} o(1) + o(1) \cdot O(1) = o(1), \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

by the SLLN and Lemma 2.4.1. Hence, the a.s. uniform convergence of $\frac{M^{(1)}(t_1, t_2)}{n^{1/2}}$ in t_1, t_2 follows over the regions as in (3.55).

Remark 3.3.2 If $\lambda_1 = \lambda_2$ and the sum is, for example, of the form $\sum_{i=1}^{[n(\lambda_2 - \lambda_1)]} \frac{Y_{[n\lambda_1] + i}}{[n(\lambda_2 - \lambda_1)]}$, then this term is not defined due to division by 0. Since this sums runs from $i = 1$ to 0, it involves no terms and we define sums of this nature to be zero.

Proposition 3.3.1 $m_{\lambda_1, \lambda_2}(t_1, t_2) \equiv 0$ if and only if $\sigma^2 = \tau^2 = \rho^2$.

Proof.

From equation (3.51), it is easily seen that if $\sigma^2 = \tau^2 = \rho^2$, then $m_{\lambda_1, \lambda_2}(t_1, t_2) = 0$. Conversely, assume $m_{\lambda_1, \lambda_2}(t_1, t_2) \equiv 0$. From equation (3.51), it is seen that

$$m_{\lambda_1, \lambda_2}(t_1, t_2) = A_j t_1 t_2 + B_j t_2 + C_j t_1 + D_j$$

for $1 \leq j \leq 6$. The coefficients, A_j, B_j, C_j and D_j , are functions of σ^2, τ^2, ρ^2 and λ_1 and λ_2 . If $m_{\lambda_1, \lambda_2}(t_1, t_2) = 0$, this must hold for all (t_1, t_2) satisfying $0 < t_1 < t_2 < 1$

which implies that $A_j = B_j = C_j = D_j = 0$ for $1 \leq j \leq 6$. Moreover, since the coefficients are functions of λ_1 and λ_2 , $A_j = B_j = C_j = D_j = 0$ for $1 \leq j \leq 6$ and for all λ_1, λ_2 satisfying $0 < \lambda_1 \leq \lambda_2 < 1$. From this, we obtain seventeen equations in three unknown parameters. In particular, they are

$$\begin{aligned}
A_2 &= \sigma^2 - \tau^2 = 0, \\
A_3 &= \sigma^2 - \tau^2 = 0, \\
A_5 &= \tau^2 - \rho^2 = 0, \\
B_1 &= \sigma^2 - \rho^2 - \lambda_1(\sigma^2 - \tau^2) - \lambda_2(\tau^2 - \rho^2) = 0, \\
B_2 &= -\lambda_1(\sigma^2 - \tau^2) + (1 - \lambda_2)(\tau^2 - \rho^2) = 0, \\
B_4 &= (1 - \lambda_2)(\tau^2 - \rho^2) = 0, \\
B_5 &= \tau^2 - \rho^2 = 0, \\
C_2 &= \sigma^2 - \tau^2 = 0, \\
C_3 &= (1 - \lambda_1)(\sigma^2 - \tau^2) = 0, \\
C_4 &= \sigma^2 - \tau^2 = 0, \\
C_5 &= \sigma^2 - \tau^2 - \lambda_2(\tau^2 - \rho^2) = 0, \\
C_6 &= \sigma^2 - \tau^2 = 0, \\
D_2 &= \sigma^2 - \tau^2 = 0, \\
D_3 &= \sigma^2 - \tau^2 = 0, \\
D_4 &= \sigma^2 - \tau^2 = 0, \\
D_5 &= \sigma^2 - \tau^2 = 0, \\
D_6 &= \lambda_1(\sigma^2 - \tau^2) + \lambda_2(\tau^2 - \rho^2) = 0.
\end{aligned}$$

Of these seventeen equations, it is easy to see that there are only two linearly independent equations, of which two are $A_2 = 0$ and $A_5 = 0$. The conditions

$A_2 = A_5 = 0$, imply $\sigma^2 = \tau^2 = \rho^2$.

Remark 3.3.3 Under $H_O^{(2)}$, $\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{n^{1/2}\gamma} \right| \xrightarrow{a.s.} 0$, as $n \rightarrow \infty$.

Corollary 3.3.1 *Under the conditions given in Theorem 3.3.1, a test for at most two change-points which is based on large values of the statistic $\sup_{0 \leq t_1 \leq t_2 \leq 1} |M_n^{(1)}(t_1, t_2)|$ has, as $n \rightarrow \infty$, the following consistency property*

$$P\{H_O^{(2)} \text{ rejected}; H_A^{(2)} \text{ true}\} \rightarrow 1.$$

Proof.

In principle, define c_α for $0 < \alpha < 1$ via Corollary 3.1.1, and consider

$$\mathbb{P}\{H_O^{(2)} \text{ rejected}; H_A^{(2)} \text{ true}\} = \mathbb{P}\left\{\sup_{0 \leq t_1 \leq t_2 \leq 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} \right| \geq c_\alpha; H_A^{(2)} \text{ true}\right\}.$$

On account of Theorem 3.3.1 and Proposition 3.3.1, we have, as $n \rightarrow \infty$,

$$n^{1/2} \sup_{0 < t_1 < t_2 < 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma\sqrt{n}} \right| \xrightarrow{a.s.} \infty,$$

which implies the claimed consistency.

3.3.2 Population Mean Unknown

Corollary 3.3.2 *Under the conditions given in Theorem 3.3.1, with $m_{\lambda_1, \lambda_2}(t_1, t_2)$ as defined in (3.51), then as $n \rightarrow \infty$,*

$$\sup_{0 < t_1 < t_2 < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{n^{1/2}} - m_{\lambda_1, \lambda_2}(t_1, t_2) \right| \stackrel{a.s.}{=} o(1).$$

Proof.

From (3.11), we obtain

$$\begin{aligned}
\frac{\bar{M}_n^{(1)}(t_1, t_2)}{n^{1/2}} &= \frac{M_n^{(1)}(t_1, t_2)}{n^{1/2}} \\
&\quad - 2(\bar{X}_n - \mu) \left\{ \frac{(n - [(n+1)t_1])}{n} \sum_{i=1}^{[(n+1)t_2]} \frac{(X_i - \mu)}{n} \right. \\
&\quad \left. + \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_1]} \frac{(X_i - \mu)}{n} - \frac{[(n+1)t_2]}{n} \sum_{i=1}^n \frac{(X_i - \mu)}{n} \right\},
\end{aligned} \tag{3.57}$$

consequently, as $n \rightarrow \infty$,

$$\begin{aligned}
\sup_{0 < t_1 < t_2 < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1)}(t_1, t_2)}{n^{1/2}} \right| \\
\leq 2|\bar{X}_n - \mu| \left\{ \sup_{0 < t_2 < 1} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_i - \mu)}{n} \right| \right. \\
\quad \left. + \sup_{0 < t_1 < 1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{(X_i - \mu)}{n} \right| + \left| \sum_{i=1}^n \frac{(X_i - \mu)}{n} \right| \right\} \\
\stackrel{\text{a.s.}}{=} o(1) \cdot (o(1) + o(1) + o(1)) = o(1).
\end{aligned} \tag{3.58}$$

In order to see this, the right hand side bound in (3.58) will have to be examined over all the regions of the domain of definition in (3.51). We will do this only over $0 < t_1 < t_2 \leq \lambda_1 \leq \lambda_2 < 1$. In the latter case, we have

$$\begin{aligned}
\sup_{0 < t_2 \leq \lambda_1 < 1} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_i - \mu)}{n} \right| \\
\leq \sup_{0 < t_2 \leq \lambda_1 < 1} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_i - \mu)}{\sqrt{2n\sigma^2 \log \log n}} \right| \frac{\sqrt{2n\sigma^2 \log \log n}}{n} \\
\stackrel{\text{a.s.}}{=} O(1) \cdot o(1) = o(1), \text{ as } n \rightarrow \infty,
\end{aligned}$$

by the Hartman-Wintner LIL. The sum involving t_1 is a.s. $o(1)$ in sup-norm, as $n \rightarrow \infty$, by the same argument. The last term in (3.58) does not involve t_1 or

t_2 and therefore is a.s. $o(1)$, as $n \rightarrow \infty$, by the SLLN, on separating the sum into constituent parts

$$\sum_{i=1}^n = \sum_{i=1}^{[n\lambda_1]} + \sum_{i=[n\lambda_1]+1}^{[n\lambda_2]} + \sum_{i=[n\lambda_2]+1}^n$$

with respects to (3.1). The same holds for $2|\bar{X}_n - \mu|$ in front of the bracketed term.

Continuing along these lines over all the regions as in (3.51) and then using also Theorem 3.3.1, we conclude

$$\begin{aligned} \sup_{0 < t_1 < t_2 < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{n^{1/2}} - m_{\lambda_1, \lambda_2}(t_1, t_2) \right| &\leq \sup_{0 < t_1 < t_2 < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1)}(t_1, t_2)}{n^{1/2}} \right| \\ &\quad + \sup_{0 < t_1 < t_2 < 1} \left| \frac{M_n^{(1)}(t_1, t_2)}{n^{1/2}} - m_{\lambda_1, \lambda_2}(t_1, t_2) \right| \\ &\stackrel{\text{a.s.}}{=} o(1), \text{ as } n \rightarrow \infty. \end{aligned}$$

Corollary 3.3.3 *Under the conditions given in Theorem 3.3.1, a test for at most two change-points which is based on large values of the statistic $\sup_{0 \leq t_1 \leq t_2 \leq 1} |\bar{M}_n^{(1)}(t_1, t_2)|$ has, as $n \rightarrow \infty$, the following consistency property*

$$P\{H_O^{(2)} \text{ rejected}; H_A^{(2)} \text{ true}\} \rightarrow 1.$$

Proof. This follows from Corollary 3.3.2 and Proposition 3.3.1.

3.3.3 Small Changes in the Population Mean

Corollary 3.3.4 *Assume $H_A^{(2)}$, (3.2) and (3.14) all hold, and at least one of $0 < \sigma^2 \neq \tau^2 < \infty$ or $0 < \sigma^2 \neq \rho^2 < \infty$ or $0 < \tau^2 \neq \rho^2 < \infty$ must hold. Let $\gamma_A^2 := \mathbb{E}Y_{i,n}^2 - (\mathbb{E}Y_{i,n})^2$, $Y_{i,n} := (X_{i,n} - \mu_{in})^2$, $X_{i,n}$ be as defined in (3.13), and, mutatis mutandis à la (3.13), write γ_A as in Remark 3.3.1. Let $k_1^* = [n\lambda_1]$ and $k_2^* = [n\lambda_2]$ with $0 < \lambda_1 \leq \lambda_2 < 1$ fixed. Then, as $n \rightarrow \infty$,*

$$\sup_{0 < t_1 < t_2 < 1} \left| \frac{M_n^{(1*)}(t_1, t_2)}{n^{1/2}} - m_{\lambda_1, \lambda_2}(t_1, t_2) \right| = o_P(1)$$

with $m_{\lambda_1, \lambda_2}(t_1, t_2)$ given in (3.51).

Proof.

In the proof of Lemma 3.1.2, we obtained equation (3.20), which is reproduced here with $M_n^{(1*)}(t_1, t_2)$ as in (3.15). Namely, as in (3.20), we have

$$\begin{aligned} \frac{\bar{M}_n^{(1)}(t_1, t_2)}{n^{1/2}} &= \frac{M_n^{(1*)}(t_1, t_2)}{n^{1/2}} + \frac{1}{n} \left\{ D_n^{(2)}(t_1, t_2) + D_n^{(3)}(t_1, t_2) + D_n^{(4)}(t_1, t_2) \right. \\ &\quad \left. + D_n^{(5)}(t_1, t_2) \right\}. \end{aligned}$$

It will be shown that

$$\sup_{0 < t_1 < t_2 < 1} \left| \frac{D_n^{(i)}(t_1, t_2)}{n} \right| = o_P(1), \quad \text{for } i = 2, 3, 4, 5, \quad (3.59)$$

as $n \rightarrow \infty$.

Consider

$$\begin{aligned} \sup_{0 < t_1 < t_2 < 1} \left| \frac{D_n^{(2)}(t_1, t_2)}{n} \right| &\leq 2 \sup_{0 < t_2 < 1} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_i - \mu_{in})(\mu_{in} - \bar{\mu})}{n} \right| \\ &\quad + 2 \sup_{0 < t_2 < 1} \frac{[(n+1)t_1]}{n} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n} \right| \\ &\quad + 2 \sup_{0 < t_1 < 1} \frac{[(n+1)t_2]}{n} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n} \right| \\ &\quad + 2 \sup_{0 < t_2 < 1} \frac{[(n+1)t_2]}{n} \left| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n} \right|. \quad (3.60) \end{aligned}$$

We have

$$\sup_{0 < t_2 < 1} \left| \frac{\sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n} \right| = o_P(1), \quad (3.61)$$

as $n \rightarrow \infty$.

To show (3.61) holds over all regions of the domain of definition of (3.51), we have by Theorem 1.0.4 that

$$\begin{aligned}
\mathbb{P} \left\{ \sup_{1 \leq [(n+1)t_2] \leq n} \left| \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu_{in})(\mu_{in} - \bar{\mu}) \right| \geq \varepsilon n \right\} \\
\leq \frac{1}{n^2 \varepsilon^2} \sum_{i=1}^n \mathbb{E} [((X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}))^2] \\
\leq \frac{1}{n^2 \varepsilon^2} \max_{1 \leq i \leq n} (\mu_{in} - \bar{\mu})^2 \sum_{i=1}^n \mathbb{E} (X_{i,n} - \mu_{in})^2 \\
\leq \frac{1}{n \varepsilon^2} \max\{\sigma^2, \rho^2, \tau^2\} \max_{1 \leq i \leq n} (\mu_{in} - \bar{\mu})^2 \\
= n^{-1} \cdot o(n^{-1/2}) \\
= o(n^{-3/2}) \\
= o(1), \tag{3.62}
\end{aligned}$$

as $n \rightarrow \infty$, which holds regardless of the location of the λ_i 's, where in the last inequality Remark 3.3.1, *mutatis mutandis*, was also used. As a consequence of this, we conclude

$$\sup_{0 < t_1 < t_2 < 1} \left| \frac{D_n^{(2)}(t_1, t_2)}{n} \right| = o_P(1), \tag{3.63}$$

as $n \rightarrow \infty$.

Continuing,

$$\sup_{0 < t_1 < t_2 < 1} \left| \frac{D_n^{(3)}(t_1, t_2)}{n} \right| = o_P(1), \tag{3.64}$$

as $n \rightarrow \infty$, by (3.25). As this term does not involve any random variables, (3.64) holds for all regions of the domain of definition of (3.51).

Moreover,

$$\begin{aligned}
\frac{D_n^{(4)}(t_1, t_2)}{n} &= 2(\bar{X}_n - \bar{\mu}) \left\{ \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})}{n} - \frac{[(n+1)t_1]}{n} \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})}{n} \right. \\
&\quad \left. - \frac{[(n+1)t_2]}{n} \sum_{i=[(n+1)t_2]}^n \frac{(X_{i,n} - \mu_{in})}{n} \right\}, \tag{3.65}
\end{aligned}$$

which implies

$$\begin{aligned}
\sup_{0 < t_1 < t_2 < 1} \left| \frac{D_n^{(4)}(t_1, t_2)}{n} \right| &\leq 2|\bar{X}_n - \mu| \left\{ \sup_{0 < t_2 < 1} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})}{n} \right| \right. \\
&\quad \left. + \sup_{0 < t_1 < 1} \left| \sum_{i=1}^{[(n+1)t_1]} \frac{(X_{i,n} - \mu_{in})}{n} \right| + \left| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})}{n} \right| \right\} \\
&= o_P(1) \cdot O_P(1) = o_P(1), \tag{3.66}
\end{aligned}$$

as $n \rightarrow \infty$, by consistency of $\bar{X}_n$, under $H_A^{(2)}$, and (3.62) with $(\mu_{in} - \bar{\mu}) = 1$. As a result of (3.62), (3.66) holds over all regions of (3.51).

Consider now

$$\sup_{0 < t_1 < t_2 < 1} \left| \frac{D^{(5)}(t_1, t_2)}{n} \right| = o_P(1), \tag{3.67}$$

as $n \rightarrow \infty$, by (3.29). Again, the bound in (3.67) also holds over all regions in the domain of definition of (3.51).

Equations (3.63), (3.64), (3.66) and (3.67) imply (3.59), and we conclude with

$$\sup_{0 < t_1 < t_2 < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1*)}(t_1, t_2)}{n^{1/2}} \right| = o_P(1), \tag{3.68}$$

as $n \rightarrow \infty$.

Equation (3.68) along with Corollary 3.3.2 imply

$$\begin{aligned}
\sup_{0 < t_1 < t_2 < 1} \left| \frac{M_n^{(1*)}(t_1, t_2)}{n^{1/2}} - m_{\lambda_1, \lambda_2}(t_1, t_2) \right| &\leq \sup_{0 < t_1 < t_2 < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1*)}(t_1, t_2)}{n^{1/2}} \right| \\
&\quad + \sup_{0 < t_1 < t_2 < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{n^{1/2}} - m_{\lambda_1, \lambda_2}(t_1, t_2) \right| \\
&= o_P(1),
\end{aligned}$$

as $n \rightarrow \infty$.

Corollary 3.3.5 *Under the conditions given in Corollary 3.3.4, the test for at most two change-points which is based on large values of the statistic $\sup_{0 \leq t_1 \leq t_2 \leq 1} |M_n^{(1*)}(t_1, t_2)|$ has, as $n \rightarrow \infty$, the following consistency property*

$$P\{H_O^{(2)} \text{ rejected}; H_A^{(2)} \text{ true}\} \rightarrow 1.$$

Proof. This follows from Corollary 3.3.4 and Proposition 3.3.1.

Chapter 4

Two Change-points with Weights

4.1 Asymptotic Results Under the Null Hypothesis $H_O^{(2)}$

In Section 3.1 we derived the asymptotic distribution of test statistics (cf. Corollaries 3.1.1, 3.1.2 and 3.1.3) that can be used in combination with Section 3.2 to test $H_O^{(2)}$ versus $H_A^{(2)}$ in a consistent way (cf. Section 3.3). Arguing along the lines of Section 2.1 of [5] (cf. also Chapter 7 of this thesis) one can conclude that these test statistics tend to be sensitive to changes occurring near $\frac{n}{2}$, while relatively insensitive to those occurring near zero and n . This suggests that, by employing appropriate weight functions, one might be able to improve the power of the so far proposed tests (cf. Chapter 7) when testing for at most two changes in the variance.

Following now the works of M. Csörgő, S. Csörgő, L. Horváth and D. M. Mason [8], M. Csörgő, Q.-M. Shao and B. Szyszkowicz [4] and B. Szyszkowicz [33], we introduce some definitions and quote three theorems concerning the nature of weight functions that will play a crucial role in our further investigations.

Definition 4.1.1 Let Q be the class of positive functions on $(0, 1)$ which are non-decreasing in a neighbourhood of zero and non-increasing in a neighbourhood of

one, where a function q defined on $(0,1)$ is called positive if

$$\inf_{\delta \leq t \leq 1-\delta} q(t) > 0 \quad \text{for all } \delta \in (0, 1/2).$$

Let $\{W(t); t \geq 0\}$ be a standard Wiener process, and $\{B(t); 0 \leq t \leq 1\}$ a Brownian bridge as in (1.3), i.e., $B(t) = W(t) - tW(1)$.

Definition 4.1.2 A function $q \in Q$ will be called a local function of a Brownian bridge if $q(t)$ and $q(1-t)$ are local functions of a Wiener process on $(0,1/2]$, i.e., if we have

$$\begin{aligned} & \limsup_{t \downarrow 0} \frac{|W(t)|}{q(t)} < \infty \text{ a.s.} \\ \text{and} & \limsup_{t \downarrow 0} \frac{|W(t)|}{q(1-t)} < \infty \text{ a.s.} \end{aligned} \quad (4.1)$$

Definition 4.1.3 A function $q \in Q$ will be called a Chibisov-O'Reilly function of a Brownian bridge if $q(t)$ and $q(1-t)$ are Chibisov-O'Reilly functions of a Wiener process on $(0,1/2]$, i.e., if we have

$$\begin{aligned} & \lim_{t \downarrow 0} \frac{|W(t)|}{q(t)} = 0 \text{ a.s.} \\ \text{and} & \lim_{t \downarrow 0} \frac{|W(t)|}{q(1-t)} = 0 \text{ a.s.} \end{aligned} \quad (4.2)$$

Definition 4.

$$I(q, c) := \int_0^1 (t(1-t))^{-1} \exp(-c(t(1-t))^{-1} q^2(t)) dt, \quad c > 0. \quad (4.3)$$

Theorem A Whenever the integral $I(q, c) < \infty$ for $c > 0$ and $q \in Q$ then

$$\lim_{t \downarrow 0} \frac{q(t)}{t^{1/2}} = \lim_{t \uparrow 1} \frac{q(t)}{(1-t)^{1/2}} = \infty. \quad (4.4)$$

Theorem B A function $q \in Q$ is a local function of a Brownian bridge if and only if the integral $I(q, c) < \infty$ for some $c > 0$.

Theorem C *A function $q \in Q$ is a Chibisov–O’Reilly function of a Brownian bridge if and only if the integral $I(q, c) < \infty$ for all $c > 0$.*

For proofs of theorems A, B and C, we refer to M. Csörgő, S. Csörgő, L. Horváth and D. M. Mason [9] (cf. also M. Csörgő, Q. Shao and B. Szyszkowicz [4]).

Remark 4.1.1 The limiting two-time parameter Gaussian process $\{\Gamma(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$ that figures in Theorems 3.1.1, 3.1.2, 3.1.3, 3.1.4 and is defined in Lemma 1.0.3 is tied down to zero when $t_2 = 0$ (and hence also $t_1 = 0$) and again to zero when $t_1 = 1$ (and hence also $t_2 = 1$). Not surprisingly then, it is easily seen that $\Gamma(t_1, t_2)$ can be also written in terms of a Brownian bridge as well, over the region $0 \leq t_1 \leq t_2 \leq 1$. Namely, we have

$$\begin{aligned}
 \Gamma(t_1, t_2) &:= t_2 W(t_1) + (1 - t_1) W(t_2) - t_2 W(1) \\
 &= t_2 W(t_1) - t_1 W(t_2) + W(t_2) - t_2 W(1) \\
 &= t_2 W(t_1) - t_1 W(t_2) + B(t_2) \\
 &= t_2 W(t_1) - t_2 t_1 W(1) + t_2 t_1 W(1) - t_1 W(t_2) + B(t_2) \\
 &= t_2 B(t_1) - t_1 (W(t_2) - t_2 W(1)) + B(t_2) \\
 &= t_2 B(t_1) - t_1 B(t_2) + B(t_2) \\
 &= (1 - t_1) B(t_2) + t_2 B(t_1), \quad 0 \leq t_1 \leq t_2 \leq 1, \quad (4.5)
 \end{aligned}$$

where $B(\cdot)$ is a Brownian bridge.

Thus, if we are to study the two-time parameter partial sum process in (3.5) in sup-norm metric, then we are to deal with (4.5) in such a metric in the limit. This is the reason for introducing $q \in Q$ to be a local function of a Brownian bridge as in Definition 4.1.2 and, similarly, that it be a Chibisov–O’Reilly function of a

Brownian bridge as in Definition 4.1.3. Indeed, as an immediate consequence of these considerations, if we let $t_1 = t_2 = t \in [0, 1]$ in (3.5) and let it be zero at $t_1 = t_2 = t = 1$, we arrive at

$$M_n^{(1)}(t, t) := \begin{cases} n^{-1/2}(\sum_{i=1}^{[(n+1)t]} Y_i - \frac{[(n+1)t]}{n} \sum_{i=1}^n Y_i), & 0 \leq t < 1, \\ 0, & t = 1 \end{cases} \quad (4.6)$$

and, as a special case of B. Szyszkowicz ([34, Theorem 2.1]), conclude (cf. [3, Theorem 2.1.1, Theorem 2.4.11, Corollary 2.4.2])

Theorem D Assume $H_O^{(2)}$ with $k_1^* = k_2^* = k^* \geq n$, (3.1) and let $q \in Q$. Then we can define a sequence of Brownian bridges $\{B_n(t); 0 \leq t \leq 1\}$ such that, as $n \rightarrow \infty$,

$$(i) \sup_{0 < t < 1} \frac{\left| \frac{1}{\sigma} M_n^{(1)}(t, t) - B_n(t) \right|}{q(t)} = \begin{cases} o_P(1) & \text{if and only if } I(q, c) < \infty \text{ for all } c > 0 \\ O_P(1) & \text{if and only if } I(q, c) < \infty \text{ for some } c > 0. \end{cases}$$

(ii) Let $\{B(t); 0 \leq t \leq 1\}$ be a Brownian Bridge. Then

$$\sup_{0 < t < 1} \frac{1}{\sigma} \frac{|M_n^{(1)}(t, t)|}{q(t)} \xrightarrow{\mathcal{D}} \sup_{0 < t < 1} \frac{|B(t)|}{q(t)}$$

if and only if $I(q, c) < \infty$ for some $c > 0$.

Statement (ii) of Theorem D is also concluded in E. Gombay, L. Horváth and M. Hušková [17, Theorem 1.1] for a slightly different, asymptotically equivalent form of $M_n^{(1)}(t, t)$ of (4.6). Namely, (cf. [17, (1.3)]) they study the problem of testing for one change in the variance (cf. (3.1) and (3.2) and $H_O^{(2)}$ versus $H_A^{(2)}$ with $k_1^* = k_2^* = k^*$) in terms of tests that are based on functionals of

$$M_n(t) := \begin{cases} n^{-1/2} \left(\sum_{1 \leq i \leq (n+1)t} Y_i - t \sum_{1 \leq i \leq n} Y_i \right), & 0 \leq t \leq \frac{n}{n+1}, \\ n^{-1/2} \left(\sum_{1 \leq i \leq n} Y_i - t \sum_{1 \leq i \leq n} Y_i \right), & \frac{n}{n+1} < t \leq 1, \end{cases} \quad (4.7)$$

where, just like in (4.6), $Y_i = (X_i - \mu)^2$, $i = 1, \dots, n$, under $H_O^{(2)}$ with $k_1^* = k_2^* = k^* \geq 0$. In particular we note that, via Szyszkowicz [34, Theorem 2.1]),

Theorem D continues to hold true in terms of $M_n(t)$ as well, i.e., on replacing $M_n^{(1)}(t, t)$ in both its statements by $M_n(t)$ as in (4.6), (i) and (ii) of Theorem D remain true as $n \rightarrow \infty$.

In this chapter, we are concerned with the problem of extending (i) of Theorem D to the case of $\{M_n^{(1)}(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$ as in (3.5) for $0 \leq t_1 \leq t_2 < 1$, and zero for $t_1 = 1$. We will also deal with statement (ii) of Theorem D but within our context. Consequently, just like $\Gamma(t_1, t_2)$ (cf. (4.5)), $M_n^{(1)}(t_1, t_2)$ is now tied down to zero when $t_2 = 0$, and again to zero when $t_1 = 1$, and thus, as noted above, it yields $M_n^{(1)}(t, t)$ as in (4.6)

Instead of taking this route however, we will slightly change the definition of $M_n^{(1)}(t_1, t_2)$ (cf. (4.10)) so that it should yield $M_n(t)$ of (4.7) (cf. (4.11)), the slightly changed version of $M_n^{(1)}(t_1, t_2)$, that is used in E. Gombay, L. Horváth and M. Hušková (1996).

4.1.1 Population Mean Known

Let $g, h \in Q$ and, via $\Gamma(t_1, t_2)$ of (4.5), define the Gaussian processes

$$\Gamma^w(t_1, t_2) := (1 - t_1) \frac{B(t_2)}{g(t_2)} + t_2 \frac{B(t_1)}{h(t_1)} \quad \text{for } 0 < t_1 \leq t_2 < 1, \quad (4.8)$$

a weighted version of $\Gamma(t_1, t_2)$, and, writing $\Gamma_n(t_1, t_2)$ of (1.5) à la (4.5),

$$\Gamma_n^w(t_1, t_2) := (1 - t_1) \frac{B_n(t_2)}{g(t_2)} + t_2 \frac{B_n(t_1)}{h(t_1)} \quad \text{for } 0 < t_1 \leq t_2 < 1, \quad (4.9)$$

a weighted version of $\Gamma_n(t_1, t_2)$, and where $B_n(t_i) := \frac{W(t_i) - t_i W(n)}{\sqrt{n}}$ for $i = 1, 2$.

Let $S((n+1)t) := \sum_{i=1}^{[(n+1)t]} Y_i$ and $S(n) := \sum_{i=1}^n Y_i$. We redefine the sequence of partial sum process specified by equation (3.5) in the following way,

$$M_n^{(1)}(t_1, t_2) = \begin{cases} \frac{(1-t_1)(S((n+1)t_2)-t_2S(n))}{n^{1/2}} + \frac{t_2(S((n+1)t_1)-t_1S(n))}{n^{1/2}} & 0 \leq t_1 \leq t_2 \leq \frac{n}{n+1} \\ \frac{(1-t_1)(1-t_2)S(n)}{n^{1/2}} + \frac{t_2(S((n+1)t_1)-t_1S(n))}{n^{1/2}} & 0 \leq t_1 \leq \frac{n}{n+1} < t_2 \leq 1 \\ \frac{(1-t_1)(1-t_2)S(n)}{n^{1/2}} + \frac{t_2(1-t_1)S(n)}{n^{1/2}} & \frac{n}{n+1} < t_1 \leq t_2 \leq 1. \end{cases} \quad (4.10)$$

This version reduces to $M_n(t)$ (cf. (4.6)) of Gombay, Horváth and Hušková [17, (1.3)], for when $t_1 = t_2 = t$, then we have

$$M_n(t) := M_n^{(1)}(t, t) = \begin{cases} \frac{(S((n+1)t)-tS(n))}{n^{1/2}} & 0 \leq t \leq \frac{n}{n+1} \\ \frac{S(n)}{n^{1/2}} - \frac{tS(n)}{n^{1/2}} & \frac{n}{n+1} < t \leq 1. \end{cases} \quad (4.11)$$

Using definition (4.10), fashioned after (4.8), with g and $h \in Q$, we define the corresponding weighted version, as shown below.

$$M_n^{(1)w}(t_1, t_2) := \begin{cases} \frac{(1-t_1)(S((n+1)t_2)-t_2S(n))}{g(t_2)n^{1/2}} + \frac{t_2(S((n+1)t_1)-t_1S(n))}{h(t_1)n^{1/2}} & 0 < t_1 \leq t_2 \leq \frac{n}{n+1} \\ \frac{(1-t_1)(1-t_2)S(n)}{g(t_2)n^{1/2}} + \frac{t_2(S((n+1)t_1)-t_1S(n))}{h(t_1)n^{1/2}} & 0 < t_1 \leq \frac{n}{n+1} < t_2 < 1 \\ \frac{(1-t_1)(1-t_2)S(n)}{g(t_2)n^{1/2}} + \frac{t_2(1-t_1)S(n)}{h(t_1)n^{1/2}} & \frac{n}{n+1} < t_1 \leq t_2 < 1, \end{cases} \quad (4.12)$$

Theorem 4.1.1 Assume $H_0^{(2)}$, (3.2) hold and $\infty > \gamma > 0$. Then, on the probability space of Theorem D, we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\} \stackrel{D}{=} \{\Gamma(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$ for each $n \geq 1$, such that the following hold true.

a) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for all $c > 0$ then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| = o_P(1).$$

b) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| = O_P(1).$$

c) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{M_n^{(1)w}(t_1, t_2)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq t_2 < 1} |\Gamma^w(t_1, t_2)|.$$

where $M_n^{(1)w}(t_1, t_2)$, $\Gamma_n^w(t_1, t_2)$ and $\Gamma^w(t_1, t_2)$ are defined in (4.12), (4.9) and (4.8) respectively.

Proof.

With $\{B_n(t); 0 \leq t \leq 1\}$ as in Theorem D, let $\Gamma_n(t_1, t_2) := (1 - t_1)B_n(t_1) + t_2B_n(t_1)$, $0 \leq t_1 \leq t_2 \leq 1$. Clearly then, we have for each $n \geq 1$ $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\} \stackrel{\mathcal{D}}{=} \{\Gamma(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$. Without loss of generality, it will be assumed that $\mathbb{E}Y_1 = 0$ and $\gamma^2 = 1$.

Consider

$$\begin{aligned} & \sup_{0 < t_1 \leq t_2 < 1} \left| M_n^{(1)w}(t_1, t_2) - \Gamma_n^w(t_1, t_2) \right| \\ &= \sup_{0 < t_1 \leq t_2 \leq \frac{n}{n+1}} \left| (1 - t_1) \frac{(S((n+1)t_2)) - t_2 S(n)}{g(t_2)n^{1/2}} \right. \\ & \quad \left. + t_2 \frac{(S((n+1)t_1)) - t_1 S(n)}{h(t_1)n^{1/2}} - (1 - t_1) \frac{B_n(t_2)}{g(t_2)} - t_2 \frac{B_n(t_1)}{h(t_1)} \right| \\ & \quad + \sup_{0 < t_1 \leq \frac{n}{n+1} < t_2 < 1} \left| (1 - t_1) \frac{(S(n) - t_2 S(n))}{g(t_2)n^{1/2}} \right. \\ & \quad \left. + t_2 \frac{(S((n+1)t_1) - t_1 S(n))}{h(t_1)n^{1/2}} - (1 - t_1) \frac{B_n(t_2)}{g(t_2)} - t_2 \frac{B_n(t_1)}{h(t_1)} \right| \\ & \quad + \sup_{\frac{n}{n+1} < t_1 \leq t_2 < 1} \left| (1 - t_1)(1 - t_2) \frac{S(n)}{g(t_2)n^{1/2}} + t_2(1 - t_1) \frac{S(n)}{h(t_1)n^{1/2}} \right. \\ & \quad \left. - (1 - t_1) \frac{B_n(t_2)}{g(t_2)} - t_2 \frac{B_n(t_1)}{h(t_1)} \right| \\ &=: I_1(n) + I_2(n) + I_3(n). \end{aligned} \tag{4.13}$$

First we show that

$$I_1(n) = \begin{cases} o_P(1) & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for all } c > 0 \\ O_P(1) & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \end{cases} \quad (4.14)$$

as $n \rightarrow \infty$. To this end consider

$$\begin{aligned} I_1(n) \leq & \sup_{0 < t_1 \leq t_2 \leq \frac{n}{n+1}} (1 - t_1) \left| \frac{S((n+1)t_2) - t_2 S(n)}{n^{1/2}} - B_n(t_2) \right| / g(t_2) \\ & + \sup_{0 < t_1 \leq t_2 \leq \frac{n}{n+1}} t_2 \left| \frac{S((n+1)t_1) - t_1 S(n)}{n^{1/2}} - B_n(t_1) \right| / h(t_1). \end{aligned} \quad (4.15)$$

Clearly, the two terms on the right hand side are asymptotically identical in distribution. Moreover,

$$\begin{aligned} & \sup_{0 < t_1 \leq \frac{n}{n+1}} \left| \frac{S((n+1)t_1) - t_1 S(n)}{n^{1/2}} - B_n(t_1) \right| / h(t_1) \\ & = \begin{cases} o_P(1) & \text{if } I(h, c) < \infty \text{ for all } c > 0, \\ O_P(1) & \text{if } I(h, c) < \infty \text{ for some } c > 0 \end{cases} \end{aligned} \quad (4.16)$$

as $n \rightarrow \infty$, which follows by Theorem D, part (i).

Next we show that

$$I_2(n) = \begin{cases} o_P(1) & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for all } c > 0, \\ O_P(1) & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \end{cases} \quad (4.17)$$

as $n \rightarrow \infty$.

Consider the following.

$$\begin{aligned} I_2(n) \leq & \sup_{0 < t_1 \leq \frac{n}{n+1} < t_2 < 1} \frac{(1 - t_1)(1 - t_2) |S(n)|}{g(t_2) \sqrt{n}} + \sup_{0 < t_1 \leq \frac{n}{n+1} < t_2 < 1} \frac{(1 - t_1) |B_n(t_2)|}{g(t_2)} \\ & + \sup_{0 < t_1 \leq \frac{n}{n+1} < t_2 < 1} t_2 \frac{\left| \frac{S((n+1)t_1) - t_1 S(n)}{n^{1/2}} - B_n(t_1) \right|}{h(t_1)}. \end{aligned} \quad (4.18)$$

We have

$$\sup_{\frac{n}{n+1} < t_2 < 1} \frac{(1-t_2)^{1/2}}{g(t_2)} \sup_{\frac{n}{n+1} < t_2 < 1} (1-t_2)^{1/2} \frac{|S(n)|}{\sqrt{n}} = o(1) \cdot o(1) \cdot O_P(1) = o(1), \quad (4.19)$$

as $n \rightarrow \infty$, since $g \in Q$ and $I(g, c) < \infty$ for some $c > 0$ implies, by Theorem A, equation (4.4).

Turning to the second term on the right hand side of (4.18), we show that

$$\begin{aligned} & \sup_{0 < t_1 \leq \frac{n}{n+1} < t_2 < 1} (1-t_1) \frac{|B_n(t_2)|}{g(t_2)} \\ &= \begin{cases} o_P(1) & \text{if } I(g, c) < \infty \text{ for all } c > 0 \\ O_P(1) & \text{if } I(g, c) < \infty \text{ for some } c > 0, \end{cases} \end{aligned} \quad (4.20)$$

as $n \rightarrow \infty$. In order to do so, we must establish first that

$$\sup_{\frac{n}{n+1} < t_2 < 1} \frac{|B_n(t_2)|}{g(t_2)} = \begin{cases} o_P(1) & \text{if } I(g, c) < \infty \text{ for all } c > 0 \\ O_P(1) & \text{if } I(g, c) < \infty \text{ for some } c > 0, \end{cases} \quad (4.21)$$

as $n \rightarrow \infty$.

Since $\{B(t); 0 \leq t \leq 1\} \stackrel{D}{=} \{B(1-t); 0 \leq t \leq 1\}$, we have

$$\sup_{\frac{n}{n+1} < t_2 < 1} \frac{|B_n(t_2)|}{g(t_2)} \stackrel{D}{=} \sup_{\frac{n}{n+1} < t_2 < 1} \frac{|B(t_2)|}{g(t_2)} \stackrel{D}{=} \sup_{\frac{n}{n+1} < t_2 < 1} \frac{|B(1-t_2)|}{g(t_2)} \quad (4.22)$$

and

$$\begin{aligned} \sup_{\frac{n}{n+1} < t_2 < 1} \frac{|B(1-t_2)|}{g(t_2)} &\leq \sup_{\frac{n}{n+1} < t_2 < 1} \left(\frac{|W(1-t_2)|}{g(t_2)} + \frac{(1-t_2)}{g(t_2)} |W(1)| \right) \\ &= I_2^{(1)}(n) + I_2^{(2)}(n) \end{aligned} \quad (4.23)$$

As $I(g, c) < \infty$ for some $c > 0$, we have, via Theorem A, (4.4), and consequently that

$$I_2^{(2)}(n) \stackrel{a.s.}{=} o(1) \cdot O(1) = o(1), \text{ as } n \rightarrow \infty. \quad (4.24)$$

Using Theorems B and C, we establish

$$I_2^{(1)}(n) \stackrel{a.s.}{=} \begin{cases} o(1) & \text{if } I(g, c) \text{ and } I(h, c) < \infty \text{ for all } c > 0 \\ O(1) & \text{if } I(g, c) \text{ and } I(h, c) < \infty \text{ for some } c > 0, \end{cases} \quad (4.25)$$

as $n \rightarrow \infty$.

As a result of (4.23), (4.24) and (4.25), via (4.22) we have (4.21), and consequently (4.20). Combining (4.16) and (4.19) with (4.20), we establish the result in (4.17).

