Adults' Complex Mental Addition:

Flexibility, Adaptivity, and Efficiency

by

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Abstract

The goal of this program of research was to investigate how adults use solution procedures for complex mental addition. In a series of four experiments, I examined procedure use in relation to Lemaire and Siegler's (1995) four dimensions of "strategic competence": procedure repertoire, frequency, efficiency, and adaptivity. Adults solved 2- plus 1-digit and 2- plus 2-digit addition problems (e.g., 59 + 6; 45 + 74) varying in complexity (i.e., unit carries and decade carries), orientation (i.e., horizontal and vertical) and, in the case of 2- plus 1-digit problems, operand order. Using trial-by-trial verbal protocols, two experiments determined the procedure repertoire and frequency. Five common calculation procedures (with some differences for 2- plus 1-digit and 2- plus 2-digit problems) were reported. Adults consistently divided into two groups according to problem-solving style. *Flexible* solvers used a variety of procedures, varying them according to problem complexity, orientation, and operand order. *Stable* solvers used one procedure to solve virtually all problems.

In a further two experiments, the choice/no-choice paradigm (Lemaire & Siegler, 1995) was employed to measure relative procedure efficiency and to test whether people's procedure choices were influenced by procedure performance characteristics (i.e., speed and accuracy) as well as problem characteristics. Finally, overall performance, procedure choice, relative procedure efficiency, and adaptivity of problem-solving were examined in relation to people's arithmetic fluency and problem-solving style. Flexible and stable individuals were equally adaptive in their problem-solving in the sense that they demonstrated similar performance efficiency in both the choice and no-choice conditions and they chose their 'best' procedure (as measured in the no-choice condition) equally often when they had the choice. I discuss the implications of the present results for the concept of adaptive expertise in complex mental arithmetic, until now defined as the flexible use of a variety of procedures (e.g., Baroody & Dowker, 2003; Hatano, 1988; Shrager, & Siegler, 1998). For these relatively skilled adults, it did not appear to be procedure variation that made problem-solving adaptive. The results suggest a role for consistent use of an efficient procedure as a possible 'end point' in the development of arithmetic expertise.

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CHAPTER 1

Multi-digit mental arithmetic is a complex cognitive task involving multiple steps and a variety of procedures. People performing complex calculations may call upon various cognitive processes, including working memory and mental representation of digits, numerical magnitude, rules, and heuristics. Performing these complex calculations without external aids is a skill whose value was at one time taken for granted but has come into question with the advent of the personal computer and the portable calculator. However, the ability to perform mental calculations quickly and accurately is frequently helpful in everyday life (e.g., checking for the correct change, calculating a tip, scheduling time).

The importance of teaching mental calculation skills has been debated in elementary education in North America and Europe since at least the 1960s. Consequently, mathematics education reforms in North America and Europe over the past 20 years have had different emphases. In North America, mental calculation has been de-emphasized in favour of concept development and discovery learning. The use of calculators in elementary and secondary mathematics education has increased over this time. New methods of teaching arithmetic in the primary and junior grades have moved away from rote learning of 'math facts' and repeated practice of mental arithmetic procedures toward exercises (and texts) in which children are guided to discover mathematics concepts and problem-solving methods for themselves. In contrast, many European elementary mathematics curricula are based on the belief that flexible mental calculation skills contribute to better fundamental conceptual knowledge in arithmetic and that such skills and knowledge lead to greater success in more advanced mathematics learning. Young elementary school children in Belgium, Italy, and the Netherlands, for example, have been instructed and practiced in mental calculation procedures for solving multi-digit problems. These differences in training and experience appear to have persistent effects on the selection of procedures and on the execution of mental processes involved in multi-digit arithmetic (Imbo & LeFevre, 2009).

The goal of the present research was to investigate how adults perform complex mental arithmetic and, more specifically, to explore the use of multi-step procedures in adults' complex mental arithmetic performance. It is well-established that both children and adults use a variety of solution procedures to solve simple arithmetic problems (e.g., Bisanz & LeFevre, 1990; Lefevre, Sadesky, & Bisanz, 1996; Lemaire & Siegler, 1995) and children have also reported using a variety of procedures for complex arithmetic (e.g., Beishuizen, Van Putten, & Van Mulken, 1997; Fuson, 1990; Lucangeli, Tressoldi, Bendotti, Bonanomi, & Siegel, 2003). Further, children are expected to develop in the direction of increasing effectiveness in their approach to arithmetic calculation, choosing the best procedure for the application and executing procedures more effectively (e.g., Lemaire & Siegler, 1995). Current models of procedure choice (e.g., SCADS, Shrager & Siegler, 1998) posit that, in general, people will choose the procedure that has proven most effective in solving a given problem (or type of problem) in the past. Indeed, children and adults have both been shown to adapt their solution approach to both the nature of the problem and the relative effectiveness of available procedures (Lemaire, Arnaud, & Lecacheur, 2004; Lemaire & Lecacheur, 2002; Siegler & Lemaire, 1997). Thus, as a group, adults were expected to report using a variety of procedures to solve complex mental arithmetic problems and to be influenced by a number of factors, including characteristics of the problem and characteristics of the various procedures, in choosing a procedure to solve a given problem. However, it was also hypothesized that individual differences would influence adults' approaches to complex mental computation.

Two different types of individual differences were explored in the present research: individual differences in efficiency, mainly indexed by speed of solutions, and individual differences in solution styles. Efficiency differences (referred to as 'skill' in reference to performance on paper-and-pencil arithmetic problems) were observed across four experiments. Less skilled participants were slower and sometimes less accurate than more-skilled participants. The more interesting results, however, concerned solution styles. Some participants selected from among a variety of procedures when given the opportunity (flexible), whereas others predominately used a single procedure (stable).

Mental arithmetic is a valuable domain within which to explore cognition, both because it is an activity that most people engage in on a regular basis and because arithmetic calculation requires many important tasks of cognition such as encoding and memory access. Thus, research on how people solve mental arithmetic problems can illuminate a variety of theoretical and applied questions. The present research explores the ways in which adults approach the task of mentally calculating multi-digit arithmetic problems. It answers several important questions about the strategies adults use in complex arithmetic and whether or not these strategies serve to optimize their performance in this domain. In Chapter 2, I describe how individual differences in arithmetic skill were assessed and used to compare participants across four experiments. Fluency (i.e., speed and accuracy) on a multi-digit paper-and-pencil arithmetic task was examined as a factor that is likely to have an impact on adults' performance in complex mental arithmetic and that may also influence their choice of solution procedures. In Chapter 3, two experiments are presented in which participants were given mental addition problems and asked to report their solution procedures. In Experiment 1, participants solved two-digit plus one-digit addition problems that varied in complexity (i.e., presence of a carry operation) and the manner of presentation (i.e., order and orientation). In Experiment 2, participants solved two-digit plus two-digit addition problems that varied in complexity and orientation. Their verbatim reports were used to refine a list of procedures that was used in the two experiments presented in Chapter 3. In Experiment 3, participants again solved two-digit plus two-digit addition problems and reported their procedures, this time by choosing from the list developed in Experiment 2. In Experiment 4, a subset of the participants from Experiment 3 returned and solved similar problems. On this occasion, they were instructed to use each of three procedures on one of three sets of problems. Experiments 3 and 4 together were based on the choice/no-choice paradigm (Siegler & Lemaire, 1997), used to measure characteristics (i.e., speed and accuracy) of the three procedures and examine their relationship to procedure choices.

Solution Procedures and Problem-Solving Strategies

Multi-digit mental arithmetic involves the coordination of multiple sub-tasks. For example, when people solve arithmetic problems such as 34 + 47 or 12 x 43 they use many steps, which may include accessing basic facts, maintaining intermediate results in memory, recalling the syntax of place value, and regrouping (i.e., carrying or borrowing) across hundreds, decades, and units (Ashcraft, Donley, Halas, & Vakali, 1992; DeStefano

& LeFevre, 2004; Widaman, Geary, Cormier, & Little, 1989). There are a variety of ways of combining the multiple steps that go into solving mental arithmetic problems. Such goal-directed sequences of steps or sub-procedures are often referred to in the literature as *strategies* (e.g., Ashcraft, 1990; Beishuizen, 1993; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Geary, Brown, & Samaranayake, V.A., 1991; Imbo & LeFevre, 2009; Lemaire & Siegler, 1995; Siegler, 1987a; 1987b; 1999; Siegler, & Shrager, 1984; Shrager & Siegler, 1998). However, it is important in the context of mental arithmetic to distinguish between a *procedure* as a sequence of cognitive tasks undertaken to solve an arithmetic problem and a *strategy* as a way of choosing how a problem, or problems, will be solved.

Bisanz and LeFevre (1990) make an important distinction between a strategy and a procedure. In this view, procedures are "mental activities, or sequences of activities, that occur over time, accomplish a goal, and can be stored in memory" (p. 215). A strategy, on the other hand, is a process that results in the selection and implementation of one of a number of available procedures to achieve a goal. In the case of solving complex mental arithmetic problems, adults may store a variety of calculation procedures in memory and choose among them to solve problems. Choosing among procedures can be considered a problem-solving strategy, or style, and using one procedure on all problems is another. In the present research, the term *style*, rather than *strategy*, will be used to refer to an individual's overall approach to problem-solving, in order to avoid confusion when making comparisons with other research. Selection among procedures may occur in response to a variety of factors such as the type of problem and how the problem is presented and would, presumably, be intended to maximize the chances of quick and successful calculation. Many researchers assume that using a variety of different procedures and varying the procedures on the basis of characteristics of the problems and characteristics of the procedures is the most adaptive approach to arithmetic calculation (e.g., Imbo & LeFevre, 2009; Lemaire et al., 2004; Lemaire & Siegler, 1995; Siegler & Lemaire, 1997).

Solution Procedures in Simple Arithmetic

Research in the domain of simple arithmetic (e.g., 3 + 4, 15 – 7, 4 x 9, or 63 ÷ 9) has shown that children as young as 4 and 5 years old use a variety of procedures to solve simple arithmetic problems and that the flexible use of multiple procedures continues as children's arithmetic skills improve over time (e.g., Baroody, 1987; Bisanz & LeFevre, 1990; Carpenter & Moser, 1984; Lemaire & Siegler, 1995; Siegler, 1987b; 1999; Siegler & Shrager, 1984; Siegler & Robinson, 1982; Woods, Resnick, & Groen, 1975). Adults have also been shown to use a variety of solution procedures in simple arithmetic (e.g., LeFevre, Bisanz et al., 1996; LeFevre, Sadesky, & Bisanz, 1996). The most efficient way that people answer questions such as 3 + 4 is by retrieving stored "facts" (e.g., 3 + 4 = 7) from memory. However, both children and adults also solve simple arithmetic problems using procedures that involve a combination of memory retrieval and other steps. Such solution procedures include counting (e.g., solving 9 + 2 by counting 9, 10, 11), transformation (e.g., 5 + 6 = [5 + 5] + 1), repeated addition (e.g., 3 x 4 = 4 + 4 + 4), and reference to another operation (e.g., recasting 63 ÷ 9 as 9 x _ = 63; Campbell & Xue, 2001; Hecht, 2002; LeFevre, Bisanz et al., 1996; LeFevre & Morris, 1999; LeFevre, Sadesky, & Bisanz, 1996; Lemaire & Siegler, 1995).

It is important to consider what development occurs between children's spontaneous invention of solution procedures and their early arithmetic education (Ambrose, Baek, & Carpenter, 2003; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Carpenter, & Moser, 1984), on the one hand, and adult procedurai behaviour in the arithmetic domain, on the other. According to the Strategy Choice and Discovery Simulation (SCADS) model (Shrager & Siegler, 1998), the repeated successful use of both retrieval and non-retrieval procedures in simple arithmetic over time strengthens the problem-answer associations in memory and, thus, contributes to increasing use of retrieval as children get older. Retrieval is, in fact, used more by adults than by children (e.g., Siegler & Shrager, 1984). Thus, with experience, people increasingly use the most efficient method for performing simple addition. Lemaire and Siegler (1995) proposed four dimensions of "strategic competence" that should change over time: a) which strategies/procedures are used (*repertoire*), b) when each strategy/procedure is used (*frequency*), c) how well each strategy/procedure is executed (*efficiency*), and d) how strategies/procedures are chosen (*adaptivity*). Changes in these dimensions may explain the "global improvements in speed and accuracy that characterize learning" (p. 84). They observed that, even over the year that they were in second grade, children changed their relative frequency of use of various single-digit multiplication procedures, chose better ways to execute the procedures they used, became better at choosing the best procedure for each problem type, and executed procedures faster and with fewer errors. They found that the factors that determined the speed and accuracy (i.e., effectiveness) of a procedure also predicted the frequency of use of that procedure. They concluded that children increasingly choose more effective ways of executing all procedures by learning to estimate what works best for each problem or type of problem.

Learning to choose the fastest and most accurate way to solve each problem leads children to increasingly better performance overall. By the time they are adults, then, they should be adaptive procedure users. Shrager and Siegler (1998) generated the SCADS computational model to explain the choice of problem-solving procedures in simple arithmetic. According to the model, people store data about the speed and accuracy with which a problem is solved using a particular procedure. Procedure choice is based on the relative strengths of the procedures in solving specific problems and types of problems. These associative strengths are based on the past speed and accuracy of each procedure in solving the problem at hand as well as problems in general and problems that have similar features. During problem-solving, when a procedure's association strength exceeds a predefined confidence criterion, that procedure is executed. Thus, the SCADS model demonstrates the increasingly efficient use of increasingly effective arithmetic solution procedures in simple arithmetic over time based on associative strengths that are continually adjusted with computational experience.

Solution Procedures in Multi-digit Arithmetic

In many European countries, including Holland, Germany, England, and Wales, the elementary education system emphasizes mental arithmetic skills for multi-digit arithmetic (Beishuizen, 1993; Blöte, Klein, & Beishuizen, 2000). The school curriculum encourages flexibility in the use of mental calculation procedures based on the belief that mental arithmetic flexibility contributes to better computational skills in general and to a greater understanding of the properties of numbers, such as their decimal structure and how they can be decomposed and put back together while maintaining their value (i.e., additive composition or part-whole relations). In North American schools, written multi-digit arithmetic is introduced earlier and less emphasis is placed on learning a variety of mental arithmetic procedures (Beishuizen, 1993; Ferrini-Mundy & Schmidt, 2005; Fuson, 1990; Resnick & Omanson, 1987; Ontario Mathematics Curriculum Grades 1-8, 2005). Nonetheless, children in both Europe and North America use some variety of solution procedures (e.g., Fuson, 1990; Lucangeli et al., 2003). Further, research has shown that preschool children and children in early elementary school invent multi-digit arithmetic procedures before they are formally taught (e.g., Carpenter, et al., 1998; Carpenter & Moser, 1984) and children improve in their use of calculation procedures as their math skills develop (e.g., Lemaire & Siegler, 1995).

If the result of experience with multi-digit calculation procedures is an increasing knowledge of their relative efficacy and a consequent tendency to choose procedures according to their effectiveness on a particular problem (as in SCADS for simple arithmetic), then adults can be expected to have reached a more or less "expert" level of adaptive procedure use. In other words, they should have gained sufficient knowledge about the relative effectiveness of various procedures to be quite good at selecting the most efficient procedure when solving multi-digit problems.

Digit Versus Holistic Procedures

The research on children's use of procedures in multi-digit addition has identified several distinguishable solution methods that fall into two main categories. First, there are procedures in which the addends are treated as concatenations of single digits for the purpose of calculation. The prototypical exemplar of such a procedure is the column-by-column algorithm that is taught in school for paper-and-pencil calculation, customarily applied from right to left to arithmetic problems presented vertically on the page, aligned in columns. For addition, this procedure involves a series of simple sums. If the sum in any of these simple additions is greater than nine, a value is "carried" over and added to the sum in the next simple addition to the left. This sub-procedure is commonly referred to as a *carry operation.* In the present research, this addition procedure is referred to as the *digit algorithm*, to emphasize the digit-by-digit representation of the numbers. The word "columnar", often used to describe this procedure, was avoided here in order not to imply that problems presented horizontally are mentally restructured into vertical format to be solved in this way. It has not, as yet, been established that a digit-by-digit algorithm necessarily requires that the numbers be mentally represented in vertical columns (cf. Hayes, 1973).

The other category of solution procedures for multi-digit addition comprises those in which the operands are represented and manipulated in a more holistic manner. In 2-plus 2-digit addition problems, for example, operands are thought of as groups of decades and units and are decomposed in various ways into tens and ones and recomposed to arrive at the answer. For example, in the problem 43 + 26, the second operand is thought of throughout calculation as 20 and 6, or two 10s and 6 ones, not as the digits 2 and 6 side-by-side. Likewise, 43 is represented as 40 and 3 throughout calculation, and not as 4 and 3 side-by-side. Thus, the problem may be solved as 43 + 10 + 10 + 6 (i.e., 43, 53, 63, 69) or as 40 + 20 + 3 + 6 (i.e., 40, 60, 63, 69).

A variety of such procedures have been identified in the developmental and education literature. For example, Dutch elementary school mathematics programs include instruction in holistic mental arithmetic procedures for multi-digit addition and subtraction in early elementary school (Beishuizen, 1993; Beishuizen et al., 1997). Consequently, young school children employ procedures such as *decomposition in tens and ones* (termed "1010" by these authors), which involves breaking both operands into tens and ones (e.g., 62 + 25 becomes [60 + 20] + [2 + 5]), and *partial decomposition* (termed "N10"), which involves counting up or down by tens and then ones from one intact operand (e.g., 62 + 25 becomes 62 +10 +10 + 5, or 62 + 20 + 5), to perform two-digit mental addition. Children find the partial decomposition procedure more difficult to learn but researchers consider it more effective in the long run than decomposition of both operands because the latter leads to more errors with carry problems, at least in children (Beishuizen, 1993; Beishuizen et al., 1997). Similarly, children in Italy reported using retrieval, counting, the digit algorithm, decomposition, partial decomposition, and a transformation procedure that involves simplifying the problem by borrowing from one operand what is needed to bring the other up to a multiple of ten, then adding on what is left (e.g., 37 + 48 becomes 37 + 3 + [48 – 3] = 40 + 45; Lucangeli et al., 2003). Even in American schools, where digit algorithms are emphasized through the early introduction of written addition (Ferrini-Mundi & Schmidt, 2005; Fuson, 1990; Resnick & Omanson, 1987), children have been observed to use both the digit algorithm and holistic procedures for mental addition (Fuson, 1990).

Very little research has been conducted to determine whether adults use similar procedures to those reported by children doing complex mental addition. However, Lemaire and Arnaud (2008) gathered trial-by-trial procedure reports from younger and older adults and listed nine solution methods. Some of these appeared to be variations of the same procedure. For example, their participants reported both "rounding the first operand down" (e.g., 12 + 46 = [10 + 46] + 2) and "rounding the second operand down" (e.g., 12 + 46 = [12 + 40] + 6). These both appear to be instances of Beishuizen and colleagues' (Beishuizen, 1993; Beishuizen et al., 1997; Blöte, et al., 2000) N10 procedure, a partial decomposition procedure in which one operand is kept intact and the other is broken down into tens and ones. Lemaire and Arnaud's participants also reported using retrieval, decomposition ("rounding both operands down"), a digit algorithm ("columnar retrieval"), and several types of transformations, which usually involve both addition and subtraction (e.g., 23 + 49 = [23 + 50] – 1). Thus, there is evidence that adults continue to use the same set of procedures that children use for multi-digit addition.

Procedure Choice

Lucangeli et al. (2003) observed that "the choice of a strategy appears to be more a result of experience with it and frequency of its use than its level of effectiveness" (p. 518). This conclusion is at odds with Shrager and Siegler's (1998) SCADS model, which predicts that over time children increasingly choose procedures according to the effectiveness of those procedures on the type of problem being solved. Shrager and Siegler proposed that people estimate the likely effectiveness of each procedure on each problem from experience and choose the fastest and most accurate. Structural features of problems (e.g., presence of a carry; operands that are multiples of 10) should be inherently related to the effectiveness of one procedure relative to another and so these features should predict what procedure will be used. The best predictor of the frequency of use of a procedure should be related to its advantages relative to others and to the difficulty of executing it correctly. Thus, one question that arises is whether procedure selection among adults is consistent with the SCADS model or is more in line with the observations of Lucangeli and colleagues. In other words, do adults simply use the procedures they have been taught, or do they choose the best procedure for a given problem among a variety of procedures?

Problem Characteristics and Procedure Choice

One characteristic that has been found to predict procedure choice in simple arithmetic is problem complexity. Both adults (LeFevre, Sadesky, & Bisanz, 1996; Penner-Wilger, Leth-Steensen, & LeFevre, 2002) and children (Siegler, 1988) select non-retrieval procedures more often on larger multiplication problems (e.g., 8 x 9) than on smaller problems (e.g., 2 x 3). Children use repeated addition on the most difficult problems (Lemaire & Siegler, 1995). In multi-digit arithmetic, adults have consistently been shown to be faster and more accurate when solving problems without a carry than those requiring a carry operation (e.g., Ashcraft, et al., 1992; Frensch & Geary, 1993; Imbo, Vandierendonck, & DeRammelaere, 2007). Problems become more difficult as the number of carries in the problem increases (e.g., carries in both the ones position and the tens position; Imbo et al., 2007) and adults have reported using different procedures for harder than for easier problems (Lemaire & Arnaud, 2008). Thus, I hypothesized that people would take a different approach to solving the more difficult carry problems than no-carry problems and that adults' selection of procedures to solve multi-digit arithmetic problems would vary with the complexity of the problem. This research project was an investigation of adults' use of solution procedures for multi-digit mental arithmetic, with a focus on addition problems such as 32 + 27 and 46 + 7. I conducted a series of studies in which adults' procedure use across a variety of multi-digit problems was manipulated and assessed. Complexity was varied by including problems with and without a carry in the units and problems with and without a carry in the decades.

In addition to complexity, the manner in which the problems were presented was varied. Two- plus two-digit and two- plus one-digit problems were presented both vertically and horizontally and two- plus one-digit problems were presented with the larger operand first and with the smaller operand first. Available solution procedures appear to be better suited to one presentation over another. For example, vertical presentation, with the ones digits and the tens digits aligned, may lead people to use a digit procedure. On the other hand, a digit procedure might be more difficult to use on a problem presented horizontally. Such problems may lead people to use holistic procedures. In fact, this conjecture is supported by Trbovich and LeFevre's (2003) observation that adults used visual working memory more on vertical problems and auditory working memory more on horizontal problems. The digit algorithm, which involves adding both the ones digits and the tens digits as simple sums and keeping track of carried digits, is likely to be relatively more dependent on visual working memory. Holistic procedures, on the other hand, are likely to depend more on auditory memory to keep track of intermediate sums (Noël, Désert, Aubrun, & Seron, 2001).

Individual Differences and Procedure Choice

Geary et al. (1991) found that, in the normal course of early development in mathematics, most children shifted to using more efficient procedures, like retrieval, and to executing procedures more quickly; however, math-disabled children did not demonstrate these changes within the age range that was tested (i.e., Grades 1 – 2). Performance and procedure use in mental computation and other math-related tasks is also related to age (Luwel et al., 2005), math anxiety (Ashcraft & Kirk, 2001) and calculation fluency (e.g., LeFevre & Kulak, 1994; Smith-Chant & LeFevre, 2003). Thus, although problem characteristics such as complexity and orientation of presentation may influence procedure choice through their interaction with procedure effectiveness, individual differences are also likely to be important in understanding how people choose solution procedures in mental arithmetic. In the present research, arithmetic fluency (i.e., efficiency as indexed by speed and accuracy) and problem-solving style were examined as individual differences that may affect the way in which adults use procedures in complex mental arithmetic.

The Choice/No-Choice Paradigm

In order to determine whether adults choose mental arithmetic procedures based on their effectiveness, it is necessary to measure the performance characteristics (i.e., speed and accuracy) of those procedures. The standard paradigm used to assess the effectiveness of procedures involves presenting a set of problems, noting procedure use on each one, then calculating the median speed and percent accuracy that accompany the use of each procedure. However, this approach confounds procedure effectiveness, individual characteristics, and procedure choice. For example, Beishuizen and colleagues (Beishuizen, 1993; Wolters, Beishuizen, Broers, & Knoppert, 1990) selected consistent decomposition ("1010") and partial decomposition ("N10") users to solve arithmetic problems. They found that the partial decomposition method was faster and more accurate but it was also found that children who consistently used partial decomposition were stronger students overall and had better basic fact knowledge (Wolters et al., 1990) than the other group. Thus, their estimates of procedure effectiveness were confounded with students' ability.

Siegler and Lemaire (1997) proposed that, if the speed and accuracy of procedures is measured when people have chosen which one to use, these measures will be biased by selection effects involving features of the problems on which procedures are used and characteristics of the people who choose each procedure most often. That is, we would overestimate the speed and accuracy of procedures that are used disproportionately on easy problems or by high-skill individuals. On the other hand, the speed and accuracy of procedures that are used disproportionately on difficult problems or by low-skill individuals would be underestimated. Hence, an apparent procedure advantage will not have a straightforward interpretation. To solve this problem, Siegler and Lemaire developed the "choice/no-choice" paradigm to examine the performance characteristics of problem-solving procedures and to evaluate how procedure choices vary with these performance characteristics.

In the *choice* condition of the choice/no-choice paradigm, participants are presented with a task and allowed to choose the solution method they will use. Usually, they are required to choose from two or three typical or common procedures. In contrast, in the *no-choice* condition, all participants are instructed to use a particular procedure on all types of problems. The speed and accuracy of procedures are estimated in the no-choice condition and people's procedure choices in the choice condition are examined in relation to these speed and accuracy measures to determine whether or not procedure choice can be predicted from procedure effectiveness.

The choice/no choice paradigm has been implemented in a number of ways. Siegler and Lemaire (1997) presented adults with the choice of a) mental calculation, b) using a calculator, or c) using pencil and paper to solve multi-digit multiplication problems. All participants then completed three no-choice conditions, counterbalanced for order, one in which they were instructed to use mental addition, one in which they were instructed to use a calculator, and one in which they were instructed to use pencil and paper. Thus, the no-choice condition was a repeated measure and all of the procedures available in the choice condition were evaluated in no-choice conditions. Siegler and Lemaire found procedure effectiveness (i.e., speed and accuracy as determined in the no-choice conditions) to be the strongest predictor of procedure frequency on a given problem, followed by problem characteristics. They also found that participants performed better when they had a choice than when they did not.

Lemaire and Lecacheur (2002) varied the paradigm to investigate the procedures used by adults and children in a complex addition estimation task. In this study, choice/no-choice condition was a between-groups variable. Participants were given a limited choice between two of several procedures commonly used in estimation (LeFevre, Greenham, & Waheed, 1993; Lemaire, Lecacheur, & Farioli, 2000), although note that most of the participants expressed a preference for a third procedure which was not allowed. Overall, participants' procedure selection was influenced by problem characteristics and relative procedure effectiveness. Lemaire and Lecacheur also observed age-related changes in strategic behaviour. It seems, however, that these results are limited by the fact that what was intended to be a choice condition may have been for many participants effectively a no-choice or limited-choice condition in which a preferred procedure was unavailable.

Using another variant of the choice/no choice procedure, Abbate and Di Nuovo (1998) examined the use of complex mental addition procedures in adults. Participants reported the procedure they had used on a block of 15 addition problems with and without a carry operation. For the second block of 15 problems, the participants who had reported using "non-visual procedures" (e.g., decomposition) were instructed to use the "visual columnar strategy" (i.e., digit algorithm) reported by the others. The individuals who spontaneously chose the digit algorithm were instructed to continue using the digit algorithm. In other words, all participants used the digit algorithm on all problems in the no-choice condition. In the no-choice condition, the participants who had changed solution procedures (i.e., from the holistic to the digit algorithm) performed more poorly on the carry problems than those who did not change (i.e., continued to use the digit algorithm). There was no difference in performance between the two groups on the easier problems. The results highlight the importance of considering procedure choice in light of problem complexity and individual differences. The design is incomplete, however, in that only the holistic procedure users were required to change from their preferred solution method. Futhermore, Abbatte and DiNuovo assumed that people are stable procedure users, that is, that they use only one procedure on all problems. This assumption seems unlikely, based on the variability in procedure use that has been observed in children and in adults solving simple arithmetic problems (e.g., Smith-Chant & LeFevre, 2003).

