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Earthquake Analysis of Bridge Systems by Ritz Vectors

by

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M.Sc. (Civil Engineering)

Technical University of Cracow, Poland, 1989

A thesis submitted to

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Earthquake Analysis of Bridge Systems by Ritz Vectors

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in partial fulfillment of the requirements for
the degree of Master of Engineering

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May 1997

*To my parents,
my wife,
and my son*

Abstract

The evaluation of the dynamic response of a large structural system subjected to dynamic loads is very computationally expensive and time consuming. The transformation techniques based on Rayleigh-Ritz approximation can reduce the size of the problem and thus the amount of calculations required for the solution.

This thesis presents a number of Rayleigh-Ritz algorithms that can be used in dynamic analysis of bridge systems. Recent advances in frequency and time history response analysis of structures by the Lanczos method and other Ritz analysis procedures are discussed. Various techniques to generate the special set of load-dependent Ritz vectors for the dynamic modelling and analysis of structural systems are described. A wide range of different Ritz algorithms and analysis procedures have been implemented in a comprehensive computer program for bridge structures. The effectiveness of the various Ritz analysis procedures are investigated by utilizing the procedures to analyze typical bridge systems. The vibrational mode shapes and natural frequencies, and the dynamic response of the bridge systems subjected to earthquake ground motions are calculated and discussed.

The accuracy and efficiency of the Ritz algorithms are greatly influenced by the selection of the Ritz vectors used for the discretization and modelling of the structural behaviour. In the present research work, a new cut-off criterion based on the concept of effective mass participation is developed to generate the truncated set of Ritz vectors for multi-directional seismic response analysis. The effectiveness of the new cut-off criterion is verified by examples.

Numerical results obtained in this study show that the Lanczos method is more effective than the subspace iteration procedure in the frequency analysis of bridge systems. The subspace iteration method, despite the fact that it has been traditionally accepted as an efficient tool for dynamic analysis of complex structural systems by computer, is found to be less efficient in terms of time and memory usage than the Lanczos method. In a comparison study of the Lanczos and Wilson's methods for obtaining approximate eigensolutions, it is found that both methods are efficient and are comparable with respect to requirement in computer time. Both methods give good approximate results to the vibration modes with the lowest vibrational frequencies. Convergence of estimates for higher modes tends not to be in sequential order and spread over a wide frequency range. In another form of implementation of the Lanczos and Wilson's methods considered in this study, the load-dependent Ritz vector method is found to have an even faster convergence rate and is numerically more efficient for dynamic response time history analysis of bridges in comparison to using mode shapes by traditional modal superposition method. For the determination of the number of Ritz vectors needed to be included in the analysis for accurate results, a revised cut-off criterion based on the concept of effective mass participation in dynamic response for the multi-directional seismic response analysis has been developed in this thesis. Numerical results show that the new cut-off criterion is efficient and accurate.

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Table of Contents

Abstract	iv
Acknowledgments	vi
Table of Contents	vii
List of Tables	xii
List of Figures	xiv
1 Introduction	1
1.1 General	1
1.2 Problem definition	2
1.3 Objectives and scope	4
2 Free Vibration Analysis	7

2.1	General	7
2.2	General eigenvalue problem	8
2.2.1	Eigenproblem formulation	8
2.2.2	Properties of eigensolution	9
2.3	The Rayleigh-Ritz approximation procedure	11
2.3.1	General	11
2.3.2	Subspace iteration method	13
2.3.3	Lanczos method	15
3	Lanczos Method	18
3.1	General	18
3.2	Historical development of Lanczos algorithm	19
3.3	Derivation of the simple Lanczos algorithm	22
3.4	Loss of orthogonality and convergence	27
3.4.1	Mechanisms of loss of orthogonality	27
3.4.2	Accuracy and residual norm	28
3.4.3	Paige theorem	29

3.5	Misconvergence	30
3.6	Multiple eigenvalues	31
3.7	Reorthogonalization procedures	32
3.7.1	General	32
3.7.2	Full reorthogonalization	33
3.7.3	Selective orthogonalization	33
3.7.4	Partial reorthogonalization	35
3.7.5	Combined selective and partial reorthogonalization	38
4	Dynamic Response Analysis	47
4.1	General	47
4.2	Modal analysis of earthquake response	49
4.3	Load dependent Ritz vectors	54
4.3.1	Derived Ritz vectors	54
4.3.2	Wilson's Ritz vectors	58
4.4	Cut-off criterion for multi-directional earthquake loading	62
4.5	Comparison of WYDs and Lanczos methods	66

5	Computer Implementation of Ritz Vector Algorithms	70
5.1	General	70
5.2	NEABS program organization	71
5.3	Modifications to NEABS program	72
6	Numerical Applications of Ritz Vector Analysis Procedures	78
6.1	General	78
6.2	Dynamic analysis of bridges	79
6.3	Dynamic analysis of a building model	81
6.4	Dynamic analysis of Blackstone River Bridge	82
6.4.1	Analysis model of Blackstone River Bridge	82
6.4.2	Frequency analysis of Blackstone River Bridge	82
6.4.3	Dynamic response analysis of Blackstone River Bridge	83
6.5	Dynamic analysis of Macdonald-Cartier Bridge	86
6.5.1	Analysis model of Macdonald-Cartier Bridge	86
6.5.2	Frequency analysis of Macdonald-Cartier Bridge	87
6.5.3	Dynamic response analysis of Macdonald-Cartier Bridge	88

6.6	Dynamic analysis of Alexandra Bridge	90
6.6.1	Analysis model of Alexandra Bridge	90
6.6.2	Frequency analysis of Alexandra Bridge	92
6.6.3	Dynamic response analysis of Alexandra Bridge	92
7	Summary, Conclusion and Recommendations	148
7.1	Summary	148
7.2	Conclusions	150
7.3	Recommendations for Future Work	153
	References	154

List of Tables

2.1	Subspace iteration procedure	17
3.1	The Krylov subspace	42
3.2	The simple Lanczos algorithm	43
4.1	Algorithm for generating of WYD Ritz vectors	68
4.2	LWYD algorithm for generating of Ritz vectors	69
6.1	Estimates of 15 frequencies of shear beam frame	95
6.2	Estimates of 15 frequencies of Blackstone River Bridge	96
6.3	15 exact frequencies of Blackstone River Bridge	97
6.4	Ritz algorithms comparison for Blackstone River Bridge, Load case 1	98

6.5	Ritz algorithms comparison for Blackstone River Bridge,	
	Load case 2	99
6.6	Estimates of 15 frequencies of Macdonald-Cariter Bridge	100
6.7	Ritz algorithms comparison for Macdonald-Cartier Bridge,	
	Load case 1	101
6.8	Ritz algorithms comparison for Macdonald-Cartier Bridge,	
	Load case 2	102
6.9	Estimates of 15 frequencies of Alexandra Bridge	103
6.10	Ritz algorithms comparison for Alexandra Bridge,	
	Load case 1	104
6.11	Ritz algorithms comparison for Alexandra Bridge (Part 1),	
	Load case 2	105
6.12	Ritz algorithms comparison for Alexandra Bridge (Part 2),	
	Load case 2	106
6.13	Ritz algorithms comparison for Alexandra Bridge,	
	Load case 3	107

List of Figures

3.1	Lanczos algorithm with selective reorthogonalization	44
3.2	Lanczos algorithm with partial reorthogonalization	45
3.3	Combine selective and partial reorthogonalization procedure	46
5.1	Subroutine organization chart of NEABS program	75
5.2	Ritz module subroutine organization chart	76
5.3	Lanczos module subroutine organization chart	77
6.1	Imperial Valley Earthquake of 1940	108
6.2	Saguenay Earthquake of 1988	109
6.3	Directions of seismic action (seismic load cases)	110
6.4	Fictitious offshore platform modelled as a shear beam structure . . .	111

6.5	Elevation, cross-sectional dimensions and finite element model of Blackstone River Bridge	112
6.6	Displacement time history response of Blackstone River Bridge. Load case 1	113
6.7	Base shear time history response of Blackstone River Bridge. Load case 1	114
6.8	Base moment time history response of Blackstone River Bridge. Load case 1	115
6.9	Displacement time history response of Blackstone River Bridge, Load case 2	116
6.10	Base shear time history response of Blackstone River Bridge (Part 1), Load case 2	117
6.11	Base shear time history response of Blackstone River Bridge (Part 2), Load case 2	118
6.12	Base moment time history response of Blackstone River Bridge (Part 1), Load case 2	119
6.13	Base moment time history response of Blackstone River Bridge (Part 2), Load case 2	120
6.14	Cross-sectional dimensions of Macdonald-Cartier Bridge	121
6.15	Layout of major structural members of Macdonald-Cartier Bridge . .	122

6.16	Finite element model of Macdonald-Cartier Bridge	123
6.17	Displacement time history response of Macdonald-Cartier Bridge (Part 1), Load case 1	124
6.18	Displacement time history response of Macdonald-Cartier Bridge (Part 2), Load case 1	125
6.19	Displacement time history response of Macdonald-Cartier Bridge (Part 1), Load case 2	126
6.20	Displacement time history response of Macdonald-Cartier Bridge (Part 2), Load case 2	127
6.21	Base shear time history response of Macdonald-Cartier Bridge at Pier 3 (Part 1), Load case 1	128
6.22	Base shear time history response of Macdonald-Cartier Bridge at Pier 3 (Part 2), Load case 1	129
6.23	Base moment time history response of Macdonald-Cartier Bridge at Pier 3 (Part 1), Load case 1	130
6.24	Base moment time history response of Macdonald-Cartier Bridge at Pier 3 (Part 2), Load case 1	131
6.25	Base shear time history response of Macdonald-Cartier Bridge at Pier 3 (Part 1), Load case 2	132

6.26	Base shear time history response of Macdonald-Cartier Bridge at Pier 3 (Part 2), Load case 2	133
6.27	Base moment time history response of Macdonald-Cartier Bridge at Pier 3 (Part 1), Load case 2	134
6.28	Base moment time history response of Macdonald-Cartier Bridge at Pier 3 (Part 2), Load case 2	135
6.29	Elevation and cross-sectional dimensions of Alexandra Bridge	136
6.30	Truss element mesh of Alexandra Bridge	137
6.31	Axial force time history response of Alexandra Bridge Truss member 404, Load case 1	138
6.32	Axial force time history response of Alexandra Bridge Truss member 404 (Part 1), Load case 2	139
6.33	Axial force time history response of Alexandra Bridge Truss member 404 (Part 2), Load case 2	140
6.34	Axial force time history response of Alexandra Bridge Truss member 404 (Part 3), Load case 2	141
6.35	Displacement time history response of Alexandra Bridge (Part 1), Load case 1	142
6.36	Displacement time history response of Alexandra Bridge (Part 2), Load case 1	143

6.37 Displacement time history response of Alexandra Bridge	
(Part 1), Load case 2	144
6.38 Displacement time history response of Alexandra Bridge	
(Part 2), Load case 2	145
6.39 Axial force time history response of Alexandra Bridge	
Truss member 404 (Part 1), Load case 3	146
6.40 Axial force time history response of Alexandra Bridge	
Truss member 404 (Part 2), Load case 3	147

Chapter 1

Introduction

1.1 General

The application of Ritz vectors in the response analysis of structures subjected to dynamic loads has attracted a lot of interest from structural engineers and researchers, because of the efficiency of the method in analyzing large complex structural systems. This thesis gives a general overview of the numerical techniques involved in the generation and application of the Ritz vectors for dynamic analysis of bridge structures. These algorithms have been implemented in a comprehensive computer structural analysis package, specially developed for the analysis of complex bridge systems. Frequency and time history response analyses of a number of typical highway bridges have been carried out to illustrate the capability of the implemented Ritz algorithms. The efficiency and accuracy of the Ritz methods for dynamic analysis of bridge systems have been investigated. A new cut-off criterion

for the generation of the Ritz vectors used to discretize and model the structural behaviour is developed.

1.2 Problem definition

The dynamic behaviour of many engineering structures, especially those that can be severely affected by strong earthquakes, is an important factor for consideration in the design process of these structures. The behaviour and performance of structures during severe earthquakes are essential elements in the safety assessment of existing structures in seismically active areas. Typically, earthquake response analysis of structures is very time consuming. The amount of computation involved is not only affected by the size of the problem, but also by the time varying characteristics of the loads and the changes in the structural and material properties of the systems in the case of nonlinear time history analysis. To investigate the dynamic behaviour of multiple-degrees-of-freedom (MDOF) structural systems, the incremental form of the dynamic equations of motion of the systems expressed in geometric coordinates is usually solved by a step-by-step time integration procedure. Although accurate results can be obtained using this approach, the amount of computation involved in the analysis of a large system can be prohibitively expensive. Therefore in recent years, considerable research efforts have been focused on the development of new and efficient analysis procedures or algorithms for frequency and dynamic response analysis of structural systems which reduce the size of the problem, and yet provide accurate and satisfactory results.

The determination of the free vibration properties of engineering structures is

a very important step in the understanding of the dynamic behaviour and performance of the structures. There are a number of numerical techniques today that can determine the vibration frequencies and the corresponding vibration mode shapes of linear elastic structural systems. Among these numerical techniques, the subspace iteration method is generally considered a very reliable procedure which has been implemented in many structural analysis computer programs. In recent years, significant improvement and advances have been made on the numerical stability problem of the traditional Lanczos method which have made the non-iterative Lanczos procedure a powerful and efficient alternative tool to the subspace iteration method for obtaining partial eigensolutions of large complex structural systems. Unfortunately, the Lanczos method is not as widely known to the civil engineering community as the subspace iteration method.

In another class of approach as for the vibration analysis of structural systems, the reduction methods are generally based on the idea of coordinate transformation. The traditional modal superposition method may be considered an example of this approach. In modal analysis, the geometric coordinates of the system are transformed to the generalized modal coordinates. For the analysis of a complex system, the procedure involves the solution of a large eigenproblem and the extraction of appropriate number of the first few vibration modes. However, the modal analysis procedures do not explicitly consider the spatial distribution and frequency content of the applied loading in determining the number of vibration modes required for accurate results. The determination of the frequencies and shapes of the required vibration modes for accurate solution of a large MDOF system can be very costly. Furthermore in the modal analysis procedures, significant amount of computational effort may be wasted to generate vibration modes that are orthogonal to the load

vectors, and thus do not contribute to the dynamic response of the structure.

Other reduction techniques, such as those based on the Rayleigh-Ritz approximation concept, offer more efficient alternative ways to reduce the size of the dynamic problem, provided the spatial distribution of the applied dynamic loads remains unchanged throughout the entire duration of dynamic response. Fortunately, most time-varying loads acting on civil engineering structures, including those caused by earthquake excitation, have that characteristic in spatial distribution. Typically, the analysis of a large complex structural system subjected to dynamic loads by Ritz algorithms using a set of load-dependent Ritz vectors generated by the Lanczos or Wilson's method requires less computational effort than the traditional means of modal analysis using vibration modes as the transformation basis. The load-dependent Ritz vector approach generally has better convergence rate in determining the dynamic response of complex structural systems.

1.3 Objectives and scope

The objective of this thesis is to review and improve the existing numerical techniques and algorithms used in the generation of Ritz type vectors for efficient dynamic analysis of large complex bridge systems. In the present study, a comprehensive computer tool for dynamic analysis is developed which incorporates several state-of-the-art Ritz algorithms. The accuracy and computational efficiency of the Ritz vector algorithms in determining free vibration properties of bridges are evaluated. The relative effectiveness of these algorithms in reducing the size of the dynamic problem of structures with emphasis on bridges is also investigated.

The research work presented in this study is divided into two parts: theoretical development and numerical applications. First, recent advances in the solution techniques of eigenvalue problem are reviewed. The theory of the Lanczos method and other related developments in the eigensolution techniques by Ritz methods are discussed. The Lanczos and Wilson's methods are described in detail as two forms of implementation of the more general Ritz vector algorithm and as alternatives to the classical modal superposition approach for dynamic response analysis. Improved versions of the Lanczos and Wilson's algorithms have been implemented in the computer program NEABS, Nonlinear Earthquake Analysis of Bridge Systems, to study the dynamic behaviour of bridge systems. The modal superposition procedure has been introduced in the NEABS program and the direct time-step integration analysis procedure has been extensively modified to allow dynamic time history response analysis of systems by load-dependent Ritz vectors. The accuracy and efficiency of the Ritz vector reduction methods are evaluated and assessed based on the number of Ritz vectors necessary in the analysis to provide the required accuracy in the solution. The efficiency in the representation of the structural mass that participates in the dynamic response and computation time are considered in the assessment. A new cut-off criterion for multi-directional earthquake load based on the concept of effective mass participation has been developed and verified by numerous numerical examples.

The thesis is divided into seven chapters. Chapter 1 gives an introduction to the studied problem. The objectives and scope of the research study are presented. Chapter 2 presents useful background information and definitions on eigenvalue problem. It also discusses the different approaches for the solution of the eigenproblem based on the Rayleigh-Ritz approximation. Chapter 3 describes the basic

Lanczos algorithm primarily developed with the objective to obtain an accurate eigensolution by reducing the bandwidth and the size of the system matrices. The derivation of the simple Lanczos algorithm is presented and the mechanisms involved in the loss of orthogonality among the generated Lanczos vectors are explained. The reorthogonalization procedures that improve the stability problem of the original Lanczos method are described. Chapter 4 presents the dynamic response analysis procedures by Ritz vectors. The concept of load-dependent Ritz vectors is introduced. The properties of the derived Ritz vectors obtained from the Lanczos method and the Wilson's Ritz vectors are discussed. Error estimates on the number of Ritz vectors necessary to achieve a satisfactory level of accuracy in dynamic response analysis are established. Finally, the similarities as well as differences among the different implementations of the Lanczos and Wilson's methods are discussed. Chapter 5 presents information about the NEABS computer program adopted in the present study for dynamic analysis of bridge systems. The implementation of the Ritz vectors algorithms in the NEABS program are explained. Chapter 6 presents numerical examples on the use of Ritz vectors in frequency and dynamic response analysis of bridge systems. Numerical results are compared with those obtained by other researchers. The main focus here is to examine the computational performance of the different implemented Ritz algorithms in dynamic response analysis of bridge systems under earthquake loads. Finally, a summary, conclusions of the present research work and suggestions for future work are presented in Chapter 7.

Chapter 2

Free Vibration Analysis

2.1 General

The evaluation of transient response of a structure subjected to dynamic loads often involves the solution of an eigenvalue problem. For a large structural system, obtaining solutions to the eigenvalue problem represents a major cost in the solution process. In this chapter, the definition and properties of the eigenproblem are reviewed, followed by discussions on the numerical methods which are based on the Rayleigh-Ritz approximation.

2.2 General eigenvalue problem

2.2.1 Eigenproblem formulation

After discretization, the equation of motion for the finite element model of a structural system is expressed as follows

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t) \quad (2.1)$$

where the vectors $\ddot{\mathbf{u}}$, $\dot{\mathbf{u}}$ and $\mathbf{u}$ are the acceleration, velocity, and displacement vectors which describe the response of the structure, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the square $n \times n$ mass, damping, and stiffness matrices, and $\mathbf{p}(t)$ is the vector of the time-dependent forcing function.

When the applied load vector in Equation 2.1 is equal to zero, the equation below describes the free vibration response of the system

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0 \quad (2.2)$$

in which case the dynamic response may be initiated when one or more degrees-of-freedom of the system are set in motion due to initial displacements, velocities or both.

For the case when damping is neglected, the finite element discretization of the structure leads to the general undamped, free vibration equation of motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0 \quad (2.3)$$

In the solution of the free vibration response, the harmonic displacement response is assumed to be of the general form

$$\mathbf{u}(t) = \boldsymbol{\phi} e^{i\omega t} \quad (2.4)$$

When Equation 2.4 is substituted into Equation 2.3. it becomes

$$\mathbf{K}\phi = \lambda\mathbf{M}\phi \quad . \quad \lambda = \omega^2 \quad (2.5)$$

In the above equation, ϕ is an eigenvector. λ is an eigenvalue and ω is the natural frequency of vibration. Equation 2.5 is referred to as the general linear eigenvalue problem and it may also be expressed in another form given by

$$(\mathbf{A} - \nu\mathbf{I})\phi = 0 \quad (2.6)$$

where $\mathbf{I}$ is an identity matrix of the same order of $\mathbf{A}$ and

$$\mathbf{A} = \mathbf{K}^{-1}\mathbf{M} \quad (2.7)$$

$$\nu = \frac{1}{\lambda} \quad (2.8)$$

Put in the form of Equation 2.6, the free vibration eigenvalue problem is reformat-
ted as a standard symmetric eigenvalue problem. The eigenvalues of the standard
eigenproblem are the reciprocals of the eigenvalues of the generalized eigenproblem.
The eigenvectors for both forms remain the same as Φ .

2.2.2 Properties of eigensolution

Before the discussion on the numerical techniques for obtaining the eigensolution.
it is useful to give an overview on the fundamental characteristics and properties of
the eigensolution. In order to have a non-trivial solution, that is $\phi \neq 0$, the matrix
($\mathbf{K} - \lambda\mathbf{M}$) must be singular, i.e.

$$\det(\mathbf{K} - \lambda\mathbf{M}) = 0 \quad (2.9)$$

By expanding the determinant as represented by Equation 2.9, the characteristic polynomial is obtained as follows

$$p(\lambda) = \det(\mathbf{K} - \lambda\mathbf{M}) \quad (2.10)$$

where the order of the polynomial is equal to the order of $\mathbf{K}$ and $\mathbf{M}$. There are n solution roots for Equation 2.9, which are the eigenvalues of the eigenproblem. The eigenvalues are the squares of the natural frequencies. For each eigenvalue λ_i , a corresponding unique non-trivial eigenvector (mode shape) ϕ_i can be found by using Equation 2.5 or Equation 2.6 of which the magnitude may be arbitrarily specified. Thus, an eigenvector is defined only by its direction in the n -dimensional space. Furthermore, the eigenvectors have the orthogonal property with respect to both the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$. When appropriately normalized, the eigenvectors satisfy the following properties

$$\phi_i^T \mathbf{M} \phi_j = \delta_{ij} \quad (2.11)$$

$$\phi_i^T \mathbf{K} \phi_j = \lambda_i \delta_{ij} \quad (2.12)$$

where δ_{ij} is the Kronecker delta. The eigenvectors can be grouped into a modal matrix Φ , where $\Phi = [\phi_1, \phi_2, \dots, \phi_n]$. The orthogonality property can then be expressed in the following form

$$\Phi^T \mathbf{M} \Phi = \mathbf{I} \quad (2.13)$$

$$\Phi^T \mathbf{K} \Phi = \Lambda \quad (2.14)$$

where Λ is the spectral matrix which is a diagonal matrix formed by the eigenvalues λ .

