ADJOINT SENSITIVITY ANALYSIS ALGORITHMS FOR GENERAL CIRCUITS WITH DISTRIBUTED MULTICONDUCTOR TRANSMISSION LINES

by
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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Applied Sciences

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September 2012

Your file Votre référence
ISBN: 978-0-494-93507-1

Our file Notre référence
ISBN: 978-0-494-93507-1

Abstract

Adjoint sensitivity analysis is a widely-used technique to evaluate few circuit responses with respect to many circuit parameters. Existing adjoint sensitivity analysis for circuits containing lossy multiconductor transmission lines (MTL) is dependent on the details of the specific macromodel being used for MTL. In this thesis, a generalized algorithm based on the variational approach is developed for the first-order time-domain adjoint sensitivity analysis of lossy MTLs with respect to electrical and/or physical parameters. Also, a novel second-order adjoint sensitivity analysis algorithm based on the variational approach is developed for the circuits containing distributed MTL circuits in both frequency and time-domains. While the proposed algorithms inherit all the advantages of the adjoint sensitivity analysis, they are independent of the specifics of the MTL macromodel used. In addition, the application of the proposed second-order sensitivity analysis algorithm for delay optimization is presented. Several numerical examples are presented to demonstrate the validity and accuracy of the proposed algorithms.

Dedicated to my Family

Acknowledgments

I would like to thank my supervisors, Professor Michel Nakhla and Professor Ram Achar for their guidance and motivation throughout this research. They provided the initial inspiration for this work and gave critical guidance throughout the process.

I would like to acknowledge and thank my colleagues Ashok Narayanan, Behzad Nouri, Douglas Paul, Mina Farhan, Andrew Charest, Jiwon Kang, Dragan Trifkovic, Mehrdad Rufuie, Harjot Dhindsa and Harjinder Saini for the helpful discussions and relaxing chats which made my stay at Carleton a pleasant and enjoyable experience.

I would like to thank all the members of faculty, staff and students in electronics department. A special thanks are due to Nagui Mikhail and Scott Bruce for their technical support.

Lastly, I give a special thanks to my parents Rajinder Singh Saini and Baljeet Kaur Saini, sister Navjot Kaur Saini and wife Rajwinder Kaur Girn, for their endless support and strong encouragement.

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List of Symbols

$\boldsymbol{\xi}_n(t)$, $\Xi_n(s)$	MNA variable of the n_{th} adjoint system in time and frequency domain, respectively.
$\mathbf{x}(t), \mathbf{X}(s)$	MNA variable of the adjoint system in time and frequency domain, respectively.
$\mathbf{b}(t), \mathbf{B}(s)$	Source vector in the original system in time and frequency domain, respectively.
$\mathbf{C}, \mathbf{G}$	Constant memory and memoryless element of a MNA system.
$\mathbf{M}(s)$	Matrix formed by $\mathbf{G}$, $\mathbf{C}$ and $\mathbf{A}_k$ of MNA in frequency domain.
$\mathbf{z}(t), \mathbf{Z}(s)$	Derivative of MNA variable of original system w.r.t. a parameter of a circuit in time and frequency domain, respectively.
$\mathbf{j}_{\lambda_i}(t)$, $\mathbf{J}_{\lambda_i}(s)$	Source related to an i_{th} parameter (λ_i) of the original circuit in time and frequency domain, respectively.
$\mathbf{v}_k(t)$, $\mathbf{V}_k(s)$	Voltage across the k_{th} transmission line in the original system in time and frequency domain, respectively.

$\mathbf{i}_k(t), \mathbf{I}_k(s)$	Current across the k_{th} transmission line in the original system in time and frequency domain, respectively.
$\boldsymbol{\psi}_k^n(t)$, $\boldsymbol{\Psi}_k^n(s)$	Voltage across the k_{th} transmission line in the n^{th} adjoint system in time and frequency domain, respectively.
$\boldsymbol{\phi}_k^n(t)$, $\boldsymbol{\Phi}_k^n(s)$	Current across the k_{th} transmission line in the n_{th} adjoint system in time and frequency domain, respectively.
$\mathbf{y}_k(t)$, $\mathbf{Y}_k(s)$	Admittance parameter of k_{th} transmission line in time and frequency domain, respectively.
$\mathbf{a}_k(t)$, $\mathbf{A}_k(s)$	MNA size admittance parameter of k_{th} transmission line in time and frequency domain, respectively.
$\mathbf{D}_k$	Selector matrix of the k_{th} transmission line.
i, j	Indexes for the selector matrix $\mathbf{D}_k$.
$\mathbf{f}(\mathbf{x}(t))$	Vector of nonlinear element in MNA equation in time-domain.
$\mathcal{F}\{.\}$	Fourier transform operator.
$\mathcal{F}^{-1}\{.\}$	Inverse Fourier transform operator.
W	Objective (or cost) function, which is used both in time and frequency domain.
w	Function of the MNA variable of the original system and all the parameters $\boldsymbol{\lambda}$, which is used both in time and frequency domain.
s	Laplace variable $s = i\omega$.

λ	Parameter of a circuit.
$\boldsymbol{\lambda}$	Vector of all the parameters of a circuit.
m, n	Vector indexes for the parameter vector $\boldsymbol{\lambda}$ of a circuit.
$\mathbf{U}_{n\times n}$	$n \times n$ Identity matrix.
d	Length of a transmission line.
$\mathbf{R}_k, \mathbf{L}_k, \mathbf{G}_k, \mathbf{C}_k$	Per-unit-length resistance, inductance, conductance, capacitance of the k^{th} transmission line.
$\mathbf{Z}_k$	Per-unit-length impedance of k^{th} transmission line
$\boldsymbol{\Omega}_k$	Per-unit-length admittance of k^{th} transmission line
p, q	Matrix indexes for the per-unit-length matrix.
γ_i	The i_{th} eigenvalue of $\mathbf{Z}_k\boldsymbol{\Omega}_k$
$\boldsymbol{\beta}_i$	The i_{th} eigenvector of $\mathbf{Z}_k\boldsymbol{\Omega}_k$
$\mathbf{S}_q$	Related to $\mathbf{Y}_k$
$\mathbf{S}_v$	Related to $\mathbf{Y}_k$
$\boldsymbol{\Gamma}$	Related to $\mathbf{Y}_k$
$\mathbf{E}_1, \mathbf{E}_2$	Diagonal matrices related to $\mathbf{Y}_k$
k_i, l_i, m_i	Related to $\frac{\partial^2 \mathbf{E}_1}{\partial \lambda_m \partial \lambda_n}$ and $\frac{\partial^2 \mathbf{E}_2}{\partial \lambda_m \partial \lambda_n}$
T_d	Delay time
$\boldsymbol{\alpha}$	Vector of first-order sensitivities w.r.t. all the parameters of a circuit.
$\boldsymbol{\Lambda}$	Hessian matrix of second-order sensitivities w.r.t. all the parameters of a

circuit.

$\mathbf{u}$	Related to second-order sensitivity algorithm in time-domain.
$\mathbf{r}$	Related to second-order sensitivity algorithm in time-domain.
$\mathbf{k}$	Related to second-order sensitivity algorithm in time-domain.
$\mathbf{e}$	Related to second-order sensitivity algorithm in time-domain.
$\mathbf{p}$	Related to second-order sensitivity algorithm in time-domain.
$\mathbf{q}_n$	Related to second-order sensitivity algorithm in time-domain where n could be 1,2 and 3.

Abbreviations

FFT	Fast Fourier Transform
IFFT	Inverse Fast Fourier Transform
MoC	Method of Characteristics
MRA	Matrix Rational Approximation
MTL	Multi-conductor Transmission Line
p. u. l.	Per Unit Length
TL	Transmission Line
VLSI	Very Large Scale Integration
w.r.t.	with respect to
MNA	Modified Nodal Analysis
RLCG	Resistance-Inductance-Capacitance-Conductance

Chapter 1

Introduction

1.1 Background and motivation

The continually increasing operating frequencies and sharper edge rates coupled with the higher density/complexity of modern integrated circuits and microwave applications have made interconnect analysis and optimization a challenging task. Effects such as reflections, crosstalk, and propagation delays associated with the interconnects have become critical factors, and improperly designed interconnects can result in increased signal delay, ringing, and inadvertent false switching [1]. Interconnections are ubiquitous, being present at various levels of the design hierarchy such as on-chip, packaging structures, multichip modules (MCMs), printed circuit boards (PCBs), and backplanes. At higher frequencies, lumped interconnect models become inadequate and distributed multi-conductor transmission line (MTL) models based on Telegrapher's equations become necessary.

Interconnects play an important role in determining the performance of modern multi-function designs. These performance metrics include operating frequency, density, power-consumption and area. For example, shortening interconnects can reduce the problem of delay and reflections while providing for higher density. However, a higher density can lead to excessive crosstalk between adjacent interconnects. Designers must make proper trade-offs, often between conflicting design requirements, to obtain the best possible performance. Therefore, efficient and accurate sensitivity analysis of circuit response w.r.t. interconnect parameters becomes significantly important in identifying the critical design components in tolerance assignments and in optimizing the overall interconnect performance [2–11].

Sensitivity analysis of circuits in terms of first-order derivatives (or sensitivities) can be found in the literature [2–13]. [3] presents a new algorithm for direct sensitivity analysis of transmission line circuits modeled using the Matrix Rational Approximation [14] and subsequently expanded its scope in [4] to include delay based MTL macromodels. Recently, a generic method for direct sensitivity analysis of distributed interconnects independent of the details of the macromodel used [5] was proposed. Using direct sensitivity analysis [2–5], the sensitivity of all the outputs w.r.t. a single parameter can be obtained. However, designers are typically interested in knowing the sensitivity of a particular output w.r.t. several parameters. In this case, direct sensitivity approach becomes computationally expensive as the number of solutions required increases with the number of parameters.

To address this difficulty, adjoint sensitivity analysis [12] was first introduced for lumped circuits by Director and Rohrer, based on Tellegen's theorem [15] [16]. Using the adjoint analysis [12], sensitivities of a single output w.r.t. all parameters of interest can be evaluated at once, providing more practical and significant computational advantages. Also, an alternate method based on variational approach can be found for adjoint sensitivity analysis of lumped circuits [7], [13] and for the special case of single lossless transmission line macromodels [8].

In gradient based optimization, it is important not only to evaluate the first-order sensitivities but also the second-order sensitivities to speed-up the nonlinear optimization iterations [17]. To achieve this speed-up, the second-order time-domain adjoint sensitivity analysis based on Tellegen's theorem [15] [16] for a linear circuit (excluding multiconductor transmission lines) was developed in [17].

One application of the sensitivity analysis is in delay optimization. Several sensitivity based (gradient based) delay optimization approaches are available in literature [17], [18] for circuits containing transmission lines. Among them [17] uses the lumped approximation whereas [18] uses the moment matching model. It has been demonstrated in [17] that the second-order sensitivity is desirable in gradient-based delay optimization for the nonlinear optimization iterations.

Next section presents the list of contributions of this thesis.

1.2 Contributions

The objective of this thesis is to develop an efficient algorithm to evaluate first and second-order sensitivities of circuits including distributed transmission line in time-domain. In this thesis, an efficient algorithm based on variational approach for adjoint sensitivity analysis of multiconductor transmission line circuits is presented. An important additional advantage of the proposed algorithm is that its formulation is independent of the specifics of the MTL macromodel used (i.e., the proposed method can be adopted in conjunction with a wide variety of MTL macromodels that are available in the literature; without requiring to re-derive the constituting sensitivity relations for each case). The specific contributions made in this thesis are listed below.

1. For a non-linear circuit including MTL, an analytic formulation of the adjoint circuit equations and first-order sensitivity equations are derived in the time domain (chapter 4) [19].
2. For a linear circuit including MTL, an analytic formulation of the adjoint circuit equations and second-order sensitivity equations are derived in the

frequency domain (chapter 5).

3. For a non-linear circuit including MTL, an analytic formulation of the adjoint circuit equations and second-order sensitivity equations are derived in the time domain (chapter 6).

4. A new equation for evaluating the second-order sensitivity of delay is proposed (chapter 7). Furthermore, the application of the second-order frequency domain adjoint sensitivity of MTL in evaluating the sensitivities of delay is introduced.

1.3 Organization of the thesis

This thesis is organized as follows. Chapter 2 provides a review of sensitivity analysis algorithms with emphasis on formulation of circuit and objective (or cost) function. Chapter 3 provides a review of first-order time-domain adjoint sensitivity analysis for nonlinear circuits. Chapter 4 presents the details of the proposed first-order time-domain adjoint sensitivity analysis including distributed transmission lines. Chapter 5 presents the details of the proposed second-order frequency-domain adjoint sensitivity analysis including distributed transmission lines. Chapter 6 presents the details of the proposed second-order time-domain adjoint sensitivity analysis including distributed transmission lines. Chapter 7 presents the details of the proposed second-order delay sensitivity analysis in time-domain for linear circuit. Chapter 8 provides the conclusions and several possible future research work that can be undertaken from the concepts that are developed in this thesis.

Chapter 2

Review of sensitivity analysis algorithms

In many situation, designers must make proper trade-offs, often between conflicting design requirements, to obtain the best possible performance. Therefore, efficient and accurate sensitivity analysis of circuit response w.r.t. circuit parameters becomes significantly important in identifying the critical design components in tolerance assignments and in optimizing the overall circuit performance. For example, in gradient based optimizations, sensitivities of an objective (or cost) function are required to speed-up the nonlinear optimization iterations [17]. In this chapter, the review of sensitivity analysis algorithms are presented.

The remainder of this chapter is arranged as follows. In Section 2.1, a formulation of circuit equations is provided. Section 2.2 shows the formulation of the objective function. Sections 2.3 and 2.4 review sensitivity approaches using perturbation and direct sensitivity techniques.

2.1 Formulation of circuit equations

Consider a general circuit consisting of linear and nonlinear components. The corresponding modified nodal analysis (MNA) equations [3] can be written in the time-domain as

$$\mathbf{C}\frac{d\mathbf{x}(t)}{dt} + \mathbf{G}\mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t)) = \mathbf{b}(t) \tag{2.1}$$

where

- $\mathbf{x}(t) \in \Re^{n\times 1}$ is the vector of unknowns corresponding to node voltages, independent and dependent voltage source currents, and inductor currents; $\mathbf{b}(t) \in \Re^{n\times 1}$ is an input vector with entries determined by the independent current and voltage sources; $\mathbf{f}(\mathbf{x}(t)) \in \Re^{n\times 1}$ is a vector related to nonlinear elements and n is the total number of MNA variables;
- $\mathbf{G}$, $\mathbf{C} \in \Re^{n\times n}$ are constant matrices describing the lumped memory-less and memory elements, respectively.

In optimization algorithms, the objective function is required using $\mathbf{x}(t)$ of (2.1).

2.2 Formulation of the objective function

Consider the MNA equation (2.1) and an objective function $W(t)$ whose value is to be optimized at time $t = T$ as given by

$$W(T) = \int_0^T w(\mathbf{x}(t), \boldsymbol{\lambda}) dt \tag{2.2}$$

where $w(\mathbf{x}(t), \boldsymbol{\lambda})$ is a scalar function and $\boldsymbol{\lambda}$ is a vector of N_λ parameters of a circuit given by [13]

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{N_\lambda} \end{bmatrix} \tag{2.3}$$

In order to perform the required optimization, we need the sensitivity (or derivative) of the objective function, which can be obtained by differentiating (2.2) w.r.t. a particular parameter λ_m as

$$\frac{\partial W(T)}{\partial \lambda_m} = \int_0^T \left(\left[\frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \mathbf{x}} \right] \mathbf{z}(t) + \frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \lambda_m} \right) dt \tag{2.4}$$

where $m \in \{1, \cdots, N_\lambda\}$, and

$$\mathbf{z}(t) = \frac{\partial \mathbf{x}(t)}{\partial \lambda_m} \tag{2.5}$$

is the sensitivity vector of circuit variables w.r.t. a parameter λ_m.

Several techniques are available to evaluate $\mathbf{z}(t)$ [20] [5]. The technique based on perturbation is reviewed in Section 2.3. In Section 2.4, a details of the direct sensitivity analysis technique are reviewed.

2.3 Perturbation technique

The first-order derivative of $\mathbf{x}(\lambda_m)$ can be approximated by a forward difference approximation given by [20]

$$\mathbf{z}(t) = \frac{\partial \mathbf{x}(\lambda_m)}{\partial \lambda_m} \doteq \frac{\mathbf{x}(\lambda_m + h) - \mathbf{x}(\lambda_m)}{h} + O(h) \tag{2.6}$$

or a centered difference approximation given by

$$\mathbf{z}(t) = \frac{\partial \mathbf{x}(\lambda_m)}{\partial \lambda_m} \doteq \frac{\mathbf{x}(\lambda_m + h) - \mathbf{x}(\lambda_m - h)}{2h} + O(h^2) \tag{2.7}$$

where $h > 0$ is a small value and the big O notation refers to the order of error in the approximation.

Similarly, (2.7) can be extended to second-order and given here for completion, which will be used in chapters 5 and 6. The second-order derivatives of $\mathbf{x}(\lambda_m, \lambda_n)$ can be approximated by a centered difference approximation given by

$$\frac{\partial^2 \mathbf{x}(\lambda_m, \lambda_n)}{\partial \lambda_m \partial \lambda_n} \doteq \frac{\begin{bmatrix} \mathbf{x}(\lambda_m + h, \lambda_n + k) - \mathbf{x}(\lambda_m + h, \lambda_n - k) \\ -\mathbf{x}(\lambda_m - h, \lambda_n + k) + \mathbf{x}(\lambda_m - h, \lambda_n - k) \end{bmatrix}}{4hk} \tag{2.8}$$

and

$$\frac{\partial^2 \mathbf{x}(\lambda_m, \lambda_n)}{\partial \lambda_m^2} \doteq \frac{\mathbf{x}(\lambda_m + h, \lambda_n) - 2\mathbf{x}(\lambda_m, \lambda_n) + \mathbf{x}(\lambda_m - h, \lambda_n)}{h^2} \tag{2.9}$$

where $k > 0$ is a small value. The next section reviews the direct sensitivity analysis technique to evaluate $\mathbf{z}(t)$.

2.4 Direct sensitivity analysis technique

Using the direct sensitivity approach [5], $\mathbf{z}(t)$ in (2.4) can be obtained by differentiating (2.1) as follows:

$$\mathbf{C}\frac{d\mathbf{z}(t)}{dt} + \mathbf{G}\mathbf{z}(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}\mathbf{z}(t) = \mathbf{j}_{\lambda_m}(t) \tag{2.10}$$

where

$$\mathbf{j}_{\lambda_m}(t) = \frac{\partial \mathbf{b}(t)}{\partial \lambda_m} - \frac{\partial \mathbf{C}}{\partial \lambda_m}\frac{d\mathbf{x}(t)}{dt} - \frac{\partial \mathbf{G}}{\partial \lambda_m}\mathbf{x}(t) - \frac{\partial \mathbf{f}(t)}{\partial \lambda_m} \tag{2.11}$$

As can be seen from (2.10) and (2.11), evaluating $\mathbf{z}(t)$ w.r.t. N_λ variables requires the solution of (2.10) and (2.11) N_λ times, making the process computationally expensive. In order to address this difficulty, the adjoint sensitivity analysis was proposed in the literature [12] [13]. The next chapter provides a review of adjoint sensitivity analysis.

Chapter 3

Review of adjoint sensitivity analysis algorithms

Using direct sensitivity analysis [2–5], the sensitivity of all the outputs w.r.t. a single parameter can be obtained. However, designers are typically interested in knowing the sensitivity of a particular output w.r.t. several parameters. In this case, the direct sensitivity approach becomes computationally expensive as the number of solutions required increases with the number of parameters.

To address this difficulty, adjoint sensitivity analysis [12] was first introduced for lumped circuits by Director and Rohrer, based on Tellegen's theorem [15] [16]. Using the adjoint analysis [12], sensitivities of a single output w.r.t. all parameters of interest can be evaluated at once, providing more practical and significant computational advantages. Also, an alternate method based on variational approach can be found for adjoint sensitivity analysis [13].

To perform the optimization, we need the derivative of the objective function $\frac{\partial W(T)}{\partial \lambda_m}$ which can be obtained by using adjoint sensitivity analysis and is reviewed in this chapter.

The rest of this chapter is arranged as follows. Section 3.1 reviews an adjoint sensitivity analysis based on Tellegen's theorem [12]. Section 3.2 reviews an adjoint sensitivity analysis based on variational approach [13].

3.1 Adjoint sensitivity analysis based on Tellegen's theorem

Consider two circuits, which are topologically same [12]. Let $v_B(t)$ and $i_B(t)$ be the voltage and current, respectively, of a branch in first circuit. Let $\psi_B(\tau)$ and $\phi_B(\tau)$ be the voltage and current, respectively, of the same branch in second circuit. Assuming the first circuit as an original circuit, the second circuit will become an adjoint circuit later on. Symbol t is defined as the time scale of the original circuit. Similarly, τ is defined as the time scale of the adjoint circuit.

Using Tellegen's theorem [15] [16], the product of current and voltages becomes

$$\sum_B v_B(t)\phi_B(\tau) = 0 \tag{3.1}$$

$$\sum_B \psi_B(\tau)i_B(t) = 0 \tag{3.2}$$

where t and τ are the arbitrary time points for the original and adjoint circuits, respectively. For small variations in the original circuit we get

$$\sum_B \left[v_B(t) + \delta v_B(t)\right]\phi_B(\tau) = 0 \tag{3.3}$$

$$\sum_B \psi_B(\tau)\left[i_B(t) + \delta i_B(t)\right] = 0 \tag{3.4}$$

where $\delta v_B(t)$ and $\delta i_B(t)$ denote a small change in voltage and current, respectively. Substituting (3.1) in (3.3) gives

$$\sum_B \delta v_B(t)\phi_B(\tau) = 0 \tag{3.5}$$

Similarly, substituting (3.2) in (3.4) gives

$$\sum_B \psi_B(\tau)\delta i_B(t) = 0 \tag{3.6}$$

Adding (3.5) and (3.6) yields

$$\sum_B [\delta v_B(t)\phi_B(\tau) - \psi_B(\tau)\delta i_B(t)] = 0 \tag{3.7}$$

Expanding and integrating (3.7) w.r.t. t from 0 to T yields

$$\begin{aligned}\int_0^T \Bigg[&\sum_V (\delta v_V(t)\phi_V(\tau) - \psi_V(\tau)\delta i_V(t)) + \\ &\sum_I (\delta v_I(t)\phi_I(\tau) - \psi_I(\tau)\delta i_I(t)) + \\ &\sum_R (\delta v_R(t)\phi_R(\tau) - \psi_R(\tau)\delta i_R(t)) + \\ &\sum_C (\delta v_C(t)\phi_C(\tau) - \psi_C(\tau)\delta i_C(t)) + \\ &\sum_L (\delta v_L(t)\phi_L(\tau) - \psi_L(\tau)\delta i_L(t)) + \\ &\sum_{NR} (\delta v_{NR}(t)\phi_{NR}(\tau) - \psi_{NR}(\tau)\delta i_{NR}(t)) \Bigg] dt = 0\end{aligned} \tag{3.8}$$

where V, I, R, C, L and NR denote the branch containing voltage source, current source, resistor, capacitor, inductor and nonlinear resistor, respectively. An illustration of these branches is shown in Fig. 3.1.

Next, taking first-order Taylor series expansion of the objective function (2.2) gives,

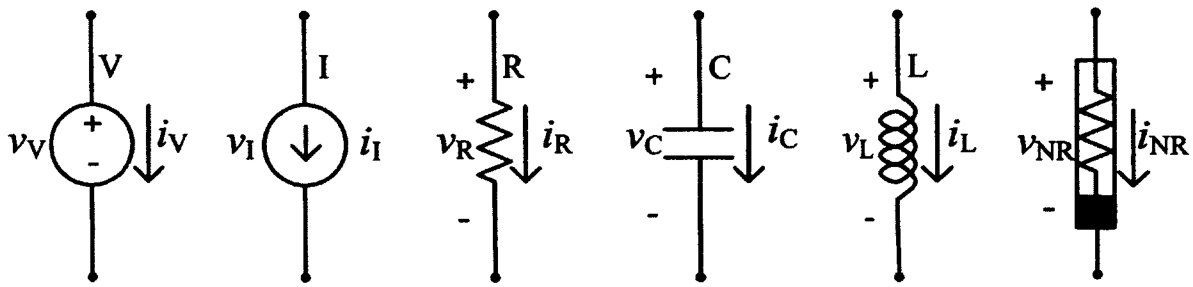

Figure 3.1: Illustration of branches in a circuit

$$\delta W(T) = \frac{\partial W(T)}{\partial \lambda_1}\delta\lambda_1 + \frac{\partial W(T)}{\partial \lambda_2}\delta\lambda_2 \cdots + \frac{\partial W(T)}{\partial \lambda_{N_\lambda}}\delta\lambda_{N_\lambda} \tag{3.9}$$

The goal of the adjoint sensitivity analysis technique based on Tellegen's theorem is to bring (3.8) into the form of (3.9).

Next two sections review the sensitivities of $W(T)$ w.r.t. linear and nonlinear components.

3.1.1 Derivation of sensitivities w.r.t. linear components

Consider the capacitor terms (fourth summation) in (3.8). To simplify the analysis, consider only one capacitor and drop the subscript C. Then we have the integral

$$\int_0^T [\delta v(t)\phi(\tau) - \psi(\tau)\delta i(t)]\, dt \tag{3.10}$$

Assuming, the charge $q(t)$ on a capacitor is a function (g) of capacitance C and voltage $v(t)$, which can be written as

$$q(t) = g(C, v(t)) = Cv(t) \tag{3.11}$$

Taking the first-order Taylor series expansion of (3.11) gives

$$\delta q(t) = \frac{\partial g(t)}{\partial C}\delta C + \frac{\partial g(t)}{\partial v(t)}\delta v(t) \tag{3.12}$$

Simplifying (3.12) yields

$$\delta q(t) = v(t)\delta C + C\delta v(t) \tag{3.13}$$

Then, differentiating (3.13) w.r.t. t yields

$$\delta i(t) = \frac{dv(t)}{dt}\delta C + C\frac{d\delta v(t)}{dt} \tag{3.14}$$

Substituting (3.14) into (3.10) gives

$$\int_0^T \left[\delta v(t)\phi(\tau) - \psi(\tau)\frac{dv(t)}{dt}\delta C - \psi(\tau)C\frac{d\delta v(t)}{dt}\right] dt \tag{3.15}$$

Doing integration by parts on the third term in (3.15) gives

$$\int_0^T \left[\phi(\tau) + \frac{d\psi(\tau)}{dt}C\right] \delta v(t)dt - \int_0^T \left[\psi(\tau)\frac{dv(t)}{dt}\delta C\right] dt - \psi(\tau)C\delta v(t)|_0^T \tag{3.16}$$

Recall, the goal is to convert (3.8) into the form of (3.9). Therefore, all terms except δC and $\delta v(0)$ terms need to vanish.

Let $\tau = T - t$ and therefore, $\frac{d\tau}{dt} = -1$. Then first integrand of (3.16) becomes

$$\phi(\tau) + \frac{d\psi(\tau)}{d\tau}\frac{d\tau}{dt}C = \phi(\tau) - \frac{d\psi(\tau)}{d\tau}C \tag{3.17}$$

Note that, if we chose $\phi(\tau) = \frac{d\psi(\tau)}{d\tau}C$ then the first integral in (3.16) will vanish. In other words, a capacitor in the original circuit is replaced by the same capacitor in the adjoint circuit. Rewriting (3.16) gives

$$-\int_0^T \left[\psi(\tau)\frac{dv(t)}{dt}\delta C\right] dt - \psi(0)C\delta v(T) + \psi(T)C\delta v(0) \tag{3.18}$$

Also, if we chose the initial condition of capacitor, $\psi(0)$, of the adjoint circuit to be zero, then the middle term in (3.18) will vanish. Rewriting (3.18) gives

$$\int_0^T \left[-\psi(\tau)\frac{dv(t)}{dt}\delta C\right] dt + \psi(T)C\delta v(0) \tag{3.19}$$

Following the above process for all the capacitors gives

$$\sum_C \left\{\left[-\int_0^T \psi_C(\tau)\frac{dv_C(t)}{dt}dt\right]\delta C + \left[\psi_C(T)C\right]\delta v_C(0)\right\} \tag{3.20}$$

Using a similar technique as described above, every inductor and resistor in the original circuit can be replaced by the same inductor and resistor in the adjoint circuit. Then the corresponding equations related to inductors and resistors can be written as

$$\sum_L \left\{ \left[\int_0^T \phi_L(\tau) \frac{di_L(t)}{dt} dt \right] \delta L + \left[-\phi_L(T) L \right] \delta i_L(0) \right\} \tag{3.21}$$

and

$$\sum_R \left[\int_0^T \phi_R(\tau) i_R(t) dt \right] \delta R, \tag{3.22}$$

respectively.

Substituting $\tau = T - t$ and comparing equations (3.20), (3.21) and (3.22) with the form in (3.9) yields

$$\frac{\partial W(T)}{\partial C} = - \int_0^T \psi_C(T-t) \frac{dv_C(t)}{dt} dt \tag{3.23}$$

$$\frac{\partial W(T)}{\partial L} = \int_0^T \phi_L(T-t) \frac{di_L(t)}{dt} dt \tag{3.24}$$

$$\frac{\partial W(T)}{\partial R} = \int_0^T \phi_R(T-t) i_R(t) dt \tag{3.25}$$

$$\frac{\partial W(T)}{\partial v_C(0)} = \psi_C(T)C \tag{3.26}$$

$$\frac{\partial W(T)}{\partial i_L(0)} = -\phi_L(T)L \tag{3.27}$$

The sensitivity of an objective function w.r.t. each component and initial condition can be found by using closed-form relations (3.23) to (3.27) [12] [17]. For example, the sensitivity w.r.t. a capacitance C at a time point T is given in terms of an integration of the voltage $\psi_C(\tau)$ across the capacitor in the adjoint circuit and the derivative of the voltage $v_C(t)$ across the capacitor in the original circuit. Moreover, the sensitivity w.r.t. an initial condition related to a capacitor in the original circuit is given by (3.26).

In the next section, the derivation of sensitivities w.r.t. nonlinear components is presented.

3.1.2 Derivation of sensitivities w.r.t. nonlinear components

Consider a nonlinear resistor in Fig. 3.2, whose current $i_{NR}(t)$ is given as

$$i_{NR}(t) = g\left(i_a(t), v_b(t), v_{NR}(t), \lambda\right) \tag{3.28}$$

where $v_{NR}(t)$ is the voltage across its own branch, $i_a(t)$ is the external branch current,

$v_b(t)$ is the external voltage difference and λ is the parameter. Taking the first-order Taylor series expansion of (3.28) gives

$$\delta i_{NR}(t) = \frac{\partial g(t)}{\partial i_a(t)}\delta i_a(t) + \frac{\partial g(t)}{\partial v_b(t)}\delta v_b(t) + \frac{\partial g(t)}{\partial v_{NR}(t)}\delta v_{NR}(t) + \frac{\partial g(t)}{\partial \lambda}\delta\lambda \qquad (3.29)$$

Using Tellegen's theorem on three branches in Fig. 3.2 yields

$$\begin{aligned}\int_0^T &[(\delta v_a(t)\phi_a(\tau) - \psi_a(\tau)\delta i_a(t)) + \\ &(\delta v_b(t)\phi_b(\tau) - \psi_b(\tau)\delta i_b(t)) + \\ &(\delta v_{NR}(t)\phi_{NR}(\tau) - \psi_{NR}(\tau)\delta i_{NR}(t))]\, dt \qquad (3.30)\end{aligned}$$

Since $v_a(t)$ and $i_b(t)$ are always zero, $\delta v_a(t) = 0$ and $\delta i_b(t) = 0$. Substituting these in (3.30) gives

$$\begin{aligned}\int_0^T &[-\psi_a(\tau)\delta i_a(t) + \delta v_b(t)\phi_b(\tau) + \\ &(\delta v_{NR}(t)\phi_{NR}(\tau) - \psi_{NR}(\tau)\delta i_{NR}(t))]\, dt \qquad (3.31)\end{aligned}$$

Substituting (3.29) in (3.31) yields

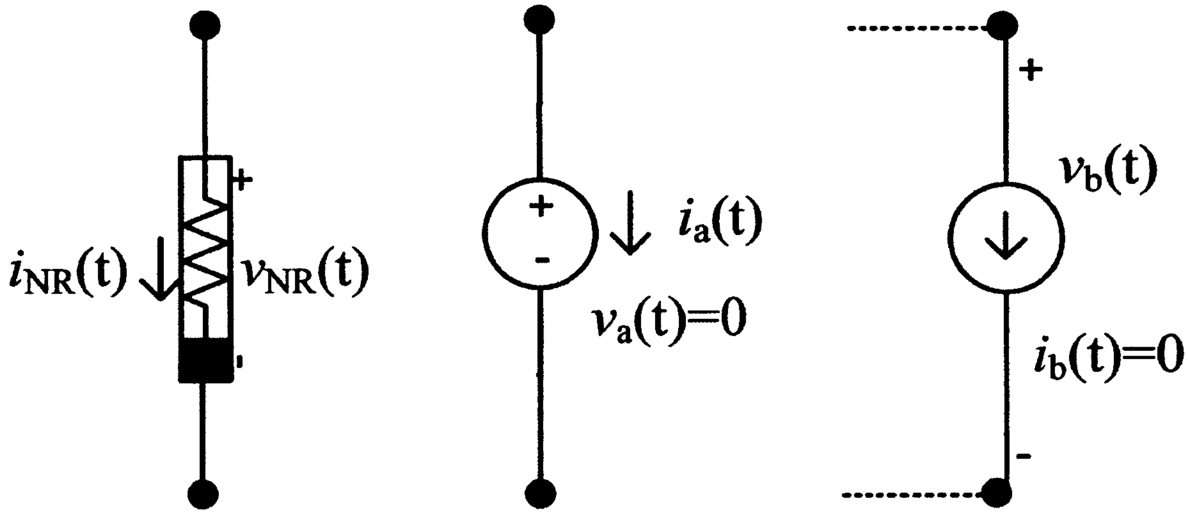

Figure 3.2: Illustrative example of a nonlinear resistor

$$\begin{aligned}\int_0^T \Bigg(&\left[-\psi_a(\tau) - \psi_{NR}(\tau)\frac{\partial g(t)}{\partial i_a}\right] \delta i_a(t) + \\ &\left[\phi_b(\tau) - \psi_{NR}(\tau)\frac{\partial g(t)}{\partial v_b}\right] \delta v_b(t) + \\ &\left[\phi_{NR}(\tau) - \psi_{NR}(\tau)\frac{\partial g(t)}{\partial v_{NR}}\right] \delta v_{NR}(t) + \\ &\left[-\psi_{NR}(\tau)\frac{\partial g(t)}{\partial \lambda}\right] \delta\lambda \Bigg)\, dt \end{aligned} \tag{3.32}$$

Recall, the goal is to convert (3.8) into the form of (3.9). Therefore, we have to get rid of other terms except the $\delta\lambda$ term. To achieve this, let us choose,

$$\psi_a(\tau) = -\psi_{NR}(\tau)\frac{\partial g(t)}{\partial i_a} \tag{3.33}$$

$$\phi_b(\tau) = \psi_{NR}(\tau)\frac{\partial g(t)}{\partial v_b} \tag{3.34}$$

$$\phi_{NR}(\tau) = \psi_{NR}(\tau)\frac{\partial g(t)}{\partial v_{NR}} \tag{3.35}$$

Equation (3.33) means that if in the original circuit a nonlinear resistor is dependent on an external branch current then in the adjoint circuit a voltage controlled voltage source is added in that branch (see Fig. 3.3). Equation (3.34) means that if in the original circuit a nonlinear resistor is dependent on an external node voltage difference then in the adjoint circuit a voltage controlled current source is added between those nodes. Similarly, (3.35) means a nonlinear resistor in the original circuit is replaced by voltage controlled current source in the adjoint circuit.

Once the simulation of the original and the adjoint circuit are done, the sensitivity of an objective function w.r.t. λ of nonlinear resistor can be found by using

$$\frac{\partial W(T)}{\partial \lambda} = -\int_0^T \psi_{NR}(T-t)\frac{\partial g(t)}{\partial \lambda}dt \tag{3.36}$$

It is to be noted that without loss of generality, (3.33) to (3.36) can be used even if a nonlinear resistor is dependent on multiple currents, voltages and parameters.

The next section presents the derivation of sources for an adjoint circuit.

3.1.3 Derivation of sources for an adjoint circuit

By substituting the sensitivity equations (3.23), (3.24), (3.25) and (3.36) into (3.8) and by ignoring the initial condition terms ((3.26) and (3.27)) to simplify the analysis, we get

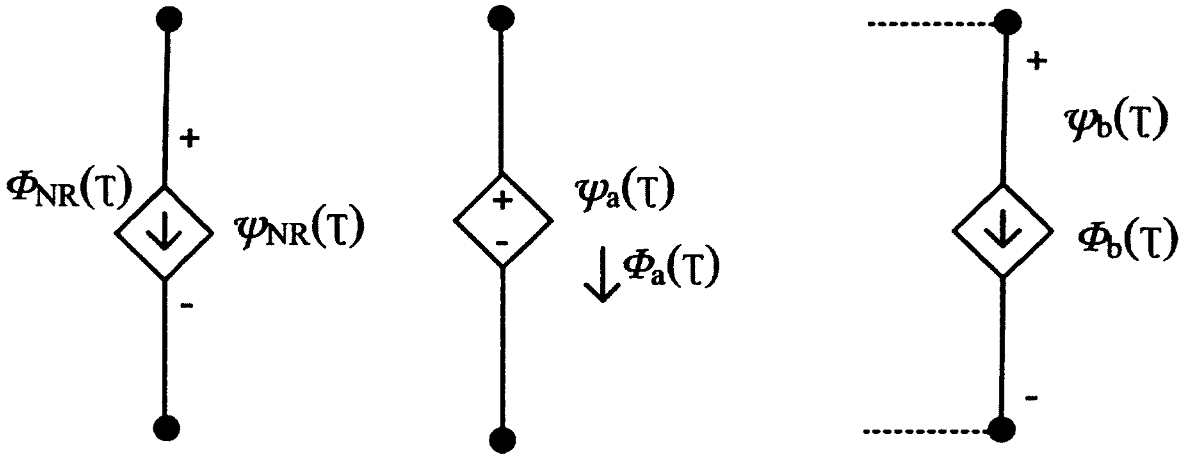

Figure 3.3: Adjoint branches corresponding to Fig. 3.2

$$
\begin{aligned}
\int_0^T \Bigg[& \sum_V (\delta v_V(t)\phi_V(\tau) - \psi_V(\tau)\delta i_V(t)) + \\
& \sum_I (\delta v_I(t)\phi_I(\tau) - \psi_I(\tau)\delta i_I(t)) + \\
& \sum_R (\phi_R(\tau) i_R(t) \delta R) + \\
& \sum_C (-\psi_C(\tau)\frac{dv_C(t)}{dt}\delta C) + \\
& \sum_L (\phi_L(\tau)\frac{di_L(t)}{dt}\delta L) + \\
& \sum_{NR} (-\psi_{NR}(\tau)\frac{\partial g(t)}{\partial \lambda}\delta\lambda) \Bigg] dt = 0
\end{aligned}
\tag{3.37}
$$

Assuming the sources $v_V(t)$ and $i_I(t)$ remains unperturbed in the original circuit results in

$$\delta v_V(t) = 0; \qquad \delta i_I(t) = 0 \tag{3.38}$$

Substituting (3.38) in (3.37) gives

$$\begin{aligned}
&\int_0^T \left[\sum_V \psi_V(\tau)\delta i_V(t) - \sum_I \delta v_I(t)\phi_I(\tau) \right] dt = \\
&\int_0^T \left[\sum_R (i_R(t)\phi_R(\tau)\delta R) \right] dt + \\
&\int_0^T \left[\sum_C -\psi_C(\tau)\frac{dv_C(t)}{dt}\delta C \right] dt + \\
&\int_0^T \left[\sum_L \frac{di_L(t)}{dt}\phi_L(\tau)\delta L \right] dt + \\
&\int_0^T \left[\sum_{NR} (-\psi_{NR}(\tau)\frac{\partial g(t)}{\partial \lambda}\delta\lambda) \right] dt
\end{aligned} \tag{3.39}$$

Next, by assuming w in (2.2) is a function of currents through all the voltage sources and voltages across all the current sources gives

$$w(t) = w(\{i_V(t)\ \forall\ V\}, \{v_I(t)\ \forall\ I\}) \tag{3.40}$$

Taking the first-order Taylor series expansion of (2.2) with (3.40) results in

$$\delta W(T) = \int_0^T \left[\sum_V \frac{\partial w(t)}{\partial i_V} \delta i_V(t) - \sum_I \delta v_I(t) \frac{\partial w(t)}{\partial v_I} \right] dt \tag{3.41}$$

Setting the RHS of (3.41) equal to the LHS of (3.39) results in

$$\begin{aligned} &\delta W(T) = \\ &\int_0^T \left[\sum_R (i_R(t)\phi_R(\tau)\delta R) \right] dt + \\ &\int_0^T \left[\sum_C -\psi_C(\tau) \frac{dv_C(t)}{dt} \delta C \right] dt + \\ &\int_0^T \left[\sum_L \frac{di_L(t)}{dt} \phi_L(\tau) \delta L \right] dt + \\ &\int_0^T \left[\sum_{NR} (-\psi_{NR}(\tau) \frac{\partial g(t)}{\partial \lambda} \delta \lambda) \right] dt \end{aligned} \tag{3.42}$$

which is in the same form as (3.9). Using the RHS of (3.41), the LHS of (3.39) and $t = T - \tau$, the voltage and current sources in an adjoint circuit become

$$\psi_V(\tau) = \frac{\partial w(T-\tau)}{\partial i_V} \tag{3.43}$$

$$\phi_I(\tau) = \frac{\partial w(T-\tau)}{\partial v_I} \tag{3.44}$$

In the next section, the properties of an adjoint circuit are discussed.