Imbo and LeFevre (2009) had all participants solve complex addition problems varying in complexity (e.g., 34 + 21 and 16 + 38) in three conditions, a choice condition and two no-choice conditions. On a trial-by-trial basis, participants in the choice condition reported whether they had used the *units-tens* (UT) procedure (i.e., adding the units first followed by the tens) or the *tens-units* (TU) procedure. They found that people's approach to procedure choice varied across cultures (i.e., Belgians, Canadians, and Chinese). This study had the advantage of obtaining procedure reports for all trials but limited participants' choice of procedures. Furthermore, both the digit algorithm and various holistic procedures can be implemented from right to left or from left to right i.e., units first or tens first), so the actual procedure being used was indeterminate. For example, 43 + 25 may be solved using decomposition as either 40 + 20 + 3 + 5 or as 20 + 40 + 5 + 3 (or, for that matter, as 40 + 20 + 5 + 3). Thus, the use of the UT/TU dichotomy may obscure some important differences in performance and procedure choice between using digit and holistic procedures.

In summary, Siegler and Lemaire's (1997) "choice/no-choice" paradigm has proven to be an effective method to examine the performance characteristics of solution procedures. It has been implemented in a variety of ways. In most cases, the choice condition has been constrained such that participants are asked to choose among two or three procedures. These may not include procedures that people would choose in a free choice situation. Consequently, any analysis of factors affecting people's procedure choices is also constrained and generalization to people's free choice in everyday situations is limited. In addition, only Imbo and LeFevre (2009) have taken into account the possible interaction of problem characteristics and individual differences in measuring procedure effectiveness in the no-choice condition. For example, decomposition may be more effective than the digit algorithm in solving horizontal problems for people with strong overall math fluency but not for individuals with weaker math skills. With these considerations in mind, the present research evaluated the performance characteristics of solution procedures and the effectiveness of adults' procedure choices using the choice/no-choice paradigm with an unconstrained choice condition and three no-choice conditions based on the most frequently reported solution procedures for 2-digit plus 2-digit addition. Procedure effectiveness was measured within problem characteristics and individual differences.

Adaptivity

The fourth dimension in Lemaire and Siegler's (1995) model of 'strategic competence' is the *adaptivity* with which people choose among available problem-solving procedures. Procedural adaptivity has captured the interest of a number of researchers in recent years (e.g., Imbo & LeFevre, 2009; Lemaire et al., 2004; Lemaire & Lecacheur, 2002; Luwel, Lemaire, & Verschaffel, 2005; Luwel, Verschaffel, Oghena, & DeCorte, 2003; Siegler & Lemaire, 1997; Torbeyns, Verschaffell, & Ghesquière, 2004a; 2004b; Torbeyns, Vanderveken, Verschaffell, & Ghesquière, 2006; Verschaffell, Torbeyns, DeSmedt, Luwel, & Van Dooren, 2007). In some instances, the terms adaptivity and flexibility are used interchangeably (Heirdsfield & Cooper, 2002), suggesting that any flexible use of a variety of procedures is adaptive. Others consider procedure choice adaptive to the extent that procedure selection is adapted to task demands (e.g., arithmetic operation, problem type) and procedure characteristics (e.g., Imbo & LeFevre, 2009; Luwel, Lemaire, & Verschaffel, 2005; Torbeyns et al., 2004).

However, the majority of studies of adaptivity incorporate the assumption that to be adaptive a choice must be in some way advantageous to the individual. More specifically, adaptivity in procedure choice refers to the ability to choose the most efficient procedure in a given situation (e.g., Imbo & LeFevre, 2009; Luwel et al., 2005; Torbeyns et al., 2004a; 2004b). This is usually the procedure that produces the correct answer (or closest estimate) in the shortest time (e.g., Lemaire & Siegler, 1995).

The adaptivity concept has been operationalized in a number of ways, using the choice/no-choice method. One way has been to examine the correlations, on an item-by-item basis, between the relative speed and accuracy of procedures in the no-choice condition and the frequency of procedure use in the choice condition (e.g., Siegler & Lemaire, 1997; Torbeyns et al., 2006). Another method has been to measure the proportion of questions in the choice condition on which participants chose their 'best' procedure on a given type of problem, as determined by speed and accuracy of the procedure on that type of problem in the no-choice condition (Imbo & LeFevre, 2009; Torbeyns et al., 2004a). In the present research, the latter approach was used to measure the extent to which adults chose procedures on the basis of procedure characteristics. The concept of adaptivity as procedural flexibility based on problem characteristics and procedure characteristics was examined.

The Present Research

It is clear from the above discussion that, overall, both children and adults use a variety of solution procedures when performing simple arithmetic. The specific procedure used to solve a given problem is related to the task (e.g., estimation, verification, production), the operation (e.g., addition, multiplication), and characteristics of the problem, such as complexity. We also know that children use a variety of procedures to solve more complex multi-digit mental arithmetic problems. In Europe, mental arithmetic is stressed in elementary school and various holistic solution procedures are taught for multi-digit addition. Flexible procedure use is encouraged, as it is considered to reflect more sophisticated mathematical ability and is believed to contribute to a better-developed number sense (Beishuizen, 1993; Blöte, et al., 2000; Ferrini-Mundy & Schmidt, 2005). Most of the previous research regarding solution procedures for complex mental arithmetic has involved young children who are in the process of learning and refining arithmetic procedures. Longitudinal studies show that, overall, children use a variety of procedures early on. As learning progresses, several changes occur. There is a shift in the relative frequency of use of each procedure toward an increasing use of faster and more accurate approaches and an increase in the use of more effective ways to execute procedures. Also, with experience, children become faster and more accurate in the execution of all computation procedures and better at choosing the best one for each problem. An important question, therefore, is what is the final outcome of this process of change? That is, do adults continue to use a variety of solution procedures when they perform complex mental arithmetic and do they use only the most effective procedure for each problem? Further, do individual differences, for example, in overall arithmetic skill, contribute to adults' procedure choices?

The present research is an investigation of adults' use of solution procedures in performing complex mental addition. Complex addition is defined here as two-digit plus one-digit sums (e.g., 23 + 5; Experiment 1) and two-digit plus two-digit sums (e.g., 34 + 28; Experiments 2 through 4). The experiments described below were designed to define a comprehensive set of procedures that adults use to solve multi-digit mental addition problems and to determine a) whether or not all adults use a variety of procedures; b) to what extent procedure choices are related to problem characteristics (i.e., problem complexity; orientation of presentation - horizontal vs. vertical) and individual differences in arithmetic skill; c) whether some procedures are more effective overall for multi-digit addition; d) whether the speed and accuracy of procedures varies with problem characteristics; and e) whether adults choose procedures based on their effectiveness. I employed the choice/no-choice paradigm to test whether adult's mental addition procedure choices are related to procedure characteristics (speed and accuracy), problem characteristics (complexity and orientation), and individual difference characteristics (arithmetic skill).

CHAPTER 2

In this chapter, I describe how individual differences in arithmetic skill were assessed and used to compare participants across four experiments. In the present research, fluency (i.e., speed and accuracy) on a multi-digit paper-and-pencil arithmetic task was examined as a factor that is likely to have an impact on adults' performance in complex mental arithmetic and that may also influence their choice of solution procedures.

Several researchers have argued the importance of taking individual differences into consideration in studying complex cognitive tasks such as mental arithmetic (e.g., Ashcraft & Kirk, 2001; LeFevre & Kulak, 1994; Luwel et al. 2005; Torbeyns et al., 2004b; Smith-Chant & LeFevre, 2003). For example, performance and procedure use in mental computation and other math-related tasks has been shown to be related to age (Luwel et al., 2005), math ability/disability (Geary, 1991; Torbeyns et al., 2004b), math anxiety (Ashcraft & Kirk, 2001) and calculation fluency (e.g., LeFevre & Kulak, 1994; Smith-Chant & LeFevre, 2003).

Smith-Chant and LeFevre (2003) divided adults into three groups on the basis of arithmetic fluency, or skill (i.e., high, average, and low), measured as speed and accuracy of execution of multi-digit arithmetic procedures using pencil and paper. These authors demonstrated that participants' speed and accuracy and their use of solution procedures in solving single-digit mental multiplication varied with skill. For example, participants in the low-skill group used a greater variety of procedures than did the average- and high-skill participants and were more likely to vary their choice of procedure with variation in instructional requirements. They also found that response latencies and errors on a multiplication production task increased as arithmetic skill decreased. Similarly, LeFevre & Kulak (1994) found evidence that speed of obligatory activation of simple addition facts was related to arithmetic skill. Thus, among other individual differences, arithmetic skill is related to simple arithmetic calculation processes. On these and other complex cognitive tasks, averaging across skill levels may result in misleading conclusions regarding the underlying processes involved. Therefore, I measured arithmetic fluency in the present experiments and, using Smith-Chant and LeFevre's three-category skill grouping, I examined the relations between performance and procedure choice in complex mental arithmetic, on the one hand, and skill, on the other.

In this chapter, I examine whether Smith-Chant and LeFevre's three-way grouping of participants by arithmetic skill successfully separates distinct skill groups in the present research and I determine the similarity of the distribution of skill across the samples in Experiments 1, 2, and 3.

Method

In the first three experiments, participants completed two subtests of the Kit of Factor-Referenced Cognitive Tests (the 'French Kit'; French, Ekstrom, & Price, 1963), as a measure of arithmetic skill.

Participants

All of the participants in this research completed the fluency test, for a total of 108 individuals: 24 in Experiment 1, 24 in Experiment 2, and 60 in Experiment 3. Participants in Experiment 4 were a subgroup of the participants in Experiment 3 and did not complete the French Kit again for this experiment.

Materials

Each arithmetic subtest consists of two pages of multi-digit problems arranged in rows. The addition problems have three terms, each either a one- or a two-digit number (e.g., 34 + 56 + 27; 47 + 8 + 92). On each page of the addition subtest, the problems are arranged in columnar form (i.e., vertically) in six rows of 10 problems (i.e., 60 per page for a total of 120 problems). Each page of the subtraction-multiplication subtest also has six rows of 10 problems presented vertically. Beginning with the first, every other row consists of two- by two-digit subtraction problems (e.g., 34 - 18); alternate rows consist of two- by one-digit multiplication problems (e.g., 54 x 6). Hence, there are a total of 120 problems on the two pages of the subtraction-multiplication subtest.

Procedure

Participants completed the addition subtest of the French Kit, Parts 1 and 2 (one page each). Before beginning the addition subtest, they completed 10 practice addition problems. Two minutes were allowed for each page and they were instructed to work as quickly and accurately as possible, one page at a time. They then solved 20 more practice problems, 10 subtraction and 10 multiplication, and completed the multiplication-subtraction subtest with instructions to work as quickly and accurately as possible, one page at a time. Again, they were allowed two minutes per page. A total score for each participant was calculated by summing the number correct across the four pages of the French Kit, with a possible maximum score of 240.

Results

To determine whether arithmetic skill levels varied across the three experiments in which participants completed the French Kit and whether the performance of a subgroup of Italian participants differed from the overall group in a way that would affect interpretation of the results, total scores were analyzed using two separate one-way ANOVAs, one including all participants and one excluding the twelve Italian participants in Experiment 2. The results are shown in Figure 1. There was no significant difference in scores across the three experiments, either with the Italians, $F(2,105) = 0.428$, $MS_e = 586$, $p = 0.65$, or without them, $F(2,93) = 0.444$, $MS_e = 642$, $p = 0.64$. An independent samples t-test, with degrees of freedom adjusted for inequality of variance, indicated no significant difference in scores between the Canadian and Italian groups in Experiment 2, $t(16) = 1.19$, mean difference = 8.9, $p = 0.25$.

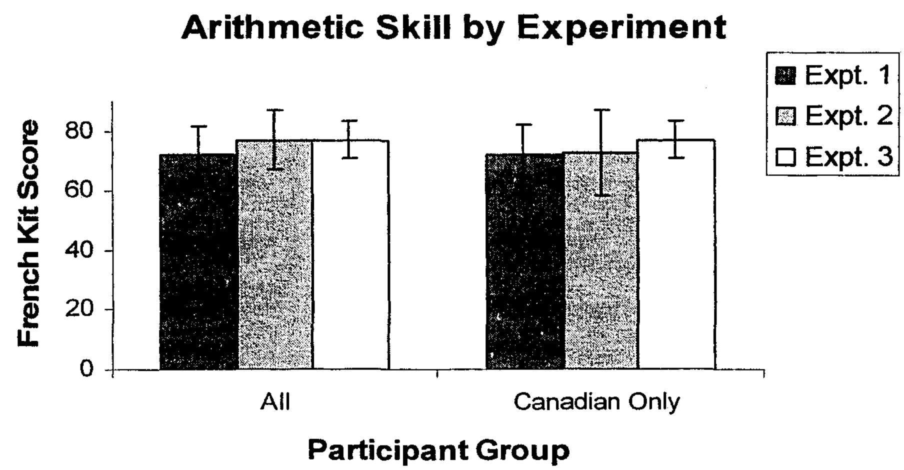

Figure 1. **Mean French Kit scores across Experiments 1, 2, and 3 for all participants and for Canadian participants only. Error bars represent 95% confidence intervals, based on the *MSe* within groups.**

Table 1

Means and Standard Deviations of French Kit Scores by Skill Group for the Overall Sample

Skill Group		Full Sample			Canadians Only	
	N	*M*	*SD*	*N*	*M*	*SD*
Low	45	53	11.5	43	52	11.5
Average	36	83	6.4	27	83	6.2
High	27	106	13.7	26	106	14.0
Total	108	76	24.1	96	75	25.2

Note. The full sample includes the Italian (n = 12) and the Canadian participants (n = 96).

In the overall sample (n = 108), participants' mean skills in multi-digit arithmetic were in the average range, between 70 and 90 (M = 76, SD = 24.1), with a range of 27 to 150. Participants were divided into three skill groups based on the criteria used by Smith-Chant and LeFevre (2003): low-skill participants had total French Kit scores less than 70, average-skill participants had scores between 70 and 89, and high-skill participants had scores of 90 or above. The distribution of scores by skill group is shown in Table 1. Forty-two percent of the participants were low-skill, with 33% qualifying as average and 25% as high skill, respectively. As expected, there was a significant difference in arithmetic skill across these groups, both with Italian participants included, $F(2,105) = 215.13$, $MS_e = 116$, $p < .001$, and with Italian participants excluded, $F(2,93) = 199.21$, $MS_e = 123$, $p < .001$. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). All pairwise comparisons were significant in both analyses. There was no difference in the distributions of low, average, and high skill participants across experiments, χ^2 (4, N=108) = 4.43, p = 0.35.

Discussion

Thus, Smith-Chant and LeFevre's (2003) three-way grouping of participants by arithmetic skill effectively delineated distinct levels of arithmetic fluency in the overall sample. Therefore, these skill designations were used in each of the experiments that follow. Further, the distribution of skill across the samples in Experiments 1, 2, and 3 was similar.

CHAPTER 3

Research indicates that children and adults use a variety of procedures for simple mental calculation and that children also report using several procedures to solve multi-digit calculations. Children's procedures fall into two main categories, digit-oriented and holistic, based on the form of representation of the operands and the cognitive processes involved. In this chapter, I discuss two experiments in which a self-report paradigm was employed in order to determine what procedures adults use to solve multi-digit addition problems mentally. In each experiment, responses were recorded verbatim and grouped according to similarity. From this information, a list of commonly reported calculation procedures was developed for future use. Two further goals of these experiments were to discover whether adults are flexible mental addition problem-solvers (i.e., whether some or all adults use a variety of calculation procedures) and, then, for those who are flexible, to examine how procedure characteristics and individual differences influence their choice of procedures in solving these problems. Most of the participants were educated in Canada; however, in Experiment 2, I had the opportunity to collect the same data from groups of Canadians and Italians. Thus, I was able to compare the procedural repertoires, performance efficiency, and procedure use of a group of North Americans and a group of Europeans.

In the first experiment, Canadian adults solved one-digit plus two-digit mental addition problems and reported their solution procedures. The problems were similar to those used by Trbovich and LeFevre (2003). Trbovich and LeFevre found that adults relied more on phonological than on visual working memory when solving problems presented horizontally and more on visual than on phonological working memory when solving problems presented vertically. Based on the observed differential involvement of these components of working memory, they hypothesized that adults may use different calculation methods to solve addition problems depending on the orientation in which they are presented. In the first experiment, presentation orientation (horizontal vs. vertical) was varied. I hypothesized that, among those who employ a variety of calculation procedures, vertically-presented problems would be solved predominantly using the digit algorithm, which can be expected to rely largely on visually-based cognitive processes. On the other hand, I expected horizontally-presented problems to be solved predominantly using holistic procedures such as decomposition and transformation which appear to be more dependent on verbally-based cognitive processes. Trbovich and LeFevre demonstrated that order of presentation (i.e., one-digit addend first vs. two-digit addend first) affected two-plus-one digit addition performance. In addition, problem complexity (i.e., carry vs. no carry) has consistently been shown to influence arithmetic performance. Therefore, order of presentation and complexity were varied to test whether these problem characteristics influenced adults' choice of computation procedures.

In the second experiment, groups of Canadian and Italian adults solved two-digit plus two-digit mental addition problems and reported their solution procedures. Presentation orientation and problem complexity were varied, again to determine whether these problem characteristics influenced adults' choice of computation procedures.

Experiment 1

Trbovich and LeFevre (2003) proposed that people's differential use of phonological and visual components of working memory for horizontal and vertical problems may be attributable to the use of different procedures based on orientation. It has also been established that people are slower on horizontal than vertical problems, especially when the 1-digit operand is presented first (e.g. Brysbaert et al., 1998; Trbovich & LeFevre, 2003), and on more complex problems (e.g., Imbo, Vandierendonck, & DeRammelaere, 2007). These response-time differences also may be related to the procedures people use to solve problems with varying characteristics. The goal of this experiment was to determine what procedures adults use to mentally solve 2- plus 1-digit addition problems (similar to those used by Trbovich and LeFevre) and to test whether their procedure choices are related to the problem characteristics orientation, operand order, and complexity.

Method

Participants

Twenty-four undergraduate psychology students educated in Canada (8 females), ranging in age from 19 to 25 years (median = 20 years), received course credit for their participation.

Materials

Addition Task. Participants solved 48 different 1-digit plus 2-digit addition problems with sums less than 100, 24 with a carry and 24 without (see Table 2 for examples and Appendix B1 for a complete list). Ties (e.g., 37 + 7) and zeros were excluded. Each problem was solved in both orders (1-digit first and 2-digit first), thus the list included 96 addition problems. For all participants, these 96 problems were presented both horizontally and vertically. Participants, therefore, solved 192 problems, divided into two blocks of 96, horizontal and vertical.

Table 2

Experiment 1: Sample Two-digit Plus One-digit Problems

	No-Carry		Carry	
	Horizontal	Vertical	Horizontal	Vertical
One-Digit First	2+73 =	5 + 43	9+13 =	8 +34
Two-Digit First	24+3 =	32 + 6	28+9 =	62 + 9

Skill Test. As discussed in Chapter 2, participants completed the addition and multiplication-subtraction subtests from the Kit of Factor-Referenced Cognitive Tests (French, et al., 1963) as a measure of arithmetic skill.

Procedure

Participants were tested individually in a single session lasting between 1½ and 2 hours. The experiment was programmed in E-Prime. Presentation orientation was blocked and participants were randomly assigned to orientation conditions so that one half received all horizontal problems first and the other half received all vertical problems first. Order was randomized within blocks.

The addition problems were presented on two computer monitors; the participant watched one monitor while the experimenter watched the other. Problems were presented in Courier New 30-point font and instructions were in Courier New 24-point font. Participants were asked to solve each problem as quickly and accurately as possible. In the vertical presentation, the one-digit addend was aligned with the unit value of the two-digit addend.

Each participant first completed 10 practice trials. During the practice session, participants were encouraged to report as much detail as possible about how they solved each problem and were given trial-by-trial feedback regarding the level of detail required. To begin each practice and experimental trial, the experimenter pressed the spacebar. An asterisk appeared in the centre of the screen, was displayed for 1 second, then flashed off and on twice at 200-ms intervals. On what would have been the fourth appearance of the asterisk, the addition problem appeared, centered where the asterisk had been. The problem remained on the screen until the participant responded. The participant solved the problem and gave the numeric answer orally. Participants responded using a headset microphone attached to a serial response box. Latencies were recorded using a voice-activated timing switch. When the participant reported the sum, the words "How did you solve the problem?" appeared in the centre of the screen. Numeric responses and detailed procedure reports were recorded by the experimenter.

Results

Skill Test

The distribution of scores in this sample was similar to that in the overall group. Participants' mean arithmetic skills overall were average ($M = 72$, $SD = 20.4$), with a range of 40 to 99. The distribution of individuals by skill group (as described earlier) is shown in Table 3. Half of the participants were in the low-skill group, with 25% qualifying as average and 25% as high skill, respectively.

Table 3

Experiment 1: Means and Standard Deviations for French Kit Scores by Skill Group

Skill	N	M	*SD*	Range
Low	12	53	8.3	40 – 65
Average	6	86	3.4	80 – 90
High	6	95	3.1	91 – 99
Total	24	71	20.4	40 – 99

Addition Task

The 24 participants solved 192 addition problems each, for a total of 4608 problems; 196 (4.3%) were invalid due to either inadvertent voice key triggers or equipment failures. For each participant, latencies less than 300 ms or greater than 3 standard deviations above the participant's mean latency (n = 52; 1.1%) were also considered invalid and were excluded from the analyses. The remaining latencies ranged from 486 ms to 9477 ms. Of the 4360 valid responses, 253 (5.8%) were errors. Percentage of incorrect responses and median latencies for correct responses were calculated for each participant in each operand order by problem orientation by complexity condition. Error percentages and median correct latencies were analyzed in separate 2 (operand order: 1-digit first, 2-digit first) by 2 (problem orientation: horizontal, vertical) by 2 (complexity: carry, no-carry) by 3 (skill: low, average, high) mixed ANOVAS with skill as a between-groups variable. Unless otherwise indicated, the alpha level for the analyses was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violations of sphericity assumptions. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). Ninety-five percent confidence intervals (CIs) are presented on figures. A summary of the ANOVAs is shown in Table 4.

Table 4

Experiment 1: Analysis of Variance for Latencies and Percent Errors for Adults Solving Two-digit Plus One-digit Addition Problems Varying in Order, Orientation, and Complexity

		Latencies		% Errors	
Source	*Df*	*MS*	*F*	*MS*	*F*
		Between			
Skill	2		4.45*		1.31
Error (Skill)	21	2,238,051		131.60	
		Within			
Operand Order	1		10.12^{**}		4.70^{*}
Order x Skill	2		0.25		1.05
Error (Order)	21	24,319		23.50	
Orientation	1		2.43		0.32
Orientation x Skill	2		0.12		0.11
Error (Orientation)	21	176,500		33.10	
Complexity	1		30.82^{**}		18.06^{**}
Complexity x Skill	2		3.70^{*}		0.69
Error (Complexity)	21	569,596		53.70	
Order x Orientation	1		10.21^{**}		0.62
Order x Orientation x Skill	2		0.09		0.70

Error (Order x Orientation)	21	21,972		10.60	
Order x Complexity	1		0.16		0.02
Order x Complexity x Skill	2		0.79		2.71
Error (Order x Complexity)	21	25,013		15.00	
Orientation x Complexity	1		0.08		7.83*
Orientation x Complexity x Skill	2		0.97		1.21
Error (Orientation x Complexity)	21	84,949		21.10	
Order x Orient. X Complexity	1		0.01		1.89
Order x Orient. X Complex. X Skill	2		0.16		1.98
Error (Order x Orient. X Complexity)	21	29,101		15.70	

Latencies. Overall, computation speed increased with arithmetic skill level (see Table 5). Both high-skill and average-skill adults solved problems faster than those in the low-skill group but there was no significant difference in latencies between the average- and high-skill groups.

Table 5

Experiment 1: Mean Latencies and Percent Errors (Standard Error in Parentheses) by Skill Group for Adults Solving Multi-digit Addition Problems

Arithmetic Skill	Latency	% Error
Low	1991 (153)	6.0 (1.8)
Average	1356 (216)	7.7 (1.7)
High	1338 (216)	4.0 (1.7)

Latencies were also influenced by problem characteristics. Problems without a carry operation were solved more quickly than carry problems (1,243 vs. 1,880 ms). A significant interaction between complexity and skill resulted from the fact that the carry effect was larger for low-skill participants (mean difference 1032 ms) than for average- or high-skill partic: ants (mean difference 464 ms and 417 ms, respectively). The main effect of orientation was not significant; however, there was a significant interaction between order and orientation (see Figure 2) such that participants solved 2- plus 1-digit problems more quickly when the problems were presented horizontally (1,513 vs. 1,685 ms), $F(1,21) = 7.32$, $p = .01$, whereas there was no difference in latencies across operand order in the vertical presentation (1,510 vs. 1,537 ms). These results involving operand order and orientation replicate the patterns observed by Brysbaert et al. (1998) and Trbovich and LeFevre (2003).

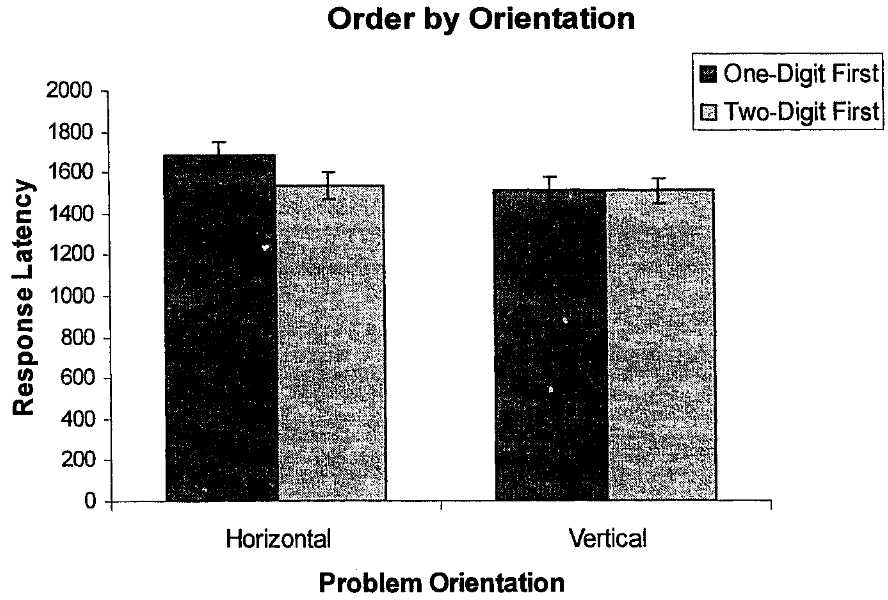

Figure 2. **Experiment 1: Mean response latencies for adults solving multi-digit addition problems presented horizontally or vertically and with the 1-digit or the 2-digit addend first. Error bars represent 95% confidence intervals, based on the MS_e within groups.**

Errors. Accuracy did not vary significantly across skill levels (see Table 5). However, problem characteristics did influence the rate of errors on these problems. Participants made fewer errors when problems were presented with the 2-digit addend first than when they were presented with the 1-digit addend first (5.1% vs. 6.7%). They also made fewer errors on no-carry than on carry problems (3.5% vs. 8.3%). Although there was no main effect for orientation, a significant orientation by complexity interaction (see Figure 3) occurred because the increase in the rate of errors with increased complexity (i.e., the carry) was greater for vertical problems than for horizontal problems (mean difference 6.7% vs. 2.8%, respectively).