It is noted that the eigenvalues are all real values when both $\mathbf{K}$ and $\mathbf{M}$ are symmetric. In most cases the matrices $\mathbf{K}$ and $\mathbf{M}$ are positive definite, so the eigenvalues

are greater than zero. However, if the system is capable of rigid body motions (where the frequency of oscillation for the rigid body mode is zero), $\mathbf{K}$ will be positive semidefinite, and the eigenvalues will be greater than or equal to zero. In a case when the mass of the system is lumped at nodes, and $\mathbf{M}$ is a diagonal matrix with some zero diagonal elements, the corresponding eigenvalues of the zero elements are taken to be infinity. For the case of multiple roots for λ_i , where m is the multiplicity of the eigenvalue, the m eigenvectors associated with the eigenvalue λ_i are still linearly independent, and any linear combination of those vectors is also a valid mode shape for the eigenvalue λ_i .

2.3 The Rayleigh-Ritz approximation procedure

2.3.1 General

The Rayleigh-Ritz method was originally developed to reduce the size of the dynamics problem. A partial solution of the sparse, large eigenvalue problem is obtained. The Rayleigh-Ritz method can be looked upon as the Ritz extension to the Rayleigh method. The main idea of the Rayleigh method is based on the principle of conservation of energy [1], which states that the total energy in a freely vibrating system must remain constant if there is no damping force to dissipate it. By equating the energy expression for the maximum strain energy to that for the maximum kinetic energy, a scalar ϱ may be defined by the ratio as follows

$$\varrho = \frac{\mathbf{u}^T \mathbf{K} \mathbf{u}}{\mathbf{u}^T \mathbf{M} \mathbf{u}} \quad (2.15)$$

where $\mathbf{K}$ and $\mathbf{M}$ represent the stiffness and mass matrices of order n , and $\mathbf{u}$ is an arbitrary vector also of the order n . The scalar ϱ is known as the Rayleigh quotient. The Rayleigh quotient represents an upper bound estimate to the lowest eigenvalue or the fundamental frequency.

In the Rayleigh-Ritz method, a coordinate transformation is performed to reduce the number of degrees of freedom of the system, which reduces the computational effort required in dynamic response analysis. The basic assumption of the Rayleigh-Ritz method is that the dynamic response may be represented by set of space-dependent functions amplified by the time-dependent generalized coordinates [2], [3], as follows

$$\mathbf{u}(x, t) = \mathbf{\Psi}(x)\mathbf{z}(t) \quad (2.16)$$

where $\mathbf{\Psi}(x)$ is referred here as the matrix of Ritz vectors, and $\mathbf{z}(t)$ is the vector of generalized coordinates. The size of $\mathbf{\Psi}$ matrix is determined by the total number of degrees-of-freedom n and the number of Ritz vectors q used in the transformation. where it is common that $q \ll n$. Thus, $\mathbf{\Psi}$ is the $(n \times q)$ matrix storing the Ritz base vectors. Applying the transformation of Equation 2.16 to the equation of motion given in Equation 2.1, it leads to

$$\mathbf{M}^* \ddot{\mathbf{u}} + \mathbf{C}^* \dot{\mathbf{u}} + \mathbf{K}^* \mathbf{u} = \mathbf{p}^*(t) \quad (2.17)$$

where $\mathbf{M}^* = \mathbf{\Psi}^T \mathbf{M} \mathbf{\Psi}$, $\mathbf{C}^* = \mathbf{\Psi}^T \mathbf{C} \mathbf{\Psi}$, $\mathbf{K}^* = \mathbf{\Psi}^T \mathbf{K} \mathbf{\Psi}$, and $\mathbf{p}^* = \mathbf{\Psi}^T \mathbf{p}$.

It is noted here that matrices $\mathbf{M}^*$, $\mathbf{C}^*$, $\mathbf{K}^*$ are the projections of $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ onto the subspace spanned by the q Ritz vectors ψ_i , $i = 1, \dots, q$. The eigenproblem for the reduced system is then expressed as follows

$$(\mathbf{K}^* - \mathbf{\Theta} \mathbf{M}^*) \mathbf{Z} = 0 \quad (2.18)$$

where $\mathbf{\Theta}$ is a diagonal matrix of the eigenvalues of the transformed eigenproblem.

and $\mathbf{Z}$ is the matrix storing the $\mathbf{M}^*$ -orthonormal eigenvectors of the reduced system $\mathbf{z}_1, \dots, \mathbf{z}_q$. Approximations to the eigenvectors of the original problem are then obtained as follows

$$\Phi^* = \Psi \mathbf{Z} \quad (2.19)$$

As noted by Bathe [4], the eigenvalues calculated from the Ritz analysis are the upper bounds on the corresponding exact eigenvalues of the system. The quality of the approximate solution very much depends on the adequacy of the assumed set of vectors Ψ . It is not so important that each Ritz vector is close to a required eigenvector, but rather that linear combinations of the Ritz vectors should produce good approximations of the eigenproblem $\mathbf{K}\Phi = \Lambda \mathbf{M}\Phi$.

2.3.2 Subspace iteration method

The subspace iteration method can be considered as a type of the Rayleigh-Ritz analysis procedure. It is useful in determining the dominant eigenvalues and corresponding eigenvectors of a large sparse eigenproblem. This method involves a simultaneous iteration process in which all the trial vectors are subjected to inverse iteration followed by orthogonalization procedures. The Gram-Schmidt orthogonalization is applied to prevent convergence of all the trial vectors to the most dominant eigenvector. The orthogonalization procedure assures convergence to the first p eigenvalues and eigenvectors. The algorithm of the subspace iteration method is summarized in Table 2.1.

Aspects of the subspace iteration method can be found in the reference by Bathe and Wilson [5]. The matrices $\mathbf{K}_{k+1}^*$ and $\mathbf{M}_{k+1}^*$ after each iteration tend towards a diagonal form and the generalized Jacobi method can be used as a very efficient

technique for the solution of the reduced eigenproblem. An important aspect of the subspace iteration method concerns the selection of the starting vectors and the number of the trial vectors that should be included in the iteration. It is generally observed that a larger number of iteration vectors increases the rate of convergence. However, the disadvantage of this is that it increases the amount of computation in each iteration. If p is the number of the required eigenvalues and q is a number of iteration vectors, it is found in practice that the method is most efficient when $q = \min\{2p, p + 8\}$.

Furthermore, the proper choice of the starting vectors is a very important step in the subspace iteration method. If there are properly chosen, they can reduce the number of iterations and thus reduce the total cost of obtaining the eigensolution. The starting vectors should be chosen in a such way that they excite the degrees-of-freedom with large mass and small stiffness. In general, the first Ritz vector in the first column of the matrix Ψ_1 , can be the diagonal of $\mathbf{M}$ to ensure excitation of all the degrees-of-freedom of the system. For other columns of Ψ_1 , unit vectors may be selected with entries $+1$ at the degrees-of-freedom with the smallest ratio of k_{ii}/m_{ii} [4], [6].

The Ritz eigenvalues Θ are upper bounds to the eigenvalues of the original eigenvalue problem

$$\lambda_j \leq \theta_j \quad (2.20)$$

for $j = 1, 2, \dots, p$ where $\theta_1 \leq \theta_2 \leq \dots \leq \theta_p$.

The higher Ritz eigenvalues give poorer approximations, while the lower ones give better approximations and they often tend to converge first. The eigenvalues from

the reduced eigenproblem are deemed to be converged when

$$\frac{|\theta_i^{(k+1)} - \theta_i^{(k)}|}{\theta_i^{(k)}} \leq tol \quad \text{for } i = 1, 2, \dots, m \quad (2.21)$$

where $\theta_i^{(k)}$ is the i^{th} diagonal element of Θ_k at the k^{th} iteration and $m < p$.

Further modification in the subspace iteration method were introduced by Bathe and Ramaswamy [7] to accelerate the computation. The basic steps in the original procedure are supplemented by a shifting technique, vector overrelaxation and the Lanczos method to generate a set of starting vectors.

Finally, it is worth while to mention that the subspace iteration method is widely used by structural engineers and implemented in many computer programs for dynamic response analysis.

2.3.3 Lanczos method

The Lanczos method was primarily developed as an improvement to the Rayleigh-Ritz method in obtaining the eigensolution. In recent years, this technique has evolved to become one of the most effective procedures for determination a few eigenpairs of large sparse problems [8].

The main idea behind Lanczos method is to construct a set of $\mathbf{M}$ -orthonormal vectors by applying the Gram-Schmidt orthogonalization process to the Krylov sequence. This set of vectors, denoted by $\mathbf{Q}$, transforms the generalized eigenproblem $\mathbf{K}\phi = \lambda\mathbf{M}\phi$ into the standard form of a tridiagonal matrix $\mathbf{T}_m$, where

$$\mathbf{T}_m = \mathbf{Q}_m^T (\mathbf{M}\mathbf{K}^{-1}\mathbf{M}) \mathbf{Q}_m, \quad i = 1, \dots, m \quad (2.22)$$

The standard eigenproblem as given in Equation 2.6 can be expressed in terms of $\mathbf{T}_m$ as follows

$$\mathbf{T}_m \mathbf{s} = \frac{1}{\lambda} \mathbf{I}_m \mathbf{s} \quad (2.23)$$

where $\mathbf{s}$ is an eigenvector of Equation 2.23. It is noted that the eigenvalues of $\mathbf{T}_m$ are the reciprocals of the eigenvalues of the problem $\mathbf{K}\phi = \lambda \mathbf{M}\phi$, and the eigenvectors are related as follows

$$\Phi = \mathbf{Q}\mathbf{S}. \quad (2.24)$$

The derivation and a more detailed study of the Lanczos method are presented in the next chapter.

Table 2.1: Subspace iteration procedure**I. Initialization**

1. Form $\mathbf{K}$ and $\mathbf{M}$
2. Factorize

$$\mathbf{K} = \mathbf{L}\mathbf{D}\mathbf{L}^T$$

3. Select starting iteration vectors Ψ_1

II. Iteration

1. For $k = 1, 2, \dots$, form

$$\mathbf{K}\bar{\Psi}_{k+1} = \mathbf{M}\Psi_k$$

and solve for $\bar{\Psi}_{k+1}$

2. Compute reduced matrices (the projection of matrices $\mathbf{K}$ and $\mathbf{M}$ in an $q \times q$ space)

$$\mathbf{K}_{k+1}^* = \bar{\Psi}_{k+1}^T \mathbf{K} \bar{\Psi}_{k+1}$$

$$\mathbf{M}_{k+1}^* = \bar{\Psi}_{k+1}^T \mathbf{M} \bar{\Psi}_{k+1}$$

3. Solve the reduced eigenproblem of order $q \times q$

$$\mathbf{K}_{k+1}^* \mathbf{Z}_{k+1} = \mathbf{M}_{k+1}^* \mathbf{Z}_{k+1} \Lambda_{k+1}$$

4. Find an improved approximation to the eigenvectors of original system

$$\Psi_{k+1} = \bar{\Psi}_{k+1} \mathbf{Z}_{k+1}$$

5. Check convergence, if convergence reached required level of tolerance stop, otherwise continue
6. Go to step II-1.

Chapter 3

Lanczos Method

3.1 General

In this chapter development of the Lanczos method is briefly reviewed. The derivation and the reorthogonalization techniques of the different implementations of the Lanczos method are discussed. Recently, with significant progress and improvement made in the numerical stability of the algorithm, the Lanczos method has become popular for application in the solution of the eigenproblem of large sparse matrices. In the formulation of the Lanczos method, the reorthogonalization procedure is a very important part of the algorithm which has significant influences on the convergence and accuracy of the solution. A proper understanding of the reorthogonalization process is essential in the study of the behaviour and performance of the Lanczos method and its practical application. The reorthogonalization procedures are discussed in detail in this chapter.

3.2 Historical development of Lanczos algorithm

The Lanczos method was first presented in 1950 by Lanczos [9]. The procedure was originally developed to find a few eigenvectors of a large sparse symmetric matrix. In practice, however, the algorithm was often used instead to reduce the matrix to a tridiagonal form. After the introduction of the Householder reduction method [10] in 1955 which was a more efficient scheme for tridiagonalization of matrices than the Lanczos algorithm, the Lanczos method had lost its advantage and fell into a period of obscurity. Evidently, one of the main reasons for the Lanczos method to fall into obscurity was the lack of computational stability of the algorithm. It was noticed that round-off errors have a very dramatic effect on the performance of the algorithm. The original Lanczos algorithm is especially sensitive to the orthogonality requirement among the Lanczos vectors. Hence, it has problems with maintaining proper independence among the generated Lanczos vectors.

In 1952, Lanczos [11] implemented into his algorithm a full orthogonalization procedure, in which at each step of the generation process, the newly generated Lanczos vector is orthogonalized against all the previous ones using the Gram-Schmidt orthogonalization procedure. This version of the algorithm is very expensive in terms of the usage of computer memory as well as in computation time. Because of these drawbacks, practical applications of the Lanczos method remained limited until further improvement and progress were achieved. In the 1970s, interest in the Lanczos algorithm was revived because of advances in electronic computation allowing the solution of large complex engineering problems which often involved the solution of large sparse matrices.

In 1970 Ojalvo & Newman [12] applied the algorithm with full reorthogonalization to solve a large sparse matrix eigenproblem in obtaining the lower natural frequencies and mode shapes. They followed Lanczos idea [9] to truncate the tridiagonalization process at some point to produce a reduced tridiagonal matrix $\mathbf{T}_m$ of order $m \times m$, where m is much less than the size of the original problem n . In this aspect, the Lanczos algorithm seemed to be very competitive compared with other methods to find good approximations to the dominant eigensolutions of the given eigenproblem. However, the cost of reorthogonalization at each iteration step started to dominate the overall computational effort. In 1971, Paige [13] presented many important findings regarding the simple Lanczos algorithm. From his detailed study, important understanding and insight on the behaviour of the algorithm were achieved. More importantly, the mechanisms of the numerical stability problem of the Lanczos process were established. After Paige's discoveries and his theoretical explanations of the behaviour of the algorithm [13] - [15], the research on the Lanczos method began to concentrate on two directions.

The first direction of research is focused on so called 'Paige-style' Lanczos algorithm. Paige states that the simple Lanczos algorithm could still give good results without reorthogonalization. However, this approach produces a tridiagonal matrix $\mathbf{T}$ whose dimensions are often larger than the original problem. It also generates multiple copies of the required eigenvalues. There is also no certain way to identify and reject the spurious copies, especially when the original problem contains multiple roots in the eigenvalue problem. Furthermore, it is difficult to assess the amount of storage needed during computations. Therefore, the focus of the research at the time was directed towards identification of good eigenvalues and rejection the spurious ones. Research in this area was conducted by Cullum & Willoughby [16].

In later research, Golub, Underwood and Wilkinson [17] suggest to use the Lanczos method iteratively. The algorithm periodically restarts with a new starting vector chosen as some linear combination of the Ritz vectors defined in the current subspace. Golub [18], Underwood [19] as well as Cullum and Donath [20] implement this concept in the iterative block Lanczos algorithm, which is further developed by Golub and Underwood [21] and also by Nour-Omid and Clough [22]. The block Lanczos algorithm is a natural extension to the simple Lanczos method. Similar to the subspace iteration method, the idea of the block Lanczos algorithm is to employ a set of orthonormal vectors $\mathbf{Q}_1$, where $\mathbf{Q}_1$ is an $n \times p$ matrix, instead of just a single starting vector $\mathbf{q}_1$. As a result, a block tridiagonal matrix $\mathbf{T}_m$ is obtained with blocks of the size $p \times p$. The block Lanczos method has the advantage of finding multiple eigenvalues (up to the block size p), while the simple Lanczos algorithm can not detect multiplicity in computed eigenvalues.

The second direction of research since the 1970s concentrates on improvement to the reorthogonalization procedures, since they are considered in their original form too expensive for large matrices. Parlett and Scott [23] and later Simon [24, 25] present new semi-orthogonalization techniques and their implementations in the Lanczos algorithm which preserve robust linear independence among the Lanczos vectors. Parlett and Scott introduce the concept of selective reorthogonalization scheme of which the main idea is to maintain some level of orthogonality, and only reorthogonalize when it is necessary with respect to the converged eigenvectors. Another approach proposed by Simon [24, 25] emphasizes maintaining semiorthogonality among the Lanczos vectors, and the algorithm reorthogonalizes, when it is necessary the new Lanczos vector against the previous Lanczos vectors. Parlett and Nour-Omid [26] and Nour-Omid [27] came up with the idea of combining both

selective and partial orthogonalization techniques by applying them simultaneously.

3.3 Derivation of the simple Lanczos algorithm

In the solution of the eigenvalue problem of a structural system defined by the stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$, a vector $\mathbf{r}$ is selected as the starting initial vector. Next, the sequence of vectors $\{\mathbf{r}, \mathbf{K}^{-1}\mathbf{M}\mathbf{r}, (\mathbf{K}^{-1}\mathbf{M})^2\mathbf{r}, \dots, (\mathbf{K}^{-1}\mathbf{M})^n\mathbf{r}\}$ known as the Krylov sequence is generated. The Gram-Schmidt orthogonalization procedure is applied to the Krylov sequence. As a result, a set of $\mathbf{M}$ -orthonormal vectors $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ is constructed for use in the Rayleigh-Ritz reduction of the original problem to obtain a standard eigenproblem of the reduced transformed system in a tridiagonal form.

Assume for the present that the first j Lanczos vectors $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_j\}$ have been obtained, and the next Lanczos vector $\mathbf{q}_{j+1}$ is to be found. The first j vectors are orthogonal and satisfy the orthonormality condition $\mathbf{q}_i^T \mathbf{M} \mathbf{q}_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. To calculate $\mathbf{q}_{j+1}$, the vector $\boldsymbol{\vartheta}_j = (\mathbf{K}^{-1}\mathbf{M})^j \mathbf{r}$ must be orthogonalized against all previous j Lanczos vectors. From the definition of $\boldsymbol{\vartheta}_j$, the vector $\boldsymbol{\vartheta}_j$ can be expressed in terms of $\boldsymbol{\vartheta}_{j-1}$ as follows

$$\boldsymbol{\vartheta}_j = (\mathbf{K}^{-1}\mathbf{M})^j \mathbf{r} = \mathbf{K}^{-1}\mathbf{M}(\mathbf{K}^{-1}\mathbf{M})^{j-1} \mathbf{r} = \mathbf{K}^{-1}\mathbf{M}\boldsymbol{\vartheta}_{j-1} \quad (3.1)$$

Now, the vector $\boldsymbol{\vartheta}_{j-1}$ which has been $\mathbf{M}$ -orthonormalized against the first $(j-1)$ Lanczos vectors to obtain $\mathbf{q}_j$ can be written as a linear combination of the first j Lanczos vectors as $\boldsymbol{\vartheta}_{j-1}$ is a vector in the Krylov subspace defined by the j Lanczos

vectors. Therefore

$$\boldsymbol{\vartheta}_{j-1} = \sum_{i=1}^j a_i \mathbf{q}_i \quad (3.2)$$

where a_i is the component of $\boldsymbol{\vartheta}_{j-1}$ along $\mathbf{q}_i$. Table 3.1 illustrates different sets of the Krylov sequences and helps to identify which sequence of vectors span the same Krylov subspace at a given j step. Using Equation 3.1 and Equation 3.2, the following expressions are obtained

$$\boldsymbol{\vartheta}_j = \sum_{i=1}^j a_i \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_i \quad (3.3)$$

and

$$\boldsymbol{\vartheta}_j = a_j \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j + \sum_{i=1}^{j-1} a_i \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_i \quad (3.4)$$

As can be observed from Table 3.1, each vector $\mathbf{K}^{-1} \mathbf{M} \mathbf{q}_i$ can be expressed as a linear combination of the first $i + 1$ Lanczos vectors as follows

$$\mathbf{K}^{-1} \mathbf{M} \mathbf{q}_i = \sum_{k=1}^{i+1} b_k \mathbf{q}_k \quad (3.5)$$

Consequently, Equation 3.4 can be rewritten in a form

$$\boldsymbol{\vartheta}_j = a_j \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j + \sum_{i=1}^{j-1} a_i \sum_{k=1}^{i+1} b_k \mathbf{q}_k \quad (3.6)$$

$$\boldsymbol{\vartheta}_j = a_j \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j + \sum_{i=1}^j c_i \mathbf{q}_i \quad (3.7)$$

Because the vector $\boldsymbol{\vartheta}_j$ is $\mathbf{M}$ -orthonormalized against the preceding j Lanczos vectors, the components of $\boldsymbol{\vartheta}_j$ along each of the $\mathbf{q}_i$'s is removed, so that the second summation term in the last equation should be equal to zero, thus

$$\boldsymbol{\vartheta}_j = a_j \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j \quad (3.8)$$

Equation 3.8 indicates that the next vector $\mathbf{q}_{j+1}$ can be obtained by first calculating the vector $\bar{\mathbf{r}}_j$ from the previous vector $\mathbf{q}_j$

$$\bar{\mathbf{r}}_j = \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j \quad (3.9)$$

In general, the vector $\bar{\mathbf{r}}_j$ may contain components from all the previous Lanczos vectors, such that $\bar{\mathbf{r}}_j$ can be expressed as a linear combination of all the previous Lanczos vectors $\mathbf{q}_j$ plus an orthogonal component denoted by the vector $\mathbf{r}_j$.

$$\bar{\mathbf{r}}_j = \mathbf{r}_j + \alpha_j \mathbf{q}_j + \beta_j \mathbf{q}_{j-1} + \gamma_j \mathbf{q}_{j-2} + \dots \quad (3.10)$$

