Compressed Sensing of Block Sparse Signals with Application to Wideband Spectrum Sensing

by

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A thesis submitted to the
Faculty of Graduate and Postdoctoral Affairs
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Electrical and Computer Engineering

Ottawa-Carleton Institute for Electrical and Computer Engineering
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Carleton University
Ottawa, Ontario
August, 2015

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acceptance of the thesis

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Doctor of Philosophy in Electrical and Computer Engineering

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August, 2015

Abstract

Wideband spectrum sensing (WSS) is a significant challenge in cognitive radios (CRs) due to requirement of very high-speed analog-to-digital converters (ADCs), operating at or above the Nyquist rate. To alleviate the need for high sampling rates, compressed sensing can be applied to WSS. Compressed sensing is a signal processing technique to recover high dimensional signals from a fewer measurements than conventional sampling theory demands. In compressed sensing, the superior algorithms are the ones that can recover the signal with fewer measurements.

In many applications of compressed sensing such as WSS, the signal has a block sparse structure which can be used to reduce the required sampling rate. Here, we propose a family of iterative algorithms for the recovery of block sparse signals, referred to as *iterative reweighted ℓ_2/ℓ_1 minimization* algorithms (IR- ℓ_2/ℓ_1). As an example, we apply the proposed algorithms to WSS. Our simulation and analytical results on the recovery of both ideally and approximately block sparse signals show that the proposed iterative algorithms have significant advantages in terms of accuracy and the number of required measurements over the existing recovery methods at a small cost in computational complexity.

The approximate message passing (AMP) algorithm of Donoho *et al.* has attracted much attention due to its remarkable performance/complexity trade-off. In this thesis, we also propose a weighting/reweighting scheme to improve the performance of AMP algorithm for the recovery of block sparse signals with known borders. For the case of unknown borders, we propose an iterative algorithm which combines a border detection scheme with a recovery algorithm for block sparse signals with known borders. Simulation results, both in noiseless and noisy scenarios, show a considerably better performance/complexity trade-off compared to other state-of-the-art recovery algorithms.

Although the recovery methods can be applied to WSS for the spectrum reconstruction to detect the spectrum holes, to further reduce the sampling rate, one can

find a sufficiently large fraction of spectrum holes without spectrum reconstruction. Here, we first propose a very low-complexity zero-block detection scheme that can detect a large fraction of spectrum holes from the sub-Nyquist samples, even when the undersampling ratio is very small. We then propose an iterative low-complexity scheme for the reliable detection of zero blocks in a block sparse signal. This scheme is based on the application of verification-based (VB) recovery algorithms in compressed sensing to block sparse signals. To apply both schemes to WSS, we devise a block sparse sensing matrix by designing a novel analog-to-information converter (AIC). The AIC, the sensing matrix and the VB algorithms can be optimized such that the largest number of zero blocks for a given number of measurements can be detected. These works introduce a new paradigm in the recovery of block sparse signals, where one is interested in partial detection of the complement of the support set, reliably, rather than the full recovery of the signal or its support.

Acknowledgments

Firstly, I would like to express my appreciation and sincere gratitude to my advisor Prof. Amir H. Banihashemi for his motivation, patience, continuous support, and immense knowledge. His vision and creativeness has inspired this research. His guidance helped me in all the time of research and writing of this thesis.

Besides my advisor, I would like to thank the rest of my committee members: Prof. Shahram Shirani, Prof. Michel Barbeau, Prof. Martin Bouchard, and Dr. Sreeraman Rajan, for their comments and feedback on this thesis.

Also I thank all my friends in our research group, particularly, Yasser, Neda, and Anoosheh. Thank you all.

I would like to thank my beloved parents and my sisters, Fahimeh and Hadiseh, for their endless love, continuous encouragement and constant support throughout my life and my studies. This accomplishment would not have been possible without them. I would like to dedicate this dissertation to my wonderful family for their love, support, and self-sacrifices.

Finally my deepest gratitude goes to my best friend, Hoda, for her endless encouragement and support. Besides our friendship, she is also my collaborator, my colleague, and my best classmate. Every time I have an idea, Hoda is the first one to review it and give valuable suggestions and insightful feedback. She is always there cheering me up and stood by me through the good times and bad.

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Nomenclature

ADC	Analog To Digital Converter
AIC	Analog To Information Converter
AMP	Approximate Message-Passing
AMP-JS	AMP With James-Stein's Estimator
AMP-ST	AMP with Soft Thresholding function
AMP-BST	AMP with Block Soft Thresholding function
BMI	Branches of Mixers and Integrators
CR	Cognitive Radio
CS	Compressed Sensing
D1M	Degree One Measurement
EC	Edge Correction
ED	Edge Detection
EM	Equal Measurement
GB	Guard Band
i.i.d.	Independent and identically distributed
IRLS	Iterative Reweighted Least Squares
$IR\ell_1$	Iterative Reweighted ℓ_1 Minimization
$IR\text{-}\ell_2/\ell_1$	Iterative Reweighted ℓ_2/ℓ_1 Minimization
ISD	Iterative Support Detection

LASSO	Least Absolute Shrinkage and Selection Operator
LP	Low Pass
MN	Measurement Node
NB-VB	Node-Based Verification-Based
PC-SBL	Pattern-Coupled Sparse Bayesian Learning
PU	Primary User
RIP	Restricted Isometry Property
RAMP-WST	Reweighted AMP with Weighted Soft Thresholding function
RAMP-WBST	Reweighted AMP with Weighted Block Soft Thresholding function
SNR	Signal to Noise Ratio
SU	Secondary User
VB	Verification Based
VN	Variable Node
WSS	Wideband Spectrum Sensing
ZM	Zero Measurement
ZMD	Zero Measurement Detection

Notational Conventions

Notation	Description
N	dimension of the signal
M	number of measurements
α	sparsity ratio
K	number of non-zero elements
L	number of blocks
P	number of non-zero blocks
$\mathbf{u}_i$	the i -th sub-vector of block sparse signal
B_i	the length of the i -th block
B	the length of the blocks when all the blocks have the same length
$\mathbf{n}$	measurement noise vector
$\mathbf{A}$	sensing matrix
Ψ	basis matrix
Φ	measurement matrix
$\mathbf{F}$	unitary fourier matrix
$\mathbf{d}$	binary spectrum occupancy
σ_n	noise standard deviation
S^c	complement of set S
ε	bounds the ℓ_2 norm of the noise in the measurements

Notation	Description
$\Re(.)$	real part of a complex value
$\Im(.)$	imaginary part of a complex value
$\ \mathbf{x}\ _p$	ℓ_p norm of vector $\mathbf{x}$ defined as $\left(\sum_{i=1}^N x_i ^p\right)^{1/p}$
$\ \mathbf{x}\ _0$	ℓ_0 norm of vector $\mathbf{x}$. Number of non-zero elements of $\mathbf{x}$
$\mathcal{N}(\mathbf{A})$	null space of matrix $\mathbf{A}$
$\mathbf{A}^T$	transpose of matrix $\mathbf{A}$
$\mathbb{C}^N$	vector spaced of complex valued N dimensional vectors
$\mathbb{R}^N$	vector spaced of real valued N dimensional vectors
$\zeta(.,.)$	the thresholding function used in AMP algorithms. The argument(s) after “;” is(are) its parameter(s)
$(a)_+$	a if $a > 0$ and zero otherwise
$\mathcal{N}(.,.)$	Gaussian distribution. First and second arguments are mean and standard deviation, respectively
div	divergence operator
dim(.)	dimension of a vector
Supp($\mathbf{u}_{GB}$)	the support set of the GB
P_D	Probability of detection
P_{FA}	probability of false alarm
$P_D^{(zm)}$	Probability of detection for zero measurements
$P_{FA}^{(zm)}$	probability of false alarm for zero measurements
P_{ZD}	probability of zero block detection
P_{WAD}	probability of wrong zero block detection

Notation	Description
$E(.)$	expected value of a random variable
$P(.)$	probability of an event
$\mathbf{A}_m$	the m -th row of matrix $\mathbf{A}$
$\mathbf{A}_m^l$	the l -th block of the m -th row of matrix $\mathbf{A}$
$^+\mathbf{x}$	positive frequency part of $\mathbf{x}$
$\check{a}$	indicate the updated version of variable a
$\boldsymbol{\xi}$	mean square error vector corresponding to the measurement vector
$\mathcal{M}$	set of MNs
$\mathcal{V}$	set of VNs
$\mathcal{M}_z$	set of zero MNs
$\mathcal{V}_z$	set of zero VNs
$\mathcal{M}_{nz}$	set of non-zero MNs
$\mathcal{V}_{nz}$	set of non-zero VNs
$\mathcal{M}(v)$	the set of MNs neighboring to the VN v
$\mathcal{V}(m)$	the set of VNs neighboring to the MN m
$\mathcal{V}(a, b)$	the set of common VNs neighboring both a and b
$\mathcal{E}_{S_1, S_2}$	the edge set which connects the set $S_1 \subseteq \mathcal{V}$ to the set $S_2 \subseteq \mathcal{M}$
$\lambda(x)$	degree distribution of VNs
$\rho(x)$	degree distribution of MNs
$d(m)$	degree of node m
d_M	degree of all MNs
d_V	degree of all VNs

Chapter 1

Introduction

1.1 Compressed Sensing

Compressed sensing (CS) is a signal processing technique that can recover an unknown signal with high probability from a small set of linear projections, called *measurements*, conditioned on the sparsity of the signal in some domain (see [1–3], for details).

Let $\bar{\mathbf{r}} \in \mathbb{C}^N$ denote an N dimensional complex signal, which is sparse in domain Ψ , where Ψ is an $N \times N$ *basis matrix*. In particular, we assume that representation $\bar{\mathbf{x}}$ of $\bar{\mathbf{r}}$ in domain Ψ is a K -sparse signal, i.e., $\bar{\mathbf{x}} \in \mathbb{C}^N$ has only K non-zero elements ($K \ll N$). Further assume that Φ is an $M \times N$ random *measurement matrix* with complex elements in general. For instance, let the entries of Φ be independent and identically distributed (i.i.d.) real random variables from a zero-mean Gaussian distribution with variance $1/N$. Mathematically, the M -dimensional measurement vector $\mathbf{y}$ is given by $\mathbf{y} = \Phi \bar{\mathbf{r}} = \Phi \Psi \bar{\mathbf{x}}$. In compressed sensing, the problem is to find the sparsest vector $\hat{\mathbf{x}}$ of dimension N given $\mathbf{y}$, i.e.,

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{s.t. } \mathbf{y} = \mathbf{A} \mathbf{x}, \quad (1.1)$$

where $\mathbf{A} = \Phi \Psi$ is called the *sensing matrix*.¹ Note that $\bar{\mathbf{x}}$ denotes the true signal,

¹In (1.1) and in the rest of the thesis, we use the ℓ_p -norm with the definition

$$\|\mathbf{x}\|_p = \begin{cases} |\{i : x_i \neq 0, i = 1, \dots, N\}| & p = 0 \\ (\sum_{i=1}^N |x_i|^p)^{1/p} & 0 < p < \infty \\ \max_{i=1, \dots, N} |x_i| & p = \infty. \end{cases}$$

while $\hat{\mathbf{x}}$ and $\mathbf{x}$ denote the estimated signal, which is the solution of the optimization problem, and the variable in the optimization problem, respectively. It is well-known that in the setting of Problem (1.1), the K -sparse signal $\bar{\mathbf{x}}$ can be recovered with high probability, using only $M = K + 1$ measurements if Φ is randomly generated independently of $\bar{\mathbf{x}}$ [4]. In other words, the solution of ℓ_0 -norm minimization (ℓ_0 minimization, in brief) is, with high probability, the true signal, i.e., $\hat{\mathbf{x}} = \bar{\mathbf{x}}$. The problem in (1.1), however, is NP-hard [5].

1.1.1 Convex Optimization

Fortunately, the ℓ_0 minimization problem in (1.1) can be relaxed to an ℓ_1 minimization problem:

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}, \quad (1.2)$$

that can be solved in polynomial time using linear programming techniques and yet can provide the same solution as in (1.1) under certain conditions [1–3, 5].

In the presence of noise, Problem (1.2) can be written as

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t. } \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \varepsilon, \quad (1.3)$$

where ε bounds the ℓ_2 -norm of the noise in the measurements. If the measurements are corrupted by a zero-mean Gaussian noise with variance σ_n^2 , a likely upper bound on the ℓ_2 -norm of the noise is $\varepsilon = \sigma_n \sqrt{M + 2\sqrt{2M}}$ (details can be found in [6], Section 3.3). Problem (1.3) is referred to as LASSO [7] and is a convex problem (a second order cone programming problem [8]) that can be solved efficiently in polynomial time, using convex programming tools [9], [10].

It is proved that when matrix $\mathbf{A}$ is a Gaussian random matrix, the solution of Problem (1.2), $\hat{\mathbf{x}}$, coincides with the true signal $\bar{\mathbf{x}}$ (solution of (1.1)), if $M = O(K \log(N/K))$ [1–3, 5], which is much smaller than N , but can still be a large number compared to K .

There are a number of iterative methods such as *iterative reweighted least squares* (IRLS) [11], [12], *iterative reweighted ℓ_1 minimization* ($\text{IR}\ell_1$) [6], [13] and *iterative support detection* (ISD) [14] that in many cases, can recover the sparse signal with far fewer measurements than the standard ℓ_1 minimization for both noisy and noiseless cases [15].

In a reconstruction procedure, one can also reduce the number of required measurements by employing a priori information on the support region of the sparse signal [16]. In [12], it is shown that a priori information can be used by the IRLS algorithm to improve the reconstruction performance in terms of the number of required measurements, the number of iterations and the convergence time. Zhang and Qiu [17] incorporated a priori information in the orthogonal matching pursuit algorithm in a spectrum sensing application to improve the sensing performance.

1.1.2 Approximate Message Passing Algorithms

A popular category of recovery algorithms for compressed sensing are those based on *message-passing algorithms*, see, e.g., [18–29]. In particular, the approximate message-passing (AMP) algorithm of [26] has attracted much attention due to its remarkable performance/complexity trade-off.

In [30], a (re)weighting scheme was proposed for AMP. This scheme which is based on an estimate for the support set of the signal, is devised to solve a weighted ℓ_1 problem, where the weights are set to $\omega < 1$ for the signal elements in the support set and to 1, otherwise. This results in a (re)weighted AMP (WAMP) algorithm whose threshold values are (re)weighted by $\omega < 1$ for the signal elements in the (estimated) support set.

1.1.3 Verification-based Algorithms

There also exists a category of CS recovery algorithms called *verification-based* (VB) algorithms that are based on sparse sensing matrices, see, e.g., [29,31,32]. The number of operations to calculate the measurements would be considerably lower using the sparse sensing matrices compared to the dense ones. In addition, the complexity of VB algorithms is only of $\mathcal{O}(N)$.

VB algorithms can be formulated as message-passing algorithms on a bipartite graph [29]. In such a graph, signal elements are represented by one set of nodes and the measurements are represented by another set. As the message-passing algorithm proceeds in iterations, signal elements are verified according to certain verification rules. The verified elements then remain unchanged in subsequent iterations. In this thesis, we are interested in a class of VB algorithms called *Node-based* VB (NB-VB) algorithms due to their superior performance compared to the other class of VB

algorithms, *message-based* VB algorithms [29].

1.2 Block Sparse Signals

In this thesis, we consider the reconstruction of *block sparse signals* (i.e., signals whose non-zero elements occur in clusters). Such signals appear in various applications such as spectrum sensing [33], equalization of sparse communication channels [34], DNA microarrays [35], source localization [36], color imaging [37], and in general when dealing with multi-band signals [38]. Compressed sensing of block sparse signals has been considered in a number of recent publications [39–47], where different assumptions about the sensing matrix were made and different techniques including convex and non-convex relaxations were used for the recovery of the signal.

1.2.1 With Knowledge of Block Borders

In [44], ℓ_2/ℓ_1 -norm minimization (ℓ_2/ℓ_1 minimization, in brief) was proposed as a technique to reconstruct block sparse signals. The main result of [44] on the recovery of block sparse signals using the ℓ_2/ℓ_1 relaxation was derived based on the null-space characterization of the sensing matrix $\mathbf{A}$. In particular, it was shown in [44] that in the noiseless scenario, perfect recovery using a Gaussian sensing matrix is possible with high probability as the signal length tends to infinity. In [45], also, ℓ_2/ℓ_1 minimization was proposed for the block sparse signal recovery. The notion of block restricted isometry property (RIP) was then used to derive the conditions for perfect recovery. Eldar *et al.* [46] used yet another condition, based on block-coherence measure, to guarantee successful recovery of block sparse signals through ℓ_2/ℓ_1 minimization. In [47], ℓ_q/ℓ_1 minimization with $q \geq 2$, and an alternate convex program, denoted by P'_{ℓ_q/ℓ_1} , for the recovery of block sparse signals were considered. In the same work, conditions for the recovery of the signal based on the notions of mutual and cumulative subspace coherence were also derived.²

AMP has also been applied to the recovery of block sparse signals [48, 49]. In this context, it is demonstrated in [48] that AMP with James-Stein’s shrinkage estimator (AMP-JS) can outperform ℓ_2/ℓ_1 minimization for large block sizes.

²It was demonstrated in [47] that for some cases, P'_{ℓ_2/ℓ_1} can outperform ℓ_2/ℓ_1 minimization. In the context of the iterative algorithms proposed here, where we use a weighted version of ℓ_2/ℓ_1 minimization in each iteration, we observed no notable difference in the performance of iterative algorithms when ℓ_2/ℓ_1 minimization was replaced with P'_{ℓ_2/ℓ_1} .

To recover the block sparse signals in the context of compressed sensing, many other works have been devoted to enhancing the recovery performance using the block sparse structure of the signals, see, e.g., block-based orthogonal matching pursuit [50], and block-based sparse Bayesian learning [51].

In all the above literature, i.e., [28, 44, 48, 50, 51], the information about the block borders of the block sparse signal is assumed to be available a priori.

1.2.2 Without Knowledge of Block Borders

In practice, however, the block borders are often unknown. Recently, a number of recovery algorithms based on Bayesian learning have been applied to block-sparse signals with unknown block borders [52–56]. Although, these Bayesian algorithms operate in the absence of the a priori knowledge of the block borders, they do assume a certain a priori distribution for the signal which includes a dependency model for neighboring signal elements. To the best of our knowledge, the best performing Bayesian algorithm in this context is the pattern-coupled sparse Bayesian learning (PC-SBL) of [55]. In [55] a reweighting scheme is also proposed to improve the performance of $\text{IR}\ell_1$ for the recovery of block sparse signals with unknown block borders. In this scheme, each weight is not only a function of its corresponding signal element, but also a function of the two immediate neighboring elements. This coupling mechanism is shown to encourage block sparse solutions [55].

1.3 Compressed Wideband Spectrum Sensing

In a wireless communication environment, many of the primary users (PUs) do not use their licensed frequency bands at all times. The surveys show that the maximum frequency utilization of the allocated spectrum can be less than 10% [57]. To increase the frequency utilization in such environments, secondary users (SUs) equipped with cognitive radios (CRs) [58] can be deployed to use the available spectrum (*spectrum holes*) without interfering with the PUs. Spectrum sensing, as the task of finding the spectrum holes, is thus one of the important functions performed by CRs [33].

As an integral part of spectrum sensing, an analog to digital converter (ADC) is often used to sample the signal at or above the Nyquist rate and then processing the signal to detect the spectrum holes. In wideband scenarios,³ this will require very

³The term “wideband” is used rather loosely in this thesis to refer to a scenario where ADC at

high-speed ADCs, which are challenging to implement.

To alleviate the need for high sampling rates, compressed sensing (CS) [3] has been recently applied to WSS [17, 33, 59–63]. Compressed Sensing is a signal processing technique that can recover an unknown signal with high probability from a small set of linear projections, called *measurements*, if the signal is sparse in some domain. Linear projections of the signal and its sparse representation are described by two matrices called *measurement matrix* and *sensing matrix*, respectively.

At the core of a CS scheme is an analog to information converter (AIC), which operates at sub-Nyquist rates, see, e.g., [64]. In wideband compressed spectrum sensing, the goal is to decrease the number of measurements required to reconstruct a spectrum with a given sparsity ratio. This will reduce the sampling rate of the AIC.

In the literature, the spectrum is often reconstructed by solving the ℓ_1 -norm minimization problem, see, e.g., [33, 59]. Due to the nature of the spectrum sensing problem, the signal in the frequency domain is block sparse. Moreover, in many scenarios, the allocated frequency bands to PUs are fixed and thus, this information can be available to the CRs. The problem of wideband spectrum sensing therefore fits well within the framework of block sparse signal recovery. Compressed sensing of block sparse signals has been considered in a number of recent publications, see, e.g., [44, 46–48, 60]. In [60], instead of ℓ_1 -norm minimization, ℓ_2/ℓ_1 -norm minimization is applied to the compressed spectrum sensing. In this technique, which is used for the reconstruction of block sparse signals [44], the required number of measurements is much less than ℓ_1 -norm minimization. In all the above literature [33, 59, 60], first, the spectrum is fully recovered after acquiring the compressed measurements. Then, the spectrum holes are detected from the reconstructed spectrum by applying an energy detection algorithm. The CR, however, does not need to necessarily reconstruct the entire spectrum to be able to detect the spectrum holes [65]. As demonstrated in [65], the occupied channels can be directly detected from the compressed measurements, thereby bypassing the spectrum reconstruction stage and reducing the computational complexity.

In practice, one may not need to detect *all* the spectrum holes, just a sufficiently large fraction of them may suffice. In a scenario such as a cooperative CR network, from the standpoint of the SUs, however, it is important to find as many vacant sub-channels as possible with a given number of measurements [66].

Nyquist rate would be challenging to implement.

To the best of our knowledge, all compressed WSS methods in the literature are based on dense sensing matrices (matrices that have few, or no, zero entries), and use linear or convex optimization, which has complexity between $\mathcal{O}(N^2)$ and $\mathcal{O}(N^3)$, where N is the dimension of the signal. In mobile CRs with limited power and computational resources, it is impractical to use CS with such costly computations for the reconstruction of the spectrum. In addition, the processing time required for signal reconstruction can impose a significant delay in the spectrum sensing. These motivate us to look for a low-complexity scheme for spectrum sensing with the goal of detecting a large portion of spectrum holes.

1.4 Organization and Contribution of the Thesis

1.4.1 Chapter 2

In Chapter 2, we present the notations and definitions that will be used throughout the thesis. We also provide the background on the compressed sensing of block sparse signals. To this end, we review the ℓ_2/ℓ_1 minimization problem and its theoretical guarantees. We also review the AMP algorithms and represent two thresholding functions which are used in this work.

1.4.2 Chapter 3

In this chapter, we propose a family of iterative algorithms for the recovery of block sparse signals when the block borders are known. These algorithms, referred to as *iterative reweighted ℓ_2/ℓ_1 minimization* algorithms (IR- ℓ_2/ℓ_1), solve a weighted ℓ_2/ℓ_1 minimization in each iteration. As an example, we apply the proposed algorithms to wideband spectrum sensing. Our simulation and analytical results on the recovery of both ideally and approximately block sparse signals show that the proposed iterative algorithms have significant advantages in terms of accuracy and the number of required measurements over non-iterative approaches as well as existing iterative methods. In particular, we demonstrate that, by increasing the block length, the recovery performance of the proposed algorithms approaches the Wu-Verdú theoretical limit. The improvement in performance comes at a rather small cost in complexity increase. Further improvement in performance is achieved by using a priori information about the location of non-zero blocks, even if such a priori information is not

perfectly reliable. This chapter is based on the following two papers [67, 68].

1.4.3 Chapter 4

In this chapter, we propose a weighting/reweighting scheme for block sparse signal recovery with known borders to improve the performance of the so-called approximate message passing (AMP) algorithm of Donoho *et al.*. We consider the application of AMP for the recovery of sparse signals from an under-determined system of linear equations, and variants of AMP for the recovery of block sparse signals. For the scenarios in which the block borders are not available, we also propose an iterative algorithm which combines a border detection scheme with a recovery algorithm for block sparse signals with known borders. In each iteration of the proposed algorithm, the block borders are detected based on the obtained estimate of the signal and this information is used by the recovery algorithm in the next iteration to improve the accuracy of the signal estimation. In this work, due to its remarkable performance/complexity trade-off, we use the so-called approximate message passing (AMP) algorithm with James-Stein's shrinkage estimator as the recovery algorithm. The simulation results show that the proposed algorithms can be superior to the state-of-the-art algorithms in terms of both the recovery performance and computational complexity. This chapter is based on the following two papers [69, 70].

1.4.4 Chapter 5

In this chapter, we propose a very low-complexity zero-block detection scheme that can detect a large fraction of zero blocks from the sub-Nyquist samples, even when the undersampling ratio is very small ($M \ll L$). The scheme is based on a block sparse sensing matrix, which is implemented through the design of a novel analog-to-information converter (AIC). The proposed scheme identifies some measurements as being zero and then verifies the blocks associated with them as being zero. Analytical and simulation results are presented that demonstrate the effectiveness of the proposed method in reliable detection of spectrum holes with complexity much lower than existing schemes, such that the proposed scheme has a linear complexity in L . This work also introduces a paradigm in compressed sensing where one is interested in reliable detection of (some of the) zero blocks rather than the recovery of the whole block sparse signal. An important application of the results would be in cognitive

radios with limited computational resources. This chapter is mainly based on [71].

1.4.5 Chapter 6

In this chapter, a low-complexity scheme for the reliable detection of zero blocks in a block sparse signal is proposed. The scheme is based on the application of verification based (VB) recovery algorithms in compressed sensing to block sparse signals, and is described in the context of wideband spectrum sensing (WSS). To apply VB algorithms to WSS, we need a block sparse sensing matrix which is designed in Chapter 5. The AIC, the sensing matrix and the VB algorithms are then optimized such that the largest number of zero blocks for a given number of measurements can be detected. The analysis and simulations demonstrate significant improvement in performance/complexity over the existing block sparse recovery schemes within this new framework. Some of the results mentioned in this chapter were published in [72].

1.4.6 Chapter 7

In this chapter, we discuss some of the open problems in the area of block sparse recovery and compressed wideband spectrum sensing, with some guidelines for further research.

Chapter 2

Background

2.1 Notations and Definitions

In this thesis, we use capital bold letters to refer to matrices and lowercase bold letters for vectors. In addition, the bold symbols $\mathbf{1}$ and $\mathbf{0}$ refer to the vectors of all ones and all zeros, respectively. For the real and imaginary parts of a complex argument, we use $\Re(\cdot)$ and $\Im(\cdot)$, respectively. In addition, $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^{-1}$ and $(\cdot)^\dagger$ denote the transpose, conjugate, matrix inversion and conjugate transpose, respectively. We assume all vectors are column vectors unless specified otherwise.

Let $\mathbb{C}$ and $\mathbb{R}$ be the fields of complex and real numbers, respectively. For a matrix $\mathbf{A}$ with N columns, we denote its null space by $\mathcal{N}(\mathbf{A}) = \{\boldsymbol{\eta} \in \mathbb{C}^N | \mathbf{A}\boldsymbol{\eta} = \mathbf{0}\}$.

Definition 1: Suppose signal $\bar{\mathbf{x}}$ consists of L sub-vectors, $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \dots, \bar{\mathbf{u}}_L\}$, where the sub-vectors can be either zero or non-zero (a non-zero vector can have both zero and non-zero elements). The signal $\bar{\mathbf{x}} = [\bar{\mathbf{u}}_1^T, \bar{\mathbf{u}}_2^T, \dots, \bar{\mathbf{u}}_L^T]^T$ is called *ideally P -block sparse* (or *P -block sparse*, in brief) if at most $P \ll L$ out of L sub-vectors are non-zero.

We use the notations $f_0 < \dots < f_L$ for the boundaries of the blocks, i.e., the sub-vector $\bar{\mathbf{u}}_i$, $i \in \{1, \dots, L\}$, which is the i -th segment of $\bar{\mathbf{x}}$, corresponds to the elements whose indices are in the range $f_{i-1}, \dots, f_i - 1$. Let B_i denote the length of the block i . We thus have $\sum_{i=1}^L B_i = N$.

In this thesis, we use the terms *block*, *sub-vector* and *cluster*, interchangeably.

Definition 2: Signal $\bar{\mathbf{x}}$ is called *approximately P -block sparse with parameter δ* , if it consists of L sub-vectors $\{\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_L\}$ such that $\sum_{i=P+1}^L \|\bar{\mathbf{u}}_{\pi(i)}\|_2 \leq \delta$, for some $\delta \geq 0$, where $P \ll L$ and π is a permutation of the set $\{1, \dots, L\}$ such that $\|\bar{\mathbf{u}}_{\pi(1)}\|_2 \geq \|\bar{\mathbf{u}}_{\pi(2)}\|_2 \geq \dots \geq \|\bar{\mathbf{u}}_{\pi(L)}\|_2$. We call π a *proper permutation* with respect to

the set $\{\|\bar{\mathbf{u}}_i\|_2\}_{1 \leq i \leq L}$.¹

For a given subset Γ of $\{1, \dots, L\}$, $\mathbf{x}_\Gamma$ denotes the sub-vector of $\mathbf{x}$ consisting of $\mathbf{u}_i$ vectors with $i \in \Gamma$. For an index set $S \subset \{1, 2, \dots, L\}$, S^c and $|S|$ denote the complement of S with respect to $\{1, 2, \dots, L\}$ and the size of set S , respectively.

Let $\mathcal{G}(\mathcal{V} \cup \mathcal{M}, \mathcal{E}_{\mathcal{V}, \mathcal{M}})$ denote a bipartite graph where $\mathcal{V} \cup \mathcal{M}$ is the node set and $\mathcal{E}_{\mathcal{V}, \mathcal{M}}$ is the edge set, so that, every edge in $\mathcal{E}_{\mathcal{V}, \mathcal{M}}$ connects a node in $\mathcal{V}$ to a node in $\mathcal{M}$. We refer to the sets $\mathcal{V}$ and $\mathcal{M}$ as *variable nodes* (VNs) and *measurement nodes* (MNs), respectively. In a bipartite graph, $\mathcal{E}_{S_1, S_2} \subseteq \mathcal{E}_{\mathcal{V}, \mathcal{M}}$ denotes the set of the edges that connect the set $S_1 \subseteq \mathcal{V}$ to the set $S_2 \subseteq \mathcal{M}$. In the VB algorithms, such an edge set is updated in each iteration. We use $\mathcal{E}_{S_1, S_2}^{(j)}$ to denote such an edge set in the j -th iteration. In addition, notations $\mathcal{V}(m)$ and $\mathcal{M}(v)$ are used to denote the set of VNs neighboring to the measurement node m and the set of MNs neighboring to the variable node v , respectively. For two measurements a and b , $\mathcal{V}(a, b) = \mathcal{V}(a) \cap \mathcal{V}(b)$ denotes the set of common VNs neighboring both a and b . We use the notation $\mathcal{V}^c(a, b)$ for the complement set of $\mathcal{V}(a, b)$ with respect to $\mathcal{V}$.

