Weakly Nonlinear Wave Interactions in Some Two-Dimensional Initial-Value Problems in Geophysical Fluid Dynamics

Stephanie Cates

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Master of Science

School of Mathematics and Statistics
Ottawa-Carleton Institute of Mathematics and Statistics

Carleton University
Ottawa, Ontario, Canada

August 11, 2010

Your file Votre référence
ISBN: 978-0-494-71574-1
Our file Notre référence
ISBN: 978-0-494-71574-1

Abstract

In this thesis, we study initial-value problems describing wave propagation in two-dimensional geophysical flows. We consider two different problems; Rossby waves on a horizontal plane and internal gravity waves on a vertical plane. In each case, we consider waves that are periodic and sinusoidal in both space directions; allowing us to use the techniques of Fourier analysis. Firstly, we solve the linear problem and find the solutions to be periodic. Linearization is justified as long as the wave amplitude is small enough. Next, we solve the weakly nonlinear problem using asymptotic approximations. We look for a solution in powers of the small amplitude parameter. The leading order solutions are the linear solutions already derived. The nonlinear terms are found to be products of the leading order terms, giving rise to higher wavenumber components. We investigate the development of these components which represents a transfer of wave energy to smaller scales.

Acknowledgements

I would like to express my deepest appreciation to my supervisor, Professor Lucy Campbell, for her endless patience, support, and encouragement throughout my entire thesis project. Without her guidance and countless hours of help, this thesis would not have been possible.

I would also like to thank the numerous mathematics Professors who have contributed to my intellectual growth over the past 6 years at Carleton University. Their passion and devotion to teaching, and their friendly and helpful nature has inspired me to enter the teaching profession. Thank you for everything that you have done to make Carleton feel like a second home.

Lastly, I would like to thank my family. There are not enough words to express how fortunate I am to have such a wonderful, supportive family who would do anything for me. I would like to thank my sister, Cassie, for being someone I can always count on. I would like to thank my Grandma for her continuous love and support. Finally, I would like to thank my parents, Bart and Lisa, for their understanding, endless love and guidance, and encouragement when it was needed most. Thank you mom for always listening and for being someone I can talk to about anything, and thank you dad for always believing that I could accomplish any task I pursued.

Dedication

Dedicated in loving memory of my Papa (1940-2008); my biggest fan. I am forever grateful for his never ending love and support and for always believing in me.

Contents

Chapter 1

Introduction

1.1 Waves in Fluids

Fluid mechanics is the branch of science that governs fluid flows. A fluid is a substance that does not have a preferred shape and deforms, or flows, continuously when shear stress is applied. Shear stress is a force that is applied parallel to one of the faces of an object. A fluid may be further classified as either a liquid or a gas. While a gas expands to fill the container it occupies, a liquid merely takes on the shape of its container but does not fill it. Waves commonly occur in fluids. A wave is a moving oscillation that transports information between two points in space and time, without carrying matter along. Waves transfer energy from one point to another. They are generated due to the existence of a restoring force balanced with an inertial force. The restoring force acts to bring the system back to its original state while the inertial force extends the system past this undisturbed state.

Large-scale fluids in nature- the Earth's atmosphere or oceans, for example- are referred to as geophysical fluids. There are various types of waves within such fluids. These include surface gravity waves, internal gravity waves, and Rossby waves. Surface gravity waves are most common in everyday life and occur at the surface of a liquid with gravity acting as the restoring force. These waves are observed at the surface of oceans, rivers, and at beaches. Internal gravity waves occur in the interior of a density-stratified fluid as a result of upward buoyancy forces and the restoring force of gravity. Lastly, Rossby waves (or planetary waves) result from the effects of the Earth's rotation, or the Coriolis force. These types of waves were first described in the Earth's atmosphere by the Swedish meteorologist Carl-Gustaf Rossby in 1939.

There are various properties that make waves interesting to study. For instance, waves propagate in space and time, can have large or small amplitudes, they sometimes break when their amplitude is too large, and they can reflect at boundaries. Furthermore, in the atmosphere and ocean, interactions between waves and the background fluid flow can have profound effects on the general circulation. This can in turn affect global weather and climate conditions. Thus, it is important to understand the mechanics involved in wave generation, propagation, and evolution because it gives us a deeper understanding of the natural phenomena surrounding our everyday lives.

Waves in fluids can be studied mathematically by representing the waves as perturbations to a basic background state and modelling the fluid flow using the governing equations for fluid dynamics which are based on the principles of conservation of mass, momentum, and energy. If the wave amplitude is sufficiently small relative to the magnitude of the background flow, we can linearize the equations. We can sometimes make further simplifications to the equations based on the particular problem or configuration we wish to study. Wave phenomena are highly nonlinear in general but for small amplitudes we can obtain some insight by studying relatively simple weakly nonlinear models. By introducing a small parameter, ε ($\varepsilon \ll 1$), usually related to the wave amplitude and taking the limit as the parameter approaches zero, we obtain a linear problem. We can solve this problem exactly or find an asymptotic solution which is valid in the particular regime of interest. We then obtain a nonlinear correction to the linear solution in powers of the small parameter.

In linear mathematical studies, we often assume the waves to be sinusoidal in at least one of the space coordinates and/or in time. This assumption allows us to make use of the techniques of Fourier analysis, such as Fourier series or Fourier transforms.

As an illustration, the function $\cos(kx+ly+mz-\omega t)$ represents a sinusoidal wave in three-dimensional space with rectangular coordinates x, y, z and time t. Here, k, l, m are the wavenumbers and $\vec{k}=(k,l,m)$ is the wavenumber vector. The wavelengths are $\frac{2\pi}{k}$, $\frac{2\pi}{l}$ and $\frac{2\pi}{m}$, respectively. ω represents the frequency of the waves and the period of oscillation is $\frac{2\pi}{\omega}$. It is more convenient to write the above cosine function as $e^{\imath(kx+ly+mx-\omega t)}+e^{-\imath(kx+ly+mx-\omega t)}$, because it is easier to work with complex functions especially when taking products.

Nonlinear or weakly nonlinear problems involve products of different sinusoidal functions. For example, two waves $e^{\imath k_1 x}+$ c.c. (complex conjugate) and $e^{\imath k_2 x}+$ c.c., with wavenumbers k_1 and k_2, respectively, interact to give $e^{\imath(k_1+k_2)x}+e^{\imath(k_1-k_2)x}+$ c.c. Thus, the wavenumbers are sums and differences of the original two wavenumbers. Also, interactions of a wave $e^{\imath kx}+e^{-\imath kx}$ (with wavenumber k) with itself results in a zero wavenumber component as well as higher "harmonics" such as $\pm 2k$, $\pm 3k$, and so on. The higher wavenumber components have shorter wavelengths. Therefore, wave-wave interactions give rise to small-scale disturbances within the fluid flow.

In this study, we make use of the techniques of Fourier analysis and weakly nonlinear analysis to examine wave interactions and the development of higher harmonics in some specific geophysical flow configurations. In particular, internal gravity waves in a Boussinesq flow and Rossby waves on a β-plane. In a Boussinesq flow, the density variations are neglected in all terms except those involving gravity, in a vertical plane, whereas a β-plane takes into account the effects of the Coriolis force on a rectangular strip in a two-dimensional plane tangent to the Earth's surface. These are both two-dimensional problems which, when linearized and under certain simplifications, are tractable to some degree of analysis. These two-dimensional problems describe a simple set-up for studying Rossby waves and internal gravity waves separately. A three-dimensional configuration would allow both types of waves to be studied together. Topics such as interactions between the two types of waves could be considered. Howevever, such a configuration would be too complicated to allow for analytical solutions but could be studied using numerical methods.

We are interested in analyzing how the waves evolve with time and how higher wavenumber components develop. We consider a specific initial-value problem in which the waves are generated by a spatially sinusoidal forcing function at initial time and propagate forward in time. In particular, we consider the following initial condition

$$\psi(x, y, 0) = 2[\cos(kx) + \cos(ly) + \cos(kx + ly) + \cos(kx - ly)] \tag{1.1}$$

where $\psi(x, y, t)$ is the perturbation (wave) to some fluid variable. One could work with velocity components, pressure, or density, but in our study ψ represents the streamfunction which we define in Section 2.1. The function given in (1.1) can be decomposed into Fourier modes and written as

$$e^{\imath kx} + e^{\imath ly} + e^{\imath(kx+ly)} + e^{\imath(kx-ly)} + \text{c.c.} \tag{1.2}$$

This gives rise to the following eight wavenumbers

$$\vec{k} = (k, 0), (0, l), (k, l), (k, -l), (-k, 0), (0, -l), (-k, -l), (-k, l)$$

In a linear problem with the initial condition given in (1.1), the above eight wavenumbers are the only ones that will be present, for all time. On the other hand, in a nonlinear problem, other wavenumbers will be generated, such as $(\pm 2k, 0), (0, \pm 2l), (\pm 2k, \pm 2l)$ and so on. In a weakly nonlinear problem defined in terms of a small parameter, ε say, which defines the wave amplitude, the $O(1)$ problem will only have the initial wavenumbers but higher wavenumbers will be generated at $O(\varepsilon)$. In this study, we work out the solutions of both the $O(1)$ and the $O(\varepsilon)$ problems and thus obtain an approximate asymptotic solution. Our study shows the cascade of wave energy in wavenumber space from large scales (small wavenumbers) to smaller scales (higher wavenumbers).

The development of small-scale features in a fluid flow is of interest to atmospheric and oceanographic scientists and meterologists to study and predict the behaviour of the atmosphere and ocean using large-scale general circulation models. Small-scale

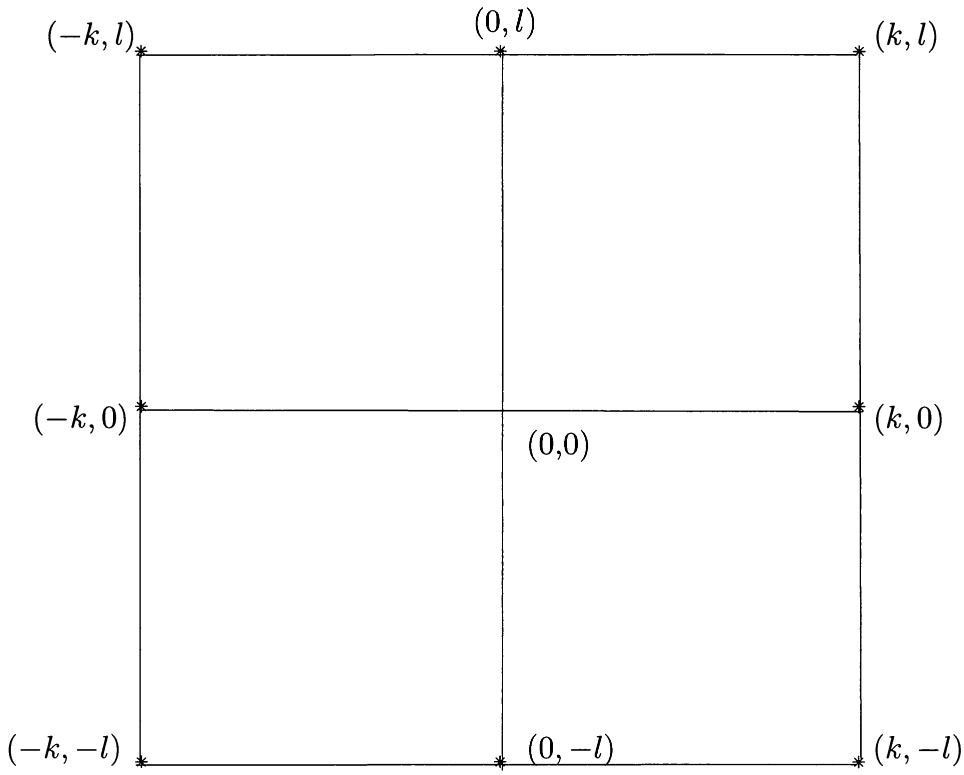


Figure 1.1: Plot showing the original eight wavenumbers

features arising from wave interactions are often unresolved by the models because they are smaller than or of the order of magnitude of the grid spacing of the models. For example, the drag force resulting from internal gravity wave interactions is represented in large-scale models by so-called gravity wave drag parameterizations. Analytical studies of simplified configurations involving wave interactions can be helpful in improving these parameterizations.

1.2 Overview of the Thesis

An overview of the thesis is as follows. The background information, key definitions, and standard fluid dynamics derivations are found in the appendices at the end of the thesis. Appendix A contains basic asymptotics definitions and notation. Appendix

B outlines some of the notation used throughout the thesis. In Appendix C, the governing equations of fluid dynamics based on the principles of conservation of mass, momentum and energy are derived. In Appendix D, one will find a brief discussion on dispersion relations followed by details concerning the specific dispersion relations for both the β-plane and Boussinesq problems.

The thesis begins with a chapter devoted to geophysical fluid dynamics in which the conservation law equations from Appendix C act as the starting point. In this chapter (Chapter 2), the reader will learn what distinguishes geophysical fluids from other types of fluids and will be taken through the assumptions made under the Boussinesq approximation. Furthermore, the reader will see how the Boussinesq approximation can be applied to the conservation law equations. The concept of a streamfunction, a crucial component of our study, is also described in this chapter. Once the set of Boussinesq equations are obtained, we discuss how the Coriolis force alters these equations. By expanding the Coriolis parameter, f, as a Taylor series, we begin our derivation of the β-plane equation. Chapter 2 ends with the barotropic vorticity equation, the starting point for our study.

Chapter 3 marks the beginning of the description of my master's research. Since the barotropic vorticity equation from Chapter 2 is in terms of dimensional quantities, we begin Chapter 3 with a discussion of how to nondimensionalize this equation. Having a dimensionless equation is of importance when we introduce a small parameter, ε, to guarantee that the terms multiplied by ε are in fact small relative to the other terms. We then linearize this dimensionless equation by expressing the total streamfunction, Ψ, as the sum of the background or mean streamfunction and a small perturbation to the streamfunction. The resulting equation models the perturbed problem. To obtain insight into the behaviour of this equation, we take a Fourier transform to turn the PDE into a system of ODEs, which is easier to work with. The system of ODEs has the form $\frac{\partial \vec{X}}{\partial t} = A\vec{X} + \varepsilon f(\vec{X})$. Firstly, we truncate the Fourier spectrum to only include wavenumbers of $\pm 1, 0$ and then we solve the linear problem when $\varepsilon = 0$ by applying a specific initial condition. The linear solutions are found to be exponential functions of t with complex exponents. Most of the solutions are periodic in time as given by the disperion relations derived in Appendix D, and they contain exponentially decaying factors. We then use perturbation theory to solve the nonlinear problem. We conclude the chapter by analyzing how wavenumbers of ± 2 alter the nonlinear analysis of the previous section.

Chapter 4 adopts similar techniques to those used in Chapter 3 but this time for the Boussinesq problem. Using the assumptions made under the Boussinesq approximation, we follow a standard derivation to come up with two equations; one modelling how Ψ changes with time and one demonstrating how T, the temperature, changes with time. These equations are coupled, meaning that the Ψ equation depends on T and vice versa. Then, following the procedure described by Saltzman [11] we express T as the sum of an average in the x direction and a departure from T. We also express the total streamfunction as the sum of the background streamfunction and a small perturbation from the streamfunction, just as we did for the β-plane problem. After some manipulation, we end up with two perturbed equations for the Boussinesq problem. Once again, we perform a Fourier transform to convert the PDEs to ODEs and we truncate the spectrum to include wavenumbers of $\pm 1, 0$ only. Since there are two sets of coupled ODEs, this problem is more difficult than the Rossby wave problem. We then go on to solve the linear problem and use perturbation theory to solve the nonlinear problem. The chapter concludes with a brief discussion on the effects of higher wavenumbers to the nonlinear problem.

Chapter 2

Geophysical Fluid Dynamics

In this chapter, we will describe how the rotation of the Earth- as well as vertical density stratification- affects fluid dynamical systems, closely following the information in Chapter 14 of Kundu and Cohen [9]. This is a standard procedure which can be found in any book on geophysical fluid dynamics, such as Holton [8] or Pedlosky [10]. Our starting point will be the equations that describe the conservation of mass, momentum and energy in a fluid. More precisely, equations (C.11), (C.43), and (C.54), which are derived in Appendix C.

Geophysical fluid dynamics (GFD) concerns fluid motion in the Earth's atmosphere and ocean. To study GFD, we take into account the effects of density stratification (layering) in the fluid and the effects of the Earth's rotation. The natural coordinate system for studying the motion of the atmosphere and the ocean is a coordinate frame that rotates with the Earth. This gives rise to the Coriolis force. The density stratification within a given medium gives rise to the buoyancy force, which is represented in terms involving the acceleration due to gravity, g, in the equations of motion. These conditions distinguish geophysical fluid dynamics from other branches of fluid dynamics. As we shall see in the sections that follow, the equations of motion are very much the same as what we have derived in the appendices but with an additional term added to take into account the Coriolis force.

2.1 The Boussinesq Approximation

In 1903, Boussinesq proposed that under certain conditions we may neglect the density changes in a fluid in all terms except the gravitational ones (where the density, ρ, is multiplied by g) (see [4]). A formal discussion of the conditions under which the Boussinesq approximation hold is given by Spiegel and Veronis [12]. Under the Boussinesq approximation, the following assumptions are made:

1. The variation of the density, ρ, in space and time is small relative to the overall magnitude of the density, ie. $\left|\frac{D\rho}{Dt}\right| \ll |\rho|$ and so we set $\frac{1}{\rho}\frac{D\rho}{Dt} \approx 0$.
2. The variation of the density in the vertical direction, $\frac{\partial\rho}{\partial z}$, is small.
3. The fluid is almost incompressible, ie. the variation of the density with pressure is small and we set $\frac{\partial\rho}{\partial p} \approx 0$.
4. The properties of the fluid such as the viscosity, heat conductivity, and heat capacity at constant volume or pressure are approximately constant throughout the fluid, so we set the viscosity coefficient, μ, the thermal conductivity coefficient, κ, the heat capacity at constant volume, C_{v}, and the heat capacity at constant pressure, C_{p}, to a constant and we neglect the viscous dissipation.

In this section, we will explore how the Boussinesq approximation simplifies the three governing equations of motion (continuity equation, momentum equation, and thermal energy equation).

We begin with the continuity equation (C.11). Under the Boussinesq approximation, we assume that

$$\frac{1}{\rho}\frac{D\rho}{Dt} \ll \nabla \cdot \vec{u}$$

(by assumption 1) and we let ρ be a constant. Thus, the continuity equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{u} = 0$$

becomes

$$\nabla \cdot \vec{u} = 0 \tag{2.1}$$

under the Boussinesq approximation. Equation (2.1) is said to be in *incompressible form*. An alternate form of (2.1) is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.2}$$

In two dimensions, (2.1) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.3}$$

This enables us to define a *streamfunction*, ψ, by

$$\frac{\partial \psi}{\partial y} = u, \quad \frac{\partial \psi}{\partial x} = -v \tag{2.4}$$

This is valid since

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial \psi}{\partial x}\right) = 0$$

Equivalently, we could define the streamfunction as

$$\frac{\partial \psi}{\partial y} = -u, \quad \frac{\partial \psi}{\partial x} = v$$

Before progressing further, we will introduce the concept of a *streamline.* At any moment in time and at every point in a fluid flow, there is a velocity vector with a definite direction and magnitude. Streamlines are instantaneous curves that are everywhere tangent to the velocity vectors at any given time. In other words, they are tangent to the direction field. For unsteady flows, the velocity changes with time and so the streamline pattern also changes with time. Let $d\vec{s} = (dx, dy, dz)$ be an element of arc length along a streamline and let $\vec{u} = (u, v, w)$ be the local velocity vector. Then,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{2.5}$$

along a streamline. So, in two dimensions, (2.5) becomes

$$\frac{dx}{u} = \frac{dy}{v}$$

and thus

$$udy - vdx = 0$$

Now, in terms of streamfunctions, we have

$$\frac{dx}{\frac{\partial\psi}{\partial y}} = \frac{dy}{\frac{-\partial\psi}{\partial x}}$$

Or, equivalently,

$$\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0$$

which implies that $d\psi = 0$ along a streamline. We now calculate $\frac{d\psi}{dx}$ by dividing the above expression by dx.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0$$

Thus, $\frac{d\psi}{dx} = 0$. Similarly, $\frac{d\psi}{dy} = 0$. Thus, we conclude that ψ is constant along a streamline. As a result, the instantaneous streamlines in a fluid flow are given by the curves where $\psi = C$, where C is a constant. In other words, they are the level curves of the streamfunction. Different values of C result in different streamlines.

Consider an arbitrary line element $d\vec{x}$ in a fluid flow, where $d\vec{x} = (dx, dy)$. Now consider two streamlines given by $\vec{x}$ and $\vec{x} + d\vec{x}$. A natural question that arises is 'what is the volume rate of flow across the line segment joining the two streamlines?' Firstly, the volume rate of flow in the y-direction is vdx and $-udy$ in the x-direction. Thus, the total rate of flow is

$$vdx + (-udy) = -\frac{\partial\psi}{\partial u}dx - \frac{\partial\psi}{\partial y}dy = -d\psi$$

Since the volume rate of flow must be a positive value, $d\psi < 0$. Therefore, the sign of ψ is such that facing the direction of fluid motion ψ increases to the left.

The streamfunction is useful for plotting streamlines but can also be used to simplify the equations of motion. For example, consider the following momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Now, define a streamfunction, ψ, such that

$$\psi_y = u, \quad -\psi_x = v$$

Then, the above momentum equation simplifies to

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

an equation in terms of a single unknown parameter.

We now study how the Boussinesq approximation simplifies the momentum equation. Recall from equation (C.41) in Appendix C that the momentum equation is given by

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\left(\nabla \cdot \vec{u}\right)\delta_{ij}\right]$$

Under the Boussinesq approximation, we treat μ as a constant (by assumption 4) which means that we can take μ outside of the derviative in the above equation to give

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\left[\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial}{\partial x_j}\left(\nabla \cdot \vec{u}\right)\delta_{ij}\right]$$

which simplifies to

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\left[\nabla^2 \vec{u_i} + \frac{1}{3}\frac{\partial}{\partial x_i}\left(\nabla \cdot \vec{u}\right)\right]$$

Now, because of the incompressible continuity equation (2.1), $\nabla \cdot \vec{u} = 0$, the above equation simplifies to

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 u \tag{2.6}$$

We consider a static reference state in which the density is ρ_0 everywhere and the pressure is $p_0(z)$, so that

$$\nabla p_0 = \left(\frac{\partial p_0}{\partial x}, \frac{\partial p_0}{\partial y}, \frac{\partial p_0}{\partial z}\right) = \left(0, 0, \frac{dp_0}{dz}\right) = \rho_0\left(0, 0, -g\right) = \rho_0 \vec{g}$$

Now, write p and ρ in terms of reference variables to give

$$\begin{aligned} p &= p_0(z) + p'(x, y, z, t) \\ \rho &= \rho_0 + \rho'(x, y, z, t) \end{aligned}$$

and assume that $p_0 \gg p'$, $\rho_0 \gg \rho'$. Substituting this information into (2.6) gives

$$(\rho_0 + \rho')\frac{D\vec{u}}{Dt} = -\nabla p_0 - \nabla p' + \rho_0\vec{g} + \rho'\vec{g} + \mu\nabla^2\vec{u}$$

which simplifies to

$$(\rho_0 + \rho')\frac{D\vec{u}}{Dt} = -\nabla p' + \rho'\vec{g} + \mu\nabla^2\vec{u}$$

since $\nabla p_0 = \rho_0\vec{g}$. Dividing by ρ_0 gives

$$\left(1 + \frac{\rho'}{\rho_0}\right)\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\vec{g} + \nu\nabla^2\vec{u} \tag{2.7}$$

where

$$\nu = \frac{\mu}{\rho_0} \tag{2.8}$$

In the above equation,

$$\frac{\rho'}{\rho_0} \ll 1$$

so we can neglect it. It represents a small correction to the inertia term $\frac{D\vec{u}}{Dt}$. However, this ratio also appears in the buoyancy term $\frac{\rho'}{\rho_0}\vec{g}$. This term is very important and cannot be neglected because, in particular, these density changes are the driving force behind convective motion when a fluid is heated. Therefore, we are able to neglect density variations in the momentum equation except in terms where ρ is multiplied by $\vec{g}$. The resulting simplified momentum equation under the Bossinesq approximation is thus

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\vec{g} + \nu\nabla^2\vec{u} \tag{2.9}$$

Lastly, we describe how the Boussinesq approximation applies to the thermal energy equation. Recall from equation (C.54) in Appendix C that the thermal energy equation is

$$\rho\frac{De}{Dt} = -\nabla \cdot \vec{g} - p(\nabla \cdot \vec{u}) + \phi$$

Although $\nabla \cdot \vec{u} = 0$ under the Bossinesq approximation for the continuity equation, we are not justified in doing this for the thermal energy equation since the term $p(\nabla \cdot \vec{u})$ is not negligible compared to the other terms. The term $p(\nabla \cdot \vec{u})$ is only negligible for fluids that are completely incompressible.

From the continuity equation and assumption 1, we know that

$$-\nabla \cdot \vec{u} = \frac{1}{\rho}\frac{D\rho}{Dt} \ll 1$$

then,

$$-p(\nabla \cdot \vec{u}) = \frac{p}{\rho}\frac{D\rho}{Dt} \approx \frac{p}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \frac{DT}{Dt} \tag{2.10}$$

since

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial p}\frac{Dp}{Dt} + \frac{\partial \rho}{\partial T}\frac{DT}{Dt}$$

and we neglect the $\frac{\partial \rho}{\partial p}\frac{Dp}{Dt}$ term using the third Boussinesq assumption.

For a perfect gas, for which $p = \rho RT$ and $C_\mathrm{p} - C_\mathrm{v} = R$, we have

$$\begin{aligned}
-p\nabla \cdot \vec{u} &= \frac{p}{\rho}\frac{D\rho}{Dt} \\
&= \frac{p}{\rho}\frac{D}{Dt}\left(\frac{p}{RT}\right) \\
&\approx \frac{p}{\rho}\frac{p}{R}\frac{D}{Dt}\left(\frac{1}{T}\right) \\
&= \frac{p}{\rho}\frac{p}{R}\left(\frac{-1}{T^2}\right)\frac{DT}{Dt} \\
&= \frac{-\rho RT\rho RT}{R\rho T^2}\frac{DT}{Dt} \\
&= -R\rho\frac{DT}{Dt} \\
&= -(C_\mathrm{p} - C_\mathrm{v})\rho\frac{DT}{Dt}
\end{aligned}$$

Note that we have made use of (2.10) with ρ replaced with $\frac{p}{RT}$.

The energy equation then becomes

$$\rho\frac{De}{Dt} = -\nabla \cdot \vec{g} - (C_\mathrm{p} - C_\mathrm{v})\rho\frac{DT}{Dt} + \phi$$

Now, for a perfect gas, $e = C_\mathrm{v}T$ and $C_\mathrm{p} = (\frac{\partial e}{\partial T})_\mathrm{v}$. Substituting these expressions into the above equation gives

$$C_\mathrm{v}\rho\frac{DT}{Dt} = -\nabla \cdot \vec{g} - (C_\mathrm{p} - C_\mathrm{v})\rho\frac{DT}{Dt} + \phi$$

After simplifying, we obtain

$$C_{\mathrm{p}}\rho\frac{DT}{Dt} = -\nabla\cdot\vec{g} + \phi \tag{2.11}$$

Note that if the $p\nabla\cdot\vec{u}$ term had been dropped from the energy equation, the C_{p} in equation (2.11) would be replaced with C_{v}.