In Equation 3.10, the terms with $\alpha_j, \beta_j, \gamma_j, \dots$ are the components of $\bar{\mathbf{r}}_j$ in the directions of the previous Lanczos vectors. These coefficients can be evaluated from the orthonormality condition of the Lanczos vectors. Multiplying both sides of Equation 3.10 by $\mathbf{q}_j^T \mathbf{M}$ gives

$$\mathbf{q}_j^T \mathbf{M} \bar{\mathbf{r}}_j = \mathbf{q}_j^T \mathbf{M} \mathbf{r}_j + \alpha_j \mathbf{q}_j^T \mathbf{M} \mathbf{q}_j + \beta_j \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-1} + \gamma_j \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-2} + \dots \quad (3.11)$$

The first term on the right-hand side disappears by definition. Furthermore, all terms after the second one in Equation 3.11 vanish due to $\mathbf{M}$ -orthogonality. Therefore, by applying the normalizing definition to the second term, Equation 3.11 is reduced to an expression for the amplitude of $\bar{\mathbf{r}}_j$ along $\mathbf{q}_j$

$$\alpha_j = \mathbf{q}_j^T \mathbf{M} \bar{\mathbf{r}}_j \quad (3.12)$$

Similarly, the component of $\bar{\mathbf{r}}_j$ along $\mathbf{q}_{j-1}$ may be found by pre-multiplying Equation 3.10 by $\mathbf{q}_{j-1}^T \mathbf{M}$. In this case, all terms except the third vanish by orthogonality, and the coefficient of β_j is unity, therefore

$$\beta_j = \mathbf{q}_{j-1}^T \mathbf{M} \bar{\mathbf{r}}_j \quad (3.13)$$

Next, using Equation 3.9 to eliminate $\bar{\mathbf{r}}_j$ and applying transposition, Equation 3.13 can be expressed in another form

$$\begin{aligned} \beta_j &= \mathbf{q}_{j-1}^T \mathbf{M} \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j \\ &= \bar{\mathbf{r}}_{j-1}^T \mathbf{M} \mathbf{q}_j \\ &= \mathbf{q}_j^T \mathbf{M} \bar{\mathbf{r}}_{j-1} \end{aligned} \quad (3.14)$$

By expanding $\bar{\mathbf{r}}_{j-1}$ in terms of the preceding Lanczos vectors and the vector $\mathbf{r}_{j-1}$. Equation 3.14 becomes

$$\beta_j = \mathbf{q}_j^T \mathbf{M} \mathbf{r}_{j-1} + \alpha_{j-1} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-1} + \beta_{j-1} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-2} + \gamma_{j-1} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-3} + \dots \quad (3.15)$$

All terms except the first vanish on the right-hand side due to $\mathbf{M}$ -orthogonality, thus

$$\beta_j = \mathbf{q}_j^T \mathbf{M} \mathbf{r}_{j-1} \quad (3.16)$$

Now, as the vector $\mathbf{q}_j$ is calculated by mass normalizing $\mathbf{r}_{j-1}$, where β_j is the normalizing factor, $\mathbf{q}_j$ can be expressed as follows

$$\mathbf{q}_j = \frac{1}{\beta_j} \mathbf{r}_{j-1} \quad (3.17)$$

The normalizing factor β_j is obtain as follows

$$\beta_j^2 = \mathbf{r}_{j-1}^T \mathbf{M} \mathbf{r}_{j-1} \quad (3.18)$$

The same procedures for α_j and β_j can be applied to obtain γ_j

$$\gamma_j = \mathbf{q}_j^T \mathbf{M} \mathbf{r}_{j-2} + \alpha_{j-2} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-2} + \beta_{j-2} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-3} + \gamma_{j-2} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-4} + \dots \quad (3.19)$$

Since from Equation 3.17 $\mathbf{r}_{j-2} = \beta_{j-1} \mathbf{q}_{j-1}$, all terms on the right-hand side vanish due to orthogonality. Therefore, γ_j is equal to zero. All further terms in the expression for $\bar{\mathbf{F}}_j$ in Equation 3.10 also vanish, which can be showed in the same way as for γ_j .

Finally, the above derivations indicate that the orthogonalization procedure used to generate the Lanczos vector need be applied only to the previous two vectors. Hence, the simple Lanczos algorithm is based on the three term recurrence formula

$$\mathbf{r}_j = \beta_{j+1} \mathbf{q}_{j+1} = \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j - \alpha_j \mathbf{q}_j - \beta_j \mathbf{q}_{j-1} \quad (3.20)$$

The steps of the simple Lanczos algorithm are summarized in Table 3.2.

Equation 3.20 can be arranged in a global matrix form as follows

$$[\mathbf{K}^{-1}\mathbf{M}][\mathbf{Q}_m] - [\mathbf{Q}_m][\mathbf{T}_m] = [0 \ 0 \ \dots \ \mathbf{r}_m] = \mathbf{r}_m \mathbf{e}_m^T \quad (3.21)$$

where $\mathbf{Q}_m$ is an $n \times m$ matrix with columns $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$, $\mathbf{e}_m$ is the last column of the identity matrix $\mathbf{I}_m$, and $\mathbf{T}_m$ is a tridiagonal matrix made up of the coefficients α_j and β_j

$$\mathbf{T}_m = \begin{bmatrix} \alpha_1 & \beta_2 & & & & \\ \beta_2 & \alpha_2 & \beta_3 & & & \\ & \beta_3 & . & . & & \\ & & . & . & . & \\ & & & . & . & \beta_m \\ & & & & \beta_m & \alpha_m \end{bmatrix} \quad (3.22)$$

The set of generated Lanczos vectors, $\mathbf{Q}_m$, can then be used in the Rayleigh-Ritz procedure to obtain the best approximation to the eigenvalues and eigenvectors of the transformed generalized eigenproblem

$$(\mathbf{K}^{-1}\mathbf{M} - \nu\mathbf{I})\phi = 0 \quad (3.23)$$

where the relation between the eigenvalue ν of the transformed problem and the original eigenvalue λ is governed by Equation 2.8. In the formulation here, the pair $\{\theta_i^{(m)}, \mathbf{y}_i^{(m)}\}$ denotes a Ritz pair which provides the best approximation to the eigenpair $\{\nu, \phi\}$. The Ritz vector $\mathbf{y}_i^{(m)}$ is expressed in terms of the Lanczos vectors $\mathbf{Q}_m$ and a coefficient vector $\mathbf{s}_i^{(m)}$.

$$\mathbf{y}_i^{(m)} = \mathbf{Q}_m \mathbf{s}_i^{(m)}, \quad i = 1, 2, \dots, m \quad (3.24)$$

By using Equation 3.24, Equation 3.23 is rewritten as follows

$$(\mathbf{K}^{-1}\mathbf{M} - \theta_i^{(m)}\mathbf{I})\mathbf{Q}_m \mathbf{s}_i^{(m)} = 0 \quad (3.25)$$

Multiplying by $\mathbf{Q}_m^T \mathbf{M}$ and noting the orthonormal properties of the Lanczos vectors $\mathbf{Q}_m^T \mathbf{M} \mathbf{Q}_m = \mathbf{I}_m$, the eigenproblem of Equation 3.25 is transformed to a tridiagonal form

$$\mathbf{T}_m \mathbf{s}_i^{(m)} - \theta_i^{(m)} \mathbf{s}_i^{(m)} = 0 \quad (3.26)$$

where

$$\mathbf{T}_m = \mathbf{Q}_m^T \mathbf{M} \mathbf{K}^{-1} \mathbf{M} \mathbf{Q}_m \quad (3.27)$$

and $\mathbf{s}_i^{(m)}$ is the eigenvector of the reduced tridiagonal system. As the number of Lanczos vectors increases, the eigenvalues of $\mathbf{T}_m$, $\theta_i^{(m)}$, converge to the eigenvalues of the transformed problem represented by Equation 3.23.

3.4 Loss of orthogonality and convergence

3.4.1 Mechanisms of loss of orthogonality

The loss of orthogonality among Lanczos vectors, thus their linear independence to working accuracy has been attributed in the past to cancellation and accumulation of round-off error for example as in the calculation of $\mathbf{r}_j$ by Equation 3.20. As indicated in the research by Parlett [28] and Simon [29, 30], round-off errors occur when elementary operations are carried out in floating-point computation with finite precision. In fact, if cancellation is the sole cause for the loss of orthogonality, the only measure to remediate the problem would be application of full reorthogonalization scheme after the generation of each Lanczos vector. However, there is another mechanism responsible for the loss of orthogonality in the Lanczos algorithm which has been described in detail by Parlett [28, 31] and Simon [29, 30].

Suppose after j steps of Lanczos algorithm, the vectors $\mathbf{q}_1, \dots, \mathbf{q}_j$ are obtained that are perfectly orthogonal. At the step $j + 1$, a small error in computation occurs such that $\mathbf{q}_{j+1}$ is no longer orthogonal to the previous Lanczos vectors. The algorithm is continued to construct the subsequent Lanczos vectors. Although these subsequent Lanczos vectors are perfectly orthogonal to $\mathbf{q}_{j+1}$ and $\mathbf{q}_j$, they will not be orthogonal to the previous vectors. According to Simon [30], the error once introduced is propagated to the future Lanczos vectors. Thus, the loss of orthogonality among the Lanczos vectors can result from the propagation of local error that occurs during computation. Depending on the distribution of the $\mathbf{q}_j$'s, the error may be amplified in subsequent Lanczos steps.

3.4.2 Accuracy and residual norm

According to Parlett [28], the residual norm

$$\|\mathbf{K}^{-1}\mathbf{M}\mathbf{y} - \theta\mathbf{y}\| \quad (3.28)$$

is a good indicator on the accuracy of the Rayleigh-Ritz approximation pair $(\theta, \mathbf{y})$. Considering that the Ritz vectors are mass normalized, such that $\|\mathbf{y}\|_{\mathbf{M}} = (\mathbf{y}^T \mathbf{M} \mathbf{y})^{1/2} = 1$, the norm of the residual vector given by $\mathbf{r} = \mathbf{K}^{-1}\mathbf{M}\mathbf{y} - \theta\mathbf{y}$ can be employed to provide an estimate on the accuracy of θ by means of the inequality

$$|\nu - \theta| \leq \|\mathbf{r}\|_{\mathbf{M}} \quad (3.29)$$

To check for the convergence of the Ritz vectors, the norm of the residual vector can be calculated from Equations 3.21 and 3.26.

$$\begin{aligned} \|\mathbf{K}^{-1}\mathbf{M}\mathbf{y}_i^{(j)} - \theta_i^{(j)}\mathbf{y}_i^{(j)}\|_{\mathbf{M}} &= \|\mathbf{r}_j \mathbf{e}_j^T \mathbf{s}_i^{(j)}\|_{\mathbf{M}} \\ &= \|\mathbf{r}_j\|_{\mathbf{M}} |\mathbf{e}_j^T \mathbf{s}_i^{(j)}| \end{aligned} \quad (3.30)$$

Finally, Equation 3.30 can be written in the form

$$\|\mathbf{K}^{-1}\mathbf{M}\mathbf{y}_i^{(j)} - \theta_i^{(j)}\mathbf{y}_i^{(j)}\|_{\mathbf{M}} = \beta_{j+1}|s_{ij}| \quad (3.31)$$

where s_{ij} is the bottom element of the normalized eigenvector $\mathbf{s}_i^{(j)}$ of the tridiagonal matrix $\mathbf{T}_j$, and β_{j+1} is a coefficient obtained during the Lanczos process. Equation 3.31 indicates that there is no need to evaluate the Ritz vector $\mathbf{y}_i$ in order to assess convergence. The Ritz vector may be calculated after the required accuracy is reached [27], [28].

3.4.3 Paige theorem

As discussed at the beginning of this chapter, Paige [13, 15] conducted a detailed study on the stability behaviour of the Lanczos algorithm from which significant advances on the understanding of the simple Lanczos algorithm were made. An important finding of the study is the relationship between convergence and the loss of orthogonality. After some steps in the Lanczos algorithm, the mutual orthogonality of the computed Lanczos vectors $\mathbf{Q}_j$ breaks down as

$$\mathbf{Q}_j^T \mathbf{M} \mathbf{Q}_j \neq \mathbf{I}_j \quad (3.32)$$

Paige, instead of evaluating $\mathbf{Q}_j^T \mathbf{M} \mathbf{q}_{j+1}$, monitored the inner product of each Ritz vector $\mathbf{y}_i^{(j)}$ with $\mathbf{q}_{j+1}$

$$\mathbf{y}_i^{(j)T} \mathbf{M} \mathbf{q}_{j+1} = \mathbf{s}_i^{(j)T} \mathbf{Q}_j^T \mathbf{M} \mathbf{q}_{j+1} \quad (3.33)$$

In exact arithmetic the right-hand side of Equation 3.33 is zero, but it is not so in finite precision. Paige [13] in his research discovered the following relation

$$\mathbf{y}_i^{(j)T} \mathbf{M} \mathbf{q}_{j+1} = \frac{\gamma_{ij} \epsilon \|\mathbf{T}_j\|}{\beta_{i+1} |s_{ij}|} \quad (3.34)$$

where s_{ij} is the bottom j^{th} element of the i^{th} eigenvector $\mathbf{s}_i^{(j)}$ of the tridiagonal matrix $\mathbf{T}_j$. ϵ is the precision of the arithmetic. γ_{ij} is a scalar quantity close to unity. The term $\gamma_{ij}\epsilon\|\mathbf{T}_j\|$ reflects the magnitude of the round-off error which is in the order ϵ [8], [28], [31]. The left-hand side of Equation 3.34 is typically near null. The only time that this is not the case is when the denominator approaches ϵ , then it becomes non-zero which indicates the loss of orthogonality among the Lanczos vectors. On the other hand, the term $\beta_{i+1}|s_{ij}|$ is also an indicator of convergence of the Ritz pair, as shown in Equation 3.31. Therefore, it means that the loss of orthogonality occurs when a Ritz eigenvalue is converged to working accuracy.

According to Paige's observation as stated in Paige's theorem, the Lanczos vector $\mathbf{q}_{j+1}$ tilts more towards the most accurate Ritz vector than towards any other converging Ritz vector [31]. Therefore, the orthogonality of the Lanczos vector is lost in the direction of the converging Ritz vector. In this case, the converging Ritz vector must therefore be a linear combination of previous Lanczos vectors. Finally, it can be stated that the loss of orthogonality is initiated when a Ritz value converges.

Loss of orthogonality $\Rightarrow$ convergence of a Ritz value

3.5 Misconvergence

The simplest convergence criterion for the eigenvalue can be derived from the change in the Ritz value $\theta^{(j)}$ at a given step, that is

$$\frac{|\theta^{(j-1)} - \theta^{(j)}|}{|\theta^{(j)}|} \leq tol \quad (3.35)$$

However, such a convergence criterion has been shown to be unreliable. The Ritz converged value that has reached working accuracy can suddenly change in the next few steps. This behaviour is called misconvergence. According to Nour-Omid [27], the situation of misconvergence is avoided when the value $\beta_{i+1}|s_{ij}|$ in Equation 3.31 is used for monitoring the convergence of the Ritz eigenpair. The criterion $\rho_{mi} = \beta_{i+1}|s_{ij}|$ is a more reliable convergence criterion for the Lanczos method.

3.6 Multiple eigenvalues

According to Parlett and Scott [23] and Parlett [28, 31], the simple Lanczos algorithm examines only the subspace spanned by the vectors $\mathbf{q}_1, \mathbf{K}^{-1}\mathbf{M}\mathbf{q}_1, \dots, (\mathbf{K}^{-1}\mathbf{M})^j\mathbf{q}_1$ and in theory it is unable to detect any eigenvector which is orthogonal to $\mathbf{q}_1$. Furthermore, the algorithm can not produce another copy of an eigenvector for a multiple eigenvalue. For an eigenvalue with multiplicity m , it possesses m non-unique eigenvectors and any linear combination of them is also an eigenvector. Now suppose a second eigenvector, obtained as a linear combination of the m non-unique eigenvectors, is also orthogonal to the Lanczos vector $\mathbf{q}_1$. Since the eigenvector is orthogonal to all subsequent Lanczos vectors, the Lanczos algorithm can not find this eigenvector. However, in finite precision, each step of the Lanczos procedure will produce, due to round-off, components in the direction of the required eigenvectors. After one eigenvector of the multiple eigenvalue has been found, the components in other orthogonal directions will grow as the algorithm continues until a second eigenvector which is orthogonal to the first one has converged. Therefore, the Lanczos algorithm in finite precision is capable of finding multiple eigenvectors, although they are found sequentially instead of simultaneously.

3.7 Reorthogonalization procedures

3.7.1 General

In the simple Lanczos algorithm, the orthogonality among Lanczos vectors diminishes after a number of steps. Depending on the level of orthogonality κ_j among the Lanczos vectors, where κ_j at the j^{th} step is expressed as follows

$$\kappa_j = \max_{1 \leq i \leq j-1} |\mathbf{q}_j^T \mathbf{M} \mathbf{q}_i| \quad (3.36)$$

different reorthogonalization strategies can be used.

In general, there are two reorthogonalization approaches. The first one is to preserve the level of orthogonality among Lanczos vectors to working precision ϵ . This approach is represented by the full reorthogonalization procedure. The second approach is the semi-orthogonalization strategy, in which orthogonality is maintained only to the level $\kappa_j \leq \sqrt{\epsilon}$, as shown by theoretical studies and numerical results to be an optimal tolerance. Research results have shown that the use of a more restrictive tolerance $\kappa_j < \sqrt{\epsilon}$ would lead only to an increase in the number of steps where reorthogonalization is required, without gaining any improvement in terms of accuracy.

The full reorthogonalization procedure and three semi-orthogonalization schemes of the selective, partial and combined selective and partial methods, are discussed in this section.

3.7.2 Full reorthogonalization

The procedure is based on the idea of reorthogonalizing the vector $\mathbf{q}_{j+1}$ against all previous Lanczos vectors $\mathbf{Q}_i$ in order to maintain orthogonality among the generated series within the round-off accuracy ϵ . The round-off accuracy ϵ is a unit round-off error which is typically the smallest number that can be represented by the computer, such that $|1 + \epsilon| > 1$. An extra step is added to the simple Lanczos algorithm, as shown in Table 3.2, after the calculation of $\mathbf{r}_j$

$$\mathbf{r}'_j = \mathbf{r}_j - \sum_{k=1}^j (\mathbf{r}_j^T \mathbf{M} \mathbf{q}_k) \mathbf{q}_k \quad (3.37)$$

As a consequence of this change, all $\mathbf{q}$'s must be kept in fast memory and the computation cost of each step becomes very expensive.

3.7.3 Selective orthogonalization

Parlett and Scott [23] introduced the idea of maintaining orthogonality among Lanczos vectors by orthogonalizing the $\mathbf{q}$'s against the few vectors of nearly converged Ritz vectors. They named the procedure selective orthogonalization. In this procedure, the modification to the simple Lanczos algorithm is based on Paige's work [13] and reorthogonalization is applied to a selected group of vectors. As the Lanczos algorithm proceeds, the loss of orthogonality among the Lanczos vectors starts to deteriorate when one of the Ritz vectors begins to converge. The other unconverged Ritz vectors are mutually orthogonal. However, these unconverged vectors and the Lanczos vector $\mathbf{q}_{j+1}$ have large components in the direction of the Ritz vectors that are nearly converged. The growth of the components of the Lanczos vectors along a converged Ritz vector is referred here as the return of a banished Ritz vector.

Thus, the first step of selective orthogonalization is to compute and monitor the parameter presented in Equation 3.31, $\rho_{mi} = \beta_{j+1}|s_{ij}|$ [32]. If $\rho_{mi} \leq \text{tol}$, which indicates the convergence of a Ritz value θ_i , the algorithm pauses and the corresponding Ritz vector $\mathbf{y}_i$ is computed. The component of this vector along $\mathbf{q}_{j+1}$ is then purged. It might also be necessary to orthogonalize $\mathbf{q}_{j+2}$ against the converged Ritz vector $\mathbf{y}_i$. It then follows from the three term recurrence formula given in Equation 3.20 that all subsequent $\mathbf{q}$'s would remain orthogonal to the previous Lanczos vectors.

According to Parlett [28, 31] the state of orthogonality between a converged Ritz vector, $\mathbf{y}_i$, and the current Lanczos vector, $\mathbf{q}_j$, can be measured by the component $\mathbf{y}_i$ along $\mathbf{q}_j$. From the definition of τ_j as

$$\tau_j = \mathbf{y}_i^T \mathbf{M} \mathbf{q}_j \quad (3.38)$$

and multiplying Equation 3.20 by $\mathbf{y}_i^T \mathbf{M}$, the following recurrence relationship for τ_{j+1} can be obtained

$$\beta_{j+1} \mathbf{y}_i^T \mathbf{M} \mathbf{q}_{j+1} \simeq \theta_i \mathbf{y}_i^T \mathbf{M} \mathbf{q}_j - \alpha_j \mathbf{y}_i^T \mathbf{M} \mathbf{q}_j - \beta_j \mathbf{y}_i^T \mathbf{M} \mathbf{q}_{j-1} \quad (3.39)$$

$$\beta_{j+1} \tau_{j+1} \simeq \theta_i \tau_j - \alpha_j \tau_j - \beta_j \tau_{j-1} \quad (3.40)$$

$$\tau_{j+1} \simeq \frac{(\theta_i - \alpha_j) \tau_j - \beta_j \tau_{j-1}}{\beta_{j+1}} \quad (3.41)$$

The three term recurrence for τ_{j+1} is updated at each step with τ_0 set to zero at the first step. Whenever τ_{j+1} becomes greater than the tolerance, $\tau_{j+1} \geq \sqrt{\epsilon}$, it signals that the component of the corresponding Ritz vector $\mathbf{y}_i$ has grown too much along $\mathbf{q}_{j+1}$. The vector $\mathbf{y}_i$ is then explicitly deflated out of $\mathbf{q}_{j+1}$, in other words orthogonalized against it. After orthogonalization, τ_{j+1} is set back to ϵ . The procedure of the modified simple Lanczos algorithm with the selective orthogonalization scheme is summarized in Figure 3.1.