In a bipartite graph, a node in $\mathcal{V}$ ($\mathcal{M}$) has degree i ($i \geq 1$), if it is connected to i nodes in $\mathcal{M}$ ($\mathcal{V}$). We use $d(v)$ to denote the degree of v . Let the polynomials $\lambda(x) = \sum_i \lambda_i x^i$ and $\rho(x) = \sum_i \rho_i x^i$, represent the degree distributions of VNs and MNs, respectively, where λ_i and ρ_i denote the fraction of degree- i nodes in VNs and MNs, respectively. If the degree of all nodes in $\mathcal{M}$ and $\mathcal{V}$ are $d(m) = d_M, \forall m \in \mathcal{M}$ and $d(v) = d_V, \forall v \in \mathcal{V}$, respectively, the graph is called *bi-regular* (*regular*, in brief). Otherwise, it is called *irregular*. In a regular graph, $\rho(x) = x^{d_M}$ and $\lambda(x) = x^{d_V}$. A regular graph with variable degree d_V and measurement degree d_M is referred to as (d_V, d_M) -regular.

For block sparse signals, we represent each block with a VN. Each variable node thus represents a number of complex-valued elements. A VN is a zero VN, if all the elements of the corresponding block are zero; otherwise, it is a non-zero VN. The value of each MN is a linear combination of the elements of the neighboring VNs, and is assumed to be a real number. Furthermore, let $\mathcal{V}_z \subseteq \mathcal{V}$, $\mathcal{V}_{nz} \subseteq \mathcal{V}$, $\mathcal{M}_z \subseteq \mathcal{M}$ and $\mathcal{M}_{nz} \subseteq \mathcal{M}$ refer to the sets of zero VNs, non-zero VNs, zero MNs and non-zero MNs, respectively. Clearly, $\mathcal{V}_{nz}$ ($\mathcal{M}_{nz}$) is the complement of the set $\mathcal{V}_z$ ($\mathcal{M}_z$) with respect to $\mathcal{V}$ ($\mathcal{M}$). For the iterative algorithm discussed in this thesis, the sensing graph (sensing matrix) is updated in each iteration, sometimes more than once. We use the

¹To apply this definition to random signals, we require the signal to satisfy this condition with probability (close to) 1.

variable with a breve to indicate that it is an updated version, e.g., $\breve{\mathcal{M}}$ denotes an updated version of $\mathcal{M}$.

Following [29], we use the term “verification” for the recovery of signals using VB algorithms. In the context of the proposed algorithm, this term is used to indicate that a VN (corresponding to a block) is identified as being zero or non-zero. General terms “detection” or “recovery,” on the other hand, are used when the exact value of a variable node (block) is obtained. With a slight abuse of terminology, we may use either one of the three terms when we are only concerned about zero blocks. We declare “success,” when a recovery algorithm recovers all the signal elements. Therefore, the probability of success is the probability that an algorithm can fully recover the signal. In order to quantify the performance of the proposed zero-block detection algorithm, we define two measures:

- (i) *Probability of zero-block detection (P_{ZD})*: the average ratio of the number of zero-blocks that are detected correctly to the total number of zero-blocks that exist in the spectrum, where the average is taken over the ensembles of sensing graphs and input signals.
- (ii) *Probability of wrong zero-block detection (P_{WZD})*: the average ratio of the number of non-zero blocks that are falsely detected as zero to the total number of the blocks detected as zero, where the average is taken over the ensembles of sensing graphs and input signals.

2.2 Block Sparse Compressed Sensing Background

2.2.1 Convex Optimization

To exploit the block structure of a block sparse signal $\bar{\mathbf{x}}$ in the recovery process, the following ℓ_2/ℓ_1 minimization was proposed in [44]

$$\begin{aligned} \hat{\mathbf{x}} &= [\hat{\mathbf{u}}_1^T, \hat{\mathbf{u}}_2^T, \dots, \hat{\mathbf{u}}_L^T]^T \\ &= \arg \min_{\mathbf{u}_1, \dots, \mathbf{u}_L} \sum_{i=1}^L \|\mathbf{u}_i\|_2 \quad \text{s.t.} \quad \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \varepsilon, \end{aligned} \quad (2.1)$$

where ε bounds the ℓ_2 -norm of the noise in measurements. The problem in (2.1) is a relaxation of the following ℓ_2/ℓ_0 minimization problem

$$\begin{aligned}\bar{\mathbf{x}} &= [\bar{\mathbf{u}}_1^T, \bar{\mathbf{u}}_2^T, \dots, \bar{\mathbf{u}}_L^T]^T \\ &= \arg \min_{\mathbf{u}_1, \dots, \mathbf{u}_L} |\{i : \|\mathbf{u}_i\|_2 \neq 0, i = 1, \dots, L\}| \\ &\text{s.t. } \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \varepsilon.\end{aligned}$$

In the rest of the thesis, we use the terms “true block sparse signal” and “solution of ℓ_2/ℓ_0 minimization,” interchangeably, to refer to $\bar{\mathbf{x}}$.

The simulation results in [44] demonstrated that ℓ_2/ℓ_1 minimization outperformed the ℓ_1 minimization in the case of block sparse signals. The following theorem introduces a necessary and sufficient condition on the sensing matrix $\mathbf{A}$ so that $\hat{\mathbf{x}}$ and $\bar{\mathbf{x}}$ coincide for any ideally P -block sparse signal $\bar{\mathbf{x}}$.

Theorem 2.1: [44] Let $\bar{\mathbf{x}} = [\bar{\mathbf{u}}_1^T, \dots, \bar{\mathbf{u}}_L^T]^T$ be the true P -block sparse signal and $\hat{\mathbf{x}} = [\hat{\mathbf{u}}_1^T, \dots, \hat{\mathbf{u}}_L^T]^T$ be a minimizer of (2.1) for $\varepsilon = 0$. Further, assume that $\mathbf{A}$ is an $M \times N$ sensing matrix. Then, signals $\hat{\mathbf{x}}$ and $\bar{\mathbf{x}}$ coincide for any P -block sparse $\bar{\mathbf{x}}$ if and only if for all index sets $S \subset \{1, 2, \dots, L\}$ with $|S| \leq P$ and for all $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}) \setminus \mathbf{0}$, we have

$$\sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 < \gamma \sum_{i \in S^c} \|\boldsymbol{\eta}_i\|_2, \quad (2.2)$$

for some $0 < \gamma < 1$.

Here, $\boldsymbol{\eta}_i$ is the i -th block of $\boldsymbol{\eta}$ (the block boundaries of $\boldsymbol{\eta}$ are the same as those of $\bar{\mathbf{x}}$). In the following, we use the notation $\bar{\gamma}$ as the infimum of all values of γ that satisfy (2.2).

In practice, the signals are not ideally block sparse. In [73], it was proved that when $\bar{\mathbf{x}}$ is approximately P -block sparse, the difference between the solution of ℓ_2/ℓ_1 minimization with $\varepsilon = 0$, $\hat{\mathbf{x}}$, and the true signal, $\bar{\mathbf{x}}$, in ℓ_2/ℓ_1 norm (i.e., $\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$), is bounded from above.

Theorem 2.2: [73] Let $\bar{\mathbf{x}}$ be the true approximately P -block sparse signal with parameter δ , and $\hat{\mathbf{x}}$ be a minimizer of (2.1) for $\varepsilon = 0$. Further, assume that $\mathbf{A}$ is an

$M \times N$ sensing matrix. Then, for $0 < \bar{\gamma} < 1$, we have

$$\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 \leq 2 \left(\frac{1 + \bar{\gamma}}{1 - \bar{\gamma}} \right) \delta, \quad (2.3)$$

if and only if for all index sets $S \subset \{1, 2, \dots, L\}$ with $|S| \leq P$ and for all $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}) \setminus \mathbf{0}$, we have

$$\sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 < \bar{\gamma} \sum_{i \in S^c} \|\boldsymbol{\eta}_i\|_2, \quad (2.4)$$

where $\boldsymbol{\eta}_i$ is the i -th block of $\boldsymbol{\eta}$ with the block boundaries the same as those of $\bar{\mathbf{x}}$.

2.2.2 Approximate Message Passing

Starting from $\mathbf{x}^0 = \mathbf{0}$ and $\mathbf{z}^t = \mathbf{y}$, the AMP algorithm proceeds iteratively as follows:

$$\mathbf{v}^t = \mathbf{x}^t + \mathbf{A}^T \mathbf{z}^t, \quad (2.5)$$

$$\mathbf{x}^{t+1} = \zeta(\mathbf{v}^t; \tau^t, \sigma_t), \quad (2.6)$$

$$\mathbf{z}^{t+1} = \mathbf{y} - \mathbf{A} \mathbf{x}^{t+1} + b_{t+1} \mathbf{z}^t, \quad (2.7)$$

where $\mathbf{x}^t$ is the estimate of $\mathbf{x}$ in the t -th iteration, $\mathbf{z}^t$ is the residual vector in that iteration, $\zeta(\cdot) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is the thresholding function, τ^t is a positive threshold, and σ_t is the noise standard deviation of $\mathbf{v}^t$ estimated from $\mathbf{z}^t$ by

$$\sigma_t = \frac{1}{\Phi^{-1}(0.75)} \text{median}\{|\mathbf{z}^t|\}. \quad (2.8)$$

In (2.8), we have $\Phi(x) = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{x^2}{2}}$ and in (2.7), the parameter b_t is determined by

$$b_t = \frac{1}{N} \text{div} \zeta(\mathbf{v}^{t-1}; \tau^{t-1}, \sigma_{t-1}), \quad (2.9)$$

where “div” denotes the divergence operator. The threshold parameter in the t -th iteration, τ^t , is computed iteratively as follows

$$\tau^t = \tau^{t-1} b_t, \quad (2.10)$$

with the initial value $\tau^0 = 1$.

The thresholding function, $\zeta(\cdot)$, plays an important role in the performance of AMP. It is shown in [26] that in the case of Gaussian sensing matrix, the solution of LASSO coincides with that of AMP when $\zeta(\cdot)$ is a *soft thresholding* function. In soft thresholding, the i -th element of $\zeta^{\text{ST}}(\mathbf{x}; \tau)$, i.e., ζ_i^{ST} , is given by

$$\zeta_i^{\text{ST}} = \left(1 - \frac{\tau}{|x_i|}\right)_+ x_i, \quad i = 1, \dots, N, \quad (2.11)$$

where $(x)_+ = \max(x, 0)$ and τ is a positive parameter.

In block sparse signal recovery with known block borders, the soft thresholding function can be generalized to the block soft thresholding function [48]. The block soft thresholding function, $\zeta^{\text{BST}}(\mathbf{x}; \tau, \sigma)$, consists of L components $\zeta_i^{\text{BST}} : \mathbb{R}^{B_i} \rightarrow \mathbb{R}^{B_i}$, defined as

$$\zeta_i^{\text{BST}} = \left(1 - \frac{\tau\sigma}{\|\mathbf{u}_i\|_2}\right)_+ \mathbf{u}_i, \quad i = 1, \dots, L, \quad (2.12)$$

where $\mathbf{u} \in \mathbb{R}^B$, $\tau \in \mathbb{R}_+$, and $(z)_+ = \max(z, 0)$. It was shown in [48] that, for Gaussian sensing matrices and at large system limit, the solution of AMP-BST coincides with that of ℓ_2/ℓ_1 minimization (group-LASSO).

However, it is proved in [48] that AMP-JS outperforms the AMP with block soft thresholding, particularly for large block lengths. The James-Stein function, $\zeta^{\text{JS}}(\mathbf{x}; \sigma)$, consists of L components $\zeta_i^{\text{JS}} : \mathbb{R}^{B_i} \rightarrow \mathbb{R}^{B_i}$, $B_i \geq 3$, defined as

$$\zeta_i^{\text{JS}} = \left(1 - \frac{\sigma^2(B_i - 2)}{\|\mathbf{u}_i\|_2^2}\right)_+ \mathbf{u}_i, \quad i = 1, \dots, L. \quad (2.13)$$

Clearly, to apply (2.13) and (2.12), one needs to know the block borders.

2.3 A Small Note on Sensing Matrix

Throughout this thesis, the entries of the sensing matrix are generated from a zero mean Gaussian distribution in an i.i.d. fashion. However, the variance is not always the same. Since scaling does not have any influence on the null space property of a matrix, the performance of the most recovery methods is not sensitive to variance values. However, AMP algorithms are one of the recovery algorithms that are sensitive to the choice of the sensing matrix. Therefore, in Chapter 4, we select the variance

of the sensing matrix from the order of $1/M$, such that each column of the sensing matrix is normalized. In Chapters 5 and 6, due to the nature of the proposed AIC, the sensing matrix is a complex matrix where the real and imaginary parts of the elements of non-zero blocks of the sensing matrix are selected from a zero mean Gaussian distribution with variance one, in an i.i.d. fashion. However, without loss of generality, the ℓ_2 -norm of each block is normalized in order to simplify the equations.

Chapter 3

Iterative Reweighted ℓ_2/ℓ_1 Recovery Algorithms

3.1 Introduction

In this chapter, we propose a family of iterative algorithms to recover block sparse signals. In each iteration, the proposed algorithms solve a weighted ℓ_2/ℓ_1 minimization with the weights being updated in each iteration based on the solution of the previous iteration, hence the name “iterative reweighted (IR) ℓ_2/ℓ_1 minimization.” Different algorithms within the family differ in the way that the weights are updated from one iteration to the next. In this chapter, we mainly focus on two algorithms from this family, referred to as $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$. In the former algorithm, the weights are updated inversely proportional to the ℓ_2 -norm of the corresponding blocks in the solution of the previous iteration. In the latter algorithm, however, some weights are set to zero, if the ℓ_2 -norm of the corresponding blocks in the previous iteration is larger than a certain threshold. Otherwise, the weights are set to one. In general, all the $\text{IR-}\ell_2/\ell_1$ algorithms perform the standard ℓ_2/ℓ_1 minimization in the first iteration and then improve the solution in subsequent iterations. Our results show that the iterative improvement to standard ℓ_2/ℓ_1 minimization is attainable for noiseless scenarios and for cases where measurements are corrupted with noise whose power relative to the signal is below a certain threshold. For such cases, we demonstrate through simulation results that significant improvements in terms of the number of required measurements for recovery can be obtained compared to the standard ℓ_2/ℓ_1 minimization in both noiseless and noisy cases. Particularly, as the length of the blocks increases, the required number of measurements for the reconstruction of the

block sparse signal using IR- ℓ_2/ℓ_1 algorithms approaches the theoretical limit [74]. In addition, we demonstrate by a number of examples that the proposed algorithms can outperform James-Stein AMP [48] by a rather large margin in noiseless scenarios and in noisy scenarios with high signal-to-noise ratio (SNR) and small number of measurements (small under-sampling ratios). In general, our numerical results show that the proposed approaches can reconstruct block sparse signals better than existing AMP variants [27, 28, 48]. That said, current AMP variants might be using denoisers that are not well suited to block sparse signals.

Moreover, we analyze IR- ℓ_2/ℓ_1 algorithms and give theoretical guarantees on the success of these algorithms in recovering block sparse signals. We also show that at the presence of a priori information on the support region of the block sparse signal, the performance of both IR⁽¹⁾- ℓ_2/ℓ_1 and IR⁽²⁾- ℓ_2/ℓ_1 algorithms can be improved, even if such information is not perfectly reliable. In general, our theoretical and simulation results demonstrate the superior performance of the proposed iterative recovery algorithms compared to the existing schemes. This often comes at a rather small cost in computational complexity.

In many practical scenarios, $\bar{\mathbf{x}}$ is not ideally block sparse (i.e., $\bar{\mathbf{x}}$ has very few or no zero elements but has many close to zero elements in clusters). Such signals are called *approximately block sparse signals*. In such cases, it is often impossible to recover the signal $\bar{\mathbf{x}}$ accurately from a relatively small number of measurements M . In such scenarios, we establish upper bounds on the signal reconstruction error of the proposed iterative algorithms and show improvements over the bounds derived for the standard ℓ_2/ℓ_1 minimization.

As an example, we apply the proposed recovery algorithms to the problem of wideband compressed spectrum sensing [33], [60, 75, 76], and demonstrate through simulation results that significant improvements in the probability of detection, for a given probability of false alarm, can be achieved over the ℓ_2/ℓ_1 minimization.

3.2 Iterative Reweighted (IR-) ℓ_2/ℓ_1 Minimization

Although ℓ_1 minimization can solve ℓ_0 minimization problem under certain conditions, it requires a larger number of measurements for the signal reconstruction due to its dependency on the magnitude of the signal. If we can also reduce (eliminate) the dependency of the ℓ_2/ℓ_1 minimization (2.1) on the ℓ_2 -norm of the blocks by

exploiting a weighting strategy, the reconstruction performance can be improved. In the following, we first present weighted ℓ_2/ℓ_1 as the core of iterative algorithms for the reconstruction of block sparse signals. We then propose the iterative algorithms when no a priori information is available. Furthermore, we modify the reconstruction process to use the a priori information about the support region of the block sparse signal. We also analyze the success of weighted ℓ_2/ℓ_1 minimization in the recovery of ideally block sparse signals where the measurements are noiseless. We then generalize the analysis to the case of approximately block sparse signals.

3.2.1 Weighted ℓ_2/ℓ_1 minimization

Consider the following weighted ℓ_2/ℓ_1 minimization problem:

$$\begin{aligned}\hat{\mathbf{x}} &= [\hat{\mathbf{u}}_1^T, \hat{\mathbf{u}}_2^T, \dots, \hat{\mathbf{u}}_L^T]^T \\ &= \arg \min_{\mathbf{u}_1, \dots, \mathbf{u}_L} \sum_{i=1}^L \omega_i \|\mathbf{u}_i\|_2 \quad \text{s.t.} \quad \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \varepsilon, \end{aligned} \quad (3.1)$$

where ω_i 's are the weights that can be considered as free parameters.

The signal reconstruction can be improved if the values of the weights are properly set. In particular, in this thesis, we use the formulation in (3.1) iteratively, where the weights in the current iteration are determined based on the estimate of the signal obtained in the previous iteration. In each iteration, the convex problem (3.1) is solved by using convex programming tools [9, 10, 77]. The computational complexity of solving (3.1) is essentially the same as that of unweighted (standard) ℓ_2/ℓ_1 minimization. Therefore, for an algorithm that solves (3.1) iteratively, the computational complexity is roughly j_{max} times that of the standard ℓ_2/ℓ_1 minimization, where j_{max} is the maximum number of iterations.

Our focus in this chapter will be mainly on two incarnations of the iterative algorithm: $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$.

$\text{IR}^{(1)}\text{-}\ell_2/\ell_1$

In this algorithm, the weights are updated inversely proportional to the ℓ_2 -norm of their corresponding blocks in the previous iteration.

IR⁽²⁾- ℓ_2/ℓ_1

In this algorithm, the weights only take values 0 and 1. A weight will be zero if the ℓ_2 -norm of the corresponding block in the previous iteration is larger than a certain threshold. Otherwise, the weight will be set at one. The terms with zero weight disappear from the summation in the formulation of (3.1). The summation is thus performed over an index set Γ , which is a subset of $\{1, \dots, L\}$. We use the term *truncated ℓ_2/ℓ_1 minimization* to refer to (3.1) in this scenario.

In the rest of the thesis, we often use the following version of (3.1), in which all the weights appearing in the summation are non-zero:

$$\begin{aligned} \hat{\mathbf{x}} &= [\hat{\mathbf{u}}_1^T, \hat{\mathbf{u}}_2^T, \dots, \hat{\mathbf{u}}_L^T]^T \\ &= \arg \min_{\mathbf{u}_1, \dots, \mathbf{u}_L} \sum_{i \in \Gamma} \omega_i \|\mathbf{u}_i\|_2 \quad \text{s.t.} \quad \|\mathbf{y} - \mathbf{Ax}\|_2 \leq \varepsilon, \end{aligned} \quad (3.2)$$

where $\Gamma \subseteq \{1, \dots, L\}$ is a given index set.

3.2.2 Iterative Reweighted ℓ_2/ℓ_1 Minimization with no A Priori Information

Consider the weighted ℓ_2/ℓ_1 minimization problem in (3.1). In the absence of any a priori information, we propose an iterative reconstruction scheme, IR- ℓ_2/ℓ_1 , where the ℓ_2 -norm of estimated blocks in one iteration is used to select the corresponding weights in the next iteration. Each iteration of IR- ℓ_2/ℓ_1 thus consists of two steps: (i) updating the weights, and (ii) signal reconstruction. In Step (i), some weights may be selected to be zero, effectively reducing the support set from $\{1, \dots, L\}$ to a subset of it, Γ . In the following, we refer to this as “support detection” and limit the operations of Step (i) to support detection and two specific update rules for non-zero weights. We initialize the proposed iterative algorithms with the solution to the standard ℓ_2/ℓ_1 minimization and demonstrate that by properly choosing the update rules for the index set Γ and non-zero weights, the initial solution can be much improved iteratively.

The algorithmic framework of IR- ℓ_2/ℓ_1 is given in Algorithm 3.2.2:

Algorithm 3.2.2: Iterative reweighted ℓ_2/ℓ_1 minimization (IR- ℓ_2/ℓ_1)

Inputs: Φ (or $\mathbf{A}$), $\mathbf{y}$, ρ_0 , β , ε , j_{max} and $f_0, f_1 \dots, f_L$

Output: $\hat{\mathbf{x}} = [\hat{\mathbf{u}}_1^T, \hat{\mathbf{u}}_2^T, \dots, \hat{\mathbf{u}}_L^T]^T$

1. Initialization step:

- (a) Set the iteration number $j \leftarrow 1$ and $\rho \leftarrow \rho_0$;
- (b) $\Gamma^{(1)} \leftarrow \{1, \dots, L\}$;
- (c) $\omega_i^{(1)} = 1, i \in \{1, \dots, L\}$;
- (d) Obtain $\hat{\mathbf{x}}^{(1)}$ by solving (3.2) for $\Gamma = \Gamma^{(1)}$ and $\omega_i = \omega_i^{(1)}$;

2. **While** $j < j_{max}$, **do**:

(a) Support detection:

Determine $\Gamma^{(j+1)}$ using $\hat{\mathbf{x}}^{(j)}$;

(b) Update the non-zero weights:

Determine $\omega_i^{(j+1)}$, for $i \in \Gamma^{(j+1)}$ using $\hat{\mathbf{x}}^{(j)}$;

(c) Signal reconstruction:

Obtain $\hat{\mathbf{x}}^{(j+1)}$ by solving (3.2) for $\Gamma = \Gamma^{(j+1)}$ and $\omega_i = \omega_i^{(j+1)}$;

(d) Update ρ :

if $\frac{\|\hat{\mathbf{x}}^{(j+1)} - \hat{\mathbf{x}}^{(j)}\|_2}{\|\hat{\mathbf{x}}^{(j)}\|_2} < \sqrt{\frac{\rho}{100}}$ **then** $\rho \leftarrow \rho/10$;

(e) $j \leftarrow j + 1$.

In Algorithm 3.2.2, the superscript (j) is used to refer to a variable, a vector or a set in iteration j .

In support detection, we update $\Gamma^{(j+1)}$ such that $\Gamma^{(j+1)} \leftarrow \{i : \|\hat{\mathbf{u}}_i^{(j)}\|_2 < \epsilon^{(j)}\}$. We set the threshold $\epsilon^{(j)}$ for the support detection in iteration j at $\epsilon^{(j)} = \max_i \|\hat{\mathbf{u}}_i^{(j)}\|_2 / \beta^j$ for $i = 1, \dots, L$, with $\beta > 1$. Note that this selection reduces the threshold as the iterations proceed. The choice of β affects the efficiency of the algorithm. If β is selected to be too large, $\epsilon^{(j)}$ values will be too small, resulting in the estimated index set Γ to be too small, making the reconstruction step ineffective due to excessive truncation. If, on the other hand, β is selected to be too small, then too many

iterations will be required to accurately reconstruct the block sparse signal, decreasing the speed of recovery.

For updating the non-zero weights, we choose the weights as

$$\omega_i^{(j+1)} = \frac{1}{\|\hat{\mathbf{u}}_i^{(j)}\|_2 + \rho}, \quad i \in \Gamma^{(j+1)}. \quad (3.3)$$

To justify the choice for non-zero weights in (3.3), we note that in (3.2), large weights will discourage non-zero blocks whereas small weights will encourage such blocks. Therefore, the weights could be chosen inversely proportional to the true ℓ_2 -norm of the blocks. With the lack of knowledge about true ℓ_2 -norms, here, we use the updating rule of (3.3), in which ℓ_2 -norms of estimated blocks in the previous iteration are used instead of true ℓ_2 -norms, and ρ is a regularization constant to prevent instability. We will prove that these weights guarantee $\hat{\mathbf{x}} = \bar{\mathbf{x}}$ under certain conditions.

One may also note that if the ℓ_2 -norm of the i -th block in the j -th iteration, $\|\hat{\mathbf{u}}_i^{(j)}\|_2$, is small, the index of this block would be in the set $\Gamma^{(j+1)}$ and $\omega_i^{(j+1)}$ would be large. So the blocks with small ℓ_2 -norm are pushed towards becoming zero. This thus yields a block sparse solution.

At each iteration, the set of block indices, Γ , is updated in Step 2(a). The size of this set however may not necessarily decrease with iterations. After updating Γ , all the components of $\mathbf{x}$ are updated in Step 2(c), including both the components corresponding to Γ and those that correspond to Γ^c .

Since, the overall optimization problem is non-convex, the proposed algorithm would produce local minima. To avoid being trapped in an undesirable local minimum, choosing a suitable starting point is important. As a good starting point, in the first iteration (i.e., $j = 1$), the algorithm finds the solution of the standard ℓ_2/ℓ_1 minimization (i.e., in the initialization step, we set $\Gamma^{(1)} \leftarrow \{1, \dots, L\}$ and $\omega_i^{(1)} = 1, \forall i \in \{1, 2, \dots, L\}$). In this iteration, non-zero and zero blocks are weighted the same since there is no a priori information about the zero or non-zero blocks available.

Since an all-zero sub-vector $\hat{\mathbf{u}}_i^{(j)}, i \in \{1, 2, \dots, L\}$ can cause instability if the parameter ρ is equal to zero, we choose ρ to be strictly positive. Similar to [11] and [12], to avoid getting trapped in a local minimum, we start by setting ρ to be a relatively large value (e.g., ρ_0 can be selected close to the standard deviation of

ℓ_2 -norm of non-zero blocks) and reduce its value by a factor of 10,¹ in the $(j + 1)$ -th iteration if

$$\frac{\|\hat{\mathbf{x}}^{(j+1)} - \hat{\mathbf{x}}^{(j)}\|_2}{\|\hat{\mathbf{x}}^{(j)}\|_2} < \sqrt{\frac{\rho}{100}}. \quad (3.4)$$

Two special cases:

IR⁽¹⁾- ℓ_2/ℓ_1

In this case, Algorithm 3.2.2 is executed without the support detection at Step 2(a), i.e., in all iterations, we set $\Gamma^{(j)} \leftarrow \{1, \dots, L\}$.²

By choosing the non-zero weights based on (3.3), this iterative algorithm attempts to find a local minimum of the objective function $\sum_{i=1}^L \|\mathbf{u}_i\|_2 / (\|\mathbf{u}_i\|_2 + \rho)$ that more closely than the objective function of the ℓ_2/ℓ_1 minimization resembles the ℓ_0 minimization. In early iterations, the signal estimate is inaccurate, however, the locations of the active blocks with high ℓ_2 -norm are likely to be identified. In the following iterations, the influence of those high-energy blocks is down-weighted. This implies that the active blocks with smaller levels of energy would be up-weighted and can thus be detected easier.

IR⁽²⁾- ℓ_2/ℓ_1

In this case, Algorithm 3.2.2 is executed without the weight-updating step 2(b), i.e., in all iterations, we set $\omega_i^{(j)} = 1, \forall i \in \Gamma^{(j)}$.³ In other words, from the inaccurate estimate of the signal in the early iterations, the weights corresponding to the high-energy blocks are likely to be set to 0. Otherwise, they will be set to 1. In fact, it is highly probable that Γ^c includes the indices of the high-energy blocks and, hence, $\mathbf{x}_\Gamma$ would be sparser than $\mathbf{x}$. This implies that the low-energy blocks can be detected easier.

By choosing a suitable value for β , this algorithm often converges very fast (in less than 4 iterations). Unlike the case for IR⁽²⁾- ℓ_2/ℓ_1 , the convergence speed of

¹The value 10 is selected empirically. Our experiments show that the performance of the proposed algorithms is not very sensitive to the small changes in the value of the decreasing rate of ρ per iteration.

²IR⁽¹⁾- ℓ_2/ℓ_1 can be considered as the adaptation of IR ℓ_1 [6] applied to ℓ_2/ℓ_1 minimization rather than ℓ_1 minimization.

³IR⁽²⁾- ℓ_2/ℓ_1 can be considered as the adaptation of ISD [14] applied to ℓ_2/ℓ_1 minimization rather than ℓ_1 minimization.

$\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ cannot be adjusted by any parameter in the algorithm. Therefore, in some applications where the computational time is important, with setting $j_{\max}$ to a small number, $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ would outperform $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$. This will be verified in Section 3.4.

3.2.3 Iterative Reweighted ℓ_2/ℓ_1 Minimization with A Priori Information about the Support Region of the Sparse Signal

In practical applications such as magnetic resonance imaging (MRI) and wideband spectrum sensing, the partial information about the support region of the sparse signal may be available a priori. In general, in the applications whose sparsity patterns are varying slowly with time, the support estimate can be used from the previous time slot. In [78], the authors showed that in MRI, the maximum support changes for two subsequent sequences are less than 2%. In wideband spectrum sensing also the support of the block sparse signal changes slowly over time. Moreover, in the context of spectrum sensing, some of the frequency bands are heavily used by the users such as local radio stations, local TV stations, etc. [17], and this information can be available at the recovery terminal.

Here, we assume that the index set of non-zero blocks is partially known, possibly with some error. We use such side-information to improve the recovery performance via changing the initial step of $\text{IR}\text{-}\ell_2/\ell_1$ algorithms.

Let $\mathcal{Q}$ be a subset of $\{1, 2, \dots, L\}$ containing the indices of the blocks that are known to be non-zero, i.e., $\|\bar{\mathbf{u}}_p\|_2 \neq 0, \forall p \in \mathcal{Q}$. This prior information, however, may not be perfectly reliable. To measure the reliability of the side information, we define *reliability*, $\mathcal{R}$, as the ratio of the expected number of correctly identified components of $\mathcal{Q}$ (i.e., the components that represent the true non-zero blocks) to the size of $\mathcal{Q}$. In some applications such as spectrum sensing, when the knowledge about the statistical behavior of the signal occupancy or the changing rate of the support is available, the value of $\mathcal{R}$ would also be available. For instance, if the average changes of the support from a time slot to the next is 5%, set $\mathcal{Q}$ can include the indices of occupied blocks in the previous time slot and $\mathcal{R}$ would be 0.95. It is clear that in the applications in which the support changes slowly over time, $\mathcal{R}$ is close to 1.