Under the Boussinesq approximation, the viscous dissipation of energy, ϕ, is negligible. Now, assume that $\vec{g}$ satisfies Fourier's law of heat conduction, meaning that

$$\vec{g} = -k\nabla T$$

Then, (2.11) becomes

$$C_p\rho\frac{DT}{Dt} = -\nabla(-k\nabla T) = k\nabla^2 T$$

Finally, dividing by $C_{\mathrm{p}}\rho$ yields

$$\frac{DT}{Dt} = \kappa\nabla^2 T \tag{2.12}$$

where

$$\kappa = \frac{k}{C_{\mathrm{p}}\rho} \tag{2.13}$$

is the *thermal diffusivity.*

In summary, the Boussinesq approximation applies if the flow speed is much less than the speed of sound, propagation of sound waves is not considered, the vertical scale of the fluid flow is not too large, and the temperature differences within the fluid are small. We are justified in treating the density term as a constant in both the continuity and momentum equations except in the gravity terms. The properties of a fluid, such as μ, k, C_{p}, and C_{v} are also treated as constants under the Boussinesq approximation. Neglecting Coriolis forces, the following set of equations correspond to the Boussinesq approximation

$$\begin{aligned}
\nabla \cdot \vec{u} &= 0 \\
\frac{D\vec{u}}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \nabla^2 u \\
\frac{Dv}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu \nabla^2 v \\
\frac{Dw}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0} + \nu \nabla^2 w \\
\frac{DT}{Dt} &= \kappa \nabla^2 T
\end{aligned} \tag{2.14}$$

Under the Boussinesq approximation, the density and temperature are related by a linear relation given by

$$\rho = \rho_0 \left[1 - \frac{1}{T_0}(T - T_0) \right] \tag{2.15}$$

where ρ_0 and T_0 are the reference density and temperature, respectively. To prove this, we note that for a perfect gas ρ satisfies

$$\rho = \frac{p}{RT}, \text{ or } \rho T = \frac{p}{R}$$

We then write ρ and T in terms of their reference states as

$$\rho = \rho_0 + \rho', \; T = T_0 + T'$$

Multiplying ρ and T together gives

$$(\rho_0 + \rho')(T_0 + T') = \frac{p_0 + p'}{R}$$

Now, after linearizing around the reference states, we obtain

$$\rho_0 T_0 + \rho_0 T' + \rho' T_0 + \rho' T' = \frac{p_0}{R} + \frac{p'}{R}$$

After simplifying, we find an expression for ρ':

$$\rho' = -\frac{1}{T_0}\rho_0 T'$$

and thus

$$\rho - \rho_0 = -\frac{1}{T_0}\rho_0 (T - T_0)$$

Finally,

$$\rho = \rho_0 \left[1 - \frac{1}{T_0}(T - T_0)\right]$$

Thus, under the Boussinesq approximation, the density, ρ, and the temperature, T, are related by a linear expression. This means that the temperature equation in (2.14) can be written equivalently in terms of ρ as

$$\frac{D\rho}{Dt} = \kappa \nabla^2 \rho \tag{2.16}$$

2.2 The Coriolis Force

The *Coriolis force* deflects a moving object when viewed from a rotating frame of reference, such as the Earth. More specifically, the Coriolis force deflects particles to the right of its intended path in the northern hemisphere and to the left in the southern hemisphere. The Coriolis force is proportional to the speed of rotation of the Earth and acts in a direction that is perpendicular to the rotation axis and to the velocity of Earth within the rotating frame.

Let us begin by regarding the atmosphere or ocean as a thin layer on a rotating sphere. The depth scale of flow, H, is on the order of magnitude of a few kilometers, whereas the horizontal scale, L, is on the order of magnitude of a few hundred or even thousand kilometers. In other words, $H << L$. Let U and W be the horizontal and vertical velocities, respectively. Then, after scaling, the continuity equation,

$$u_x + v_y + w_z = 0$$

becomes

$$\frac{U}{L} + \frac{U}{L} + \frac{W}{H} = 0$$

So, in order for $\frac{U}{L}$ and $\frac{W}{H}$ to balance, we must have

$$\frac{U}{L} \sim \frac{W}{H} \Rightarrow \frac{W}{U} \sim \frac{H}{L} \ll 1$$

Thus, $W \ll U$.

In order to study large-scale geophysical flows, we need to use spherical coordinates. However, if the horizontal length scale, L, is much smaller than the radius of the Earth ($\approx$ 6371 km), we can ignore the curvature of the Earth and use a *local* Cartesian system on a tangent plane. In doing so, we have essentially taken a thin strip of the Earth and, if we were to cut it open and stretch it out, we would have a rectangular domain. In this Cartesian system, x is the longitude which increases eastward, y is the latitude which increases northward, and z is the height, or altitude, which increases upward. The velocity components corresponding to this system are u, v, and w, respectively.

The Earth rotates at a rate of

$$\Omega = \frac{2\pi \text{rad}}{\text{day}} = 0.73 \times 10^{-4} s^{-1}$$

around the polar axis in a counterclockwise direction when looking down at the north pole. In the local coordinate system (x, y, z), the components of the Earth's angular velocity, $\vec{\Omega}$, are

$$\vec{\Omega} = (0, |\vec{\Omega}| \cos\theta, |\vec{\Omega}| \sin\theta) = (0, \Omega\cos\theta, \Omega\sin\theta)$$

where $\Omega = |\vec{\Omega}|$ and θ is the latitude. The x coordinate of $\vec{\Omega}$ is zero since the x-axis is oriented into the plane of the page and $\vec{\Omega}$ does not have a component in this direction.

We now want to find a way to represent the Coriolis force in terms of the angular velocity. We consider a frame of reference (x_1, x_2, x_3) rotating at a uniform angular velocity Ω with respect to a fixed frame (X_1, X_2, X_3). In our case, $(x_1, x_2, x_3) = (x, y, z)$, the local coordinate system, and (X_1, X_2, X_3) is a fixed external frame. Let $\vec{P}$ be a vector in the rotating frame. Thus,

$$\vec{P} = P_1\hat{i}_1 + P_2\hat{i}_2 + P_3\hat{i}_3$$

where $\hat{i}_1$, $\hat{i}_2$, and $\hat{i}_3$ are the unit vectors in the rotating frame. To a fixed observer in the fixed frame, the unit vectors $\hat{i}_1$, $\hat{i}_2$, and $\hat{i}_3$ change with time. To this observer,

$$\left(\frac{d\vec{P}}{dt}\right)_F = \frac{d}{dt}\left(P_1\hat{i}_1 + P_2\hat{i}_2 + P_3\hat{i}_3\right) = \hat{i}_1\frac{dP_1}{dt} + \hat{i}_2\frac{dP_2}{dt} + \hat{i}_3\frac{dP_3}{dt} + P_1\frac{d\hat{i}_1}{dt} + P_2\frac{d\hat{i}_2}{dt} + P_3\frac{d\hat{i}_3}{dt}$$

where the subscript F signifies the fixed frame. On the other hand, to a rotating observer,

$$\left(\frac{d\vec{P}}{dt}\right)_R = \hat{\imath}_1\frac{dP_1}{dt} + \hat{\imath}_2\frac{dP_2}{dt} + \hat{\imath}_3\frac{dP_3}{dt}$$

where the subscript R signifies the rotating frame. Thus,

$$\left(\frac{d\vec{P}}{dt}\right)_F = \left(\frac{d\vec{P}}{dt}\right)_R + P_1\frac{d\hat{\imath}_1}{dt} + P_2\frac{d\hat{\imath}_2}{dt} + P_3\frac{d\hat{\imath}_3}{dt} \tag{2.17}$$

The extra terms appear in the rate of change for the fixed observer and not the rotating observer because the rotating observer moves along with the rotataing frame and would not notice the unit vector changing and thus $\frac{d\hat{\imath}}{dt} = 0$ in the rotating frame.

We would now like to define the rates of change $\frac{d\hat{\imath}}{dt}$ in terms of Ω since $\frac{d\hat{\imath}}{dt}$ does not give insight into how the angle changes with time. We note that in time dt, each unit vector $\hat{\imath}$ traces a cone with radius $\sin\alpha$, where α is the constant angle between $\hat{\imath}$ and Ω. Suppose that in time dt the tip of the unit vector travels at an angle of $d\theta$ along the circumference of the base of the cone. If the unit vector is $\hat{\imath}$ at intitial time t and $\hat{\imath} + d\hat{\imath}$ at time $t + dt$, then the magnitude of $d\hat{\imath}$ is the arc length. We recall that

$$\text{arc length} = rd\theta$$

so

$$d\hat{\imath} = \sin\alpha d\theta$$

Thus,

$$\frac{d\hat{\imath}}{dt} = \sin\alpha\frac{d\theta}{dt}$$

But,

$$\frac{d\theta}{dt} = |\vec{\Omega}|$$

since it is the rate of change of the angle with time. Therefore,

$$\frac{d\hat{\imath}}{dt} = |\vec{\Omega}|\sin\alpha = |\vec{\Omega}||\hat{\imath}|\sin\alpha \tag{2.18}$$

The direction of $\frac{d\hat{i}}{dt}$ is perpendicular to the plane containing $\vec{\Omega}$ and $\hat{\imath}$. We recall from the definition of the cross product that

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \vec{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\vec{n}$ is perpendicular to $\vec{a}$ and $\vec{b}$. Thus,

$$\frac{d\hat{\imath}_j}{dt} = \vec{\Omega} \times \hat{\imath}_j \tag{2.19}$$

for each $j = 1, 2, 3$. Now, returning to (2.17) and replacing $\frac{d\hat{\imath}_j}{dt}$ with (2.19), we obtain

$$\left(\frac{d\vec{P}}{dt}\right)_F = \left(\frac{d\vec{P}}{dt}\right)_R + \vec{\Omega} \times \vec{P} \tag{2.20}$$

We are now interested in obtaining an expression for the acceleration in the fixed frame when viewed from within the fixed frame. The velocity vector is $\vec{u} = \frac{d\vec{r}}{dt}$. Now, set $\vec{P} = \vec{r}$, the position vector of a point within the rotating frame. Replacing the velocity vector in (2.20) and differentiating with respect to t gives the following equation

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{2.21}$$

If we differentiate $\vec{u}$ with respect to t, we obtain the acceleration, $\frac{d\vec{u}_F}{dt}$. Replacing $\vec{P}$ with $\vec{u}_F$ in (2.20) we obtain the following acceleration equation

$$\left(\frac{d\vec{u}_F}{dt}\right)_F = \left(\frac{d\vec{u}_F}{dt}\right)_R + \vec{\Omega} \times \vec{u_F} \tag{2.22}$$

The term $\left(\frac{d\vec{u}_F}{dt}\right)_F$ represents the acceleration in the fixed frame as seen from the fixed frame whereas the term $\left(\frac{d\vec{u}_F}{dt}\right)_R$ represents the acceleration in the fixed frame as seen from the rotating frame. We now wish to obtain an acceleration equation in which the right hand side depends completely on terms from within the rotating frame. To do so, we substitute (2.21) into (2.22) to obtain the following

$$\vec{a}_F = \vec{a}_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{2.23}$$

where $\vec{a}_F = \left(\frac{d\vec{u}_F}{dt}\right)_F$, $\vec{a}_R = \left(\frac{d\vec{u}_R}{dt}\right)_R$, and $\left(\frac{d\vec{r}}{dt}\right)_R = \vec{u}_R$. In the above equation, $\vec{a}_F$ is the acceleration in the fixed frame, $\vec{a}_R$ is the acceleration in the rotating frame, $2\vec{\Omega} \times \vec{u}_R$ is called the Coriolis acceleration, and $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ is called the centripetal acceleration. By definition of the cross product, the Coriolos acceleration acts in a direction that is perpendicular to the plane containing $\vec{\Omega}$ and $\vec{u}$. Using trigonometry and making use of

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

one can show that the centripetal acceleration simplifies to $-|\vec{\Omega}|^2\vec{R}$ where $\vec{R}$ is the vector that is perpendicular to the axis of rotation. Furthermore, it acts vertically since $-|\vec{\Omega}|^2\vec{R}$ acts in a direction opposite to $\vec{R}$.

From now on, we will drop the subscripts because we have expressed the acceleration in the fixed frame in terms of components in the rotating frame. In other words, from here onwards we have replaced $\vec{u}_R$ with $\vec{u}$ and $\vec{a}_R$ with $\vec{a}$. Moreover, the terms within the rotating frame are what we can actually see and measure on Earth and the terms within the fixed frame are of no use to us.

We are now able to return to the equations of motion. Under the Boussinesq approximation, recall from (2.9) that the momentum equation was

$$\rho_0 \frac{D\vec{u}}{Dt} = -\nabla p + \rho\vec{g} + \mu\nabla^2 u$$

as was deduced earlier in equation (2.9). We replace

$$\rho_0 \frac{D\vec{u}}{Dt}$$

with

$$\rho_0 \left(\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} - |\vec{\Omega}|^2\vec{R} \right)$$

by making use of (2.23). Thus, the momentum equation under the Boussinesq approximation becomes

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} - |\vec{\Omega}|^2\vec{R} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\vec{u} + \frac{\rho}{\rho_0}\vec{g}_{\text{Newtonian}}$$

We often absorb the centripetal force into the gravitational force term so that the effective gravity force is equal to the sum of the Newtonian gravity and the centripetal force. In doing so, we obtain

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\vec{u} - 2\vec{\Omega}\times\vec{u} + \frac{\rho}{\rho_0}\vec{g} \tag{2.24}$$

where

$$\vec{g} = \vec{g}_{\text{Newtonian}} + |\vec{\Omega}|^2\vec{R}$$

We will now define the Coriolis acceleration in terms of a Coriolis parameter, f, which will be derived shortly. Recall that

$$\vec{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta)$$

where θ is the latitude, and $\vec{u} = (u, v, w)$. Thus, the Coriolis acceleration, $2\vec{\Omega}\times\vec{u}$, becomes

$$2(w\Omega\cos\theta - v\Omega\sin\theta, u\Omega\sin\theta, -u\Omega\cos\theta)$$

when taking the cross product of $\vec{\Omega}$ and $\vec{u}$. We require $w << v$ since we showed that $W << U$ at the beginning of this chapter. Thus, $w\cos\theta << v\sin\theta$. Then, the Coriolis acceleration is given by the following

$$[-(2\Omega\sin\theta)v, (2\Omega\sin\theta)u, -(2\Omega\cos\theta)u] \tag{2.25}$$

In general, the vertical component of the Coriolis acceleration, $-2\Omega\cos\theta u$, is negligible compared with the dominant terms in the vertical equation of motion, in particular $\frac{g\rho}{\rho_0}$ and $\frac{1}{\rho_0}\frac{dp}{dz}$, as was discussed in the Boussinesq analysis of the previous section. As a result, the Coriolis acceleration is approximated by

$$(-fv, fu, 0) \tag{2.26}$$

where

$$f = 2\Omega\sin\theta \tag{2.27}$$

is called the *planetary vorticity* or the *Coriolis parameter.* The sign of f changes depending on its location, for instance, $f > 0$ in the northern regions and $f < 0$

in the southern regions. After applying the approximated Coriolis acceleration, the momentum equation of motion from (2.24) becomes

$$\begin{aligned} \frac{Du}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u + fv \\ \frac{Dv}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v - fu \\ \frac{Dw}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\nabla^2 w - \frac{g\rho}{\rho_0} \end{aligned} \tag{2.28}$$

There are further approximations that can be made with respect to the Coriolis acceleration. We will briefly discuss two such approximations which lead to the *f-plane model* and the *β-plane model.* Both approximations are for flows on a horizontal plane and only involve the u and v equations in (2.28). In both cases the density, ρ_0, is constant. The Coriolis force is the Coriolis acceleration multiplied by the constant density, ρ_0.

The Coriolis parameter, $f = 2\Omega \sin\theta$, varies with the latitude θ. If the length and time scales are small enough we can treat f as a constant, say $f = f_0$. This is called the *f-plane model.*

A more accurate approximation involves expanding f using a Taylor series about a central latitude, θ_0 and approximating f by a linear function of y. More specifically,

$$f = f_0 + \beta y$$

where $\beta = (\frac{df}{dy})_{\theta_0}$ = a constant, and f_0 is also a constant. This is called the *β-plane model.*

2.3 Derivation of the β-Plane Equation

In this section, we will derive the β-plane equation which will be the starting point for this thesis project.

We begin with the governing equations of motion from Appendix C for a two-dimensional fluid in a horizontal plane defined by rectangular coordinates x and y.

Under these conditions, $\vec{u} = (u, v)$, $\vec{g} = (0, 0)$, and $\rho = \rho(x, y, t)$. As we saw earlier, under the Boussinesq approximation, the continuity equation becomes $\nabla \cdot \vec{u} = 0$, and in two dimensions is

$$u_x + v_y = 0 \tag{2.29}$$

The Boussinesq approximation and Coriolis force applied to the momentum equation lead to the following set of equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u + fv \tag{2.30}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v - fu \tag{2.31}$$

where $\nu = \frac{\mu}{\rho_0}$ and the definition of $\frac{D\vec{u}}{Dt}$ from Appendix B has been applied. Notice that the above two equations do not have a gravitational term since the only non-zero gravity component is in the z direction. Finally, the Boussinesq approximation applied to the energy equation yields the following equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\nabla^2 T \tag{2.32}$$

We now make the β-plane approximation, where $f = f_0 + \beta y$, and substitute this into equations (2.30) and (2.31). We proceed by differentiating equation (2.30) with respect to y which gives

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) + u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) + \nu\frac{\partial}{\partial y}(\nabla^2 u) + f_0\frac{\partial v}{\partial y} + \beta v + \beta y\frac{\partial v}{\partial y} \tag{2.33}$$

Similarly, differentiating equation (2.31) with respect to x yields

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) + \nu\frac{\partial}{\partial x}(\nabla^2 v) - f_0\frac{\partial u}{\partial x} - \beta u\frac{\partial u}{\partial x} \tag{2.34}$$

We now define a streamfunction Ψ such that

$$u = -\Psi_y, \quad v = \Psi_x \tag{2.35}$$

After subtracting (2.34) from (2.33) and making use of (2.35) we obtain

$$\frac{\partial}{\partial t}\nabla^2\Psi + \Psi_x\frac{\partial}{\partial y}\left(\nabla^2\Psi\right) - \Psi_y\frac{\partial}{\partial x}\left(\nabla^2\Psi\right) + \beta\Psi_x - \nu\nabla^4\Psi = 0 \tag{2.36}$$

We have also made use of the fact that we can change the order of differentiation for the streamfunction because it is a continuous function. The quantity

$$\nabla^2\Psi = \Psi_{xx} + \Psi_{yy} = v_x - u_y \tag{2.37}$$

is called the *vorticity*, where $\Psi = \Psi(x, y, t)$ is the total streamfunction. Equation (2.36) is called the *barotropic vorticity equation*. A fluid is said to be *barotropic* if the pressure is a function of density only and density is a function of pressure only, meaning that surfaces of constant pressure are also surfaces of constant density. A two-dimensional fluid on a horizontal plane (with no vertical variations) is, by definition, barotropic.

Chapter 3

Initial Value Problem for Rossby Waves on a β-Plane

Now that we have derived all of the governing equations of motion as well as other fundamental approximations from the field of fluid dynamics, we are able to begin our problem.

Our first problem of interest involves Rossby waves (or planetary waves) on a β-plane. Rossby waves are very important in large-scale meteorological applications. They are generated as a result of the variation of the Coriolis force with latitude. These types of waves are important to study because they give us insight into climate changes and patterns in both the atmosphere and ocean.

3.1 Nondimensionalization

In our discussion of fluids thus far, we have assumed that the properties of a fluid are measured in terms of pre-defined units, such as m, s, K, etc. We require that all of the terms, variables, and parameters in a given equation have the same dimensions. To do so, we use nondimensionalization, a technique which involves introducing new dimensionless variables into a given equation. Nondimensionalization allows us to compare different problems that may have different physical scales. For example, we could compare flows with different length scales, flow speeds, or fluid properties after nondimensionalizing. Following nondimensionalization, two very different problems may be *dynamically similar* and if we solve one problem we can predict the outcome of the other problem without solving it directly. Flows are dynamically similar if their nondimensional parameters are equal.

In this section, we describe the process of nondimensionalization applied to the barotropic vorticity equation (2.36) which describes how Rossby waves propagate on a β-plane.

Let x, y represent displacement defined in a two-dimensional Cartesian coordinate system in a horizontal plane, t represent time, u, v represent the x and y components of the velocity, respectively, ρ represent density, p represent pressure, and g represent acceleration. Also, let L_x and L_y represent typical length scales in the x and y directions, respectively. Lastly, let U and V represent typical velocity scales in the x and y directions, respectively, and T represent a typical time scale.

We consider equation (2.36) written in terms of dimensional quantities which we denote with asterisks (for convenience, we did not include asterisks in the derivation of equation (2.36) in the previous Chapter)

$$\frac{\partial}{\partial t^*}\nabla^{*2}\Psi^* + \Psi^*_x\frac{\partial}{\partial y^*}\left(\nabla^{*2}\Psi^*\right) - \Psi^*_y\frac{\partial}{\partial x^*}\left(\nabla^{*2}\Psi^*\right) + \beta^*\Psi^*_x - \nu^*\nabla^{*4}\Psi^* = 0 \qquad (3.1)$$

where $\nabla^{*2} = \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}}$ is the dimensional Laplacian operator. We note that each term in this equation has units of s^{-2} and thus the dimensional planetary vorticity gradient, β^*, and dimensional viscosity coefficient, ν^*, must have units of $m^{-1}s^{-1}$ and m^2s^{-1}, respectively in order to satisfy this.

Following the notation used by Campbell and Maslowe [5], we define the corresponding nondimensional variables and parameters (without asterisks) by

$$
\begin{aligned}
x &= \frac{x^*}{L_x} & y &= \frac{y^*}{L_y} & t &= \frac{t^*}{T} = \frac{Ut^*}{L_x} \\
u &= \frac{u^*}{U} & \Psi &= \frac{\Psi^*}{L_y U} & \beta &= \frac{\beta^*}{B} \\
\nu &= \frac{\nu^*}{V}
\end{aligned}
\tag{3.2}
$$

where $B = \frac{U}{L_y^2}$, $V = \frac{UL_y^2}{L_x}$, Ψ represents the total streamfunction and β is the gradient of planetary vorticity in the y-direction. Also, L_x and L_y are typical length scales in the x (east-west) and y (north-south) directions, respectively. In configurations with waves, L_x and L_y are generally assumed to be of the order of magnitude of the wavelengths in the x and y directions. We note that B and V are determined from the nondimensionalization process. From the above set of equations, we conclude that the nondimensional variable is a ratio of the dimensional variable and the reference variable.

Derivatives with respect to the nondimensional variables are related to their dimensional counterparts by

$$
\frac{\partial}{\partial t^*} = \frac{U}{L_x}\frac{\partial}{\partial t} \qquad \frac{\partial}{\partial y^*} = \frac{1}{L_y}\frac{\partial}{\partial y} \qquad \frac{\partial}{\partial x^*} = \frac{1}{L_x}\frac{\partial}{\partial x} \tag{3.3}
$$

After making the appropriate substitutions from (3.2) and (3.3), we arrive at the following equation

$$
\begin{aligned}
&\frac{U^2 L_y}{L_x}\frac{\partial}{\partial t}\left(\frac{1}{L_x^2}\frac{\partial^2}{\partial x^2} + \frac{1}{L_y^2}\frac{\partial^2}{\partial y^2}\right)\Psi + \frac{U^2 L_y}{L_x}\Psi_x\frac{\partial}{\partial y}\left(\frac{1}{L_x^2}\frac{\partial^2}{\partial x^2} + \frac{1}{L_y^2}\frac{\partial^2}{\partial y^2}\right)\Psi \\
&\quad -\frac{U^2 L_y}{L_x}\Psi_y\frac{\partial}{\partial x}\left(\frac{1}{L_x^2}\frac{\partial^2}{\partial x^2} + \frac{1}{L_y^2}\partial^2\partial y^2\right)\Psi + \frac{BUL_y}{L_x}\beta\Psi_x \\
&\quad -VL_yU\nu\left(\frac{1}{L_x^2}\frac{\partial^2}{\partial x^2} + \frac{1}{L_y^2}\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{L_x^2}\frac{\partial^2}{\partial x^2} + \frac{1}{L_y^2}\frac{\partial^2}{\partial y^2}\right)\Psi = 0
\end{aligned}
\tag{3.4}
$$

To simplify the above equation, we divide by $\frac{U^2}{L_y L_x}$. In doing so, we arrive at the dimensionless barotropic vorticity equation

$$
\frac{\partial}{\partial t}\nabla^2\Psi + \Psi_x\frac{\partial}{\partial y}\left(\nabla^2\Psi\right) - \Psi_y\frac{\partial}{\partial x}\left(\nabla^2\Psi\right) + \beta\Psi_x - \nu\nabla^4\Psi = 0 \tag{3.5}
$$

where the nondimensional Laplacian operator is defined as

$$\nabla^2 = \delta\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3.6}$$

where $\delta = \frac{L_y^2}{L_x^2}$ is the aspect ratio which gives a measure of the magnitude of the length scale in the x direction to the length scale in the y direction. In our study, we will consider the case where $\delta \sim O(1)$ and thus set $\delta = 1$, meaning that $L_x = L_y$. In practice in geophysical flows, wavelengths in the north-south direction are generally shorter than in the east-west direction, so $L_y \ll L_x$. In that case, δ could be considered a small parameter for the purpose of asymptotics. The limit $\delta \to 0$ is called the long wave limit. By setting $\delta = 0$ or $\delta \ll 1$, we neglect, or reduce, some of the variation in the x direction. Here we allow $\delta \sim O(1)$, assuming that variations in the x direction are as important as in the y direction.

3.2 Fourier Analysis

Our analysis begins with the nondimensional barotropic vorticity equation (3.5) from the previous section. Equation (3.5) can be linearized by expressing the total streamfunction, Ψ, as the sum of the background or mean streamfunction and a small perturbation (wave) of the streamfunction which is assumed to be periodic in both x and y. In other words,

$$\Psi(x, y, t) = \bar{\psi}(y) + \varepsilon\psi(x, y, t) \tag{3.7}$$

where $\varepsilon << 1$. Let $\bar{\psi}(y) = -\bar{u}y$, where $\bar{u}$ is a constant. Thus, the background or mean velocity is $(\bar{u}, 0)$. We now substitute (3.7) into the dimensionless barotropic vorticity equation (3.5) where $\bar{\psi}(y)$ is replaced with $-\bar{u}y$. After dividing by ε, which is justified since $\varepsilon \neq 0$, we obtain the following equation for the perturbed problem

$$\frac{\partial}{\partial t}\nabla^2\psi + \bar{u}\nabla^2\psi_x + \beta\psi_x - \nu\nabla^4\psi + \varepsilon(\psi_x\nabla^2\psi_y - \psi_y\nabla^2\psi_x) = 0 \tag{3.8}$$

As $\varepsilon \to 0$, the above problem becomes linear. Linearization is justified since $\varepsilon << 1$, meaning that the wave amplitude is small relative to the magnitude of the mean flow.

Before proceeding any further, we will define and describe our problem more precisely. We consider (3.8) (in nondimensional variables and parameters) in the domain $0 < x < 2\pi$, $0 < y < 2\pi$, $t > 0$. We also assume that we have periodic boundary conditions, meaning that $\psi(0, y, t) = \psi(2\pi, y, t)$ and $\psi(x, 0, t) = \psi(x, 2\pi, t)$, and we also have periodic boundary conditions on the first derivatives of ψ. We specify an initial condition at $t = 0$, such as

$$\psi(x, y, 0) = A_1 \cos x + A_2 \cos y + A_3 \cos(x + y) + A_4 \cos(x - y) \tag{3.9}$$

which is a periodic function of x and y. Here A_1, A_2, A_3, A_4 are constants. Thus, the initial wave energy corresponds to wavenumbers of $\vec{k} = (k, l) = (\pm 1, 0), (0, \pm 1), \pm(1, 1), \pm(1, -1)$

We begin our analysis by taking a discrete Fourier Transform of the perturbed equation (3.8) (a PDE). In other words, we express ψ as a double Fourier series,

$$\psi(x, y, t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \hat{\psi}(k, l, t) e^{i(kx+ly)} \tag{3.10}$$

where k and l are the wave numbers in the x and y directions, respectively. The complex Fourier coefficients, $\hat{\psi}$, are given by

$$\hat{\psi}(k, l, t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \psi(x, y, t) e^{-i(kx+ly)} dx dy \tag{3.11}$$

The notation used here follows that of Haberman [7], but similar Fourier analysis arguments can be found in any text on Partial Differential Equations.

Taking a Fourier Transform of the linear terms in equation (3.8) is straightforward; however, when taking a Fourier Transform of the nonlinear terms, we must use a convolution. The Fourier Transform of a product of two functions, F and G, is defined to be

$$\mathcal{F}(F \cdot G) = \sum_{\bar{k}} \sum_{\bar{l}} \hat{F}(k - \bar{k}, l - \bar{l}) \hat{G}(\bar{k}, \bar{l}) \tag{3.12}$$

where $\mathcal{F}$ represents the discrete Fourier Transform.

For example,

$$\mathcal{F}(\psi_x \nabla^2 \psi_y) = \sum_{\bar{k}} \sum_{\bar{l}} \bar{l}(k - \bar{k})(\bar{k}^2 + \bar{l}^2) \hat{\psi}(k - \bar{k}, l - \bar{l}) \hat{\psi}(\bar{k}, \bar{l})$$

where $F = \psi_x$, $G = \nabla^2\psi_y$, $\mathcal{F}(\psi_x) = ik\hat{\psi} = \hat{F}$, and $\mathcal{F}(\nabla^2\psi_y) = -il(k^2+l^2)\hat{\psi} = \hat{G}$.