For the last term in (4.13), we claim that

$$I_3(n) = \begin{cases} o_P(1) & \text{if } I(g, c) \text{ and } I(h, c) < \infty \text{ for all } c > 0 \\ O_P(1) & \text{if } I(g, c) \text{ and } I(h, c) < \infty \text{ for some } c > 0, \end{cases} \quad (4.26)$$

as $n \rightarrow \infty$. Consider

$$\begin{aligned} I_3(n) \leq & \sup_{\frac{n}{n+1} < t_2 < 1} (1 - t_2) \frac{|S(n)|}{g(t_2)n^{1/2}} + \sup_{\frac{n}{n+1} < t_1 < 1} (1 - t_1) \frac{|S(n)|}{h(t_1)n^{1/2}} \\ & + \sup_{\frac{n}{n+1} < t_2 < 1} \frac{|B_n(t_2)|}{g(t_2)} + \sup_{\frac{n}{n+1} < t_1 < 1} \frac{|B_n(t_1)|}{h(t_1)}. \end{aligned} \quad (4.27)$$

We have

$$\begin{aligned} \sup_{\frac{n}{n+1} < t_2 < 1} \frac{(1 - t_2)^{1/2}}{g(t_2)} \sup_{\frac{n}{n+1} < t_2 < 1} (1 - t_2)^{1/2} \frac{|S(n)|}{\sqrt{n}} & \stackrel{a.s.}{=} o(1) \cdot o(1) \cdot O_P(1) \\ & = o(1), \end{aligned} \quad (4.28)$$

as $n \rightarrow \infty$, by arguments similar to those used in (4.19). Similar arguments to those used in (4.20) also yield that, as $n \rightarrow \infty$, $\sup_{\frac{n}{n+1} < t_1 < 1} \frac{|B_n(t_1)|}{h(t_1)} = o_P(1)$ if $I(h, c) < \infty$ for all $c > 0$. The respective $O_P(1)$ statement in (4.26) also follows from (4.20), and this completes our claim for $I_3(n)$ as in (4.26). In light

of (4.14), (4.17) and (4.26), the proof of statements a) and b) of Theorem 4.1.1 is now complete.

Having established statements a) and b) of Theorem 4.1.1, there remains statement c) of this theorem to prove. To do so, we make use of the following proposition.

Proposition 4.1.1 *Assume $H_O^{(2)}$, (3.1) and (3.2) hold. Let $g, h \in Q$ satisfy $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$ and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 < t_1 \leq t_2 \leq 1\}$ as in Theorem 4.1.1 such that, as $n \rightarrow \infty$,*

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{M_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| = o_P(1). \quad (4.29)$$

Proof. This follows from B. Szyszkowicz [31, Theorem 1.1] via symmetrizing the partial sum process $M_n^{(1)}(t_1, t_2)$ of (4.10) along the lines of the proof of Theorem 5.2.1 of B. Szyszkowicz [33]. We turn now to the proof of statement c) of Theorem 4.1.1. Clearly, (4.29) implies, with $\Gamma^w(t_1, t_2)$ as in (4.8), that

$$\lim_{n \rightarrow \infty} \left| \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|M_n^{(1)w}(t_1, t_2)|}{\gamma} \leq x \right\} - \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} |\Gamma^w(t_1, t_2)| \leq x \right\} \right| = 0, \quad (4.30)$$

$\infty < x < \infty$. Let $A_n = \{\omega; \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} |\Gamma^w(t_1, t_2)| \leq x\}$. We note that A_n is non-increasing sequence of events in n . As a result of the continuity of measures, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}\{A_n\} = \mathbb{P} \left\{ \sup_{0 < t_1 \leq t_2 < 1} |\Gamma^w(t_1, t_2)| \leq x \right\}. \quad (4.31)$$

As a direct consequence of (4.30) and (4.31), we are able to conclude that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|M_n^{(1)w}(t_1, t_2)|}{\gamma} \leq x \right\} = \mathbb{P} \left\{ \sup_{0 < t_1 \leq t_2 < 1} |\Gamma^w(t_1, t_2)| \leq x \right\},$$

and with this result statement c) of Theorem 4.1.1 follows as claimed.

Remark 4.1.2 With $g = h = 1$, a) of Theorem 4.1.1 reduces to Theorem 3.1.1

4.1.2 Population Mean Unknown

The weighted partial sum process developed in (4.10), Section 4.1.1 depends on the unknown population mean μ (cf. (3.1), (3.5) and recall $Y_i := (X_i - \mu)^2$, $i = 1, \dots, n$, $n \geq 1$), and in cases where it is unknown, cannot be used to construct a statistic. A natural solution is to substitute the sample mean, an estimate of μ , for the population mean in (4.10). As we are interested in functionals of $M_n^{(1)}(t_1, t_2)$ (cf. (4.10)), it would be of interest to know if such substitutions effect the limiting distribution. Such substitutions, as is frequently the case, tend to increase the randomness of functionals of this process, and then cause the thus altered process to have a limiting distribution different from that of the functional of the original partial sum process $M_n^{(1)}(t_1, t_2)$ (cf. Chapter 2, Section 2.2 for an example where the limiting distribution of T_n and $\hat{T}_n$ are different). In what follows, we explore the extent to which, if any, the limiting distribution of $\bar{M}_n^{(1)}(t_1, t_2)$ defined in (4.32), changes from that of $M_n^{(1)}(t_1, t_2)$ of (4.10).

Let $\bar{Y}_i$ be as defined in (3.9), i.e., $\bar{Y}_i := (X_i - \bar{X}_n)^2$, $i = 1, \dots, n$, $n \geq 1$. When the sample mean $\bar{X}_n$ is substituted for the population mean μ , we define the following altered partial sum process.

$$\bar{M}_n^{(1)}(t_1, t_2) = \begin{cases} \frac{(1-t_1)(\sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} \bar{Y}_i - t_2 \sum_{i=1}^n \bar{Y}_i)}{n^{1/2}} + \frac{t_2(\sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} \bar{Y}_i - t_1 \sum_{i=1}^n \bar{Y}_i)}{n^{1/2}} & 0 \leq t_1 \leq t_2 \leq \frac{n}{n+1} \\ \frac{(1-t_1)(1-t_2) \sum_{i=1}^n \bar{Y}_i}{n^{1/2}} + \frac{t_2(\sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} \bar{Y}_i - t_1 \sum_{i=1}^n \bar{Y}_i)}{n^{1/2}} & 0 \leq t_1 \leq \frac{n}{n+1} < t_2 \leq 1 \\ \frac{(1-t_1)(1-t_2) \sum_{i=1}^n \bar{Y}_i}{n^{1/2}} + \frac{t_2(1-t_1) \sum_{i=1}^n \bar{Y}_i}{n^{1/2}} & \frac{n}{n+1} < t_1 \leq t_2 \leq 1. \end{cases} \quad (4.32)$$

In view of definition (4.32), we define the corresponding weighted version with $g, h \in Q$,

$$\bar{M}_n^{(1)w}(t_1, t_2) := \begin{cases} \frac{(1-t_1)(\sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} \bar{Y}_i - t_2 \sum_{i=1}^n \bar{Y}_i)}{g(t_2)n^{1/2}} + \frac{t_2(\sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} \bar{Y}_i - t_1 \sum_{i=1}^n \bar{Y}_i)}{h(t_1)n^{1/2}} & 0 < t_1 \leq t_2 \leq \frac{n}{n+1} \\ \frac{(1-t_1)(1-t_2) \sum_{i=1}^n \bar{Y}_i}{g(t_2)n^{1/2}} + \frac{t_2(\sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} \bar{Y}_i - t_1 \sum_{i=1}^n \bar{Y}_i)}{h(t_1)n^{1/2}} & 0 < t_1 \leq \frac{n}{n+1} < t_2 < 1 \\ \frac{(1-t_1)(1-t_2) \sum_{i=1}^n \bar{Y}_i}{g(t_2)n^{1/2}} + \frac{t_2(1-t_1) \sum_{i=1}^n \bar{Y}_i}{h(t_1)n^{1/2}} & \frac{n}{n+1} < t_1 \leq t_2 < 1. \end{cases} \quad (4.33)$$

We note, for future reference, that $\bar{M}_n^{(1)}(t_1, t_2)$ can be expressed as a function of $M_n^{(1)}(t_1, t_2)$ (cf. (4.10)) plus some cross-product terms, as shown below. After some algebra, we arrive at

$$\bar{M}_n^{(1)}(t_1, t_2) = \begin{cases} M_n^{(1)}(t_1, t_2) - \frac{2(1-t_1)(\bar{X}_n - \mu) \sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} (X_i - \mu)}{n^{1/2}} \\ - \frac{2t_2(\bar{X}_n - \mu) \sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} (X_i - \mu)}{n^{1/2}} + \frac{t_2 n (\bar{X}_n - \mu)^2}{n^{1/2}} \\ + \frac{(1-t_1)[(n+1)t_2](\bar{X}_n - \mu)^2}{n^{1/2}} + \frac{t_2[(n+1)t_1](\bar{X}_n - \mu)^2}{n^{1/2}} & 0 \leq t_1 \leq t_2 \leq \frac{n}{n+1} \\ M_n^{(1)}(t_1, t_2) - \frac{2t_2(\bar{X}_n - \mu) \sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} (X_i - \mu)}{n^{1/2}} + \frac{t_2[(n+1)t_1](\bar{X}_n - \mu)^2}{n^{1/2}} \\ - \frac{n(1-t_1-t_2)(\bar{X}_n - \mu)^2}{n^{1/2}} & 0 \leq t_1 \leq \frac{n}{n+1} < t_2 \leq 1 \\ M_n^{(1)}(t_1, t_2) - \frac{(1-t_1)(\bar{X}_n - \mu)^2 n}{n^{1/2}} & \frac{n}{n+1} < t_1 \leq t_2 \leq 1. \end{cases} \quad (4.34)$$

With this decomposition, we are now able to establish the following lemma.

Lemma 4.1.1 Assume $H_O^{(2)}$, (3.1), (3.2) hold, $\infty > \gamma > 0$ and $g, h \in Q$ satisfy integral conditions $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1)w}(t_1, t_2) \right| = o_P(1).$$

Proof.

$$\begin{aligned} \sup_{0 < t_1 \leq t_2 < 1} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1)w}(t_1, t_2) \right| &\leq \sup_{0 < t_1 \leq t_2 < \frac{n}{n+1}} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1)w}(t_1, t_2) \right| \\ &\quad + \sup_{0 < t_1 \leq \frac{n}{n+1} < t_2 < 1} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1)w}(t_1, t_2) \right| \\ &\quad + \sup_{\frac{n}{n+1} < t_1 \leq t_2 < 1} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1)w}(t_1, t_2) \right| \\ &= I_1(n) + I_2(n) + I_3(n) \end{aligned} \quad (4.35)$$

$$= o_P(1), \quad n \rightarrow \infty. \quad (4.36)$$

For verifying (4.36), we begin by showing that

$$I_2(n) = o_P(1), \quad \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \quad (4.37)$$

as $n \rightarrow \infty$, and argue each term in (4.35) in turn. Equation (4.34) suggests a way to majorize $I_2(n)$. Using (4.34), we have

$$\begin{aligned} I_2(n) &\leq |\bar{X}_n - \mu| \sup_{0 < t_1 \leq \frac{n}{n+1}} \left| \frac{\sum_{i=1}^{[(n+1)t_1]} (X_i - \mu)}{h(t_1)n^{1/2}} \right| + n(\bar{X}_n - \mu)^2 \sup_{0 < t_1 \leq \frac{n}{n+1}} \frac{t_1}{n^{1/2}h(t_1)} \\ &\quad + (\bar{X}_n - \mu)^2 n \sup_{\frac{n}{n+1} < t_2 < 1} \frac{1 - t_2}{n^{1/2}g(t_2)} + \\ &= I_2^{(1)}(n) + I_2^{(2)}(n) + I_2^{(3)}(n) \end{aligned} \quad (4.38)$$

$$= o_P(1), \text{ as } n \rightarrow \infty. \quad (4.39)$$

We argue the terms in (4.38) individually.

$$I_2^{(1)}(n) = |n^{1/2}(\bar{X}_n - \mu)| \sup_{\frac{1}{n+1} \leq t_1 \leq \frac{n}{n+1}} \left| \frac{\sum_{i=1}^{[(n+1)t_1]} (X_i - \mu)}{n^{1/2}} \right| \sup_{\frac{1}{n+1} \leq t_1 \leq \frac{n}{n+1}} \frac{1}{n^{1/2}h(t_1)}$$

$$\begin{aligned}
&= O_P(1) \cdot O_P(1) \cdot o(1) \\
&= o_P(1), \text{ as } n \rightarrow \infty.
\end{aligned} \tag{4.40}$$

We argue now the $o(1)$ term in (4.40). If $\delta \in (0, 1/2)$ is chosen so that $h(t_1)$ is monotone non-decreasing in $[\frac{1}{n+1}, \delta]$ and monotone non-increasing on $(1 - \delta, \frac{n}{n+1}]$, then

$$\begin{aligned}
&\sup_{\frac{1}{n+1} \leq t_1 \leq \frac{n}{n+1}} \frac{1}{n^{1/2} h(t_1)} \\
&\leq \sup_{\frac{1}{n+1} \leq t_1 < \delta} \frac{1}{n^{1/2} h(t_1)} + \sup_{\delta \leq t_1 \leq 1 - \delta} \frac{1}{n^{1/2} h(t_1)} \\
&\quad + \sup_{1 - \delta < t_1 \leq \frac{n}{n+1}} \frac{1}{n^{1/2} h(t_1)} \\
&\leq \frac{1}{n^{1/2} h(1/(n+1))} + \frac{1}{n^{1/2} \inf_{\delta \leq t_1 \leq 1 - \delta} h(t_1)} \\
&\quad + \frac{1}{n^{1/2} h(n/(n+1))} \\
&= o(1), \text{ as } n \rightarrow \infty,
\end{aligned} \tag{4.41}$$

as $I(h, c) < \infty$ for some $c > 0$, implies (cf. Theorem A)

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{(n+1)^{1/2}} (n+1)^{1/2} h(1/(n+1)) = \lim_{n \rightarrow \infty} \frac{n^{1/2}}{(n+1)^{1/2}} \frac{h(1/(n+1))}{\frac{1}{(n+1)^{1/2}}} = \infty, \tag{4.42}$$

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{(n+1)^{1/2}} (n+1)^{1/2} h(n/(n+1)) = \lim_{n \rightarrow \infty} \frac{n^{1/2}}{(n+1)^{1/2}} \frac{h(n/(n+1))}{(1 - \frac{n}{n+1})^{1/2}} = \infty, \tag{4.43}$$

and note that $h(t_1)$ is a positive function (cf. Definition 4.1.1). We note, as well, that the remaining two terms in (4.40) are $O_P(1)$ and $O_P(1)$, as $n \rightarrow \infty$, by the CLT and Donsker's Theorem (restated on $D[0, 1]$) respectively.

Continuing, we have

$$I_2^{(2)}(n) = o_P(1), \quad \text{if } I(h, c) < \infty \text{ for some } c > 0, \tag{4.44}$$

as $n \rightarrow \infty$, employing again the argument used in (4.41), (4.4) as well as the CLT.

As to the final term in (4.38), we have

$$\begin{aligned} I_2^{(3)}(n) &\leq |n^{1/2}(\bar{X}_n - \mu)(\bar{X}_n - \mu)| \sup_{\frac{n}{n+1} < t_2 < 1} \frac{1 - t_2}{g(t_2)} \\ &= O_P(1) \cdot o(1) \cdot o(1), \text{ as } n \rightarrow \infty, \end{aligned} \quad (4.45)$$

which follows by the CLT, the SLLN and reasoning as in (4.19). Combining (4.40), (4.44) along with (4.45), we have (4.39).

For the third term in (4.35),

$$I_3(n) = o_P(1), \quad \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \quad (4.46)$$

as $n \rightarrow \infty$, by (4.45) applied to the only term in (4.34) that corresponds to this region.

As to the first term in (4.35), to show

$$I_1(n) = o_P(1), \quad \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \quad (4.47)$$

as $n \rightarrow \infty$, we must separate the region $0 < t_1 \leq t_2 \leq \frac{n}{n+1}$ into regions $0 < t_1 \leq t_2 < \frac{1}{n+1}$, $0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}$ and $\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}$, and argue each case separately. Over the region $\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}$, we have

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1)}(t_1, t_2) \right| = o_P(1), \quad (4.48)$$

as $n \rightarrow \infty$, on applying the argument used in (4.40) and (4.41) twice, once to each of the partial sums in (4.34) that corresponds to this region. As to the three remaining terms in (4.34) corresponding to this region, (i.e., the ones which are of the form $\frac{(\bar{X}_n - \mu)^2}{n^{1/2}}$), we apply (4.41) along with the CLT to conclude

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{2n(\bar{X}_n - \mu)^2}{n^{1/2}g(t_2)} = o_P(1)$$

and

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{n(\bar{X}_n - \mu)^2}{n^{1/2}h(t_1)} = o_P(1),$$

as $n \rightarrow \infty$.

Continuing, we have also that

$$\sup_{0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \left| \bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1)}(t_1, t_2) \right| = o_P(1), \quad (4.49)$$

as $n \rightarrow \infty$. (4.49) follows as a result of (4.40) and (4.41) applied to the term

$$\sup_{0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{2(1-t_1)|(\bar{X}_n - \mu)| \sum_{i=1}^{[(n+1)t_2]} (X_i - \mu)|}{n^{1/2}g(t_2)} = o_P(1),$$

as $n \rightarrow \infty$, in (4.34), (4.41) along with the SLLN applied to

$$\sup_{0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{t_1[(n+1)t_2](\bar{X}_n - \mu)^2}{n^{1/2}g(t_2)} = o_P(1),$$

as $n \rightarrow \infty$, (4.41) along with the CLT applied to

$$\sup_{0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{nt_2(\bar{X}_n - \mu)^2}{n^{1/2}g(t_2)} = o_P(1),$$

as $n \rightarrow \infty$, in (4.34). As the above terms inherit the $o_P(1)$ property as $n \rightarrow \infty$, we have the claimed result in (4.49).

For (t_1, t_2) restricted to the region $0 < t_1 \leq t_2 < \frac{1}{n+1}$, we have

$$\sup_{0 < t_1 \leq t_2 < \frac{1}{n+1}} \left| \bar{M}_n^{(1)}(t_1, t_2) - M_n^{(1)}(t_1, t_2) \right| = o_P(1), \quad (4.50)$$

as $n \rightarrow \infty$, by (4.4), the CLT and SLLN applied to the only term in (4.34).

As a result of (4.48), (4.49) and (4.50), we now have (4.47). Moreover, (4.37), (4.46) and (4.47) establish the claim in (4.36) and our proof of Lemma 4.1.1 is now complete.

Theorem 4.1.2 Assume $H_O^{(2)}$, (3.1), (3.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$, as in Theorem 4.1.1, such that the following hold.

a) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for all $c > 0$, then, as $n \rightarrow \infty$

$$\sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| = o_P(1).$$

b) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$

$$\sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| = O_P(1),$$

c) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+2}} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq t_2 < 1} |\Gamma^w(t_1, t_2)|.$$

where $\bar{M}_n^{(1)w}(t_1, t_2)$, $\Gamma_n^w(t_1, t_2)$ and $\Gamma^w(t_1, t_2)$ are defined in (4.34), (4.9) and (4.8) respectively.

Proof. We have

$$\begin{aligned} & \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \\ & \leq \sup_{0 < t_1 \leq t_2 < 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1)w}(t_1, t_2) \right| \\ & \quad + \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \\ & = \begin{cases} o_P(1), & \text{if } I(g(t), c) < \infty \text{ and } I(h(t), c) < \infty \text{ for all } c > 0, \\ O_P(1), & \text{if } I(g(t), c) < \infty \text{ and } I(h(t), c) < \infty \text{ for some } c > 0, \end{cases} \end{aligned}$$

as $n \rightarrow \infty$, by Lemma 4.1.1 and statements a) and b) of Theorem 4.1.1. Statements

a) and b) of Theorem 4.1.2 are proven.

To prove statement c) consider

$$\begin{aligned}
& \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \\
& \leq \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1)w}(t_1, t_2)}{\gamma} \right| \\
& \quad + \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{M_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \\
& = o_p(1), \tag{4.51}
\end{aligned}$$

as $n \rightarrow \infty$, by Lemma 4.1.1 (Lemma 4.1.1 requires the integral conditions hold only for some $c > 0$), and Proposition 4.1.1. Equation (4.51) implies

$$\lim_{n \rightarrow \infty} \left| \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|\bar{M}_n^{(1)w}(t_1, t_2)|}{\gamma} \leq x \right\} - \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} |\Gamma^w(t_1, t_2)| \leq x \right\} \right| = 0,$$

$\infty < x < \infty$. Using the continuity property of measures, as in the proof of Proposition 4.1.1, we conclude with

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|\bar{M}_n^{(1)w}(t_1, t_2)|}{\gamma} \leq x \right\} = \mathbb{P} \left\{ \sup_{0 < t_1 \leq t_2 < 1} |\Gamma^w(t_1, t_2)| \leq x \right\},$$

and statement c) of Theorem 4.1.2 follows as claimed.

Remark 4.1.3 With $g = h = 1$, statement a) of Theorem 4.1.2 reduces to Theorem 3.1.2.

4.1.3 Small Changes in the Population Mean

Consider the sequence of independent random variables as described in (3.13) and satisfying condition (3.14). Let $Y_{i,n}^* := (X_{i,n} - \mu_{in})^2$ $i = 1, \dots, n$, $n \geq 1$, and define a new partial sum process

$$M_n^{(1*)}(t_1, t_2) := \begin{cases} \frac{(1-t_1)(\sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} Y_{i,n}^* - t_2 \sum_{i=1}^n Y_{i,n}^*)}{n^{1/2}} + \frac{t_2(\sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} Y_{i,n}^* - t_1 \sum_{i=1}^n Y_{i,n}^*)}{n^{1/2}} & 0 \leq t_1 \leq t_2 \leq \frac{n}{n+1} \\ \frac{(1-t_1)(1-t_2) \sum_{i=1}^n Y_{i,n}^*}{n^{1/2}} + \frac{t_2(\sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} Y_{i,n}^* - t_1 \sum_{i=1}^n Y_{i,n}^*)}{n^{1/2}} & 0 \leq t_1 \leq \frac{n}{n+1} < t_2 \leq 1 \\ \frac{(1-t_1)(1-t_2) \sum_{i=1}^n Y_{i,n}^*}{n^{1/2}} + \frac{t_2(1-t_1) \sum_{i=1}^n Y_{i,n}^*}{n^{1/2}} & \frac{n}{n+1} < t_1 \leq t_2 \leq 1 \end{cases} \quad (4.52)$$

and the corresponding weighted version with $g, h \in Q$,

$$M_n^{(1*)w}(t_1, t_2) := \begin{cases} \frac{(1-t_1)(\sum_{i=1}^{\lfloor (n+1)t_2 \rfloor} Y_{i,n}^* - t_2 \sum_{i=1}^n Y_{i,n}^*)}{g(t_2)n^{1/2}} + \frac{t_2(\sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} Y_{i,n}^* - t_1 \sum_{i=1}^n Y_{i,n}^*)}{h(t_1)n^{1/2}} & 0 < t_1 \leq t_2 \leq \frac{n}{n+1} \\ \frac{(1-t_1)(1-t_2) \sum_{i=1}^n Y_{i,n}^*}{g(t_2)n^{1/2}} + \frac{t_2(\sum_{i=1}^{\lfloor (n+1)t_1 \rfloor} Y_{i,n}^* - t_1 \sum_{i=1}^n Y_{i,n}^*)}{h(t_1)n^{1/2}} & 0 < t_1 \leq \frac{n}{n+1} < t_2 < 1 \\ \frac{(1-t_1)(1-t_2) \sum_{i=1}^n Y_{i,n}^*}{g(t_2)n^{1/2}} + \frac{t_2(1-t_1) \sum_{i=1}^n Y_{i,n}^*}{h(t_1)n^{1/2}} & \frac{n}{n+1} < t_1 \leq t_2 < 1. \end{cases} \quad (4.53)$$

Consider now the partial sum processes $M_n^{(1*)}(t_1, t_2)$ and $\bar{M}_n^{(1)}(t_1, t_2)$ and their weighted counterparts, $M_n^{(1*)w}(t_1, t_2)$ and $\bar{M}_n^{(1)w}(t_1, t_2)$ defined in (4.52), (4.32), (4.53) and (4.33) respectively. The representation of our partial sum process (4.52) does allow a decomposition of $\bar{M}_n^{(1)}(t_1, t_2)$ similar to that of $\tilde{M}_n(t)$, the partial sum process developed by E. Gombay, L. Horváth and M. Hušková [17, page 157-158], that is used in the proof of their Theorem 2.5.

In what follows, we make use of this and decompose $\bar{M}_n^{(1)}(t_1, t_2)$ (cf. (4.32)) into a function of $M_n^{(1*)}(t_1, t_2)$ (cf. (4.52)) plus some additional cross-product terms as well. To keep the algebraic details to a minimum, we provide this decomposition for only the regions $0 < t_1 \leq t_2 < \frac{1}{n+1}$ and $0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}$. Since $\bar{M}_n^{(1)}(t_1, t_2)$, for the remaining regions, consists of sums of a similar nature, they will not be reproduced here. We begin by factoring $\bar{M}_n^{(1)}(t_1, t_2)$ into $M_n^{(1*)}(t_1, t_2)$ plus

some cross-product terms in the region $0 < t_1 \leq t_2 < \frac{1}{n+1}$.

$$\begin{aligned}
n^{1/2} \bar{M}_n^{(1)}(t_1, t_2) &= -(1-t_1)t_2 \sum_{i=1}^n (X_{i,n} - \bar{\mu} + \bar{\mu} - \bar{X}_n)^2 \\
&\quad - t_1 t_2 \sum_{i=1}^n (X_{i,n} - \bar{\mu} + \bar{\mu} - \bar{X}_n)^2 \\
&= -(1-t_1)t_2 \left(\sum_{i=1}^n (X_{i,n} - \bar{\mu})^2 - 2(\bar{X} - \bar{\mu}) \sum_{i=1}^n (X_{i,n} - \bar{\mu})^2 \right. \\
&\quad \left. + n(\bar{X} - \bar{\mu})^2 \right) \\
&\quad - t_1 t_2 \left(\sum_{i=1}^n (X_{i,n} - \bar{\mu})^2 - 2(\bar{X} - \bar{\mu}) \sum_{i=1}^n (X_{i,n} - \bar{\mu})^2 \right. \\
&\quad \left. + n(\bar{X} - \bar{\mu})^2 \right) \\
&= -(1-t_1)t_2 \left(\sum_{i=1}^n (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})^2 \right. \\
&\quad \left. - 2(\bar{X} - \mu) \sum_{i=1}^n (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu}) + n(\bar{X}_n - \bar{\mu})^2 \right) \\
&\quad - t_1 t_2 \left(\sum_{i=1}^n (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu})^2 \right. \\
&\quad \left. - 2(\bar{X} - \mu) \sum_{i=1}^n (X_{i,n} - \mu_{in} + \mu_{in} - \bar{\mu}) \right. \\
&\quad \left. + n(\bar{X}_n - \bar{\mu})^2 \right) \\
&= -(1-t_1)t_2 \left(\sum_{i=1}^n (X_{i,n} - \mu_{in})^2 \right. \\
&\quad \left. + 2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) + \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2 \right. \\
&\quad \left. - 2(\bar{X}_n - \bar{\mu}) \sum_{i=1}^n (X_i - \mu_{in}) + n(\bar{X}_n - \bar{\mu})^2 \right) \\
&\quad - t_1 t_2 \left(\sum_{i=1}^n (X_{i,n} - \mu_{in})^2 + 2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right. \\
&\quad \left. + \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2 - 2(\bar{X}_n - \bar{\mu}) \sum_{i=1}^n (X_{i,n} - \mu_{in}) \right. \\
&\quad \left. + n(\bar{X}_n - \bar{\mu})^2 \right) \\
&= n^{1/2} M_n^{(1*)}(t_1, t_2)
\end{aligned}$$

$$\begin{aligned}
& - (1 - t_1)t_2 \left(2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right. \\
& \quad \left. + \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2 - n(\bar{X}_n - \bar{\mu})^2 \right) \\
& - t_1 t_2 \left(2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right. \\
& \quad \left. + \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2 - n(\bar{X}_n - \bar{\mu})^2 \right) \quad (4.54)
\end{aligned}$$

We factor $\bar{M}_n^{(1)}(t_1, t_2)$ (cf. (4.32)), for the region $0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}$, but state the final decomposition only, without any algebraic justification.

$$\begin{aligned}
n^{1/2} \bar{M}_n^{(1)}(t_1, t_2) &= n^{1/2} M_n^{(1*)}(t_1, t_2) \\
&+ 2(1 - t_1) \left(\sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right. \\
& \quad \left. - t_2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right) \\
&+ (1 - t_1) \left(\sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2 - t_2 \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2 \right) \\
&- 2(1 - t_1)(\bar{X}_n - \bar{\mu}) \left(\sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in}) - t_2 \sum_{i=1}^n (X_{i,n} - \mu_{in}) \right) \\
&- 2(1 - t_1)(\bar{X}_n - \bar{\mu}) \left(\sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu}) - t_2 \sum_{i=1}^n (\mu_{in} - \bar{\mu}) \right) \\
&+ (1 - t_1)[[(n+1)t_2] - nt_2](\bar{X}_n - \bar{\mu})^2 \\
&- t_1 t_2 \left(2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) + \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2 \right. \\
& \quad \left. - n(\bar{X}_n - \bar{\mu})^2 \right) \quad (4.55)
\end{aligned}$$

If this decomposition were continued, by applying a similar process as used to arrive at (4.55) to the remaining regions, we would have obtained a similar decomposition of $\bar{M}_n^{(1)}(t_1, t_2)$ as $M_n^{(1*)}(t_1, t_2)$ plus some cross-product terms. We note that this decomposition (cf. (4.55)) corresponds closely with the decomposition of $\tilde{M}_n(t)$ in

terms of $A_1(t) + A_2(t) + A_3(t) + A_4(t) + A_5(t) + A_6(t)$ (cf. E. Gombay, L. Horváth and M. Hušková [17, page 157-158]).

Lemma 4.1.2 *Assume $H_O^{(2)}$, (3.2), (3.13), (3.14) and $g, h \in Q$ satisfy integral conditions $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$*

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1*)w}(t_1, t_2) \right| = o_P(1).$$

Proof.

Consider

$$\begin{aligned} \sup_{0 < t_1 \leq t_2 < 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1*)w}(t_1, t_2) \right| &\leq \sup_{0 < t_1 \leq t_2 < \frac{1}{n+1}} \frac{1}{\gamma} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1*)w}(t_1, t_2) \right| \\ &+ \sup_{0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{1}{\gamma} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1*)w}(t_1, t_2) \right| \\ &+ \sup_{0 < t_1 \leq \frac{n}{n+1} < t_2 < 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1*)w}(t_1, t_2) \right| \\ &+ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{1}{\gamma} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1*)w}(t_1, t_2) \right| \\ &+ \sup_{\frac{n}{n+1} < t_1 \leq t_2 < 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1*)w}(t_1, t_2) \right| \\ &= J_1(n) + J_2(n) + J_3(n) + J_4(n) + J_5(n) \quad (4.56) \end{aligned}$$

$$= o_P(1), \text{ as } n \rightarrow \infty. \quad (4.57)$$

In verifying (4.57), we will show only the following.

$$J_1(n) = J_2(n) = o_P(1), \text{ if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \quad (4.58)$$

as $n \rightarrow \infty$. Since the remaining terms in (4.56) (i.e., $i = 3, 4, 5$) inherit the same property as $J_1(n)$, we have

$$J_i(n) = o_P(1), \text{ if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \quad (4.59)$$

as $n \rightarrow \infty$, which follows either directly from arguments used in (4.58) or in some cases suitably modified to accommodate the different regions. Most often these modifications entail the use of one of the conditions, depending on the region under consideration, that are given in (4.4).

Using (4.54), we majorize $J_1(n)$ as shown below.

$$\begin{aligned}
J_1(n) &\leq \left| \frac{2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \sup_{0 < t_1 \leq t_2 < \frac{1}{n+1}} \frac{t_2(1-t_1)}{g(t_2)} \\
&\quad + \left| \frac{\sum_{i=1}^n (\mu_{in} - \bar{\mu})^2}{n^{1/2}} + \frac{n(\bar{X}_n - \bar{\mu})^2}{n^{1/2}} \right| \sup_{0 < t_1 \leq t_2 < \frac{1}{n+1}} \frac{t_2(1-t_1)}{g(t_2)} \\
&\quad + \left| \frac{2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \sup_{0 < t_1 \leq t_2 < \frac{1}{n+1}} \frac{t_2 t_1}{h(t_1)} \\
&\quad + \left| \frac{\sum_{i=1}^n (\mu_{in} - \bar{\mu})^2}{n^{1/2}} + n^{1/2}(\bar{X}_n - \bar{\mu})^2 \right| \sup_{0 < t_1 \leq t_2 < \frac{1}{n+1}} \frac{t_2 t_1}{h(t_1)} \\
&= J_1^{(1)}(n) + J_1^{(2)}(n) + J_1^{(3)}(n) + J_1^{(4)}(n) \tag{4.60}
\end{aligned}$$

$$= o_P(1), \text{ as } n \rightarrow \infty. \tag{4.61}$$

To establish (4.61), we begin with the following term.

$$\begin{aligned}
J_1^{(1)}(n) &= \left| \frac{\sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \sup_{0 < t_1 \leq t_2 < \frac{1}{n+1}} \frac{t_2(1-t_1)}{g(t_2)} \\
&= o_P(1), \text{ if } I(g, c) < \infty \text{ some } c > 0, \tag{4.62}
\end{aligned}$$

as $n \rightarrow \infty$. Clearly then,

$$\begin{aligned}
J_1^{(1)}(n) &\leq \left| \frac{\sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \sup_{0 < t_2 < \frac{1}{n+1}} \frac{t_2}{g(t_2)} \\
&= o_P(1) \cdot o(1) = o_P(1), \text{ as } n \rightarrow \infty. \tag{4.63}
\end{aligned}$$

which follows by (4.4) and (3.24). Following this line of reasoning, we conclude also,

$$J_1^{(3)}(n) = o_P(1), \tag{4.64}$$

as $n \rightarrow \infty$.

For the remaining terms on the right hand side bound of (4.60), use of the Lindeberg-Feller CLT (cf. Theorem 1.0.7) as in (3.30) and (3.31), or equation (3.25) or consistency of $\bar{X}_n$ must be used, and in all cases (4.4) must be applied. With this we have shown that $J_1^{(i)}(n)$ for $i = 2, 4$, also inherit the property specified in (4.62). With this last statement, we conclude $J_1(n)$ has the property asserted in (4.58).

Now, we turn our attention to the region $0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}$ with the purpose of establishing that the cross-product terms are $o_P(1)$ in sup-norm, as $n \rightarrow \infty$. As previously mentioned, the decomposition of $\bar{M}_n^{(1)}(t_1, t_2)$ as shown in (4.55) follows that of E. Gombay, L. Horváth and M. Hušková [17] in the proof of their Theorem 2.5.

As was done for $J_1(n)$, we majorize $J_2(n)$ by the supremum of each of the sums in (4.55), over the region $0 < t_1 < \frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}$. We show the following.

$$J_2(n) = o_P(1) \quad \text{if } I(g, c) < \infty \quad \text{and} \quad I(h, c) < \infty \quad \text{for some } c > 0, \quad (4.65)$$

as $n \rightarrow \infty$.

Consider,

$$\begin{aligned} J_2(n) \leq & \sup_{\frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{1}{n^{1/2}g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right. \\ & \left. - t_2 \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right| \\ & + \sup_{\frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{1}{n^{1/2}g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu})^2 \right. \\ & \left. - t_2 \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2 \right| \\ & + |\bar{X}_n - \mu| \sup_{\frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{1}{n^{1/2}g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} (\mu_{in} - \bar{\mu}) \right| \end{aligned}$$

$$\begin{aligned}
& \left| -t_2 \sum_{i=1}^n (\mu_{in} - \bar{\mu}) \right| \\
& + \sup_{\frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{|\bar{X}_n - \bar{\mu}|}{n^{1/2}g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} (X_{i,n} - \mu_{in}) \right. \\
& \quad \left. - t_2 \sum_{i=1}^n (X_{i,n} - \mu_{in}) \right| \\
& + (\bar{X}_n - \mu)^2 \sup_{\frac{1}{n+1} < t_2 \leq \frac{n}{n+1}} \frac{([(n+1)t_2] - nt_2)}{n^{1/2}g(t_2)} \\
& + \sup_{0 < t_1 \leq \frac{1}{n+1}} \frac{t_1}{n^{1/2}h(t_1)} \left| \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \right| \\
& + \left(\sum_{i=1}^n (\mu_{in} - \bar{\mu})^2 + n|\bar{X}_n - \bar{\mu}| \right) \sup_{0 < t_1 < \frac{1}{n+1}} \frac{t_1}{n^{1/2}h(t_1)} \\
& = J_2^{(1)}(n) + J_2^{(2)}(n) + J_2^{(3)}(n) + J_2^{(4)}(n) + J_2^{(5)}(n) + J_2^{(6)}(n) + J_2^{(7)}(n).
\end{aligned} \tag{4.66}$$

As was done for $J_1(n)$ (cf. (4.62) and (4.63)), we establish, for $i = 1, \dots, 7$, that

$$J_2^{(i)}(n) = o_P(1), \quad \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \tag{4.67}$$

as $n \rightarrow \infty$,

$$\begin{aligned}
J_2^{(1)}(n) &= \sup_{\frac{1}{n+1} \leq t_2 \leq \frac{1}{2}} \frac{1}{g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right. \\
& \quad \left. - t_2 \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
& + \sup_{\frac{1}{2} \leq t_2 \leq \frac{n}{n+1}} \frac{1}{g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right. \\
& \quad \left. - t_2 \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
& = J_2^{(1a)}(n) + J_2^{(1b)}(n)
\end{aligned} \tag{4.68}$$

Let $\delta \in (0, 1/2)$ and $\delta > \frac{1}{n+1}$, then

$$\begin{aligned}
J_2^{(1a)}(n) &= \sup_{\frac{1}{n+1} \leq t_2 < \delta} \frac{1}{g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right. \\
&\quad \left. - t_2 \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
&\quad + \sup_{\delta \leq t_2 \leq \frac{1}{2}} \frac{1}{g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right. \\
&\quad \left. - t_2 \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
&= J_{2_1}^{(1a)}(n) + J_{2_2}^{(1a)}(n). \tag{4.69}
\end{aligned}$$

In what follows, we deal with each term of (4.69) separately.

$$\begin{aligned}
J_{2_1}^{(1a)}(n) &\leq \sup_{\frac{1}{n+1} \leq t_2 < \delta} \frac{1}{g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
&\quad + \sup_{\frac{1}{n+1} \leq t_2 < \delta} \frac{t_2}{g(t_2)} \left| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
&= o_P(1), \tag{4.70}
\end{aligned}$$

as $n \rightarrow \infty$.

To establish (4.70), we use the Hájek-Rényi inequality, which provides a simple and direct proof. For the first term on the right hand bound in (4.70), we have

$$\begin{aligned}
&\sup_{\frac{1}{n+1} \leq t_2 < \delta} \frac{1}{g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
&\leq \sup_{0 < t_2 < \delta} \frac{t_2^{1/2}}{g(t_2)} \sup_{1 \leq (n+1)t_2 \leq (n+1)\delta} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{[t_2(n+1)]^{1/2}} \right| \sqrt{\frac{n+1}{n}} \\
&\leq \sup_{0 < t_2 < \delta} \frac{t_2^{1/2}}{g(t_2)} \max_{1 \leq k \leq (n+1)\delta} \left| \sum_{k=1}^k \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{k^{1/2}} \right| \sqrt{\frac{n+1}{n}} \\
&= O(1) \cdot o_P(1) = o_P(1), \tag{4.71}
\end{aligned}$$

as $n \rightarrow \infty$. To see (4.71), we note that the $O(1)$ term follows from (4.4), while the $o_P(1)$ term follows from the explanation provided in (4.72), which is given below.

We appeal now to the Hájek-Rényi inequality to account for the $o_P(1)$ term in (4.71). By the Hájek-Rényi inequality, we have

$$\begin{aligned} \mathbb{P} \left\{ \max_{1 \leq k \leq (n+1)\delta} \left| \sum_{i=1}^k \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{k^{1/2}} \right| \geq \varepsilon \right\} &\leq \frac{\max_{1 \leq i \leq n} (\mu_{in} - \bar{\mu})^2}{\varepsilon^2} \sum_{k=1}^{(n+1)\delta} \frac{\sigma^2}{k} \\ &= \frac{\sigma^2(\log(n+1) + C + O(\frac{1}{n}))}{n^{1/2}} \cdot n^{1/2} o(n^{-1/2}) \frac{1}{\varepsilon^2} \\ &= o(1), \end{aligned} \quad (4.72)$$

as $n \rightarrow \infty$, where C is the Euler constant.