Based on the a priori information, we modify Algorithm 3.2.2 by initializing the

weights as follows

$$\omega_i^{(1)} = \begin{cases} 1, & \text{if } i \notin \mathcal{Q} \\ 1 - \mathcal{R} + \tau\xi(\mathcal{R} - 1), & \text{otherwise,} \end{cases} \quad (3.5)$$

where $0 \leq \tau \ll 1$ is a non-negative small constant and $\xi(\cdot)$ is defined by $\xi(x) = 1$ if $x = 0$; and $= 0$, otherwise. If $\mathcal{R} = 1$, it means that all the blocks with indices in $\mathcal{Q}$ are non-zero. The corresponding weights are thus assigned the value τ . We experimentally found that when τ is a small positive constant (e.g., 10^{-2}), IR- ℓ_2/ℓ_1 algorithms have better performance than when $\tau = 0$.⁴ The weighting strategy in the next iterations is the same as the strategy presented in Algorithm 3.2.2.

Our results show that the required number of measurements for the reconstruction decreases by increasing the value of $\mathcal{R}$, and is always smaller than what is needed at the absence of any a priori information.

3.2.4 Sufficient Conditions for the Recovery of Sparse Signals by Weighted ℓ_2/ℓ_1 Minimization: Noiseless Scenario ($\varepsilon = 0$)

In this part, we introduce a sufficient exact recovery condition on the sensing matrix $\mathbf{A}$ so that the solution of (3.2) coincides with the true signal $\bar{\mathbf{x}}$, if for a given $\Gamma \subseteq \{1, \dots, L\}$, $\bar{\mathbf{x}}_\Gamma$ is a P -block sparse signal.

Theorem 3.1: Let $\bar{\mathbf{x}}$ be the true block sparse signal (solution of ℓ_2/ℓ_0 minimization) so that for a given set $\Gamma \subseteq \{1, \dots, L\}$, $\bar{\mathbf{x}}_\Gamma$ is a P -block sparse sub-vector of $\bar{\mathbf{x}}$ and $\hat{\mathbf{x}}$ is a minimizer of (3.2) for $\varepsilon = 0$. Further assume that $\mathbf{A}$ is an $M \times N$ sensing matrix and $\omega_i = 1/(\rho + \|\bar{\mathbf{u}}_i\|_2), \forall i \in \Gamma$, where $\rho > 0$ is a regularization constant. If for all index sets $S \subset \Gamma$ with $|S| \leq P$ and for all $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}) \setminus \mathbf{0}$, we have

$$\sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 < \bar{\gamma} \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2, \quad (3.6)$$

for some $0 < \bar{\gamma} < 1$, where $\boldsymbol{\eta}_i$ is the i -th block of $\boldsymbol{\eta}$ with the block boundaries the

⁴This is particularly the case when the blocks with indices in $\mathcal{Q}$ have relatively small ℓ_2 norms compared to non-zero blocks whose indices are not in $\mathcal{Q}$. In such circumstances, not having the blocks with indices in $\mathcal{Q}$ as part of the objective function will result in over-estimating the ℓ_2 norm of those blocks in the first iteration and will consequently deteriorate the performance of the algorithm.

same as those of $\bar{\mathbf{x}}$, then $\hat{\mathbf{x}}$ will be the unique minimizer of (3.2) and $\hat{\mathbf{x}} = \bar{\mathbf{x}}$.

Proof:

*Lemma 3.1:*⁵ The true signal $\bar{\mathbf{x}} = [\bar{\mathbf{u}}_1^T, \dots, \bar{\mathbf{u}}_L^T]^T$ uniquely solves (3.2) for a given index set Γ if and only if

$$\sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i\|_2 < \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2, \quad \forall \boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}), \quad \boldsymbol{\eta} \neq \mathbf{0}. \quad (3.7)$$

Let $S = \Gamma \cap J$, where J is the index set of non-zero blocks of $\bar{\mathbf{x}}$. Thus, we can write

$$\begin{aligned} \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2 &= \sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2 + \sum_{i \in \Gamma \cap S^c} \omega_i \|\mathbf{0} + \boldsymbol{\eta}_i\|_2 \\ &= \sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2 - \sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i\|_2 + \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 \\ &\quad + \sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i\|_2 - \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 + \sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2 \\ &= \sum_{i \in S} \omega_i \underbrace{(\|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2 - \|\bar{\mathbf{u}}_i\|_2 + \|\boldsymbol{\eta}_i\|_2)}_{\geq 0} \\ &\quad + \sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i\|_2 + \left(\sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2 - \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 \right). \end{aligned} \quad (3.8)$$

Since $\sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i\|_2 = \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i\|_2$ (by the definition of S), it is sufficient to have $\sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2 > \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2$ for (3.7) to hold true. To show that this inequality is valid, we can write

$$\sum_{i \in S} \left(\frac{\rho}{\rho + \|\bar{\mathbf{u}}_i\|_2} \right) \|\boldsymbol{\eta}_i\|_2 \stackrel{(a)}{<} \sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 \stackrel{(b)}{<} \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2, \quad (3.9)$$

where (3.9a) holds true because $\rho/(\rho + \|\bar{\mathbf{u}}_i\|_2) < 1$ for $i \in S$ and (3.9b) follows from (3.6) as $0 < \bar{\gamma} < 1$. By dividing both sides of (3.9) by $\rho > 0$, and also using the fact that $\|\bar{\mathbf{u}}_i\|_2 = \mathbf{0}$ for $i \in \Gamma \cap S^c$, we can write

$$\sum_{i \in S} \frac{1}{\rho + \|\bar{\mathbf{u}}_i\|_2} \|\boldsymbol{\eta}_i\|_2 < \sum_{i \in \Gamma \cap S^c} \frac{1}{\rho + \|\bar{\mathbf{u}}_i\|_2} \|\boldsymbol{\eta}_i\|_2. \quad (3.10)$$

⁵The proof is provided in Appendix A.

Therefore,

$$\sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2 > \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2. \quad (3.11)$$

■

*Proposition 3.1:*⁶ Assume $\Gamma \subseteq \{1, \dots, L\}$ is a given set and $\mathbf{A}$ is an $M \times N$ matrix. If for all index sets $S \subset \Gamma$ and for all $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}) \setminus \mathbf{0}$, we have

$$\sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 < \bar{\gamma} \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2, \quad (3.12)$$

for some $0 < \bar{\gamma} < 1$, where $\boldsymbol{\eta}_i$ is the i th block of $\boldsymbol{\eta}$, then (A.3) implies the linear independence of all the columns of $\mathbf{A}$ in blocks whose indices are in Γ^c .

One should note that the statement of Theorem 3.1 is for a fixed set of weights ω_i and a fixed index set Γ . It therefore does not take into account the evolution of weights and the set Γ throughout iterations.

In the following, we discuss the result of Theorem 3.1 for two special cases of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$.

$\text{IR}^{(1)}\text{-}\ell_2/\ell_1$

Since in $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, an (untruncated) weighted ℓ_2/ℓ_1 minimization is solved in each iteration (i.e., $\Gamma = \{1, \dots, L\}$), the null space property in (3.6) can be rewritten as

$$\sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 < \bar{\gamma} \sum_{i \in S^c} \|\boldsymbol{\eta}_i\|_2. \quad (3.13)$$

Comparing the null space property of the sensing matrix in (3.13) and Inequality (2.2) for the exact recovery of block sparse signals by ℓ_2/ℓ_1 minimization, we can see that the sufficient conditions on matrix $\mathbf{A}$ for exact reconstruction are the same for both untruncated weighted ℓ_2/ℓ_1 minimization and ℓ_2/ℓ_1 minimization. One should note that Condition (2.2) is also a necessary condition on matrix $\mathbf{A}$ for exact recovery by ℓ_2/ℓ_1 minimization (see the proof in [44]). This is, however, not the case for the untruncated weighted ℓ_2/ℓ_1 minimization, i.e., Condition (3.13) is sufficient but not necessary for the success of untruncated weighted ℓ_2/ℓ_1 minimization. From the proof, we can see that Inequality (3.11) is also another sufficient condition on matrix

⁶The proof is provided in Appendix A.

$\mathbf{A}$ for the exact reconstruction and it is not difficult to see that Inequality (3.11) with $\Gamma = \{1, \dots, L\}$ may hold true depending on the values of ω_i , even if (3.13) is not satisfied. In other words, if $\mathbf{A}$ does not satisfy (3.13), ℓ_2/ℓ_1 minimization will fail, yet, weighted ℓ_2/ℓ_1 minimization may still succeed. This implies that $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ minimization would outperform the ℓ_2/ℓ_1 minimization in the recovery of block sparse signals. In fact, our simulation results, presented in Section 3.4, demonstrate that for a given sparsity level, the number of required measurements for an accurate recovery by $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ is significantly smaller than that of the ℓ_2/ℓ_1 minimization.

$\text{IR}^{(2)}\text{-}\ell_2/\ell_1$

In Theorem 3.1, with setting $\omega_i = 1, \forall i \in \Gamma$, the exact recovery condition on $\mathbf{A}$ for the truncated ℓ_2/ℓ_1 minimization is also (3.6). This theorem implies that depending on the set Γ , truncated ℓ_2/ℓ_1 minimization, compared to ℓ_2/ℓ_1 minimization, can reconstruct the signals with a larger number of non-zero blocks. In the exact reconstruction of block sparse signals by the standard ℓ_2/ℓ_1 minimization, the total number of non-zero blocks in the signal can not be greater than P (i.e., exact recovery is possible only when $\bar{\mathbf{x}}$ is a P -block sparse signal). In the truncated ℓ_2/ℓ_1 minimization, however, it is sufficient for $\bar{\mathbf{x}}_\Gamma$ to be a P -block sparse signal, which means $\bar{\mathbf{x}}$ itself can have more than P non-zero blocks.

In the $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ algorithm, the set Γ is updated in each iteration so that the indices of the blocks with the largest ℓ_2 -norm are most likely to be in the set Γ^c . This implies that, as the iterations progress, the signal constrained to the support set Γ is likely to be sparser than the original signal, thus improving the performance compared to the standard ℓ_2/ℓ_1 minimization. This will be verified with our simulation results, presented in Section 3.4.

3.2.5 Error Bound on the Recovery of Approximately Block Sparse Signals with Weighted ℓ_2/ℓ_1 Minimization: Noiseless Scenario ($\varepsilon = 0$)

In this section, we will consider the case of so-called “approximately” block sparse signals and prove that when $\bar{\mathbf{x}}_\Gamma$ is approximately P -block sparse, for a given set Γ , the difference between the solution of weighted ℓ_2/ℓ_1 minimization with $\varepsilon = 0$, $\hat{\mathbf{x}}$, and the true signal, $\bar{\mathbf{x}}$, in ℓ_2/ℓ_1 norm (i.e., $\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$), is bounded from above.

Theorem 3.2: Let $\bar{\mathbf{x}}$ be the true signal (solution of ℓ_2/ℓ_0 minimization) so that for a given set $\Gamma \subseteq \{1, \dots, L\}$, $\bar{\mathbf{x}}_\Gamma$ is an approximately P -block sparse sub-vector of $\bar{\mathbf{x}}$ with parameter δ_Γ . Further, assume that $\mathbf{A}$ is an $M \times N$ sensing matrix such that for all sets $\Upsilon \subseteq \{1, \dots, L\}$ with $|\Upsilon| = |\Gamma|$, all sets $S \subset \Upsilon$ with $|S| = P$ and all $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}) \setminus \mathbf{0}$, we have

$$\sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 < \bar{\gamma} \sum_{i \in \Upsilon \cap S^c} \|\boldsymbol{\eta}_i\|_2, \quad (3.14)$$

for some $0 < \bar{\gamma} < 1$, where $\boldsymbol{\eta}_i$ is the i -th block of $\boldsymbol{\eta}$ with the block boundaries the same as those of $\bar{\mathbf{x}}$. Let π be the proper permutation with respect to the set $\{\|\bar{\mathbf{u}}_i\|_2\}_{i \in \Gamma}$. If $\hat{\mathbf{x}}$ is a minimizer of (3.2) for $\varepsilon = 0$ with $\omega_i = 1/(\rho + \|\bar{\mathbf{u}}_i\|_2)$, $\forall i \in \Gamma$, where $\rho > 0$ is a regularization constant, then

$$\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 \leq 2C\delta_\Gamma, \quad (3.15)$$

with

$$C = \frac{\rho + \kappa}{\rho(1 - \Omega)} \left((1 + \bar{\gamma}) + \bar{\gamma} \max\left\{1, \frac{|\Gamma^c|}{P}\right\} \right), \quad (3.16)$$

$\Omega = \bar{\gamma}(\rho + \kappa)/(\rho + \mu)$, $\mu = \|\bar{\mathbf{u}}_{\pi(P)}\|_2$ and $\kappa = \|\bar{\mathbf{u}}_{\pi(P+1)}\|_2$.

Proof:

Outline: Since $\Gamma \subseteq \{1, \dots, L\}$, we can write $\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 = \sum_{i \in \Gamma} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 + \sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$. We first find upper bounds on $\sum_{i \in \Gamma} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$ and $\sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$, respectively. We then combine them to obtain an upper bound on $\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$.

Upper bound on $\sum_{i \in \Gamma} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$: Since $\hat{\mathbf{x}}$ is the solution of (3.2), we have $\sum_{i \in \Gamma} \omega_i \|\hat{\mathbf{u}}_i\|_2 \leq \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i\|_2$. In addition, let $\boldsymbol{\eta} = \hat{\mathbf{x}} - \bar{\mathbf{x}}$, which means $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A})$. Therefore we have $\sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2 \leq \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i\|_2$. Further, for the set $S \subset \Gamma$, we

can write

$$\begin{aligned}
 \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i\|_2 &\geq \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2 \\
 &= \sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2 + \sum_{i \in \Gamma \cap S^c} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2 \\
 &\geq \sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i\|_2 - \sum_{i \in \Gamma \cap S^c} \omega_i \|\bar{\mathbf{u}}_i\|_2 + \sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2 \\
 &\quad - \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2, \tag{3.17}
 \end{aligned}$$

where the last inequality follows from the triangle inequality. Since $\sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i\|_2 = \sum_{i \in S} \omega_i \|\bar{\mathbf{u}}_i\|_2 + \sum_{i \in \Gamma \cap S^c} \omega_i \|\bar{\mathbf{u}}_i\|_2$ and similarly, $\sum_{i \in \Gamma} \omega_i \|\boldsymbol{\eta}_i\|_2 = \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 + \sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2$, Inequality (3.17) can be rewritten as follows

$$2 \sum_{i \in \Gamma \cap S^c} \omega_i \|\bar{\mathbf{u}}_i\|_2 \geq \sum_{i \in \Gamma} \omega_i \|\boldsymbol{\eta}_i\|_2 - 2 \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2. \tag{3.18}$$

Now, let S be the set $\{\pi(i)\}_{1 \leq i \leq P}$ (i.e., $\Gamma \cap S^c = \{\pi(i)\}_{P+1 \leq i \leq |\Gamma|}$), then we have

$$\sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 = \sum_{i \in S} \frac{1}{\rho + \|\bar{\mathbf{u}}_i\|_2} \|\boldsymbol{\eta}_i\|_2 \leq \frac{1}{\rho + \mu} \sum_{i \in S} \|\boldsymbol{\eta}_i\|_2, \tag{3.19}$$

since for all $i \in S$, $\|\bar{\mathbf{u}}_i\|_2 \geq \mu$ (by the definition of μ). On the other hand, for all $i \in S^c$, $\|\bar{\mathbf{u}}_i\|_2 \leq \kappa$ (where $\kappa = \|\bar{\mathbf{u}}_{\pi(P+1)}\|_2$, by definition). Thus, we have

$$\sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2 = \sum_{i \in \Gamma \cap S^c} \frac{1}{\rho + \|\bar{\mathbf{u}}_i\|_2} \|\boldsymbol{\eta}_i\|_2 \geq \frac{1}{\rho + \kappa} \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2. \tag{3.20}$$

Let $\Upsilon = \Gamma$ in (3.14). By combining (3.14), (3.19) and (3.20), it can be shown that

$$\sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 < \Omega \sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2, \tag{3.21}$$

where $\Omega = \bar{\gamma}(\rho + \kappa)/(\rho + \mu)$. Again by rewriting $\sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2$ as $\sum_{i \in \Gamma} \omega_i \|\boldsymbol{\eta}_i\|_2 - \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2$ in the right hand side of (3.21), it can be easily shown that

$$\frac{1 - \Omega}{\Omega} \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 < \sum_{i \in \Gamma} \omega_i \|\boldsymbol{\eta}_i\|_2 - 2 \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2, \tag{3.22}$$

since $\kappa \leq \mu$ and $0 < \bar{\gamma} < 1$ and thus $0 < \Omega < 1$. Now, (3.22) can be written as

$$\begin{aligned} \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 &< \frac{\Omega}{1 - \Omega} \left(\sum_{i \in \Gamma} \omega_i \|\boldsymbol{\eta}_i\|_2 - 2 \sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 \right) \\ &\leq \frac{2\Omega}{1 - \Omega} \sum_{i \in \Gamma \cap S^c} \omega_i \|\bar{\mathbf{u}}_i\|_2, \end{aligned} \quad (3.23)$$

where the last inequality follows from (3.18). Therefore, we can write

$$\begin{aligned} \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2 &\stackrel{(a)}{\leq} (\rho + \kappa) \sum_{i \in \Gamma \cap S^c} \omega_i \|\boldsymbol{\eta}_i\|_2 \\ &\stackrel{(b)}{\leq} (\rho + \kappa) \left(\sum_{i \in S} \omega_i \|\boldsymbol{\eta}_i\|_2 + 2 \sum_{i \in \Gamma \cap S^c} \omega_i \|\bar{\mathbf{u}}_i\|_2 \right) \\ &\stackrel{(c)}{\leq} (\rho + \kappa) \left(\frac{2\Omega}{1 - \Omega} \sum_{i \in \Gamma \cap S^c} \omega_i \|\bar{\mathbf{u}}_i\|_2 + 2 \sum_{i \in \Gamma \cap S^c} \omega_i \|\bar{\mathbf{u}}_i\|_2 \right) \\ &\stackrel{(d)}{\leq} \frac{2(\rho + \kappa)}{1 - \Omega} \frac{1}{\rho} \sum_{i \in \Gamma \cap S^c} \|\bar{\mathbf{u}}_i\|_2, \end{aligned} \quad (3.24)$$

where (3.24a), (3.24b), (3.24c) and (3.24d), respectively, follow from (3.20), (3.17), (3.23) and the fact that $\omega_i \leq 1/\rho, \forall i$. By setting $\Upsilon = \Gamma$ in (3.14) and using (3.24), we have

$$\begin{aligned} \sum_{i \in \Gamma} \|\boldsymbol{\eta}_i\|_2 &< (1 + \bar{\gamma}) \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2 \\ &< \frac{2(1 + \bar{\gamma})(\rho + \kappa)}{\rho(1 - \Omega)} \sum_{i \in \Gamma \cap S^c} \|\bar{\mathbf{u}}_i\|_2. \end{aligned} \quad (3.25)$$

Finally, by setting $\boldsymbol{\eta}_i = \hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i$, in (3.25), the upper bound on $\sum_{i \in \Gamma} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$ is derived as

$$\sum_{i \in \Gamma} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 < \frac{2(1 + \bar{\gamma})(\rho + \kappa)}{\rho(1 - \Omega)} \sum_{i \in \Gamma \cap S^c} \|\bar{\mathbf{u}}_i\|_2. \quad (3.26)$$

Upper bound on $\sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$: We study two cases:

Case 1. $|\Gamma^c| \leq P$. We can find $S_1 \subset S$ such that $|S_1 \cup \Gamma^c| = P$. Therefore, we can

write:

$$\sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 \leq \sum_{i \in S_1 \cup \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 \stackrel{(a)}{<} \bar{\gamma} \sum_{i \in \Gamma \cap S^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2, \quad (3.27)$$

where (3.27a) follows from the null space property for $\mathbf{A}$ in (3.14). Clearly, $\Upsilon = S_1 \cup \Gamma^c \cup (\Gamma \cap S^c)$ and $|\Upsilon| = |\Gamma|$.

Case 2. $|\Gamma^c| > P$. Let $S_2 \subset \Gamma^c$ be the index set of the P largest $\|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$ for $i \in \Gamma^c$. Therefore, we have

$$\frac{1}{|\Gamma^c|} \sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 < \frac{1}{P} \sum_{i \in S_2} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2. \quad (3.28)$$

On the other hand, from the null space property (3.14), by setting $\Upsilon = S_2 \cup (\Gamma \cap S^c)$, we have

$$\sum_{i \in S_2} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 < \bar{\gamma} \sum_{i \in \Gamma \cap S^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2. \quad (3.29)$$

Therefore, from (3.28) and (3.29), we obtain

$$\sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 < \bar{\gamma} \frac{|\Gamma^c|}{P} \sum_{i \in \Gamma \cap S^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2. \quad (3.30)$$

Combining Case 1, i.e., (3.27) and Case 2, i.e., (3.30), we have

$$\sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 < \max\{1, \frac{|\Gamma^c|}{P}\} \bar{\gamma} \sum_{i \in \Gamma \cap S^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2. \quad (3.31)$$

By setting $\boldsymbol{\eta}_i = \hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i$ in (3.24) and using (3.31), we obtain the following upper bound on $\sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$:

$$\sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 < \frac{2(\rho + \kappa)\bar{\gamma}}{\rho(1 - \Omega)} \max\{1, \frac{|\Gamma^c|}{P}\} \sum_{i \in \Gamma \cap S^c} \|\bar{\mathbf{u}}_i\|_2. \quad (3.32)$$

Finally, by using the two upper bounds of (3.26) and (3.32), we can write

$$\begin{aligned} \sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 &= \sum_{i \in \Gamma} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 + \sum_{i \in \Gamma^c} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 \\ &\leq 2C \sum_{i \in \Gamma \cap S^c} \|\bar{\mathbf{u}}_i\|_2, \end{aligned} \quad (3.33)$$

where

$$C = \frac{\rho + \kappa}{\rho(1 - \Omega)} \left(1 + \bar{\gamma} + \bar{\gamma} \max\left\{1, \frac{|\Gamma^c|}{P}\right\} \right). \quad (3.34)$$

Since S is assumed to be the index set of the P largest elements of $\{\|\bar{\mathbf{u}}_i\|_2\}_{i \in \Gamma}$, based on the definition of $\bar{\mathbf{x}}_\Gamma$, we have $\sum_{i \in \Gamma \cap S^c} \|\bar{\mathbf{u}}_i\|_2 \leq \delta_\Gamma$. This together with (3.33) results in (3.15). ■

Similar to Theorem 3.1, Theorem 3.2 is limited to fixed values of ω_i and a fixed set Γ . The evolution of these entities with iterations is not considered.

In the following, we discuss the result of Theorem 3.2 for the special cases of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$.

$\text{IR}^{(1)}\text{-}\ell_2/\ell_1$

When $\Gamma = \{1, \dots, L\}$, the null space property for weighted ℓ_2/ℓ_1 in (3.14) is reduced to that of ℓ_2/ℓ_1 in (2.4) and the upper bound in (3.26) on $\sum_{i \in \Gamma} \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2$ is reduced to

$$\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 \leq 2C^{(1)} \sum_{i \in S^c} \|\bar{\mathbf{u}}_i\|_2 \leq 2C^{(1)} \delta, \quad (3.35)$$

where $C^{(1)} = (1 + \bar{\gamma})(\rho + \kappa) / (\rho(1 - \Omega))$ and the last inequality follows from the assumption that S is the index set of the P largest elements of $\{\|\bar{\mathbf{u}}_i\|_2\}_{1 \leq i \leq L}$. It can be easily shown that, for sufficiently large $\bar{\gamma}$, i.e., $\bar{\gamma} > (\kappa(\rho + \mu)) / (\mu(\rho + \kappa))$, the upper bound in (2.3) is larger than that of (3.35). As can be seen in (2.4), for a given $\mathbf{A}$, $\bar{\gamma}$ is monotonically increasing in P . Therefore, larger values of $\bar{\gamma}$ are of interest. This implies that the proposed algorithm would be superior to the ℓ_2/ℓ_1 minimization in the reconstruction of approximately block sparse signals. This is also verified by our simulation results presented in Section 3.4.

$\mathbf{IR}^{(2)}\text{-}\ell_2/\ell_1$

In truncated ℓ_2/ℓ_1 minimization, since $\omega_i = 1, \forall i \in \Gamma$, we have

$$\begin{aligned}
 \sum_{i=1}^L \|\boldsymbol{\eta}_i\|_2 &= \sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 + \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2 + \sum_{i \in \Gamma^c} \|\boldsymbol{\eta}_i\|_2 \\
 &\stackrel{(a)}{\leq} \left(1 + \bar{\gamma} + \bar{\gamma} \max\{1, \frac{|\Gamma^c|}{P}\}\right) \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2 \\
 &\stackrel{(b)}{<} \frac{2}{1 - \bar{\gamma}} \left(1 + \bar{\gamma} + \bar{\gamma} \max\{1, \frac{|\Gamma^c|}{P}\}\right) \sum_{i \in \Gamma \cap S^c} \|\bar{\mathbf{u}}_i\|_2 \\
 &\stackrel{(c)}{\leq} 2C^{(2)}\delta_\Gamma,
 \end{aligned} \tag{3.36}$$

where $C^{(2)} = (1 + \bar{\gamma} + \bar{\gamma} \max\{1, |\Gamma^c|/P\})/(1 - \bar{\gamma})$. Inequality (3.36a) follows from (3.31) and the null space property given in (3.14) as $\Upsilon = \Gamma$, inequality (3.36b) follows from (3.18) when $\omega_i = 1, \forall i \in \Gamma$, and the null space property of (3.14) as $\Upsilon = \Gamma$, and finally, Inequality (3.36c) follows from the assumption that set $S \subset \Gamma$ is the index set of the P largest elements of $\{\|\bar{\mathbf{u}}_i\|_2\}_{i \in \Gamma}$.

Unlike the case for the upper bound of (3.35), it is not easy to compare the error bound of (3.36) with that of (2.3). Although, $C^{(2)}$ is clearly larger than $(1 + \bar{\gamma})/(1 - \bar{\gamma})$, it is not as easy to compare δ with δ_Γ . Since $\bar{\mathbf{x}}_\Gamma$ is approximately P -block sparse, $\bar{\mathbf{x}}$ would be an approximately P' -block sparse with $P' > P$. Therefore, according to the definition of δ and δ_Γ , we can not easily compare them. In $\mathbf{IR}^{(2)}\text{-}\ell_2/\ell_1$ algorithm, Γ is iteratively updated so that the blocks with larger ℓ_2 -norm are likely to be in Γ . Therefore, we expect that δ_Γ would be much smaller than δ such that the effect of $C^{(2)}$ would be dominated by the effect of δ_Γ . Particularly, when the ℓ_2 -norm of non-zero blocks decays fast enough if sorted in descending order, $\delta_\Gamma \ll \delta$. This would imply that the upper bound in (3.36) can be smaller than (2.3). Thus, $\mathbf{IR}^{(2)}\text{-}\ell_2/\ell_1$ may outperform the ℓ_2/ℓ_1 minimization. Our simulation results in Section 3.4, in fact, verify the superior performance of $\mathbf{IR}^{(2)}\text{-}\ell_2/\ell_1$ minimization in the reconstruction of approximately block sparse signals compared to the standard ℓ_2/ℓ_1 minimization.

3.3 Wideband Compressed Spectrum Sensing

The problem of wideband spectrum sensing therefore fits well within the framework of block sparse signal recovery discussed here and we can apply the iterative ℓ_2/ℓ_1 minimization algorithms, i.e., $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$, to this problem in order to reduce the sampling rate of the AIC.

Suppose that $r(t)$ is the wideband analog signal received and down-converted by a cognitive radio and that it spans the frequency range of $[0, W] = [0, f_L]\Delta$, where W is the bandwidth of $r(t)$ and Δ is the frequency resolution.⁷ We assume that $r(t)$ occupies L non-overlapping consecutive spectrum bands, referred to as “sub-channels”. We denote the boundaries of the sub-channels by $0 = f_0 < f_1 < \dots < f_L$. Without loss of generality, we also assume that the sub-channels have equal bandwidth $B = W/L$ Hz.

The signal $r(t)$ is assumed to be sparse in the frequency domain. Let vector $\bar{\mathbf{r}} \in \mathbb{R}^{2N}$ denote the discrete representation of the analog signal $r(t)$, where $2N$ is the required number of samples for the recovery of the signal without aliasing. Vector $\bar{\mathbf{x}} = \mathbf{F}\bar{\mathbf{r}}$ is the spectrum of $\bar{\mathbf{r}}$, where $\bar{\mathbf{x}} \in \mathbb{C}^{2N}$ has only $2K$ non-zero elements ($K \ll N$) in $2P$ clusters and $\mathbf{F}$ is the $2N \times 2N$ unitary Fourier matrix whose (i, j) -th entry is $\frac{1}{\sqrt{N}}e^{-\pi(i-1)(j-1)/N}$, for $1 \leq i, j \leq 2N$. Since $\bar{\mathbf{r}}$ is a real signal, the spectrum is conjugate symmetric. Due to this symmetry, we discard the negative frequency samples and regenerate them from the positive ones.

In compressed spectrum sensing, M measurements ($M \ll 2N$) are obtained through linear combinations of the elements of $\bar{\mathbf{r}}$, e.g., using an AIC module [79]. In the presence of noise, we have $\mathbf{y} = \Phi\bar{\mathbf{r}} + \mathbf{z} = \Phi\mathbf{F}^{-1}\bar{\mathbf{x}} + \mathbf{z}$, where $\mathbf{z}$ is a noise vector whose elements are i.i.d. zero-mean Gaussian random variables with variance σ_n^2 . In this work, we are interested in the recovery of the spectrum from the measurements. We would also like to detect the spectrum holes after the reconstruction of the spectrum, i.e., we are interested in detecting which of the L sub-channels are (un)occupied. For this, we pass the estimate $\hat{\mathbf{x}}$ through a thresholding device to determine the $L \times 1$ binary spectrum occupancy vector, $\hat{\mathbf{d}}$, as follows:

$$\hat{d}_l = \begin{cases} 1, & \text{if } \hat{s}_l \geq \lambda; \\ 0, & \text{otherwise,} \end{cases} \quad (3.37)$$

⁷Frequency resolution is referred to as the distance between two frequency samples.

Figure 3.1: Probability of success of (a) $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, and (b) $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ for different number of iterations versus M for $N = 256$, $L = 32$ and Signal Model I. Solid and dashed curves correspond to $P = 8$ and $P = 3$, respectively.

where λ is a properly-selected decision threshold and

$$\hat{s}_l = \frac{1}{f_l - f_{l-1}} \sum_{i=f_{l-1}}^{f_l-1} |\hat{x}_i|^2, \text{ for } l = 1, \dots, L. \quad (3.38)$$

One can evaluate the detection performance of a given method by computing the detection probability of occupied sub-channels as

$$P_D = \mathbf{d}^T \hat{\mathbf{d}} / (\mathbf{1}^T \mathbf{d}), \quad (3.39)$$

and the false alarm probability as

$$P_{FA} = (\mathbf{1} - \mathbf{d})^T \hat{\mathbf{d}} / (L - \mathbf{1}^T \mathbf{d}), \quad (3.40)$$

where the size- L binary vector $\mathbf{d}$ represents the true frequency occupancy. In particular, a larger P_D and a smaller P_{FA} is desirable.

