Internal Gravity Waves and Convection Generated by a Thermal Forcing in the Atmosphere

Amir Anwar Mohamed Sayed

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School of Mathematics and Statistics
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Carleton University

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Abstract

In this thesis, we present a mathematical model that represents internal gravity waves and convection generated by a thermal forcing in the atmosphere. The goal is to investigate the mechanisms by which deep heating and convection in the lower atmosphere generate internal gravity waves. We consider a two-dimensional two-layer model of the atmosphere comprising an upper layer with stable stratification and an unstable convective lower layer. The governing equations are based on the equations for conservation of mass, momentum and energy for a fluid. The thermal forcing is represented by a nonhomogeneous term in the energy conservation equation. We study different configurations depending on the vertical structure and depth of the thermal forcing.

First, we derive exact analytical solutions for the linearized equations for the case where the perturbation amplitude is independent of time. Next, we examine the case where the perturbation amplitude is time-dependent and derive exact solutions in each layer. The linear gravity wave solution approaches the steady solution in the limit of infinite time, but the linear solution for the convection grows exponentially with time. We also carry out a weakly-nonlinear analysis of the gravity wave problem and examine the evolution of the mean flow with time due to the nonlinear interactions of the waves.

We carry out numerical simulations for each of the linear time-dependent one-layer models and then for the two-layer model. The results of the gravity wave

simulations are in very good agreement with the analytical solution. In order to generate finite-amplitude convection, it is necessary to include the viscous and heat conduction terms in the equations. The time evolution of the convection depends on the relative strength of the unstable stratification to the strength of the viscosity and heat conduction. This is in agreement with the situation that occurs in the well-known Lorenz model for atmospheric convection.

Our solutions can be used as to represent unresolved convective gravity wave drag in large-scale models of the atmosphere.

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Statistics during the course of my thesis work.

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Dedication

To my wonderful parents, my adorable wife
and my beautiful daughters Aya, Habiba and Sondos.
For their unconditional support and endless love.

List of Symbols

- $(...)^*$ dimensional quantity of $(...)$
- $(...)^*$ complex conjugate of $(...)$
- c.c. complex conjugate of preceding term
- ∇ gradient vector
- ∇^2 Laplacian operator
- $\mathcal{L}$ Laplace transform
- $\mathcal{L}^{-1}$ inverse Laplace transform
- s Laplace transform variable
- x horizontal space variable
- y latitude
- z vertical space variable or altitude
- t time variable
- $\mathbf{u}$ velocity of the fluid
- u^* x -component of the total velocity of the fluid

- $\mathcal{U}$ x -component of the nondimensional total velocity of the fluid
- u_0 background state of $u^\star$
- u' perturbation of $u^\star$
- $\bar{u}$ mean flow velocity
- u perturbation of $\mathcal{U}$
- $v^\star$ y -component of the total velocity of the fluid
- $w^\star$ z -component of the total velocity of the fluid
- $\mathcal{W}$ z -component of the nondimensional total velocity of the fluid
- w perturbation of $\mathcal{W}$
- w_0 perturbation of $\mathcal{W}$ in region 0 in the lower layer
- w_1 perturbation of $\mathcal{W}$ in region 1 in the lower layer
- w_2 perturbation of $\mathcal{W}$ in the upper layer
- $\hat{w}$ amplitude of w
- $\tilde{w}$ Laplace transform of $\hat{w}$
- Ψ total streamfunction
- ψ perturbation of streamfunction
- $\hat{\psi}$ Fourier transform of ψ
- $\tilde{\psi}$ Laplace transform of $\hat{\psi}$
- ξ total vorticity