Turning to the second term in (4.70), and arguing along the lines of (3.24) and exploiting (4.4), we conclude

$$\sup_{0 < t_2 < \delta} \frac{t_2}{g(t_2)} \left| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \leq O(1) \cdot o_P(1) = o_P(1), \quad (4.73)$$

as $n \rightarrow \infty$. Equations (4.71), (4.72) and (4.73) imply (4.70).

As to the second term in (4.69), we have

$$\begin{aligned} J_{2_2}^{(1a)}(n) &\leq \sup_{\delta \leq t_2 < \frac{1}{2}} \frac{1}{g(t_2)} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\ &\quad + \sup_{\delta \leq t_2 < \frac{1}{2}} \frac{t_2}{g(t_2)} \left| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\ &= O(1) \cdot o_P(1) + O(1) \cdot o_P(1) \\ &= o_P(1), \end{aligned} \quad (4.74)$$

as $n \rightarrow \infty$, by positivity of $g(t_2)$ (cf. Definition 4.1.1) and (3.24).

Returning to the term $J_2^{(1b)}(n)$ (cf. (4.68)), we have, where δ is specified in the statements following (4.68),

$$\begin{aligned} J_2^{(1b)}(n) &= \sup_{\frac{1}{2} \leq t_2 \leq \frac{n}{n+1}} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{g(t_2)n^{1/2}} \right. \\ &\quad \left. - t_2 \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{g(t_2)n^{1/2}} \right| \end{aligned}$$

$$\begin{aligned}
&= \sup_{\frac{1}{2} \leq t_2 \leq 1-\delta} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{g(t_2)n^{1/2}} \right. \\
&\quad \left. - t_2 \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{g(t_2)n^{1/2}} \right| \\
&\quad + \sup_{1-\delta \leq t_2 \leq \frac{n}{n+1}} \left| \sum_{i=1}^{[(n+1)t_2]} \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{g(t_2)n^{1/2}} \right. \\
&\quad \left. - t_2 \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{g(t_2)n^{1/2}} \right| \\
&= J_{2_1}^{(1b)}(n) + J_{2_2}^{(1b)}(n). \tag{4.75}
\end{aligned}$$

We shall deal again with each term of (4.75) in turn.

$$J_{2_1}^{(1b)}(n) = o_P(1), \quad \text{as } n \rightarrow \infty, \tag{4.76}$$

by (4.74), after replacing the region in (4.74) with $\frac{1}{2} < t_2 < 1 - \delta$. In order to establish the next result, we make use of the following fact,

$$\begin{aligned}
\sum_{i=1}^{[(n+1)t]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) &= \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \\
&\quad - \sum_{i=[(n+1)t]+1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}), \quad \text{for } \frac{1}{2} \leq t < 1. \tag{4.77}
\end{aligned}$$

For further details regarding this representation refer to B. Szyszkowicz [33] Theorem 5.2.1.

Returning to (4.75), we can continue along these lines.

$$\begin{aligned}
J_{2_2}^{(1b)}(n) &= \sup_{1-\delta \leq t_2 \leq \frac{n}{n+1}} \frac{1}{g(t_2)} \left| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right. \\
&\quad \left. - \sum_{i=[(n+1)t_2]+1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right. \\
&\quad \left. - t_2 \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right| \\
&= \sup_{1-\delta \leq t_2 \leq \frac{n}{n+1}} \frac{1}{g(t_2)} \left| - \sum_{i=[(n+1)t_2]+1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \right|
\end{aligned}$$

$$\begin{aligned}
& + (1 - t_2) \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \Big| \\
\leq & \sup_{1-\delta \leq t_2 \leq \frac{n}{n+1}} \frac{1}{g(t_2)} \Big| \sum_{i=[(n+1)t_2]+1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \Big| \\
& + \sup_{1-\delta \leq t_2 \leq \frac{n}{n+1}} \frac{(1 - t_2)}{g(t_2)} \Big| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \Big| \quad (4.78)
\end{aligned}$$

$$\begin{aligned}
& = O(1) \cdot o_P(1) \\
& = o_P(1), \quad (4.79)
\end{aligned}$$

as $n \rightarrow \infty$. To obtain the result in (4.79), we appeal to equations (4.80) and (4.81) as below. Consider the first term in (4.78), we have

$$\begin{aligned}
& \sup_{1-\delta \leq t_2 \leq \frac{n}{n+1}} \frac{1}{g(t_2)} \Big| \sum_{i=[(n+1)t_2]+1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \Big| \\
& \leq \sup_{1-\delta \leq t_2 \leq \frac{n}{n+1}} \frac{(1 - t_2)^{1/2}}{g(t_2)} \left(\sup_{(n+1)(1-\delta) \leq t_2(n+1) \leq n} \Big| \sum_{i=1}^{[(n+1)(1-t_2)]} \frac{(X_{[(n+1)t_2]+i,n} - \mu_{in})(\mu_{[(n+1)t_2]+i,n} - \bar{\mu})}{[(1 - t_2)(n+1)]^{1/2}} \Big| \right) \sqrt{\frac{n+1}{n}} \\
& \leq 2 \sup_{1-\delta \leq t_2 < 1} \frac{(1 - t_2)^{1/2}}{g(t_2)} \max_{1 \leq k \leq (n+1)\delta} \Big| \sum_{i=1}^k \frac{(X_{n-k+i,n} - \mu_{in})(\mu_{n-k+i,n} - \bar{\mu})}{k^{1/2}} \Big| \\
& = O(1) \cdot o_P(1) \\
& = o_P(1), \quad (4.80)
\end{aligned}$$

as $n \rightarrow \infty$, by (4.72) and (4.4).

Consider the second term in (4.78), we have

$$\begin{aligned}
& \sup_{1-\delta \leq t_2 \leq \frac{n}{n+1}} \frac{(1 - t_2)^{1/2}}{g(t_2)} \Big| \sum_{i=1}^n \frac{(X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})}{n^{1/2}} \Big| = O(1) \cdot o_P(1) \\
& = o_P(1), \quad (4.81)
\end{aligned}$$

as $n \rightarrow \infty$, by (4.4) and Chebyshev's inequality. Combining (4.80) and (4.81), we have (4.79).

Applying the Hájek-Rényi as well as the Linderberg-Feller CLT, we have also that $J_2^{(4)}(n) = o_P(1)$, as $n \rightarrow \infty$.

Returning to our main concern (4.66), we have, where δ is specified in (4.68),

$$\begin{aligned}
J_2^{(2)}(n) &\leq \sup_{\frac{1}{n+1} \leq t_2 < \delta} \frac{\frac{[(n+1)t_2]}{n}}{g(t_2)} n^{1/2} \max_{1 \leq i \leq n} (\mu_{in} - \bar{\mu})^2 \\
&\quad + \sup_{\frac{1}{n+1} \leq t_2 < \delta} \frac{t_2}{g(t_2)} n^{1/2} \max_{1 \leq i \leq n} (\mu_{in} - \mu)^2 \\
&\quad + \frac{2}{\inf_{\delta < t_2 < \frac{1}{2}} g(t_2)} n^{1/2} \max_{1 \leq i \leq n} (\mu_{in} - \mu)^2 \\
&\quad + \frac{2}{\inf_{\frac{1}{2} \leq t_2 \leq 1-\delta} g(t_2)} n^{1/2} \max_{1 \leq i \leq n} (\mu_{in} - \mu)^2 \\
&\quad + \sup_{1-\delta < t_2 \leq \frac{n}{n+1}} \frac{\frac{(n-[(n+1)t_2])}{n}}{g(t_2)} n^{1/2} \max_{1 \leq i \leq n} (\mu_{in} - \mu)^2 \\
&= o(1), \text{ if } I(g, c) < \infty \text{ for some } c > 0,
\end{aligned} \tag{4.82}$$

as $n \rightarrow \infty$, by (4.4), (3.14), positivity of $g(t_2)$ (cf. (4.41)) and the representation of the partial sum in (4.77). We can also conclude that

$$J_2^{(3)}(n) = o_P(1), \text{ if } I(g, c) < \infty \text{ for some } c > 0, \tag{4.83}$$

as $n \rightarrow \infty$. To establish this, both (3.29) and the Lindeberg-Feller CLT (cf. (3.30), (3.31)) were used.

The last three terms in (4.66), while not paired, do nevertheless share with (4.67) the $o_P(1)$, as $n \rightarrow \infty$, in sup-norm property. We establish the claimed result for the following term.

$$\begin{aligned}
&\sup_{\frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \left| \frac{(nt_2 - [(n+1)t_2])}{n^{1/2}g(t_2)} (\bar{X}_n - \bar{\mu})^2 \right| \\
&\leq \sup_{\frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{|nt_2 + t_2 - t_2 + [(n+1)t_2]|}{n^{1/2}g(t_2)} (\bar{X}_n - \bar{\mu})^2 \\
&= \sup_{\frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{|(n+1)t_2 - [(n+1)t_2] - t_2|}{n} \frac{n(\bar{X}_n - \bar{\mu})^2}{n^{1/2}g(t_2)}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\frac{1}{n+1} \leq t_2 \leq \frac{n}{n+1}} \frac{1}{n^{1/2}g(t_2)} n(\bar{X}_n - \bar{\mu})^2 \\
&= o_P(1),
\end{aligned} \tag{4.84}$$

as $n \rightarrow \infty$, by (4.41) and the Lindeberg-Feller CLT (cf. (3.30), (3.31)) as well.

With regards to the remaining two terms in (4.66), we have, upon applying (3.24) and (4.4) to $\sup_{0 < t_1 < \frac{1}{n+1}} \frac{t_1 |\sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu})|}{n^{1/2}h(t_1)}$, (4.4) and (3.25) to $\sup_{0 < t_1 < \frac{1}{n+1}} \frac{t_1 \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2}{n^{1/2}h(t_1)}$, and the Lindeberg-Feller CLT (cf. (3.30), (3.31)), consistency of $\bar{X}_n$ along with (4.4) to the term $\sup_{0 < t_1 < \frac{1}{n+1}} \frac{t_1 n(\bar{X}_n - \bar{\mu})^2}{n^{1/2}h(t_1)}$, that they all share with (4.67) the $o_P(1)$ property, as $n \rightarrow \infty$.

Equations (4.58) and (4.59) along with comments thereafter establish (4.57) and the lemma is now proved.

Theorem 4.1.3 Assume $H_O^{(2)}$, (3.2), (3.13), (3.14) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$, as in Theorem 4.1.1, such that the following hold.

a) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for all $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1*)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| = o_P(1).$$

b) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1*)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| = O_P(1),$$

c) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+2}} \left| \frac{M_n^{(1*)w}(t_1, t_2)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq t_2 < 1} |\Gamma^w(t_1, t_2)|.$$

where $M_n^{(1*)w}(t_1, t_2)$, $\Gamma_n^w(t_1, t_2)$ and $\Gamma^w(t_1, t_2)$ are defined in (4.53), (4.9) and (4.8) respectively.

Proof. We have

$$\begin{aligned}
& \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1*)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \\
& \leq \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1*)w}(t_1, t_2)}{\gamma} \right| \\
& \quad + \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \\
& = \begin{cases} o_P(1) & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for all } c > 0 \\ O_P(1) & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \end{cases}
\end{aligned}$$

as $n \rightarrow \infty$, by Lemma 4.1.2 and statements a) and b) of Theorem 4.1.2. Statements a) and b) of Theorem 4.1.3 follow as claimed.

To prove statement c) consider

$$\begin{aligned}
& \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{M_n^{(1*)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \\
& \leq \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{M_n^{(1*)w}(t_1, t_2) - \bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} \right| \\
& \quad + \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \\
& = o_P(1), \tag{4.85}
\end{aligned}$$

as $n \rightarrow \infty$, by Lemma 4.1.2 (Lemma 4.1.2 requires the integral conditions hold only for some $c > 0$) and by (4.51). Equation (4.85) implies

$$\lim_{n \rightarrow \infty} \left| \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{M_n^{(1*)w}(t_1, t_2)}{\gamma} \right| \leq x \right\} - \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} |\Gamma^w(t_1, t_2)| \leq x \right\} \right| = 0,$$

$\infty < x < \infty$. Using the continuity property of measures, as in the proof of Proposi-

tion 4.1.1, we conclude with

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|M_n^{(1*)w}(t_1, t_2)|}{\gamma} \leq x \right\} = \mathbf{P} \left\{ \sup_{0 < t_1 \leq t_2 < 1} |\Gamma^w(t_1, t_2)| \leq x \right\},$$

and statement c) of Theorem 4.1.3 follows as claimed.

Remark 4.1.4 With $g = h = 1$, statement a) of Theorem 4.1.3 reduces to Theorem 3.1.4.

4.2 Weighted Processes & Their Approximations

The processes $M_n^{(1)w}(t_1, t_2)$ and $\bar{M}_n^{(1)w}(t_1, t_2)$ (cf. (4.12) and (4.33)) introduced in Sections 4.1.1 and 4.1.2 applied different weight functions to each of the constituent partial sums. This, however, is not the preferred means by which weighted processes are generally constructed (cf. [5, (2.4)] or [6]). Rather, one would prefer to work with weighted functionals like, e.g.,

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{|M_n^{(1)}(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \quad (4.86)$$

and

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{|\bar{M}_n^{(1)}(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \quad (4.87)$$

where $M_n^{(1)}(t_1, t_2)$ and $\bar{M}_n^{(1)}(t_1, t_2)$ are defined in (4.10) and (4.32) respectively. Here, as well as throughout, $g, h \in Q$. In what follows, we use results from Sections 4.1.1, 4.1.2 and 4.1.3 to establish the asymptotics of such weighted functionals for cases: population mean known, unknown and then conclude with small changes in the population mean.

4.2.1 Population Mean Known

Corollary 4.2.1 Assume $H_O^{(2)}$, (3.1), (3.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$, as in Theorem 4.1.1, such that the following hold.

a) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for all $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} = o_P(1).$$

b) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} = O_P(1).$$

c) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+2}} \frac{\left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} \right|}{\frac{1}{2}(g(t_2) + h(t_1))} \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq t_2 < 1} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))}.$$

where $M_n^{(1)}(t_1, t_2)$, $\Gamma_n(t_1, t_2)$ and $\Gamma(t_1, t_2)$ are defined in (4.10), Lemma 1.0.6 and (4.5) respectively.

Proof.

We have

$$\begin{aligned} & \sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} \\ &= \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\frac{M_n^{(1)}(t_1, t_2)}{\gamma}}{\frac{1}{2}(g(t_2) + h(t_1))} - \frac{\Gamma_n(t_1, t_2)}{\frac{1}{2}(g(t_2) + h(t_1))} \right| \\ &= \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1)w}(t_1, t_2)}{\gamma} - \Gamma_n^w(t_1, t_2) \right| \end{aligned} \tag{4.88}$$

$$= \begin{cases} o_P(1), & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for all } c > 0, \\ O_P(1), & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \end{cases}$$

as $n \rightarrow \infty$, where both $M_n^{(1)w}(t_1, t_2)$ and $\Gamma_n^w(t_1, t_2)$ (cf. (4.12) and (4.9)) are weighted by $q(t_1, t_2) := \frac{1}{2}(g(t_2) + h(t_1))$. Now the claim follows by Theorem 4.1.1, after noting that the term in (4.88) is majorized by two times the terms that majorize (4.13). The weight functions used in the majorization of (4.13) are $(1/2)$ times those that are specified in Theorem 4.1.1, i.e., in terms of the above $q(0, t_2) = \frac{1}{2}(g(t_2) + 0)$, and $q(t_1, 0) = \frac{1}{2}(h(t_1) + 0)$ and $q(0, t_2), q(t_1, 0) \leq \frac{1}{2}(g(t_2) + h(t_1))$ because of positivity (cf. Definition 4.1.1)

To prove statement c) we use Proposition 4.2.1, similar in nature to Proposition 4.1.1, and whose proof follows along the lines of the latter.

Proposition 4.2.1 *Assume $H_O^{(2)}$, (3.1) and (3.2) hold. Let $g, h \in Q$ satisfy $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 < t_1 \leq t_2 \leq 1\}$ as in Theorem 4.1.1 such that, as $n \rightarrow \infty$,*

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right| = o_P(1). \quad (4.89)$$

Clearly, Proposition 4.2.1 implies

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|M_n^{(1)}(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} \right. \\ \left. - \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} \right| = 0, \end{aligned} \quad (4.90)$$

$\infty < x < \infty$. By continuity of probability measures, as used in the proof of Theorem 4.1.1 statement c), we have, as $n \rightarrow \infty$, that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|M_n^{(1)}(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} = \mathbb{P} \left\{ \sup_{0 < t_1 \leq t_2 < 1} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\},$$

and with this result statement c) of Corollary 4.2.1 is now shown.

4.2.2 Population Mean Unknown

Corollary 4.2.2 *Assume $H_O^{(2)}$, (3.1), (3.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$, as in Theorem 4.1.1, such that the following hold.*

a) *Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for all $c > 0$, then, as $n \rightarrow \infty$,*

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} = o_P(1).$$

b) *Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,*

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} = O_P(1).$$

c) *Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,*

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+2}} \frac{\left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} \right|}{\frac{1}{2}(g(t_2) + h(t_1))} \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq t_2 < 1} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))}.$$

where $\bar{M}_n^{(1)}(t_1, t_2)$, $\Gamma_n(t_1, t_2)$ and $\Gamma(t_1, t_2)$ are defined in (4.32), Lemma 1.0.6 and (4.5) respectively.

Proof.

We have

$$\begin{aligned} & \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2)}{\frac{1}{2}(g(t_2) + h(t_1))} \right| \\ &= \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{\bar{M}_n^{(1)w}(t_1, t_2) - M_n^{(1)w}(t_1, t_2)}{\gamma} \right| \end{aligned} \tag{4.91}$$

$$\begin{aligned}
& + \sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{M_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} \\
& = \begin{cases} o_P(1), & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for all } c > 0, \\ O_P(1), & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \end{cases}
\end{aligned}$$

as $n \rightarrow \infty$, where $M_n^{(1)w}(t_1, t_2)$, $\bar{M}_n^{(1)w}(t_1, t_2)$, and $\Gamma_n(t_1, t_2)$ (cf. (4.12), (4.33) and (1.5)) are weighted by $q(t_1, t_2) := \frac{1}{2}(g(t_2) + h(t_1))$. Now the claim follows by Lemma 4.1.1 and statements a) and b) of Corollary 4.2.1, after noting that the term in (4.91) is majorized by two times the terms that majorize (4.35). The weight functions used in the majorization of (4.35) are $(1/2)$ times those that are specified in Lemma 4.1.1, i.e., in terms of the above $q(0, t_2) = \frac{1}{2}(g(t_2) + 0)$ and $q(t_1, 0) = \frac{1}{2}(h(t_1) + 0)$ and $q(0, t_2), q(t_1, 0) \leq \frac{1}{2}(g(t_2) + h(t_1))$ because of positivity (cf. Definition 4.1.1)

By Proposition 4.2.1 and Lemma 4.1.1 (Lemma 4.1.1 requires the integral conditions hold only for some $c > 0$), where the weight function used in Lemma 4.1.1 is now $q(t_1, t_2) := \frac{1}{2}(g(t_2) + h(t_1))$, and after noting the comments made regarding the majorization of (4.91), we can conclude

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left| \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|\bar{M}_n^{(1)}(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} \right. \\
& \quad \left. - \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} \right| = 0,
\end{aligned} \tag{4.92}$$

$\infty < x < \infty$. By continuity of probability measures, as used in the proof of Theorem 4.1.1 statement c), we have, as $n \rightarrow \infty$, that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|\bar{M}_n^{(1)}(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} = \mathbb{P} \left\{ \sup_{0 < t_1 \leq t_2 < 1} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\},$$

and with this result statement c) of Corollary 4.2.3 is now shown.

4.2.3 Small Changes in the Population Mean

Corollary 4.2.3 Assume $H_O^{(2)}$, (3.2), (3.13), (3.14) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, t_2); 0 \leq t_1 \leq t_2 \leq 1\}$, as in Theorem 4.1.1, such that the following hold.

a) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for all $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} = o_P(1).$$

b) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} = O_P(1).$$

c) Let $g, h \in Q$. If $I(g, c) < \infty$ and $I(h, c) < \infty$ for some $c > 0$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+2}} \frac{\left| \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} \right|}{\frac{1}{2}(g(t_2) + h(t_1))} \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq t_2 < 1} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))}.$$

where $M_n^{(1*)}(t_1, t_2)$, $\Gamma_n(t_1, t_2)$ and $\Gamma(t_1, t_2)$ are defined in (4.52), Lemma 1.0.6 and 4.5 respectively.

Proof.

We have

$$\begin{aligned} & \sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{M_n^{(1*)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} \\ &= \sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1*)w}(t_1, t_2) - \bar{M}_n^{(1)w}(t_1, t_2)}{\gamma} \right| \\ & \quad + \sup_{0 < t_1 \leq t_2 < 1} \frac{\left| \frac{\bar{M}_n^{(1)}(t_1, t_2)}{\gamma} - \Gamma_n(t_1, t_2) \right|}{\frac{1}{2}(g(t_2) + h(t_1))} \end{aligned} \tag{4.93}$$

$$= \begin{cases} o_P(1), & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for all } c > 0, \\ O_P(1), & \text{if } I(g, c) < \infty \text{ and } I(h, c) < \infty \text{ for some } c > 0, \end{cases}$$

as $n \rightarrow \infty$, where $M_n^{(1*)w}(t_1, t_2)$, $\bar{M}_n^{(1)w}(t_1, t_2)$, and $\Gamma_n(t_1, t_2)$ (cf. (4.53), (4.33) and (1.5)) are weighted by $q(t_1, t_2) := \frac{1}{2}(g(t_2) + h(t_1))$. Now the claim follows by Lemma 4.1.2 and statements a) and b) of Corollary 4.2.2, after noting that the term in (4.93) is majorized by two times the terms used in the majorization of (4.56). The weight functions used in the majorization of (4.56) are $(1/2)$ times those that are specified in Lemma 4.1.2, i.e., in terms of the above $q(0, t_2) = \frac{1}{2}(g(t_2) + 0)$ and $q(t_1, 0) = \frac{1}{2}(h(t_1) + 0)$, and $q(0, t_2), q(t_1, 0) \leq \frac{1}{2}(g(t_2) + h(t_1))$ because of positivity (cf. Definition 4.1.1)

By Proposition 4.2.1, Lemma 4.1.1 and Lemma 4.1.2 (Lemmas 4.1.1 and 4.1.2 require the integral conditions hold only for some $c > 0$), where the weight function used in Lemmas 4.1.1 and 4.1.2 are now $q(t_1, t_2) := \frac{1}{2}(g(t_2) + h(t_1))$, after noting what was said in the latter paragraph regarding the majorization of (4.93), we can conclude

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|M_n^{(1*)}(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} \right. \\ \left. - \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} \right| = 0, \end{aligned} \quad (4.94)$$

$\infty < x < \infty$. By continuity of probability measures, as used in the proof of Theorem 4.1.1 statement c), we have, as $n \rightarrow \infty$, that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{\frac{1}{n+1} \leq t_1 \leq t_2 \leq \frac{n}{n+1}} \frac{|M_n^{(1*)}(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\} = \mathbb{P} \left\{ \sup_{0 < t_1 \leq t_2 < 1} \frac{|\Gamma(t_1, t_2)|}{\frac{1}{2}(g(t_2) + h(t_1))} \leq x \right\},$$

and with this result statement c) of Corollary 4.2.3 is now shown.

Chapter 5

Multiple Change-points

5.1 Asymptotic Results Under the Null Hypothesis $H_O^{(s)}$

Statistical inference has hypothesis testings as one of its main components. The Neyman-Pearson Lemma dogmatized the process by which a researcher tests a hypothesis. Given the probability of a type I error; the probability of rejecting the null hypothesis when it is true, the Lemma finds that rejection region with highest power. Also, the possibility to construct uniformly most powerful tests is one of the achievements of this Lemma

However, one hypothesis test often leads to another. With reference to this paper, if one were to find a change in variance at time k^* , one might then test for the presence of a second change in variance by employing the test for a second time. This process can be continued s times, if at the $s - 1$ th test, a change-point is detected. While the Neyman-Pearson Lemma allows control of a type I error on a test-by-test bases, it does not permit control of experiment-wide error. In particular, if one were to test for the presence of s changes in the variance, at least s tests would be necessary, when restricting to the traditional AMOC test statistic. Even though this has no effect on the type I error, it does affect the experiment-wide

error rate. If, for example, the error rate was set at the usual 0.05 percent, then the probability of making no error would be 0.95^s , assuming of course that these s tests were independent. Since the probability of no error is 0.95^s , the probability of making at least one error in s independent tests is $1 - 0.95^s$, which is much larger than the 0.05 test-by-test error rate. Because of the increase in the chance of error, one would like to develop a test that controls both test-by-test as well as experiment-wide error rates. The following section sets out to construct one such test statistic.

5.1.1 Population Mean Known

Given a set of observations $\{X_1, \dots, X_n\}$, for $n = 1, 2, \dots$, one might be interested in testing for at most s changes in their variance at s distinct but unknown times.

With positive constants $\sigma_1, \dots, \sigma_{s+1}$, let

$$X_i = \begin{cases} \mu + \sigma_1 \epsilon_i, & 1 \leq i \leq k_1^*, \\ \mu + \sigma_2 \epsilon_i, & k_1^* < i \leq k_2^*, \\ \dots & \dots \\ \mu + \sigma_{s+1} \epsilon_i, & k_s^* < i \leq n, \end{cases} \quad (5.1)$$

where

ϵ_i are independent and identically

distributed random variables with $\mathbb{E}\epsilon_i = 0, \mathbb{E}\epsilon_i^2 = 1$

and $\mathbb{E}\epsilon_i^4 < \infty, \quad i = 1, \dots, n.$ (5.2)

The values of the parameters; $\mu, \sigma_1^2, \dots, \sigma_{s+1}^2$ and $k_1^*, \dots, k_{s+1}^*$ are unknown. Assume at least one of the inequalities hold: $\sigma_i^2 \neq \sigma_j^2, \quad i \neq j$ and $i, j = 1, \dots, s+1$, the no change in variance null hypothesis can be formulated as

$$H_O^{(s)} : k_1^* > n$$

against the at most s changes in variance alternative

$$H_A^{(s)} : 1 \leq k_1^* \leq k_2^* \leq \cdots \leq k_s^* < n.$$

We shall use $M_n^{(1)}(t_1, \dots, t_s)$, the s -time parameter stochastic process developed in equation (2.17), on which to fashion a statistic that can be used to test $H_O^{(s)}$ against $H_A^{(s)}$, the AMSC in variance alternative.

As we had in Theorem 3.1.1, and in view of (2.17), we have

Theorem 5.1.1 *Assume $H_O^{(s)}$, (5.1), (5.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \cdots \leq t_s \leq 1\}_{n \in \mathbb{N}}$ such that, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq \cdots \leq t_s \leq 1} \left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| = o_P(1). \quad (5.3)$$

where, for each $n \geq 1$,

$$\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 < \cdots \leq t_s \leq 1\} \stackrel{\mathcal{D}}{=} \{\Gamma(t_1, \dots, t_s); 0 \leq t_1 < \cdots \leq t_s \leq 1\},$$

and $\Gamma(t_1, \dots, t_s)$ and $\Gamma_n(t_1, \dots, t_s)$ are defined in Definition 1.0.3 and Lemma 1.0.5, respectively.

Proof.

Using the definitions of $M_n^{(1)}(t_1, \dots, t_s)$ and $\Gamma_n(t_1, \dots, t_s)$ as given in (2.17) and Lemma 1.0.5, respectively, via (5.3), we have, where now the random variables Y_i for $i = 1, \dots, n$ and the standard Wiener process $\{W(nt); 0 \leq t \leq 1\}$ are viewed to be on the same probability space which is that of Major (1979),

$$\begin{aligned}
I(n) = & \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \sum_{i=1}^s \left(\frac{([(n+1)t_{l+1}] - [(n+1)t_{l-1}])}{n} \sum_{i=1}^{[(n+1)t_l]} \frac{(Y_i - \sigma^2)}{\gamma n^{1/2}} \right) \right. \\
& - \frac{[(n+1)t_s]}{n} \sum_{i=1}^s \frac{(Y_i - \sigma^2)}{\gamma n^{1/2}} - \sum_{i=1}^s (t_{l+1} - t_{l-1}) \frac{W(nt_l)}{n^{1/2}} \\
& \left. + t_s \frac{W(n)}{n^{1/2}} \right| \quad (5.4)
\end{aligned}$$

We note that Lemma 2.4.2 was used in (5.4) to show that the following term, which was added and subtracted,

$$\sum_{l=1}^s \frac{([(n+1)t_{l+1}] - [(n+1)t_{l-1}])[(n+1)t_l]\sigma^2}{n^{3/2}} - \frac{[(n+1)t_s]n\sigma^2}{n^{3/2}} = 0.$$

Consider

$$\begin{aligned}
I(n) \leq & \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \sum_{l=1}^s \left| \frac{[(n+1)t_{l+1}]}{n} - t_{l+1} \right| \left| \sum_{i=1}^{[(n+1)t_l]} \frac{Y_i - \sigma^2}{n^{1/2}\gamma} \right| \\
& + \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \sum_{l=1}^s \left| \frac{[(n+1)t_{l-1}]}{n} - t_{l-1} \right| \left| \sum_{i=1}^{[(n+1)t_l]} \frac{Y_i - \sigma^2}{n^{1/2}\gamma} \right| \\
& + \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \sum_{l=1}^s |t_{l+1} - t_{l-1}| \left| \sum_{i=1}^{[(n+1)t_l]} \frac{Y_i - \sigma^2}{n^{1/2}\gamma} - \frac{W(nt_l)}{n^{1/2}} \right| \\
& + \left| \sum_{i=1}^n \frac{Y_i - \sigma^2}{n^{1/2}\gamma} - \frac{W(n)}{n^{1/2}} \right| \\
\leq & \sum_{l=1}^s \sup_{0 < t_{l+1} < 1} \left| \frac{[(n+1)t_{l+1}]}{n} - t_{l+1} \right| \sup_{0 \leq t_l \leq 1} \left| \sum_{i=1}^{[(n+1)t_l]} \frac{(Y_i - \sigma^2)}{n^{1/2}\gamma} \right| \\
& + \sum_{l=1}^s \sup_{0 \leq t_{l-1} \leq 1} \left| \frac{[(n+1)t_{l-1}]}{n} - t_{l-1} \right| \sup_{0 < t_l < 1} \left| \sum_{i=1}^{[(n+1)t_l]} \frac{(Y_i - \sigma^2)}{n^{1/2}\gamma} \right|
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l=1}^s \sup_{0 \leq t_l \leq 1} \left| \sum_{i=1}^{[(n+1)t_l]} \frac{(Y_i - \sigma^2)}{n^{1/2}\gamma} - \frac{W(nt_l)}{n^{1/2}} \right| \\
& + \left| \sum_{i=1}^n \frac{(Y_i - \sigma^2)}{n^{1/2}\gamma} - \frac{W(n)}{n^{1/2}} \right|.
\end{aligned} \tag{5.5}$$

We establish now the following result

$$\begin{aligned}
I(n) &= 2s \cdot o(1) \cdot O_P(1) + s \cdot o_P(1) + o_P(1) \\
&= o_P(1), \text{ as } n \rightarrow \infty.
\end{aligned} \tag{5.6}$$

In verifying (5.6), we appeal again to the theorem of Major (1979) (Theorem S.2.2.1 in conjunction with (S.2.2.2) of Csörgő and Révész [10]) a Wiener process $\{W(t); 0 \leq t \leq \infty\}$ can be defined in such a way that, as $n \rightarrow \infty$,

$$\sup_{0 \leq t_l \leq 1} \left| \sum_{i=1}^{[(n+1)t_l]} \frac{Y_i}{\gamma n^{1/2}} - \frac{W(nt)}{n^{1/2}} \right| = o_P(1), \text{ for } l = 1, \dots, s.$$

The remaining terms in (5.5) that are not of the same form as above inherit also the property asserted in (5.6) by Lemma 2.4.1 and Donsker's Theorem (stated on $D[0, 1]$).

An immediately consequence of Theorem 5.1.1 is the following corollary.

Corollary 5.1.1 *Under the conditions given in Theorem 5.1.1, we have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} |\Gamma(t_1, \dots, t_s)|.$$

5.1.2 Population Mean Unknown

The partial sum process developed in (2.17) depends on the population mean and as such cannot be used to construct a statistic in the case where it is unknown. If we replace the population mean with a consistent estimator, it can then be used to construct a statistic. In what follows, we explore to what extent, if any, this has on the asymptotic distribution of the thus altered partial sum process.

Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Under $H_O^{(s)}$, $\bar{X}_n$ is a consistent estimator of μ , the population mean, Let

$$\bar{Y}_i = (X_i - \bar{X}_n)^2, \quad i = 1, \dots, n, \quad n \geq 1.$$

In view of (2.17), we introduce the following s -time parameter partial sum process.

$$\begin{aligned} \bar{M}_n^{(1)}(t_1, \dots, t_s) &:= n^{-1/2} \left\{ \sum_{l=1}^s \left(\frac{[(n+1)t_{l+1}] - [(n+1)t_{l-1}]}{n} \sum_{i=1}^{[(n+1)t_l]} \bar{Y}_i \right) \right. \\ &\quad \left. - \frac{[(n+1)t_s]}{n} \sum_{i=1}^n \bar{Y}_i \right\}, \quad 0 < t_1 \leq \dots \leq t_s < 1. \end{aligned} \quad (5.7)$$

In this regard, we have

Remark 5.1.1 *We have the following representation of $\bar{M}_n^{(1)}(t_1, \dots, t_s)$.*

$$\begin{aligned} \bar{M}_n^{(1)}(t_1, \dots, t_s) &= M_n^{(1)}(t_1, \dots, t_s) \\ &\quad - 2 \sum_{l=1}^s \frac{[(n+1)t_{l+1}] - [(n+1)t_{l-1}]}{n} \sum_{i=1}^{[(n+1)t_l]} \frac{(X_i - \mu)}{n^{1/2}} (\bar{X}_n - \mu) \\ &\quad + 2 \frac{[(n+1)t_s]}{n} \sum_{i=1}^n \frac{(X_i - \mu)}{n^{1/2}} (\bar{X}_n - \mu). \end{aligned}$$

Lemma 3.1.1 of Chapter 3 and Remark 5.1.1 suggest the following relationship between the above two s -time parameter stochastic processes.

Lemma 5.1.1 *Assume $H_O^{(s)}$, (5.1), (5.2) hold and $\infty > \gamma > 0$. Then, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq t_2 \leq 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)}(t_1, \dots, t_s) - M_n^{(1)}(t_1, \dots, t_s) \right| = o_P(1).$$

Proof.

This follows from Remark 5.1.1 and arguments similar to those used to establish Lemma 3.1.1.

With the aid of Lemma 5.1.1, and similar to the role Lemma 3.1.3 had in the proof of Theorem 3.1.2, we can establish the corresponding theorem.

Theorem 5.1.2 *Assume $H_O^{(s)}$, (5.1), (5.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq t_s \leq 1\}_{n \in \mathbb{N}}$ such that, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| = o_P(1)$$

where, for each $n \geq 1$,

$$\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 < \dots \leq t_s \leq 1\} \stackrel{D}{=} \{\Gamma(t_1, \dots, t_s); 0 \leq t_1 < \dots \leq t_s \leq 1\},$$

and $\Gamma(t_1, \dots, t_s)$ and $\Gamma_n(t_1, \dots, t_s)$ are defined in Definition 1.0.3 and Lemma 1.0.5, respectively.

Proof.

In the probability space of Major (1979) (Theorem S.2.2.1 in conjunction with (S.2.2.2) of Csörgő and Révész [10]), we have

$$\begin{aligned} \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| & \leq \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \frac{M_n^{(1)}(t_1, \dots, t_s)}{\gamma} \right| \\ & \quad + \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| \\ & = o_P(1) + o_P(1) = o_P(1), \text{ as } n \rightarrow \infty, \end{aligned}$$

by Lemma 5.1.1 and Theorem 5.1.1 .

Corollary 5.1.2 *Under the conditions given in Theorem 5.1.2, we have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} |\Gamma(t_1, \dots, t_s)|.$$

5.1.3 Small Changes in the Population Mean

Along the lines of equation (2.6) of E. Gombay, L. Horváth, M. Hušková [17], we consider a generalisation of their model. We assume here that

$$X_{i,n} = \begin{cases} \mu_{in} + \sigma_1 \epsilon_i, & 1 \leq i \leq k_1^*, \\ \mu_{in} + \sigma_2 \epsilon_i, & k_1^* < i \leq k_2^*, \\ \vdots & \vdots \\ \mu_{in} + \sigma_{s+1} \epsilon_i, & k_s^* < i \leq n, \end{cases} \quad (5.8)$$

where $\mu_{in}, \dots, \mu_{nn}$ are unknown constants, the ϵ_i 's are i.i.d random variables that satisfy the conditions given in (5.2).

As in (2.7) of [17], let

$$\max_{1 \leq i \leq n} (\mu_{in} - \bar{\mu})^2 = o(n^{-1/2}), \quad (5.9)$$

where $\bar{\mu} = \frac{1}{n} \sum_{1 \leq i \leq n} \mu_{in}$. In addition, let $Y_{i,n}^* = (X_{i,n} - \mu_{in})^2$, $i = 1, \dots, n$, $n \geq 1$. In terms of the newly defined $Y_{i,n}$, we define a third s -time parameter partial sum process.

$$\begin{aligned} M_n^{(1*)}(t_1, \dots, t_s) &:= n^{-1/2} \left\{ \sum_{l=1}^s \left(\left(\frac{[(n+1)t_{l+1}] - [(n+1)t_{l-1}]}{n} \right) \sum_{i=1}^{[(n+1)t_l]} Y_{i,n}^* \right) \right. \\ &\quad \left. - \frac{[(n+1)t_s]}{n} \sum_{i=1}^n Y_{i,n}^* \right\} \quad 0 \leq t_1 \leq \dots \leq t_s \leq 1. \end{aligned} \quad (5.10)$$

We can use $M_n^{(1*)}(t_1, \dots, t_s)$ (cf. (5.10)) on which to fashion a statistic that is suited to testing $H_O^{(s)}$. Recall (5.7) for the sake of the following remark.

Remark 5.1.2 We have the following representation of $\bar{M}_n^{(1)}(t_1, \dots, t_s)$.

$$\begin{aligned} n^{1/2} \bar{M}_n^{(1)}(t_1, \dots, t_s) &= n^{1/2} M_n^{(1*)}(t_1, \dots, t_s) + D_n^{(2)}(t_1, \dots, t_s) + D_n^{(3)}(t_1, \dots, t_s) \\ &\quad + D_n^{(4)}(t_1, \dots, t_s) + D_n^{(5)}(t_1, \dots, t_s), \end{aligned}$$

where

$$\begin{aligned} D_n^{(2)}(t_1, \dots, t_s) &:= 2 \sum_{l=1}^s \frac{([(n+1)t_{l+1}] - [(n+1)t_{l-1}])}{n} \sum_{i=1}^{[(n+1)t_l]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}), \\ &\quad - 2 \frac{[(n+1)t_s]}{n} \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}), \\ D_n^{(3)}(t_1, \dots, t_s) &:= \sum_{l=1}^s \frac{([(n+1)t_{l+1}] - [(n+1)t_{l-1}])}{n} \sum_{i=1}^{[(n+1)t_l]} (\mu_{in} - \bar{\mu})^2 \\ &\quad - \frac{[(n+1)t_s]}{n} \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2, \\ D_n^{(4)}(t_1, \dots, t_s) &:= 2(\bar{X}_n - \bar{\mu}) \left\{ \sum_{l=1}^s \frac{([(n+1)t_{l+1}] - [(n+1)t_{l-1}])}{n} \sum_{i=1}^{[(n+1)t_l]} (X_{i,n} - \mu_{in}) \right. \\ &\quad \left. - \frac{[(n+1)t_s]}{n} \sum_{i=1}^n (X_{i,n} - \mu_{in}) \right\}, \\ D_n^{(5)}(t_1, \dots, t_s) &:= 2(\bar{X}_n - \bar{\mu}) \left\{ \sum_{l=1}^s \frac{([(n+1)t_{l+1}] - [(n+1)t_{l-1}])}{n} \sum_{i=1}^{[(n+1)t_l]} (\mu_{in} - \bar{\mu}) \right. \\ &\quad \left. - \frac{[(n+1)t_s]}{n} \sum_{i=1}^n (\mu_{in} - \bar{\mu}) \right\}. \end{aligned}$$

Along similar lines as Lemma 3.1.2, we use Remark 5.1.2 to assert the following lemma.