3.1.4 Properties of an adjoint circuit

If the original circuit contains a nonlinear resistor, then it is replaced by a voltage controlled current source in the adjoint circuit given by (3.35). Substituting $t = T - \tau$ in (3.35) gives

$$\phi_{NR}(\tau) = \psi_{NR}(\tau)\frac{\partial g(T-\tau)}{\partial v_{NR}} \tag{3.45}$$

Equation (3.45) signifies three properties of the adjoint circuit:

1. $\phi_{NR}(\tau)$ is a linear function of $\psi_{NR}(\tau)$. Therefore, every nonlinear component in the original circuit is replaced by a linear component in adjoint circuit. Therefore, the adjoint circuit is always a linear system.

2. The gain is a function of time τ, which makes the adjoint circuit a time-

varying system.

3. If the sensitivity at a time point T is required, then the adjoint system is valid only from time 0 to T.

Therefore, if there is a nonlinear component in the original circuit, then the adjoint circuit will be a Linear-Time-Varying system. It is to be noted that if the sensitivity at another time point $\tilde{T}$ is required, then the adjoint system is valid from 0 to $\tilde{T}$, such that

$$\phi_{NR}(\tau) = \psi_{NR}(\tau)\frac{\partial g(\tilde{T} - \tau)}{\partial v_{NR}} \tag{3.46}$$

In this section, the adjoint sensitivity analysis based on Tellegen's theorem was reviewed. In the next section, the adjoint sensitivity analysis based on the variational approach is reviewed.

3.2 Adjoint sensitivity analysis based on variational approach

The variational approach has been presented in the literature [13] as an alternative approach for adjoint sensitivity analysis. Using the variational approach [13], we define an auxiliary variable $\boldsymbol{\xi}_a(\tau) \in \Re^{n\times 1}$. Multiplying (2.10) by $\boldsymbol{\xi}_a^t(\tau)$ and integrating w.r.t. $t \in [0, T]$ gives

$$\int_0^T \boldsymbol{\xi}_a^t(\tau)\left[\mathbf{C}\frac{d\mathbf{z}(t)}{dt} + \mathbf{G}\mathbf{z}(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}\mathbf{z}(t) - \mathbf{j}_{\lambda_m}(t)\right] dt = 0 \tag{3.47}$$

Substituting $\tau = T - t$ in (3.47) and using integration by parts, we get

$$\int_0^T \left[\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau}\mathbf{C} + \boldsymbol{\xi}_a^t(\tau)\mathbf{G} + \boldsymbol{\xi}_a^t(\tau)\frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}\right]\mathbf{z}(t)dt = -\boldsymbol{\xi}_a^t(\tau)\mathbf{C}\mathbf{z}(t)|_0^T + \int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_{\lambda_m}(t)dt \tag{3.48}$$

It is to be noted that, contrary to (2.10) and (2.11), (3.48) allows us to avoid calculating $\mathbf{z}(t)$ explicitly. This is possible if $\boldsymbol{\xi}_a(\tau)$ can be found such that (using (3.48) and (2.4)):

$$\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau}\mathbf{C} + \boldsymbol{\xi}_a^t(\tau)\mathbf{G} + \boldsymbol{\xi}_a^t(\tau)\frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}} = \frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \mathbf{x}} \tag{3.49}$$

where $\boldsymbol{\xi}_a(\tau)$ is referred to as the adjoint MNA variable vector [21]. Taking the transpose of both sides and substituting $t = T - \tau$ in (3.49) gives

$$\mathbf{C}^t \frac{d\boldsymbol{\xi}_a(\tau)}{d\tau} + \mathbf{G}^t \boldsymbol{\xi}_a(\tau) + \left[\frac{\partial \mathbf{f}(T-\tau)}{\partial \mathbf{x}}\right]^t \boldsymbol{\xi}_a(\tau) = \left[\frac{\partial w(T-\tau)}{\partial \mathbf{x}}\right]^t \qquad \tau \in [0, T] \quad (3.50)$$

Next, the solution of (3.50) is not unique until the initial conditions are defined, which can be obtained by substituting (3.48) in (2.4)

$$\frac{\partial W(T)}{\partial \lambda_m} = -\boldsymbol{\xi}_a^t(0)\mathbf{C}\mathbf{z}(T) + \boldsymbol{\xi}_a^t(T)\mathbf{C}\mathbf{z}(0) + \int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_{\lambda_m}(t)dt + \int_0^T \frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \lambda_m} dt \tag{3.51}$$

To avoid calculating $\mathbf{z}(T)$ explicitly, $\boldsymbol{\xi}_a^t(0)$ is selected to be equal to zero. This implies that, an adjoint system is simulated forward-in-time, from 0 to T, with initial conditions set to zero (i.e. $\boldsymbol{\xi}_a^t(0) = 0$).

The sensitivity function in (3.51) can be obtained in terms of the solution of the original system (2.1) and the corresponding adjoint equation (3.50) as

$$\frac{\partial W(T)}{\partial \lambda_m} = \boldsymbol{\xi}_a^t(T)\mathbf{C}\mathbf{z}(0) + \int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_{\lambda_m}(t)dt + \int_0^T \frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \lambda_m} dt \tag{3.52}$$

where $\mathbf{z}(0)$ is found from the DC analysis of the original system (2.1) as

$$\mathbf{G}\mathbf{x}_{DC} + \mathbf{f}(\mathbf{x}_{DC}) = \mathbf{b}(0) \tag{3.53}$$

3.3 Comparison between Tellegen's theorem-based and variational approaches

If λ_m is a capacitance, then $\mathbf{j}_{\lambda_m}(t)$ in (2.11) becomes

$$\mathbf{j}_C(t) = -\frac{\partial \mathbf{C}(t)}{\partial C} * \frac{d\mathbf{x}(t)}{dt} \tag{3.54}$$

and the middle term of the RHS in (3.52) becomes

$$\int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_C(t)dt = -\int_0^T \psi_C(\tau) \times \frac{dv_C(t)}{dt}dt \tag{3.55}$$

where $\psi_C(\tau)$ and $v_C(t)$ can be found by simulating the adjoint and original systems, respectively. Substituting $\tau = T - t$ in (3.55) gives

$$\frac{\partial W(T)}{\partial C} = -\int_0^T \psi_C(T-t) \times \frac{dv_C(t)}{dt}dt \tag{3.56}$$

which is the same as (3.23) from Tellegen's approach.

If λ_m is an initial voltage on a capacitor, then the first term of the RHS in (3.52) becomes

$$\boldsymbol{\xi}_a^t(T)\mathbf{C}\mathbf{z}(0) = \boldsymbol{\xi}_a^t(T)\mathbf{C}\frac{\partial \mathbf{x}(0)}{\partial v_C(0)} = \psi_C(T) \times C \tag{3.57}$$

which is the same as (3.26) from Tellegen's approach.

If λ_m is a parameter of a nonlinear resistor, then $\mathbf{j}_{\lambda_m}(t)$ in (2.11) becomes

$$\mathbf{j}_\lambda(t) = -\frac{\partial \mathbf{f}(t)}{\partial \lambda} \tag{3.58}$$

Assuming the current ($g(t)$) through a nonlinear resistor is a function of the parameter λ, then the middle term of the RHS in (3.52) becomes

$$\int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_\lambda(t)dt = -\int_0^T \psi_{NR}(\tau) \times \frac{\partial g(t)}{\partial \lambda}dt \tag{3.59}$$

where $\psi_{NR}(\tau)$ and $\frac{\partial g(t)}{\partial \lambda}$ can be found by simulating the adjoint and original systems, respectively. Substituting $\tau = T - t$ in (3.59) gives

$$\frac{\partial W(T)}{\partial \lambda} = -\int_0^T \psi_{NR}(T-t) \times \frac{\partial g(t)}{\partial \lambda}dt \tag{3.60}$$

which is the same as (3.36) from Tellegen's approach.

Using this comparison, it is clear that the two approaches are equivalent. It is to be noted that in subsequent chapters the variational approach is used.

Chapter 4

Proposed variational approach based first-order adjoint sensitivity analysis in time domain for networks with MTLs

In this chapter, an efficient algorithm based on variational approach for adjoint sensitivity analysis of multiconductor transmission line circuits is presented ([6] describes a brief conceptual description of the related work based on Tellegen's theorem). An important additional advantage of the proposed algorithm is that its formulation is independent of the specifics of the MTL macromodel used (i.e., the proposed method can be adopted in conjunction with a wide variety of MTL macromodels that are available in the literature; without requiring to re-derive the constituting sensitivity relations for each case).

The rest of this chapter is organized as follows. In Section 4.1, a formulation of circuit equations is provided. The proposed adjoint method based on variational approach is described in Section 4.2. The application of the proposed approach in terms of numerical examples is demonstrated in Section 4.3 and conclusions are presented in Section 4.4.

4.1 Formulation of circuit equations

Consider a general circuit consisting of [3] linear and nonlinear components and distributed transmission lines. The corresponding modified nodal analysis (MNA) equations can be written in the time-domain as

$$\mathbf{C}\frac{d\mathbf{x}(t)}{dt} + \mathbf{G}\mathbf{x}(t) + \sum_{k=1}^{N_t} \mathbf{a}_k(t) * \mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t)) = \mathbf{b}(t) \tag{4.1}$$

where

$$\mathbf{a}_k(t) = \mathbf{D}_k \mathbf{y}_k(t) \mathbf{D}_k^t \tag{4.2}$$

and

- $\mathbf{x}(t) \in \Re^{n\times 1}$ is the vector of unknowns corresponding to node voltages, independent and dependent voltage source currents, and inductor currents; $\mathbf{b}(t) \in \Re^{n\times 1}$ is an input vector with entries determined by the independent current and voltage sources; $\mathbf{f}(\mathbf{x}(t)) \in \Re^{n\times 1}$ is a vector related to nonlinear elements and n is the total number of MNA variables;
- $\mathbf{G}$, $\mathbf{C} \in \Re^{n\times n}$ are constant matrices describing the lumped memory-less and memory elements, respectively;
- $\mathbf{D}_k = [d_{i,j}], d_{i,j} \in \{0, 1\}$ is a selector matrix that maps the vector of terminal currents $\mathbf{i}_k(t)$ entering the transmission line k into the nodal space of the circuit, where $i \in \{1, \cdots, n\}, j \in \{1, \cdots, 2N_k\}$, N_k is number of coupled lines in the k^{th} transmission line and N_t is the total number of distributed transmission lines in the circuit;
- $\mathbf{y}_k(t) \in \Re^{2N_k\times 2N_k}$ is the time-domain admittance matrix of the k^{th} transmission line, where $\mathbf{i}_k(t)$ and $\mathbf{v}_k(t) \in \Re^{2N_k\times 1}$ are the vectors corresponding to terminal currents and voltages, respectively; $\mathbf{v}_k(t) = \mathbf{D}_k^t\mathbf{x}(t)$ and '$*$' denotes the convolution operator.

The distributed elements are typically best represented in the Laplace domain and do not have a direct representation in time-domain. In order to overcome this problem, several time-domain macromodels have been proposed in the literature, such as MoC (which does not guarantee passivity [22] [23]), MRA (which guarantees passivity [14]) and DEPACT (which provides delay extraction along with guaranteed passivity [24]).

For evaluating sensitivities of circuit responses, the conventional direct sensitivity approach is based on differentiating (4.1) w.r.t. a specific parameter λ_m. One of the major challenges with the direct sensitivity approach is that if the circuit contains N_λ number of parameters then to evaluate circuit sensitivities, perturbed set of equations have to be solved N_λ times, which can become prohibitively CPU expensive.

To address this problem, the adjoint sensitivity concept for lumped circuits was introduced in [12]. The initial derivation of an adjoint sensitivity [12] was based on Tellegen's theorem [15], [16]. The main advantage of using the adjoint sensitivity approach is that the sensitivity w.r.t. all the parameters of interest can be found by simulating only two systems: original and adjoint. Also, an alternate method based on variational approach can be found for adjoint sensitivity analysis of lumped circuits [7], [13] and for the special case of single lossless transmission line macromodels [8].

The main focus of this chapter is to extend the adjoint approach for time-domain sensitivity analysis of general lossy MTL networks. The new method is based on the variational approach and the related details are given in the subsequent sections.

4.2 Development of the proposed adjoint sensitivity concept for distributed multiconductor transmission lines

In this section, details of the proposed time-domain adjoint sensitivity analysis for circuits including the general case of distributed MTL networks are given.

4.2.1 Problem formulation

Using the direct sensitivity approach [5], $\mathbf{z}(t)$ in (2.4) is obtained by differentiating (4.1) as follows:

$$\mathbf{C}\frac{d\mathbf{z}(t)}{dt} + \mathbf{G}\mathbf{z}(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}\mathbf{z}(t) + \sum_{k=1}^{N_t} [\mathbf{a}_k(t) * \mathbf{z}(t)] = \mathbf{j}_{\lambda_m}(t) \tag{4.3}$$

where

$$\mathbf{j}_{\lambda_m}(t) = \frac{\partial \mathbf{b}(t)}{\partial \lambda_m} - \frac{\partial \mathbf{C}}{\partial \lambda_m}\frac{d\mathbf{x}(t)}{dt} - \frac{\partial \mathbf{G}}{\partial \lambda_m}\mathbf{x}(t) - \sum_{k=1}^{N_t}\left[\frac{\partial \mathbf{a}_k(t)}{\partial \lambda_m} * \mathbf{x}(t)\right] - \frac{\partial \mathbf{f}(t)}{\partial \lambda_m} \tag{4.4}$$

As can be seen from (4.3) and (4.4), evaluating $\mathbf{z}(t)$ w.r.t. N_λ variables requires the solution of (4.3) and (4.4) N_λ times, making the process computationally expensive.

In order to address this difficulty, the next section provides a new adjoint sensitivity approach for circuits including distributed transmission line interconnect networks.

4.2.2 Proposed adjoint sensitivity approach for distributed transmission line interconnects

Using the variational approach [13], we define an auxiliary variable $\boldsymbol{\xi}_a(\tau) \in \Re^{n\times 1}$. Multiplying (4.3) by $\boldsymbol{\xi}_a^t(\tau)$ and integrating w.r.t. $t \in [0, T]$ gives

$$\int_0^T \boldsymbol{\xi}_a^t(\tau)\left[\mathbf{C}\frac{d\mathbf{z}(t)}{dt} + \mathbf{G}\mathbf{z}(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}\mathbf{z}(t) + \sum_{k=1}^{N_t}\left[\mathbf{a}_k(t) * \mathbf{z}(t)\right] - \mathbf{j}_{\lambda_m}(t)\right] dt = 0 \quad (4.5)$$

Substituting $\tau = T - t$ in (4.5) and using integration by parts, we get

$$\int_0^T \left[\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau}\mathbf{C} + \boldsymbol{\xi}_a^t(\tau)\mathbf{G} + \boldsymbol{\xi}_a^t(\tau)\frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}} + \sum_{k=1}^{N_t}\left[\boldsymbol{\xi}_a^t(\tau) * \mathbf{a}_k(\tau)\right]\right]\mathbf{z}(t)dt =$$

$$-\boldsymbol{\xi}_a^t(\tau)\mathbf{C}\mathbf{z}(t)|_0^T + \int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_{\lambda_m}(t)dt \quad (4.6)$$

It is to be noted that, contrary to (4.3) and (4.4), (4.6) allows us to avoid calculating $\mathbf{z}(t)$ explicitly. This is possible if $\boldsymbol{\xi}_a(\tau)$ can be found such that (using (4.6) and (2.4)):

$$\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau}\mathbf{C} + \boldsymbol{\xi}_a^t(\tau)\mathbf{G} + \boldsymbol{\xi}_a^t(\tau)\frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}} + \sum_{k=1}^{N_t}\left[\boldsymbol{\xi}_a^t(\tau) * \mathbf{a}_k(\tau)\right] = \frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \mathbf{x}} \tag{4.7}$$

where $\boldsymbol{\xi}_a(\tau)$ is referred to as the adjoint MNA variable vector [21]. Taking the transpose of both sides and substituting $t = T - \tau$ in (4.7) gives

$$\begin{aligned} \mathbf{C}^t\frac{d\boldsymbol{\xi}_a(\tau)}{d\tau} + \mathbf{G}^t\boldsymbol{\xi}_a(\tau) + \left[\frac{\partial \mathbf{f}(T-\tau)}{\partial \mathbf{x}}\right]^t \boldsymbol{\xi}_a(\tau) & \\ + \sum_{k=1}^{N_t}\left[\mathbf{D}_k\boldsymbol{\phi}_k(\tau)\right] = \left[\frac{\partial w(T-\tau)}{\partial \mathbf{x}}\right]^t & \qquad \tau \in [0, T] \end{aligned} \tag{4.8}$$

where

$$\boldsymbol{\phi}_k(\tau) = \mathbf{y}_k^t(\tau) * \boldsymbol{\psi}_k(\tau) \tag{4.9}$$

$$\boldsymbol{\psi}_k(\tau) = \mathbf{D}_k^t\boldsymbol{\xi}_a(\tau) \tag{4.10}$$

It is to be noted that $\boldsymbol{\psi}_k(\tau)$ and $\boldsymbol{\phi}_k(\tau)$ are the terminal voltage and current vectors of the k^{th} transmission line in the adjoint system, respectively.

Next, (4.9) can be written in the Laplace domain as

$$\mathbf{\Phi}_k(s) = \mathbf{Y}_k^t(s)\mathbf{\Psi}_k(s) \tag{4.11}$$

where $\mathbf{Y}_k^t(s)$ is the Laplace transform of $\mathbf{y}_k^t(\tau)$. The adjoint system is defined by (4.8) and (4.11). However, the solution of (4.8) is not unique until the initial conditions are defined, which can be obtained by substituting (4.6) in (2.4)

$$\frac{\partial W(T)}{\partial \lambda_m} = -\boldsymbol{\xi}_a^t(0)\mathbf{C}\mathbf{z}(T) + \boldsymbol{\xi}_a^t(T)\mathbf{C}\mathbf{z}(0) + \int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_{\lambda_m}(t)dt + \int_0^T \frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \lambda_m}dt \tag{4.12}$$

To avoid calculating $\mathbf{z}(T)$ explicitly, $\boldsymbol{\xi}_a^t(0)$ is selected to be equal to zero. This implies that, an adjoint system is simulated forward-in-time, from 0 to T, with initial conditions set to zero (i.e. $\boldsymbol{\xi}_a^t(0) = 0$).

The sensitivity function in (4.12) can be obtained in terms of the solution of the original system (4.1) and the corresponding adjoint equations (4.8) -(4.11) as

$$\frac{\partial W(T)}{\partial \lambda_m} = \boldsymbol{\xi}_a^t(T)\mathbf{C}\mathbf{z}(0) + \int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_{\lambda_m}(t)dt + \int_0^T \frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \lambda_m}dt \tag{4.13}$$

where $\mathbf{z}(0)$ is found from the DC analysis of the original system (4.1) as

$$\mathbf{G}\mathbf{x}_{DC} + \mathbf{f}(\mathbf{x}_{DC}) + \sum_{k=1}^{N_t} [\mathbf{A}_k(s=0)\mathbf{x}_{DC}] = \mathbf{b}(0) \tag{4.14}$$

where $\mathbf{A}_k(s)$ is the Laplace transform of $\mathbf{a}_k(t)$.

If λ_m is an electrical or physical parameter of the k^{th} transmission line, then $\mathbf{j}(t)$ in (4.4) becomes

$$\mathbf{j}_{\lambda_m}(t) = -\frac{\partial \mathbf{a}_k(t)}{\partial \lambda_m} * \mathbf{x}(t) \tag{4.15}$$

and the middle term of the RHS in (4.13) becomes

$$\int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_{\lambda_m}(t)dt = -\int_0^T \boldsymbol{\psi}_k^t(\tau) \times \left[\frac{\partial \mathbf{y}_k(t)}{\partial \lambda_m} * \mathbf{v}_k(t)\right] dt \tag{4.16}$$

where $\boldsymbol{\psi}_k^t(\tau)$ and $\mathbf{v}_k(t)$ can be found by simulating the adjoint and original systems, respectively.

Next, the details of evaluation of $\frac{\partial \mathbf{y}_k(t)}{\partial \lambda_m}$ in (4.16) in a closed-form is given in the following section.

4.2.3 Evaluation of the sensitivity of the MTL admittance matrix

Consider the k^{th} transmission line containing N_k coupled lines with $\mathbf{R}_k(s)$, $\mathbf{L}_k(s)$, $\mathbf{G}_k(s)$, $\mathbf{C}_k(s) \in \Re^{N_k \times N_k}$ as per-unit-length (p.u.l.) resistance, inductance, conductance and capacitance matrices, respectively. The p.u.l. parameters can be constant or frequency-dependent. The corresponding p.u.l. impedance $\mathbf{Z}_k(s)$ and admittance $\boldsymbol{\Omega}_k(s)$ matrices of the MTL network are given by [10]

$$\mathbf{Z}_k(s) = \mathbf{R}_k(s) + s\mathbf{L}_k(s) \tag{4.17}$$

$$\boldsymbol{\Omega}_k(s) = \mathbf{G}_k(s) + s\mathbf{C}_k(s) \tag{4.18}$$

The frequency-domain admittance matrix $\mathbf{Y}_k(s)$ of the MTL [10] is given by

$$\mathbf{Y}_k(s)\mathbf{V}_k(s) = \mathbf{I}_k(s) \tag{4.19}$$

$$\mathbf{Y}_k(s) = \begin{bmatrix} \mathbf{S}_q\mathbf{E}_1\mathbf{S}_v^{-1} & \mathbf{S}_q\mathbf{E}_2\mathbf{S}_v^{-1} \\ \mathbf{S}_q\mathbf{E}_2\mathbf{S}_v^{-1} & \mathbf{S}_q\mathbf{E}_1\mathbf{S}_v^{-1} \end{bmatrix}, \tag{4.20}$$

$$\mathbf{S}_q = \mathbf{Z}_k^{-1}\mathbf{S}_v\boldsymbol{\Gamma}. \tag{4.21}$$

$\mathbf{E}_1(s)$ and $\mathbf{E}_2(s)$ are diagonal matrices defined as

$$\mathbf{E}_1(s) = diag\left\{\frac{1+e^{-2\gamma_i D}}{1-e^{-2\gamma_i D}}, i = 1...N_k\right\} \tag{4.22}$$

$$\mathbf{E}_2(s) = diag\left\{\frac{2}{e^{-\gamma_i D}-e^{\gamma_i D}}, i = 1...N_k\right\} \tag{4.23}$$

where D is the length of the transmission line and

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_{N_k} \end{bmatrix} \tag{4.24}$$

with γ_i being the i^{th} eigenvalue of the $\mathbf{Z}_k\mathbf{\Omega}_k$ and

$$\mathbf{S}_v = \begin{bmatrix} \boldsymbol{\beta}_1 & \dots & \boldsymbol{\beta}_{N_k} \end{bmatrix} \tag{4.25}$$

is the corresponding eigenvector matrix related by

$$\left(\gamma_i^2\mathbf{U} - \mathbf{Z}_k\mathbf{\Omega}_k\right)\boldsymbol{\beta}_i = 0. \tag{4.26}$$

Next, (4.20) can be rewritten as

$$\mathbf{Y}_k \begin{bmatrix} \mathbf{S}_v & 0 \\ 0 & \mathbf{S}_v \end{bmatrix} = \begin{bmatrix} \mathbf{S}_q & 0 \\ 0 & \mathbf{S}_q \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_1 \end{bmatrix} \tag{4.27}$$

Differentiating (4.27) w.r.t. λ_m gives

$$\frac{\partial \mathbf{Y}_k}{\partial \lambda_m} \begin{bmatrix} \mathbf{S}_v & 0 \\ 0 & \mathbf{S}_v \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{S}_q}{\partial \lambda_m} & 0 \\ 0 & \frac{\partial \mathbf{S}_q}{\partial \lambda_m} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_1 \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{S}_q & 0 \\ 0 & \mathbf{S}_q \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{E}_1}{\partial \lambda_m} & \frac{\partial \mathbf{E}_2}{\partial \lambda_m} \\ \frac{\partial \mathbf{E}_2}{\partial \lambda_m} & \frac{\partial \mathbf{E}_1}{\partial \lambda_m} \end{bmatrix} - \mathbf{Y}_k \begin{bmatrix} \frac{\partial \mathbf{S}_v}{\partial \lambda_m} & 0 \\ 0 & \frac{\partial \mathbf{S}_v}{\partial \lambda_m} \end{bmatrix} \tag{4.28}$$

To proceed further with the evaluation of $\frac{\partial \mathbf{Y}_k(s)}{\partial \lambda_m}$ in (4.28), individual derivatives, namely, $\frac{\partial \mathbf{S}_v}{\partial \lambda_m}$, $\frac{\partial \mathbf{S}_q}{\partial \lambda_m}$, $\frac{\partial \mathbf{E}_1}{\partial \lambda_m}$ and $\frac{\partial \mathbf{E}_2}{\partial \lambda_m}$ are required and are evaluated as follows.

C.1) Evaluation of $\frac{\partial \mathbf{S}_q}{\partial \lambda_m}$

Differentiating (4.21) gives

$$\mathbf{Z}_k \frac{\partial \mathbf{S}_q}{\partial \lambda_m} = \frac{\partial \mathbf{S}_v}{\partial \lambda_m} \mathbf{\Gamma} + \mathbf{S}_v \frac{\partial \mathbf{\Gamma}}{\partial \lambda_m} - \frac{\partial \mathbf{Z}_k}{\partial \lambda_m} \mathbf{S}_q \tag{4.29}$$

Differentiating (4.26) w.r.t. λ_m gives

$$\frac{\partial \gamma_i^2}{\partial \lambda_m}\boldsymbol{\beta}_i + \gamma_i^2 \frac{\partial \boldsymbol{\beta}_i}{\partial \lambda_m} - \frac{\partial \left(\mathbf{Z}_k \boldsymbol{\Omega}_k\right)}{\partial \lambda_m}\boldsymbol{\beta}_i - \mathbf{Z}_k \boldsymbol{\Omega}_k \frac{\partial \boldsymbol{\beta}_i}{\partial \lambda_m} = 0 \tag{4.30}$$

Normalizing the magnitude of $\boldsymbol{\beta}_i$ and differentiating w.r.t. λ_m gives

$$\boldsymbol{\beta}_i^t \frac{\partial \boldsymbol{\beta}_i}{\partial \lambda_m} = 0 \tag{4.31}$$

Next, equations (4.30) and (4.31) can be written in a matrix form as

$$\begin{bmatrix} \gamma_i^2 \mathbf{U} - \mathbf{Z}_k \boldsymbol{\Omega}_k & \boldsymbol{\beta}_i \\ \\ \boldsymbol{\beta}_i^t & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \boldsymbol{\beta}_i}{\partial \lambda_m} \\ \\ \frac{\partial \gamma_i^2}{\partial \lambda_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial (\mathbf{Z}_k \boldsymbol{\Omega}_k)}{\partial \lambda_m} \boldsymbol{\beta}_i \\ \\ 0 \end{bmatrix} \tag{4.32}$$

Using (4.25) and (4.32), $\frac{\partial \mathbf{S}_v}{\partial \lambda_m}$ can be evaluated.

$\frac{\partial \mathbf{S}_q}{\partial \lambda_m}$ can be obtained by using (4.17), (4.24), (4.25) and (4.32) in (4.29).

C.2) Evaluation of $\frac{\partial \mathbf{E}_1}{\partial \lambda_m}$ and $\frac{\partial \mathbf{E}_2}{\partial \lambda_m}$

Differentiating (4.22) and (4.23) w.r.t. λ_m results in

$$\frac{\partial \mathbf{E}_1}{\partial \lambda_m} = diag \left\{ \frac{-4\left(\frac{\partial \gamma_i}{\partial \lambda_m} D + \gamma_i \frac{\partial D}{\partial \lambda_m}\right) e^{-2\gamma_i D}}{(1 - e^{-2\gamma_i D})^2} \right\} \tag{4.33}$$

$$\frac{\partial \mathbf{E}_2}{\partial \lambda_m} = diag\left\{\frac{2\left(\frac{\partial \gamma_i}{\partial \lambda_m} D + \gamma_i \frac{\partial D}{\partial \lambda_m}\right)\left(e^{-\gamma_i D} + e^{\gamma_i D}\right)}{\left(e^{-\gamma_i D} - e^{\gamma_i D}\right)^2}\right\} \tag{4.34}$$

Using (4.32), (4.33) and (4.34), $\frac{\partial \mathbf{E}_1}{\partial \lambda_m}$ and $\frac{\partial \mathbf{E}_2}{\partial \lambda_m}$ can be evaluated.

4.2.4 Evaluation of the sensitivity of the objective function

The sensitivity of the admittance matrix $\left(\frac{\partial \mathbf{Y}_k(s)}{\partial \lambda_m}\right)$ can be evaluated by substituting (4.30)-(4.34) into (4.28).

Next, using the sensitivity of the admittance matrix and (4.16), the sensitivity of the objective function (4.13) is computed as follows:

$$\frac{\partial W(T)}{\partial \lambda_m} = \boldsymbol{\xi}^t(T)\mathbf{C}\mathbf{z}(0) + \int_0^T \boldsymbol{\psi}_k^t(\tau)\mathbf{q}(t)dt + \int_0^T \frac{\partial w(\mathbf{x}(t), \boldsymbol{\lambda})}{\partial \lambda_m} dt \tag{4.35}$$

where

$$\mathbf{q}(t) = \mathcal{F}^{-1}\left\{\frac{\partial \mathbf{Y}_k(s)}{\partial \lambda_m} \times \mathbf{V}_k(s)\right\} \tag{4.36}$$

$$\mathbf{V}_k(s) = \mathcal{F}\left\{\mathbf{v}_k(t)\right\} \tag{4.37}$$

$\mathcal{F}\{\ \}$ and $\mathcal{F}^{-1}\{\ \}$ denote the FFT and IFFT operators, respectively.

4.2.5 Summary of the computational steps

A summary of the computational steps for the proposed adjoint sensitivity analysis are given below:

- *Step 1*: Simulate the original circuit (4.1) from $t = 0$ to $t = T$ to get $\mathbf{x}(t)$.
- *Step 2*: Replace all the nonlinear elements in the original circuit with linear time-varying elements $\left[\frac{\partial \mathbf{f}(T-\tau)}{\partial \mathbf{x}}\right]^t$. The independent sources for adjoint circuit are evaluated using $\left[\frac{\partial w(T-\tau)}{\partial \mathbf{x}}\right]^t$. Simulate the adjoint circuit equations (4.8) to (4.11) from $\tau = 0$ to $\tau = T$ with zero initial conditions to get $\boldsymbol{\xi}(\tau)$.
- *Step 3*: Use $\mathrm{x}(t)$ from *Step 1*, $\boldsymbol{\xi}(\tau)$ from *Step 2* and (4.13) to find the sensitivity of the objective function (2.4).

It is to be noted that, the concept of self-adjoint can also be used in case the circuit under-consideration consists entirely of lossless transmission lines [25], [26]. However, practical high-speed interconnect circuits contain lossy transmission lines in addition to active and nonlinear components. Moreover, in many cases, the p.u.l. parameters of the transmission lines can be frequency dependent.

The proposed approach can be easily adapted to the case of TLs with frequency-dependent (FD) p.u.l. parameters. This can be accomplished by representing the p.u.l. FD parameters by rational functions using techniques such as vectorfit [27], [28], [29] and subsequently synthesizing them as lumped equivalent circuits.

4.2.6 Comparison with perturbation and direct sensitivity analysis techniques

In this section, a brief discussion of the computational cost of the proposed technique versus the perturbation [11] and direct [2–5] based sensitivity techniques is given.

Using the perturbation technique (2.6), the MNA equations described by (4.1) and (4.2) are solved at each time point twice corresponding to the nominal and perturbed value of the parameter under consideration. Hence, for N_λ parameters, the main computational cost for the perturbation technique is

$$C_P = (N_\lambda + 1) \sum_{i=1}^{N_p} \left[N_{NR}^i (C_{LU} + C_{F/B}) \right] \tag{4.38}$$

where N_p is the number of time points, N_{NR}^i is the number of Newton-Raphson iterations at the time point t_i, C_{LU} and $C_{F/B}$ are the computational times for LU decomposition and forward-backward substitutions, respectively.

Using the direct sensitivity approach, (4.1) and (4.3) can be solved simultaneously avoiding the need for an additional LU decomposition at each time point while solving (4.3). However, an extra forward-backward substitution is needed at every time point for each parameter under consideration. Hence, for N_λ parameters, the main computational cost using the direct approach is

$$C_D = \sum_{i=1}^{N_p} \left[N_{NR}^i (C_{LU} + C_{F/B}) \right] + (N_\lambda N_p) C_{F/B} \tag{4.39}$$

Similarly, the adjoint approach requires the solution of (4.1) and (4.8). However, in contrast to the direct approach where (4.3) is parameter-dependent, (4.8) is independent of any specific parameter. As a result, (4.8) needs to be solved only once independent of the number of parameters. Hence, for N_λ parameters, the main computational cost using the adjoint approach is

$$C_A = \sum_{i=1}^{N_p} \left[N_{NR}^i (C_{LU} + C_{F/B}) \right] + N_p C_{F/B} + N_\lambda C_I, \tag{4.40}$$

if the LU factors are stored while solving (4.1); otherwise

$$\begin{aligned} C_A = & \sum_{i=1}^{N_p} \left[N_{NR}^i (C_{LU} + C_{F/B}) \right] + N_p (C_{LU} + C_{F/B}) \\ & + N_\lambda C_I, \end{aligned} \tag{4.41}$$

where C_I is the computational cost associated with evaluating the numerical integration in (4.13). In the computational results presented in the next section, we used

(4.41) to avoid the additional memory required to store the LU factors.

In addition to extending all the advantages of the adjoint sensitivity analysis approach to distributed transmission lines, the proposed method is generic and is compatible with any MTL macromodel.

4.3 Numerical examples

In this section, four numerical examples are presented to demonstrate the validity and accuracy of the proposed method. Here, Examples 1 and 2 demonstrate the validity and accuracy of the proposed method. Example-3 illustrates the adoption of the proposed algorithm in an optimization problem. Example-4 illustrates the computational savings using the proposed algorithm. In all the examples a relative sensitivity is defined as a change in the objective function due to 1% change in the parameter as

$$S_{\lambda}^{W}(t) = \frac{\partial W(t)}{\partial \lambda} \frac{\lambda}{100} \tag{4.42}$$

4.3.1 Example 1

The circuit considered in this experiment is shown in Fig. 4.1 [11]. It contains three lossy coupled transmission lines numbered by subcircuits 1, 2 and 3. The lengths of the transmission lines in the subcircuits 1, 2 and 3 are 0.2m, 0.5m and 0.3m, respectively. The electrical parameters of the TL #1 are $L_1 = 600nH/m$, $C_1 = 1nF/m$, $R_1 = 1\Omega/m$ and $G_1 = 5mS/m$. The parameters of TL #2 are:

$$\mathbf{L_2} = \begin{bmatrix} 600 & 50 \\ 50 & 600 \end{bmatrix} nH/m \qquad \mathbf{C_2} = \begin{bmatrix} 1.2 & -0.11 \\ -0.11 & 1.2 \end{bmatrix} nF/m$$

$$\mathbf{R_2} = \begin{bmatrix} 2.25 & 0.225 \\ 0.225 & 2.25 \end{bmatrix} \Omega/m \qquad \mathbf{G_2} = \begin{bmatrix} 7.5 & 0 \\ 0 & 7.5 \end{bmatrix} mS/m$$

The parameters of TL #3 are:

$$\mathbf{L_3} = \begin{bmatrix} 1 & 0.11 & 0.03 & 0 \\ 0.11 & 1 & 0.11 & 0.03 \\ 0.03 & 0.11 & 1 & 0.11 \\ 0 & 0.03 & 0.11 & 1 \end{bmatrix} \mu H/m$$

$$\mathbf{C}_3 = \begin{bmatrix} 1.5 & -0.17 & -0.03 & 0 \\ -0.07 & 1.5 & -0.07 & -0.03 \\ -0.03 & -0.07 & 1.5 & -0.07 \\ 0 & -0.03 & -0.07 & 1.5 \end{bmatrix} nF/m$$

$$\mathbf{R}_3 = \begin{bmatrix} 3.5 & 0.35 & 0.035 & 0 \\ 0.35 & 3.5 & 0.35 & 0.035 \\ 0.035 & 0.35 & 3.5 & 0.35 \\ 0 & 0.035 & 0.35 & 3.5 \end{bmatrix} \Omega/m$$

$$\mathbf{G}_3 = \begin{bmatrix} 10 & 1 & 0.1 & 0 \\ 1 & 10 & 1 & 0.1 \\ 0.1 & 1 & 10 & 1 \\ 0 & 0.1 & 1 & 10 \end{bmatrix} mS/m$$

The applied voltage is a trapezoidal pulse with rise and fall time of 1ns, a pulse width of 7ns and a magnitude of 2V. All nonlinear resistors follow $I = V^3$ relation. The output response and the relative sensitivity are computed for the voltage at node

V_{out} in Fig. 4.1. The voltage response at node V_{out} is shown in Fig. 4.2. The corresponding adjoint circuit is shown in Fig. 4.3. The time-domain relative sensitivities w.r.t. lumped resistor R_1, $R_1^{(1,1)}$ of TL #1 and $C_3^{(3,3)}$ of TL #3 were computed using the proposed method. A comparison of the relative sensitivity obtained using the proposed approach and the direct approach [5] are shown in Figs. 4.4, 4.5 and 4.6. As seen, the results from both the approaches match accurately.