3.4 Simulation Results

In this section, we present some simulation results to demonstrate the effectiveness of IR- ℓ_2/ℓ_1 algorithms with focusing on IR⁽¹⁾- ℓ_2/ℓ_1 and IR⁽²⁾- ℓ_2/ℓ_1 as recovery algorithms for block sparse signals in both noiseless and noisy scenarios. We also provide comparisons with other recovery algorithms. Subsections 3.4.1 and 3.4.2 of this section are concerned with the recovery of signals with real elements in noiseless and noisy scenarios, respectively. Subsection 3.4.3 is devoted to the application of the proposed algorithms to the problem of wideband spectrum sensing in which the signal is complex. In Subsection 3.4.1, we first discuss the recovery of ideally block sparse signals, followed by that of approximately block sparse signals and recovery at the presence of a priori information about the location of non-zero blocks. In the first part itself, the performance of the proposed algorithms, the role of block length on the performance, the effect of signal model on recovery, comparison with other recovery algorithms, and other reweighted iterative algorithms are discussed, respectively.

We consider a signal with dimension N , which consists of L non-overlapping, consecutive and equal-sized blocks. The measurement matrix is randomly constructed with i.i.d. elements, where each element is a zero-mean Gaussian random variable with variance $1/N$. We also assume that the basis matrix Ψ is the $N \times N$ identity matrix $\mathbf{I}_N$. Therefore, $\mathbf{A} = \Phi$. For IR⁽²⁾- ℓ_2/ℓ_1 , we use $\beta = 4$ for noiseless scenarios and $\beta = 2$ for noisy scenarios.

Each experiment is repeated 1000 times and each trial for an experiment consists of (a) randomly creating a signal $\bar{\mathbf{x}}$, (b) constructing a random $M \times N$ measurement matrix, (c) creating M measurements accordingly (and adding noise to each measurement in noisy scenarios), and (d) obtaining $\hat{\mathbf{x}}$ using the reconstruction algorithm.

3.4.1 Noiseless Scenarios

In noiseless scenarios, we declare “success” in the recovery of a signal when $\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 / \sum_{i=1}^L \|\bar{\mathbf{u}}_i\|_2 \leq 10^{-3}$.

1) Ideally Block Sparse Signals

To model a P -block sparse signal, we select P blocks (out of L) randomly and assign the entries of these blocks using a zero-mean Gaussian distribution with variance σ_s^2 in an i.i.d. fashion. The entries of the other blocks are set to zero. We set $\sigma_s^2 = 1$,

Figure 3.2: Probability of success of (a) IR⁽¹⁾- ℓ_2/ℓ_1 , and (b) IR⁽²⁾- ℓ_2/ℓ_1 for different number of iterations versus M for two different block lengths of 8 and 32 ($N = 1024$, $K = 256$ and Signal Model I). Solid and dashed curves correspond to $L = 128$ (block length of $N/L = 8$) and $L = 32$ (block length of $N/L = 32$), respectively.

and then normalize the signal such that $\sum_{i=1}^L \|\bar{\mathbf{u}}_i\|_2 = 1$. This model is referred to as Model I.

a) Performance of IR- ℓ_2/ℓ_1 Algorithms: In this part, we consider signals with $N = 256$ and $L = 32$. Fig. 3.1 shows the probability of success of IR⁽¹⁾- ℓ_2/ℓ_1 and IR⁽²⁾- ℓ_2/ℓ_1 versus M for different number of iterations and two sparsity ratios of 25% ($K = 64$ and $P = 8$) and $\approx 10\%$ ($K = 24$ and $P = 3$). We observe that most of the benefit comes from the first few iterations. Therefore, the added computational cost compared to the standard ℓ_2/ℓ_1 minimization is rather small. Furthermore, as seen in these figures, the convergence speed of IR⁽²⁾- ℓ_2/ℓ_1 is, in general, higher than that of IR⁽¹⁾- ℓ_2/ℓ_1 . While IR⁽²⁾- ℓ_2/ℓ_1 converges in 2-3 iterations, this number for IR⁽¹⁾- ℓ_2/ℓ_1 is about 5-6. We also see that for a fixed N and a fixed ratio of M/K , the probability of success increases with M (or K) in both iterative algorithms. For example, in Fig. 3.1(a), when $M = 30$ and $P = 3$, the probability of success is 0.6 after 5 iterations while when $M = 80$ and $P = 8$, the probability of success is more than 0.9 after 5 iterations. In Fig. 3.1(b), also, the probability of success is 0.3 after 2 iterations with $M = 30$ and $P = 3$, while the probability of success is 0.65 after 2 iterations with $M = 80$ and $P = 8$.

b) Effect of Block Length on Recovery Performance: Fig. 3.2 demonstrates the performance of IR⁽¹⁾- ℓ_2/ℓ_1 and IR⁽²⁾- ℓ_2/ℓ_1 when $N = 1024$, $K = 256$ and thus the

sparsity ratio $K/N = P/L$ is 0.25. Two different block lengths N/L of 8 and 32 are considered in Fig. 3.2. As demonstrated in [44], increasing the block length leads to an improvement in the recovery performance of ℓ_2/ℓ_1 minimization. Since IR- ℓ_2/ℓ_1 algorithms execute ℓ_2/ℓ_1 minimization in the first iteration, a better performance of the latter would imply a better performance for the former. In addition, this general trend is somehow expected since between the two signals with the same dimension and sparsity, the one with smaller blocks on average contains more uncertainty. In fact, Fig. 3.2 shows that the performance of both iterative algorithms improves with increasing the block length. For instance, for the larger block length of 32, in the 8th iteration, IR⁽¹⁾- ℓ_2/ℓ_1 succeeds with overwhelming probability in reconstruction of the block sparse signals when the number of measurements is only 4% more than the theoretical limit of K , i.e., 266 (Fig. 3.2(a)). In addition, with $M = K = 256$, the probability of success of 0.97 can be achieved when the block length is 32 ($P = 8$).

Moreover, Fig. 3.2(b) shows that IR⁽²⁾- ℓ_2/ℓ_1 converges even faster as the length of the blocks increases. This is due to the fact that by increasing the block length, the difference between the ℓ_2 -norm of non-zero blocks decreases on average.

c) *Effect of Signal Model on IR- ℓ_2/ℓ_1 Algorithms:* To investigate the effect of signal model on the performance of the recovery algorithms, we use a second signal model, referred to as Model II. Similar to Model I, P out of L blocks are chosen randomly as non-zero blocks. The elements of the p -th non-zero block, however, follow a zero mean Gaussian distribution with variance σ_p^2 (i.e., $\mathcal{N}(0, \sigma_p^2)$), where σ_p is the p -th element of a random permutation of the sequence $\{i^{-1/\theta}\}_{i=1}^P$ for a fixed θ . Clearly, increasing θ results in decreasing the decaying rate of the ℓ_2 -norm of non-zero blocks. We set $\theta = 0.7$, and normalize $\bar{\mathbf{x}}$ so that $\sum_{i=1}^L \|\bar{\mathbf{u}}_i\|_2 = 1$. In Figs. 3.3(a) and 3.3(b), we have shown ℓ_2 -norm of blocks of a realization of a block sparse signal with $N = 256$, $L = 32$ and $P = 8$ that follows Model I and II, respectively. We observe that in Model II, ℓ_2 -norm of the non-zero blocks decays faster than Model I, if sorted in descending order. We found that when the ℓ_2 -norm of non-zero blocks decays faster, the convergence speed of IR⁽²⁾- ℓ_2/ℓ_1 would be slower while the convergence speed of IR⁽¹⁾- ℓ_2/ℓ_1 does not change much. Therefore, we set j_{max} to 5 for both IR⁽¹⁾- ℓ_2/ℓ_1 and IR⁽²⁾- ℓ_2/ℓ_1 for Signal Model II. Fig. 3.4 shows the probability of success of iterative reweighted algorithms and the standard ℓ_2/ℓ_1 minimization for the two signal models. Unlike the ℓ_2/ℓ_1 minimization, the performance of the proposed algorithms depends on the signal model. In particular, IR⁽²⁾- ℓ_2/ℓ_1 performs better on Model-II signals,

Figure 3.3: ℓ_2 -norm of the blocks in one realization of the signal for Signal Models I and II ($N = 256$, $K = 64$, $L = 32$ and $P = 8$).

while the trend for $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ is opposite. In fact, the results show that for signals with Model II, $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ outperforms $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$.

In the $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ algorithm, the threshold $\epsilon^{(j)}$ decreases as the iterations proceed. Therefore, for Signal Model I, after a very small number of iterations, there would not be any non-zero blocks left in $\bar{\mathbf{x}}_\Gamma$. This thus implies that the performance will not be improved in the subsequent iterations. For Signal Model II, however, the performance would still be improved by increasing the number of iterations since it is more likely to have some non-zero blocks left in $\bar{\mathbf{x}}_\Gamma$. On the other hand, a slight deterioration in the recovery performance of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ is expected with Signal Model II. This is due to the zero blocks that are likely to be estimated as non-zero blocks with small ℓ_2 -norm, in the early iterations. In the subsequent iterations, those zero blocks and non-zero blocks with small ℓ_2 -norm are weighted the same. However, for the recovery of Model-I signals, non-zero blocks are likely to have much smaller weights than zero blocks.

d) Comparison with Other Recovery Algorithms: In Fig. 3.5, we compare the performance of iterative reweighted ℓ_2/ℓ_1 minimization algorithms to that of ℓ_1 minimization [80], $\text{IR}\ell_1$ [6], ISD [14], IRLS [11], standard ℓ_2/ℓ_1 minimization [44] and James-Stein AMP (AMP-JS) [48] for $N = 256$, $L = 32$, and $P = 8$. Based on the results of Fig. 3.1, we select the maximum number of iterations (i.e., $j_{\max}$) for $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ to be 5 and for $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ to be 3. We found that the selected parameters (the number of iterations and β) for $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ are also good choices for $\text{IR}\ell_1$ and ISD, respectively. We set the parameter ρ in $\text{IR}\ell_1$ the same as that in

Figure 3.4: Probability of success of different algorithms for Signal Models I and II versus M ($N = 256$, $K = 64$, $L = 32$ and $P = 8$). Solid and dashed lines correspond to Model I and Model II, respectively.

$\text{IR}^{(1)}-\ell_2/\ell_1$, i.e., we set ρ_0 to the standard deviation of nonzero signal components and reduce its value by a factor of 10 in each iteration if Inequality (3.4) holds. The parameters of IRLS are set to those recommended in [11]. AMP-JS is performed with the maximum number of iterations equal to 300. The algorithm stops if the maximum number of iterations is reached or if the normalized ℓ_2 -norm of the difference between successive estimates, i.e., $\|\hat{\mathbf{x}}^{(i)} - \hat{\mathbf{x}}^{(i-1)}\|_2 / \|\hat{\mathbf{x}}^{(i)}\|_2$, is less than 10^{-5} . Fig. 3.5 shows that the two iterative algorithms outperform all the other algorithms, including standard ℓ_2/ℓ_1 and AMP-JS, by a large margin. In particular, we note that the smallest value of M that results in the probability of success of almost one is 92 and 96 for $\text{IR}^{(1)}-\ell_2/\ell_1$ and $\text{IR}^{(2)}-\ell_2/\ell_1$, respectively. This value, however, is greater than 110 for all the other algorithms, including ℓ_2/ℓ_1 minimization and AMP-JS.

To have a more clear picture of the comparison between the proposed algorithms and AMP-JS, we have shown the probability of success curves of the three algorithms for Signal Model I with $N = 1024$ and $K = 256$ and two block lengths of $N/L = 8$ and 32 in Fig. 3.6. The curves for standard ℓ_2/ℓ_1 minimization are also included for reference. Fig. 3.6 shows that while the performance of AMP-JS improves by increasing the block length, it is still far from the performance of the proposed algorithms. More results for $N = 5120$ and $K = 1280$ are presented in Fig. 3.7, where $\text{IR}^{(1)}-\ell_2/\ell_1$

Figure 3.5: Probability of success of different algorithms versus M ($N = 256$, $K = 64$, $L = 32$, $P = 8$ and Signal Model I).

and AMP-JS are compared at different block lengths. The maximum number of iterations for AMP-JS in Fig. 3.7 is selected to be 1000, and the stopping criterion for the normalized ℓ_2 -norm of the difference between successive estimates is set at 10^{-6} .⁸ For $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, we use $j_{\max} = 10$ for block lengths 16 and 32, and $j_{\max} = 8$ for block length 64. Fig. 3.7 shows that the performance of AMP-JS improves with increasing the block length but saturates at about $N/L = 32$. In all block lengths, $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ outperforms AMP-JS, and the performance of the former is practically the same as the theoretical limit even for the block length of 16.

To compare the computational complexity of different recovery algorithms, in Tables 3.1-3.3, we have reported the average runtime of the algorithms for scenarios corresponding to Figs. 3.5-3.7, respectively. All algorithms that involve solving a convex optimization problem (ℓ_1 , $\text{IR}\ell_1$, ISD, ℓ_2/ℓ_1 , $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$) are implemented using SpaRSA [77, 81]. In such implementations, the maximum number of iterations and the stopping criterion for the normalized ℓ_2 -norm of the difference

⁸One should note that for the AMP-JS results presented in Fig. 3.7, we declare success if the normalized error is less than 0.01 rather than 0.001 used for $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$. If such an advantage is not given to AMP-JS, its performance measured by the probability of success will deteriorate noticeably. For example, for the scenario depicted in Fig. 3.7, with the normalized error threshold of 0.001 for success, AMP-JS with $N/L \geq 32$ requires about $M = 1340$ to achieve a success probability of 0.6 instead of $M = 1300$ as shown in Fig. 3.7.

Figure 3.6: Probability of success of different recovery algorithms versus M for block lengths of $N/L = 8$ (solid line) and 32 (dashed line) ($N = 1024$, $K = 256$ and Signal Model I). For $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$, $j_{\max}$ is selected to be 8 and 3, respectively.

Figure 3.7: Probability of success of AMP-JS and $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ for different block lengths versus M ($N = 5120$, $K = 1280$ and Signal Model I).

between successive estimates are set at the same values as those of AMP-JS in each scenario, i.e., 300 and 10^{-5} , respectively, for the scenarios in Figs. 3.5 and 3.6, and 1000 and 10^{-6} , respectively, for the scenario of Fig. 3.7. The reported runtimes are based on running the algorithms in MATLAB 7.11.0 environment on a desktop computer with an Intel Core i7 2.8 GHz CPU and 8 GB of RAM, and under the Microsoft Windows 7 operating system. The number of measurements for the results of Tables 3.1-3.3 are selected to be $M = 170$, 520 and 1330, respectively. These are the smallest values that result in success probability of about one for all the algorithms involved in each scenario. The results of Table 3.1 show that among all recovery algorithms, the one that performs the best, i.e., $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, has also the largest runtime. This

runtime, however, is not much larger than those of the other algorithms. In particular, the average runtime of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ is only about 3 and 5 times those of ℓ_2/ℓ_1 minimization and AMP-JS, respectively. The results of Tables 3.2 and 3.3 show that the runtime of all algorithms reduces with increasing the block length. These results also demonstrate that, for $N = 1024$, the average runtime of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ is about 5 times that of AMP-JS for both block lengths 8 and 32. This ratio increases to about 9 to 10 for $N = 5120$ at different block lengths.

e) Other $\text{IR}\text{-}\ell_2/\ell_1$ Algorithms: Clearly, in $\text{IR}\text{-}\ell_2/\ell_1$ algorithms, the reweighting strategy plays an important role in the reconstruction performance. So far in this chapter, we focused on two special cases of these algorithms. In general, however, there may be other reweighting strategies that result in better reconstruction performance depending for example on the signal distribution or the desirable trade-off between performance and the speed of convergence. For instance, one can execute Algorithm 1 with both support detection and weight-updating steps (i.e., Steps 2(a) and 2(b)). We refer to this option as $\text{IR}^{(3)}\text{-}\ell_2/\ell_1$, and consider the case with $\beta = 3$. The probability of success of this algorithm applied to block sparse signals with $N = 256$, $L = 32$ and $P = 8$ is depicted in Fig. 3.8 versus M for Signal Models I and II when the maximum number of iterations is set to 3 and 5, respectively. For comparison, we have also presented the results for $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$. To have a fair comparison, for each signal model, we use the same number of iterations for all algorithms. Fig. 3.8 demonstrates that for both signal models, $\text{IR}^{(3)}\text{-}\ell_2/\ell_1$ outperforms the other two algorithms. Note that since $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ needs more iterations to converge, by setting $j_{\max}$ to a small number, its performance would be inferior to $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(3)}\text{-}\ell_2/\ell_1$. However, if $j_{\max}$ is selected to be a larger number, $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ would outperform the other two iterative algorithms.

2) Approximately Block Sparse Signals

An approximately block sparse signal is generated by two Gaussian distributions as in [82]. In this model, we select P blocks randomly and assign the entries of these blocks using a zero-mean Gaussian distribution with variance σ_s and the entries of the rest of the blocks are assigned using another zero-mean Gaussian distribution with variance σ'_s , where $\sigma'_s < \sigma_s$. For each set of blocks, the entries are selected in an i.i.d. fashion, and independent of the entries of the other set's blocks. Here, we set $\sigma_s = 1$

Figure 3.8: Probability of success of different iterative reweighted ℓ_2/ℓ_1 recovery algorithms versus M ($N = 256$, $L = 32$ and $P = 8$). Solid lines correspond to Signal Model I with $j_{max} = 3$ and dashed lines correspond to Signal Model II with $j_{max} = 5$.

and $\sigma'_s = 0.01$, then normalize the signal such that $\sum_{i=1}^L \|\bar{\mathbf{u}}_i\|_2 = 1$.

In Fig. 3.9, we have provided the normalized reconstruction error (i.e., $\sum_{i=1}^L \|\hat{\mathbf{u}}_i - \bar{\mathbf{u}}_i\|_2 / \sum_{i=1}^L \|\bar{\mathbf{u}}_i\|_2$) of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ and the standard ℓ_2/ℓ_1 minimization when the signal is approximately block sparse with $N = 256$, $L = 32$ and $P = 8$. The parameter j_{max} for $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ is set to 5 and 3, respectively. Fig. 3.9 demonstrates that for smaller values of M , the reconstruction error of the iterative methods is, in general, much smaller than that of the standard method. The anomaly is for values of M less than about 70, where $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ performs inferior to the standard ℓ_2/ℓ_1 . This can be attributed to the low reliability of support detection in the first iteration that consequently deteriorates the performance of $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ in subsequent iterations.

3) Block Sparse Signal Recovery with A Priori Information

In the following experiment, we study the effect of a priori information about non-zero blocks on the performance of the proposed algorithms. We use Signal Model I with $N = 256$, $L = 32$, $P = 8$ and set $\tau = 0.01$ to initialize the weights. Figs. 3.10(a) and (b) show the probability of success of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ for different combinations of $|\mathcal{Q}|$ and $\mathcal{R}$, respectively. The parameter j_{max} for the two algorithms are selected to be 5 and 3, respectively. Note that the solid curves belong to the cases where the a priori information is perfectly reliable ($\mathcal{R} = 1$). In both figures, for a given value of $\mathcal{R}$ ($|\mathcal{Q}|$), increasing $|\mathcal{Q}|$ ($\mathcal{R}$) improves the success probability. (In

particular, for both algorithms, if $|\mathcal{Q}| = P = 8$ and $\mathcal{R} = 1$, the success probability tends to one for $M > K = 64$.) This is expected as increasing $|\mathcal{Q}|$ ($\mathcal{R}$) increases the total amount of reliable a priori information. The figure also shows that any a priori information improves the performance compared to the case with no a priori information ($|\mathcal{Q}| = 0$).

3.4.2 Noisy Scenarios

In this subsection, we consider the case where measurements are contaminated with noise. For the measurement matrix and the generation of results based on random experiments, we follow the same assumptions as those in Subsection A. The only difference is that, we add i.i.d. zero-mean Gaussian noise with variance σ_n^2 to measurements. For the signal, we use Model I but without any normalization.

Based on the results presented in Subsection V.A, in the absence of noise, increasing the number of iterations improves the performance of the proposed iterative algorithms. In the presence of noise, however, this is not generally the case. In particular, for a given SNR, there is a certain threshold I_{th} on the number of iterations beyond which the performance starts to deteriorate. This implies that there is a threshold SNR_{th} on the SNR values for which the proposed algorithms provide an improvement over the standard ℓ_2/ℓ_1 minimization, i.e., for SNR values less than SNR_{th} , $I_{th} = 1$. In the following, for each SNR value, we select $j_{max} = I_{th}$ for the iterative algorithms.

We consider a block sparse signal with $N = 256$, $L = 32$ and $P = 8$. In Figs. 3.11(b)-(d), we have reported the performance of the proposed algorithms in comparison with the standard ℓ_2/ℓ_1 minimization and AMP-JS for $\sigma_n = 0.01, 0.05$ and 0.1 , respectively. In Fig. 3.11(a), curves for noiseless scenario are presented for reference. The figures show that although, $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ performs inferior to $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ for smaller values of M , it performs practically the same as $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ for larger values of M . Moreover, the figures demonstrate that the proposed algorithms consistently outperform the standard ℓ_2/ℓ_1 minimization, although the amount of improvement decreases with decreasing SNR. As expected, $j_{max} = I_{th}$ has also a decreasing trend with SNR for both iterative algorithms. Compared to AMP-JS, although the proposed algorithms are superior in the high SNR region and smaller number of measurements (small under-sampling ratios), they are outperformed by AMP-JS at lower

SNR values and larger number of measurements.

To study the effect of block length on the relative performance of different algorithms in the presence of noise, we also consider a block sparse signal with $N = 1024, L = 32$ and $P = 8$ (same sparsity ratio of 0.25 as in Fig. 3.11 but larger block length of 32 instead of 8). Results similar to those of Fig. 3.11 are presented in Fig. 3.12 for this case. In comparison with the results of Fig. 3.11, one can see that increasing the block length works in favor of the proposed algorithms and increases the performance gap between these algorithms, on the one hand, and standard ℓ_2/ℓ_1 minimization and AMP-JS, on the other hand. In particular, the proposed algorithms outperform AMP-JS for all SNR values and almost all values of M simulated in Fig. 3.12. The exception is for the largest M values at the right hand side of Figs. 3.12 (c) and (d).

3.4.3 Wideband Compressed Spectrum Sensing

a) Noiseless Scenario:

To apply $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ in the context of wideband spectrum sensing, we consider a spectrum where only 25% of the sub-channels are occupied ($N = 128, K = 32, L = 16$ and $P = 4$). To model the spectrum, we randomly select P out of L sub-channels (from positive frequency samples) and assign the same amplitude to all the spectrum components of each sub-channel. This amplitude, for each sub-channel, is selected randomly using a uniform distribution in the range of $[\nu, 1 + \nu]$ where $\nu > 0$ is the minimum level of the spectrum in an active sub-channel. In our simulations, we set $\nu = 0.1$. We also assign the phase of the spectrum components using a uniform distribution in the range of $[0, 2\pi)$ in an i.i.d. fashion. Note that the whole spectrum in each sub-channel has the same amplitude but not the same phase. An example of a noise free $|\bar{\mathbf{x}}|$ is presented in Fig. 3.13(a) (spectrum plots are displayed for positive frequencies). For this signal, the results of recovery by ℓ_1 minimization, $\text{IR}\ell_1$, IRLS, ISD, ℓ_2/ℓ_1 minimization, $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ for $M = 80$ are presented in Figures 3.13(b)-(h), respectively. These results indicate that the first four algorithms essentially fail in the reconstruction of the input signal when $M = 80$. As can be seen, ℓ_2/ℓ_1 minimization is capable of rough reconstruction of the spectrum. However, it fails in the reconstruction of the blocks with small ℓ_2 -norm. Accurate reconstruction is achieved by both proposed algorithms.

b) Noisy Scenario:

Here, we consider the signal model described in Part 3.4.3(a) and assume that the measurements are contaminated with a Gaussian noise with variance σ_n^2 . To identify the occupied sub-channels, we obtain the binary spectrum occupancy vector by using the thresholding technique in (3.37). The threshold λ in (3.37) is found by fixing P_{FA} to 0.01 for each M . For this fixed value of P_{FA} , P_D is plotted versus M for ℓ_2/ℓ_1 minimization and the two proposed algorithms in Fig. 3.14 for values of $\sigma_n = 0.05$ and 0.1. The parameter j_{max} is selected to be 3 and 2, for $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$, respectively. As can be seen, for both values of σ_n , $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ outperforms the other algorithms over the whole range of M values. Although $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ performs inferior to $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ for smaller values of M , it performs practically the same as $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ for values of M larger than 75 for $\sigma_n = 0.05$ and larger than 80 for $\sigma_n = 0.1$. Both proposed algorithms are substantially superior to the standard ℓ_2/ℓ_1 minimization.

3.5 Conclusion

In this chapter, we proposed a general iterative framework for the recovery of block sparse signals. The family of algorithms defined by this framework are based on iterative application of weighted ℓ_2/ℓ_1 minimization and are thus referred to as iterative reweighted ℓ_2/ℓ_1 minimization ($\text{IR}\text{-}\ell_2/\ell_1$). Different algorithms within the family differ in the way that the weights are updated from one iteration to the next. We focused particularly on two special cases in this family, i.e., $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$. We analyzed both algorithms and discussed their application to block sparse signals in the absence and presence of a priori information about the support of the signal. We derived theoretical guarantees and showed that an accurate reconstruction was possible if the sensing matrix satisfied certain conditions. In the case of approximately block sparse signals, reconstruction error bounds for both algorithms were also derived. Moreover, we applied the proposed iterative algorithms to the problem of wideband spectrum sensing.

Our results demonstrated that both proposed algorithms can achieve a significant

improvement over other recovery algorithms including the standard ℓ_2/ℓ_1 minimization and the recently popular approximate message-passing (AMP) in noiseless scenarios and high SNR regimes. Moreover, our results showed that as the length of the blocks increases, the required number of measurements for the reconstruction of block sparse signals by the proposed algorithms tends to the theoretical limit of the number of non-zero elements. The improvement in performance was at the cost of a modest increase in complexity. Our results also indicated that the relative performance of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ depended on the signal model as well as the number of iterations. The former algorithm would be desirable for signals with slow decaying distributions for the ℓ_2 -norm of the blocks and when a larger number of iterations can be allowed. The latter algorithm, however, is more attractive due to its faster convergence speed particularly for fast decaying signal distributions.

Table 3.1: Average runtime of the recovery algorithms of Fig. 3.5 for $M = 170$.

Algorithm	Average Runtime (sec.)
ℓ_1 minimization	0.028
$\text{IR}\ell_1$	0.074
ISD	0.059
IRLS	0.127
AMP-JS	0.021
ℓ_2/ℓ_1 minimization	0.033
$\text{IR}^{(2)}\text{-}\ell_2/\ell_1$	0.081
$\text{IR}^{(1)}\text{-}\ell_2/\ell_1$	0.095

Table 3.2: Average runtime of the recovery algorithms of Fig. 3.6 for $M = 520$.

Algorithm	Average Runtime (sec.)	
	$N/L = 32$	$N/L = 8$
AMP-JS	0.064	0.131
ℓ_2/ℓ_1 minimization	0.182	0.246
$\text{IR}^{(2)}\text{-}\ell_2/\ell_1$	0.245	0.485
$\text{IR}^{(1)}\text{-}\ell_2/\ell_1$	0.325	0.713

Table 3.3: Average runtime of the recovery algorithms of Fig. 3.7 for $M = 1330$.

Algorithm	Average Runtime (sec.)		
	$N/L = 64$	$N/L = 32$	$N/L = 16$
AMP-JS	8.066	8.876	12.329
$\text{IR}^{(1)}\text{-}\ell_2/\ell_1$	69.948	87.288	108.589

Figure 3.9: Normalized reconstruction error for $\text{IR}^{(1)}-\ell_2/\ell_1$, $\text{IR}^{(2)}-\ell_2/\ell_1$ and the standard ℓ_2/ℓ_1 minimization for approximately block sparse signals versus M ($N = 256$, $L = 32$ and $P = 8$).

Figure 3.10: Probability of success of (a) $\text{IR}^{(1)}-\ell_2/\ell_1$, and (b) $\text{IR}^{(2)}-\ell_2/\ell_1$ versus M at the presence of a priori information on the non-zero block locations ($N = 256$, $L = 32$, $P = 8$ and Signal Model I).

(a) Noiseless

 (b) Noisy with $\sigma_n = 0.01$ (SNR = 33.98 dB)

 (c) Noisy with $\sigma_n = 0.05$ (SNR = 20.0 dB)

 (d) Noisy with $\sigma_n = 0.1$ (SNR = 13.98 dB)

Figure 3.11: Normalized reconstruction error of different recovery algorithms versus M for various SNR values ($N = 256$, $L = 32$, $P = 8$ and Signal Model I). For $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, $j_{\max}$ is selected to be 5, 5, 3, 3 in (a), (b), (c) and (d), respectively. For $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$, $j_{\max}$ is selected to be 3, 3, 2, 2 in (a), (b), (c) and (d), respectively.

(a) Noiseless

 (b) Noisy with $\sigma_n = 0.01$ (SNR = 33.98 dB)

 (c) Noisy with $\sigma_n = 0.05$ (SNR = 20.0 dB)

 (d) Noisy with $\sigma_n = 0.1$ (SNR = 13.98 dB)

Figure 3.12: Normalized reconstruction error of different recovery algorithms versus M for various SNR values ($N = 1024$, $L = 32$, $P = 8$ and Signal Model I). For $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, $j_{\max}$ is selected to be 5, 3, 3, 2 in (a), (b), (c) and (d), respectively. For $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$, $j_{\max}$ is selected to be 5, 3, 2, 2 in (a), (b), (c) and (d), respectively.

Figure 3.13: Spectrum reconstruction for one realization of the spectrum with parameters $N = 128$, $K = 32$, $L = 16$, $P = 4$ and $M = 80$: (a) input spectrum; reconstructed spectrum using: (b) ℓ_1 minimization; (c) $\text{IR}\ell_1$; (d) ISD; (e) IRLS; (f) ℓ_2/ℓ_1 minimization; (g) $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$; (h) $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$. For both $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$ and $\text{IR}\ell_1$, $j_{\max}$ is selected to be 5. For both $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$ and ISD, $j_{\max}$ is selected to be 3.

Figure 3.14: Probability of detection versus M for $N = 128$, $K = 32$, $L = 16$, $P = 4$ ($P_{FA} = 0.01$). Solid and dashed lines correspond to $\sigma_n = 0.05$ and $\sigma_n = 0.1$, respectively.