Lemma 5.1.2 Assume $H_O^{(s)}$, (5.2), (5.8) and (5.9) hold and $\infty > \gamma > 0$. Then, as $n \rightarrow \infty$,

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \frac{1}{\gamma} \left| \bar{M}_n^{(1)}(t_1, \dots, t_s) - M_n^{(1*)}(t_1, \dots, t_s) \right| = o_P(1).$$

Proof. This follows from arguments similar to those used in the proof of Lemma 3.1.2 (cf. (3.22), (3.25), (3.26), (3.28)).

With the aid of Lemma 5.1.2, and along the lines of Theorem 3.1.3, we have the following in terms of a s -time parameter Gaussian process.

Theorem 5.1.3 *Assume $H_O^{(s)}$, (5.2), (5.8) and (5.9) hold and $\infty > \gamma > 0$, then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\}_{n \in \mathbb{N}}$ such that, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| = o_P(1).$$

where, for each $n \geq 1$,

$$\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\} \stackrel{D}{=} \{\Gamma(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\}.$$

and $\Gamma(t_1, \dots, t_s)$ and $\Gamma_n(t_1, \dots, t_s)$ are defined in Definition 1.0.3 and Lemma 1.0.5, respectively.

Proof. In the probability space of Major (1979) (Theorem S.2.2.1 in conjunction with (S.2.2.2) of Csörgő and Révész [10]), we have

$$\begin{aligned} \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| & \leq \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} \right| \\ & \quad + \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| \\ & = o_P(1) + o_P(1) = o_P(1), \text{ as } n \rightarrow \infty, \end{aligned}$$

by Lemma 5.1.2 and Theorem 5.1.2.

The following corollary is an immediate consequence of Theorem 5.1.3.

Corollary 5.1.3 *Under the conditions given in Theorem 5.1.3, we have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} |\Gamma(t_1, \dots, t_s)|.$$

Let $Y_{i,n} = (X_{i,n} - \bar{\mu})^2$, $i = 1, \dots, n$, $n \geq 1$. In view of (2.17) and in terms of our newly defined $Y_{i,n}$, let

$$\begin{aligned} M_n^{(3)}(t_1, \dots, t_s) &:= n^{-1/2} \left\{ \sum_{l=1}^s \left(\frac{[(n+1)t_{l+1}] - [(n+1)t_{l-1}]}{n} \right)^{[(n+1)t_l]} \sum_{i=1}^{[(n+1)t_l]} Y_{i,n} \right. \\ &\quad \left. - \frac{[(n+1)t_s]}{n} \sum_{i=1}^n Y_{i,n} \right\}, \quad 0 \leq t_1 \leq \dots \leq t_s \leq 1. \end{aligned} \quad (5.11)$$

Remark 5.1.3 We have the following representation of $M_n^{(3)}(t_1, \dots, t_s)$.

$$\begin{aligned} n^{1/2} M_n^{(3)}(t_1, \dots, t_s) &= n^{1/2} M_n^{(1*)}(t_1, \dots, t_s) \\ &\quad + 2 \sum_{l=1}^s \left(\frac{[(n+1)t_{l+1}] - [(n+1)t_{l-1}]}{n} \right)^{[(n+1)t_l]} \sum_{i=1}^{[(n+1)t_l]} (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \\ &\quad + \sum_{l=1}^s \left(\frac{[(n+1)t_{l+1}] - [(n+1)t_{l-1}]}{n} \right)^{[(n+1)t_l]} \sum_{i=1}^{[(n+1)t_l]} (\mu_{in} - \bar{\mu})^2 \\ &\quad - 2 \frac{[(n+1)t_s]}{n} \sum_{i=1}^n (X_{i,n} - \mu_{in})(\mu_{in} - \bar{\mu}) \\ &\quad - \frac{[(n+1)t_s]}{n} \sum_{i=1}^n (\mu_{in} - \bar{\mu})^2. \end{aligned}$$

For the sake of the next lemma, recall (5.10), (5.11) and Remark 5.1.3.

Lemma 5.1.3 *Assume $H_o^{(s)}$, (5.2), (5.8) and (5.9) hold and $\infty > \gamma > 0$. Then, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \frac{1}{\gamma} \left| M_n^{(3)}(t_1, \dots, t_s) - M_n^{(1*)}(t_1, \dots, t_s) \right| = o_P(1).$$

Proof. This follows from Remark 5.1.3 and arguments similar to those used to establish Lemma 3.1.3.

With the aid of Lemma 5.1.3, and along the same lines of Theorem 3.1.4, we have the following in terms of a s -time parameter Gaussian process.

Theorem 5.1.4 *Assume $H_O^{(s)}$, (5.2), (5.8) and (5.9) all hold and $\infty > \gamma > 0$, then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\}_{n \in \mathbb{N}}$ such that, as $n \rightarrow \infty$,*

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(3)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| = o_P(1)$$

where, for each $n \geq 1$,

$$\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\} \stackrel{D}{=} \{\Gamma(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\}.$$

and $\Gamma(t_1, \dots, t_s)$ and $\Gamma_n(t_1, \dots, t_s)$ are defined in Definition 1.0.3 and Lemma 1.0.5, respectively.

Proof. In the probability space of Major (1979) (Theorem S.2.2.1 in conjunction with (S.2.2.2) of Csörgő and Révész [10]), we have

$$\begin{aligned} \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(3)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| &\leq \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(3)}(t_1, \dots, t_s)}{\gamma} - \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} \right| \\ &\quad + \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right| \\ &= o_P(1) + o_P(1) = o_P(1), \text{ as } n \rightarrow \infty, \end{aligned}$$

by Lemma 5.1.3 and Theorem 5.1.3 .

The following corollary is an immediate consequence of Theorem 5.1.4.

Corollary 5.1.4 *Under the conditions given in Theorem 5.1.4, we have, as $n \rightarrow \infty$, that*

$$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(3)}(t_1, \dots, t_s)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} |\Gamma(t_1, \dots, t_s)|.$$

5.2 Estimating γ^2 Under the Null Hypothesis $H_O^{(s)}$

The functionals developed thus far (cf. (5.3), (5.7) and (5.10)) depend on the unknown parameter γ (cf. Corollaries 5.1.1 - 5.1.2) and as such are not yet statistics. Transforming them into statistics requires estimating γ^2 . We propose two estimators. The first is a method of moment type estimator, while the second, with a view to $H_A^{(s)}$, incorporates the possibility of s changes by including estimates of the $s + 1$ variances, as can be seen in (5.13). To this end, we have

$$\hat{\gamma}_{1,n}^2 = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^4}{n} - \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n} \right)^2 \quad (5.12)$$

and

$$\hat{\gamma}_{2,n}^2 = n^{-1} \left\{ \sum_{i=1}^n (X_i - \bar{X}_n)^4 - \sum_{l=0}^s \frac{1}{k_{l+1} - k_l} \left(\sum_{i=1}^{k_{l+1} - k_l} (X_{k_l+i} - \bar{X}_n)^2 \right)^2 \right\}, \quad (5.13)$$

$$\text{if } 1 \leq k_1 \leq \dots \leq k_s < n,$$

$$= \hat{\gamma}_{1,n}^2,$$

$$\text{if } k_1 = \dots = k_s = n.$$

As was the case in Chapter 3, Section 3.2, we must estimate the unknown parameters $k_1, \dots, k_s$. We propose, below, the following estimator,

$$\begin{aligned} (\hat{k}_1, \dots, \hat{k}_s) &:= \min \left\{ (k_1, \dots, k_s); \left| M_n^{(1)} \left(\frac{k_1}{n+1}, \dots, \frac{k_s}{n+1} \right) \right| \right. \\ &\quad \left. = \max_{1 \leq m_1 \leq \dots \leq m_s < n} \left| M_n^{(1)} \left(\frac{m_1}{n+1}, \dots, \frac{m_s}{n+1} \right) \right| \right\}, \quad (5.14) \end{aligned}$$

where $\min\{(k_1, \dots, k_s); \dots\}$ is used to denote the point $(k_1, \dots, k_s)$ where the maximum occurs which has the smallest distance using euclidean norm from the origin. Once the estimates $\hat{k}_1, \dots, \hat{k}_s$ of parameters $k_1, \dots, k_s$ have been determined, they could then be substituted for the corresponding parameters in formula (5.13) to arrive at

$$\tilde{\gamma}_{2,n,\hat{k}_1,\dots,\hat{k}_s}^2 = n^{-1} \left\{ \sum_{i=1}^n (X_i - \bar{X}_n)^4 - \sum_{l=1}^s \frac{1}{\hat{k}_{l+1} - \hat{k}_l} \left(\sum_{i=1}^{\hat{k}_{l+1} - \hat{k}_l} (X_{\hat{k}_l+i} - \bar{X}_n)^2 \right)^2 \right\}. \quad (5.15)$$

Theorem 5.2.1 *If $H_O^{(s)}$ (5.1), (5.2) hold, then, as $n \rightarrow \infty$, we have*

$$|\hat{\gamma}_{1,n}^2 - \gamma^2| = o_P(1)$$

and

$$\max_{1 \leq k_1 \leq \dots \leq k_s \leq n} |\hat{\gamma}_{2,n,k_1,\dots,k_s}^2 - \gamma^2| = o_P(1).$$

Proof. The first statement was proved in Theorem 3.2.1, while the last statement can be proved by repeating arguments similar to those used to establish the corresponding statement in Theorem 3.2.1.

As Theorem 5.2.1 establishes the uniform consistency of $\hat{\gamma}_{2,n,k_1,\dots,k_s}^2$ in $k_1, \dots, k_s$, $1 \leq k_1 \leq \dots \leq k_s < n$, we are then able to conclude that $\tilde{\gamma}_{2,n,\hat{k}_1,\dots,\hat{k}_s}^2$ is a consistent estimator of γ^2 as well.

Corresponding to Remark 3.2.1, we have the remark below regarding $\tilde{\gamma}_{2,n,\hat{k}_1,\dots,\hat{k}_s}^2$.

Remark 5.2.1 Since $\hat{\gamma}_{1,n}^2$ and $\tilde{\gamma}_{2,n,\hat{k}_1,\dots,\hat{k}_s}^2$ are both consistent estimators of γ^2 under $H_O^{(s)}$ (cf. Theorem 5.2.1), we can replace γ with either one of $\hat{\gamma}_{1,n}$ or $\tilde{\gamma}_{2,n,\hat{k}_1,\dots,\hat{k}_s}$ in Corollaries 5.1.1 and 5.1.2, and they will continue to hold true by Slutsky's theorem.

5.3 Asymptotic Results Under the Alternative Hypothesis $H_A^{(s)}$

This section is concerned with the asymptotic behaviour of $M_n^{(1)}(t_1, \dots, t_s)$ (cf. (2.17)) and $\bar{M}_n^{(1)}(t_1, \dots, t_s)$ (cf. (5.7)), under $H_A^{(s)}$ (cf. (2.4)). Using their limiting behaviour under $H_A^{(s)}$, we will establish the consistency of the tests of $H_O^{(s)}$ (cf. Corollaries 5.1.1, 5.1.2, 5.1.3) under the three cases: population mean known, unknown and then under small changes in the mean. Under $H_A^{(s)}$, the limiting function will depend on the unknown location of s change-points $k_1^* := [n\lambda_1], \dots, k_s^* := [n\lambda_s]$ where $0 < \lambda_1 \leq \dots \leq \lambda_s < 1$ are fixed.

5.3.1 Population Mean Known

Let

$$S(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js})$$

$$:= \left(\sum_{i=1}^{[(n+1)x_1]} + \sum_{i=[(n+1)x_1]+1}^{[(n+1)x_2]} + \dots + \sum_{i=[(n+1)x_{j-1}s+1]+1}^{[(n+1)x_{js}]} + \sum_{i=[(n+1)x_{js}]+1}^{[(n+1)t_j]} \right) Y_i, \quad (5.16)$$

where $0 < x_1 \leq x_2 \leq \dots \leq x_s \leq \dots \leq x_{js} \leq t_j < 1$ and $j = 1, \dots, s+1$. At most s of these variables will assume one of the values of λ_i . The variables $x_1, \dots, x_{js}$, are used to partition the block $\{Y_1, \dots, Y_{[(n+1)t_j]}\}$ into sub-blocks that do not contain a change-point. Since the location of the change-points are unknown, the possibility of s changes in the block $\{Y_1, \dots, Y_{[(n+1)t_j]}\}$ must be considered. Partitioning $\{Y_1, \dots, Y_{[(n+1)t_j]}\}$ into sub-blocks; $\{Y_{[(n+1)x_1]+1}, \dots, Y_{[(n+1)x_2]}\}, \dots, \{Y_{[(n+1)x_{(j-1)s+1}], \dots, Y_{[(n+1)t_j]}\}$ leaves a partition with no changes in the variance within each sub-block. For this partition, via the SLLN, one can establish the

asymptotic properties of (5.16). In particular, we have

$$\begin{aligned}
& \frac{1}{n} S(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}) \\
&= \frac{1}{n} \left(\sum_{i=1}^{[(n+1)x_1]} + \sum_{i=[(n+1)x_1]+1}^{[(n+1)x_2]} + \dots + \sum_{i=[(n+1)x_{js-1}]+1}^{[(n+1)x_{js}]} + \sum_{i=[(n+1)x_{js}]+1}^{[(n+1)t_j]} \right) Y_i \\
&= \frac{[(n+1)x_1]}{n} \sum_{i=1}^{[(n+1)x_1]} Y_i \\
&\quad + \frac{([(n+1)x_2] - [(n+1)x_1])}{n} \sum_{i=1}^{[(n+1)x_2] - [(n+1)x_1]} \frac{Y_{[(n+1)x_1]+i}}{[(n+1)x_2] - [(n+1)x_1]} + \dots \\
&\quad \dots + \frac{([(n+1)t_j] - [(n+1)x_{js}])}{n} \sum_{i=1}^{[(n+1)t_j] - [(n+1)x_{js}]} \frac{Y_{[(n+1)x_{js}]+i}}{[(n+1)t_j] - [(n+1)x_{js}]} \\
&= \frac{[(n+1)x_1]}{n} \sum_{i=1}^{[(n+1)x_1]} Y_i + \frac{([(n+1)(x_2 - x_1)])}{n} \sum_{i=1}^{[(n+1)(x_2 - x_1)]} \frac{Y_{[(n+1)x_1]+i}}{[(n+1)(x_2 - x_1)]} + \dots \\
&\quad \dots + \frac{([(n+1)(t_j - x_{js}])}{n} \sum_{i=1}^{[(n+1)(t_j - x_{js})]} \frac{Y_{[(n+1)x_{js}]+i}}{[(n+1)(t_j - x_{js})]}
\end{aligned}$$

$$\stackrel{a.s.}{\longrightarrow} x_1 \sigma_{l(x_1)}^2 + (x_2 - x_1) \sigma_{l(x_2)}^2 + \dots + (x_s - x_{s-1}) \sigma_{l(x_s)}^2 + \dots$$

$$\dots + (x_{(j-1)s+1} - x_{(j-1)s}) \sigma_{l(x_{(j-1)s+1})}^2 + \dots + (t_j - x_{js}) \sigma_{l(t_j)}^2, \quad \text{as } n \rightarrow \infty, \quad (5.17)$$

where $x_0 := 0$, and

$$l(z) := \begin{cases} 1 & 0 < z \leq \lambda_1 \\ 2 & \lambda_1 < z \leq \lambda_2 \\ \vdots & \\ s & \lambda_{s-1} < z \leq \lambda_s \\ s+1 & \lambda_s < z \leq 1. \end{cases}$$

Let us define

$$\begin{aligned} \eta(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}) \\ := \sum_{i=1}^{js} (x_i - x_{i-1}) \sigma_{l(x_i)}^2 + (t_j - x_{js}) \sigma_{l(t_j)}^2, \end{aligned} \quad (5.18)$$

$$x_0 := 0, \quad 0 < x_1 \leq \dots \leq x_{js} \leq t_j < 1, \quad \text{for } j = 1, \dots, s+1,$$

where $t_{s+1} = 1$. Using (5.17) and (5.18), we can conclude that

$$\begin{aligned} \frac{1}{n} S(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}) \\ \xrightarrow{\text{a.s.}} \eta(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}), \\ x_0 := 0, \quad 0 < x_1 \leq \dots \leq x_{js} \leq t_j < 1 \quad \text{for } j = 1, \dots, s+1, \end{aligned}$$

as $n \rightarrow \infty$, where $t_{s+1} := \frac{n}{n+1}$ and $t_{s+1} := 1$ in the partial sum and limiting function respectively.

Remark 5.3.1 Assume $H_A^{(s)}$, put $Y_i := (X_i - \mu)^2$, and define (cf. (5.1))

$$\gamma_A^2 := \begin{cases} \gamma_1^2 := \mathbb{E}Y_i^2 - (\sigma_1^2)^2, & 1 \leq i \leq k_1^* \\ \gamma_2^2 := \mathbb{E}Y_i^2 - (\sigma_2^2)^2, & k_1^* < i \leq k_2^* \\ \vdots \\ \gamma_{s+1}^2 := \mathbb{E}Y_i^2 - (\sigma_{s+1})^2 & k_s^* < i \leq n \end{cases}$$

Then, we write

$$\gamma_A = \gamma_1 I\{1 \leq i \leq k_1^*\} + \gamma_2 I\{k_1^* < i \leq k_2^*\} + \dots + \gamma_{s+1} I\{k_s^* < i \leq n\},$$

and note that, given $H_A^{(s)}$ with $k_1^*, \dots, k_s^*$ as in Theorem 5.3.1,

$$\text{Var}(Y_i/\gamma_A) = (\mathbb{E}Y_i^2 - (\mathbb{E}Y_i)^2)/\gamma_A^2 = 1, \quad i = 1, \dots, n,$$

and Theorem 1.0.2 is applicable in the indicated $s+1$ regions for i in the definition of γ_A^2 above.

To show that the a.s. convergence in (5.17) is uniform, we establish the following lemma.

Lemma 5.3.1 Assume $H_A^{(s)}$ and (5.2) hold, at least one of $\sigma_i^2 \neq \sigma_j^2$, $i \neq j$ for $i, j = 1, \dots, s+1$, $0 < \sigma_i^2 < \infty$, $i = 1, \dots, s+1$. Let $\gamma_A^2 := \mathbb{E}Y_i^2 - (\mathbb{E}Y_i)^2$, where $Y_i := (X_i - \mu)^2$, X_i be as given in (5.1), and let γ_A be as defined in Remark 5.3.1. If $k_i^* = [n\lambda_i]$ for $i = 1, \dots, s$ and $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_s < 1$ are fixed, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_j < x_{js+1} < 1} \left| \frac{1}{n} S(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}) - \eta(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}) \right| \stackrel{a.s.}{=} o(1), \quad (5.19)$$

$x_0 := 0$, $0 < x_1 \leq \dots \leq x_{js} \leq t_j \leq x_{js+1} < 1$ for $j = 1, \dots, s$, and the x_i 's for $i = 1, \dots, ss+1$ are fixed.

Proof.

Recall equations (5.16), (5.17), (5.18), (5.19) and consider the following.

$$\begin{aligned} & \sup_{0 < t_j < x_{js+1} < 1} \left| \frac{1}{n} S(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}) - \eta(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}) \right| \\ & \leq \sup_{0 < t_j < x_{js+1} < 1} \left| \frac{1}{n} S(t_j, x_1, \dots, x_s, \dots, x_{(j-1)s+1}, \dots, x_{js}) - \sum_{k=1}^{js} \frac{[(n+1)x_k] - [(n+1)x_{k-1}]}{n} \sigma_{l(x_k)}^2 + \frac{[(n+1)t_j] - [(n+1)x_{js}]}{n} \sigma_{l(t_j)}^2 \right| \\ & \quad + \left| \sum_{k=1}^{js} \frac{[(n+1)x_k] - [(n+1)x_{k-1}]}{n} \sigma_{l(x_k)}^2 - \frac{[(n+1)t_j] - [(n+1)x_{ks}]}{n} \sigma_{l(t_j)}^2 - \sum_{k=1}^{js} (x_k - x_{k-1}) \sigma_{l(x_k)}^2 + (t_j - x_{js}) \sigma_{l(t_j)}^2 \right| \\ & \leq \sum_{k=1}^{js} \frac{[(n+1)x_k] - [(n+1)x_{k-1}]}{n} \left| \sum_{i=[(n+1)x_{k-1}]+1}^{[(n+1)x_k]} \frac{Y_i - \sigma_{l(x_k)}^2}{[(n+1)x_k] - [(n+1)x_{k-1}]} \right| \end{aligned}$$

$$\begin{aligned}
& + \sup_{0 < t_j < x_{js+1} < 1} \left| \frac{([(n+1)t_j] - [(n+1)x_{js}])}{n} \right| \left| \sum_{i=1}^{[(n+1)(t_j - x_{js})]} \frac{Y_{[(n+1)x_{js}] + i} - \sigma_{l(t_j)}^2}{n^{1/2}[(n+1)(t_j - x_{js})]} \right| \\
& + \sum_{k=1}^{js} \left| \left(\frac{[(n+1)x_k] - [(n+1)x_{k-1}]}{n} - (x_k - x_{k-1}) \right) \right| \sigma_{l(x_k)}^2 \\
& + \sup_{0 < t_j < x_{js+1} < 1} \left| \left(\frac{([(n+1)t_j] - [(n+1)x_{js}])}{n} - (t_j - x_{js}) \right) \right| \sigma_{l(t_j)}^2 \\
& =: I_1(n) + I_2(n) + I_3(n) + I_4(n) \\
& \stackrel{a.s.}{=} o(1), \quad \text{as } n \rightarrow \infty. \tag{5.20}
\end{aligned}$$

To verify the claim in (5.20), we have

$$I_1(n) \stackrel{a.s.}{=} o(1), \quad \text{as } n \rightarrow \infty, \tag{5.21}$$

after decomposing the partial sum along the lines of (5.17) and applying the SLLN to each partial sum,

$$\begin{aligned}
I_2(n) & \leq \sup_{0 < t_j - x_{js} < x_{js+1} - x_{js} < 1} \left| \sum_{i=1}^{[(n+1)(t_j - x_{js})]} \frac{Y_{[(n+1)x_{js}] + i} - \sigma_{l(t_j)}^2}{\gamma_A \sqrt{[(n+1)(t_j - x_{js})] \log \log n}} \right| \gamma_A \sqrt{\frac{\log \log n}{n}} \\
& \stackrel{a.s.}{=} o(1), \quad \text{as } n \rightarrow \infty, \tag{5.22}
\end{aligned}$$

which follows by Theorem 1.0.2. Moreover, as $n \rightarrow \infty$,

$$\begin{aligned}
I_3(n) & \leq \sum_{j=1}^{js} \left| \left(\frac{[(n+1)x_j] - [(n+1)x_{j-1}]}{n} - (x_j - x_{j-1}) \right) \right| \sigma_{l(x_j)}^2 \\
& \quad + \sup_{0 < t_j < x_{js+1} < 1} \left| \left(\frac{([(n+1)t_j] - [(n+1)x_{js}])}{n} - (t_j - x_{js}) \right) \right| \sigma_{l(t_j)}^2 \\
& = o(1), \tag{5.23}
\end{aligned}$$

as $n \rightarrow \infty$, by Lemma 2.4.1. With equations (5.21), (5.22) and (5.23) our claim in (5.20) now follows.

Remark 5.3.2

$$\begin{aligned}
 n^{1/2} M_n^{(1)}(t_1, \dots, t_s) = & \\
 & \sum_{l=1}^s \frac{[(n+1)t_{l+1}] - [(n+1)t_{l-1}]}{n} S(t_l, x_1, \dots, x_s, \dots, x_{(l-1)s+1}, \dots, x_{ls}) \\
 & - \frac{[(n+1)t_s]}{n} S(t_{s+1}, x_1, \dots, x_s, \dots, x_{ss+1}, \dots, x_{ss+s}), \tag{5.24}
 \end{aligned}$$

$$0 < x_1 \leq x_2 \leq \dots \leq x_s \leq t_1 \leq x_{s+1} \leq \dots \leq x_{2s} \leq t_2 \leq x_{2s+1} \leq \dots \leq x_{ss+s} < 1,$$

where $t_o := 0$ and $t_{s+1} := \frac{n}{n+1}$.

Since the location of the change-points $k_1^* := [n\lambda_1], \dots, k_s^* := [n\lambda_s]$, where $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_s < 1$ are unknown, we define, as in Orasch [24], the functions $a_i = a_i(\lambda_1, \dots, \lambda_s, t_1, \dots, t_s)$, $1 \leq i \leq s(s+1)$, which will be used in the definition of the limiting function of (5.24).

$$\begin{aligned}
 a_1 &:= \begin{cases} \lambda_1, & \lambda_1 \leq t_1 \\ c_1, & \text{otherwise} \end{cases} \\
 a_2 &:= \begin{cases} \lambda_2, & \lambda_1 \leq \lambda_2 \leq t_1 \\ c_2, & \text{otherwise} \end{cases} \\
 &\vdots \\
 a_s &:= \begin{cases} \lambda_s, & \lambda_1 \leq \dots \leq \lambda_s \leq t_1 \\ c_s, & \text{otherwise} \end{cases} \\
 a_{s+1} &:= \begin{cases} \lambda_1, & t_1 < \lambda_1 \leq t_2 \\ \lambda_2, & \lambda_1 \leq t_1 < \lambda_2 \leq t_2 \\ \vdots & \\ \lambda_s, & \lambda_1 \leq \dots \leq \lambda_{s-1} \leq t_1 < \lambda_s \leq t_2 \\ c_{s+1}, & \text{otherwise} \end{cases} \tag{5.25}
 \end{aligned}$$

$$\begin{aligned}
a_{s+2} &:= \begin{cases} \lambda_2, & t_1 < \lambda_1 \leq \lambda_2 \leq t_2 \\ \lambda_3, & \lambda_1 \leq t_1 < \lambda_2 \leq \lambda_3 \leq t_2 \\ \vdots & \\ \lambda_s, & \lambda_1 \leq \dots \leq \lambda_{s-2} \leq t_1 < \lambda_{s-1} \leq \lambda_s \leq t_2 \\ c_{s+2}, & \text{otherwise} \end{cases} \\
&\vdots \\
a_{s+s} &:= \begin{cases} \lambda_s, & t_1 < \lambda_1 \leq \dots \leq \lambda_s \leq t_2 \\ c_{s+s}, & \text{otherwise} \end{cases} \\
&\vdots \\
a_{2s+1} &:= \begin{cases} \lambda_1, & t_2 < \lambda_1 \leq t_3 \\ \lambda_2, & \lambda_1 \leq t_2 < \lambda_2 \leq t_3 \\ \vdots & \\ \lambda_s, & \lambda_1 \leq \dots \leq \lambda_{s-1} \leq t_2 < \lambda_s \leq t_3 \\ c_{2s+1}, & \text{otherwise} \end{cases} \\
&\vdots \\
a_{2s+s} &:= \begin{cases} \lambda_s, & t_2 < \lambda_1 \leq \dots \leq \lambda_s \leq t_3 \\ c_{2s+s}, & \text{otherwise} \end{cases} \\
&\vdots \\
a_{ss+1} &:= \begin{cases} \lambda_1, & t_s < \lambda_1 < 1 \\ \lambda_2, & \lambda_1 \leq t_s < \lambda_2 < 1 \\ \vdots & \\ \lambda_s, & \lambda_1 \leq \dots \leq \lambda_{s-1} \leq t_s < \lambda_s < 1 \\ c_{2s+1}, & \text{otherwise} \end{cases} \\
&\vdots \\
a_{ss+s} &:= \begin{cases} \lambda_s, & t_s < \lambda_1 \leq \dots \leq \lambda_s < 1 \\ c_{ss+s}, & \text{otherwise} \end{cases}
\end{aligned}$$

where $0 < a_1 \leq a_2 \leq \dots \leq a_{ss+s} < 1$ and $c_i \in [0, 1]$, $1 \leq i \leq (s+1)s$.

To understand this notation better, we derive (5.16) and (5.18) for the case $s = 2$, and for $0 < \lambda_1 \leq t_1 < \lambda_2 \leq t_2 < 1$.

$$S(t_1, a_1, a_2) = \sum_{i=1}^{[(n+1)a_1]} Y_i + \sum_{i=[(n+1)a_1]+1}^{[(n+1)a_2]} Y_i + \sum_{l=[(n+1)a_2]+1}^{[(n+1)t_1]} Y_l$$

$$\begin{aligned}
S(t_2, a_1, a_2, a_3, a_4) &= \sum_{i=1}^{[(n+1)a_1]} Y_i + \sum_{i=[(n+1)a_1]+1}^{[(n+1)a_2]} Y_i, \\
&+ \sum_{i=[(n+1)a_2]+1}^{[(n+1)a_3]} Y_i + \sum_{i=[(n+1)a_3]+1}^{[(n+1)a_4]} Y_i + \sum_{i=[(n+1)a_4]+1}^{[(n+1)t_2]} Y_i,
\end{aligned}$$

and

$$\begin{aligned}
S(t_3, a_1, \dots, a_6) &= \sum_{i=1}^{[(n+1)a_1]} Y_i + \sum_{i=[(n+1)a_1]+1}^{[(n+1)a_2]} Y_i, \\
&+ \sum_{i=[(n+1)a_2]+1}^{[(n+1)a_3]} Y_i + \sum_{i=[(n+1)a_3]+1}^{[(n+1)a_4]} Y_i + \sum_{i=[(n+1)a_4]+1}^{[(n+1)a_5]} Y_i \\
&+ \sum_{i=[(n+1)a_5]+1}^{[(n+1)a_6]} Y_i + \sum_{i=[(n+1)a_6]+1}^{[(n+1)t_3]} Y_i.
\end{aligned}$$

This choice of the λ_i 's, imply the following: $a_1 = \lambda_1$, $a_2 = c_2$, $a_3 = \lambda_2$, $a_4 = c_4$, $a_5 = c_5$, and $a_6 = c_6$. Using these values for the a_i 's, we have

$$\begin{aligned}
S(t_1, a_1, a_2) &= \sum_{i=1}^{[(n+1)\lambda_1]} Y_i + \sum_{i=[(n+1)\lambda_1]+1}^{[(n+1)t_1]} Y_i, \\
S(t_2, a_1, a_2, a_3, a_4) &= \sum_{i=1}^{[(n+1)\lambda_1]} Y_i + \sum_{i=[(n+1)\lambda_1]+1}^{[(n+1)\lambda_2]} Y_i + \sum_{i=[(n+1)\lambda_2]+1}^{[(n+1)t_2]} Y_i, \\
S(t_3, a_1, \dots, a_6) &= \sum_{i=1}^{[(n+1)\lambda_1]} Y_i + \sum_{i=[(n+1)\lambda_1]+1}^{[(n+1)\lambda_2]} Y_i + \sum_{i=[(n+1)\lambda_2]+1}^{[(n+1)t_3]} Y_i,
\end{aligned}$$

$$\eta(t_1, a_1, a_2) = \lambda_1 \sigma_1^2 + (t_1 - \lambda_1) \sigma_2^2,$$

$$\eta(t_2, a_1, a_2, a_3, a_4) = \lambda_1 \sigma_1^2 + (\lambda_2 - \lambda_1) \sigma_2^2 + (t_2 - \lambda_2) \sigma_3^2,$$

and

$$\eta(t_3, a_1, \dots, a_6) = \lambda_1 \sigma_1^2 + (\lambda_2 - \lambda_1) \sigma_2^2 + (t_3 - \lambda_2) \sigma_3^2.$$

We combine the above results, as done below, to obtain the limiting function of (5.24).

$$\begin{aligned}
& \sum_{l=1}^2 (t_{l+1} - t_{l-1}) \eta(t_l, a_1, \dots, a_{(l-1)s+1}) \\
&= t_2 \eta(t_l, a_1, a_2) + (t_3 - t_1) \eta(t_2, a_1, a_2, a_3, a_4) - t_2 \eta(t_3, a_1, \dots, a_6) \\
&= t_2 \lambda_1 \sigma_1^2 + t_2 (t_1 - \lambda_1) \sigma_2^2 + (t_3 - t_1) [\lambda_1 \sigma_1^2 + (\lambda_2 - \lambda_1) \sigma_2^2 + (t_2 - \lambda_2) \sigma_3^2] \\
&\quad - t_2 \lambda_1 \sigma_1^2 - t_2 (\lambda_2 - \lambda_1) \sigma_2^2 - t_2 (t_3 - \lambda_2) \sigma_3^2 \\
&= t_2 t_1 (\sigma_2^2 - \sigma_3^2) + t_1 [\lambda_1 (\sigma_2^2 - \sigma_1^2) + \lambda_2 (\sigma_3^2 - \sigma_2^2)] \\
&\quad + t_2 [\lambda_1 (\sigma_2^2 - \sigma_1^2) + \lambda_2 (\sigma_3^2 - \sigma_2^2)] + \lambda_2 (\sigma_2^2 - \sigma_3^2) + \lambda_1 (\sigma_1^2 - \sigma_2^2), \quad (5.26)
\end{aligned}$$

where $\sigma_1^2 = \sigma^2$, $\sigma_2^2 = \tau^2$ and $\sigma_3^2 = \rho^2$. We see that (5.26) corresponds with that of (3.51). With this notation, we are able now to establish following result regarding the limiting function of (5.24).

Theorem 5.3.1 Assume $H_A^{(s)}$ and (5.2) hold, at least one of $\sigma_i^2 \neq \sigma_j^2$, $i \neq j$, $i, j = 1, \dots, s+1$, $0 < \sigma_i^2 < \infty$, $i = 1, \dots, s+1$. Let $\gamma_A^2 := \mathbb{E}Y_i^2 - (\mathbb{E}Y_i)^2$, where $Y_i := (X_i - \mu)^2$, X_i be as given in (5.1), and let γ_A be as defined in Remark 5.3.1. If $k_i^* = [n\lambda_i]$ for $i = 1, \dots, s$ and $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_s < 1$ fixed, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{1}{n^{1/2}} M_n^{(1)}(t_1, \dots, t_s) - m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) \right| \stackrel{a.s.}{=} o(1). \quad (5.27)$$

where

$$\begin{aligned}
& m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) \\
&:= \sum_{l=1}^s (t_{l+1} - t_{l-1}) \eta(t_l, a_1, \dots, a_s, \dots, a_{(l-1)s+1}, \dots, a_{ls}) \\
&\quad - t_s \eta(t_{s+1}, a_1, \dots, a_s, \dots, a_{ss+1}, \dots, a_{ss+s}), \quad (5.28) \\
& 0 < a_1 \leq \dots \leq a_s \leq t_1 < a_{s+1} \leq \dots \leq a_{ss+1} \leq \dots \leq a_{ss+s} < 1,
\end{aligned}$$

where $\eta(t_l, a_1, \dots, a_s, \dots, a_{(l-1)s+1}, \dots, a_{ls})$ and $a_1, \dots, a_{ss+s}$ are defined in (5.18) and (5.25) respectively, and where $t_0 := 0$ and $t_{s+1} := 1$.

Proof.

Consider

$$\begin{aligned}
& \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} \right. \\
& \quad \left. - \sum_{j=1}^s (t_{j+1} - t_{j-1}) \eta(t_1, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js}) \right. \\
& \quad \left. - t_s \eta(t_{s+1}, a_1, \dots, a_s, \dots, a_{ss+1}, \dots, a_{ss+s}) \right| \\
& \leq \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \sum_{j=1}^s \left(\frac{[(n+1)t_{j+1}]}{n} \right. \\
& \quad \left. - \frac{[(n+1)t_{j-1}]}{n} \right) \left| \left(\frac{S(t_j, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js})}{n} \right. \right. \\
& \quad \left. \left. - \eta(t_j, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js}) \right) \right| \\
& + \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \frac{[(n+1)t_s]}{n} \left| \frac{S(t_{s+1}, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{ss+s})}{n} \right. \\
& \quad \left. - \eta(t_{s+1}, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{ss+s}) \right| \\
& + \sup_{0 < t_1 < t_2 < \dots \leq t_s < 1} \left| \sum_{j=1}^s \left(\frac{[(n+1)t_{j+1}]}{n} \right. \right. \\
& \quad \left. \left. - t_{j+1} \right) \eta(t_j, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js}) \right| \\
& + \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \sum_{j=1}^s \left(\frac{[(n+1)t_{j-1}]}{n} \right. \right. \\
& \quad \left. \left. - t_{j-1} \right) \eta(t_j, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js}) \right| \\
& + \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \left(\frac{[(n+1)t_s]}{n} \right. \right. \\
& \quad \left. \left. - t_s \right) \eta(t_{s+1}, a_1, \dots, a_s, \dots, a_{ss+1}, \dots, a_{ss+s}) \right|
\end{aligned}$$

$$\begin{aligned}
&= I_1(n) + I_2(n) + I_3(n) + I_4(n) + I_5(n) \\
&\stackrel{a.s.}{=} o(1),
\end{aligned} \tag{5.29}$$

as $n \rightarrow \infty$.

In verifying (5.29), we show, for $i = 1, 2, 3, 4, 5$, that

$$I_i(n) \stackrel{a.s.}{=} o(1), \tag{5.30}$$

as $n \rightarrow \infty$. When $i = 1$, we have

$$\begin{aligned}
I_1(n) &\leq \sum_{j=1}^s \sup_{0 < t_j < a_{js+1} < 1} \left| \frac{S(t_j, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js})}{n} \right. \\
&\quad \left. - \eta(t_j, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js}) \right| \\
&\stackrel{a.s.}{=} o(1), \text{ as } n \rightarrow \infty,
\end{aligned} \tag{5.31}$$

by Lemma 5.3.1.

Applying the SLLN to each sub-block of the sum in $I_2(n)$, as carried out in (5.17), we conclude that

$$I_2(n) \stackrel{a.s.}{=} o(1), \tag{5.32}$$

as $n \rightarrow \infty$. With regards to $I_3(n)$, $I_4(n)$ and $I_5(n)$, we have

$$I_3(n) = I_4(n) = s \cdot o(1) \cdot O(1) = o(1) \tag{5.33}$$

and

$$I_5(n) = o(1) \cdot O(1) = o(1), \tag{5.34}$$

as $n \rightarrow \infty$. The above results all follow from Lemma 2.4.1 and the fact that the sums contained in $I_3(n)$, $I_4(n)$ and $I_5(n)$ are functions of s not n , and are therefore $O(1)$, as $n \rightarrow \infty$. Equations (5.31), (5.32), (5.33) and (5.34) imply (5.30), and consequently (5.29).

Proposition 5.3.1 $m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) = 0$ if and only if $\sigma_1^2 = \dots = \sigma_{s+1}^2$.

Proof.