- ζ perturbation of vorticity
- $\hat{\zeta}$ Fourier transform of ζ
- ρ^* dimensional density of the fluid
- ϱ nondimensional density of the fluid
- ρ_0 reference state of ρ^*
- ρ' perturbation density of ρ^*
- $\bar{\rho}$ mean density or background density
- ρ perturbation density of ϱ
- $\bar{\rho}_1$ background density of the lower layer
- $\bar{\rho}_2$ background density of the upper layer
- ρ_{01} magnitude of $\bar{\rho}_1$ at $z = 0$
- ρ_{02} magnitude of $\bar{\rho}_2$ at $z = 0$
- $\hat{\rho}$ Fourier transform of ρ
- T temperature of the fluid
- T_0 reference state of the temperature
- T' perturbation of the temperature
- p^* dimensional pressure of the fluid
- p' perturbation of p^*
- $\mathcal{P}$ nondimensional pressure of the fluid

- $\bar{p}$ hydrostatic pressure
- p perturbation of $\mathcal{P}$
- $\mathcal{F}$ total thermal forcing
- F thermal forcing perturbation
- $\hat{F}$ Fourier transform of the thermal forcing F
- h_1 vertical level in the lower layer at which the thermal forcing is centered
- a parameter that specifies the horizontal extent of the wave packet thermal forcing
- b parameter that specifies the vertical depth of the thermal forcing
- k x -component of the wavevector
- l y -component of the wavevector
- m z -component of the wavevector
- ω frequency of the wave
- L_x typical length scale in the horizontal or zonal direction
- L_z typical length scale in the vertical direction
- U typical scale for the horizontal velocity
- W typical scale for the vertical velocity
- R typical scale for the density
- δ square of the vertical-to-horizontal aspect ratio
- $\mathbf{g}$ acceleration due to gravity

- h lower boundary of the two-layer model
- H scale height of the atmosphere
- N^2 square of the buoyancy frequency
- χ quantity defined by $\chi = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}$ and equals N^2 in the upper layer
- $c = c_x$ phase velocity in the x direction
- c_z phase velocity in the z direction
- $\mathbf{C}_g$ group velocity
- C_{gx} group velocity in the x direction
- C_{gz} group velocity in the z direction
- C_p heat capacity at constant pressure
- C_V heat capacity at constant volume
- e internal energy
- κ thermal diffusivity
- $\mathbf{q}$ heat flux vector per unit area
- Q diabatic heat
- φ viscous dissipation
- μ constant of viscosity
- ν kinematic viscosity
- ε characteristic parameter of the nonlinearity of the governing equations

- $\overline{(\dots)}$ average over a wavelength $\frac{2\pi}{k}$ of $(\dots)$
- $\psi^{(0)}$ leading order term in the perturbation series of the streamfunction
- $\psi^{(1)}$ first nonlinear correction term in the perturbation series of the streamfunction
- $\phi^{(0)}$ first-wavenumber component of the streamfunction perturbation
- $\phi_0^{(1)}$ zero-wavenumber component of the streamfunction perturbation
- $\phi_2^{(1)}$ second-wavenumber component (second harmonic) of the streamfunction perturbation
- $\rho^{(0)}$ leading order term in the perturbation series of the density
- $\rho^{(1)}$ first nonlinear correction term in the perturbation series of the density
- $\mathcal{R}^{(0)}$ first-wavenumber component of the density perturbation
- $\mathcal{R}_0^{(1)}$ zero-wavenumber component of the density perturbation
- $\mathcal{R}_2^{(1)}$ second-wavenumber component of the density perturbation
- $\bar{u}_0$ zero-wavenumber component of the mean horizontal velocity
- $\bar{F}$ average of the vertical flux of horizontal momentum

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Chapter 1

Introduction

1.1 Internal Gravity Waves and Convection

The Earth's atmosphere is a fluid that is continually being acted upon by the force of gravity. There is therefore a distinct decrease in density with altitude, so the atmosphere is almost always stably stratified. A stably stratified fluid is one in which the fluid density decreases with height. The atmosphere, due to the atmosphere stratification, is capable of sustaining a large number of waves [see, for example, Beer (1974), Holton (2004) and Sutherland (2010)].