Taking a Fourier Transform of (3.8) gives the following ordinary differential equation

$$\frac{\partial}{\partial t}\hat{\psi}(k,l,t) = \left[-\bar{u}ik + \frac{i\beta k}{k^2+l^2} - \nu(k^2+l^2)\right]\hat{\psi}(k,l,t)$$
$$+\frac{\varepsilon}{k^2+l^2}\left[\sum_{\bar{k}}\sum_{\bar{l}}(\bar{l}k - l\bar{k})(\bar{k}^2+\bar{l}^2)\hat{\psi}(k-\bar{k}, l-\bar{l}, t)\hat{\psi}(\bar{k},\bar{l},t)\right] \tag{3.13}$$

where $(k,l) \neq (0,0)$. Note that $\hat{\psi}(0,0,t) = 0$ so we can omit it henceforth. If we substitute $(k,l) = (0,0)$ into (3.13), we find that $0 = 0$ which does not tell us anything. However, if we return to the original u and v equations in (2.30) and (2.31) then take a Fourier Transform and set $(k,l) = (0,0)$, we obtain

$$\frac{\partial\hat{u}}{\partial t}(0,0,t) = 0 \text{ and } \frac{\partial\hat{v}}{\partial t}(0,0,t) = 0$$

Thus, $\hat{u}$ and $\hat{v}$ are both constant. Furthermore, since the initial condition specified in (3.9) gives $\hat{\psi}(0,0,0) = 0$, both $\hat{u}$ and $\hat{v}$ are zero and will remain zero indefinitely. Therefore, $\hat{\psi}(0,0,t) = 0$.

We also require that

$$\begin{aligned} \hat{\psi}(k,-l) &= \hat{\psi}^*(-k,l) & \qquad \hat{\psi}(-k,0) &= \hat{\psi}^*(k,0) \\ \hat{\psi}(-k,-l) &= \hat{\psi}^*(k,l) & \qquad \hat{\psi}(0,-l) &= \hat{\psi}^*(0,l) \end{aligned} \tag{3.14}$$

where $\hat{\psi}^*$ is the complex conjugate of $\hat{\psi}$. These conditions are necessary because the Fourier coefficients, $\hat{\psi}$, are complex-valued but ψ is required to be real and thus the coefficients corresponding to negative k and l values should be complex conjugates of the corresponding positive k and l values.

For the special case when $\varepsilon = 0$, equation (3.8) is linear and no new wavenumbers can be generated besides $k, l = -1, 0$ and 1. If $\varepsilon \neq 0$ but $\varepsilon \ll 1$, we can represent ψ as a perturbation series:

$$\psi \sim \psi^{(0)} + \varepsilon\psi^{(1)} + O(\varepsilon^2)$$

The $O(1)$ term is the linear solution comprising of the original wavenumbers. Higher wavenumbers appear at $O(\varepsilon)$. We consider the linear, $O(1)$, problem first, then the weakly nonlinear problem which incorporates both $O(1)$ and $O(\varepsilon)$ terms and truncate the spectrum to only consider the original wavenumber values of ± 1 and 0. We then include the full spectrum of wavenumbers present at $O(\varepsilon)$.

Truncation of the Fourier spectrum to the original wavenumbers produces the following eight differential equations

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\psi}(1,0,t) &= (-i\bar{u}+i\beta-\nu)\hat{\psi}(1,0,t)+\varepsilon[\hat{\psi}(1,1,t)\hat{\psi}(0,-1,t) \\
&\quad -\hat{\psi}(1,-1,t)\hat{\psi}(0,1,t)] \\
\frac{\partial}{\partial t}\hat{\psi}(0,1,t) &= -\nu\hat{\psi}(0,1,t)+\varepsilon[\hat{\psi}(1,0,t)\hat{\psi}(-1,1,t)-\hat{\psi}(1,1,t)\hat{\psi}(-1,0,t)] \\
\frac{\partial}{\partial t}\hat{\psi}(1,1,t) &= \left(-i\bar{u}+\frac{1}{2}i\beta-2\nu\right)\hat{\psi}(1,1,t) \\
\frac{\partial}{\partial t}\hat{\psi}(1,-1,t) &= \left(-i\bar{u}+\frac{1}{2}i\beta-2\nu\right)\hat{\psi}(1,-1,t) \\
\frac{\partial}{\partial t}\hat{\psi}(-1,0,t) &= (i\bar{u}-i\beta-\nu)\hat{\psi}(-1,0,t)+\varepsilon[\hat{\psi}(0,1,t)\hat{\psi}(-1,-1,t) \\
&\quad -\hat{\psi}(0,-1,t)\hat{\psi}(-1,1,t)] \\
\frac{\partial}{\partial t}\hat{\psi}(0,-1,t) &= -\nu\hat{\psi}(0,-1,t)+\varepsilon[\hat{\psi}(1,-1,t)\hat{\psi}(-1,0,t)-\hat{\psi}(1,0,t)\hat{\psi}(-1,-1,t)] \\
\frac{\partial}{\partial t}\hat{\psi}(-1,-1,t) &= \left(i\bar{u}-\frac{1}{2}i\beta-2\nu\right)\hat{\psi}(-1,-1,t) \\
\frac{\partial}{\partial t}\hat{\psi}(-1,1,t) &= \left(i\bar{u}-\frac{1}{2}i\beta-2\nu\right)\hat{\psi}(-1,1,t)
\end{aligned} \tag{3.15}$$

Notice that the first, second, fifth and sixth equations are nonlinear while the others are linear. This means that the components corresponding to wavenumbers of $(\pm 1, 0)$ and $(0, \pm 1)$ are affected by the other wavenumber components, while the components corresponding to wavenumbers of $\pm(1,1)$ and $\pm(1,-1)$ evolve independently.

To simplify our notation let

$$
\begin{aligned}
\hat{\psi}(1,0,t) &= X_1(t) & \hat{\psi}(0,1,t) &= X_2(t) \\
\hat{\psi}(1,1,t) &= X_3(t) & \hat{\psi}(1,-1,t) &= X_4(t) \\
\hat{\psi}(-1,0,t) &= X_5(t) & \hat{\psi}(0,-1,t) &= X_6(t) \\
\hat{\psi}(-1,-1,t) &= X_7(t) & \hat{\psi}(-1,1,t) &= X_8(t)
\end{aligned}
\tag{3.16}
$$

Making use of (3.14), we obtain the following conjugate relations

$$
\begin{aligned}
\hat{\psi}(-1,-1,t) &= \hat{\psi}^*(1,1,t) & \hat{\psi}(-1,0,t) &= \hat{\psi}^*(1,0,t) \\
\hat{\psi}(0,-1,t) &= \hat{\psi}^*(0,1,t) & \hat{\psi}(1,-1,t) &= \hat{\psi}^*(-1,1,t)
\end{aligned}
\tag{3.17}
$$

As a result,

$$
\begin{aligned}
\hat{\psi}(-1,0,t) &= \hat{\psi}^*(1,0,t) = X_5(t) \\
\hat{\psi}(0,-1,t) &= \hat{\psi}^*(0,1,t) = X_6(t) \\
\hat{\psi}(-1,-1,t) &= \hat{\psi}^*(1,1,t) = X_7(t) \\
\hat{\psi}(-1,1,t) &= \hat{\psi}^*(1,-1,t) = X_8(t)
\end{aligned}
\tag{3.18}
$$

Substituting (3.16) into (3.15) and making use of (3.17) and (3.18) gives the following simplified set of differential equations

$$
\begin{aligned}
\frac{d}{dt}X_1 &= (-i\bar{u} + i\beta - \nu)X_1 + \varepsilon[X_3X_6 - X_4X_2] \\
\frac{d}{dt}X_2 &= -\nu X_2 + \varepsilon[X_1X_8 - X_3X_5] \\
\frac{d}{dt}X_3 &= \left(-i\bar{u} + \frac{1}{2}i\beta - 2\nu\right) X_3 \\
\frac{d}{dt}X_4 &= \left(-i\bar{u} + \frac{1}{2}i\beta - 2\nu\right) X_4 \\
\frac{d}{dt}X_5 &= (i\bar{u} - i\beta - \nu)X_5 + \varepsilon[X_2X_7 - X_6X_8] \\
\frac{d}{dt}X_6 &= -\nu X_6 + \varepsilon[X_4X_5 - X_1X_7] \\
\frac{d}{dt}X_7 &= \left(i\bar{u} - \frac{1}{2}i\beta - 2\nu\right) X_7 \\
\frac{d}{dt}X_8 &= \left(i\bar{u} - \frac{1}{2}i\beta - 2\nu\right) X_8
\end{aligned}
\tag{3.19}
$$

We note that the last four equations are indeed the complex conjugates of the first four equations. By setting $\varepsilon = 0$, the above differential equations can be written as a linear system of the form

$$\frac{\partial \vec{X}}{\partial t} = A\vec{X} \tag{3.20}$$

where

$$\begin{aligned} \vec{X} &= \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \\ A &= \begin{pmatrix} -i\bar{u} + i\beta - \nu & 0 & 0 & 0 \\ 0 & -\nu & 0 & 0 \\ 0 & 0 & -i\bar{u} + \frac{1}{2}i\beta - 2\nu & 0 \\ 0 & 0 & 0 & -i\bar{u} + \frac{1}{2}i\beta - 2\nu \end{pmatrix} \end{aligned} \tag{3.21}$$

Notice that for the linear system we only need the first four equations of (3.19) since the linear part of these equations only depend on X_1, X_2, X_3, and X_4. Similarly, we can write (3.19) as a system of nonlinear equations which has the form

$$\frac{\partial \vec{X}}{\partial t} = A\vec{X} + \varepsilon f(\vec{X}) \tag{3.22}$$

where

$$
\begin{aligned}
\vec{X} &= \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{pmatrix} \\
A &= \begin{pmatrix}
-\imath\bar{u}+\imath\beta-\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\nu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\imath\bar{u}+\frac{1}{2}\imath\beta-2\nu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\imath\bar{u}+\frac{1}{2}\imath\beta-2\nu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \imath\bar{u}-\imath\beta-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\nu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \imath\bar{u}-\frac{1}{2}\imath\beta-2\nu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \imath\bar{u}-\frac{1}{2}\imath\beta-2\nu
\end{pmatrix} \\
f(\vec{X}) &= \begin{pmatrix} X_3X_6 - X_4X_2 \\ X_1X_8 - X_3X_5 \\ 0 \\ 0 \\ X_2X_7 - X_6X_8 \\ X_4X_5 - X_1X_7 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
\tag{3.23}
$$

For the nonlinear system, we require X_1 through X_8 since the first four equations in (3.19) do not depend solely on X_1 through X_4 as in the linear case.

We will now determine the fixed point(s) of both the linear and nonlinear systems defined above. But first, we will introduce the notion of a *fixed point.* A fixed point occurs when the first derivative with respect to t is zero. In other words, a fixed point occurs when the velocity is zero. If $f(x^*) = 0$, then x^* is a fixed point. For the linear case, by setting the derivatives of X_1 through X_4 to zero we obtain

$$X_1 = X_2 = X_3 = X_4 = 0$$

Thus, the only fixed point is the origin. Similarly, for the nonlinear case, the only fixed point is at the origin, given by

$$X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0$$

This origin refers to the state of zero wave amplitude which is obviously an equilibrium solution. In other words, if the wave amplitude is zero initially then it will remain zero for all time.

3.3 Solution of the Linear β-Plane Problem

We will now find the general solution of the linear system (3.20) where $\vec{X}$ and A are defined by (3.21). These linear equations have the form

$$\frac{d}{dt}X_j = (a_j + ib_j)X_j \tag{3.24}$$

where a_j and b_j are constants and $j = 1, .., 4$. We can thus use the separable method to solve the differential equations. As a result, the solutions are of the form

$$X_j = X_j(0)e^{a_j t}e^{ib_j t} \tag{3.25}$$

where $X_j(0)$ is determined from the initial condition. Accordingly, the general solution of (3.20) is

$$\begin{aligned}
X_1 &= X_1(0)e^{-\nu t}e^{i(-\bar{u}+\beta)t} &(3.26)\\
X_2 &= X_2(0)e^{-\nu t} &(3.27)\\
X_3 &= X_3(0)e^{-2\nu t}e^{i\left(-\bar{u}+\frac{1}{2}\beta\right)t} &(3.28)\\
X_4 &= X_4(0)e^{-2\nu t}e^{i\left(-\bar{u}+\frac{1}{2}\beta\right)t} &(3.29)
\end{aligned}$$

The solutions are periodic due to the $e^{\imath b_j t}$ term and they have a component of exponential decay from the $e^{a_j t}$ term. Thus, as $t \to \infty$ the solutions oscillate indefinitely in space and the amplitude approaches zero. We could have also found the dispersion relation of the linear part of (3.8) to find the general solution, as outlined in Appendix D.

Let us consider a specified initial condition for the perturbed PDE (3.8) and assume that we have periodic boundary conditions in both the x and y directions, namely

$$\psi(x, y, 0) = 2[\cos k_0 x + \cos l_0 y + \cos(k_0 x + l_0 y) + \cos(k_0 x - l_0 y)] \tag{3.30}$$

at $t = 0$. For simplicity, let $k_0 = 1$ and $l_0 = 1$. A contour plot of this function in x-y space is shown in Figure 3.1 along with contour plots of each of its components.

The intial condition can be rewritten using the complex exponential definition of the cosine function to give

$$\psi(x, y, 0) = e^{\imath x} + e^{\imath y} + e^{\imath(x+y)} + e^{\imath(x-y)} + \text{c.c.} \tag{3.31}$$

where c.c. stands for complex conjugate. The term $e^{\imath x}$ corresponds to $\hat{\psi}(1, 0) = X_1$ since $k = 1$ and $l = 0$. Similarly, the term $e^{\imath y}$ corresponds to $\hat{\psi}(0, 1) = X_2$ since $k = 0$ and $l = 1$. Accordingly, $e^{\imath(x+y)}$ corresponds to $\hat{\psi}(1, 1) = X_3$ since $k = l = 1$ and the term $e^{-\imath(x+y)}$ corresponds to $\hat{\psi}(-1, -1) = X_7$ since $k = l = -1$. Thus,

$$X_j(0) = 1, \ \forall j = 1, .., 8$$

since the coefficients (or amplitude) of $e^{\imath x}$, $e^{\imath y}$, $e^{\imath(x+y)}$, $e^{\imath(xy)}$, and the complex conjugates are all 1.

With the initial condition (3.31), the solution of the linearized equation is given by

$$\hat{\psi}(1, 0) = X_1 = e^{-\nu t} e^{\imath(-\bar{u}+\beta)t} \tag{3.32}$$

$$\hat{\psi}(0, 1) = X_2 = e^{-\nu t} \tag{3.33}$$

$$\hat{\psi}(1, 1) = X_3 = e^{-2\nu t} e^{\imath\left(-\bar{u}+\frac{1}{2}\beta\right)t} \tag{3.34}$$

$$\hat{\psi}(1, -1) = X_4 = e^{-2\nu t} e^{\imath\left(-\bar{u}+\frac{1}{2}\beta\right)t} \tag{3.35}$$

Each Fourier mode of the solution oscillates with time according to $e^{-\imath\omega t}$ where ω is the complex frequency given by the dispersion relation (D.6) in Appendix D. In the above four equations, ω has the form $\omega = \lambda_2 - i\nu$, where λ_2 is real. Since the imaginary part of ω is negative, each Fourier mode includes an exponentially decaying factor of $e^{-\nu t}$ or $e^{-2\nu t}$ which represents the effect of the viscosity. In an inviscid problem ($\nu = 0$) the amplitude of the solution remains at a constant value of 1 for all time. For the wavenumber component $(0,1)$, the real part of the frequency is zero and so there are no oscillations in time. The special parameter configurations $\beta = \bar{u}$ and $\beta = 2\bar{u}$ give solutions with only one or two oscillatory components. For example, if $\beta = \bar{u}$, only X_3 and X_4 will be oscillatory while X_1 and X_2 will be comprised of exponentially decaying factors only.

3.4 Solution of the Nonlinear β-Plane Problem

We now look to solve the nonlinear system (3.22) where $\vec{X}$, A, and $f(\vec{X})$ are defined in (3.23). To do so, we will use perturbation theory. Perturbation theory uses iterations for obtaining approximate solutions to problems involving a small parameter, ε, in our case. The goal is to decompose a difficult problem into an infinite number of relatively easy ones. We begin by assuming that the solution of our system of differential equations is in the form of a perturbation series in powers of ε and we compute the coefficients of the series. We then sum the series to obtain the solution of our original problem. In practice, we take only the first few terms of the series to obtain an approximate solution which is asymptotic to the exact solution for $\varepsilon \ll 1$. The notation used here follows that of Bender and Orszag [3].

Firstly, we substitute

$$X_\jmath(t) \sim X_\jmath^{(0)}(t) + \varepsilon X_\jmath^{(1)}(t) + O(\varepsilon^2) \tag{3.36}$$

into (3.19), where $X_\jmath^{(n)}(t)$ $(n \geq 0)$ are the coefficients of the perturbation series. The equations for $j = 3, 4, 7, 8$ are linear so their exact solutions are given by the solutions found in the previous section ((3.34), (3.35)) where X_7 and X_8 are the complex conjugates of X_3 and X_4, respectively. For these four j values, the terms at $O(\varepsilon)$ and higher are all zero. Thus, we only need to carry out the substitution (3.36) for $j = 1, 2, 5$ and 6.

The solutions for $j = 3, 4, 7$ and 8 are

$$X_3(t) = e^{-2\nu t} e^{\imath\left(-\bar{u}+\frac{1}{2}\beta\right)t} \tag{3.37}$$

$$X_4(t) = e^{-2\nu t} e^{\imath\left(-\bar{u}+\frac{1}{2}\beta\right)t} \tag{3.38}$$

$$X_7(t) = e^{-2\nu t} e^{\imath\left(\bar{u}-\frac{1}{2}\beta\right)t} \tag{3.39}$$

$$X_8(t) = e^{-2\nu t} e^{\imath\left(\bar{u}-\frac{1}{2}\beta\right)t} \tag{3.40}$$

since $X_\jmath(0) = 1$ according to the initial condition (3.31). The equations for $j = 1, 2, 5, 6$ are nonlinear, so for these j values there will be nonzero terms at $O(\varepsilon)$ and higher.

To obtain an approximate solution, we will find the first two coefficients, $X_\jmath^{(0)}$ and $X_\jmath^{(1)}$. To find $X_\jmath^{(0)}$ we look at the zeroth-order problem; that is, at $O(\varepsilon^0) = O(1)$. At $O(1)$, the equations for $j = 1, 2, 5$ and 6 are linear and of the form

$$\frac{d}{dt} X_\jmath^{(0)} = C_\jmath X_\jmath^{(0)} \tag{3.41}$$

where $C_\jmath$ is a complex constant. The solution is of the form

$$X^{(0)}(t) = e^{C_\jmath t} \tag{3.42}$$

since the initial condition specified earlier gives $X^{(0)}(0) = 1$. In particular, the solutions at $O(1)$ are

$$X_1^{(0)}(t) = e^{-\nu t} e^{\imath(-\bar{u}+\beta)t} \tag{3.43}$$

$$X_2^{(0)}(t) = e^{-\nu t} \tag{3.44}$$

$$X_5^{(0)}(t) = e^{-\nu t} e^{\imath(\bar{u}-\beta)t} \tag{3.45}$$

$$X_6^{(0)}(t) = e^{-\nu t} \tag{3.46}$$

which are the linear solutions we found in the previous section (and their complex conjugates).

Next, we look at the $O(\varepsilon)$ problem to determine $X_j^{(1)}$. At $O(\varepsilon)$, the equations for $j = 1, 2, 5$ and 6 are nonlinear and of the form

$$\frac{d}{dt}X_j^{(1)} - a_1 X_j^{(1)} = \text{products of } X_j^{(0)} \text{ functions} \tag{3.47}$$

where a_1 is a complex constant that depends on j. For convenience, we have left out the subscript j on a_1. Since $X_j^{(0)}$ are the exponential functions we found at $O(1)$, the above equation becomes

$$\frac{d}{dt}X_j^{(1)} - a_1 X_j^{(1)} = e^{a_2 t}e^{a_3 t} - e^{a_4 t}e^{a_5 t} \tag{3.48}$$

where a_2, a_3, a_4 and a_5 are complex constants. This is a linear equation with respect to $X_j^{(1)}$ and can thus be solved using any known method used to solve linear differential equations. Finding an integrating factor, $I(t)$, is one approach that will be used here. The integrating factor is

$$I(t) = e^{-a_1 t} \tag{3.49}$$

After multiplying both sides of equation (3.47) by $I(t)$, then integrating and simplifying, we arrive at the following solution

$$X_j^{(1)}(t) = \left(\frac{1}{a_2 + a_3 - a_1}\right) e^{(a_2+a_3)t} - \left(\frac{1}{a_4 + a_5 - a_1}\right) e^{(a_4+a_5)t} + De^{a_1 t} \tag{3.50}$$

where D is a constant that is obtained after applying the initial condition $X_j^{(1)}(0) = 0$. The constant D is of the form

$$\frac{a_2 + a_3 - a_4 - a_5}{(a_2 + a_3 - a_1)(a_4 + a_5 - a_1)} \tag{3.51}$$

Accordingly, the solutions for $j = 1, 2, 5$ and 6 are

$$X_1^{(1)}(t) = 0 \tag{3.52}$$

$$X_2^{(1)}(t) = Ae^{-3\nu t}e^{\imath\left(\frac{1}{2}\beta\right)t} - A^* e^{-3\nu t}e^{-\imath\left(\frac{1}{2}\beta\right)t} + Be^{-\nu t} \tag{3.53}$$

$$X_5^{(1)}(t) = 0 \tag{3.54}$$

$$X_6^{(1)}(t) = A^* e^{-3\nu t}e^{-\imath\left(\frac{1}{2}\beta\right)t} - Ae^{-3\nu t}e^{\imath\left(\frac{1}{2}\beta\right)t} + B^* e^{-\nu t} \tag{3.55}$$

where $A = \left(\frac{1}{-2\nu+\frac{1}{2}\imath\beta}\right)$, $B = \left(\frac{\imath\beta}{4\nu^2+\frac{1}{4}\beta^2}\right)$, A^* and B^* are the complex conjugates of A and B, respectively. Note that the $O(\varepsilon)$ solutions for $\hat{\psi}(1,0,t)$ and $\hat{\psi}(-1,0,t)$ are both zero. Making use of (3.36), we obtain the approximate solutions of $X_\jmath(t)$ when $\jmath = 1, 2, 5$ and 6. They are

$$X_1(t) \sim e^{-\nu t} e^{\imath(-\bar{u}+\beta)t} + O(\varepsilon^2) \tag{3.56}$$

$$X_2(t) \sim e^{-\nu t} + \varepsilon \left[A e^{-3\nu t} e^{\imath\left(\frac{1}{2}\beta\right)t} - A^* e^{-3\nu t} e^{-\imath\left(\frac{1}{2}\beta\right)t} + B e^{-\nu t} \right] + O(\varepsilon^2) \tag{3.57}$$

$$X_5(t) \sim e^{-\nu t} e^{\imath(\bar{u}-\beta)t} + O(\varepsilon^2) \tag{3.58}$$

$$X_6(t) \sim e^{-\nu t} + \varepsilon \left[A^* e^{-3\nu t} e^{-\imath\left(\frac{1}{2}\beta\right)t} - A e^{-3\nu t} e^{\imath\left(\frac{1}{2}\beta\right)t} + B^* e^{-\nu t} \right] + O(\varepsilon^2) \tag{3.59}$$

So in the end we have found that at $O(\varepsilon)$ the only nonzero contributions result from the wavenumbers $(0,1)$ and $(0,-1)$. In particular, from the terms that are periodic in the y direction but independent of x. For the wavenumber $(0,1)$, the $O(\varepsilon)$ term arises from products of X_1, X_8, X_3 and X_5. In other words, from interactions between the pairs of wavenumbers $(1,0)$ and $(-1,1)$, and $(1,1)$ and $(-1,0)$. For the wavenumber $(0,-1)$, the $O(\varepsilon)$ term arises from products of X_4, X_5, X_1 and X_7; thus interactions between the pairs of wavenumbers $(1,-1)$ and $(-1,0)$, and $(1,0)$ and $(-1,-1)$. This is shown schematically in Figure 3.2. Arrows are used to demonstrate the interactions which generate the $O(\varepsilon)$ term. A star represents wavenumbers with nonzero $O(\varepsilon)$ terms while an open circle represents wavenumbers with no $O(\varepsilon)$ terms.

3.5 Higher Wavenumber Terms for the Nonlinear β-Plane Problem

Now, instead of truncating the Fourier spectrum at k, $l = \pm 1$, suppose we extend the spectrum to k, $l = \pm 2$. In doing so, we will have more convolution terms to consider and the number of differential equations in (3.19) jumps from eight to twenty four. As a result, there will be more terms at $O(\varepsilon)$ in the perturbation series which will in turn give a more accurate solution to the β-plane problem.