As shown in equation (5.18),

$$\begin{aligned}
 & \eta(t_j, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js}) \\
 &= \sum_{i=1}^{(j-1)s+s} (a_i - a_{i-1}) \sigma_{l(a_i)}^2 + (t_j - a_{js}) \sigma_{l(t_j)}^2. \\
 &= a_1 \sigma_{l(a_1)}^2 + (a_2 - a_1) \sigma_{l(a_2)}^2 + \dots \\
 &\quad \dots + (t_j - a_{js}) \sigma_{l(t_j)}^2 \\
 &= a_1 (\sigma_{l(a_1)}^2 - \sigma_{l(a_2)}^2) + \dots \\
 &\quad \dots + a_{js} (\sigma_{l(t_s)}^2 - \sigma_{l(a_{js})}^2) + t_j \sigma_{l(t_j)}^2 \\
 &= t_j \sigma_{l(t_j)}^2, \quad j = 1, \dots, s+1 \quad (\text{since } \sigma_1^2 = \dots = \sigma_{s+1}^2).
 \end{aligned}$$

Upon substitution of $\eta(t_j, a_1, \dots, a_s, \dots, a_{(j-1)s+1}, \dots, a_{js}) = t_j \sigma_{l(t_j)}^2$ into (5.28), we have

$$\begin{aligned}
 m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) &= \sum_{l=1}^s (t_{l+1} - t_{l-1}) t_l \sigma_{l(t_l)}^2 - t_s \sigma_{l(t_{s+1})}^s \\
 &= t_2 t_1 \sigma_{l(t_1)}^2 + (t_3 - t_1) t_2 \sigma_{l(t_2)}^2 + \dots \\
 &\quad \dots + (t_{s+1} - t_{s-1}) t_s \sigma_{l(t_s)}^s - t_s \sigma_{l(t_{s+1})}^2 \\
 &= t_2 t_1 (\sigma_{l(t_1)}^2 - \sigma_{l(t_2)}^2) + \dots \\
 &\quad \dots + t_s t_{s-1} (\sigma_{l(t_{s-1})}^2 - \sigma_{l(t_s)}^2) + t_s (\sigma_{l(t_s)}^2 - \sigma_{l(t_{s+1})}^2) \\
 &= 0 \quad (\text{since } \sigma_1^2 = \dots = \sigma_{s+1}^2).
 \end{aligned}$$

Conversely, assume $m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) = 0$ then

$$\sum_{l=1}^s (t_{l+1} - t_{l-1}) \eta(t_l, a_1, \dots, a_s, \dots, a_{(l-1)s+1}, \dots, a_{ls})$$

$$\begin{aligned}
& -t_s \eta(t_{s+1}, a_1, \dots, a_s, \dots, a_{ss+1}, \dots, a_{ss+s}) \\
& = 0. \quad (5.35)
\end{aligned}$$

This implies

$$\begin{aligned}
& t_2(a_1 \sigma_{l(a_1)}^2 + (a_2 - a_1) \sigma_{l(a_2)}^2 + \dots + (a_s - a_{s-1}) \sigma_{l(a_s)}^2) + t_2(t_1 - a_s) \sigma_{l(t_1)}^2 \\
& + (t_3 - t_1)(a_1 \sigma_{l(a_1)}^2 + (a_2 - a_1) \sigma_{l(a_2)}^2 + \dots + (a_{2s} - a_{2s-1}) \sigma_{l(a_{2s})}^2) + (t_3 - t_1)(t_2 - a_{2s}) \sigma_{l(t_2)}^2 \\
& + (t_4 - t_2)(a_1 \sigma_{l(a_1)}^2 + (a_2 - a_1) \sigma_{l(a_2)}^2 + \dots + (a_{3s} - a_{3s-1}) \sigma_{l(a_{3s})}^2) + (t_4 - t_2)(t_3 - a_{3s}) \sigma_{l(t_3)}^2 \\
& \quad \vdots \\
& \quad \vdots \\
& + (t_s - t_{s-2})(a_1 \sigma_{l(a_1)}^2 + (a_2 - a_1) \sigma_{l(a_2)}^2 + \dots + (a_{(s-2)s+s} - a_{(s-2)s+s-1}) \sigma_{l(a_{(s-2)s+s})}^2) \\
& \quad + (t_s - t_{s-2})(t_{s-1} - a_{(s-2)s+s}) \sigma_{l(t_{s-1})}^2 \\
& + (t_{s+1} - t_{s-1})(a_1 \sigma_{l(a_1)}^2 + (a_2 - a_1) \sigma_{l(a_2)}^2 + \dots + (a_{(s-1)s+s} - a_{(s-1)s+s-1}) \sigma_{l(a_{(s-1)s+s})}^2) \\
& \quad + (t_{s+1} - t_{s-1})(t_s - a_{(s-1)s+s}) \sigma_{l(t_s)}^2 \\
& - t_s(a_1 \sigma_{l(a_1)}^2 + (a_2 - a_1) \sigma_{l(a_2)}^2 + \dots + (a_{ss+s} - a_{ss+s-1}) \sigma_{l(a_{ss+s})}^2) \\
& \quad + (t_{s+1} - a_{ss+s}) \sigma_{l(t_{s+1})}^2 \\
& = 0. \quad (5.36)
\end{aligned}$$

Collecting terms involving t_i and $t_i t_j$ in equation (5.36), we obtain the resulting equation

$$\begin{aligned}
& -t_1[a_1(\sigma_{l(a_1)}^2 - \sigma_{l(a_2)}^2) + \cdots + a_{2s}(\sigma_{l(a_{2s})}^2 - \sigma_{l(t_2)}^2)] - t_2[-a_s(\sigma_{l(a_s)}^2 - \sigma_{l(t_1)}^2) \\
& + a_{s+1}(\sigma_{l(a_{s+1})}^2 - \sigma_{l(a_{s+2})}^2) + \cdots + a_{3s}(\sigma_{l(a_{3s})}^2 - \sigma_{l(t_3)}^2)] - t_3[-a_{2s}(\sigma_{l(a_{2s})}^2 - \sigma_{l(t_2)}^2) \\
& + a_{2s+1}(\sigma_{l(a_{2s+1})}^2 - \sigma_{l(a_{2s+2})}^2) + \cdots + a_{4s}(\sigma_{l(a_{4s})}^2 - \sigma_{l(t_4)}^2)] - t_4[-a_{3s}(\sigma_{l(a_{3s})}^2 - \sigma_{l(t_3)}^2) \\
& + a_{3s+1}(\sigma_{l(a_{3s+1})}^2 - \sigma_{l(a_{3s+2})}^2) + \cdots + a_{5s}(\sigma_{l(a_{5s})}^2 - \sigma_{l(t_5)}^2)] \cdots \\
& \cdots - t_{s-1}[-a_{(s-2)s}(\sigma_{l(a_{(s-2)s+1})}^2 - \sigma_{l(t_{s-2})}^2) \\
& + a_{(s-2)s+1}(\sigma_{l(a_{(s-2)s+1})}^2 - \sigma_{l(a_{(s-2)s+2})}^2) + \cdots + a_{(s-1)s+s}(\sigma_{l(a_{(s-1)s+s})}^2 - \sigma_{l(t_s)}^2)] \\
& - t_s[a_{(s-2)s+s}(\sigma_{l(a_{(s-2)s+s})}^2 - \sigma_{l(t_{s-1})}^2) \\
& + a_{(s-1)s+1}(\sigma_{l(a_{(s-1)s+1})}^2 - \sigma_{l(a_{(s-1)s+2})}^2) + \cdots + a_{ss+s}(\sigma_{l(a_{ss+s})}^2 - \sigma_{l(t_{s+1})}^2)] \\
& + t_{s+1}[a_1(\sigma_{l(a_1)}^2 - \sigma_{l(a_2)}^2) + a_2(\sigma_{l(a_2)}^2 - \sigma_{l(a_3)}^2) + \cdots + a_{ss+s}(\sigma_{l(a_{ss+s})}^2 - \sigma_{l(t_s)}^2)] \\
& + t_2t_1(\sigma_{l(t_1)}^2 - \sigma_{l(t_2)}^2) + t_3t_2(\sigma_{l(t_2)}^2 - \sigma_{l(t_3)}^2) + t_4t_3(\sigma_{l(t_3)}^2 - \sigma_{l(t_4)}^2) \\
& + \cdots + t_st_{s-1}(\sigma_{l(t_{s-1})}^2 - \sigma_{l(t_s)}^2) + t_{s+1}t_s(\sigma_{l(t_s)}^2 - \sigma_{l(a_{ss+s})}^2) \tag{5.37} \\
& = 0.
\end{aligned}$$

We can write the above as $-t_1A_1 - t_2A_2 - \cdots - t_{s+1}A_{s+1} + t_2t_1B_1 + t_3t_2B_2 + \cdots + t_{s+1}t_sB_s = 0$.

Assume not i.e., that is assume $m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) = 0$ but at least one of $\sigma_i^2 \neq \sigma_j^2$, $i \neq j$ for $i, j = 1, \dots, s+1$. For argument sake let $\sigma_1^2 \neq \sigma_2^2$ and $\sigma_3^2 = \cdots = \sigma_{s+1}^2 = \sigma_2^2$. Furthermore set $0 < \lambda_1 < t_1$, and $\lambda_2 = \cdots = \lambda_s = \lambda_1$. These imply $a_1 = \lambda_1$ and $a_2 = c_2, \dots, a_i = c_i, \dots, a_{ss+s} = c_{ss+s}$. In terms of (5.37), we have

$$\begin{aligned}
& -t_1\lambda_1(\sigma_1^2 - \sigma_2^2) + t_{s+1}\lambda_1(\sigma_1^2 - \sigma_2^2) \\
& = -t_1\lambda_1(\sigma_1^2 - \sigma_2^2) + \lambda_1(\sigma_1^2 - \sigma_2^2) \quad (t_{s+1} := 1) \\
& \neq 0
\end{aligned}$$

for any $0 < t_1 < 1$. As this can never be 0 for any possible value of t_1 satisfying

$0 < t_1 < 1$, we have a contradiction.

5.3.2 Population Mean Known

Corollary 5.3.1 *Under the conditions given in Theorem 5.3.1, a test for at most s change-points which is based on large values of the statistic*

$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} |M_n^{(1)}(t_1, \dots, t_s)|$ *has, as $n \rightarrow \infty$, the following consistency property*

$$\mathbb{P}\{H_O^{(s)} \text{ Rejected}; H_A^{(s)} \text{ True}\} \rightarrow 1.$$

Proof.

In principle, define c_α for $0 < \alpha < 1$ via Corollary 5.1.1 and consider

$$\mathbb{P}\{H_O^{(s)} \text{ Rejected}; H_A^{(s)} \text{ True}\} = \mathbb{P}\left\{\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} \left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{\gamma} \right| \geq c_\alpha; H_A^{(s)} \text{ True}\right\}.$$

On account of Theorem 5.1.1 and Proposition 5.3.1, we have, as $n \rightarrow \infty$,

$$\sqrt{n} \sup_{0 < t_1 < \dots < t_s < 1} \left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{\sqrt{n}\gamma} \right| \xrightarrow{\text{a.s.}} \infty,$$

which implies the as claimed consistency.

5.3.3 Population Mean Unknown

Corollary 5.3.2 *Under the conditions given in Theorem 5.3.1, we have, as $n \rightarrow \infty$,*

$$\sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} - m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) \right| \stackrel{\text{a.s.}}{=} o(1).$$

Proof. From Remark 5.1.1, and after dividing by $n^{1/2}$, we obtain

$$\begin{aligned} \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} &= \frac{M_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} \\ &\quad - 2 \sum_{i=1}^s \frac{[(n+1)t_{i+1}] - [(n+1)t_{i-1}]}{n} \sum_{i=1}^{[(n+1)t_i]} \frac{(X_i - \mu)(\bar{X}_n - \mu)}{n} \\ &\quad + 2 \frac{[(n+1)t_s]}{n} \sum_{i=1}^n \frac{(X_i - \mu)(\bar{X}_n - \mu)}{n}. \end{aligned}$$

Consequently, as $n \rightarrow \infty$, we have

$$\begin{aligned}
& \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s) - M_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} \right| \\
& \leq 2 \left| \bar{X}_n - \mu \right| \left\{ \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \sum_{l=1}^s \frac{([(n+1)t_{l+1}] - [(n+1)t_{l-1}])}{n} \left| \sum_{i=1}^{[(n+1)t_l]} \frac{(X_i - \mu)}{n} \right| \right. \\
& \quad \left. + \left| \bar{X}_n - \mu \right| \right\} \\
& \stackrel{\text{a.s.}}{=} o(1) \cdot o(1) + o(1) \cdot o(1) = o(1).
\end{aligned} \tag{5.38}$$

To see the bound in (5.38), the partial sum on right hand side of (5.38) will have to be decomposed along the lines of the partial sum in (5.16). For those partial sums whose limits contain only x_j 's, the SLLN can be applied to each sub-block to obtain the corresponding $o(1)$ a.s., as $n \rightarrow \infty$, property in (5.38). For the only partial sum whose limit contains t_j , we apply the Hartman-Wintner LIL. A similar decomposition must be also undertaken for the term $|\bar{X}_n - \mu|$. This then establishes the corresponding $o(1)$ terms in (5.38), as $n \rightarrow \infty$. With this explanation, via Theorem 5.3.1, we conclude

$$\begin{aligned}
& \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}}(t_1, \dots, t_s) - m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) \right| \\
& \leq \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} \right. \\
& \quad \left. - \frac{M_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} \right| \\
& \quad + \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} \right. \\
& \quad \left. - m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) \right| \\
& \stackrel{\text{a.s.}}{=} o(1) + o(1) = o(1),
\end{aligned}$$

as $n \rightarrow \infty$.

Corollary 5.3.3 *Under the conditions given in Theorem 5.3.1, a test for at most s change-points which is based on large values of the statistic $\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} |\bar{M}_n^{(1)}(t_1, \dots, t_s)|$ has, as $n \rightarrow \infty$, the following consistency property*

$$P\{H_O^{(s)} \text{ Rejected}; H_A^{(s)} \text{ True}\} \rightarrow 1.$$

Proof. This follows from Corollary 5.3.2 and Proposition 5.3.1.

5.3.4 Small Changes in the Population Mean

Corollary 5.3.4 *Assume $H_A^{(s)}$, conditions (5.2), (5.9) hold, at least one of $\sigma_i^2 \neq \sigma_j^2$, $i \neq j$, $i, j = 1, \dots, s+1$, $0 < \sigma_i^2 < \infty$ for $i = 1, \dots, s+1$. Let $\gamma_A^2 := \mathbb{E}Y_{i,n}^2 - (\mathbb{E}Y_{i,n})^2$, where $Y_{i,n} := (X_{i,n} - \mu)^2$, X_i be as given in (5.1), and, mutatis mutandis à la (5.8), write γ_A as in Remark 5.3.1. If $k_i^* = [n\lambda_i]$ for $i = 1, \dots, s$ and $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_s < 1$ are fixed, then, as $n \rightarrow \infty$, we have*

$$\sup_{0 < t_1 < \dots < t_s < 1} \left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{n^{1/2}} - m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) \right| = o_P(1), \quad (5.39)$$

where $M_n^{(1*)}(t_1, \dots, t_s)$ and $m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s)$ are defined in (5.10) and (5.28) respectively.

Proof.

Using the representation of $\bar{M}_n^{(1)}(t_1, \dots, t_s)$ given in Remark 5.1.2, and after dividing by n , we arrive at

$$\begin{aligned} \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} &= \frac{M_n^{(1*)}(t_1, \dots, t_s)}{n^{1/2}} + \frac{D^{(2)}(t_1, \dots, t_s)}{n} + \frac{D^{(3)}(t_1, \dots, t_s)}{n} \\ &\quad + \frac{D^{(4)}(t_1, \dots, t_s)}{n} + \frac{D^{(5)}(t_1, \dots, t_s)}{n}, \end{aligned}$$

with $M_n^{(1*)}(t_1, \dots, t_s)$ as in (5.10). With this representation, we conclude that

$$\begin{aligned}
& \sup_{0 < t_1 < \dots < t_s < 1} \left| \frac{\tilde{M}_n^{(1)}(t_1, \dots, t_s) - M_n^{(1*)}(t_1, \dots, t_s)}{n^{1/2}} \right| \\
& \leq \sum_{i=2}^5 \sup_{0 < t_1 < \dots < t_s < 1} \left| \frac{D^{(i)}(t_1, \dots, t_s)}{n} \right| \\
& = o_P(1), \quad \text{as } n \rightarrow \infty,
\end{aligned} \tag{5.40}$$

where $D^{(i)}(t_1, \dots, t_s)$ for $i = 2, \dots, 5$ are specified in Remark 5.1.2.

In order to see the right hand bound in (5.40), we appeal to (3.63), (3.25), (3.66) and (3.67), as they can be applied here. With this explanation, via Corollary 5.3.2, we conclude

$$\begin{aligned}
& \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{n^{1/2}} - m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) \right| \\
& \leq \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{\tilde{M}_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} - \frac{M_n^{(1*)}(t_1, \dots, t_s)}{n^{1/2}} \right| \\
& \quad + \sup_{0 < t_1 < t_2 < \dots < t_s < 1} \left| \frac{\tilde{M}_n^{(1)}(t_1, \dots, t_s)}{n^{1/2}} - m_{\lambda_1, \dots, \lambda_s}(t_1, \dots, t_s) \right| \\
& = o_P(1) + o(1) = o_P(1),
\end{aligned}$$

as $n \rightarrow \infty$.

Corollary 5.3.5 *Under the conditions given in Corollary 5.3.4, a test for at most s change-points which is based on large values of the statistic*

$\sup_{0 \leq t_1 \leq \dots \leq t_s \leq 1} |M_n^{(1*)}(t_1, \dots, t_s)|$ *has, as $n \rightarrow \infty$, the following consistency property*

$$P\{H_O^{(s)} \text{ Rejected}; H_A^{(s)} \text{ True}\} \rightarrow 1.$$

Proof. This follows from Corollary 5.3.4 and Proposition 5.3.1.

Chapter 6

Multiple Change-points with Weights

6.1 Asymptotic Results Under the Null Hypothesis $H_O^{(s)}$

We extend here the weighted two-time parameter partial sum processes $M_n^{(1)}(t_1, t_2)$ and $\frac{M_n^{(1)}(t_1, t_2)}{\frac{1}{2}(g(t_2)+h(t_1))}$ and their variants (cf. (4.10), (4.32), (4.86) and (4.87)) as well as their corresponding limiting distributions (cf. Theorems 4.1.1, 4.1.2 and 4.1.3 and Corollaries 4.2.1, 4.2.2 and 4.2.3) to the weighted s -time parameter partial sum process. As the proofs of theorems and corollaries to come involve application of arguments similar to those used in the proofs of Theorems 4.1.1, 4.1.2 and 4.1.3, and Corollaries 4.2.1, 4.2.2 and 4.2.3, detailed proofs will not be provided. As in the previous chapters (cf. Chapters 3, 4 and 5), we develop the weighted s -time parameter partial sum process for the case where the population mean is known, unknown and then conclude with small changes in the population means.

Before we begin our development of such s -time parameter partial sum processes, we first prove the following lemma regarding the s -time parameter Gaussian process. The following lemma can be motivated by comments similar to those made in Remark 4.1.1, suitably altered, to apply to the s -time parameter Gaussian process as defined

in Definition 1.0.3, which asserts that it can be expressed as a sum of Brownian bridges, where a Brownian Bridge is defined in Lemma 1.0.3.

Lemma 6.1.1 *The s -time parameter Gaussian process in equation (1.2) can be expressed as a linear combination of Brownian bridges.*

$$\Gamma(t_1, \dots, t_s) = \sum_{l=1}^s (t_{l+1} - t_{l-1}) B(t_l) \quad 0 \leq t_1 \leq t_2 \leq \dots \leq t_s \leq 1 \quad (6.1)$$

where $t_{s+1} := 1$.

Proof.

Equation (1.2) can be rewritten as

$$\begin{aligned} \Gamma(t_1, \dots, t_s) &= (t_2 - t_0)W(t_1) + (t_3 - t_1)W(t_2) + \dots + (t_{s+1} - t_{s-1})W(t_s) - t_s W(1) \\ &= \sum_{l=1}^s t_{l+1} W(t_l) - t_{l-1} W(t_l) \\ &= \sum_{l=1}^s (t_{l+1} W(t_l) - t_{l+1} t_l W(1) + t_l t_{l-1} W(1) - t_{l-1} W(t_l)) \\ &\quad + \sum_{l=1}^s (t_{l+1} - t_{l-1}) t_l W(1) - t_s W(1) \\ &= \sum_{l=1}^s \{(t_{l+1} (W(t_l) - t_l W(1)) - t_{l-1} (W(t_l) - t_l W(1)))\} \\ &= \sum_{l=1}^s (t_{l+1} - t_{l-1}) (W(t_l) - t_l W(1)), \end{aligned}$$

since $\sum_{l=1}^s (t_{l+1} - t_{l-1}) t_l = t_s$.

We introduce its weighted counterparts, with $g_l \in Q$, for $l = 1, \dots, s$,

$$\Gamma^w(t_1, \dots, t_s) = \sum_{l=1}^s (t_{l+1} - t_{l-1}) \frac{(W(t_l) - t_l W(1))}{g_l(t_l)} \quad (6.2)$$

and

$$\Gamma_n^w(t_1, \dots, t_s) = \sum_{l=1}^s (t_{l+1} - t_{l-1}) \frac{(W(nt_l) - t_l W(n))}{n^{1/2} g_l(t_l)}. \quad (6.3)$$

6.1.1 Population Mean Known

As in Chapter 4, Section 4.1.1, and in view of (4.10) and (4.12), we introduce the corresponding s -time parameter partial sum processes.

$$M_n^{(1)}(t_1, \dots, t_s) := n^{-1/2} \begin{cases} \sum_{\ell=1}^s (t_{\ell+1} - t_{\ell-1}) S((n+1)t_\ell) - t_\ell S(n) & 0 \leq t_1 \leq \dots \leq t_s \leq \frac{n}{n+1} \\ \sum_{\ell=1}^{s-1} (t_{\ell+1} - t_{\ell-1}) t_\ell S(n) + (t_{s+1} - t_{s-1})(1 - t_s) S(n) & 0 \leq t_1 \leq t_2 \leq \dots \leq t_{s-1} \leq \frac{n}{n+1} < t_s \leq 1 \\ \vdots & \vdots \\ \sum_{\ell=1}^s (t_{\ell+1} - t_{\ell-1})(1 - t_\ell) S(n) & \frac{n}{n+1} < t_1 \leq \dots \leq t_s \leq 1. \end{cases} \quad (6.4)$$

and the weighted version of (6.4), with $g_l \in Q$, for $s = 1, \dots, s$,

$$M_n^{(1)w}(t_1, \dots, t_s) := n^{-1/2} \begin{cases} \sum_{\ell=1}^s \frac{(t_{\ell+1} - t_{\ell-1})}{g_l(t_l)} (S((n+1)t_\ell) - t_\ell S(n)) & 0 < t_1 \leq t_2 \leq \dots \leq t_s \leq \frac{n}{n+1} \\ \sum_{\ell=1}^{s-1} \frac{(t_{\ell+1} - t_{\ell-1})}{g_l(t_l)} S((n+1)t_\ell) - t_\ell S(n) + (1 - t_{s-1})(1 - t_s) \frac{t_s}{g_s(t_s)} S(n) & 0 < t_1 \leq \dots \leq t_{s-1} \leq \frac{n}{n+1} < t_s < 1 \\ \vdots & \vdots \\ \sum_{\ell=1}^s \frac{(t_{\ell+1} - t_{\ell-1})}{g_l(t_l)} (1 - t_\ell) S(n) & \frac{n}{n+1} < t_1 \leq \dots \leq t_s < 1. \end{cases} \quad (6.5)$$

With Theorem 4.1.1 as a reference, we extend that theorem here to the above weighted s -time parameter partial sum processes.

Theorem 6.1.1 Assume $H_0^{(s)}$, (5.1), (5.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_n); 0 \leq t_1 \leq \dots \leq t_s < 1\}$ such that the following hold.

- a) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for all $c > 0$, and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \left| \frac{M_n^{(1)w}(t_1, \dots, t_s)}{\gamma} - \Gamma_n^w(t_1, \dots, t_s) \right| = o_P(1).$$

- b) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \left| \frac{M_n^{(1)w}(t_1, \dots, t_s)}{\gamma} - \Gamma_n^w(t_1, \dots, t_s) \right| = O_P(1),$$

- c) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} t_1 \leq \dots \leq t_s \leq \frac{n}{n+1}} \left| \frac{M_n^{(1)w}(t_1, \dots, t_s)}{\gamma} \right| \xrightarrow{P} \sup_{0 < t_1 \leq \dots \leq t_s < 1} |\Gamma^w(t_1, \dots, t_s)|$$

where $M_n^{(1)w}(t_1, \dots, t_s)$, $\Gamma_n^w(t_1, \dots, t_s)$, and $\Gamma^w(t_1, \dots, t_s)$ are defined (6.5), (6.3) and (6.2) respectively.

Proof. Repeat the arguments used in the proof of Theorem 4.1.1.

6.1.2 Population Mean Unknown

Let $\bar{Y}_i = (X_i - \bar{X}_n)^2$ $i = 1, \dots, n$, $n \geq 1$, where $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}$. As in (4.32) and (4.33), we define the corresponding s -time partial sum processes.

$$\bar{M}_n^{(1)}(t_1, \dots, t_s) := n^{-1/2} \begin{cases} \sum_{\ell=1}^s (t_{\ell+1} - t_{\ell-1}) \left(\sum_{i=1}^{[(n+1)t_\ell]} \bar{Y}_i - t_\ell \sum_{i=1}^n \bar{Y}_i \right) & 0 \leq t_1 \leq \dots \leq t_s \leq \frac{n}{n+1} \\ \sum_{\ell=1}^{s-1} (t_{\ell+1} - t_{\ell-1}) t_\ell \sum_{i=1}^n \bar{Y}_i + (1 - t_{s-1})(1 - t_s) \sum_{i=1}^n \bar{Y}_i & 0 \leq t_1 \leq t_2 \leq \dots \leq t_{s-1} \leq \frac{n}{n+1} < t_s \leq 1 \\ \vdots & \vdots \\ \sum_{\ell=1}^s (t_{\ell+1} - t_{\ell-1})(1 - t_\ell) \sum_{i=1}^n \bar{Y}_i & \frac{n}{n+1} < t_1 \leq \dots \leq t_s \leq 1. \end{cases} \quad (6.6)$$

and corresponding weighted version, with $g_\ell \in Q$, for $s = 1, \dots, s$,

$$\bar{M}_n^{(1)w}(t_1, \dots, t_s) := n^{-1/2} \left\{ \begin{array}{ll} \sum_{\ell=1}^s \frac{(t_{\ell+1} - t_{\ell-1})}{g_\ell(t_\ell)} \left(\sum_{i=1}^{[(n+1)t_\ell]} \bar{Y}_i - t_\ell \sum_{i=1}^n \bar{Y}_i \right) & 0 < t_1 \leq \dots \leq t_s \leq \frac{n}{n+1} \\ \sum_{\ell=1}^{s-1} \frac{(t_{\ell+1} - t_{\ell-1})t_\ell}{g_\ell(t_\ell)} \sum_{i=1}^n \bar{Y}_i + \frac{(1 - t_{s-1})(1 - t_s)}{g_s(t_s)} \sum_{i=1}^n \bar{Y}_i & 0 < t_1 \leq \dots \leq t_{s-1} \leq \frac{n}{n+1} < t_s < 1 \\ \vdots & \vdots \\ \sum_{\ell=1}^s \frac{(t_{\ell+1} - t_{\ell-1})(1 - t_\ell)}{g_\ell(t_\ell)} \sum_{i=1}^n \bar{Y}_i & \frac{n}{n+1} < t_1 \leq \dots \leq t_s < 1. \end{array} \right. \quad (6.7)$$

With Theorem 4.1.2 as a reference, we extend that theorem here to the above weighted s -time partial sum process.

Theorem 6.1.2 Assume $H_0^{(s)}$, (5.1), (5.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_n); 0 \leq t_1 \leq \dots \leq t_s < 1\}$ such that the following hold.

- a) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for all $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \left| \frac{\bar{M}_n^{(1)w}(t_1, \dots, t_s)}{\gamma} - \Gamma_n^w(t_1, \dots, t_s) \right| = o_P(1).$$

- b) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \left| \frac{\bar{M}_n^{(1)w}(t_1, \dots, t_s)}{\gamma} - \Gamma_n^w(t_1, \dots, t_s) \right| = O_P(1),$$

c) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq \dots \leq t_s \leq \frac{n}{n+1}} \left| \frac{\bar{M}_n^{(1)w}(t_1, \dots, t_s)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq \dots \leq t_s < 1} |\Gamma^w(t_1, \dots, t_s)|$$

where $\bar{M}_n^{(1)w}(t_1, \dots, t_s)$, $\Gamma_n^w(t_1, \dots, t_s)$, and $\Gamma^w(t_1, \dots, t_2)$ are defined (6.7),

(6.3) and (6.2) respectively.

Proof. Repeat the argument used in the proof of Theorem 4.1.2.

6.1.3 Small Changes in the Population Mean

Consider the model specified in (5.8) along with condition (5.9). As in Section 4.1.3, let $Y_{i,n}^* := (X_{i,n} - \mu_{in})^2$, $i = 1, \dots, n$, $n \geq 1$, and substitute $Y_{i,n}^*$ for Y_i in the s -time parameter partial sum process $M_n^{(1)}(t_1, \dots, t_s)$ (cf. (6.4)). As in Chapter 4, Section 4.1.3 equations (4.32) and (4.33), we develop here their s -time parameter partial sum processes counterparts, as defined below.

$$M_n^{(1)*}(t_1, \dots, t_s) := n^{-1/2} \begin{cases} \sum_{\ell=2}^s (t_{\ell+1} - t_{\ell-1}) \left(\sum_{i=1}^{[(n+1)t_s]} Y_{i,n}^* - t_s \sum_{i=1}^n Y_{i,n}^* \right) & 0 \leq t_1 \leq \dots \leq t_s \leq \frac{n}{n+1} \\ \sum_{\ell=1}^{s-1} (t_{\ell+1} - t_{\ell-1}) \left(\sum_{i=1}^{[(n+1)t_\ell]} Y_{i,n}^* - t_\ell \sum_{i=1}^n Y_{i,n}^* \right) + (1 - t_{s-1})(1 - t_s) \sum_{i=1}^n Y_{i,n}^* & 0 \leq t_1 \leq t_2 \leq \dots \leq t_{s-1} \leq \frac{n}{n+1} < t_s \leq 1 \\ \vdots & \vdots \\ \sum_{\ell=1}^s (t_{\ell+1} - t_{\ell-1})(1 - t_\ell) \sum_{i=1}^n Y_{i,n}^* & \frac{n}{n+1} \leq t_1 \leq \dots \leq t_s \leq 1. \end{cases} \quad (6.8)$$

and the corresponding weighted version, with $g_\ell \in Q$, for $s = 1, \dots, s$,

$$M_n^{(1*)w}(t_1, \dots, t_s) :=$$

$$n^{-1/2} \left\{ \begin{array}{ll} \sum_{\ell=1}^s \frac{(t_{\ell+1} - t_{\ell-1})}{g_\ell(t_\ell)} \left(\sum_{i=1}^{[(n+1)t_\ell]} Y_{i,n}^* - t_\ell \sum_{i=1}^n Y_{i,n}^* \right) & 0 < t_1 \leq \dots \leq t_s \leq \frac{n}{n+1} \\ \sum_{\ell=1}^{s-1} \frac{(t_{\ell+1} - t_{\ell-1})}{g_\ell(t_\ell)} \left(\sum_{i=1}^{[(n+1)t_\ell]} Y_{i,n}^* - t_\ell \sum_{i=1}^n Y_{i,n}^* \right) + \frac{(1 - t_{s-1})}{g_s(t_s)} (1 - t_s) \sum_{i=1}^n Y_{i,n}^* & 0 < t_1 \leq \dots \leq t_{s-1} \leq \frac{n}{n+1} < t_s < 1 \\ \vdots & \vdots \\ \sum_{\ell=1}^s \frac{(t_{\ell+1} - t_{\ell-1})}{g_\ell(t_\ell)} (1 - t_\ell) \sum_{i=1}^n Y_{i,n}^* & \frac{n}{n+1} < t_1 \leq \dots \leq t_s < 1. \end{array} \right. \quad (6.9)$$

With Theorem 4.1.3 as a reference, we extend that theorem here to the above weighted s -time parameter partial sum process.

Theorem 6.1.3 Assume $H_0^{(s)}$, (5.2), (5.8), (5.9) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\}$ such that the following hold.

a) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty \forall c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 \leq \dots \leq t_s < 1} \left| \frac{M_n^{(1*)w}(t_1, \dots, t_s)}{\gamma} - \Gamma_n^w(t_1, \dots, t_s) \right| = o_P(1).$$

b) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 < 1} \left| \frac{M_n^{(1*)w}(t_1, \dots, t_s)}{\gamma} - \Gamma_n^w(t_1, \dots, t_s) \right| = O_P(1).$$

c) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq \dots \leq t_s \leq \frac{n}{n+1}} \left| \frac{M_n^{(1*)w}(t_1, \dots, t_s)}{\gamma} \right| \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq \dots \leq t_s < 1} |\Gamma^w(t_1, \dots, t_s)|$$

where $M_n^{(1*)w}(t_1, \dots, t_s)$, $\Gamma_n^w(t_1, \dots, t_s)$ and $\Gamma^w(t_1, \dots, t_s)$ are defined in (6.9), (6.3) and (6.2) respectively.

Proof. Repeat the argument used in the proof of Theorem 4.1.3.

6.2 Weighted Processes & Their Approximations

6.2.1 Population Mean Known

As we did in Chapter 4, Section 4.2, Theorems 6.1.1 - 6.1.3 are used to study the asymptotics of weighted versions of the partial sum processes developed in (6.4), (6.6) and (6.8). In particular, we are interested in exploring the limiting distribution of functionals of the partial sum processes, i.e., functionals like

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \frac{|M_n^{(1)}(t_1, \dots, t_s)|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} \quad (6.10)$$

and

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \frac{|\tilde{M}_n^{(1)}(t_1, \dots, t_s)|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)}. \quad (6.11)$$

Weighted functionals of this nature are more consistent with current research (cf. [5, (2.4)] or [6]). Here as well as throughout this section $g_\ell \in Q$, for $i = 1, \dots, s$.

Corollary 6.2.1 *Assume $H_0^{(s)}$, (5.1), (5.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\}$ such that the following hold.*

a) Let $g_\ell \in Q$, and for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for all $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 \leq \dots \leq t_s < 1} \frac{\left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} = o_P(1).$$

b) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \frac{\left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} = O_P(1),$$

c) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq \dots \leq t_s < \frac{n}{n+1}} \frac{\left| \frac{M_n^{(1)}(t_1, \dots, t_s)}{\gamma} \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq \dots \leq t_s < 1} \frac{\left| \Gamma(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)}.$$

where $M_n^{(1)}(t_1, \dots, t_s)$, $\Gamma_n(t_1, \dots, t_s)$ and $\Gamma(t_1, \dots, t_s)$ are defined in (6.4), Lemma 1.0.5 and (6.1), respectively.

Proof. This follow from Theorem 6.1.1.

6.2.2 Population Mean Unknown

Corollary 6.2.2 Assume $H_0^{(s)}$, (5.1), (5.2) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\}$ such that the following hold.

a) Let $g_\ell(t_\ell) \in Q$. If $I(g_\ell, c) < \infty$ for all $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 \leq \dots \leq t_s} \frac{\left| \frac{\tilde{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} = o_P(1).$$

b) Let $g_\ell(t_\ell) \in Q$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq t_2 \leq \dots \leq t_s} \frac{\left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} = O_P(1),$$

c) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq \dots \leq t_s \leq \frac{n}{n+2}} \frac{\left| \frac{\bar{M}_n^{(1)}(t_1, \dots, t_s)}{\gamma} \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq \dots \leq t_s < 1} \frac{\left| \Gamma(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)}.$$

where $\bar{M}_n^{(1)}(t_1, \dots, t_s)$, $\Gamma_n(t_1, \dots, t_s)$ and $\Gamma(t_1, \dots, t_s)$ are defined in (6.6), Lemma 1.0.5 and (6.1) respectively.

Proof. This follows from Theorem 6.1.2.

6.2.3 Small Changes in the Population Mean

Consider the model specified in (5.8) and satisfying condition (5.9). Let $M_n^{(1*)}(t_1, \dots, t_s)$ be as defined in (6.8).

Corollary 6.2.3 Assume $H_0^{(s)}$, (5.1), (5.8), (5.9) hold and $\infty > \gamma > 0$. Then we can define a sequence of Gaussian processes $\{\Gamma_n(t_1, \dots, t_s); 0 \leq t_1 \leq \dots \leq t_s \leq 1\}$ such that the following hold.

a) Let $g_\ell \in Q$. If $I(g_\ell, c) < \infty$ for all $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \frac{\left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} = o_P(1).$$

b) Let $g_\ell(t_\ell) \in Q$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{0 < t_1 \leq \dots \leq t_s < 1} \frac{\left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} - \Gamma_n(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} = O_P(1),$$

c) Let $g_\ell \in Q$, for $\ell = 1, \dots, s$. If $I(g_\ell, c) < \infty$ for some $c > 0$ and for $\ell = 1, \dots, s$, then, as $n \rightarrow \infty$,

$$\sup_{\frac{1}{n+1} \leq t_1 \leq \dots \leq t_s \leq \frac{n}{n+1}} \frac{\left| \frac{M_n^{(1*)}(t_1, \dots, t_s)}{\gamma} \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)} \xrightarrow{\mathcal{D}} \sup_{0 < t_1 \leq \dots \leq t_s < 1} \frac{\left| \Gamma(t_1, \dots, t_s) \right|}{\frac{1}{s} \sum_{\ell=1}^s g_\ell(t_\ell)}.$$

where $M_n^{(1*)}(t_1, \dots, t_s)$, $\Gamma_n(t_1, \dots, t_s)$ and $\Gamma(t_1, \dots, t_s)$ are defined in (6.8), Lemma 1.0.5 and (6.1) respectively.

Proof. This follow from Theorem 6.1.3.

Chapter 7

Simulation of Weighted Functionals

As noted already in the Abstract, this chapter is based on joint research undertaken with Markus Orasch and the author of this thesis (cf. [25] manuscript in preparation).

In Chapter 2, Section 2.2, we discussed $\hat{T}_n$ (cf. (2.1)) to test the AMOC in mean null hypothesis which is restated here for convenient reference (cf. (2.3), (2.4)).

$H_O^{(1)} : X_1, \dots, X_n$ are independent identically distributed random variables with $\mathbb{E}X_i = \mu$ and variance $\sigma^2 = \mathbb{E}(X_1 - \mu)^2 < \infty$

against the *at most one change in the mean* alternative

$H_A^{(1)} : X_1, \dots, X_n$ are independent random variables and there is an integer $k^* = k(n)$ $1 \leq k^* < n$, such that $\mu_1 = \mathbb{E}X_1 = \dots = \mathbb{E}X_{k^*} \neq \mu_2 = \mathbb{E}X_{k^*+1} = \dots = \mathbb{E}X_n$, and variance $\sigma^2 = \mathbb{E}(X_1 - \mu)^2$.

We assume throughout this chapter that the variance $\text{Var}(X_1) = \sigma^2 = 1$. It was mentioned previously (cf. Chapter 2, Section 2.2) that $\hat{T}_n$ (cf. (2.1)) is noted for being insensitive to detecting a change in mean occurring on the tails, i.e., for values of k near 1 or n (cf., e.g., Csörgő and Horváth [5], [6], [7]). As suggested in the

latter works (cf. Chapter 2, Section 2.2) and carried out in some detail in Chapter 4, introducing weight functions may provide one such remedy to this problem.

Thus far little if any research, either theoretical or empirical in nature, has been done to investigate the extent to which weight functions can be used to correct the mentioned deficiency of $\hat{T}_n$ when used in testing $H_O^{(1)}$. What follows now is one of the first attempts to quantify this conjecture regarding the use of weight functions in change-point detection (cf. Csörgő and Horváth [5]). To begin this investigation, we introduce a number of statistics with the purpose of studying their empirical power for testing the null hypothesis $H_O^{(1)}$. These statistics will be formed as weighted functionals of (2.2).

As is frequently the case in statistics and hence also non-parametric statistics, many limiting distributions are not readily available in tabulated format. Consequently, we must numerically simulate the limiting distributions of our test statistics in order to investigate their empirical power. After simulation, we collect the results, in tabulated format, and place them in the Appendix of this thesis. As a by-product of this simulation, we are also able to tabulate the simulated distribution of the argmax estimator of k^* (cf. (7.1)). It is hoped that both results will be of benefit to applied researchers.

To establish the asymptotic results that follow regarding our family of weighted statistics, we choose to parametrize this family in terms of $M_n^{(1)}(t, t)$ (cf. (4.6)) as in B. Szyszkowicz [34, Theorem 2.1] and appeal to statement (ii) of Theorem D in support of our claim. However, in what follows, for simulation purposes, we refer here as well as throughout to appropriately weighted versions of the family of statistics as specified in (2.2).

Accordingly then (cf. also Gombay, Horváth and Hušková [17, (1.9)]), using

weight functions as in (ii) of Theorem D on page 69, and setting $k = [(n+1)t]$, we estimate k^* via (cf. (2.2))

$$\hat{k}^* = \operatorname{argmax} \frac{|M_n(\frac{k}{n+1}, \frac{k}{n+1})|}{q(\frac{k}{n})}. \quad (7.1)$$

In the case of testing for at most one change in the mean, and as was discussed in Chapter 2 (cf. (2.2)), it seems natural to compare the mean of the first k observations (before the change) to the mean of the last $n-k$ observations (after the change), where $1 \leq k < n$. This leads one to consider functionals of the family of statistics (cf. M. Csörgő and L. Horváth [5, Section 2], and B. Szyszkowicz [34, Remark 2.2])

$$\frac{1}{k} \sum_{i=1}^k X_i - \frac{1}{n-k} \sum_{i=k+1}^n X_i = \frac{n}{k(n-k)} \left(S(k) - \frac{k}{n} S(n) \right), \quad 1 \leq k < n,$$

where $S(k) := \sum_{i=1}^k X_i$. Rewriting the above equation, we have

$$\frac{n}{k(n-k)} \left(S(k) - \frac{k}{n} S(n) \right) = \frac{1}{n \frac{k}{n} (1 - \frac{k}{n})} \left(S(k) - \frac{k}{n} S(n) \right), \quad 1 \leq k < n.$$

This suggests using $\frac{k}{n}(1 - \frac{k}{n})$ as a desirable weight function. Multiplying the above sequence of stochastic processes by $n^{1/2}$ and taking the maximum, we have a sequence of statistics, namely

$$T_n^{(1)} := \max_{1 \leq k < n} \frac{|S(k) - \frac{k}{n} S(n)|}{\frac{k}{n}(1 - \frac{k}{n}) n^{1/2}}. \quad (7.2)$$

Unfortunately, this sequence converges in probability to ∞ as $n \rightarrow \infty$, even if the no change in mean null hypothesis were true.