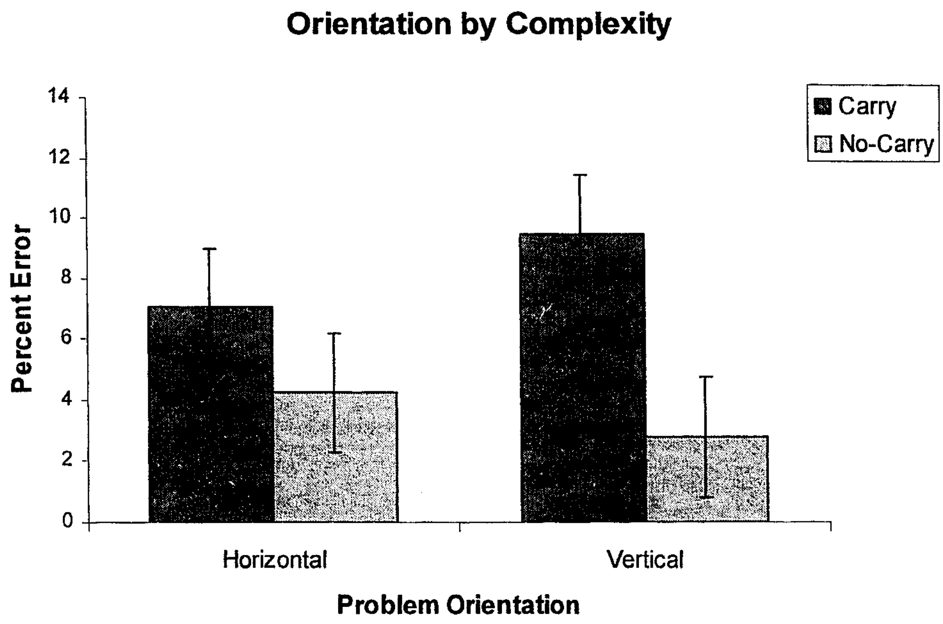

Figure 3. **Experiment 1: Mean percent errors for adults solving multi-digit addition problems with and without a carry, presented horizontally and vertically. Error bars represent 95% confidence intervals, based on the MS_e within groups.**

Summary: Latencies and Errors

Adults' performance on this 2- plus 1-digit mental addition task varied with overall arithmetic skill and with problem characteristics. Accuracy did not vary significantly across skill levels but low-skill solvers stood out as much slower and much more affected by increasing complexity in the problems than average- and high-skill solvers. Regardless of arithmetic skill, people were more accurate on problems when the 2-digit addend was presented first. They were also faster on 2- plus 1-digit problems when they were presented horizontally but not when problems were vertical. Further, participants were faster on no-carry problems across orientations and more accurate on no-carry problems in vertical, but not horizontal orientation.

These differences in speed and accuracy may be attributable in part to variation in choice of solution procedures. For example, if adults tend to use the standard digit algorithm to solve vertical problems but use holistic procedures, such as decomposition (e.g., 37 + 5 = [37 + 3] + 2) or counting (37 + 5 = 38, 39, 40, 41, 42), to solve horizontal problems, as Trbovich and LeFevre (2003) have suggested, the storage and recall of the carry digit necessitated by the digit algorithm may make vertical carry problems more difficult and error-prone. Procedure reports were examined to determine what procedures adults used to solve mental addition problems involving a 1-digit and a 2-digit addend and whether their procedure choices varied with arithmetic skill or problem characteristics.

Procedure Reports

The reported procedures were grouped according to similarity and five common procedures were coded. These were called *retrieval*, *digit algorithm*, *decomposition*, *transformation,* and *counting from the larger addend* (see Table 6 for descriptions and frequencies of procedures). The essential difference between decomposition and the digit algorithm is that when an operand is decomposed into tens and ones a magnitude representation of the tens is maintained throughout calculation, whereas, when using the digit algorithm, the tens digit is thought of as a single digit (e.g., for decomposition, 25 is represented as 20 and 5; for the digit algorithm, 25 is represented as a 2 and a 5).

Two participants' responses referred only to how the unit digits had been summed and could not, therefore, be categorized in terms of multi-digit addition procedures. Thus, for the analyses of procedure choice, twenty-two participants solved a total of 4224 problems; 182 (4.3%) were invalid due to either inadvertent voice key triggers or equipment failures. In addition, for each participant, latencies less than 300 ms or greater than 3 standard deviations above the participant's mean latency ($n = 47$; 1.1%) were considered invalid and were excluded from the analyses. The remaining latencies ranged from 486 ms to 9477 ms. Of the 3995 valid numeric responses, 118 (3.0%) procedure reports were unclear and could not be coded and 28 (0.7%) were idiosyncratic procedures or "I don't know" responses. These were excluded from the following analyses. Of the 3849 remaining cases, 213 (5.5%) were errors. Trials on which participants made errors were included in the analyses of procedure choice.

Table 6

Experiment 1: Descriptions and Frequencies of Reported Solution Procedures for One-Plus Two-digit Addition Problems

Procedure	% Frequency			Description
	Flex.	Stab.	Both	
Digit Algorithm	38	96	43	Participants treated the 2-digit addend as a concatenation of single digits. They added the 1-digit addend to the unit digit of the larger addend. In the case of a carry problem, they added 1 to the tens digit of the larger addend and put the resulting tens digit and ones digit together.
Decomposition	26	0	23	The larger addend was decomposed into tens and ones before adding. e.g., 25 + 4 = **20 + (5 + 4) = 20 + 9** = 29
Transformation	21	0	19	The addends were transformed in some way to make the addition easier or more familiar. e.g., 54 + 8 = **54 + 6 + (8 – 6) = 60 + 2** = 62; 54 + 5 = **54 + (6 – 1) = (54 + 6) – 1 = 60 – 1** = 59
Count	10	3	10	Participants counted up from the larger addend to the sum. e.g., 28 + 5 was solved by counting 29, 30, 31, 32, 33
Retrieval	5	2	5	Participants "just knew" the answer without calculating.

As shown in Table 6, the digit algorithm was the most frequently reported single procedure. Nevertheless, participants reported using holistic solution procedures (i.e., transformation, decomposition, or counting) on more than half (52%) of the addition problems. Retrieval was reported on only a small proportion of the problems and these tended to be the simplest problems (i.e., problems with no carry presented horizontally with the 2-digit operand first such as 32 + 3). Decomposition was the most popular of the holistic calculation methods, followed by transformation and counting. Two of the participants used the digit algorithm almost exclusively; that is, they reported using the digit algorithm more than 90% of the time and they used no other single procedure more than 5% of the time. Participants who used any one procedure more than 90% of the time and no other procedure more than 5% of the time and, thus, appeared not to choose solution procedures on any basis other than an overall preference for one approach will be referred to as *stable problem-solvers*. Participants who chose from among two or more solution procedures will be called *flexible problem-solvers*. In this sample, twenty people were flexible problem-solvers. All 20 reported using three or more different procedures to solve problems. Eighteen (90%) reported using each of 2 or more procedures on at least 15% of problems (see Appendix A1, Table A1.1).

Blöte and colleagues (e.g., Blöte et al., 2000; Blöte, Van der Burg, & Klein, 2001) consider flexible problem-solving to be the more desirable problem-solving style. They proposed that the flexible use of a variety of procedures in children's arithmetic problem-solving reflects better conceptual numerical knowledge. It remains to be established whether the flexible use of a variety of procedures in complex mental arithmetic leads to more effective (i.e., faster and/or more accurate) problem-solving. It was not feasible to compare the speed and accuracy of calculation between flexible and stable solvers in this experiment because there were so few in the stable group.

Only those 20 people who were flexible problem-solvers were included in the analysis of procedure choice to examine whether arithmetic skill or characteristics of the problem such as complexity, operand order, or orientation of presentation were related to people's choice of procedure. Percentage of use of each of the five procedures listed in Table 6 was calculated for each participant in each operand order by orientation by complexity condition. Retrieval (5% of problems) and counting (10% of problems) were then eliminated from the analyses. Thus, the remaining percentages were independent (i.e., did not sum to 100%). Percentage use of the three remaining procedures was analyzed in a 3 (procedure: digit algorithm, transformation, decomposition) by 2 (operand order: 1-digit first, 2-digit first) by 2 (problem orientation; horizontal, vertical) by 2 (complexity: carry, no-carry) by 3 (skill: low, average, high) mixed ANOVA with skill as the only between-groups variable. Only interactions involving the procedure variable are discussed, as these are the effects that reflect variations in procedure use with various factors. Unless otherwise indicated, the alpha level for the analyses was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violations of sphericity. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). Ninety-five percent confidence intervals (CIs) are presented on figures. A summary of the ANOVA is shown in Appendix A1 (Table A1.2).

Table 7

Experiment 1: Mean Percent Use of Solution Procedures across Arithmetic Skill (Standard Deviation in Brackets)

	Low Skill (n = 12)	Average Skill (n = 6)	High Skill (n = 6)
Digit Algorithm	42 (9.2)	23 (15.1)	41 (13.6)
Transformation	24 (4.9)	38 (8.1)	20 (7.2)
Decomposition	19 (6.6)	21 (11.0)	25 (9.8)

Mean percentages of reported use of the three solution procedures across levels of arithmetic skill are shown in Table 7. Participants appeared to choose solution procedures differently based on their level of arithmetic skill. For example, high- and low-skill participants reported using the digit algorithm more and holistic procedures less than did average-skill participants. However, there were no statistically significant main or interaction effects on procedure choice involving arithmetic skill. Thus, on the whole, adults at different levels of arithmetic skill did not demonstrate statistically significant differences in procedure choice.

Table 8

Experiment 1: Mean Percent Use of Solution Procedures Across Operand Order, Orientation of Presentation, and Problem Complexity (Standard Deviation in Brackets)

	Carry		No-Carry	
	Horizontal	Vertical	Horizontal	Vertical
Digit Algorithm				
1-plus-2	19 (8.0)	39 (9.4)	37 (9.9)	49 (10.0)
2-plus-1	21 (7.3)	35 (8.8)	38 (8.7)	45 (9.6)
Transformation				
1-plus-2	50 (7.6)	41 (7.7)	36 (9.9)	1 (0.4)
2-plus-1	46 (7.3)	42 (7.5)	2 (0.8)	1 (0.5)
Decomposition				
1-plus-2	19 (5.1)	12 (4.1)	0 (0)	35 (10.2)
2-plus-1	22 (6.6)	14 (5.0)	32 (9.0)	39 (10.9)

There were significant interaction effects of procedure use with operand order, $F(1.2, 21.3) = 9.69$, $MSE = 525.7$, orientation, $F(2, 34) = 6.51$, $MSE = 904.6$, and complexity, $F(2, 34) = 11.89$, $MSE = 2031.4$. As shown in Table 8, participants overall reported using transformation more on 1-digit-first problems than on 2-digit-first problems and decomposition more on 2-digit-first problems than on 1-digit-first problems. There was no difference in the percentage use of the digit algorithm between the two operand orders. As hypothesized, they reported transformation, a holistic procedure, more often for horizontal than vertical problems and the digit algorithm more often for vertical than horizontal problems. The digit algorithm was used on 13% more no-carry than carry problems but this difference was not statistically significant.

Transformation was also reported significantly more on carry than on no-carry problems. This result is not surprising, as the most common use of transformation is to "round up" the larger addend to the next decade number by breaking up the smaller addend into the amount needed for rounding up and the remainder (e.g., 45 + 7 = 45 + 5 + 2 = 50 + 2 = 52) and this approach is only relevant on carry problems. A carry problem such as 26 + 8 may also be solved by transformation as (26 + 6) + 2. However, this approach was less common. On no-carry problems, transformation was reported when a participant rounded the larger number to the next decade and then subtracted an amount for correction (e.g., 74 + 5 = 74 + 6 – 1 = 80 – 1 = 79). This approach was used only on no-carry problems with sums that ended in '9' and appeared to come into play predominantly on horizontal problems with the smaller addend first, probably the least familiar presentation for 2- plus 1-digit addition problems.

There were significant three-way interactions of procedure choice with operand order and complexity, $F(1.4, 23) = 11.56$, $MSE = 271.4$; orientation and complexity, $F(2, 34) = 6.50$, $MSE = 620.3$; and order and orientation, $F(1.2, 19.6) = 13.01$, $MSE = 359.9$ (see Appendix A1 for details and Figures A1.1 through A1.3). However, these and the two-way interactions were qualified by a significant four-way interaction of procedure choice with order, orientation, and complexity, $F(1.3, 21.4) = 7.87$, $MSE = 328.09$. As shown in Figure 4, procedure use was influenced by the orientation and complexity of the problems but these effects varied with the operand order. On one hand, the two orders of presentation appeared to elicit similar patterns of use of the digit algorithm. On the other hand, the patterns of use of decomposition and transformation varied across operand order.

Across operand orders, the digit algorithm was used less often for carry than for no-carry problems and the discrepancy was more pronounced for horizontal problems. In vertical orientation, decomposition was used less often for carry than for no-carry problems for both the 2-digit-first and the 1-digit-first problems. In contrast, for horizontal problems, decomposition was used less often for carry than no-carry 2-digit-first problems and more often for carry than no-carry 1-digit-first problems. Transformation was used very little for no-carry problems in general, with the exception of horizontal no-carry problems presented with the one-digit operand first. Transformation was used to solve 36% of these problems, compared with 1% to 2% of other no-carry problems. Thus, as the use of the digit algorithm decreased from no-carry to carry problems and from vertical to horizontal problems, the use of holistic procedures increased but not always in the same way. As participants shifted away from the digit algorithm with increasing problem complexity, they increased their use of transformation procedures. The use of decomposition varied in somewhat more complex ways.

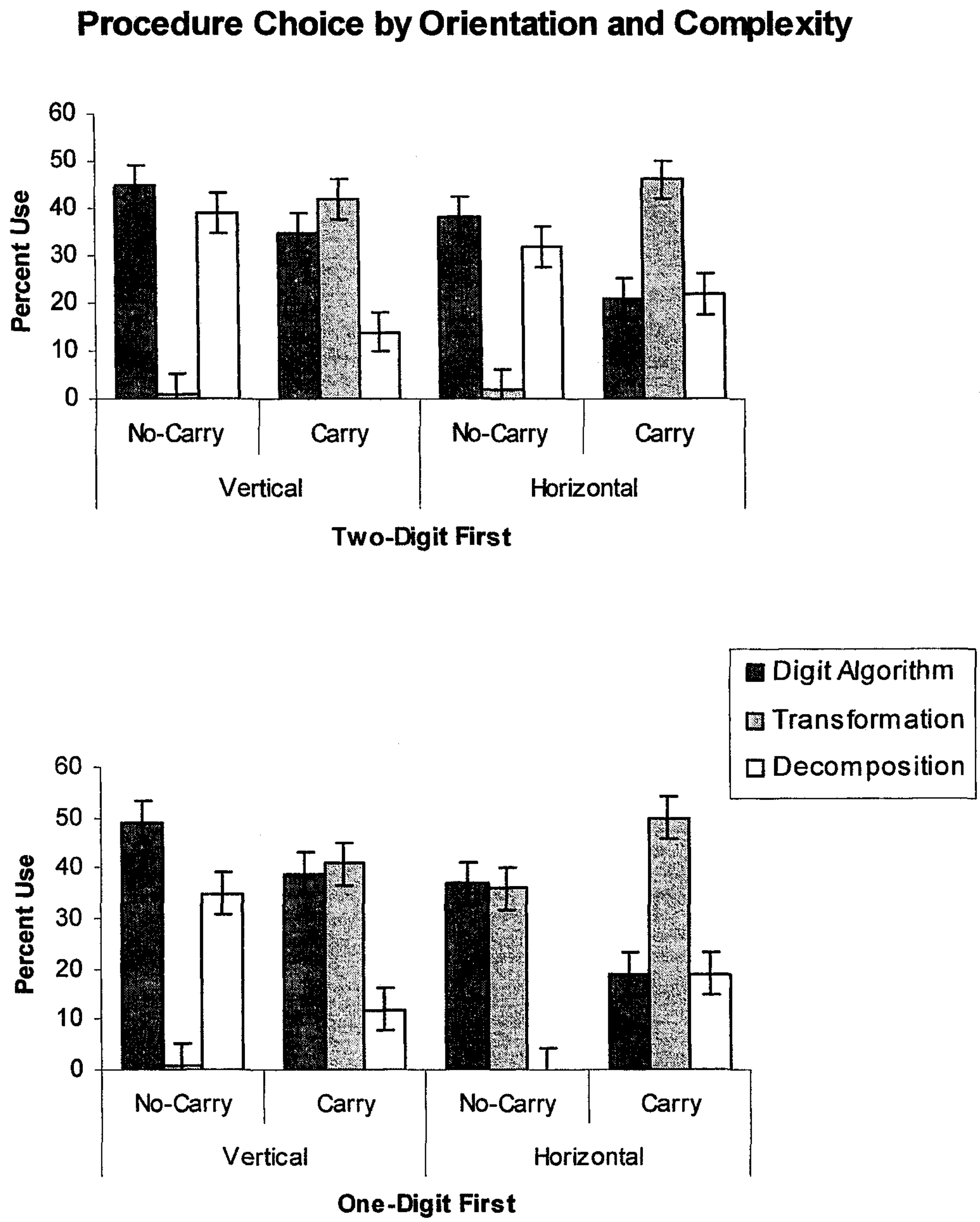

Figure 4. **Experiment 1: Mean percent use of the digit algorithm, transformation, and decomposition as a function of orientation, complexity, and operand order of double-digit plus single-digit addition problems. Error bars represent 95% confidence intervals, based on the MS_e from the four-way interaction.**

Summary: Procedure Reports

It is apparent that adults solved complex mental addition problems in different ways. Stable problem-solvers demonstrated a preference for one calculation procedure and used it to solve virtually all problems, regardless of problem characteristics. Flexible problem-solvers chose among a variety of calculation methods depending on characteristics of the problems. For example, in this experiment, flexible adult problem-solvers used the digit algorithm more for vertical than for horizontal problems and less for carry than for no-carry problems. In general, as the use of the digit algorithm decreased, the use of transformation increased. These results supported the view that solutions to multi-digit arithmetic problems vary with characteristics of the individual (flexible versus stable) and characteristics of the problems. In this experiment, most of the participants were flexible problem-solvers. However, I hypothesized that the pattern would be different if the problems were more difficult.

To further examine the solution procedures used to solve multi-digit addition problems, in the second experiment adults were presented with more difficult problems to solve. Problem characteristics were once again manipulated, this time with 2-digit plus 2-digit addition problems.

Experiment 2

One goal of this experiment was to determine what procedures adults use when solving 2- plus 2-digit addition problems with varying problem characteristics (i.e., orientation and complexity) and compile a concise list of solution procedures to be used in subsequent studies to provide participants with a response choice. Having such a list will eliminate the need in future experiments to record verbatim reports. This method, in which participants report their problem-solving approach by choosing from an established list of procedures, has been widely used in research regarding simple arithmetic (e.g., Campbell & Xue, 2001; Campbell & Penner-Wilger, 2006; LeFevre, DeStefano, Penner-Wilger, & Daley, 2006; LeFevre, Sadeskey, & Bisanz, 1996; LeFevre, Bisanz et al., 1996). Two further goals of this study were to replicate the previous finding of different problem-solving styles (i.e., stable and flexible) and to determine whether arithmetic skill and problem characteristics influence adults' choice of computation procedures on these more difficult multi-digit problems.

In this experiment, I also had the opportunity to compare procedural repertoires and styles between cultures (Italians and Canadians) that have differing approaches to early elementary mathematics education. I hypothesized that, for the Italian participants, the flexible use of holistic mental calculation procedures would have been stressed over paper-and-pencil algorithms for solving multi-digit arithmetic problems (Lucangeli et al., 2003) and, thus, there would be more flexible problem-solvers among the Italians than the Canadians, for whom early mathematics education would have focused more on using the standard digit algorithm on paper.

Method

Participants

Participants were drawn from undergraduate psychology student populations at two different universities. Twelve participants who were English-speaking and educated in Canada (7 females) and twelve participants who were Italian-speaking and educated in Italy (6 females) received course credit for their participation. Canadians ranged in age from 18 to 26 (median = 22 years) and Italians ranged in age from 20 to 32 (median = 23 years).

Materials

Addition Task. Participants solved 96 different two-digit plus two-digit addition problems, 48 vertically and 48 horizontally. Half of the problems had a carry in the units. Half of the unit-carry problems and half of the no-unit-carry problems had a carry in the tens. Thus, there were four levels of complexity: i) problems with no carry in the ones or the tens (*no-carry*), ii) problems with a carry in the tens only (*decade-carry*), iii) problems with a carry in the ones only (*unit-carry*), and iv) problems with a carry in the tens and in the ones (*double-carry*). Zeros and all forms of ties (e.g., 32 + 32; 34 + 54; 38 + 34) were excluded (See Table 9 for examples and Appendix B2 for a complete list). Note that, whereas in Experiment 1 participants solved each different problem four times, in Experiment 2 each problem was solved only once. Thus, for each participant, vertical and horizontal sets comprised different problems.

Table 9

Experiment 2: Sample Two-digit Plus Two-digit Problems

	No Unit Carry		Unit Carry	
	Horizontal	Vertical	Horizontal	Vertical
No Decade Carry	12+73 =	14 +74	13+47 =	28 +34
Decade Carry	24+83 =	32 +96	28+93 =	62 +79

Skill Test. As in Experiment 1, participants completed the addition and multiplication-subtraction subtests from the Kit of Factor-Referenced Cognitive Tests (French, Ekstrom, & Price, 1963).

Procedure

Participants were tested individually in a single session lasting about 1 hour. The experiment was programmed in E-Prime. The 96 addition problems were presented in two different ways, as list A and list B. Each list contained the full set of problems. Half of list A was presented horizontally and half was presented vertically. In list B, the presentation orientation was reversed so that the horizontal problems from list A were presented vertically in list B and the vertical problems from list A were presented horizontally in list B. Within nationality, participants were randomly assigned to list A or B. Horizontal and vertical presentation were blocked. Half of the Italians and half of the Canadians were randomly assigned to solve the horizontal block first. Within each block, problems were presented in random order. The addition problems were presented on two computer monitors; the participant watched one monitor while the experimenter watched the other. Problems were presented in Courier New 30-point font and instructions were in Courier New 24-point font. Participants were asked to solve each problem as quickly and accurately as possible.

Every participant first completed 10 practice trials. During the practice session, participants were encouraged to report as much detail as possible about how they solved each problem and were given trial-by-trial feedback regarding the level of detail required. To begin each practice and experimental trial, the experimenter pressed the spacebar. An asterisk appeared in the centre of the screen, was displayed for 1 second, then flashed on and off twice at 200-millisecond intervals. On what would have been the fourth appearance of the asterisk, the addition problem appeared, centered where the asterisk had been. The problem remained on the screen until the participant responded. The participant solved the problem and gave the numeric answer orally. When the participant reported the sum, the words "How did you solve the problem?" appeared in the centre of the screen. Numeric responses and detailed procedure reports were recorded by the experimenter. Latencies were not recorded.

Results

Skill Test

Participants' arithmetic skills overall were average ($M = 77$, $SD = 18.6$), with a range of 27 to 112. The distribution of individuals and scores by skill group and nationality is shown in Table 10. One-third of the participants were in the low skill group, with 46% qualifying as average and 21% as high skill, respectively. Half of the Canadian participants' scores were within the low-skill range and Canadians comprised 75% of the low-skill group. On the other hand, 67% of the Italian participants' scores were within the average-skill range and Italians made up 73% of the average-skill group. The range of scores for Canadians was 27 to 112 compared with a more narrow range of 65 to 102 for the Italians. Recall from Chapter 2 that the difference in overall skill scores between the Italian and Canadian groups was not significant. Further, the distribution of participants in skill groups was independent of nationality, χ^2 (2, N=24) = 4.47; p = .107. Nonetheless, there was a trend toward the Canadian participants demonstrating lower arithmetic skills than the Italians.

Table 10

Experiment 2: Means and Standard Deviations for French Kit Scores by Skill Group and Nationality

Skill Group	Full Sample				Canadians			Italians		
	N	M	*SD*	Range	N	M	*SD*	N	M	*SD*
Low	8	57	14.1	27 – 69	6	55	15.6	2	65	0.0
Average	11	81	5.3	75 – 88	3	81	6.6	8	81	5.2
High	5	100	7.7	91 – 112	3	100	10.7	2	100	2.8
TOTAL	24	77	18.6	27 – 112	12	73	23.4	12	81.5	11.4

Addition Task

Twenty-four participants solved 96 addition problems each, for a total of 2304 problems. Of those, 194 (8.4%) were errors. Percentage of incorrect responses was calculated for each participant in each orientation by complexity condition. In preliminary analyses, gender was unrelated to performance and was therefore omitted from further discussion. Error percentages were analyzed in a 2 (problem orientation; horizontal, vertical) by 4 (complexity: no-carry, decade-carry, unit-carry, double-carry) by 3 (skill: low, average, high) by 2 (nationality: Canadian, Italian) mixed ANOVA with skill and nationality as between groups variables. Unless otherwise indicated, the alpha level for the analyses was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violation of sphericity. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). Ninety-five percent confidence intervals (CIs) are presented on figures. A summary of the ANOVA is shown in Table 11.

Table 11

Experiment 2: Analysis of Variance for Percent Errors for Italian and Canadian Adults Solving Two-digit Plus Two-digit Addition Problems Varying in Complexity and Orientation of Presentation

		% Error	
Source	*df*	*MS*	*F*
Between			
Skill	2		16.40***
Nationality	1		5.17*
Skill x Nationality	2		6.03*
Error	18	49.3	
Within			
Orientation	1		8.43**
Orientation x Skill	2		0.19
Orientation x Nationality	1		0.28
Orientation x Skill x Nationality	2		0.77
Error (Orientation)	18	68.6	
Complexity	1.9		15.76**
Complexity x Skill	6		2.42*
Complexity x Nationality	3		1.80
Complexity x Skill x Nationality	6		1.28
Error (Complexity)	54	65.1	
Orientation x Complexity	3		1.49
Orientation x Complexity x Skill	6		0.48
Orientation x Complexity x Nationality	3		0.87
Orientation x Complexity x Skill x Nationality	6		1.18
Error (Orientation x Complexity)	54	81.4	

Errors. Percentage of errors varied with arithmetic skill such that, whereas average- and high-skill adults had similar error rates (7.4% vs. 5.6%, respectively), the low-skill group stood out as significantly less accurate (13.5%). Error rate also varied with nationality in that Italians made more errors than Canadians (10.2% vs. 7.5%, respectively). It must be noted here that nationality is confounded with problem-solving style in that most of the Italian participants were flexible and most of the Canadians were stable. Therefore, the difference in accuracy may actually have been between flexible and stable problem-solvers. However, in a separate orientation by complexity ANOVA with skill and style as between group variables, there were no main effects or interactions involving problem-solving style.

A significant interaction between skill and nationality was attributable to the fact that the difference in accuracy between Italians and Canadians was evident in the low-skill group (mean difference 8.3%) but not in the average- or high-skill groups (mean difference 1% in both). There were also significant main effects for problem orientation and complexity. More errors were made on horizontal than on vertical problems (10.8% vs. 6.9%, respectively) and errors increased significantly with the complexity of the problems (see Table 12). Both a carry in the decades and a carry in the units influenced the rate of errors. Across levels of unit complexity, more errors were made on problems with a decade carry than those without and across levels of decade complexity, more errors were made on problems with a unit carry than those without.

Table 12

Experiment 2: Mean percent errors for adults (both Italian and Canadian) solving two-digit plus two-digit addition problems varying in complexity

Unit Complexity	Decade Complexity	
	No-carry	Carry
No-carry	3.9	7.5
Carry	7.7	16.3

As shown in Figure 5, complexity interacted with skill. Both low- and average-skill participants made more errors on problems with a unit carry than on those without, across decade complexity. However, whereas low-skill participants' were less accurate on problems with a decade carry than on those without when there was a unit carry, average-skill participants' accuracy was not affected significantly by a decade carry, across unit complexity. In contrast, high-skill participants made more errors on problems with than without a decade carry and were not affected by a unit carry. Thus, the accuracy levels of participants in different skill groups were differentially affected by aspects of the complexity of the problems.