Just as we did before, we will now define X_9 through X_{24} in terms of $\hat{\psi}$.

$$
\begin{aligned}
\hat{\psi}(2,0,t) &= X_9(t) & \hat{\psi}(0,2,t) &= X_{10}(t) \\
\hat{\psi}(1,2,t) &= X_{11}(t) & \hat{\psi}(2,1,t) &= X_{12}(t) \\
\hat{\psi}(2,2,t) &= X_{13}(t) & \hat{\psi}(1,-2,t) &= X_{14}(t) \\
\hat{\psi}(2,-1,t) &= X_{15}(t) & \hat{\psi}(2,-2,t) &= X_{16}(t) \\
\hat{\psi}(-2,0,t) &= X_{17}(t) & \hat{\psi}(0,-2,t) &= X_{18}(t) \\
\hat{\psi}(-1,-2,t) &= X_{19}(t) & \hat{\psi}(-2,-1,t) &= X_{20}(t) \\
\hat{\psi}(-2,-2,t) &= X_{21}(t) & \hat{\psi}(-1,2,t) &= X_{22}(t) \\
\hat{\psi}(-2,1,t) &= X_{23}(t) & \hat{\psi}(-2,2,t) &= X_{24}(t)
\end{aligned}
\tag{3.60}
$$

Furthermore, similar to the conjugate relations defined in (3.17), we have the following additional relations

$$
\begin{aligned}
\hat{\psi}(2,-1) &= \hat{\psi}^*(-2,1) & \hat{\psi}(-2,0) &= \hat{\psi}^*(2,0) \\
\hat{\psi}(2,-2) &= \hat{\psi}^*(-2,2) & \hat{\psi}(-1,-2) &= \hat{\psi}^*(1,2) \\
\hat{\psi}(1,-2) &= \hat{\psi}^*(-1,2) & \hat{\psi}(-2,-1) &= \hat{\psi}^*(2,1) \\
\hat{\psi}(-2,-2) &= \hat{\psi}^*(2,2) & \hat{\psi}(0,-2) &= \hat{\psi}^*(0,2)
\end{aligned}
\tag{3.61}
$$

By substituting (3.60) and (3.61) into (3.13), we obtain the set of twenty four differential equations that result when truncating the Fourier spectrum at k, $l = \pm 2$.

$$\begin{aligned}
\frac{d}{dt}X_1 &= (-i\bar{u}+i\beta-\nu)X_1 \\
&\quad +\varepsilon[X_3X_6 - X_4X_2 + 3X_7X_{12} - 3X_8X_{15} + 6X_{13}X_{19} - 6X_{16}X_{22} \\
&\quad +2X_{11}X_{18} - 2X_{10}X_{14}] \\
\frac{d}{dt}X_2 &= -\nu X_2 + \varepsilon[X_1X_8 - X_3X_5 + 3X_4X_{22} - 3X_7X_{11} + 6X_{15}X_{24} - 6X_{13}X_{20} \\
&\quad +2X_9X_{23} - 2X_{12}X_{17}] \\
\frac{d}{dt}X_3 &= \left(-i\bar{u}+\frac{1}{2}i\beta-2\nu\right)X_3 + \frac{\varepsilon}{2}[4X_6X_{11} - 4X_5X_{12} + 4X_4X_{10} - 4X_8X_9] \\
\frac{d}{dt}X_4 &= \left(-i\bar{u}+\frac{1}{2}i\beta-2\nu\right)X_4 + \frac{\varepsilon}{2}[4X_5X_{15} - 4X_2X_{14} + 4X_7X_9 - 4X_3X_{18}] \\
\frac{d}{dt}X_5 &= (i\bar{u}-i\beta-\nu)X_5 \\
&\quad +\varepsilon[X_2X_7 - X_6X_8 + 3X_3X_{20} - 3X_4X_{23} + 6X_{11}X_{21} - 6X_{14}X_{24} \\
&\quad +2X_{10}X_{19} - 2X_{18}X_{22}] \\
\frac{d}{dt}X_6 &= -\nu X_6 + \varepsilon[X_4X_5 - X_1X_7 + 3X_8X_{14} - 3X_3X_{19} + 6X_{16}X_{23} - 6X_{12}X_{21} \\
&\quad +2X_{15}X_{17} - 2X_9X_{20}] \qquad (3.62) \\
\frac{d}{dt}X_7 &= \left(i\bar{u}-\frac{1}{2}i\beta-2\nu\right)X_7 + \frac{\varepsilon}{2}[4X_2X_{19} - 4X_1X_{20} + 4X_8X_{18} - 4X_4X_{17}] \\
\frac{d}{dt}X_8 &= \left(i\bar{u}-\frac{1}{2}i\beta-2\nu\right)X_8 + \frac{\varepsilon}{2}[4X_1X_{23} - 4X_6X_{22} + 4X_3X_{17} - 4X_7X_{10}] \\
\frac{d}{dt}X_9 &= \left(-2i\bar{u}+\frac{1}{2}i\beta-4\nu\right)X_9 + \frac{\varepsilon}{4}[16X_{18}X_{13} - 16X_{10}X_{16} + 8X_6X_{12} - 8X_2X_{15}] \\
\frac{d}{dt}X_{10} &= -4\nu X_{10} + \frac{\varepsilon}{4}[8X_{22}X_1 - 8X_5X_{11} + 16X_9X_{24} - 16X_{13}X_{17}] \\
\frac{d}{dt}X_{11} &= \left(-i\bar{u}+\frac{1}{5}i\beta-5\nu\right)X_{11} + \frac{\varepsilon}{5}[6X_1X_{10} - X_2X_3 + 4X_9X_{22} - 9X_8X_{12} \\
&\quad -14X_5X_{13}] \\
\frac{d}{dt}X_{12} &= \left(-2i\bar{u}+\frac{2}{5}i\beta-5\nu\right)X_{12} + \frac{\varepsilon}{5}[X_1X_3 + 9X_4X_{11} - 4X_{10}X_{15} - 6X_2X_9 \\
&\quad +14X_6X_{13}]
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}X_{13} &= \left(-2i\bar{u}+\frac{1}{4}i\beta-8\nu\right)X_{13}+\frac{\varepsilon}{8}[8X_1X_{11}-8X_2X_{12}] \\
\frac{d}{dt}X_{14} &= \left(-i\bar{u}+\frac{1}{5}i\beta-5\nu\right)X_{14}+\frac{\varepsilon}{5}[X_6X_4+2X_1X_{18}+14X_5X_{16}+9X_7X_{15} \\
&\quad -4X_{19}X_9] \\
\frac{d}{dt}X_{15} &= \left(-2i\bar{u}+\frac{2}{5}i\beta-5\nu\right)X_{15}+\frac{\varepsilon}{5}[4X_{18}X_{12}+6X_6X_9-14X_2X_{16}-9X_3X_{14} \\
&\quad -X_4X_1] \\
\frac{d}{dt}X_{16} &= \left(-2i\bar{u}+\frac{1}{4}i\beta-8\nu\right)X_{16}+\frac{\varepsilon}{8}[8X_6X_{15}-8X_1X_{14}] \\
\frac{d}{dt}X_{17} &= \left(2i\bar{u}-\frac{1}{2}i\beta-4\nu\right)X_{17}+\frac{\varepsilon}{4}[16X_{10}X_{21}-16X_{18}X_{24}+8X_2X_{20}-8X_6X_{23}] \\
\frac{d}{dt}X_{18} &= -4\nu X_{18}+\frac{\varepsilon}{4}[16X_{17}X_{16}-16X_9X_{21}+8X_5X_{14}-8X_1X_{19}] \qquad (3.63) \\
\frac{d}{dt}X_{19} &= \left(i\bar{u}-\frac{1}{5}i\beta-5\nu\right)X_{19}+\frac{\varepsilon}{5}[6X_5X_{18}-X_7X_6+4X_{14}X_{17}-9X_4X_{20} \\
&\quad -14X_1X_{21}] \\
\frac{d}{dt}X_{20} &= \left(2i\bar{u}-\frac{2}{5}i\beta-5\nu\right)X_{20}+\frac{\varepsilon}{5}[X_7X_5+9X_8X_{19}-4X_{18}X_{23}-6X_6X_{17} \\
&\quad +14X_2X_{21}] \\
\frac{d}{dt}X_{21} &= \left(2i\bar{u}-\frac{1}{4}i\beta-8\nu\right)X_{21}+\frac{\varepsilon}{8}[8X_5X_{19}-8X_6X_{20}] \\
\frac{d}{dt}X_{22} &= \left(i\bar{u}-\frac{1}{5}i\beta-5\nu\right)X_{22}+\frac{\varepsilon}{5}[X_8X_2-6X_5X_{10}+14X_1X_{24}+9X_3X_{23} \\
&\quad -4X_{11}X_{17}] \\
\frac{d}{dt}X_{23} &= \left(2i\bar{u}-\frac{2}{5}i\beta-5\nu\right)X_{23}+\frac{\varepsilon}{5}[4X_{10}X_{20}+6X_2X_{17}-14X_6X_{24}-9X_7X_{22} \\
&\quad -X_8X_5] \\
\frac{d}{dt}X_{24} &= \left(2i\bar{u}-\frac{1}{4}i\beta-8\nu\right)X_{24}+\frac{\varepsilon}{8}[8X_2X_{23}-8X_5X_{22}]
\end{aligned}
$$

As before, we seek solutions for each X_j in (3.62) and (3.63) as a perturbation series in powers of ε. The initial condition is still (3.31) which means that $X_j(0) = 1$ for $j = 1, ..., 8$ and $X_j(0) = 0$ for $j = 9, ..., 24$. Thus, at leading order, $X_j^{(0)}(t) = 0 \ \forall\ j = 9, ..., 24$ and the solutions for $j = 1, ..., 8$ are the same as those derived in Section 3.4 (see (3.43), (3.44), (3.37), (3.38), (3.45), (3.46), (3.39), and (3.40)).

At $O(\varepsilon)$, we have equations that are of the form (3.47) for X_j. In the equations corresponding to $j = 1, ..., 8$, all of the terms on the right-hand side involve products of the X_j's with $j = 9, ..., 24$ and thus are zero. The only nonzero terms are the ones involving products of the X_j's with $j = 1, ..., 8$. These terms are the same as those that appeared in Section 3.4 and thus the $O(\varepsilon)$ solutions for $j = 1, ..., 8$ remain unchanged. For the equations in (3.62) and (3.63) corresponding to $j = 9, ..., 24$, the only nonzero terms on the right-hand side of (3.47) result from j values of $11, 12, 14, 15$ and their complex conjugates $19, 20, 22, 23$. For $j = 9, 10, 13, 16, 17, 18, 21, 24$, at $O(\varepsilon)$ the equations are of the form

$$\frac{d}{dt}X_j^{(1)} = g_j X_j^{(1)} \tag{3.64}$$

where g_j is a complex constant, with a corresponding solution of $X_j(t) = 0$ after applying the initial condition $X_j^{(1)}(0) = 0$. For $j = 11, 12, 14, 15, 19, 20, 22, 23$, at $O(\varepsilon)$ the equations are of the form

$$\frac{d}{dt}X_j^{(1)}(t) - d_1 X_j^{(1)} = C_1 e^{d_2 t} e^{d_3 t} \tag{3.65}$$

where d_1, d_2, d_3 are complex constants that depend on j and C_1 is a real constant also dependent on j. Once again, the j subscripts have been omitted for convenience. To solve the above differential equation, we will use the integrating factor method just as before. The integrating factor is

$$I(t) = e^{-d_1 t} \tag{3.66}$$

After multiplying (3.65) by $I(t)$, integrating and simplifying, we obtain the following solution

$$X_j^{(1)}(t) = A'_j e^{(d_2+d_3)t} + M_j e^{d_1 t} \tag{3.67}$$

where $A'_j = \left(\frac{C_1}{d_2+d_3-d_1}\right)$ and $M_j = -A'_j$ is the constant obtained by applying the initial condition $X_j^{(1)}(0) = 0$. Consequently, the solutions for $j = 11, 12, 14, 15, 19, 20, 22, 23$

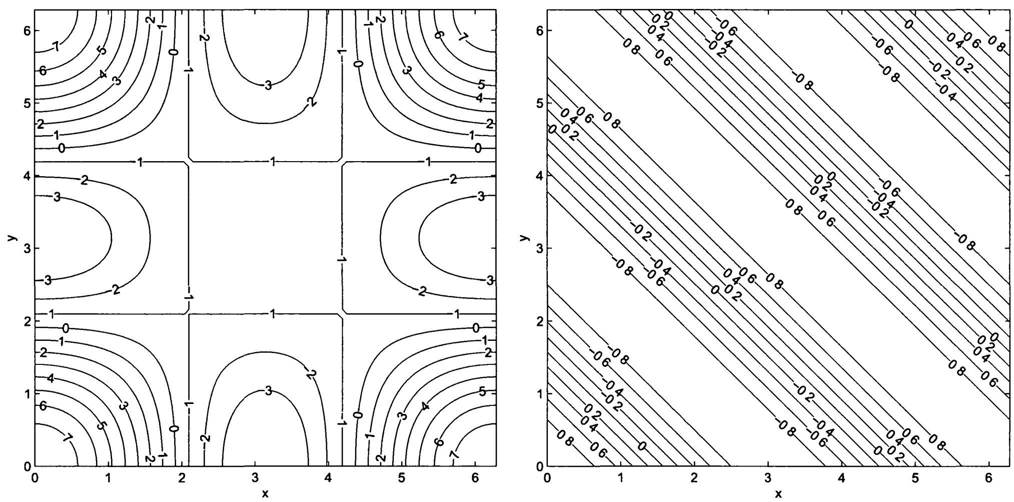
are

$$
\begin{aligned}
X_{11}^{(1)}(t) &= -C' e^{-3\nu t} e^{\imath\left(-\bar{u}+\frac{1}{2}\beta\right)t} + M e^{-5\nu t} e^{\imath\left(-\bar{u}+\frac{1}{5}\beta\right)t} \\
X_{12}^{(1)}(t) &= C'' e^{-3\nu t} e^{\imath\left(-2\bar{u}+\frac{3}{2}\beta\right)t} + M e^{-5\nu t} e^{\imath\left(-2\bar{u}+\frac{2}{5}\beta\right)t} \\
X_{14}^{(1)}(t) &= C' e^{-3\nu t} e^{\imath\left(-\bar{u}+\frac{1}{2}\beta\right)t} + M e^{-5\nu t} e^{\imath\left(-\bar{u}+\frac{1}{5}\beta\right)t} \\
X_{15}^{(1)}(t) &= -C'' e^{-3\nu t} e^{\imath\left(-2\bar{u}+\frac{3}{2}\beta\right)t} + M e^{-5\nu t} e^{\imath\left(-2\bar{u}+\frac{2}{5}\beta\right)t} \\
X_{19}^{(1)}(t) &= -B' e^{-3\nu t} e^{\imath\left(\bar{u}-\frac{1}{2}\beta\right)t} + M e^{-5\nu t} e^{\imath\left(\bar{u}-\frac{1}{5}\beta\right)t} \\
X_{20}^{(1)}(t) &= B'' e^{-3\nu t} e^{\imath\left(2\bar{u}-\frac{3}{2}\beta\right)t} + M e^{-5\nu t} e^{\imath\left(2\bar{u}-\frac{2}{5}\beta\right)t} \\
X_{22}^{(1)}(t) &= B' e^{-3\nu t} e^{\imath\left(\bar{u}-\frac{1}{2}\beta\right)t} + M e^{-5\nu t} e^{\imath\left(\bar{u}-\frac{1}{5}\beta\right)t} \\
X_{23}^{(1)}(t) &= -B'' e^{-3\nu t} e^{\imath\left(2\bar{u}-\frac{3}{2}\beta\right)t} + M e^{-5\nu t} e^{\imath\left(2\bar{u}-\frac{2}{5}\beta\right)t}
\end{aligned} \tag{3.68}
$$

where

$$
\begin{aligned}
C' &= \frac{1}{\left(10\nu + \frac{3}{2}i\beta\right)} & \qquad C'' &= \frac{1}{\left(10\nu + \frac{11}{2}i\beta\right)} \\
B' &= \frac{1}{\left(10\nu - \frac{3}{2}i\beta\right)} & \qquad B'' &= \frac{1}{\left(10\nu - \frac{11}{2}i\beta\right)}
\end{aligned} \tag{3.69}
$$

Note that with $\nu \neq 0$, the terms in (3.68) approach zero faster than the leading order terms $X_{\jmath}^{(0)}$ for $\jmath = 1, ..., 8$ because of the presence of $e^{-3\nu t}$ and $e^{-5\nu t}$. Also, higher wavenumber terms, $k, l > 2$, do not appear until higher order in ε, so for $\varepsilon \ll 1$ the terms we have worked out give a good approximation to the exact solution with error $\sim O(\varepsilon^2)$. The wavenumber components that are nonzero at $O(\varepsilon)$ are illustrated with a sketch in Figure 3.3. A square represents the original eight wavenumbers, a star represents the wavenumbers which have nonzero terms at $O(\varepsilon)$, and no label marker represents a wavenumber with no $O(\varepsilon)$ terms.



(a) (b)

(c) (d)

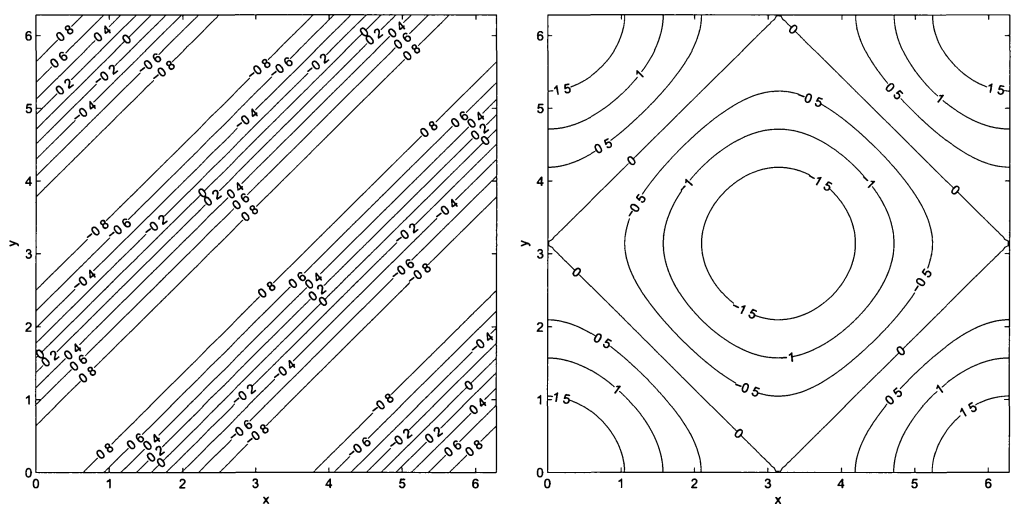


Figure 3.1: (a) Contour Plot of the initial condition for the β-plane problem: $f(x,y) = 2[\cos(x+y) + \cos x + \cos y + \cos(x-y)]$. The initial condition for the Boussinesq problem is the same except $f = f(x,z)$. (b) Contour plot of $f(x,y) = \cos(x+y)$ (c) Contour plot of $f(x,y) = \cos(x-y)$ (d) Contour plot of $f(x,y) = \cos x + \cos y$

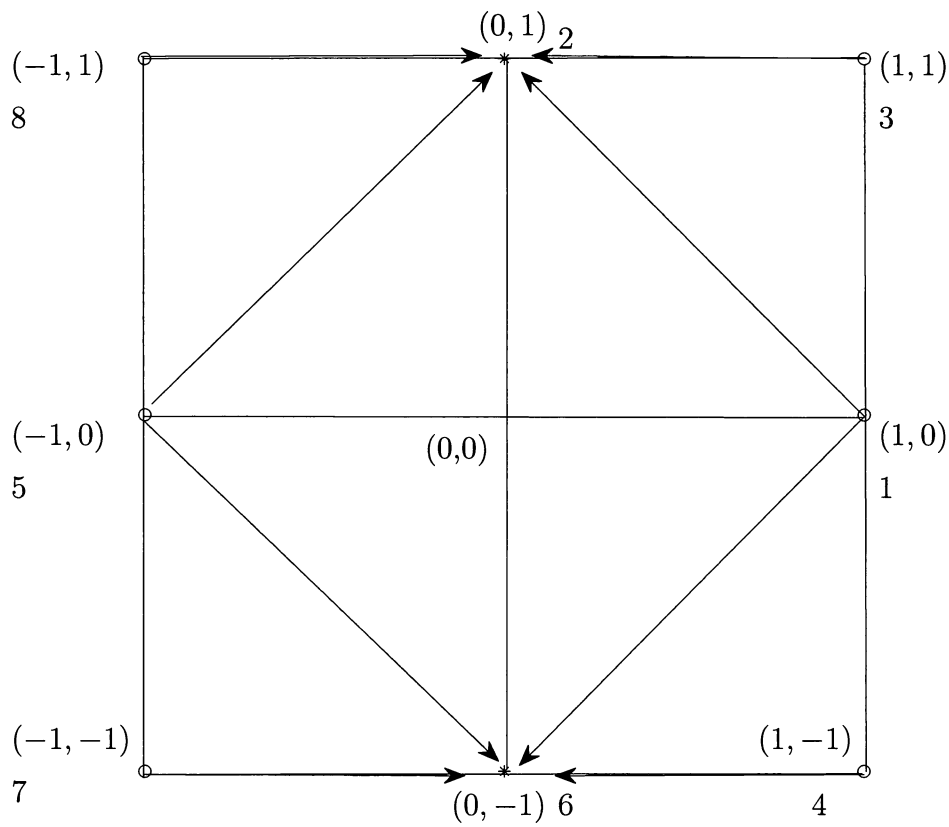


Figure 3.2: Plot showing how $O(\varepsilon)$ terms are generated for the original eight wavenumbers in the β-plane problem. Of these eight wavenumbers, the only ones that give nonzero contributions at $O(\varepsilon)$ are the wavenumbers (0,1) and (0,-1) which are denoted by stars. The (0,1) component is generated by interactions between the $O(1)$ components corresponding to (-1,1), (1,1), (-1,0), and (1,0) as shown by the arrows. Similarly, the (0,-1) component is generated by interactions between the $O(1)$ components corresponding to (1,-1), (-1,0), (1,0), and (-1,-1). The numerical labels correspond to the numbering of the wavenumbers. For instance, 1 represents X_1. The $O(\varepsilon)$ terms for the ψ (X) equations are the same for the Boussinesq problem so this figure applies to that problem as well.

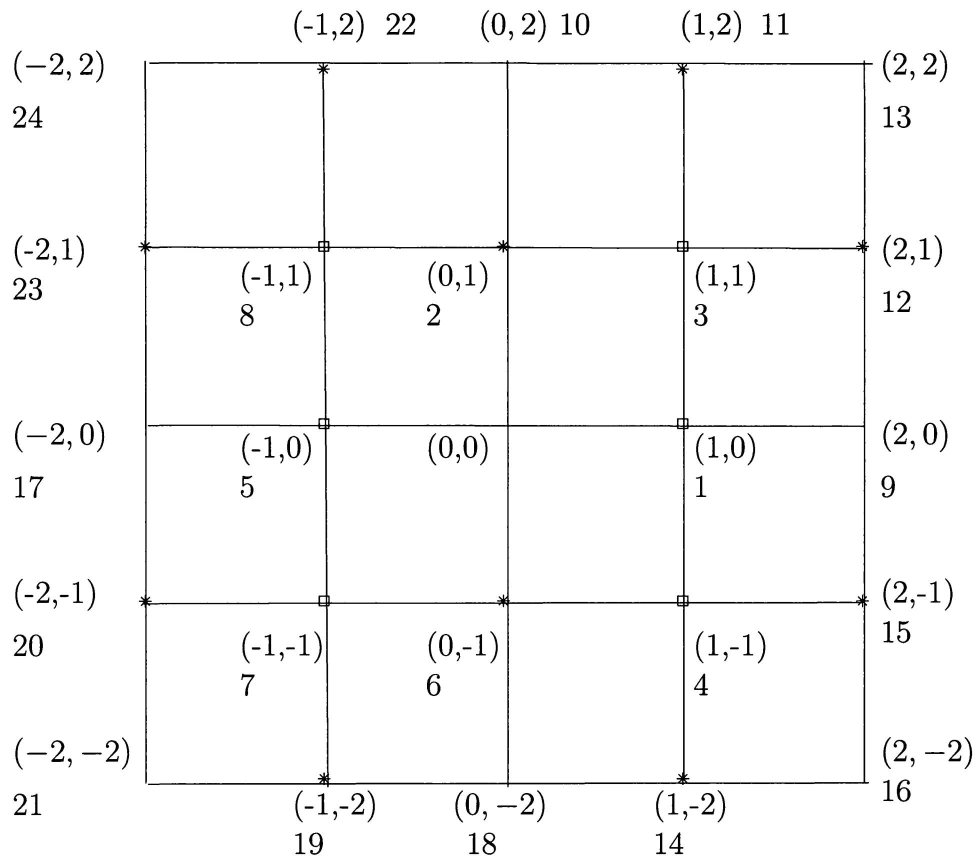


Figure 3.3: Plot showing which components are nonzero at $O(\varepsilon)$ for the higher order wavenumber problem for the β-plane case. The nonzero components are denoted by stars, the original eight wavenumber components are denoted with a square, and the components with no contribution at $O(\varepsilon)$ are not given a label marker. The numerical labels correspond to the numbering of the wavenumbers. For instance, 22 represents X_{22}. This diagram can also be used for the ψ (X) equations in the Boussinesq problem.

Chapter 4

Initial Value Problem for Gravity Waves in a Boussinesq Fluid

4.1 Fourier Analysis for the Boussinesq Problem

The other problem studied here is that of internal gravity waves (or buoyancy waves) in a two-dimensional Boussinesq fluid. Atmospheric gravity waves exist when the atmosphere is stratified in such a way that when a fluid is displaced vertically it will experience buoyancy oscillations, since the buoyancy force is the restoring force that generates gravity waves. In the ocean, where low- density water lies above high-density water, internal gravity waves propagate along the boundary. This problem is of particular importance in the context of atmospheric and oceanic modelling given the fact that, in the real atmosphere and ocean, gravity waves are small-scale (wavelengths generally < 100 kilometers) compared with Rossby waves (wavelengths on the planetary scale of thousands of kilometers). Thus, it is more difficult to simulate gravity waves in large-scale models.

We begin our analysis of the Boussinesq problem by deriving equations for the perturbed problem following a process similar to that used for the β-plane configuration. As before, we begin with the governing equations of motion from Appendix C and Chapter 2 for a two-dimensional fluid in rectangular coordinates, but this time we use x and z instead of x and y. Under these conditions, $\vec{u} = (u, w)$, $\vec{g} = (0, -g)$, and $\rho = \rho(x, z, t)$. As we saw in previous chapters, the continuity equation becomes $\nabla \cdot \vec{u} = 0$ under the Boussinesq approximation and in two dimensions is

$$u_x + w_z = 0 \tag{4.1}$$

The Boussinesq approximation applied to the momentum equation leads to the following set of equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{4.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0} + \nu\nabla^2 w \tag{4.3}$$

where $\nu = \frac{\mu}{\rho_0}$. Notice this time that the w equation has a gravity term, whereas in the β-plane problem there weren't any gravitational terms. Now, applying the Boussinesq approximation to the energy equation generates the following equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \kappa\nabla^2 T \tag{4.4}$$

Analogous to what we did for the Rossby wave equations (2.30) and (2.31) in Chapter 3, we differentiate equation (4.2) with respect to z and equation (4.3) with respect to x, which generates the following equations

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial t}\right) + u\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) + \nu\frac{\partial}{\partial z}(\nabla^2 u) \tag{4.5}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) + w\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) &= -\frac{1}{\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) - \frac{1}{\rho_0}\frac{\partial}{\partial x}(\rho g) \\ &\quad + \nu\frac{\partial}{\partial x}(\nabla^2 w)\end{aligned} \tag{4.6}$$

We now define a streamfunction Ψ such that

$$u = -\Psi_z, \quad w = \Psi_x \tag{4.7}$$

Subtracting equation (4.5) from equation (4.6) and applying the streamfunction (4.7) produces the following equation

$$\frac{\partial}{\partial t}\nabla^2\Psi + \Psi_x\nabla^2\Psi_z - \Psi_z\nabla^2\Psi_x + \frac{g}{\rho_0}\rho_x - \nu\nabla^4\Psi = 0 \tag{4.8}$$

We can write ρ_x in terms of T since under the Boussinesq approximation there is a linear relation between ρ and T (see 2.15). This means that $\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial T}\frac{\partial T}{\partial x}$, where $\frac{\partial \rho}{\partial T}$ is a constant which can either be set to 1 or be absorbed into ρ_0. As a result, equation (4.8) becomes

$$\frac{\partial}{\partial t}\nabla^2\Psi + \Psi_x\nabla^2\Psi_z - \Psi_z\nabla^2\Psi_x + \frac{g}{\rho_0}T_x - \nu\nabla^4\Psi = 0 \tag{4.9}$$

We then apply the streamfunction to equation (4.4) as follows

$$\frac{\partial T}{\partial t} + \Psi_x T_z - \Psi_z T_x - \kappa\nabla^2 T = 0 \tag{4.10}$$

We consider equations (4.9) and (4.10) (in nondimensional variables and parameters) in the domain where $0 < x < 2\pi$, $0 < z < 2\pi$, and $t > 0$. The process of nondimensionalization is analogous to that described in Chapter 3. We make use of reference length scales L_x, L_z, a reference velocity U, a reference time scale $\frac{L_x}{U}$, a reference streamfunction $L_z U$, a reference value for the temperature, and reference values of the dimensional viscosity and heat conduction constants. In the nondimensionalized equations, the Laplacian operator is

$$\nabla^2 = \frac{L_z^2}{L_x^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \tag{4.11}$$

and as before we can define $\delta = \frac{L_z^2}{L_x^2}$; the square of the aspect ratio, and consider the case when $\delta = 1$.

We introduce a temperature perturbation by following the notation of Saltzman [11], we let

$$T(x,z,t) = \bar{T}(z,t) + \varepsilon T'(x,z,t) \tag{4.12}$$

where $\bar{T}(z,t)$ is an average in the x-direction, $T'(x,z,t)$ is a departure from T, and $\varepsilon << 1$. In addition, $\bar{T}(z,t)$ can be expanded as a linear variation between the upper and lower boundaries and a departure from the linear variation as follows

$$\bar{T}(z,t) = \bar{T}(0,t) - \frac{\Delta T}{H}z + \varepsilon\bar{T}''(z,t) \tag{4.13}$$

where $\Delta T = \bar{T}(0) - \bar{T}(H)$, H is the height of the fluid, $\bar{T}(0,t) - \frac{\Delta T}{H} z$ is the linear variation between the boundaries, and $\bar{T}''(z,t)$ is the departure from the linear variation. After applying (4.13) to (4.12), we obtain the following expression for $T(x,z,t)$

$$T(x,z,t) = \bar{T}(0,t) - \frac{\Delta T}{H} z + \varepsilon \bar{T}''(z,t) + \varepsilon T'(x,z,t) \tag{4.14}$$

We now let

$$\theta(x,z,t) = \bar{T}''(z,t) + T'(x,z,t)$$

and replace this in (4.14) to obtain

$$T(x,z,t) = \bar{T}(0,t) - \frac{\Delta T}{H} z + \varepsilon \theta(x,z,t) \tag{4.15}$$

Similar to what we did in the β-plane problem, we let

$$\Psi(x,z,t) = \bar{\psi}(z) + \varepsilon \psi(x,z,t) \tag{4.16}$$

where $\bar{\psi}(z) = -\bar{u}z$ and $\bar{u}$ is called the mean velocity and is constant. Also, $\varepsilon << 1$. Substituting (4.15) and (4.16) into (4.9) generates the first perturbed equation

$$\frac{\partial}{\partial t}\nabla^2\psi + \bar{u}\nabla^2\psi_x + \frac{g}{\rho_0}\theta_x - \nu\nabla^4\psi + \varepsilon(\psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x) = 0 \tag{4.17}$$

Likewise, substituting (4.15) and (4.16) into (4.10) generates the second perturbed equation

$$\frac{\partial \theta}{\partial t} - \frac{\Delta T}{H}\psi_x + \bar{u}\theta_x - \kappa\nabla^2\theta + \varepsilon(\psi_x\theta_z - \psi_z\theta_x) = 0 \tag{4.18}$$

We have now derived two equations (4.17) and (4.18), which describe the evolution of the streamfunction and temperature perturbations. We will proceed to solve these equations using similar procedures to those outlined in the β-plane problem.