In response, a statistician might suggest to renormalize $\frac{1}{\sqrt{n}}(S(k) - \frac{k}{n} S(n))$ by its standard deviation, which is $\sqrt{\frac{k}{n}(1 - \frac{k}{n})}$ for $1 \leq k < n$. Even for this suggestion, the sequence of statistics, under $H_O^{(1)}$,

$$T_n^{(2)} := \max_{1 \leq k < n} \frac{|S(k) - \frac{k}{n} S(n)|}{\left(\frac{k}{n}(1 - \frac{k}{n})\right)^{1/2} n^{1/2}} \quad (7.3)$$

converges in probability to ∞ as $n \rightarrow \infty$. Gombay, Horváth and Hušková [17, Theorem 1.2] show that, when $T_n^{(2)}$ is properly scaled and centered, it has a non-degenerate asymptotic distribution.

Using Donsker's theorem on $D[0,1]$, we have, as $n \rightarrow \infty$, that under $H_O^{(1)}$, $(S([nt]) - \frac{[nt]}{n}S(n))/n^{1/2}$ behaves like a Brownian Bridge $\{B(t); 0 \leq t \leq 1\}$. Hence

$$T_n^{(3)} := \max_{1 \leq k \leq n} \frac{|S(k) - \frac{k}{n}S(n)|}{n^{1/2}} \xrightarrow{\mathcal{D}} \sup_{0 \leq t \leq 1} |B(t)|. \quad (7.4)$$

The limiting distribution in (7.4) is the well-known limiting distribution of the two-sided Kolmogorov-Smirnov statistic (cf. Kolmogorov (1933a, English version 1933b)), and it has been widely tabulated.

Using a weight function (cf. (7.5)) suggested by Csörgő and Horváth [5], then under H_O , as $n \rightarrow \infty$, we have

$$\begin{aligned} T_n^{(4)} &:= \max_{1 \leq k \leq n} \frac{|S(k) - \frac{k}{n}S(n)|}{\left(\frac{k}{n}(1 - \frac{k}{n}) \log \log \frac{1}{\frac{k}{n}(1 - \frac{k}{n})}\right)^{1/2} n^{1/2}} \\ &\xrightarrow{\mathcal{D}} \sup_{0 < t < 1} \frac{|B(t)|}{\left(t(1 - t) \log \log \frac{1}{t(1 - t)}\right)^{1/2}}, \end{aligned} \quad (7.5)$$

where again $\{B(t); 0 \leq t \leq 1\}$ is a Brownian bridge (cf. Lemma 1.0.3). We note that convergence in (7.5) follows from statement (ii) of Theorem D on assuming the existence of only two moments for X_1 .

The weighted Brownian bridge from (7.5) was first studied by M. Csörgő, S. Csörgő, L. Horváth and D. M. Mason [8, Remark 4.2.4], while its distribution has been tabulated by Eastwood and Eastwood [15]. The motivation for the above mentioned statistics is for them to be used to construct tests with power larger than their unweighted counterparts.

7.1 Constructing Test Statistics

With a view to constructing statistics that are more sensitive to a change in the mean on the tails and in the middle as well, the three weight functions introduced thus far (cf. (7.2), (7.3) and (7.5)) will be combined to produce more statistics. Before doing this however, we explore the following relationship between two of the weight functions as this provides useful insight into what follows.

Whenever k is chosen to satisfy the inequality $na < k < nb$, where n refers to the sample size and a and b are specified below, we have

$$\left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \log \log \frac{1}{\frac{k}{n} \left(1 - \frac{k}{n} \right)} \right)^{1/2} < \left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \right)^{1/2}. \quad (7.6)$$

Namely, if one solves the resulting quadratic equation for its roots, a and b , their values are found to be

$$(a, b) := \left(\frac{1 - \sqrt{1 - 4e^{-e}}}{2}, \frac{1 + \sqrt{1 - 4e^{-e}}}{2} \right) = (0.071033 \dots, 0.928966 \dots),$$

then (7.6) holds true. Otherwise, when k is chosen to satisfy $1 \leq k \leq na$ or $nb \leq k < n$, the opposite of (7.6) is true. This result implies the following relationship

$$\frac{|S(k) - \frac{k}{n}S(n)|}{\left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \log \log \frac{1}{\frac{k}{n} \left(1 - \frac{k}{n} \right)} \right)^{1/2} n^{1/2}} \geq \frac{|S(k) - \frac{k}{n}S(n)|}{\left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \right)^{1/2} n^{1/2}}, \quad (7.7)$$

for $na < k < nb$. Furthermore, we note that for each k , $1 \leq k < n$,

$$\min \left(\frac{|S(k) - \frac{k}{n}S(n)|}{\left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \log \log \frac{1}{\frac{k}{n} \left(1 - \frac{k}{n} \right)} \right)^{1/2} n^{1/2}}, \frac{|S(k) - \frac{k}{n}S(n)|}{\left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \right)^{1/2} n^{1/2}} \right) \geq \frac{|S(k) - \frac{k}{n}S(n)|}{n^{1/2}}$$

and

$$\max \left(\frac{|S(k) - \frac{k}{n}S(n)|}{\left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \log \log \frac{1}{\frac{k}{n} \left(1 - \frac{k}{n} \right)} \right)^{1/2} n^{1/2}}, \frac{|S(k) - \frac{k}{n}S(n)|}{\left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \right)^{1/2} n^{1/2}} \right) \leq \frac{|S(k) - \frac{k}{n}S(n)|}{\frac{k}{n} \left(1 - \frac{k}{n} \right) n^{1/2}}.$$

Using the above inequalities, we make the following observations; $T_n^{(2)}$ and $T_n^{(4)}$ are both larger than $T_n^{(3)}$ (cf. (7.4)) and smaller than $T_n^{(1)}$ (cf. (7.2)). Using the

fact that $T_n^{(4)} = O_P(1)$, as $n \rightarrow \infty$, along with the above inequalities, the following new statistics are developed.

As a result of the above considerations, we have the following statistics

$$T_n^{(5)} := \max_{1 \leq k < n} \begin{cases} \frac{|S(k) - \frac{k}{n}S(n)|}{n^{1/2}}, & \text{if } k \in (na, nb) \\ \frac{|S(k) - \frac{k}{n}S(n)|}{(\frac{k}{n}(1 - \frac{k}{n}) \log \log \frac{1}{\frac{k}{n}(1 - \frac{k}{n})})^{1/2} n^{1/2}}, & \text{if } k \in [1, na] \cup [nb, n] \end{cases} \quad (7.8)$$

$$T_n^{(6)} := \max_{1 \leq k < n} \begin{cases} \frac{|S(k) - \frac{k}{n}S(n)|}{(\frac{k}{n}(1 - \frac{k}{n}))^{1/2} n^{1/2}}, & \text{if } k \in (na, nb) \\ \frac{|S(k) - \frac{k}{n}S(n)|}{(\frac{k}{n}(1 - \frac{k}{n}) \log \log \frac{1}{\frac{k}{n}(1 - \frac{k}{n})})^{1/2} n^{1/2}}, & \text{if } k \in [1, na] \cup [nb, n] \end{cases} \quad (7.9)$$

$$T_n^{(7)} := \max_{1 \leq k < n} \begin{cases} \frac{|S(k) - \frac{k}{n}S(n)|}{\frac{k}{n}(1 - \frac{k}{n}) n^{1/2}}, & \text{if } k \in (na, nb) \\ \frac{|S(k) - \frac{k}{n}S(n)|}{(\frac{k}{n}(1 - \frac{k}{n}) \log \log \frac{1}{\frac{k}{n}(1 - \frac{k}{n})})^{1/2} n^{1/2}}, & \text{if } k \in [1, na] \cup [nb, n] \end{cases} \quad (7.10)$$

$$T_n^{(8)} := \max_{1 \leq k < n} \begin{cases} \frac{|S(k) - \frac{k}{n}S(n)|}{n^{1/2}}, & \text{if } k \in (na, nb) \\ \frac{|S(k) - \frac{k}{n}S(n)|}{(\frac{k}{n}(1 - \frac{k}{n}))^{1/2} n^{1/2}}, & \text{if } k \in [1, na] \cup [nb, n] \end{cases} \quad (7.11)$$

and

$$T_n^{(9)} := \max_{1 \leq k < n} \begin{cases} \frac{|S(k) - \frac{k}{n}S(n)|}{n^{1/2}}, & \text{if } k \in (na, nb), \\ \frac{|S(k) - \frac{k}{n}S(n)|}{\frac{k}{n}(1 - \frac{k}{n}) n^{1/2}}, & \text{if } k \in [1, na] \cup [nb, n]. \end{cases} \quad (7.12)$$

As the following statistics combine $T_n^{(1)}$ through $T_n^{(4)}$, one would expect them to be more sensitive to a change in the mean that occurs on the tails as well as in the middle. These statistics, excluding $T_n^{(8)}$ and $T_n^{(9)}$, have limiting distributions whose nature will be explored below.

It follows from statement (ii) of Theorem D that, on assuming the existence of only two moments for X_1 under $H_O^{(1)}$, the above statistics have the following limiting distributions as $n \rightarrow \infty$:

$$T_n^{(5)} \xrightarrow{D} \sup_{0 < t < 1} \frac{|B(t)|}{q(t)}, \quad (7.13)$$

where

$$\frac{|B(t)|}{q(t)} := \begin{cases} \frac{|B(t)|}{(t(1-t) \log \log \frac{1}{t(1-t)})^{1/2}}, & \text{if } t \in (0, a] \cup [b, 1) \\ |B(t)| & \text{if } t \in (a, b) \end{cases}$$

$$T_n^{(6)} \xrightarrow{\mathcal{D}} \sup_{0 < t < 1} \frac{|B(t)|}{q(t)}, \quad (7.14)$$

where

$$\frac{|B(t)|}{q(t)} := \begin{cases} \frac{|B(t)|}{(t(1-t) \log \log \frac{1}{t(1-t)})^{1/2}}, & \text{if } t \in (0, a] \cup [b, 1) \\ \frac{|B(t)|}{(t(1-t))^{1/2}}, & \text{if } t \in (a, b) \end{cases}$$

and lastly

$$T_n^{(7)} \xrightarrow{\mathcal{D}} \sup_{0 < t < 1} \frac{|B(t)|}{q(t)}, \quad (7.15)$$

where

$$\frac{|B(t)|}{q(t)} := \begin{cases} \frac{|B(t)|}{(t(1-t) \log \log \frac{1}{t(1-t)})^{1/2}}, & \text{if } t \in (0, a] \cup [b, 1) \\ \frac{|B(t)|}{(t(1-t))^{1/2}}, & \text{if } t \in (a, b). \end{cases}$$

As most of the above distributions have yet to be recorded in any format, we must now tabulate the distribution of $\sup_{0 < t < 1} \frac{|B(t)|}{q(t)}$ in (7.13), (7.14) and (7.15). Statistics $T_n^{(1)}$, $T_n^{(2)}$, $T_n^{(8)}$ and $T_n^{(9)}$ tend to ∞ in probability, as $n \rightarrow \infty$, and consequently have no limiting distribution to simulate. It is expected that from the statistics $T_n^{(i)}$, $1 \leq i \leq 9$, either $T_n^{(5)}$ or $T_n^{(6)}$ should produce a test with the highest empirical power.

7.2 Pseudocode

To tabulate the CDF of the limiting distribution of the above constructed statistics $T^{(3)}, \dots, T^{(7)}$, respectively given by equations (7.4), (7.5), (7.13), (7.14) and (7.15), we must tabulate the CDF of the following random variables,

$$T^{(j)} = \sup_{0 < t < 1} \frac{|B(t)|}{q(t)} \text{ for } j = 3, 4, 5, 6, 7,$$

where j refers to the CDF of the j th statistic.

For those $T_n^{(j)}$ s with a limiting distribution, we have, for sufficiently large n , that $T_n^{(j)}$ has approximately the same CDF as $\sup_{0 < t < 1} \frac{|B(t)|}{q(t)}$, with $q(t)$ appropriately

specified. This will prove to be useful in the simulation of the CDF of $\sup_{0 < t < 1} \frac{|B(t)|}{q(t)}$. To this end, we now propose an algorithm. Let N represent the sample size and M the number of Monte Carlo iterations.

- 1) Generate N independent standard normal random variables (i.e. $N(0, 1)$).
- 2) On the m^{th} iteration, calculate

$$\frac{|S_m(k) - \frac{k}{N} S_m(N)|}{q(\frac{k}{N}) N^{1/2}},$$

for $k = 1, \dots, N - 1$.

- 3) On the m^{th} iteration, record the value of

$$T_{N,m}^{(j)} = \max_{1 \leq k < N} \frac{|S_m(k) - \frac{k}{N} S_m(N)|}{q(\frac{k}{N}) N^{1/2}}$$

and

$$\hat{k}^* = \operatorname{argmax} \frac{|S_m(k) - \frac{k}{N} S_m(N)|}{q(\frac{k}{N}) N^{1/2}}.$$

- 4) Repeat 1) through 3) M times.

With the recorded values obtained in 3), tabulate the empirical CDF of $T_N^{(j)}$, $\hat{k}^*$ and $\frac{\hat{k}^*}{N}$. Most of the tabulated CDFs are located in the Appendix of this thesis.

To show that the simulated cumulative distribution functions do, in the limit, converge to their respective theoretical cumulative distribution functions, we define

$$G_{N,M}(x) := \frac{\sum_{m=1}^M I(T_{N,m}^{(j)} \leq x)}{M},$$

$$G_N(x) := \mathbb{P}\{T_N^{(j)} \leq x\}$$

and

$$G(x) := \mathbb{P}\left\{\sup_{0 < t < 1} \frac{|B(t)|}{q(t)} \leq x\right\},$$

where $I(\cdot)$ is the indicator function.

Proposition 7.2.1 *We have that $\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} G_{N,M}(x) = G(x)$, for each fixed $x \in \mathbb{R}$, where N and M represent the sample size and number of Monte Carlo iterations respectively, and $\mathbb{E}|X_1|^2 < \infty$.*

Proof. By the triangle inequality, we have

$$|G(x) - G_{N,M}(x)| \leq |G_N(x) - G_{N,M}(x)| + |G(x) - G_N(x)|.$$

Moreover,

$$\lim_{M \rightarrow \infty} |G_N(x) - G_{N,M}(x)| = 0, \quad \text{a.s.,}$$

by the Glivenko-Cantelli theorem and

$$\lim_{N \rightarrow \infty} |G_N(x) - G(x)| = 0, \quad x \in [0, \infty)$$

by statement (ii) of Theorem D.

7.3 Tabulated Values

Tables 7.1, 7.2, 7.3 and 7.4, tabulate the simulated CDF of the r.v.'s $\sup_{0 < t < 1} |B(t)|$ and

$\sup_{0 < t < 1} \frac{|B(t)|}{\left(t(1-t) \log \log \frac{1}{t(1-t)}\right)^{1/2}}$ for sample sizes $N = 10,000$ and $N = 100,000$, with $M = 15,000$ Monte Carlo iterations. We note that the results produced by the simulation correspond closely to those produced by Eastwood and Eastwood [15, Section 4.3, Tabulation of Weighted Functionals] which indicates that our simulation does indeed behave as expected and in accordance with Proposition 7.2.1. Moreover, when the simulated results are compared to those results tabulated by Smirnov [29], they are found to correspond very closely.

The Appendix contains the tabulated results from the simulation for the remaining random variables. Again, the CDFs were simulated using two sample sizes:

$N = 10,000$ and $N = 100,000$, for each random variable. We also include an estimate of the "distribution" of those statistics for which no limiting CDF's exist and note that this is possible only do to the nature of simulations where an infinite sample size cannot be entertained. In addition to these simulated CDFs, the simulated CDFs of $\hat{k}^*$ and $\frac{\hat{k}^*}{N}$ are also provided, but only for those statistics for which a limiting CDF exists, and only when $N = 100,000$.

Table 7.1:

N = 10,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P}\{\sup_{0 \leq t \leq 1} B(t) \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.435	0.01	0.720	0.34	0.943	0.67
0.466	0.02	0.727	0.35	0.951	0.68
0.481	0.03	0.733	0.36	0.959	0.69
0.500	0.04	0.740	0.37	0.968	0.70
0.514	0.05	0.746	0.38	0.976	0.71
0.525	0.06	0.752	0.39	0.984	0.72
0.536	0.07	0.758	0.40	0.993	0.73
0.545	0.08	0.764	0.41	1.002	0.74
0.554	0.09	0.770	0.42	1.011	0.75
0.563	0.10	0.777	0.43	1.021	0.76
0.571	0.11	0.783	0.44	1.031	0.77
0.579	0.12	0.789	0.45	1.042	0.78
0.587	0.13	0.795	0.46	1.055	0.79
0.595	0.14	0.801	0.47	1.067	0.80
0.602	0.15	0.808	0.48	1.080	0.81
0.610	0.16	0.815	0.49	1.093	0.82
0.617	0.17	0.822	0.50	1.107	0.83
0.624	0.18	0.828	0.51	1.121	0.84
0.630	0.19	0.835	0.52	1.135	0.85
0.637	0.20	0.842	0.53	1.150	0.86
0.643	0.21	0.849	0.54	1.164	0.87
0.649	0.22	0.856	0.55	1.181	0.88
0.655	0.23	0.862	0.56	1.200	0.89
0.661	0.24	0.870	0.57	1.220	0.90
0.667	0.25	0.876	0.58	1.247	0.91
0.673	0.26	0.883	0.59	1.270	0.92
0.679	0.27	0.890	0.60	1.292	0.93
0.685	0.28	0.898	0.61	1.321	0.94
0.692	0.29	0.905	0.62	1.353	0.95
0.698	0.30	0.913	0.63	1.400	0.96
0.703	0.31	0.920	0.64	1.446	0.97
0.709	0.32	0.927	0.65	1.510	0.98
0.715	0.33	0.935	0.66	1.618	0.99

Table 7.2:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P}\{\sup_{0 \leq t \leq 1} B(t) \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.436	0.01	0.729	0.34	0.948	0.67
0.464	0.02	0.735	0.35	0.957	0.68
0.484	0.03	0.740	0.36	0.966	0.69
0.504	0.04	0.746	0.37	0.975	0.70
0.518	0.05	0.753	0.38	0.983	0.71
0.530	0.06	0.759	0.39	0.992	0.72
0.542	0.07	0.765	0.40	1.000	0.73
0.550	0.08	0.771	0.41	1.009	0.74
0.558	0.09	0.778	0.42	1.017	0.75
0.567	0.10	0.784	0.43	1.027	0.76
0.575	0.11	0.790	0.44	1.037	0.77
0.583	0.12	0.796	0.45	1.048	0.78
0.590	0.13	0.802	0.46	1.060	0.79
0.598	0.14	0.808	0.47	1.073	0.80
0.605	0.15	0.814	0.48	1.086	0.81
0.612	0.16	0.821	0.49	1.101	0.82
0.619	0.17	0.827	0.50	1.113	0.83
0.626	0.18	0.833	0.51	1.127	0.84
0.634	0.19	0.840	0.52	1.143	0.85
0.641	0.20	0.847	0.53	1.156	0.86
0.648	0.21	0.854	0.54	1.172	0.87
0.655	0.22	0.860	0.55	1.191	0.88
0.661	0.23	0.867	0.56	1.212	0.89
0.668	0.24	0.874	0.57	1.232	0.90
0.674	0.25	0.881	0.58	1.253	0.91
0.681	0.26	0.888	0.59	1.275	0.92
0.687	0.27	0.895	0.60	1.302	0.93
0.693	0.28	0.902	0.61	1.335	0.94
0.699	0.29	0.910	0.62	1.366	0.95
0.705	0.30	0.918	0.63	1.408	0.96
0.711	0.31	0.925	0.64	1.459	0.97
0.717	0.32	0.932	0.65	1.528	0.98
0.723	0.33	0.939	0.66	1.640	0.99

Table 7.3: Brownian Bridge

N = 10,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \sup_{0 < t < 1} \frac{ B(t) }{\sqrt{t(1-t) \log \log(t(1-t))^{-1}}} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.60	0.01	2.50	0.34	3.19	0.67
1.70	0.02	2.51	0.35	3.21	0.68
1.76	0.03	2.53	0.36	3.24	0.69
1.80	0.04	2.55	0.37	3.26	0.70
1.85	0.05	2.57	0.38	3.29	0.71
1.88	0.06	2.59	0.39	3.32	0.72
1.91	0.07	2.61	0.40	3.36	0.73
1.94	0.08	2.63	0.41	3.39	0.74
1.97	0.09	2.64	0.42	3.41	0.75
2.00	0.10	2.66	0.43	3.44	0.76
2.03	0.11	2.68	0.44	3.48	0.77
2.05	0.12	2.70	0.45	3.52	0.78
2.07	0.13	2.72	0.46	3.56	0.79
2.10	0.14	2.74	0.47	3.60	0.80
2.12	0.15	2.76	0.48	3.64	0.81
2.15	0.16	2.78	0.49	3.68	0.82
2.17	0.17	2.80	0.50	3.72	0.83
2.19	0.18	2.82	0.51	3.77	0.84
2.21	0.19	2.84	0.52	3.82	0.85
2.24	0.20	2.87	0.53	3.87	0.86
2.26	0.21	2.89	0.54	3.92	0.87
2.28	0.22	2.91	0.55	3.98	0.88
2.29	0.23	2.93	0.56	4.05	0.89
2.31	0.24	2.95	0.57	4.12	0.90
2.33	0.25	2.98	0.58	4.19	0.91
2.35	0.26	3.00	0.59	4.25	0.92
2.37	0.27	3.02	0.60	4.35	0.93
2.39	0.28	3.05	0.61	4.45	0.94
2.41	0.29	3.07	0.62	4.57	0.95
2.42	0.30	3.09	0.63	4.72	0.96
2.44	0.31	3.11	0.64	4.89	0.97
2.46	0.32	3.13	0.65	5.16	0.98
2.48	0.33	3.16	0.66	5.53	0.99

Table 7.4: Brownian Bridge

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \sup_{0 < t < 1} \frac{ B(t) }{\sqrt{t(1-t) \log \log(t(1-t))^{-1}}} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.64	0.01	2.51	0.34	3.21	0.67
1.74	0.02	2.53	0.35	3.23	0.68
1.79	0.03	2.54	0.36	3.26	0.69
1.84	0.04	2.56	0.37	3.29	0.70
1.89	0.05	2.58	0.38	3.31	0.71
1.92	0.06	2.60	0.39	3.34	0.72
1.95	0.07	2.62	0.40	3.37	0.73
1.98	0.08	2.64	0.41	3.40	0.74
2.00	0.09	2.66	0.42	3.43	0.75
2.03	0.10	2.68	0.43	3.46	0.76
2.05	0.11	2.70	0.44	3.50	0.77
2.08	0.12	2.72	0.45	3.53	0.78
2.10	0.13	2.73	0.46	3.57	0.79
2.12	0.14	2.75	0.47	3.61	0.80
2.15	0.15	2.77	0.48	3.65	0.81
2.17	0.16	2.79	0.49	3.70	0.82
2.19	0.17	2.81	0.50	3.74	0.83
2.22	0.18	2.83	0.51	3.80	0.84
2.24	0.19	2.85	0.52	3.85	0.85
2.26	0.20	2.87	0.53	3.91	0.86
2.28	0.21	2.89	0.54	3.97	0.87
2.29	0.22	2.91	0.55	4.03	0.88
2.31	0.23	2.94	0.56	4.09	0.89
2.33	0.24	2.96	0.57	4.16	0.90
2.35	0.25	2.98	0.58	4.24	0.91
2.37	0.26	3.01	0.59	4.32	0.92
2.38	0.27	3.03	0.60	4.42	0.93
2.40	0.28	3.06	0.61	4.52	0.94
2.42	0.29	3.08	0.62	4.62	0.95
2.44	0.30	3.11	0.63	4.75	0.96
2.46	0.31	3.13	0.64	4.96	0.97
2.47	0.32	3.16	0.65	5.17	0.98
2.49	0.33	3.18	0.66	5.64	0.99

7.4 The Empirical Power Function for Change in Mean

7.4.1 Simulation

Consider the non-parametric two-sample problem as described in the first paragraph of Chapter 2, Section 2.2 in which $X_1, \dots, X_{k^*}$ and $X_{k^*+1}, \dots, X_n$ are independent random samples with CDFs $F(x)$ and $F(x - \Delta)$, respectively. We wish to test the hypothesis

$$\begin{array}{ll}
 H_O : & X_k \text{ has CDF } F(x) \quad \text{for } k = 1, \dots, n \\
 \text{versus} & \\
 H_A : & X_k \text{ has CDF } F(x) \quad \text{for } k = 1, \dots, k^* \\
 & X_k \text{ has CDF } F(x - \Delta) \quad \text{for } k = k^* + 1, \dots, n
 \end{array} \tag{7.16}$$

where $\Delta > 0$.

To estimate the power of tests based on each of the nine statistics, n random variables were generated from seven distributions. Δ assumed one of three possible values: $\Delta = 0.5, 0.75$ and 1 , while $k^* + 1 = [0.5n]$ for changes in the middle and $k^* + 1 = [0.95n]$ for changes on the tail. We note that, due to the symmetry of the change in mean statistics, it does not matter whether the change in the mean occurs for $k^* + 1 = [0.95n]$ or $k^* + 1 = [0.05n]$. The distributions used for the purpose of this simulation, and listed in the order they appear in the tables, are Chi-squared, Binomial, Normal, Student's - t, Exponential, Uniform and Logistic.

After generating n random variables from each distribution, those r.v.'s corresponding to values of $k \leq k^*$, (i.e. no change in the mean) are to have mean zero and variance one. While those r.v.'s for which $k \geq k^* + 1$ (i.e. after the change in mean) are to have mean set to one of the three possible values of Δ and a variance of 1. The power function for the test of hypothesis (7.16) was estimated by repeatedly generating random variables (r.v.'s) of size n from each distribution, adjusting

the r.v.'s as mentioned above, calculating the test statistic, and then determining whether the value exceeded the appropriate critical level which, for purposes of this simulation, was set at the 5% level. To ensure an accurate estimate of the power function was obtained, the above process was repeated 500 times.

To ensure that our conclusions are not dependent upon the sample size, the power of the tests was calculated for five different sample sizes. For the purpose of this simulation, the sample size n assumed each of the values; 100, 250, 500, 750 and 1,000. Before the results of the simulation are discussed, two important definitions will be given that will clarify what is meant by the tail as well as the middle of the sample.

Definition 7.4.1 The middle of the sample will refer to those values of the sample for which the index k is an element of the open interval $(n \cdot a, n \cdot b)$, where n refers to the sample size and a and b are defined in (7.7).

Definition 7.4.2 The tail of the sample will refer to those values of the sample for which the index k is an element of the interval $[1, n \cdot a] \cup [n \cdot b, n]$, where n refers to the sample size and a and b are defined in (7.7).

Tables 7.5 through 7.7 calculate the empirical power for a change in the mean that occurs in the middle for values of $\Delta = 0.5, 0.75$ and 1, while Tables 7.8 through 7.9 calculate the empirical power for a change in the mean that occurs on the tail, again using values of $\Delta = 0.5, 0.75$ and 1, While the last table, Table 7.10, estimates the significance level of each test.

Before any comparative analysis is completed, we perform some simple consistency checks. Specifically, as the sample size increases from $n = 100$ to $n = 1,000$, one would expect the empirical power of the tests to increase. Indeed, for all statistics the power increases. As theory suggests, statistics $T_n^{(1)}$, $T_n^{(8)}$ and $T_n^{(9)}$ approach infinity, in probability, as their sample sizes approach infinity. Consequently, any test based on them would have both a significance level and power of 1. In the case

of the above three statistics, the empirical powers (cf. Tables 7.5 - 7.9) of these three tests do approach 1, while the significance levels for $T_n^{(1)}$ and $T_n^{(9)}$ are substantially larger than the 0.05 level (cf. 7.10). Theory also suggests that $T_n^{(3)}$ should be sensitive to changes in the mean that occur in the middle of the sample. From tables 7.5 - 7.7, we see that $T_n^{(3)}$ is very sensitive to a change in the mean on the middle of the sample for all distributions but the Logistic and only when $N = 100$ (i.e., empirical power of 0.182, Table 7.5).

7.4.2 Empirical Power of Nine Test Statistics for Change on Middle

Table 7.5: Empirical Power: Middle

Change on Middle									
Change in Mean; $N = 100, M = 500, \Delta = 0.5$									
Distribution	T1	T2	T3	T4	T5	T6	T7	T8	T9
Chi-sqrd	0.026	0.304	0.566	0.590	0.056	0.438	0.122	0.076	0.026
Binomial	0.000	0.268	0.588	0.602	0.026	0.432	0.086	0.030	0.009
Normal	0.006	0.344	0.618	0.636	0.040	0.464	0.110	0.060	0.006
Student's t	0.030	0.280	0.590	0.606	0.062	0.420	0.114	0.074	0.003
Exponential	0.060	0.300	0.558	0.570	0.076	0.444	0.124	0.106	0.006
Uniform	0.000	0.290	0.580	0.600	0.048	0.440	0.110	0.050	0.000
Logistic	0.038	0.054	0.182	0.196	0.034	0.106	0.038	0.070	0.038
N = 250									
Chi-sqrd	0.142	0.924	0.986	0.980	0.310	0.964	0.674	0.166	0.142
Binomial	0.000	0.916	0.986	0.988	0.360	0.958	0.656	0.154	0.000
Normal	0.150	0.944	0.992	0.992	0.310	0.980	0.662	0.156	0.152
Student's t	0.178	0.936	0.990	0.992	0.310	0.968	0.654	0.214	0.178
Exponential	0.164	0.938	0.976	0.980	0.342	0.964	0.688	0.182	0.164
Uniform	0.138	0.940	0.994	0.996	0.304	0.972	0.636	0.140	0.132
Logistic	0.196	0.328	0.612	0.638	0.068	0.470	0.154	0.112	0.196
N = 500									
Chi-sqrd	0.454	0.982	0.998	0.990	0.610	0.990	0.880	0.580	0.454
Binomial	0.384	0.994	1.000	1.000	0.636	1.000	0.880	0.596	0.384
Normal	0.494	0.990	1.000	1.000	0.660	1.000	0.912	0.598	0.494
Student's t	0.446	0.990	1.000	1.000	0.604	0.996	0.880	0.580	0.446
Exponential	0.292	0.982	1.000	1.000	0.636	0.998	0.872	0.580	0.292
Uniform	0.530	0.988	0.998	1.000	0.638	0.998	0.894	0.530	0.530
Logistic	0.402	0.498	0.758	0.782	0.100	0.630	0.224	0.136	0.402
N = 750									
Chi-sqrd	0.794	1.000	1.000	1.000	0.960	1.000	0.994	0.994	0.794
Binomial	0.766	1.000	1.000	1.000	0.924	1.000	0.990	0.984	0.766
Normal	0.760	1.000	1.000	1.000	0.918	1.000	0.986	0.994	0.760
Student's t	0.750	1.000	1.000	1.000	0.926	1.000	0.986	0.986	0.750
Exponential	0.630	0.998	1.000	1.000	0.910	1.000	0.990	0.986	0.630
Uniform	0.852	1.000	1.000	1.000	0.938	1.000	0.992	0.986	0.850
Logistic	0.682	0.726	0.926	0.928	0.158	0.830	0.400	0.286	0.682
N = 1,000									
Chi-sqrd	0.756	1.000	1.000	1.000	0.990	1.000	0.998	0.994	0.756
Binomial	0.732	1.000	1.000	1.000	0.992	1.000	1.000	0.996	0.732
Normal	0.768	1.000	1.000	1.000	0.986	1.000	1.000	0.990	0.768
Student's t	0.738	1.000	1.000	1.000	0.990	1.000	1.000	0.992	0.738
Exponential	0.706	1.000	1.000	1.000	0.992	1.000	1.000	0.990	0.704
Uniform	0.834	1.000	1.000	1.000	0.996	1.000	1.000	0.994	0.840
Logistic	0.698	0.862	0.966	0.968	0.226	0.932	0.536	0.310	0.698

Table 7.6: Empirical Power: Middle

Change on Middle									
Change in Mean: $N = 100, M = 500, \Delta = 0.75$									
Distribution	T1	T2	T3	T4	T5	T6	T7	T8	T9
Chi-sqrd	0.006	0.708	0.902	0.908	0.128	0.824	0.330	0.130	0.006
Binomial	0.000	0.722	0.910	0.914	0.132	0.840	0.344	0.112	0.000
Normal	0.018	0.672	0.888	0.900	0.120	0.808	0.296	0.148	0.019
Student's t	0.046	0.726	0.908	0.926	0.128	0.834	0.346	0.132	0.046
Exponential	0.040	0.690	0.896	0.908	0.150	0.810	0.304	0.150	0.040
Uniform	0.000	0.692	0.898	0.918	0.134	0.814	0.314	0.128	0.000
Logistic	0.028	0.144	0.414	0.444	0.056	0.268	0.078	0.070	0.028
$N = 250$									
Chi-sqrd	0.232	1.000	1.000	1.000	0.952	1.000	0.998	0.520	0.230
Binomial	0.068	1.000	1.000	1.000	0.938	1.000	0.998	0.496	0.067
Normal	0.250	1.000	1.000	1.000	0.934	1.000	0.994	0.494	0.250
Student's t	0.246	1.000	1.000	1.000	0.956	1.000	1.000	0.464	0.245
Exponential	0.230	1.000	1.000	1.000	0.946	1.000	0.998	0.486	0.225
Uniform	0.218	1.000	1.000	1.000	0.946	1.000	1.000	0.470	0.216
Logistic	0.188	0.732	0.920	0.938	0.160	0.866	0.382	0.196	0.185
$N = 500$									
Chi-sqrd	0.584	1.000	1.000	1.000	0.996	1.000	1.000	0.840	0.584
Binomial	0.542	1.000	1.000	1.000	1.000	1.000	1.000	0.828	0.541
Normal	0.590	1.000	1.000	1.000	1.000	1.000	1.000	0.884	0.592
Student's t	0.596	1.000	1.000	1.000	0.996	1.000	1.000	0.838	0.597
Exponential	0.490	1.000	1.000	1.000	0.998	1.000	1.000	0.838	0.492
Uniform	0.584	1.000	1.000	1.000	0.998	1.000	1.000	0.868	0.582
Logistic	0.474	0.928	0.990	0.994	0.320	0.980	0.634	0.344	0.473
$N = 750$									
Chi-sqrd	0.742	1.000	1.000	1.000	1.000	1.000	1.000	0.982	0.741
Binomial	0.756	1.000	1.000	1.000	1.000	1.000	1.000	0.968	0.753
Normal	0.826	1.000	1.000	1.000	1.000	1.000	1.000	0.972	0.823
Student's t	0.774	1.000	1.000	1.000	1.000	1.000	1.000	0.966	0.771
Exponential	0.666	1.000	1.000	1.000	1.000	1.000	1.000	0.962	0.666
Uniform	0.790	1.000	1.000	1.000	1.000	1.000	1.000	0.978	0.790
Logistic	0.622	0.994	0.998	0.998	0.676	0.998	0.982	0.480	0.659
$N = 1,000$									
Chi-sqrd	0.886	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.886
Binomial	0.998	1.000	1.000	1.000	1.000	1.000	1.000	0.994	0.997
Normal	0.904	1.000	1.000	1.000	1.000	1.000	1.000	0.994	0.901
Student's t	0.890	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.892
Exponential	0.800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.801
Uniform	0.950	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.950
Logistic	0.746	0.998	1.000	1.000	0.908	1.000	0.994	0.664	0.748

With regards to the remaining test statistics $T_n^{(4)}$ through to $T_n^{(7)}$ (cf. (7.5), (7.8), (7.9), (7.10)), Table 7.5 suggests that all, but $T_n^{(5)}$ and $T_n^{(7)}$, are sensitive to a change in the mean that occurs on the middle. $T_n^{(5)}$ and $T_n^{(7)}$ are insensitive to a change in mean that occurs on the middle but only for values of $\Delta = 0.5$ and 0.75 , and when $N = 100$ (cf. Tables 7.5 and 7.6). $T_n^{(7)}$ is more sensitive than $T_n^{(5)}$ for values of $\Delta = 0.5, 0.75, 1$, and when $N = 100$ (cf. Table 7.7). Nevertheless, both $T_n^{(5)}$ and $T_n^{(7)}$ are sensitive to a change in the mean that occurs on the middle when $N \geq 250$ and for all values of Δ considered in the simulation. The empirical power also indicates, unexpectedly, that $T_n^{(4)}$ and $T_n^{(6)}$ are just as sensitive as $T_n^{(3)}$ to a change in the mean that occurs in the middle (cf. Tables 7.5 - 7.7), again for all values of Δ considered in the simulation.

Table 7.7: Empirical Power: Middle

Change on Middle									
Change in Mean: $N = 100, M=500, \Delta = 1$									
Distribution	T1	T2	T3	T4	T5	T6	T7	T8	T9
Chi-sqrd	0.026	0.956	0.998	0.998	0.398	0.986	0.732	0.366	0.026
Binomial	0.000	0.958	0.994	0.996	0.440	0.980	0.758	0.382	0.000
Normal	0.020	0.960	0.998	0.998	0.382	0.994	0.740	0.352	0.020
Student's t	0.028	0.948	0.990	0.990	0.374	0.990	0.686	0.366	0.026
Exponential	0.042	0.962	0.998	0.998	0.384	0.988	0.736	0.372	0.042
Uniform	0.000	0.964	1.000	1.000	0.398	0.988	0.718	0.338	0.000
Logistic	0.020	0.340	0.634	0.652	0.086	0.506	0.160	0.088	0.020
$N = 250$									
Chi-sqrd	0.208	1.000	1.000	1.000	0.994	1.000	1.000	0.994	0.208
Binomial	0.000	1.000	1.000	1.000	0.994	1.000	1.000	0.990	0.000
Normal	0.262	1.000	1.000	1.000	0.986	1.000	1.000	0.990	0.262
Student's t	0.244	1.000	1.000	1.000	0.994	1.000	1.000	0.992	0.244
Exponential	0.158	1.000	1.000	1.000	0.992	1.000	1.000	0.994	0.158
Uniform	0.288	1.000	1.000	1.000	0.990	1.000	1.000	0.984	0.288
Logistic	0.178	0.872	0.970	0.978	0.232	0.934	0.552	0.234	0.178
$N = 500$									
Chi-sqrd	0.544	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.542
Binomial	0.774	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.778
Normal	0.550	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.550
Student's t	0.550	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.550
Exponential	0.554	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.554
Uniform	0.594	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.594
Logistic	0.442	1.000	1.000	1.000	0.820	1.000	0.960	1.000	0.422
$N = 750$									
Chi-sqrd	0.804	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.740
Binomial	0.828	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.742
Normal	0.820	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.746
Student's t	0.828	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.750
Exponential	0.800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.736
Uniform	0.846	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.774
Logistic	0.624	1.000	1.000	1.000	0.980	1.000	1.000	0.978	0.624
$N = 1,000$									
Chi-sqrd	0.992	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.844
Binomial	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.854
Normal	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.866
Student's t	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.840
Exponential	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.832
Uniform	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.874
Logistic	0.732	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.732

7.4.3 Estimated Power of Nine Test Statistics for Change on Tail

Tables 7.8 and 7.9 record values of the empirical power for a change in the mean that occurs on the tail, again for values of $\Delta = 0.5, 0.75$ and 1 . Each of the three tables tabulate the empirical power for $N = 100, 250, 500, 750$ and $1,000$. For such changes, it will be shown that $T_n^{(5)}$ and $T_n^{(6)}$ are most sensitive when compared to the other statistics.