3.7.4 Partial reorthogonalization

Partial orthogonalization procedure was proposed by Simon [24, 25] as an alternative semi-orthogonalization scheme. The level of orthogonality of the vector $\mathbf{q}_{j+1}$ with the previous Lanczos vectors is maintained by monitoring the inner product of $\mathbf{q}_{j+1}\mathbf{M}\mathbf{Q}_j$. When it is necessary the vector $\mathbf{q}_{j+1}$ is reorthogonalized against the Lanczos vectors $\mathbf{Q}_j$. In partial reorthogonalization, convergence of a Ritz value is the mechanism of the loss of orthogonality that requires the removal of the components of the previous Lanczos vectors in $\mathbf{q}_{j+1}$.

In finite precision computation, a round-off error is introduced at each step. The three term recurrence formula at the j^{th} step, expressed by Equation 3.20, is modified as follows

$$\beta_{j+1}\mathbf{q}_{j+1} = \mathbf{K}^{-1}\mathbf{M}\mathbf{q}_j - \alpha_j\mathbf{q}_j - \beta_j\mathbf{q}_{j-1} - \mathbf{f}_j \quad (3.42)$$

where the vector $\mathbf{f}_j$ accounts for the rounding error at the j^{th} step. This error is usually small but it will cause the orthogonality relation

$$\mathbf{Q}_j^T\mathbf{M}\mathbf{Q}_j = \mathbf{I}_j \quad (3.43)$$

to fail completely after a while.

Defining w_{ji} as

$$w_{ji} = \mathbf{q}_j^T\mathbf{M}\mathbf{q}_i \quad (3.44)$$

where

$$w_{jj} = 1 \quad \text{for } j = 1, \dots, i \quad (3.45)$$

and

$$w_{j(j-1)} = \mathbf{q}_j^T\mathbf{M}\mathbf{q}_{j-1} \quad \text{for } j = 2, \dots, i \quad (3.46)$$

After transformation procedures are performed using Equation 3.42, the following expression is obtained

$$\beta_{j+1}w_{(j+1)i} = \beta_{i+1}w_{j(i+1)} + (\alpha_i - \alpha_j)w_{ji} + \beta_iw_{j(i-1)} - \beta_jw_{(j-1)i} + \mathbf{q}_j^T \mathbf{M} \mathbf{f}_i - \mathbf{q}_i^T \mathbf{M} \mathbf{f}_j \quad (3.47)$$

for $1 \leq i < j$, where $w_{j(i+1)} = w_{(i+1)j}$ and also $w_{i0} \equiv 0$. It can be noticed from the above equation that the inner product $w_{(j+1)i} = \mathbf{q}_{j+1}^T \mathbf{M} \mathbf{q}_i$ is the summation of the weighted inner products from the previous two Lanczos steps and the round-off terms $\mathbf{q}_j^T \mathbf{M} \mathbf{f}_i - \mathbf{q}_i^T \mathbf{M} \mathbf{f}_j$. It was noticed by Simon [29, 30] that the growth of $|\mathbf{q}_j^T \mathbf{M} \mathbf{q}_i|$ follows a certain pattern and by no means is irregular or random. It can be shown that the error introduced at step j will be amplified through subsequent steps to a such extent that orthogonality is lost among Lanczos vectors.

Whenever the term w_{ji} approaches the threshold, $w_{ji} \geq \sqrt{\epsilon}$, the round-off error terms in Equation 3.47 $\mathbf{q}_j^T \mathbf{M} \mathbf{f}_i - \mathbf{q}_i^T \mathbf{M} \mathbf{f}_j$, which are at the full machine precision level ϵ , do not contribute significantly to the total value of $w_{(j+1)i}$. These round-off terms are only important for very small w_{ji} . When the round-off terms are neglected, Equation 3.47 becomes

$$\beta_{j+1}w_{(j+1)i} = \beta_{i+1}w_{j(i+1)} + (\alpha_i - \alpha_j)w_{ji} + \beta_iw_{j(i-1)} - \beta_jw_{(j-1)i} \quad (3.48)$$

It still gives a good estimate to $w_{(j+1)i}$ and is also a very good indicator when the orthogonality is lost.

Simon [29] conducted some study and analysis regarding the above expression. He was the first to apply Equation 3.48 in a reorthogonalization process. Equation 3.48 can also be written in a vector form

$$\beta_{j+1}\mathbf{w}_{j+1} = \mathbf{T}_j\mathbf{w}_j - \alpha_j\mathbf{w}_j - \beta_j\mathbf{w}_{j-1} \quad (3.49)$$

where $\mathbf{w}_{j+1} = \mathbf{q}_{j+1}^T \mathbf{M} \mathbf{Q}_j$ and $\mathbf{T}_j$ is a tridiagonal matrix from the Lanczos algorithm. Hence, the computation of the estimates $\mathbf{w}_j$ involves only the simple updating procedure for two vectors of length j . Furthermore, it is not necessary to carry out orthogonalization at every step, so that the overall amount of computation for the reorthogonalization process is reduced.

Simon [30] suggested after Parlett [28, 31] to perform the Gram-Schmidt reorthogonalization at the j^{th} and the $(j+1)^{\text{th}}$ steps whenever $\|\mathbf{q}_{j+1}^T \mathbf{M} \mathbf{Q}_j\| \geq \sqrt{\epsilon}$. The reason for this additional reorthogonalization comes from observation that at step j if the three term recurrence formula, Equation 3.49, indicates need of reorthogonalization, then the Lanczos vector $\mathbf{q}_{j+1}$ should be orthogonalized against all previous vectors $\mathbf{Q}_j$. From Equation 3.48, the magnitude of $w_{(j+1)i}$ after reorthogonalization can be assumed to be in the order of ϵ , $w_{(j+1)i} = \mathbf{q}_{j+1}^T \mathbf{M} \mathbf{q}_i = O(\epsilon)$ for all $i \leq j$. In the next step, the following expression can be derived

$$\beta_{j+2} w_{(j+2)i} = -\beta_{j+1} w_{ji} + O(\epsilon) \quad (3.50)$$

As can be observed from Equation 3.48, if the value $|w_{(j+1)i}| \geq \sqrt{\epsilon}$ before the reorthogonalization, the term w_{ji} which gives an estimate of the level of orthogonality of the Lanczos vector $\mathbf{q}_j$ with the previous Lanczos vectors is likely to be of comparable magnitude in the order of $\sqrt{\epsilon}$. Consequently, Equation 3.50 indicates that one reorthogonalization will not necessarily help to reduce the magnitude of $w_{(j+2)i}$ to the round-off level because of the contribution from w_{ji} . Therefore, two successive reorthogonalizations at steps j and $j+1$ are needed to ensure

$$\beta_{j+2} w_{(j+2)i} = O(\epsilon) \quad (3.51)$$

It can then be safely assumed that for at least the next few Lanczos steps the generated Lanczos vectors will remain orthogonal.

The three term recurrence formula. Equation 3.49, can be used to determine the level of orthogonality among the Lanczos vectors. The procedure is a very economical because it avoids the explicit calculation of the inner product w_{ji} , resulting in considerable saving in the amount of computation. It is worth while to mention that in order to start the recurrence, w_0 is set to zero and $w_1 = q_2^T M q_1$. The flowchart of the simple Lanczos algorithm modified to accommodate the partial reorthogonalization procedure is shown in Figure 3.2.

3.7.5 Combined selective and partial reorthogonalization

Both reorthogonalization schemes, selective and partial, adopt the semiorthogonality condition and neither method seems to have a clear advantage over the other. Parlett and Nour-Omid [26] introduced the idea of combining the two methods in their implementation of the Lanczos algorithm, making the orthogonalization procedure more reliable. In the early stage of a Lanczos run, the selective scheme gives accurate results when only one or two Ritz values converge. In subsequent steps, the reorthogonalization procedure becomes complicated as the calculation of the Lanczos vectors progress when a number of Ritz values converge at the same time resulting in the loss of semiorthogonality.

Nour-Omid [27] suggests to monitor the magnitude of the elements in the vector w_{j+1} . If the largest absolute value is less than $\sqrt{\epsilon}$, no action is undertaken regarding reorthogonalization. Otherwise, because there are two possible cases that may be responsible for the loss of orthogonality, two different reorthogonalization procedures are adopted depending on the reason of the loss of orthogonality

- (1) orthogonalization against previous Lanczos vectors due to convergence of a Ritz value.
- (2) orthogonalization against Ritz vectors due to growth of components of converged eigenvectors along the Lanczos vector.

In the event of a converged Ritz vector in case 2, the three term recurrence for τ_{j+1} is also updated to determine if the loss of orthogonality occurs due to the growth of the Ritz vector along the Lanczos vector.

If the absolute value of some τ_{j+1} is greater than the level of semiorthogonality, $\sqrt{\epsilon}$, the new Lanczos vector is orthogonalized against those Ritz vectors whose components have grown too much. However, because the orthogonalization of $\mathbf{q}_{j+1}$ against Ritz vectors $\mathbf{y}_k$ changed the state of orthogonality among the Lanczos vectors, the vector $\mathbf{w}_{j+1}$ also has to be modified. The modified Lanczos vector $\bar{\mathbf{q}}_{j+1}$ is expressed as follows

$$\bar{\mathbf{q}}_{j+1} = \mathbf{q}_{j+1} - \mathbf{y}_k^T \mathbf{M} \mathbf{q}_{j+1} \mathbf{y}_k \quad (3.52)$$

Thus.

$$\begin{aligned} \bar{\mathbf{w}}_{j+1} &= \mathbf{Q}_j^T \mathbf{M} \bar{\mathbf{q}}_{j+1} \\ &= \mathbf{Q}_j^T \mathbf{M} (\mathbf{q}_{j+1} - \mathbf{y}_k^T \mathbf{M} \mathbf{q}_{j+1} \mathbf{y}_k) \\ &= \mathbf{w}_{j+1} - \mathbf{y}_k^T \mathbf{M} \mathbf{q}_{j+1} \mathbf{s}_k \end{aligned} \quad (3.53)$$

and since

$$\mathbf{y}_k^T \mathbf{M} \mathbf{q}_{j+1} = \mathbf{s}_k^T \mathbf{Q}_j^T \mathbf{M} \mathbf{q}_{j+1} = \mathbf{s}_k^T \mathbf{w}_{j+1} \quad (3.54)$$

Equation 3.53 becomes

$$\bar{\mathbf{w}}_{j+1} = \mathbf{w}_{j+1} - \mathbf{s}_k^T \mathbf{w}_{j+1} \mathbf{s}_k \quad (3.55)$$

It can be observed from the above equation that the modification to the vector $\mathbf{w}_{j+1}$ is equal to the orthogonalization of $\mathbf{w}_{j+1}$ against the eigenvector of the tridiagonal matrix $\mathbf{T}_{j+1}$.

The elements of the modified vector $\bar{\mathbf{w}}_{j+1}$ are examined again and if any one of them is greater than $\sqrt{\epsilon}$, then the loss of orthogonality is also due to the convergence of a Ritz value and the orthogonalization procedure with respect to the previous Lanczos vectors is performed. Figure 3.3 gives a flow chart on how the two reorthogonalization methods can be used together in the reorthogonalization process.

In the conclusion to this chapter, it can be said that the Lanczos algorithm is based on a three term recurrence relation to construct the set of orthogonal transformation vectors called Lanczos vectors. The recurrence formula is very economical to evaluate the reduction of the eigenproblem to a tridiagonal form, which gives good approximations to a few eigenvalues of a large sparse matrix system.

Generally, the Lanczos method considered as a Rayleigh-Ritz procedure is more efficient than the subspace iteration method. The main difference between the Lanczos method and the subspace iteration method is that in the Lanczos algorithm each step of the iteration sequence yields one Lanczos vector, whereas in the subspace iteration method the iteration step serves only to improve the approximation to the true vibration mode shape. The Lanczos method is vulnerable to numerical instability, which leads to the loss of orthogonality between the Lanczos vectors. This situation can not in general be prevented because as Paige [13] - [15] has shown. the loss of orthogonality is also a signal of convergence of a Ritz value. As a preventive measure to avoid the loss of orthogonality, a costly full reorthogonalization

procedure has been used. Since then, the semi-orthogonalization scheme has been developed and found to be a very effective tool to deal with the problem.

The main reason behind the introduction of the semi-orthogonalization procedure is to maintain a strong linear independence among the Lanczos vectors while at the same time to reduce the number of reorthogonalization steps as compared to the full orthogonalization scheme. Parlett [28] and Scott [32] have stated that the semiorthogonality is sufficient requirement in finite precision arithmetic to provide improvement to the simple Lanczos algorithm. It has the advantage that the calculation of redundant Ritz pairs is avoided. Furthermore, the number of required Lanczos steps is kept to the minimum and multiple eigenvalues can be found in a sequential order.

Both full and partial orthogonalization procedures orthogonalize with respect to the Lanczos vectors. Furthermore, partial orthogonalization may be considered as a relaxed or less restrictive version of full orthogonalization. Selective orthogonalization is an entirely different approach, it orthogonalizes with the converging Ritz vectors. The direct theoretical comparison of selective and partial reorthogonalization methods is rather difficult, but both methods have been proved to be equally effective.

Table 3.1: The Krylov subspace

$k = 1$	2	...	i	$i + 1$	...	j	$j + 1$
ϑ_0	ϑ_1	...	ϑ_{i-1}	ϑ_i	...	ϑ_{j-1}	ϑ_j
$\mathbf{r}$	$\mathbf{K}^{-1}\mathbf{M}\mathbf{r}$	...	$(\mathbf{K}^{-1}\mathbf{M})^{i-1}\mathbf{r}$	$(\mathbf{K}^{-1}\mathbf{M})^i\mathbf{r}$	...	$(\mathbf{K}^{-1}\mathbf{M})^{j-1}\mathbf{r}$	$(\mathbf{K}^{-1}\mathbf{M})^j\mathbf{r}$
$\mathbf{r}_0$	$\mathbf{r}_1$	...	$\mathbf{r}_{i-1}$	$\mathbf{r}_i$	...	$\mathbf{r}_{j-1}$	$\mathbf{r}_j$
$\mathbf{q}_1$	$\mathbf{q}_2$	...	$\mathbf{q}_i$	$\mathbf{q}_{i+1}$	...	$\mathbf{q}_j$	$\mathbf{q}_{j+1}$
$\mathbf{q}_1$	$\mathbf{K}^{-1}\mathbf{M}\mathbf{q}_1$	...	$\mathbf{K}^{-1}\mathbf{M}\mathbf{q}_{i-1}$	$\mathbf{K}^{-1}\mathbf{M}\mathbf{q}_i$	...	$\mathbf{K}^{-1}\mathbf{M}\mathbf{q}_{j-1}$	$\mathbf{K}^{-1}\mathbf{M}\mathbf{q}_j$

Table 3.2: The simple Lanczos algorithm

1. Set $\mathbf{q}_0 = 0$

2. Select a starting iteration vector $\mathbf{r}_0$ and calculate

$$\beta_1 = (\mathbf{r}_0^T \mathbf{M} \mathbf{r}_0)^{1/2}$$

$$\mathbf{q}_1 = \frac{\mathbf{r}_0}{\beta_1}$$

3. For $j = 1, \dots, n$

$$\bar{\mathbf{r}}_j = \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j$$

$$\alpha_j = \bar{\mathbf{r}}_j^T \mathbf{M} \mathbf{q}_j$$

$$\mathbf{r}_j = \bar{\mathbf{r}}_j - \alpha_j \mathbf{q}_j - \beta_j \mathbf{q}_{j-1}$$

$$\beta_{j+1} = (\mathbf{r}_j^T \mathbf{M} \mathbf{r}_j)^{1/2}$$

$$\mathbf{q}_{j+1} = \frac{\mathbf{r}_j}{\beta_{j+1}}$$

Figure 3.1: Lanczos algorithm with selective reorthogonalization

Figure 3.2: Lanczos algorithm with partial reorthogonalization

Figure 3.3: Combine selective and partial reorthogonalization procedure

Chapter 4

Dynamic Response Analysis

4.1 General

The Rayleigh-Ritz method introduced in chapter 2 to obtain an approximate eigensolution can also be employed in dynamic response analysis of a structural system. Recently because of many advantages and efficiency of the method, the Ritz analysis procedure has become more popular as a very important tool for studying the behaviour of structures under dynamic loads.

Traditionally, the dynamic response of a linear multi-degree-of-freedom (MDOF) system can be obtained by the modal superposition method. The advantage of this method is that the size of the original problem is reduced as the dynamic response can usually be described adequately by the contributions from only the first few lower modes. The determination of the dynamic response of a structural system even though it may have hundreds of degrees-of-freedom is reduced to the

solution of a set of only a few uncoupled modal equations by using the mode shapes to discretize the behaviour of the system. In the context of the Ritz method of analysis, the vibration mode shapes can be considered a special type of Rayleigh-Ritz vectors with the standard mode superposition method considered as a form of Ritz analysis. For a large complex system, the need to obtain the eigenvectors for use in the modal analysis may be very inefficient. Furthermore, the determination and use of the mode shapes do not take into account the spatial distribution of the dynamic load. This is a major advantage of the Rayleigh-Ritz procedure as compared to the standard mode superposition method.

In comparison, Rayleigh-Ritz vectors are required only to satisfy the geometric boundary conditions of the problem, whereas mode shapes must satisfy not only the geometric boundary conditions but also more rigorous conditions in that they are also solutions to the eigenvalue problem of the structural system. Therefore, it is much less costly to generate a set of Rayleigh-Ritz vectors for the reduction of the system than the mode shapes. The disadvantages are the resulting transformed matrices are no longer uncoupled and usually full. However, for a large complex system, the advantage of the Rayleigh-Ritz method usually more than compensates for the disadvantage. A special form of Rayleigh-Ritz procedure is so called load-dependent Ritz vector algorithms which transform the dynamic problem, as expressed in Equation 2.1, to the new system expressed in Equation 2.17. The use of the load-dependent Ritz vectors as generalized coordinates in the modelling of the problem improves numerical efficiency and reduces the computational effort required in the analysis. Typically, fewer Ritz vectors are needed to describe the dynamic response of the system with the same or even higher level of accuracy in the results than required by the modal analysis method. This is in addition to the advantage

that the need for the solution of the large eigenvalue problem is avoided.

After a brief introduction to modal superposition analysis in earthquake response, the various load-dependent Ritz vectors algorithms are presented in this chapter. The concept of the derived Ritz vectors (DRV) obtained from the Lanczos algorithm, where the starting vector is a static displacement vector, is presented. Next, discussions on the original Wilson's method (WYD) and improved Wilson's method (LWYD) to generate load-dependent Ritz vectors are followed. Finally, the similarities and differences among the different Ritz vector algorithms and analysis procedures are discussed.

4.2 Modal analysis of earthquake response

The modal superposition method is considered a special case of the Ritz method, therefore, a summary of modal analysis procedures presented below can be useful for later discussions of Ritz analysis.

The standard mode superposition method is based on the transformation of the geometric displacement $\mathbf{u}(s, t)$ of the considered structural system to a generalized form

$$\mathbf{u}(s, t) = \Phi(s)\mathbf{z}(t) \quad (4.1)$$

In Equation 4.1, the matrix Φ constitutes the set of mode shapes (eigenvectors) which serve to transform the generalized coordinate vector $\mathbf{z}(t)$ to the displacement vector $\mathbf{u}(s, t)$ in the geometric coordinates. Substituting Equation 4.1 into the ma-

trix equation of motion for a structural system given in Equation 2.1 yields

$$\mathbf{M}\Phi\ddot{\mathbf{z}} + \mathbf{C}\Phi\dot{\mathbf{z}} + \mathbf{K}\Phi\mathbf{z} = \mathbf{p}(t) \quad (4.2)$$

Multiplying both sides of Equation 4.2 by Φ^T , the following equation is obtained

$$\mathbf{M}^*\ddot{\mathbf{z}} + \mathbf{C}^*\dot{\mathbf{z}} + \mathbf{K}^*\mathbf{z} = \mathbf{p}^* \quad (4.3)$$

where $\mathbf{M}^*$, $\mathbf{C}^*$, $\mathbf{K}^*$ are respectively the generalized mass, damping and stiffness matrices, $\mathbf{p}^*$ is the generalized external force vector, defined as follows

$$\begin{aligned} \mathbf{M}^* &= \Phi^T \mathbf{M} \Phi \\ \mathbf{C}^* &= \Phi^T \mathbf{C} \Phi \\ \mathbf{K}^* &= \Phi^T \mathbf{K} \Phi \\ \mathbf{p}^* &= \Phi^T \mathbf{p} \end{aligned} \quad (4.4)$$

Special considerations are required for damped systems in the representation of the damping matrix $\mathbf{C}$. The definition of the damping matrix $\mathbf{C}$ usually presents some difficulties due to the complexity of the damping mechanisms involved in a structural system. To simplify the problem, assumptions are usually made concerning the damping mechanism in that the damping force is taken proportional to the velocity as in viscous damping. Moreover, the damping matrix is assumed to be proportional to $\mathbf{M}$ and $\mathbf{K}$ where $\mathbf{C}$ can be expressed as follows

$$\mathbf{C} = a_0 \mathbf{M} + a_1 \mathbf{K} \quad (4.5)$$

The damping having the property of Equation 4.5 is called Rayleigh damping [1], [3]. The constant coefficients a_0 and a_1 are respectively the proportionality constants with $\mathbf{M}$ and $\mathbf{K}$ of the structural system.

In general, the time-varying load vector $\mathbf{p}(t)$ can be written as the product of a spatially distributed loading function $\mathbf{f}(s)$, which remains unchanged with time, and a time-varying scalar amplitude function $g(t)$ as follows

$$\mathbf{p}(s, t) = \mathbf{f}(s)g(t) \quad (4.6)$$

Using Equation 4.6 for the load vector, Equation 4.3 becomes

$$\mathbf{M}^* \ddot{\mathbf{u}} + \mathbf{C}^* \dot{\mathbf{u}} + \mathbf{K}^* \mathbf{u} = \Phi^T \mathbf{f}(s)g(t) \quad (4.7)$$

After solving the above equation for the response of each modal coordinate, the response in the original geometric coordinates can be obtained from Equation 4.1.