Note that the two PDEs (4.17) and (4.18) are coupled; in (4.17) there is a term involving θ_x and in (4.18) there is a term involving ψ_x as well as nonlinear terms involving products of ψ and θ derivatives. Saltzman's procedure for introducing the temperature perturbation results in constants $\frac{g}{\rho_0}$ and $-\frac{\Delta T}{H}$ multiplying θ_x and ψ_x in equations (4.17) and (4.18). Thus, both equations have constant coefficients throughout which simplifies the process of taking Fourier transforms in both x and y directions.

We consider (4.17) and (4.18) in the rectangular domain $0 < x < 2\pi$, $0 < z < 2\pi$ with periodic boundary conditions at $x = 0, 2\pi$ and $z = 0, 2\pi$, and we specify the following initial conditions which generate Fourier modes with wavenumber pairs $(k, m) = (\pm 1, 0), (0, \pm 1), \pm(1, 1), \pm(1, -1)$.

$$\psi(x, z, 0) = 2[\cos x + \cos z + \cos(x + z) + \cos(x - z)] \tag{4.19}$$

$$\theta(x, z, 0) = 2[\cos x + \cos z + \cos(x + z) + \cos(x - z)] \tag{4.20}$$

We now perform a Fourier decomposition of both ψ and θ to transform the perturbed equations (4.17) and (4.18) into ODEs. We define the inverse Fourier Transform of ψ and θ as follows

$$\psi(x, z, t) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \hat{\psi}(k, m, t) e^{\imath(kx+mz)} \tag{4.21}$$

$$\theta(x, z, t) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \hat{\theta}(k, m, t) e^{\imath(kx+mz)} \tag{4.22}$$

where k and m are the wave numbers in the x and z directions, respectively. The complex Fourier coefficients $\hat{\psi}$ and $\hat{\theta}$ are defined by (3.11) with l replaced with m and y replaced with z. Taking a Fourier Transform of equation (4.17) produces the following ODE

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\psi}(k, m, t) &= -[i\bar{u}k + \nu(k^2 + m^2)]\hat{\psi}(k, m, t) + \frac{igk}{\rho_0(k^2 + m^2)}\hat{\theta}(k, m, t) \\ &+ \frac{\varepsilon}{k^2 + m^2}\left[\sum_{\bar{k}}\sum_{\bar{m}}(\bar{k}^2 + \bar{m}^2)(\bar{m}k - m\bar{k})\hat{\psi}(k - \bar{k}, m - \bar{m}, t)\hat{\psi}(\bar{k}, \bar{m}, t)\right]\end{aligned} \tag{4.23}$$

where $(k, m) \neq (0, 0)$. We have made use of the convolution defined in (3.12) to obtain (4.23). Equivalently, taking a Fourier Transform of (4.18) leads to the following ODE

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\theta}(k, m, t) &= i\frac{\Delta T}{H}k\hat{\psi}(k, m, t) - [i\bar{u}k + \kappa(k^2 + m^2)]\hat{\theta}(k, m, t) \\ &+ \varepsilon\left[\sum_{\bar{k}}\sum_{\bar{m}}(\bar{m}k - m\bar{k})\hat{\psi}(k - \bar{k}, m - \bar{m}, t)\hat{\theta}(\bar{k}, \bar{m}, t)\right], \quad (k, m) \neq (0, 0)\end{aligned} \tag{4.24}$$

As we did before, we truncate the spectrum and only allow values of -1, 1 and 0 for k, m, $\bar{k}$, and $\bar{m}$. Using similar arguments as before in the β-plane problem, we note that both $\hat{\psi}(0,0,t) = 0$ and $\hat{\theta}(0,0,t) = 0$. Truncation of the Fourier spectrum leads to the following eight differential equations for $\hat{\psi}$.

$$
\begin{aligned}
\frac{\partial}{\partial t}\hat{\psi}(1,0,t) &= (-i\bar{u}-\nu)\hat{\psi}(1,0,t) + \frac{ig}{\rho_0}\hat{\theta}(1,0,t) \\
&\quad +\varepsilon[\hat{\psi}(1,1,t)\hat{\psi}(0,-1,t) - \hat{\psi}(1,-1,t)\hat{\psi}(0,1,t)] \\
\frac{\partial}{\partial t}\hat{\psi}(0,1,t) &= -\nu\hat{\psi}(0,1,t) + \varepsilon[\hat{\psi}(1,0,t)\hat{\psi}(-1,1,t) - \hat{\psi}(1,1,t)\hat{\psi}(-1,0,t)] \\
\frac{\partial}{\partial t}\hat{\psi}(1,1,t) &= (-i\bar{u}-2\nu)\hat{\psi}(1,1,t) + \frac{ig}{2\rho_0}\hat{\theta}(1,1,t) \\
\frac{\partial}{\partial t}\hat{\psi}(1,-1,t) &= (-i\bar{u}-2\nu)\hat{\psi}(1,-1,t) + \frac{ig}{2\rho_0}\hat{\theta}(1,-1,t) \\
\frac{\partial}{\partial t}\hat{\psi}(-1,0,t) &= (i\bar{u}-\nu)\hat{\psi}(-1,0,t) - \frac{ig}{\rho_0}\hat{\theta}(-1,0,t) \\
&\quad +\varepsilon[\hat{\psi}(0,1,t)\hat{\psi}(-1,-1,t) - \hat{\psi}(0,-1,t)\hat{\psi}(-1,1,t)] \\
\frac{\partial}{\partial t}\hat{\psi}(0,-1,t) &= -\nu\hat{\psi}(0,-1,t) + \varepsilon[\hat{\psi}(1,-1,t)\hat{\psi}(-1,0,t) - \hat{\psi}(1,0,t)\hat{\psi}(-1,-1,t)] \\
\frac{\partial}{\partial t}\hat{\psi}(-1,-1,t) &= (i\bar{u}-2\nu)\hat{\psi}(-1,-1,t) - \frac{ig}{2\rho_0}\hat{\theta}(-1,-1,t) \\
\frac{\partial}{\partial t}\hat{\psi}(-1,1,t) &= (i\bar{u}-2\nu)\hat{\psi}(-1,1,t) - \frac{ig}{2\rho_0}\hat{\theta}(-1,1,t)
\end{aligned}
\tag{4.25}
$$

Note as before, the first, second, fifth and sixth equations are nonlinear and the others are linear. Also, the second equation does not have a θ term in the linear part of the equation. Now let

$$
\begin{aligned}
\hat{\theta}(1,0,t) &= Y_1(t) & \hat{\theta}(0,1,t) &= Y_2(t) \\
\hat{\theta}(1,1,t) &= Y_3(t) & \hat{\theta}(1,-1,t) &= Y_4(t) \\
\hat{\theta}(-1,0,t) &= Y_5(t) & \hat{\theta}(0,-1,t) &= Y_6(t) \\
\hat{\theta}(-1,-1,t) &= Y_7(t) & \hat{\theta}(-1,1,t) &= Y_8(t)
\end{aligned}
\tag{4.26}
$$

Making use of (3.14) with $\hat{\psi}$ replaced with $\hat{\theta}$, we obtain the following conjugate relations

$$\hat{\theta}(-1,-1,t) = \hat{\theta}^*(1,1,t) \qquad \hat{\theta}(-1,0,t) = \hat{\theta}^*(1,0,t) \tag{4.27}$$
$$\hat{\theta}(0,-1,t) = \hat{\theta}^*(0,1,t) \qquad \hat{\theta}(1,-1,t) = \hat{\theta}^*(-1,1,t)$$

As a result,

$$\begin{aligned}
\hat{\theta}(-1,0,t) &= \hat{\theta}^*(1,0,t) = Y_5(t) \\
\hat{\theta}(0,-1,t) &= \hat{\theta}^*(0,1,t) = Y_6(t) \\
\hat{\theta}(-1,-1,t) &= \hat{\theta}^*(1,1,t) = Y_7(t) \\
\hat{\theta}(-1,1,t) &= \hat{\theta}^*(1,-1,t) = Y_8(t)
\end{aligned} \tag{4.28}$$

where $\hat{\theta}^*$ is the complex conjugate of $\hat{\theta}$. Now making use of (4.26), (4.27), (4.28), (3.16) and the conjugate relations in (3.17) and (3.18), we arrive at the following differential equations written in terms of X_j and Y_j, where $j = 1, ..., 8$.

$$\begin{aligned}
\frac{d}{dt}X_1 &= (-i\bar{u} - \nu)X_1 + \frac{ig}{\rho_0}Y_1 + \varepsilon[X_3X_6 - X_4X_2] \\
\frac{d}{dt}X_2 &= -\nu X_2 + \varepsilon[X_1X_8 - X_3X_5] \\
\frac{d}{dt}X_3 &= (-i\bar{u} - 2\nu)X_3 + \frac{ig}{2\rho_0}Y_3 \\
\frac{d}{dt}X_4 &= (-i\bar{u} - 2\nu)X_4 + \frac{ig}{2\rho_0}Y_4 \\
\frac{d}{dt}X_5 &= (i\bar{u} - \nu)X_5 - \frac{ig}{\rho_0}Y_5 + \varepsilon[X_2X_7 - X_6X_8] \\
\frac{d}{dt}X_6 &= -\nu X_6 + \varepsilon[X_4X_5 - X_1X_7] \\
\frac{d}{dt}X_7 &= (i\bar{u} - 2\nu)X_7 - \frac{ig}{2\rho_0}Y_7 \\
\frac{d}{dt}X_8 &= (i\bar{u} - 2\nu)X_8 - \frac{ig}{2\rho_0}Y_8
\end{aligned} \tag{4.29}$$

Note that the last four equations are the complex conjugates of the first four. Similarly, truncation of the Fourier spectrum leads to the following eight differential equations for $\hat{\theta}$.

$$
\begin{aligned}
\frac{\partial}{\partial t}\hat{\theta}(1,0,t) &= i\frac{\Delta T}{H}\hat{\psi}(1,0,t) - (i\bar{u}+\kappa)\hat{\theta}(1,0,t) \\
&\quad +\varepsilon[\hat{\psi}(1,-1,t)\hat{\theta}(0,1,t) - \hat{\psi}(1,1,t)\hat{\theta}(0,-1,t) + \hat{\psi}(0,-1,t)\hat{\theta}(1,1,t) \\
&\quad -\hat{\psi}(0,1,t)\hat{\theta}(1,-1,t)] \\
\frac{\partial}{\partial t}\hat{\theta}(0,1,t) &= -\kappa\hat{\theta}(0,1,t) \\
&\quad +\varepsilon[\hat{\psi}(1,1,t)\hat{\theta}(-1,0,t) + \hat{\psi}(1,0,t)\hat{\theta}(-1,1,t) - \hat{\psi}(-1,1,t)\hat{\theta}(1,0,t) \\
&\quad -\hat{\psi}(-1,0,t)\hat{\theta}(1,1,t)] \\
\frac{\partial}{\partial t}\hat{\theta}(1,1,t) &= i\frac{\Delta T}{H}\hat{\psi}(1,1,t) - (i\bar{u}+2\kappa)\hat{\theta}(1,1,t) \\
&\quad +\varepsilon[\hat{\psi}(1,0,t)\hat{\theta}(0,1,t) - \hat{\psi}(0,1,t)\hat{\theta}(1,0,t)] \\
\frac{\partial}{\partial t}\hat{\theta}(1,-1,t) &= i\frac{\Delta T}{H}\hat{\psi}(1,-1,t) - (i\bar{u}+2\kappa)\hat{\theta}(1,-1,t) \\
&\quad +\varepsilon[\hat{\psi}(0,-1,t)\hat{\theta}(1,0,t) - \hat{\psi}(1,0,t)\hat{\theta}(0,-1,t)] \\
\frac{\partial}{\partial t}\hat{\theta}(-1,0,t) &= -i\frac{\Delta T}{H}\hat{\psi}(-1,0,t) - (-i\bar{u}+\kappa)\hat{\theta}(-1,0,t) \qquad (4.30) \\
&\quad +\varepsilon[\hat{\psi}(0,1,t)\hat{\theta}(-1,-1,t) + \hat{\psi}(-1,1,t)\hat{\theta}(0,-1,t) - \hat{\psi}(0,-1,t)\hat{\theta}(-1,1,t) \\
&\quad -\hat{\psi}(-1,-1,t)\hat{\theta}(0,1,t)] \\
\frac{\partial}{\partial t}\hat{\theta}(0,-1,t) &= -\kappa\hat{\theta}(0,-1,t) \\
&\quad +\varepsilon[\hat{\psi}(-1,-1,t)\hat{\theta}(1,0,t) + \hat{\psi}(-1,0,t)\hat{\theta}(1,-1,t) - \hat{\psi}(1,-1,t)\hat{\theta}(-1,0,t) \\
&\quad -\hat{\psi}(1,0,t)\hat{\theta}(-1,-1,t)] \\
\frac{\partial}{\partial t}\hat{\theta}(-1,-1,t) &= -i\frac{\Delta T}{H}\hat{\psi}(-1,-1,t) - (-i\bar{u}+2\kappa)\hat{\theta}(-1,-1,t) \\
&\quad +\varepsilon[\hat{\psi}(-1,0,t)\hat{\theta}(0,-1,t) - \hat{\psi}(0,-1,t)\hat{\theta}(-1,0,t)] \\
\frac{\partial}{\partial t}\hat{\theta}(-1,1,t) &= -i\frac{\Delta T}{H}\hat{\psi}(-1,1,t) - (-i\bar{u}+2\kappa)\hat{\theta}(-1,1,t) \\
&\quad +\varepsilon[\hat{\psi}(0,1,t)\hat{\theta}(-1,0,t) - \hat{\psi}(-1,0,t)\hat{\theta}(0,1,t)]
\end{aligned}
$$

Note that in contrast to the $\hat{\psi}$ equations all of the $\hat{\theta}$ equations are nonlinear. As a result, the Boussinesq problem is more complicated and very different qualitatively than the β-plane problem. Also note that the second and sixth equations do not have a $\hat{\psi}$ term in the linear part of the equation. Now, rewriting the above equations in terms of X_j and Y_j generates the following set of differential equations

$$\begin{aligned}
\frac{d}{dt}Y_1 &= i\frac{\Delta T}{H}X_1 - (i\bar{u} + \kappa)Y_1 + \varepsilon[X_4Y_2 - X_2Y_4 + X_6Y_3 - X_3Y_6] \\
\frac{d}{dt}Y_2 &= -\kappa Y_2 + \varepsilon[X_3Y_5 - X_5Y_3 + X_1Y_8 - X_8Y_1] \\
\frac{d}{dt}Y_3 &= i\frac{\Delta T}{H}X_3 - (i\bar{u} + 2\kappa)Y_3 + \varepsilon[X_1Y_2 - X_2Y_1] \\
\frac{d}{dt}Y_4 &= i\frac{\Delta T}{H}X_4 - (i\bar{u} + 2\kappa)Y_4 + \varepsilon[X_6Y_1 - X_1Y_6] \\
\frac{d}{dt}Y_5 &= -i\frac{\Delta T}{H}X_5 - (-i\bar{u} + \kappa)Y_5 + \varepsilon[X_2Y_7 - X_7Y_2 + X_8Y_6 - X_6Y_8] \\
\frac{d}{dt}Y_6 &= -\kappa Y_6 + \varepsilon[X_7Y_1 - X_1Y_7 + X_5Y_4 - X_4Y_5] \\
\frac{d}{dt}Y_7 &= -i\frac{\Delta T}{H}X_7 - (-i\bar{u} + 2\kappa)Y_7 + \varepsilon[X_5Y_6 - X_6Y_5] \\
\frac{d}{dt}Y_8 &= -i\frac{\Delta T}{H}X_8 - (-i\bar{u} + 2\kappa)Y_8 + \varepsilon[X_2Y_5 - X_5Y_2]
\end{aligned} \tag{4.31}$$

4.2 Solution of the Linear Boussinesq Problem

In this section, we will find the solution of the linear Boussinesq problem just as we did for the β-plane problem. To do so, we set $\varepsilon = 0$ in (4.29) and (4.31). Note that, in doing so, the first four equations for both $\frac{d}{dt}X_j$ and $\frac{d}{dt}Y_j$ only depend on X_1 through X_4 and Y_1 through Y_4. Thus, we only need to find the solution for these equations since the last four equations are the complex conjugates of these first four equations. Now, the linearized equations are of the form

$$\frac{d}{dt}X_j = p_jX_j + q_jY_j \tag{4.32}$$

$$\frac{d}{dt}Y_j = r_jX_j + s_jY_j \tag{4.33}$$

for $j = 1, 3, 4$ and of the form

$$\frac{d}{dt}X_j = p_jX_j \tag{4.34}$$

$$\frac{d}{dt}Y_j = s_jY_j \tag{4.35}$$

for $j = 2$, where p_j, q_j, r_j, s_j are complex-valued constants. The above two equations (4.34) and (4.35) have the following exponential solutions

$$X_2(t) = e^{-\nu t} \tag{4.36}$$

$$Y_2(t) = e^{-\kappa t} \tag{4.37}$$

Note that the initial conditions $X_2(0) = 1$ and $Y_2(0) = 1$ were applied to obtain the above solutions.

For $j = 1, 3, 4$, the corresponding X_j and Y_j equations are coupled and thus each pair can be written as a system as follows

$$\frac{d}{dt}\vec{Z}_j(t) = A\vec{Z}_j \tag{4.38}$$

where

$$\vec{Z}_j = \begin{pmatrix} X_j \\ Y_j \end{pmatrix} \text{ and } A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{4.39}$$

To determine the solution of the system defined above, we need to find both the eigenvalues and eigenvectors. Note that we have dropped the j subscripts for simplicity. The general solution is of the form

$$\vec{Z}_j(t) = C_1\vec{v}_1 e^{\lambda^+ t} + C_2\vec{v}_2 e^{\lambda^- t} \tag{4.40}$$

where C_1 and C_2 are the constants to be determined after applying the initial conditions, λ_+ and λ_- are the eigenvalues, and $\vec{v}_1$ and $\vec{v}_2$ are the eigenvectors. Recall that the characteristic polynomial is

$$\lambda^2 - tr(A)\lambda + det(A) = 0 \tag{4.41}$$

where $tr(A)$ is the trace and $det(A)$ is the determinant of the matrix A. In our case,

$$tr(A) = p + s \;\text{ and }\; det(A) = ps - qr \tag{4.42}$$

Thus, our characteristic polynomial is

$$\lambda^2 - (p+s)\lambda + (ps - qr) = 0 \tag{4.43}$$

which has roots, or eigenvalues, of the form

$$\lambda^{\pm} = \frac{p+s\pm\sqrt{(p-s)^2+4qr}}{2} \tag{4.44}$$

To find the eigenvectors, we search for a vector $\vec{v}$ that satisfies

$$(A-\lambda I)\vec{v} = 0 \tag{4.45}$$

Let us consider λ^{+} from (4.44). Then, we want to find $\vec{v}_1 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ that satisfies the following equation

$$\begin{pmatrix} \frac{p-s-\sqrt{(p-s)^2+4qr}}{2} & q \\ r & \frac{-p+s-\sqrt{(p-s)^2+4qr}}{2} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.46}$$

By choosing $w_1 = 2q$, we find that $w_2 = -p+s+\sqrt{(p-s)^2+4qr}$. Thus,

$$\vec{v}_1 = \begin{pmatrix} 2q \\ -p+s+\sqrt{(p-s)^2+4qr} \end{pmatrix} \tag{4.47}$$

Similarly, we find that $\vec{v}_2$ has the following form

$$\vec{v}_2 = \begin{pmatrix} 2q \\ -p+s-\sqrt{(p-s)^2+4qr} \end{pmatrix} \tag{4.48}$$

Now, after substituting the eigenvalues and eigenvectors into (4.40), applying the initial condition $\vec{Z}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and a bit of algebraic manipulation, we find the constants to be

$$C_1 = \frac{p-s+\sqrt{(p-s)^2+4qr}+2q}{4q\sqrt{(p-s)^2+4qr}} \tag{4.49}$$

$$C_2 = \frac{-p+s+\sqrt{(p-s)^2+4qr}-2q}{4q\sqrt{(p-s)^2+4qr}} \tag{4.50}$$

For simplicity, we will define χ to be

$$\chi = \sqrt{(p-s)^2+4qr} \tag{4.51}$$

Putting all of this together, we arrive at the solution to the initial value problem

$$\vec{Z}_j(t) = \frac{1}{4q\chi}\left[g_1 e^{\left(\frac{p+s+\chi}{2}\right)t} + g_2 e^{\left(\frac{p+s-\chi}{2}\right)t}\right] \tag{4.52}$$

where $g_1 = \begin{pmatrix} 2q(p-s+\chi+2q) \\ (p-s+\chi+2q)(-p+s+\chi) \end{pmatrix}$, $g_2 = \begin{pmatrix} 2q(-p+s+\chi-2q) \\ (-p+s+\chi-2q)(-p+s-\chi) \end{pmatrix}$

Now that we have found the solution for arbitrary values of p, q, r, and s, we can easily find the solutions for $j = 1, 3, 4$. The eigenvalues are

$$\lambda_1^{\pm} = \frac{-2i\bar{u} - \nu - \kappa \pm \sqrt{(\kappa-\nu)^2 - \frac{4g\Delta T}{\rho_0 H}}}{2} \tag{4.53}$$

$$\lambda_3^{\pm} = -i\bar{u} - \nu - \kappa \pm \sqrt{(\kappa-\nu)^2 - \frac{g\Delta T}{2\rho_0 H}} \tag{4.54}$$

$$\lambda_4^{\pm} = -i\bar{u} - \nu - \kappa \pm \sqrt{(\kappa-\nu)^2 - \frac{g\Delta T}{2\rho_0 H}} \tag{4.55}$$

and the corresponding solutions are

$$\vec{Z}_1(t) = g_3[\vec{U}_1 e^{(\lambda_1^+)t} + \vec{U}_2 e^{(\lambda_1^-)t}] \tag{4.56}$$

$$\vec{Z}_3(t) = g_4[\vec{U}_3 e^{(\lambda_3^+)t} + \vec{U}_4 e^{(\lambda_3^-)t}] \tag{4.57}$$

$$\vec{Z}_4(t) = g_4[\vec{U}_3 e^{(\lambda_4^+)t} + \vec{U}_4 e^{(\lambda_4^-)t}] \tag{4.58}$$

where

$$
\begin{aligned}
g_3 &= \frac{\rho_0}{4ig\sqrt{(\kappa-\nu)^2 - \frac{4g\Delta T}{\rho_0 H}}} \\
\vec{U}_1 &= \left[\frac{\frac{2ig}{\rho_0}\left(\kappa-\nu+\sqrt{(\kappa-\nu)^2-\frac{4g\Delta T}{\rho_0 H}}+\frac{2ig}{\rho_0}\right)}{\left(\kappa-\nu+\sqrt{(\kappa-\nu)^2-\frac{4g\Delta T}{\rho_0 H}}+\frac{2ig}{\rho_0}\right)\left(\nu-\kappa+\sqrt{(\kappa-\nu)^2-\frac{4g\Delta T}{\rho_0 H}}\right)}\right] \\
\vec{U}_2 &= \left[\frac{\frac{2ig}{\rho_0}\left(\nu-\kappa+\sqrt{(\kappa-\nu)^2-\frac{4g\Delta T}{\rho_0 H}}-\frac{2ig}{\rho_0}\right)}{\left(\nu-\kappa+\sqrt{(\kappa-\nu)^2-\frac{4g\Delta T}{\rho_0 H}}-\frac{2ig}{\rho_0}\right)\left(\nu-\kappa+\sqrt{(\kappa-\nu)^2-\frac{4g\Delta T}{\rho_0 H}}\right)}\right] \\
g_4 &= \frac{\rho_0}{4ig\sqrt{(\kappa-\nu)^2 - \frac{g\Delta T}{2\rho_0 H}}} \\
\vec{U}_3 &= \left[\frac{\frac{2ig}{\rho_0}\left(\kappa-\nu+\sqrt{(\kappa-\nu)^2-\frac{g\Delta T}{2\rho_0 H}}+\frac{ig}{2\rho_0}\right)}{4\left(\kappa-\nu+\sqrt{(\kappa-\nu)^2-\frac{g\Delta T}{2\rho_0 H}}+\frac{ig}{2\rho_0}\right)\left(\nu-\kappa+\sqrt{(\kappa-\nu)^2-\frac{g\Delta T}{2\rho_0 H}}\right)}\right] \\
\vec{U}_4 &= \left[\frac{\frac{2ig}{\rho_0}\left(\nu-\kappa+\sqrt{(\kappa-\nu)^2-\frac{g\Delta T}{2\rho_0 H}}-\frac{ig}{2\rho_0}\right)}{4\left(\nu-\kappa+\sqrt{(\kappa-\nu)^2-\frac{g\Delta T}{2\rho_0 H}}-\frac{ig}{2\rho_0}\right)\left(\nu-\kappa-\sqrt{(\kappa-\nu)^2-\frac{g\Delta T}{2\rho_0 H}}\right)}\right]
\end{aligned}
$$

and $\lambda_1^{\pm}$, $\lambda_3^{\pm}$, $\lambda_4^{\pm}$ are given by (4.53), (4.54), (4.55). Note that these exponents ((4.53)-(4.55)) could also have been obtained from the dispersion relation (D.14) derived in Appendix D. In the absence of viscosity and heat conduction ($\nu = 0$, $\kappa = 0$), the solutions are periodic in time with constant amplitude. On the other hand, nonzero ν and κ result in exponential decay. The constant $\frac{g\Delta T}{\rho_0 H} = N^2$ gives a measure of the extent of the stratification of the fluid; where N is called the *Brunt-Väisälä frequency*, or *buoyancy frequency*. In other words, it gives a measure of how rapidly the density and temperature decrease with height. If $(\kappa-\nu)^2 < 4\frac{k^2N^2}{(k^2+m^2)^3}$, the viscosity and heat conduction act to lengthen the period of the oscillation. Conversely, if $(\kappa-\nu)^2 > 4\frac{k^2N^2}{(k^2+m^2)^3}$, then there is an additional exponentially growing or decaying factor.

4.3 Solution of the Nonlinear Boussinesq Problem

In a similar manner to the technique used to solve the nonlinear β-plane problem, we will use perturbation theory to find the solution of the nonlinear Boussinesq problem. However, this problem is more complex and we have three different cases to consider. Firstly, we will consider j values of 1 and 5, when both the X_j's and Y_j's are nonlinear. Secondly, we will look at j values of 2 and 6, when the X_j and Y_j equations are uncoupled. Lastly, we will consider $j = 3, 4, 7, 8$, where the X_j's are linear while the Y_j's are nonlinear.

We begin by substituting (3.36) and

$$Y_j(t) \sim Y_j^{(0)}(t) + \varepsilon Y_j^{(1)}(t) + O(\varepsilon^2) \tag{4.59}$$

into (4.29) and (4.31) for $j = 1, 5$. At $O(1)$ the solutions are simply the linear solutions that we worked out in the previous section. For $j = 1$, the solution $\vec{Z}_1$ is given by (4.56). Since X_5 is the complex conjugate of X_1 and Y_5 is the complex conjugate of Y_1, the solution $\vec{Z}_5(t) = \begin{pmatrix} X_5 \\ Y_5 \end{pmatrix}$ is the complex conjugate of $\vec{Z}_1$.