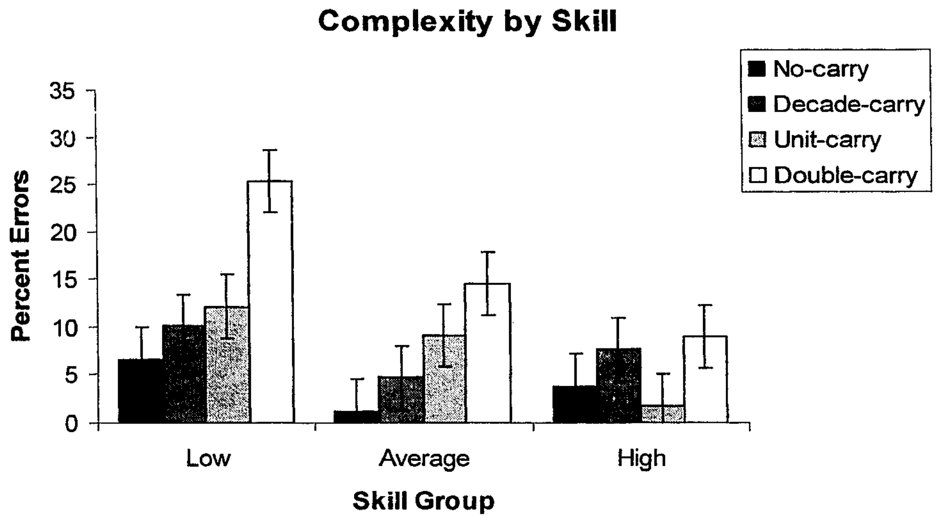

Figure 5. **Experiment 2: Mean percent errors for adults solving multi-digit addition problems varying in complexity and orientation of presentation. Error bars represent 95% confidence intervals, based on the MS_e from the two-way interaction.**

Summary: Errors

Accuracy in 2- plus 2-digit mental addition was affected by participants' nationality and arithmetic skill as well as by the complexity and orientation of the problem. Overall, Italians made more errors than Canadians, however, the discrepancy in errors between the Italian and Canadian groups was most pronounced at the low-skill level. Low-skill adults overall were less accurate than average- or high-skill adults. Further, skill was related to how participants' performance was affected by different aspects of the complexity of the problems. Low- and average-skill adults' accuracy decreased on problems with a unit carry, whereas high-skill participants were not affected by the presence of a carry in the units. In contrast, for the high-skill group, accuracy was affected by the presence of a carry in the decades. Decade carries did not affect the accuracy of the average-skill group but did make carry problems more difficult for those in the low-skill group. Finally, more errors were made on horizontal than on vertical problems across skill and nationality and across levels of complexity.

Procedure Reports

The reported procedures were grouped according to similarity and, once again, five common procedures were coded. These were called *retrieval*, *digit algorithm, partial decomposition, decomposition*, and *transformation* (see Table 13 for descriptions and frequencies of procedures). Decomposition and transformation were somewhat different than the procedures by the same names used to solve the simpler 2- plus 1-digit problems of Experiment 1. However, the underlying logic was the same for each. Decomposition in both cases was based on the breakdown of 2-digit numbers into units and decades. In decomposition with 2-digit plus 2-digit problems, both addends were decomposed into units and decades (with the magnitude of the decade number maintained throughout calculation) and the sum constructed in one of two ways. Either the units were added together separately from the decades, then the unit sum and the decade sum were added together, or the decades were added together, then the unit values were added on sequentially. In partial decomposition, one addend was kept intact and the other was decomposed. Transformation in both cases involved recasting the problem in the form of an easier problem to solve, usually rounding one addend up to the next decade number by adding part of the other addend, then adding on the remainder (see Table 13 for examples).

All responses were categorized in terms of these five multi-digit procedures. Thus, for the analyses of procedure choice, twenty-four participants solved a total of 2304 problems. Nine (less than 1%) were idiosyncratic procedures or "don't know" responses. These were excluded from the following analyses. Trials on which participants made errors were included in the analyses of procedure choice.

Table 13

Experiment 2: Descriptions and Frequencies of Reported Solution Procedures for Two-digit Plus Two-digit Addition Problems (Italian and Canadian)

Procedure	% Frequency			Description
	Can.	Ital.	Both	
Digit Algorithm	93	41	67	Participant treated each addend as a concatenation of single digits and performed a series of simple additions (right to left or left to right).
Decomposition	1	50	25	Both addends were decomposed into decades and units and the sum was constructed in one of several ways, e.g., $25 + 34 = (20 + 30) + (5 + 4)$ $= 50 + 9$ $= 59$
Partial Decomposition	3	1	2	One addend was decomposed into decades and units and added to the other (intact) addend. e.g., $25 + 34 = (25 + 30) + 4$ $= 55 + 4$ $= 59$
Transformation	2	3	3	The addends were transformed in some way to make the addition easier or more familiar. e.g., $57 + 35 = (57 + 3) + (35 - 3)$ $= 60 + 32$ $= 92$
Retrieval	1	5	3	Participant "just knew" the answer without calculating.

Across cultures, the digit algorithm was the most frequently reported single procedure (67%; see Table 13) overall. Participants reported using holistic procedures (i.e., transformation, partial decomposition, or decomposition) in solving 30% of 2- plus 2-digit addition problems. They reported retrieval on only 3% of problems overall. Transformation and partial decomposition were also reported infrequently (3% and 2%, respectively). However, the picture for Canadians was quite different from that for Italians. On the whole, Canadians were stable problem-solvers, with an overwhelming preference for the digit algorithm (93%). In contrast, Italians were more flexible. Three Italian participants were stable problem-solvers, two preferring the digit algorithm and one preferring decomposition, compared with 9 stable problem-solvers in the Canadian group, all using the digit algorithm all or most of the time. The relative frequencies of flexible vs. stable problem-solvers varied significantly with nationality, $\chi^2(1, N = 24) = 6.0$. Italians reported using decomposition half of the time and the digit algorithm somewhat less (41%). Retrieval, partial decomposition, and transformation were used very little by either group.

This pattern of results supports the view that, because European elementary mathematics education stresses flexible mental calculation and explicitly teaches procedures, individuals educated in Europe are more likely to have a greater repertoire of solution procedures for multi-digit mental arithmetic problems. Frequencies of procedures reported by flexible and stable problem-solvers are shown in Table 15. Eight of the 12 flexible solvers (67%) reported using three or more different procedures to solve problems. All reported using each of at least two procedures on more than 10% of problems (see Appendix A2, Table A2.1 and Table A2.2).

Table 14 shows the distribution of flexible and stable problem-solvers across skill groups. The relative frequencies of flexible versus stable problem-solvers did not vary significantly with arithmetic skill, $\chi^2(1, N = 24) = 1.5, p = .57$.

Table 14

Experiment 2: Frequencies of Flexible and Stable Problem-solvers Across Skill Levels

	Skill Group		
Style	Low	Average	High
Flexible	3	7	2
Stable	5	4	3

Results of a one-way ANOVA indicated that flexible and stable problem-solvers were equally accurate (9.2% vs. 7.6% errors, respectively) when solving 2- plus 2-digit addition problems, $F\,(1, 22) = 0.98, p = 0.33, MSE = 14.74$.

Table 15

Experiment 2: Frequencies of Solution Procedures for Two-digit Plus Two-digit Addition Problems across Problem-solving Style

	Percentage Frequency	
Procedure	Flexible	Stable
Retrieval	5	0
Digit Algorithm	44	90
Partial Decomposition	3	1
Decomposition	42	9
Transformation	5	0

Only those 12 people who were flexible problem-solvers were included in the analysis of procedure choice to examine whether arithmetic skill or characteristics of the problem such as complexity or orientation of presentation influenced people's choice of procedure in solving 2- plus 2-digit problems. Percentage of use of each of the five procedures listed in Table 13 was calculated for each participant in each orientation by complexity condition. Retrieval (3% of problems) was then eliminated from the analysis. Thus, the remaining percentages were independent (i.e., did not sum to 100%). Percentage use of the four remaining procedures was analyzed in a 2 (problem orientation; horizontal, vertical) by 4 (complexity: no-carry, decade-carry, unit-carry, double-carry) by 4 (procedure: digit algorithm, decomposition, transformation, partial decomposition) by 3 (skill: low, average, high) mixed ANOVA with skill as the only between-groups variable. Unless otherwise indicated, the alpha level for the analyses was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violation of sphericity. Only interactions involving the procedure variable were examined, as these are the effects that reflect variations in procedure use with various factors. Only the procedure by orientation interaction approached significance, $F(3, 27) = 2.52$, $MS_e = 1058$, $p = 0.08$. People maintained a consistent level of decomposition use on vertical and horizontal problems (39% and 36%, respectively) but, from horizontal to vertical, they shifted their use of transformation and partial decomposition (from 22%, combined, to 6%) toward increased use of the digit algorithm (from 35% to 51%).

Summary: Procedure Reports

Adults reported using five clearly delineated procedures to solve 2- plus 2-digit addition problems, although some were consistently used more frequently than others. Once again, participants divided into flexible and stable problem-solvers. Most of the flexible solvers in this experiment were among the Italians, supporting the hypothesis that Europeans, influenced by an emphasis in elementary school on flexible mental calculation, are more likely to use a variety of procedures to perform complex arithmetic. Arithmetic fluency was not related to procedure choice. Further, problem characteristics did not have a clear effect on people's choice of procedures. However, there was a tendency for procedure choice to vary with the orientation of the problem, shifting between transformation and partial decomposition on the one hand, and the digit algorithm on the other. Although solution approaches varied with problem-solving style, flexible and stable solvers were equally accurate; however, there was no opportunity to compare their solution speed as latencies were not measured in this experiment.

Discussion: Experiments 1 and 2

Trbovich and LeFevre (2003) proposed that presenting complex mental addition problems vertically may lead adults to use the standard digit algorithm to solve them and, therefore, to rely more heavily on a visual component of working memory. They further hypothesized that adults are more likely to use a variety of holistic procedures (e.g., transformation, decomposition) to solve problems that are presented horizontally. These procedures are likely to involve greater reliance on auditory aspects of working memory to monitor the sequence of steps in the procedure and to store intermediate sums. The results of Experiments 1 and 2 support these hypotheses. On both 2- plus 1-digit problems and 2-

plus 2-digit, participants shifted from greater use of holistic procedures on horizontal problems to greater use of the digit algorithm on vertical problems. It is not surprising that adults used the digit algorithm more frequently for vertical than for horizontal problems. When addition problems are presented vertically, the digits are aligned in such a way as to encourage adding each column as a simple sum. On the other hand, to solve horizontal problems using the digit algorithm requires the solver to mentally select the unit digits together and the decade digits together (when there are two). This would be expected to load heavily on spatial working memory. In fact, several participants commented that this was particularly difficult to do for horizontal problems with the smaller addend first in 2-plus 1-digit additions.

For 2- plus 1-digit problems, people also tended to shift from the digit algorithm to transformation with increasing problem complexity. This shift may be explained by the relation between solution procedures and the problem-size effect. That is, people may use the digit algorithm more often when the single-digit sums are easier to retrieve. Carry problems have large single-digit additions, which adults retrieve more slowly and less accurately than small additions (Campbell & Xue, 2001; LeFevre, Sadesky, & Bisanz, 1996; Smith-Chant & LeFevre, 2003). Participants' greater use of the digit algorithm for no-carry than for carry problems may also be attributable to the increased working memory load resulting from the storage and recall of the carry digit necessitated by the digit algorithm. The digit algorithm was chosen least often for horizontal carry problems, on which the manipulation of a carry digit would likely be the most difficult.

More generally, the results indicated that adults' procedure choices for multi-digit addition were influenced by problem characteristics as well as by individual characteristics but not, in these results, by overall arithmetic skill. The complex patterns that emerged in these results make it clear that procedure choices must be viewed in the context of interactions among different problem characteristics. In addition, individual differences in problem-solving style must be taken into account. It appears that some individuals use only one procedure to solve all problems, regardless of problem characteristics. The performance of these individuals should be examined separately from that of individuals who use a variety of solution procedures. In the following two experiments, the choice/no-choice paradigm was used to further investigate the relations between complex mental addition procedure choice and individual and problem characteristics as well as the relation between procedure choice and performance characteristics of the procedures.

CHAPTER 4

In this chapter, I discuss two experiments in which, together, a choice/no-choice paradigm was employed to examine the influence of individual differences, problem characteristics, and procedure characteristics on adults' choice of solution procedures in performing multi-digit mental addition. The first experiment (Experiment 3) comprised the Choice condition. Adults solved 2-digit plus 2-digit addition problems and reported their solution procedures by selecting from an established list, developed from the verbatim reports in Experiment 2. As participants were allowed to choose freely how they solved these problems, it was anticipated that they would once again, as in Experiments 1 and 2, divide into two groups on the basis of solution style (i.e., flexible vs. stable). Participants were recruited from a variety of areas of study with the intention of finding a larger number of flexible problem-solvers among Canadian university students than were present in Experiment 2. It was reasoned that students in math-, science-, and business-related fields of study would have taken more secondary school and university level mathematics courses and thus have more mathematics experience. Further, it was hypothesized that these more math-experienced students were more likely to be flexible math problem-solvers. One goal of this experiment was to compare the computational efficiency (i.e., speed and accuracy) of flexible versus stable problem-solvers. The second goal of this experiment was to determine the influence of individual differences (i.e., arithmetic skill) and problem characteristics (i.e., orientation and complexity) on adults' procedure choices.

Three no-choice conditions were implemented in Experiment 4. In each condition, participants were told to solve the problems using one of three procedures chosen by the experimenter from Experiment 3. Thus, participants used each procedure to solve all problems, regardless of orientation or complexity. The three most frequently reported procedures from Experiment 3, partial decomposition, decomposition and the digit algorithm, were included. Partial decomposition is considered by Beishuizen and colleagues (e.g., Beishuizen et al., 1997; Blöte et al., 2000) to be the most efficient procedure to use for solving 2- plus 2-digit problems. One goal of this experiment was to obtain measures of procedure efficiency that were not affected by selection biases (Siegler & Lemaire, 1997). The speed and accuracy of each procedure was measured overall and relative to problem characteristics. Using these measures, the performance characteristics of the three procedures were compared overall, for particular problem types, and for particular types of problem-solvers.

Experiment 3

One goal of this experiment was to confirm the presence of flexible and stable mental calculators among adults. It was hypothesized that a sample of adults recruited across areas of study that vary in the number of mathematics courses required both for entry and for graduation, would vary across math problem-solving styles. Another goal was to test whether procedure choice among flexible problem-solvers is related to problem characteristics and individual difference characteristics. It was hypothesized that people would choose different procedures based on the orientation and complexity of the problem and that how they made these choices would vary with arithmetic skill. A final goal of this experiment was to implement a self-report paradigm in which participants reported their solution procedures for complex addition problems by choosing alternatives from a list.

Method

Participants

Sixty students, 59 undergraduate students and one high school student (27 females) were recruited. They represented a variety of areas of study (e.g., history, social science, math, science, commerce, and engineering) and ranged in age from 15 to 45 (median = 21). They were paid $10 each for their participation. Forty participants reported receiving their elementary education in Canada. The rest were educated in a variety of countries such as England, Bulgaria, and Greece as well as China, Korea, Vietnam, and a number of African countries.

Materials

Addition Task. Participants solved 48 different 2- plus 2-digit addition problems. Half of the problems had a carry in the ones. Half of the unit-carry problems and half of the no-unit-carry problems had a carry in the tens. Thus, as in Experiment 2, there were four levels of complexity: i) problems with no carry in the ones or tens (*no-carry*) ii) problems with a carry in the tens only (*decade-carry*), iii) problems with a carry in the ones only (*unit-carry*), and iv) problems with a carry in the tens and in the ones (*double-carry*). Zeros and all forms of ties (e.g., 32 + 32; 34 + 54; 38 + 34) were excluded (See Table 16 for examples and Appendix B3 for a complete list).

Table 16

Experiment 3: Sample Two-digit Plus Two-digit Addition Problems

	No Unit Carry		Unit Carry	
	Horizontal	Vertical	Horizontal	Vertical
No Decade Carry	12+73 =	14 +74	13+47 =	28 +34
Decade Carry	24+83 =	32 +96	28+93 =	62 +79

Skill Test. As in the previous two experiments, participants completed the addition and multiplication-subtraction subtests from the Kit of Factor-Referenced Cognitive Tests (French, Ekstrom, & Price, 1963).

Procedure

Participants were tested individually in a single session lasting approximately 1 hour. Half of the addition problems were randomly assigned to list A and the other half were assigned to list B. Participants were randomly assigned to orientation conditions so that one half saw the questions from list A presented vertically and those from list B presented horizontally and the rest saw list B in vertical orientation and list A in horizontal orientation. Each participant was shown a list of mental addition procedures, with descriptions, prepared as a result of Experiment 2 (i.e., retrieval, digit algorithm, transformation, partial decomposition, decomposition, and other). Each procedure was explained to the participant, who then solved 16 practice problems and 48 experimental problems. The experiment was programmed in E-Prime. The addition problems were presented on two computer monitors; the participant watched one monitor while the experimenter watched the other. Problems were presented in Courier New 30-point font and instructions were in Courier New 24-point font. Each problem remained on the screen until the participant responded using a headset microphone attached to a serial response box. Participants were asked to solve each problem as quickly and accurately as possible. On each trial, the participant solved the problem, gave the answer orally, and then reported the procedure used by choosing one from the list. Latencies were recorded using a voice-activated timing switch. Responses and procedure reports were recorded by the experimenter.

Results

Skill Test

The distribution of scores in this sample was, once again, similar to that in the overall group. Participants' arithmetic skills overall were average ($M = 77$, $SD = 27.3$). The distribution of individuals by skill group is shown in Table 17. Forty-two percent of the participants were in the low-skill group, with 27% qualifying as average skill and 32% as high skill, respectively.

Table 17

Experiment 3: Means and Standard Deviations for French Kit Scores by Skill Group

Skill	N	Mean	SD	Range
Low	25	51	12.1	29-68
Average	16	81	6.3	70-89
High	19	109	15.3	90-150
Total	60	77	27.3	29-150

Arithmetic Task

Sixty participants solved 48 two-digit plus two-digit addition problems each, for a total of 2880 problems; 175 (6%) were invalid due to inadvertent voice key triggers, equipment failures, or latencies less than 300 ms or greater than 3 standard deviations above the mean latency (by participant). The remaining latencies ranged from 771 ms to 22,284 ms. Of the 2,705 valid responses, 314 (11.6%) were errors. Median correct latencies and percent of incorrect responses were calculated for each participant in each orientation by complexity condition and were analyzed in separate 2 (orientation; horizontal, vertical) by 4 (complexity: no-carry, decade-carry, unit-carry, double-carry) by 3 (skill: high, average, low) by 2 (problem-solving style: flexible, stable) ANOVAs with skill and style as between-group variables. Unless otherwise indicated, the alpha level was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violations of sphericity. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). Ninety-five percent confidence intervals (CIs) are presented on figures. A summary of the ANOVA is shown in Table 18.

Table 18

Experiment 3: Analysis of Variance for Latencies and Percent Errors for Adults Solving Two-digit Plus Two-digit Addition Problems Varying in Complexity and Orientation

	Latency			% Error		
Source	*df*	*MS*	*F*	*Df*	*MS*	*F*
		Between				
Skill	2		505.56**	2		4.28*
Style	1		23.81	1		3.15
Skill x Style	2		1.14	2		2.27
Error	54	10,560,000		54	415.67	
		Within				
Orientation	1		27.55**	1		1.11
Orientation x Skill	2		2.16	2		2.49
Orientation x Style	1		0.27	1		0.13
Orientation x Skill x Style	2		0.25	2		0.53
Error (Orientation)	54	563,680		54	140.19	
Complexity	1.9†		63.58**	2.5†		23.87**
Complexity x Skill	6		8.37**	6		1.07
Complexity x Style	3		0.31	3		0.69
Complexity x Skill x Style	6		0.69	6		2.11
Error (Complexity)	162	1,136,138		162	186.40	

Orientation x Complexity	2.4†		1.21	2.4†		0.69
Orientation x Complexity x Skill	6		1.40	3		0.67
Orientation x Complexity x Style	3		1.05	3		0.07
Orientation x Complexity x Skill x Style	6		1.23	6		0.60
Error (Orientation x Complexity)	162	533,469		162	202.21	

†Greenhouse-Geisser adjustment of df and Mean Square Error for violation of sphericity assumption. In no case did the adjustment make a difference to significance.

Latercies. In the choice condition, the average- and high-skill groups did not differ significantly with respect to response speed (means of 3194, 2490 ms, respectively) but the low-skill participants were significantly slower than both (4824 ms). Flexible and stable problem-solvers were similar in their overall response latency.

Participants were quicker to solve vertical than horizontal problems (mean latency 3314 and 3692 ms, respectively). Latencies also increased significantly with each increase in complexity (mean latency 2453, 3366, 3806, 4387 ms for no-carry, decade-carry, unit-carry, and double-carry, respectively) and, overall, adults' calculation speed was affected by both unit and decade carries. However, a significant complexity by skill interaction reflected the fact that the three skill groups were differentially affected by unit carries and decade carries (See Figure 6). For low- and average-skill participants, performance was slowed both by a unit carry and by a decade carry. On the other hand, the solution latencies of the high-skill participants were affected only by the presence of a unit carry. That is, for this group, whereas the discrepancies between no-carry and unit-carry problems and between decade-carry and double-carry problems were significant, the discrepancies between no-carry and decade-carry problems and between unit-carry and double-carry problems were not.

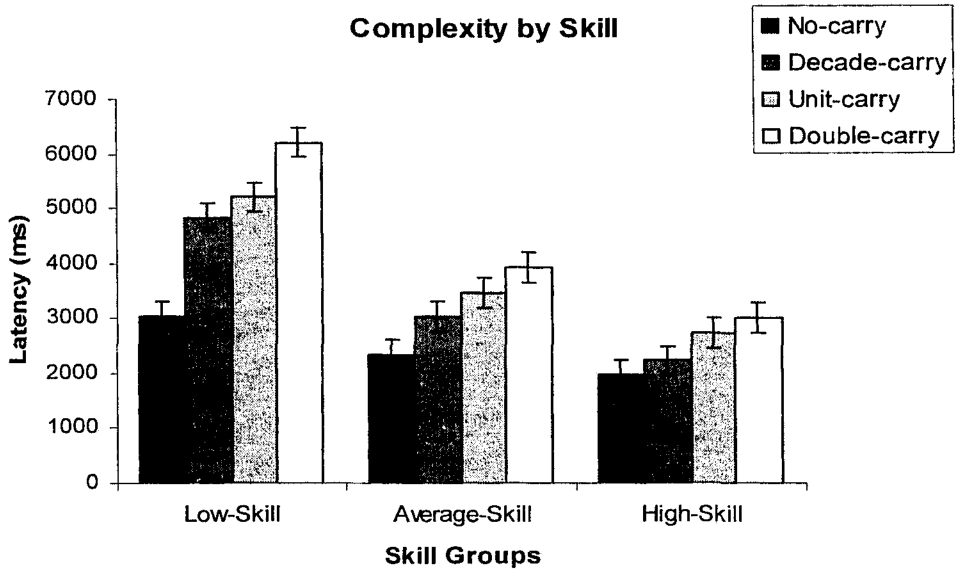

Figure 6. **Experiment 3: Mean latencies across skill levels for adults solving two-digit plus two-digit addition problems varying in complexity. Error bars represent 95% confidence intervals, based on the MS_e from the two-way interaction.**

High-skill participants were significantly more accurate (7% errors) than those in the average- and low-skill groups, who did not differ with respect to accuracy (13% errors each). Flexible problem-solvers made more errors than stable problem-solvers (13% vs. 9%); however, this discrepancy only approached significance, $F(1, 54) = 3.15$, $p = .08$, $MSE = 1307.8$.

Problem orientation did not influence accuracy. However, as shown in Figure 7, the presence of a decade carry in these problems had a striking effect on accuracy. Significantly more errors were made on decade carry problems whether or not there was a carry in the units. For problems with no carry in the decades, significantly more errors were made on those with a carry in the units than on those without. However, for problems with a carry in the decades, a unit carry made no difference to accuracy. For 2- plus 2-digit problems that were solved using the digit-based procedure, decade carries differed from unit carries. A carry in the unit position necessitated solving a larger single-digit sum, maintaining a carry value in memory, and adding three digits in the decade position, whereas a carry in the decade position required only solving a large single-digit sum. It appears that accuracy was more affected by the large sum than by the carry digit. However, latencies were affected by both. These results may reflect a speed-accuracy tradeoff in that participants may have slowed down in order to keep the carry value in memory when they used the digit algorithm.

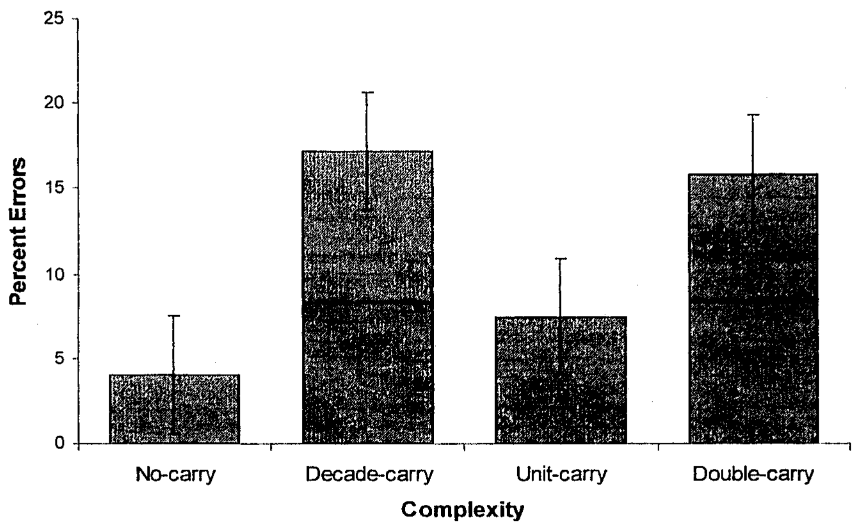

***Figure 7.* Experiment 3: Mean percent errors for adults solving two-digit plus two-digit addition problems varying in complexity. Error bars represent 95% confidence intervals, based on the MS_e within groups.**

Summary: Latencies and Errors

Speed and accuracy in 2- plus 2-digit mental addition were influenced in varying ways by participants' problem-solving style and arithmetic skill as well as by the complexity and orientation of the problems. For example, flexible and stable problem-solvers were equally quick but there was a tendency for the stable group to be more accurate. Further, adults with weaker arithmetic skills were slower mental calculators than those with average or stronger arithmetic skills and high-skill calculators were more accurate than those with lower or average skills. Both speed and accuracy of mental addition were hampered by increasing complexity in the problems. However, the degree and pattern of the effects on speed of decade and unit carries varied across levels of arithmetic skill. Finally, vertical problems were solved more quickly than horizontal problems but accuracy was consistent across orientation.

Procedure Reports

Trials on which participants made errors were included in the analyses of procedure choice. Table 20 shows the reported frequencies of procedures. Thirty-three of the 60 participants (55%) were flexible problem-solvers. Similarly, 33 participants were from math-related areas of study such as engineering, science, commerce, and mathematics. Although a variety of participants was sought in order to ensure increased numbers of flexible problem-solvers in the sample, the relative frequencies of flexible vs. stable problem-solvers was independent of study area, $\chi^2(1, N = 60) = 0.2, p = .66$.

Further, problem-solving style (i.e., flexible vs. stable) was independent of arithmetic skill, $\chi^2(2, N = 60) = 1.8, p = .42$ (see Table 19 for the distribution of flexible and stable problem-solvers across skill groups) and participants who received their elementary education in Canada ($n = 40$) were no more likely to have a stable style than those who did not (n = 20), $\chi^2(1, N = 60) = 0.3, p = .58$.

Table 19

Experiment 3: Distribution of Flexible and Stable Problem-solvers across Skill Levels

	Skill Group		
Style	Low	Average	High
Flexible	12	11	10
Stable	13	5	9

Table 20

Experiment 3: Frequencies of Solution Procedures for Two-digit Plus Two-digit Addition Problems

	% Frequency		
Procedure	Flexible	Stable	Both
Digit Algorithm	57	87	72
Decomposition	18	7	13
Partial Decomposition	15	3	9
Transformation	8	1	5
Retrieval	2	2	1

Once again, the digit algorithm was the most frequently reported single procedure (see Table 20) overall. Participants reported using holistic procedures (i.e., transformation, partial decomposition, or decomposition) in solving 27% of 2- plus 2-digit addition problems. They reported retrieval on only 1% of problems overall. Transformation and partial decomposition were reported more frequently than in Experiment 2 (5% vs. 9%, respectively). Twenty-three of the 33 flexible problem-solvers (70%) reported using three or more different procedures to solve problems. All reported using each of at least 2 procedures on more than 6% of problems (see Appendix A3, Table A3.1).

Only those 33 people who were flexible problem-solvers were included in the analysis of procedure choice to examine whether arithmetic skill or characteristics of the problem (orientation and complexity) influenced people's choice of procedure.