In modal response analysis, the number of excited mode shapes L is usually much less than the size of the problem. As a measure of the degree of participation of a particular mode in the dynamic response of a structural system, the modal participation factor can be used which is defined as follows

$$\Gamma_i = \frac{\phi_i^T \mathbf{f}(s)}{\phi_i^T \mathbf{M} \phi_i} \quad (4.8)$$

Alternatively, another important measure is an estimate error in the representation of the dynamic load. The spatial distribution of the dynamic load, $\mathbf{f}(s)$, can be expressed in terms of the mode shapes as follows

$$\mathbf{f} = \sum_{i=1}^N \phi_i \mathbf{f}^T \mathbf{M} \phi_i \quad (4.9)$$

with the mode shape ϕ_i normalized to unity with respect to the mass matrix. In mode superposition analysis, a truncated series of L mode shapes is used with $L < N$. The higher order components of the load as presented by the higher modes of mode $(L + 1)$ to mode N are omitted. Therefore, an error vector in the load representation $\mathbf{e}_L$ exists, which is given as follows

$$\mathbf{e}_L = \mathbf{f} - \sum_{i=1}^L \phi_i \mathbf{f}^T \mathbf{M} \phi_i \quad (4.10)$$

Consequently, an error norm ε can be defined as follows

$$\varepsilon = \frac{|\mathbf{f}^T \mathbf{e}_L|}{\mathbf{f}^T \mathbf{f}} \quad (4.11)$$

which can be used as a cut-off criterion for determining the number of mode shapes that should be included in modal analysis. When the number of mode shapes L used in the modal analysis increases, ε_L becomes small and at some point the higher modes can be truncated.

The support motions during strong ground shaking of an earthquake induce dynamic vibration in the structural system. The equations of motion of a MDOF system can be expressed as follows

$$\mathbf{M}\ddot{\mathbf{u}}^t + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0 \quad (4.12)$$

where $\mathbf{u}^t$ represents the vector of the total displacement which is equal to the sum of the relative displacements $\mathbf{u}(t)$ and a rigid body displacements $\mathbf{u}_s(t)$ of the system

$$\mathbf{u}^t(t) = \mathbf{u}(t) + \mathbf{u}_s(t) \quad (4.13)$$

The rigid body displacements, also known as the quasi-static displacements, can be expressed in terms of an influence coefficient vector $\mathbf{r}$ which represents the displacements of structure due to a unit displacement of the support, and the displacement of support $u_g(t)$ due to ground motion.

$$\mathbf{u}_s(t) = \mathbf{r}u_g(t) \quad (4.14)$$

and

$$\mathbf{u}^t(t) = \mathbf{u}(t) + \mathbf{r}u_g(t) \quad (4.15)$$

Therefore, the equation of motion for a MDOF system has the form

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = -\mathbf{M}\ddot{\mathbf{u}}_s = -\mathbf{M}\mathbf{r}\ddot{u}_g \quad (4.16)$$

The effective force vector as generated by the earthquake ground motion is

$$\mathbf{p}_{eff}(t) = -\mathbf{M}\mathbf{r}\ddot{u}_g(t) \quad (4.17)$$

Similar to the generalized loading case in Equation 4.8, the modal participation factor for earthquake loading can be defined as follows

$$\Gamma_i = \frac{\phi_i^T \mathbf{M} \mathbf{r}}{\phi_i^T \mathbf{M} \phi_i} \quad (4.18)$$

Equations 4.9 to 4.11, which give the error estimates, are also valid for the case of earthquake loading in which the spatial load vector $\mathbf{f}(s)$ is expressed in terms of the structure mass matrix $\mathbf{M}$ and the distribution vector $\mathbf{r}$.

$$\mathbf{f}(s) = \mathbf{M} \mathbf{r} \quad (4.19)$$

The error norm in Equation 4.11 for earthquake loading thus becomes

$$\varepsilon = \frac{\mathbf{r}^T \mathbf{M} \mathbf{r} - \sum_{i=1}^L (\phi_i^T \mathbf{M} \mathbf{r})^2}{\mathbf{r}^T \mathbf{M} \mathbf{r}} \quad (4.20)$$

When the mass matrix $\mathbf{M}$ is diagonal, the term $\mathbf{r}^T \mathbf{M} \mathbf{r}$ represents the total mass of the system that participates in the dynamic response due to the support motion. the term $\sum_{i=1}^L (\phi_i^T \mathbf{M} \mathbf{r})^2$ represents the part of the system mass considered in the modal analysis.

The dynamic equations of motion of the system can be solved by a number of numerical schemes, including the Duhamel integral method and the direct time step integration procedure. In the direct time integration approach, the Newmark β methods which include the average and linear acceleration methods have been widely used in time history analysis. The accuracy of the Newmark β methods very much depends on the size of the time step chosen for the numerical integration. Details of these methods are described in the reference by Newmark [33].

4.3 Load dependent Ritz vectors

In the context of the present research, the mode superposition method can be looked upon as a special case of the Rayleigh-Ritz method in which the mode shapes are the Ritz vectors employed in the transformation of the original system. Within the context of the Rayleigh-Ritz method, it is not necessary to employ the very restrictive eigenvectors to discretize the system which lead to an uncoupled set of dynamic equations of motion. Numerical studies have shown that good approximation to the dynamic response and behaviour of the system can still be obtained by the use of less rigorous Ritz vectors for the transformation and reduction of the dynamic system.

The concept of the recently developed load-dependent Ritz vectors is presented here and two procedures, the Lanczos and Wilson's methods, for the generation of this special type of Ritz vectors are also discussed. Both, the Lanczos and Wilson's algorithms, are based on the idea of constructing a M -orthonormal set of vectors which takes into consideration the static effect due to the spatial distribution of the dynamic loads.

4.3.1 Derived Ritz vectors

The Lanczos vectors discussed in chapter three may be considered as a set of Ritz vectors. The orthogonality properties of the Lanczos vectors lead to a tridiagonal form of equations of motion for the transformed generalized dynamic problem. When the initial vector in the Lanczos algorithm is taken as the static displacement vector

of the system obtained from application of the spatially distributed dynamic loads as static loads, the constructed Lanczos coordinates are called the derived Ritz vectors (DRV) after Nour-Omid and Clough [34].

The derived Ritz vectors have the important characteristic that the first DRV vector represents a static response to the dynamic load so that there is no need for a static correction in the analysis. In present discussion, the symbol ψ denotes the DRV vectors so that Equation 3.27 has the form

$$\mathbf{T}_i = \Psi_i^T \mathbf{M} \mathbf{K}^{-1} \mathbf{M} \Psi_i \quad (4.21)$$

Substituting

$$\mathbf{u}(t) = \Psi \mathbf{z}(t) \quad (4.22)$$

and multiplying Equation 2.1 by $\Psi^T \mathbf{M} \mathbf{K}^{-1}$, it leads to

$$\Psi_i^T \mathbf{M} \mathbf{K}^{-1} \mathbf{M} \Psi_i \ddot{\mathbf{z}}(t) + \Psi_i^T \mathbf{M} \mathbf{K}^{-1} \mathbf{C} \Psi_i \dot{\mathbf{z}}(t) + \Psi_i^T \mathbf{M} \Psi_i \mathbf{z}(t) = \mathbf{p}_i(t) \quad (4.23)$$

where

$$\mathbf{p}_i = \Psi_i^T \mathbf{M} \mathbf{K}^{-1} \mathbf{p}(t) \quad (4.24)$$

Assuming the damping can be represented as a linear combination of the mass and stiffness matrices as defined in Equation 4.5, the equation of motion given in Equation 4.23 can be written in a tridiagonal form

$$\mathbf{T}_i \ddot{\mathbf{z}} + [a_0 \mathbf{T}_i + a_1 \mathbf{I}_i] \dot{\mathbf{z}} + \mathbf{z} = \mathbf{p}_i \quad (4.25)$$

When the applied load has the form of Equation 4.6 and considering the starting vector as the static displacement due to $\mathbf{f}(s)$

$$\mathbf{r}_0 = \beta_1 \psi_1 = \mathbf{K}^{-1} \mathbf{f}(s) \quad (4.26)$$

the forcing function $\mathbf{p}_i$ can be expressed as follows

$$\begin{aligned}
 \mathbf{p}_i &= \Psi_i^T \mathbf{M} \mathbf{K}^{-1} \mathbf{p}(t) \\
 &= \Psi_i^T \mathbf{M} \mathbf{K}^{-1} \mathbf{f}(s) g(t) \\
 &= \Psi_i^T \mathbf{M} \psi_1 \beta_1 g(t) \\
 &= \Psi_i^T \mathbf{M} \Psi_i \mathbf{e}_1 \beta_1 g(t) \\
 &= \beta_1 \mathbf{e}_1 g(t)
 \end{aligned} \tag{4.27}$$

where β_1 is the off-diagonal element of a tridiagonal matrix $\mathbf{T}_i$ and $\mathbf{e}_1$ is the first column of the identity matrix $\mathbf{I}_i$. Consequently, Equation 4.25 has the form

$$\mathbf{T}_i \ddot{\mathbf{z}}(t) + [a_0 \mathbf{T}_i + a_1 \mathbf{I}_i] \dot{\mathbf{z}} + \mathbf{z}(t) = \beta_1 \mathbf{e}_1 g(t) \tag{4.28}$$

which shows that only the first Ritz coordinate is subjected directly to the excitation. Further, the orthogonality condition eliminates any direct excitation to the other coordinates. But they are put into motion by their tridiagonal coupling to the adjacent Ritz vectors.

The solution to the equations of motion given by Equation 4.25 or Equation 4.28 may be obtained either through the step-by-step integration method or by first solving the eigenproblem and then use the eigenvectors of $\mathbf{T}_i$ in a mode superposition analysis. In order to determine how many DRV vectors are needed to obtain the required accuracy in a dynamic response analysis, a participation factor is calculated as follows

$$\gamma_j = \psi_j^T \mathbf{f}. \tag{4.29}$$

The participation factor associated with each DRV vector may be obtained through the recurrence relationship. Multiplying Equation 3.20 by $\mathbf{f}$ gives

$$\beta_{j+1} \mathbf{f}^T \psi_{j+1} = \mathbf{f}^T \mathbf{K}^{-1} \mathbf{M} \psi_j - \alpha_j \mathbf{f}^T \psi_j - \beta_j \mathbf{f}^T \psi_{j-1} \tag{4.30}$$

Furthermore, the first term on the right-hand side of Equation 4.30 disappears because from Equation 4.27

$$\psi_j^T \mathbf{M} \mathbf{K}^{-1} \mathbf{f} = \psi_j^T \mathbf{M} \psi_1 \beta_1 \quad (4.31)$$

and thus for $j \neq 1$

$$\mathbf{f}^T \mathbf{K}^{-1} \mathbf{M} \psi_j = \beta_1 \psi_1^T \mathbf{M} \psi_j = 0 \quad (4.32)$$

Consequently, Equation 4.30 has the new form

$$\beta_{j+1} \mathbf{f}^T \psi_{j+1} = -\alpha_j \mathbf{f}^T \psi_j - \beta_j \mathbf{f}^T \psi_{j-1} \quad (4.33)$$

Now, from the definition of the participation factor given in Equation 4.29, a recurrence formula is obtained from Equation 4.33 as follows

$$\gamma_{j+1} = -\frac{(\alpha_j \gamma_j + \beta_j \gamma_{j-1})}{\beta_{j+1}}. \quad (4.34)$$

The participation factor γ_{j+1} , for $j \geq 2$, can be calculated at a negligible cost during each step of the algorithm. The starting value γ_1 is equal to $\psi_1^T \mathbf{f}$.

The DRV algorithm is terminated when the magnitude of the participation factor of the latest vector drops below a specified tolerance [1].

Another cut-off criterion may be based on the error vector defined by Equation 4.10. Replacing the mode shapes by DRV vectors in Equation 4.10, the error vector $\mathbf{e}_L$ is obtained as follows

$$\mathbf{e}_L = \mathbf{f} - \sum_{i=1}^L \psi_i \mathbf{f}^T \mathbf{M} \psi_i \quad (4.35)$$

The error vector in load representation $\mathbf{e}_L$ given in Equation 4.35 can be used to determine the number of Ritz vectors required in the analysis.

In the standard mode superposition method, the level of participation of a particular eigenvector in the dynamic response analysis depends on the characteristics of the dynamic loading. In particular, the spatial distribution of the dynamic load plays an important role. The eigenvectors that are nearly orthogonal to the loading vector are negligibly excited. They contribute very little to the total response of the system. The effort involved in the computation of those mode shapes unnecessarily increase the cost of the analysis.

Finally, it should be noticed that the DRV vectors provides a more realistic transformation basis for the reduction of a large structural system by recognizing the spatially distribution of the applied loads and thus fewer number of DRV vectors than the number of mode shapes are usually needed to reach an adequate level of accuracy in the analysis.

4.3.2 Wilson's Ritz vectors

Another similar Ritz analysis approach for dynamic response analysis which generates a set of Ritz vectors that also takes into consideration the spatial distribution of the applied load was introduced by Wilson *et al.* [2]. The vectors obtained by Wilson in his method have been referred to as the load-dependent Ritz vectors. Wilson's method is mathematically similar to the Lanczos algorithm. The vectors generated by both methods span the same Krylov subspace. In the first Wilson's method, each newly generated Wilson's Ritz vector is orthogonalized against all preceding Wilson's Ritz vectors, whereas in the Lanczos algorithm the orthogonalization procedure, which is based on the three term recurrence formula given in Equation 3.20, affects only the two preceding Lanczos vectors.

The original Wilson algorithm, also called the WYD method which stands for the developers of the method Wilson, Yuan and Dickens [2], is based on the idea of selecting an orthonormalized Ritz basis in the Krylov space that accounts for the spatial distribution of the load. The method employs Gram-Schmidt orthogonalization to obtain a set of orthogonal vectors.

Examining the algorithm of the WYD method as shown in Table 4.1, it is noted that the first Wilson's Ritz vector represents a static response to the load vector f while ignoring the dynamic effect. In the second step, a load vector proportional to the dynamic inertia force resulting from the first Ritz vector of the form $\omega^2 Mx_1$ is applied to generate the next Ritz vector. The static solution at each step of the Wilson's algorithm neglects the dynamic effect which is later used as the load vector in the next step for the generation of the next load dependent Ritz vector. As such, the generated vectors consist of vectors excited by the input loading. Thus, the derived Ritz vectors or Wilson's Ritz vectors represent the major part of the response, whereas in the case of modal analysis using the exact modes, which are determined independent of the loading, there is no guarantee that the included modes actually participate in the dynamic response of the system.

From previous studies and applications, the Gram-Schmidt orthogonalization procedure is sometimes known to be numerically unstable. In view of this, the original WYD algorithm has been modified by Leger [3] with improvement on the numerical stability of the algorithm. The improvement at each step is achieved by using an improved static vector obtained from the one-step inverse iteration procedure instead of the previous Wilson's vector, x_{i-1} . The algorithm of this modified version of the Wilson's method referred here as the LWYD algorithm is shown in Table 4.2. First, an initial vector u_1 is obtained from the static deflected shape. Using this

static deflected shape as the initial vector in the one-step inverse iteration. a new Wilson's vector $\mathbf{x}_1$ is then obtained. At each subsequent step, the initial vector $\mathbf{u}_1$ is updated and all components of the previous vectors $\mathbf{X}$ in it are removed using the Gram-Schmidt orthogonalization procedure. The updated static vector $\mathbf{u}_i$ is then used in the recursive relation to generate additional Wilson's Ritz vectors. Leger [35, 36] suggests that by using the modified initial static deflected shape in each step, the rounding errors that may accumulate in the original algorithm are partially eliminated and a better control on the static correction is achieved.

The cut-off criteria for generating the Wilson's Ritz vectors in the Wilson's method is developed based on two considerations:

- (1) representation of spatial distribution of load vector
- (2) representation of mass distribution.

Those two criteria are useful in the determination of the proper number of Wilson's Ritz vectors that should be included in a dynamic response analysis to obtain the required accuracy in a truncated system. For the present purpose, the spatially distributed load vector $\mathbf{f}(s)$ can be expressed as follows

$$\mathbf{f} = \sum_{i=1}^N \mathbf{x}_i \mathbf{f}^T \mathbf{M} \mathbf{x}_i \quad (4.36)$$

and the error in the representation of the load when only L Wilson's Ritz vectors are used is given by

$$\mathbf{e}_L = \mathbf{f} - \mathbf{f}_L \quad (4.37)$$

where

$$\mathbf{f}_L = \sum_{i=1}^L \mathbf{x}_i \mathbf{f}^T \mathbf{M} \mathbf{x}_i \quad (4.38)$$

Finally, a scalar representation of the error which can be treated as a cut-off criterion is defined as follows

$$\varepsilon_L = \frac{|\mathbf{f}^T \mathbf{e}_L|}{\mathbf{f}^T \mathbf{f}} \quad (4.39)$$

The second cut-off criterion based on the representation of the mass distribution can be formulated as follows

$$\xi_L = \frac{\sum_{i=1}^L (\overline{\mathbf{M}}^T \mathbf{x}_i)^2}{\sum_{i=1}^N \overline{M}_i} \quad (4.40)$$

where $\overline{\mathbf{M}}$ is a vector whose i^{th} term is calculated as follows

$$\overline{M}_i = \sum_{j=1}^N m_{ij} \quad (4.41)$$

where m_{ij} is the (i, j) element of matrix $\mathbf{M}$. The two criteria discussed here can be applied simultaneously as either one of the two can be used as the criterion to terminate the Wilson's algorithm.

Xia and Humar [37] has developed another algorithm similar to the Wilson's methods which is based on the consideration of the frequency spectrum of the applied load. The method produces so called frequency dependent Ritz vectors for determining the response of a structural system subjected to high frequency loading. This technique combines the two Wilson's method, WYD and LWYD, with a shifting procedure, where the special error norms are developed to terminate the Ritz vector generation process when the required accuracy of the solution is achieved.

4.4 Cut-off criterion for multi-directional earthquake loading

As it was mentioned earlier, a commonly used criterion to measure the truncation error of the Rayleigh-Ritz analysis is the effective participating mass.

It is suggested by Leger *et al.* [3] and Joo *et al.* [38] that since the support movement is direction dependent, the criterion expressed in Equation 4.40 should be applied in each of the three directions separately in a general three dimensional analysis.

The influence vector $\mathbf{r}$ can be expressed in the Ritz vector subspace as follows

$$\mathbf{r} = \mathbf{X}\boldsymbol{\beta} \quad (4.42)$$

where $\boldsymbol{\beta}$ represents the generalized coordinates of the Ritz vector for $\mathbf{r}$. The i^{th} component of $\boldsymbol{\beta}$ can be evaluated by multiplying both sides of Equation 4.42 by $\mathbf{x}_i^T \mathbf{M}$ and applying the mass orthonormality condition

$$\beta_i = \mathbf{x}_i^T \mathbf{M} \mathbf{r} \quad (4.43)$$

Now, the effective participating mass M_i^{eff} associated with the i^{th} Ritz vector can be expressed as follows

$$M_i^{eff} = \mathbf{r}^T \mathbf{M} \mathbf{x}_i \mathbf{x}_i^T \mathbf{M} \mathbf{r} = \beta_i^2 \quad (4.44)$$

The mass participating factors for ground motion applied in one specific direction (when the first L Ritz vectors are used in the analysis) can be written as follows

$$\xi_L^1 = \frac{\sum_{i=1}^L (M_i^1)^{eff}}{\mathbf{r}_1^T \mathbf{M} \mathbf{r}_1} \quad \text{and} \quad (M_i^1)^{eff} = \mathbf{r}_1^T \mathbf{M} \mathbf{x}_i \mathbf{x}_i^T \mathbf{M} \mathbf{r}_1 = \beta_{1i}^2 \quad (4.45)$$

$$\xi_L^2 = \frac{\sum_{i=1}^L (M_i^2)^{eff}}{\mathbf{r}_2^T \mathbf{M} \mathbf{r}_2} \quad \text{and} \quad (M_i^2)^{eff} = \mathbf{r}_2^T \mathbf{M} \mathbf{x}_i \mathbf{x}_i^T \mathbf{M} \mathbf{r}_2 = \beta_{2i}^2 \quad (4.46)$$

$$\xi_L^3 = \frac{\sum_{i=1}^L (M_i^3)^{eff}}{\mathbf{r}_3^T \mathbf{M} \mathbf{r}_3} \quad \text{and} \quad (M_i^3)^{eff} = \mathbf{r}_3^T \mathbf{M} \mathbf{x}_i \mathbf{x}_i^T \mathbf{M} \mathbf{r}_3 = \beta_{3i}^2 \quad (4.47)$$

where $(M_i^1)^{eff}$, $(M_i^2)^{eff}$, and $(M_i^3)^{eff}$ are the effective participating masses associated with the i^{th} Ritz vector in 1-, 2-, and 3-direction of ground excitation, respectively. The vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are respectively the influence vectors corresponding to the displacement at each active degree-of-freedom of the structure due to a unit base displacement in the 1-, 2-, and 3-direction of ground motion.

The effectiveness of the truncated series of Ritz vectors in representing the dynamic load can be evaluated by consideration of the effective mass of the structure that participates in the dynamic response of the system. The analysis results are exact if the number of Ritz vectors used in the analysis account for all the mass of the structure that participates in the vibrational response of the system. However, accurate results can still be obtained even though the effective mass accounted for by the truncated series of Ritz vectors used in the analysis is less than the total participating mass. The typical range of effective mass that should be included in the analysis in order to guarantee high accuracy in the analysis results is 85%-95% of the total participating mass.