At $O(\varepsilon)$, the equations for $j = 1, 5$ are of the form

$$\frac{d}{dt}X_j^{(1)}(t) = p_j X_j^{(1)} + q_j Y_j^{(1)} + f_1(t) \tag{4.60}$$
$$\frac{d}{dt}Y_j^{(1)}(t) = r_j X_j^{(1)} + s_j Y_j^{(1)} + f_2(t) \tag{4.61}$$

where $f_1(t)$ is a nonlinear function (dependent on j) of the $X_k^{(0)}$ $(k \neq j)$ solutions found at $O(1)$ and $f_2(t)$ is a nonlinear function (dependent on j) of the $X_k^{(0)}$ and $Y_k^{(0)}$ $(k \neq j)$ solutions found at $O(1)$. The above two equations can be written as a system in the following way

$$\frac{d}{dt}\vec{Z} = A\vec{Z} + \vec{f}(t) \tag{4.62}$$

where

$$\vec{Z} = \begin{pmatrix} X_j^{(1)} \\ Y_j^{(1)} \end{pmatrix} \text{ and } A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \text{ and } \vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{4.63}$$

The j subscripts have been dropped once again for simplicity. Now, to solve the nonlinear system (4.62), we make use of the following theorem which is stated and proved in many ODE books, such as Walter [13] (p. 193):

Theorem 4.3.1. *The initial value problem*

$$\frac{d}{dt}\vec{\mathbf{Z}} = A\vec{\mathbf{Z}} + \vec{\mathbf{f}}(t),\ \vec{\mathbf{Z}}(t_0) = \vec{Z}_0 \tag{4.64}$$

(A constant) has the solution

$$\vec{\mathbf{Z}}(t) = e^{A(t-t_0)}\vec{Z}_0 + \int_{t_0}^{t} e^{A(t-s)}\vec{\mathbf{f}}(s)ds \tag{4.65}$$

where $\Phi(t) = e^{A(t-t_0)}$ is the fundamental matrix with $\Phi(t_0) = I$.

Since our initial conditions are $X_j^{(1)}(0) = 0$ and $Y_j^{(1)}(0) = 0$, $t_0 = 0$ and $\vec{Z}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in our case. By applying the above theorem, we find that

$$\vec{Z}(t) = \int_0^t e^{A(t-s)}\vec{f}(s)ds \tag{4.66}$$

Thus, in order to find the solution of this nonlinear problem, we must find the fundamental matrix. Since we have already worked out the eigenvalues and eigenvectors for the linearized Boussinesq problem, we will find the fundamental matrix by making use of the following property

$$e^{At} = \Phi(t)\Phi^{-1}(0) \tag{4.67}$$

Making use of the eigenvalues and eigenvectors found in (4.44), (4.47), and (4.48), we find $\Phi(t)$ to be

$$\Phi(t) = \begin{bmatrix} j_1 e^{(\lambda_+)t} & j_2 e^{(\lambda_-)t} \end{bmatrix} \tag{4.68}$$

where $j_1 = \begin{pmatrix} 2q \\ -p+q+\chi \end{pmatrix}$, $j_2 = \begin{pmatrix} 2q \\ -p+s-\chi \end{pmatrix}$. Finally, replacing τ with 0, making use of (4.67) and (4.51), and simplifying, we obtain the fundamental matrix

$$e^{At} = \frac{-1}{4q\chi}\begin{bmatrix} 2q(-p+s-\chi)e^{(\lambda^+)t}+2q(p-s-\chi)e^{(\lambda^-)t} & -4q^2e^{(\lambda^+)t}+4q^2e^{(\lambda^-)t} \\ (-p+s+\chi)(-p+s-\chi)e^{(\lambda^+)t}+(-p+s-\chi)(p-s-\chi)e^{(\lambda^-)t} & -2q(-p+s+\chi)e^{(\lambda^+)t}+2q(-p+s-\chi)e^{(\lambda^-)t} \end{bmatrix} \tag{4.69}$$

To avoid complicated expressions, we will simply state the values of p, q, r, s for $j = 1, 5$ and note that the fundamental matrices satisfy (4.69).

$$p = -i\bar{u} - \nu \qquad q = i\frac{g}{\rho_0} \qquad r = i\frac{\Delta T}{H} \qquad s = -i\bar{u} - \kappa \tag{4.70}$$

$$p = i\bar{u} - \nu \qquad q = -i\frac{g}{\rho_0} \qquad r = -i\frac{\Delta T}{H} \qquad s = i\bar{u} - \kappa \tag{4.71}$$

where (4.70) corresponds to $j = 1$ and (4.71) to $j = 5$.

The solutions of this nonlinear problem for $j = 1, 5$ satisfy (4.66) with the following f vectors

$$\begin{aligned} \text{for } j = 1 : \vec{f} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \text{for } j = 5 : \vec{f} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \tag{4.72}$$

Thus, there are no terms at $O(\varepsilon)$ for j values of 1 and 5.

Next we consider the second case; when $j = 2, 6$. For these two j values, $r = q = 0$, so equations (4.60) and (4.61) are uncoupled and we can thus work out their solutions separately. At $O(1)$, the solutions are once again the linear solutions derived in the previous section. More precisely, the solutions for X_2 and Y_2 are (4.36) and (4.37) respectively. Since both the X_6 and Y_6 equations have the same linear part as X_2 and Y_2, respectively, their solutions are the same as in (4.36) and (4.37) with $j = 2$ replaced with $j = 6$. At $O(\varepsilon)$ the equations for X_j and Y_j are of the form

$$\frac{d}{dt}X_j^{(1)} - a_1 X_j^{(1)} = \text{products of } X_j^{(0)} \text{ functions} \tag{4.73}$$

$$\frac{d}{dt}Y_j^{(1)} - \alpha_1 Y_j^{(1)} = \text{products of } X_j^{(0)} \text{ and } Y_j^{(0)} \text{ functions} \tag{4.74}$$

analogous to the nonlinear β-plane equation (3.47). Since the nonlinear X_2 and X_6 equations for the Boussinesq problem are the same as those in the β-plane problem, the solutions for $X_2^{(1)}$ and $X_6^{(1)}$ are given by (3.53) and (3.55), respectively. Following the integrating factor method outlined in section 3.4, the solution of equation (4.74) becomes

$$Y_j^{(1)}(t) = \left(\frac{n_1}{c_1 + c_2 - \alpha_1}\right) e^{(c_1+c_2)t} - \left(\frac{n_2}{c_3 + c_4 - \alpha_1}\right) e^{(c_3+c_4)t} + \left(\frac{n_3}{c_5 + c_6 - \alpha_1}\right) e^{(c_5+c_6)t} - \left(\frac{n_4}{c_7 + c_8 - \alpha_1}\right) e^{(c_7+c_8)t} + De^{\alpha_1 t} \quad (4.75)$$

where $c_1, ..., c_8$ and $n_1, ..., n_4$ are the complex constants (all dependent on j) arising from the linear exponential solutions of the previous section (see equations (4.56)-(4.58)), and D satisfies

$$D = \left(\frac{n_2}{c_3 + c_4 - \alpha_1}\right) - \left(\frac{n_1}{c_1 + c_2 - \alpha_1}\right) + \left(\frac{n_4}{c_5 + c_6 - \alpha_1}\right) - \left(\frac{n_3}{c_7 + c_8 - \alpha_1}\right) \quad (4.76)$$

Note that we do not explicitly determine the solution for Y_2 or Y_6 to avoid writing lengthy, complicated solutions. For this case, since both the X and Y solutions are nonzero, both contribute terms at $O(\varepsilon)$.

We now consider the final case; when $j = 3, 4, 7, 8$. Once again, at $O(1)$ the solutions are simply the linear solutions derived in the previous section, where $\vec{Z}_7$ and $\vec{Z}_8$ are the complex conjugates of $\vec{Z}_3$ and $\vec{Z}_4$, respectively. At $O(\varepsilon)$, the equations are of the following form

$$\frac{d}{dt}X_j^{(1)}(t) = p_j X_j^{(1)} + q_j Y_j^{(1)} \quad (4.77)$$

$$\frac{d}{dt}Y_j^{(1)}(t) = r_j X_j^{(1)} + s_j Y_j^{(1)} + \bar{f}(t) \quad (4.78)$$

where $\bar{f}(t)$ is a nonlinear function (dependent on j) of the X_k and Y_k ($k \neq j$) solutions found at $O(1)$. As before, this can be written as the same system in (4.62) and (4.63) with $\vec{f}$ replaced with $\begin{pmatrix} 0 \\ \bar{f} \end{pmatrix}$. Once again, the fundamental matrices satisfy (4.69) with the following p, q, r, s values (where the j subscripts have been omitted)

$$p = -i\bar{u} - 2\nu \qquad q = i\frac{g}{2\rho_0} \qquad r = i\frac{\Delta T}{H} \qquad s = -i\bar{u} - 2\kappa \quad (4.79)$$

$$p = i\bar{u} - 2\nu \qquad q = -i\frac{g}{2\rho_0} \qquad r = -i\frac{\Delta T}{H} \qquad s = i\bar{u} - 2\kappa \quad (4.80)$$

where (4.79) corresponds to $j = 3$ and $j = 4$, and (4.80) to $j = 7$ and $j = 8$. The complete solution of this nonlinear problem also satisfies (4.66) with the following f vectors

$$\begin{aligned}
\text{for } j = 3: \vec{f} &= \begin{pmatrix} 0 \\ b_1 - b_2 \end{pmatrix} \\
\text{for } j = 4: \vec{f} &= \begin{pmatrix} 0 \\ b_3 - b_4 \end{pmatrix} \\
\text{for } j = 7: \vec{f} &= \begin{pmatrix} 0 \\ b_5 - b_6 \end{pmatrix} \\
\text{for } j = 8: \vec{f} &= \begin{pmatrix} 0 \\ b_7 - b_8 \end{pmatrix}
\end{aligned} \tag{4.81}$$

where $b_1, ..., b_8$ are products of exponential solutions found at leading order (in Section 4.2). We note that each $b_1, ..., b_8$ contains a factor of either $e^{-\nu t}$ or $e^{-\kappa t}$, another element that distinguishes this case from the first one (where the f vectors were the zero vector). Also, the first component of $\vec{f}$ is zero, while the second component is nonzero, meaning that for these four j values of 3,4,7,8, it is only the $\hat{\theta}$ or Y equations that contribute nonzero components at $O(\varepsilon)$.

In summary, for the $\hat{\psi}$ or X equations the only terms that arise at $O(\varepsilon)$ correspond to j values of 2 and 6, or wavenumbers of (0,1) and (0,-1). These were the same wavenumbers that produced nonzero terms at $O(\varepsilon)$ for the β-plane problem. Thus, Figure 3.2 can also be used to illustrate how terms are generated at $O(\varepsilon)$ for the X equations of the Boussinesq problem, where a star represents a wavenumber with nonzero terms at $O(\varepsilon)$ while an open circle represents a wavenumber with no contribution at $O(\varepsilon)$. In contrast, for the $\hat{\theta}$ or Y equations there are additional terms that arise at $O(\varepsilon)$. As before, j values of 2 and 6 produce nonzero terms at $O(\varepsilon)$ as well as j values of 3,4,7,8. The generation of these $O(\varepsilon)$ terms is shown schematically in Figure 4.1. Once again, a star represents a wavenumber with nonzero terms at $O(\varepsilon)$, while an open circle represents a wavenumber with no terms at $O(\varepsilon)$.

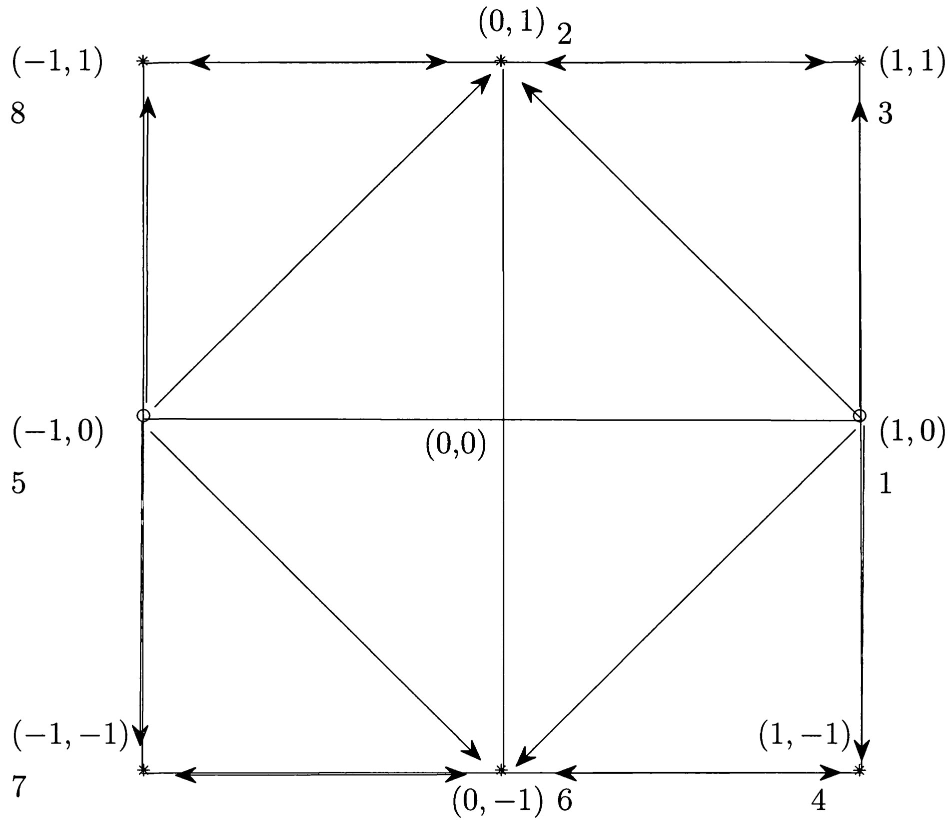


Figure 4.1: Plot showing how $O(\varepsilon)$ terms are generated for the original eight wavenumbers in the Y equations for the Boussinesq problem. Of these eight wavenumbers, the following have nonzero contributions at $O(\varepsilon)$: (0,1), (1,1), (1,-1), (0,-1), (-1,-1) and (-1,1), which are denoted by stars. Arrows are used to demonstrate how the wavenumbers interact to produce the terms at $O(\varepsilon)$. The numerical labels correspond to the numbering of the wavenumbers. For instance, 1 represents Y_1.

4.4 Higher Wavenumber Terms for the Nonlinear Boussinesq Problem

Analogous to what we did for the β-plane problem, we will extend the Fourier spectrum to include modes of ± 2. In doing so, the number of equations for the gravity wave problem increases from sixteen to fourty eight.

We begin by defining $Y_9, ..., Y_{24}$ in terms of $\hat{\theta}$ as follows

$$
\begin{aligned}
\hat{\theta}(2,0,t) &= X_9(t) & \hat{\theta}(0,2,t) &= X_{10}(t) \\
\hat{\theta}(1,2,t) &= X_{11}(t) & \hat{\theta}(2,1,t) &= X_{12}(t) \\
\hat{\theta}(2,2,t) &= X_{13}(t) & \hat{\theta}(1,-2,t) &= X_{14}(t) \\
\hat{\theta}(2,-1,t) &= X_{15}(t) & \hat{\theta}(2,-2,t) &= X_{16}(t) \\
\hat{\theta}(-2,0,t) &= X_{17}(t) & \hat{\theta}(0,-2,t) &= X_{18}(t) \\
\hat{\theta}(-1,-2,t) &= X_{19}(t) & \hat{\theta}(-2,-1,t) &= X_{20}(t) \\
\hat{\theta}(-2,-2,t) &= X_{21}(t) & \hat{\theta}(-1,2,t) &= X_{22}(t) \\
\hat{\theta}(-2,1,t) &= X_{23}(t) & \hat{\theta}(-2,2,t) &= X_{24}(t)
\end{aligned}
\tag{4.82}
$$

$X_9, ..., X_{24}$ satisfy the equations defined in the Rossby wave problem (3.60). Also, the typical conjugate relations hold true in this problem as well.

By making the appropriate substitutions from this section into the ODEs for $\hat{\psi}$ and $\hat{\theta}$, (4.23) and (4.24), respectively, we generate the following twenty four equations for $\hat{\psi}$.

$$
\begin{aligned}
\frac{d}{dt}X_1 &= (-i\bar{u}-\nu)X_1 + i\frac{g}{\rho_0}Y_1 \\
&\quad +\varepsilon[X_3X_6 - X_4X_2 + 3X_7X_{12} - 3X_8X_{15} + 6X_{13}X_{19} - 6X_{16}X_{22} \\
&\quad +2X_{11}X_{18} - 2X_{10}X_{14}] \\
\frac{d}{dt}X_2 &= -\nu X_2 + \varepsilon[X_1X_8 - X_3X_5 + 3X_4X_{22} - 3X_7X_{11} + 6X_{15}X_{24} \\
&\quad -6X_{13}X_{20} + 2X_9X_{23} - 2X_{12}X_{17}] \\
\frac{d}{dt}X_3 &= (-i\bar{u}-2\nu)X_3 + i\frac{g}{2\rho_0}Y_3 + \frac{\varepsilon}{2}[4X_6X_{11} - 4X_5X_{12} + 4X_4X_{10} - 4X_8X_9] \\
\frac{d}{dt}X_4 &= (-i\bar{u}-2\nu)X_4 + i\frac{g}{2\rho_0}Y_4 + \frac{\varepsilon}{2}[4X_5X_{15} - 4X_2X_{14} + 4X_7X_9 - 4X_3X_{18}] \\
\frac{d}{dt}X_5 &= (i\bar{u}-\nu)X_5 - i\frac{g}{\rho_0}Y_5 \\
&\quad +\varepsilon[X_2X_7 - X_6X_8 + 3X_3X_{20} - 3X_4X_{23} + 6X_{11}X_{21} - 6X_{14}X_{24} \\
&\quad +2X_{10}X_{19} - 2X_{18}X_{22}] \qquad (4.83) \\
\frac{d}{dt}X_6 &= -\nu X_6 + \varepsilon[X_4X_5 - X_1X_7 + 3X_8X_{14} - 3X_3X_{19} + 6X_{16}X_{23} - 6X_{12}X_{21} \\
&\quad +2X_{15}X_{17} - 2X_9X_{20}] \\
\frac{d}{dt}X_7 &= (i\bar{u}-2\nu)X_7 - i\frac{g}{2\rho_0}Y_7 + \frac{\varepsilon}{2}[4X_2X_{19} - 4X_1X_{20} + 4X_8X_{18} - 4X_4X_{17}] \\
\frac{d}{dt}X_8 &= (i\bar{u}-2\nu)X_8 - i\frac{g}{2\rho_0}Y_8 + \frac{\varepsilon}{2}[4X_1X_{23} - 4X_6X_{22} + 4X_3X_{17} - 4X_7X_{10}] \\
\frac{d}{dt}X_9 &= (-2i\bar{u}-4\nu)X_9 + i\frac{g}{2\rho_0}Y_9 + \frac{\varepsilon}{4}[16X_{18}X_{13} - 16X_{10}X_{16} + 8X_6X_{12} - 8X_2X_{15}] \\
\frac{d}{dt}X_{10} &= -4\nu X_{10} + \frac{\varepsilon}{4}[8X_{22}X_1 - 8X_5X_{11} + 16X_9X_{24} - 16X_{13}X_{17}] \\
\frac{d}{dt}X_{11} &= (-i\bar{u}-5\nu)X_{11} + i\frac{g}{5\rho_0}Y_{11} + \frac{\varepsilon}{5}[6X_1X_{10} - X_2X_3 + 4X_9X_{22} - 9X_8X_{12} \\
&\quad -14X_5X_{13}] \\
\frac{d}{dt}X_{12} &= (-2i\bar{u}-5\nu)X_{12} + i\frac{2g}{5\rho_0}Y_{12} + \frac{\varepsilon}{5}[X_1X_3 + 9X_4X_{11} - 4X_{10}X_{15} - 6X_2X_9 \\
&\quad +14X_6X_{13}]
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}X_{13} &= (-2i\bar{u}-8\nu)X_{13}+i\frac{g}{4\rho_0}Y_{13}+\frac{\varepsilon}{8}[8X_1X_{11}-8X_2X_{12}] \\
\frac{d}{dt}X_{14} &= (-i\bar{u}-5\nu)X_{14}+i\frac{g}{5\rho_0}Y_{14}+\frac{\varepsilon}{5}[X_6X_4+2X_1X_{18}+14X_5X_{16}+9X_7X_{15} \\
&\quad -4X_{19}X_9] \\
\frac{d}{dt}X_{15} &= (-2i\bar{u}-5\nu)X_{15}+i\frac{2g}{5\rho_0}Y_{15}+\frac{\varepsilon}{5}[4X_{18}X_{12}+6X_6X_9-14X_2X_{16}-9X_3X_{14} \\
&\quad -X_4X_1] \\
\frac{d}{dt}X_{16} &= (-2i\bar{u}-8\nu)X_{16}+i\frac{g}{4\rho_0}Y_{16}+\frac{\varepsilon}{8}[8X_6X_{15}-8X_1X_{14}] \\
\frac{d}{dt}X_{17} &= (2i\bar{u}-4\nu)X_{17}-i\frac{g}{2\rho_0}Y_{17}+\frac{\varepsilon}{4}[16X_{10}X_{21}-16X_{18}X_{24}+8X_2X_{20}-8X_6X_{23}] \\
\frac{d}{dt}X_{18} &= -4\nu X_{18}+\frac{\varepsilon}{4}[16X_{17}X_{16}-16X_9X_{21}+8X_5X_{14}-8X_1X_{19}] \qquad (4.84) \\
\frac{d}{dt}X_{19} &= (i\bar{u}-5\nu)X_{19}-i\frac{g}{5\rho_0}Y_{19}+\frac{\varepsilon}{5}[6X_5X_{18}-X_7X_6+4X_{14}X_{17}-9X_4X_{20} \\
&\quad -14X_1X_{21}] \\
\frac{d}{dt}X_{20} &= (2i\bar{u}-5\nu)X_{20}-i\frac{2g}{5\rho_0}Y_{20}+\frac{\varepsilon}{5}[X_7X_5+9X_8X_{19}-4X_{18}X_{23}-6X_6X_{17} \\
&\quad +14X_2X_{21}] \\
\frac{d}{dt}X_{21} &= (2i\bar{u}-8\nu)X_{21}-i\frac{g}{4\rho_0}Y_{21}+\frac{\varepsilon}{8}[8X_5X_{19}-8X_6X_{20}] \\
\frac{d}{dt}X_{22} &= (i\bar{u}-5\nu)X_{22}-i\frac{g}{5\rho_0}Y_{22}+\frac{\varepsilon}{5}[X_8X_2-6X_5X_{10}+14X_1X_{24}+9X_3X_{23} \\
&\quad -4X_{11}X_{17}] \\
\frac{d}{dt}X_{23} &= (2i\bar{u}-5\nu)X_{23}-i\frac{2g}{5\rho_0}Y_{23}+\frac{\varepsilon}{5}[4X_{10}X_{20}+6X_2X_{17}-14X_6X_{24}-9X_7X_{22} \\
&\quad -X_8X_5] \\
\frac{d}{dt}X_{24} &= (2i\bar{u}-8\nu)X_{24}-i\frac{g}{4\rho_0}Y_{24}+\frac{\varepsilon}{8}[8X_2X_{23}-8X_5X_{22}]
\end{aligned}
$$

We also have twenty four equations for $\hat{\theta}$, in terms of Y. These equations are omitted because they are long and complicated. However, they follow the same pattern as the Y_j equations for $j = 1, ..., 8$ in (4.31) with similar nonlinear components as in the above X_j equations.

At $O(1)$, for $j = 1, ..., 8$, we have the same solutions that were derived in the previous two sections. On the other hand, for $j = 9, ..., 24$, at $O(1)$ the equations have the following form

$$\frac{d}{dt}X_j^{(0)} = p_j X_j^{(0)} + q_j Y_j^{(0)}$$
$$\frac{d}{dt}Y_j^{(0)} = r_j X_j^{(0)} + s_j Y_j^{(0)} \tag{4.85}$$

where p_j, q_j, r_j, s_j are complex constants. The general solution of (4.85) is

$$\begin{pmatrix} X_j^{(0)} \\ Y_j^{(0)} \end{pmatrix} = A_1 \vec{v}_1 e^{\lambda^+ t} + A_2 \vec{v}_2 e^{\lambda^- t} \tag{4.86}$$

where A_1, A_2 are constants dependent on j found after applying the initial condition. Note that this equation has the same form as equations (4.56)-(4.58). However, the initial condition is $X_j^{(0)}(0) = 0 = Y_j^{(0)}(0)$ for this scenario. Thus, $A_1 = A_2 = 0$ and therefore $X_j^{(0)} = Y_j^{(0)} = 0$ for $j = 9, ..., 24$.

At $O(\varepsilon)$ for $j = 1, ..., 8$, the equations for $X_j^{(1)}$ and $Y_j^{(1)}$ are the same as in Section 4.1 given by (4.29) and thus the solutions are also the same. At $O(\varepsilon)$ for $j = 9, ..., 24$, half of the nonlinear components are zero because they also involve products of $X_j^{(0)}$ and $Y_j^{(0)}$ with j values of $9, ..., 24$. The following eight j values produce nonzero products for both the $X_j^{(1)}$ and $Y_j^{(1)}$ equations: $j = 11, 12, 14, 15, 19, 20, 22, 23$. For the remaining eight j values, the solutions will be zero at $O(\varepsilon)$. These are the same j values that produced nonzero terms at $O(\varepsilon)$ for the Rossby wave problem for $j = 9, ..., 24$.

Chapter 5

Conclusions

This concludes our study of initial-value problems describing wave propagation in two-dimensional geophysical flows. We considered two different problems: Rossby waves on a horizontal β-plane and internal gravity waves on a vertical plane for a Boussinesq fluid. The β-plane took into account the effects of the Coriolis force on a rectangular strip in two dimensions, while in a Boussinesq fluid the density variations were neglected in all terms except those involving gravity. In both problems, we considered waves that were sinusoidal and periodic in both space directions. This enabled us to use the techniques of Fourier analysis; in particular, to express our perturbations using a Fourier series representation and then take a Fourier transform to convert the PDEs to a system of ODEs. We began our analysis by solving the linear problems which produced periodic solutions. We were justified in allowing linearization so long as the wave amplitude was small enough. We then went on to solve the nonlinear problems using asymptotic analysis techniques. In doing so, we searched for solutions in powers of the small amplitude parameter. The leading order solutions were found to be the solutions already developed in the linear problem. The nonlinear terms were found to be products of the leading order terms. This produced higher wavenumber components which represented a transfer of wave energy to smaller scales. Lastly, we examined the development of these higher harmonics and the impact they had on the previously derived nonlinear solutions.

This study was of importance because of its relevance to atmospheric modelling and oceanographic studies. Without the Earth's atmosphere and oceans, life on our planet would not exist. By studying both Rossby waves and internal gravity waves, we gain insight into the behaviour of waves in both the atmosphere and ocean. This information can then be used to predict global weather and climate conditions. It is thus important to understand the mechanics behind wave generation, propagation, interaction, and evolution because it gives us a deeper understanding and appreciation of the natural phenomenon involved in our lives.

Chapter 3 was devoted to the study of Rossby waves on a β-plane. After nondimensionalizing, using Fourier analysis, and truncating the Fourier spectrum to include only the original 8 wavenumbers of $(1,0),(0,1),(1,1),(1,-1),(-1,0),(0,-1),(-1,-1)$, and $(-1,1)$, we produced a system of 8 differential equations where the last 4 equations were the complex conjugates of the first 4. At this stage, we noticed that the equations for X_1, X_2, X_5, X_6 were nonlinear, while the equations for X_3, X_4, X_7, X_8 were strictly linear. This meant that the wavenumbers $(\pm 1, 0)$ and $(0, \pm 1)$ were affected by the other wavenumber components, whereas the components with wavenumbers of $\pm(1,1)$ and $\pm(1,-1)$ progressed independently.

We then went on to solve the linear β-plane problem. Only $X_1, ..., X_4$ were needed to fully describe the linear problem as $X_5, ..., X_8$ were the complex conjugates of the first four X's. The solutions for X_1, X_3, X_4 were composed of both a periodic element and an element of exponential decay, while X_2 was found to have only an element of exponential decay (since the real part of the frequency was zero). This decay resulted from the viscosity term. If we had an inviscid problem ($\nu = 0$), then the amplitude of the solutions would remain at a constant value of 1 for all time. For X_1, X_3, X_4, as $t \rightarrow \infty$, the solutions oscillate indefinitely in space and the amplitude approaches zero. These solutions could have also been derived using the dispersion relations described in Appendix D.

Next we solved the nonlinear β-plane problem using the perturbation series technique. This time, we needed all eight X equations to fully describe the nonlinear problem. The X_j equations for $j = 3, 4, 7, 8$ were linear and thus their exact solutions were given by the linear solutions derived in section 3.3. As a result, for these four j values, the terms at $O(\varepsilon)$ and higher are zero. Therefore, we were only required to carry out the perturbation method for $j = 1, 2, 5, 6$. At leading order, we found the same linear exponential solutions that were derived in Section 3.3 (and their complex conjugates). At $O(\varepsilon)$, we found that the only nonzero contributions resulted from wavenumbers of $(0, 1)$ and $(0, -1)$. For the wavenumber $(0, 1)$, the $O(\varepsilon)$ term arose from interactions between the following pairs of wavenumbers: $(1, 0)$ and $(-1, 1)$, and $(1, 1)$ and $(-1, 0)$, while the $O(\varepsilon)$ term for the wavenumber $(0, -1)$ developed from interactions between wavenumber pairs of $(1, -1)$ and $(-1, 0)$, and $(1, 0)$ and $(-1, -1)$.