Chapter 4

Approximate Message Passing Algorithms

4.1 Introduction

In this chapter, we propose to weigh the threshold values of AMP for different signal elements in each iteration, in accordance with the size of that element in the previous iteration. The weights are selected similar to those in $\text{IR}\ell_1$. This results in an iterative reweighted AMP algorithm that performs similar to $\text{IR}\ell_1$, but with much lower complexity. The sparse signal recovery can be considered as a special case of the recovery of block sparse signals where the block size of all the blocks is one. We also propose a reweighing scheme for block-based AMP with block soft thresholding [48], where the block thresholds are updated in each iteration following a scheme similar to that of $\text{IR-}\ell_2/\ell_1$ algorithm. This results in an algorithm which performs similarly to $\text{IR-}\ell_2/\ell_1$ but with much lower complexity. Note that similar to $\text{IR-}\ell_2/\ell_1$ algorithms, the information about the block boundaries is assumed to be known a priori.¹

Motivated by the attractive performance/complexity trade-off of AMP, we also propose another modification to the AMP algorithm which makes it effective in dealing with block sparse signals with unknown block borders. The proposed algorithm performs a few iterations of the original AMP to obtain a rough estimate of the signal. This estimate is then used to detect the block borders, which are subsequently employed within the block-AMP framework to improve the signal estimate. The accuracy of both the border detection and the signal recovery is improved through iterations by a properly designed border detection/correction technique. Since it is

¹The results of Section 4.2 were established as a joint work with Neda Haghighatpanah, another Ph.D. student. The author has tested and implemented the AMP algorithms with soft thresholding, block soft thresholding, weighted soft thresholding, and weighted block soft thresholding functions, in this chapter.

shown in [48] that AMP-JS outperforms the other variants of block-AMP (in the presence of border knowledge of blocks), particularly, for large block lengths, we also use JS estimator in this chapter. The simulation results show that the proposed algorithm either outperforms or performs closely to the state-of-the-art PC-SBL algorithm in both noiseless and noisy scenarios, but has a much lower complexity.

It is important to note that the proposed border detection scheme can be used with any recovery algorithm for block sparse signals with known borders, in an iterative framework. In this chapter, we have focused on AMP-JS as the recovery algorithm.

4.2 Block-AMP with Known Block Borders

In general, there is a correspondence between convex optimization methods and specific AMP algorithms [83], such that the solution of a convex optimization coincides with the solution of the AMP algorithm with a specific thresholding function. Such a correspondence was shown between ℓ_1 minimization (LASSO) and AMP-ST in [84,85], and between AMP-BST and ℓ_2/ℓ_1 minimization (group LASSO) in [48]. Similarly, there is a correspondence between the weighted ℓ_2/ℓ_1 minimization of (3.1) and AMP with the weighted block soft thresholding (WBST) function $\zeta^{\text{WBST}}(\mathbf{x}; \tau, \sigma)$ which consists of L components $\zeta_i^{\text{WBST}} : \mathbb{R}^{B_i} \rightarrow \mathbb{R}^{B_i}$, defined as:

$$\zeta_i^{\text{WBST}}(\mathbf{u}_i; \tau, \sigma) = (1 - \frac{\tau\sigma\omega_i}{\|\mathbf{u}_i\|_2})_+ \mathbf{u}_i, \quad i = 1, \dots, L. \quad (4.1)$$

This correspondence can be used to efficiently solve the weighted ℓ_2/ℓ_1 minimization in each iteration of IR- ℓ_2/ℓ_1 using AMP-WBST. We thus propose the implementation presented in Algorithm 4.2 for the reweighted AMP-WBST where the reweighting (updating of weights) is performed according to

$$\omega_i^t = \frac{1}{\|\hat{\mathbf{u}}_i^{t-1}\|_2 + \rho}, \quad i = 1, \dots, L, \quad (4.2)$$

with the initial value $\omega_i^0 = 1$, for $i = 1, \dots, L$.

Algorithm 4.2: Reweighted AMP-WBST (RAMP-WBST)

Inputs: $\mathbf{A}$, $\mathbf{y}$, c_t and t_{max}

Output: $\hat{\mathbf{x}}$

```

Set  $t \leftarrow 0$  and  $\tau^0 \leftarrow 1$ 
Set  $\mathbf{x}^t = \mathbf{0}$ ,  $\mathbf{z}^t = \mathbf{y}$ , and  $\boldsymbol{\omega}^t = \mathbf{1}$ 
while  $t < t_{max}$  or  $\frac{\|\mathbf{y} - \mathbf{A}\mathbf{x}^t\|_2}{\|\mathbf{y}\|_2} > c_t$ , do

     $\mathbf{v}^t = \mathbf{x}^t + \mathbf{A}^T \mathbf{z}^t$ 
     $\mathbf{x}^{t+1} = \zeta^{\text{WBST}}(\mathbf{v}^t; \tau^t, \sigma_t)$ 
     $b_{t+1} = \frac{1}{N} \text{div } \zeta^{\text{WBST}}(\mathbf{v}^t; \tau^t, \sigma_t)$ 
     $\mathbf{z}^{t+1} = \mathbf{y} - \mathbf{A}\mathbf{x}^{t+1} + b_{t+1}\mathbf{z}^t$ 
     $\tau^{t+1} = b_{t+1}\tau^t$ 

     $\omega_i^{t+1} = \frac{1}{\|\mathbf{u}_i^{t+1}\|_2 + \rho}$ ,  $\forall i \in \{1 \cdots, L\}$ 2
     $t \leftarrow t + 1$ 

end while
 $\hat{\mathbf{x}} = \mathbf{x}^t$ 

```

Remark 4.1: Although the signal estimate is inaccurate in the early iterations of Algorithm 4.2, the estimated blocks with large ℓ_2 -norm are likely to be nonzero. Based on (4.2), such blocks are assigned smaller weights (thresholds), while, for the blocks with smaller ℓ_2 -norm, larger weights (thresholds) are chosen. This would mean that the latter blocks are pushed toward zero in the subsequent iterations, thus encouraging a sparser solution.

Remark 4.2: If $B_i = 1, \forall i \in \{1, \dots, L\}$, we are dealing with a sparse signal, rather than a block sparse one. In this case, Algorithm 4.2 represents a reweighted AMP-WST (RAMP-WST) that corresponds to $\text{IR}\ell_1$.

4.3 Block-AMP with Unknown Block Borders

For the recovery of block sparse signals in the absence of knowledge about the block borders, AMP-JS cannot be directly applied. In the previous section, we applied a reweighting strategy to AMP-BST to encourage the block structure in a signal. In this section, however, the main idea is to equip AMP with a mechanism to detect the block borders and then use this mechanism along with AMP-JS to improve the accuracy of signal recovery iteratively. In the proposed algorithm, in early iterations, we implement the standard AMP with soft thresholding. After t_0 iterations of AMP,

²Note that the parameter ρ is set similar to the $\text{IR-}\ell_2/\ell_1$ algorithm.

an (inaccurate) signal estimate is obtained. We then apply a border (edge) detection algorithm to this signal estimate and obtain an initial estimate for the block border locations. (Hereafter, we use the terms “block border” and “edge,” interchangeably.) We then apply AMP-JS using the estimated (and probably inaccurate) block borders and improve the accuracy of both the signal and the block borders iteratively. To estimate the block borders accurately as the iterations go on, we employ an edge detection technique along with guard bands and an edge correction technique. These are presented in the following subsections along with the pseudocode of the proposed algorithm.

4.3.1 Edge Detection (ED)

In [86], Canny considered a computational approach to the edge detection problem and showed that an optimal edge detector can be approximated by a first-order derivative of Gaussian operator $G'(x)$, where

$$G(x) = \exp\left(-\frac{x^2}{\sigma^2}\right), \quad (4.3)$$

and σ is the scaling parameter. An edge can then be marked (detected) at the location of the local maxima (or minima) in the output of the convolution of $G'(x)$ with the signal. The operator $G'(x)$ is also suitable for edge detection in noisy signals due to the noise smoothing effect of the Gaussian operator. The level of noise suppression is determined by the value of σ , which should be properly chosen. If σ is selected to be too large, many edges might be ignored (not detected). If, on the other hand, the value of σ is selected to be too small, a large number of false edges might be detected. In this work, to avoid the false detection of edges within non-zero blocks where there is a large difference between the values of adjacent signal elements with different signs, we apply the Gaussian edge detector to the absolute value of the signal elements.

4.3.2 Guard Band (GB)

An inaccurate edge detection in one iteration can deteriorate the performance of the recovery algorithm in subsequent iterations. Consider a case where a few non-zero elements of a non-zero block are erroneously detected to be part of a zero block.

Depending on the energy of the non-zero elements, the zero block may be detected as non-zero and the energy of the non-zero elements would be distributed throughout the zero block. This error will then propagate in the upcoming iterations.

To avoid this phenomenon, and to limit the range of signal elements that would be affected by a false edge detection, we form a narrow guard band in the neighborhood of each detected border. To this end, we replace each detected border by two borders, each on one side of the detected border and equally distanced from it. We place the new borders q elements apart from the detected border. Therefore, the length of the guard bands will be $2q$. Since the detected edges are likely to be in the neighborhood of the true borders, the true borders are likely to be in the guard bands.

Note that the length of each block should not be less than 3 in AMP-JS. Since, after inserting guard bands between the blocks, the blocks will be shorter, there might be some blocks with length less than 3. In this case, we randomly remove one of the borders of the blocks with length less than 3 and merge them with their adjacent block. The new block borders are then used as input to AMP-JS for the next iteration.

4.3.3 Edge Correction (EC)

Using guard bands limits the propagation of errors caused by false edge detection. However, all the elements of a guard band are likely to be estimated as non-zero even if some of them are in fact equal to zero. Therefore, after implementing some iterations using guard bands, one would need to replace back each of them with a single border.

One should note that guard bands by their nature are almost always located in between a zero (lower energy) and a non-zero (higher energy) block. To enhance the accuracy of the edge detection in the presence of guard bands, moving from the lower energy block towards the higher energy block, we select the border to be at a location where the signal amplitude is increased beyond a certain threshold. This threshold is chosen to be $T = \frac{\|\mathbf{u}_L\|_2 + \|\mathbf{u}_R\|_2}{\dim(\mathbf{u}_L) + \dim(\mathbf{u}_R)}$, where $\mathbf{u}_L$ and $\mathbf{u}_R$ denote the left and right blocks of the guard band $\mathbf{u}_{GB}$, respectively, and $\dim(\cdot)$ denotes the dimension of a vector. To distinguish between this enhanced edge detection and the process described in Subsection 4.3.1, we refer to the new edge detection as “edge correction (EC).” The pseudocode for edge correction is as follows:

if $\frac{\|\mathbf{u}_L\|_2}{\dim(\mathbf{u}_L)} > \frac{\|\mathbf{u}_R\|_2}{\dim(\mathbf{u}_R)}$ **then**

$\tilde{i} = \{\max i : |x_i| > T, i \in \text{Supp}(\mathbf{u}_{GB})\}$
 For $i \in \text{Supp}(\mathbf{u}_{GB})$ and $i \leq \tilde{i}$, assign x_i as part of $\mathbf{u}_L$.
 For $i \in \text{Supp}(\mathbf{u}_{GB})$ and $i > \tilde{i}$, assign x_i to $\mathbf{u}_R$.
 else
 $\tilde{i} = \{\min i : |x_i| > T, i \in \text{Supp}(\mathbf{u}_{GB})\}$
 For $i \in \text{Supp}(\mathbf{u}_{GB})$ and $i \geq \tilde{i}$, assign x_i as part of $\mathbf{u}_R$.
 For $i \in \text{Supp}(\mathbf{u}_{GB})$ and $i < \tilde{i}$, assign x_i to $\mathbf{u}_L$.
 end if

Remark 4.3: In the above pseudocode, the vector $\text{Supp}(\mathbf{u}_{GB})$ is the support set of the guard band. The process described after finding the value of $\tilde{i}$ in the pseudocode, is to select the new border and to partition the elements of $\mathbf{u}_{GB}$ between $\mathbf{u}_L$ and $\mathbf{u}_R$.

4.3.4 Proposed Algorithm

The algorithmic framework of the proposed recovery scheme is given in Algorithm 4.3.4.

Algorithm 4.3.4: Edge detection based AMP-JS without the knowledge of block borders (ED-based AMP-JS)

Inputs: $\mathbf{A}$, $\mathbf{y}$ and t_{max}

Output: $\hat{\mathbf{x}}$

$t \leftarrow 0$
 $\mathbf{x}^t = \mathbf{0}$ and $\mathbf{z}^t = \mathbf{y}$
while $t < t_{max}$ or $\frac{\|\mathbf{y} - \mathbf{A}\mathbf{x}^t\|_2}{\|\mathbf{y}\|_2} > 10^{-8}$ **do**
 $\mathbf{v}^t = \mathbf{x}^t + \mathbf{A}^T \mathbf{z}^t$
if $t < t_0$ **then**
 Choose τ as the M -th largest element of $|\mathbf{v}^t|$ [87]
 $\mathbf{x}^{t+1} = \zeta^{\text{ST}}(\mathbf{v}^t)$
 $b_{t+1} = \frac{1}{N} \text{div}(\zeta^{\text{ST}}(\mathbf{v}^t))$
else

```

    Obtain the block borders using ED over  $|\mathbf{x}^t|$ 
    if  $t < t_1$  then
        Obtain new borders using GB
    else
        Obtain new borders using EC
    end if
     $\mathbf{x}^{t+1} = \zeta^{\text{JS}}(\mathbf{v}^t)$ 
     $b_{t+1} = \frac{1}{N} \text{div}(\zeta^{\text{JS}}(\mathbf{v}^t))$ 
end if
     $\mathbf{z}^{t+1} = \mathbf{y} - \mathbf{A}\mathbf{x}^{t+1} + b_{t+1}\mathbf{z}^t$ 
     $t \leftarrow t + 1$ 

end while

 $\hat{\mathbf{x}} = \mathbf{x}^t$ 

```

Remark 4.4: For making the edge correction more effective, we start with the GBs of length 4 and increase the length of the GBs by 2 every Δt iterations. We start from smaller GBs, because it is more likely to have the true edges in the close vicinity of the detected edges. However, by this choice, some edges may fall outside the GBs. To detect such edges, we increase the length of the GBs as the iterations progress. Please note that, for brevity, this step is not shown in Algorithm 4.3.4.

Remark 4.5: The complexity of the proposed algorithm is dominated by that of AMP-JS, which is $\mathcal{O}(MN)$. The complexity of the edge detection method is $\mathcal{O}(N \log(N))$ [86], which is less than $\mathcal{O}(MN)$ as long as M grows faster than $\log(N)$. On the other hand, the complexity of SBL algorithms such as PC-SBL, is dominated by that of an $N \times N$ matrix inversion, which can be converted to an $M \times M$ matrix inversion [55]. Such an inversion has complexity $\mathcal{O}(M^3)$. As we will see in the following section, the runtime of the proposed algorithm can be significantly lower than that of PC-SBL.

4.4 Simulation Results

In this section, some simulation results are presented to show the effectiveness of the proposed modification of AMP algorithms in both noiseless and noisy scenarios. We also compare the performance of the proposed algorithms with other recovery

algorithms including the state-of-the-art AMP-JS and the pattern-coupled sparse Bayesian learning (PC-SBL) algorithm of [55].

We consider a signal of dimension N which consists of L consecutive blocks of length $B_i, i = 1, \dots, L$. To model a P -block sparse signal, we select P blocks (out of L) randomly and assign the entries of these blocks using a zero-mean Gaussian distribution with variance one in an i.i.d. fashion. The entries of the other blocks are then set to zero.

In our simulations, each experiment is repeated 500 times with the following steps: (a) randomly creating the sparse signal $\mathbf{x}$, (b) constructing the random measurement matrix of size $M \times N$, (c) creating the corresponding measurement vector $\mathbf{y}$ (and adding noise to vector $\mathbf{y}$ in noisy scenarios), (d) recovering $\mathbf{x}$ using the reconstruction algorithm, (e) calculating the reconstruction error and/or making decision regarding success/failure of the recovery algorithm. Each reported result in the curves is an average over the 500 trials.

For all recovery algorithms of this section, we declare success if $\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_2}{\|\hat{\mathbf{x}}\|_2} < 0.01$. We use SpaRSA [77, 81], to solve all the convex optimizations. The simulations are run in MATLAB environment on a desktop computer with an Intel Core i7 2.8 GHz CPU and 8 GB of RAM, and under the Microsoft Windows 7 operating system.

4.4.1 Known Block Borders

In this subsection, we present some simulation results to demonstrate the effectiveness of the reweighted AMP algorithm when the information about the block borders is available. We set $t_{max} = 500$ in all AMP algorithms of this subsection. In Algorithm 4.2, we choose $c_t = 10^{-8}$. For both $\text{IR}\ell_1$ and $\text{IR-}\ell_2/\ell_1$ algorithms, we set the number of iterations to 5. Other parameters for these algorithms are set following the recommendations in [6] and Chapter 3, respectively. The elements of the measurement matrix are drawn in an i.i.d. fashion from a zero-mean Gaussian distribution with variance $\frac{1}{1.05M}$,³ where M is the number of measurements.

Here, we consider a block sparse signal with $N = 256$, $B_i = B = 8, i = 1, \dots, L = 32$, and sparsity ratio of 0.25 ($K = 64, P = 8$). For this signal, the success probability of RAMP-WBST versus the number of measurements M is plotted in Fig. 4.1,

³It is known that the performance of AMP is sensitive to the choice of the measurement matrix. This particular value of variance is found empirically to result in a good performance for AMP in the experiments performed in this paper.

along with the curves for a number of other recovery algorithms. As expected, the performances of ℓ_1 minimization and AMP-ST are identical. So are the performances of ℓ_2/ℓ_1 minimization and AMP-BST. AMP-JS outperforms all the above algorithms but is still inferior to IR- ℓ_2/ℓ_1 algorithm of Chapter 3. The proposed RAMP-WBST performs practically the same as IR- ℓ_2/ℓ_1 but with much lower complexity. For example, the average runtime of the two algorithms for $M = 90$ is 0.184 sec. and 0.628 sec., respectively. We have also included the performance of the WAMP algorithm of [30] with the optimum value of ω to get the best performance ($\omega = 0.3$) in Fig. 4.1. As can be seen, this algorithm performs the same as IR ℓ_1 and inferior to the proposed RAMP-WST.

In Fig. 4.2, we have compared the performance of the proposed RAMP-WBST with two known AMP algorithms for the recovery of block sparse signals with known borders: AMP-BST and AMP-JS. The results are for a block sparse signal with equal-length blocks, $N = 1024$, $K = 256$, and three different block lengths $B = 4, 8$, and 32. Fig. 4.2 shows that the performance of each algorithm improves by increasing the block length, and that the proposed algorithm outperforms the other two algorithms in every block length. The performance gap is particularly large for shorter block lengths. Note that all algorithms have basically the same computational complexity.

For the noisy case, we consider the measurements to be contaminated with the additive white Gaussian noise with zero mean and variance σ_n^2 . For the signal, we consider one with equal-sized blocks of length $B = 4$ and with $N = 1024, K = 256$. The average normalized reconstruction error (i.e., $20 \log (\|\hat{\mathbf{x}} - \mathbf{x}\|_2 / \|\hat{\mathbf{x}}\|_2)$) of the proposed reweighted AMP algorithm, AMP-JS and AMP-BST are plotted in Fig. 4.3 versus M , for three SNR values of 40, 24, and 15 dB. Fig. 4.3 shows that, for this block length, RAMP-WBST outperforms both known block-based variants of AMP. One should however note that as the block length is increased, AMP-JS overtakes both AMP-BST and the proposed algorithm, particularly for larger values of M .

4.4.2 Unknown Block Borders

In this subsection, we present some simulation results to demonstrate the effectiveness of the ED-based AMP-JS algorithm when the information about the block borders is

Figure 4.1: Probability of success of different recovery algorithms versus M for a block sparse signal with $N = 256$, $L = 32$, $P = 8$, $B = 8$.

Figure 4.2: Probability of success of RAMP-WBST, AMP-BST and AMP-JS versus M for a block sparse signal with known borders, equal-length blocks, $N = 1024$, and $K = 256$. The solid, dashed and dotted lines correspond to block lengths $B = 32$, 8 and 4, respectively.

unavailable. In this part, we generate a block sparse signal of dimension 1000, which consists of 24 non-overlapping blocks of dimension 10, 24 blocks of dimension 15 and 20 blocks of dimension 20. The blocks are randomly distributed in the signal. For

Figure 4.3: Average normalized reconstruction error versus M for AMP-BST, AMP-JS, and RAMP-WBST ($N = 1024, K = 256, B = 4$). The solid, dashed, and dotted lines correspond to the SNR values of 40, 24, and 15 dB, respectively.

each set of blocks of the same length, we randomly select three of them to be non-zero (i.e., $K = 135$). The measurement matrix is randomly constructed with i.i.d. elements from a zero mean Gaussian distribution with variance $1/(1.1M)$. Since the guard bands should be narrow and at least 3-element wide, we select their width to be $2q = 4$. We also set the scaling parameter of the Gaussian operator to be $\sigma = 7$.

We set $t_{max} = 200$ for all the iterative algorithms in this subsection. In Algorithm 4.3.4, we set $t_0 = 10$ and $t_1 = 100$. We also set $\Delta t = 30$. If $t_1 = t_{max}$, then no edge correction is performed and when $t_1 = t_0$, no GB is used. When “if and else statement” with regard to GB and EC is removed from the algorithm, only edge detection is performed.

Fig. 4.4 shows the success probability of different variants of the ED-based AMP-JS algorithm versus the number of measurements M . We set $t_{max} = 200$ for all the iterative algorithms. As expected, the best performance is achieved when a combination of edge detection (ED), guard bands (GB) and edge correction (EC) is employed. The second best performance with a rather small gap to the first, is due to edge detection plus edge correction, indicating that edge correction plays an important role in the combination. For the rest of simulations, we report the performance of the ED-based AMP-JS algorithm where all three techniques are used. For comparison, in Fig. 4.4, we have also included the performance curves for AMP with soft thresholding,

Figure 4.4: Probability of success of different recovery algorithms versus M ($N = 1000$, $K = 135$ with 24, 24 and 20 blocks of length 10, 15 and 20, respectively, where 3 blocks of each length are randomly selected to be non-zero).

AMP-JS with a priori knowledge of the block borders and the-state-of-the-art PC-SBL algorithm of [55]. Fig. 4.4 shows that our algorithm significantly outperforms AMP with soft thresholding but, expectedly, performs inferior to AMP-JS with the knowledge of block borders. Compared to PC-SBL, the ED-based AMP-JS algorithm performs at least as good, in larger values of M , and much better, for smaller number of measurements.

To compare the runtime of the ED-based AMP-JS algorithm with that of PC-SBL, we consider the case with $M = 300$. For this case, the average runtime of our algorithm is 0.70 second versus 15.78 seconds for PC-SBL. This is a speed improvement by a factor of about 22.

In Fig. 4.5, we have provided the average value of the average normalized reconstruction error of the ED-based AMP-JS block-based algorithm, PC-SBL and AMP-JS with known block borders versus M for two SNR values of 25 dB and 40 dB. As seen from this figure, for smaller number of measurements, PC-SBL outperforms the ED-based AMP-JS algorithm, but is inferior to the AMP-JS with known borders. For larger number of measurements, all algorithms perform practically the same. The results of Fig. 4.5 provide information about the average reconstruction error of different algorithms. To obtain more information about the distribution of

Figure 4.5: Average normalized reconstruction error of the ED-based AMP-JS algorithm, PC-SBL and AMP-JS with known block borders versus M . The solid and dashed lines correspond to SNR= 40 dB and 25 dB, respectively. ($N = 1000$, $K = 135$ with 24, 24 and 20 blocks of length 10, 15 and 20, respectively, where 3 blocks of each length are randomly selected to be non-zero.)

reconstruction error of the algorithms, we have plotted in Fig. 4.5 the success probability of the ED-based AMP-JS algorithm and PC-SBL versus the success threshold δ for $M = 300$, and two SNR values of 25 dB and 40 dB. The success is declared when the reconstruction error is below δ . Fig. 4.6 shows that the ED-based AMP-JS algorithm has a higher probability of success compared to PC-SBL for smaller threshold values. The trend however reverses for larger thresholds. Fig. 4.6 also shows that the reconstruction error for PC-SBL is more concentrated around its average value compared to that of the ED-based AMP-JS algorithm.

In terms of the comparison between both ED-based AMP-JS algorithms and PC-SBL, we also noticed that while the performance of the ED-based AMP-JS algorithms is rather independent of the signal energy, that of the PC-SBL is highly sensitive to changes in the signal energy both in noiseless and noisy scenarios (for a given SNR). For example, if in our simulations, we normalize the signal to have unit energy, the performance of the ED-based AMP-JS algorithms do not change. This normalization, however, degrades the performance of PC-SBL significantly.

Figure 4.6: Probability of success of the ED-based AMP-JS algorithm and PC-SBL versus success threshold for two SNR values. The solid and dashed lines correspond to SNR= 40 dB and 25 dB, respectively. ($N = 1000$, $M = 300$, $K = 135$ with 24, 24 and 20 blocks of length 10, 15 and 20, respectively, where 3 blocks of each length are randomly selected to be non-zero.)

4.5 Conclusion

In this chapter, we proposed a number of weighting/reweighting schemes to improve the performance of AMP algorithms. We applied the reweighting schemes to the AMP-ST for regular signals, and to the AMP-BST for block sparse signals with known block borders. We also proposed a block-based AMP algorithm for the recovery of block sparse signals with unknown block borders. At the core of this algorithm is a border detection/correction scheme that can be used in combination with any recovery algorithm of block sparse signals with known borders, within an iterative framework. In this work, we considered AMP with James-Stein's shrinkage estimator as the recovery algorithm. We demonstrated through simulations that the proposed algorithms provide an attractive performance/complexity trade-off in comparison with the state-of-the-art recovery algorithms that are based on Bayesian learning.

Chapter 5

Ultra Low-Complexity Zero Block Detection

In this chapter, we propose an ultra low-complexity zero block detection scheme, which is based on a block sparse sensing matrix, and apply it to the wideband spectrum sensing. To this end, we design a measurement matrix as part of the AIC, such that the resulting sensing matrix has a block sparse structure. As will be seen later, the structure of the sensing matrix affects the performance of zero detection scheme. We analyze the proposed zero-block detection scheme for regular sensing matrices (those with the same number of non-zero blocks in each row and column) and then generalize the results to the case of irregular matrices. We show that one can optimize the structure of the sensing matrix to maximize the fraction of spectrum holes that can be detected for a given number of measurements. We consider both noiseless and noisy scenarios in this work, and analyze the performance of the proposed algorithm in both cases. Our analytical and simulation results demonstrate the effectiveness of the proposed algorithm in reliable detection of spectrum holes (even when the number of measurements is relatively small) with complexity that is negligible compared to the existing block sparse recovery algorithms such as AMP-JS.

5.1 System Model and Problem Statement

Suppose that $r(t)$ is the wideband analog signal consists of L sub-channels, similar to the model described in Section 3.3.

We assume that the vector $\mathbf{r} \in \mathbb{R}^N$ is the discrete representation of the analog signal $r(t)$, where N corresponds to the Nyquist rate, i.e., N is the required number

of samples for the recovery of the signal without aliasing. The signal $r(t)$ is also assumed to be sparse in the frequency domain. Vector $\mathbf{x} = \mathbf{F}\mathbf{r}$ is the spectrum of $\mathbf{r}$ with only K non-zero elements ($K \ll N$). Since $r(t)$ is a real signal, the spectrum $\mathbf{x}$ is conjugate symmetric. We use the notation $^+\mathbf{x}$ to denote the positive frequency part of $\mathbf{x}$. Due to the symmetry, we only focus on the positive frequency elements, $^+\mathbf{x}$.

In the spectrum sensing context, the spectrum of the signal is block sparse. Suppose that at any given time each sub-channel is occupied with probability $\alpha \ll 1$ independent of the other sub-channels. We refer to α as the *sparsity ratio*. We can write $^+\mathbf{x} = [\mathbf{u}_1^T, \dots, \mathbf{u}_L^T]^T$, where the sub-vector $\mathbf{u}_i$ is the i -th segment of $^+\mathbf{x}$, which corresponds to the elements in the range $[f_{i-1}, f_i)$. In this model, we assume that, the magnitude of non-zero elements of $\mathbf{x}$ follows a distribution, f .

Mathematically, in the noiseless scenario, the measurement vector $\mathbf{y} \in \mathbb{R}^M$ is given by, $\mathbf{y} = \mathbf{\Phi}\mathbf{r} = \mathbf{\Phi}\mathbf{F}^{-1}\mathbf{x} = \mathbf{A}\mathbf{x}$, where $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$ is the measurement matrix and $\mathbf{A} = \mathbf{\Phi}\mathbf{F}^{-1}$ is the sensing matrix. Since $\mathbf{A}$ is the multiplication of the real matrix $\mathbf{\Phi}$ by $\mathbf{F}^{-1}$, each row of $\mathbf{A}$ is conjugate symmetric. Therefore, similar to the vector $\mathbf{x}$, we consider only half of the elements of the rows of $\mathbf{A}$. Let $^+\mathbf{A}$ denote the $M \times N/2$ matrix including the last $N/2$ elements of each row of $\mathbf{A}$.¹ In this thesis, we consider the case where $\mathbf{A}$ is a random block sparse matrix in row,² i.e., each row of $^+\mathbf{A}$ consists of L blocks, in correspondence with the blocks of $^+\mathbf{x}$, and that only a few blocks out of L blocks are non-zero and the rest of the blocks are zero.³ (We will explain in Section 5.2 how a sensing matrix with this structure can be constructed.) Let row-vector $^+\mathbf{A}_m^{(l)}$ be the l -th block of the m -th row of $^+\mathbf{A}$. Also, assume that the magnitudes of all elements of a non-zero block $^+\mathbf{A}_m^{(l)}$ follow a distribution g . The m -th element of the measurement vector $\mathbf{y}$ is given by

$$y_m = 2\Re \left(\sum_{l=1}^L ^+\mathbf{A}_m^{(l)} \cdot \mathbf{u}_l \right), \quad (5.1)$$

¹Depending on whether N is odd or even, there will be $(N+1)/2$ or $N/2$ elements corresponding to non-negative frequencies, respectively. In this thesis, for simplicity, we assume N is an even number.

²Here, the randomness refers to the locations of non-zero blocks. However, we will see later, as we discuss the design of AIC, that the non-zero elements are also random.

³The reason for the sparsity choice is to keep the complexity low. As becomes evident later, this choice also improves the performance of the recovery algorithm.

Figure 5.1: An example of a sensing graph with $M = 2$ and $L = 3$.

where “.” denotes the inner product. Now the problem is to find a fraction μ of spectrum holes from a given number of measurements M , where the goal is to maximize μ . One should note that in this work, we are interested in very small values of M , i.e., $M \ll L$. This is in contrast with the existing CS frameworks, where one is often interested in M values that are small in comparison with N (not L).

The relationships in (5.1) for $m = 1, \dots, M$, can be represented by a bipartite graph with L VNs and M MNs. This bipartite graph is referred to as *sensing graph*. In the sensing graph, a VN represents a sub-vector consisting of $B = N/2L$ complex elements. There exists an edge between a variable node and a measurement node if and only if the sub-channel signal corresponding to the variable node appears in the linear combination (5.1) corresponding to the measurement node. An example of a sensing graph is presented in Fig. 5.1.

To evaluate the performance of the recovery algorithms, in this chapter, we use two different measures of under-sampling: (1) the ratio M/N , which is conventionally used in CS literature, and (2) the ratio M/L , which is useful in dealing with block sparse signals. We refer to these measures as “variable” and “block” under-sampling ratios, respectively. In our model, the latter is larger than the former by a factor of $2B$.