Percentage of use of each of the five procedures listed in Table 20 was calculated for each participant in each orientation by complexity condition. Retrieval (2% of problems) and transformation (8% of problems) were then eliminated from the analyses. Thus, the remaining percentages were independent (i.e., did not sum to 100%). Percentage use of the three remaining procedures was analyzed in a 3 (procedure: digit algorithm, partial decomposition, decomposition) by 2 (problem orientation: horizontal, vertical) by 4 (complexity: no-carry, decade-carry, unit-carry, double-carry) by 3 (skill: high, average, low) mixed ANOVA with skill as the only between-groups variable. Only interactions involving the procedure variable are discussed. Unless otherwise indicated, the alpha level for the analyses was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violations of sphericity. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). Ninety-five percent confidence intervals (CIs) are presented on figures. A summary of the ANOVA is shown in Appendix A3 (Table A3.2).

Among these flexible mental calculators, arithmetic skill did not appear to influence procedure choice. However, significant procedure by orientation, F (1.3, 60) = 12.01, MSE = 996.32, and procedure by complexity, F (3, 180) = 4.56, MSE = 585.36, interactions indicated that, across skill levels, choices varied with orientation and complexity (see Figures 8 and 9). Although the use of partial decomposition did not change with the orientation of the problem, the digit algorithm was used more for vertical than horizontal problems and decomposition was used more for horizontal than vertical problems. For the most part, the use of the digit algorithm and partial decomposition decreased and the use of decomposition increased as problems became more complex.

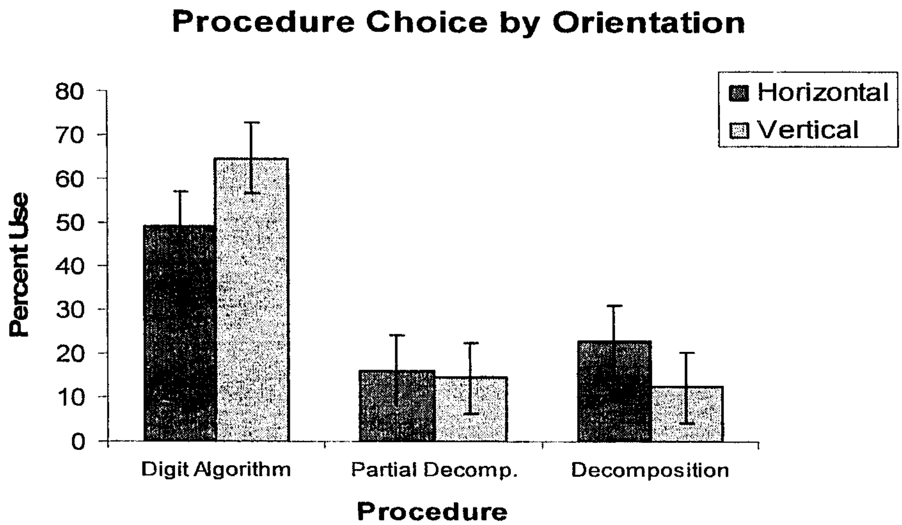

Figure 8. **Experiment 3: Mean percent procedure choice for adults solving two-digit plus two-digit addition problems varying in orientation. Error bars represent 95% confidence intervals, based on the MS_e for the two-way interaction.**

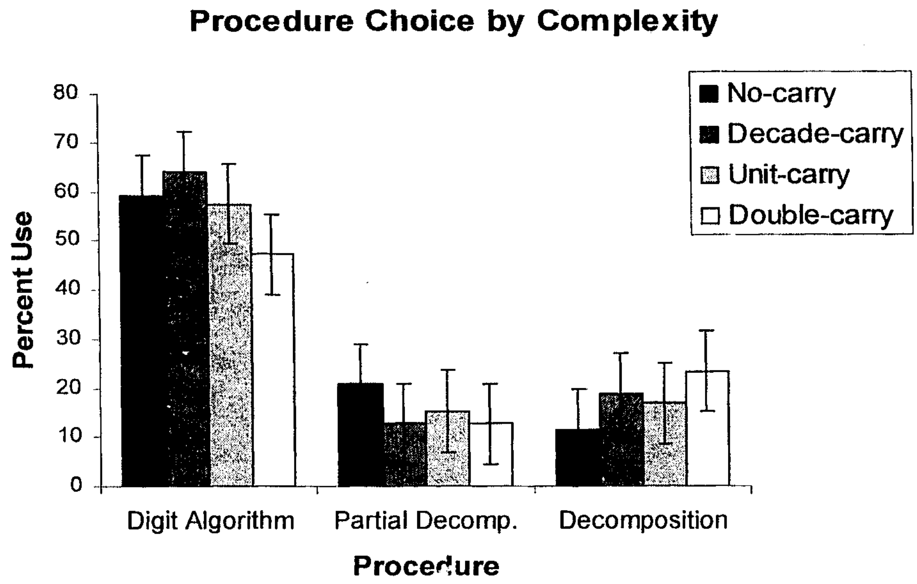

Figure 9. **Experiment 3: Mean percent procedure choice for adults solving two-digit plus two-digit addition problems varying in complexity. Error bars represent 95% confidence intervals, based on the MS_e for the two-way interaction.**

Summary: Procedure Reports

Adults' procedure choices for 2- plus 2-digit addition were not influenced by overall arithmetic skill but procedure selection was nevertheless variable across individuals. As in Experiments 1 and 2, some participants were flexible and chose from among a variety of procedures, whereas others relied mainly on a single solution approach. Approximately half of the sample in this experiment was composed of flexible problem-solvers, who varied their procedure choices on the basis of problem characteristics. Specifically, these flexible solvers used the digit algorithm more frequently and decomposition less frequently for vertical than for horizontal problems. Further, these solvers shifted from use of the digit algorithm and partial decomposition to decomposition as the problems increased in complexity. Thus, these results support the hypotheses that, among adults, there are both flexible and stable problem-solvers and that flexible solvers vary their procedure choices according to characteristics of the problem to be solved. Problem-solving style was not related to area of study or educational background. In Experiment 4, many of the individuals who participated in Experiment 3 returned and solved problems using three of the procedures reported here and I obtained measures of the relative speed and accuracy of these procedures.

Siegler and Lemaire (1997) argued that, to obtain measures of procedure characteristics such as speed and accuracy that are not confounded by selection biases, participants must complete a *no-choice* condition, in which they are instructed to use a particular procedure for all problems. Therefore, in Experiment 4, participants from the choice condition in Experiment 3 solved problems in three *no-choice* conditions, one in which they were trained and instructed to use the digit algorithm for all problems, one in which they were trained and instructed to use decomposition for all problems, and a third in which they were trained and instructed to use partial decomposition for all problems. The speed and accuracy of each solution procedure overall and in interaction with problem characteristics and individual characteristics were determined across the three no-choice conditions. These measures were used together with the results of Experiment 3 to examine the influence of performance characteristics on people's selection of procedures when they were free to choose. They were also compared with similar measures of speed and accuracy in the choice condition (Experiment 3) in order to compare participants' mental computation effectiveness when they solve problems the way that they prefer to relative to when the solution method is constrained.

Experiment 4

Experiment 3 constituted the Choice condition of the Choice/No-choice paradigm. In Experiment 4, three No-choice conditions were implemented. The first goal of this experiment was to obtain unbiased measures, as described by Siegler and Lemaire (1997), of the performance characteristics (speed and accuracy) of the three solution procedures that were reported most frequently in Experiment 3. These were decomposition, partial decomposition, and the digit algorithm. These speed and accuracy measures were used to examine the relationship between the performance characteristics of the procedures and adults' choices among these three procedures in the Choice condition (Experiment 3).

The second goal of the present experiment was to compare the relative effectiveness of these three procedures for complex addition problems with varying characteristics. Beishuizen and colleagues (Beishuizen, 1993; Beishuizen, VanPutten, & VanMulken, 1997; Blöte et al., 2000) proposed that, for children solving multi-digit mental addition problems, holistic solution procedures are more effective than the standard digit algorithm and partial decomposition (i.e., their 'N10' procedure) is more effective than decomposition (their '1010' procedure). In Experiment 3, adults reported using the digit algorithm and decomposition much more frequently than partial decomposition. This finding suggests that these participants were less practiced in partial decomposition than in the other two procedures. Therefore, it was hypothesized that they would calculate less effectively overall with partial decomposition than with either the digit algorithm or decomposition. There was no reason to suppose that either decomposition or the digit algorithm would be superior in performance to the other overall. However, it was expected that the speed and accuracy of procedures would vary across problem complexity and orientation and across levels of arithmetic skill. Finally, it was hypothesized that stable problem-solvers would calculate most effectively using the digit algorithm, with which they had more practice than with the other two procedures. No previous work has compared these procedures (for adults) in a no-choice condition. The no-choice condition allows a test of Beishuzen's claim that holistic procedures are more effective than the digit algorithm, and that, within the holistic procedures, partial decomposition is more effective than decomposition.

A third goal of this experiment was to compare the performance of flexible and stable problem-solvers using all three procedures across different types of problems. In previous studies using the choice/no-choice paradigm, stable problem-solvers (usually those who rely exclusively on the digit algorithm) and flexible problem-solvers have not been considered separately. In no-choice conditions, the stable solvers are required to use procedures they have little or no practice with and the flexible solvers use procedures in circumstances in which they would not normally choose them. Thus, this experiment provided an opportunity to examine the ability of stable solvers to be flexible and to compare their calculation effectiveness using unpracticed and possibly unfamiliar procedures with that of those who have more experience with several procedures but are not allowed to choose among them. Lemaire and colleagues have not addressed this complication of the no-choice design.

Previous work examining procedure choice on complex cognitive tasks has shown that adults and children adapt their choices to procedure characteristics as well as to problem characteristics (e.g., Kerkman, & Siegler, 1993; Lemaire, Arnaud, & Lecacheur, 2004; Luwel et al., 2005). Choice of solution procedure has also been shown to vary with individual characteristics such as age (Lemaire & Lecacheur, 2002a; Lemaire & Lecacheur, 2002b; Luwel et al., 2005), cultural background (Imbo & LeFevre, 2009), arithmetic skill (Imbo & Vandierendonck, 2007), and gender (Carr & Davis, 2001; Carr & Jessup, 1997; Carr, Jessup, & Fulller, 1999). In the present analysis, speed and accuracy of procedures were examined in interaction with presentation orientation, problem complexity, and arithmetic skill.

Method

Participants

Thirty-six participants (60%; 17 females) from the choice condition (i.e., Experiment 3) returned to take part in Experiment 4. They ranged in age from 15 to 45 (median age = 19.5) and were paid $10 each for their participation. Those who did not return either declined when asked (during Experiment 3) or could not subsequently be reached. Nineteen (53%) of the returning participants were flexible problem-solvers. This is similar to the 55% flexible solvers in the original sample (Experiment 3). Thus, attrition did not appear to be related to problem-solving style. All returning stable problem-solvers preferred the digit algorithm. The reported frequencies of procedures are shown in Table 21.

Table 21

Experiment 4: Frequencies of Solution Procedures for Two-digit Plus Two-digit Addition Problems Reported by Flexible and Stable Problem-solvers and by the Combined Group.

	% Procedure Frequency		
Procedure	Flexible	Stable	Flexible and Stable Combined
Retrieval	2	0	1
Digit Algorithm	48	99	72
Partial Decomposition	17	0.5	9
Decomposition	25	0	13
Transformation	8	0.5	5

The distribution of style (flexible, stable) by arithmetic skill (low, average, high) is shown in Table 22. The relative frequencies of flexible versus stable problem-solvers did not vary significantly with skill, $\chi^2(2, N = 36) = 1.70, p = .43$.

Table 22

Experiment 4: Distribution of Flexible and Stable Problem-solvers across Skill Levels

	Skill Group		
Style	Low	Average	High
Flexible	6	7	6
Stable	9	4	4

Materials

Addition Task. Participants solved 216 different 2- plus 2-digit addition problems. As in Experiments 2 and 3, there were four levels of complexity: no-carry, decade-carry, unit carry, and double-carry. Zeros and all forms of ties (e.g., 32 + 32; 34 + 54; 38 + 34) were excluded (See Appendix B4 for a complete list of problems). Addition problems were presented in random order in all three no-choice conditions.

Skill Test. These participants had completed the addition and subtraction-multiplication sub-tests of the Kit of Factor-Referenced Cognitive Tests (French, Ekstrom, & Price, 1963) when they participated in Experiment 3.

Procedure

Participants were tested individually in a single session lasting approximately 1 hour. The 216 addition problems were presented in two different ways, as list A and list B. Each list contained the full set of problems. Half of list A was presented horizontally and half was presented vertically. In list B, the presentation orientation was reversed so that horizontal problems from list A were presented vertically in list B and vertical problems from list A were presented horizontally in list B. Each list was divided into three sets of 72 problems for the three procedure conditions, digit algorithm, decomposition, and partial decomposition. The procedure conditions were presented in all six possible orders (i.e., all permutations of the three procedures). Thus, there were 12 conditions (2 orientations by 6 orders of procedure) to which participants were assigned randomly so that the presentation of problems to participants was balanced for orientation and order in which the procedures were practiced. The experiment was programmed in E-Prime.

In the *no-choice-digit algorithm* condition, participants were told that they must use the digit algorithm for all problems. The algorithm was demonstrated and the participant solved 16 practice problems step-by-step out loud, with feedback, before solving the 72 experimental problems. Twice during the experimental problems, the participant was asked if s/he was using the digit procedure. The *no-choice-decomposition* and the *no-choice-partial decomposition* conditions were conducted in the same way, varying only the solution method that was required. The problems were presented on two computer monitors with participants watching one monitor while the experimenter watched the other. Problems were presented in Courier New 30-point font and instructions were in Courier New 24-point font. Each problem remained on the screen until the participant responded using a headset microphone attached to a serial response box. Participants were asked to solve the problems as quickly and accurately as possible. The participant solved the problem and gave the numeric answer orally. Latencies were recorded using a voice-activated timing switch. Responses were recorded by the experimenter.

Results

Skill Test

In this subgroup of participants from Experiment 3, arithmetic skill overall was average ($M = 75$, $SD = 26.7$), with a range of 29 to 132. The distribution of individuals by skill group is shown in Table 23. Forty-two percent of the participants were in the low skill group, 30% were average, and 28% were high skill, respectively.

Table 23

Experiment 4: Means, Standard Deviations, and Ranges for Math Fluency Scores by Skill Group

Skill	N	Mean	SD	Range
Low	15	49	12.2	29 – 66
Average	11	81	5.3	71 – 89
High	10	107	14.3	90 – 132

Arithmetic Task

The 36 participants solved 216 different 2- plus 2-digit addition problems each, for a total of 7,776 problems; 150 (1.9%) were invalid due to either inadvertent voice key triggers or equipment failures. For each participant, latencies less than 300 ms or greater than 3 standard deviations above the participant's mean latency (n = 156; 2.0%) were also considered invalid and were excluded from the analyses. The remaining latencies ranged from 961 ms to 25,285 ms. Of the 7,470 valid responses, 559 (7.5%) were errors. Median correct latencies and percentage of errors were calculated for each participant in each procedure by problem orientation by problem complexity condition. Median correct latencies and error percentages were analyzed in separate 3 (procedure: digit algorithm, partial decomposition, decomposition) by 2 (orientation; horizontal, vertical) by 4 (complexity: no-carry, decade-carry, unit-carry, double-carry) by 3 (skill: high, average, low) by 2 (problem-solving style: flexible, stable) ANOVAs with skill and style as between-group variables. Unless otherwise indicated, the alpha level was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violations of sphericity. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). Ninety-five percent confidence intervals (CIs) are presented on figures. A summary of the ANOVA is shown in Appendix A4 (Table A4.1).

Individual Differences

Latencies. Participants' problem-solving style (i.e., flexible or stable when given a choice) made no difference to their overall speed of solving problems in the no-choice condition. However, there was a significant main effect in response latencies for skill, $F\,(2, 30) = 12.32$, $MSE = 53{,}180{,}000$. Both high- and average-skill participants solved problems significantly more quickly than did low-skill participants (3,267, 3,687, and 5,995 ms, respectively). High-skill participants responded more quickly than did average skill participants, however, this speed advantage (420 ms) was not significant.

Errors. There were no main effects in errors for either problem-solving style or arithmetic skill.

Problem Characteristics

Latencies. There were significant main effects of problem orientation, $F\,(1, 30) = 28.30$, $MSE = 658{,}442$, and complexity, $F\,(2, 59) = 64.10$, $MSE = 4{,}801{,}632$, on speed of problem-solving. Overall, participants solved vertical problems more quickly than horizontal problems (mean difference = 300 ms) and, from fastest to slowest, complexities were: no-carry, decade-carry, unit-carry, and double-carry (3048, 4110, 4763, 5343 ms, respectively). The mean differences in latencies decreased as complexity increased (i.e., the slope of change decreased) but all differences were significant. A complexity by skill interaction, $F(6, 90) = 7.56$, $MSE = 3{,}125{,}124$, can be explained in terms of two differences in performance among the skill groups. First, the effect of increasing complexity on computation speed decreased as arithmetic skill increased. Whereas low-skill participants' latencies increased by 3,813 ms from the least to the most complex problems, latencies increased by 1,885 and 1,187 ms for average- and high-skill participants, respectively. Second, whereas low-skill participants' performance was slowed significantly by each increase in problem complexity (i.e., by both a unit carry and a decade carry) and average-skill participants' performance was slowed by the presence of a unit carry and by a decade carry when there was no unit carry, the speed of participants in the high-skill group was affected by a unit carry only.

Errors. There was a significant main effect of problem orientation, $F(1, 30) = 8.38$, $MSE = 104.27$ on addition errors. Overall, participants solved vertical problems more accurately than horizontal problems (mean difference = 2.3%). This discrepancy was most evident in the performance of flexible problem-solvers; that is, whereas both flexible and stable problem-solvers computed vertical problems more accurately (mean difference 3.6% and 0.9%, respectively), the difference was only significant for flexible problem-solvers. The orientation by style interaction approached significance, $F(1, 30) = 3.57$, $p = .07$, $MSE = 104.27$.

There was also a significant main effect of problem complexity, $F(2.2, 66) = 20.38$, $MSE = 194.27$. From most to least accurate, complexities were: no-carry, decade-carry, unit-carry, double-carry (3.0, 7.1, 9.0, 12.3% errors, respectively). Both unit carries and decade carries had a significant effect on accuracy overall. In errors, the complexity by skill interaction approached significance, $F(6, 90) = 2.08$, $p = .06$, $MSE = 142.51$. As with latencies, low-skill participants' error rate increased more than did the error rates in the average- or high-skill groups. Low-skill individuals showed a significant increase in errors with both a unit carry and a decade carry. On the other hand, average-skill participants made more errors with a unit carry but not with a decade carry and the accuracy of participants in the high-skill group was affected by a unit carry only when there was no decade carry and by a decade carry only when there was no unit carry. For the high-skill group, in other words, once there was a carry in either the units or the decades, a further increase in complexity did not affect accuracy.

Procedure Performance Characteristics

Latencies. Latencies varied across the three procedures, $F(2, 60) = 13.96$, $MSE = 5{,}168{,}567$. Overall, participants calculated the most quickly using decomposition (3,891 ms), followed by the digit algorithm (4,173 ms), and then partial decomposition (4,884 ms). Both decomposition and the digit algorithm were significantly faster than partial decomposition but the difference in speed between decomposition and the digit algorithm (281 ms) only approached significance ($p = .09$).

There was a significant interaction of procedure with problem-solving style, $F(2, 60) = 4.75$, $MSE = 5{,}168{,}567$. As shown in Figure 10, the pattern for stable problem-solvers was similar to that of the overall group in that partial decomposition (5231 ms) was significantly slower than both decomposition and the digit algorithm; however, the difference between decomposition (3,969 ms) and the digit algorithm (4,161 ms) did not approach significance. On the other hand, although flexible solvers appeared to be faster using decomposition (3,622 ms) than either partial decomposition or the digit algorithm (4,536 and 4,377 ms, respectively), there were no significant speed differences among procedures. It is noteworthy that, although all of the stable problem-solvers involved in this experiment used the digit algorithm almost exclusively when they had a choice, they solved problems equally quickly using decomposition when asked to do so. Comparing across procedures, there was no evidence of significant differences between flexible and stable solvers in problem-solving speed on any procedure.

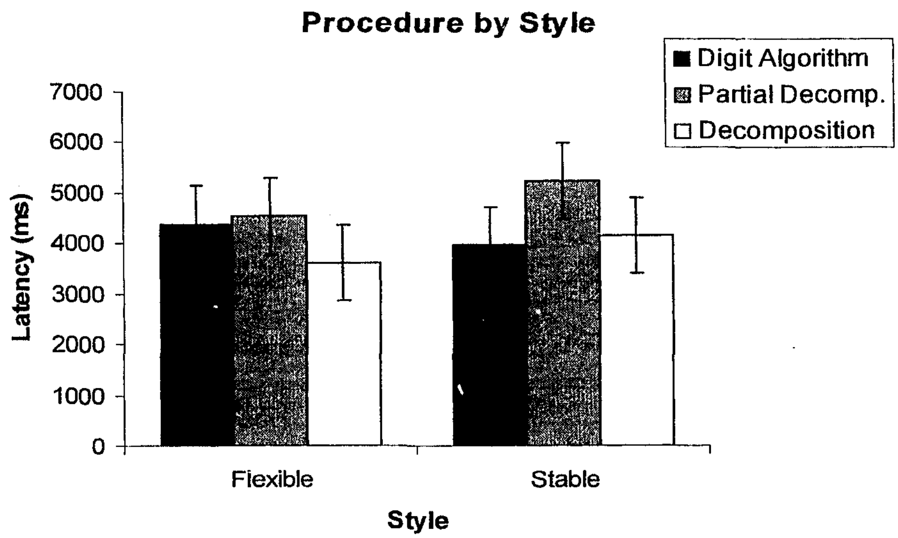

Figure 10. **Experiment 4: Mean latencies of solution procedures across problem-solving styles for adults solving two-digit plus two-digit addition problems in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the two-way interaction.**

Further, procedure interacted with arithmetic skill, $F(4, 60) = 3.65$, MSE = 5,168,567 and problem complexity, $F(3.9, 116) = 4.64$, MSE = 1,369,423, and there was a three-way interaction among procedure, skill, and complexity, $F(12, 180) = 2.31$, MSE = 878,955. As shown in Figure 11, for average- and high-skill adults, partial decomposition was a slower computation method than both decomposition and the digit algorithm. For the low skill group, partial decomposition and the digit algorithm were similar (6,728 and 6,241 ms, respectively) and decomposition (5,016 ms) was faster than both. For average-skill individuals, decomposition and the digit algorithm were similar (3,468 and 3,374 ms, respectively) and partial decomposition (4219 ms) was slower than both. Finally, for those in the high-skill group, the digit algorithm (2,903 ms) was faster than partial decomposition (3,705 ms); decomposition (3,191 ms), between the two, was not significantly different from either. Comparing across procedures, low-skill participants were slower than both of the other groups on all three procedures; average- and high-skill participants did not differ significantly on any.

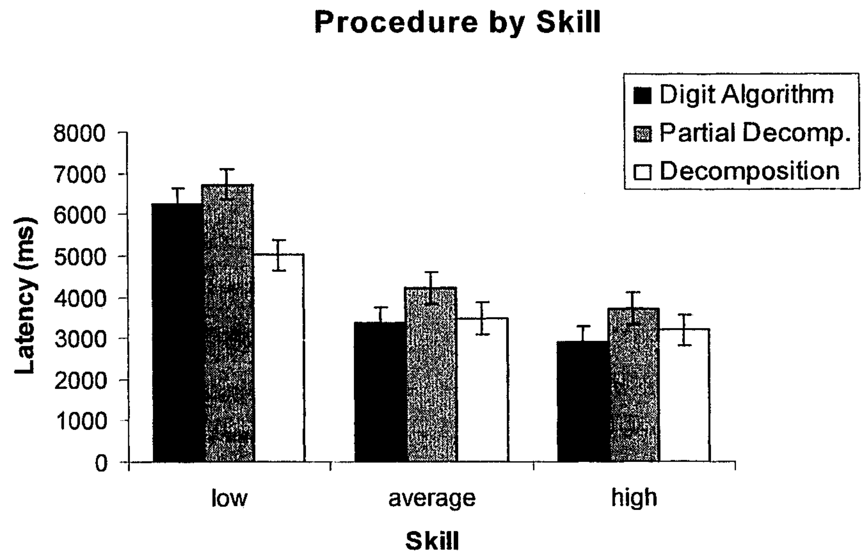

Figure 11. **Experiment 4: Mean latencies of solution procedures across levels of arithmetic skill for adults solving two-digit plus two-digit addition problems in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the two-way interaction.**

The procedure by complexity interaction is illustrated in Figure 12. Although the speed of performance of all procedures was negatively affected by both unit carries and decade carries, latencies increased differentially for different procedures. The most dramatic difference was that a unit carry had a greater effect on the digit algorithm than on the other two procedures so that whereas, for the simplest problems, partial decomposition was the least efficient procedure, with increasing complexity the digit algorithm became equally inefficient relative to decomposition. As a result, for problems with no unit carry (no-carry and decade-carry), decomposition and the digit algorithm were equivalent and were both significantly faster than partial decomposition and, for problems with a unit carry (unit-carry and double-carry), decomposition was faster than both partial decomposition and the digit algorithm.

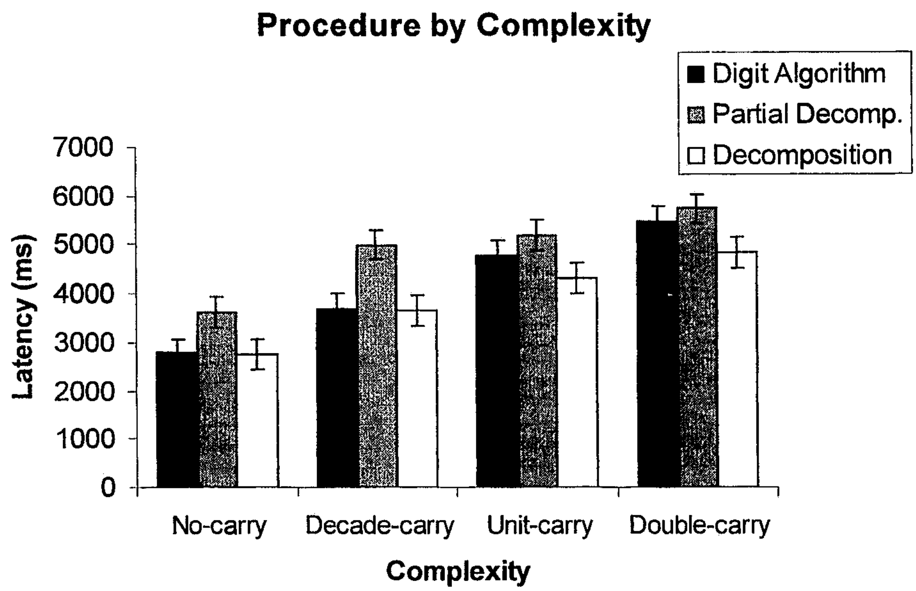

Figure 12. **Experiment 4: Mean latencies of solution procedures for adults solving two-digit plus two-digit addition problems varying in complexity in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the two-way interaction.**

A significant three-way interaction among procedure, complexity, and skill is illustrated in Figure 13. Problem-solving latencies for low-skill participants increased significantly for all procedures with each increase in complexity, whereas for those in the average-skill group, although overall latencies increased with complexity, a decade carry had no effect when there was already a unit carry (i.e., latencies were not significantly different between the unit-carry problems and the double-carry problems), regardless of the procedure used. High-skill participants, on the other hand, were differentially affected by the complexity of the problem depending on the procedure they used. For problems with no decade carry, a unit carry slowed performance with all procedures, whereas for problems with a decade carry, a unit carry slowed performance only with decomposition and the digit algorithm.