Lastly, we considered the effects of extending the Fourier spectrum to ± 2 for the Rossby wave problem. In doing so, the number of differential equations increased from eight to twenty four, giving us a more accurate solution of the β-plane problem. We employed perturbation methods once again in order to solve this problem. At leading order, we found $X_j = 0$ for all new j values ($j = 9, ..., 24$), and, for $j = 1, ..., 8$, the solutions were the same as those from the previous nonlinear section (section 3.4). At $O(\varepsilon)$, for $j = 1, ..., 8$, the solutions once again remained unaltered from the previous nonlinear solutions. For $j = 9, ..., 24$, at $O(\varepsilon)$, many of the terms were zero because they involved products of the leading order X_j's for $j = 9, ..., 24$, which were all zero. The only nonzero contributions at $O(\varepsilon)$ came from j values of 11,12,14,15,19,20,22,23 and these solutions were found to approach zero faster than those of the original wavenumber problem.

Chapter 4 described our second problem; internal gravity waves in a Boussinesq fluid. This problem started in the same way as the β-plane problem did. We discussed nondimensionalization, took Fourier transforms, and truncated the spectrum to include only the original eight wavenumbers. However, this problem was more difficult because there were two sets of coupled differential equations to work with.

Thus, for the original wavenumber problem, instead of having eight equations, we had sixteen: eight for X_j and eight for Y_j. The same four X_j equations as in the Rossby wave problem were nonlinear for the Boussinesq problem ($j = 1, 2, 5, 6$). On the other hand, all of the Y_j equations were nonlinear.

As before, we solved the linear Boussinesq problem first. Similar to the Rossby problem, we found that the solutions for $j = 1, 3, 4$ involved both exponential decay and oscillatory components, while the solution for $j = 2$ only contained an exponetially decaying factor. Once again, in the absence of viscosity and heat conduction, the solutions for $j = 1, 3, 4$ remain periodic in time with constant amplitude, while nonzero ν and κ result in components of exponential decay.

Next we tackled the nonlinear Boussinesq problem using similar techniques to those applied to the β-plane problem. We had three different cases to consider: firstly, when both the X_j's and Y_j's were coupled and nonlinear ($j = 1, 5$), secondly, when both the X_j's and Y_j's were uncoupled and nonlinear ($j = 2, 6$), and lastly, when the X_j's and Y_j's were coupled where the X_j equations were linear while the Y_j equations were nonlinear ($j = 3, 4, 7, 8$). For all three cases, at leading order, the solutions were once again the linear solutions developed in the previous section (Section 4.2). For the first case, at $O(\varepsilon)$, there were no contributions at $O(\varepsilon)$. For the second case, the nonzero contributions came from $j = 2, 6$ for both the X and Y components, corresponding to wavenumbers of $(0, 1)$ and $(0, -1)$; the same nonzero contributors as in the Rossby problem. For the third and final case, at $O(\varepsilon)$, there were nonzero contributions that arose from the Y components corresponding to j values of 3,4,7,8; a factor that distinguishes the Boussinesq problem from the β-plane problem.

Lastly, we extended the Fourier spectrum to ± 2 for the gravity wave problem which increased the number of equations from sixteen to fourty eight. At leading order, for $j = 1, ..., 8$, the solutions from the previous nonlinear section (section 4.3) remained unchanged, whereas for $j = 9, ..., 24$ both X_j and Y_j were found to be zero upon applying the initial condition. At $O(\varepsilon)$, for $j = 1, ..., 8$, the solutions once again remained the same from the previous section. For $j = 9, ..., 24$, at $O(\varepsilon)$, half of the nonlinear components were found to be zero because they involved products of the leading order X_j and Y_j's which were zero. The only eight j values that produced nonzero terms at $O(\varepsilon)$ for $j = 9, ..., 24$ were 11,12,14,15,19,20,22,23; the same eight j values that gave rise to nonzero terms for the β-plane problem.

In summary, for both the Rossby wave problem and the internal gravity wave problem, at leading order we found linear solutions for the original forced wavenumbers of 0, ± 1. These solutions were exponential functions of t with complex exponents. These solutions were periodic in t for j values of 1,3,4, as given by the dispersion relations derived in Appendix D. The exponentially decaying factors resulted due to the viscosity and/or heat conduction (as in the Boussinesq case). At $O(\varepsilon)$, some of the forced wavenumber components had $O(\varepsilon)$ corrections due to the interactions of other wavenumber components, while others did not since they were unaltered by other wavenumbers. Higher wavenumber components with $k, l, m = \pm 2$ also developed at $O(\varepsilon)$ for certain wavenumber pairs.

This study could be extended by considering the following possible future topics. Firstly, if $\delta \ll 1$, then there would be two small parameters to consider, δ and ε. Asymptotic methods could be used to solve this problem. Secondly, if $\varepsilon \sim O(1)$, the problem would become fully nonlinear and no longer be considered weakly nonlinear. This would result in the development of more higher wavenumbers and would require numerical methods to solve it. With the initial conditions and the periodic boundary conditions used in our study, the waves generated were sinusoidal functions of both space variables. However, observations of Rossby waves and internal gravity waves in the atmosphere and ocean show that they generally take the form of spatially localized wave packets (see [5] for more details). In mathematical terms, a wave packet could be modelled by using a spatially localized initial condition by multiplying each of the cosines in (1.1) by a Gaussian function, $e^{-\mu^2 x^2}$ or $e^{-\mu^2 y^2}$, for instance. Note that as $x, y \to \pm\infty$, the Gaussian functions approach zero. We would choose μ to be a small parameter so that the wave packet "envelope" would vary slowly compared with the wavelength of oscillation. This problem is a multi-scale problem and would require a continuous Fourier integral transform instead of a Fourier series to be solved. We would look for solutions in powers of the small parameters μ and ε (and possibly δ as mentioned above). The solutions we derived in this study would correspond to the limit of $\mu \to 0$ and thus could be used as a starting point for such analyses. Lastly, one could consider a full three-dimensional problem in which both Rossby waves and internal gravity waves are allowed to interact with one another. This problem would also require numerics in order to be solved. The approximate solutions we have derived for our simplified two-dimensional problem can give some guidance as to how to proceed in finding the solution of any of these more complicated problems.

Appendix A

Asymptotics Definitions

In this Appendix, we present some typical definitions from the field of asymptotics. These definitions can be found in any text concerning the study of asymptotics and we will follow the notation used in Bender and Orszag [3].

Definition A.0.1. The notation $f(x) \ll g(x)$ as $x \to x_0$ means

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

where the symbol "$\ll$" stands for 'much smaller than'.

Definition A.0.2. The notation $f(x) \sim g(x)$ (as $x \to x_0$) means $f(x) - g(x) \ll g(x)$ or

$$\lim_{x \to x_0} \frac{f(x) - g(x)}{g(x)} = 0$$

or, equivalently,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

where the symbol "$\sim$" means asymptotic to or similar to.

Note that if $f(x) \sim g(x)$ as $x \to x_0$, then $g(x) \sim f(x)$ as $x \to x_0$. Also, if $f(x) \ll g(x)$, then $g(x) \gg f(x)$.

Definition A.0.3. In general, we say that $f(x) \sim O(g(x))$ as $x \to x_0$ if $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ is bounded, meaning that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = A$$

where $0 < |A| < \infty$. We say "$f(x)$ is at most of order $g(x)$ as $x \to x_0$" or "$f(x)$ is 'O' of $g(x)$ as $x \to x_0$.

Definition A.0.4. We say that $f(x) \sim o(g(x))$ as $x \to x_0$ if

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$$

We say "$f(x)$ is 'o' of $g(x)$ as $x \to x_0$.

Definition A.0.5. The power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to be asymptotic to the function $y(x)$ as $x \to x_0$, written as

$$y(x) \sim \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \text{as } x \to x_0$$

if

$$y(x) - \sum_{n=0}^{N} a_n(x - x_0)^n \ll (x - x_0)^N \quad \text{as } x \to x_0 \quad \text{for every} N$$

or, equivalently,

$$y(x) - \sum_{n=0}^{N} a_n(x - x_0)^n \sim a_M(x - x_0)^M \quad \text{as } x \to x_0$$

where a_M is the first non-zero coefficient after a_N.

Definition A.0.6. A perturbation series is a series in powers of a small parameter, such as ε, for instance. In other words, a series of the form

$$y(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n$$

where the a_n are functions of the independent variables. This series could be divergent as $n \to \infty$. ε must be chosen such that when $\varepsilon = 0$, the problem is solvable.

Appendix B

Notation

In this section, we will present the notation that will be used throughout this paper. The notation closely follows that of Kundu and Cohen [9].

B.1 Einstein Summation Convention

Let $\vec{x} = (x_1, x_2, x_3)$, where the components of $\vec{x}$ are $x_\imath$ with $i = 1, 2, 3$. Define the coordinate axes by x_1, x_2, and x_3 (analogous to x, y, and z). We then define the rotated coordinate axes by x'_1, x'_2, and x'_3. Let $C_{\imath\jmath}$ be the cosine of the angle between $x_\imath$ and $x'_\jmath$. C_{12} is the angle between x_1 and x'_2, for instance. Note that $C_{\imath\jmath} \neq C_{\jmath\imath}$. Using geometry, one can show that

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{B.1}$$

We can write (B.1) in a more compact way as

$$x'_\jmath = \sum_{\imath=1}^{3} x_\imath C_{\imath\jmath} \tag{B.2}$$

Note that, on the right-hand side of (B.2), the index i appears twice and the summation is carried out over all values of this repeated index. The index j appears on both sides of the equation and it is called the *free index*. We now introduce a simplifed notation called the *Einstein summation convention*. This convention says that whenever an index appears twice in a term, a summation over the repeated index is implied. Thus, (B.2) can be written as

$$x'_j = x_i C_{ij} \tag{B.3}$$

Using this notation, the multiplication of two matrices, A and B, where $A = A_{ij}$ and $B = B_{ij}$, is given by $P_{ij} = A_{ik} B_{kj}$. Since the index k is repeated, a summation over k is implied.

B.2 Derivatives with respect to time

There are two different ways to describe fluid motion, using the *Eulerian* or the *Lagrangian* description. For the Eulerian description, we consider what happens at a given spatial position, $\vec{x}$. The independent variables are $\vec{x} = (x_1, x_2, x_3)$ and t. The dependent variables, or flow variables, are functions of $\vec{x}$ and t, $F(\vec{x}, t)$. On the other hand, for the Lagrangian description, we follow the motion of each individual fluid particle and assign each particle with a label, $\vec{a}$, which is taken to be the position vector of the particle's location at $t = 0$. In this situation, the independent variables are $\vec{a}$ and t and the dependent variables are functions of $\vec{a}$ and t. Velocity, $\vec{v}(\vec{a}, t)$, is an example of a dependent or flow variable. At a given point, the position vector is $\vec{r} = \vec{r}(\vec{a}, t)$ which is a dependent variable, unlike $\vec{x}$ in the Eulerian case. For our purposes, we will use the Eulerian description.

In the Lagrangian description, $\frac{\partial F}{\partial t}$ denotes the *total* rate of change of a quantity $\vec{F}(\vec{a}, t)$ as seen by a moving fluid particle with label $\vec{a}$. On the other hand, in the Eulerian description, $\frac{\partial F}{\partial t}$ represents the *local* rate of change of a quantity $\vec{F}(\vec{x}, t)$ at a point $\vec{x}$. To measure the total rate of change following a moving fluid particle, we consider a fluid property,

$$\vec{F}(\vec{x}, t) = \vec{F}(\vec{a}, t)$$

at the same position and time given by the above two descriptions. We will assume that the time and length scales are the same in both descriptions. The position vector of the particle at time t is given by $\vec{x} = \vec{r}(\vec{a}, t)$. Thus,

$$\vec{F}(\vec{x}, t) = \vec{F}(\vec{r}(\vec{a}, t))$$

Now, by differentiating and applying the chain rule, we find that the rate of change following a particle is given by

$$\frac{\partial F}{\partial t} + \nabla F \cdot \frac{\partial \vec{r}}{\partial t}$$

where $\frac{\partial F}{\partial t}$ represents the local rate of change and $\nabla F \cdot \frac{\partial \vec{r}}{\partial t}$ represents the rate of change for the particle itself. Since $\frac{\partial r}{\partial t} = \vec{u}$, the above expression becomes

$$\frac{\partial F}{\partial t} + \vec{u} \cdot \nabla F \equiv \frac{DF}{Dt} \tag{B.4}$$

This is called the *material derivative* or *particle derivative* and is denoted by $\frac{D}{Dt}$.

Appendix C

Conservation Laws

C.1 Introduction

In this appendix, we present a derivation of the governing equations of fluid dynamics, which form the starting point for our study. The derivations are standard and can be found in any fluid dynamics text (such as Acheson [1], Holton [8], or Pedlosky [10]). We will follow the fluid dynamics text of Kundu and Cohen [9].

The study of fluid dynamics is governed by the laws of conservation of mass, momentum and energy. These laws can be stated in one of two forms; *differential* form or *integral* form. Differential form is applicable at a point, and integral form is applicable to an extended region. In integral form, the laws relate to either a *fixed* volume, V, or a *material* volume, $\mathcal{V}$. A fixed volume is fixed in space and consists of different fluid particles while a material volume consists of the same fluid particles and the bonding surface moves with the fluid. This chapter provides the necessary groundwork needed before moving on to our specific problem.

C.2 Time Derivatives of Volume Integrals

In the next few sections, as we begin to derive the conservation laws, it will become important to compute time derivatives of volume integrals. In order to do so, we must apply *Leibniz's Theorem.*

Consider the following time derivative of a volume integral

$$\frac{d}{dt}\int_{V(t)} F dV$$

where F $= F(\vec{x}, t)$ is a tensor of any order, and $V(t)$ is any region (fixed or moving). Leibniz's theorem allows us to compute the above integral. In general, we assume that the surfaces of the volume are moving with the same velocity.

In one dimension, suppose that the region V is bounded by the lines $x = a$ and $x = b$. Furthermore, suppose that the lines are moving. Then, $a = a(t)$ and $b = b(t)$. Leibniz's theorem states that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)dx = \int_a^b \frac{\partial F}{\partial t}dx + \frac{db}{dt}F(b,t) - \frac{da}{dt}F(a,t) \tag{C.1}$$

assuming a, b and F are continuous. Leibniz's theorem can be generalized to higher dimensions as follows

$$\frac{d}{dt}\int_{V(t)} F(\vec{x},t)dV = \int_{V(t)} \frac{\partial F}{\partial t}dV + \int_{A(t)} F\vec{u}_A \cdot d\vec{A} \tag{C.2}$$

where $\vec{u}_A$ is the velocity of the boundary and $A(t)$ is the surface of $V(t)$. For a fixed volume, V, $\vec{u}_A = 0$ and thus equation (C.2) becomes

$$\frac{d}{dt}\int_{V} F(\vec{x},t)dV = \int_{V} \frac{\partial F}{\partial t}dV \tag{C.3}$$

For a material volume, $\mathcal{V}$, the boundary moves with the fluid, so $\vec{u}_A = \vec{u}$, where $\vec{u}$ is the fluid velocity. Equation (C.2) then becomes

$$\frac{D}{Dt}\int_{\mathcal{V}} F(\vec{x},t)d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial F}{\partial t}d\mathcal{V} + \int_{A} F\vec{u} \cdot d\vec{A} \tag{C.4}$$

Equation (C.4) is called *Reynold's Transport Theorem.*

C.3 Conservation of Mass and the Continuity Equation

In this section, we will derive the continuity equation in two different ways; both express the principle of conservation of mass.

We will begin by considering a fixed volume, V, in space. The total mass inside this volume is given by the following volume integral

$$\int_V \rho dV$$

where ρ is the density. The rate of increase of mass inside V is represented by

$$\frac{d}{dt}\int_V \rho(\vec{x},t)dV = \int_V \frac{\partial \rho}{\partial t}dV \tag{C.5}$$

The right-hand side of equation (C.5) holds true due to Leibniz's theorem (C.3) with $F = \rho$ and assuming that ρ is continuous. Conservation of mass tells us that the rate of increase of mass inside V must be balanced by the rate of mass flow out of V. We must now come up with an expression to represent the rate of mass flow out of the volume V.

Let $d\vec{A} = \vec{n}dA$ represent an outer element on the surface A of V, where $\vec{n}$ is the unit vector normal to the surface of all area elements. Let $\vec{u} = \frac{d\vec{l}}{dt}$ be the velocity of the fluid, where $\vec{l}$ is the displacement of the fluid. In the time interval dt, the displacement of fluid is $d\vec{l}$ and the volume of fluid flowing through the element dA is

$$d\vec{l} \cdot \vec{n}dA$$

Thus, the mass flowing through dA in time dt is given by

$$\rho d\vec{l} \cdot \vec{n}dA$$

So, the mass flowing through dA in unit time is

$$\rho\frac{d\vec{l}}{dt}\cdot\vec{n}dA = \rho\vec{u}\cdot\vec{n}dA$$

And so, the outward flux through dA is given by

$$\rho\vec{u}\cdot dA$$

Now, the rate of mass flow out of V can be expressed as the surface integral of the outward flux through dA

$$\int_A \rho\vec{u}\cdot d\vec{A} \tag{C.6}$$

According to the principle of conservation of mass, we now have

$$\int_V \frac{\partial\rho}{\partial t}dV = -\int_A \rho\vec{u}\cdot d\vec{A} \tag{C.7}$$

We must now transform the surface integral on the right hand side of (C.7) into a volume integral using *Gauss' Divergence Theorem* which states that the flux of a vector field out of a closed surface is equal to the integral of the divergence of that vector field over the volume enclosed by the surface. In other words

$$\int_V (\nabla\cdot\vec{F})dV = \int_{\partial V} \vec{F}\cdot d\vec{A} \tag{C.8}$$

Thus, applying (C.8) to (C.7) with $F = \rho\vec{u}$ we have

$$\int_V \frac{\partial\rho}{\partial t}dV = -\int_V \nabla\cdot(\rho\vec{u})dV$$

Hence,

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right) dV = 0 \tag{C.9}$$

The above relation holds for *any* volume and this is only possible if the integrand vanishes at every point. As a result,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \tag{C.10}$$

Equation (C.10) is called the *continuity equation*, which represents the differential form of the law of conservation of mass. We can express (C.10) in a different form as follows. By applying the gradient product rule we can rewrite $\nabla \cdot (\rho\vec{u})$ as

$$\rho\nabla \cdot \vec{u} + \vec{u} \cdot \nabla\rho$$

Thus, (C.10) becomes

$$\frac{\partial\rho}{\partial t} + \vec{u} \cdot \nabla\rho + \rho\nabla \cdot \vec{u} = 0$$

The above expression is equivalent to

$$\frac{D\rho}{Dt} + \rho\nabla\vec{u} = 0$$

which then becomes

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{u} = 0 \tag{C.11}$$

after dividing through by ρ. The derivative $\frac{D\rho}{Dt}$ in the above equation is called the rate of change of density following a fluid particle. This derivative can be nonzero due to changes in pressure, temperature or fluid properties. A fluid is called *incompressible* if its density does not change with pressure. In our case, we will assume that

$$\frac{1}{\rho}\frac{D\rho}{Dt} \approx 0$$

After making this approximation, equation (C.11) becomes

$$\nabla \cdot \vec{u} = 0 \tag{C.12}$$

Equation (C.12) is the incompressible form of the continuity equation (C.11).

C.4 Conservation of Momementum

In this section, we will derive the conservation of momentum equations through the use of Cauchy's equation as well as the Navier-Stokes equations.

Consider the motion of an infinitesimal fluid element of fixed mass, with volume $d\mathcal{V}$. Newton's second law states that the net force $\vec{F}$ on the element is equal to its mass times its acceleration. In other words, the rate of change of the total momentum of the element. In mathematical notation, this can be expressed as

$$\vec{F} = ma, \text{ or } \vec{F} = \frac{D\vec{M}}{Dt}$$

where $\vec{M} = m\vec{u}$. So,

$$\vec{F} = m\frac{D\vec{u}}{Dt}$$

and thus,

$$\vec{F} = \rho\frac{D\vec{u}}{Dt}$$

C.4.1 Forces that Act on a Fluid; Normal and Shear Stresses

Before we proceed any further, it is important to discuss the types of forces that act on a fluid.

There are three main types of forces that act on a fluid: *body forces*, *surface forces*, and *line forces*. Body forces arise from the medium being placed in a force field (gravitational, magnetic, electrostatic, etc.) with no physical contact. Surface forces are exerted on an area element by the surroundings through direct contact. Surface forces can be resolved into normal and tangential components;

$$\vec{F} = (F_n, F_s)$$

where F_n is the normal component to the area and $\vec{F_s}$ is the tangential component to the area. Lastly, line forces are surface tension forces that act along a line and appear at the interface between two liquids or between a liquid and a gas. They do not appear in equations of motion; they only appear in the boundary conditions.

Now, consider an element of area dA in a fluid with force $d\vec{F}$. Then

$$d\vec{F} = (dF_n, dF_s)$$

The normal and shear stress (force per unit area) on the element are

$$\tau_n = \frac{dF_n}{dA} \text{ and } \tau_s = \frac{dF_s}{dA}$$

respectively.

The stress at a point in a material requires nine components for complete specification, since two directions are involved in its description. One direction specifies the orientation of the surface and the other direction specifies the direction of the force. Let τ_{ij} denote the j-component of the force on a surface whose outward normal points in the i-direction, and i, j = 1,2,3.

For a point on the surface with normal vector $\hat{n} = (n_1, n_2, n_3)$, the stress has nine components and can be written as a matrix as follows

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

Now, suppose we have a general two-dimensional element (not necessarily rectangular). We want to determine the components of the stress on the hypotenuse (side AC). $\hat{n} = (n_1, n_2)$ is the normal vector to side AC with

$$n_1 = \cos\theta_1, \;\; n_2 = \cos\theta_2$$

Firstly, however, we need to find the force on side AC, which we will denote by

$$\vec{F} = (F_1, F_2)$$

where F_1 is the force in the x_1 direction and F_2 is the force in the x_2 direction. For forces in the x_1 direction to balance, we must have

$$F_1 = \tau_{11} dx_2 + \tau_{21} dx_1 \tag{C.13}$$

Similarly, for forces to balance in the x_2 direction, we must have

$$F_2 = \tau_{12} dx_2 + \tau_{22} dx_1 \tag{C.14}$$

Force per unit area is given by the following expressions

$$\begin{aligned} \vec{f} &= \frac{F}{ds} = (f_1, f_2) = \left(\tfrac{F_1}{ds}, \ \tfrac{F_2}{ds}\right) \\ f_1 &= \frac{F_1}{ds} = \tau_{11}\frac{dx_2}{ds} + \tau_{21}\frac{dx_1}{ds} \end{aligned} \tag{C.15}$$

Now, using trigonometry, we have

$$\cos\theta_1 = \frac{dx_2}{ds} \text{ and } \cos\theta_2 = \frac{dx_1}{ds},$$

so (C.15) becomes

$$f_1 = \tau_{11}\cos\theta_1 + \tau_{21}\cos\theta_2 = \tau_{11}n_1 + \tau_{21}n_2 \tag{C.16}$$

Similarly,

$$f_2 = \tau_{12}\cos\theta_1 + \tau_{22}\cos\theta_2 = \tau_{12}n_1 + \tau_{22}n_2 \tag{C.17}$$

Now, putting (C.16) and (C.17) together gives

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{C.18}$$

which has the form $\vec{f} = \tau \cdot \hat{n}$. In other words, the force per unit area on the surface is equal to the stress tensor, τ times the direction of the orientation of the surface.

In (C.18), the diagonal elements of the stress tensor τ are the normal stresses (τ_{11} and τ_{22}) which act in the directions n_1 and n_2 respectively. On the other hand, the non-diagonal elements are the tangential (shear) stresses (τ_{12} and τ_{21}) which are perpendicular to n_1 and n_2 respectively.

We can extend the above discussion to higher dimensions as well. In three dimensions, the equivalent version of (C.18) is

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \tag{C.19}$$

C.4.2 Cauchy's Equation of Motion

Now that we have investigated normal and shear stresses, we can derive Cauchy's equation of motion.

Consider a material volume $\mathcal{V}$. Newton's second law, $\vec{F} = ma$, states that the rate of change of momentum is equal to the sum of the body and surface forces throughout the fluid. We do not need to consider internal stresses within the fluid at this time. The rate of change of momentum is represented by the following expression

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho u_i d\mathcal{V} \tag{C.20}$$

Now, using Liebnitz's Theorem (C.1), (C.20) becomes

$$\int_{\mathcal{V}} \rho \frac{Du_i}{Dt} d\mathcal{V} \tag{C.21}$$

We will now decompose (C.21) into the sum of the body forces and the surface forces. Thus, (C.21) becomes

$$\int_{\mathcal{V}} \rho g_i d\mathcal{V} + \int_A \tau_{ij} dA_j \tag{C.22}$$

where the first integral represents the body force and the second integral represents the surface force. Also, g is the body force per unit mass and so ρg is the body force per unit volume. The surface force on area element $d\vec{A}$ is

$$dA\hat{n} \cdot \tau = d\vec{A} \cdot \tau$$

since $\hat{n} \cdot \tau$ is the force per unit area on the surface (C.18). We must now transform the surface integral in (C.22) into a volume integral using Gauss' Divergence Theorem as follows

$$\int_A \tau_{ij} dA_j = \int_{\mathcal{V}} \frac{\partial \tau_{ij}}{\partial x_j} d\mathcal{V}$$

Substituting this volume integral into (C.22) gives

$$\int_{\mathcal{V}} (\rho \frac{Du_i}{Dt} - \rho g_i - \frac{\partial \tau_{ij}}{\partial x_j}) d\mathcal{V} = 0 \tag{C.23}$$

Since (C.23) must hold for any volume, the integrand must vanish at every point. Hence,

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j} \tag{C.24}$$

The above equation (C.24) is Cauchy's equation of motion.

C.4.3 Newtonian Fluids

A fluid at rest only has normal components of stress acting on the surface and the stress does not depend on the orientation of that surface. This means that $\tau_{ij} = 0$ for each $i \neq j$. Thus, the only nonzero stresses arise when $i = j$. Therefore, τ_{ij} is proportional to the Kronecker delta which is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and can be expressed in tensor form as

$$\vec{\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We write

$$\tau_{ij} = -p\delta_{ij}$$

for a fluid at rest, where p is the *thermodynamic pressure.*

For a moving fluid, however, there are additional components of stress due to viscosity that we must consider. Thus, in this case, we write the stress as

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij} \tag{C.25}$$

where σ_{ij} is the stress due to viscosity (this includes shear stresses). We will now take an in-depth look at the general form of the tensor σ_{ij}.

Processes of internal friction (viscosity) result when different fluid particles move with different velocities in which there is a relative motion between various parts of the fluid. Thus, σ_{ij} must depend on $\frac{\partial u_i}{\partial x_j}$, $\frac{\partial^2 u_i}{\partial x_j^2}$; space derivatives of the velocity.

If the velocity gradients are small, we assume that the effects of viscosity result only from the first derivatives and ignore higher derivatives. Hence, σ_{ij} depends only on $\frac{\partial u_i}{\partial x_j}$. We further assume that σ_{ij} is a linear function of $\frac{\partial u_i}{\partial x_j}$. Now, since viscosity results from fluid particles moving with different velocities, $\sigma_{ij} = 0$ for $\vec{u} = C$, where C is a constant. Thus, $\sigma_{ij} = 0$ for $\frac{\partial u_i}{\partial x_j} = 0$. As a result, there can be no terms in σ_{ij} that are independent of $\frac{\partial u_i}{\partial x_j}$. So, we say that σ_{ij} is proportional to

$$\frac{\partial u_i}{\partial x_j} = \sum_i \sum_j \frac{\partial u_i}{\partial x_j}$$

For each i, j, we can write the tensor $\frac{\partial u_i}{\partial x_j}$ as

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{C.26}$$

where $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the *symmetric* part and $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ is the *antisymmetric* part.