Theory indicates that $T_n^{(3)}$ is not sensitive to changes on the tail and, as a consequence, one would expect the empirical power function for tests based on this statistic to be low. Table 7.8 with $\Delta = 0.5$ confirms that tests based on $T_n^{(3)}$ do in fact have low empirical power, regardless of the underlying distribution and sample size. Even for values of $\Delta = 0.75$ or 1 and for $N \leq 500$, the empirical power is low

(cf. Tables 7.9).

Table 7.8: Empirical Power: Tail

Change on Tail									
Change in Mean: $N = 100, M=500, \Delta = 0.5$									
Distribution	T1	T2	T3	T4	T5	T6	T7	T8	T9
Chi-sqrd	0.032	0.036	0.048	0.040	0.072	0.072	0.066	0.128	0.032
Binomial	0.000	0.014	0.034	0.050	0.046	0.026	0.036	0.070	0.000
Normal	0.022	0.036	0.058	0.054	0.062	0.048	0.060	0.098	0.022
Student's t	0.024	0.044	0.058	0.048	0.082	0.072	0.068	0.086	0.024
Exponential	0.052	0.088	0.050	0.062	0.130	0.134	0.090	0.138	0.052
Uniform	0.000	0.022	0.056	0.064	0.050	0.056	0.054	0.068	0.000
Logistic	0.028	0.006	0.036	0.036	0.016	0.026	0.026	0.060	0.028
$N = 250$									
Chi-sqrd	0.180	0.108	0.060	0.062	0.142	0.148	0.158	0.196	0.180
Binomial	0.000	0.094	0.048	0.050	0.176	0.138	0.178	0.200	0.000
Normal	0.154	0.072	0.054	0.056	0.154	0.148	0.172	0.216	0.154
Student's t	0.190	0.100	0.060	0.056	0.190	0.152	0.178	0.240	0.190
Exponential	0.190	0.144	0.056	0.068	0.208	0.182	0.198	0.276	0.190
Uniform	0.128	0.080	0.060	0.068	0.174	0.136	0.160	0.224	0.128
Logistic	0.172	0.042	0.072	0.068	0.088	0.082	0.074	0.104	0.172
$N = 500$									
Chi-sqrd	0.468	0.294	0.092	0.110	0.460	0.370	0.410	0.496	0.468
Binomial	0.458	0.298	0.116	0.124	0.452	0.370	0.388	0.520	0.458
Normal	0.498	0.270	0.112	0.122	0.420	0.330	0.390	0.508	0.498
Student's t	0.456	0.240	0.098	0.104	0.414	0.310	0.360	0.490	0.456
Exponential	0.364	0.294	0.096	0.132	0.414	0.350	0.354	0.500	0.364
Uniform	0.544	0.258	0.096	0.092	0.444	0.380	0.382	0.500	0.544
Logistic	0.430	0.080	0.044	0.044	0.122	0.100	0.114	0.202	0.430
$N = 750$									
Chi-sqrd	0.674	0.564	0.244	0.288	0.758	0.698	0.780	0.648	0.674
Binomial	0.634	0.592	0.214	0.268	0.762	0.698	0.762	0.684	0.634
Normal	0.672	0.558	0.222	0.264	0.758	0.668	0.772	0.692	0.672
Student's t	0.642	0.584	0.226	0.256	0.782	0.706	0.812	0.682	0.642
Exponential	0.490	0.566	0.202	0.278	0.758	0.668	0.780	0.648	0.490
Uniform	0.730	0.554	0.232	0.280	0.766	0.680	0.766	0.690	0.730
Logistic	0.602	0.124	0.068	0.078	0.232	0.182	0.258	0.250	0.602
$N = 1,000$									
Chi-sqrd	0.828	0.622	0.234	0.292	0.804	0.716	0.716	0.836	0.828
Binomial	1.000	0.640	0.212	0.264	0.782	0.700	0.698	0.820	1.000
Normal	0.806	0.588	0.534	0.782	0.996	0.992	0.984	0.844	0.804
Student's t	0.778	0.622	0.190	0.244	0.810	0.686	0.686	0.846	0.778
Exponential	0.658	0.640	0.194	0.266	0.794	0.724	0.724	0.798	0.656
Uniform	0.858	0.582	0.190	0.238	0.770	0.664	0.670	0.802	0.801
Logistic	0.730	0.160	0.082	0.082	0.290	0.226	0.256	0.318	0.730

$T_n^{(4)}$, unexpectedly, also has low empirical power, as can be confirmed for $\Delta = 0.5$ and $N \leq 1,000$ (cf. Table 7.8) and $\Delta = .75$ and $N \leq 500$ (cf. Table 7.9). Its empirical power does improve for values of $\Delta = .75$ and 1, and for $N > 500$ (cf. Table 7.9).

Statistic $T_n^{(5)}$, a combination of statistics $T_n^{(3)}$ and $T_n^{(4)}$, should be more sensitive than $T_n^{(4)}$. Tables 7.8 and 7.9 confirm that the empirical power of $T_n^{(5)}$ is larger than $T_n^{(4)}$ and indeed larger than the remaining statistics. Statistic $T_n^{(6)}$, a combination of $T_n^{(2)}$ and $T_n^{(4)}$, is only marginally less sensitive than $T_n^{(5)}$ as Tables 7.8 and 7.9 indicate. $T_n^{(5)}$ does appear more sensitive than $T_n^{(6)}$ for a change on the tail when $\Delta = .5$ and for $N \geq 500$, otherwise their empirical power's are similar.

Table 7.9: Empirical Power: Tail

Change on Tail									
Change in Mean: $N = 100, M=500, \Delta = 0.75$									
Distribution	T1	T2	T3	T4	T5	T6	T7	T8	T9
Chi-sqrd	0.040	0.104	0.064	0.070	0.152	0.158	0.118	0.182	0.050
Binomial	0.000	0.056	0.056	0.058	0.170	0.134	0.110	0.178	0.000
Normal	0.016	0.064	0.074	0.064	0.158	0.128	0.120	0.164	0.016
Student's t	0.022	0.074	0.058	0.054	0.130	0.110	0.098	0.160	0.022
Exponential	0.072	0.108	0.054	0.068	0.184	0.138	0.112	0.238	0.072
Uniform	0.000	0.080	0.078	0.074	0.152	0.132	0.108	0.154	0.000
Logistic	0.016	0.030	0.048	0.050	0.052	0.058	0.062	0.088	0.016
N = 250									
Chi-sqrd	0.232	0.310	0.112	0.128	0.480	0.400	0.468	0.520	0.232
Binomial	0.068	0.304	0.108	0.122	0.466	0.404	0.456	0.496	0.068
Normal	0.025	0.324	0.112	0.120	0.466	0.400	0.442	0.494	0.250
Student's t	0.246	0.266	0.108	0.108	0.448	0.354	0.408	0.464	0.246
Exponential	0.230	0.330	0.126	0.154	0.454	0.420	0.434	0.486	0.230
Uniform	0.218	0.288	0.108	0.120	0.486	0.384	0.456	0.470	0.218
Logistic	0.188	0.084	0.046	0.050	0.158	0.098	0.138	0.196	0.188
N = 500									
Chi-sqrd	0.584	0.898	0.352	0.538	0.972	0.922	0.908	0.840	0.584
Binomial	0.542	0.902	0.358	0.550	0.956	0.930	0.890	0.828	0.543
Normal	0.592	0.892	0.358	0.554	0.964	0.932	0.878	0.884	0.590
Student's t	0.596	0.896	0.336	0.530	0.960	0.918	0.874	0.838	0.596
Exponential	0.490	0.878	0.368	0.494	0.946	0.916	0.894	0.838	0.490
Uniform	0.584	0.886	0.356	0.512	0.962	0.936	0.906	0.868	0.584
Logistic	0.474	0.276	0.118	0.130	0.442	0.360	0.374	0.344	0.474
N = 750									
Chi-sqrd	0.742	0.892	0.388	0.560	0.956	0.924	0.894	0.982	0.742
Binomial	0.756	0.880	0.336	0.512	0.944	0.902	0.866	0.968	0.756
Normal	0.828	0.920	0.334	0.508	0.972	0.942	0.910	0.972	0.828
Student's t	0.774	0.884	0.312	0.532	0.962	0.918	0.868	0.968	0.774
Exponential	0.666	0.902	0.342	0.530	0.966	0.938	0.918	0.962	0.666
Uniform	0.790	0.864	0.402	0.554	0.948	0.908	0.868	0.978	0.790
Logistic	0.622	0.274	0.126	0.128	0.450	0.340	0.384	0.480	0.622
N = 1,000									
Chi-sqrd	0.886	0.974	0.542	0.774	0.994	0.982	0.972	0.998	0.886
Binomial	0.996	0.970	0.514	0.756	0.992	0.982	0.966	0.994	0.998
Normal	0.904	0.982	0.516	0.752	0.998	0.994	0.970	0.994	0.904
Student's t	0.890	0.972	0.526	0.760	0.998	0.984	0.960	0.996	0.890
Exponential	0.800	0.980	0.518	0.766	0.996	0.996	0.976	1.000	0.800
Uniform	0.950	0.976	0.532	0.776	0.992	0.984	0.966	0.998	0.950
Logistic	0.746	0.418	0.122	0.146	0.638	0.506	0.520	0.664	0.746
Change in Mean: $N = 100, M=500, \Delta = 1$									
Distribution	T1	T2	T3	T4	T5	T6	T7	T8	T9
Chi-sqrd	0.050	0.178	0.098	0.102	0.290	0.262	0.228	0.298	0.050
Binomial	0.000	0.152	0.084	0.084	0.308	0.238	0.212	0.306	0.000
Normal	0.034	0.162	0.086	0.088	0.296	0.220	0.218	0.324	0.036
Student's t	0.046	0.138	0.068	0.070	0.282	0.214	0.198	0.306	0.044
Exponential	0.102	0.196	0.082	0.102	0.282	0.252	0.212	0.320	0.103
Uniform	0.002	0.160	0.082	0.080	0.286	0.228	0.170	0.306	0.004
Logistic	0.034	0.056	0.062	0.066	0.102	0.074	0.072	0.108	0.036
N = 250									
Chi-sqrd	0.370	0.628	0.190	0.262	0.770	0.716	0.606	0.790	0.360
Binomial	0.532	0.610	0.208	0.278	0.778	0.704	0.600	0.794	0.532
Normal	0.316	0.612	0.186	0.224	0.776	0.702	0.610	0.774	0.316
Student's t	0.340	0.612	0.208	0.274	0.800	0.714	0.600	0.792	0.340
Exponential	0.324	0.582	0.188	0.254	0.746	0.688	0.620	0.750	0.324
Uniform	0.290	0.606	0.194	0.220	0.768	0.702	0.600	0.728	0.290
Logistic	0.190	0.140	0.074	0.076	0.260	0.262	0.600	0.272	0.187
N = 500									
Chi-sqrd	0.748	0.946	0.408	0.638	0.988	0.978	0.952	0.980	0.746
Binomial	0.714	0.954	0.444	0.670	0.990	0.970	0.960	0.990	0.714
Normal	0.730	0.958	0.412	0.630	0.990	0.972	0.964	0.988	0.713
Student's t	0.740	0.968	0.438	0.644	0.998	0.980	0.972	0.988	0.715
Exponential	0.652	0.944	0.476	0.662	0.984	0.962	0.962	0.994	0.651
Uniform	0.754	0.946	0.438	0.698	0.990	0.972	0.954	0.992	0.754
Logistic	0.490	0.338	0.124	0.146	0.524	0.424	0.438	0.570	0.499
N = 750									
Chi-sqrd	0.748	1.000	0.740	0.932	1.000	1.000	1.000	0.984	0.748
Binomial	0.710	0.998	0.726	0.940	1.000	0.998	1.000	0.992	0.710
Normal	0.746	1.000	0.752	0.940	1.000	1.000	0.994	0.992	0.746
Student's t	0.734	0.998	0.754	0.930	0.998	0.998	0.998	0.980	0.734
Exponential	0.644	0.998	0.714	0.926	1.000	1.000	0.998	0.994	0.644
Uniform	0.724	0.994	0.742	0.944	0.998	0.996	0.994	0.990	0.724
Logistic	0.470	0.570	0.200	0.248	0.782	0.672	0.678	0.578	0.470
N = 1,000									
Chi-sqrd	0.980	1.000	0.902	0.982	1.000	1.000	1.000	1.000	0.980
Binomial	1.000	1.000	0.918	0.990	1.000	1.000	1.000	1.000	0.999
Normal	0.986	1.000	0.920	0.988	1.000	1.000	0.998	1.000	0.985
Student's t	0.982	1.000	0.908	0.986	1.000	1.000	0.998	1.000	0.983
Exponential	0.974	1.000	0.900	0.998	1.000	1.000	1.000	1.000	0.974
Uniform	0.990	1.000	0.924	0.990	1.000	1.000	1.000	1.000	0.999
Logistic	0.806	0.724	0.234	0.326	0.882	0.794	0.768	0.894	0.804

7.4.4 Estimated Alpha Level of Nine Test Statistics

Table 7.10 estimates the significance level for each of the nine test statistics, and is used to confirm the accuracy of our simulation. The significance level used throughout was the standard 5 percent level. For most of the test statistics, $T_n^{(2)}$ through to $T_n^{(7)}$, the estimated significance level was less than the 5 percent level. However, for the Student's-t distribution, all statistics, regardless of the values for N and Δ , had an estimated significance level equal to or larger than 5 percent. It ranged from a low of 0.046 to a high of 0.12. This does suggest that the change in mean statistics have a larger tendency, on average, to falsely reject H_0 for distributions with large tails, than would be expected at the five percent level.

Table 7.10: Empirical Power: No Change

No Change									
N = 100									
Distribution	T1	T2	T3	T4	T5	T6	T7	T8	T9
Chi-sqrd	0.026	0.016	0.018	0.024	0.024	0.026	0.026	0.030	0.026
Binomial	0.000	0.008	0.038	0.044	0.020	0.014	0.000	0.002	0.000
Normal	0.014	0.012	0.032	0.028	0.024	0.026	0.032	0.034	0.014
Student's t	0.014	0.046	0.062	0.066	0.066	0.082	0.058	0.038	0.014
Exponential	0.056	0.048	0.034	0.038	0.062	0.062	0.036	0.082	0.056
Uniform	0.000	0.006	0.028	0.032	0.004	0.020	0.016	0.006	0.000
Logistic	0.026	0.020	0.042	0.042	0.034	0.038	0.028	0.052	0.280
N = 250									
Chi-sqrd	0.144	0.020	0.048	0.054	0.026	0.056	0.040	0.038	0.144
Binomial	0.000	0.004	0.028	0.026	0.018	0.026	0.044	0.016	0.000
Normal	0.132	0.014	0.030	0.032	0.022	0.032	0.034	0.040	0.132
Student's t	0.158	0.068	0.088	0.090	0.070	0.108	0.108	0.054	0.158
Exponential	0.148	0.096	0.046	0.050	0.090	0.090	0.056	0.126	0.148
Uniform	0.006	0.008	0.048	0.038	0.018	0.046	0.028	0.018	0.006
Logistic	0.138	0.030	0.028	0.038	0.034	0.036	0.044	0.072	0.138
N = 500									
Chi-sqrd	0.414	0.030	0.044	0.040	0.038	0.040	0.034	0.068	0.414
Binomial	0.000	0.016	0.040	0.038	0.016	0.042	0.042	0.024	0.000
Normal	0.414	0.018	0.046	0.040	0.028	0.042	0.048	0.060	0.414
Student's t	0.422	0.076	0.086	0.082	0.076	0.092	0.080	0.082	0.422
Exponential	0.228	0.064	0.046	0.046	0.056	0.068	0.042	0.092	0.288
Uniform	0.534	0.012	0.042	0.038	0.032	0.032	0.046	0.050	0.532
Logistic	0.372	0.020	0.046	0.044	0.044	0.046	0.038	0.072	0.372
N = 750									
Chi-sqrd	0.604	0.032	0.048	0.050	0.042	0.050	0.070	0.076	0.604
Binomial	0.274	0.014	0.054	0.058	0.026	0.028	0.040	0.040	0.274
Normal	0.598	0.036	0.044	0.042	0.046	0.058	0.048	0.066	0.598
Student's t	0.604	0.064	0.078	0.076	0.088	0.084	0.074	0.076	0.604
Exponential	0.398	0.068	0.048	0.054	0.070	0.064	0.040	0.122	0.398
Uniform	0.678	0.012	0.042	0.044	0.030	0.036	0.040	0.048	0.678
Logistic	0.548	0.044	0.054	0.054	0.046	0.064	0.068	0.100	0.594
N = 1,000									
Chi-sqrd	0.628	0.028	0.036	0.038	0.034	0.042	0.038	0.088	0.629
Binomial	0.268	0.014	0.054	0.046	0.030	0.042	0.052	0.040	0.268
Normal	0.592	0.032	0.050	0.052	0.054	0.058	0.056	0.072	0.592
Student's t	0.572	0.086	0.094	0.100	0.110	0.118	0.124	0.074	0.572
Exponential	0.338	0.048	0.042	0.050	0.054	0.048	0.048	0.140	0.338
Uniform	0.714	0.022	0.052	0.056	0.032	0.036	0.038	0.048	0.714
Logistic	0.608	0.022	0.036	0.034	0.046	0.038	0.054	0.100	0.603

7.5 Conclusion

Based on evidence provided by the empirical power function for testing the no change in mean hypothesis using nine test statistics, one can conclude that statistic $T_n^{(6)}$ has highest empirical power among the nine statistics used to test for changes in the middle and on the tails as well (cf. Tables 7.5, 7.8 and 7.9). Even though $T_n^{(6)}$ does appear best, statistic $T_n^{(5)}$ has marginally larger empirical power than $T_n^{(6)}$ for changes on the tail (cf. Tables 7.8 and 7.9). Its performance for a change in the middle, however, was not as good as $T_n^{(6)}$ and in some cases it was completely insensitive (cf. Table 7.8).

7.6 Appendix

This final section provides in tabular form the cumulative distributions for those change in mean test statistics developed in Chapter 7 (cf. Section 7.1) that have a limiting distribution. These can be found in Tables 7.12 through 7.23. The cumulative distribution functions for the estimate of the location of change $\hat{k}^*$ and $\frac{\hat{k}^*}{N}$ have their distribution tabulated in Tables 7.24 through 7.28 and Tables 7.29 through 7.33, respectively. The final set of tables, Tables 7.34 through Tables 7.40, tabulate the cumulative distribution for weighted functionals of a Wiener process. Some of these tables tabulate the cumulative distribution for functionals of weighted Brownian bridges that correspond to the limiting distribution of similar functionals of weighted partial sum process studied by Gombay, Horváth and Hušková [17, (1.9)].

Table 7.11:

N = 10,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \{T_{10,000}^{(1)} \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
12.18	0.01	31.99	0.34	61.39	0.67
13.97	0.02	32.57	0.35	62.76	0.68
14.97	0.03	33.19	0.36	63.98	0.69
15.85	0.04	33.83	0.37	65.40	0.70
16.49	0.05	34.45	0.38	66.83	0.71
17.06	0.06	35.17	0.39	68.12	0.72
17.75	0.07	35.83	0.40	69.48	0.73
18.32	0.08	36.45	0.41	70.84	0.74
18.84	0.09	37.18	0.42	72.11	0.75
19.32	0.10	37.82	0.43	73.68	0.76
19.78	0.11	38.56	0.44	75.16	0.77
20.26	0.12	39.29	0.45	76.79	0.78
20.75	0.13	39.89	0.46	78.83	0.79
21.23	0.14	40.58	0.47	80.54	0.80
21.71	0.15	41.30	0.48	82.30	0.81
22.24	0.16	42.09	0.49	84.42	0.82
22.74	0.17	42.90	0.50	86.35	0.83
23.22	0.18	43.88	0.51	88.38	0.84
23.70	0.19	44.78	0.52	90.62	0.85
24.14	0.20	45.64	0.53	92.93	0.86
24.58	0.21	46.62	0.54	95.86	0.87
25.15	0.22	47.64	0.55	98.17	0.88
25.66	0.23	48.67	0.56	100.60	0.89
26.10	0.24	49.70	0.57	103.66	0.90
26.60	0.25	50.77	0.58	107.04	0.91
27.12	0.26	51.84	0.59	110.90	0.92
27.67	0.27	53.05	0.60	114.54	0.93
28.30	0.28	54.09	0.61	118.96	0.94
28.85	0.29	55.24	0.62	124.35	0.95
29.39	0.30	56.36	0.63	130.28	0.96
30.04	0.31	57.48	0.64	137.93	0.97
30.66	0.32	58.73	0.65	149.50	0.98
31.32	0.33	59.90	0.66	165.00	0.99

Table 7.12:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \{T_{100,000}^{(1)} \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
123.91	0.01	298.42	0.34	489.82	0.67
140.25	0.02	303.00	0.35	497.13	0.68
149.70	0.03	307.75	0.36	505.18	0.69
158.88	0.04	312.29	0.37	513.82	0.70
165.92	0.05	317.00	0.38	521.96	0.71
171.87	0.06	322.36	0.39	529.91	0.72
177.87	0.07	327.19	0.40	539.08	0.73
182.96	0.08	332.07	0.41	548.26	0.74
187.88	0.09	337.21	0.42	556.14	0.75
192.83	0.10	341.73	0.43	564.72	0.76
197.97	0.11	346.44	0.44	574.00	0.77
202.63	0.12	351.55	0.45	585.80	0.78
208.09	0.13	356.28	0.46	595.77	0.79
212.06	0.14	361.52	0.47	606.54	0.80
216.30	0.15	366.73	0.48	617.45	0.81
221.38	0.16	372.42	0.49	630.13	0.82
225.82	0.17	377.50	0.50	644.18	0.83
231.37	0.18	382.61	0.51	657.88	0.84
235.49	0.19	388.48	0.52	672.88	0.85
239.73	0.20	394.28	0.53	686.67	0.86
243.82	0.21	401.56	0.54	703.40	0.87
248.41	0.22	408.05	0.55	719.71	0.88
252.53	0.23	415.05	0.56	738.00	0.89
256.67	0.24	421.31	0.57	755.46	0.90
260.68	0.25	427.71	0.58	772.00	0.91
264.61	0.26	434.76	0.59	795.71	0.92
268.61	0.27	441.07	0.60	820.25	0.93
272.95	0.28	447.22	0.61	848.83	0.94
277.07	0.29	453.59	0.62	884.89	0.95
281.06	0.30	460.46	0.63	923.00	0.96
284.92	0.31	466.70	0.64	971.50	0.97
289.22	0.32	474.47	0.65	1030.00	0.98
293.86	0.33	482.50	0.66	1120.00	0.99

Table 7.13:

N = 10,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ T_{10,000}^{(2)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.54	0.01	2.27	0.34	2.72	0.67
1.63	0.02	2.28	0.35	2.74	0.68
1.68	0.03	2.29	0.36	2.76	0.69
1.73	0.04	2.30	0.37	2.78	0.70
1.76	0.05	2.32	0.38	2.80	0.71
1.79	0.06	2.33	0.39	2.81	0.72
1.82	0.07	2.34	0.40	2.83	0.73
1.84	0.08	2.35	0.41	2.85	0.74
1.87	0.09	2.37	0.42	2.87	0.75
1.89	0.10	2.38	0.43	2.89	0.76
1.91	0.11	2.39	0.44	2.92	0.77
1.93	0.12	2.41	0.45	2.94	0.78
1.95	0.13	2.42	0.46	2.96	0.79
1.97	0.14	2.43	0.47	2.98	0.80
1.99	0.15	2.45	0.48	3.01	0.81
2.00	0.16	2.46	0.49	3.03	0.82
2.02	0.17	2.48	0.50	3.06	0.83
2.04	0.18	2.49	0.51	3.08	0.84
2.05	0.19	2.50	0.52	3.11	0.85
2.07	0.20	2.52	0.53	3.14	0.86
2.09	0.21	2.53	0.54	3.17	0.87
2.10	0.22	2.54	0.55	3.21	0.88
2.12	0.23	2.56	0.56	3.25	0.89
2.13	0.24	2.57	0.57	3.29	0.90
2.15	0.25	2.58	0.58	3.34	0.91
2.16	0.26	2.60	0.59	3.39	0.92
2.17	0.27	2.62	0.60	3.44	0.93
2.19	0.28	2.63	0.61	3.51	0.94
2.20	0.29	2.65	0.62	3.60	0.95
2.21	0.30	2.66	0.63	3.69	0.96
2.23	0.31	2.68	0.64	3.80	0.97
2.24	0.32	2.69	0.65	3.98	0.98
2.25	0.33	2.71	0.66	4.30	0.99

Table 7.14:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ T_{100,000}^{(2)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.65	0.01	2.37	0.34	2.80	0.67
1.74	0.02	2.39	0.35	2.81	0.68
1.79	0.03	2.40	0.36	2.83	0.69
1.83	0.04	2.41	0.37	2.84	0.70
1.87	0.05	2.42	0.38	2.86	0.71
1.90	0.06	2.44	0.39	2.88	0.72
1.93	0.07	2.45	0.40	2.89	0.73
1.96	0.08	2.46	0.41	2.91	0.74
1.98	0.09	2.47	0.42	2.93	0.75
2.00	0.10	2.48	0.43	2.94	0.76
2.02	0.11	2.50	0.44	2.96	0.77
2.04	0.12	2.51	0.45	2.98	0.78
2.06	0.13	2.52	0.46	3.00	0.79
2.08	0.14	2.53	0.47	3.02	0.80
2.10	0.15	2.54	0.48	3.04	0.81
2.12	0.16	2.56	0.49	3.07	0.82
2.13	0.17	2.57	0.50	3.09	0.83
2.15	0.18	2.58	0.51	3.11	0.84
2.16	0.19	2.59	0.52	3.14	0.85
2.18	0.20	2.61	0.53	3.17	0.86
2.20	0.21	2.62	0.54	3.20	0.87
2.21	0.22	2.63	0.55	3.23	0.88
2.22	0.23	2.65	0.56	3.26	0.89
2.24	0.24	2.66	0.57	3.30	0.90
2.25	0.25	2.67	0.58	3.33	0.91
2.26	0.26	2.69	0.59	3.38	0.92
2.28	0.27	2.70	0.60	3.42	0.93
2.29	0.28	2.71	0.61	3.48	0.94
2.30	0.29	2.73	0.62	3.54	0.95
2.32	0.30	2.74	0.63	3.62	0.96
2.33	0.31	2.75	0.64	3.70	0.97
2.34	0.32	2.77	0.65	3.83	0.98
2.36	0.33	2.78	0.66	4.06	0.99

Table 7.15:

N = 10,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \{T_{10,000}^{(3)} \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.435	0.01	0.720	0.34	0.943	0.67
0.466	0.02	0.727	0.35	0.951	0.68
0.481	0.03	0.733	0.36	0.959	0.69
0.500	0.04	0.740	0.37	0.968	0.70
0.514	0.05	0.746	0.38	0.976	0.71
0.525	0.06	0.752	0.39	0.984	0.72
0.536	0.07	0.758	0.40	0.993	0.73
0.545	0.08	0.764	0.41	1.002	0.74
0.554	0.09	0.770	0.42	1.011	0.75
0.563	0.10	0.777	0.43	1.021	0.76
0.571	0.11	0.783	0.44	1.031	0.77
0.579	0.12	0.789	0.45	1.042	0.78
0.587	0.13	0.795	0.46	1.055	0.79
0.595	0.14	0.801	0.47	1.067	0.80
0.602	0.15	0.808	0.48	1.080	0.81
0.610	0.16	0.815	0.49	1.093	0.82
0.617	0.17	0.822	0.50	1.107	0.83
0.624	0.18	0.828	0.51	1.121	0.84
0.630	0.19	0.835	0.52	1.135	0.85
0.637	0.20	0.842	0.53	1.150	0.86
0.643	0.21	0.849	0.54	1.164	0.87
0.649	0.22	0.856	0.55	1.181	0.88
0.655	0.23	0.862	0.56	1.200	0.89
0.661	0.24	0.870	0.57	1.220	0.90
0.667	0.25	0.876	0.58	1.247	0.91
0.673	0.26	0.883	0.59	1.270	0.92
0.679	0.27	0.890	0.60	1.292	0.93
0.685	0.28	0.898	0.61	1.321	0.94
0.692	0.29	0.905	0.62	1.353	0.95
0.698	0.30	0.913	0.63	1.400	0.96
0.703	0.31	0.920	0.64	1.446	0.97
0.709	0.32	0.927	0.65	1.510	0.98
0.715	0.33	0.935	0.66	1.618	0.99

Table 7.16:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \{T_{100,000}^{(3)} \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.436	0.01	0.729	0.34	0.948	0.67
0.464	0.02	0.735	0.35	0.957	0.68
0.484	0.03	0.740	0.36	0.966	0.69
0.504	0.04	0.746	0.37	0.975	0.70
0.518	0.05	0.753	0.38	0.983	0.71
0.530	0.06	0.759	0.39	0.992	0.72
0.542	0.07	0.765	0.40	1.000	0.73
0.550	0.08	0.771	0.41	1.009	0.74
0.558	0.09	0.778	0.42	1.017	0.75
0.567	0.10	0.784	0.43	1.027	0.76
0.575	0.11	0.790	0.44	1.037	0.77
0.583	0.12	0.796	0.45	1.048	0.78
0.590	0.13	0.802	0.46	1.060	0.79
0.598	0.14	0.808	0.47	1.073	0.80
0.605	0.15	0.814	0.48	1.086	0.81
0.612	0.16	0.821	0.49	1.101	0.82
0.619	0.17	0.827	0.50	1.113	0.83
0.626	0.18	0.833	0.51	1.127	0.84
0.634	0.19	0.840	0.52	1.143	0.85
0.641	0.20	0.847	0.53	1.156	0.86
0.648	0.21	0.854	0.54	1.172	0.87
0.655	0.22	0.860	0.55	1.191	0.88
0.661	0.23	0.867	0.56	1.212	0.89
0.668	0.24	0.874	0.57	1.232	0.90
0.674	0.25	0.881	0.58	1.253	0.91
0.681	0.26	0.888	0.59	1.275	0.92
0.687	0.27	0.895	0.60	1.302	0.93
0.693	0.28	0.902	0.61	1.335	0.94
0.699	0.29	0.910	0.62	1.366	0.95
0.705	0.30	0.918	0.63	1.408	0.96
0.711	0.31	0.925	0.64	1.459	0.97
0.717	0.32	0.932	0.65	1.528	0.98
0.723	0.33	0.939	0.66	1.640	0.99

Table 7.17:

N = 10,000 M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ T_{10,000}^{(4)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.60	0.01	2.50	0.34	3.19	0.67
1.70	0.02	2.51	0.35	3.21	0.68
1.76	0.03	2.53	0.36	3.24	0.69
1.80	0.04	2.55	0.37	3.26	0.70
1.85	0.05	2.57	0.38	3.29	0.71
1.88	0.06	2.59	0.39	3.32	0.72
1.91	0.07	2.61	0.40	3.36	0.73
1.94	0.08	2.63	0.41	3.39	0.74
1.97	0.09	2.64	0.42	3.41	0.75
2.00	0.10	2.66	0.43	3.44	0.76
2.03	0.11	2.68	0.44	3.48	0.77
2.05	0.12	2.70	0.45	3.52	0.78
2.07	0.13	2.72	0.46	3.56	0.79
2.10	0.14	2.74	0.47	3.60	0.80
2.12	0.15	2.76	0.48	3.64	0.81
2.15	0.16	2.78	0.49	3.68	0.82
2.17	0.17	2.80	0.50	3.72	0.83
2.19	0.18	2.82	0.51	3.77	0.84
2.21	0.19	2.84	0.52	3.82	0.85
2.24	0.20	2.87	0.53	3.87	0.86
2.26	0.21	2.89	0.54	3.92	0.87
2.28	0.22	2.91	0.55	3.98	0.88
2.29	0.23	2.93	0.56	4.05	0.89
2.31	0.24	2.95	0.57	4.12	0.90
2.33	0.25	2.98	0.58	4.19	0.91
2.35	0.26	3.00	0.59	4.25	0.92
2.37	0.27	3.02	0.60	4.35	0.93
2.39	0.28	3.05	0.61	4.45	0.94
2.41	0.29	3.07	0.62	4.57	0.95
2.42	0.30	3.09	0.63	4.72	0.96
2.44	0.31	3.11	0.64	4.89	0.97
2.46	0.32	3.13	0.65	5.16	0.98
2.48	0.33	3.16	0.66	5.53	0.99

Table 7.18:

N = 100,000 M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ T_{100,000}^{(4)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.64	0.01	2.51	0.34	3.21	0.67
1.74	0.02	2.53	0.35	3.23	0.68
1.79	0.03	2.54	0.36	3.26	0.69
1.84	0.04	2.56	0.37	3.29	0.70
1.89	0.05	2.58	0.38	3.31	0.71
1.92	0.06	2.60	0.39	3.34	0.72
1.95	0.07	2.62	0.40	3.37	0.73
1.98	0.08	2.64	0.41	3.40	0.74
2.00	0.09	2.66	0.42	3.43	0.75
2.03	0.10	2.68	0.43	3.46	0.76
2.05	0.11	2.70	0.44	3.50	0.77
2.08	0.12	2.72	0.45	3.53	0.78
2.10	0.13	2.73	0.46	3.57	0.79
2.12	0.14	2.75	0.47	3.61	0.80
2.15	0.15	2.77	0.48	3.65	0.81
2.17	0.16	2.79	0.49	3.70	0.82
2.19	0.17	2.81	0.50	3.74	0.83
2.22	0.18	2.83	0.51	3.80	0.84
2.24	0.19	2.85	0.52	3.85	0.85
2.26	0.20	2.87	0.53	3.91	0.86
2.28	0.21	2.89	0.54	3.97	0.87
2.29	0.22	2.91	0.55	4.03	0.88
2.31	0.23	2.94	0.56	4.09	0.89
2.33	0.24	2.96	0.57	4.16	0.90
2.35	0.25	2.98	0.58	4.24	0.91
2.37	0.26	3.01	0.59	4.32	0.92
2.38	0.27	3.03	0.60	4.42	0.93
2.40	0.28	3.06	0.61	4.52	0.94
2.42	0.29	3.08	0.62	4.62	0.95
2.44	0.30	3.11	0.63	4.75	0.96
2.46	0.31	3.13	0.64	4.96	0.97
2.47	0.32	3.16	0.65	5.17	0.98
2.49	0.33	3.18	0.66	5.64	0.99

Table 7.19:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \{T_{100,000}^{(5)} \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.21	0.01	1.79	0.34	2.17	0.67
1.27	0.02	1.80	0.35	2.18	0.68
1.32	0.03	1.81	0.36	2.20	0.69
1.35	0.04	1.82	0.37	2.21	0.70
1.37	0.05	1.83	0.38	2.23	0.71
1.40	0.06	1.85	0.39	2.24	0.72
1.42	0.07	1.86	0.40	2.25	0.73
1.45	0.08	1.87	0.41	2.27	0.74
1.47	0.09	1.88	0.42	2.28	0.75
1.49	0.10	1.89	0.43	2.30	0.76
1.50	0.11	1.90	0.44	2.32	0.77
1.52	0.12	1.91	0.45	2.34	0.78
1.54	0.13	1.92	0.46	2.36	0.79
1.55	0.14	1.93	0.47	2.38	0.80
1.57	0.15	1.94	0.48	2.40	0.81
1.58	0.16	1.95	0.49	2.42	0.82
1.59	0.17	1.96	0.50	2.44	0.83
1.61	0.18	1.97	0.51	2.46	0.84
1.62	0.19	1.98	0.52	2.48	0.85
1.63	0.20	2.00	0.53	2.51	0.86
1.64	0.21	2.01	0.54	2.53	0.87
1.65	0.22	2.02	0.55	2.56	0.88
1.67	0.23	2.03	0.56	2.59	0.89
1.68	0.24	2.04	0.57	2.62	0.90
1.69	0.25	2.05	0.58	2.65	0.91
1.70	0.26	2.07	0.59	2.70	0.92
1.71	0.27	2.08	0.60	2.74	0.93
1.72	0.28	2.09	0.61	2.78	0.94
1.74	0.29	2.10	0.62	2.84	0.95
1.75	0.30	2.11	0.63	2.92	0.96
1.76	0.31	2.13	0.64	3.01	0.97
1.77	0.32	2.14	0.65	3.14	0.98
1.78	0.33	2.15	0.66	3.35	0.99

Table 7.20:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \{T_{100,000}^{(6)} \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.39	0.01	2.03	0.34	2.44	0.67
1.47	0.02	2.05	0.35	2.45	0.68
1.52	0.03	2.06	0.36	2.47	0.69
1.56	0.04	2.07	0.37	2.48	0.70
1.59	0.05	2.08	0.38	2.50	0.71
1.62	0.06	2.09	0.39	2.52	0.72
1.64	0.07	2.10	0.40	2.53	0.73
1.67	0.08	2.11	0.41	2.55	0.74
1.69	0.09	2.13	0.42	2.57	0.75
1.71	0.10	2.14	0.43	2.59	0.76
1.73	0.11	2.15	0.44	2.60	0.77
1.75	0.12	2.16	0.45	2.62	0.78
1.76	0.13	2.17	0.46	2.64	0.79
1.78	0.14	2.18	0.47	2.67	0.80
1.79	0.15	2.19	0.48	2.69	0.81
1.81	0.16	2.21	0.49	2.71	0.82
1.82	0.17	2.22	0.50	2.73	0.83
1.84	0.18	2.23	0.51	2.76	0.84
1.85	0.19	2.24	0.52	2.78	0.85
1.86	0.20	2.25	0.53	2.82	0.86
1.88	0.21	2.27	0.54	2.84	0.87
1.89	0.22	2.28	0.55	2.87	0.88
1.90	0.23	2.29	0.56	2.90	0.89
1.91	0.24	2.30	0.57	2.94	0.90
1.93	0.25	2.31	0.58	2.97	0.91
1.94	0.26	2.33	0.59	3.02	0.92
1.95	0.27	2.34	0.60	3.07	0.93
1.96	0.28	2.36	0.61	3.12	0.94
1.97	0.29	2.37	0.62	3.18	0.95
1.99	0.30	2.38	0.63	3.26	0.96
2.00	0.31	2.40	0.64	3.36	0.97
2.01	0.32	2.41	0.65	3.48	0.98
2.02	0.33	2.43	0.66	3.68	0.99

Table 7.21:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ T_{100,000}^{(7)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
3.00	0.01	5.14	0.34	6.72	0.67
3.22	0.02	5.18	0.35	6.78	0.68
3.36	0.03	5.22	0.36	6.85	0.69
3.48	0.04	5.26	0.37	6.91	0.70
3.58	0.05	5.30	0.38	6.98	0.71
3.67	0.06	5.34	0.39	7.04	0.72
3.76	0.07	5.39	0.40	7.10	0.73
3.84	0.08	5.44	0.41	7.18	0.74
3.92	0.09	5.48	0.42	7.25	0.75
3.98	0.10	5.53	0.43	7.32	0.76
4.04	0.11	5.58	0.44	7.40	0.77
4.09	0.12	5.62	0.45	7.49	0.78
4.16	0.13	5.67	0.46	7.57	0.79
4.21	0.14	5.71	0.47	7.65	0.80
4.26	0.15	5.76	0.48	7.75	0.81
4.32	0.16	5.81	0.49	7.83	0.82
4.36	0.17	5.86	0.50	7.93	0.83
4.41	0.18	5.90	0.51	8.03	0.84
4.46	0.19	5.95	0.52	8.14	0.85
4.51	0.20	6.00	0.53	8.24	0.86
4.56	0.21	6.04	0.54	8.36	0.87
4.60	0.22	6.09	0.55	8.47	0.88
4.65	0.23	6.14	0.56	8.61	0.89
4.70	0.24	6.19	0.57	8.75	0.90
4.74	0.25	6.24	0.58	8.88	0.91
4.79	0.26	6.28	0.59	9.06	0.92
4.84	0.27	6.33	0.60	9.24	0.93
4.87	0.28	6.38	0.61	9.47	0.94
4.92	0.29	6.44	0.62	9.71	0.95
4.96	0.30	6.49	0.63	10.04	0.96
5.00	0.31	6.54	0.64	10.45	0.97
5.05	0.32	6.60	0.65	10.95	0.98
5.10	0.33	6.66	0.66	11.78	0.99