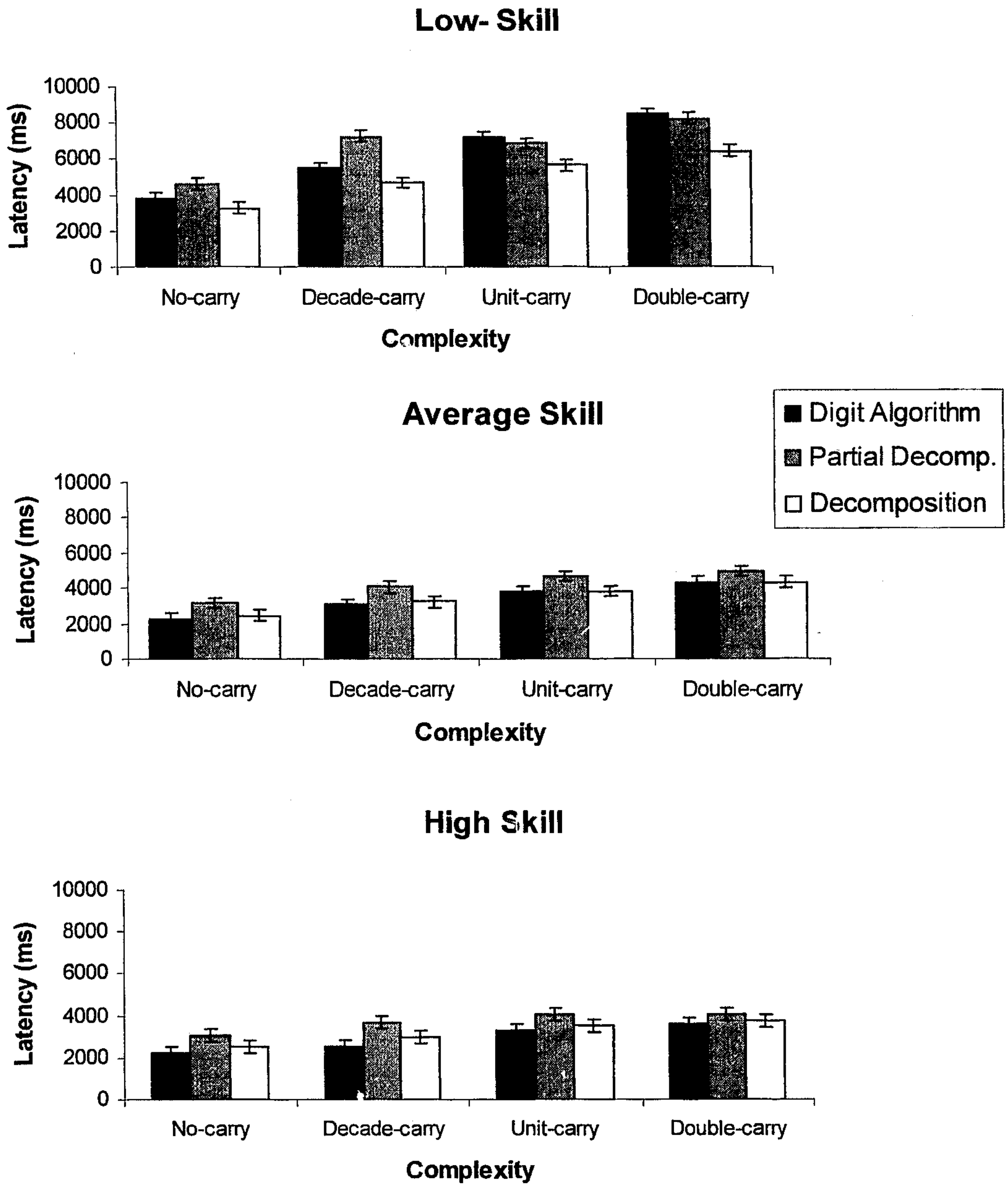

Figure 13. **Experiment 4: Mean latencies of solution procedures across levels of arithmetic skill for adults solving two-digit plus two-digit addition problems varying in complexity in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the two-way interaction.**

Further, the relative efficiency of procedures changed across levels of complexity in different ways across skill groups. For low-skill participants, the digit algorithm became less efficient relative to the other procedures as complexity increased. The result was that, for problems with no unit carry (no-carry and decade-carry), decomposition was the fastest, the digit algorithm was next, and partial decomposition was the slowest, with all discrepancies significant. For problems with a unit carry, decomposition was faster than the other two procedures, which did not differ significantly from each other. For average-skill participants, at all levels of complexity decomposition and the digit algorithm were equivalent and partial decomposition was significantly slower than both. Thus, increasing complexity did not have as great an effect on speed of performance of partial decomposition and the digit algorithm for these people as it did for their low-skill counterparts. For high-skill participants, problems without a decade carry (no-carry and unit-carry) were solved equally quickly with decomposition and the digit algorithm and significantly more slowly with partial decomposition. For problems with only a decade carry, the digit algorithm was the fastest, decomposition was next, and partial decomposition was the slowest, with all discrepancies significant. For the most complex problems, with both a unit carry and a decade carry, the digit algorithm was faster than partial decomposition; decomposition, between the two, did not differ significantly from either. For this group, partial decomposition leveled out with increasing complexity and latencies for the digit algorithm increased at a faster rate than did those for decomposition. Thus, although increasing complexity decreased the speed of execution of all procedures for adults at all skill levels, its effect was greatest on the digit algorithm, especially for low-skill individuals. The average-skill group demonstrated the greatest consistency in latency increases across procedures with increasing complexity.

Comparing across procedure and level of complexity, participants in the low-skill group were significantly slower than those in the other two skill groups in all conditions, whereas average- and high-skill participants solved problems with equivalent speed in all conditions.

A two-way interaction between procedure and orientation approached significance, $F(2, 60) = 2.95$, $MSE = 503{,}666$, $p = .06$. As shown in Figure 14, across orientations, decomposition and the digit algorithm were faster than partial decomposition. For vertical problems, decomposition and the digit algorithm were equally fast but, for horizontal problems, decomposition was faster than the digit algorithm. For both orientations, partial decomposition was significantly slower than the other two procedures.

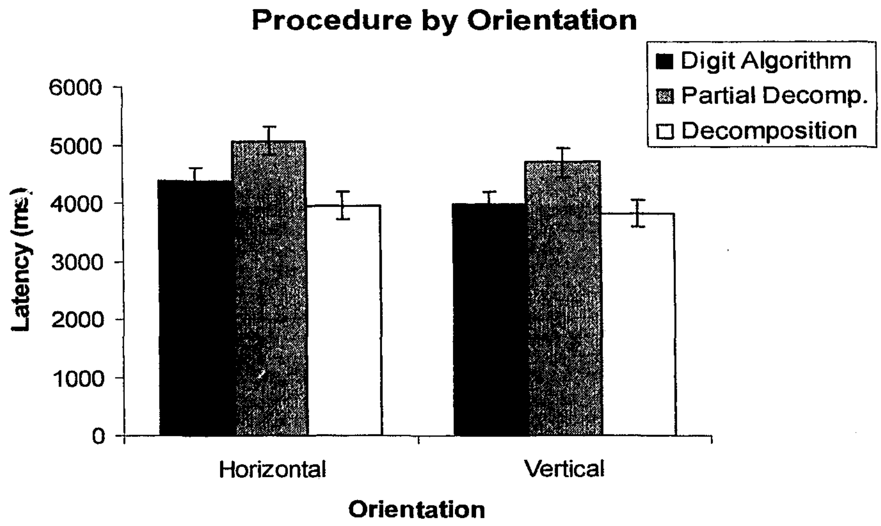

Figure 14. **Experiment 4: Mean latencies of solution procedures for adults solving two-digit plus two-digit addition problems varying in spatial orientation of presentation in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the two-way interaction.**

This pattern was qualified by a significant four-way interaction among procedure, orientation, style, and arithmetic skill, $F(4, 60) = 2.96$, $MSE = 503{,}666$ (See Figure 15). The high-skill flexible solvers used all three procedures with equal speed for both horizontal and vertical problems. For the average-skill flexible solvers, across orientation, decomposition and the digit algorithm were equally fast and partial decomposition was significantly slower than both. For the low-skill flexible group decomposition was the fastest procedure for both horizontal and vertical problems. They used partial decomposition and the digit algorithm equally quickly to solve vertical problems but, when they solved horizontal problems, partial decomposition was slower than decomposition and the digit algorithm was slower than partial decomposition. The stable problem-solvers were always slower using partial decomposition than the other two procedures, with the low-skill group showing the biggest differences. Across problem orientation, the digit algorithm was faster than decomposition for the high-skill group and equivalent in speed to decomposition for the average-skill group. For the low-skill group, decomposition was faster than the digit algorithm, even though this is a group that would use the digit algorithm when given the choice.

Thus, for the most part, partial decomposition, used relatively little in the choice condition by flexible problem-solvers (17%) and almost never by stable problem-solvers (0.5%) was the slowest. High-skill flexible solvers, who are accustomed to using a variety of procedures, did equally well with all three and high-skill stable solvers, who usually use only the digit algorithm, were fastest using that procedure. Regardless of problem-solving style, people with average arithmetic skills did equally well with decomposition and the digit algorithm. The greatest differences among procedures in the no-choice condition were seen in the performance of the low-skill problem-solvers, who were fastest using decomposition, whether they usually use a variety of procedures or only the digit algorithm.

Errors. There were no significant main or interaction effects on errors involving differences among procedures.

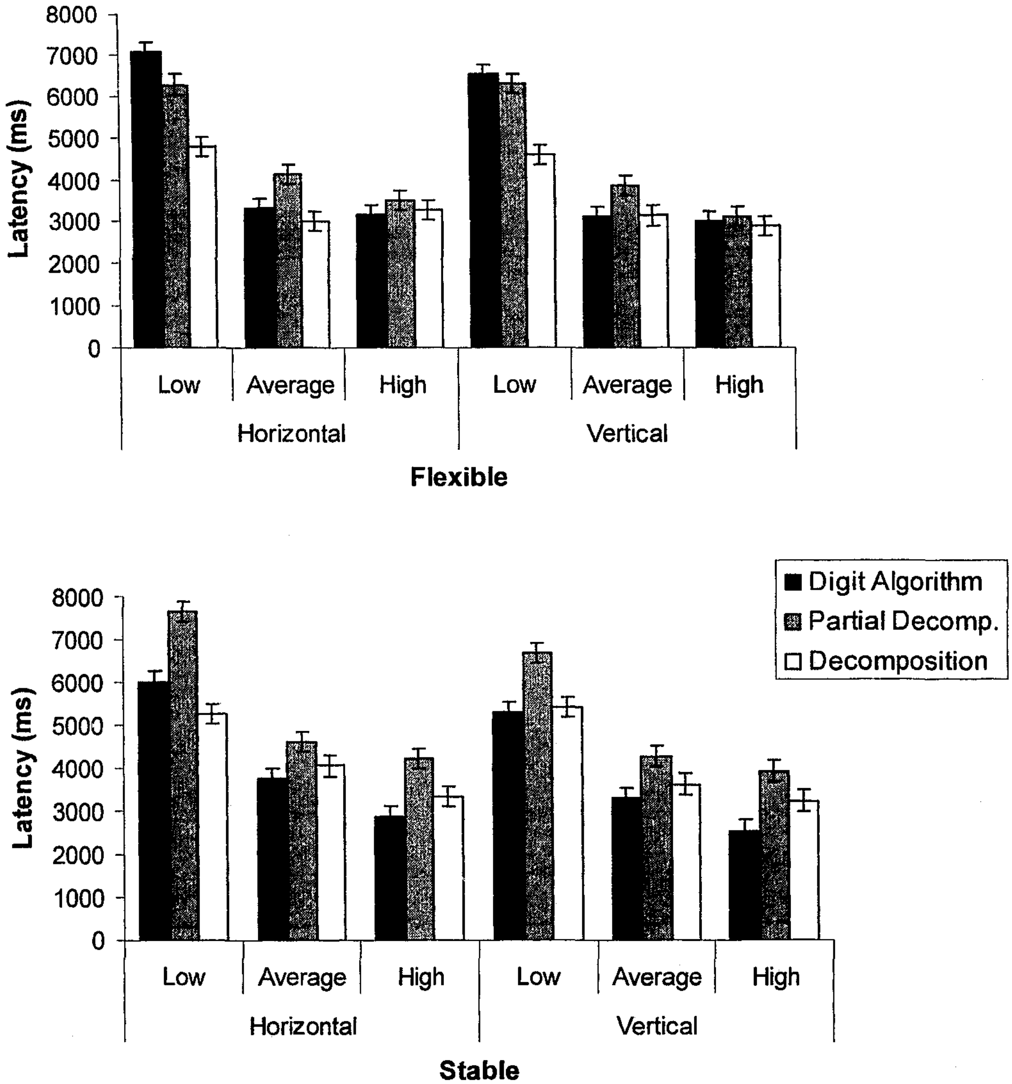

Figure 15. **Experiment 4: Mean latencies of solution procedures across problem-solving style and level of arithmetic skill for adults solving horizontal and vertical two-digit plus two-digit addition problems using various procedures in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the four-way interaction.**

Summary: Latencies and Errors

As expected, adults' performance on the multi-digit mental addition task was related to their overall arithmetic skill. There was no significant difference in response speed between average and high skill participants; however, both groups solved problems more quickly than did the low skill group. The results were consistent with previous studies in that both speed and accuracy of performance were sensitive to characteristics of the problem. Horizontal problems were harder to solve than vertical problems and performance (speed and accuracy) declined as complexity increased. In general, the negative effect (on latencies) of complexity decreased with increasing arithmetic skill.

The results suggested that there is no overall 'best' procedure for adults for solving multi-digit addition problems and that the effectiveness of the procedures examined here varied with level of arithmetic skill. However, contrary to the suggestions of Beishuzen and colleagues (Beishuizen, 1993; Beishuizen, VanPutten, & VanMulken, 1997; Blöte et al., 2000), partial decomposition consistently emerged as the slowest, from 514 to 802 ms slower than the next slowest procedure, depending on arithmetic skill. Low-skill participants calculated more quickly (1,026 ms) with decomposition than with the digit algorithm, whereas average- and high-skill participants were slightly, but not significantly, faster when using the digit algorithm than decomposition (69 and 287 ms, respectively). The advantage of the digit algorithm and decomposition was consistent but was less pronounced for problems presented vertically and for problems that required a carry operation in the unit digits. Thus, although multi-digit problems were solved equally accurately with all three solution procedures, calculation speed varied with procedure but in different ways depending on problem characteristics and individual differences. Decomposition appeared to be a more efficient procedure but was only significantly better for less-skilled solvers. Decomposition was as good as, or better than, the digit algorithm for stable individuals, even though they always used the digit algorithm in the choice condition. For flexible problem-solvers, decomposition showed a consistent advantage for both low- and average-skill participants. Thus, procedures are efficient both because they are inherently better under some conditions, and because they are more practiced or more familiar for some solvers. Requiring individuals to use a relatively unpracticed procedure (e.g., flexible, less-skilled participants using the digit algorithm) resulted in relatively worse performance than when they used a more practiced procedure. Thus, although the no-choice condition may provide unbiased information about relative procedure efficiency in some ways, participants' previous experience (or lack of experience) with procedures they are forced to use provides another source of bias.

Choice/No-Choice Comparisons

Performance With and Without a Choice

In order to determine whether or not adults' mental calculation performance was better (i.e., faster and/or more accurate) when they chose how to solve the problems or when they were told what procedures to use, median correct latencies and percentage of errors were calculated for each participant in each choice/no-choice by problem orientation by problem complexity condition. Median correct latencies and error percentages were analyzed in separate 2 (choice condition: choice, no-choice) by 2 (orientation; horizontal, vertical) by 4 (complexity: no-carry, decade-carry, unit-carry, double-carry) by 3 (skill: high, average, low) by 2 (problem-solving style: flexible, stable) ANOVAs with skill and style as between-group variables. Unless otherwise indicated, the alpha level was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violations of sphericity. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). Ninety-five percent confidence intervals (CIs) are presented on figures.

Latencies. There was a main effect of choice condition on latencies, $F(1, 30) = 36.51$, $MSE = 2{,}147{,}179$, as well as a choice condition by skill interaction, $F(2, 30) = 7.70$, $MSE = 2{,}147{,}179$. Across problem-solving styles, people solved problems almost 800 ms faster in the choice condition (3490 ms) than in the no-choice condition (4264 ms). However, as illustrated in Figure 16, this discrepancy was due to differences in performance across conditions for the low-skill group only. Average- and high-skill individuals did equally well whether or not they chose the procedure(s) to use.

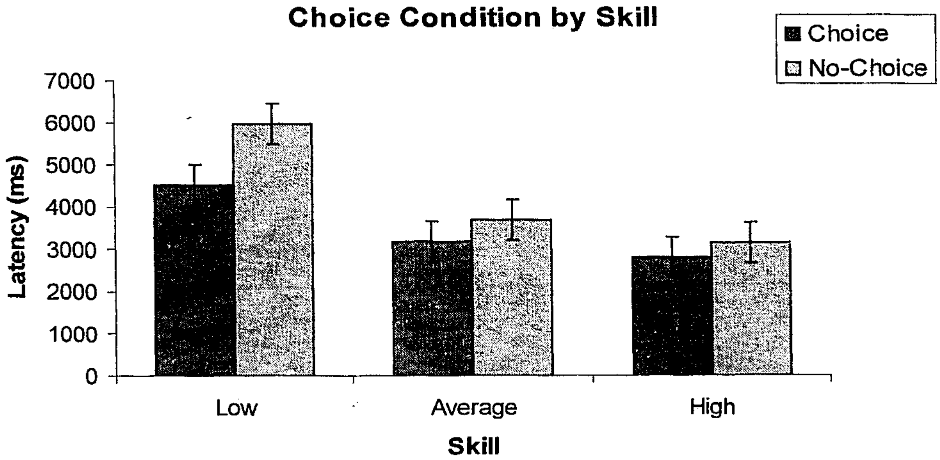

Figure 16. **Experiments 3 and 4: Mean latencies of solution procedures across choice condition and level of arithmetic skill for adults solving horizontal and vertical two-digit plus two-digit addition problems in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the four-way interaction.**

Errors. There was a main effect of choice condition on errors, $F(1, 30) = 45.00$, $MSE = 156.4$. Across skill groups, people solved problems more accurately in the choice condition (6% errors) than in the no-choice condition (13% errors). As illustrated in Figure 17, this error discrepancy was significant for flexible solvers and not for stable solvers but the choice condition by style interaction only approached significance, $F(1, 30) = 2.93$, $MSE = 156.4$, $p = .09$.

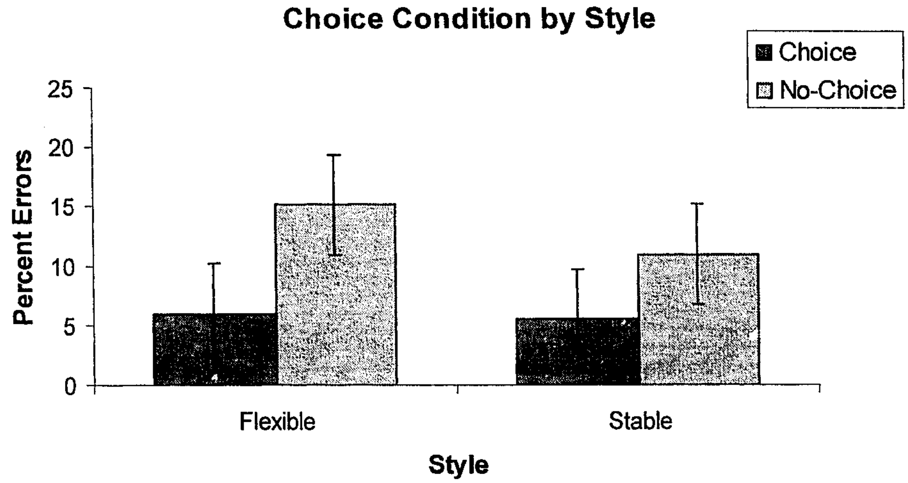

Figure 17. **Experiments 3 and 4: Percent errors across choice condition and problem-solving style for adults solving horizontal and vertical two-digit plus two-digit addition problems in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the four-way interaction.**

There was also a choice condition by orientation interaction, $F(1, 30) = 18.52$, $MSE = 80.0$, such that the freedom to choose what procedure(s) to use in solving problems made a significant difference to accuracy for horizontal problems (11% difference) but not for vertical problems (4% difference; See Figure 18).

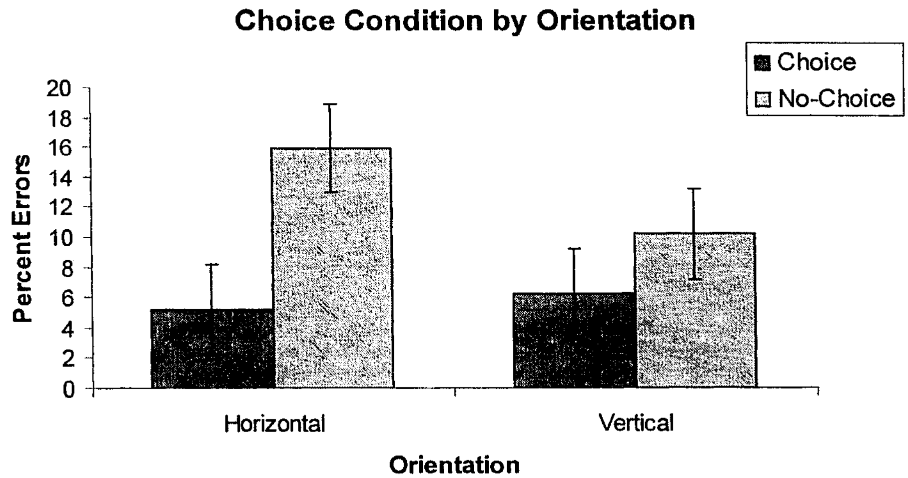

Figure 18. **Experiments 3 and 4: Percent errors across choice condition and orientation for adults solving horizontal and vertical two-digit plus two-digit addition problems in the no-choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the four-way interaction.**

The four-way interaction among choice condition, orientation, style, and skill approached significance, $F(2,30) = 2.93$, $MSE = 80.0$, $p = .07$ (See Figure 19). On vertical problems, there was a choice advantage for low-skill people but not for the average- or high-skill groups, whether flexible or stable. Flexible and stable problem-solvers experienced similar error rates in the choice condition but stable solvers were more accurate in the no-choice condition, resulting in a smaller choice advantage. On horizontal problems, participants at all skill levels and across styles were significantly more accurate in the choice condition. In the average- and high-skill groups, stable solvers were more accurate than flexible solvers in the no-choice condition and, in the low-skill group, stable solvers were more accurate than flexible solvers in both choice conditions.

Thus, the increased accuracy in the choice condition was largely attributable to performance on horizontal problems, although low-skill participants also experienced a choice advantage on vertical problems. Accuracy rates on horizontal problems varied, not only with whether or not participants chose solution procedures, but also with arithmetic skill and problem-solving style. Overall, although the stable and flexible groups were equally accurate when they solved problems the way they wanted to, and even though flexible solvers had more experience using a variety of procedures, error rates increased more for flexible than for stable solvers when they were told how to solve problems.

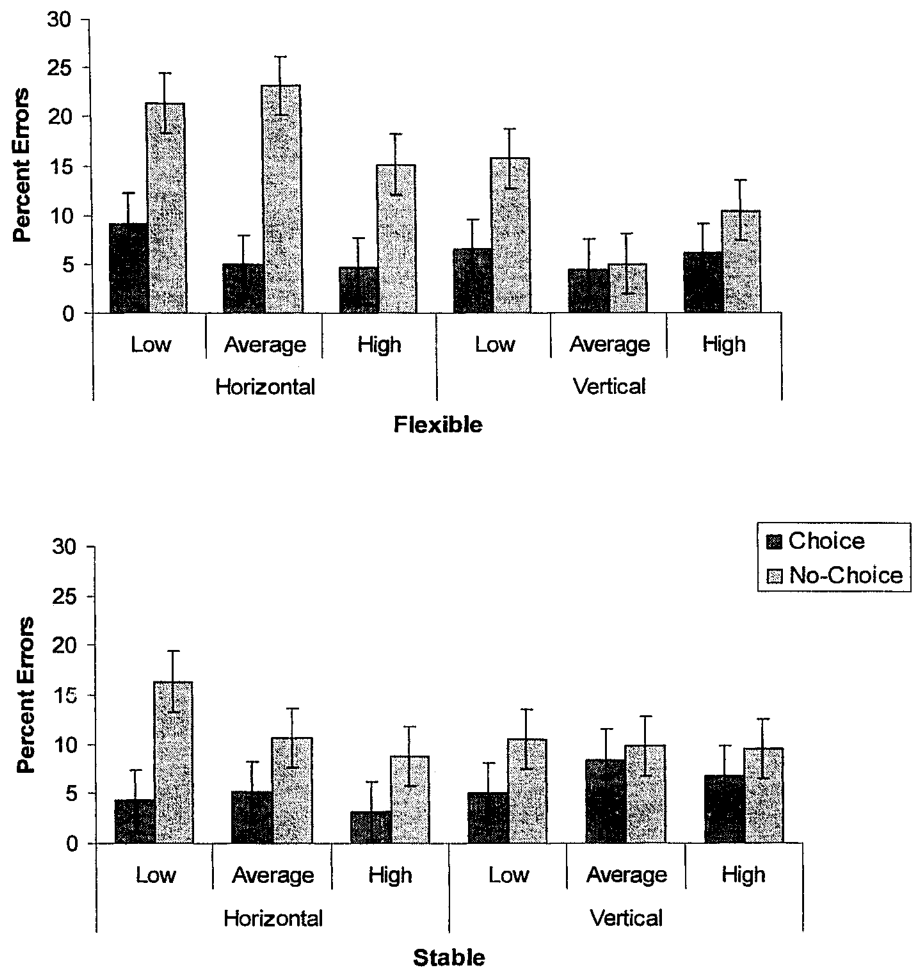

Figure 19. **Choice/No-Choice: Percent errors across problem-solving style and level of arithmetic skill for adults solving horizontal and vertical two-digit plus two-digit addition problems in the choice and no-choice conditions. Error bars represent 95% confidence intervals, based on the MS_e for the four-way interaction.**

Summary: Choice/No-Choice

Thus, adults in the low-skill group solved multi-digit problems faster and flexible problem-solvers at all skill levels made fewer errors when they chose which procedure(s) to use than when they were told how to solve the problems. In addition, the freedom to choose what procedure(s) to use made a significant difference to accuracy for horizontal, but not for vertical, problems.

Adaptivity

In existing research in the domain of mental arithmetic, people are considered to be adaptive when they vary their procedure choices in response to problem characteristics and/or procedure characteristics in order to optimize performance efficiency (i.e., speed and accuracy). Implicit in this definition is the assumption that variability in choices (that is, flexibility) is necessarily better than stability. According to this view, stable problem-solvers are 'inflexible' and, therefore, not adaptive in their problem-solving. In the present research, it has been established that only some people use a variety of procedures for multi-digit mental addition. Others (stable problem-solvers) use one procedure to solve all, or almost all, problems, regardless of factors related to the problems or to other available procedures. Hence, according to the definition of adaptivity as variation in choice of procedures, only flexible solvers can be considered adaptive. However, if an individual's best procedure is the one s/he consistently uses, then presumably that person would be perfectly adaptive, while only ever using a single procedure.

All of the stable problem-solvers who took part in this no-choice experiment had chosen the digit algorithm exclusively or almost exclusively in the choice condition (93% - 100%) for an overall average of 99% use of the digit algorithm (See Table 21). They clearly did not always choose the procedure that optimizes their performance in each orientation by complexity condition. For example, they were faster using decomposition than the digit algorithm in the no-choice condition overall. Further, high-skill stable people were faster using decomposition and average-skill stable people were equally fast using decomposition and the digit algorithm across orientation, whereas low-skill stable people were faster with decomposition on horizontal problems but equally fast with decomposition and the digit algorithm on vertical problems. Finally, all of the low-skill adults found the digit algorithm increasingly difficult with increasing problem complexity. In spite of these performance variations, stable problem-solvers did not vary their procedure choice with problem orientation, complexity, or level of arithmetic skill. Nonetheless, they were as fast and at least as accurate as the flexible group in solving 2-plus 2-digit arithmetic problems in both choice and no-choice conditions.