4.3.2 Example 2

In this experiment, a nonlinear circuit is considered (Fig. 4.7), containing a lossy coupled transmission line with a length of 0.05m. The physical parameters of the transmission line are shown in Fig. 4.8.

The applied voltage is a trapezoidal pulse with rise and fall time of 0.5ns, a pulse width of 2ns and a magnitude of 5V. The nonlinear resistors follow the relation $I_1 = k_1 V_1^3$ and $I_2 = k_2 V_2^3$, where $k_1 = k_2 = 1 \times 10^{-3}$. The output response and the relative sensitivity are computed for the voltage at node V_{active} in Fig. 4.7. The voltage responses at nodes V_{active} and V_{victim} are shown in Figs. 4.9 and 4.10, respectively. The corresponding adjoint circuit is shown in Fig. 4.11. The time-domain relative sensitivity of V_{victim} w.r.t. different parameters are computed using the proposed approach and are compared with the direct approach [5] in Figs. 4.12 and 4.13.

It can be seen that both the approaches match reasonably well. The observed mi-

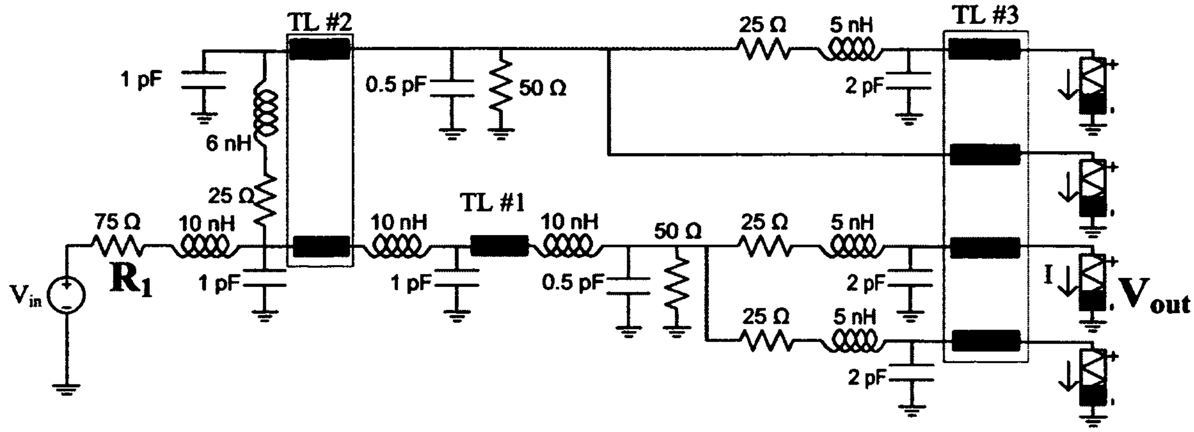

Figure 4.1: Circuit containing three lossy coupled transmission lines (Example 1)

nor deviation is due to the numerical tolerance settings for integrating the differential equations (4.3) and (4.4) and in calculating the convolution integrals (4.13) that are associated with the two approaches under consideration.

To observe the effect of the degree of nonlinearity on the accuracy of the proposed method, the relationship for the I_2 was changed to $I_2 = k_2 V_2^5$ and the experiment was repeated. Even for this case, as shown in Fig. 4.14, the computed relative sensitivities from both the direct and proposed approaches were in good agreement.

4.3.3 Example 3

The nonlinear circuit considered in this example is shown in Fig. 4.15 containing a lossy coupled transmission line with a length of 0.05m. The physical parameters of the transmission line are shown in Fig. 4.8.

The applied voltage is a trapezoidal pulse with rise and fall time of 0.5ns, a

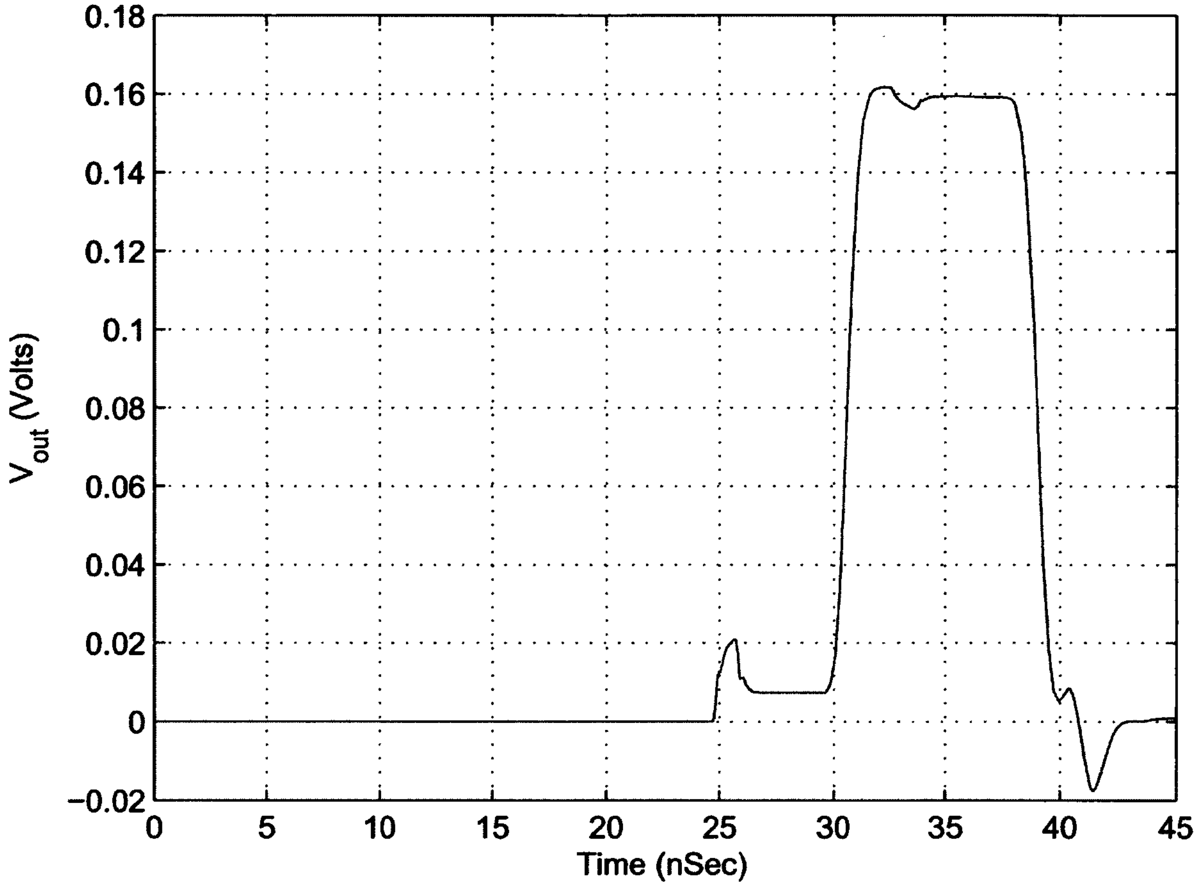

Figure 4.2: Transient response of the circuit shown in Fig. 4.1 at node V_{out}

pulse width of 2ns and a magnitude of 5V. The nonlinear resistor follows the relation $I_1 = k_1 V_1^3$, where $k_1 = 1 \times 10^{-3}$.

Objective function considered is the average power dissipated by the load Z_L,

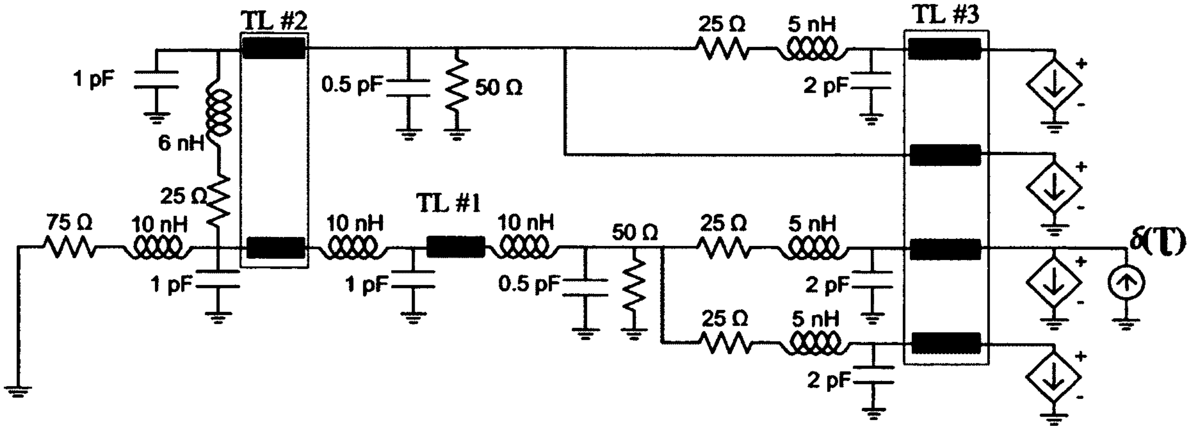

Figure 4.3: Adjoint circuit corresponding to Fig. 4.1 with an impulse source $\delta(\tau)$

$$P_{Diss}(4ns) = \frac{1}{4ns}\int_0^{4ns} [v_a(t)i_a(t)]\,dt \tag{4.43}$$

The power dissipated by the load in the original circuit is $35.05mW$. The corresponding proposed adjoint circuit is shown in Fig. 4.16. The result of the proposed adjoint approach is compared with that from the perturbation method. The relative sensitivity w.r.t. the physical parameters w, t and s are shown in Table 4.1, which further validates the accuracy of the proposed method.

4.3.4 Example 4

To demonstrate the efficiency of the proposed method, a relatively large circuit consisting of 3626 lumped components (resistors, inductors and capacitors) and a TL

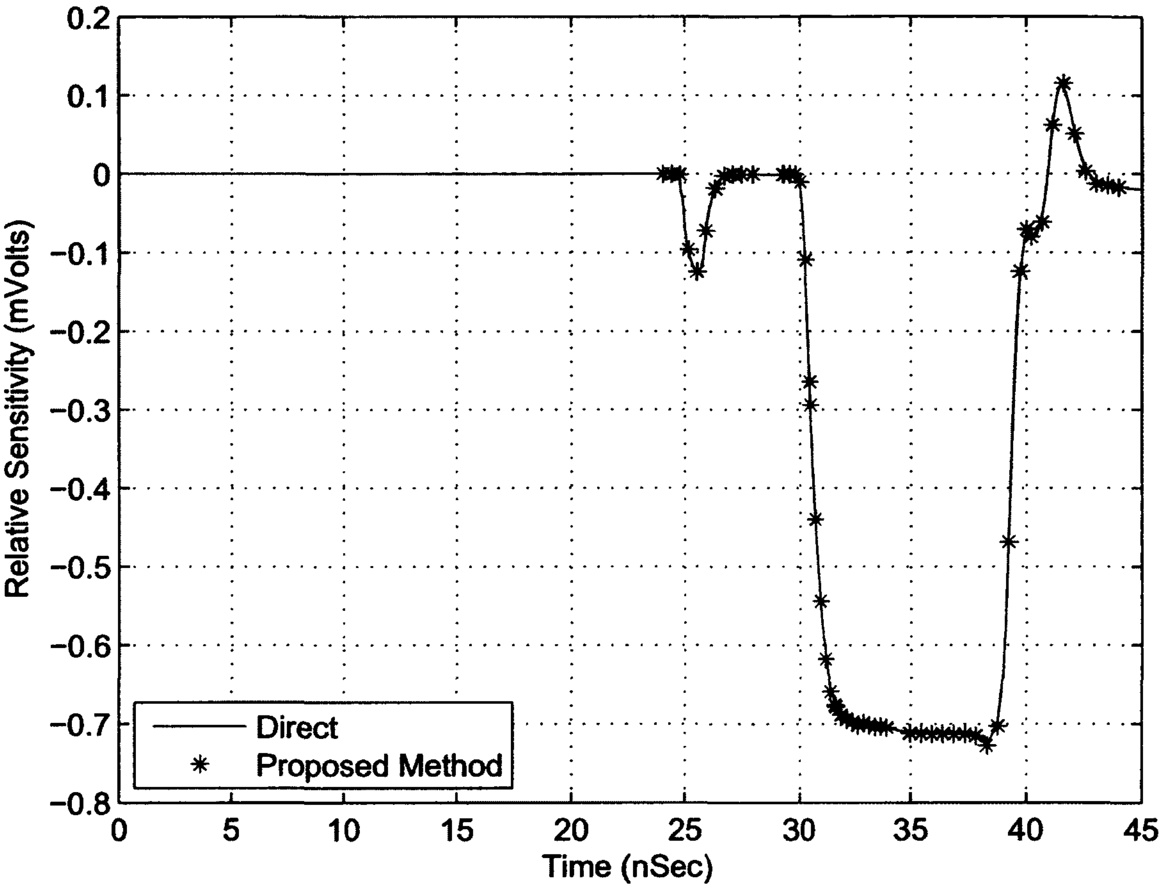

Figure 4.4: Relative sensitivity of the output voltage V_{out} w.r.t. R_1 of Fig. 4.1

network is considered. For sensitivity calculations we selected at random 200 parameters corresponding to lumped components and 4 p.u.l. parameters corresponding to TL network.

The computational cost comparison using proposed adjoint sensitivity with direct and perturbation techniques is given in Table 4.2.

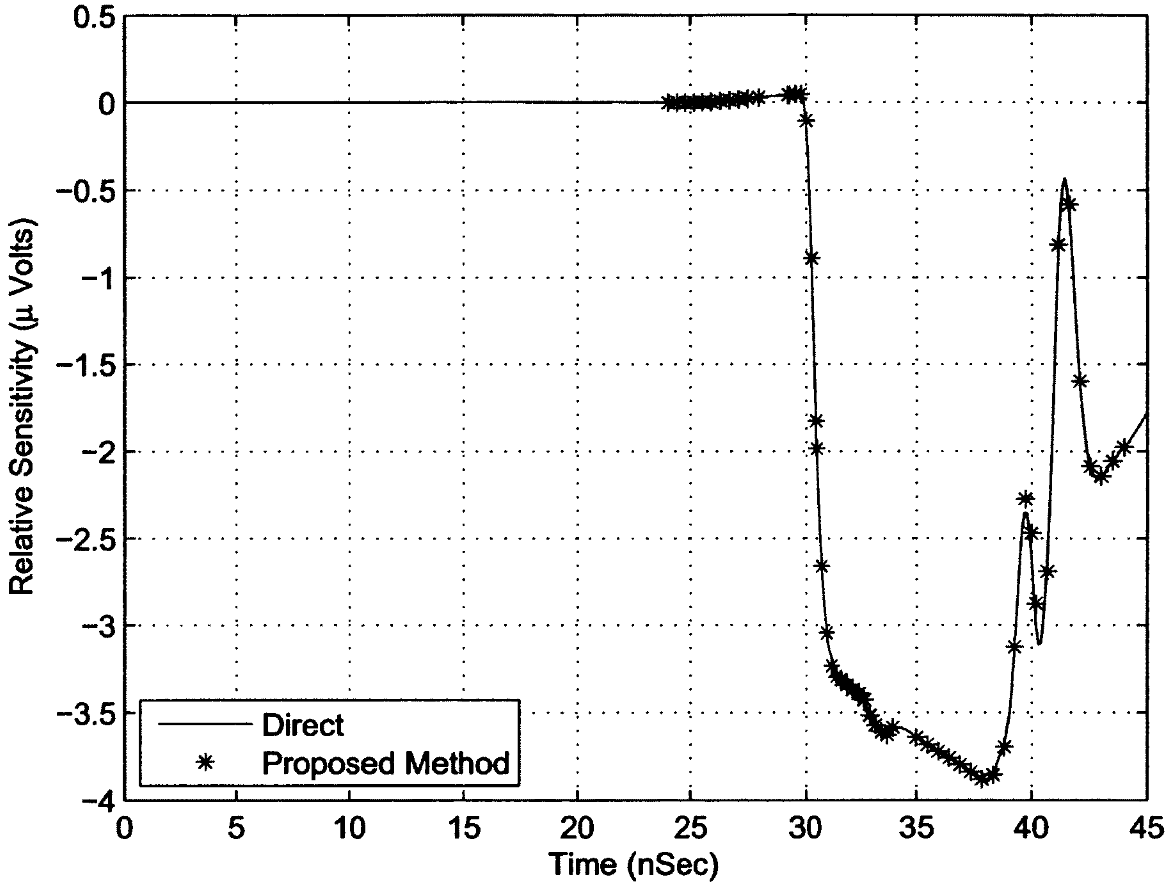

Figure 4.5: Relative sensitivity of the output voltage V_{out} w.r.t. $R_1^{(1,1)}$ of TL #1 of Fig. 4.1

4.4 Conclusions

A generalized approach for first-order time-domain adjoint sensitivity analysis of lossy distributed MTLs in the presence of nonlinear terminations is described. The method is based on the variational approach and enables sensitivity analysis of interconnect structures w.r.t. both electrical and physical parameters while providing significant

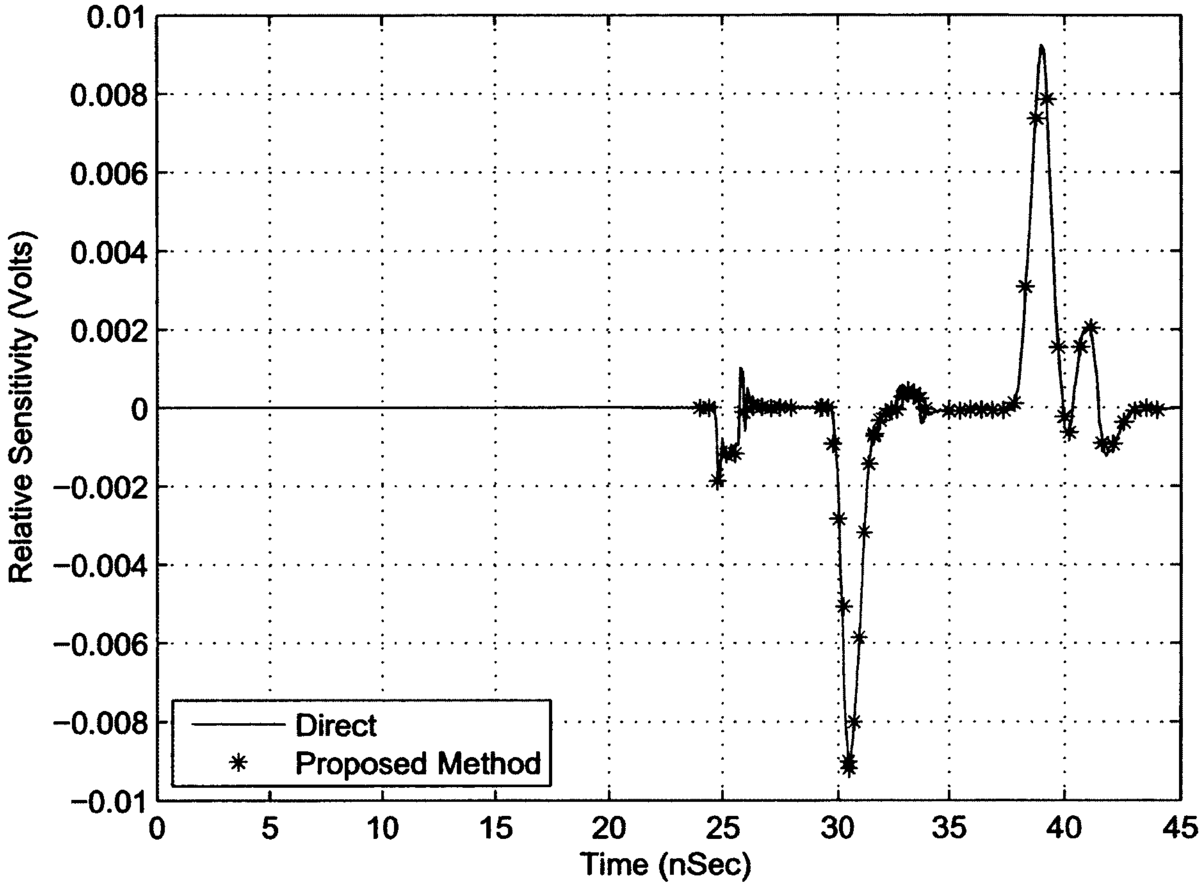

Figure 4.6: Relative sensitivity of the output voltage V_{out} w.r.t. $C_3^{(3,3)}$ of TL #3 of Fig. 4.1

computational cost advantages. While the new approach provides all the advantages of an adjoint sensitivity analysis, its formulation is independent of the specifics of the MTL macromodel used. This makes the proposed method applicable to a wide variety of macromodels available in the literature. The proposed approach can also be easily extended to include the case of frequency-dependent per-unit length parameters.

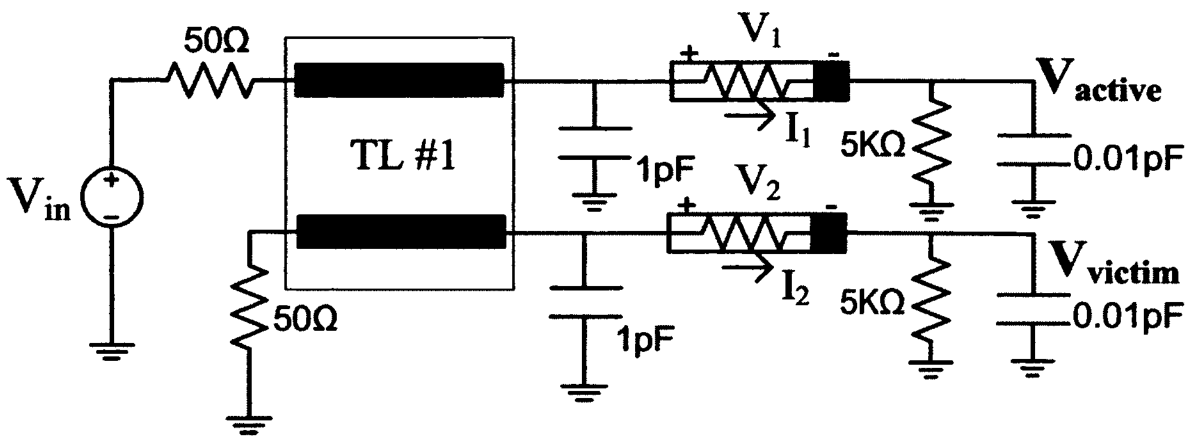

Figure 4.7: Circuit containing a lossy coupled transmission line (Example 2)

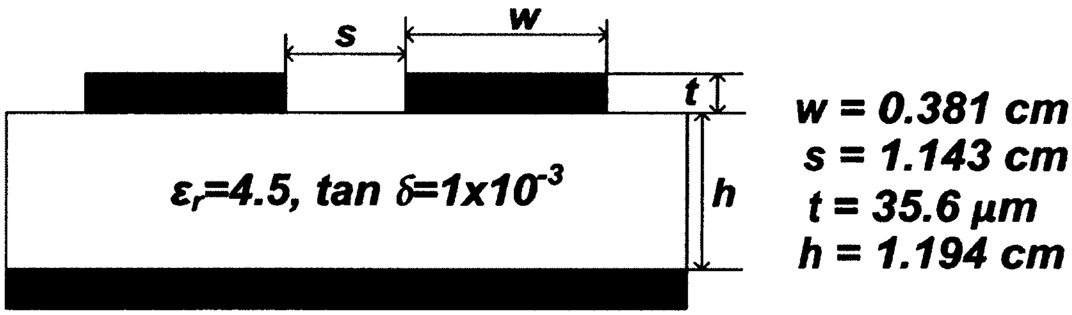

Figure 4.8: Physical parameters of lossy coupled transmission lines of Fig. 4.7

Table 4.1: Sensitivity of dissipated power

Relative Sens.	Perturbation (Watt)	Adjoint (Watt)	Relative Difference
S_w^W	3.65E-06	3.28E-06	10.18%
S_t^W	4.85E-07	4.78E-07	1.34%
S_s^W	1.83E-06	1.80E-06	2.02%

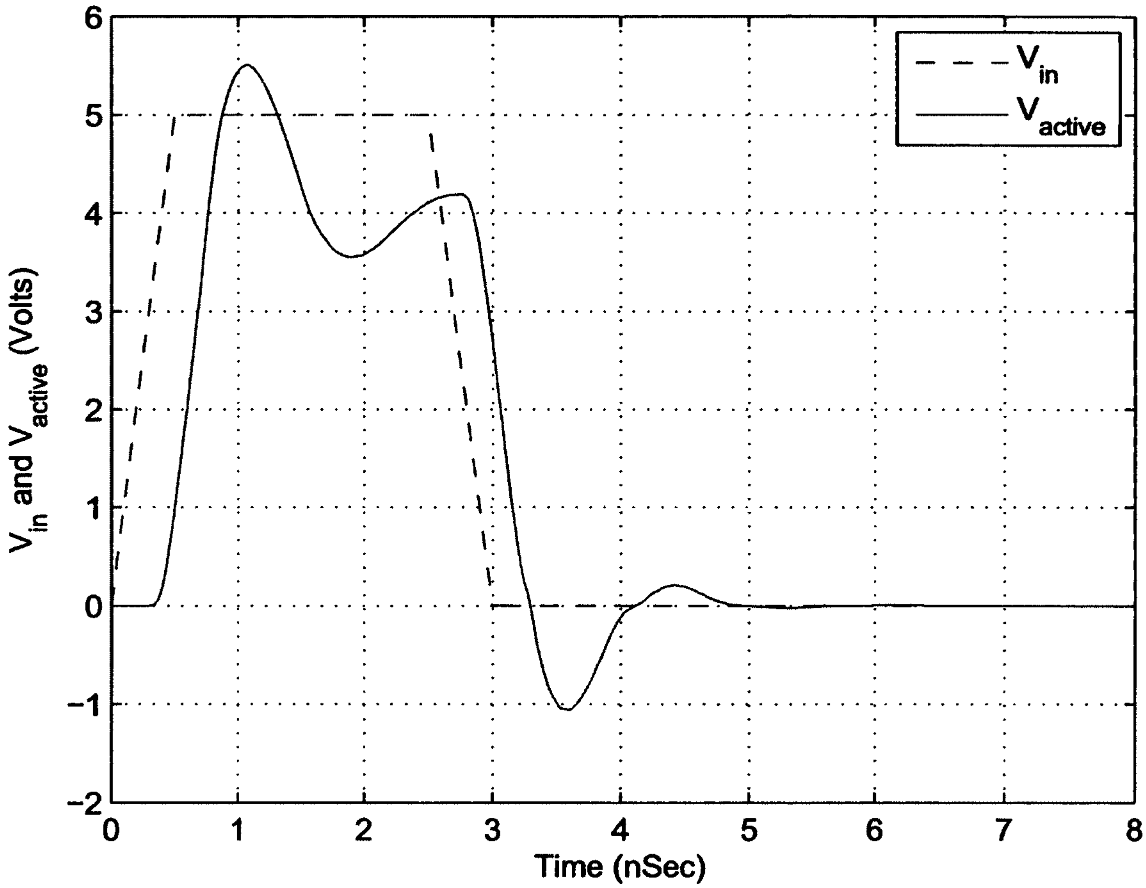

Figure 4.9: Transient response of the circuit shown in Fig. 4.7 at nodes V_{in} and V_{active}

Table 4.2: Computational cost comparison for Example 4

Method	CPU Time (Sec)	Speed-up
Perturbation	246	1
Direct Sensitivity	21.2	11.6
Proposed Adjoint Sensitivity	2.51	98

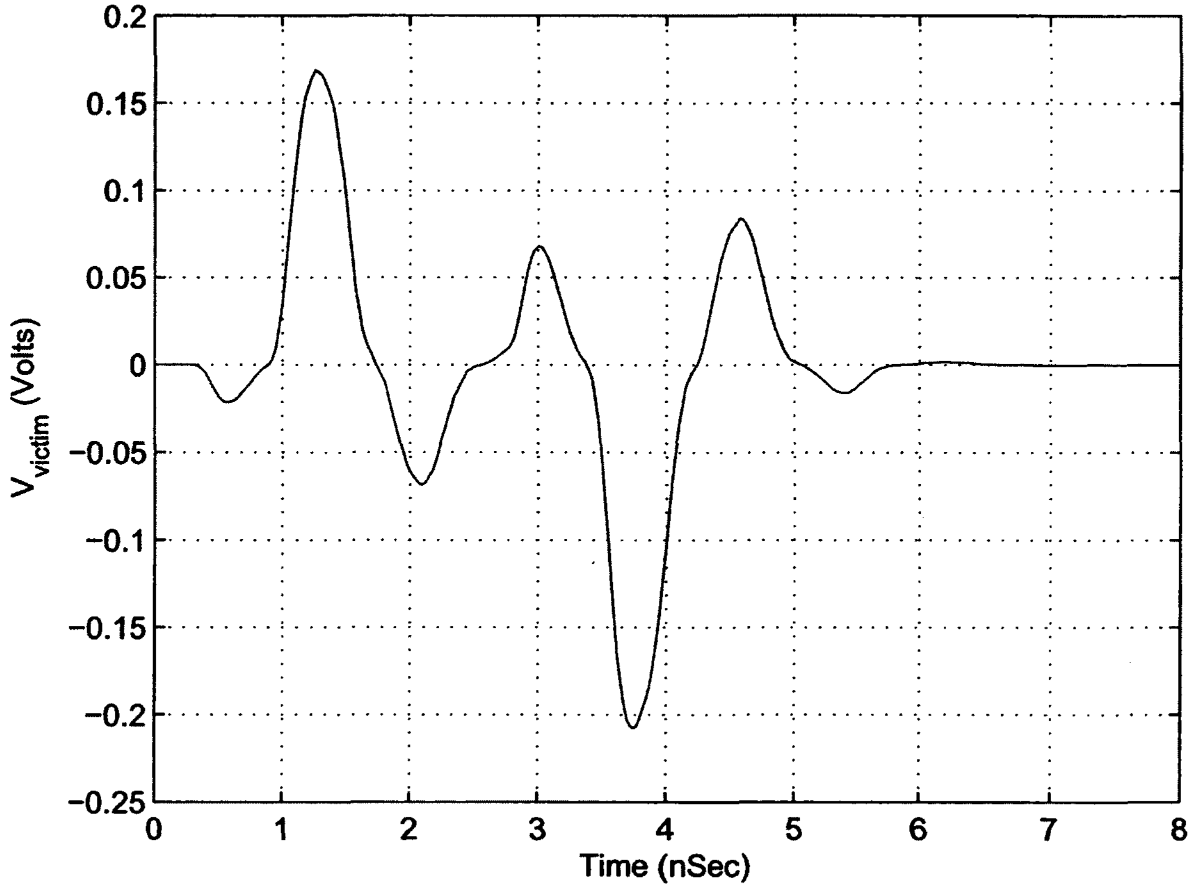

Figure 4.10: Transient response of the circuit shown in Fig. 4.7 at node V_{victim}

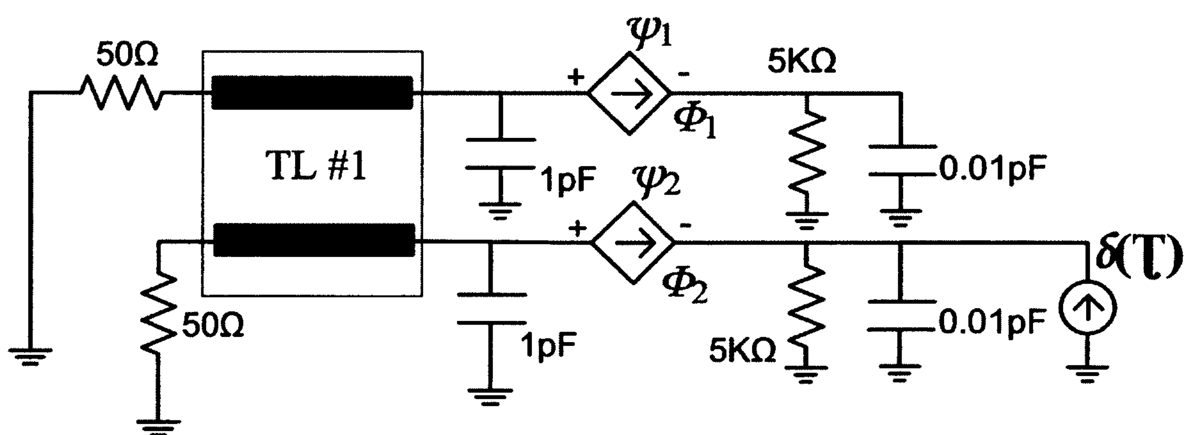

Figure 4.11: Adjoint circuit corresponding Fig. 4.7 with an impulse source $\delta(\tau)$

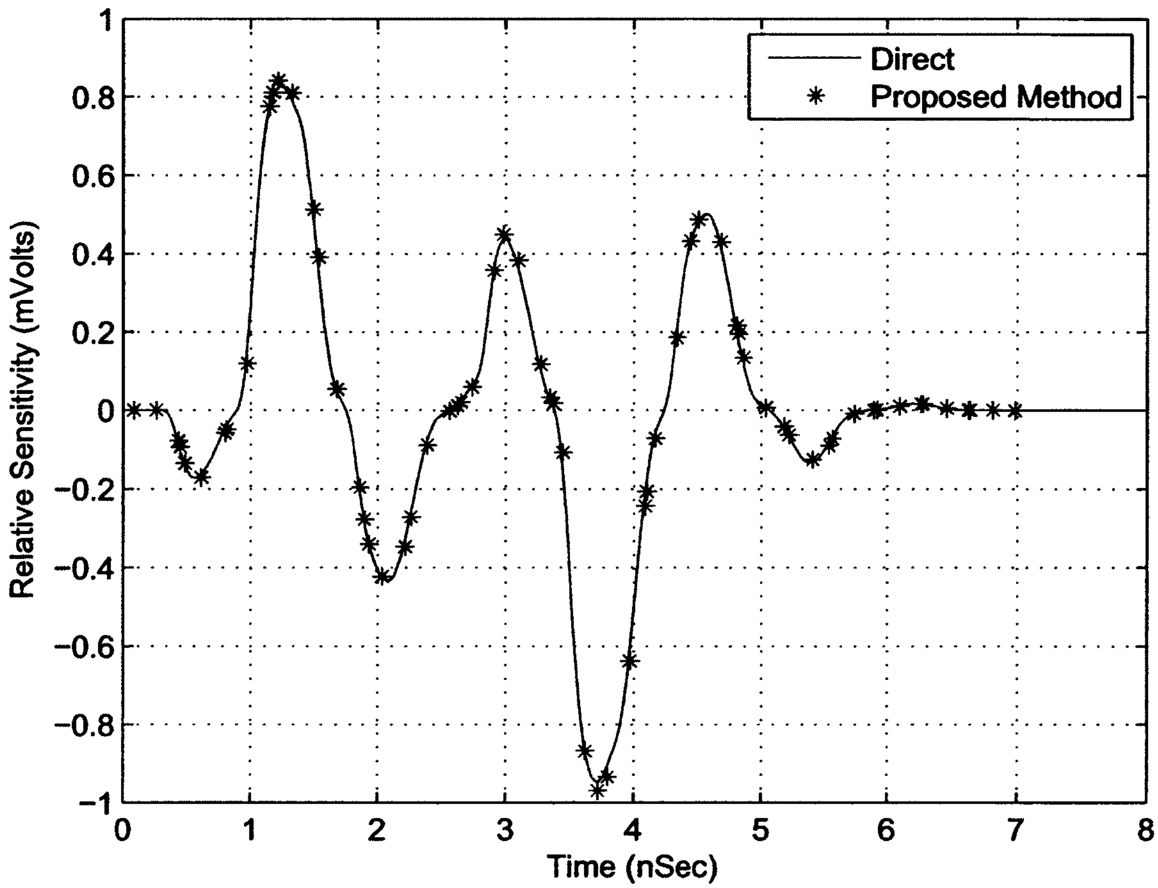

Figure 4.12: Relative sensitivity of output voltage V_{victim} w.r.t. k_2 for circuit in Fig. 4.7

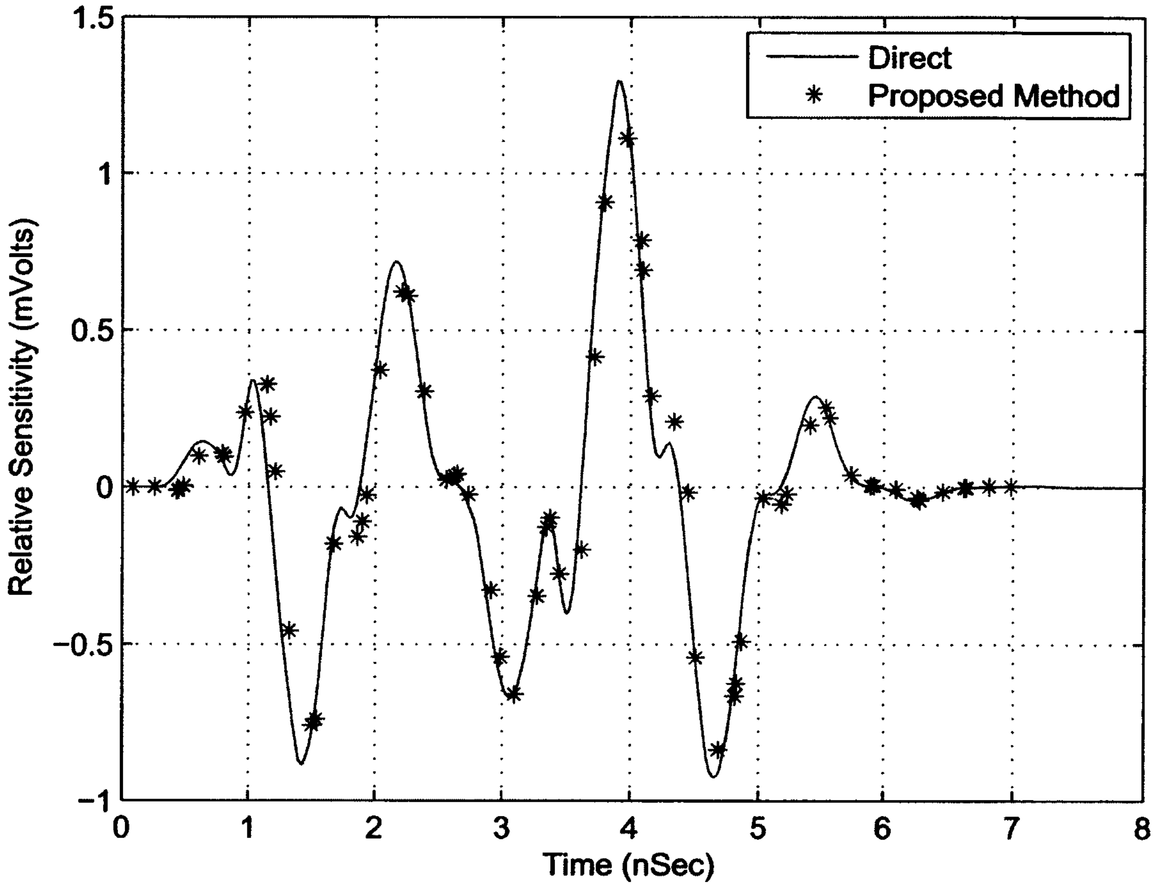

Figure 4.13: Relative sensitivity of output voltage V_{victim} w.r.t. width, w, of TL in Fig. 4.7