Figure 5.2: (a) The general structure of an AIC, (b) Sampling waveforms of the AIC for a block sparse sensing matrix, (c) An alternate structure to generate the sampling waveforms.

5.2 Design of an AIC with Block Sparse Sensing Matrix

The general structure of an AIC consisting of M parallel branches of mixers and integrators (BMIs) are shown in Fig. 5.2(a). In this section, our objective is to generate the sampling waveforms $\phi_i(t)$ of Fig. 5.2(a) such that the resulting sensing matrix is block sparse.

Since $\mathbf{A} = \mathbf{\Phi}\mathbf{F}^{-1}$, we have $\mathbf{A}_m^\dagger = (\mathbf{F}^{-1})^\dagger \mathbf{\Phi}_m^\dagger$ for all $m \in \{1, 2, \dots, M\}$, where $\mathbf{A}_m$ and $\mathbf{\Phi}_m$ are the m -th row-vectors of $\mathbf{A}$ and $\mathbf{\Phi}$, respectively. In addition, the Fourier matrix $\mathbf{F}$ is a unitary matrix, and thus $\mathbf{F}^{-1} = \mathbf{F}^\dagger$. As a result, for the real column-vector $\mathbf{\Phi}_m^T$, we have $\mathbf{A}_m^\dagger = \mathbf{F}\mathbf{\Phi}_m^T$. In other words, each row-vector of $\mathbf{A}$ is the conjugate of the Fourier transform of each row-vector of $\mathbf{\Phi}$. This implies that if the rows of the measurement matrix are block sparse in the frequency domain, then $\mathbf{A}$ will have block sparse rows.

Fig. 5.2(b) demonstrates how the sampling waveforms $\phi_i(t)$, $i = 1, \dots, M$, in Fig. 5.2(a) are generated to result in a block sparse sensing matrix. In Fig. 5.2(b), the input signal, $p_c(t)$, is a pseudo-random Bernoulli sequence, with the rate equal to or higher than B , where B is the bandwidth of each sub-channel.⁴ The input signal is then filtered by a low-pass (LP) filter with the cut-off frequency equal to $B/2$ so that the output signal of this filter is a baseband signal with frequency

⁴ $p_c(t)$ can also be generated by any other noise generator with a rate equal to or higher than B .

range of $[-B/2, B/2]$. Finally, this signal is modulated by the carrier frequencies $f_c^{(1)}, f_c^{(2)}, \dots, f_c^{(L)}$, where $f_c^{(i)} = (f_i + f_{i-1})/2$ is the center frequency of the i -th sub-channel. The resulted waveforms $v_l(t)$, $l = 1, \dots, L$, can be then linearly combined based on a properly designed mapping, to generate M sampling waveforms $\phi_m(t)$, $m = 1, \dots, M$, needed in Fig. 5.2(a) to create the M measurements. One may note that for having a sparse sensing matrix, a mapping with sparse linear combinations is required.

Considering the sensing graph described in Section 5.1, we can make a correspondence between the variable nodes and waveforms $v_l(t)$, $l = 1, \dots, L$, and between the measurement nodes and sampling waveforms $\phi_m(t)$, $m = 1, \dots, M$, and construct the mapping in Fig. 5.2(b) based on such correspondence. We thus have

$$\phi_m(t) = \sum_{l \in \mathcal{V}(m)} v_l(t),$$

where $|\mathcal{V}(m)| = d(m)$. Therefore, $\tilde{\phi}_m(f) = \sum_{l \in \mathcal{V}(m)} \tilde{v}_l(f)$, where $\tilde{\phi}_m(f)$ and $\tilde{v}_l(f)$ are the Fourier transforms of $\phi_m(t)$ and $v_l(t)$, respectively. It is thus easy to see that the m -th row of $\mathbf{A}$, which consists of the elements of $\tilde{\phi}_m^*(f)$, is block sparse with $d(m)$ non-zero blocks in ${}^+\mathbf{A}_m$.

As seen in Fig. 5.2(b), only one pseudo-random sequence and one LP filter is used. This thus implies that all non-zero blocks in the sensing graph are the same, i.e., ${}^+\mathbf{A}_a^{(v)} = {}^+\mathbf{A}_b^{(u)} \neq \mathbf{0}$, for all $a, b \in \mathcal{M}$ and $v, u \in \mathcal{V}$. This property is essential for adopting the VB algorithms in the WSS application, described in the next chapter.

In Fig. 5.2(b), the required number of multipliers is L . An alternate approach for generating the sampling waveforms with only M multipliers is shown in Fig. 5.2(c).

5.3 Proposed Spectrum Hole Detection Scheme

5.3.1 Noiseless Scenario: Proposed Scheme and Its Analysis

A.1. Proposed Scheme

The following lemma is the foundation of the proposed zero block detection in the absence of noise.

Lemma 5.1: If $y_m = 0$ in (5.1), the sub-vectors of ${}^+\mathbf{x}$ corresponding to the non-zero blocks of ${}^+\mathbf{A}_m$, are zero, with probability one.

Proof: The proof simply follows from the continuity of at least one the distributions f or g , described in Section 5.1. ■

Based on Lemma 5.1, the proposed scheme detects a zero block in the sensing graph, if that block is connected to a zero measurement node. We refer to this simple detection scheme for zero blocks (vacant sub-channels) as *zero measurement detection (ZMD)*. In order to quantify the performance of the proposed method, we obtain the probability of zero block detection (P_{ZD}) and the probability of wrong zero block detection (P_{WZD}).

Since P_{WZD} indicates the level of interference to the PUs, it is an important measure from the standpoint of PUs. Note that in the noiseless case, based on Lemma 5.1, $P_{WZD} = 0$ and the objective is to design a sensing graph in order to maximize P_{ZD} .

A.2. Analysis

In this section, we first consider bi-regular sensing graphs. In such graphs, the MN $m \in \mathcal{M}_z$, if and only if, all d_M neighboring VNs belong to $\mathcal{V}_z$. Therefore, the probability that m is zero is calculated by

$$P(m \in \mathcal{M}_z) = (1 - \alpha)^{d_M}. \quad (5.2)$$

Therefore, $E(|\mathcal{M}_z|) = M(1 - \alpha)^{d_M}$.

On the other hand, the zero VN, v , is recovered by the ZMD method, if and only if, at least one of the neighboring MNs is in $\mathcal{M}_z$. By definition, P_{ZD} is the ratio of the expected number of detected zero variables to the total number of zero VNs. Therefore, it is clear that P_{ZD} is the probability that the zero VN, v , is recovered by ZMD, i.e., $P_{ZD} = P(|\mathcal{E}_{v, \mathcal{M}_z}| \geq 1 | v \in \mathcal{V}_z) = 1 - P(|\mathcal{E}_{v, \mathcal{M}_z}| = 0 | v \in \mathcal{V}_z)$.

For calculating this probability, let the edge e connect a VN in $\mathcal{V}_z$ to a MN in $\mathcal{M}$. We first calculate the probability p_0 that the edge e connects a VN in $\mathcal{V}_z$ to a MN in $\mathcal{M}_z$. Let t_e and h_e be the tail node and the head node of the edge e in $\mathcal{M}$ and $\mathcal{V}$,

respectively. Therefore, we can write

$$\begin{aligned}
 p_0 &= \text{P}(t_e \in \mathcal{M}_z | h_e \in \mathcal{V}_z) \\
 &= \frac{\text{P}(t_e \in \mathcal{M}_z, h_e \in \mathcal{V}_z)}{\text{P}(h_e \in \mathcal{V}_z)} = \frac{\text{P}(t_e \in \mathcal{M}_z)}{\text{P}(h_e \in \mathcal{V}_z)} \\
 &= \frac{E(|\mathcal{E}_{\mathcal{V}, \mathcal{M}_z}|) / |\mathcal{E}_{\mathcal{V}, \mathcal{M}}|}{E(|\mathcal{E}_{\mathcal{V}_z, \mathcal{M}}|) / |\mathcal{E}_{\mathcal{V}, \mathcal{M}}|} = \frac{E(|\mathcal{E}_{\mathcal{V}, \mathcal{M}_z}|)}{E(|\mathcal{E}_{\mathcal{V}_z, \mathcal{M}}|)} = \frac{d_M E(|\mathcal{M}_z|)}{d_V |\mathcal{V}_z|} \\
 &= \frac{d_M M (1 - \alpha)^{d_M}}{d_V L (1 - \alpha)} = (1 - \alpha)^{d_M - 1}, \tag{5.3}
 \end{aligned}$$

where the last equality comes from the fact that in bi-regular graphs, $d_V = M d_M / L$. Thus, the probability that the zero variable node v does not have any connection to $\mathcal{M}_z$, i.e., all d_V neighboring MNs belong to $\mathcal{M}_{nz}$, is $(1 - p_0)^{d_V}$. Therefore,

$$\begin{aligned}
 P_{ZD} &= 1 - \text{P}(|\mathcal{E}_{v, \mathcal{M}_z}| = 0 | v \in \mathcal{V}_z) \\
 &= 1 - (1 - p_0)^{d_V} = 1 - \left(1 - (1 - \alpha)^{d_M - 1}\right)^{d_V}. \tag{5.4}
 \end{aligned}$$

We also can simply generalize the analysis of bi-regular sensing graphs to the case of irregular ones with the constraint that the VNs and MNs follow the degree distributions $\lambda(x)$ and $\rho(x)$, respectively. After following the same steps as the ones for regular graphs, we can obtain P_{ZD} as

$$P_{ZD} = 1 - \sum_i \lambda_i (1 - p'_0)^i, \tag{5.5}$$

where $p'_0 = \sum_i i \rho_i (1 - \alpha)^{i-1} / \sum_i i \rho_i$.

5.3.2 Noisy Scenario: Proposed Scheme and Its Analysis

B.1. Proposed Scheme

In practice, the measurements are corrupted by noise, i.e., $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, where $\mathbf{n}$ is an $M \times 1$ vector whose elements are independent and identically distributed (i.i.d.) zero-mean Gaussian random variables with variance σ_n^2 , $\mathcal{N}(0, \sigma_n^2)$. (Notation $\mathcal{N}(\cdot, \cdot)$ is used for the Gaussian distribution function, in which the first and second arguments denote the mean and the variance of the distribution, respectively.) Therefore, the

m -th measurement can be written as

$$y_m = 2\Re \left(\sum_{i \in \mathcal{V}(m)} {}^+ \mathbf{A}_m^{(i)} \cdot \mathbf{u}_i \right) + n_m. \quad (5.6)$$

Noise has a destructive effect on the application of ZMD method. In other words, if the noise-free measurement is equal to zero, its noisy version is not equal to zero with probability one. This will, in effect, disable ZMD method as described for noiseless scenarios. Here, we propose a test to detect zero measurements and we also show that in the high SNR regime, the test is translated to a thresholding approach, and in this regime, we obtain the optimum threshold, analytically. Using the thresholding technique, the problem is reduced to one similar to the noiseless scenario.

We consider an irregular sensing graph. For the analysis, we assume that the real and imaginary parts of non-zero elements of the sub-vectors $\mathbf{u}_i$ both come from a Gaussian distribution $\mathcal{N}(0, \sigma_s^2)$, in an i.i.d. fashion. We consider $d(m)$ hypotheses $\mathcal{H}_i$ for $i = 0, \dots, d(m)$, where $\mathcal{H}_i$ represents the scenario that the measurement m is connected to i non-zero blocks and $d(m) - i$ zero-blocks. Therefore, the a priori probability of $\mathcal{H}_i$ is

$$P(\mathcal{H}_i) = \binom{d(m)}{i} \alpha^i (1 - \alpha)^{d(m)-i}, \quad i = 0, \dots, d(m). \quad (5.7)$$

Furthermore, suppose that only one of the neighboring blocks of the m -th MN (e.g., the j -th block) out of $d(m)$ blocks is non-zero and $d(m) - 1$ neighboring blocks are zero (i.e., Hypothesis $\mathcal{H}_1$ is true). In this case, we have

$$\begin{aligned} y_m &= 2\Re \left({}^+ \mathbf{A}_m^{(j)} \cdot \mathbf{u}_j \right) + n_m \\ &= 2 \sum_{i=1}^D \Re({}^+ \mathbf{A}_{m,i}^{(j)}) \cdot \Re(\mathbf{u}_j^i) - \Im({}^+ \mathbf{A}_{m,i}^{(j)}) \cdot \Im(\mathbf{u}_j^i) + n_m, \end{aligned} \quad (5.8)$$

where ${}^+ \mathbf{A}_{m,i}^{(j)}$ and $\mathbf{u}_j^i$ are the i -th element of the sub-vector ${}^+ \mathbf{A}_m^{(j)}$ and the i -th element of the sub-vector $\mathbf{u}_j$, respectively. The probability density function of y_m under this assumption ($\mathcal{H}_1$) is

$$P(y_m | \mathcal{H}_1) = \mathcal{N}(0, 4\|{}^+ \mathbf{A}_m^{(j)}\|_2^2 \sigma_s^2 + \sigma_n^2). \quad (5.9)$$

Based on the structure of the AIC presented in Section 5.2, $\|{}^+\mathbf{A}_m^{(j)}\|_2^2 = cte$, $\forall j$. Without loss of generality, we assume that $\|{}^+\mathbf{A}_m^{(j)}\|_2^2 = 1$. Therefore, $P(y_m|\mathcal{H}_1) = \mathcal{N}(0, 4\sigma_s^2 + \sigma_n^2)$. Similarly, the probability density function of y_m under the hypothesis $\mathcal{H}_i$ is given by

$$P(y_m|\mathcal{H}_i) = \mathcal{N}(0, 4i\sigma_s^2 + \sigma_n^2), \quad i = 0, \dots, d(m). \quad (5.10)$$

In ZMD, we need to determine whether the noise-free version of y_m , i.e., y_m^{nf} , is zero or not. If y_m^{nf} is zero, $\mathcal{H}_0$ is true with probability one. Otherwise, $\mathcal{H}_i$ for $i = 1, \dots, d(m)$ is true ($\bar{\mathcal{H}}_0$ is true), i.e., $\mathcal{H}_0$ is false. To make a decision on whether y_m^{nf} is zero or not, we use the likelihood ratio test (LRT). The LR is defined by

$$\Lambda(y_m) = \frac{P(y_m|\bar{\mathcal{H}}_0)}{P(y_m|\mathcal{H}_0)}, \quad (5.11)$$

and the test is performed as follows:

$$\begin{aligned} \text{if } \Lambda(y_m) < c & : \quad \text{accept } \mathcal{H}_0; \\ \text{if } \Lambda(y_m) \geq c & : \quad \text{reject } \mathcal{H}_0 \text{ (accept } \bar{\mathcal{H}}_0), \end{aligned} \quad (5.12)$$

where c is a properly selected threshold.

We can calculate $P(y_m|\bar{\mathcal{H}}_0)$ in (5.11) by

$$P(y_m|\bar{\mathcal{H}}_0) = \sum_{i=0}^{d(m)} P(y_m|\mathcal{H}_i, \bar{\mathcal{H}}_0)P(\mathcal{H}_i|\bar{\mathcal{H}}_0). \quad (5.13)$$

Using the facts that $P(\mathcal{H}_0|\bar{\mathcal{H}}_0) = 0$ and $P(\mathcal{H}_i, \bar{\mathcal{H}}_0) = P(\mathcal{H}_i)$, for $i \geq 1$, Equation (5.13) can be rewritten as

$$P(y_m|\bar{\mathcal{H}}_0) = \frac{1}{P(\bar{\mathcal{H}}_0)} \sum_{i=1}^{d(m)} P(y_m|\mathcal{H}_i)P(\mathcal{H}_i). \quad (5.14)$$

Therefore, using (5.7), (5.10), (5.11) and (5.14), we have

$$\begin{aligned}\Lambda(y_m) &= \frac{\sum_{i=1}^{d(m)} \binom{d(m)}{i} \alpha^i (1-\alpha)^{d(m)-i} \mathcal{N}(0, 4i\sigma_s^2 + \sigma_n^2)}{\sum_{i=1}^{d(m)} \binom{d(m)}{i} \alpha^i (1-\alpha)^{d(m)-i} \mathcal{N}(0, \sigma_n^2)} \\ &= \frac{\sum_{i=1}^{d(m)} \binom{d(m)}{i} \alpha^i (1-\alpha)^{d(m)-i} \sqrt{\frac{\sigma_n^2}{4i\sigma_s^2 + \sigma_n^2}} \exp\left\{\frac{2i\sigma_s^2 y_m^2}{(4i\sigma_s^2 + \sigma_n^2)\sigma_n^2}\right\}}{1 - (1-\alpha)^{d(m)}}.\end{aligned}\tag{5.15}$$

In the high SNR regime, where $\sigma_s^2 \gg \sigma_n^2$, we can neglect σ_n^2 against σ_s^2 in (6.6). We can thus simplify the decision for $\mathcal{H}_0$ (i.e., $\Lambda(y_m) < c$) to

$$|y_m| < c', \tag{5.16}$$

where c' is determined by the target P_{WZD} as described in the next section. In other words, if $|y_m| < c'$, we decide $\hat{y}_m^{nf} = 0$ and if $|y_m| \geq c'$, we decide $\hat{y}_m^{nf} \neq 0$.

B.2. Analysis

Based on the likelihood ratio test, the measurement values are partitioned into two regions:

$$\mathcal{R}_0 = \{y_m : |y_m| < c'\}, \text{ and } \mathcal{R}_1 = \{y_m : |y_m| \geq c'\}. \tag{5.17}$$

Based on the correct or the erroneous detection of zero/non-zero measurements, we can partition the set of measurements $\mathcal{M}$ into four different sets: $\mathcal{M}_z^C$, $\mathcal{M}_z^W$, $\mathcal{M}_{nz}^C$ and $\mathcal{M}_{nz}^W$. The set $\mathcal{M}_z^W$ represents the set of non-zero measurements which are detected erroneously as zero, $\mathcal{M}_z^W = \{y_m : y_m \in \mathcal{R}_0, \bar{\mathcal{H}}_0\}$. Similarly, $\mathcal{M}_z^C = \{y_m : y_m \in \mathcal{R}_0, \mathcal{H}_0\}$, $\mathcal{M}_{nz}^W = \{y_m : y_m \notin \mathcal{R}_0, \mathcal{H}_0\}$ and $\mathcal{M}_{nz}^C = \{y_m : y_m \notin \mathcal{R}_0, \bar{\mathcal{H}}_0\}$. In the following, we first consider the case where the sensing graph is regular.

The probability of false alarm (for zero measurements) $P_{FA}^{(zm)}$ is defined as the probability that a measurement y_m is detected as zero while at least one of its neighboring VNs belongs to $\mathcal{V}_{nz}$, i.e., $P_{FA}^{(zm)} = \int_{y \in \mathcal{R}_0} P(y|\bar{\mathcal{H}}_0) dy$. For the signal model considered here, by using (5.14), we have

$$\begin{aligned}
 P_{FA}^{(zm)} &= \frac{\sum_{i=1}^{d_M} \binom{d_M}{i} \alpha^i (1-\alpha)^{d_M-i} \int_{y \in \mathcal{R}_0} P(y|\mathcal{H}_i) dy}{1 - (1-\alpha)^{d(m)}} \\
 &= \frac{\sum_{i=1}^{d_M} \binom{d_M}{i} \alpha^i (1-\alpha)^{d_M-i} \operatorname{erf}\left(\frac{c'}{\sqrt{2(4i\sigma_s^2 + \sigma_n^2)}}\right)}{1 - (1-\alpha)^{d(m)}}, \tag{5.18}
 \end{aligned}$$

where $\operatorname{erf}(x) := (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$.

Probability of detection (for zero measurements), $P_D^{(zm)}$, is defined as the probability that a zero measurement is detected correctly as zero, i.e., $P_D^{(zm)} = \int_{y \in \mathcal{R}_0} P(y|\mathcal{H}_0) dy$. For our signal model, $P_D^{(zm)}$ is given by

$$P_D^{(zm)} = \operatorname{erf}\left(\frac{c'}{\sqrt{2\sigma_n^2}}\right). \tag{5.19}$$

Clearly, $E(|\mathcal{M}_z^C|) = MP(\mathcal{H}_0)P_D^{(zm)} = M(1-\alpha)^{d_M}P_D^{(zm)}$ and $E(|\mathcal{M}_z^W|) = MP(\bar{\mathcal{H}}_0)P_{FA}^{(zm)} = M(1-P(\mathcal{H}_0))P_{FA}^{(zm)} = M(1-(1-\alpha)^{d_M})P_{FA}^{(zm)}$.

Now, we obtain P_{WZD} and P_{ZD} of the VNs based on the $P_D^{(zm)}$ and $P_{FA}^{(zm)}$ of the MNs. Based on the definition, we have

$$\begin{aligned}
 P_{WZD} &= P(v \in \mathcal{V}_{nz} | |\mathcal{E}_{v, \mathcal{M}_z^C \cup \mathcal{M}_z^W}| \geq 1) \\
 &= \frac{P(|\mathcal{E}_{v, \mathcal{M}_z^C \cup \mathcal{M}_z^W}| \geq 1 | v \in \mathcal{V}_{nz}) P(v \in \mathcal{V}_{nz})}{P(|\mathcal{E}_{v, \mathcal{M}_z^C \cup \mathcal{M}_z^W}| \geq 1)}. \tag{5.20}
 \end{aligned}$$

Since, with probability one, $\mathcal{E}_{v, \mathcal{M}_z^C} = \emptyset$ for $v \in \mathcal{V}_{nz}$, we can write

$$\begin{aligned}
 P(|\mathcal{E}_{v, \mathcal{M}_z^C \cup \mathcal{M}_z^W}| \geq 1 | v \in \mathcal{V}_{nz}) &= P(|\mathcal{E}_{v, \mathcal{M}_z^W}| \geq 1 | v \in \mathcal{V}_{nz}) \\
 &= 1 - P(|\mathcal{E}_{v, \mathcal{M}_z^W}| = 0 | v \in \mathcal{V}_{nz}) \\
 &= 1 - (1 - P(t_e \in \mathcal{M}_z^W | h_e \in \mathcal{V}_{nz}))^{d_v}, \tag{5.21}
 \end{aligned}$$

where $P(t_e \in \mathcal{M}_z^W | h_e \in \mathcal{V}_{nz}) = P_{FA}^{(zm)}$.

For the denominator of (5.20), we have

$$\begin{aligned}
 P(|\mathcal{E}_{v, \mathcal{M}_z^C \cup \mathcal{M}_z^W}| \geq 1) &= 1 - P(|\mathcal{E}_{v, \mathcal{M}_z^C \cup \mathcal{M}_z^W}| = 0) \\
 &= 1 - (1 - P(t_e \in \mathcal{M}_z^C \cup \mathcal{M}_z^W))^{d_v}, \tag{5.22}
 \end{aligned}$$

where

$$\begin{aligned} P(t_e \in \mathcal{M}_z^C \cup \mathcal{M}_z^W) &= \frac{E(|\mathcal{M}_z^C|) + E(|\mathcal{M}_z^W|)}{M} \\ &= (1 - \alpha)^{d_M} P_D^{(\text{zm})} + (1 - (1 - \alpha)^{d_M}) P_{FA}^{(\text{zm})}. \end{aligned} \quad (5.23)$$

Since, $P(v \in \mathcal{V}_{nz}) = \alpha$, from (5.21) and (5.22), P_{WZD} is calculated by

$$P_{WZD} = \alpha \frac{1 - \left(1 - P_{FA}^{(\text{zm})}\right)^{d_V}}{1 - \left(1 - (1 - \alpha)^{d_M} P_D^{(\text{zm})} - (1 - (1 - \alpha)^{d_M}) P_{FA}^{(\text{zm})}\right)^{d_V}}. \quad (5.24)$$

The probability of zero detection, P_{ZD} , can be calculated by

$$\begin{aligned} P_{ZD} &= P(|\mathcal{E}_{v, \mathcal{M}_z^C \cup \mathcal{M}_z^W}| \geq 1 | v \in \mathcal{V}_z) \\ &= 1 - P(|\mathcal{E}_{v, \mathcal{M}_z^C \cup \mathcal{M}_z^W}| = 0 | v \in \mathcal{V}_z) \\ &= 1 - \left(1 - P(t_e \in \mathcal{M}_z^C \cup \mathcal{M}_z^W | h_e \in \mathcal{V}_z)\right)^{d_V}, \end{aligned} \quad (5.25)$$

where

$$\begin{aligned} P(t_e \in \mathcal{M}_z^C \cup \mathcal{M}_z^W | h_e \in \mathcal{V}_z) &= \frac{P(t_e \in \mathcal{M}_z^C \cup \mathcal{M}_z^W, h_e \in \mathcal{V}_z)}{P(h_e \in \mathcal{V}_z)} \\ &= \frac{E(|\mathcal{E}_{\mathcal{V}_z, \mathcal{M}_z^C}|) + E(|\mathcal{E}_{\mathcal{V}_z, \mathcal{M}_z^W}|)}{E(|\mathcal{E}_{\mathcal{V}, \mathcal{M}}|)} \cdot \frac{1}{(1 - \alpha)} \\ &= \frac{E(|\mathcal{M}_z^C|) d_M + E(|\mathcal{M}_z^W|) \sum_{i=1}^{d_M} (d_M - i) \binom{d_M}{i} \alpha^i (1 - \alpha)^{d_M - i} / P(\bar{\mathcal{H}}_0)}{M d_M (1 - \alpha)} \\ &= (1 - \alpha)^{d_M - 1} P_D^{(\text{zm})} + (1 - (1 - \alpha)^{d_M - 1}) P_{FA}^{(\text{zm})}. \end{aligned} \quad (5.26)$$

Therefore,

$$P_{ZD} = 1 - \left(1 - (1 - \alpha)^{d_M - 1} P_D^{(\text{zm})} - (1 - (1 - \alpha)^{d_M - 1}) P_{FA}^{(\text{zm})}\right)^{d_V}. \quad (5.27)$$

When the measurements are noiseless, $P_D^{(\text{zm})} = 1$ and $P_{FA}^{(\text{zm})} = 0$. With these values for $P_D^{(\text{zm})}$ and $P_{FA}^{(\text{zm})}$, P_{WZD} in (5.24) will be zero and P_{ZD} in (5.27) is reduced to (5.4). With a large value for the threshold c' , all the measurements are detected as zero, which results in $P_D^{(\text{zm})} = 1$ and $P_{FA}^{(\text{zm})} = 1$. We see from Equations (5.24) and

(5.27) that, in this case, $P_{WZD} = \alpha$ and $P_{ZD} = 1$, which are expected.

For given values of d_M , d_V , σ_s and σ_n , the threshold c' can be chosen to provide a target P_{WZD} . We can then obtain P_{ZD} using this threshold level. It is important to note that while $P_D^{(zm)}$ and $P_{FA}^{(zm)}$ depend on the signal model, P_{WZD} and P_{ZD} do not. Therefore, Equations (5.24) and (5.27) can be applied to other signal models.

The above results can be generalized to irregular sensing graphs (matrices). The derivation of P_{ZD} and P_{WZD} is similar to (5.20)-(5.27). For irregular sensing matrices, we have

$$P_{WZD} = \frac{\alpha \left(1 - \sum_i \lambda_i \left(1 - P_{FA}^{(zm)} \right)^i \right)}{1 - \sum_i \lambda_i \left(1 - P_{FA}^{(zm)} - (P_D^{(zm)} - P_{FA}^{(zm)}) \sum_j \rho_j (1 - \alpha)^j \right)^i},$$

and

$$P_{ZD} = 1 - \sum_i \lambda_i \left(1 - P_{FA}^{(zm)} - (P_D^{(zm)} - P_{FA}^{(zm)}) \frac{\sum_j j \rho_j (1 - \alpha)^{j-1}}{\sum_j j \rho_j} \right)^i.$$

In noiseless scenario, when $P_D^{(zm)} = 1$ and $P_{FA}^{(zm)} = 0$, from the above equations, we have $P_{WZD} = 0$ and P_{ZD} reduces to (5.5).

5.3.3 Complexity of the Proposed Algorithm

In terms of the complexity, the proposed scheme has very low complexity as it only needs to identify the zero measurement nodes and verify the adjacent variable nodes as vacant sub-channels. This is negligible in comparison with any of the existing algorithms for the recovery of block sparse signals. In particular, one of the most powerful algorithms for the recovery of block sparse signals, in terms of performance/complexity trade-off, is AMP with James-Stein's estimator (AMP-JS) [48]. In terms of complexity, the complexity of AMP-JS is $\mathcal{O}(MN)$, which is much higher than the minimal complexity of the proposed scheme.

5.4 Simulation Results

In this section, we present some simulation results to demonstrate the effectiveness of the proposed method for detecting the spectrum holes in both noiseless and noisy

Figure 5.3: P_{ZD} obtained by both analysis (dashed lines) and simulations (solid lines) versus d_M for different sparsity ratios for bi-regular graphs with $L = 1000$ and $M = 500$. The stars are the maximum P_{ZD} for each sparsity ratio.

cases. We consider an N dimensional signal with the positive frequency range of $[0 : N/2]$, containing L non-overlapping channels of equal bandwidth $\frac{N}{2L}$. We use an N -point unitary Fourier matrix to map the input signal from the time to the frequency domain. To model the block sparse spectrum, each block is independently selected to be non-zero with probability α (from positive frequency elements). We assign the real and imaginary parts of the elements of non-zero blocks from a Gaussian distribution $\mathcal{N}(0, \sigma_s^2)$ in an i.i.d. fashion. In our simulations, we set $\sigma_s = 1$. It is important to note that the choice of these continuous distributions does not affect the performance of the ZMD method, in either noiseless or noisy scenarios.

5.4.1 Noiseless Measurements

Fig. 5.3 shows P_{ZD} , obtained both analytically and by simulations, versus d_M for different sparsity ratios. In this figure, P_{ZD} of ZMD is depicted for the regular graphs with $L = 1000$ and $M = 500$. Thus, $d_V = d_M/2$. As seen from this figure, simulation and analytical results match perfectly. Furthermore, for each sparsity ratio, the optimum d_M is depicted in terms of maximizing P_{ZD} . We see that for smaller sparsity ratios, larger d_M is required to maximize P_{ZD} and with increasing the sparsity ratio, the optimum value of d_M decreases. In particular, for larger sparsity ratios, the best degree distribution is $(d_V, d_M) = (1, 2)$.

Fig. 5.4 shows P_{ZD} versus number of measurements, M , for different algorithms in

Figure 5.4: P_{ZD} versus M for a signal with $\alpha = 0.25$ and $L = 500$, for different algorithms and sensing graphs.

a case with $L = 500$ and $\alpha = 0.25$. Note that the x-axis is in log scale. In this figure, we see P_{ZD} of the regular graphs with $d_V = 1$ when d_M is obtained by $d_V L/M$ for each M , as well as the regular graphs with $d_M = 4$ when d_V is obtained by $d_M M/L$ for each M . We have also plotted the curve corresponding to a graph, where each measurement node is connected to only one variable node and each variable node is connected to at most one measurement node. This case is denoted by “1to1” in Fig. 5.4.