We also note that $\sigma_{ij} = 0$ when the entire fluid is in uniform motion, meaning that all particles of the fluid are rotating at the same speed. When a fluid is in uniform rotation,

$$\vec{u} = \vec{\Omega} \times \vec{x} = \begin{vmatrix} i & j & k \\ \Omega_1 & \Omega_2 & \Omega_3 \\ x_1 & x_2 & x_3 \end{vmatrix} \tag{C.27}$$

where $\vec{\Omega}$ is the angular velocity. From the previous equation, we have

$$\begin{aligned} u_1 &= \Omega_2 x_3 - \Omega_3 x_2 \\ u_2 &= -\Omega_1 x_3 + \Omega_3 x_1 \\ u_3 &= \Omega_1 x_2 - \Omega_2 x_1 \end{aligned} \tag{C.28}$$

Using (C.28), we can show that

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0, \text{ and } \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \neq 0$$

For example, for $i = 1$, $j = 2$, $\frac{\partial u_1}{\partial x_2} = -\Omega_3$ and $\frac{\partial u_2}{\partial x_1} = \Omega_3$. Thus, $\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0$ but $\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \neq 0$. This means that for a fluid in uniform motion the symmetric part of the tensor $\frac{\partial u_i}{\partial x_j}$ is zero when the antisymmetric part is nonzero. Since σ_{ij} is also zero for a fluid in uniform motion, the only possibility is that σ_{ij} depends only on the symmetric part. Thus, when the fluid is in uniform rotation (when $\sigma_{ij} = 0$), σ_{ij} only depends on the symmetric part of the tensor defined in (C.26) and not on the antisymmetric part.

From the above investigation, we will now define σ_{ij} in the following way

$$\sigma_{ij} = K_{ijmn}\frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right) \tag{C.29}$$

where K_{ijmn} is a fourth order tensor with 81 components, and (C.29) is summed over m and n for each i, j.

We now assume that the fluid is *isotropic*, meaning that it is the same at every point (homogeneous). Also, K_{ijmn} is an isotropic tensor and thus its components are unchanged by a rotation of the frame of reference or of the coordinate system. Some important properties of isotropic tensors include the fact that the only second order isotropic tensor is δ_{ij} (the Kronecker delta function), and all isotropic tensors of even order can be written as the sum of products of delta tensors (source [2]). As a result, the fourth order tensor K_{ijmn} must be of the following form

$$K_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta jn + \gamma\delta_{in}\delta_{jm} \tag{C.30}$$

As we mentioned earlier, σ_{ij} is symmetric. So, K_{ijmn} is also symmetric in i and j. This means that $K_{ijmn} = K_{jimn}$. Now, applying the symmetry of K_{ijmn} to equation (C.30) gives

$$K_{ijmn} = K_{jimn} = \lambda\delta_{ji}\delta_{mn} + \mu\delta_{jm}\delta in + \gamma\delta_{jn}\delta_{im}$$

and this only holds true if $\mu = \gamma$. Hence, equation (C.30) becomes

$$K_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu[\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}] \tag{C.31}$$

The above equation (C.31) is a symmetric tensor in m and n so we can interchange m and n and get the same result. Thus, the equation can be simplified to

$$K_{ijmn} = \lambda\delta_{ij}\delta_{mn} + 2\mu\delta_{im}\delta_{jn} \tag{C.32}$$

We are now in a position to come up with the general form of the tensor σ_{ij}. By substituting (C.32) into (C.29) we get the following

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{mn}\frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right) + 2\mu[\delta_{im}\delta_{jn}\frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)]$$

We now replace m by i and n by j to get the following result

$$\sigma_{ij} = \lambda\delta_{ij}\left(\frac{\partial u_m}{\partial x_m}\right) + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{C.33}$$

Note that

$$\frac{\partial u_m}{\partial x_m} = \sum_m \frac{\partial u_m}{\partial x_m} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \vec{u}$$

Now that we have a general expression for σ_{ij}, we can come up with a more complete expression for (C.25). Thus, the complete stress tensor is

$$\tau_{ij} = -p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \lambda\delta_{ij}(\nabla \cdot \vec{u}) \tag{C.34}$$

Let us now consider the diagonal terms of the above equation (C.34);

$$\tau_{ii} = -p\delta_{ii} + 2\mu\left(\frac{\partial u_i}{\partial x_i}\right) + \lambda\delta_{ii}\left(\frac{\partial u_i}{\partial x_i}\right)$$

Since we are summing over i on both sides of this equation, the Einstein summation convention (B.3) can be applied. Before doing so, we note that

$$\delta_{ii} = \sum_{i=1}^{N=3} \delta_{ii} = \sum_{i=1}^{N=3} 1 = 3$$

and so

$$\tau_{ii} = -3p + (2\mu + 3\lambda)\nabla \cdot \vec{u} \tag{C.35}$$

Using equation (C.35), we can come up with an expression for the thermodynamic pressure;

$$p = -\frac{1}{3}\tau_{ii} + \left(\frac{2}{3}\mu + \lambda\right)\nabla \cdot \vec{u} \tag{C.36}$$

Now define

$$\tilde{p} = -\frac{1}{3}\tau_{ii}$$

where $\tilde{p}$ is the *mean* or *mechanical* pressure. Equation (C.36) then becomes

$$p = \tilde{p} + \left(\frac{2}{3}\mu + \lambda\right) \nabla \cdot \vec{u} \tag{C.37}$$

or

$$p - \tilde{p} = \left(\frac{2}{3}\mu + \lambda\right) \nabla \cdot \vec{u}$$

The latter equation represents the difference between the thermodynamic and mechanical pressures.

For an *incompressible* fluid, $\nabla \cdot \vec{u} = 0$ and so $p = \tilde{p}$. In other words, for an incompressible fluid, the thermodynamic and mechanical pressures are equal. The stress tensor for an incompressible fluid is

$$\tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

On the other hand, for *compressible* fluids, the pressure is given by equation (C.37) and $p = \tilde{p}$ only when the constant

$$\frac{2}{3}\mu + \lambda = 0 \tag{C.38}$$

This is called the *Stokes' Assumption*. Another way to express the Stokes' assumption is

$$\lambda = -\frac{2}{3}\mu$$

and if we substitute this expression into equation (C.34) we get

$$\begin{aligned} \tau_{ij} &= -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}(\nabla \cdot \vec{u}) \\ &= -[p + \frac{2}{3}\mu(\nabla \cdot \vec{u})]\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \end{aligned} \tag{C.39}$$

Fluids that obey the above equation are called *Newtonian Fluids*. Furthermore, a Newtonian fluid is one that obeys the linear Newtonian friction law, meaning that σ_{ij} is a linear function of $\frac{\partial u_i}{\partial x_j}$. This linear relationship is quite accurate for fluids such as air and water. Nevertheless, some liquids, particularly liquids in the chemical industry, exhibit non-Newtonian behaviour. As a result, σ_{ij} may be a nonlinear function of $\frac{\partial u_i}{\partial x_j}$. Some reasons for this nonlinearity include the fact that σ_{ij} depends on its *history* (past behaviour of the fluid) which causes the fluid to have *memory*; an elastic property as well as a viscous property. Non-Newtonian fluids are often said to be *viscoelastic*.

C.4.4 Navier-Stokes Equation

All of the work in the previous section has enabled us to produce the governing equation for a Newtonian fluid. We do so by applying the Stokes' assumption ($\lambda = -\frac{2}{3}\mu$) to (C.34) and then substituting (C.40) into Cauchy's equation (C.24). We must also substitute an expression for $\frac{\partial \tau_{ij}}{\partial x_j}$ into Cauchy's equation. Taking the partial derivative of (C.40) with respect to x_j gives

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\left(\nabla \cdot \vec{u}\right)\delta_{ij}\right] \tag{C.40}$$

In the above equation,

$$\frac{\partial p}{\partial x_j}\delta_{ij} = \frac{\partial p}{\partial x_i}$$

and μ is the viscosity coefficient. Now, making the substitutions mentioned above gives the following governing equation for Newtonian fluids

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_j} + \rho g_i + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\left(\nabla \cdot \vec{u}\right)\delta_{ij}\right] \tag{C.41}$$

Equation (C.41) is a general form of the *Navier-Stokes equation*. The viscosity coefficient, μ, depends on the temperature of the medium. For liquids, μ decreases with temperature, and for fluids μ increases with temperature. If the temperature differences are small within the fluid, μ can be treated as a constant and in this case μ can be taken outside of the derivative in (C.41) as follows

$$\rho\frac{Du_i}{Dt} = \frac{\partial p}{\partial x_i} + \rho g_i + \mu\left[\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial}{\partial x_j}\left(\nabla \cdot \vec{u}\right)\delta_{ij}\right]$$

Now, since the terms in the square brackets simplify to

$$\begin{aligned}
\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{2}{3}\frac{\partial^2 u_j}{\partial x_i \partial x_j} &= \frac{\partial^2 u_j}{\partial x_j^2} + \frac{1}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) \\
&= \frac{\partial^2 u_j}{\partial x_j^2} + \frac{1}{3}\frac{\partial}{\partial x_i}\left(\nabla \cdot \vec{u}\right)
\end{aligned}$$

we can further simplify (C.41) to

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \left[\nabla^2 u_i + \frac{1}{3} \frac{\partial}{\partial x_i} (\nabla \cdot \vec{u}) \right] \tag{C.42}$$

where

$$\nabla^2 u_i \equiv \frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}$$

is the Laplacian of u_i. Equation (C.42) is a more simplified version of the Navier-Stokes equation and represents the conservation of momentum. For incompressible fluids, since $\nabla \cdot \vec{u} = 0$, (C.42) reduces to

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \nabla^2 u_i \tag{C.43}$$

The above equation can also be written in vector form as follows

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{u} \tag{C.44}$$

The above equation is the Navier-Stokes equation for an incompressible fluid. If viscous effects are negligible, then $\mu = 0$ and equation (C.44) reduces to

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{g} \tag{C.45}$$

This is called the *Euler equation.*

C.5 Conservation of Energy

In this section, we will derive various forms of the mechanical energy equation. We will also discuss the first law of thermodynamics and, through the use of the mechanical energy equation, we will derive the thermal energy equation.

C.5.1 Mechanical Energy Equation

There are various types of energy to consider within a fluid. For instance, the thermal energy (internal energy), and mechanical energy which can be further divided into kinetic and potential energy. An expression for the kinetic energy is given by

$$\frac{mu_i^2}{2}$$

If we are interested in the kinetic energy per unit volume, the expression becomes

$$\frac{m}{V}\frac{u_i^2}{2} = \frac{\rho u_i^2}{2}$$

and per unit mass is

$$\frac{u_i^2}{2}$$

We can use the momentum equation (C.24) to derive an equation for the mechanical energy of a fluid by multiplying this equation by u_i (and summing over i) as follows:

$$\rho u_i \frac{Du_i}{Dt} = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j}$$

This equation can be rewritten as

$$\rho \frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j} \tag{C.46}$$

This is the simplest form of the mechanical energy equation. In this equation, we have written u_i^2 for $\sum_i u_i u_i$.

Another form of the mechanical energy equation can be found by multiplying the continuity equation (C.10) by $\frac{\rho u_i^2}{2}$, which gives

$$\frac{1}{2}\rho u_i^2 \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right] = 0$$

Now, if we add this equation to (C.46), we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i^2\right) + \frac{\partial}{\partial x_j}\left[u_j \frac{1}{2}\rho u_i^2\right] = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j}$$

Using vector notation and defining

$$E \equiv \frac{1}{2}\rho u_i^2$$

as the kinetic energy per unit volume gives

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{u}E) = \rho \vec{u} \cdot \vec{g} + \vec{u} \cdot (\nabla \cdot \tau) \tag{C.47}$$

The second term in the above equation ($\nabla \cdot (\vec{u}E)$) represents the divergence of the kinetic energy flux $\vec{u}E$. *Flux divergence* terms like this emerge in energy balances and it signifies the net loss at a point due to the divergence of the flux. Flux divergence terms are also called *transport* terms because they transfer quantities from one region to another without affecting the entire field in which they exist.

Let us now shift our attention back to the first mechanical energy equation we derived in (C.46). Let us consider the term $u_i \frac{\partial \tau_{ij}}{\partial x_j}$ from (C.46). u_i represents the velocity and $\frac{\partial \tau_{ij}}{\partial x_j}$ represents the net force at a point due to stress differences on opposing faces of an element. The force due to the differences in stress accelerates the fluid at that point and increases the kinetic energy. The total work done by surface forces (stress) on a fluid element is $\frac{\partial}{\partial x_j}(u_i \tau_{ij})$ and this can be decomposed into two components:

$$\frac{\partial}{\partial x_j} = u_i \frac{\partial \tau_{ij}}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j} \tag{C.48}$$

Part of the work is used to accelerate the fluid and in turn increase the kinetic energy $\left(u_i \frac{\partial \tau_{ij}}{\partial x_j}\right)$, and the other part is used to deform the element without accelerating it, which increases the internal energy $\left(\tau_{ij} \frac{\partial u_i}{\partial x_j}\right)$.

We will now use the equation for total work to derive an alternate version of the mechanical energy equation. Firstly, we will rearrange (C.48) and isolate for $u_i \frac{\partial \tau_{ij}}{\partial x_j}$ and then substitute this expression into (C.46) to obtain the following

$$\rho \frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \rho u_i g_i + \frac{\partial}{\partial x_j}(u_i \tau_{ij}) - \tau_{ij}\left(\frac{\partial u_i}{\partial x_j}\right)$$

We will now substitute the expression for τ_{ij} with Stokes' assumption applied from the previous section (C.40) into the above equation to obtain

$$\rho \frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \rho u_i g_i + \frac{\partial}{\partial x_j}(u_i \tau_{ij}) - \left[-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu \nabla \cdot \vec{u}\delta_{ij}\right]\left(\frac{\partial u_i}{\partial x_j}\right)$$

After the above simplifications have been applied, we get the following equation

$$\rho \frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \rho \vec{g}\vec{u} + \frac{\partial}{\partial x_j}(u_i \tau_{ij}) + p(\nabla \cdot \vec{u}) - \phi \tag{C.49}$$

where

$$\phi = 2\mu\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}(\nabla \cdot \vec{u})\delta_{ij}\right]^2$$

Now, by defining

$$E = \frac{\rho u_i^2}{2}$$

and substituting this expression into (C.49), we obtain the following equation

$$\frac{DE}{Dt} = \rho \vec{g}\vec{u} + \frac{\partial}{\partial x_j}(u_i \tau_{ij}) + p(\nabla \cdot \vec{u}) - \phi \tag{C.50}$$

where the first term on the right-hand side of the equation represents the rate of work done by the body force (gravity in this case), the second term represents the total rate of work done by the surface force, the third term represents the rate of work done by volume expansion, and the last term is the rate of viscosity dissipation.

We will now derive an alternate version of the mechanical energy equation in integral form by integrating (C.50) over a fixed volume, V. After we make use of the material derivative, (C.50) becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(u_i E) = \rho \vec{g}\vec{u} + \frac{\partial}{\partial x_j}(u_i \tau_{ij}) + p(\nabla \cdot \vec{u}) - \phi$$

Finally, we apply the Divergence Theorem to the above expression to obtain the required integral equation

$$\frac{d}{dt}\int EdV + \int E\vec{u}dA = \int \rho \vec{g}\vec{u}dV + \int u_i \tau_{ij} dA_j + \int p(\nabla \cdot \vec{u})dV - \int \phi dV \tag{C.51}$$

The first term on the left-hand side of this equation represents the rate of change of kinetic energy and the second term represents the rate of outflow across the boundary of the volume. The first term on the right-hand side of the equation represents the work done by body forces, the second term signifies the work done by surface forces because $\tau_{ij} d\vec{A}_j$ is the force in the i direction and $u_i \tau_{ij} d\vec{A}_j$ is the scalar product of the force with the velocity vector, the third term is the work done by volume expansion and the last term represents the viscous dissipation.

Thus far, we have derived various forms of the mechanical energy equation using momentum conservation. However, in fluid flows involving temperature variations, we require a different principle for energy conservation that takes into account the thermodynamics of the problem.

The total energy of a system is the sum of the internal energy, e, and the kinetic energy, E. The total energy is also called the *stored* energy. The internal energy per unit mass is given by

$$e = C_V T$$

where C_V is the specific heat at constant volume. The kinetic energy per unit volume is given by

$$E = \frac{\rho u_i^2}{2}$$

and so $\frac{u_i^2}{2}$ represents the kinetic energy per unit mass.

We now consider a moving material volume, $\mathcal{V}$, with surface, A. The total energy is given by

$$\int_{\mathcal{V}} \rho(e + \frac{1}{2}u_i^2)d\mathcal{V}$$

The quantity $e + \frac{1}{2}u_i^2$ is often called the stored energy.

The first law of thermodynamics states that the rate of change of stored energy is equal to the sum of the rate of work done and the rate of heat added to the material volume. In other words,

$$de = dQ + W$$

where de is the change in internal energy, dQ is the change in heat, and W is the work done on the system. Now define $\vec{q}$ to be the heat flux vector per unit area. Then, the first law of thermodynamics can be expressed as

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho \left(e + \frac{1}{2}u_i^2\right) d\mathcal{V} = \int_{\mathcal{V}} \rho g_i u_i d\mathcal{V} + \int_A \tau_{ij} u_i dA_j - \int_A q_i d\vec{A}_i \tag{C.52}$$

The first term of the above equation represents rate of change of energy, the second term is the work done by body forces, the third term is the work done by surface forces and the last term is the heat flux through the surface. We note that the last term has a negative sign since the vector $d\vec{A}$ acts outward and $\int \vec{q}d\vec{A}$ represents the heat outflow. Thus, $-\int \vec{q}d\vec{A}$ represents the heat flux into $\mathcal{V}$.

We now write the left-hand side of equation (C.52) as

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho \left(e + \frac{1}{2}u_i^2\right) d\mathcal{V} = \int_{\mathcal{V}} \rho \frac{D}{Dt}\left(e + \frac{1}{2}u_i^2\right) d\mathcal{V}$$

We are able to do this because $\mathcal{V}$ is a material volume. We have made use of the fact that

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho f d\mathcal{V} = \int_{\mathcal{V}} \rho \frac{Df}{Dt} d\mathcal{V}$$

which can be derived from (C.4) (Reynolds transport theorem). On the right- hand side of (C.52), we change the surface integrals to volume integrals using the Divergence theorem and we obtain the following result

$$\int_{\mathcal{V}} \rho \frac{D}{Dt}\left(e + \frac{1}{2}u_i^2\right) d\mathcal{V} = \int_{\mathcal{V}} \left[\rho g_i u_i + \frac{\partial}{\partial x_j}(\tau_{ij}u_i) - \frac{\partial q_i}{\partial x_i}\right] d\mathcal{V}$$

Thus,

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}u_i^2\right) d\mathcal{V} = \rho g_i u_i + \frac{\partial}{\partial x_j}\left(\tau_{ij}u_i\right) - \frac{\partial q_i}{\partial x_i} \tag{C.53}$$

This is the first law of thermodynamics in differential form, which consists of both mechanical and thermal energy terms. The mechanical energy is composed of both the kinetic and potential energy terms, and $\rho\frac{De}{Dt}$ is the thermal energy term.

We will now derive the thermal energy equation. We begin by subtracting the mechanical energy equation (C.50) (with E replaced by $\frac{1}{2}u_i^2$) from (C.53) to obtain

$$\rho \frac{De}{Dt} = -p(\nabla \cdot \vec{u}) + \phi - \frac{\partial q_i}{\partial x_i}$$

Now, rewriting this equation in vector form, gives the desired result:

$$\rho \frac{De}{Dt} = -p(\nabla \cdot \vec{u}) - \nabla \cdot \vec{q} + \phi \tag{C.54}$$

The first term represents the rate of change of internal energy, the second term represents the rate of volume compression, the third term represents the rate of heat flux and the last term represents the rate of viscous dissipation. This equation is called the *thermal energy equation* or the *heat equation* and it tells us that the internal energy increases due to the convergence of heat, volume compression and heating due to viscous dissipation.

Appendix D

Dispersion Relations

D.1 Dispersion Relations Defined

In this Appendix, we will briefly define what a dispersion relation is, following the notation in Section 14.2 of Haberman [7], and then derive the dispersion relations for both the β-plane and Boussinesq problems, in the subsequent sections.

A *linear dispersive system* (in a two-dimensional configuration defined in terms of coordinates x and y) is one which admits a solution of the form

$$\psi(\vec{x}, t) = Ae^{\imath(kx+ly-\omega t)} \tag{D.1}$$

where k and l are the wave numbers in the x and y directions, respectively, A is the amplitude, and $\omega = \omega(k, l)$ is the frequency. Furthermore, a dispersive system has a phase speed of $\vec{c} = (c_x, c_y)$ where

$$c_x = \frac{\omega(k,l)}{k} \tag{D.2}$$

$$c_y = \frac{\omega(k,l)}{l} \tag{D.3}$$

are the components of the phase speed in the x and y directions, respectively. The phase speed is not constant but depends on k and l. If the phase speed is constant, the solution is said to be nondispersive. We can also add to the definition of a dispersive solution the fact that

$$\nabla\omega = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}\right) \neq \text{constant vector}$$

By substituting (D.1) into the partial differential equation that describes the evolution of ψ in space and time, a relation between ω, k and l emerges. This relation is called a *dispersion relation* and has the form

$$G(\omega, k, l) = 0 \tag{D.4}$$

where the function G is determined by the equation for ψ.

D.2 Dispersion Relation for β-Plane Problem

We will now find the dispersion relation of the linear part of equation (3.8). In other words, when $\varepsilon = 0$. After substituting

$$\psi = Ae^{i(kx+ly-\omega t)}$$

into the linear part of (3.8) and simplifying, we obtain

$$[i\omega(k^2+l^2) - i\bar{u}k(k^2+l^2) + i\beta k - \nu(k^2+l^2)^2]Ae^{i(kx+ly-\omega t)} = 0 \tag{D.5}$$

After simplfying, we obtain the following dispersion relation

$$\omega = \bar{u}k - \frac{\beta k}{k^2+l^2} - i\nu(k^2+l^2) \tag{D.6}$$

We note that the dispersion relation can be written as

$$\omega = \lambda_2 + i\lambda_1 \tag{D.7}$$

where λ_1 and λ_2 are analagous to a and b defined in (3.24). The real part of ω is the frequency of the wave. The imaginary part produces an exponentially decaying factor of $e^{\lambda_1 t} = e^{-\nu(k^2+l^2)}$ which shows that the viscosity acts to damp the amplitude of the wave to zero as $t \to \infty$. Comparing

$$\psi = Ae^{i(kx+ly-\omega t)} = Ae^{-i\omega t}e^{i(kx+ly)} \tag{D.8}$$

to the definition of the inverse Fourier transform (3.10), we see that

$$\hat{\psi} = Ae^{-\imath\omega t} \tag{D.9}$$

Since the $X_\jmath$'s in (3.16) are defined in terms of $\hat{\psi}$, the dispersion relation is another way to find the general solution of (3.20). Take X_2 for instance, where $k = 0$ and $l = 1$. Then, $\omega = -i\nu$ and thus

$$\hat{\psi}(0,1,t) = Ae^{-\nu t}$$

As a result,

$$X_2 = Ae^{-\nu t}$$

where A is the same as the constant $X_\jmath(0)$ which is determined from the initial condition.

D.3 Dispersion Relation for Boussinesq Problem

Akin to what was done for the β-plane problem, we will also find the dispersion relations for the linear part of equations (4.17) and (4.18). We begin by substituting

$$\psi = Ae^{\imath(kx+mz-\omega t)} \quad \text{and} \quad \theta = Be^{\imath(kx+mz-\omega t)} \tag{D.10}$$

into both linear equations. After simplifying, we obtain the following two equations in two unknowns (A and B)

$$[i\omega(k^2+m^2) - i\bar{u}k(k^2+m^2) - \nu(k^2+m^2)^2]A - i\frac{g}{\rho_0}kB = 0 \tag{D.11}$$

$$[-i\omega + i\bar{u}k + \kappa(k^2+m^2)]B - i\frac{\Delta T}{H}kA = 0 \tag{D.12}$$

To eliminate B, for instance, we multiply (D.11) by $-i(\omega-\bar{u}k)+\kappa(k^2+m^2)$ and (D.12) by $-i\frac{g}{\rho_0}k$ and add the two equations together. The following quadratic equation for $(\omega - \bar{u}k)$ results:

$$\left[(k^2+m^2)(\omega-\bar{u}k)^2 + i(k^2+m^2)^2(\kappa+\nu)(\omega-\bar{u}k) - \kappa\nu(k^2+m^2)^3 - \frac{\Delta T g}{H\rho_0}k^2\right]A = 0 \tag{D.13}$$

After using the quadratic formula and a bit of algebraic manipulation, we arrive at the following dispersion relation for the Boussinesq problem:

$$\omega = \bar{u}k - i\frac{(\kappa+\nu)}{2}(k^2+m^2) \pm \frac{\sqrt{4\frac{\Delta T g}{H\rho_0}k^2(k^2+m^2) - (k^2+m^2)^4(\kappa-\nu)^2}}{2(k^2+m^2)} \tag{D.14}$$

Once again, we note that the dispersion relations can be decomposed into the sum of a real and imaginary part. For the case of an inviscid and nonconducting fluid ($\nu = 0$, $\kappa = 0$), the dispersion relation is

$$\omega = \bar{u}k \pm \frac{\sqrt{\frac{g\Delta T}{H\rho_0}}k}{\sqrt{k^2+m^2}} \tag{D.15}$$

The ratio $\sqrt{\frac{g\Delta T}{H\rho_0}}$ is called the *Brunt-Väisälä frequency* or *buoyancy frequency*, and is usually denoted by N. It gives a measure of the extent of the stratification of the fluid, or, how rapidly the density and temperature decrease with height. The dispersion relation in terms of N is

$$\omega = \bar{u}k - i\frac{(\kappa+\nu)}{2}(k^2+m^2) \pm \frac{\sqrt{N^2k^2 - \frac{(k^2+m^2)^3(\kappa-\nu)^2}{4}}}{\sqrt{k^2+m^2}} \tag{D.16}$$

By comparing ψ and θ in (D.10) to the inverse Fourier transforms defined in (4.22), we find that

$$\hat{\psi}(k,m,t) = Ae^{-\imath\omega t} \quad \text{and} \quad \hat{\theta}(k,m,t) = Be^{-\imath\omega t} \tag{D.17}$$

This demonstrates that dispersion relations offer an alternative method to solve the linear equations (4.32) and (4.33). For example, let us consider X_2 and Y_2, where $k = 0$ and $m = 1$. Then, $\omega = -i\nu$ for X_2 and $\omega = -i\kappa$ for Y_2. Thus,

$$X_2 = Ae^{-\nu t} \quad \text{and} \quad Y_2 = Be^{-\kappa t}$$

where A is the same as $X_\jmath(0)$ and B is the same as $Y_\jmath(0)$, which are determined from the specified initial conditions.

Bibliography

[1] Acheson, D.J., 1990, *Elementary fluid dynamics*, Clarendon Press, Oxford.

[2] Aris, R., 1962, *Vectors, tensors, and the basic equations of fluid mechanics*, Dover.

[3] Bender, C.M., and Orszag, S.A., 1999, *Advanced mathematical methods for scientists and engineers, asymptotic methods and perturbation theory*, Springer, New York.

[4] Boussinesq, J., 1903, *Theorie analytique de la chaleur*, Vol. II., Gauthier-Villars, Paris.

[5] Campbell, L.J., and Maslowe, S.A., 1998, Forced Rossby wave packets in barotropic shear flows with critical layers, *Dyn. Atmos. Oceans* **28**, 9-37.

[6] Cushman-Roisin, B., 1994, *Introduction to geophysical fluid dynamics*, Prentice Hall, New Jersey.

[7] Haberman, R., 2004, *Applied partial differential equations with Fourier series and boundary value problems*, 4th ed., Pearson Prentice Hall, New Jersey.

[8] Holton, J.R., 1979, *An introduction to dynamic meteorology*, 2nd ed., Academic Press, New York.

[9] Kundu, P.K., and Cohen, I.M., 2004, *Fluid mechanics*, 3rd ed., Academic Press, California.

[10] Pedlosky, J., 1987, *Geophysical fluid dynamics*, 2nd ed., Springer-Verlag, New York.

[11] Saltzman, B., 1962, Finite amplitude free convection as an initial value problem, *J. Atmos. Sci.* **19**, 329-341.

[12] Spiegel, E.A., and Veronis, G., 1960, On the Boussinesq approximation for a compressible fluid *Astrophys. J.*, **131**, 442-447.

[13] Walter, W., 1998, *Ordinary differential equations*, Springer-Verlag, New York.