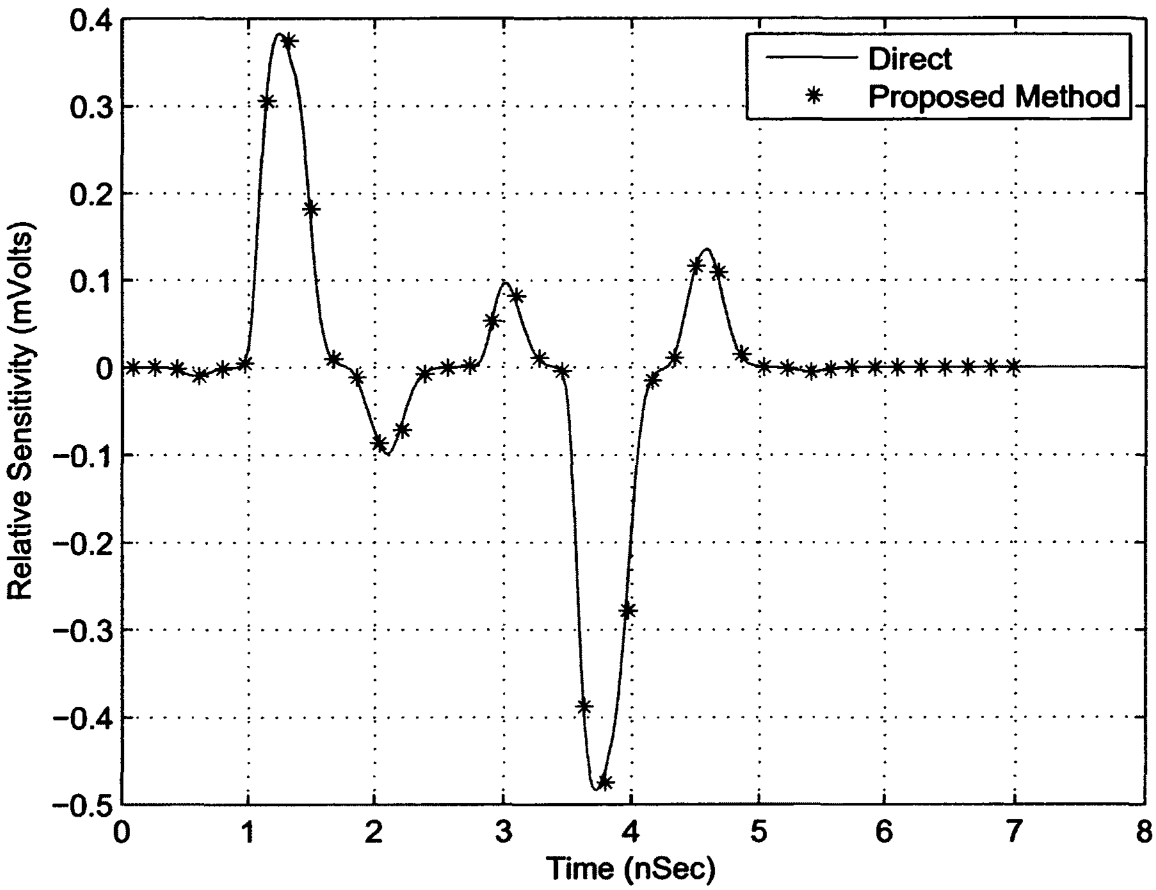

Figure 4.14: Relative sensitivity of output voltage V_{victim} w.r.t. k_2 for circuit in Fig. 4.7 and $I_2 = k_2 V_2^5$

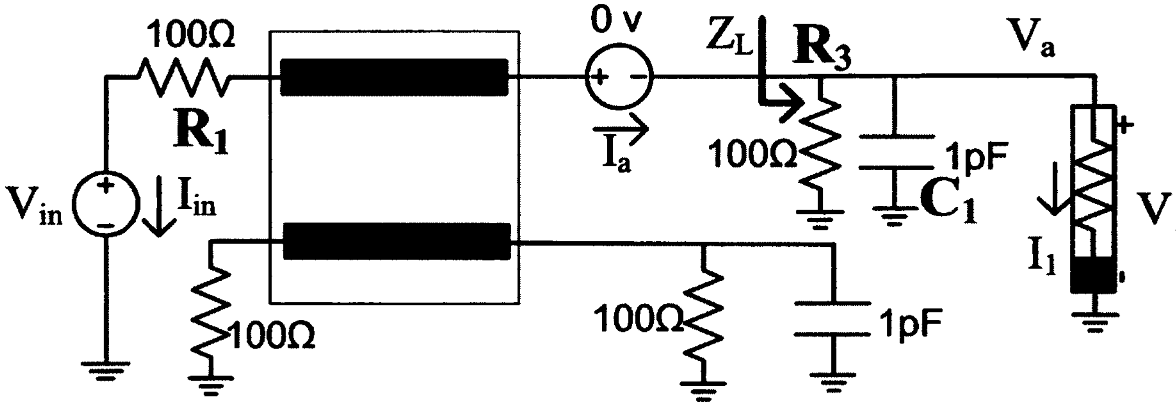

Figure 4.15: Circuit containing a lossy coupled transmission line (Example 3)

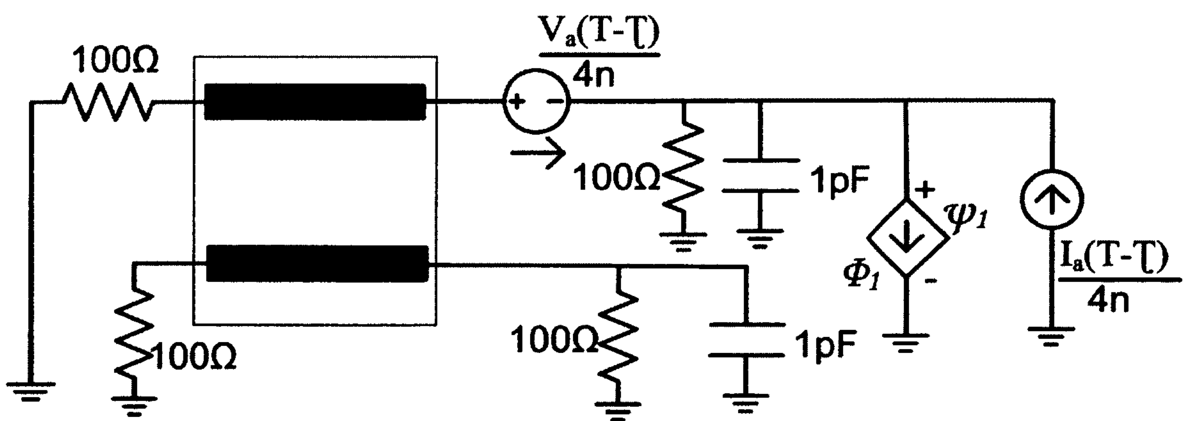

Figure 4.16: Adjoint circuit corresponding to Fig. 4.15

Chapter 5

Proposed variational approach based second-order adjoint sensitivity analysis for networks with MTLs in frequency domain

Sensitivity analysis in frequency domain is often required in gradient based optimizations [30] [31] [32] [33], in evaluating group delay sensitivities [34], noise analysis [35] and many other applications [21]. The first-order frequency-domain adjoint sensitivity analysis based on Tellegen's theorem for a linear lumped circuit was introduced by Rohrer [30]. In gradient based optimization problems, it was not only important to get first-order sensitivities but also second-order sensitivities [31]. To achieve this, [30] was extended to second order in [31]. It was further extended to include two-port transmission lines in [32].

Also, an alternate method for adjoint sensitivity analysis was introduced by Branin [35]. The first-order frequency-domain adjoint sensitivity analysis based on Branin's method [35] for circuits including lossy multiconductor transmission lines was described in [33] [34]. In this chapter, the second-order frequency-domain adjoint sensitivity based on the variational approach [13] for a linear circuit containing lossy multiconductor transmission lines is presented.

The rest of this chapter is arranged as follows. Section 5.1 formulates the circuit equation in frequency domain. Section 5.2 presents the development of the proposed second-order adjoint sensitivity analysis in frequency domain. Section 5.3 presents numerical examples validating the proposed method.

5.1 Formulation of circuit equations

Consider a general circuit consisting of linear components and distributed transmission lines. The corresponding modified nodal analysis (MNA) equations [33] in frequency domain can be written as

$$\mathbf{M}(s)\mathbf{X}(s) = \mathbf{B}(s) \tag{5.1}$$

where

$$\mathbf{M}(s) = s\mathbf{C} + \mathbf{G} + \sum_{k=1}^{N_t} \mathbf{A}_k(s) \tag{5.2}$$

$$\mathbf{A}_k(s) = \mathbf{D}_k \mathbf{Y}_k(s) \mathbf{D}_k^t \tag{5.3}$$

and

- $\mathbf{X}(s) \in C^{n\times 1}$ is the vector of unknowns corresponding to node voltages, independent and dependent voltage source currents, and inductor currents; $\mathbf{B}(s) \in C^{n\times 1}$ is an input vector with entries determined by the independent current and voltage sources and n is the total number of MNA variables;
- $\mathbf{G}$, $\mathbf{C} \in \Re^{n\times n}$ are constant matrices describing the lumped memory-less and memory elements, respectively;
- $\mathbf{D}_k = [d_{i,j}], d_{i,j} \in \{0, 1\}$ is a selector matrix which maps the vector of terminal currents $\mathbf{I}_k(s)$ entering the transmission line k, into the nodal space of the circuit, where $i \in \{1, \cdots, n\}, j \in \{1, \cdots, 2N_k\}$, N_k is number of coupled lines in the k^{th} transmission line and N_t is the total number of distributed transmission lines in the circuit;
- $\mathbf{Y}_k(s) \in C^{2N_k\times 2N_k}$ is the Laplace-domain admittance matrix of the k^{th} transmission line, where $\mathbf{I}_k(s)$ and $\mathbf{V}_k(s) \in C^{2N_k\times 1}$ are the corresponding termi-

nal currents and voltages, respectively, related by $\mathbf{I}_k(s) = \mathbf{Y}_k(s)\mathbf{V}_k(s)$ and $\mathbf{V}_k(s) = \mathbf{D}_k^t\mathbf{X}(s)$.

The next section presents the development of second-order adjoint sensitivity in the frequency domain.

5.2 Development of the proposed second-order adjoint sensitivity in the frequency domain

This section is organized as follows. Problem formulation is given in Section 5.2.1. Development of the first-order adjoint sensitivity is presented in Section 5.2.2. It is further extended to second-order adjoint sensitivity in Section 5.2.3. The second-order sensitivity of the MTL admittance matrix is presented in Section 5.2.4.

5.2.1 Problem formulation

Consider (5.1) and an objective function $W(s)$ whose value is to be optimized at frequency s as given by

$$W(s) = w(\mathbf{X}(s), \boldsymbol{\lambda}) \tag{5.4}$$

where $w(\mathbf{X}(s), \boldsymbol{\lambda})$ is a scalar function and $\boldsymbol{\lambda} \in \Re^{1\times N_\lambda}$ is a vector of N_λ parameters of a circuit given by

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{N_\lambda} \end{bmatrix} \tag{5.5}$$

To perform optimization [21], we need the derivative of $W(s)$, which can be obtained by differentiating (5.4) w.r.t. a particular circuit parameter λ_m as

$$\frac{\partial W(s)}{\partial \lambda_m} = \left[\frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \mathbf{X}} \right] \mathbf{Z}_1(s) + \frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \lambda_m} \tag{5.6}$$

where $m \in \{1, \cdots, N_\lambda\}$, and

$$\mathbf{Z}_1(s) = \frac{\partial \mathbf{X}(s)}{\partial \lambda_m} \tag{5.7}$$

is the sensitivity vector of circuit variables w.r.t. a parameter λ_m.

Using the direct sensitivity approach, $\mathbf{Z}_1(s)$ is obtained by differentiating (5.1) w.r.t. λ_m as

$$\mathbf{M}(s)\mathbf{Z}_1(s) = \mathbf{J}_{\lambda_m}(s) \tag{5.8}$$

where

$$\mathbf{J}_{\lambda_m}(s) = \frac{\partial \mathbf{B}(s)}{\partial \lambda_m} - \frac{\partial \mathbf{M}(s)}{\partial \lambda_m}\mathbf{X}(s) \tag{5.9}$$

As can be seen from (5.8) and (5.9), evaluating $\mathbf{Z}_1(s)$ w.r.t. N_λ variables requires the solution of (5.8) and (5.9) N_λ times, making the process computationally expensive.

In the subsequent of this chapter, development of the proposed second-order sensitivity analysis in the frequency-domain for networks including transmission lines is described. The new method is based on the variational approach [13]. In Section 5.2.2 variational approach for first-order sensitivity analysis for networks including transmission lines is described. Subsequently, this is extended for second-order sensitivity analysis for networks including transmission lines is described in Section 5.2.3.

5.2.2 Variational approach based first-order adjoint sensitivity analysis for distributed transmission line interconnects in the frequency domain

Using the variational approach [13], we define an auxiliary variable $\Xi_a(s) \in C^{n\times 1}$. Multiplying (5.8) by $\Xi_a^t(s)$ gives

$$\left[\Xi_a^t(s)\mathbf{M}(s)\right]\mathbf{Z}_1(s) = \Xi_a^t(s)\mathbf{J}_{\lambda_m}(s) \tag{5.10}$$

It is to be noted that, contrary to (5.8) and (5.9), (5.10) allows us to avoid calculating $\mathbf{Z}_1(s)$ explicitly. This is possible if $\Xi_a(s)$ can be found such that (using (5.10) and (5.6)):

$$\Xi_a^t(s)\mathbf{M}(s) = \left[\frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \mathbf{X}}\right] \tag{5.11}$$

where $\Xi_a(s)$ is referred to as the a^{th} adjoint MNA variable vector [21]. Taking the transpose of both sides in (5.11) gives

$$\mathbf{M}^t(s)\Xi_a(s) = \left[\frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \mathbf{X}}\right]^t \tag{5.12}$$

Next, substituting (5.10) in (5.6) gives

$$\frac{\partial W(s)}{\partial \lambda_m} = \underbrace{\Xi_a^t(s)\mathbf{J}_{\lambda_m}(s)}_{\text{Term 1}} + \frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \lambda_m} \tag{5.13}$$

Once the original (5.1) and adjoint (5.12) systems are simulated, the sensitivity of an objective function w.r.t. λ_m (5.13) can be found by using $\mathbf{X}(s)$ and $\Xi_a(s)$. It is to be noted that in the direct sensitivity approach, (5.8) is dependent on a parameter λ_m whereas in the adjoint sensitivity approach, (5.12) is independent of the parameter λ_m. Therefore, (5.8) has to be solved N_λ times whereas (5.12) has to be solved only once. If λ_m is the electrical or physical parameter of the k^{th} transmission line, then $\mathbf{J}_{\lambda_m}(s)$ in (5.9) becomes

$$\mathbf{J}_{\lambda_m}(s) = -\frac{\partial \mathbf{A}_k(s)}{\partial \lambda_m}\mathbf{X}(s) \tag{5.14}$$

and (5.13) becomes

$$\begin{aligned}\frac{\partial W(s)}{\partial \lambda_m} &= -\,\Xi_a^t(s)\frac{\partial \mathbf{A}_k(s)}{\partial \lambda_m}\mathbf{X}(s) + \frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \lambda_m} \\ &= -\,[\boldsymbol{\Psi}_k^a(s)]^t\,\frac{\partial \mathbf{Y}_k(s)}{\partial \lambda_m}\mathbf{V}_k(s) + \frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \lambda_m}\end{aligned} \tag{5.15}$$

where $\boldsymbol{\Psi}_k^a(s) \in C^{2N_k \times 1}$ is the vector of terminal voltages across the k^{th} transmission line in the a^{th} adjoint circuit.

This section described the first-order adjoint sensitivity for distributed transmission lines using the variational approach. The variational approach can also be used to extend adjoint sensitivity to second order. The next section provides its extension for second-order adjoint sensitivity analysis for circuits including distributed transmission line interconnect networks.

5.2.3 Variational approach based second-order adjoint sensitivity analysis for distributed transmission line interconnects in frequency domain

In gradient optimization problems, it was not only important to get first-order sensitivity but also second-order sensitivity [31]. The second-order sensitivity of $W(s)$, can be obtained by either differentiating (5.6) [21] [36] or (5.13) [37] w.r.t. a particular circuit parameter λ_n. Differentiating (5.13) w.r.t. λ_n gives

$$\begin{aligned}\frac{\partial^2 W(s)}{\partial\lambda_m\partial\lambda_n} &= \underbrace{\frac{\partial\boldsymbol{\Xi}_a^t(s)}{\partial\lambda_n}\mathbf{J}_{\lambda_m}(s)}_{\text{Term 2}} + \underbrace{\boldsymbol{\Xi}_a^t(s)\frac{\partial\mathbf{J}_{\lambda_m}(s)}{\partial\lambda_n}}_{\text{Term 3}} \\ &+ \left[\frac{\partial^2 w(\mathbf{X}(s),\boldsymbol{\lambda})}{\partial\lambda_m\partial\mathbf{X}}\right]\mathbf{Z}_2(s) + \frac{\partial^2 w(\mathbf{X}(s),\boldsymbol{\lambda})}{\partial\lambda_m\partial\lambda_n}\end{aligned} \tag{5.16}$$

where $n \in \{1, \cdots, N_\lambda\}$ and

$$\mathbf{Z}_2(s) = \frac{\partial\mathbf{X}(s)}{\partial\lambda_n} \tag{5.17}$$

is the sensitivity vector of circuit variables w.r.t. a parameter λ_n. It is to be noted that [37] presents a special case of the objective function (5.4) where $W(s)$ is the linear function of $\mathbf{X}(s)$. Here, a more generic objective function is considered. From now onwards to simplify presentation $w(\mathbf{X}(s), \boldsymbol{\lambda})$ will be represented as w.

Next, *Term 2* of the RHS in (5.16) requires the derivative of the a^{th} adjoint variable, $\frac{\partial \Xi_a^t(s)}{\partial \lambda_n}$, which can be obtained by differentiating (5.11) w.r.t. λ_n as

$$\frac{\partial \Xi_a^t(s)}{\partial \lambda_n}\mathbf{M}(s) = -\Xi_a^t(s)\frac{\partial \mathbf{M}(s)}{\partial \lambda_n} + \mathbf{Z}_2^t(s)\frac{\partial^2 w}{\partial \mathbf{X}^2} + \frac{\partial^2 w}{\partial \mathbf{X}\partial \lambda_n} \tag{5.18}$$

Equation (5.18) can be used to evaluate $\frac{\partial \Xi_a^t(s)}{\partial \lambda_n}$ explicitly, which is computationally expensive. In order to avoid evaluating $\frac{\partial \Xi_a^t(s)}{\partial \lambda_n}$ explicitly, the variational approach can be used. Using the variational approach, we define an auxiliary variable $\Xi_b(s) \in C^{n\times 1}$. Multiplying (5.18) by $\Xi_b(s)$ gives

$$\frac{\partial \Xi_a^t(s)}{\partial \lambda_n}\mathbf{M}(s)\Xi_b(s) = -\Xi_a^t(s)\frac{\partial \mathbf{M}(s)}{\partial \lambda_n}\Xi_b(s) + \mathbf{Z}_2^t(s)\frac{\partial^2 w}{\partial \mathbf{X}^2}\Xi_b(s) + \frac{\partial^2 w}{\partial \mathbf{X}\partial \lambda_n}\Xi_b(s) \tag{5.19}$$

It is to be noted that, contrary to (5.18), (5.19) allows us to avoid calculating $\frac{\partial \Xi_a^t(s)}{\partial \lambda_n}$ explicitly. This is possible if $\Xi_b(s)$ can be found such that (using (5.19) and *Term 2* in (5.16)):

$$\mathbf{M}(s)\Xi_b(s) = \mathbf{J}_{\lambda_m}(s) \tag{5.20}$$

where $\Xi_b(s)$ is the b^{th} adjoint MNA variable.

Term 3 of the RHS in (5.16) also requires $\frac{\partial \mathbf{J}_{\lambda_m}(s)}{\partial \lambda_n}$, which can be obtained by differentiating (5.9) w.r.t. λ_n as

$$\frac{\partial \mathbf{J}_{\lambda_m}(s)}{\partial \lambda_n} = \frac{\partial^2 \mathbf{B}(s)}{\partial \lambda_m \partial \lambda_n} - \frac{\partial \mathbf{M}(s)}{\partial \lambda_m}\mathbf{Z}_2(s) - \frac{\partial^2 \mathbf{M}(s)}{\partial \lambda_m \partial \lambda_n}\mathbf{X}(s) \tag{5.21}$$

Substituting (5.19) and (5.21) for *Term 2* and *Term 3* of the RHS in (5.16) results in

$$\begin{aligned}\frac{\partial^2 W(s)}{\partial \lambda_m \partial \lambda_n} &= -\Xi_a^t(s)\frac{\partial \mathbf{M}(s)}{\partial \lambda_n}\Xi_b(s) + \frac{\partial^2 w}{\partial \mathbf{X} \partial \lambda_n}\Xi_b(s) + \\ &\Xi_a^t(s)\frac{\partial^2 \mathbf{B}(s)}{\partial \lambda_m \partial \lambda_n} - \Xi_a^t(s)\frac{\partial^2 \mathbf{M}(s)}{\partial \lambda_m \partial \lambda_n}\mathbf{X}(s) + \frac{\partial^2 w}{\partial \lambda_m \partial \lambda_n} \\ &\underbrace{+\left\{\Xi_b^t\frac{\partial^2 w}{\partial \mathbf{X}^2} - \Xi_a^t\frac{\partial \mathbf{M}(s)}{\partial \lambda_m} + \left[\frac{\partial^2 w}{\partial \lambda_m \partial \mathbf{X}}\right]\right\}\mathbf{Z}_2(s)}_{\text{Term 4}}\end{aligned} \tag{5.22}$$

Term 4 of the RHS in (5.22) requires the value of $\mathbf{Z}_2(s)$. Using the direct sensitivity approach, $\mathbf{Z}_2(s)$ is obtained by differentiating (5.1) w.r.t. λ_n as

$$\mathbf{M}(s)\mathbf{Z}_2(s) = \mathbf{J}_{\lambda_n}(s) \tag{5.23}$$

where

$$\mathbf{J}_{\lambda_n}(s) = \frac{\partial \mathbf{B}(s)}{\partial \lambda_n} - \frac{\partial \mathbf{M}(s)}{\partial \lambda_n}\mathbf{X}(s) \tag{5.24}$$

Equation (5.23) can be used to evaluate $\mathbf{Z}_2(s)$ explicitly, which is computationally expensive. In order to avoid evaluating $\mathbf{Z}_2(s)$ explicitly, the variational approach can be used. Using the variational approach, we define an auxiliary variable $\Xi_c(s) \in C^{n\times 1}$. Multiplying (5.23) by $\Xi_c^t(s)$ gives

$$\Xi_c^t(s)\mathbf{M}(s)\mathbf{Z}_2(s) = \Xi_c^t(s)\mathbf{J}_{\lambda_n}(s) \tag{5.25}$$

It is to be noted that, contrary to (5.23) and (5.24), (5.25) allows us to avoid calculating $\mathbf{Z}_2(s)$ explicitly. This is possible if $\Xi_c(s)$ can be found such that (using (5.25) and *Term 4* of the RHS in (5.22)):

$$\Xi_c^t(s)\mathbf{M}(s) = \Xi_b^t \frac{\partial^2 w}{\partial \mathbf{X}^2} - \Xi_a^t \frac{\partial \mathbf{M}(s)}{\partial \lambda_m} + \left[\frac{\partial^2 w}{\partial \lambda_m \partial \mathbf{X}}\right] \tag{5.26}$$

where $\Xi_c(s)$ is referred to as the c^{th} adjoint MNA variable vector. Taking the transpose of both sides in (5.11) gives

$$\mathbf{M}^t(s)\Xi_c(s) = \frac{\partial^2 w}{\partial \mathbf{X}^2}\Xi_b - \frac{\partial \mathbf{M}^t(s)}{\partial \lambda_m}\Xi_a + \left[\frac{\partial^2 w}{\partial \lambda_m \partial \mathbf{X}}\right]^t \tag{5.27}$$

Next, substituting (5.25) with (5.24) in (5.22) gives

$$\begin{aligned}\frac{\partial^2 W(s)}{\partial \lambda_m \partial \lambda_n} &= -\Xi_a^t(s)\frac{\partial \mathbf{M}(s)}{\partial \lambda_n}\Xi_b(s) + \frac{\partial^2 w}{\partial \mathbf{X}\partial \lambda_n}\Xi_b(s) + \\ &\Xi_a^t(s)\frac{\partial^2 \mathbf{B}(s)}{\partial \lambda_m \partial \lambda_n} - \Xi_a^t(s)\frac{\partial^2 \mathbf{M}(s)}{\partial \lambda_m \partial \lambda_n}\mathbf{X}(s) + \frac{\partial^2 w}{\partial \lambda_m \partial \lambda_n} \\ &+\Xi_c^t(s)\frac{\partial \mathbf{B}(s)}{\partial \lambda_n} - \Xi_c^t(s)\frac{\partial \mathbf{M}(s)}{\partial \lambda_n}\mathbf{X}(s)\end{aligned} \tag{5.28}$$

If λ_m and λ_n are an electrical or physical parameter of k^{th} transmission line, then (5.28) becomes

$$\frac{\partial^2 W(s)}{\partial\lambda_m\partial\lambda_n} = -\left[\mathbf{\Psi}_k^a(s)\right]^t \frac{\partial \mathbf{Y}_k(s)}{\partial\lambda_n}\mathbf{\Psi}_k^b(s) + \frac{\partial^2 w}{\partial\mathbf{X}\partial\lambda_n}\Xi_b(s) +$$
$$\Xi_a^t(s)\frac{\partial^2\mathbf{B}(s)}{\partial\lambda_m\partial\lambda_n} - \left[\mathbf{\Psi}_k^a(s)\right]^t \frac{\partial^2 \mathbf{Y}_k(s)}{\partial\lambda_m\partial\lambda_n}\mathbf{V}_k(s) + \frac{\partial^2 w}{\partial\lambda_m\partial\lambda_n}$$
$$+\Xi_c^t(s)\frac{\partial\mathbf{B}(s)}{\partial\lambda_n} - \left[\mathbf{\Psi}_k^c(s)\right]^t \frac{\partial \mathbf{Y}_k(s)}{\partial\lambda_n}\mathbf{V}_k(s) \tag{5.29}$$

where $\mathbf{\Psi}_k^a(s)$, $\mathbf{\Psi}_k^b(s)$ and $\mathbf{\Psi}_k^c(s) \in C^{2N_k\times 1}$ are the terminal voltages across the k^{th} transmission line in a^{th}, b^{th} and c^{th} adjoint systems, respectively.

Next, the details of evaluation of $\frac{\partial^2 \mathbf{Y}_k(s)}{\partial\lambda_m\partial\lambda_n}$ in (5.29) in a closed-form are given in the following section. The details of evaluation of $\frac{\partial \mathbf{Y}_k(s)}{\partial\lambda_m}$ in a closed-form are already given in Section 4.2.3.

5.2.4 Evaluation of the second-order sensitivity of the MTL admittance matrix

Differentiating (4.28) w.r.t. λ_n results in [34]

$$\frac{\partial^2\mathbf{Y}_k}{\partial\lambda_m\partial\lambda_n}\begin{bmatrix}\mathbf{S}_v & 0\\ & \\ 0 & \mathbf{S}_v\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\mathbf{S}_q}{\partial\lambda_m\partial\lambda_n} & 0\\ & \\ 0 & \frac{\partial^2\mathbf{S}_q}{\partial\lambda_m\partial\lambda_n}\end{bmatrix}\begin{bmatrix}\mathbf{E}_1 & \mathbf{E}_2\\ & \\ \mathbf{E}_2 & \mathbf{E}_1\end{bmatrix} +$$

$$
\begin{bmatrix} \frac{\partial \mathbf{S}_q}{\partial \lambda_m} & 0 \\ 0 & \frac{\partial \mathbf{S}_q}{\partial \lambda_m} \end{bmatrix}
\begin{bmatrix} \frac{\partial \mathbf{E}_1}{\partial \lambda_n} & \frac{\partial \mathbf{E}_2}{\partial \lambda_n} \\ \frac{\partial \mathbf{E}_2}{\partial \lambda_n} & \frac{\partial \mathbf{E}_1}{\partial \lambda_n} \end{bmatrix}
+
\begin{bmatrix} \frac{\partial \mathbf{S}_q}{\partial \lambda_n} & 0 \\ 0 & \frac{\partial \mathbf{S}_q}{\partial \lambda_n} \end{bmatrix}
\begin{bmatrix} \frac{\partial \mathbf{E}_1}{\partial \lambda_m} & \frac{\partial \mathbf{E}_2}{\partial \lambda_m} \\ \frac{\partial \mathbf{E}_2}{\partial \lambda_m} & \frac{\partial \mathbf{E}_1}{\partial \lambda_m} \end{bmatrix}
$$

$$
+
\begin{bmatrix} \mathbf{S}_q & 0 \\ 0 & \mathbf{S}_q \end{bmatrix}
\begin{bmatrix} \frac{\partial^2 \mathbf{E}_1}{\partial \lambda_m \partial \lambda_n} & \frac{\partial^2 \mathbf{E}_2}{\partial^2 \lambda_m \partial \lambda_n} \\ \frac{\partial^2 \mathbf{E}_2}{\partial^2 \lambda_m \partial \lambda_n} & \frac{\partial^2 \mathbf{E}_1}{\partial \lambda_m \partial \lambda_n} \end{bmatrix}
- \frac{\partial \mathbf{Y}_k}{\partial \lambda_n}
\begin{bmatrix} \frac{\partial \mathbf{S}_v}{\partial \lambda_m} & 0 \\ 0 & \frac{\partial \mathbf{S}_v}{\partial \lambda_m} \end{bmatrix}
$$

$$
- \mathbf{Y}_k
\begin{bmatrix} \frac{\partial^2 \mathbf{S}_v}{\partial \lambda_m \partial \lambda_n} & 0 \\ 0 & \frac{\partial^2 \mathbf{S}_v}{\partial \lambda_m \partial \lambda_n} \end{bmatrix}
- \frac{\partial \mathbf{Y}_k}{\partial \lambda_m}
\begin{bmatrix} \frac{\partial \mathbf{S}_v}{\partial \lambda_n} & 0 \\ 0 & \frac{\partial \mathbf{S}_v}{\partial \lambda_n} \end{bmatrix}
\qquad (5.30)
$$

The first-order derivatives, $\frac{\partial \mathbf{S}_v}{\partial \lambda_m}$, $\frac{\partial \mathbf{S}_q}{\partial \lambda_m}$, $\frac{\partial \mathbf{E}_1}{\partial \lambda_m}$ and $\frac{\partial \mathbf{E}_2}{\partial \lambda_m}$, can be found by using the algorithm given in Section 4.2.3. To proceed further with the evaluation of $\frac{\partial^2 \mathbf{Y}_k(s)}{\partial \lambda_m \partial \lambda_m}$ in (5.30), individual derivatives (namely $\frac{\partial^2 \mathbf{S}_v}{\partial \lambda_m \partial \lambda_n}$, $\frac{\partial^2 \mathbf{S}_q}{\partial \lambda_m \partial \lambda_n}$, $\frac{\partial^2 \mathbf{E}_1}{\partial \lambda_m \partial \lambda_n}$ and $\frac{\partial^2 \mathbf{E}_2}{\partial \lambda_m \partial \lambda_n}$) are required and are evaluated as follows.

5.2.4.1) Evaluation of $\frac{\partial^2 \mathbf{S}_v}{\partial \lambda_m \partial \lambda_n}$

Differentiating (4.32) w.r.t. λ_n results in

$$\begin{bmatrix} \gamma_i^2\mathbf{U} - \mathbf{Z}_k\mathbf{\Omega}_k & \boldsymbol{\beta}_i \\ \\ \boldsymbol{\beta}_i^t & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^2\boldsymbol{\beta}_i}{\partial\lambda_m\partial\lambda_n} \\ \\ \frac{\partial^2\gamma_i^2}{\partial\lambda_m\partial\lambda_n} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial^2(\mathbf{Z}_k\mathbf{\Omega}_k)}{\partial\lambda_m\partial\lambda_n}\boldsymbol{\beta}_i + \frac{\partial(\mathbf{Z}_k\mathbf{\Omega}_k)}{\partial\lambda_m}\frac{\partial\boldsymbol{\beta}_i}{\partial\lambda_n} \\ \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} \frac{\partial\gamma_i^2}{\partial\lambda_n}\mathbf{U} - \frac{\partial(\mathbf{Z}_k\mathbf{\Omega}_k)}{\partial\lambda_n} & \frac{\partial\boldsymbol{\beta}_i}{\partial\lambda_n} \\ \\ \frac{\partial\boldsymbol{\beta}_i^t}{\partial\lambda_n} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial\boldsymbol{\beta}_i}{\partial\lambda_m} \\ \\ \frac{\partial\gamma_i^2}{\partial\lambda_m} \end{bmatrix} \tag{5.31}$$

Using (4.25), (4.32) and (5.31), $\frac{\partial^2\mathbf{S}_v}{\partial\lambda_m\partial\lambda_n}$ can be evaluated.

5.2.4.2) Evaluation of $\frac{\partial^2\mathbf{S}_q}{\partial\lambda_m\partial\lambda_n}$

Differentiating (4.29) w.r.t. λ_n results in

$$\begin{aligned} \mathbf{Z}_k\frac{\partial^2\mathbf{S}_q}{\partial\lambda_m\partial\lambda_n} &= \frac{\partial^2\mathbf{S}_v}{\partial\lambda_m\partial\lambda_n}\mathbf{\Gamma} + \frac{\partial\mathbf{S}_v}{\partial\lambda_m}\frac{\partial\mathbf{\Gamma}}{\partial\lambda_n} + \frac{\partial\mathbf{S}_v}{\partial\lambda_n}\frac{\partial\mathbf{\Gamma}}{\partial\lambda_m} \\ &+\mathbf{S}_v\frac{\partial^2\mathbf{\Gamma}}{\partial\lambda_m\partial\lambda_n} - \frac{\partial\mathbf{Z}_k}{\partial\lambda_m}\frac{\partial\mathbf{S}_q}{\partial\lambda_n} - \frac{\partial\mathbf{Z}_k}{\partial\lambda_n}\frac{\partial\mathbf{S}_q}{\partial\lambda_m} \end{aligned} \tag{5.32}$$

Using (4.24), (4.25), (5.31) and (5.32), $\frac{\partial^2 \mathbf{S}_v}{\partial \lambda_m \partial \lambda_n}$ can be evaluated.

5.2.4.3) Evaluation of $\frac{\partial^2 \mathbf{E}_1}{\partial \lambda_m \partial \lambda_n}$ and $\frac{\partial^2 \mathbf{E}_2}{\partial \lambda_m \partial \lambda_n}$

Differentiating (4.33) and (4.34) w.r.t. λ_n results in

$$\frac{\partial^2 \mathbf{E}_1}{\partial \lambda_m \partial \lambda_n} = diag \left\{ \frac{-4 \begin{bmatrix} (k_i - 2l_i m_i)\, e^{-2\gamma_i D} \left(1 - e^{-2\gamma_i D}\right) \\ -4\,(l_i m_i)\, e^{-4\gamma_i D} \end{bmatrix}}{\left(1 - e^{-2\gamma_i D}\right)^3} \right\} \tag{5.33}$$

$$\frac{\partial^2 \mathbf{E}_2}{\partial \lambda_m \partial \lambda_n} = diag \left\{ \frac{2 \begin{bmatrix} k_i \left(e^{-2\gamma_i D} - e^{2\gamma_i D}\right) \\ -l_i m_i \left(e^{-\gamma_i D} - e^{\gamma_i D}\right)^2 \\ +2\,(l_i m_i) \left(e^{-\gamma_i D} + e^{\gamma_i D}\right)^2 \end{bmatrix}}{\left(e^{-\gamma_i D} - e^{\gamma_i D}\right)^3} \right\} \tag{5.34}$$

where

$$k_i = \frac{\partial^2 \gamma_i}{\partial \lambda_m \partial \lambda_n} D + \frac{\partial \gamma_i}{\partial \lambda_m} \frac{\partial D}{\partial \lambda_n} + \frac{\partial \gamma_i}{\partial \lambda_n} \frac{\partial D}{\partial \lambda_m} \tag{5.35}$$

$$l_i = \frac{\partial \gamma_i}{\partial \lambda_n} D + \gamma_i \frac{\partial D}{\partial \lambda_n} \tag{5.36}$$

$$m_i = \frac{\partial \gamma_i}{\partial \lambda_m} D + \gamma_i \frac{\partial D}{\partial \lambda_m} \tag{5.37}$$

$$\frac{\partial^2 D}{\partial \lambda_m \partial \lambda_n} = 0 \tag{5.38}$$

Using (4.32), (5.31) and (5.33)-(5.38), $\frac{\partial^2 \mathbf{E}_1}{\partial \lambda_m \partial \lambda_n}$ and $\frac{\partial^2 \mathbf{E}_2}{\partial \lambda_m \partial \lambda_n}$ can be evaluated, respectively.

The next section summarizes the steps required for sensitivity analysis w.r.t. all the parameters of a circuit.

5.2.5 Summary of the computational steps

A summary of the computational steps for the proposed second-order adjoint sensitivity analysis of networks including MTLs in frequency domain is given below:

- *Step 1*: Simulate the original system (5.1) to get $\mathbf{X}(s)$.
- *Step 2*: Simulate the a^{th} adjoint system (5.12) to get $\Xi_a(s)$.
- *Step 3*: The first-order sensitivity can be found by using $\mathbf{X}(s)$ from *Step 1* and $\Xi_a(s)$ from *Step 2* in (5.13).
- *Step 4*: Simulate the b^{th} adjoint system (5.20) to get $\Xi_b(s)$.

- *Step 5*: Simulate the c^{th} adjoint system (5.27) to get $\Xi_c(s)$.
- *Step 6*: The second-order sensitivity can be found by using $\mathbf{X}(s)$ from *Step 1*, $\Xi_a(s)$ from *Step 2*, $\Xi_b(s)$ from *Step 4* and $\Xi_c(s)$ from *Step 5* in (5.28).

The next section presents an algorithm to evaluate sensitivity w.r.t. all the parameters of a circuit.