Table 7.22:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ T_{100,000}^{(8)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.61	0.01	2.51	0.34	3.36	0.67
1.70	0.02	2.53	0.35	3.40	0.68
1.76	0.03	2.55	0.36	3.44	0.69
1.82	0.04	2.57	0.37	3.49	0.70
1.87	0.05	2.59	0.38	3.54	0.71
1.91	0.06	2.61	0.39	3.59	0.72
1.94	0.07	2.63	0.40	3.64	0.73
1.97	0.08	2.65	0.41	3.71	0.74
2.00	0.09	2.67	0.42	3.77	0.75
2.03	0.10	2.69	0.43	3.84	0.76
2.05	0.11	2.71	0.44	3.90	0.77
2.08	0.12	2.73	0.45	3.97	0.78
2.10	0.13	2.76	0.46	4.04	0.79
2.13	0.14	2.78	0.47	4.12	0.80
2.15	0.15	2.80	0.48	4.20	0.81
2.17	0.16	2.82	0.49	4.29	0.82
2.19	0.17	2.84	0.50	4.39	0.83
2.21	0.18	2.87	0.51	4.49	0.84
2.23	0.19	2.89	0.52	4.62	0.85
2.25	0.20	2.92	0.53	4.73	0.86
2.27	0.21	2.94	0.54	4.87	0.87
2.29	0.22	2.96	0.55	4.98	0.88
2.31	0.23	2.99	0.56	5.11	0.89
2.33	0.24	3.02	0.57	5.26	0.90
2.35	0.25	3.05	0.58	5.41	0.91
2.36	0.26	3.08	0.59	5.58	0.92
2.38	0.27	3.11	0.60	5.77	0.93
2.40	0.28	3.14	0.61	5.98	0.94
2.42	0.29	3.17	0.62	6.23	0.95
2.44	0.30	3.21	0.63	6.53	0.96
2.46	0.31	3.24	0.64	6.89	0.97
2.48	0.32	3.28	0.65	7.37	0.98
2.49	0.33	3.32	0.66	8.25	0.99

Table 7.23:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ T_{100,000}^{(9)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
178.14	0.01	1249.65	0.34	2782.46	0.67
211.07	0.02	1289.08	0.35	2843.82	0.68
240.82	0.03	1329.63	0.36	2909.37	0.69
265.06	0.04	1370.25	0.37	2966.41	0.70
288.88	0.05	1411.29	0.38	3025.64	0.71
316.76	0.06	1450.00	0.39	3095.18	0.72
341.03	0.07	1489.27	0.40	3168.88	0.73
362.00	0.08	1536.33	0.41	3230.33	0.74
386.36	0.09	1573.58	0.42	3306.64	0.75
410.17	0.10	1619.14	0.43	3379.27	0.76
436.71	0.11	1663.36	0.44	3445.64	0.77
465.33	0.12	1707.00	0.45	3520.33	0.78
493.58	0.13	1750.60	0.46	3589.08	0.79
522.63	0.14	1792.53	0.47	3663.43	0.80
556.22	0.15	1838.67	0.48	3743.43	0.81
588.60	0.16	1888.88	0.49	3825.18	0.82
621.21	0.17	1937.50	0.50	3920.75	0.83
652.00	0.18	1979.86	0.51	4014.50	0.84
685.45	0.19	2025.21	0.52	4125.33	0.85
718.38	0.20	2076.44	0.53	4229.73	0.86
751.41	0.21	2120.93	0.54	4351.29	0.87
789.67	0.22	2169.50	0.55	4449.86	0.88
832.00	0.23	2217.53	0.56	4568.67	0.89
869.00	0.24	2265.18	0.57	4702.00	0.90
900.25	0.25	2317.00	0.58	4838.00	0.91
934.89	0.26	2359.69	0.59	4988.25	0.92
971.38	0.27	2407.00	0.60	5164.00	0.93
1012.95	0.28	2457.46	0.61	5354.50	0.94
1049.38	0.29	2506.33	0.62	5574.50	0.95
1087.79	0.30	2550.50	0.63	5842.83	0.96
1132.00	0.31	2606.45	0.64	6164.50	0.97
1169.40	0.32	2663.00	0.65	6594.50	0.98
1213.04	0.33	2722.00	0.66	7378.67	0.99

Table 7.24:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ k_{100,000}^{(3*)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
8087.50	0.01	38956.25	0.34	62420.83	0.68
10283.34	0.02	39691.67	0.35	63180.00	0.69
11790.00	0.03	40420.00	0.36	63906.25	0.70
13425.00	0.04	41083.34	0.37	61787.50	0.67
14737.50	0.05	41775.00	0.38	64625.00	0.71
16000.00	0.06	42437.50	0.39	65300.00	0.72
17290.00	0.07	43108.33	0.40	66043.75	0.73
18357.14	0.08	43756.25	0.41	66811.11	0.74
19329.17	0.09	44440.00	0.42	67550.00	0.75
20270.83	0.10	45239.29	0.43	68420.00	0.76
21100.00	0.11	45978.13	0.44	69188.89	0.77
22115.00	0.12	46616.67	0.45	70028.13	0.78
23100.00	0.13	47289.29	0.46	70720.00	0.79
23970.00	0.14	48070.00	0.47	71509.09	0.80
24795.83	0.15	48787.50	0.48	72234.38	0.81
25591.67	0.16	49566.66	0.49	73145.00	0.82
26516.67	0.17	50329.54	0.50	73997.22	0.83
27345.00	0.18	50943.75	0.51	74708.34	0.84
28300.00	0.19	51704.17	0.52	75583.34	0.85
29097.50	0.20	52400.00	0.53	76515.00	0.86
29858.34	0.21	53078.57	0.54	77435.00	0.87
30679.17	0.22	53859.38	0.55	78250.00	0.88
31381.25	0.23	54525.00	0.56	79300.00	0.89
32104.17	0.24	55100.00	0.57	80212.50	0.90
32700.00	0.25	55761.11	0.58	81135.00	0.91
33355.00	0.26	56508.34	0.59	82018.75	0.92
34096.43	0.27	57206.25	0.60	83143.75	0.93
34869.45	0.28	57792.86	0.61	84266.66	0.94
35575.00	0.29	58443.18	0.62	85391.67	0.95
36244.45	0.30	59104.17	0.63	86591.66	0.96
37020.83	0.31	59800.00	0.64	87987.50	0.97
37660.00	0.32	60386.11	0.65	89445.00	0.98
38256.25	0.33	61150.00	0.66	91712.50	0.99

Table 7.25:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ k_{100,000}^{(4*)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.00	0.01	29551.00	0.34	56526.00	0.67
0.00	0.02	30857.25	0.35	57287.36	0.68
0.00	0.03	31826.00	0.36	57991.00	0.69
0.00	0.04	32729.57	0.37	58723.73	0.70
0.00	0.05	33712.11	0.38	59514.64	0.71
0.00	0.06	34767.67	0.39	60251.00	0.72
0.00	0.07	35688.50	0.40	61095.44	0.73
0.00	0.08	36619.75	0.41	61964.34	0.74
0.00	0.09	37461.00	0.42	62844.75	0.75
0.00	0.10	38254.57	0.43	63784.34	0.76
0.00	0.11	39126.00	0.44	64744.75	0.77
0.00	0.12	39960.09	0.45	65762.11	0.78
0.00	0.13	40727.92	0.46	66876.00	0.79
0.00	0.14	41622.43	0.47	68036.00	0.80
0.00	0.15	42351.00	0.48	69181.00	0.81
0.00	0.16	43146.45	0.49	70301.00	0.82
45.81	0.17	43797.43	0.50	71519.75	0.83
464.04	0.18	44571.83	0.51	72701.00	0.84
1406.56	0.19	45427.67	0.52	74231.00	0.85
3076.00	0.20	46116.62	0.53	75692.66	0.86
5171.00	0.21	46795.44	0.54	77467.68	0.87
7801.00	0.22	47608.69	0.55	79191.00	0.88
10034.33	0.23	48337.36	0.56	80813.51	0.89
12351.00	0.24	49132.82	0.57	82717.66	0.90
14792.66	0.25	49934.34	0.58	84641.00	0.91
17163.51	0.26	50694.34	0.59	86711.00	0.92
19176.00	0.27	51351.00	0.60	88993.85	0.93
20901.00	0.28	52128.27	0.61	91251.00	0.94
22638.50	0.29	52838.50	0.62	93613.50	0.95
24092.66	0.30	53576.00	0.63	96026.00	0.96
25517.67	0.31	54363.50	0.64	97751.00	0.97
27041.00	0.32	55017.67	0.65	99094.75	0.98
28362.11	0.33	55741.00	0.66	99746.46	0.99

Table 7.26:

$N = 100,000, M = 15,000$					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ k_{100,000}^{(5*)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.00	0.01	2241.87	0.34	95164.91	0.67
0.00	0.02	2487.00	0.35	95467.32	0.68
0.00	0.03	2755.29	0.36	95780.09	0.69
0.00	0.04	3032.58	0.37	96086.00	0.70
0.00	0.05	3303.14	0.38	96329.50	0.71
0.00	0.06	3602.50	0.39	96589.00	0.72
0.00	0.07	3895.38	0.40	96888.37	0.73
2.70	0.08	4186.00	0.41	97156.60	0.74
8.48	0.09	4458.14	0.42	97421.43	0.75
14.27	0.10	4748.50	0.43	97665.21	0.76
20.05	0.11	5026.88	0.44	97900.55	0.77
25.83	0.12	5367.67	0.45	98105.29	0.78
32.66	0.13	5637.25	0.46	98305.64	0.79
48.23	0.14	5884.60	0.47	98492.11	0.80
64.97	0.15	6143.50	0.48	98680.00	0.81
87.03	0.16	6361.00	0.49	98869.75	0.82
117.74	0.17	6569.70	0.50	99036.56	0.83
158.64	0.18	6748.69	0.51	99178.27	0.84
201.29	0.19	6890.71	0.52	99310.75	0.85
251.77	0.20	7000.38	0.53	99415.74	0.86
313.10	0.21	7074.14	0.54	99516.77	0.87
387.00	0.22	37051.00	0.55	99598.73	0.88
484.18	0.23	92897.98	0.56	99670.09	0.89
559.00	0.24	92951.82	0.57	99738.39	0.90
657.81	0.25	93052.82	0.58	99793.17	0.91
776.59	0.26	93207.36	0.59	99833.50	0.92
928.27	0.27	93406.00	0.60	99873.59	0.93
1076.71	0.28	93594.33	0.61	99901.41	0.94
1250.41	0.29	93817.56	0.62	99931.96	0.95
1396.00	0.30	94037.25	0.63	99952.51	0.96
1603.63	0.31	94287.67	0.64	99967.32	0.97
1831.00	0.32	94587.25	0.65	99978.08	0.98
2017.32	0.33	94895.29	0.66	99988.85	0.99

Table 7.27:

$N = 100,000, M = 15,000$					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ k_{100,000}^{(6*)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.00	0.01	1130.17	0.34	61976.00	0.67
0.00	0.02	1513.50	0.35	65176.00	0.68
0.00	0.03	2018.86	0.36	68251.00	0.69
0.00	0.04	2588.50	0.37	70751.00	0.70
0.00	0.05	3182.25	0.38	73376.00	0.71
0.00	0.06	3984.33	0.39	75438.50	0.72
0.00	0.07	4782.25	0.40	77551.00	0.73
0.00	0.08	5607.25	0.41	79801.00	0.74
0.00	0.09	6476.00	0.42	81759.34	0.75
0.00	0.10	7284.33	0.43	83582.25	0.76
0.00	0.11	8065.29	0.44	85426.00	0.77
0.00	0.12	8926.00	0.45	86859.33	0.78
0.00	0.13	9951.00	0.46	88241.00	0.79
0.00	0.14	11176.00	0.47	89426.00	0.80
0.00	0.15	12488.50	0.48	90521.00	0.81
0.00	0.16	14234.34	0.49	91441.00	0.82
0.00	0.17	15938.50	0.50	92361.00	0.83
0.00	0.18	17767.67	0.51	93113.50	0.84
0.00	0.19	19538.50	0.52	93857.25	0.85
0.00	0.20	21538.50	0.53	94691.00	0.86
0.00	0.21	23838.50	0.54	95276.00	0.87
0.00	0.22	26217.66	0.55	96051.00	0.88
0.00	0.23	28967.66	0.56	96681.00	0.89
0.00	0.24	31794.75	0.57	97332.25	0.90
0.00	0.25	34592.66	0.58	97934.33	0.91
0.00	0.26	37563.50	0.59	98459.33	0.92
9.42	0.27	40388.50	0.60	98903.08	0.93
31.62	0.28	43117.66	0.61	99233.81	0.94
88.50	0.29	46517.66	0.62	99480.17	0.95
186.42	0.30	50042.66	0.63	99680.46	0.96
349.81	0.31	53363.50	0.64	99827.09	0.97
531.00	0.32	56376.00	0.65	99911.91	0.98
777.56	0.33	59126.00	0.66	99964.84	0.99

Table 7.28:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ k_{100,000}^{(7*)} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.00	0.01	11512.54	0.34	88373.73	0.67
7106.76	0.02	11979.57	0.35	88755.17	0.68
7116.48	0.03	12576.00	0.36	89204.13	0.69
7126.42	0.04	13164.89	0.37	89545.45	0.70
7147.37	0.05	13901.00	0.38	89849.68	0.71
7176.38	0.06	14688.50	0.39	90091.00	0.72
7205.28	0.07	15557.25	0.40	90327.25	0.73
7242.86	0.08	16534.34	0.41	90574.75	0.74
7281.68	0.09	17717.67	0.42	90785.52	0.75
7324.84	0.10	18776.00	0.43	90961.71	0.76
7381.99	0.11	20251.00	0.44	91182.00	0.77
7451.00	0.12	22546.00	0.45	91357.25	0.78
7526.42	0.13	25101.00	0.46	91512.00	0.79
7594.54	0.14	28576.00	0.47	91658.81	0.80
7684.13	0.15	32551.00	0.48	91785.09	0.81
7779.23	0.16	37776.00	0.49	91906.00	0.82
7884.52	0.17	44401.00	0.50	92026.00	0.83
7978.03	0.18	52463.50	0.51	92126.93	0.84
8076.00	0.19	58742.66	0.52	92242.25	0.85
8183.58	0.20	65201.00	0.53	92323.32	0.86
8316.39	0.21	70538.50	0.54	92414.94	0.87
8453.78	0.22	73801.00	0.55	92484.33	0.88
8604.00	0.23	76851.00	0.56	92551.67	0.89
8762.96	0.24	79032.25	0.57	92609.61	0.90
8931.88	0.25	81090.29	0.58	92669.75	0.91
9126.00	0.26	82501.00	0.59	92715.69	0.92
9354.95	0.27	83776.00	0.60	92757.87	0.93
9573.73	0.28	84701.00	0.61	92794.42	0.94
9845.05	0.29	85609.33	0.62	92824.62	0.95
10122.67	0.30	86369.75	0.63	92848.92	0.96
10438.50	0.31	86929.57	0.64	92869.92	0.97
10767.67	0.32	87511.71	0.65	92882.54	0.98
11103.78	0.33	87963.50	0.66	92891.80	0.99

Table 7.29:

$N = 100,000, M = 15,000$					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \frac{k_{100,000}^{(3*)}}{N} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.0809	0.01	0.3896	0.34	0.6242	0.68
0.1028	0.02	0.3969	0.35	0.6318	0.69
0.1179	0.03	0.4042	0.36	0.6391	0.70
0.1343	0.04	0.4108	0.37	0.6179	0.67
0.0809	0.05	0.4178	0.38	0.6463	0.71
0.1028	0.06	0.4244	0.39	0.6530	0.72
0.1179	0.07	0.4311	0.40	0.6604	0.73
0.1343	0.08	0.4376	0.41	0.6681	0.74
0.1933	0.09	0.4444	0.42	0.6755	0.75
0.2027	0.10	0.4524	0.43	0.6842	0.76
0.2110	0.11	0.4598	0.44	0.6919	0.77
0.2212	0.12	0.4662	0.45	0.7003	0.78
0.2310	0.13	0.4729	0.46	0.7072	0.79
0.2397	0.14	0.2397	0.47	0.7151	0.80
0.2480	0.15	0.4879	0.48	0.7223	0.81
0.2559	0.16	0.4957	0.49	0.7315	0.82
0.0809	0.17	0.5033	0.50	0.7400	0.83
0.1028	0.18	0.5094	0.51	0.7471	0.84
0.1179	0.19	0.5170	0.52	0.7558	0.85
0.1343	0.20	0.5240	0.53	0.7652	0.86
0.0809	0.21	0.5308	0.54	0.7744	0.87
0.1028	0.22	0.5386	0.55	0.7825	0.88
0.1179	0.23	0.5453	0.56	0.7930	0.89
0.1343	0.24	0.5510	0.57	0.8021	0.90
0.1933	0.25	0.5576	0.58	0.8114	0.91
0.2027	0.26	0.5651	0.59	0.8202	0.92
0.2110	0.27	0.5721	0.60	0.8314	0.93
0.2212	0.28	0.5779	0.61	0.8427	0.94
0.2310	0.29	0.5844	0.62	0.8539	0.95
0.2397	0.30	0.2397	0.63	0.8659	0.96
0.2480	0.31	0.5980	0.64	0.8799	0.97
0.2559	0.32	0.6039	0.65	0.8945	0.98
0.2559	0.33	0.6115	0.66	0.9171	0.99

Table 7.30:

$N = 100,000, M = 15,000$					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \frac{k_{100,000}^{(4*)}}{N} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.0000	0.02	0.3086	0.35	0.5729	0.68
0.0000	0.03	0.3183	0.36	0.5799	0.69
0.0000	0.04	0.3273	0.37	0.5872	0.70
0.0000	0.05	0.3371	0.38	0.3371	0.71
0.0000	0.06	0.3477	0.39	0.6025	0.72
0.0000	0.07	0.3569	0.40	0.6110	0.73
0.0000	0.08	0.3662	0.41	0.6196	0.74
0.0000	0.09	0.3746	0.42	0.6284	0.75
0.0000	0.10	0.3825	0.43	0.6378	0.76
0.0000	0.11	0.3913	0.44	0.6474	0.77
0.0000	0.12	0.3996	0.45	0.6576	0.78
0.0000	0.13	0.4073	0.46	0.6688	0.79
0.0000	0.14	0.4162	0.47	0.6804	0.80
0.0000	0.15	0.4235	0.48	0.6918	0.81
0.0000	0.16	0.4315	0.49	0.7030	0.82
0.0005	0.17	0.4380	0.50	0.7152	0.83
0.0046	0.18	0.4457	0.51	0.7270	0.84
0.0141	0.19	0.4543	0.52	0.7423	0.85
0.0308	0.20	0.4612	0.53	0.7569	0.86
0.0517	0.21	0.4680	0.54	0.7747	0.87
0.0780	0.22	0.4761	0.55	0.7919	0.88
0.1003	0.23	0.4834	0.56	0.8081	0.89
0.1235	0.24	0.4913	0.57	0.8272	0.90
0.1479	0.25	0.4993	0.58	0.8464	0.91
0.1716	0.26	0.5069	0.59	0.8671	0.92
0.1918	0.27	0.5135	0.60	0.8899	0.93
0.2090	0.28	0.5213	0.61	0.9125	0.94
0.2264	0.29	0.5284	0.62	0.9361	0.95
0.2409	0.30	0.5358	0.63	0.9603	0.96
0.2552	0.31	0.5436	0.64	0.9775	0.97
0.2704	0.32	0.5502	0.65	0.9909	0.98
0.2836	0.33	0.5574	0.66	0.9975	0.99

Table 7.31:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \frac{k_{100,000}^{(5*)}}{N} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.00	0.01	0.022	0.34	0.9516	0.67
0.00	0.02	0.025	0.35	0.9547	0.68
0.00	0.03	0.028	0.36	0.9578	0.69
0.00	0.04	0.030	0.37	0.9609	0.70
0.00	0.05	0.033	0.38	0.9633	0.71
0.00	0.06	0.036	0.39	0.9659	0.72
0.00	0.07	0.039	0.40	0.9689	0.73
0.00	0.08	0.042	0.41	0.9716	0.74
0.00	0.09	0.045	0.42	0.9742	0.75
0.00	0.10	0.047	0.43	0.9767	0.76
0.00	0.11	0.050	0.44	0.9790	0.77
0.00	0.12	0.054	0.45	0.9811	0.78
0.00	0.13	0.056	0.46	0.9831	0.79
0.00	0.14	0.059	0.47	0.9849	0.80
0.00	0.15	0.061	0.48	0.9868	0.81
0.00	0.16	0.064	0.49	0.9887	0.82
0.00	0.17	0.066	0.50	0.9904	0.83
0.00	0.18	0.067	0.51	0.9918	0.84
0.00	0.19	0.069	0.52	0.9931	0.85
0.00	0.20	0.070	0.53	0.9942	0.86
0.00	0.21	0.071	0.54	0.9952	0.87
0.00	0.22	0.371	0.55	0.9960	0.88
0.00	0.23	0.929	0.56	0.9967	0.89
0.01	0.24	0.930	0.57	0.9974	0.90
0.01	0.25	0.931	0.58	0.9979	0.91
0.01	0.26	0.932	0.59	0.9983	0.92
0.01	0.27	0.934	0.60	0.9987	0.93
0.01	0.28	0.936	0.61	0.9990	0.94
0.01	0.29	0.938	0.62	0.9993	0.95
0.01	0.30	0.940	0.63	0.9995	0.96
0.02	0.31	0.943	0.64	0.9997	0.97
0.02	0.32	0.946	0.65	0.9998	0.98
0.02	0.33	0.949	0.66	0.9999	0.99

Table 7.32:

N = 100,000, M = 15,000					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \frac{k_{100,000}^{(6*)}}{N} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.0000	0.01	0.0113	0.34	0.6198	0.67
0.0000	0.02	0.0151	0.35	0.6518	0.68
0.0000	0.03	0.0202	0.36	0.6825	0.69
0.0000	0.04	0.0259	0.37	0.7075	0.70
0.0000	0.05	0.0318	0.38	0.7338	0.71
0.0000	0.06	0.0398	0.39	0.7544	0.72
0.0000	0.07	0.0478	0.40	0.7755	0.73
0.0000	0.08	0.0561	0.41	0.7980	0.74
0.0000	0.09	0.0648	0.42	0.8176	0.75
0.0000	0.10	0.0728	0.43	0.8358	0.76
0.0000	0.11	0.0807	0.44	0.8543	0.77
0.0000	0.12	0.0893	0.45	0.8686	0.78
0.0000	0.13	0.0995	0.46	0.8824	0.79
0.0000	0.14	0.1118	0.47	0.8943	0.80
0.0000	0.15	0.1249	0.48	0.9052	0.81
0.0000	0.16	0.1423	0.49	0.9144	0.82
0.0000	0.17	0.1594	0.50	0.9236	0.83
0.0000	0.18	0.1777	0.51	0.9311	0.84
0.0000	0.19	0.1954	0.52	0.9386	0.85
0.0000	0.20	0.2154	0.53	0.9469	0.86
0.0000	0.21	0.2384	0.54	0.9528	0.87
0.0000	0.22	0.2622	0.55	0.9605	0.88
0.0000	0.23	0.2897	0.56	0.9668	0.89
0.0000	0.24	0.3179	0.57	0.9733	0.90
0.0000	0.25	0.3459	0.58	0.9793	0.91
0.0000	0.26	0.3756	0.59	0.9846	0.92
0.0000	0.27	0.4039	0.60	0.9890	0.93
0.0000	0.28	0.4312	0.61	0.9923	0.94
0.0000	0.29	0.4652	0.62	0.9948	0.95
0.0000	0.30	0.5004	0.63	0.9968	0.96
0.0000	0.31	0.5336	0.64	0.9983	0.97
0.0000	0.32	0.5638	0.65	0.9991	0.98
0.0000	0.33	0.5913	0.66	0.9996	0.99

Table 7.33:

$N = 100,000, M = 15,000$					
$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \frac{k_{100,000}^{(7*)}}{N} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.0000	0.01	0.1151	0.34	0.8837	0.67
0.0711	0.02	0.1198	0.35	0.8876	0.68
0.0712	0.03	0.1258	0.36	0.8920	0.69
0.0713	0.04	0.1316	0.37	0.8955	0.70
0.0715	0.05	0.1390	0.38	0.8985	0.71
0.0718	0.06	0.1469	0.39	0.9009	0.72
0.0721	0.07	0.1556	0.40	0.9033	0.73
0.0724	0.08	0.1653	0.41	0.9057	0.74
0.0728	0.09	0.1772	0.42	0.9079	0.75
0.0732	0.10	0.1878	0.43	0.9096	0.76
0.0738	0.11	0.2025	0.44	0.9118	0.77
0.0745	0.12	0.2255	0.45	0.9136	0.78
0.0753	0.13	0.2510	0.46	0.9151	0.79
0.0759	0.14	0.2858	0.47	0.9166	0.80
0.0768	0.15	0.3255	0.48	0.9179	0.81
0.0778	0.16	0.3778	0.49	0.9191	0.82
0.0788	0.17	0.4440	0.50	0.9203	0.83
0.0798	0.18	0.5246	0.51	0.9213	0.84
0.0808	0.19	0.5874	0.52	0.9224	0.85
0.0818	0.20	0.6520	0.53	0.9232	0.86
0.0832	0.21	0.7054	0.54	0.9241	0.87
0.0845	0.22	0.7380	0.55	0.9248	0.88
0.0860	0.23	0.7685	0.56	0.9255	0.89
0.0876	0.24	0.7903	0.57	0.9261	0.90
0.0893	0.25	0.8109	0.58	0.9267	0.91
0.0913	0.26	0.8250	0.59	0.9272	0.92
0.0935	0.27	0.8378	0.60	0.9276	0.93
0.0957	0.28	0.8470	0.61	0.9279	0.94
0.0985	0.29	0.8561	0.62	0.9282	0.95
0.1012	0.30	0.8637	0.63	0.9285	0.96
0.1044	0.31	0.8693	0.64	0.9287	0.97
0.1077	0.32	0.8751	0.65	0.9288	0.98
0.1110	0.33	0.8796	0.66	0.9289	0.99

Table 7.34:

$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \sup_{0 < t < 1} \frac{W(t) - tW(1)}{(t(1-t))^{1/16}} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.482	0.01	0.795	0.34	1.032	0.67
0.514	0.02	0.802	0.35	1.040	0.68
0.534	0.03	0.808	0.36	1.049	0.69
0.551	0.04	0.814	0.37	1.058	0.70
0.566	0.05	0.820	0.38	1.067	0.71
0.578	0.06	0.826	0.39	1.076	0.72
0.589	0.07	0.833	0.40	1.086	0.73
0.600	0.08	0.839	0.41	1.095	0.74
0.610	0.09	0.845	0.42	1.105	0.75
0.619	0.10	0.852	0.43	1.116	0.76
0.629	0.11	0.858	0.44	1.128	0.77
0.638	0.12	0.865	0.45	1.140	0.78
0.646	0.13	0.873	0.46	1.152	0.79
0.655	0.14	0.880	0.47	1.165	0.80
0.663	0.15	0.887	0.48	1.177	0.81
0.671	0.16	0.893	0.49	1.190	0.82
0.678	0.17	0.900	0.50	1.203	0.83
0.686	0.18	0.908	0.51	1.218	0.84
0.693	0.19	0.916	0.52	1.232	0.85
0.700	0.20	0.924	0.53	1.248	0.86
0.707	0.21	0.932	0.54	1.265	0.87
0.714	0.22	0.940	0.55	1.284	0.88
0.722	0.23	0.948	0.56	1.306	0.89
0.729	0.24	0.955	0.57	1.330	0.90
0.735	0.25	0.962	0.58	1.355	0.91
0.742	0.26	0.969	0.59	1.381	0.92
0.749	0.27	0.977	0.60	1.410	0.93
0.756	0.28	0.984	0.61	1.443	0.94
0.763	0.29	0.992	0.62	1.483	0.95
0.769	0.30	0.999	0.63	1.531	0.96
0.775	0.31	1.008	0.64	1.591	0.97
0.781	0.32	1.016	0.65	1.653	0.98
0.788	0.33	1.024	0.66	1.795	0.99

Table 7.35:

$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \sup_{0 < t < 1} \frac{W(t) - tW(1)}{(t(1-t))^{5/16}} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.818	0.01	1.27	0.34	1.593	0.67
0.857	0.02	1.28	0.35	1.604	0.68
0.887	0.03	1.29	0.36	1.616	0.69
0.913	0.04	1.30	0.37	1.629	0.70
0.936	0.05	1.31	0.38	1.642	0.71
0.957	0.06	1.32	0.39	1.655	0.72
0.975	0.07	1.33	0.40	1.669	0.73
0.993	0.08	1.34	0.41	1.684	0.74
1.008	0.09	1.34	0.42	1.697	0.75
1.022	0.10	1.35	0.43	1.711	0.76
1.039	0.11	1.36	0.44	1.726	0.77
1.051	0.12	1.37	0.45	1.743	0.78
1.062	0.13	1.38	0.46	1.758	0.79
1.074	0.14	1.39	0.47	1.772	0.80
1.086	0.15	1.40	0.48	1.787	0.81
1.097	0.16	1.41	0.49	1.803	0.82
1.109	0.17	1.42	0.50	1.819	0.83
1.121	0.18	1.43	0.51	1.840	0.84
1.131	0.19	1.44	0.52	1.862	0.85
1.142	0.20	1.45	0.53	1.886	0.86
1.152	0.21	1.46	0.54	1.908	0.87
1.162	0.22	1.47	0.55	1.932	0.88
1.171	0.23	1.48	0.56	1.959	0.89
1.180	0.24	1.49	0.57	1.987	0.90
1.189	0.25	1.50	0.58	2.021	0.91
1.197	0.26	1.51	0.59	2.065	0.92
1.206	0.27	1.52	0.60	2.109	0.93
1.215	0.28	1.53	0.61	2.148	0.94
1.225	0.29	1.54	0.62	2.201	0.95
1.234	0.30	1.55	0.63	2.268	0.96
1.244	0.31	1.56	0.64	2.345	0.97
1.253	0.32	1.57	0.65	2.449	0.98
1.262	0.33	1.58	0.66	2.624	0.99

Table 7.36:

$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \sup_{0 < t < 1} \frac{ W(t) - tW(1) }{(t(1-t))^{3/16}} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.620	0.01	0.991	0.34	1.273	0.67
0.654	0.02	0.998	0.35	1.283	0.68
0.681	0.03	1.005	0.36	1.292	0.69
0.698	0.04	1.014	0.37	1.302	0.70
0.714	0.05	1.022	0.38	1.311	0.71
0.730	0.06	1.029	0.39	1.321	0.72
0.745	0.07	1.036	0.40	1.333	0.73
0.759	0.08	1.044	0.41	1.345	0.74
0.771	0.09	1.053	0.42	1.357	0.75
0.783	0.10	1.062	0.43	1.370	0.76
0.794	0.11	1.070	0.44	1.384	0.77
0.806	0.12	1.078	0.45	1.397	0.78
0.816	0.13	1.086	0.46	1.411	0.79
0.826	0.14	1.095	0.47	1.424	0.80
0.836	0.15	1.103	0.48	1.438	0.81
0.845	0.16	1.111	0.49	1.454	0.82
0.855	0.17	1.120	0.50	1.469	0.83
0.864	0.18	1.130	0.51	1.484	0.84
0.873	0.19	1.140	0.52	1.500	0.85
0.881	0.20	1.148	0.53	1.519	0.86
0.889	0.21	1.155	0.54	1.540	0.87
0.897	0.22	1.163	0.55	1.563	0.88
0.906	0.23	1.172	0.56	1.589	0.89
0.915	0.24	1.180	0.57	1.621	0.90
0.924	0.25	1.188	0.58	1.649	0.91
0.932	0.26	1.197	0.59	1.679	0.92
0.941	0.27	1.207	0.60	1.713	0.93
0.949	0.28	1.217	0.61	1.755	0.94
0.957	0.29	1.226	0.62	1.798	0.95
0.964	0.30	1.236	0.63	1.852	0.96
0.971	0.31	1.245	0.64	1.920	0.97
0.978	0.32	1.253	0.65	2.005	0.98
0.985	0.33	1.263	0.66	2.166	0.99

Table 7.37:

$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \sup_{0 < t < 1} \frac{ W(t) - tW(1) }{(t(1-t))^{7/16}} \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.146	0.01	1.734	0.34	2.112	0.67
1.206	0.02	1.744	0.35	2.123	0.68
1.251	0.03	1.754	0.36	2.136	0.69
1.284	0.04	1.764	0.37	2.149	0.70
1.315	0.05	1.776	0.38	2.164	0.71
1.345	0.06	1.787	0.39	2.180	0.72
1.370	0.07	1.797	0.40	2.194	0.73
1.390	0.08	1.809	0.41	2.209	0.74
1.408	0.09	1.822	0.42	2.227	0.75
1.426	0.10	1.833	0.43	2.244	0.76
1.443	0.11	1.844	0.44	2.261	0.77
1.458	0.12	1.855	0.45	2.278	0.78
1.473	0.13	1.865	0.46	2.297	0.79
1.489	0.14	1.876	0.47	2.315	0.80
1.504	0.15	1.887	0.48	2.333	0.81
1.517	0.16	1.897	0.49	2.353	0.82
1.531	0.17	1.909	0.50	2.375	0.83
1.545	0.18	1.921	0.51	2.397	0.84
1.558	0.19	1.931	0.52	2.421	0.85
1.572	0.20	1.941	0.53	2.446	0.86
1.586	0.21	1.953	0.54	2.472	0.87
1.599	0.22	1.964	0.55	2.502	0.88
1.611	0.23	1.976	0.56	2.532	0.89
1.623	0.24	1.988	0.57	2.563	0.90
1.633	0.25	2.001	0.58	2.595	0.91
1.644	0.26	2.014	0.59	2.631	0.92
1.655	0.27	2.026	0.60	2.675	0.93
1.666	0.28	2.037	0.61	2.727	0.94
1.676	0.29	2.048	0.62	2.784	0.95
1.688	0.30	2.060	0.63	2.856	0.96
1.700	0.31	2.073	0.64	2.931	0.97
1.711	0.32	2.086	0.65	3.080	0.98
1.722	0.33	2.101	0.66	3.282	0.99

Table 7.38:

$G(x) \stackrel{\text{def}}{=} \mathbf{P} \left\{ \sup_{0 \leq t \leq 1} (1-t)W(t) + t(W(1) - W(t)) \leq x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.26	0.01	0.51	0.34	0.71	0.67
0.29	0.02	0.51	0.35	0.72	0.68
0.30	0.03	0.52	0.36	0.73	0.69
0.32	0.04	0.52	0.37	0.74	0.70
0.33	0.05	0.53	0.38	0.74	0.71
0.34	0.06	0.53	0.39	0.75	0.72
0.35	0.07	0.54	0.40	0.76	0.73
0.35	0.08	0.55	0.41	0.77	0.74
0.36	0.09	0.55	0.42	0.78	0.75
0.37	0.10	0.56	0.43	0.79	0.76
0.38	0.11	0.56	0.44	0.80	0.77
0.38	0.12	0.57	0.45	0.81	0.78
0.39	0.13	0.57	0.46	0.82	0.79
0.39	0.14	0.58	0.47	0.83	0.80
0.40	0.15	0.59	0.48	0.85	0.81
0.41	0.16	0.59	0.49	0.86	0.82
0.41	0.17	0.60	0.50	0.87	0.83
0.42	0.18	0.60	0.51	0.88	0.84
0.42	0.19	0.61	0.52	0.89	0.85
0.43	0.20	0.62	0.53	0.91	0.86
0.43	0.21	0.62	0.54	0.93	0.87
0.44	0.22	0.63	0.55	0.94	0.88
0.45	0.23	0.63	0.56	0.96	0.89
0.45	0.24	0.64	0.57	0.98	0.90
0.46	0.25	0.65	0.58	1.00	0.91
0.46	0.26	0.65	0.59	1.03	0.92
0.47	0.27	0.66	0.60	1.05	0.93
0.47	0.28	0.67	0.61	1.08	0.94
0.48	0.29	0.67	0.62	1.12	0.95
0.48	0.30	0.68	0.63	1.16	0.96
0.49	0.31	0.69	0.64	1.21	0.97
0.49	0.32	0.69	0.65	1.29	0.98
0.50	0.33	0.70	0.66	1.40	0.99

Table 7.39:

$G(x) \stackrel{\text{def}}{=} \mathbf{P} \{ \sup_{0 < t_1 < t_2 < 1} t_2 W(t_1) + (1 - t_1)(W(t_1) - t_2 W(1)) \leq x \}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.529	0.01	0.847	0.34	1.096	0.67
0.561	0.02	0.854	0.35	1.105	0.68
0.584	0.03	0.861	0.36	1.114	0.69
0.602	0.04	0.868	0.37	1.124	0.70
0.617	0.05	0.874	0.38	1.134	0.71
0.630	0.06	0.881	0.39	1.143	0.72
0.642	0.07	0.888	0.40	1.153	0.73
0.654	0.08	0.895	0.41	1.163	0.74
0.666	0.09	0.902	0.42	1.174	0.75
0.676	0.10	0.908	0.43	1.186	0.76
0.685	0.11	0.915	0.44	1.198	0.77
0.694	0.12	0.921	0.45	1.211	0.78
0.702	0.13	0.929	0.46	1.223	0.79
0.711	0.14	0.936	0.47	1.234	0.80
0.719	0.15	0.943	0.48	1.247	0.81
0.727	0.16	0.950	0.49	1.261	0.82
0.734	0.17	0.957	0.50	1.276	0.83
0.742	0.18	0.965	0.51	1.291	0.84
0.749	0.19	0.973	0.52	1.305	0.85
0.756	0.20	0.981	0.53	1.319	0.86
0.762	0.21	0.988	0.54	1.336	0.87
0.769	0.22	0.996	0.55	1.355	0.88
0.776	0.23	1.004	0.56	1.377	0.89
0.783	0.24	1.013	0.57	1.400	0.90
0.789	0.25	1.021	0.58	1.423	0.91
0.795	0.26	1.029	0.59	1.451	0.92
0.801	0.27	1.038	0.60	1.476	0.93
0.808	0.28	1.046	0.61	1.509	0.94
0.815	0.29	1.053	0.62	1.543	0.95
0.821	0.30	1.061	0.63	1.591	0.96
0.827	0.31	1.071	0.64	1.646	0.97
0.834	0.32	1.080	0.65	1.725	0.98
0.840	0.33	1.088	0.66	1.837	0.99

Table 7.40:

$G(x) \stackrel{\text{def}}{=} \mathbf{P} \{ \sup_{0 < t_1 < t_2 < 1} (t_2 - 2t_1)W(t_1) + (1 + t_1 - 2t_2)(W(t_2) + t_2W(1)) \leq x \}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.382	0.01	0.681	0.34	0.947	0.67
0.410	0.02	0.687	0.35	0.956	0.68
0.429	0.03	0.694	0.36	0.966	0.69
0.444	0.04	0.701	0.37	0.976	0.70
0.458	0.05	0.708	0.38	0.987	0.71
0.469	0.06	0.715	0.39	0.998	0.72
0.480	0.07	0.722	0.40	1.010	0.73
0.489	0.08	0.728	0.41	1.021	0.74
0.499	0.09	0.735	0.42	1.033	0.75
0.508	0.10	0.742	0.43	1.046	0.76
0.517	0.11	0.749	0.44	1.059	0.77
0.526	0.12	0.757	0.45	1.071	0.78
0.534	0.13	0.764	0.46	1.085	0.79
0.542	0.14	0.771	0.47	1.101	0.80
0.549	0.15	0.778	0.48	1.116	0.81
0.556	0.16	0.786	0.49	1.132	0.82
0.563	0.17	0.794	0.50	1.148	0.83
0.570	0.18	0.802	0.51	1.166	0.84
0.577	0.19	0.810	0.52	1.184	0.85
0.584	0.20	0.818	0.53	1.203	0.86
0.590	0.21	0.826	0.54	1.224	0.87
0.597	0.22	0.835	0.55	1.249	0.88
0.604	0.23	0.843	0.56	1.275	0.89
0.611	0.24	0.852	0.57	1.299	0.90
0.618	0.25	0.861	0.58	1.328	0.91
0.626	0.26	0.870	0.59	1.360	0.92
0.633	0.27	0.878	0.60	1.397	0.93
0.640	0.28	0.887	0.61	1.436	0.94
0.646	0.29	0.897	0.62	1.485	0.95
0.653	0.30	0.906	0.63	1.537	0.96
0.660	0.31	0.915	0.64	1.601	0.97
0.667	0.32	0.926	0.65	1.700	0.98
0.674	0.33	0.937	0.66	1.853	0.99

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