In Fig. 5.4, we have also plotted P_{ZD} of AMP-JS, which is known as the-state-of-the-art in block sparse signal recovery, for two different block lengths 3 and 5. For AMP-JS, the sensing matrix is selected to be a dense matrix randomly constructed with i.i.d. elements, where each element is a zero-mean Gaussian random variable with variance $1/M$. In the AMP-JS, there is a thresholding function based on the ℓ_2 -norm of each block. Thus, the zero and non-zero blocks can be detected easily. Since in the noiseless scenario, unlike the case for the proposed ZMD scheme, the P_{WZD} of AMP-JS is not necessarily zero, P_{ZD} of AMP-JS is plotted only for the M values in which $P_{WZD} < 0.1\%$. We see that with increasing the block length, the required number of measurements by AMP-JS is increased for a target P_{ZD} . However, the performance of the proposed method does not depend on the block length. The comparison shows the superior performance of the proposed scheme, particularly for regular graphs with $d_V = 1$. For very low number of measurements, the 1to1 graph outperforms the others.

Figure 5.5: The probabilities $P_D^{(zm)}$, $P_{FA}^{(zm)}$, P_{ZD} and P_{WZD} versus the threshold, c' , when the sensing graph is regular with degree distribution $(1, 2)$ for $L = 1000$, $\alpha = 0.25$ and $\text{SNR} = 25$ dB. The markers and the dashed lines correspond to the simulation and analytical results, respectively.

5.4.2 Noisy Measurements

In Fig. 5.5, the probabilities of false alarm ($P_{FA}^{(zm)}$), detection ($P_D^{(zm)}$), wrong zero detection (P_{WZD}) and zero detection (P_{ZD}) are plotted versus the threshold c' . Both analytical and simulation results are presented. The sensing graphs are regular graphs with degrees $d_V = 1$ and $d_M = 2$. The number of blocks L is 1000 and $\alpha = 0.25$, and the curves are plotted for $\text{SNR} = 25$ dB. As seen in this figure, the simulation and analytical results match closely.

Fig. 5.6 shows P_{ZD} versus sparsity ratio for different noise levels when P_{WZD} is limited below 2%. The curve for the noiseless scenario is also provided for reference. The sensing graphs are regular with degree distribution equal to $(1, 2)$, $L = 1000$ and $M = 500$. Fig. 5.6 shows that the performance improves with increasing SNR, and at each SNR, the P_{ZD} drops rapidly when the sparsity ratio is increased beyond a certain threshold. Before such a threshold is reached, the performance is practically the same as that of the noiseless case, but, when this threshold is passed, the P_{ZD} curve demonstrates a waterfall behavior down towards $P_{ZD} = 0$. This waterfall region (for a given SNR) corresponds to the values of α for which the threshold c' has to be decreased rapidly (with increasing α) to maintain P_{WZD} below 2%. As a consequence, P_{ZD} is decreased sharply.

Figure 5.6: P_{ZD} versus sparsity ratio for different noise levels with regular sensing graphs with the degree distribution of (1,2) for $L = 1000$ and $M = 500$.

5.5 Conclusion

A novel zero-block detection scheme in the context of wideband spectrum sensing was proposed. For this, an analog to information converter (AIC) with block sparse sensing matrix was designed. Analytical and simulation results for both noiseless and noisy scenarios were presented. The results demonstrated the effectiveness of the proposed scheme in reliable detection of spectrum holes with minimal complexity even in scenarios where the number of measurements were relatively small. In particular, it was shown that, at the presence of noise, performance similar to the noiseless case can be achieved at higher SNR values and smaller sparsity ratios. Probably one of the most important contributions of this work was to introduce a paradigm in compressed sensing in which one is interested in reliable detection of (some of the) zero blocks rather than the recovery of the whole block sparse signal.

Chapter 6

Low-Complexity Detection of Zero Blocks

Motivated by the low-complexity of VB algorithms, our goal in this chapter is to devise similar algorithms for the problem of compressed WSS. To apply the NB-VB algorithms to the problem of WSS, however, we are faced with a number of challenges:

- 1) In the application of the original VB algorithms to sparse signals, each element of the signal is represented by a node. The spectrum however is block sparse, and to take advantage of this structure, we need to represent each block of the spectrum with one node. In such a model, the original verification rules¹ are not directly applicable. To use the VB rules in the context of WSS, the AIC module designed in chapter 5 is required. Taking advantage of the block sparsity of the spectrum, we can then reduce the required number of measurements by $\frac{1}{\text{Block Size}}$, compared to the case where VB algorithms are applied to a sparse (not necessarily block-sparse) signal.
- 2) In the WSS application, the measurements are real while the spectrum and the measurement matrix are complex. Under such condition, we show that the VB algorithms are not able to recover the exact value of the non-zero elements of the spectrum. A modification of the framework is thus required to use VB algorithms for the detection of zero-blocks without any need to recover the non-zero blocks.
- 3) Although VB algorithms provide an attractive performance/complexity trade-off, they are very sensitive to the presence of noise in the measurements. In fact, the verification rules of VB algorithms are not directly applicable to noisy

¹We will talk about the verification rules later in the chapter.

measurements. However, in the WSS, the measurements are corrupted with noise. One thus needs to develop techniques to handle noisy measurements.

- 4) In original VB algorithms, the objective is to recover all the signal elements. For WSS, and in the context of this thesis, however, we are only interested in reliable detection of (some of the) zero-blocks, not full reconstruction of the spectrum. Given the number of measurements, the goal in this chapter is thus to detect as many zero-blocks as reliably possible. Neither the full reconstruction of the spectrum nor the detection of all spectrum holes is the target.

In this chapter, we address all the above challenges in using VB algorithms in the context of WSS.

In this work, given a number of measurements M , we are interested in a recovery algorithm that can detect a fraction μ of zero-blocks *reliably*.² The value M can be much smaller than what would be needed to recover the whole signal reliably, and μ , which is a function of M , is generally smaller than one but tends to one as M is increased.

To apply NB-VB algorithms in this context, using the proposed AIC in Chapter 5, we modify the VB rules to make them applicable to the proposed framework. Furthermore, we extend the proposed VB algorithm to the case of noisy measurements. We then demonstrate that for a given number of measurements, the performance of the proposed VB scheme is superior to the existing recovery algorithms including AMP-JS.

6.1 NB-VB Algorithms for Recovery of Block Sparse Signals: Noiseless Measurements

Although, the problem of WSS should be considered in the presence of noise, we first study the noiseless case in this section. This helps to better understand the noisy case. In addition, the discussions of this section would be applicable to other block sparse CS scenarios, where zero-block detection in the absence of noise may be of interest.

²Reliable detection of zero-blocks means that the probability that a detected zero-block is non-zero is negligible.

NB-VB algorithms are a class of recovery algorithms for CS that can recover a sparse signal by iterative message-passing on a sparse sensing graph [29,31,32]. In [29], the NB-VB algorithms were analyzed over regular sensing graphs in the asymptotic regime, where N tends to infinity. It was demonstrated in [29] that a *threshold phenomenon* happens in the asymptotic regime, in that, there exists a threshold on the sparsity ratio, where almost all the signals with sparsity ratio below that threshold can be recovered successfully, while almost all the signals with sparsity ratio above that threshold cannot be. The threshold, which is attributed to an ensemble of sensing graphs with a certain degree distribution, was obtained in [29] using a technique called *density evolution*. The analysis of [29] was then extended to irregular sensing graphs in [88] and it was shown that properly designed irregular graphs can substantially outperform the regular ones with the same under-sampling ratio. It is important to note that VB algorithms have so far been only applied to sparse (not necessarily block sparse) signal recovery.

In the following, we first describe the original NB-VB algorithms for the recovery of real-valued sparse signals. We focus on the best performing NB-VB algorithm dubbed as SBB in [29]. In each iteration of this algorithm, the VNs (signal elements) are verified based on the following three simple rules:³

1. **Rule 1:** If a MN is zero, then all neighboring VNs are verified as zero.
2. **Rule 2:** If the degree of a MN is one, then the value of its unique neighboring VN is verified with the value of the MN.
3. **Rule 3:** If two MNs neighboring a VN, v , have the same non-zero value, all uncommon neighboring VNs to these MNs are verified with zero and if v is their unique common VN, v is verified with the common value of the MNs.

Once a VN is verified, all its adjacent edges are removed from the graph, and if the value of the verified VN is non-zero, this value is subtracted from the neighboring measurements. The iterations continue until all the variable nodes are recovered or the algorithm is stalled as no new variable node can be verified using the three rules. In the above algorithm, the continuity of signal distribution or the continuity of the distribution of non-zero weights of the graph edges guarantees that the probability of false verification is equal to zero [29].

³In the description of the verification rules, we assume that the edge weights are one (the measurement matrix entries are 0 and 1).

In the standard VB algorithms, a success would be declared if the signal is fully recovered through the iterative verification rules. If the algorithm is stalled while some of the VNs are still not verified, it would normally be considered a failure. In the context of WSS, however, even if the iterations are stalled at some point with some VNs still unverified, we do not consider this a failure as long as some zero VNs are identified. Our focus in WSS is thus reliable detection of zero blocks (vacant sub-channels). This can be initially performed by using Rule 1. To increase the number of verified zero-blocks iteratively, however, we do need to verify some non-zero blocks by applying Rules 2 and 3. The application of these rules to the problem of WSS poses some challenges.

In WSS, the signal is block sparse with complex elements. This implies that, even if each block contains only one complex element, that element cannot be recovered by either Rule 2 or Rule 3. To see this, suppose the m -th MN has degree one and is connected to the i -th VN, which consists of only one complex-valued element. We thus have

$$\begin{aligned} y_m &= 2\Re\left({}^+\mathbf{A}_m^{(i)} \cdot \mathbf{u}_i\right) \\ &= 2\Re\left({}^+\mathbf{A}_m^{(i)}\right) \cdot \Re(\mathbf{u}_i) - 2\Im\left({}^+\mathbf{A}_m^{(i)}\right) \cdot \Im(\mathbf{u}_i). \end{aligned} \quad (6.1)$$

This linear equation with two unknowns, $\Re(\mathbf{u}_i)$ and $\Im(\mathbf{u}_i)$, however, is under-determined and does not have a unique solution. The VN $\mathbf{u}^{(i)}$ thus cannot be recovered from the degree-1 MN m . To apply the NB-VB algorithm based on Rules 1-3 in the context of WSS, we need to modify these rules as explained in the following:⁴

1) *Zero Measurement (ZM)*:⁵ This rule is in principle the same as Rule 1, i.e., if a MN is zero, then all neighboring VNs are verified as zero blocks. (These variable nodes and their adjacent edges are then removed from the graph.)

ZM rule is applicable because if y_m in (5.1) has zero value, then the blocks of ${}^+\mathbf{x}$ corresponding to the non-zero blocks of ${}^+\mathbf{A}_m$, are zero with probability one. This simply follows from the fact that at least one of the distributions f or g , described in Section 5.1, is continuous.

In the following, we discuss how Rule 2 can be modified for the WSS application. Suppose y_a is a non-zero MN, and only the v -th block of ${}^+\mathbf{A}_a$ is non-zero (i.e., $|\mathcal{V}(a)| =$

⁴We consider the case where the sensing matrix is block sparse with the block boundaries matching those of the signal (as described in Section 5.2).

⁵This rule is the zero block detection scheme of Chapter 5 without iterations.

1). Therefore, we have

$$y_a = 2\Re \left({}^+\mathbf{A}_a^{(v)} \cdot \mathbf{u}_v \right). \quad (6.2)$$

This thus implies that $\mathbf{u}_v$ is a non-zero block. However, as we discussed before, the value of $\mathbf{u}_v$ cannot be resolved and remains unknown. Since the value of the elements of the VN can not be determined, the effect of this VN cannot be, in general, removed from the neighboring MNs. The exception to this is when the VN v has the same contribution in all its neighboring MNs, i.e., ${}^+\mathbf{A}_a^{(v)} = {}^+\mathbf{A}_m^{(v)}, \forall a, m \in \mathcal{M}(v)$. In this case, we can write

$$y_m = 2\Re \left({}^+\mathbf{A}_m^{(v)} \cdot \mathbf{u}_v \right) + \sum_{i \in \mathcal{V}(m) \setminus v} 2\Re \left({}^+\mathbf{A}_m^{(i)} \cdot \mathbf{u}_i \right), \quad (6.3)$$

and thus Equation (6.3) can be written as

$$y_m = y_a + \sum_{i \in \mathcal{V}(m) \setminus v} 2\Re \left({}^+\mathbf{A}_m^{(i)} \cdot \mathbf{u}_i \right), \quad \forall m \in \mathcal{M}(v). \quad (6.4)$$

Therefore, to remove the effect of the non-zero VN, v , one can simply subtract y_a from the value of all neighboring MNs, i.e., $y_m \leftarrow y_m - y_a, \forall m \in \mathcal{M}(v)$. This also explains the choice that we made in designing the AIC to make sure all the non-zero blocks of the sensing matrix are identical. Rule 2 is thus changed as follows:

2) *Degree One Measurement (D1M)*: If the degree of a MN a with value y_a is one, then its unique neighboring VN, v , is verified as being a non-zero block. In this case, the value of all neighboring measurement nodes of v is subtracted by y_a . (VN v and all its adjacent edges are then removed from the graph.)

Related to Rule 3, one can see that in the proposed framework, if the blocks of the sensing matrix are selected randomly, the probability of having two equal measurement nodes is zero. To be able to adopt this verification rule, we thus need the same assumption that all the blocks of the sensing matrix be identical. In this case, measurements can be equal. If they are, and if they have a unique common VN as their neighbor, then that variable node can be identified as non-zero and its contribution can be deducted from all its neighboring measurement nodes. We thus have the following adaptation of Rule 3 for block sparse signals.

3) *Equal Measurement (EM)*: If two MNs neighboring a VN, v , have the same

non-zero value y_a , all uncommon neighboring VNs to these MNs are verified as zero. Furthermore, if v is the unique common VN of the two MNs, v is verified as being non-zero, and all the measurements neighboring v are updated by subtracting their values by y_a . (VN v and all its adjacent edges are then removed from the graph.)

The general framework of the algorithm in the noiseless scenario is given in the following.

Algorithm 6.1: NB-VB zero-block detection in noiseless scenarios

Inputs: $\mathbf{A}$ ($\mathcal{E}_{\mathcal{V},\mathcal{M}}$), $\mathbf{y}$

Output: Verified variable nodes as zero and non-zero

Initializing step:

$j = 0$.

$\check{\mathcal{M}} = \mathcal{M}$ and $\check{\mathcal{V}} = \mathcal{V}$.

$\check{\mathbf{y}} = \mathbf{y}$.

$\mathcal{E}_{\check{\mathcal{V}},\check{\mathcal{M}}}^{(j)} = \mathcal{E}_{\mathcal{V},\mathcal{M}}$.

do:

$j \leftarrow j + 1$.

Update $\mathcal{E}_{\check{\mathcal{V}},\check{\mathcal{M}}}^{(j)} = \mathcal{E}_{\check{\mathcal{V}},\check{\mathcal{M}}}^{(j-1)}$.

ZM:

for all $\{m \in \check{\mathcal{M}} : \check{y}_m = 0\}$ **do:**

Verify all $\{v \in \check{\mathcal{V}}(m)\}$ as zero VNs.

Update the sensing graph by removing all the edges of the set $\mathcal{E}_{\check{\mathcal{V}}(m),\check{\mathcal{M}}} \subseteq \mathcal{E}_{\check{\mathcal{V}},\check{\mathcal{M}}}^{(j)}$.

Update the sets $\check{\mathcal{M}}$ and $\check{\mathcal{V}}$ by removing the MNs and VNs with degree zero, respectively.

end for

D1M:

for all $\{a \in \check{\mathcal{M}} : \check{d}(a) = |\check{\mathcal{V}}(a)| = 1, \check{y}_a \neq 0\}$ **do:**

Verify $\{v \in \check{\mathcal{V}}(a)\}$ as a non-zero VN.

Update $\check{y}_m \leftarrow \check{y}_m - \check{y}_a, \forall m \in \check{\mathcal{M}}(v)$.

Update the sensing graph by removing all the edges of the set $\mathcal{E}_{\check{\mathcal{V}}(a),\check{\mathcal{M}}} \subseteq \mathcal{E}_{\check{\mathcal{V}},\check{\mathcal{M}}}^{(j)}$.

Update the sets $\check{\mathcal{M}}$ and $\check{\mathcal{V}}$ by removing the MNs and VNs with degree zero, respectively.

end for

EM:

for all $\{v \in \check{\mathcal{V}}\}$ **do:**

for all $\{a, b \in \check{\mathcal{M}}(v), a \neq b : \check{y}_a = \check{y}_b\}$ **do:**⁶

Set $\bar{\mathcal{V}}(a, b) = (\check{\mathcal{V}}(a) \cup \check{\mathcal{V}}(b)) \cap \check{\mathcal{V}}^c(a, b)$.

Verify all $\{u \in \bar{\mathcal{V}}(a, b)\}$ as zero VNs.

Update the sensing graph by removing all the edges of the set $\mathcal{E}_{u, \check{\mathcal{M}}} \subseteq \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)}, \forall u \in \bar{\mathcal{V}}(a, b)$.

if $|\check{\mathcal{V}}(a, b)| = 1$ **then:**

Verify $\{v \in \check{\mathcal{V}}(a, b)\}$ as a non-zero VN.

Update $\check{y}_m \leftarrow \check{y}_m - \check{y}_a, \forall m \in \check{\mathcal{M}}(v)$.

Update the sensing graph by removing all the edges of the set $\mathcal{E}_{v, \check{\mathcal{M}}} \subseteq \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)}$.

end if

Update the sets $\check{\mathcal{M}}$ and $\check{\mathcal{V}}$ by removing the MNs and VNs with degree zero, respectively.

end for

end for

while $\mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)} \neq \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j-1)}$

In Algorithm 6.1, notations $\check{\mathcal{V}}(\cdot)$, $\check{\mathcal{M}}(\cdot)$ and $\check{d}(\cdot)$ are used to denote the neighboring variable and measurement nodes and the degree of the nodes in the updated graph, respectively.

Remark 6.1: Using the proposed NB-VB algorithm, one can verify all the zero (and non-zero) blocks of the signal (and thus its block-wise support), if the number of measurements is sufficiently large. On the other hand, block sparse signal recovery

⁶There may be more than 2 equal MNs. However, considering only 2 of them in EM rule is enough and the other MNs will be updated to zero after the application of this rule. They will then be processed by ZM rule in the next iteration.

algorithms can be used in the context of WSS if the number of measurements is large enough so that the signal and thus its support can in fact be detected successfully. If the number of measurements, however, is relatively low, one of the following cases may occur: 1) the detection would be inaccurate, 2) only part of the support (not all of it) would be located, i.e., some of the blocks would remain unidentified as being either zero or non-zero. This implies that in the lack of sufficient number of measurements, the block sparse recovery algorithms would perform poorly in identifying the zero-blocks reliably. This is unlike the proposed scheme, which can still detect zero and non-zero blocks reliably even for a small number of measurements. It just detects a smaller number of them. Note that the continuity of f or g guarantees that the verified blocks are correctly verified with probability one.

Remark 6.2: The threshold values and asymptotic results of [29] and [88] are also applicable to the proposed NB-VB algorithm in the context of zero-block detection, i.e., for a given degree distribution of the sensing graph, if the sparsity ratio is below the thresholds derived in [29] and [88], for regular and irregular graphs, respectively, the proposed algorithm will succeed with probability one in detecting all the zero (and non-zero) blocks. On the other hand, if the sparsity ratio is above the threshold, then the probability of detecting *all* the zero (and non-zero) blocks is zero. In other words, in the asymptotic regime, if the sparsity ratio is below the success threshold, $P_{ZD} = 1$ and if it is beyond the success threshold, $P_{ZD} < 1$. Indeed, beyond the success threshold, the algorithm can still detect *some* zero (and non-zero) blocks reliably (just not all of them). This will be discussed in more details in Section 6.4. Note that in the noiseless case, $P_{WZD} = 0$.

Remark 6.3: In [88], irregular sensing graphs were designed that maximized the success threshold for a given under-sampling ratio and a given maximum variable node degree. Based on Remark 6.2, such sensing graphs will also be asymptotically optimal in the context of this work. The finite-length performance of such graphs will be discussed in Section 6.4.

6.2 NB-VB Algorithms for Recovery of Block Sparse Signals: Noisy Measurements

In practice, the measurements are noisy. In this case, $\mathbf{y} = \mathbf{Ax} + \mathbf{n}$, where $\mathbf{n}$ is an $M \times 1$ noise vector whose elements are assumed to be independent and identically

distributed (i.i.d.) zero-mean Gaussian random variables with variance σ_n^2 . At any stage of the verification process, the m -th measurement can be written as

$$\check{y}_m = 2\Re \left(\sum_{i \in \check{V}(m)} {}^+\mathbf{A}_m^{(i)} \cdot \mathbf{u}_i \right) + \check{n}_m = \check{y}_m^{nf} + \check{n}_m, \quad (6.5)$$

where $\check{y}_m^{nf}$ is the updated value of the m -th noise-free measurement and $\check{n}_m$ is the updated noise element. Equation (6.5) is valid throughout the verification process as variable nodes are verified iteratively and the graph is pruned accordingly.

Noise has two destructive effects on the application of NB-VB algorithms:

- 1) If the noise-free measurement is equal to zero, its noisy version is not equal to zero with probability one.
- 2) When two noise-free measurements are equal, in the noisy regime, they are not equal with probability one.

The first and the second effects will respectively disable the ZM and EM rules when the measurements are corrupted by noise. Without the ZM and EM rules, no VNs can be detected, and consequently, no MN will have a reduced degree in the updated graph. Therefore, D1M is also disabled, i.e., none of the verification rules, as presented in Section 6.1, are applicable when the measurements are corrupted by noise.

In [29], the problem of signal recovery in the presence of noise using NB-VB algorithms is briefly discussed. A thresholding technique is proposed in [29], where two thresholds, η_z and η_e , are used to identify zero and equal measurements, respectively, i.e., a measurement m is identified as zero if $|y_m| < \eta_z$, and two measurements a and b are considered equal if $|y_a - y_b| < \eta_e$. These thresholds however, are not optimized in [29]. In the simulation setting of [29], since the graph is unweighted and the non-zero elements are assumed to be even integers uniformly distributed in the range $[-1000, 1000]$, the absolute value of the non-zero noise-free measurements is greater than or equal to 2. There is also a difference of at least 2 between any two non-equal noise-free measurements. In this setting, both thresholds are set in [29] to be 1.99. Assigning this value to both thresholds is a reasonable choice in the high signal-to-noise ratio (SNR) regime considered in [29]. However, no general approach for finding these thresholds under more general settings is provided in [29]. Here, we propose

two tests to detect zero measurements and equal measurements. We also show that in the high SNR regime, the proposed tests are translated to a thresholding approach, and in this regime, we obtain the optimum thresholds analytically.

In general, the algorithm in the noisy regime is similar to that in the noiseless regime (Algorithm 6.1). The only difference is that two tests are required: one for detecting zero measurements ($\hat{y}_m^{nf} = 0, m \in \check{\mathcal{M}}$) and another one for detecting equal measurements ($\hat{y}_a^{nf} = \hat{y}_b^{nf}, a \neq b \in \check{\mathcal{M}}$). Note that we use the notation $\hat{y}_m^{nf}$ to show an estimate of the true value $\check{y}_m^{nf}$. In fact, in each iteration, we *estimate* some of the MNs as having zero or equal noise-free values. To perform the algorithm in the noisy scenario, we also define an $M \times 1$ vector, $\check{\xi}$, whose m -th element, $\check{\xi}_m$, denotes the mean squared error corresponding to the m -th measurement. The value of this error is updated every time that the value of the corresponding measurement node is updated (as a result of the application of D1M or EM rule). The updated value $\check{\xi}_m$ in each iteration is in fact equal to the sum of the variances of the independent noise components that exist in the updated value of the m -th MN. At the beginning of the proposed algorithm, all the elements of $\check{\xi}$ are initialized by σ_n^2 .

In the following, we propose the two tests for detecting zero and equal measurements.

6.2.1 ZM: Noisy Measurements

In this subsection, we propose a test to determine if the noise-free value of a MN is zero. The following analysis applies to any (updated) measurement node m with any arbitrary (updated) degree $\check{d}(m)$ in the (updated) sensing graph in any iteration of the proposed algorithm. The graph under consideration, thus, is in general irregular. For the analysis, we assume that the real and imaginary parts of non-zero elements of the sub-vectors $\mathbf{u}_i$ both come from a Gaussian distribution $\mathcal{N}(0, \sigma_s^2)$ in an i.i.d. fashion.⁷ Let us focus on MN m and suppose that $|\check{\mathcal{V}}(m)| = \check{d}(m)$ and that the noise power at this MN is $\check{\xi}_m$. With the same steps as (5.7)-(5.14), we have

⁷This assumption is made to make the analysis tractable. We however, demonstrate in Section 6.4 that the final results of the analysis would still be applicable to other signal distributions.

$$\begin{aligned}
\Lambda(\check{y}_m) &= \frac{\sum_{i=1}^{\check{d}(m)} \binom{\check{d}(m)}{i} \alpha^i (1-\alpha)^{\check{d}(m)-i} \mathcal{N}\left(0, 4i\sigma_s^2 + \check{\xi}_m\right)}{\sum_{i=1}^{\check{d}(m)} \binom{\check{d}(m)}{i} (\alpha)^i (1-\alpha)^{\check{d}(m)-i} \mathcal{N}\left(0, \check{\xi}_m\right)} \\
&= \frac{\sum_{i=1}^{\check{d}(m)} \binom{\check{d}(m)}{i} \alpha^i (1-\alpha)^{\check{d}(m)-i} \sqrt{\frac{\check{\xi}_m}{4i\sigma_s^2 + \check{\xi}_m}} \exp\left\{\frac{2i\sigma_s^2 \check{y}_m^2}{(4i\sigma_s^2 + \check{\xi}_m)\check{\xi}_m}\right\}}{1 - (1-\alpha)^{\check{d}(m)}}.
\end{aligned} \tag{6.6}$$

For minimizing the probability of error related to this decision, in the maximum a posteriori (MAP) test, c in (5.12) is selected as follows

$$c = \frac{P(\mathcal{H}_0)}{P(\bar{\mathcal{H}}_0)} = \frac{(1-\alpha)^{\check{d}(m)}}{1 - (1-\alpha)^{\check{d}(m)}}. \tag{6.7}$$

From (5.12), the decision is made in favor of zero MN (i.e., $\mathcal{H}_0$), if $\Lambda(\check{y}_m) < c$, where c is given by (6.7). Therefore, using (6.6), we decide on $\mathcal{H}_0$ if $\Lambda'(\check{y}_m, \check{\xi}_m, \check{d}(m), \alpha, \sigma_s^2) < 1$, where

$$\Lambda'(y, \xi, d, \alpha, \sigma_s^2) = \sum_{i=1}^d \binom{d}{i} \left(\frac{\alpha}{1-\alpha}\right)^i \sqrt{\frac{\xi}{4i\sigma_s^2 + \xi}} \exp\left\{\frac{2i\sigma_s^2 y^2}{(4i\sigma_s^2 + \xi)\xi}\right\}. \tag{6.8}$$

In the high SNR regime, where $\sigma_s^2 \gg \sigma_n^2$, it is implied that $\sigma_s^2 \gg \check{\xi}_m$.⁸ By neglecting $\check{\xi}_m$ against σ_s^2 , we can thus simplify the decision for $\mathcal{H}_0$ (i.e., $\Lambda'(\check{y}_m, \check{\xi}_m, \check{d}(m), \alpha, \sigma_s^2) < 1$) to

$$\sqrt{\frac{\check{\xi}_m}{\sigma_s^2}} \exp\left\{\frac{\check{y}_m^2}{2\check{\xi}_m}\right\} \sum_{i=1}^{\check{d}(m)} \frac{1}{2\sqrt{i}} \binom{\check{d}(m)}{i} \left(\frac{\alpha}{1-\alpha}\right)^i < 1, \tag{6.9}$$

or

$$|\check{y}_m| < \sqrt{\check{\xi}_m \ln \left(\frac{\sigma_s^2}{X^2(\check{d}(m), \alpha) \check{\xi}_m} \right)} \triangleq \eta_z, \tag{6.10}$$

⁸Although, the noise variances keep accumulating as the values of measurement nodes are updated, since the algorithm terminates in a rather small number of iterations, if $\sigma_s^2 \gg \sigma_n^2$, then $\sigma_s^2 \gg \check{\xi}_m$.

where

$$X(d, \alpha) := \sum_{i=1}^d \frac{1}{2\sqrt{i}} \binom{d}{i} \left(\frac{\alpha}{1-\alpha}\right)^i. \quad (6.11)$$

In other words, if $|\check{y}_m| < \eta_z$, we decide $\hat{y}_m^{nf} = 0$ and if $|\check{y}_m| \geq \eta_z$, we decide $\hat{y}_m^{nf} \neq 0$. Note that we require to know the sparsity ratio as well as signal variance to calculate the threshold.

6.2.2 EM: Noisy Measurements

In this subsection, we propose a test to determine if the noise-free values of two MNs are equal. We consider the same general assumptions made in Subsection VI.A. Suppose that there are two MNs, a and b , so that $\check{\mathcal{V}}(a, b) = \check{\mathcal{V}}(a) \cap \check{\mathcal{V}}(b) \neq \emptyset$. The values $\check{y}_a$ and $\check{y}_b$, corresponding to the MNs a and b , respectively, are given by

$$\begin{aligned} \check{y}_a &= 2\Re \left(\sum_{i \in \check{\mathcal{V}}(a)} {}^+ \mathbf{A}_a^{(i)} \cdot \mathbf{u}_i \right) + \check{n}_a, \\ \check{y}_b &= 2\Re \left(\sum_{i \in \check{\mathcal{V}}(b)} {}^+ \mathbf{A}_b^{(i)} \cdot \mathbf{u}_i \right) + \check{n}_b, \end{aligned} \quad (6.12)$$

where $\check{n}_a$ and $\check{n}_b$ are independent noise components with variances $\check{\xi}_a$ and $\check{\xi}_b$, respectively. Since $\mathbf{A}_a^{(i)} = \mathbf{A}_b^{(i)}$, $\forall a, b \in \mathcal{M}$, we can rewrite $\check{y}_a$ and $\check{y}_b$ as

$$\begin{aligned} \check{y}_a &= 2\Re \left(\sum_{i \in \check{\mathcal{V}}(a, b)} {}^+ \mathbf{A}_a^{(i)} \cdot \mathbf{u}_i \right) + 2\Re \left(\sum_{i \in \check{\mathcal{V}}(a) \cap \check{\mathcal{V}}^c(a, b)} {}^+ \mathbf{A}_a^{(i)} \cdot \mathbf{u}_i \right) + \check{n}_a, \\ \check{y}_b &= 2\Re \left(\sum_{i \in \check{\mathcal{V}}(a, b)} {}^+ \mathbf{A}_a^{(i)} \cdot \mathbf{u}_i \right) + 2\Re \left(\sum_{i \in \check{\mathcal{V}}(b) \cap \check{\mathcal{V}}^c(a, b)} {}^+ \mathbf{A}_a^{(i)} \cdot \mathbf{u}_i \right) + \check{n}_b. \end{aligned} \quad (6.13)$$

In EM rule, we need to determine whether the noise free measurements $\check{y}_a^{nf}$ and

$\check{y}_b^{nf}$ are equal or not. Let $\check{y}_{a-b} = \check{y}_a - \check{y}_b$ and $\check{n}_{a-b} = \check{n}_a - \check{n}_b$. We can write

$$\check{y}_{a-b} = 2\Re \left(\sum_{j \in \check{\mathcal{V}}(a) \cap \check{\mathcal{V}}^c(a,b)} + \mathbf{A}_a^{(j)} \cdot \mathbf{u}_j \right) - 2\Re \left(\sum_{j \in \check{\mathcal{V}}(b) \cap \check{\mathcal{V}}^c(a,b)} + \mathbf{A}_a^{(j)} \cdot \mathbf{u}_j \right) + \check{n}_{a-b}. \quad (6.14)$$

Let $\check{\xi}_{a-b}$ denote the noise variance corresponding to $\check{y}_{a-b}$. Clearly, $\check{\xi}_{a-b} = \check{\xi}_a + \check{\xi}_b$. Suppose that the MNs a and b have $\Delta = |\check{\mathcal{V}}(a,b)|$ VNs in common, and thus $\check{d}_{a-b} = \check{d}(a) + \check{d}(b) - 2\Delta$ uncommon variables. We define $\check{d}_{a-b} + 1$ hypothesis, $\mathcal{H}_i$ for $i = 0, \dots, \check{d}_{a-b}$, where $\mathcal{H}_i$ represents the scenario that i VNs out of $\check{d}(a) + \check{d}(b) - 2\Delta$ uncommon neighboring VNs of the measurements a and b are non-zero VNs. If $\check{y}_{a-b}^{nf} = \check{y}_a^{nf} - \check{y}_b^{nf} = 0$, $\mathcal{H}_0$ is true with probability one.