5.2.6 Evaluation of a Hessian matrix

The pseudocode for the first-order sensitivity analysis w.r.t. all the parameters of a circuit is given in Algorithm 5.1.

Algorithm 5.1 Calculate first-order sensitivity of an objective function

```
1: Solve: M(s)X(s) = B(s)
2: Solve: M^t(s)Ξ_a(s) = [∂w(X(s),λ)/∂X]^t
3: for m = 1 to N_λ do
4:    Solve: J_{λ_m}(s) = ∂B(s)/∂λ_m − ∂M(s)/∂λ_m
5:    Solve: ∂W(s)/∂λ_m = Ξ_a^t(s) J_{λ_m}(s) + ∂w(X(s),λ)/∂λ_m
6: end for
7: return First-order vector consisting of first-order sensitivities
```

The pseudocode for the second-order sensitivity analysis w.r.t. all the parameters of a circuit is given in Algorithm 5.2.

5.2.7 Comparison with the perturbation technique

In this section, a brief discussion of the computational cost of the proposed technique versus the perturbation-based [11] sensitivity techniques is given.

Algorithm 5.2 Calculate second-order sensitivity of an objective function

Solve: $\mathbf{M}(s)\mathbf{X}(s) = \mathbf{B}(s)$
Solve: $\mathbf{M}^t(s)\Xi_a(s) = \left[\frac{\partial w(\mathbf{X}(s),\boldsymbol{\lambda})}{\partial \mathbf{X}}\right]^t$
for $m = 1$ **to** N_λ **do**
    Solve: $\mathbf{J}_{\lambda_m}(s) = \frac{\partial \mathbf{B}(s)}{\partial \lambda_m} - \frac{\partial \mathbf{M}(s)}{\partial \lambda_m}$
    Solve: $\frac{\partial W(s)}{\partial \lambda_m} = \Xi_a^t(s)\mathbf{J}_{\lambda_m}(s) + \frac{\partial w(\mathbf{X}(s),\boldsymbol{\lambda})}{\partial \lambda_m}$
    Solve: $\mathbf{M}(s)\Xi_b(s) = \mathbf{J}_{\lambda_m}(s)$
    Solve: $\mathbf{M}^t(s)\Xi_c(s) = \frac{\partial^2 w}{\partial \mathbf{X}^2}\Xi_b - \frac{\partial \mathbf{M}^t(s)}{\partial \lambda_m}\Xi_a + \left[\frac{\partial^2 w}{\partial \lambda_m \partial \mathbf{X}}\right]^t$
    for $n = 1$ to N_λ **do**
        Solve: $\frac{\partial^2 W(s)}{\partial \lambda_m \partial \lambda_n}$ using (5.28)
    end for
end for
return Hessian matrix consisting of second-order sensitivities

Using the perturbation techniques (2.7), the MNA equations described by (5.1) are solved at each frequency point twice corresponding to the two perturbed values of the parameter under consideration. Hence, for N_λ parameters, the main computational cost for the perturbation technique for the first-order sensitivities is

$$C_P^{FO} = (2N_\lambda)N_p\left[C_{LU} + C_{F/B}\right] \tag{5.39}$$

where N_p is the number of frequency points, C_{LU} and $C_{F/B}$ are the computational times for LU decomposition and forward-backward substitutions, respectively.

Using the perturbation techniques in (2.8), the MNA equations described by (5.1) are solved at each frequency point four times corresponding to the four perturbed systems. Using the perturbation techniques in (2.9), the MNA equations described by (5.1) are solved at each frequency point three times corresponding to the two perturbed systems and the original system. Hence, for N_λ parameters, the main computational cost of evaluating the Hessian matrix using the perturbation technique is

$$C_P^{SO} = C_P^{SO,Off-Diagonal} + C_P^{SO,Diagonal} \tag{5.40}$$

where the costs for evaluating the off-diagonal and diagonal elements in Hessian matrix are

$$C_P^{SO,Off-Diagonal} = 4(N_\lambda^2 - N_\lambda) \sum_{i=1}^{N_p} \left[C_{LU} + C_{F/B} \right] \tag{5.41}$$

and

$$C_P^{SO,Diagonal} = (2N_\lambda + 1) \sum_{i=1}^{N_p} \left[C_{LU} + C_{F/B} \right], \tag{5.42}$$

respectively.

Similarly, the evaluation of first-order sensitivities using the adjoint approach requires the solution of (5.1) and (5.12). Hence, for N_λ parameters, the main computational cost using the adjoint approach is

$$C_A^{FO} = 2N_p \left[C_{LU} + C_{F/B}\right] + N_\lambda C_I^{FO} \tag{5.43}$$

where C_I^{FO} is the computational cost associated with evaluating the product in (5.13).

The evaluation of second-order sensitivities using the adjoint approach requires the solution of (5.1), (5.12) and N_λ sets of (5.20) and (5.27). Hence, for N_λ parameters, the main computational cost for evaluating Hessian matrix using the adjoint approach is [32]

$$C_A^{SO} = (2 + 2N_\lambda)N_p \left[C_{LU} + C_{F/B}\right] + N_\lambda^2 C_I^{SO} \tag{5.44}$$

where C_I^{SO} is the computational cost associated with evaluating the product in (5.28).

5.3 Numerical examples

In this section, three examples are shown to demonstrate the validity and accuracy of the proposed method. Example-1 provides the illustrative steps of the method described in Section 5.2.5. Example-2 presents the comparison of the proposed method with exact sensitivities. Example-3 presents the comparison of the proposed method with the perturbation method.

5.3.1 Example 1

This example provides the illustrative steps of the new algorithm developed in this chapter and also it analytically proves the validity of the proposed method. Consider the circuit shown in Fig. 5.1 with the admittance of the single transmission line as

$$\mathbf{Y}(s) = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \tag{5.45}$$

Further, consider the voltage at node 2 of the circuit as an objective function given by

$$W(s) = V_2(s) = \boldsymbol{\eta}^t \mathbf{X}(s) \tag{5.46}$$

where

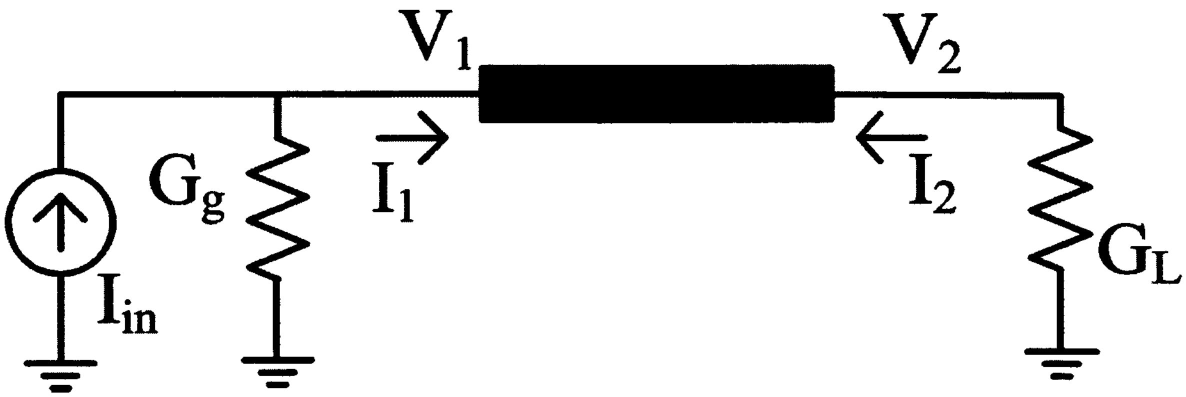

Figure 5.1: Circuit containing a single transmission line (Example 1)

$$\mathbf{X}(s) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{5.47}$$

and

$$\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.48}$$

Step 1

The original system is defined as

$$\begin{bmatrix} G_g + Y_{11} & Y_{12} \\ Y_{21} & G_L + Y_{22} \end{bmatrix} \mathbf{X}(s) = \begin{bmatrix} I_{in} \\ 0 \end{bmatrix} \tag{5.49}$$

The solution is given as

$$V_1 = \frac{-\left[Y_{22} + G_L\right] I_{in}}{-\left[Y_{11} + G_g\right]\left[Y_{22} + G_L\right] + Y_{21}Y_{12}} \tag{5.50}$$

$$V_2 = \frac{I_{in}Y_{21}}{-\left[Y_{11} + G_g\right]\left[Y_{22} + G_L\right] + Y_{21}Y_{12}} \tag{5.51}$$

Step 2

Using (5.12), the sources for the a^{th} adjoint system is given by

$$\frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \mathbf{X}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{5.52}$$

$$\left[\frac{\partial w(\mathbf{X}(s), \boldsymbol{\lambda})}{\partial \mathbf{X}}\right]^t = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.53}$$

The a^{th} adjoint system is defined as

$$\begin{bmatrix} G_g + Y_{11} & Y_{21} \\ Y_{12} & G_L + Y_{22} \end{bmatrix} \Xi_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.54}$$

where

$$\Xi_a = \begin{bmatrix} \psi_1^a \\ \\ \psi_2^a \end{bmatrix} \tag{5.55}$$

The a^{th} adjoint circuit with a unit current source is shown in Fig. 5.2. The solution is given as

$$\psi_1^a = \frac{-Y_{21}}{-Y_{21}Y_{12} + (Y_{11} + G_g)(Y_{22} + G_L)} \tag{5.56}$$

$$\psi_2^a = \frac{(Y_{11} + G_g)}{-Y_{21}Y_{12} + (Y_{11} + G_g)(Y_{22} + G_L)} \tag{5.57}$$

First-order sensitivity

The first-order sensitivity of V_2 can be found in two ways: Direct and Adjoint. Directly differentiating (5.51) w.r.t. Y_{21} results in

$$\frac{\partial V_2}{\partial Y_{21}} = \frac{-I_{in}(Y_{11} + G_g)(Y_{22} + G_L)}{[-(Y_{11} + G_g)(Y_{22} + G_L) + Y_{21}Y_{12}]^2} \tag{5.58}$$

Using (5.13) of the adjoint approach, the first-order sensitivity becomes

$$\frac{\partial W}{\partial Y_{21}} = \Xi_a^t \mathbf{J}_{Y_{21}} \tag{5.59}$$

where

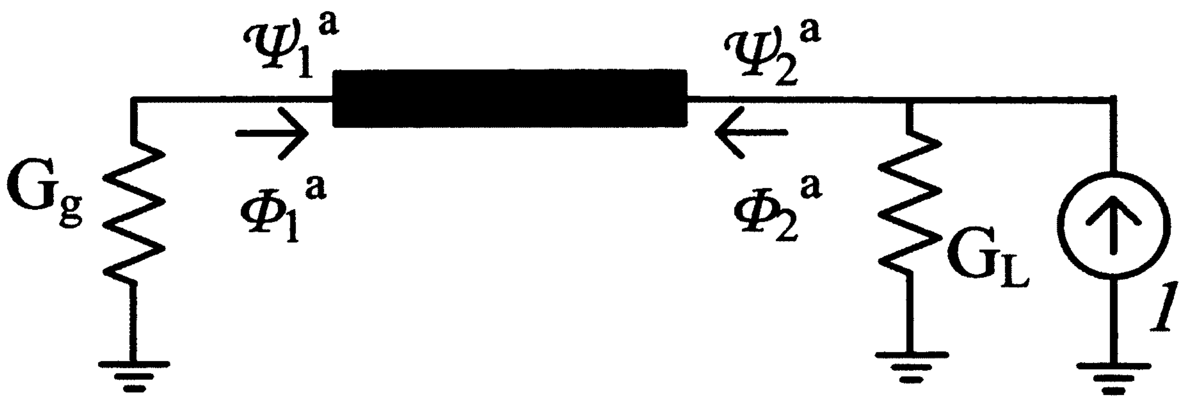

Figure 5.2: Adjoint Circuit corresponding to Fig. 5.1

$$\mathbf{J}_{Y_{21}} = \begin{bmatrix} 0 \\ V_1 \end{bmatrix} \tag{5.60}$$

Using (5.56) and (5.51) in (5.59) results in

$$\frac{\partial W(s)}{\partial Y_{21}} = \psi_2^a V_1 = \frac{-I_{in}\left(Y_{11}+G_g\right)\left(Y_{22}+G_L\right)}{\left[-\left(Y_{11}+G_g\right)\left(Y_{22}+G_L\right)+Y_{21}Y_{12}\right]^2} \tag{5.61}$$

which is the same as (5.58) of the direct approach. This validates the proposed first-order sensitivity approach. For second-order sensitivity further steps are required, which are given next.

Step 3

The b^{th} adjoint system is defined as

$$\begin{bmatrix} G_g + Y_{11} & Y_{12} \\ Y_{21} & G_L + Y_{22} \end{bmatrix} \Xi_b = \mathbf{J}_{Y_{21}} = \begin{bmatrix} 0 \\ V_1 \end{bmatrix} \tag{5.62}$$

where

$$\Xi_b = \begin{bmatrix} \psi_1^b \\ \psi_2^b \end{bmatrix} \tag{5.63}$$

The solution is given as

$$\psi_1^b = \frac{V_1 Y_{12}}{-Y_{21}Y_{12} + (Y_{11} + G_g)(Y_{22} + G_L)} \tag{5.64}$$

$$\psi_2^b = \frac{V_1 (Y_{11} + G_g)}{-Y_{21}Y_{12} + (Y_{11} + G_g)(Y_{22} + G_L)} \tag{5.65}$$

Step 4

The c^{th} adjoint system is defined as

$$\begin{bmatrix} G_g + Y_{11} & Y_{21} \\ Y_{12} & G_L + Y_{22} \end{bmatrix} \Xi_c = \begin{bmatrix} -\psi_2^a \\ 0 \end{bmatrix} \tag{5.66}$$

where

$$\Xi_c = \begin{bmatrix} \psi_1^c \\ \\ \psi_2^c \end{bmatrix} \tag{5.67}$$

The solution is given as

$$\psi_1^c = \frac{-\psi_2^a\left(Y_{22}+G_L\right)}{-Y_{21}Y_{12}+\left(Y_{11}+G_g\right)\left(Y_{22}+G_L\right)} \tag{5.68}$$

$$\psi_2^c = \frac{\psi_2^a Y_{12}}{-Y_{21}Y_{12}+\left(Y_{11}+G_g\right)\left(Y_{22}+G_L\right)} \tag{5.69}$$

Step 5

Let the second parameter be defined as

$$\lambda_2 = G_g \tag{5.70}$$

The second-order sensitivity of V_2 can be found in two ways: Direct and Adjoint. Directly differentiating (5.58) w.r.t. G_g results in

$$\frac{\partial^2 V_2}{\partial Y_{21}\partial G_g} = \frac{I_{in}\left(Y_{22}+G_L\right)\left[Y_{21}Y_{12}+\left(Y_{11}+G_g\right)\left(Y_{22}+G_L\right)\right]}{\left[-Y_{21}Y_{12}+\left(Y_{11}+G_g\right)\left(Y_{22}+G_L\right)\right]^3} \tag{5.71}$$

Using (5.28) of the adjoint approach, the second-order sensitivity becomes

$$\frac{\partial^2 V_2}{\partial Y_{21} \partial G_g} = -\psi_1^a \psi_1^b - \psi_1^c V_1$$

$$= \frac{I_{in}\left(Y_{22}+G_L\right)\left[Y_{21}Y_{12}+\left(Y_{11}+G_g\right)\left(Y_{22}+G_L\right)\right]}{\left[-Y_{21}Y_{12}+\left(Y_{11}+G_g\right)\left(Y_{22}+G_L\right)\right]^3} \tag{5.72}$$

which is the same as (5.71) of the direct approach. This validates the proposed second-order sensitivity approach.

In this example, it is analytically proved that the adjoint approach in the frequency-domain matches with the direct approach.

5.3.2 Example 2

The circuit considered in this example is a low-pass filter as shown in Fig. 5.3. The output response for the voltage at node V_{out} is shown in Fig. 5.4. The objective function is defined as

$$W(s) = \frac{GV_{out}^2(s)}{A} \tag{5.73}$$

The frequency-domain sensitivity of $W(s)$ w.r.t. different parameters are computed using the proposed approach and are compared with the exact values in Figs. 5.5 and 5.6 and a good agreement is observed among them.

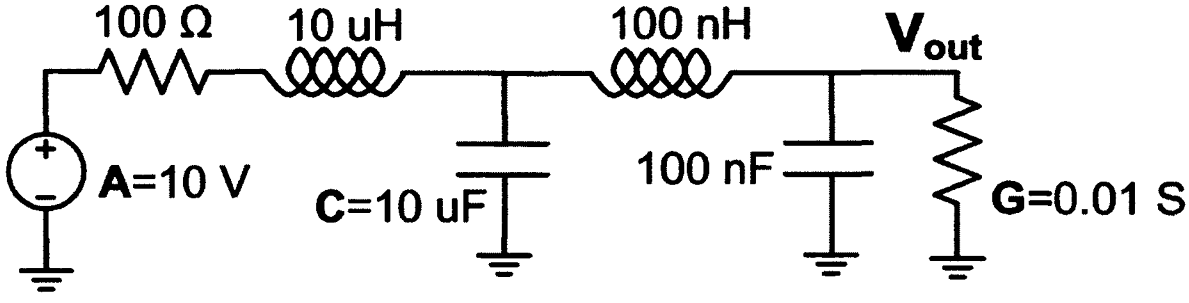

Figure 5.3: Circuit containing a low-pass filter (Example 2)

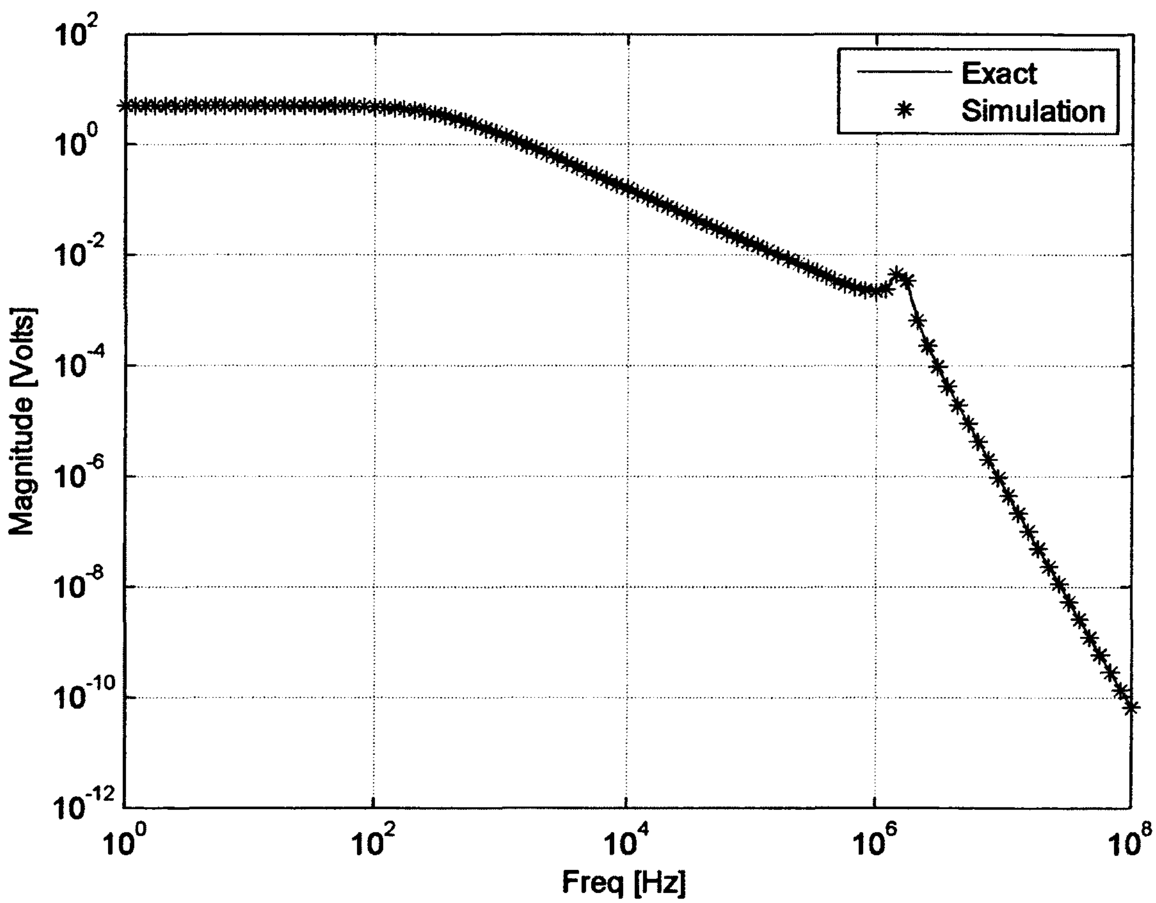

Figure 5.4: Frequency response of the circuit shown in Fig. 5.3 at node V_{out}

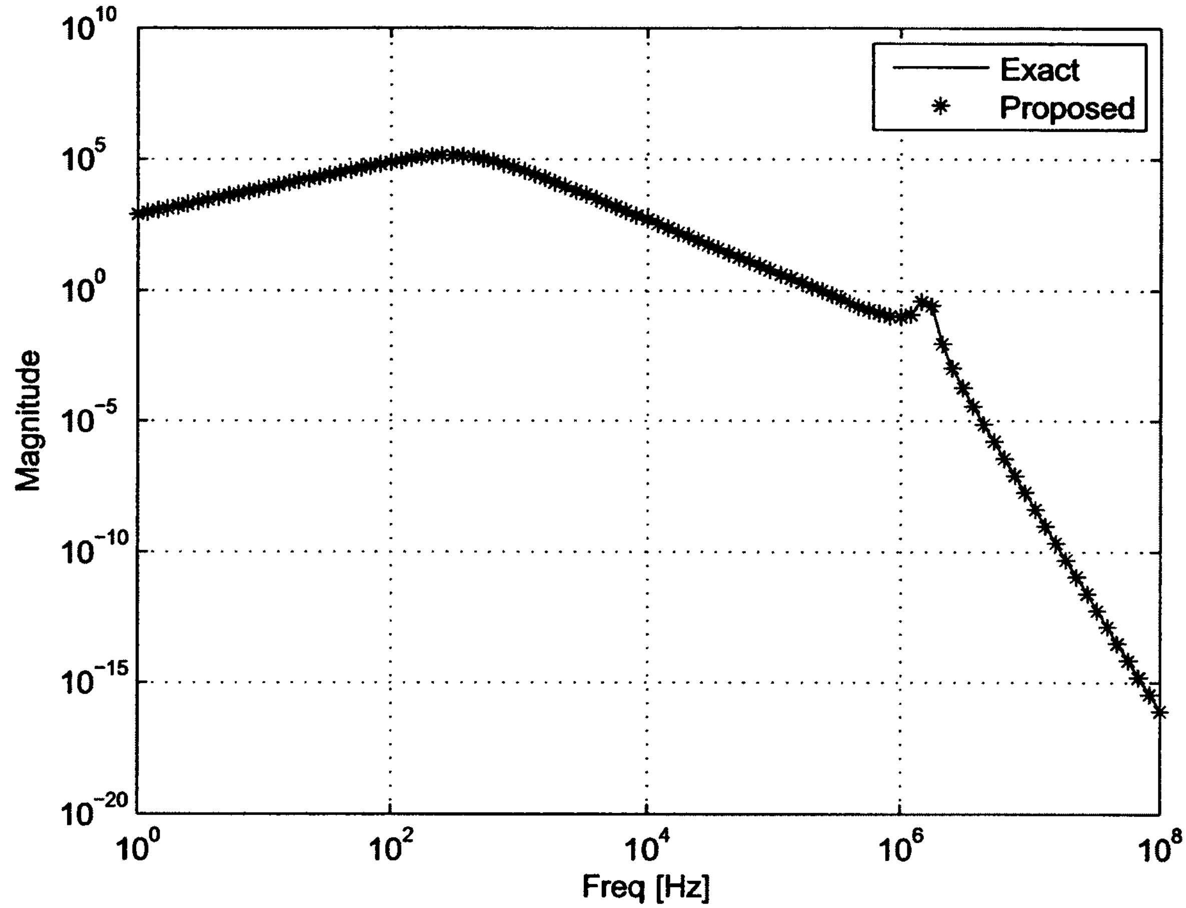

Figure 5.5: Magnitude of second-order sensitivity of $\frac{\partial^2 W(s)}{\partial G \partial C}$ for circuit in Fig. 5.3

5.3.3 Example 3

The circuit considered in this experiment is shown in Fig. 5.7 [11]. It contains three lossy coupled transmission lines numbered by subcircuits 1, 2 and 3. The lengths of the transmission lines in the subcircuits 1, 2 and 3 are 0.2m, 0.5m and 0.3m, respectively. The electrical parameters of the TL #1 are $L_1 = 600nH/m$,

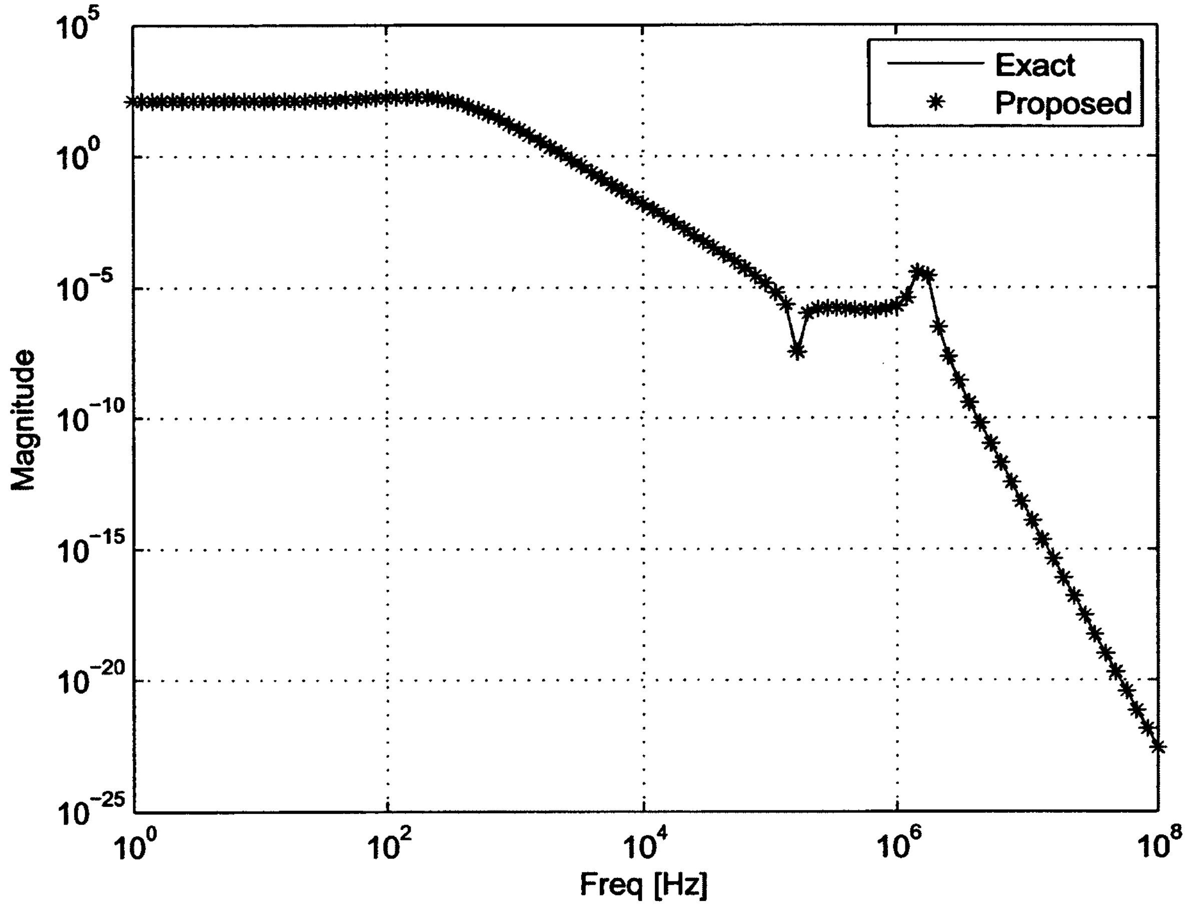

Figure 5.6: Magnitude of second-order sensitivity of $\frac{\partial^2 W(s)}{\partial G^2}$ for circuit in Fig. 5.3

$C_1 = 1nF/m$, $R_1 = 1\Omega/m$ and $G_1 = 5mS/m$. The parameters of TL #2 are:

$$\mathbf{L}_2 = \begin{bmatrix} 600 & 50 \\ 50 & 600 \end{bmatrix} nH/m \qquad \mathbf{C}_2 = \begin{bmatrix} 1.2 & -0.11 \\ -0.11 & 1.2 \end{bmatrix} nF/m$$

$$\mathbf{R}_2 = \begin{bmatrix} 2.25 & 0.225 \\ 0.225 & 2.25 \end{bmatrix} \Omega/m \qquad \mathbf{G}_2 = \begin{bmatrix} 7.5 & 0 \\ 0 & 7.5 \end{bmatrix} mS/m$$

The parameters of TL #3 are:

$$\mathbf{L}_3 = \begin{bmatrix} 1 & 0.11 & 0.03 & 0 \\ 0.11 & 1 & 0.11 & 0.03 \\ 0.03 & 0.11 & 1 & 0.11 \\ 0 & 0.03 & 0.11 & 1 \end{bmatrix} \mu H/m$$

$$\mathbf{C}_3 = \begin{bmatrix} 1.5 & -0.17 & -0.03 & 0 \\ -0.07 & 1.5 & -0.07 & -0.03 \\ -0.03 & -0.07 & 1.5 & -0.07 \\ 0 & -0.03 & -0.07 & 1.5 \end{bmatrix} nF/m$$

$$\mathbf{R}_3 = \begin{bmatrix} 3.5 & 0.35 & 0.035 & 0 \\ 0.35 & 3.5 & 0.35 & 0.035 \\ 0.035 & 0.35 & 3.5 & 0.35 \\ 0 & 0.035 & 0.35 & 3.5 \end{bmatrix} \Omega/m$$

$$\mathbf{G}_3 = \begin{bmatrix} 10 & 1 & 0.1 & 0 \\ 1 & 10 & 1 & 0.1 \\ 0.1 & 1 & 10 & 1 \\ 0 & 0.1 & 1 & 10 \end{bmatrix} mS/m$$

The applied voltage has a magnitude of 1V. The output response and the sensitivities are computed for the voltage at node V_{out} shown in Fig. 5.7. The voltage response at node V_{out} is shown in Fig. 5.8. The second-order sensitivities w.r.t. the lumped components (R_1, C_1) and the distributed parameters ($R_1^{(1,1)}$ of TL #1, $G_1^{(1,1)}$ of TL #1 and $R_3^{(1,1)}$ of TL #3) were computed using the proposed method. A comparison of the sensitivity obtained using the proposed approach and the perturbation approach (1% of each in (2.8)) are shown in Figs. 5.9, 5.10, 5.11, 5.12, 5.13, 5.14, 5.15 and 5.16. The results from both of the approaches match accurately.

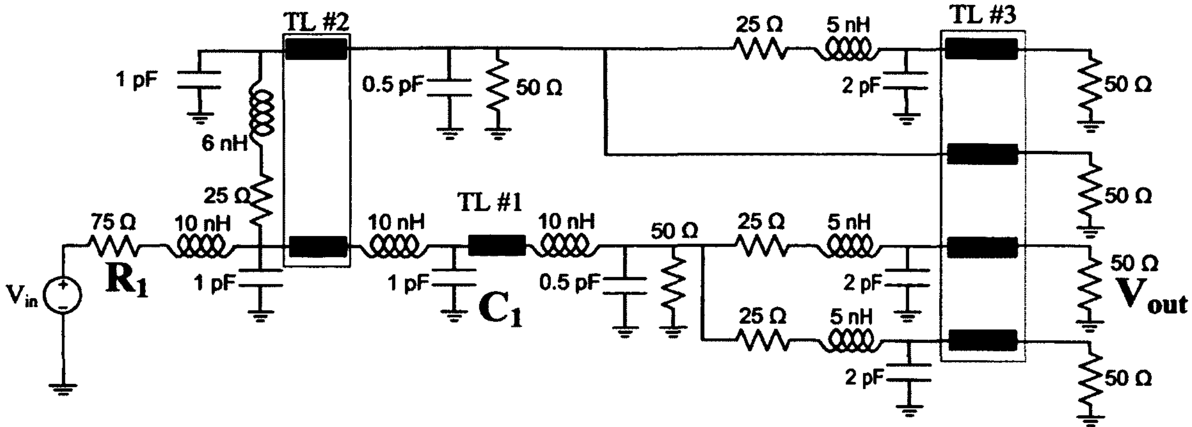

Figure 5.7: Circuit containing three lossy coupled transmission lines (Example 3)

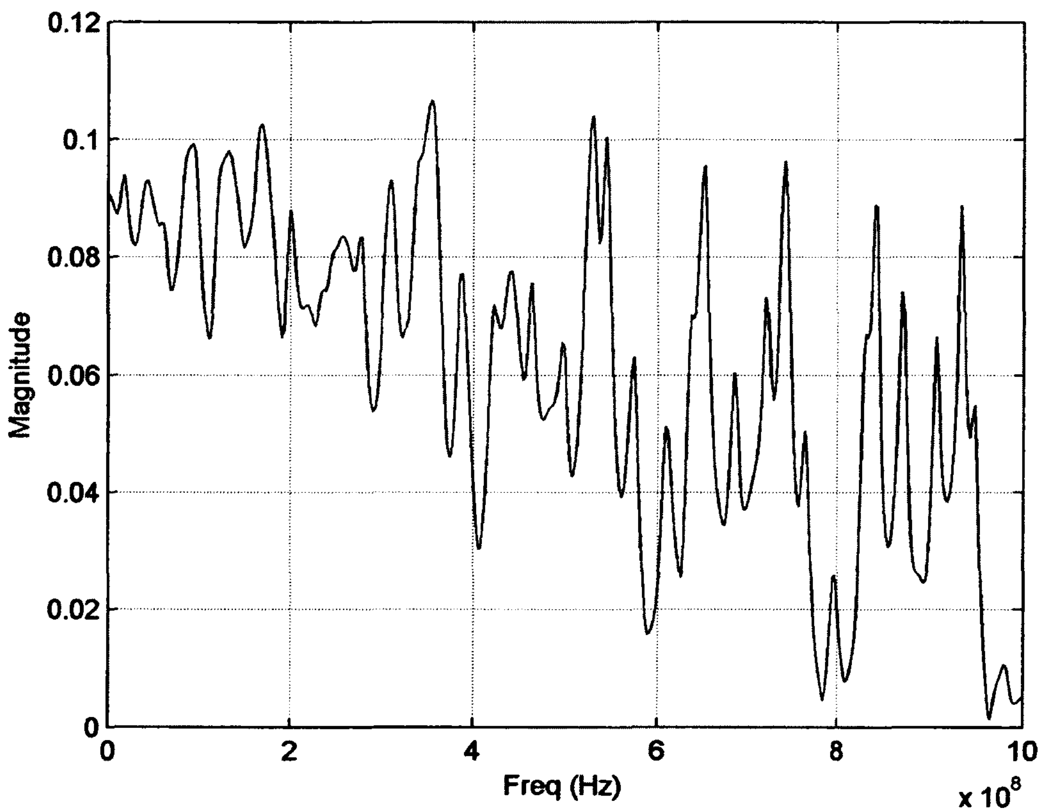

Figure 5.8: Frequency response of the circuit shown in Fig. 5.7 at node V_{out}

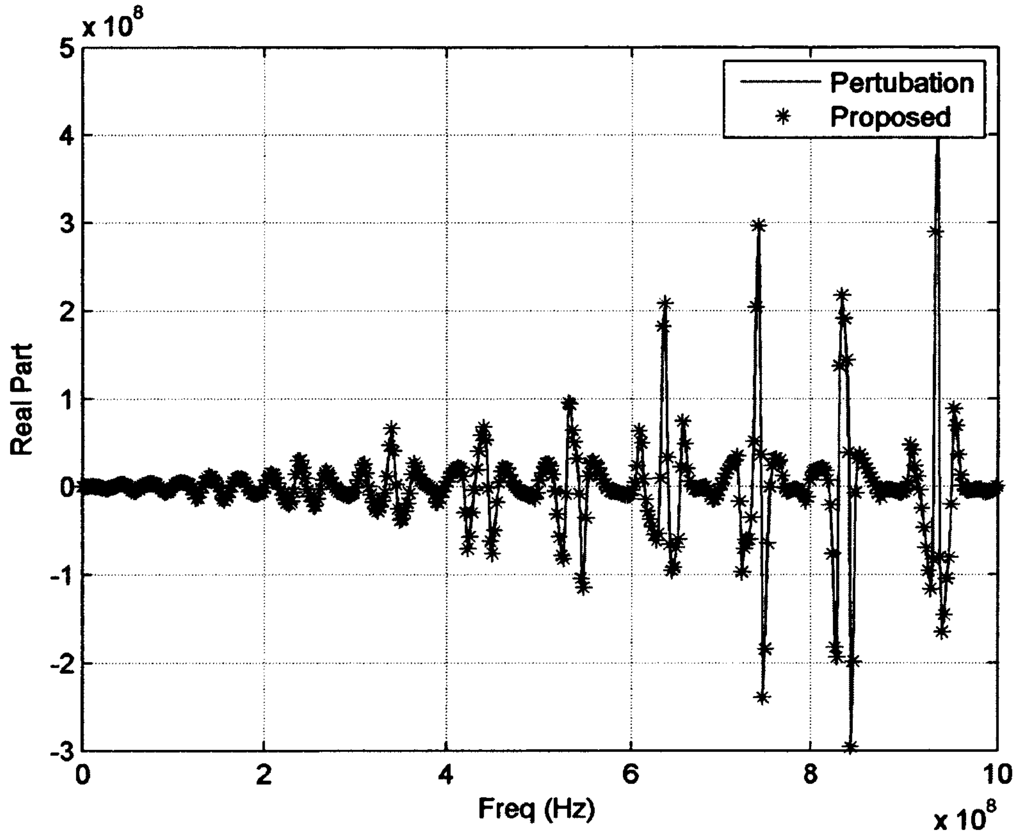

Figure 5.9: Real part of second-order sensitivity of the output voltage V_{out} w.r.t. R_1 and C_1 of Fig. 5.7