For the derivation of the effective mass of a system subjected to multi-directional support excitation, the ground motion is assumed to occur in the 1- and 2-directions simultaneously. The total mass activated by the support movement $\mathbf{r}$ is expressed as follows

$$m_r = \mathbf{r}^T \mathbf{M} \mathbf{r} \quad (4.48)$$

The spatial distribution of the dynamic load is given as follows

$$\mathbf{f}(s) = \mathbf{f}_1(s) + \mathbf{f}_2(s) \quad (4.49)$$

In the case of support motion due to an earthquake, Equation 4.49 becomes

$$\mathbf{f}(s) = \mathbf{M}\mathbf{r}_1 + \mathbf{M}\mathbf{r}_2 \quad (4.50)$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are respectively the influence vectors corresponding to the displacement at each active degree-of-freedom of the structure due to a unit base displacement in the 1- and 2-directions. The total activated mass m_r thus has the form

$$\begin{aligned} m_r &= \mathbf{r}^T \mathbf{M} \mathbf{r} \\ &= \mathbf{r}_1^T \mathbf{M} \mathbf{r}_1 + \mathbf{r}_2^T \mathbf{M} \mathbf{r}_2 + \mathbf{r}_1^T \mathbf{M} \mathbf{r}_2 + \mathbf{r}_2^T \mathbf{M} \mathbf{r}_1 \end{aligned} \quad (4.51)$$

For a diagonal mass matrix $\mathbf{M}$, Equation 4.51 has the form

$$m_r = \mathbf{r}_1^T \mathbf{M} \mathbf{r}_1 + \mathbf{r}_2^T \mathbf{M} \mathbf{r}_2 \quad (4.52)$$

The last two terms in Equation 4.51 are equal to zero, whereas the first two represent the total activated masses in the 1- and 2-directions respectively.

Now for a truncated vector basis containing L Ritz vectors, the effective participating mass associated with the i^{th} Ritz vector can be determined as follows

$$\begin{aligned} M_i^{eff} &= \mathbf{r}^T \mathbf{M} \mathbf{x}_i \mathbf{x}_i^T \mathbf{M} \mathbf{r} \\ &= (\mathbf{r}_1^T + \mathbf{r}_2^T) \mathbf{M} \mathbf{x}_i \mathbf{x}_i^T \mathbf{M} (\mathbf{r}_1 + \mathbf{r}_2) \\ &= (\beta_{1i} + \beta_{2i})(\beta_{1i} + \beta_{2i}) \\ &= \beta_{1i}^2 + \beta_{2i}^2 + \beta_{1i}\beta_{2i} + \beta_{2i}\beta_{1i} \end{aligned} \quad (4.53)$$

The term β_{1i}^2 represents the effective participating mass in the 1-direction due to ground motion in the 1-direction for the i^{th} Ritz vector. Similarly β_{2i}^2 is the effective

participating mass in the 2–direction due to ground motion in the 2–direction for the i^{th} Ritz vector. The two terms $\beta_{1i}\beta_{2i}$ and $\beta_{2i}\beta_{1i}$ represents respectively the effective participating mass in the 1–direction due to ground motion in the 2–direction and the effective participating mass in the 2–direction due to ground motion in the 1–direction for the i^{th} Ritz vector. These last two terms do not appear in Equations 4.45 - 4.47 where the mass participating factors for two directions are formulated separately.

Now, the combined mass participating factor for the activated mass by earthquake in two directions when the mass matrix is diagonal can be written as follows

$$\eta_L = \frac{\sum_{i=1}^L (\beta_{1i}^2 + \beta_{2i}^2 + 2\beta_{1i}\beta_{2i})}{\mathbf{r}_1^T \mathbf{M} \mathbf{r}_1 + \mathbf{r}_2^T \mathbf{M} \mathbf{r}_2} \quad (4.54)$$

Similarly, in a case of three directional ground motion, the mass participating factor can be formulated including the effective participating mass in one direction due to ground motion in another one. The revised form of the mass participating factor for multi-directional support motion can be defined as follows

$$\eta_L = \frac{\sum_{i=1}^L (\sum_{j=1}^M \sum_{k=1}^M \beta_{ji}\beta_{ki})}{\sum_{j=1}^M \sum_{k=1}^M \mathbf{r}_j^T \mathbf{M} \mathbf{r}_k} \quad (4.55)$$

where M is a number of direction in which mass is activated by ground motion. The mass participating factor defined in Equation 4.55 can be applied as a cut-off criterion for generation of a truncated basis composed of mode shapes or load-dependent Ritz vectors. Numerical evaluation examples of the application of the cut-off criterion expressed in Equation 4.55 are presented in chapter six.

4.5 Comparison of WYDs and Lanczos methods

The fundamentals of the WYD method and the Lanczos method are very similar. Both algorithms produce a set of orthogonal vectors which are used to discretize the dynamic behaviour of the system and to reduce the size of the problem. During the process of obtaining the Wilson's Ritz vectors or the Lanczos vectors, the Krylov sequence is orthogonalized and normalized by the Gram-Schmidt orthogonalization procedure. The procedures are non-iterative in nature. Consequently, both procedures are expected to be faster and more efficient when compared to the subspace iteration method [39]. When the starting vector in the Lanczos method is taken as a displacement vector from a static solution, the method is known as the DRV method. The first step of the Lanczos algorithm is exactly the same as in the WYD method. The vectors generated by the WYD algorithm are commonly referred to as the load-dependent Ritz vectors.

The differences between the two methods mainly occur in the fact that the generalized stiffness matrix $\mathbf{K}^*$ from the transformation by the Wilson's Ritz vectors is a full matrix in the projected subspace, whereas the Lanczos method leads to a tridiagonalized system of equations of motion for the transformed eigenvalue problem. However, as that it has been shown by Leger *et al.* [3], the inverse of $\mathbf{K}^*$ is equal to the tridiagonal $\mathbf{T}$ generated by the Lanczos algorithm.

Moreover, the response solution in Lanczos coordinates

$$\mathbf{T}\ddot{\mathbf{z}} + [a_0\mathbf{T} + a_1\mathbf{I}]\dot{\mathbf{z}} + \mathbf{z} = \mathbf{Q}^T\mathbf{M}\mathbf{K}^{-1}\mathbf{f}(s)g(t) \quad (4.56)$$

is equivalent to the solution by the WYD method

$$\mathbf{I}\ddot{\mathbf{z}} + [a_0\mathbf{I} + a_1\mathbf{K}^*]\dot{\mathbf{z}} + \mathbf{K}^*\mathbf{z} = \mathbf{X}^T\mathbf{f}(s)g(t) \quad (4.57)$$

so that one approach is the inverse of the other.

The Lanczos method is typically faster than the Wilson's method since the diagonalization of a full subspace matrix requires more time than that of a tridiagonal matrix [40].

It is important to note here that with the introduction of the updated static vectors in the LWYD algorithm, the generated load dependent Ritz vectors no longer form a Krylov space. Thus, the LWYD method is not a Lanczos type procedure.

Table 4.1: Algorithm for generating of WYD Ritz vectors

1. Given mass, stiffness matrices
 $\mathbf{M}$, $\mathbf{K}$ and load vector $\mathbf{f}$
2. Triangularize stiffness matrix

$$\mathbf{K} = \mathbf{L}\mathbf{D}\mathbf{L}^T$$

3. Solve for the first vector, $\mathbf{x}_1$

$$\mathbf{K}\mathbf{x}_1^* = \mathbf{f}$$

Solve for $\mathbf{x}_1^*$

$$\mathbf{x}_1 = \frac{\mathbf{x}_1^*}{(\mathbf{x}_1^{*T}\mathbf{M}\mathbf{x}_1^*)^{1/2}}$$

Find $\mathbf{M}$ -orthonormal vector, $\mathbf{x}_1$

4. Solve for additional vectors
Compute for $i = 2, \dots, L$

$$\mathbf{K}\mathbf{x}_i^* = \mathbf{M}\mathbf{x}_{i-1}$$

Solve for $\mathbf{x}_i^*$

$$\text{Set } \mathbf{x}_i^{(1)} = \mathbf{x}_i^*$$

Compute for $j = 1, \dots, i-1$

$$C_j = \mathbf{x}_j^T \mathbf{M} \mathbf{x}_i^{*(j)}$$

$$\mathbf{x}_i^{*(j+1)} = \mathbf{x}_i^{*(j)} - C_j \mathbf{x}_j$$

 $\mathbf{M}$ -orthogonalize

$$\mathbf{x}_i = \frac{\mathbf{x}_i^{*(i)}}{(\mathbf{x}_i^{*(i)T}\mathbf{M}\mathbf{x}_i^{*(i)})^{1/2}}$$

 $\mathbf{M}$ -normalize

5. Obtain reduced mass
and stiffness matrices

$$\mathbf{M}^* = \mathbf{X}^T \mathbf{M} \mathbf{X}$$

$$\mathbf{K}^* = \mathbf{X}^T \mathbf{K} \mathbf{X}$$

6. Solve the reduced eigenproblem

$$\mathbf{K}^* \mathbf{Z} = \mathbf{M}^* \mathbf{Z} \mathbf{\Lambda}$$

where $\mathbf{\Lambda}$ =approximate eigenvalues

7. Compute approximate eigenvectors
of the original system

$$\mathbf{X}_0 = \mathbf{X} \mathbf{Z}$$

Table 4.2: LWYD algorithm for generating of Ritz vectors

1. Given mass, stiffness matrices
M, K and load vector f
2. Triangularize stiffness matrix

$$\mathbf{K} = \mathbf{LDL}^T$$
3. Solve for the first vector, $\mathbf{x}_1$

$\mathbf{K}\mathbf{u}_1 = \mathbf{f}$ $\mathbf{K}\mathbf{x}_1^* = \mathbf{M}\mathbf{u}_1$ $\mathbf{x}_1 = \frac{\mathbf{x}_1^*}{(\mathbf{x}_1^{*T}\mathbf{M}\mathbf{x}_1^*)^{1/2}}$	Solve for $\mathbf{u}_1^*$ Solve for $\mathbf{x}_1^*$ Find M-orthonormal vector, $\mathbf{x}_1$
---	---
4. Calculate additional vectors
 Compute for $i = 2, \dots, L$

$\mathbf{a}_{i-1} = \mathbf{x}_{i-1}^T \mathbf{M}\mathbf{u}_{i-1}$ $\mathbf{u}_i = \mathbf{u}_{i-1} - \mathbf{a}_{i-1}\mathbf{x}_{i-1}$ $\mathbf{K}\mathbf{x}_i^* = \mathbf{M}\mathbf{u}_i$ <p>Compute for $j = 1, \dots, i-1$</p> $C_j = \mathbf{x}_j^T \mathbf{M}\mathbf{x}_i^{*(j)}$ $\mathbf{x}_i^{*(j+1)} = \mathbf{x}_i^{*(j)} - C_j\mathbf{x}_j$ $\mathbf{x}_i = \frac{\mathbf{x}_i^{*(i)}}{(\mathbf{x}_i^{*(i)T}\mathbf{M}\mathbf{x}_i^{*(i)})^{1/2}}$	Update load vector Solve for $\mathbf{x}_i^*$ M-orthogonalize M-normalize
---	--
5. Obtain reduced mass
and stiffness matrices

$$\mathbf{M}^* = \mathbf{X}^T \mathbf{M} \mathbf{X}$$

$$\mathbf{K}^* = \mathbf{X}^T \mathbf{K} \mathbf{X}$$
6. Solve the reduced eigenproblem:

$$\mathbf{K}^* \mathbf{Z} = \mathbf{M}^* \mathbf{Z} \mathbf{\Lambda}$$

where $\mathbf{\Lambda}$ approximate eigenvalues
7. Compute approximate eigenvectors
of the original system

$$\mathbf{X}_0 = \mathbf{X} \mathbf{Z}$$

Chapter 5

Computer Implementation of Ritz Vector Algorithms

5.1 General

This chapter gives a brief overview of the NEABS computer program specifically developed for the seismic analysis of bridge systems. The extensive modification and additions to the original NEABS program implemented in the present study are discussed. The Rayleigh-Ritz numerical algorithms discussed in the previous chapters are implemented. New modules to the NEABS program have been developed and tested. They can be utilized in frequency analysis, as well as dynamic response analysis of large complex systems. In addition to the incorporation of the Ritz type analysis capabilities in the NEABS program, the modal superposition method is introduced and the direct time step integration is modified for the time history

response analysis of reduced systems.

5.2 NEABS program organization

The original version of the computer program NEABS, Nonlinear Earthquake Analysis of Bridge Systems, was developed at the University of California, Berkeley, by Tseng and Penzien in 1973 [41] for conducting detailed linear and nonlinear dynamic analysis of bridges. Later, significant enhancements and improvements to the original version of the program were developed by Imbsen and Associates Inc. of Sacramento, California [42].

The current version of NEABS includes linear and nonlinear time history analysis using the direct step-by-step time integration method, and also frequency and response spectrum analysis modules in which mode shapes are calculated by the subspace iteration method. The organization of the NEABS program is summarized as follows:

- (1) Formulation of the static equilibrium equations of the bridge structural system (subroutine SETUP)
- (2) Solution for static dead load conditions (subroutine STATIC)
- (3) Generation of the dynamic load vector for a time step (subroutine LOADS)
- (4) Direct step-by-step time integration (subroutine INTGR)
- (5) Preparation of output results (subroutine OUTPUT).

Figure 5.1 presents an organization chart of the subroutines in the NEABS program.

5.3 Modifications to NEABS program

The Lanczos and improved Wilson's algorithms have been added to the NEABS program, which together with the subspace iteration method are contained in a new module called RITZ. Figures 5.2 and 5.3 show the organization chart of the subroutines in the module. New capabilities have been incorporated in the program by the implementation of all the Ritz analysis methods discussed earlier in this thesis. The new Lanczos and load-dependent Ritz vector algorithms can be utilized in frequency and dynamic response analysis.

In frequency analysis, three options have been implemented:

- (1) computation of exact mode shapes and frequencies by Lanczos algorithm with a combined partial and selective reorthogonalization procedure
- (2) computation of approximate mode shapes and frequencies by Lanczos algorithm with a combined partial and selective reorthogonalization procedure
- (3) computation of approximate mode shapes and frequencies by improved Wilson's algorithm.

For options (2) and (3), a specified number of Lanczos or Wilson's Ritz vectors are generated to reduce the size of the problem. The eigenproblem of the reduced system

is solved. The eigensolutions of the reduced system are then transformed back to the geometric coordinates which represent approximations to the eigensolution of the original problem.

The existing version of NEABS program performs a direct step-by-step time integration of the dynamic equilibrium equations of the entire structural system and an introduction of the reduction procedures in time history seems to be next logical step during process of implementation of Ritz vectors. Linear dynamic time history response analyses of the reduced system by the modal superposition method and the direct time step integration method have been implemented as the module INTGRM, as shown in Figure 5.1. The new features introduced in time history response analysis of linear systems are summarized as follows:

(1) Modal superposition method

- using mode shapes generated by different algorithms
 - exact mode shapes by Subspace iteration method
 - exact mode shapes by Lanczos method
 - approximate mode shapes by Lanczos method
 - approximate mode shapes by improved Wilson's method

(2) Direct time step integration of reduced system

- reduced system obtained by using Ritz vectors generated by different algorithms
 - Lanczos vectors by Lanczos method
 - DRV vectors by Lanczos method

- Wilson's Ritz vectors by improved Wilson's method
- (3) Direct time step integration of reduced system in a tridiagonal form $\mathbf{T}$ given in Equation 4.28
- tridiagonal system obtained using DRV vectors by Lanczos method

The efficiency and reliability of the reduction procedures are evaluated to ensure that the Ritz vectors used in the analysis can adequately describe the behaviour of the system. A new cut-off criterion based on the effective mass participation for multi-directional earthquake loading has been implemented to assess the effectiveness of the reduction procedures and to assess the rate of convergence.

Figure 5.1: Subroutine organization chart of NEABS program

Figure 5.2: Ritz module subroutine organization chart

Figure 5.3: Lanczos module subroutine organization chart

Chapter 6

Numerical Applications of Ritz Vector Analysis Procedures

6.1 General

The primary objectives of the numerical study reported in this chapter are to verify the computer implementation of the various Ritz vector analysis modules, to evaluate the relative efficiency of the analysis approaches, and to validate the development of a comprehensive powerful computer analysis tool for complex bridge structures. Numerical examples of frequency and dynamic response analysis due to earthquake loading are presented for typical bridge systems. The cut-off criterion developed in chapter 4 for multi-directional earthquakes is tested and verified in the examples with different load combinations acting on the bridges.

6.2 Dynamic analysis of bridges

Extensive bridge damage caused by recent earthquakes, the 1994 Northridge earthquake (California) and the 1996 Kobe earthquake (Japan) [43], have led to increased research efforts by researchers and engineers around the world to develop more effective analysis methods, new modelling approaches and design methodology for earthquake resistant design and assessment of new and existing bridge structures.

Compared to buildings most bridges seem to be relatively simple in terms of structural configuration. However, this observation may be misleading as simplicity or rather the lack of redundancy in the structural system of a bridge means that any one load path in a bridge is more critical than that in a building. For the case of a bridge subjected to earthquake loading, higher demand is therefore put on the fewer structural elements of the system to dissipate the earthquake energy. Moreover, earthquakes tend to "identify" structural weaknesses and cause damage at these locations. The lack of redundancy in a bridge system can lead to collapse due to the failure of a few or even only one element or connection. Therefore, accurate modelling of the seismic behaviour of a bridge structure is very important in the determination of its expected seismic performance.

The natural periods or frequencies of the structure identify resonant conditions, and also verify how the mass and stiffness properties of the structure influence its dynamic behaviour. The calculation of the eigensolution is an important component in the determination of the behaviour and performance of the structural system. A computational study employing different Ritz algorithms to obtain the natural frequencies of vibration of three bridges of different structural types is presented.

followed by the time history response analysis of the same bridge structures.

Acceleration records from the 1940 El Centro and 1988 Saguenay earthquakes, as shown in Figures 6.1 and 6.2, are used as input motions in the earthquake response analysis. The two adopted earthquakes represent typical west coast and east coast North American ground motions. Displacements, base shear and base moment of the analyzed bridges are presented.

Detailed assessment on the performance of the Macdonald-Cartier Bridge and the Alexandra Bridge under seismic loads have been conducted by Lau [44, 45]. Based on the evaluation envelopes of the critical response parameters obtained under the six different seismic load cases considered, two of the most critical cases are chosen here as the input motions for the present study. These two cases are shown in Figure 6.3 as load cases 1 and 2. The third load case included in the numerical study, as illustrated in Figure 6.3, is used in the verification study of the cut-off criterion η_L for a multi-directional earthquake. The seismic load case 1 is the horizontal translational ground motion of the 1940 El Centro earthquake in California applied in the transverse direction of the bridge. In load case 2, the three recorded ground motion components of the 1988 Saguenay earthquake in Quebec are applied simultaneously. The last load case considers only the two horizontal translational ground motion components of the Saguenay in the longitudinal and transverse directions of the bridge.

In the analysis, the peak horizontal ground acceleration of all the load cases is scaled to a maximum of 0.2g.

6.3 Dynamic analysis of a building model

A building model which has been used in previous numerical studies by Leger *et al.* [3] and Joo *et al.* [38] is analyzed here to verify the implementation of the different Ritz algorithms.

A multi-story building is modelled by a 40 degrees-of-freedom shear frame model, as shown in Figure 6.4. It is assumed in the model that there are no masses at nodes 37, 38, and 39.

The first fifteen frequencies are calculated using different options as described in the previous chapter and the results are presented in Table 6.1. Both the subspace iteration method and the Lanczos algorithm give accurate results, which are in exact agreement with those reported by Joo *et al.* [38]. In terms of efficiency of the solution, it is noted in Table 6.1 that there is considerable saving in computational time even for such a relatively small system. The results presented in Table 6.1 also indicate that 31 Lanczos vectors are required to obtain the exact solutions of the first 15 mode shapes. The last two columns of Table 6.1 present the approximate eigen-solution results by the Lanczos method and by the improved Wilson's Ritz method employing 15 Lanczos and 15 Wilson's Ritz vectors, respectively. For this small size problem there is no significant time saving from the two approximation procedures in comparison to the exact solution by the Lanczos method. It is also noticed that the approximate eigensolutions span a wider range than the exact solutions, which illustrate the one behaviour of the Ritz algorithms and the eigensolutions do not necessary converge in exact sequential order. Nevertheless, the obtained Ritz approximate eigensolutions are good approximations to the exact solutions.

6.4 Dynamic analysis of Blackstone River Bridge

6.4.1 Analysis model of Blackstone River Bridge

The Blackstone River Bridge, located on Liard highway in Northwest Territories, is a two span concrete slab on steel girder bridge. Each span has a length of 40.30 *m*. The elevation, a typical cross section of the bridge and the finite element model are shown in Figure 6.5. The steel girders and floor beams are modelled individually by 3-dimensional beam elements. The bridge deck of 200 *mm* thick concrete slab is modelled by the grillage method by 3-D beam elements. The support conditions of the bridge are roller supports at the abutments and a fixed-end support at the bridge pier. The pier supporting the bridge superstructure is modelled by equivalent 3-dimensional beam-column elements.

The analysis model has 69 nodes with 108 elements and a total of 336 degrees-of-freedom. In the analysis, the bridge is assumed to have a 5% damping.

6.4.2 Frequency analysis of Blackstone River Bridge

The first 15 converged and approximate vibration natural frequencies for the Blackstone River Bridge are listed in Table 6.2. The results indicate that the Lanczos method is 2.5 times faster than the subspace iteration method in terms of the total CPU time for the exact solution and 3.6 times more efficient when only the CPU time of the eigensolution is considered. The spread in the spectrum of the calculated frequencies depends on the starting vector for both the Lanczos and the

Wilson's Ritz algorithms. For the bridge considered, the Lanczos method gives accurate results with only an arbitrary or random starting vector, as opposed to the Wilson's method which requires the starting vector to be a static displacement vector obtained from the spatial distribution of the load. When the starting vector is chosen randomly, the first 15 lowest vibration modes are calculated from 33 Lanczos vectors. The order of convergence of the converged vibrational frequencies in the analysis of the Blackstone River Bridge by the Lanczos method is influenced by the choice of the starting vector, as can be observed from the results shown in Table 6.3 for four different starting vectors. There are noticeable omissions in the frequency spectrum of the converged solutions.

On the other hand, when the starting vector considers the spatial distribution of mass of the structure, the calculated approximate eigensolutions are the approximations of significantly different vibration modes. The approximate solution obtained using only 15 Lanczos or Wilson's Ritz vectors is minimally more efficient in terms of the CPU time when compared with the exact solution by the Lanczos method. The wider spread in the frequency range of the approximate solutions indicates that the order of the eigenpair solutions is influenced by the number of generalized Ritz coordinates used in the reduction of the system. When the number of Lanczos or Wilson's Ritz vectors used in the analysis increases, more lower eigenpair solutions at the beginning of the spectrum converge, as shown in Table 6.2.