Putting aside the assumption that using a variety of solution procedures is a better approach overall, one might ask whether one problem-solving style is more or less effective relative to the other. For example, if stable solvers have a long history of using one procedure for all calculations, they may benefit from being highly practiced in that procedure. That is, they choose it because it is better and it is better because they are practiced at it. In both the choice and no-choice conditions, flexible and stable solvers were equally fast but there was a tendency for the stable group to be more accurate than the flexible group in the choice condition. It does not appear that using a variety of procedures, selected on the basis of problem and, possibly, procedure characteristics, yielded an advantage in this case over choosing one preferred procedure and sticking with it.

In order to compare flexible and stable problem-solvers' use of procedures in the choice condition in terms of optimizing their performance, a measure of strategy "adaptivity" was calculated for each participant in each orientation by complexity condition. The adaptivity measure was the percentage of trials on which the participant chose his/her *best* procedure as determined by performance in the no-choice condition (Imbo & LeFevre, 2009). The best procedure was determined by comparing the response latencies of the three procedures (decomposition, partial decomposition, and the digit algorithm). If, for example, a participant solved horizontal no-carry problems faster with decomposition than with the other two procedures in the no-choice condition, then decomposition was considered that individual's best procedure for horizontal no-carry problems. There were no ties. The adaptivity measure for each participant was the percentage of trials on which that individual used the best procedure in the choice condition.

I hypothesized that stable and flexible problem-solvers used the fastest procedures equally. Further, analyses thus far have shown that the pattern of speed and accuracy in the no-choice condition is not a perfect match to the pattern of procedure use in the choice condition. Therefore, I also hypothesized that both flexible and stable groups used their best procedures only to a moderate degree. Arithmetic fluency is related to participants' performance and procedure choices and was also expected to be related to degree of adaptivity. On the one hand, the low-skill group consistently stands out as different from the average- and high-skill groups. For example, they are always slower and their performance is most affected by increasing problem complexity. Their lower performance may be partly attributable to an inefficient use of solution procedures. On the other hand, the high-skill group is consistently fast, regardless of what procedure they use. Thus, they may not bother to adjust their procedure according to perceived procedure characteristics.

Adaptivity measures were analyzed in separate 2 (orientation; horizontal, vertical) by 4 (complexity: no-carry, decade-carry, unit-carry, double-carry) by 3 (skill: high, average, low) by 2 (problem-solving style: flexible, stable) ANOVAs with skill and style as between-group variables. Unless otherwise indicated, the alpha level was .05. Greenhouse-Geisser procedures were used for correcting degrees of freedom and mean square error terms under violations of sphericity. Post hoc pairwise comparisons between means were made using the confidence interval formula and method of comparison recommended by Masson and Loftus (2003). Ninety-five percent confidence intervals (CIs) are presented on figures.

Overall, participants selected their best procedure on 51% of problems in the choice condition. Stable problem-solvers (56%) appeared to use their best procedure more frequently than flexible solvers (46%), however, the difference was not significant. Both proportions were significantly greater than chance (33%), $t(35) = 21.5$, $p < .05$ and $t(35) = 12.7$, $p < .05$, respectively. Thus, both flexible and stable problem-solvers used their best procedure on about half of the significant proportion of problems, but the hypothesis that the flexible use of a variety of procedures will be more adaptive was not supported.

Adaptivity varied with complexity, $F(3, 90) = 2.91$, $MSE = 1646$, such that overall people were less likely to choose the best procedure on unit-carry than any other problems. A significant three-way complexity by orientation by skill interaction, $F(6, 90) = 3.05$, $MSE = 1081$, indicated that although overall the high skill group (60%) tended to be more adaptive than the average- and low-skill groups (47% and 46%, respectively), no group was consistent in choosing the best procedure across orientation and levels of complexity (See Figure 20). High-skill participants chose the best procedure *less* often on horizontal problems with no decade carry (especially unit-carry problems such as 26 + 57) and on the most complex vertical problems (e.g., 89 + 63). These lower percentages were, for the most part, attributable to the flexible high-skill participants, who chose their best procedure less frequently than did the stable high-skill participants in most conditions and particularly on the most complex vertical problems. Overall, in the high-skill group, the stable problem-solving approach resulted in 24% more use of the participant's best procedure. In some ways, therefore, the flexible problem solvers might be characterized as unstable, in that they chose to use less-useful procedures under higher-demand conditions (i.e., as problem complexity increased).

Those in the average-skill group chose their best procedure significantly less often on vertical problems with only a unit carry and horizontal problems with only one carry (either decade or unit). Again, the lower adaptivity scores in these conditions were attributable to the flexible participants. In this group, stable problem-solving resulted in 20% more adaptive use of procedures. The low-skill group had higher adaptivity scores for the most complex vertical problems than for other problem types. In the low-skill group, flexible problem-solving yielded 13% more use of a participant's fastest procedure. Thus, participants' use of their best procedure varied with the nature of the problem and, as expected, with the arithmetic skill of the individual. In addition, there was a tendency for the stable approach to yield as much or more use of the best procedure as did the flexible use of a variety of procedures.

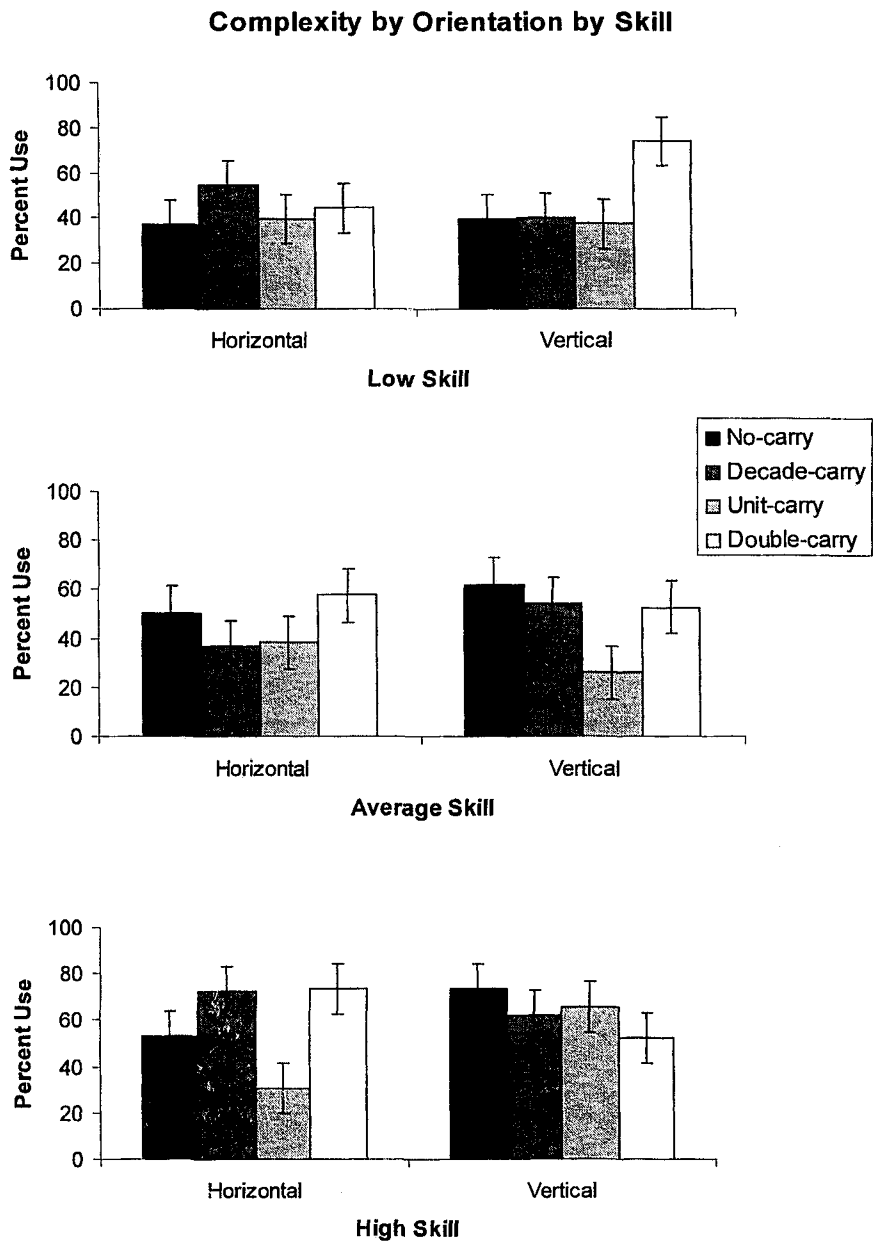

Figure 20. **Choice/No-Choice: Percent use of best procedure across level of arithmetic skill for adults solving horizontal and vertical two-digit plus two-digit addition problems in the choice condition. Error bars represent 95% confidence intervals, based on the MS_e for the four-way interaction.**

Summary: Adaptivity

Stable problem-solvers used the digit algorithm almost exclusively when they had the choice but this was not their most efficient procedure in all orientation by complexity by skill conditions in the no-choice situation. Nonetheless, their performance in both the choice and no-choice conditions was as fast and at least as accurate as that of the flexible problem-solvers. Furthermore, when adaptivity was defined as the proportion of problems solved in the choice condition using their best procedure, as determined in the no-choice condition, flexible and stable individuals were equally adaptive overall. For both groups, people's ability to choose the best procedure varied with skill level, problem-solving style, and problem complexity. In particular, flexible high-skill individuals experienced the greatest difficulty choosing their most efficient procedure to solve more complex problems. Within the average- and high-skill groups, the stable solvers appeared to be the more adaptive. The hypothesis that the flexible use of a variety of procedures is more adaptive was not supported by these results.

Discussion: Experiments 3 and 4

The results of the choice/no-choice study confirmed that the flexible/stable distinction is a consistent individual difference characteristic in adults solving multi-digit addition problems and that problem-solving style does not vary with arithmetic skill. Stable problem-solvers used the digit algorithm to solve essentially all problems and flexible problem-solvers varied their procedure choices with characteristics of the problem. For example, as expected, they used the digit algorithm more than holistic procedures for vertical problems and holistic procedures more for horizontal problems.

None of the three procedures examined stood out as the most efficient overall as the effectiveness of the procedures varied with style, skill, and problem characteristics. The most interesting result was that the stable and flexible approaches led to similar levels of adaptivity, defined as frequency of use of the best procedure given the circumstances, even though the digit algorithm was not the stable solvers' most efficient procedure in all cases when measured in the no-choice condition. The low-skill participants were consistently the slowest problem-solvers regardless of what procedure they used and whether or not they chose which procedure to use.

CHAPTER 5

Multi-digit mental arithmetic is a complex cognitive task involving multi-step procedures. A variety of these procedures is available to adults performing complex mental addition. These procedures fall into two categories based on the way the operands are represented and operated on during computation, either as strings of single digits or more holistically. Further, adults fall into two groups with regard to their complex addition problem-solving strategy, or style. Whereas some people use a variety of procedures and vary them according to a number of factors, others have developed a practice of using only one procedure and applying it to all multi-digit addition problems.

The goal of the present research was to investigate the use of multi-step procedures in adults' complex mental arithmetic performance. I explored adults' choices of multi-step solution procedures and their effect on performance in multi-digit (i.e., one- plus two-digit and two- plus two-digit) mental addition. Using trial-by-trial verbatim procedure reports, two experiments determined the procedure repertoire for complex addition of adults as a group. Several common calculation procedures were categorized as either *digit oriented* or *holistic*, according to the way in which the operands were represented and operated on during computation. In a further two experiments, a choice/no-choice paradigm (Lemaire & Siegler, 1995) was employed to test whether people's procedure choices were influenced by problem characteristics that are known to affect arithmetic performance (i.e., problem complexity and spatial orientation) and by performance characteristics (i.e., speed and accuracy) of the procedures. Finally, procedure choice was examined in relation to people's arithmetic fluency and problem-solving style. In this final chapter, I discuss the overall results in relation to Lemaire and

Siegler's (1995) four dimensions of "strategic competence": procedure repertoire, procedure frequency, procedure efficiency, and adaptivity of procedure choice. Further, the concept of adaptive expertise in problem-solving is reviewed.

Procedure Repertoire

Procedure repertoire refers to the variety of procedures that adults use to solve a certain type of problem. In this research, adults reported using a small set of procedures similar to those taught in European schools (e.g., Beishuizen, 1993) and to those reported by children in Europe and North America (Fuson, 1990; Lucangeli et al., 2003). Most of these participants appeared to have adapted the digit-based paper-and-pencil calculation procedure emphasized in North American elementary education for use as a mental computation procedure (i.e., the digit algorithm). In fact, the digit algorithm was consistently the most frequently reported method of solving complex addition problems. People also reported using a number of calculation procedures that involve a more holistic representation of the addends. For 2- plus 2-digit addition problems, these procedures were decomposition, partial decomposition, and transformation. Decomposition was the most frequently reported procedure from among them.

The important distinction between digit-based and holistic procedures is the difference in the ways in which the operands are represented during computation and, consequently, the cognitive processes involved in calculation. The digit algorithm is based on a representation of the addends as strings of single digits that disregards place value concepts and the magnitude of the addend as a whole, relying instead on digit order to preserve magnitude. This procedure employs a series of simple additions and may involve both automatic retrieval and non-retrieval procedures for simple arithmetic. On more complex problems, it requires the maintenance in working memory of intermediate sums, the relative order of the resulting answer digits, and a carry value.

On the other hand, holistic procedures are based on a representation of the addends that maintains an awareness of place value and magnitude throughout calculation. The central tenet of the place value concept is that the position of a digit within a multi-digit number dictates the value represented by the digit. For example, in the number 243, the digit 4 represents the quantity 40 (understood as one more than 39 or as 4 tens) and the digit 2 represents the quantity 200. While using holistic procedures to calculate, the individual must simultaneously maintain at least two representations of each digit in a number, whereas in using the digit algorithm, this dual representation (DeLoache, 2004) is not necessary until the answer is 'read back' as long as the relative position of the digits is preserved. Holistic procedures involve decomposing the addends in various ways and constructing the sum from the resulting parts. They require knowledge of additive composition, additive commutativity and associativity (Cowan, 2003), and the complementarity of addition and subtraction (Baroody, 2003; Resnick, 1983). Thus, use of the digit algorithm could be a primarily a rote procedure, without conceptual knowledge, whereas holistic procedures appear to require conceptual understanding to a greater degree. Notably, however, differences in the role of conceptual understanding in implementing these procedures cannot be determined from adult performance. The individual who relies on a digit algorithm does not necessarily lack conceptual understanding of magnitude but, instead, may simply be using well-practiced procedures to achieve a problem solution.

Procedure Frequency: The Flexible/Stable Distinction

In previous work, researchers have examined arithmetic performance and procedure choice for people at various developmental levels (e.g., children, young adults, and older adults; e.g., Lemaire et al. 2004; Luwel et al. 2005), with and without math disabilities (e.g., Geary et al., 1991; Torbeyns et al., 2004b), and in various cultural groups (e.g., Asian, European, and Canadian; Imbo & LeFevre, 2009) but have blended together all problem-solvers within these groups without regard for whether or not all of them use a variety of procedures for mental arithmetic. Siegler and colleagues (e.g., Siegler, 1999, 2005; Siegler & Lemaire, 1997; Siegler & Shipley, 1995) argue that cognitive variability is pervasive and continues throughout the lifespan and, on the basis of this argument, researchers should consider individual differences in problem-solving style within these groups. Accordingly, the present research shows that problem-solving style is a key distinguishing factor in the use of procedures to perform complex mental arithmetic.

A significant finding in the present research was that adults consistently divided into two groups based on problem-solving styles. Flexible problem-solvers knew and used a variety of multi-digit addition procedures, including the digit algorithm as well as a number of holistic procedures, and they selected which one to use on a given problem on the basis of the complexity of the problem and the way in which it was presented visually (i.e., order and orientation). These choices were also related to some degree to the relative speed and accuracy of available alternative procedures. Stable problem-solvers relied exclusively on one procedure (in most cases the digit algorithm) for complex mental calculations, regardless of problem characteristics and the relative effectiveness of available alternatives. This flexible/stable distinction emerged as a reliable individual difference in terms of adults' procedure choices for complex addition.

Problem-solving style is an important individual difference to consider. If the flexible use of a variety of procedures is the most adaptive way to solve arithmetic problems (e.g., Hatano, 1988) and if the "strict use of the digit algorithm is not an example of good practice" (Heirdsfield & Cooper, 2002, p.72), then stable problem-solvers should be at a disadvantage compared with their flexible counterparts. In the present research, however, flexible and stable adults were equally efficient with all procedures in the no-choice condition and were equally efficient overall in both the choice and no-choice conditions. Further, stable solvers, who had more practice with the digit algorithm than with holistic procedures, solved problems equally quickly and accurately using decomposition, compared with the digit algorithm, when asked to do so. Flexible solvers, who normally used a variety of procedures, including decomposition and the digit algorithm, were equally efficient using all three procedures.

It is tempting to conclude that, because flexible problem-solvers choose to switch between procedures that involve different number representations, they are individuals who have stronger connections between forms of number representation, thus demonstrating a stronger concept of number than stable problem-solvers (e.g., Blöte et al., 2000; Blöte et al., 2001), who choose to use only the digit algorithm or only decomposition. However, stable solvers proved to be equally well able to use both digit and holistic procedures when instructed to do so. In the no-choice condition, stable solvers were not only able to use both digit and holistic procedures after relatively brief training sessions, they solved problems as quickly and accurately when using decomposition as when using the digit algorithm. This pattern occurred even though all of the stable solvers who participated in the no-choice condition had consistently used the digit algorithm when they had the choice and thus they were, presumably, individuals who had had little practice with decomposition. For these solvers, therefore, it seems that Heirdsfield and Cooper are wrong to assume that exclusive use of the digit algorithm is necessarily a poor choice for mental arithmetic.

Procedure Efficiency

Holistic procedures such as decomposition and transformation are widely considered to be more efficient than the digit algorithm for mental computation (Beishuizen, 1993; Beishuizen et al., 1997; Blöte et al., 2000; Heirdsfield & Cooper, 1996) and Beishuizen and colleagues consider partial decomposition to be the most efficient. In the present research, however, there was no best procedure overall and partial decomposition was consistently the least efficient. The fact that it was reported relatively infrequently in the choice condition by flexible problem-solvers (17%) and almost never by stable problem-solvers (0.5%) indicates that participants had relatively little practice using it. Lack of experience using a procedure may be sufficient to increase its load on working memory, thus leading to greater inefficiency with that procedure (Heirdsfield & Cooper, 2002). In contrast, for the children studied by Beishuizen and his associates, partial decomposition was stressed in the school curriculum and thus a similar study done with adults in the Netherlands might reveal a different pattern of procedure efficiency.

Between decomposition and the digit algorithm, relative efficiency varied with arithmetic skill and problem characteristics. For example, the digit algorithm was a less efficient procedure than decomposition for adults with low arithmetic skill, whereas average- and high-skill individuals did equally well with both procedures. The digit algorithm was also less efficient for problems with a carry in the units. For these problems, the digit algorithm would require storing a carry digit and remembering to add it in the next position to the left, thus potentially placing increasing demand on working memory and slowing the calculation process. The decrease in efficiency for the digit algorithm with increasing complexity was most pronounced for the low-skill group, who began to have difficulty with this procedure when the problem had a single carry in the tens. In 2- plus 2-digit addition, these carries are somewhat less demanding than carries in the ones. They do involve large simple sums but do not require an extra digit to be stored and added in the next position. The digit algorithm may be relatively slow on these problems for low-skill individuals because this group likely uses retrieval less frequently and slower non-retrieval procedures more frequently than do more-skilled solvers on single-digit addition sums (Smith-Chant & LeFevre, 2002). High-skill individuals were most efficient using the digit algorithm for problems with no unit carry. These are the most proficient users of retrieval for simple addition, especially for smaller problems, so the digit algorithm should be faster for them on the easier problems. On more complex problems, decomposition and the digit algorithm were equally fast.

Skill

Although individual differences in problem-solving style were not related to efficiency of performance for the most part, arithmetic skill made a significant difference to relative procedure performance. The greatest differences in relative procedure efficiency in the no-choice condition were seen in the performance of the low-skill problem-solvers, who were fastest using decomposition and slowest using the digit algorithm regardless of whether they were flexible or stable problem-solvers. Participants in the low-skill group were significantly slower than those in the other two skill groups in all conditions, whereas average- and high-skill participants consistently solved problems with equivalent speed and accuracy. High-skill solvers, both flexible and stable, were fastest using the digit algorithm, whereas, regardless of problem-solving style, people with average arithmetic skills did equally well with decomposition and the digit algorithm.

Kerkman and Siegler (1993) posit that low-achievers' poor performance may result either because they make unwise choices from among available procedures or because of poor execution of the procedures. According to Siegler and Shrager (1984), on the other hand, poor students' weak performance is due to poor execution of procedures and weak factual knowledge and, regardless of what procedures they choose, they do less well than their higher-skill counterparts. The latter view was supported by this research. At least for adults, procedure choice was not related to skill. Low-skill solvers were slower than everyone else, regardless of whether they were stable or flexible problem-solvers when they had a choice and regardless of what procedure they used when they did not have a choice. This observation is consistent with Lemaire and Arnaud's (2008) finding that older adults' poorer performance relative to that of younger adults on more difficult multi-digit addition problems did not result from the two groups using different procedures on these problems.

Skill also makes a difference when examining the relations between performance and problem characteristics. Low-skill individuals stood out as slower overall but also as more vulnerable to the effects of problem characteristics such as complexity. Not only were they slower at all levels of complexity but their calculation times and error rates increased more dramatically with increases in complexity and their performance was consistently affected by carry operations both in the units and in the decades of a problem. On the other hand, the speed and accuracy of average- and high skill individuals was less affected by complexity. This finding is consistent with Walczyk and Griffith-Ross' (2006; Walczyk, 2000) discussion regarding how a lack of automaticity in sub-component skills such as number identification and simple addition may stall or slow performance on difficult math problems. Problem-solving speed for low-skill participants decreased significantly for all procedures with each increase in complexity whereas average- and high-skill participants were differentially affected by the complexity of the problem depending on the procedure they used.

Adaptivity in Complex Mental Arithmetic

It has long been accepted that adaptive expertise is based on the flexible use of a variety of meaningful procedures (e.g., Baroody & Dowker, 2003; Beishuizen, 1993; Hatano, 1988; Lemaire et al., 2004; Shrager, & Siegler, 1998). It is assumed that the use of a variety of procedures reflects an integrated network of conceptual and procedural knowledge that allows the individual to know how to perform a procedure, when to use it, and why (Bisanz & LeFevre, 1990; Hiebert & Lefevre, 1986). Hatano (1988) contrasts adaptive expertise with routine expertise, the ability to compute without understanding. The stable approach, at least for those who use the digit algorithm consistently, appears to be an expression of routine expertise. However, in the present research, stable problem-solvers solved complex addition problems as quickly and accurately as did flexible problem-solvers. This pattern of findings appears at odds with the view that expertise must be flexible to be adaptive.

Current models of procedure choice (e.g., SCADS, Shrager & Siegler, 1998) posit that, in general, people will choose adaptively. That is, they will choose the procedure that has proven most effective in solving a given problem (or type of problem) in the past. Examining the calculation performance of flexible and stable problem-solvers separately in this research has allowed me to examine more closely the concept of adaptivity in procedure choice. For a problem-solving approach to be adaptive, it must be advantageous in some way to the solver. In arithmetic calculation on an everyday basis, the most important consideration is to obtain the correct answer and the most advantageous approach is the one that will lead to the correct answer in the least time. Until now, researchers investigating procedural adaptivity (e.g., Lemaire & Lecacheur, 2002; Luwel et al., 2005; Siegler & Lemaire, 1997) have assumed that an 'adaptive' approach to problem-solving requires the flexible use of a variety of procedures. My findings suggest the need to re-examine that assumption.

Cowan (2003) presented two views of arithmetic expertise: competence as computational skill and competence as mathematical understanding and thinking. Both are relevant to complex mental arithmetic computation. However, in most research on the adaptive use of procedures to solve arithmetic problems, the focus is on computational skill. In the present research, I found that both flexible and stable individuals used the most efficient procedure for each problem type on approximately half of the problems they solved and that the stable problem-solving approach was as adaptive in that sense as the flexible approach. Further, stable solvers were able to use a variety of procedures when asked to do so and after only brief training. Therefore, it does not appear to be procedure variation that makes the problem-solving adaptive. How, then, should adaptivity be defined? It seems that there are some people for whom using one procedure consistently is adaptive (i.e., stability of choice optimizes speed and accuracy). It may be that the stable solvers were not familiar with alternative procedures. Indeed, anecdotal evidence from participants' comments during the training phase of the no-choice conditions suggested that some were learning alternative procedures for the first time. However, it may also be that over the years they have increasingly chosen the procedure that works best for them. The participants in the current research had at least 10 years of practice solving multi-digit addition problems, which may have resulted in adaptivity that did not require procedure variation (routine expertise). At any rate, it is clear that we cannot define adaptivity in problem-solving strictly as choosing well among a variety of alternatives for each problem.

It may even be that using flexibility to define adaptive performance leads to some counter-intuitive predictions. In a recent study comparing Chinese-, Canadian-, and Belgian-educated adults, Imbo and LeFevre (2009) concluded that the Chinese-educated group was the most efficient but the least 'adaptive'. Chinese participants' performance far outstripped that of the other two groups in terms of speed and accuracy and they required relatively few working memory resources to achieve these excellent results. However, when these authors defined adaptivity as selection of the 'best' procedure for each problem type (consistent with Lemaire and colleagues' work), the Chinese-educated group appeared to be the least adaptive. Although they were clearly the most skilled calculators, by definition their approach was not considered adaptive. Thus, Imbo and

LeFevre's results, like those in the present research, support a view of adaptivity that incorporates routine or stable performance as a component of adaptive expertise.

In other work on adaptivity, Rittle-Johnson and Siegler (1999) defined adaptive choices as those that increase accuracy and speed beyond what would be realized by consistent use of one procedure alone. This argument leads us into a circular logic based on the initial assumption that stable problem-solving cannot be adaptive. It may be that flexibility is more relevant when solvers have relatively little practice with a variety of procedures. Presumably the development of skill involves the refining of those procedures that are best suited to the individual and, according to this view, extensive practice could therefore lead to stable (routine) procedure use. The view that retrieval is the most efficient approach to solving single-digit arithmetic problems (Siegler & Shrager, 1984) is an example of the argument that stability and consistency have a place in arithmetic expertise. Most researchers (and the general public) are likely to agree that solving single-digit multiplication problems through memory retrieval is the 'best' approach, even if it is not always the one that they choose. Choosing from among a variety of multiplication procedures is routinely associated with less-efficient simple arithmetic performance (LeFevre, Bisanz et al., 1996; Smith-Chant & LeFevre, 2003). Thus, there appears to be a role for stable use of an efficient procedure as a possible 'end point' in the development of expertise.

Future Work

The present findings reinforce the importance of considering individual differences in research on cognitive processing in mental arithmetic. Performance and procedure selection varied with arithmetic skill but flexible and stable problem-solvers were found at all skill levels. Additional research is necessary to investigate whether this distinction in problem-solving styles varies with other individual difference characteristics, such as age or culture. Imbo and LeFevre (2009) found that Chinese-, Belgian-, and Canadian-educated young adults differed in the degree to which they solved complex addition problems from left-to-right and from right-to-left. They found differences in efficiency across cultures but were not able to relate these differences to the use of specific solution procedures or to differences in problem-solving style. Future research could investigate the issue further by comparing similar cultural groups, with varying backgrounds in elementary mathematics education, with regard to their use of the procedures reported by adults in this study, and comparing flexible and stable solvers. Asian elementary mathematics education incorporates more rote practice of procedures and memorization of facts than do education systems in Europe and North America. Thus, adults educated in Asian countries are more likely to be stable problem-solvers. If so, routine computation appears to benefit them in that they are consistently faster and more accurate than adults from the other two cultures.

A further issue of interest would be to determine whether flexible and stable problem-solvers can be found among children. A choice/no-choice study with trial-by-trial procedure reports, using the list of procedures from the current research, could be employed to study children's solution style.

Another valuable direction for future work is to investigate the involvement of different components of working memory in the implementation of digit-based and holistic procedures. Trbovich and LeFevre (2003) have demonstrated the differential involvement of various working memory components in solving horizontal and vertical problems and I have shown that adults use different procedures to solve horizontal and vertical problems. Further, I have speculated that the digit algorithm is likely to be more dependent on visual or spatial aspects of working memory and holistic procedures are likely to require more phonological working memory resources. Research using the dual-task paradigm, incorporating a complex addition task and trial-by-trial procedure reports with phonological, visual, and spatial working memory tasks would be necessary to clarify this issue.