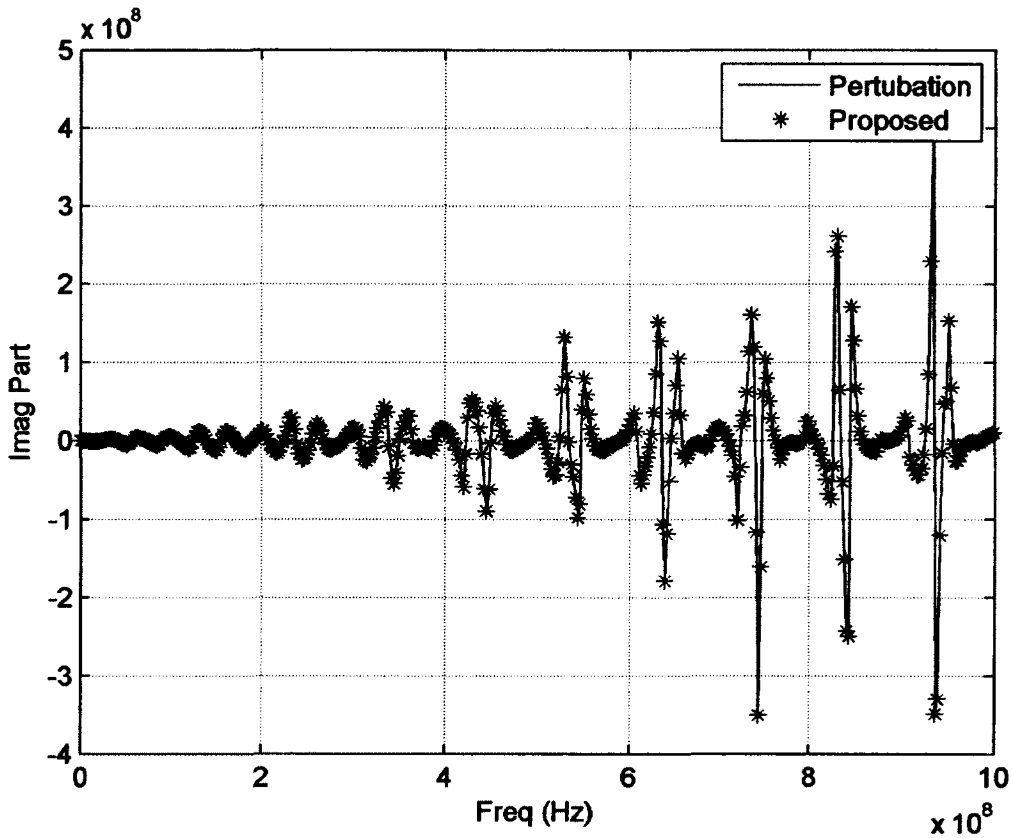

Figure 5.10: Imaginary part of second-order sensitivity of the output voltage V_{out} w.r.t. R_1 and C_1 of Fig. 5.7

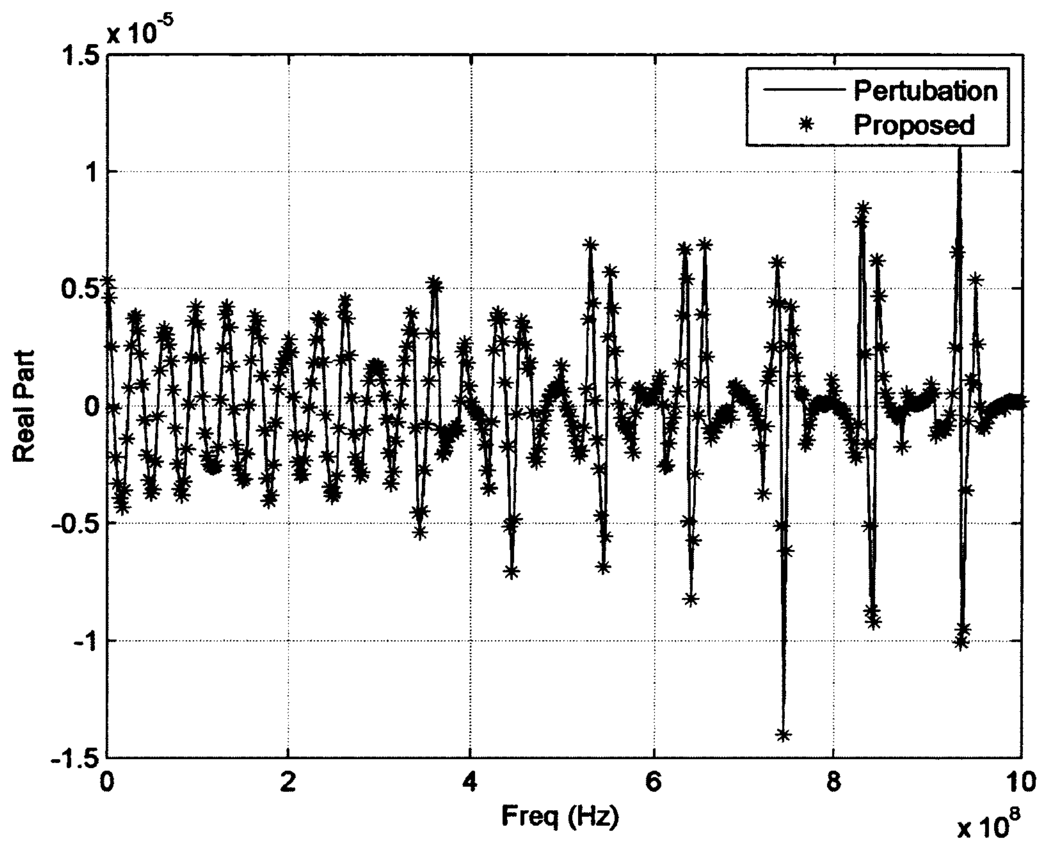

Figure 5.11: Real part of second-order sensitivity of the output voltage V_{out} w.r.t. $R_1^{(1,1)}$ of TL #1 and R_1 of Fig. 5.7

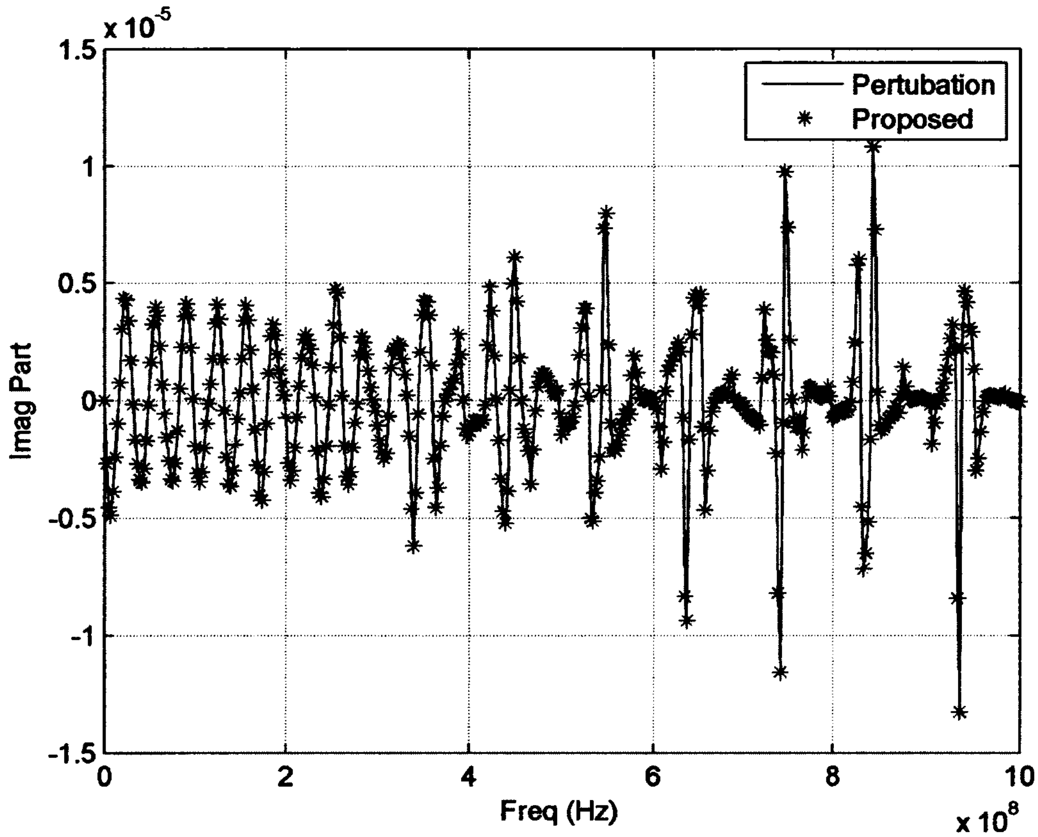

Figure 5.12: Imaginary part of second-order sensitivity of the output voltage V_{out} w.r.t. $R_1^{(1,1)}$ of TL #1 and R_1 of Fig. 5.7

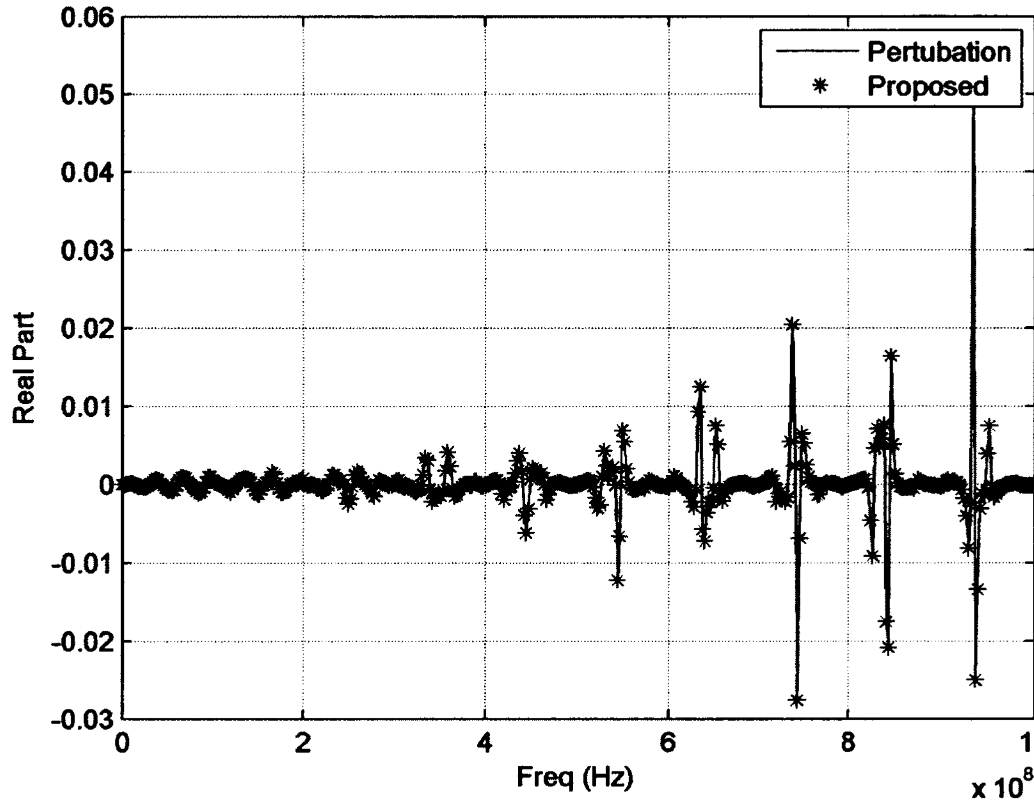

Figure 5.13: Real part of second-order sensitivity of the output voltage V_{out} w.r.t. $R_1^{(1,1)}$ and $G_1^{(1,1)}$ of TL #1 of Fig. 5.7

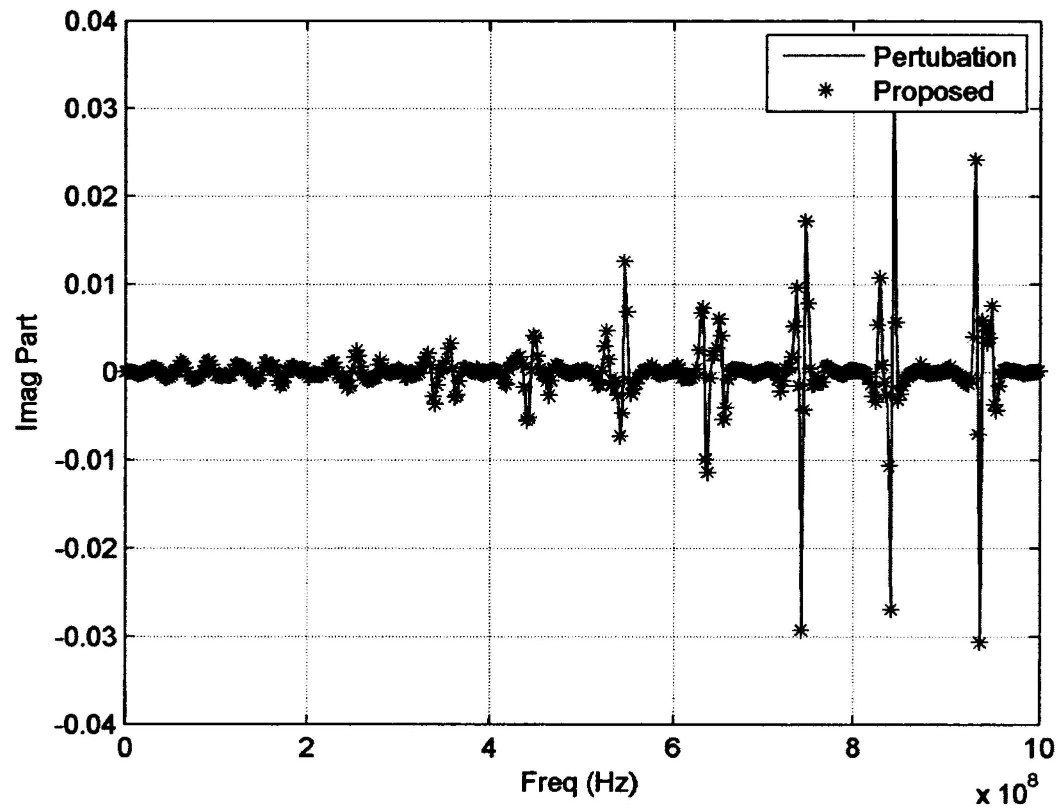

Figure 5.14: Imaginary part of second-order sensitivity of the output voltage V_{out} w.r.t. $R_1^{(1,1)}$ and $G_1^{(1,1)}$ of TL #1 of Fig. 5.7

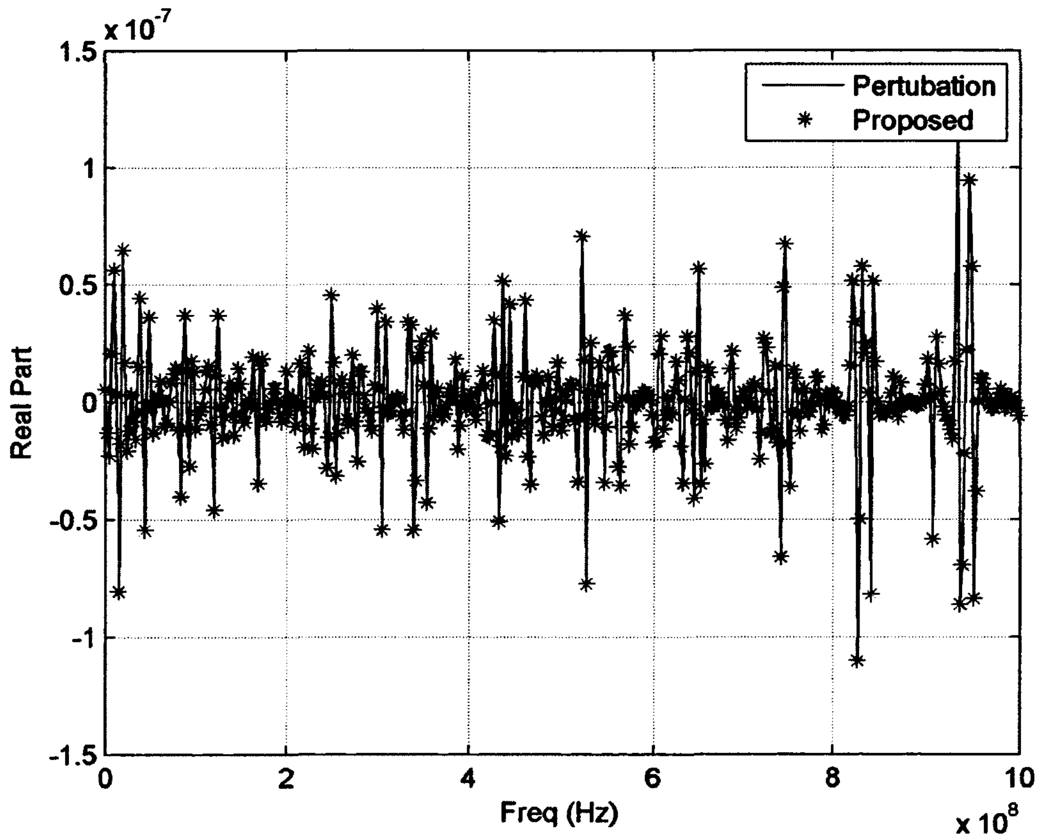

Figure 5.15: Real part of second-order sensitivity of the output voltage V_{out} w.r.t. $R_1^{(1,1)}$ of TL #1 and $R_3^{(1,1)}$ of TL #3 of Fig. 5.7

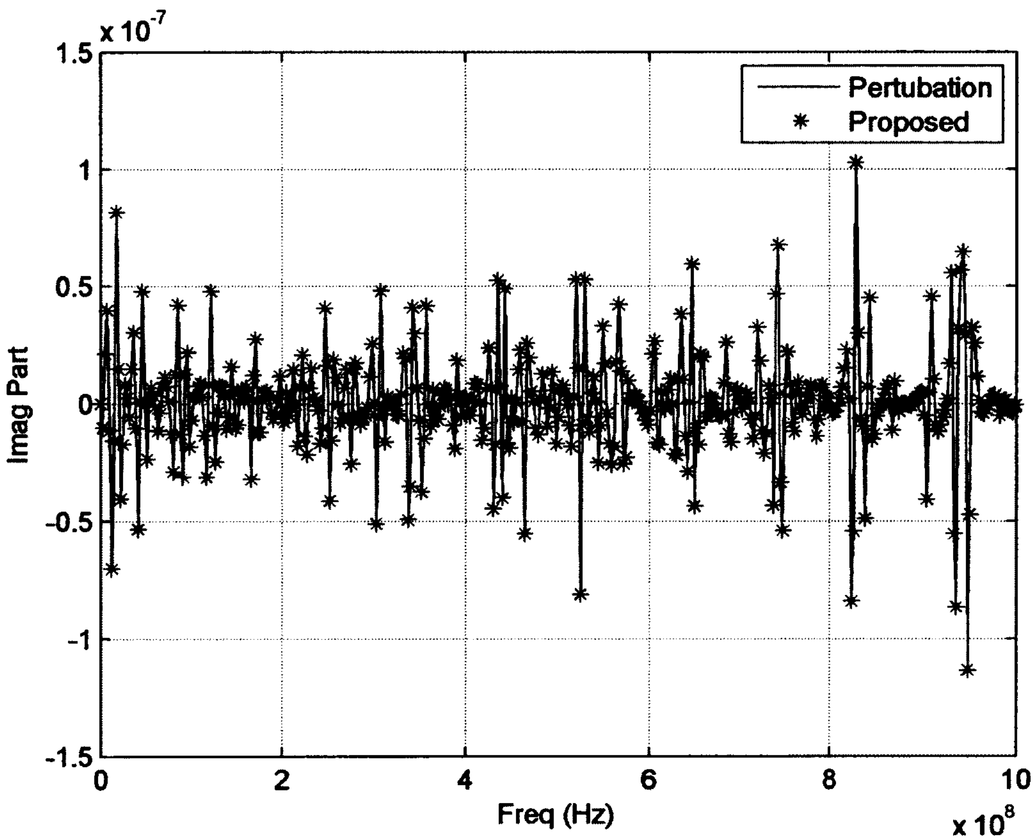

Figure 5.16: Imaginary part of second-order sensitivity of the output voltage V_{out} w.r.t. $R_1^{(1,1)}$ of TL #1 and $R_3^{(1,1)}$ of TL #3 of Fig. 5.7

Chapter 6

Proposed variational approach based second-order adjoint sensitivity analysis for networks with MTLs in time domain

In chapter 4, an efficient algorithm for the first-order time-domain adjoint sensitivity analysis of multiconductor transmission lines embedded in a nonlinear circuit was given. In gradient based optimization, it is important not only to evaluate the first-order sensitivities but also the second-order sensitivities to speed-up the nonlinear optimization iterations [17]. For this purpose, the second-order time-domain adjoint sensitivity analysis based on Tellegen's theorem [15] [16] for a linear circuit (excluding multiconductor transmission lines) was developed in [17]. In this chapter, a new method based on variational approach [13] for second-order time-domain adjoint sensitivity analysis of multiconductor transmission lines embedded in a nonlinear circuit is presented.

The rest of this chapter is organized as follows. In Section 6.1, a formulation of circuit equations is provided. Section 6.2 presents the proposed second-order adjoint sensitivity analysis technique based on the variational approach. Section 6.3 compares the proposed technique with the frequency-domain technique given in chapter 5 . Section 6.4 provides numerical examples to demonstrate the validity of the proposed approach.

6.1 Formulation of circuit equations

Consider a general circuit consisting of linear and nonlinear components and distributed transmission lines. The corresponding modified nodal analysis (MNA) equations [3] can be written in the time-domain as

$$\mathbf{C}\frac{d\mathbf{x}(t)}{dt} + \mathbf{G}\mathbf{x}(t) + \sum_{k=1}^{N_t} \mathbf{a}_k(t) * \mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t)) = \mathbf{b}(t) \tag{6.1}$$

where

$$\mathbf{a}_k(t) = \mathbf{D}_k \mathbf{y}_k(t) \mathbf{D}_k^t \tag{6.2}$$

and

- $\mathbf{x}(t) \in \Re^{n\times 1}$ is the vector of unknowns corresponding to node voltages, independent and dependent voltage source currents, and inductor currents; $\mathbf{b}(t) \in \Re^{n\times 1}$ is an input vector with entries determined by the independent current and voltage sources; $\mathbf{f}(\mathbf{x}(t)) \in \Re^{n\times 1}$ is a vector related to nonlinear elements and n is the total number of MNA variables;
- $\mathbf{G}$, $\mathbf{C} \in \Re^{n\times n}$ are constant matrices describing the lumped memory-less and memory elements, respectively;
- $\mathbf{D}_k = [d_{i,j}], d_{i,j} \in \{0, 1\}$ is a selector matrix that maps the vector of terminal currents $\mathbf{i}_k(t)$ entering the transmission line k into the nodal space of the circuit, where $i \in \{1, \cdots, n\}, j \in \{1, \cdots, 2N_k\}$, N_k is number of coupled lines in the k^{th} transmission line and N_t is the total number of distributed transmission lines in the circuit;
- $\mathbf{y}_k(t) \in \Re^{2N_k\times 2N_k}$ is the time-domain admittance matrix of the k^{th} transmission line, where $\mathbf{i}_k(t)$ and $\mathbf{v}_k(t) \in \Re^{2N_k\times 1}$ are the vectors corresponding to terminal currents and voltages, respectively; , of k^{th} transmission line. It is to be noted that $\mathbf{I}_k(s)$ is the Laplace transform of $\mathbf{i}_k(t)$. $\mathbf{v}_k(t) = \mathbf{D}_k^t\mathbf{x}(t)$ and

'*' denotes the convolution operator.

The main focus of this chapter is to develop a variational approach based method for adjoint analysis of networks including MTLs with second-order sensitivities. The related details are given in the subsequent sections.

6.2 Development of the proposed second-order adjoint sensitivity concept for distributed transmission lines

In this section, details of the proposed second-order adjoint sensitivity analysis for circuits including distributed transmission line networks are given.

6.2.1 Problem formulation

Consider (6.1) and an objective function $W(t)$ whose value is to be optimized at time $t = T$ w.r.t. circuit parameters $\boldsymbol{\lambda}$ as given by

$$W(T) = \int_0^T w(\mathbf{x}(t), \boldsymbol{\lambda})dt \tag{6.3}$$

where $w(\mathbf{x}(t), \boldsymbol{\lambda})$ is a scalar function [13] and $\boldsymbol{\lambda} \in \Re^{1\times N_\lambda}$ is a vector of N_λ parameters of a circuit given by

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{N_\lambda} \end{bmatrix} \tag{6.4}$$

To perform the required optimization, we need the sensitivity of the objective function $\frac{\partial W(T)}{\partial \lambda_m}$ where $m \in \{1, \cdots, N_\lambda\}$. The next section provides a review of the first-order adjoint sensitivity approach, which is needed to evaluate $\frac{\partial W(T)}{\partial \lambda_m}$.

6.2.2 Review of first-order adjoint sensitivity approach for distributed transmission line interconnects

It was shown in chapter 4 that the adjoint system corresponding the (6.1) is given as

$$\begin{aligned} &\mathbf{C}^t \frac{d\boldsymbol{\xi}_a(\tau)}{d\tau} + \mathbf{G}^t \boldsymbol{\xi}_a(\tau) + \left[\frac{\partial \mathbf{f}(T-\tau)}{\partial \mathbf{x}}\right]^t \boldsymbol{\xi}_a(\tau) \\ &+ \sum_{k=1}^{N_t} [\mathbf{D}_k \boldsymbol{\phi}_k^a(\tau)] = \left[\frac{\partial w(T-\tau)}{\partial \mathbf{x}}\right]^t \qquad \tau \in [0, T] \end{aligned} \tag{6.5}$$

where

$$\boldsymbol{\phi}_k^a(\tau) = \mathbf{y}_k^t(\tau) * \boldsymbol{\psi}_k^a(\tau) \tag{6.6}$$

$$\boldsymbol{\psi}_k^a(\tau) = \mathbf{D}_k^t \boldsymbol{\xi}_a(\tau) \tag{6.7}$$

with $\boldsymbol{\xi}_a(\tau)$ is referred to as the a^{th} adjoint MNA variable vector; $\boldsymbol{\psi}_k^a(\tau)$ and $\boldsymbol{\phi}_k^a(\tau)$ are the terminal voltage and current vectors of the k^{th} transmission line in the a^{th} adjoint system, respectively. In the Laplace-domain, (6.6) becomes

$$\boldsymbol{\Phi}_k^a(s) = \mathbf{Y}_k^t(s)\boldsymbol{\Psi}_k^a(s) \tag{6.8}$$

Next, the sensitivity function $\frac{\partial W(T)}{\partial \lambda_m}$ can be obtained in terms of the solution of the original system (6.1) and the corresponding adjoint equations (6.5) to (6.8) as

$$\frac{\partial W(T)}{\partial \lambda_m} = \boldsymbol{\xi}_a^t(T)\mathbf{C}\mathbf{z}_1(0) + \underbrace{\int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{j}_{\lambda_m}(t)dt}_{} + \int_0^T \frac{\partial w(t)}{\partial \lambda_m}dt \tag{6.9}$$

where

$$\begin{aligned}\mathbf{j}_{\lambda_m}(t) =& \frac{\partial \mathbf{b}(t)}{\partial \lambda_m} - \frac{\partial \mathbf{C}}{\partial \lambda_m}\frac{d\mathbf{x}(t)}{dt} - \frac{\partial \mathbf{G}}{\partial \lambda_m}\mathbf{x}(t) \\ & - \sum_{k=1}^{N_t}\left[\frac{\partial \mathbf{a}_k(t)}{\partial \lambda_m} * \mathbf{x}(t)\right] - \frac{\partial \mathbf{f}(t)}{\partial \lambda_m}\end{aligned} \tag{6.10}$$

In the following section, details of the proposed second-order adjoint sensitivity analysis approach are given.

6.2.3 Proposed second-order adjoint sensitivity approach for distributed transmission line interconnects

In this section, development of the proposed second-order adjoint sensitivity of non-linear circuits containing MTL using the variational approach [13] is presented.

Differentiating (6.9) w.r.t. a particular circuit parameter λ_n gives

$$\begin{aligned}\frac{\partial^2 W(T)}{\partial\lambda_m\partial\lambda_n} = \frac{\partial\boldsymbol{\xi}_a^t(T)}{\partial\lambda_n}\mathbf{C}\mathbf{z}_1(0) + \boldsymbol{\xi}_a^t(T)\frac{\partial\mathbf{C}}{\partial\lambda_n}\mathbf{z}_1(0) + \\ \boldsymbol{\xi}_a^t(T)\mathbf{C}\frac{\partial\mathbf{z}_1(0)}{\partial\lambda_n} + \int_0^T \frac{\partial\boldsymbol{\xi}_a^t(\tau)}{\partial\lambda_n}\mathbf{j}_{\lambda_m}(t)dt + \int_0^T \boldsymbol{\xi}_a^t(\tau)\frac{\partial\mathbf{j}_{\lambda_m}(t)}{\partial\lambda_n}dt \\ + \int_0^T \left[\frac{\partial w(t)}{\partial\lambda_m\partial\mathbf{x}}\mathbf{z}_2(t) + \frac{\partial^2 w(t)}{\partial\lambda_m\partial\lambda_n}\right]dt\end{aligned} \tag{6.11}$$

where

$$\mathbf{z}_2(t) = \frac{\partial\mathbf{x}(t)}{\partial\lambda_n} \tag{6.12}$$

is the sensitivity vector of circuit variables w.r.t. a parameter λ_n, with $n \in \{1, \cdots, N_\lambda\}$.

To simplify the analysis, it is assumed that the initial-condition parameters are not under consideration. This assumption allows us to avoid evaluating sensitivities of the DC solution [21]. Using this assumption $\mathbf{z}_1(0) = 0$, which allows us to avoid evaluating $\frac{\partial\boldsymbol{\xi}_a^t(T)}{\partial\lambda_n}$ explicitly. Rewriting (6.11) with $\mathbf{z}_1(0) = 0$ gives

$$\frac{\partial^2 W(T)}{\partial \lambda_m \partial \lambda_n} = \underbrace{\int_0^T \frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} \mathbf{j}_{\lambda_m}(t) dt}_{\text{Term 1}} + \underbrace{\int_0^T \boldsymbol{\xi}_a^t(\tau) \frac{\partial \mathbf{j}_{\lambda_m}(t)}{\partial \lambda_n} dt}_{\text{Term 2}}$$
$$+ \int_0^T \left[\frac{\partial w(t)}{\partial \lambda_m \partial \mathbf{x}} \mathbf{z}_2(t) + \frac{\partial^2 w(t)}{\partial \lambda_m \partial \lambda_n} \right] dt \tag{6.13}$$

Next, taking the transpose of both sides and substituting $\tau = T - t$ into (6.5) gives

$$\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau} \mathbf{C} + \boldsymbol{\xi}_a^t(\tau) \mathbf{G} + \boldsymbol{\xi}_a^t(\tau) \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}$$
$$+ \sum_{k=1}^{N_t} \left[\boldsymbol{\xi}_a^t(\tau) * \mathbf{a}_k(\tau) \right] = \frac{\partial w(t)}{\partial \mathbf{x}} \tag{6.14}$$

Term 1 of the RHS in (6.13) requires the derivative $\frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n}$, which can be obtained by differentiating (6.14) w.r.t. λ_n as

$$\frac{\partial^2 \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n \partial \tau} \mathbf{C} + \frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} \mathbf{G} + \frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}} + \sum_{k=1}^{N_t} \left[\frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} * \mathbf{a}_k(\tau) \right]$$
$$= \mathbf{u}^t(\tau) - \mathbf{z}_2^t(t) \mathbf{p}(\tau) + \mathbf{z}_2^t(t) \frac{\partial^2 w(t)}{\partial \mathbf{x}^2} \tag{6.15}$$

where

$$\mathbf{u}^t(\tau) = -\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau}\frac{\partial \mathbf{C}}{\partial \lambda_n} - \boldsymbol{\xi}_a^t(\tau)\frac{\partial \mathbf{G}}{\partial \lambda_n} - \boldsymbol{\xi}_a^t(\tau)\frac{\partial^2 \mathbf{f}(t)}{\partial \mathbf{x}\partial \lambda_n}$$
$$-\sum_{k=1}^{N_t}\left[\boldsymbol{\xi}_a^t(\tau) * \frac{\partial \mathbf{a}_k(\tau)}{\partial \lambda_n}\right] + \frac{\partial^2 w(t)}{\partial \mathbf{x}\partial \lambda_n}, \tag{6.16}$$

$$\mathbf{p}(\tau) = \left[\frac{\partial \mathbf{c}_1(t)}{\partial \mathbf{x}}\boldsymbol{\xi}_a(\tau), \frac{\partial \mathbf{c}_2(t)}{\partial \mathbf{x}}\boldsymbol{\xi}_a(\tau), \cdots\right] \tag{6.17}$$

and

$$\frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}} = [\mathbf{c}_1(t), \mathbf{c}_2(t), \cdots, \mathbf{c}_n(t)] \tag{6.18}$$

As can be seen from (6.15) and (6.16), evaluating $\frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n}$ w.r.t. N_λ variables requires the solution of (6.15) and (6.16) N_λ times, making the process computationally expensive. In order to avoid evaluating $\frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n}$ explicitly, we adopt the variational approach. Using the variational approach [13], we define an auxiliary variable $\boldsymbol{\xi}_b(t) \in \Re^{n\times 1}$. Multiplying (6.15) by $\boldsymbol{\xi}_b^t(t)$ and integrating w.r.t. $t \in [0, T]$ gives

$$\int_0^T \left[\frac{\partial^2 \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n \partial \tau} \mathbf{C} + \frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} \mathbf{G} + \frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}} \right.$$
$$\left. + \sum_{k=1}^{N_t} \left[\frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} * \mathbf{a}_k(\tau) \right] \right] \boldsymbol{\xi}_b(t) dt = \int_0^T \mathbf{u}^t(\tau) \boldsymbol{\xi}_b(t) dt$$
$$- \int_0^T \mathbf{z}_2^t(t) \mathbf{p}(\tau) \boldsymbol{\xi}_b(t) dt + \int_0^T \mathbf{z}_2^t(t) \frac{\partial^2 w(t)}{\partial \mathbf{x}^2} \boldsymbol{\xi}_b(t) dt \tag{6.19}$$

Substituting $\tau = T - t$ in (6.19) and using integration by parts, we get

$$\int_0^T \frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} \left[\mathbf{C} \frac{d\boldsymbol{\xi}_b(t)}{dt} + \mathbf{G} \boldsymbol{\xi}_b(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}} \boldsymbol{\xi}_b(t) \right.$$
$$\left. + \sum_{k=1}^{N_t} [\mathbf{a}_k(t) * \boldsymbol{\xi}_b(t)] \right] dt = \int_0^T \mathbf{u}^t(\tau) \boldsymbol{\xi}_b(t) dt$$
$$- \int_0^T \mathbf{z}_2^t(t) \mathbf{p}(\tau) \boldsymbol{\xi}_b(t) dt + \int_0^T \mathbf{z}_2^t(t) \frac{\partial^2 w(t)}{\partial \mathbf{x}^2} \boldsymbol{\xi}_b(t) dt$$
$$+ \frac{\partial \boldsymbol{\xi}_a^t(0)}{\partial \lambda_n} \mathbf{C} \boldsymbol{\xi}_b(T) - \frac{\partial \boldsymbol{\xi}_a^t(T)}{\partial \lambda_n} \mathbf{C} \boldsymbol{\xi}_b(0) \tag{6.20}$$

Avoiding the calculation of (6.15) is possible if $\boldsymbol{\xi}_b(t)$ can be found such that (using (6.20) and *Term 1* in (6.13)):

$$\mathbf{C}\frac{d\boldsymbol{\xi}_b(t)}{dt} + \mathbf{G}\boldsymbol{\xi}_b(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}\boldsymbol{\xi}_b(t)$$
$$+\sum_{k=1}^{N_t}\left[\mathbf{D}_k\boldsymbol{\phi}_k^b(t)\right] = \mathbf{j}_{\lambda_m}(t) \qquad t \in [0,T] \tag{6.21}$$

where

$$\boldsymbol{\phi}_k^b(t) = \mathbf{y}_k^t(t) * \boldsymbol{\psi}_k^b(t) \tag{6.22}$$

$$\boldsymbol{\psi}_k^b(t) = \mathbf{D}_k^t\boldsymbol{\xi}_b(t) \tag{6.23}$$

and $\boldsymbol{\xi}_b(t)$ is referred to as the b^{th} adjoint MNA variable vector [21]. It is to be noted that $\boldsymbol{\psi}_k^b(t)$ and $\boldsymbol{\phi}_k^b(t)$ are the terminal voltage and current vectors of the k^{th} transmission line in the b^{th} adjoint system, respectively.

Next, using the symmetry of $\mathbf{y}_k(t)$, (6.22) can be written in the Laplace domain as

$$\boldsymbol{\Phi}_k^b(s) = \mathbf{Y}_k(s)\boldsymbol{\Psi}_k^b(s) \tag{6.24}$$

The b^{th} adjoint system is defined by (6.21) and (6.24). However, the solution of (6.21) is not unique until the initial conditions are defined. To avoid calculating $\frac{\partial \boldsymbol{\xi}_a^t(T)}{\partial \lambda_n}$ in (6.20) explicitly, $\boldsymbol{\xi}_b^t(0)$ is selected to be equal to zero. This implies that, the b^{th} adjoint system is simulated forward-in-time, from 0 to T, with initial conditions set to zero (i.e. $\boldsymbol{\xi}_b^t(0) = 0$). $\frac{\partial \boldsymbol{\xi}_a^t(0)}{\partial \lambda_n}$ in (6.20) is zero since $\boldsymbol{\xi}_a(0)$ is zero in the a^{th} adjoint system. Using (6.21) and (6.20), *Term 1* of RHS in (6.13) becomes

$$\underbrace{\int_0^T \frac{\partial \boldsymbol{\xi}_a^t(\tau)}{\partial \lambda_n} \mathbf{j}_{\lambda_m}(t)dt}_{\text{Term 1}} = \int_0^T \mathbf{u}^t(\tau)\boldsymbol{\xi}_b(t)dt$$
$$- \int_0^T \mathbf{z}_2^t(t)\mathbf{p}(\tau)\boldsymbol{\xi}_b(t)dt + \int_0^T \mathbf{z}_2^t(t)\frac{\partial^2 w(t)}{\partial \mathbf{x}^2}\boldsymbol{\xi}_b(t)dt \tag{6.25}$$

Term 2 of the RHS in (6.13) also requires $\frac{\partial \mathbf{j}_{\lambda_m}(t)}{\partial \lambda_n}$, which can be obtained by differentiating (6.10) w.r.t. λ_n as

$$\frac{\partial \mathbf{j}_{\lambda_m}(t)}{\partial \lambda_n} = \mathbf{r}(t) - \frac{\partial \mathbf{C}}{\partial \lambda_m}\frac{d\mathbf{z}_2(t)}{dt} - \frac{\partial \mathbf{G}}{\partial \lambda_m}\mathbf{z}_2(t)$$
$$- \sum_{k=1}^{N_t}\left[\frac{\partial \mathbf{a}_k(t)}{\partial \lambda_m} * \mathbf{z}_2(t)\right] - \frac{\partial^2 \mathbf{f}(t)}{\partial \lambda_m \partial \mathbf{x}}\mathbf{z}_2(t) \tag{6.26}$$

where

$$\mathbf{r}(t) = \frac{\partial^2 \mathbf{b}(t)}{\partial \lambda_m \partial \lambda_n} - \frac{\partial^2 \mathbf{C}}{\partial \lambda_m \partial \lambda_n} \frac{d\mathbf{x}(t)}{dt} - \frac{\partial^2 \mathbf{G}}{\partial \lambda_m \partial \lambda_n} \mathbf{x}(t)$$
$$- \sum_{k=1}^{N_t} \left[\frac{\partial^2 \mathbf{a}_k(t)}{\partial \lambda_m \partial \lambda_n} * \mathbf{x}(t) \right] - \frac{\partial^2 \mathbf{f}(t)}{\partial \lambda_m \partial \lambda_n} \tag{6.27}$$

Substituting (6.26) in *Term 2* of the RHS in (6.13) becomes

$$\underbrace{\int_0^T \boldsymbol{\xi}_a^t(\tau) \frac{\partial \mathbf{j}_{\lambda_m}(t)}{\partial \lambda_n} dt}_{\text{Term 2}} = \int_0^T \boldsymbol{\xi}_a^t(\tau) \mathbf{r}(t) dt$$
$$+ \int_0^T \boldsymbol{\xi}_a^t(\tau) \left[-\frac{\partial \mathbf{C}}{\partial \lambda_m} \frac{d\mathbf{z}_2(t)}{dt} - \frac{\partial \mathbf{G}}{\partial \lambda_m} \mathbf{z}_2(t) \right] dt$$
$$+ \int_0^T \boldsymbol{\xi}_a^t(\tau) \left[- \sum_{k=1}^{N_t} \left[\frac{\partial \mathbf{a}_k(t)}{\partial \lambda_m} * \mathbf{z}_2(t) \right] - \frac{\partial^2 \mathbf{f}(t)}{\partial \lambda_m \partial \mathbf{x}} \mathbf{z}_2(t) \right] dt \tag{6.28}$$

Substituting (6.25) and (6.28) for the *Term 1* and *Term 2* of RHS in (6.13), respectively, and using integration by parts with $t = T - \tau$, $\mathbf{z}_2(0) = 0$ and $\boldsymbol{\xi}_a(0) = 0$ results in

$$\frac{\partial^2 W(T)}{\partial\lambda_m \partial\lambda_n} = \int_0^T \mathbf{u}^t(\tau)\boldsymbol{\xi}_b(t)dt + \int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{r}(t)dt$$
$$+ \underbrace{\int_0^T \mathbf{e}^t(\tau)\mathbf{z}_2(t)dt}_{\text{Term 3}} + \int_0^T \frac{\partial^2 w(t)}{\partial\lambda_m \partial\lambda_n}dt \tag{6.29}$$

where

$$\begin{aligned}\mathbf{e}^t(\tau) = &-\boldsymbol{\xi}_b^t(T-\tau)\mathbf{p}^t(\tau)\\ &+\boldsymbol{\xi}_b^t(T-\tau)\frac{\partial^2 w(T-\tau)}{\partial \mathbf{x}^2}\\ &-\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau}\frac{\partial \mathbf{C}}{\partial\lambda_m} - \boldsymbol{\xi}_a^t(\tau)\frac{\partial \mathbf{G}}{\partial\lambda_m} - \underbrace{\sum_{k=1}^{N_t}\left[\boldsymbol{\xi}_a^t(\tau) * \frac{\partial \mathbf{a}_k(\tau)}{\partial\lambda_m}\right]}\\ &-\boldsymbol{\xi}_a^t(\tau)\frac{\partial^2 \mathbf{f}(T-\tau)}{\partial\lambda_m \partial\mathbf{x}} + \frac{\partial^2 w(T-\tau)}{\partial\lambda_m \partial\mathbf{x}}\end{aligned} \tag{6.30}$$

It is to be noted that if λ_m is the electrical or physical parameter of a transmission line then the fifth term of the RHS in (6.30) requires the convolution operation, where $\boldsymbol{\xi}_a^t(\tau)$ is defined from $\tau = 0$ to $\tau = T$ and $\frac{\partial \mathbf{a}_k(\tau)}{\partial\lambda_m}$ can be evaluated by using IFFT on the closed-form of $\frac{\partial \mathbf{A}_k(s)}{\partial\lambda_m}$, which is given in the Section 4.2.3.