6.4.3 Dynamic response analysis of Blackstone River Bridge

The time history response for the Blackstone River Bridge is calculated by using three different transformation approaches. The two load cases 1 and 2 are considered in the analysis.

First, the subspace iteration procedure is employed to calculate the mode shapes which are then used in a modal superposition analysis. Next, the time history response is also obtained from the direct step-by-step time integration of the reduced systems by the DRV and Wilson's Ritz vectors. Comparisons of the results obtained from the three methods are presented in Figures 6.6-6.13. It is observed in Figures 6.6 and 6.9 that the use of a very small number of transformation vectors compared to the size of the original problem is sufficient to obtain reasonably accurate estimate of the displacement response of node no. 17 which location is shown in Figure 6.5. For the case of modal analysis, 5 and 10 mode shapes are used, whereas in the cases of the DRV and Wilson's methods 3 and 5 load-dependent Ritz vectors are used in the direct step-by-step time integration of the reduced systems. The CPU times required by the solutions are listed in Tables 6.4 and 6.5. The required CPU time for all three reduction procedures is significantly less than the CPU time of the exact solution by the direct step-by-step integration of the full size problem. In comparison to the displacement results, larger number of Ritz vectors are required in the analysis to give satisfactory results for the base shear and base moment. Only the results of load case 1, as shown in Figures 6.7 and 6.8, indicate that the same number of load-dependent Ritz vectors gives good approximation to the displacements as well as the internal forces. Otherwise, it is evident from the results shown

in Figures 6.10-6.13 that 37 mode shapes for load cases 1 and 2, and 14 DRV or Wilson's Ritz vectors for load case 2 are needed to obtain satisfactory results for the internal forces. The increase in the number of Ritz vectors required for good approximations of the internal forces can be explained by the fact that the internal forces involve derivatives of the displacements, thus they converge slower than the displacement itself. Furthermore, the frequency content of the ground motion can also have an effect on the number of Ritz vectors required in the analysis.

The effective mass participation factors ξ_L or η_L presented in Tables 6.4 and 6.5, show that 5 load-dependent Ritz vectors for load case 1 represent over 90% of the activated mass, whereas only 78% of the mass is accounted for by the use of 37 mode shapes. For a multi-directional earthquake of load case 2, the number of DRV and Wilson's Ritz vectors required in the analysis is increased to 14 which represents 88% of the activated mass based on the cut-off criterion η_L . Based on the same criterion η_L , 37 mode shapes represent only 82.2% of the total activated mass.

The CPU times required for the dynamic response analysis using 37 mode shapes is 20.82s for load case 1 and 18.49s for load case 2, as presented in Tables 6.4 and 6.5, which are respectively 4.5 and 4.75 times less than that required by the exact solution. The number of DRV and Wilson's Ritz vectors used in the analysis to obtain comparable results of the response is reduced by more than half. Table 6.4 shows that the results generated by the 5 Ritz vectors by the Lanczos or Wilson's algorithms requires respectively only 7.62s and 7.84s of CPU time compared to 98.9s for the exact solution. In Table 6.4, the numerical model of the Lanczos method with the tridiagonal form is solved by the direct step-by-step time integration procedure. This option is only used for the case of earthquake load applied in one direction. When the tridiagonal matrix $\mathbf{T}$ is directly used, the computational time is further

reduced by about 7% from the standard Lanczos method.

6.5 Dynamic analysis of Macdonald-Cartier Bridge

6.5.1 Analysis model of Macdonald-Cartier Bridge

The Macdonald-Cartier Bridge is one of several bridges spanning the Ottawa River. The link between Ottawa and Hull is designed as a five span composite concrete slab on steel box girder bridge. The typical cross section and elevation of bridge are shown in Figure 6.14. The longest span has a length of 159.1 *m*. The structural system of the bridge superstructure consists of main steel box girders, floor beams, stringers and cross-beams. Figure 6.15 shows the layout of the box girder and floor beam elements of the computer analysis model. The bridge deck is a 171 *mm* thick concrete slab on top of the stringers and cross-beams.

In the bridge model, each steel box girder and floor beam is modelled individually by 3-D finite element, whereas for the stringers, the cross-beams and the concrete deck, a grillage model of beam elements are used, as shown in Figure 6.16. The support conditions are modelled as roller supports at the abutments at Piers 1 and 4, and hinges at Piers 2 and 3. The four pier structures supporting the bridge superstructure are modelled by equivalent 3-D beam-column elements including the variation of the cross-sectional properties along its height.

The analysis model, which consists of the bridge superstructure and the piers.

has 594 nodes with 778 elements and a total of 3008 degrees-of-freedom. The bridge is assumed to have a 5% damping in the analysis model.

6.5.2 Frequency analysis of Macdonald-Cartier Bridge

Results for the first 15 natural vibration frequencies of the Macdonald-Cartier Bridge are presented in Table 6.6. When only the computational time of eigensolution is considered, it is 4.6 times more efficient to extract 15 exact vibration modes by the Lanczos method than by the subspace iteration procedure. Computational CPU time used by both the Lanczos and Wilson's Ritz methods to obtain the approximate solution are very similar. As can be observed from the results shown in Table 6.6, the converged approximate solutions by the Ritz algorithms do not necessarily correspond to the lowest modes. In fact, the converged results are dispersed over the frequency spectrum, especially the higher frequency results.

Results presented in Table 6.6 also suggest that when the Sturm sequence check is not included in the eigensolution computational time, it takes 0.6s, that is 6% more in CPU time, for the calculation of 39 Lanczos or 36 DRV vectors to obtain 15 converged frequencies compared to the case of 15 Lanczos, DRV or Wilson's Ritz vectors for the extraction only 6 or 7 converged eigenvalues.

Considering the above it may be noticed that the Lanczos method implemented in the NEABS program to calculate the vibrational frequencies is very effective. The Lanczos method is a quick and efficient procedure for frequency analysis of the considered bridge system.

In the computer implementation of the Lanczos algorithm, the combined selective and partial reorthogonalization procedures are implemented in the NEABS computer program. In order to apply the selective orthogonalization scheme, the mode shapes and corresponding natural vibration frequencies are calculated a few steps after convergence is first indicated. In the case of the approximate solution, a specified number of generalized Ritz coordinates are first obtained and the eigensolutions of the reduced system are then determined. The CPU time needed to generate the transformation vectors is small in comparison to the time required for the eigensolution part of the procedure, as it is the case of Wilson's method.

6.5.3 Dynamic response analysis of Macdonald-Cartier Bridge

The Macdonald-Cartier Bridge model has the largest number of degrees-of-freedom among the studied bridges with a total of 3008 degrees-of-freedom. The 3-dimensional structural model is analysed using earthquake load cases 1 and 2.

The horizontal transverse and vertical components of the displacements of node number 222 which location is shown in Figure 6.16 are plotted in Figures 6.17 and 6.18 for load case 1, and in Figures 6.19 and 6.20 for load case 2. Again, the displacement time history response shows that the results obtained by the mode shapes or the load-dependent Ritz vectors are in good agreement with the exact solutions. The results show that 15 mode shapes for load case 1 and 30 eigenvectors for load case 2 can accurately represent the displacement dynamic response of the bridge structure. However, more accurate results can be obtained by the load-

dependent Ritz vectors. Figures 6.17-6.20 show that only 10 DRV or Wilson's Ritz vectors are required to obtain results which show no significant difference from the exact results. As shown in Tables 6.7 and 6.8, it is found that the CPU time required by the Ritz algorithms range from only 8.11% to 11.76% compared to the subspace iteration method, which represent very substantial saving.

The base shear and base moment time history responses are plotted in Figures 6.21-6.28. As the results show, a significant error occurs in the time history of the internal forces. The error is still unacceptably high for load case 1 even when 100 mode shapes are used in the analysis, as shown in Figures 6.22 and 6.24. The accuracy of the results in the case of the three directional earthquake input, as illustrated in Figures 6.26 and 6.28, improves with the use of 100 eigenvectors. However, the results are still noticeably different from the exact solutions. As presented in Table 6.8, the activated mass represented by 100 mode shapes for load case 2 is 59.99% based on the cut-off criterion η_L , which is significantly higher than the 37.51% calculated for load case of one directional earthquake reported in Table 6.7. On the other hand, the effective mass participation factor, η_L (ξ_L), presented in Table 6.7 for 30 DRV vectors and 30 Wilson's Ritz vectors, is respectively 86.21% and 85.89%. According to the results shown in Table 6.8 for load case 2, 30 generated load-dependent Ritz vectors accounts for about 83% of the total activated mass. The results presented in Figures 6.26 and 6.28 confirm the accuracy of the error estimate by η_L .

It is worth while to notice that the numbers of degrees-of-freedom used to model the Macdonald-Cartier Bridge and the Blackstone River Bridge are significantly different. However, the ratio in the saving of CPU time is greater for the case of Macdonald-Cartier Bridge when compared to the numerical results of the Black-

stone River Bridge. The computational time, when load-dependent Ritz vectors are applied as the transformation basis, is reduced by 11 to 16 times in comparison to the CPU time of the exact solution, as it can be observed from Tables 6.7 and 6.8. Both the Lanczos and Wilson's methods are equally efficient, with a slight advantage given to the Lanczos method in a tridiagonal form, where it requires 65.8s less in CPU time in the time history response analysis using 38 DRV vectors, as shown in Table 6.7. In comparison, the modal analysis procedure seems to be less efficient in terms of the number of vectors required in the analysis, the representation of mass participating in the dynamic response and also CPU time usage. The calculation of the eigenvectors starts to dominate the total computational time which takes approximately 1243s for load case 1 and 1147s for load case 2 to evaluate 100 mode shapes.

6.6 Dynamic analysis of Alexandra Bridge

6.6.1 Analysis model of Alexandra Bridge

The Alexandra Bridge is located in Ottawa-Hull region. It is a steel multi-span truss bridge with adjacent approach structures at the two ends. The elevation and a typical cross section of the bridge are shown in Figure 6.29. The longest span of the main truss has a length of 169.4 *m*.

The bridge model is divided analytically into two separate parts, with the main part consisting of the suspended truss, the two cantilever trusses and the approach truss A, and the second part of the approach truss B. The superstructure of the

bridge consists of two identically parallel planes of trusses and a deck system. The truss systems are constructed of steel members capable to resist both tensile and compressive axial forces, tension members, tie bars, cross and sway bracing elements. All the structural members in the trusses are pin connected, except at a few locations, and the bottom chord truss elements are continuous members able to resist bending moment. The bridge deck consists of a 175 *mm* thick concrete slab on top of the horizontal grid system of floor beams and longitudinal deep beams framing into each other. In a previous numerical study, dynamic characteristics and behaviour of the main structure under earthquake loads have been determined.

In the analysis model, the stiffness and mass properties of all significant structural elements, including that of the concrete deck, are considered. The stiffness contributions from the steel deck grating are neglected. The members in the trusses are modelled individually by 3-dimensional finite elements. The longitudinal deep beams, floor beams, cantilever beams, stringers and the concrete deck are modelled by an equivalent beam model. The layout of the major structural elements in the 3-D finite elements analysis model is shown in Figure 6.30.

The support conditions are modelled as roller supports at the abutment on the Ottawa side and at Piers 4 and 5, and hinges at Piers 2 and 3 for the main truss and approach truss A, as shown in Figure 6.29. Because of the lack of data on the geometry and material properties of the piers only the bridge superstructure is modelled. In the dynamic response study the ground motion is assumed to be directly applied to the bridge superstructure at the bearing supports on the top of the piers and abutments, neglecting the amplification effect of the piers.

The bridge model of the main truss has 373 nodes with 803 elements and a total

of 1600 degrees-of-freedom. The bridge is assumed to have a 5% damping during dynamic response.

6.6.2 Frequency analysis of Alexandra Bridge

The frequency analysis results presented in Table 6.9 show that the modal frequencies of the Alexandra Bridge are closely spaced. The number of Lanczos vectors required to obtain the first 15 converged vibration frequencies is 45 or 48 depending on the starting vector used in the analysis. As a comparison, the number of degrees-of-freedom in the model of the Macdonald-Cartier Bridge is almost twice that of the Alexandra Bridge, only 39 Lanczos or 36 DRV vectors are required to obtain the same number of 15 converged eigenvalues as shown in Table 6.6.

It is also noticed that with an increased number of Lanczos vectors, the Lanczos method is 6.3 times faster compared to the subspace iteration procedure. The first 15 Lanczos, DRV, Wilson's Ritz vectors give the frequency estimates in a much wider range (up to mode number 71) for the Alexandra Bridge than for the Macdonald-Cartier Bridge. The highest estimated vibrational frequency for the Macdonald-Cartier Bridge corresponds to mode number 28 while using Wilson's method.

6.6.3 Dynamic response analysis of Alexandra Bridge

The theoretical background of the cut-off criterion η_L for the representation of activated mass in a loading case of multi-directional earthquake is discussed in chapter 4. A numerical example to illustrate the application and to verify the revised mass

participation factor η_L is presented here. The results of the mass participation factor reported in Table 6.10 for load case 1 agree with the results shown in Figure 6.31 for the axial force of truss member 404 which location is shown in Figure 6.29.

The mass participation factor ξ_L for unidirectional seismic excitation and η_L for multi-directional excitation are calculated respectively by Equations 4.45-4.47 and Equation 4.55. Analyzing the data provided in Tables 6.11, 6.12 for load case 2 and as well as looking at Figures 6.32-6.34 it becomes obvious that the criterion ξ_L which requires calculation of the mass participation factors in each direction separately, is confusing and it does not provide a clear answer how many generalized coordinates should be included in the analysis to attain the accurate results. The axial force time history responses for the truss member 404 presented in Figure 6.32 from 10 DRV and 10 Wilson's Ritz vectors are in good agreement with the exact solution. However, it is observed in Figure 6.32 that the use of 40 mode shapes still does not give accurate results in axial force. The mass participation factors ξ_L listed in Table 6.11 for 18 DRV vectors are 55.76%, 50.63%, and 57.55% respectively for the activated masses in the X, Y, and Z directions. The results of the dynamic response presented in Figure 6.34 using 18 DRV vectors are very close to the exact solutions. On the other hand, it is showed in Table 6.12 that the mass participation factors ξ_L for 100 mode shapes are 56.8%, 71.73%, and 75.77% respectively in the X, Y, and Z directions. In comparison, these mass participation factors are higher than those obtained in the case of using 18 DRV vectors, as given in Table 6.11. However, it may be observed from Figure 6.33 and Figure 6.34 that 100 mode shapes give poorer results in axial force compared to 18 load-dependent Ritz vectors. This is consistent according to the new cut-off criterion η_L . The total participation of activated mass is 86.68% for the case of 18 DRV vectors and only 67.06% for 100 eigenvectors. It

is evident that the proposed criterion η_L given in Equation 4.55 provides a good estimate how many generalized coordinates are necessary in the analysis procedure for accurate results. It is noted that 140 mode shapes are required to achieve similar accuracy in the results with 84.54% mass participation according to the factor η_L . The displacement time history response of node number 106 whose location is shown in Figure 6.29 is plotted in Figures 6.35-6.38. The results clearly show that the number of Ritz vectors required to obtain satisfactory results is much smaller than the traditional modal analysis techniques. It is also observed that in general more discretizing vectors are required to represent internal forces than displacements.

A numerical study on the dynamic behaviour of the Alexandra Bridge is carried out to verify the cut-off criterion η_L using load case 3, as shown in Figure 6.3. The results are presented in Table 6.13 and plotted in Figures 6.39 and 6.40. The results confirm earlier findings which indicate that mass participation in one direction caused by applied load in another direction has to be taken into consideration in the calculation of the error in mass participation.

When the load-dependent Ritz vectors are adopted in the dynamic response analysis of the Alexandra Bridge model, the CPU time required is significantly less than the other analysis algorithms for all load cases, as shown in Tables 6.10-6.13. Moreover, the number of mode shapes needed to obtain results of comparable accuracy is several times higher than the number of DRV or Wilson's vectors required. The large number of mode shapes required in the analysis leads to an increase in computational time and storage demand.

Table 6.1: Estimates of 15 frequencies of shear beam frame

Total number of DOF - 40

Mode no.	Exact value [Hz]	Converged value Subspace iteration method [Hz]	Converged value Lanczos method (31 Lanczos vectors) [Hz]	Approx. value Lanczos method (15 Lanczos vectors) [Hz]	Approx. value LWYD method (15 Wilson's Ritz vectors) [Hz]
1	0.180	0.180	0.180	0.180	0.180
2	0.533	0.533	0.533	0.533	0.533
3	0.882	0.882	0.882	0.882	0.882
4	1.243	1.243	1.243	1.243	1.243
5	1.623	1.623	1.623	1.623	1.623
6	2.016	2.016	2.016	2.016	2.016
7	2.416	2.416	2.416	2.416	2.416
8	2.816	2.816	2.816	2.816	2.816
9	3.215	3.215	3.215	3.215	3.215
10	3.611	3.611	3.611	3.611	3.611
11	4.001	4.001	4.001	4.005	4.013
12	4.386	4.386	4.386	4.439	4.494
13	4.763	4.763	4.763		
14	5.132	5.132	5.132	5.041	5.200
15	5.492	5.492	5.492		
(17)	6.182			5.987	6.309
(22)	7.701			7.606	
(23)	7.965				8.110
(24)	8.214				
Total CPU time		1.95 s	0.32 s	0.30 s	0.32 s

Table 6.2: Estimates of 15 frequencies of Blackstone River Bridge

Total number of DOF - 336

Mode no.	Exact value [Hz]	Converged value Subspace iteration method [Hz]	Converged value Lanczos* method (33 Lanczos) vectors [Hz]	Converged value Lanczos** method (31 DRV) vectors [Hz]	Approx. value Lanczos* method (15 Lanczos) vectors [Hz]	Approx. value Lanczos** method (15 DRV) vectors [Hz]	Approx. value LWYD** method (15 Wilson's) Ritz vectors [Hz]
1	1.859	1.859	1.859	1.859	1.859	1.859	1.859
2	1.865	1.865	1.865	1.865	1.865	1.865	1.865
3	1.866	1.866	1.866	1.866	1.866	1.866	1.866
4	2.587	2.587	2.587	2.587	2.587	2.746	
5	3.094	3.094	3.094	3.094	3.094		
6	3.349	3.349	3.349	3.349	3.360		
7	3.362	3.362	3.362	3.362	3.372	3.362	3.362
8	3.504	3.504	3.504	3.504	3.504	3.504	3.504
9	3.711	3.711	3.711	3.711			
10	3.754	3.754	3.754	3.754	3.813	3.754	3.754
11	3.977	3.977	3.977	3.977	4.043	3.977	3.977
12	4.513	4.513	4.513		4.565		
13	4.889	4.889	4.889	4.889		4.889	4.889
14	5.620	5.620	5.620		5.646		
15	5.650	5.650	5.650	5.650	5.677	5.650	5.650
16	7.502						
17	7.558			7.558		7.558	7.558
18	8.297				8.405		
19	8.528						
20	8.928						9.575
(22)	9.798			9.798		9.798	9.799
(23)	9.944				10.016		
(24)	10.240					10.502	10.739
(26)	10.464					14.801	16.558
(28)	11.767						
(29)	18.083					23.535	23.789
(37)	22.953						
Total CPU time		5.68 s	2.33 s	2.32 s	2.02 s	2.05 s	2.03 s
CPU time of eigensolution		4.68 s	1.29 s	1.32 s	0.97 s	1.02 s	1.01 s
		0.28 s***	0.29 s***	0.28 s***			

* - randomly chosen starting vector

** - starting vector chosen as displacement vector obtained
from a spatial distribution of load in Y - direction*** - CPU time of Sturm sequence check included
in CPU time of eigensolution

Table 6.3: 15 exact frequencies of Blackstone River Bridge

Total number of DOF - 336

Mode no.	Exact value [Hz]	Converged value Lanczos* method (33 Lanczos vectors) [Hz]	Converged value Lanczos method (31 DRV) vectors X**direction [Hz]	Converged value Lanczos method (31 DRV) vectors Y**direction [Hz]	Converged value Lanczos method (29 DRV) vectors Z**direction [Hz]
1	1.859	1.859	1.859	1.859	1.859
2	1.865	1.865	1.865	1.865	1.865
3	1.866	1.866	1.866	1.866	1.866
4	2.587	2.587	2.587	2.587	2.587
5	3.094	3.094	3.094	3.094	3.094
6	3.349	3.349	3.349	3.349	3.349
7	3.362	3.362	3.362	3.362	3.362
8	3.504	3.504	3.504	3.504	3.504
9	3.711	3.711	3.711	3.711	3.711
10	3.754	3.754		3.754	3.754
11	3.977	3.977		3.977	3.977
12	4.513	4.513	4.513		
13	4.889	4.889		4.889	4.889
14	5.620	5.620	5.620		
15	5.650	5.650		5.650	5.650
16	7.502		7.502		
17	7.558			7.558	7.558
18	8.297		8.297		
19	8.528		8.528		
20	8.928		8.928		
(22)	9.798			9.798	
(24)	10.240				10.240
Total CPU time		2.33 s	2.31 s	2.32 s	2.27 s
CPU time of eigen solution		1.29 s	1.27 s	1.32 s	1.24s
		0.29 s ***	0.29 s ***	0.28 s ***	0.30 s ***

* - randomly chosen starting vector

** - starting vector chosen as displacement vector obtained from a spatial distribution of load in indicated direction

*** - CPU time of Sturm sequence check included in CPU time of eigensolution

Table 6.4: Ritz algorithms comparison for Blackstone River Bridge,
Load case 1

		Exact solution	Mode shapes Subspace iteration method	DRV vectors Lanczos method	DRV vectors Lanczos method Tridiagonal form	Wilson's Ritz vect. LWYD method
1	No. of vectors	—	5	3	3	3
	Mass participation [%] ξ_L or η_L	—	65.37	79.42	79.42	79.12
	CPU time [s]	98.9	8.48	6.95	6.56	7.27
2	No. of vectors	—	18	4	4	4
	Mass participation [%] ξ_L or η_L	—	66.15	86.44	86.44	86.21
	CPU time [s]	98.9	10.26	7.48	7.02	7.52
3	No. of vectors	—	37	5	5	5
	Mass participation [%] ξ_L or η_L	—	77.97	91.92	91.92	91.90
	CPU time [s]	98.9	20.82	7.62	7.09	7.84

Table 6.5: Ritz algorithms comparison for Blackstone River Bridge.