It is easy to see that the problem of detecting two equal MNs, $\check{y}_a$ and $\check{y}_b$, can be considered as detecting a zero MN, $\check{y}_{a-b}$. Therefore, if $\Lambda'(\check{y}_{a-b}, \check{\xi}_{a-b}, \check{d}_{a-b}, \alpha, \sigma_s^2) < 1$, then we decide on $\mathcal{H}_0$ (i.e., $\hat{y}_a^{nf} = \hat{y}_b^{nf}$), where $\Lambda'(y, \xi, d, \alpha, \sigma_s^2)$ is given in (6.8). In the high SNR regime, the MAP test for equal MNs is derived similar to (6.10) and can be written as

$$|\check{y}_{a-b}| < \sqrt{\check{\xi}_{a-b} \ln\left(\frac{\sigma_s^2}{X^2(\check{d}_{a-b}, \alpha) \check{\xi}_{a-b}}\right)} \triangleq \eta_e. \quad (6.15)$$

In other words, if $|\check{y}_{a-b}| = |\check{y}_a - \check{y}_b| < \eta_e$, we decide on $\hat{y}_a^{nf} = \hat{y}_b^{nf}$, and if $|\check{y}_{a-b}| = |\check{y}_a - \check{y}_b| \geq \eta_e$, we decide on $\hat{y}_a^{nf} \neq \hat{y}_b^{nf}$.

6.2.3 Zero Block Detection Algorithm in Noisy Scenarios

The general framework of the proposed algorithm in the presence of noise is given in the following. Note that σ_s^2 , σ_n^2 , and α are assumed to be known a priori.

Algorithm 6.2.3: NB-VB zero-block detection in noisy scenarios

Inputs: $\mathbf{A}$ ($\mathcal{E}_{\mathcal{V}, \mathcal{M}}$), $\mathbf{y}$, σ_s^2 , σ_n^2 , α

Output: Verified variable nodes as zero and non-zero

Initializing step:

$$j = 0.$$

$$\check{\xi}_m = \xi_m = \sigma_n^2, \forall m \in \mathcal{M}.$$

$\check{\mathcal{M}} = \mathcal{M}$ and $\check{\mathcal{V}} = \mathcal{V}$.

$\check{\mathbf{y}} = \mathbf{y}$.

$\mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)} = \mathcal{E}_{\mathcal{V}, \mathcal{M}}$.

do:

$j \leftarrow j + 1$.

Update $\mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)} = \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j-1)}$.

ZM:

for all $\{m \in \check{\mathcal{M}}\}$ **do:**

if $\Lambda'(\check{y}_m, \check{\xi}_m, \check{d}(m), \alpha, \sigma_s^2) < 1$ **then:**

Verify all $\{v \in \check{\mathcal{V}}(m)\}$ as zero VNs.

Update the sensing graph by removing all the edges of the set $\mathcal{E}_{\check{\mathcal{V}}(m), \check{\mathcal{M}}} \subseteq \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)}$.

Update the sets $\check{\mathcal{M}}$ and $\check{\mathcal{V}}$ by removing the MNs and VNs with degree zero, respectively.

end if

end for

D1M:

for all $\{a \in \check{\mathcal{M}} : \check{d}(a) = |\check{\mathcal{V}}(a)| = 1\}$ **do:**

if $\Lambda'(\check{y}_a, \check{\xi}_a, \check{d}(a), \alpha, \sigma_s^2) > 1$ **then:**

Verify $\{v \in \check{\mathcal{V}}(a)\}$ as a non-zero VN.

Update $\check{y}_m \leftarrow \check{y}_m - \check{y}_a, \forall m \in \check{\mathcal{M}}(v)$.

Update $\check{\xi}_m \leftarrow \check{\xi}_m + \check{\xi}_a, \forall m \in \check{\mathcal{M}}(v)$.

Update the sensing graph by removing all the edges of the set $\mathcal{E}_{\check{\mathcal{V}}(a), \check{\mathcal{M}}} \subseteq \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)}$.

Update the sets $\check{\mathcal{M}}$ and $\check{\mathcal{V}}$ by removing the MNs and VNs with degree zero, respectively.

end if

end for

EM:⁹

⁹In simulation results presented in Section 6.4, we sort the VNs based on their degree in descending order and then check them for having equal neighboring MNs. This results in a better performance compared to the case where the VNs are sorted in ascending order of their degrees or in random.

```

for all  $\{v \in \check{\mathcal{V}}\}$  do:
    for all  $\{a, b \in \check{\mathcal{M}}(v), a \neq b\}$  do:
         $\check{\xi}_{a-b} = \check{\xi}_a + \check{\xi}_b$ .
         $\check{d}_{a-b} = \check{d}(a) + \check{d}(b) - 2|\check{\mathcal{V}}(a, b)|$ .
         $\check{y}_{a-b} = \check{y}(a) - \check{y}(b)$ .
        if  $\Lambda'(\check{y}_{a-b}, \check{\xi}_{a-b}, \check{d}_{a-b}, \alpha, \sigma_s^2) < 1$  then:
            Set  $\bar{\mathcal{V}}(a, b) = (\check{\mathcal{V}}(a) \cup \check{\mathcal{V}}(b)) \cap \check{\mathcal{V}}^c(a, b)$ .
            Verify all  $\{u \in \bar{\mathcal{V}}(a, b)\}$  as zero VNs.
            Update the sensing graph by removing all the edges of the set  $\mathcal{E}_{u, \check{\mathcal{M}}} \subseteq \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)}, \forall u \in \bar{\mathcal{V}}(a, b)$ .
            if  $|\check{\mathcal{V}}(a, b)| = 1$  then:
                Verify  $\{v \in \check{\mathcal{V}}(a, b)\}$  as a non-zero VN.
                Update  $\check{y}_m \leftarrow \check{y}_m - \frac{\check{y}_a + \check{y}_b}{2}, \forall m \in \check{\mathcal{M}}(v)$ .
                Update  $\check{\xi}_m \leftarrow \check{\xi}_m + \frac{1}{4}\check{\xi}_a + \frac{1}{4}\check{\xi}_b, \forall m \in \check{\mathcal{M}}(v)$ .
                Update the sensing graph by removing all the edges of the set  $\mathcal{E}_{v, \check{\mathcal{M}}} \subseteq \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)}$ .
            end if
            Update the sets  $\check{\mathcal{M}}$  and  $\check{\mathcal{V}}$  by removing the MNs and VNs with degree zero, respectively.
        end if
    end for
end for
while  $\mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j)} \neq \mathcal{E}_{\check{\mathcal{V}}, \check{\mathcal{M}}}^{(j-1)}$ 

```

In EM, for removing the effect of the verified non-zero VN from its neighboring MNs, since $\check{y}_a$ and $\check{y}_b$ are different and noisy, the average value of them is subtracted from $\check{y}_m$, i.e., $\check{y}_m \leftarrow \check{y}_m - 0.5\check{y}_a - 0.5\check{y}_b$. The error of the m -th measurement is also updated based on the error of both measurements a and b , i.e., $\check{\xi}_m \leftarrow \check{\xi}_m + \frac{1}{4}\check{\xi}_a + \frac{1}{4}\check{\xi}_b$. This is based on the assumption that the measurement noises are independent. Similarly, in D1M, we have $\check{y}_m \leftarrow \check{y}_m - \check{y}_a$, and thus $\check{\xi}_m \leftarrow \check{\xi}_m + \check{\xi}_a$.

6.3 Comparison with the Standard NB-VB Algorithm and Existing Block Sparse Recovery Algorithms

To compare the proposed block NB-VB algorithm with the original NB-VB algorithm (SBB) of [29], we first note that for both algorithms to have the same performance, the latter requires $2B$ times as many measurements, where B is the block size. In addition, we recall that the comparison between the two algorithms can only be made in terms of success probability, where “success” is defined as the full recovery of the signal for the latter algorithm, and the full recovery of the block-based support for the former algorithm. The proposed algorithm however, can still operate under conditions where the under-sampling ratio is much lower than what is required for the full-recovery of the block-based support. This is not the case for the original NB-VB algorithm. In terms of the complexity, the proposed algorithm has complexity $\mathcal{O}(L)$. The complexity of the original NB-VB algorithm, on the other hand, is $\mathcal{O}(N)$. This can be much larger than $\mathcal{O}(L)$ if the block size B is an increasing function of N .

One of the most powerful algorithms for the recovery of block sparse signals, in terms of performance/complexity trade-off, is AMP with James-Stein’s estimator (AMP-JS) [48]. This algorithm is shown to achieve the theoretical limit of $M = K$ as the block size tends to infinity, where M and K are the number of measurements and non-zero signal elements, respectively. Even in this limit, the proposed algorithm can outperform AMP-JS (in this scenario, the block size is an increasing function of N). To see this, we note that for a given degree distribution of the sensing graph, the block under-sampling for the proposed algorithm is fixed. This implies that the variable under-sampling ratio decreases inversely proportional to the block size B . Considering that in the limit of N tending to infinity, K increases proportional to N for the proposed algorithm, we note that the increase in M is sub-linear in K . (In fact, if L is assumed to be fixed as N tends to infinity, then M will be a constant in K .) Note that for AMP-JS, $M = K$, in the best case scenario. In terms of complexity, the complexity of AMP-JS is $\mathcal{O}(MN)$, which is in general much higher than the proposed algorithm.

6.4 Simulation Results

In this section, we present some simulation results to demonstrate the effectiveness of the proposed method for detecting the spectrum holes in both noiseless and noisy cases. We consider an N dimensional signal with the positive frequency range of $[0, \frac{N}{2}]$, containing L non-overlapping channels, each of block size $B = \frac{N}{2L}$. We use an N point unitary Fourier matrix to map the input signal from the time to the frequency domain. To model the block sparse spectrum, each block is independently selected to be non-zero with probability α (from positive frequency elements) and assign the real and imaginary parts of the elements of non-zero blocks from a Gaussian distribution $\mathcal{N}(0, \sigma_s^2)$ in an i.i.d. fashion. The $M \times N$ block sparse sensing matrix is constructed randomly such that all the non-zero blocks are identical. The real and imaginary parts of the elements of non-zero blocks of $\mathbf{A}$ are selected from a zero mean Gaussian distribution in an i.i.d. fashion and normalized such that $\|{}^+\mathbf{A}_a^{(v)}\|_2 = 1, \forall a \in \mathcal{M}, \forall v \in \mathcal{V}$.

Each experiment is repeated 500 times and each trial for an experiment consists of (a) randomly creating a signal $\mathbf{x}$, where each block has B elements, (b) constructing a random $M \times N$ block sparse sensing matrix,¹⁰ (c) creating M measurements accordingly (and adding noise to each measurement in noisy scenarios), and (d) applying the NB-VB algorithm to detect zero and non-zero blocks of the spectrum.

It is important to note that the performance of the proposed algorithm is independent of the block length B . To the best of our knowledge, however, the performance of all the existing algorithms for the recovery of block sparse signals does depend on the block size B . This results in a particularly large performance gap between the proposed algorithm and other algorithms in scenarios where B is large.

6.4.1 Noiseless Scenarios

We consider two sets of sensing graphs, each consisting of an optimized irregular and a regular graph with the same under-sampling ratio. For the first set, the regular graph has $d_V = 4$ and $d_M = 8$, and the irregular graph has the degree distribution $(\lambda(x), \rho(x)) = (0.896x^3 + 0.04x^{12} + 0.064x^{13}, x^8)$ [88]. For the second set, we have $d_V = 4$ and $d_M = 5$, for the regular graph, and $(\lambda(x), \rho(x)) = (0.9x^3 + 0.1x^{13}, 0.9375x^4 +$

¹⁰The results are not sensitive to the random choice of the sensing matrix. One can thus generate one sensing matrix randomly and use it for all the trials.

Figure 6.1: The probability of success of the NB-VB zero-detection algorithm versus sparsity ratio. Solid and dashed lines correspond to $L = 10,000$ and $L = 1000$, respectively.

$0.0625x^{20}$) for the irregular one [88]. Fig. 6.1 shows the probability of success versus the sparsity ratio for the two sets of graphs. Here, “success” is defined as the full recovery of all the zero and non-zero blocks. In Fig. 6.1, we consider $B = 8$, and for each graph, we consider two cases of $L = 1000$ and $L = 10,000$, corresponding to signal dimensions $N = 16,000$ and $160,000$, respectively. For the two sets of graphs, the variable and block under-sampling ratios are $(1/32, 1/2)$ and $(1/20, 4/5)$, respectively. In Fig. 6.1, we have also provided the threshold values for the four ensembles of the graphs. These thresholds are equal to 0.24, 0.28, 0.42 and 0.58, for regular and irregular graphs of the two sets, respectively. The results of Fig. 6.1 show that as the signal dimension increases, the transition from success rate of one to zero becomes sharper. It also demonstrates that the waterfall region of the curves in all four cases matches the success threshold, and that irregular graphs perform superior to their regular counterparts. In Fig. 6.2, we have plotted the histograms of the number of iterations of the proposed algorithm for the $(4,8)$ -regular graph at sparsity ratio 0.23 for $L = 1000$ and $10,000$. As seen in these figures, the proposed algorithm requires only a small number of iterations (< 20) to successfully verify all zero and non-zero blocks. It is important to note that if one were to recover the signal using the original NB-VB algorithm (rather than the block-based support of it using the proposed adaptation of NB-VB algorithms), one would require $2B$ times as many

Figure 6.2: Histogram of the number of iterations for a (4,8)-regular graph with (a) $L = 1000$ and (b) $L = 10,000$. ($\alpha = 0.23$)

measurements.

In Fig. 6.3, we have presented P_{ZD} for four random graphs with degree distributions as discussed before but with $L = 500$. We note that in noiseless scenarios, $P_{WZD} = 0$. The results of Fig. 6.3 show that P_{ZD} starts to decrease, rather gracefully, from one, at around the threshold, to smaller values for sparsity ratios beyond the threshold. So, even at sparsity ratios above the threshold, the proposed algorithm is still capable of detecting a rather large percentage of the zero blocks reliably. We also note that different graphs follow the same order as they did in Fig. 6.1 in terms of performance, i.e., the optimized irregular graphs perform consistently superior to their regular counterparts.

In Fig. 6.3, we have also plotted P_{ZD} of AMP-JS for variable under-sampling ratios $M/N = 1/2$ and $4/5$ and a spectrum with $L = 500$ and $B = 8$. (The variable under-sampling ratios for AMP-JS are selected to be equal to the block under-sampling ratios of the proposed scheme.) To apply AMP-JS in the context of WSS, where the signal is complex, we use the general approach of [89]. For AMP-JS, the sensing matrix is selected to be a dense matrix randomly constructed with i.i.d. elements, each with distribution $\mathcal{N}(0, 1/M)$. After the detection of the signal by AMP-JS, an energy detector is used for the detection of zero blocks. In fact, if the ℓ_2 -norm of the i -th block is greater than $\sqrt{\frac{B}{2}}\sigma_s$, the block is considered as non-zero, otherwise, zero. Since in the noiseless scenario, unlike the case of NB-VB algorithms, the P_{WZD} of AMP-JS is not necessarily zero, P_{ZD} of AMP-JS is plotted only for the sparsity ratios in which $P_{WZD} < 0.1\%$. The comparison with the proposed scheme shows the

Figure 6.3: P_{ZD} of the proposed scheme versus sparsity ratio for regular and irregular sensing matrices in comparison with AMP-JS ($L = 500$ and $B = 8$).

superior performance of AMP-JS. This however comes at the cost of $2B = 16$ times as many measurements compared to the proposed scheme, in addition to a much more complex recovery algorithm. With the number of measurements equal to that of the proposed algorithm, i.e., $M = 250$ and 400 for the block under-sampling ratios 0.5 and 0.8 , respectively, AMP-JS fails entirely in recovering the signal. If the best algorithm from the information theoretic point of view existed, it would be able to recover a signal with maximum 15 and 25 complex blocks of length 8 for the block under-sampling ratios 0.5 and 0.8 , respectively. This corresponds to sparsity ratios less than 0.03 and 0.05 , respectively.

6.4.2 Noisy Scenarios

Unlike the noiseless scenarios, in the noisy scenarios, P_{WZD} is not necessarily zero. In our simulations, we only consider cases where P_{WZD} is below 5% .

We first start by considering regular graphs in which the VNs have degree one, i.e., $d(v) = 1, \forall v \in \mathcal{V}$. This is particularly interesting as it represents the simplest sensing graphs, which require the least complexity for the detection of zero blocks. The minimum complexity is a consequence of the fact that in such graphs no message-passing is required to detect the zero blocks, i.e., all the zero blocks that can be detected using the proposed scheme will be detected in the first iteration using the ZM rule (it is easy to see that D1M and EM rules are not effective in such graphs).

Figure 6.4: P_{ZD} and P_{WZD} of the proposed algorithm versus sparsity ratio for regular sensing matrices with $d_V = 1$ ($L = 500$). Solid and dashed lines correspond to SNR = 40 dB and SNR = 25 dB, respectively.

In Fig. 6.4, we have presented the performance of such graphs in terms of P_{ZD} and P_{WZD} for values of $d_M = 2, 3, 4$ and 5. We have selected $B = 8$, $L = 500$, and considered two SNR values of 25 dB and 40 dB, where SNR is calculated by $10 \log (E(\|\mathbf{Ax}\|_2^2)/M\sigma_n^2)$. As expected, by increasing d_M , which corresponds to decreasing the under-sampling ratio, the performance degrades. By decreasing the SNR value, although P_{ZD} remains unchanged, P_{WZD} is increased. Nevertheless, for any choice of d_M and in any SNR value, the proposed scheme is capable of detecting a rather large portion of zero blocks with a rather small P_{WZD} , particularly when the sparsity ratio is low. For example, in SNR = 40 dB, when the sparsity ratio is 0.4, we can find 60% of the zero-blocks with P_{WZD} below 2%, if we use a (1,2)-regular sensing graph.

In Fig. 6.5, we compare the performance of the algorithm with the optimized irregular and corresponding regular graphs in noisy scenarios. The noiseless case is also given for reference. For this experiment, we select $L = 500$, $B = 8$ and $N = 8000$. Although, in noiseless scenarios, irregular graphs outperform regular ones, with increasing the noise level, the performance of the algorithm degrades to a larger extent for irregular graphs compared to the regular ones. This would be expected as the irregular graphs are designed to maximize the threshold in the absence of noise. In Fig. 6.5(a), we have also included the result of using the proposed algorithm on a (4,8)-regular graph at SNR = 25 dB, for a spectrum with non-Gaussian distribution. For

Figure 6.5: P_{ZD} and P_{WZD} of the proposed algorithm versus sparsity ratio for regular and irregular graphs with block under-sampling ratios of 0.5 (a) and 0.8 (b) in the presence of noise ($L = 500$, $\text{SNR} = 25$ dB and 40 dB).

the new distribution, we assign the same amplitude to all the spectrum components of each non-zero block. This amplitude, for each block, is selected randomly using a uniform distribution in the range of $[0.1, 1.1]$. We also assign a linear phase to each non-zero block, where the angle of the linear phase for each block is selected uniformly at random and in an i.i.d. fashion from $[-\pi/2, 0]$. (The whole spectrum in each block has the same amplitude and a linear phase.) We still use Equations (6.10) and (6.15) for ZM and EM rules but replace σ_s with $1/\sqrt{24}$. As can be seen from Fig. 6.5(a), with the non-Gaussian distribution, the algorithm performs almost the same as the case where the signal distribution is Gaussian.

In Figs. 6.6(a) and (b), we see the performance of the algorithm on different regular graphs with the same block under-sampling ratio of 0.5 in SNR values of 40 dB and 25 dB, respectively. In both figures, we have also presented the results for AMP-JS with the number of measurements equal to $2B = 16$ times larger than that of the proposed algorithm. The figures show that, in general, the $(2, 4)$ -regular graph has the best performance for smaller sparsity ratio values. This performance however, is surpassed by the $(1, 2)$ -regular graph at larger sparsity ratios. The comparison with AMP-JS is also similar to the noiseless case, i.e., with the same number of measurements, the proposed algorithm outperforms AMP-JS in terms of both performance

Figure 6.6: P_{ZD} and P_{WZD} of the proposed algorithm and AMP-JS versus sparsity ratio at SNR values 40 dB (a) and 25 dB (b) ($L = 500$, $B = 8$ and $M/L = 0.5$. For the proposed algorithm, different regular sensing matrices are used).

and complexity, while if one allows a significantly larger number of measurements for AMP-JS, its performance eventually catches up, and with $2B$ times more measurements surpasses that of the proposed algorithm as seen in Fig. 6.6.

6.5 Conclusion

We proposed a low-complexity compressed sensing (CS) scheme for the detection of zero blocks (unoccupied sub-channels) in the context of wideband spectrum sensing (WSS). The proposed scheme is based on the adaptation of verification based recovery algorithms, previously applied to the recovery of real sparse signals, for the case where signals are complex and block sparse. Such an adaptation required non-trivial changes to the verification rules and the treatment of the noise. To implement the proposed scheme, we require the AIC designed in Chapter 5, so that the resulting sensing matrix has a block sparse structure matching that of the signal. It was demonstrated that the AIC, and its corresponding sensing matrix, can be optimized to detect the largest fraction of zero blocks for a given number of measurements in the noiseless scenario. In noisy environments, certain thresholds in the verification algorithm are needed. We derived the optimal values of these thresholds analytically. We also presented

simulation results that demonstrated the superior performance/complexity of the proposed scheme in comparison with the existing block sparse recovery algorithms in both noiseless and noisy scenarios. In particular, taking the approximate message-passing (AMP) algorithm as an example, it was shown that while the existing block sparse recovery algorithms can perform well when the number of measurements is sufficiently large, they fail in detecting zero blocks reliably for smaller number of measurements. This is while the proposed algorithm (with a very low complexity) can always detect some zero blocks reliably even when the number of measurements is very small. The performance gap would be particularly large for larger block sizes.

Chapter 7

Future Work

7.1 Reweighting Strategies

In this thesis, we proposed a family of iterative reweighted ℓ_2/ℓ_1 minimization algorithms for the recovery of block sparse signals. Clearly, the weighting strategy plays an important role in the performance of the algorithm. In Chapter 3, three special cases of these algorithms, i.e., $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$, $\text{IR}^{(2)}\text{-}\ell_2/\ell_1$, and $\text{IR}^{(3)}\text{-}\ell_2/\ell_1$, were studied. Through the simulation results, we showed that for each signal model, one weighting strategy can outperform the others. We hope that the results presented in this chapter, particularly those presented at the end of Part 3.4.1.1, will motivate more research on other reweighting strategies within the class of $\text{IR-}\ell_2/\ell_1$ algorithms.

In Chapter 4, the weights used in the AMP framework were the same as those of $\text{IR}^{(1)}\text{-}\ell_2/\ell_1$. Similar to the case of $\text{IR-}\ell_2/\ell_1$ algorithms, different weighting strategies can be used to have a better recovery performance depending on the signal model. We did not study the other reweighting strategies for AMP algorithms. This question is left for future studies.

7.2 Analysis of the $\text{IR-}\ell_2/\ell_1$ Algorithm in Noisy Scenarios

In Chapter 3, we proved that under some conditions on the measurement matrix, in noiseless cases, the solution of the weighted ℓ_2/ℓ_1 minimization is unique and coincides with the true signal. In [13], the reweighted ℓ_1 minimization method was analyzed in the noisy case when the measurement matrix satisfied the restricted

isometry property. One may be able to extend the analysis to the case of block sparse signals when the measurement matrix has the null space property. As a result, one may be able to analyze the weighted ℓ_2/ℓ_1 minimization and obtain the reconstruction error bound when the measurements are corrupted with noise and show that the reconstruction error is decreasing as the iterations progress. Such an analysis is not trivial and remains open for future studies.

7.3 Optimization of The Sensing Graph for ZMD Scheme

We obtained the closed form of P_D and P_{WZD} in noisy cases, in Chapter 5. In this chapter, we presented the simulation results and demonstrated that the performance of the zero block detection scheme highly depends on the sensing graph structure. Future work includes the optimization of irregular degree distributions for sensing matrix (graph) to maximize P_D for a given P_{WZD} .

In Chapter 6, P_{ZD} and P_{WZD} of the proposed algorithm were demonstrated through the simulation results for different regular and irregular sensing graphs. One can analytically get P_{ZD} in noiseless scenario. (Note that $P_{WZD} = 0$ in noiseless case.) In addition, it would also be interesting to analyze the NB-VB algorithm in noisy case. The analysis would help us to determine the optimum irregular graphs to maximize P_{ZD} and minimize P_{WZD} .

7.4 Distributed Cognitive Radio Networks

To sense the spectrum in a cognitive radio network (CRN), collaborative wideband spectrum sensing can be utilized to result in more robustness [90]. However, throughout this thesis, we assumed one independent CR processed the received signal to reconstruct the spectrum or to find the zero subchannels. One can apply all the methods proposed in this work to the case of collaborative CRNs. In Chapters 5 and 6, each CR has M measurements, hence M BMIs. In mobile CRs with limited power and computational resources, it might be impractical to have M BMIs. One can apply the proposed NB-VB algorithm to the case of distributed CRNs, while each CR has only $M' \ll M$ BMIs. Suppose there are R CRs in a CRN and a fusion center to process the collected data from the CRs. Each CR needs to have $M' = \lceil M/R \rceil$ BMIs

to have M' measurements. They all send their measurements and the information about the sensing graph to the fusion center. Therefore, the fusion center receives information about M measurements and the sensing graph. In the fusion center the collected data is then processed using the proposed NB-VB algorithm. It is important to investigate the restrictions of this application. One of the restrictions is related to the fact that each VN should have the same contribution (linear combination) in the measurements of the CRs. In other words, all CRs should have an AIC with the same input signal, $p_c(t)$. For this, the same seed can be used to initialize the pseudo-random signal generators of the CRs. On the other hand, one can investigate the throughput of the CNR under the fading conditions. One of the possible questions is that what the minimum value of M' should be under different fading conditions for a given P_{ZD} and P_{WZD} .

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Appendix A

Proofs of Chapter 3

Lemma 3.1: The true signal $\bar{\mathbf{x}} = [\bar{\mathbf{u}}_1^T, \dots, \bar{\mathbf{u}}_L^T]^T$ uniquely solves (3.2) for a given index set Γ if and only if

$$\sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i\|_2 < \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i + \boldsymbol{\eta}_i\|_2, \quad \forall \boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}), \quad \boldsymbol{\eta} \neq \mathbf{0}. \quad (\text{A.1})$$

Proof: It is clear that if $\bar{\mathbf{x}}$ uniquely solves (3.2), the inequality (A.1) holds true. Now, we prove the converse by contradiction. Assume (3.7) above holds true and also assume that $\tilde{\mathbf{x}} \neq \bar{\mathbf{x}}$ is a solution of (3.2) for $\varepsilon = 0$. Therefore, $\tilde{\mathbf{x}} = \bar{\mathbf{x}} + \tilde{\boldsymbol{\eta}}$, where $\tilde{\boldsymbol{\eta}} \in \mathcal{N}(\mathbf{A})$ and $\tilde{\boldsymbol{\eta}} \neq \mathbf{0}$. We thus have

$$\sum_{i \in \Gamma} \omega_i \|\tilde{\mathbf{u}}_i\|_2 = \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i + \tilde{\boldsymbol{\eta}}_i\|_2 > \sum_{i \in \Gamma} \omega_i \|\bar{\mathbf{u}}_i\|_2, \quad (\text{A.2})$$

where the last inequality comes from (A.1). This is in contradiction with $\tilde{\mathbf{x}}$ being a solution. Therefore, $\bar{\mathbf{x}}$ is the unique solution of (3.2), when (A.1) holds true. ■

Proposition 3.1: Assume $\Gamma \subseteq \{1, \dots, L\}$ is a given set and $\mathbf{A}$ is an $M \times N$ matrix. If for all index sets $S \subset \Gamma$ and for all $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}) \setminus \mathbf{0}$, we have

$$\sum_{i \in S} \|\boldsymbol{\eta}_i\|_2 < \bar{\gamma} \sum_{i \in \Gamma \cap S^c} \|\boldsymbol{\eta}_i\|_2, \quad (\text{A.3})$$

for some $0 < \bar{\gamma} < 1$, where $\boldsymbol{\eta}_i$ is the i -th block of $\boldsymbol{\eta}$, then (A.3) implies the linear independence of all the columns of $\mathbf{A}$ in blocks whose indices are in Γ^c .

Proof: We use contradiction. Suppose that the columns of $\mathbf{A}$ in Γ^c are not independent. This means there exists a non-zero vector $\mathbf{v}$ such that $\mathbf{A}_{\Gamma^c}\mathbf{v} = \mathbf{0}$, where $\mathbf{A}_{\Gamma^c}$ is the sub-matrix of $\mathbf{A}$ consisting of the columns of the blocks in Γ^c . Now consider the vector $\boldsymbol{\eta}$ which is zero in segments whose indices are in Γ and is equal to $\mathbf{v}$ in segments whose indices are in Γ^c . Clearly $\mathbf{A}\boldsymbol{\eta} = \mathbf{0}$ and thus $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A}) \setminus \mathbf{0}$. The vector $\boldsymbol{\eta}$ however does not satisfy (A.3), because for $\boldsymbol{\eta}$, both sides of inequality are equal to zero. ■