Term 3 of the RHS in (6.29) requires the value of $\mathbf{z}_2(t)$. Using the direct sensitivity approach, $\mathbf{z}_2(t)$ is obtained by differentiating (6.1) w.r.t. λ_n as

$$\mathbf{C}\frac{d\mathbf{z}_2(t)}{dt} + \mathbf{G}\mathbf{z}_2(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}\mathbf{z}_2(t) + \sum_{k=1}^{N_t} [\mathbf{a}_k(t) * \mathbf{z}_2(t)] = \mathbf{k}(t) \tag{6.31}$$

where

$$\begin{aligned} \mathbf{k}(t) =& \frac{\partial \mathbf{b}(t)}{\partial \lambda_n} - \frac{\partial \mathbf{C}}{\partial \lambda_n}\frac{d\mathbf{x}(t)}{dt} - \frac{\partial \mathbf{G}}{\partial \lambda_n}\mathbf{x}(t) \\ & - \sum_{k=1}^{N_t}\left[\frac{\partial \mathbf{a}_k(t)}{\partial \lambda_n} * \mathbf{x}(t)\right] - \frac{\partial \mathbf{f}(t)}{\partial \lambda_n} \end{aligned} \tag{6.32}$$

Equation (6.31) can be used to evaluate $\mathbf{z}_2(t)$ explicitly, which is computationally expensive because the N_λ set of equations have to solved. In order to avoid evaluating $\mathbf{z}_2(t)$ explicitly, the variational approach is adopted. Using the variational approach, we define an auxiliary variable $\boldsymbol{\xi}_c(\tau) \in \Re^{n\times 1}$. Multiplying (6.31) by $\boldsymbol{\xi}_c^t(\tau)$ and integrating w.r.t. $t \in [0, T]$ gives

$$\begin{aligned} \int_0^T \boldsymbol{\xi}_c^t(\tau) &\left[\mathbf{C}\frac{d\mathbf{z}_2(t)}{dt} + \mathbf{G}\mathbf{z}_2(t) + \frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}\mathbf{z}_2(t) \right. \\ &\left. + \sum_{k=1}^{N_t} [\mathbf{a}_k(t) * \mathbf{z}_2(t)]\right] dt = \int_0^T \boldsymbol{\xi}_c^t(\tau)\mathbf{k}(t)dt \end{aligned} \tag{6.33}$$

Substituting $\tau = T - t$ in (6.33) and using integration by parts, we get

$$\int_0^T \left[\frac{d\boldsymbol{\xi}_c^t(\tau)}{d\tau}\mathbf{C} + \boldsymbol{\xi}_c^t(\tau)\mathbf{G} + \boldsymbol{\xi}_c^t(\tau)\frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}} \right.$$
$$\left. + \sum_{k=1}^{N_t} \left[\boldsymbol{\xi}_c^t(\tau) * \mathbf{a}_k\right] \right] \mathbf{z}_2(t)dt = \int_0^T \boldsymbol{\xi}_c^t(\tau)\mathbf{k}(t)dt$$
$$-\boldsymbol{\xi}_c^t(0)\mathbf{C}\mathbf{z}(T) - \boldsymbol{\xi}_c^t(T)\mathbf{C}\mathbf{z}(0) \tag{6.34}$$

Avoiding the calculation of (6.31) is possible if $\boldsymbol{\xi}_c(\tau)$ can be found such that (using (6.34) and *Term 3* in RHS of (6.29)):

$$\frac{d\boldsymbol{\xi}_c^t(\tau)}{d\tau}\mathbf{C} + \boldsymbol{\xi}_c^t(\tau)\mathbf{G} + \boldsymbol{\xi}_c^t(\tau)\frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}$$
$$+ \sum_{k=1}^{N_t} \left[\boldsymbol{\xi}_c^t(\tau) * \mathbf{a}_k\right] = \mathbf{e}^t(\tau) \tag{6.35}$$

where $\boldsymbol{\xi}_c(\tau)$ is referred to as the c^{th} adjoint MNA variable vector [21]. Taking the transpose of both sides and substituting $t = T - \tau$ in (6.35) gives

$$\mathbf{C}^t\frac{d\boldsymbol{\xi}_c(\tau)}{d\tau} + \mathbf{G}^t\boldsymbol{\xi}_c(\tau) + \left[\frac{\partial \mathbf{f}(T-\tau)}{\partial \mathbf{x}}\right]^t \boldsymbol{\xi}_c(\tau)$$
$$+ \sum_{k=1}^{N_t} \left[\mathbf{D}_k \boldsymbol{\phi}_k^c(\tau)\right] = \mathbf{e}(\tau) \qquad \tau \in [0, T] \tag{6.36}$$

where

$$\boldsymbol{\phi}_k^c(\tau) = \mathbf{y}_k^t(\tau) * \boldsymbol{\psi}_k^c(\tau) \tag{6.37}$$

$$\boldsymbol{\psi}_k^c(\tau) = \mathbf{D}_k^t \boldsymbol{\xi}_c(\tau) \tag{6.38}$$

and $\boldsymbol{\xi}_c(\tau)$ is referred to as the c^{th} adjoint MNA variable vector [21]. It is to be noted that $\boldsymbol{\psi}_k^c(\tau)$ and $\boldsymbol{\phi}_k^c(\tau)$ are the terminal voltage and current vectors of the k^{th} transmission line in the c^{th} adjoint system, respectively.

Next, using the symmetry of $\mathbf{y}_k(t)$, (6.37) can be written in the Laplace domain as

$$\boldsymbol{\Phi}_k^c(s) = \mathbf{Y}_k(s)\boldsymbol{\Psi}_k^c(s) \tag{6.39}$$

The c^{th} adjoint system is defined by (6.36) and (6.39). However, the solution of (6.36) is not unique until the initial conditions are defined. To avoid calculating $\mathbf{z}_2(T)$ in (6.34) explicitly, $\boldsymbol{\xi}_c^t(0)$ is selected to be equal to zero. This implies that, the c^{th} adjoint system is simulated forward-in-time, from 0 to T, with initial conditions set to zero (i.e. $\boldsymbol{\xi}_c^t(0) = 0$). $\mathbf{z}_2(0)$ in (6.34) is zero since $\mathbf{x}(0)$ is zero in the original system. Using

(6.35) and (6.34), *Term 3* of RHS of (6.29) becomes

$$\begin{aligned}\frac{\partial^2 W(T)}{\partial\lambda_m\partial\lambda_n} &= \int_0^T \mathbf{u}^t(\tau)\boldsymbol{\xi}_b(t)dt + \int_0^T \boldsymbol{\xi}_a^t(\tau)\mathbf{r}(t)dt \\ &\quad + \int_0^T \boldsymbol{\xi}_c^t(\tau)\mathbf{k}(t)dt + \int_0^T \frac{\partial^2 w(t)}{\partial\lambda_m\partial\lambda_n}dt \end{aligned} \tag{6.40}$$

If λ_m and λ_n are the electrical or physical parameters of the k^{th} transmission line then (6.40) becomes

$$\begin{aligned}\frac{\partial^2 W(T)}{\partial\lambda_m\partial\lambda_n} &= -\int_0^T \left[[\boldsymbol{\psi}_k^a(\tau)]^t * \frac{\partial\mathbf{y}_k(\tau)}{\partial\lambda_n}\right] \times \boldsymbol{\psi}_k^b(t)dt \\ &\quad - \int_0^T [\boldsymbol{\psi}_k^a(\tau)]^t \times \left[\frac{\partial^2\mathbf{y}_k(t)}{\partial\lambda_m\partial\lambda_n} * \mathbf{v}_k(t)\right] dt \\ &\quad - \int_0^T [\boldsymbol{\psi}_k^c(\tau)]^t \times \left[\frac{\partial\mathbf{y}_k(t)}{\partial\lambda_n} * \mathbf{v}_k(t)\right] dt \end{aligned} \tag{6.41}$$

The details of evaluation of $\frac{\partial\mathbf{y}_k(t)}{\partial\lambda_m}$, $\frac{\partial\mathbf{y}_k(t)}{\partial\lambda_n}$ and $\frac{\partial^2\mathbf{y}_k(t)}{\partial\lambda_m\partial\lambda_n}$ in (6.41) in a closed-form are given in the Sections 4.2.3 and 5.2.4.

6.2.4 Proposed sensitivity analysis w.r.t. a physical parameter

When studying the sensitivity of distributed networks, the derivative of the admittance matrix w.r.t. physical parameters (such as width and spacing of conductors) is required. In the case where λ_m represents a physical parameter of an interconnect, the derivative of the admittance matrix, using the chain rule can be obtained as

$$\begin{aligned}\frac{\partial \mathbf{Y}_k(s)}{\partial \lambda_m} = \sum_{p=1}^{N_k}\sum_{q=1}^{N_k}\Bigg[&\frac{\partial \mathbf{Y}_k(s)}{\partial R_k^{(p,q)}}\frac{\partial R_k^{(p,q)}(s)}{\partial \lambda_m} + \frac{\partial \mathbf{Y}_k(s)}{\partial L_k^{(p,q)}}\frac{\partial L_k^{(p,q)}(s)}{\partial \lambda_m}\\ &+\frac{\partial \mathbf{Y}_k(s)}{\partial G_k^{(p,q)}}\frac{\partial G_k^{(p,q)}(s)}{\partial \lambda_m} + \frac{\partial \mathbf{Y}_k(s)}{\partial C_k^{(p,q)}}\frac{\partial C_k^{(p,q)}(s)}{\partial \lambda_m}\Bigg]\end{aligned} \tag{6.42}$$

where the superscripts $p, q \in \{1, \cdots, N_k\}$ are matrix indexes. The second-order can be found by differentiating (6.42) w.r.t. another physical parameter λ_n as

$$\begin{aligned}\frac{\partial^2 \mathbf{Y}_k(s)}{\partial \lambda_m \partial \lambda_n} = \sum_{p=1}^{N_k}\sum_{q=1}^{N_k}\Bigg[&\frac{\partial \mathbf{Y}_k(s)}{\partial R_k^{(p,q)}}\frac{\partial^2 R_k^{(p,q)}(s)}{\partial \lambda_m \partial \lambda_n}\\ +\frac{\partial \mathbf{Y}_k(s)}{\partial L_k^{(p,q)}}\frac{\partial^2 L_k^{(p,q)}(s)}{\partial \lambda_m \partial \lambda_n} &+ \frac{\partial \mathbf{Y}_k(s)}{\partial G_k^{(p,q)}}\frac{\partial^2 G_k^{(p,q)}(s)}{\partial \lambda_m \partial \lambda_n}\\ &+\frac{\partial \mathbf{Y}_k(s)}{\partial C_k^{(p,q)}}\frac{\partial^2 C_k^{(p,q)}(s)}{\partial \lambda_m \partial \lambda_n}\Bigg]\end{aligned} \tag{6.43}$$

6.2.5 Evaluation of the sensitivity of the objective function

Next, using the first and second-order sensitivities of the admittance matrix and (6.41), the sensitivity of the objective function (6.40) is computed as follows:

$$\frac{\partial^2 W(T)}{\partial \lambda_m \partial \lambda_n} = -\int_0^T [\psi_k^a(\tau)]^t \times \mathbf{q}_1(t)dt$$
$$-\int_0^T [\psi_k^a(\tau)]^t \times \mathbf{q}_2(t)dt - \int_0^T [\psi_k^c(\tau)]^t \times \mathbf{q}_3(t)dt \tag{6.44}$$

where

$$\mathbf{q}_1(\tau) = \mathcal{F}^{-1}\left\{\frac{\partial \mathbf{Y}_k(s)}{\partial \lambda_n} \times \mathbf{\Psi}_k^b(s)\right\} \tag{6.45}$$

$$\mathbf{\Psi}_k^b(s) = \mathcal{F}\left\{\psi_k^b(t)\right\} \tag{6.46}$$

$$\mathbf{q}_2(t) = \mathcal{F}^{-1}\left\{\frac{\partial^2 \mathbf{Y}_k(s)}{\partial \lambda_m \partial \lambda_n} \times \mathbf{V}_k(s)\right\} \tag{6.47}$$

$$\mathbf{q}_3(t) = \mathcal{F}^{-1}\left\{\frac{\partial \mathbf{Y}_k(s)}{\partial \lambda_n} \times \mathbf{V}_k(s)\right\} \tag{6.48}$$

$$\mathbf{V}_k(s) = \mathcal{F}\{\mathbf{v}_k(t)\} \tag{6.49}$$

$\mathcal{F}\{\ \}$ and $\mathcal{F}^{-1}\{\ \}$ denote the FFT and IFFT operators, respectively.

6.2.6 Summary of the computational steps

A summary of the computational steps for the proposed second-order adjoint sensitivity analysis is given below:

- *Step 1*: Simulate the original circuit (6.1) from $t = 0$ to $t = T$ to get $\mathbf{x}(t)$.
- *Step 2*: Replace all the nonlinear elements in original circuit with linear time-varying elements $\left[\frac{\partial \mathbf{f}(T-\tau)}{\partial \mathbf{x}}\right]^t$. The independent sources for the a^{th} adjoint circuit are evaluated using $\left[\frac{\partial w(T-\tau)}{\partial \mathbf{x}}\right]^t$. Simulate the a^{th} adjoint circuit equations (6.5) to (6.8) from $\tau = 0$ to $\tau = T$ with zero initial conditions to get $\boldsymbol{\xi}_a(\tau)$.
- *Step 3*: Use $\mathbf{x}(t)$ from *Step 1*, $\boldsymbol{\xi}_a(\tau)$ from *Step 2* and (6.9) to find the first-order sensitivity of the objective function if required.
- *Step 4*: Replace all the nonlinear elements in the original circuit with linear time-varying elements $\frac{\partial \mathbf{f}(t)}{\partial \mathbf{x}}$. The independent sources for the b^{th} adjoint circuit are evaluated using $\mathbf{j}_{\lambda_m}(t)$. Simulate the b^{th} adjoint circuit equations (6.21) to (6.24) from $t = 0$ to $t = T$ with zero initial conditions to get $\boldsymbol{\xi}_b(t)$.

- *Step 5*: Replace all the nonlinear elements in original circuit with linear time-varying elements $\left[\frac{\partial \mathbf{f}(T-\tau)}{\partial \mathbf{x}}\right]^t$. The independent sources for the c^{th} adjoint circuit are evaluated using $\mathbf{e}(\tau)$ of (6.30). Simulate the c^{th} adjoint circuit equations (6.36) to (6.39) from $\tau = 0$ to $\tau = T$ with zero initial conditions to get $\boldsymbol{\xi}_c(\tau)$.
- *Step 6*: Use $\mathbf{x}(t)$ from *Step 1*, $\boldsymbol{\xi}_a(\tau)$ from *Step 2*, $\boldsymbol{\xi}_b(t)$ from *Step 4*, $\boldsymbol{\xi}_c(\tau)$ from *Step 5* and (6.40) to find the second-order sensitivity of the objective function.

The proposed approach can be easily adapted to the case of TLs with frequency-dependent (FD) p.u.l. parameters. This can be accomplished by representing the p.u.l. FD parameters by rational functions using techniques such as vectorfit [27], [28], [29] and subsequently synthesizing them as lumped equivalent circuits.

6.2.7 Evaluation of a Hessian matrix

The pseudocodes for the first and second-order (Hessian matrix) adjoint sensitivity analysis w.r.t. all the parameters of a circuit are given in Algorithms 6.1 and 6.2, respectively.

Algorithm 6.1 Calculation of the first-order sensitivity for the objective function

Solve: *Step 1* for $\mathbf{x}(t)$
Solve: *Step 2* for $\boldsymbol{\xi}_a(\tau)$
for $m = 1$ **to** N_λ **do**
  Solve: *Step 3* for $\frac{\partial W(T)}{\partial \lambda_m}$
end for
return Vector consisting of first-order derivatives

Algorithm 6.2 Calculation of the second-order sensitivity for the objective function

Solve: *Step 1* for $\mathbf{x}(t)$
Solve: *Step 2* for $\boldsymbol{\xi}_a(\tau)$
for $m = 1$ **to** N_λ **do**
  Solve: *Step 4* for $\boldsymbol{\xi}_b(t)$
  Solve: *Step 5* for $\boldsymbol{\xi}_c(\tau)$
  for $n = 1$ **to** N_λ **do**
    Solve: *Step 6* for $\frac{\partial^2 W(T)}{\partial \lambda_m \partial \lambda_n}$
  end for
end for
return Hessian matrix consisting of second-order sensitivities

6.2.8 Comparison with the direct sensitivity analysis technique

In this section, a brief discussion of the computational cost of the proposed technique versus the direct [2–5] sensitivity technique is given.

Using the direct sensitivity approach, (6.1) and (4.3) can be solved simultaneously avoiding the need for an additional LU decomposition at each time point while solving (4.3). However, an extra forward-backward substitution is needed at every time point for each parameter under consideration. Hence, for N_λ parameters, the main computational cost in evaluating first-order sensitivity using the direct approach is

$$C_D^{FO} = \sum_{i=1}^{N_p} \left[N_{NR}^i (C_{LU} + C_{F/B}) \right] + N_\lambda N_p C_{F/B} \tag{6.50}$$

Similarly, the adjoint approach requires the solution of (6.1) and (6.5). However, in contrast to the direct approach where (4.3) is parameter-dependent, (6.5) is independent of any specific parameter. As a result, (6.5) needs to be solved only once independent of the number of parameters. Hence, for N_λ parameters, the main computational cost in evaluating first-order sensitivity, assuming LU factors are stored while solving (6.1), is

$$C_A^{FO} = \sum_{i=1}^{N_p} \left[N_{NR}^i (C_{LU} + C_{F/B}) \right] + N_p C_{F/B} + N_\lambda C_I^{FO}, \tag{6.51}$$

where C_I^{FO} is the computational cost associated with evaluating the numerical integration in (6.9). Therefore, in evaluating the first-order sensitivity for N_λ parameters, the computational cost of using adjoint approach in (6.51) is N_λ times less than using the direct approach in (6.50).

Next, using the direct sensitivity approach, the main computational cost in evaluating the Hessian matrix (obtained by differentiating (2.4) and (4.3)) is

$$C_D^{SO} = \sum_{i=1}^{N_p} \left[N_{NR}^i (C_{LU} + C_{F/B})\right] + (N_\lambda + N_\lambda^2) N_p C_{F/B} \tag{6.52}$$

Similarly, using the adjoint approach approach, evaluating Hessian matrix requires the solution of (6.1), (6.5) and N_λ sets of (6.21) and (6.36). Hence, for N_λ parameters, the main computational cost for evaluating Hessian matrix, assuming the LU factors are stored while solving (6.1), is

$$C_A^{SO} = \sum_{i=1}^{N_p} \left[N_{NR}^i (C_{LU} + C_{F/B})\right] + (1 + 2N_\lambda) N_p C_{F/B} + N_\lambda^2 C_I^{SO} \tag{6.53}$$

where C_I^{SO} is the computational cost associated with evaluating the numerical integration in (6.40). Therefore, in evaluating the second-order sensitivity for N_λ parameters, the computational cost of using adjoint approach in (6.53) is N_λ times less than using the direct approach in (6.52).

In addition to extending all the advantages of the adjoint sensitivity analysis approach to distributed transmission lines, the proposed method is generic and is compatible with any MTL macromodel.

6.3 Comparison with second-order frequency-domain sensitivity analysis

In this section, it is shown that for linear circuits and the special case of objective function, the Steps given in 5.2.5 are equivalent to 6.2.6.

6.3.1 Problem formulation

Consider a special case of (6.1) without nonlinear components. The corresponding modified nodal analysis (MNA) equations [3] can be written in the time-domain as

$$\mathbf{C}\frac{d\mathbf{x}(t)}{dt} + \mathbf{G}\mathbf{x}(t) + \sum_{k=1}^{N_t} \mathbf{a}_k(t) * \mathbf{x}(t) = \mathbf{b}(t) \tag{6.54}$$

where

$$\mathbf{a}_k(t) = \mathbf{D}_k \mathbf{y}_k(t) \mathbf{D}_k^t \tag{6.55}$$

Assuming the initial condition of (6.54) as zero $\mathbf{b}(0) = 0$. Taking Laplace transform of (6.54) and (6.55) gives

$$\mathbf{M}(s)\mathbf{X}(s) = \mathbf{B}(s) \tag{6.56}$$

where

$$\mathbf{M}(s) = s\mathbf{C} + \mathbf{G} + \sum_{k=1}^{N_t} \mathbf{A}_k(s) \tag{6.57}$$

and $\mathbf{B}(s)$ is the Laplace-domain representation of $\mathbf{b}(t)$.

Next, consider the voltage $v_o(t)$ at a node of a circuit in time-domain as

$$v_o(t) = \boldsymbol{\eta}^t \mathbf{x}(t) \tag{6.58}$$

where $\boldsymbol{\eta} = [\eta_{i,1}], \eta_{i,1} \in \{0, 1\}$ is a selector vector that selects a row from the MNA vector $\mathbf{x}(t)$ of (6.54). In the Laplace-domain, this is

$$V_o(s) = \boldsymbol{\eta}^t \mathbf{X}(s) \tag{6.59}$$

where $\mathbf{X}(s)$ and $V_o(s)$ are the Laplace-domain representation of $\mathbf{x}(t)$ and $v_o(t)$, respectively. $V_o(s)$ can be written as

$$V_o(s) = \mathcal{L}\{v_o(t)\} \tag{6.60}$$

Differentiating (6.60) w.r.t. a particular parameter λ_m yields

$$\frac{\partial V_o(s)}{\partial \lambda_m} = \mathcal{L}\left\{\frac{\partial v_o(t)}{\partial \lambda_m}\right\} \tag{6.61}$$

Similarly, differentiating (6.61) w.r.t. another parameter λ_n gives

$$\frac{\partial^2 V_o(s)}{\partial\lambda_m\partial\lambda_n} = \mathcal{L}\left\{\frac{\partial^2 v_o(t)}{\partial\lambda_m\partial\lambda_n}\right\} \tag{6.62}$$

The next section shows that Laplace transforms of $\frac{\partial v_o(t)}{\partial\lambda_m}$, $\frac{\partial^2 v_o(t)}{\partial\lambda_m\partial\lambda_n}$, which are evaluated using steps given in Section 6.2.6, are equivalent to $\frac{\partial V_o(s)}{\partial\lambda_m}$, $\frac{\partial^2 V_o(s)}{\partial\lambda_m\partial\lambda_n}$, which are evaluated using steps given in Section 5.2.5.

6.3.2 Special case of an objective function

Consider (6.58) and a special case of an objective function $W(t)$ whose value is to be optimized at time $t = T$ as given by

$$W(T) = v_o(T) = \int_0^T w(\mathbf{x}(t), \boldsymbol{\lambda})dt \tag{6.63}$$

where

$$w(\mathbf{x}(t), \boldsymbol{\lambda}) = \delta(T-t)\boldsymbol{\eta}^t\mathbf{x}(t) \tag{6.64}$$

Further, consider the special case of an objective function in frequency-domain given by

$$W(s) = V_o(s) = w(\mathbf{X}(s), \boldsymbol{\lambda}) = \boldsymbol{\eta}^t\mathbf{X}(s) \tag{6.65}$$

Differentiating (6.64) and (6.65) w.r.t. the MNA variables in time and frequency domain gives

$$\frac{\partial w(t)}{\partial \mathbf{x}} = \delta(T - t)\boldsymbol{\eta}^t \tag{6.66}$$

and

$$\frac{\partial w(s)}{\partial \mathbf{X}} = \boldsymbol{\eta}^t, \tag{6.67}$$

respectively.

Step 1

Time-domain

The original system given by (6.54) is simulated in time domain from 0 to ∞.

Frequency-domain

The original system given by (6.56) is simulated in frequency domain from 0 to ∞.

It is to be noted that (6.56) is the Laplace transform of (6.54). Hence, (6.56) and (6.54) are equivalent.

Step 2

Time-domain

The a^{th} adjoint system corresponding to (6.54) is calculated by substituting (6.66) in (6.5) as

$$\mathbf{C}^t \frac{d\boldsymbol{\xi}_a(\tau)}{d\tau} + \mathbf{G}^t \boldsymbol{\xi}_a(\tau) + \sum_{k=1}^{N_t} \mathbf{a}_k^t(\tau) * \boldsymbol{\xi}_a(\tau) = \delta(\tau)\boldsymbol{\eta} \qquad \tau \in [0, \infty] \tag{6.68}$$

Frequency-domain

The a^{th} adjoint system corresponding to (6.56) is calculated by substituting (6.67) in (5.12) as

$$\mathbf{M}^t(s)\Xi_a(s) = \boldsymbol{\eta} \tag{6.69}$$

It is to be noted that (6.69) is the Laplace transform of (6.68). Hence, (6.69) and (6.68) are equivalent.

Step 3

In this step, the first-order sensitivity is evaluated.

Time-domain

Using $\mathbf{x}(0) = 0$ and $\mathbf{f}(t) = 0$ in (6.9), the first-order sensitivity in the time domain is given by

$$\frac{\partial W(t)}{\partial \lambda_m} = \boldsymbol{\xi}_a^t * \mathbf{j}_{\lambda_m} \tag{6.70}$$

Frequency-domain

Using (5.13), the first-order sensitivity in the objective function in given by

$$\frac{\partial W(s)}{\partial \lambda_m} = \boldsymbol{\Xi}_a^t(s)\mathbf{J}_{\lambda_m}(s) \tag{6.71}$$

It is to be noted that (6.71) is the Laplace transform of (6.70). Hence, (6.71) and (6.70) are equivalent.

Step 4

Time-domain

The b^{th} adjoint system corresponding to (6.54) is calculated by substituting $\mathbf{f}(t) = 0$ in (6.21) as

$$\mathbf{C}\frac{d\boldsymbol{\xi}_b(t)}{dt} + \mathbf{G}\boldsymbol{\xi}_b(t) + \sum_{k=1}^{N_t}\left[\mathbf{D}_k\boldsymbol{\phi}_k^b(t)\right] = \mathbf{j}_{\lambda_m}(t) \qquad t \in [0, \infty] \tag{6.72}$$

Frequency-domain

The b^{th} adjoint system corresponding to (6.56) is given by

$$\mathbf{M}(s)\boldsymbol{\Xi}_b(s) = \mathbf{J}_{\lambda_m}(s) \tag{6.73}$$

It is to be noted that (6.73) is the Laplace transform of (6.72). Hence, (6.73) and (6.72) are equivalent.

Step 5

Time-domain

The c^{th} adjoint system corresponding to (6.54) is solved by substituting $\mathbf{f}(t) = 0$ in (6.21) as

$$\mathbf{C}^t\frac{d\boldsymbol{\xi}_c(\tau)}{d\tau} + \mathbf{G}^t\boldsymbol{\xi}_c(\tau) + \sum_{k=1}^{N_t}\left[\mathbf{D}_k\boldsymbol{\phi}_k^c(\tau)\right] = \mathbf{e}(\tau) \qquad \tau \in [0, \infty] \tag{6.74}$$

where

$$\mathbf{e}^t(\tau) = -\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau}\frac{\partial \mathbf{C}}{\partial \lambda_m} - \boldsymbol{\xi}_a^t(\tau)\frac{\partial \mathbf{G}}{\partial \lambda_m} - \sum_{k=1}^{N_t}\left[\boldsymbol{\xi}_a^t(\tau) * \frac{\partial \mathbf{a}_k(\tau)}{\partial \lambda_m}\right] \tag{6.75}$$

Frequency-domain

The c^{th} adjoint system corresponding to (6.56) is given by

$$\mathbf{M}^t(s)\Xi_c(s) = -\frac{\partial \mathbf{M}^t(s)}{\partial \lambda_m}\Xi_a \tag{6.76}$$

It is to be noted that (6.76) is the Laplace transform of (6.74). Hence, (6.76) and (6.74) are equivalent.

Step 6

In this step the second-order sensitivity is evaluated.

Time-domain

Using $\mathbf{f}(t) = 0$ in (6.40) gives the second-order sensitivity of the objective function as

$$\begin{aligned}\frac{\partial^2 W(t)}{\partial\lambda_m\partial\lambda_n} =& \mathbf{u}^t * \boldsymbol{\xi}_b \\ &+\boldsymbol{\xi}_a^t * \mathbf{r} \\ &+\boldsymbol{\xi}_c^t * \mathbf{k}\end{aligned} \tag{6.77}$$

where

$$\mathbf{u}^t(\tau) = -\frac{d\boldsymbol{\xi}_a^t(\tau)}{d\tau}\frac{\partial \mathbf{C}}{\partial\lambda_n} - \boldsymbol{\xi}_a^t(\tau)\frac{\partial \mathbf{G}}{\partial\lambda_n} - \sum_{k=1}^{N_t}\left[\boldsymbol{\xi}_a^t(\tau) * \frac{\partial \mathbf{a}_k(\tau)}{\partial\lambda_n}\right], \tag{6.78}$$

$$\mathbf{r}(t) = \frac{\partial^2 \mathbf{b}(t)}{\partial\lambda_m\partial\lambda_n} - \frac{\partial^2 \mathbf{C}}{\partial\lambda_m\partial\lambda_n}\frac{d\mathbf{x}(t)}{dt} - \frac{\partial^2 \mathbf{G}}{\partial\lambda_m\partial\lambda_n}\mathbf{x}(t) - \sum_{k=1}^{N_t}\left[\frac{\partial^2 \mathbf{a}_k(t)}{\partial\lambda_m\partial\lambda_n} * \mathbf{x}(t)\right] \tag{6.79}$$

and

$$\mathbf{k}(t) = \frac{\partial \mathbf{b}(t)}{\partial\lambda_n} - \frac{\partial \mathbf{C}}{\partial\lambda_n}\frac{d\mathbf{x}(t)}{dt} - \frac{\partial \mathbf{G}}{\partial\lambda_n}\mathbf{x}(t) - \sum_{k=1}^{N_t}\left[\frac{\partial \mathbf{a}_k(t)}{\partial\lambda_n} * \mathbf{x}(t)\right] \tag{6.80}$$

Frequency-domain

Using (5.28), the second-order sensitivity of the objective function is given by

$$\begin{aligned}\frac{\partial^2 W(s)}{\partial\lambda_m\partial\lambda_n} =& - \Xi_a^t(s)\frac{\partial \mathbf{M}(s)}{\partial\lambda_n}\Xi_b(s) \\ &+ \Xi_a^t(s)\frac{\partial^2 \mathbf{B}(s)}{\partial\lambda_m\partial\lambda_n} - \Xi_a^t(s)\frac{\partial^2 \mathbf{M}(s)}{\partial\lambda_m\partial\lambda_n}\mathbf{X}(s) \\ &+ \Xi_c^t(s)\frac{\partial \mathbf{B}(s)}{\partial\lambda_n} - \Xi_c^t(s)\frac{\partial \mathbf{M}(s)}{\partial\lambda_n}\mathbf{X}(s)\end{aligned} \tag{6.81}$$

It is to be noted that (6.81) is the Laplace transform of (6.77). Hence, (6.81) and (6.77) are equivalent.

As seen in Step 3 and 6, the first and second-order sensitivities are equivalent, which validates (6.61) and (6.62). Therefore, it has been proved that, for linear circuits and a special case of objective function, the steps given in 5.2.5 are equivalent to 6.2.6.

6.4 Numerical examples

In this section, two numerical examples are presented to demonstrate the validity and accuracy of the proposed method. Here the Examples 1 and 2 correspond to the case of nonlinear and linear circuits, respectively.

6.4.1 Example 1

The circuit considered in this experiment is similar to the circuit given in Fig. 4.15 of Section 4.3.3.

The result of the proposed second-order adjoint approach is compared with that from the perturbation method. The second-order sensitivity w.r.t. the lumped parameters R_1, R_3, C_1 and a nonlinear parameter k_1 are shown in Table 6.1, which further validates the accuracy of the proposed method.

Table 6.1: Second-order sensitivity of dissipated power

Sensitivity	% Change in Parameter λ_1	% Change in Parameter λ_2	Perturbation Method	Proposed Method	Relative Difference
$\frac{\partial^2 W(T)}{\partial R_1 \partial R_3}$	1.0%	0.1%	2.710E-07	2.712E-07	0.10%
$\frac{\partial^2 W(T)}{\partial R_1 \partial C_1}$	0.1%	0.5%	-3.899E+06	-3.899E+06	0.00%
$\frac{\partial^2 W(T)}{\partial R_1 \partial k_1}$	5.0%	0.1%	1.371E-03	1.363E-03	-0.57%

6.4.2 Example 2

The circuit considered in this experiment is shown in Fig. 6.1 [11]. It contains three lossy coupled transmission lines numbered by subcircuits 1, 2 and 3. The details of these transmission lines are given in Section 5.3.3.

The applied voltage is a trapezoidal pulse with rise and fall time of 1ns, a pulse width of 7ns and a magnitude of 2V. Objective function considered is the average power dissipated by the load Z_L,

$$P_{Diss}(45ns) = \frac{1}{45ns}\int_0^{45ns} [v_a(t)i_a(t)]\,dt \tag{6.82}$$

The voltage response at node V_a is shown in Fig. 6.2. The result of the proposed adjoint approach is compared with that from the perturbation method. The first and second-order sensitivities are evaluated w.r.t. the lumped (R_1, C_1) and electrical parameters ($R_1^{(1,1)}$, $G_1^{(1,1)}$, $C_1^{(1,1)}$ of TL#1 and $C_3^{(3,3)}$ of TL#3). The first and second-order sensitivities are shown in Tables 6.2 and 6.3, respectively.