Load case 2

			Exact solution	Mode shapes Subspace iteration method		DRV vectors Lanczos method		Wilson's Ritz vect. LWYD method	
1	No. of vectors		—	5		3		3	
	Mass participation [%]	η_L	—	53.33		75.07		70.12	
		ξ_L		X	91.31	X	65.32	X	60.77
				Y	65.37	Y	43.69	Y	47.61
			Z	0.00	Z	46.65	Z	44.53	
CPU time [s]			82.88	7.57		6.09		6.39	
2	No. of vectors		—	18		5		5	
	Mass participation [%]	η_L	—	73.57		76.48		79.12	
		ξ_L		X	93.10	X	69.03	X	65.51
				Y	66.15	Y	58.57	Y	62.49
			Z	59.90	Z	50.10	Z	50.28	
CPU time [s]			82.88	11.57		6.95		7.27	
3	No. of vectors		—	37		14		14	
	Mass participation [%]	η_L	—	82.17		88.04		88.03	
		ξ_L		X	94.26	X	93.95	X	94.58
				Y	77.97	Y	87.38	Y	85.11
			Z	71.76	Z	75.64	Z	76.12	
CPU time [s]			82.88	18.49		10.07		10.71	

Table 6.6: Estimates of 15 frequencies of Macdonald-Cariter Bridge

Total number of DOF - 3008

Mode no.	Exact value [Hz]	Converged value Subspace iteration method [Hz]	Converged value Lanczos* method (39 Lanczos vectors) [Hz]	Converged value Lanczos** method (36 DRV vectors) [Hz]	Approx. value Lanczos** method (15 Lanczos vectors) [Hz]	Approx. value Lanczos** method (15 DRV vectors) [Hz]	Approx. value LWYD** method (15 Wilson's Ritz vectors) [Hz]
1	0.962	0.962	0.962	0.962	0.962	0.962	0.962
2	1.368	1.368	1.368	1.368	1.368	1.368	1.368
3	1.896	1.896	1.896	1.896	1.896	1.896	1.896
4	2.095	2.095	2.095	2.095	2.095	2.095	2.095
5	2.424	2.424	2.424	2.424	2.424	2.424	2.424
6	2.804	2.804	2.804	2.804	2.804	2.804	2.804
7	3.063	3.063	3.063	3.063	3.073		
8	3.117	3.117	3.117	3.117	3.120	3.117	3.117
9	3.126	3.126	3.126	3.126		3.126	3.126
10	3.136	3.136	3.136	3.136	3.239		4.023
11	3.969	3.969	3.969	3.969	3.971	3.981	4.024
12	4.087	4.087	4.087	4.087	4.089	4.088	
13	4.530	4.530	4.530	4.530	4.704	4.601	
14	4.689	4.689	4.689	4.689	4.879		4.716
15	4.901	4.901	4.901				
16	5.016					5.183	
17	5.236			5.236	5.503		5.270
18	5.613					5.613	5.618
(20)	6.427				6.678		6.737
(21)	7.117					7.154	
(24)	7.531					7.536	
(28)	8.560						8.757
Total CPU time		86.78 s	36.96 s	36.89 s	33.20 s	31.67 s	31.86 s
CPU time of eigen solution		63.51 s	13.69 s	13.62 s	9.18 s	9.02 s	9.34 s
		3.54 s***	3.92 s***	3.85 s***			

* - randomly chosen starting vector

** - starting vector chosen as displacement vector obtained
from a spatial distribution of load in Y - direction*** - CPU time of Sturm sequence check included
in CPU time of eigensolution

Table 6.7: Ritz algorithms comparison for Macdonald-Cartier Bridge.

Load case 1

		Exact solution	Mode shapes Subspace iteration method	DRV vectors Lanczos method	DRV vectors Lanczos method Tridiagonal form	Wilson's Ritz vect. LWYD method
1	No. of vectors	—	15	10	10	10
	Mass participation [%] ξ_L or η_L	—	3.68	81.11	81.11	80.85
	CPU time [s]	2652.8	157.12	172.01	130.52	171.86
2	No. of vectors	—	30	30	30	30
	Mass participation [%] ξ_L or η_L	—	14.17	86.21	86.21	85.89
	CPU time [s]	2652.8	253.02	195.98	163.68	196.43
3	No. of vectors	—	100	38	38	38
	Mass participation [%] ξ_L or η_L	—	37.51	90.93	90.93	90.53
	CPU time [s]	2652.8	2069.3	241.91	176.12	243.37

Table 6.8: Ritz algorithms comparison for Macdonald-Cartier Bridge,
Load case 2

			Exact solution	Mode shapes Subspace iteration method		DRV vectors Lanczos method		Wilson's Ritz vect. LWYD method	
1	No. of vectors		—	15		10		10	
	Mass participation [%]	η_L	—	23.51		45.79		46.29	
		ξ_L		X	54.27	X	55.46	X	56.12
				Y	3.68	Y	26.85	Y	27.19
			Z	12.79	Z	28.43	Z	29.31	
CPU time [s]			2366.1	131.97		79.53		81.47	
2	No. of vectors		—	30		20		20	
	Mass participation [%]	η_L	—	30.35		65.78		65.93	
		ξ_L		X	64.51	X	66.91	X	67.34
				Y	13.33	Y	33.50	Y	32.64
			Z	13.04	Z	53.75	Z	54.15	
CPU time [s]			2366.1	216.95		112.17		114.28	
3	No. of vectors		—	100		30		30	
	Mass participation [%]	η_L	—	59.99		83.01		83.41	
		ξ_L		X	78.02	X	67.92	X	68.01
				Y	37.51	Y	35.08	Y	34.95
			Z	40.53	Z	53.86	Z	54.42	
CPU time [s]			2366.1	1805.17		146.51		148.86	

Table 6.9: Estimates of 15 frequencies of Alexandra Bridge

Total number of DOF - 1600

Mode no.	Exact value [Hz]	Converged value Subspace iteration method [Hz]	Converged value Lanczos* method (48 Lanczos vectors) [Hz]	Converged value Lanczos** method (45 DRV vectors) [Hz]	Approx. value Lanczos* method (15 Lanczos vectors) [Hz]	Approx. value Lanczos** method (15 DRV vectors) [Hz]	Approx. value LWYD** method (15 Wilson's Ritz vectors) [Hz]
1	0.795	0.795	0.795	0.795	0.795	0.795	0.795
2	1.040	1.040	1.040	1.040	1.040	1.040	1.040
3	1.133	1.133	1.133	1.133			
4	1.134	1.134	1.134	1.134	1.135	1.135	1.135
5	1.159	1.159	1.159	1.159			
6	1.160	1.160	1.160	1.160	1.205	1.163	1.166
7	1.230	1.230	1.230	1.230		1.245	1.245
8	1.242	1.242	1.242	1.242	1.251	1.255	1.257
9	1.259	1.259	1.259	1.259	1.264		
10	1.272	1.272	1.272	1.272			
11	1.273	1.273	1.273	1.273	1.437	1.359	
12	1.482	1.482	1.482	1.482			1.482
13	1.483	1.483	1.483	1.483	1.490	1.483	1.549
14	1.548	1.548	1.548	1.548			1.550
15	1.552	1.552	1.552	1.552	1.552	1.552	
(23)	1.682					1.682	1.682
(24)	1.684					1.711	
(26)	1.917				2.055		
(27)	1.939				2.286		
(30)	2.362				2.380		2.425
(35)	2.497					2.503	2.716
(48)	2.853					2.952	
(55)	3.061				3.078		3.192
(59)	3.243						
(67)	3.463					3.485	
(69)	3.675						3.784
(71)	3.913				3.991	4.028	4.057
Total CPU time		80.91 s	21.04 s	20.92 s	16.56 s	16.59 s	16.48 s
CPU time of eigen solution		71.08 s	11.23 s	11.11 s	6.85 s	6.84 s	6.81 s
		2.80 s***	2.90 s**	2.90 s***			

* - randomly chosen starting vector

** - starting vector chosen as displacement vector obtained
from a spatial distribution of load in Y - direction*** - CPU time of Sturm sequence check included
in CPU time of eigensolution

Table 6.10: Ritz algorithms comparison for Alexandra Bridge,
Load case 1

		Exact solution	Mode shapes Subspace iteration method	DRV vectors Lanczos method	DRV vectors Lanczos method Tridiagonal form	Wilson's Ritz vect. LWYD method
1	No. of vectors	—	10	5	5	5
	Mass participation [%] ξ_L or η_L	—	27.84	79.31	79.31	79.32
	CPU time [s]	1231.6	70.09	36.99	35.49	36.72
2	No. of vectors	—	18	10	10	10
	Mass participation [%] ξ_L or η_L	—	35.68	85.83	85.83	85.87
	CPU time [s]	1231.6	93.76	43.92	38.46	44.06
3	No. of vectors	—	80	18	18	18
	Mass participation [%] ξ_L or η_L	—	41.46	88.59	88.59	88.79
	CPU time [s]	1231.6	397.41	57.64	46.90	57.67

Table 6.11: Ritz algorithms comparison for Alexandra Bridge (Part 1).

Load case 2

			Exact solution	Mode shapes Subspace iteration method	DRV vectors Lanczos method	Wilson's Ritz vect. LWYD method
1	No. of vectors		—	18	5	5
	Mass participation [%]	η_L	—	18.43	43.43	45.67
		ξ_L		X 0.08 Y 35.68 Z 17.43	X 17.24 Y 31.95 Z 21.95	X 26.97 Y 37.92 Z 36.03
	CPU time [s]		1063.2	84.82	31.70	31.41
2	No. of vectors		—	40	10	10
	Mass participation [%]	η_L	—	24.98	70.53	76.63
		ξ_L		X 15.33 Y 38.34 Z 17.55	X 42.53 Y 41.77 Z 50.90	X 47.99 Y 47.37 Z 52.06
	CPU time [s]		1063.2	168.13	37.28	36.81
3	No. of vectors		—	60	18	18
	Mass participation [%]	η_L	—	29.21	86.68	85.66
		ξ_L		X 16.61 Y 39.06 Z 37.20	X 55.76 Y 50.63 Z 57.55	X 55.70 Y 50.80 Z 56.67
	CPU time [s]		1063.2	271.55	47.02	46.94

Table 6.12: Ritz algorithms comparison for Alexandra Bridge (Part 2),
Load case 2

			Exact solution	Mode shapes Subspace iteration method		
1	No. of vectors		—	80		
	Mass participation [%]	η_L	—	35.71		
		ξ_L		X	26.23	
				Y	41.46	
				Z	49.31	
	CPU time [s]		1063.2	385.0		
2	No. of vectors		—	100		
	Mass participation [%]	η_L	—	67.06		
		ξ_L		X	56.80	
				Y	71.73	
				Z	75.77	
	CPU time [s]		1063.2	590.5		
3	No. of vectors		—	140		
	Mass participation [%]	η_L	—	84.54		
		ξ_L		X	84.61	
				Y	84.02	
				Z	83.57	
	CPU time [s]		1063.2	1142.7		

Table 6.13: Ritz algorithms comparison for Alexandra Bridge.

Load case 3

			Exact solution	Mode shapes Subspace iteration method		DRV vectors Lanczos method		Wilson's Ritz vect. LWYD method	
1	No. of vectors		—	60		6		6	
	Mass participation [%]	η_L	—	38.52		62.17		73.81	
		ξ_L		X	16.61	X	56.84	X	58.89
				Y	39.06	Y	33.84	Y	49.44
		Z	37.20	Z	0.75	Z	1.41		
CPU time [s]			1064.69	284.21		33.80		34.10	
2	No. of vectors		—	100		16		16	
	Mass participation [%]	η_L	—	74.37		85.33		85.43	
		ξ_L		X	53.80	X	60.89	X	61.01
				Y	71.73	Y	63.10	Y	63.23
				Z	75.77	Z	20.68	Z	20.53
CPU time [s]			1064.69	777.65		47.68		48.23

Figure 6.1: Imperial Valley Earthquake of 1940

Figure 6.2: Saguenay Earthquake of 1988

Figure 6.3: Directions of seismic action (seismic load cases)

Figure 6.4: Fictitious offshore platform modelled as a shear beam structure

(a) Bridge elevation

(b) Typical deck cross section

(c) Finite element model

Figure 6.5: Elevation, cross-sectional dimensions and finite element model of Blackstone River Bridge

Node no. 17 mid-span
Transverse translational component

Figure 6.6: Displacement time history response of Blackstone River Bridge.
Load case 1

Transverse component

Figure 6.7: Base shear time history response of Blackstone River Bridge.
Load case 1

Figure 6.8: Base moment time history response of Blackstone River Bridge.

Load case 1

Node no. 17 mid-span
Transverse translational component

Figure 6.9: Displacement time history response of Blackstone River Bridge.
Load case 2

Transverse component

Figure 6.10: Base shear time history response of Blackstone River Bridge
(Part 1), Load case 2

Transverse component

Figure 6.11: Base shear time history response of Blackstone River Bridge
(Part 2). Load case 2

Transverse component

Figure 6.12: Base moment time history response of Blackstone River Bridge
(Part 1). Load case 2

Figure 6.13: Base moment time history response of Blackstone River Bridge
(Part 2). Load case 2

(a) Bridge elevation

(b) Typical girder cross section

(c) Typical cross section

Figure 6.14: Cross-sectional dimensions of Macdonald-Cartier Bridge

Figure 6.15: Layout of major structural members of Macdonald-Cariter Bridge

Figure 6.16: Finite element model of Macdonald-Cariter Bridge

Node no. 222 at mid-span between piers 2 & 3
Transverse translational component

Figure 6.17: Displacement time history response of Macdonald-Cartier Bridge
(Part 1), Load case 1

Node no. 222 at mid-span between piers 2 & 3
Vertical translational component

Figure 6.18: Displacement time history response of Macdonald-Cartier Bridge
(Part 2), Load case 1

Node no. 222 at mid-span between piers 2 & 3
Transverse translational component

Figure 6.19: Displacement time history response of Macdonald-Cartier Bridge
(Part 1). Load case 2

Node no. 222 at mid-span between piers 2 & 3
Vertical translational component

Figure 6.20: Displacement time history response of Macdonald-Cartier Bridge
(Part 2), Load case 2

Transverse component

Figure 6.21: Base shear time history response of Macdonald-Cartier Bridge at Pier 3 (Part 1). Load case 1

Figure 6.22: Base shear time history response of Macdonald-Cartier Bridge at Pier 3 (Part 2). Load case 1

Figure 6.23: Base moment time history response of Macdonald-Cartier Bridge at Pier 3 (Part 1), Load case 1

Figure 6.24: Base moment time history response of Macdonald-Cartier Bridge at Pier 3 (Part 2). Load case 1

Figure 6.25: Base shear time history response of Macdonald-Cartier Bridge at Pier 3 (Part 1). Load case 2

Figure 6.26: Base shear time history response of Macdonald-Cartier Bridge at Pier 3 (Part 2). Load case 2

Transverse component

Figure 6.27: Base moment time history response of Macdonald-Cartier Bridge at Pier 3 (Part 1). Load case 2

Transverse component

Figure 6.28: Base moment time history response of Macdonald-Cartier Bridge at Pier 3 (Part 2). Load case 2

Figure 6.29 : Elevation and cross-sectional dimensions of Alexandra Bridge

Figure 6.30: Truss element mesh of Alexandra Bridge

Figure 6.31: Axial force time history response of Alexandra Bridge

Truss member 404. Load case 1

Figure 6.32: Axial force time history response of Alexandra Bridge
Truss member 404 (Part 1), Load case 2

Figure 6.33: Axial force time history response of Alexandra Bridge
Truss member 404 (Part 2). Load case 2

Figure 6.34: Axial force time history response of Alexandra Bridge
Truss member 404 (Part 3), Load case 2

Node no. 106 at
Mid-span on deck level between piers 2 & 3
Transverse translational component

Figure 6.35: Displacement time history response of Alexandra Bridge
(Part 1), Load case 1

Figure 6.36: Displacement time history response of Alexandra Bridge
(Part 2). Load case 1

Node no. 106 at
Mid-span on deck level between piers 2 & 3
Transverse translational component

Figure 6.37: Displacement time history response of Alexandra Bridge
(Part 1), Load case 2

Figure 6.38: Displacement time history response of Alexandra Bridge
(Part 2). Load case 2

Figure 6.39: Axial force time history response of Alexandra Bridge
Truss member 404 (Part 1), Load case 3

Figure 6.40: Axial force time history response of Alexandra Bridge
Truss member 404 (Part 2), Load case 3

Chapter 7

Summary, Conclusion and Recommendations

7.1 Summary

The research work presented in this thesis had been directed mainly toward the computer implementation and computational efficiency study of Ritz vector analysis algorithms applied in the dynamic analysis of bridges. Dynamic response analysis of large complex bridge systems is time consuming and requires large amount of CPU time and data storage. The results of this study show that the Ritz vector approach is very effective and efficient for dynamic response analysis of bridge systems. The Ritz vector algorithms implemented in this thesis include the Lanczos method with selective and partial reorthogonalization procedures as well as the improved Wilson's algorithm. The results from the numerical study show that they give accurate results

for displacement and internal force time history responses. They are much more efficient than the traditional modal analysis procedure.

The present thesis is organized into five parts. The first part is devoted to the theoretical development of the different type of Ritz vectors. The basis of general numerical techniques for the eigensolution problem is reviewed. Particular consideration is placed upon the eigensolution methods for large, sparse matrix systems, such as the subspace iteration procedure and the Lanczos method.

In the second part the Lanczos method is discussed in detail as a very effective analysis tool for frequency as well as dynamic response analysis. In particular, the convergence characteristics and behaviour of the method is discussed. The orthogonalization techniques and procedures aimed to improve the stability of the Lanczos algorithm are examined.

The development and use of a special class of Ritz vectors obtained from the spatial distribution of the dynamic load are discussed in the third part. The computational procedures for the generation of load-dependent Ritz vectors for dynamic response analysis are presented in a form similar to the Lanczos and Wilson's algorithms. The theoretical development and computational implementation of a new cut-off criterion based on the concept of the effective mass participation for multi-directional earthquake loading are presented. The validity of the new criterion is demonstrated by a numerical study.

In the fourth part of this thesis modified and improved versions of the Lanczos and Wilson's methods are implemented in NEABS, a computer program for dynamic analysis of bridge systems.

The efficiency and effectiveness of the implemented techniques are examined by means of frequency and dynamic response analysis of linear bridge systems. This is presented in the fifth and final part of this thesis.

7.2 Conclusions

Based on the results from the comparison study presented in this thesis, the following conclusions may be stated:

1. In comparison to Ritz vector analysis techniques, the subspace iteration procedure found to be a relatively expensive and time consuming technique for the analysis of a large structural systems such as bridges.
2. In comparison to the subspace iteration procedure the Lanczos method is found to be very efficient to obtain exact eigensolutions. The extent of the saving in CPU time depends on the size of the problem.
3. Approximate vibrational frequencies obtained by the Lanczos and improved Wilson's Ritz methods (LWYD) are in good agreement with the exact solution of the original system.
4. The approximated frequencies of the reduced eigenproblem spread over a wide spectrum of frequency in comparison to the exact eigenvalues. Their convergence characteristics are a function of the number of Ritz vectors used in the analysis.
5. A priori selection of the required number of mode shapes necessary to obtain satisfactory results in dynamic response analysis can not be applied to the

subspace iteration procedure. The calculation of effective mass participation factors is delayed until the iteration procedures to obtain mode shapes are completed.

6. In the Lanczos algorithm with selective reorthogonalization procedure, the exact eigenvalues and corresponding mode shapes are calculated a few steps after their convergence is indicated in the analysis. Therefore, the effective mass participation factors may be monitored at each step and terminated when the specified level of the effective mass participation is reached.
7. The Lanczos and Wilson's Ritz methods are very effective for a time history response analysis. The number of generated load-dependent Ritz vectors which accurately represent the spatial distribution of the load is significantly less than that required in modal analysis. The generation of these Ritz vectors can be terminated when the specified level of load or mass participation in the analysis is achieved.
8. The Ritz vectors approach may be characterized by a significant saving in computational time and memory storage as compared to the direct step-by-step time integration procedure.
9. The results and performance of both load-dependent Ritz vector procedures are very similar and neither the Lanczos nor Wilson's method has clear advantage over the other. When the tridiagonal form of the Lanczos method is used in the solution, the computation time is slightly reduced, which may be considered as more efficient than the other algorithms.
10. The Lanczos method with the combined selective and partial reorthogonalization procedure as well as the improved Wilson's Ritz algorithm maintain

orthogonality among the generated load-dependent Ritz vectors for large structural systems. This is an essential requirement for the proper and fast convergence of the results.

11. It is anticipated that the analysis of larger size problems than the ones considered in the present study will better demonstrate the advantage of the Lanczos method with semi-orthogonalization procedure over the Wilson's method. The further saving in the solution of large size problem is achieved through the reduced cost of the reorthogonalization process at each time step.
12. The new cut-off criterion for multi-directional earthquake loading gives accurate estimate on the number of load-dependent Ritz vectors or mode shapes that should be generated to adequately represent the total activated mass and thus will guarantee accurate results.
13. The concept of separate mass participation factor for each excitation direction does not take into an account how mass in one direction participates in the dynamic response when activated by earthquake ground motion in the other directions, which may lead to an unnecessary increase in a number of Ritz vectors used in the analysis.
14. Approximated mode shapes obtained by the Lanczos or Wilson's methods completely uncouple the dynamic equations of motion, thus, they may also be used in the response spectrum analysis of structural system.
15. Finally, the results of computational studies show that the Lanczos and Wilson's methods are equally effective in dynamic response analysis of bridge systems.

7.3 Recommendations for Future Work

In the course of this work, the need for future research and study has been identified and summarized as follows:

1. Studies are required to develop solution algorithms for nonlinear systems by load-dependent Ritz vectors.
2. Additional work is required concerning the use of the Lanczos and Wilson's methods in response spectrum analysis.
3. Additional studies are required to better understand the properties and characteristics of the proposed cut-off criterion based on the effective mass participation for multi-directional earthquake loads.

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