Finally, the results of the present research demonstrate the need to explore the concept of adaptive problem-solving and to re-examine the way in which it has been defined and operationalized for the past 20 years. Adaptive expertise, at least in multi-digit addition, does not require the flexible use of a variety of procedures. Flexible and stable (or routine) problem-solvers exhibited similar levels of efficiency in complex addition. Further research would be useful to explore how adults come to be flexible or stable mental arithmetic problem-solvers. One possibility is that people just use what they've learned (Lucangeli et al., 2003). Alternatively, the SCADS model (Shrager & Siegler, 1998) suggests that some people have experienced greater efficiency with one procedure over time and, therefore, that procedure has accumulated sufficient associative strength to be used every time. With the digit algorithm, these may be the individuals who use retrieval on simple addition facts. However, it seems unlikely that they choose the procedure each time they solve a problem. They are more likely to have made an overall 'choice' or, essentially, to have developed a routine based on either previous success or ease of execution of one procedure. Future research linking procedure use to level of mastery of the sub-skills and cognitive underpinnings of complex addition (including automaticity of number recognition, simple addition retrieval, understanding of concepts such as place value and additive composition, and working memory capacity) would help to answer questions such as: Who becomes a stable arithmetic problem-solver? What is the relation between problem-solving style and skill during learning versus later? And, more generally, What are the components of adaptive expertise?

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Appendix A1

Tables and Figures from Experiment 1

Table A1.1

Experiment 1: Distribution of procedure use by participant

Participant	Procedure				
	Retrieval	Digit Algorithm	Decomposition	Transformation	Count
Flexible					
1	38	57	5	0	0
2	8	6	49	37	0
3	4	87	4	5	0
4	7	71	2	18	2
5	1	35	0	0	64
6	0	91	0	1	8
7	1	55	0	33	11
8	8	55	21	16	0
9	1	48	31	20	0
10	0	15	34	24	27
11	1	1	74	23	1
12	1	1	55	43	0
13	14	2	65	18	1
14	2	18	61	18	1
15	1	0	0	45	54

16	1	44	27	28	0
17	10	39	31	20	0
18	8	48	1	36	7
19	2	14	49	33	2
20	1	70	0	4	25
Stable					
21	0	95	1	0	4
22	1	97	0	1	1

Table A1.2

Experiment 1: Analysis of Variance for percent use of decomposition, transformation, and the digit algorithm for adults solving multi-digit addition problems varying in complexity, order of presentation, and orientation of presentation

		% Procedure Use	
Source	*Df*	*MS*	*F*
	Between		
Skill	2		0.02
Error (Skill)	17	990.0	
	Within		
Order	1		0.23
Order x Skill	2		0.14
Error (Order)	17	17.6	
Orientation	1		6.50*
Orientation x Skill	2		0.89
Error (Orientation)	17	103.1	
Complexity	1		2.91
Complexity x Skill	2		1.80
Error (Complexity)	17	528.0	

Procedure†	1.2		1.00
Procedure x Skill†	2.4		0.66
Error (Procedure) †	20	10343.8	
Order x Orientation	1		0.20
Order x Orientation x Skill	2		0.22
Error (Order x Orientation)	17	7.6	
Order x Complexity	1		0.09
Order x Complexity x Skill	2		2.68
Error (Order x Complexity)	17	11.6	
Orientation x Complexity	1		2.89
Orientation x Complexity x Skill	2		0.35
Error (Orientation x Complexity)	17	84.9	
Order x Orientation x Complexity	1		0.32
Order x Orientation x Complexity x Skill	2		0.10
Error (Order x Orientation x Complexity)	17	11.5	
Order x Procedure†	1.2		9.69**
Order x Procedure x Skill†	2.5		0.22
Error (Order x Procedure) †	21.3	525.7	
Orientation x Procedure	2		6.51**
Orientation x Procedure x Skill	4		1.12
Error (Orientation x Procedure)	34	904.6	
Order x Orientation x Procedure†	1.2		13.01**
Order x Orientation x Procedure x Skill†	2.3		0.23
Error (Order x Orientation x Procedure) †	19.6	359.9	
Complexity x Procedure	2		11.89***
Complexity x Procedure x Skill	4		1.64
Error (Complexity x Procedure)	34	2031.4	
Order x Complexity x Procedure†	1.4		11.56**
Order x Complexity x Procedure x Skill†	2.7		0.65
Error (Order x Complexity x Procedure) †	23.0	271.4	
Orientation x Complexity x Procedure	2		6.50**
Orientation x Complexity x Procedure x Skill	4		0.18

Error (Orientation x Complexity x Procedure)	34	620.3	
Order x Orientation x Complexity x Procedure†	1.3		7.87**
Order x Orientation x Complexity x Procedure x Skill†	2.5		0.26
Error (Order x Orientation x Complexity x Procedure) †	21.4	328.0	

†Greenhouse-Geisser adjustment of df and Mean Square Error for violation of sphericity assumption. In no case did the adjustment make a difference to significance.

Three-way interactions of procedure choice with order and complexity, orientation and complexity, and order and orientation are shown in Figures A1.1 through A1.3, respectively. As shown in Figure A1.1, when the problem was presented with the 2-digit addend first, adults rarely used transformation on no-carry problems and there was no significant difference between their percentage use of decomposition and the digit algorithm. On carry problems, on the other hand, transformation was used significantly more often than the other two procedures and the digit algorithm was used significantly more often than decomposition. When the problem was presented with the 1-digit addend first, the pattern of procedure choice for carry problems was similar but for no-carry problems it was quite different. Transformation and decomposition were used equally often on the easier problems and both significantly less often than the digit algorithm.

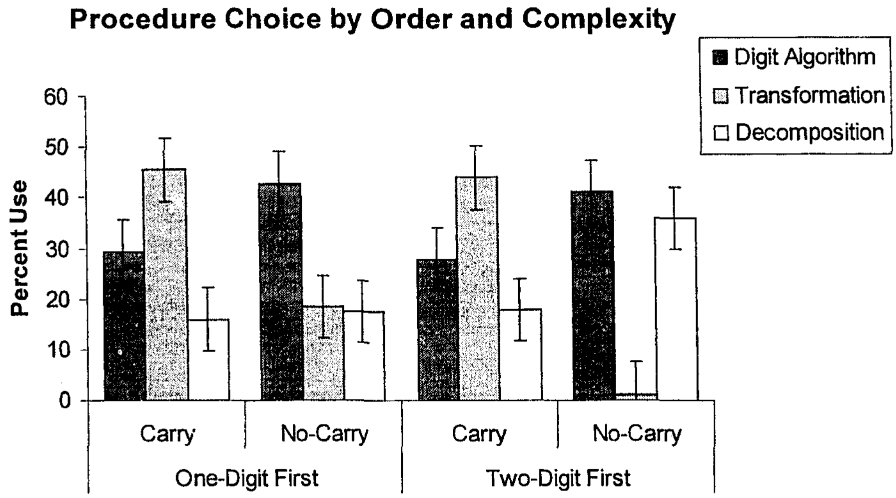

Figure A1.1. **Experiment 1: Mean percent use of the digit algorithm, transformation, and decomposition as a function of order of operand and complexity of double-digit plus single-digit addition problems. Error bars represent 95% confidence intervals, based on the MS_e from the three-way interaction.**

Similarly, the orientation in which problems were presented influenced the effect of problem complexity on people's approach to solving addition problems (see Figure A1.2). For vertical problems, participants shifted from greater use of decomposition than transformation on easier problems to greater use of transformation than decomposition on carry problems. The use of the digit algorithm was essentially the same for carry versus no-carry problems. When problems were presented horizontally, the use of transformation still increased from no-carry to carry problems but the use of decomposition remained the same and the use of the digit algorithm decreased significantly.

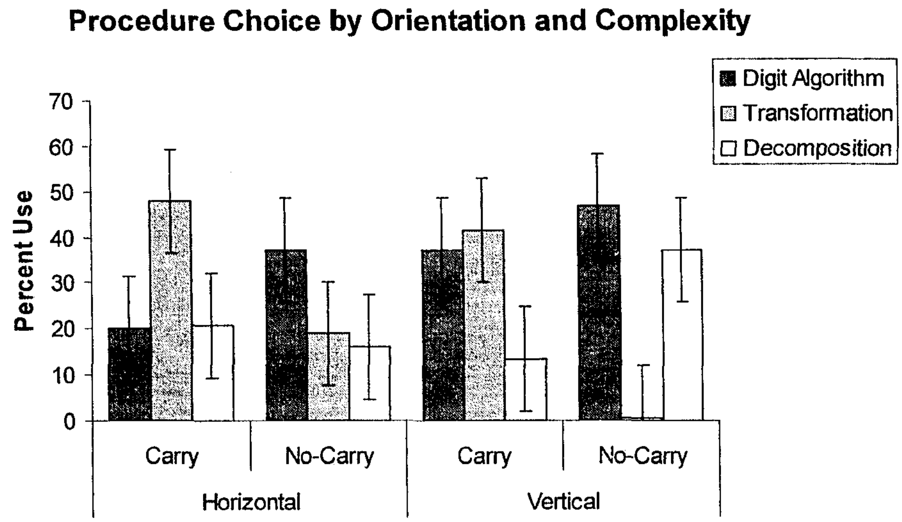

Figure A1.2. **Experiment 1: Mean percent use of the digit algorithm, transformation, and decomposition as a function of orientation of presentation and complexity of double-digit plus single-digit addition problems. Error bars represent 95% confidence intervals, based on the MS_e from the three-way interaction.**

Finally, the order and orientation of presentation of addition problems interacted in influencing solution procedure choice (see Figure A1.3). The relative percentage use of decomposition, transformation, and the digit algorithm varied between 1-digit-first problems and 2-digit-first problems when problems were presented horizontally but not when they were presented vertically.

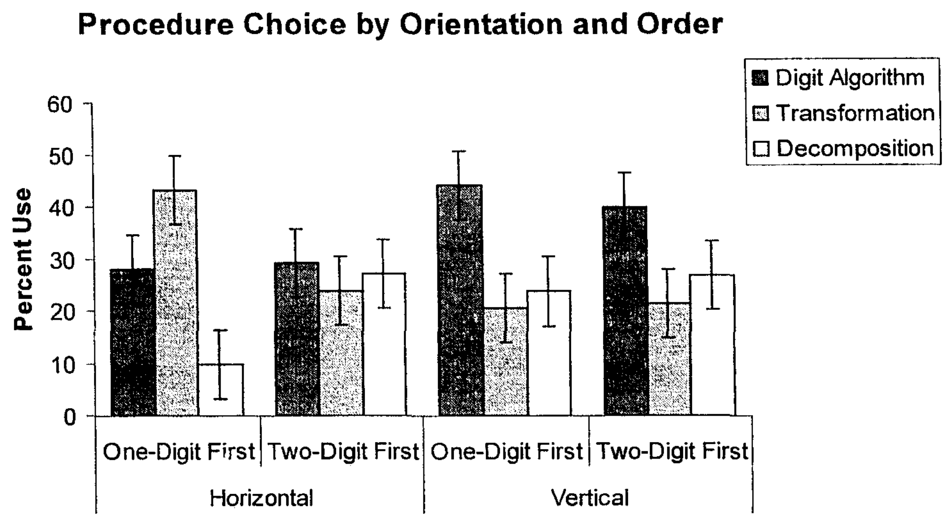

Figure A1.3. **Experiment 1: Mean percent use of the digit algorithm, transformation, and decomposition as a function of orientation and order of presentation of double-digit plus single-digit addition problems. Error bars represent 95% confidence intervals, based on the MS_e from the three-way interaction.**

Appendix A2

Tables and Figures from Experiment 2

Table A2.1

Experiment 2: Distribution of Italians' procedure use by participant

Participant	Procedure				
	Retrieval	Digit Algorithm	Partial Decomposition	Decomposition	Transformation
Flexible					
I1	1	9	6	54	30
I2	14	3	8	71	4
I3	0	24	0	76	0
I4	0	62	0	38	0
I5	0	34	0	66	0
I6	0	68	0	32	0
I7	3	12	1	84	0
I8	15	40	0	45	0
I9	19	46	0	35	0
Stable					
I10	0	95	0	5	0
I11	1	3	0	95	1
I12	1	95	0	3	1

Table A2.2

Experiment 2: Distribution of Canadians' procedure use by participant

Participa nt	Procedure				
	Retrieval	Digit Algorithm	Partial Decomposition	Decomposition	Transformation
Flexible					
C1	0	78	11	6	5
C2	0	62	15	2	21
C3	14	80	2	1	3
Stable					
C4	1	99	0	0	0
C5	0	97	2	0	1
C6	1	97	2	0	0
C7	1	93	3	1	1
C8	0	100	0	0	0
C9	0	100	0	0	0
C10	0	100	0	0	0
C11	0	100	0	0	0
C12	0	100	0	0	0

Appendix A3

Tables and Figures from Experiment 3

Table A3.1

Experiment 3: Distribution of procedure use by participant

Participant	Procedure				
	Retrieval	Digit Algorithm	Partial Decomposition	Decomposition	Transformation
Flexible					
1	11	0	81	4	4
2	0	61	15	20	4
3	0	67	2	17	14
4	16	82	0	2	0
5	0	84	0	7	9
6	0	48	4	22	26
7	0	0	31	57	12
8	13	87	0	0	0
9	2	56	0	42	0
10	0	58	38	0	4
11	0	26	7	67	0
12	0	14	0	50	36
13	0	20	41	0	39
14	0	77	0	0	23
15	2	41	6	32	19
16	0	80	0	0	20

17	8	67	10	0	15
18	0	51	0	49	0
19	0	91	9	0	0
20	0	88	8	2	2
21	0	60	5	33	2
22	4	45	7	37	7
23	0	89	7	0	4
24	0	85	9	0	6
25	0	73	27	0	0
26	0	51	0	49	0
27	0	89	11	0	0
28	0	86	2	0	12
29	0	48	4	48	0
30	0	80	0	5	15
31	0	4	56	40	0
32	0	26	74	0	0
33	0	54	46	0	0
Stable					
34	2	98	0	0	0
35	2	2	0	96	0
36	0	2	0	98	0
37	0	100	0	0	0
38	2	98	0	0	0

39	0	96	0	0	4
40	0	100	0	0	0
41	0	93	2	0	5
42	0	100	0	0	0
43	0	100	0	0	0
44	0	100	0	0	0
45	0	100	0	0	0
46	2	100	0	0	0
47	0	96	0	0	2
48	0	98	2	0	0
49	0	100	0	0	0
50	0	96	4	0	0
51	0	96	2	0	2
52	0	100	0	0	0
53	0	100	0	0	0
54	0	100	0	0	0
55	0	100	0	0	0
56	0	98	2	0	0
57	0	95	0	5	0
58	0	100	0	0	0
59	0	98	0	0	2
60	0	100	0	0	0

Table A3.2

Experiment 3: Analysis of Variance for percent choice of partial decomposition, decomposition, and the digit algorithm for adults solving two- plus two-digit addition problems varying in complexity and orientation

		% Procedure Use	
Source	*Df*	*MS*	*F*
	Between		
Skill	2		2.90
Error	30	295.26	
	Within		
Procedure	2		21.84**
Procedure x Skill	2		1.64
Error (Procedure)	60	6581.01	
Orientation	1		2.88
Orientation x Skill	2		0.03
Error (Orientation)	30	80.78	
Complexity	2†		6.69**
Complexity x Skill	6		2.72*
Error (Complexity)	90	84.14	
Procedure x Orientation	1.3†		12.01**
Procedure x Orientation x Skill	4		1.19
Error (Orientation x Procedure)	60	966.32	
Procedure x Complexity	3†		4.56**
Procedure x Complexity x Skill	12		0.58
Error (Complexity x Procedure)	180	585.36	

Orientation x Complexity†	3		0.33
Orientation x Complexity x Skill	6		0.66
Error (Orientation x Complexity)	90	29.22	
Procedure x Orientation x Complexity	6		1.69
Procedure x Orientation x Complexity x Skill	12		0.90
Error (Orientation x Complexity x Procedure)	180	335.56	

†Greenhouse-Geisser adjustment of df and Mean Square Error for violation of sphericity assumption. In no case did the adjustment make a difference to significance.

Appendix A4

Tables and Figures from Experiment 4

Table A4.1

Experiment 4: Analysis of Variance for latencies and percent errors for adults solving two-digit plus two-digit addition problems varying in complexity and orientation

		Latencies		% Errors	
Source	*Df*	*MS*	*F*	*MS*	*F*
		Between			
Style	1		0.29		1.50
Skill	2		12.32**		0.73
Style x Skill	2		0.05		0.40
Error	30	53,180,000		519.97	
		Within			
Orientation	1		28.30**		8.38**
Orientation x Style	1		2.05		3.57
Orientation x Skill	2		0.39		0.49
Orientation x Style x Skill	2		1.01		1.33
Error (Orientation)	30	658,442		104.27	
Complexity	†2(2.2)		†64.10**		†20.38
Complexity x Style	3		0.40		0.10
Complexity x Skill	6		7.56**		2.08
Complexity x Style x Skill	6		0.02		0.12
Error (Complexity)	†59(66)	†4,801,632		†194.27	
Procedure	2		13.96**		2.14
Procedure x Style	2		4.75*		0.22
Procedure x Skill	4		3.65*		0.63
Procedure x Style x Skill	4		1.27		0.37
Error (Procedure)	60	5,168,567		122.79	
Orientation x Complexity	†2.4(2.3)		†1.07		†0.77
Orient. x Complex. x Style	3		1.25		0.36

Orient. x Complex. x Skill	6		0.45		0.26
Orient. x Complex. x Style x Skill	6		0.43		0.53
Error (Orient. x Complex.)	†72(69)	†591,057		†122.15	
Orientation x Procedure	†2(1.6)		2.95		†0.59
Orient. x Proc. x Style	2		0.92		2.20
Orient. x Proc. x Skill	4		1.20		0.78
Orient. x Proc. x Style x Skill	4		2.96*		0.81
Error (Orient. x Proc.)	†60(49)	503,666		†88.15	
Complexity x Procedure	†3.9(6)		†4.64**		1.34
Complex. x Proc. x Style	6		1.60		1.22
Complex. x Proc. x Skill	12		2.31*		1.60
Complex. x Proc. x Style x Skill	12		1.28		1.61
Error (Complex. x Proc.)	†116(180)	†369,423		72.86	
Orient. x Complex. x Proc.	†3.4(6)		†1.03		1.21
Orient. x Complex. x Proc. x Style	6		1.12		1.02
Orient. x Complex. x Proc. x Skill	12		1.34		0.35
Orient. x Complex. x Proc. x Style x Skill	12		0.92		0.55
Error (Orient. x Complex. x Proc.)	†102(180)	†1,137,623		101.74	

† Greenhouse-Geisser adjustment of *df* and Mean Square Error for violation of sphericity assumption. In no case did the adjustment make a difference to significance.

Appendix B1

Addition problems used in Experiment 1

No-carry		Carry	
3 + 32 =	21 + 8 =	3 + 38 =	16 + 8 =
3 + 36 =	22 + 6 =	3 + 48 =	17 + 5 =
3 + 64 =	23 + 6 =	3 + 59 =	17 + 6 =
3 + 76 =	31 + 8 =	3 + 68 =	23 + 8 =
4 + 35 =	32 + 3 =	4 + 37 =	28 + 5 =
4 + 42 =	35 + 4 =	4 + 47 =	28 + 6 =
4 + 62 =	36 + 3 =	4 + 59 =	36 + 7 =
4 + 63 =	41 + 6 =	4 + 68 =	37 + 4 =
5 + 42 =	42 + 4 =	5 + 17 =	38 + 3 =
5 + 52 =	42 + 5 =	5 + 28 =	38 + 7 =
5 + 53 =	43 + 6 =	5 + 47 =	46 + 8 =
5 + 74 =	51 + 7 =	5 + 59 =	47 + 4 =
5 + 84 =	52 + 5 =	5 + 89 =	48 + 3 =
6 + 22 =	52 + 7 =	6 + 17 =	59 + 3 =
6 + 23 =	53 + 5 =	6 + 28 =	59 + 4 =
6 + 41 =	61 + 8 =	6 + 59 =	47 + 5 =
6 + 43 =	62 + 4 =	6 + 69 =	59 + 5 =
7 + 51 =	63 + 4 =	7 + 36 =	59 + 6 =
7 + 52 =	64 + 3 =	7 + 38 =	68 + 3 =
7 + 71 =	71 + 7 =	7 + 68 =	68 + 4 =
8 + 21 =	71 + 8 =	8 + 16 =	68 + 7 =
8 + 31 =	74 + 5 =	8 + 23 =	69 + 6 =
8 + 61 =	76 + 3 =	8 + 46 =	73 + 8 =
8 + 71 =	84 + 5 =	8 + 73 =	89 + 5 =

Appendix B2

Addition problems used in Experiment 2

No-carry	Decade-carry	Unit-carry	Double-carry
12 + 73 =	24 + 83 =	13 + 47 =	28 + 93 =
12 + 76 =	31 + 98 =	14 + 36 =	29 + 84 =
14 + 74 =	32 + 96 =	14 + 57 =	38 + 76 =
15 + 23 =	38 + 81 =	19 + 16 =	48 + 96 =
15 + 64 =	43 + 93 =	25 + 36 =	49 + 58 =
17 + 42 =	44 + 95 =	26 + 29 =	53 + 88 =
21 + 48 =	45 + 74 =	28 + 34 =	54 + 76 =
23 + 15 =	47 + 62 =	29 + 63 =	62 + 79 =
27 + 52 =	51 + 98 =	32 + 28 =	65 + 59 =
31 + 41 =	52 + 57 =	34 + 46 =	65 + 78 =
32 + 27 =	56 + 71 =	35 + 49 =	67 + 69 =
34 + 51 =	58 + 61 =	37 + 39 =	69 + 53 =
42 + 17 =	61 + 86 =	38 + 17 =	72 + 58 =
43 + 35 =	64 + 75 =	48 + 15 =	75 + 38 =
43 + 52 =	72 + 47 =	48 + 23 =	75 + 46 =
44 + 13 =	75 + 42 =	48 + 49 =	75 + 49 =
48 + 11 =	78 + 51 =	52 + 18 =	77 + 87 =
51 + 24 =	82 + 31 =	53 + 38 =	78 + 42 =
54 + 25 =	82 + 83 =	56 + 26 =	79 + 75 =
58 + 31 =	83 + 62 =	57 + 28 =	79 + 81 =
61 + 38 =	84 + 35 =	59 + 12 =	81 + 59 =
63 + 11 =	91 + 66 =	59 + 33 =	86 + 97 =
75 + 14 =	93 + 74 =	67 + 26 =	95 + 76 =
76 + 22 =	96 + 92 =	24 + 96 =	99 + 67 =

Appendix B3

Addition problems used in Experiment 3

No-carry	Decade-carry	Unit-carry	Double-carry
12 + 73 =	24 + 83 =	14 + 36 =	24 + 86 =
15 + 64 =	32 + 96 =	14 + 57 =	24 + 96 =
23 + 15 =	45 + 74 =	19 + 16 =	28 + 93 =
42 + 27 =	47 + 62 =	28 + 34 =	29 + 84 =
43 + 35 =	51 + 98 =	29 + 63 =	49 + 68 =
44 + 13 =	52 + 57 =	35 + 49 =	56 + 97 =
48 + 11 =	53 + 93 =	37 + 39 =	58 + 96 =
58 + 31 =	56 + 71 =	52 + 18 =	62 + 79 =
61 + 38 =	58 + 61 =	53 + 38 =	67 + 69 =
63 + 11 =	61 + 86 =	56 + 26 =	72 + 58 =
75 + 14 =	83 + 62 =	57 + 28 =	75 + 38 =
76 + 22 =	84 + 35 =	59 + 12 =	79 + 81 =

Appendix B4

Addition problems used in Experiment 4

No-carry	Decade-carry	Unit-carry	Double-carry
12 + 84 =	34 + 95 =	12 + 59 =	18 + 87 =
13 + 21 =	35 + 72 =	12 + 78 =	26 + 95 =
13 + 76 =	36 + 92 =	13 + 57 =	26 + 98 =
14 + 35 =	37 + 81 =	14 + 38 =	27 + 85 =
14 + 65 =	41 + 93 =	17 + 45 =	35 + 69 =
15 + 23 =	41 + 96 =	17 + 54 =	35 + 87 =
15 + 31 =	43 + 65 =	18 + 63 =	38 + 72 =
15 + 62 =	45 + 83 =	18 + 74 =	39 + 73 =
16 + 21 =	46 + 71 =	19 + 43 =	43 + 67 =
16 + 21 =	48 + 71 =	19 + 57 =	48 + 63 =
16 + 81 =	51 + 98 =	19 + 72 =	48 + 89 =
17 + 42 =	52 + 96 =	23 + 68 =	49 + 53 =
21 + 46 =	54 + 73 =	23 + 69 =	49 + 78 =
21 + 53 =	56 + 82 =	24 + 39 =	52 + 89 =
21 + 76 =	58 + 91 =	24 + 49 =	59 + 86 =
23 + 65 =	61 + 56 =	25 + 36 =	64 + 38 =
24 + 75 =	62 + 53 =	26 + 57 =	65 + 37 =
25 + 32 =	63 + 94 =	27 + 56 =	68 + 59 =
25 + 62 =	64 + 91 =	27 + 64 =	68 + 73 =
28 + 41 =	64 + 93 =	28 + 14 =	69 + 45 =
31 + 48 =	65 + 84 =	28 + 54 =	69 + 81 =
31 + 52 =	68 + 71 =	29 + 18 =	72 + 58 =
31 + 64 =	71 + 42 =	29 + 18 =	75 + 97 =
32 + 43 =	71 + 68 =	29 + 38 =	76 + 59 =
32 + 43 =	71 + 97 =	34 + 16 =	78 + 36 =
32 + 51 =	72 + 47 =	36 + 24 =	78 + 65 =
34 + 62 =	73 + 81 =	37 + 54 =	79 + 35 =

36 + 53 =	73 + 96 =	38 + 16 =	79 + 85 =
37 + 51 =	74 + 42 =	38 + 26 =	81 + 19 =
41 + 53 =	74 + 52 =	39 + 16 =	82 + 39 =
42 + 23 =	75 + 31 =	39 + 26 =	82 + 49 =
42 + 36 =	76 + 83 =	43 + 19 =	83 + 69 =
42 + 37 =	78 + 61 =	45 + 29 =	84 + 67 =
45 + 32 =	81 + 97 =	46 + 17 =	84 + 79 =
46 + 12 =	82 + 75 =	47 + 26 =	85 + 46 =
47 + 51 =	83 + 52 =	47 + 39 =	86 + 27 =
51 + 48 =	83 + 56 =	49 + 26 =	86 + 28 =
52 + 41 =	84 + 52 =	51 + 39 =	86 + 29 =
53 + 34 =	85 + 31 =	53 + 17 =	87 + 34 =
53 + 45 =	85 + 92 =	53 + 28 =	87 + 46 =
57 + 12 =	91 + 35 =	54 + 26 =	87 + 95 =
57 + 42 =	91 85 =	56 + 38 =	89 + 13 =
61 + 27 =	91 + 87 =	56 + 38 =	89 + 63 =
61 + 34 =	92 + 41 =	57 + 24 =	89 + 78 =
62 + 14 =	92 + 43 =	58 + 17 =	92 + 58 =
62 + 23 =	92 + 47 =	58 + 37 =	94 + 36 =
63 + 25 =	92 + 57 =	59 + 26 =	94 + 86 =
65 + 23 =	93 + 54 =	59 + 27 =	95 + 18 =
67 + 12 =	93 + 62 =	64 + 29 =	95 + 79 =
72 + 24 =	93 + 75 =	67 + 28 =	96 + 27 =
74 + 25 =	94 + 61 =	67 + 29 =	97 + 15 =
75 + 23 =	95 + 74 =	69 + 15 =	98 + 15 =
76 + 13 =	96 + 23 =	74 + 17 =	98 + 24 =
81 + 14 =	96 + 32 =	79 + 14 =	98 + 64 =