Table 6.2: First-order sensitivity of dissipated power

Sensitivity	% Change in Parameter	Perturbation Method	Proposed Method	Relative Difference
$\frac{\partial W(T)}{\partial C_1}$	0.1%	7.609E+04	7.230E+04	-5.24%
$\frac{\partial W(T)}{\partial R_1}$	0.1%	-2.254E-06	-2.250E-06	-0.16%
$\frac{\partial W(T)}{\partial C_3^{(3,3)}}$	1.0%	-1.300E+04	-1.308E+04	0.60%
$\frac{\partial W(T)}{\partial R_1^{(1,1)}}$	5.0%	-8.517E-07	-8.670E-07	1.76%

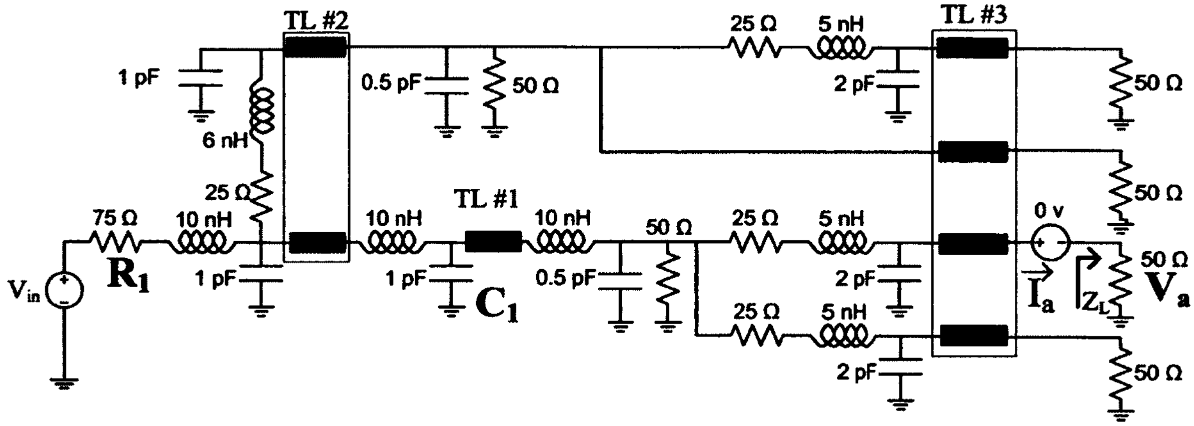

Figure 6.1: Circuit containing three lossy coupled transmission lines (Example 2)

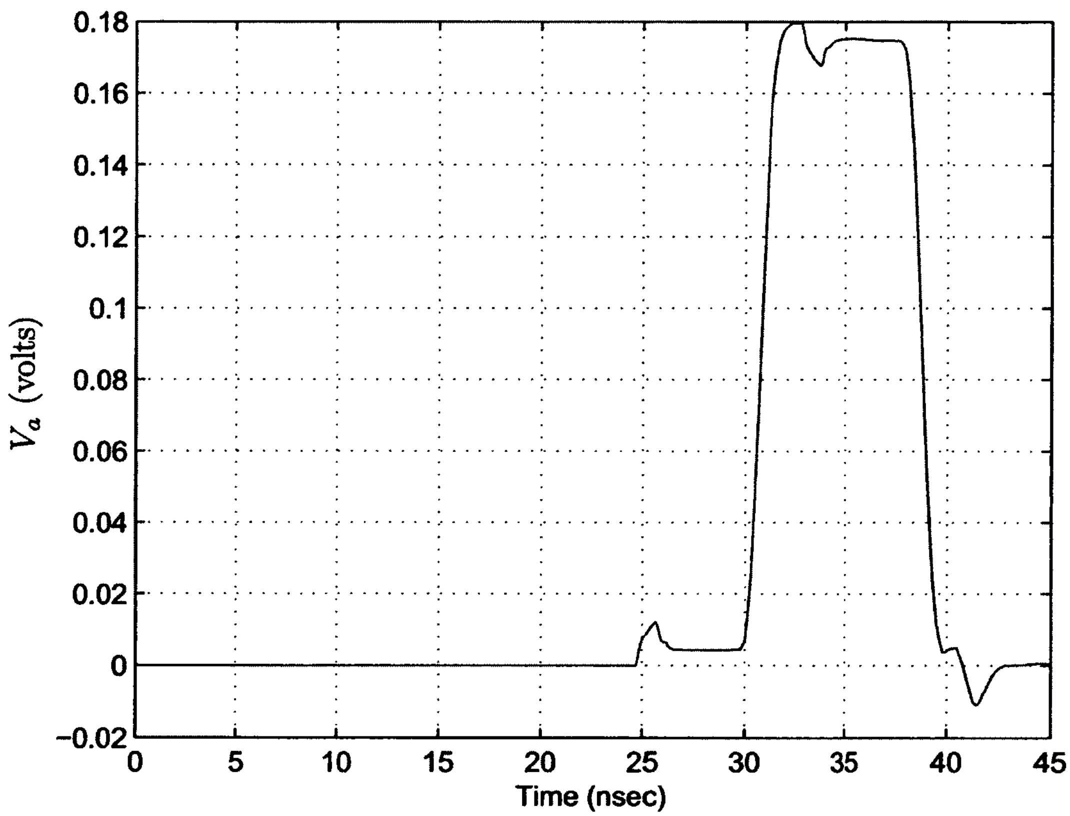

Figure 6.2: Transient response of the circuit shown in Fig. 6.1 at node V_a

Table 6.3: Second-order sensitivity of dissipated power

Sensitivity	% Change in Parameter λ_1	% Change in Parameter λ_2	Perturbation Method	Proposed Method	Relative Difference
$\frac{\partial^2 W(T)}{\partial R_1 \partial C_1}$	1.0%	50.0%	-1.561E+03	-1.448E+03	-7.79%
$\frac{\partial^2 W(T)}{\partial R_1 \partial C_3^{(3,3)}}$	1.0%	5.0%	2.855E+02	2.858E+02	0.09%
$\frac{\partial^2 W(T)}{\partial R_1 \partial R_1^{(1,1)}}$	0.1%	50.0%	1.902E-08	1.901E-08	-0.07%
$\frac{\partial^2 W(T)}{\partial R_1 \partial G_1^{(1,1)}}$	1.0%	10.0%	1.061E-05	1.059E-05	-0.20%
$\frac{\partial^2 W(T)}{\partial R_1^{(1,1)} \partial C_1}$	5.0%	5.0%	-1.660E+03	-1.659E+03	-0.08%
$\frac{\partial^2 W(T)}{\partial R_1^{(1,1)} \partial C_3^{(3,3)}}$	5.0%	1.0%	1.157E+02	1.157E+02	-0.01%
$\frac{\partial^2 W(T)}{\partial R_1^{(1,1)} \partial C_1^{(1,1)}}$	1.0%	10.0%	-3.094E+02	-3.139E+02	1.45%
$\frac{\partial^2 W(T)}{\partial G_1^{(1,1)} \partial C_3^{(3,3)}}$	10.0%	10.0%	7.771E+04	7.808E+04	0.47%

Chapter 7

Proposed second-order sensitivity for delay optimization

In this chapter, an important application of the second-order sensitivity analysis developed in this thesis for delay optimization in high-speed circuits is presented. It is to be noted that, due to the high-frequencies and rigid timing requirements, delay optimization in signal paths of modern high-speed VLSI and microwave circuits is becoming increasingly important.

Several sensitivity-based (gradient-based) delay optimization approaches can be found in literature [17], [18] for circuits containing transmission lines. Among them [17] uses the lumped approximation whereas [18] uses the moment matching model. It has been demonstrated in [17] that the second-order sensitivity is desirable in gradient-based delay optimization for the nonlinear optimization iterations.

In this chapter, a new equation for evaluating the second-order sensitivity of delay is proposed and validated with examples. Furthermore, the application of the second-order frequency-domain adjoint sensitivity analysis of MTL (Chapter 5) in evaluating the sensitivities of delay is introduced.

The rest of this chapter is organized as follows. Section 7.1 presents a formulation of the delay optimization equation, which requires sensitivities of a delay. Section 7.2 provides the proposed second-order sensitivity of a delay. Section 7.3 presents the proposed second-order sensitivity computation for linear circuits with embedded delays. Section 7.4 demonstrates the application of the proposed approach in terms of examples.

7.1 Formulation of the delay optimization

Consider the voltage $v_o(t)$ at a node of a circuit as

$$v_o(t) = \boldsymbol{\eta}^t \mathbf{x}(t) \tag{7.1}$$

where $\boldsymbol{\eta} = [\eta_{i,1}], \eta_{i,1} \in \{0, 1\}$ is a selector vector that selects a row from the MNA vector $\mathbf{x}(t)$. Further, consider a time delay T_d as a time point where the waveform, $v_o(t)$, crosses some arbitrary voltage level V_{T_d} such that

$$v_o(T_d) = \boldsymbol{\eta}^t \mathbf{x}(T_d) = V_{T_d} \tag{7.2}$$

It is to be noted that T_d is an implicit function of parameters ($\boldsymbol{\lambda}$) of the circuit and therefore, can be written as

$$T_d = T_d(\boldsymbol{\lambda}) \tag{7.3}$$

where $\boldsymbol{\lambda} \in \Re^{1 \times N_\lambda}$ is a vector of N_λ parameters of a circuit given by

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{N_\lambda} \end{bmatrix} \tag{7.4}$$

Let $\boldsymbol{\lambda}_o$ be the nominal values of a circuit. Then (7.3) can be extended using Taylor series around $\boldsymbol{\lambda}_o$ to give [17]

$$T_d(\boldsymbol{\lambda}) = T_d(\boldsymbol{\lambda}_o) + (\boldsymbol{\lambda} - \boldsymbol{\lambda}_o)\boldsymbol{\alpha}^t + \frac{1}{2}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_o)^t \boldsymbol{\Lambda}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_o) \tag{7.5}$$

where $\boldsymbol{\alpha}$ is a vector of first-order sensitivities and $\boldsymbol{\Lambda}$ is a Hessian matrix of second-order sensitivities of delay w.r.t. all the parameters of a circuit given by

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{\partial T_d}{\partial \lambda_1} & \frac{\partial T_d}{\partial \lambda_2} & \dots & \frac{\partial T_d}{\partial \lambda_{N_\lambda}} \end{bmatrix} \tag{7.6}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} \frac{\partial^2 T_d}{\partial \lambda_1^2} & \frac{\partial^2 T_d}{\partial \lambda_1 \partial \lambda_2} & \cdots & \frac{\partial^2 T_d}{\partial \lambda_1 \partial \lambda_{N_\lambda}} \\ \frac{\partial^2 T_d}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 T_d}{\partial \lambda_2^2} & \cdots & \frac{\partial^2 T_d}{\partial \lambda_2 \partial \lambda_{N_\lambda}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 T_d}{\partial \lambda_{N_\lambda} \partial \lambda_1} & \frac{\partial^2 T_d}{\partial \lambda_{N_\lambda} \partial \lambda_2} & \cdots & \frac{\partial^2 T_d}{\partial \lambda_{N_\lambda}^2} \end{bmatrix} \tag{7.7}$$

respectively. An example is shown in Appendix B to validate (7.5) using a simple RC circuit.

Details of evaluation techniques for the first ($\boldsymbol{\alpha}$) and second ($\mathbf{\Lambda}$) order sensitivities of a delay in (7.5) are given in the following section.

7.2 Proposed second-order sensitivity of a delay

Differentiating (7.2) w.r.t. a particular circuit parameter λ_m gives [17]

$$\frac{\partial v_o(T_d)}{\partial T_d}\frac{\partial T_d}{\partial \lambda_m} + \frac{\partial v_o(T_d)}{\partial \lambda_m} = 0 \tag{7.8}$$

where $m \in \{1, \cdots, N_\lambda\}$. Rearranging (7.8) yields

$$\frac{\partial T_d}{\partial \lambda_m} = -\frac{\left.\frac{\partial v_o(t)}{\partial \lambda_m}\right|_{t=T_d}}{\left.\frac{\partial v_o(t)}{\partial t}\right|_{t=T_d}} \tag{7.9}$$

where $\frac{\partial v_o(t)}{\partial \lambda_m}$ and $\frac{\partial v_o(t)}{\partial t}$ are the sensitivity of the voltage $v_o(t)$ w.r.t. λ_m and time t, respectively. Also, $\frac{\partial v_o(T_d)}{\partial \lambda_m}$ and $\frac{\partial v_o(T_d)}{\partial T_d}$ in (7.8) are the values of the $\frac{\partial v_o(t)}{\partial \lambda_m}$ and $\frac{\partial v_o(t)}{\partial t}$ at time $t = T_d$, respectively. Using (7.9) in (7.6), the first-order sensitivity of a delay w.r.t. all the N_λ parameters ($\boldsymbol{\alpha}$) can be found.

Next, differentiating (7.8) w.r.t. a particular circuit parameter λ_n gives

$$\begin{aligned}
&\frac{\partial^2 v_o(T_d)}{\partial T_d^2}\frac{\partial T_d}{\partial \lambda_n}\frac{\partial T_d}{\partial \lambda_m} + \frac{\partial^2 v_o(T_d)}{\partial T_d \partial \lambda_n}\frac{\partial T_d}{\partial \lambda_m} + \\
&\frac{\partial v_o(T_d)}{\partial T_d}\frac{\partial^2 T_d}{\partial \lambda_m \partial \lambda_n} + \frac{\partial^2 v_o(T_d)}{\partial \lambda_m \partial T_d}\frac{\partial T_d}{\partial \lambda_n} + \frac{\partial^2 v_o(T_d)}{\partial \lambda_m \partial \lambda_n} = 0
\end{aligned} \tag{7.10}$$

where $n \in \{1, \cdots, N_\lambda\}$. Rearranging (7.10) yields

$$\frac{\partial^2 T_d}{\partial \lambda_m \partial \lambda_n} = -\frac{\left(\begin{array}{c} \left.\frac{\partial^2 v_o(t)}{\partial t^2}\right|_{t=T_d} \frac{\partial T_d}{\partial \lambda_n}\frac{\partial T_d}{\partial \lambda_m} + \left.\frac{\partial^2 v_o(t)}{\partial t \partial \lambda_n}\right|_{t=T_d} \frac{\partial T_d}{\partial \lambda_m} \\ + \left.\frac{\partial^2 v_o(t)}{\partial \lambda_m \partial t}\right|_{t=T_d} \frac{\partial T_d}{\partial \lambda_n} + \left.\frac{\partial^2 v_o(t)}{\partial \lambda_m \partial \lambda_n}\right|_{t=T_d} \end{array}\right)}{\left.\frac{\partial v_o(t)}{\partial t}\right|_{t=T_d}} \tag{7.11}$$

Using (7.11) in (7.7), the second-order sensitivity of a delay w.r.t. all the N_λ parameters ($\boldsymbol{\Lambda}$) can be found. It is to be noted that the formulation in (7.11) is the proposed second-order sensitivity of a delay, which is an extension of (28) given in [17]. An example is shown in Appendix C to validate (7.9) and proposed (7.11) using a simple RC circuit.

7.3 Proposed second-order sensitivity computation for linear circuits with embedded delays

Consider a general circuit consisting of [3] linear components and distributed transmission lines. The corresponding modified nodal analysis (MNA) equations can be written in the time-domain as

$$\mathbf{C}\frac{d\mathbf{x}(t)}{dt} + \mathbf{G}\mathbf{x}(t) + \sum_{k=1}^{N_t} \mathbf{a}_k * \mathbf{x}(t) = \mathbf{b}(t) \tag{7.12}$$

where

$$\mathbf{a}_k(t) = \mathbf{D}_k \mathbf{y}_k(t) \mathbf{D}_k^t \tag{7.13}$$

Assuming the initial condition of (7.12) as zero $\mathbf{b}(0) = 0$. Taking the Laplace transform of (7.12) and (7.13) gives

$$\left[s\mathbf{C} + \mathbf{G} + \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{Y}_k(s) \mathbf{D}_k^t \right] \mathbf{X}(s) = \mathbf{B}(s) \tag{7.14}$$

where $\mathbf{B}(s)$ is the Laplace-domain representation of $\mathbf{b}(t)$.

Next, consider the voltage at a node as an objective function given by

$$V_o(s) = \boldsymbol{\eta}^t \mathbf{X}(s) \tag{7.15}$$

where $\boldsymbol{\eta} = [\eta_{i,1}], \eta_{i,1} \in \{0, 1\}$ is a selector vector, which selects a row from $\mathbf{X}(s)$. Then, using the steps given in Section 5.2.5, the first and second-order sensitivities of $V_o(s)$ w.r.t. λ_m and λ_n can be found.

Next, the first (7.8) and second-order (7.11) sensitivities of delay (T_d) can be found based on a time-domain simulation. However, for linear circuits they can also be found based on a frequency-domain simulation. Rewriting (7.8) gives

$$\frac{\partial T_d}{\partial \lambda_m} = -\frac{\kappa(T_d)}{\sigma(T_d)} \tag{7.16}$$

where

$$\kappa(t) = \mathcal{L}^{-1}\left\{\frac{\partial V_o(s)}{\partial \lambda_m}\right\} \tag{7.17}$$

$$\sigma(t) = \mathcal{L}^{-1}\left\{sV_o(s)\right\} \tag{7.18}$$

and $\mathcal{L}^{-1}\{\}$ is the inverse Laplace transform operator. Similarly, rewriting (7.10) yields

$$\frac{\partial^2 T_d}{\partial \lambda_m \partial \lambda_n} = -\frac{\rho(T_d)}{\sigma(T_d)} \tag{7.19}$$

where

$$\rho(t) = \mathcal{L}^{-1}\left\{\begin{array}{c} s^2 V_o(s)\frac{\partial T_d}{\partial \lambda_n}\frac{\partial T_d}{\partial \lambda_m} + s\frac{\partial V_o(s)}{\partial \lambda_n}\frac{\partial T_d}{\partial \lambda_m} \\ +s\frac{\partial V_o(s)}{\partial \lambda_m}\frac{\partial T_d}{\partial \lambda_n} + \frac{\partial^2 V_o(s)}{\partial \lambda_m \partial \lambda_n} \end{array}\right\} \tag{7.20}$$

It is to be noted that either the numerical Laplace transform inversion [21] or the IFFT technique can be used to evaluate $\mathcal{L}^{-1}\{\}$.

Next, the first-order sensitivity of delay w.r.t. all the parameters (α) can be found using Algorithm 5.1 and (7.16). Similarly, the second-order sensitivity of delay w.r.t. all the parameters (Λ) can be found using Algorithm 5.2 and (7.19).

The next section presents various examples to validate the proposed second-order sensitivity of delay in time-domain.

7.4 Numerical examples

In this section, three examples are presented to demonstrate the validity and accuracy of the proposed method. In Examples 1, the accuracy of the second-order sensitivities that are numerically computed using the proposed algorithm are validated against the analytically computed sensitivities, whereas in Examples 2 and 3, they are validated against the perturbation method.

7.4.1 Example 1

In this experiment, the accuracy of the second-order sensitivities that are numerically computed using the proposed algorithm are validated against the analytically computed sensitivities. For this purpose, consider a RC circuit shown in Fig. 7.1 with the input (step) voltage of 1V. The output voltage is given by v_o. The waveform of v_o crosses 0.5 v at time 0.7×10^{-4} sec as shown in Fig. 7.2.

Since this is a simple circuit, the first and second-order sensitivities of the output voltage in frequency-domain are found in closed-form. Then, the sensitivities of delay are found by using these first and second-order sensitivities in (7.16) and (7.19). Since this is a simple circuit, the exact values of sensitivities of the delay w.r.t. R and C can be found easily as given in (A.10), (A.11) and (A.12). Table 7.1 compares the values of sensitivities between the exact and proposed method.

Table 7.1: First-order delay sensitivity using exact and the proposed method (Example 1)

Sensitivity	Proposed Method	Exact	Relative Difference
$\frac{\partial T_d}{\partial R}$	7.018E-07	6.931E-07	1.23%
$\frac{\partial T_d}{\partial C}$	70.175	69.315	1.23%
$\frac{\partial^2 T_d}{\partial R \partial C}$	0.704	0.693	1.56%

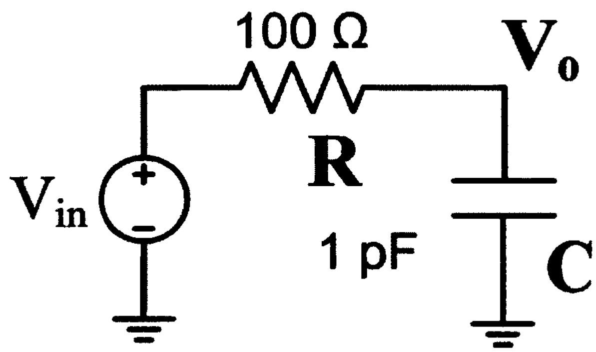

Figure 7.1: RC Circuit (Example 1 and 2)

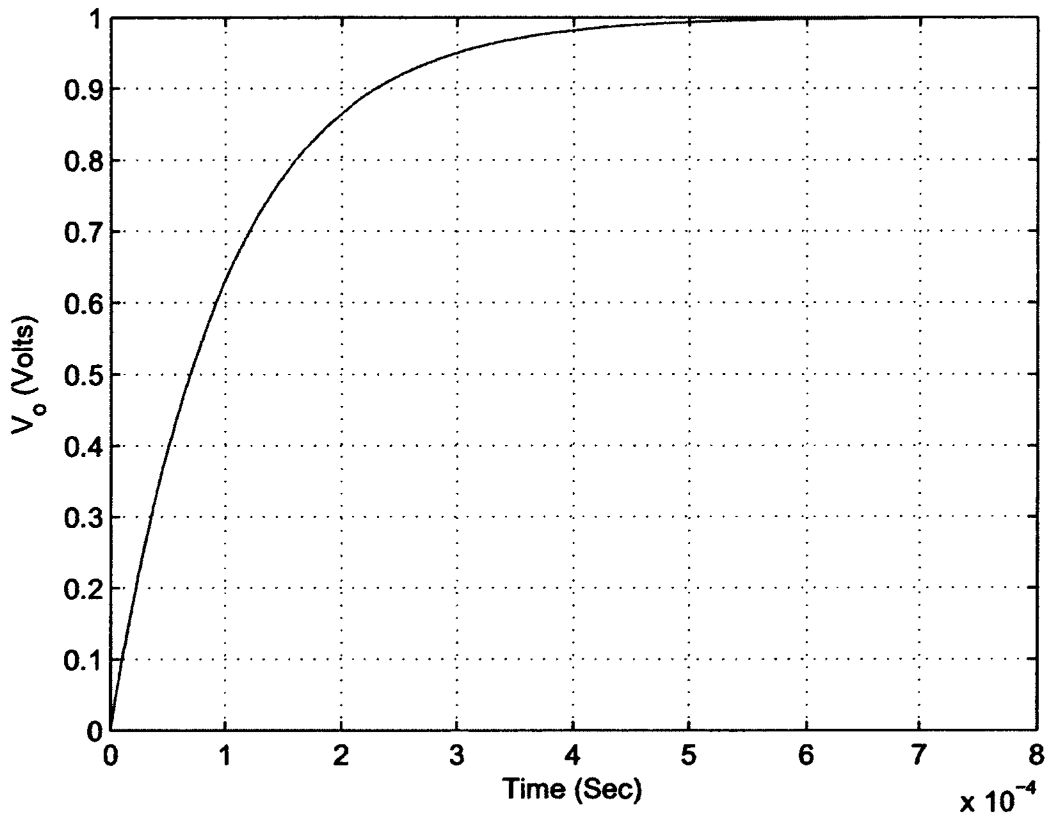

Figure 7.2: Transient response of the circuit shown in Fig. 7.1 at node v_o (Example 1)

7.4.2 Example 2

In this experiment, the accuracy of the second-order sensitivities that are numerically computed using the proposed algorithm are validated against the conventional perturbation method. For this purpose, consider a RC circuit shown in Fig. 7.1 with a trapezoidal input pulse of amplitude of 1 v, rise and fall times of 100 μsec, and a pulse width of 200 μsec. The output voltage is given by v_o. The waveform of v_o crosses 0.5 v at time 0.124 $msec$ as shown in Fig. 7.3.

The first and second-order sensitivities of the output voltage in the frequency-domain are found by using the steps given in Section 5.2.5. Then the sensitivities of delay are found by using these first and second-order sensitivities in (7.16) and (7.19) of the proposed method. Table 7.2 shows the first-order sensitivities using the proposed method and perturbation technique (10% each) in (2.7).

Table 7.2: First-order delay sensitivity using perturbation and the proposed method (Example 2)

Sensitivity	Perturbation	Proposed Method	Relative Difference
$\frac{\partial T_d}{\partial R}$	6.500E-07	6.519E-07	0.29%
$\frac{\partial T_d}{\partial C}$	65.000	65.190	0.29%

Table 7.3 shows the second-order sensitivity of delay w.r.t. R and C using the purposed method and perturbation technique (2.8).

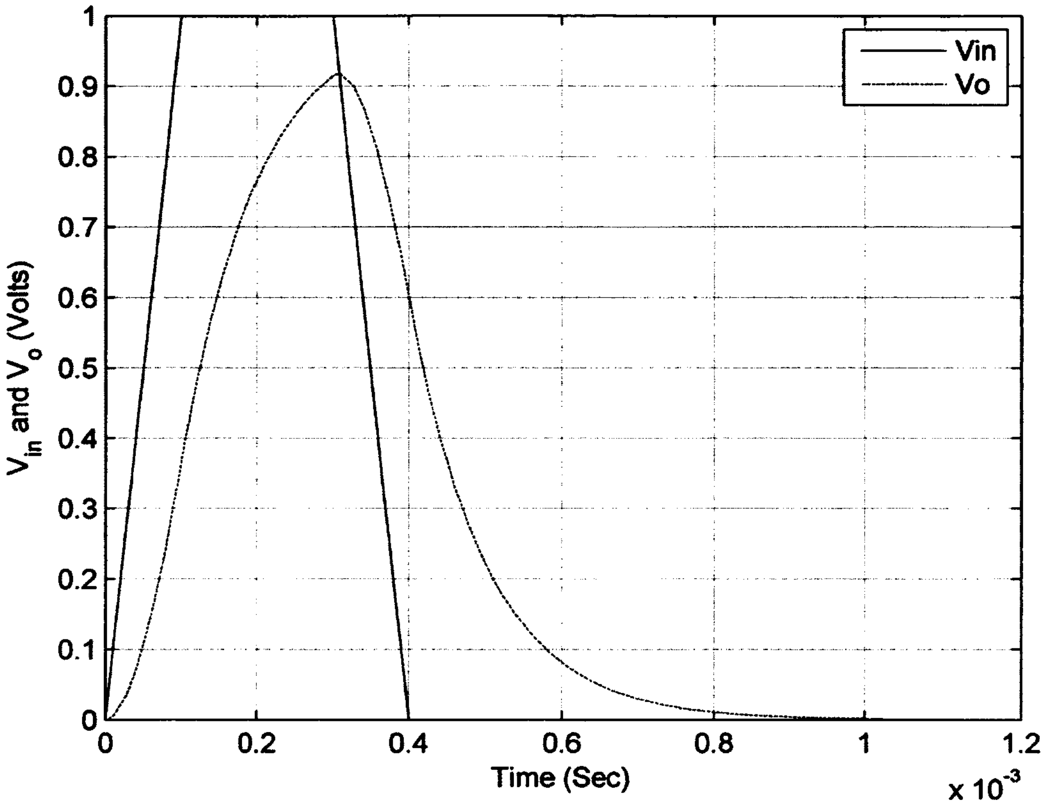

Figure 7.3: Transient response of the circuit shown in Fig. 7.1 at node v_o (Example 2)

7.4.3 Example 3

In this experiment, the accuracy of the second-order sensitivities that are numerically computed using the proposed algorithm are validated against the conventional perturbation method. The circuit considered in this experiment is similar to the circuit given in Fig. 5.7 [11] of Section 5.3.3. In this example, the voltage source is a trapezoidal pulse with an amplitude of 2 v, rise and fall time of 1 $nsec$ and a pulse width

Table 7.3: Second-order delay sensitivity using perturbation and the proposed method

Sensitivity	% Change in Parameter R	% Change in Parameter C	Perturbation Method	Proposed Method	Relative Difference
$\frac{\partial^2 T_d}{\partial R \partial C}$	50%	10%	0.700	0.694	-0.82%

of 7 *nsec*. The output voltage is given by v_o. The waveform of v_o crosses 87 *mv* at time 30.8 *nsec* as shown in Fig. 7.4.

Tables 7.4 and 7.5 show the values of the first and second-order sensitivities, respectively, using the proposed method and perturbation technique. The percentage change used in Table 7.4 for all the parameters is 50%.

Table 7.4: First-order delay sensitivity using perturbation and the proposed method (Example 3)

Sensitivity	Perturbation Method	Proposed Method	Relative Difference
$\frac{\partial T_d}{\partial R_1}$	5.013E-12	5.066E-12	1.03%
$\frac{\partial T_d}{\partial C_1}$	13.700	13.710	0.07%
$\frac{\partial T_d}{\partial R_1^{(1,1)}}$	2.000E-12	2.117E-12	5.52%
$\frac{\partial T_d}{\partial G_1^{(1,1)}}$	1.340E-09	1.369E-09	2.13%

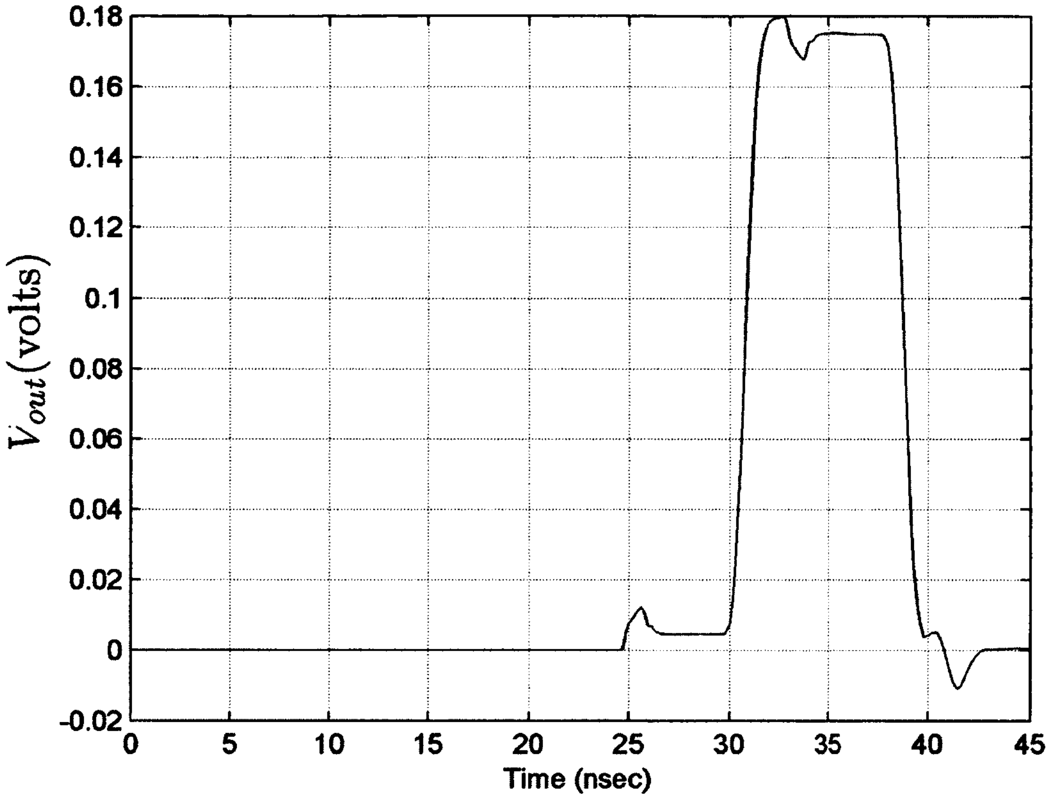

Figure 7.4: Transient response of the circuit shown in Fig. 5.7 at node V_{out} (Example 3)

Table 7.5: Second-order delay sensitivity using perturbation and the proposed method

Sensitivity	% Change in Parameter λ_1	% Change in Parameter λ_2	Perturbation Method	Proposed Method	Relative Difference
$\frac{\partial^2 T_d}{\partial R_1 \partial C_1}$	10%	50%	-3.333E-02	-3.220E-02	-3.52%
$\frac{\partial^2 T_d}{\partial R_1^{(1,1)} \partial L_1^{(1,1)}}$	10%	10%	-4.167E-06	-4.992E-06	16.53%
$\frac{\partial^2 T_d}{\partial R_1 \partial G_1^{(1,1)}}$	50%	10%	1.333E-11	1.333E-11	-0.03%
$\frac{\partial^2 T_d}{\partial R_1 \partial R_1^{(1,1)}}$	50%	50%	2.133E-14	2.092E-14	-1.96%

Chapter 8

Conclusions and future work

8.1 Conclusions

A generalized approach for first and second-order time-domain adjoint sensitivity analysis of lossy distributed MTLs in the presence of nonlinear terminations was developed. The method is based on the variational approach and enables sensitivity analysis of interconnect structures w.r.t. both electrical and physical parameters while providing significant computational cost advantages. While the new approach provides all the advantages of an adjoint sensitivity analysis, its formulation is independent of the specifics of the MTL macromodel used. This makes the proposed method applicable to a wide variety of macromodels available in the literature. The proposed approach can also be used with frequency-dependent per-unit length parameters.

Moreover, applications such as delay optimization are presented for the second-order sensitivity analysis algorithms developed in this thesis. An efficient approach to evaluate the second-order sensitivity of delay is also described.

8.2 Future work

Several suggestions for further research based on the results of this thesis are described below.

1. **Second-order sensitivity of delay in nonlinear circuits:** An efficient way is required to evaluate the terms $\frac{\partial^2 v_o(t)}{\partial t^2}$, $\frac{\partial^2 v_o(t)}{\partial t \partial \lambda_n}$ and $\frac{\partial^2 v_o(t)}{\partial \lambda_m \partial t}$ in (7.11) for a nonlinear circuit.
2. **Sensitivity of S-parameters:** The sensitivity equations ((4.35) and (5.15)) are based on the derivative of an admittance matrix. An extension of sensitivity equation that can handle derivative of S-parameter matrix would be of significant interest.
3. **Nonuniform interconnects:** Sensitivity analysis of non-uniform interconnects is an important area of interest [38] [39]. The methods described in this thesis assume uniform interconnects. An extension of these new methods to handle nonuniform lines (where the p.u.l parameter vary with the

spatial variable x) would be of significant interest.

Appendix A

Closed-form equations for RC circuit

Consider a RC circuit shown in Fig. A.1 with input (step) voltage of 1V. The output voltage v_o is given by

$$v_o = 1 - e^{\frac{-t}{RC}} \tag{A.1}$$

First and second-derivatives of (A.1) w.r.t. t are

$$\frac{\partial v_o}{\partial t} = \frac{e^{\frac{-t}{RC}}}{RC} \tag{A.2}$$

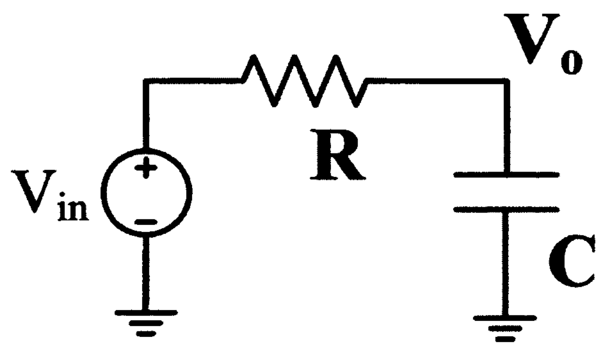

Figure A.1: RC Circuit

$$\frac{\partial^2 v_o}{\partial t^2} = -\frac{e^{\frac{-t}{RC}}}{R^2C^2} \tag{A.3}$$

First and second-derivatives of (A.1) w.r.t. R and C gives

$$\frac{\partial v_o}{\partial R} = -e^{\frac{-t}{RC}}\frac{t}{R^2C} \tag{A.4}$$

$$\frac{\partial v_o}{\partial C} = -e^{\frac{-t}{RC}}\frac{t}{RC^2} \tag{A.5}$$

$$\frac{\partial^2 v_o}{\partial R \partial C} = e^{\frac{-t}{RC}}\frac{t}{R^2C^2} - e^{\frac{-t}{RC}}\frac{t^2}{R^3C^3} \tag{A.6}$$

Differentiating (A.4) and (A.5) w.r.t. t yields

$$\frac{\partial^2 v_o}{\partial R \partial t} = e^{\frac{-t}{RC}}\frac{t}{R^3C^2} - e^{\frac{-t}{RC}}\frac{1}{R^2C} \tag{A.7}$$

$$\frac{\partial^2 v_o}{\partial C \partial t} = e^{\frac{-t}{RC}} \frac{t}{R^2 C^3} - e^{\frac{-t}{RC}} \frac{1}{RC^2} \tag{A.8}$$

Let a delay T_d be defined as the time t at which output voltage v_o is 0.5V. Substituting $v_o = 0.5V$ in (A.1) and rearranging yields

$$T_d = -RC(ln(0.5)) \tag{A.9}$$

Differentiating (A.9) w.r.t. R and C yields

$$\frac{\partial T_d}{\partial R} = -C(ln(0.5)) \tag{A.10}$$

$$\frac{\partial T_d}{\partial C} = -R(ln(0.5)) \tag{A.11}$$

$$\frac{\partial^2 T_d}{\partial R \partial C} = -(ln(0.5)) \tag{A.12}$$

$$\frac{\partial^2 T_d}{\partial R^2} = 0 \tag{A.13}$$

$$\frac{\partial^2 T_d}{\partial C^2} = 0 \tag{A.14}$$

Appendix B

Validation of Taylor series expansion of delay

Consider a RC circuit shown in Fig. A.1 with input (step) voltage of 1V. The parameters $\boldsymbol{\lambda}$ of RC circuit are

$$\boldsymbol{\lambda} = \begin{bmatrix} R & C \end{bmatrix} \tag{B.1}$$

The nominal values of RC circuit are given as

$$\boldsymbol{\lambda_o} = \begin{bmatrix} 100 & 1e^{-6} \end{bmatrix} \tag{B.2}$$

Using (A.10) to (A.14), the first derivative vector $\boldsymbol{\alpha}$ and Hessian matrix $\boldsymbol{\Lambda}$ becomes

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{\partial T_d}{\partial R} & \frac{\partial T_d}{\partial C} \end{bmatrix} = \begin{bmatrix} -1e^{-6}(ln(0.5)) & -100(ln(0.5)) \end{bmatrix} \tag{B.3}$$

and

$$\boldsymbol{\Lambda} = \begin{bmatrix} \frac{\partial^2 T_d}{\partial R^2} & \frac{\partial^2 T_d}{\partial R \partial C} \\ \frac{\partial^2 T_d}{\partial C \partial R} & \frac{\partial^2 T_d}{\partial C^2} \end{bmatrix} = \begin{bmatrix} 0 & -ln(0.5) \\ -ln(0.5) & 0 \end{bmatrix} \tag{B.4}$$

Substituting (B.3) in the first-order expansion of (7.5) gives

$$T_d^1(\boldsymbol{\lambda}) = 69.3\mu - (R - 100)1\mu(ln(0.5)) - (C - 1\mu)100(ln(0.5)) \tag{B.5}$$

Substituting (B.3) and (B.4) in the second-order expansion of (7.5) gives

$$\begin{aligned} T_d^2(\boldsymbol{\lambda}) =& 69.3\mu - (R - 100)1\mu(ln(0.5)) - (C - 1\mu)100(ln(0.5)) \\ & - (R - 100)(C - 1\mu)(ln(0.5)) \end{aligned} \tag{B.6}$$

The first $T_d^1(\boldsymbol{\lambda})$ and second $T_d^2(\boldsymbol{\lambda})$ order Taylor series expansions of delay $T_d(R, C)$ given in (B.5) and (B.6) are compared with exact delay given in (A.9) with different combinations of R and C. The comparison is shown in Table B.1. Note that the second-order Taylor series expansion of (A.9) captures the whole behavior. This is expected because the delay in (A.9) is only the product of R and C.

Table B.1: Comparison between exact delay and the Taylor series expansion

R	C	Exact Delay	First-Order Delay (T_d^1)	Rel. Diff. in (T_d^1)	2^{nd}-Order Delay (T_d^2)	Rel. Diff. in (T_d^2)
101	1.01μ	7.071E-05	7.070E-05	0.01%	7.071E-05	0.00%
110	1.1μ	8.387E-05	8.318E-05	0.83%	8.387E-05	0.00%
99	0.99μ	6.794E-05	6.793E-05	0.01%	6.794E-05	0.00%
90	0.9μ	5.614E-05	5.545E-05	1.23%	5.614E-05	0.00%

In this section, it has been shown that the second-order Taylor series expansion of delay (7.5) is valid.

Appendix C

Validation of second-order sensitivity of delay

Consider a RC circuit shown in Fig. A.1 with input (step) voltage of 1V. The output voltage $v_o(t)$ is given by (A.1). For a simple RC circuit, (A.9) to (A.14) allows us to find exact delay time and its first and second-order sensitivities.

Next, (7.9) allows us to find the first-order sensitivity of delay w.r.t. R. Using (A.2) and (A.4) at $t = T_d$ in (7.9) gives

$$\frac{\partial T_d}{\partial R} = -C(ln(0.5)), \tag{C.1}$$

which is same as (A.10).

Similarly, (7.11) allows us to find the second-order sensitivity w.r.t. R and C.

Using (A.3), (A.6), (A.7), (A.8) at $t = T_d$ with (A.10) and (A.11) in (7.11) gives

$$\frac{\partial^2 T_d}{\partial R \partial C} = -(ln(0.5)), \tag{C.2}$$

which is same as (A.12).

In this section, it has been shown that the equation of second-order sensitivity of delay (7.11) is valid.

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