The atmosphere always contains waves forced by buoyancy and gravity as well as waves that are influenced by the curvature of the Earth and its rotation (the Coriolis effects). The waves that result from the Coriolis force are called planetary waves or Rossby waves, after the Swedish meteorologist Carl-Gustaf Rossby (1898-1957). These are large-scale waves with wavelengths thousands of kilometers long. The waves that propagate due to upward buoyancy force and the restoring force of gravity are called buoyancy waves or internal gravity waves. The buoyancy force results from the variation of fluid density and temperature with height and it acts to lift a parcel of fluid upwards, while the downward gravitational force acts to bring

the fluid back to its original state. The competing effects of the two forces result in oscillatory motions or waves in the fluid. The frequency of gravity waves is generally much larger than the Coriolis frequency. So in studying the propagation of the internal gravity waves, the effect of the Earth's rotation on the waves can be neglected. There are also inertia-gravity waves which result from a combination of forces; the Coriolis force and the gravitational and buoyancy forces. This study will focus on the internal gravity waves which occur in the interior of a fluid and we will simply refer to these internal gravity waves as gravity waves.

We cannot see atmospheric gravity waves, like many waves that are invisible, however we can see their effects in the clouds. When gravity waves occur at the free surface of a liquid, they are called surface gravity waves. These waves are observed at the surface of oceans, rivers and at beaches. In a fluid such as the ocean, which is bounded both above and below, gravity waves propagate primarily in the horizontal plane since vertically traveling waves are reflected from the boundaries to form standing waves. However, in a fluid that has no upper boundaries, such as the atmosphere, gravity waves may propagate vertically as well as horizontally [Holton (2004)].

Atmospheric gravity waves occur naturally in the atmosphere and oceans and produce large-scale effects on the general circulation of the atmosphere and oceans. They are often associated with the formation of clear air turbulence. They occur on a variety of spatial and temporal scales and at all altitudes in the atmosphere. However the major wave effects occur in the middle atmosphere between about 10 and 110 km altitudes because of decreasing density [Fritts and Alexander (2003)].

The most observable and important sources of the gravity waves are topography and convection. The mechanisms for the generation of topographic gravity waves have been studied extensively [see, for example, Baines (1995) and Wurtele, Sharman and Datta (1996)] but the convective generation mechanisms are less well understood. This thesis research project is a mathematical study of the generation of the gravity waves by convection. Convection is a movement of fluid from one place to another

place. All motion that can be attributed to the action of a steady gravitational field upon variations of density in a fluid may be called convective motion and thus all the kinetic energy of the Earth's atmosphere and oceans and the bulk of that of many fluid systems in the known universe results from convection. Convection is described by an equation of the form

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F = 0,$$

where $F(\mathbf{x}, t)$ is any flow variable, $\mathbf{u}$ is the velocity vector, the vector $\mathbf{x}$ is the position vector of a point and t represents time. The time derivative $\frac{\partial F}{\partial t}$ is the local rate of change of F at a given location. The term $\mathbf{u} \cdot \nabla F$ is the change in F that results from the movement of a fluid parcel from one place to another place where the value of F is different. Thus the notation $\frac{DF}{Dt}$ represents the rate of change of F following a fluid particle moving with velocity $\mathbf{u}$.

In the atmosphere, whether or not convection will occur depends on the lapse rate. The lapse rate is the rate at which the temperature of the atmosphere decreases with height. Convection occurs only when the lapse rate exceeds a certain value known as the dry adiabatic lapse rate. As a fluid parcel rises, the pressure falls, the parcel expands, and thus its temperature falls. If the temperature of the surroundings decreases more quickly with height, a rising parcel will be warmer than its surroundings and will therefore continue to rise, therefore the situation will be unstable and convection occurs. Convection carries heat upward and thus reduces the lapse rate until it falls to the equilibrium value, in this case convection no longer occurs [see, for example, Gill (1982)].

The atmosphere is divided into distinct layers because of its distinctive characteristics in different height ranges. Immediately above the Earth's surface is the troposphere where our weather occurs. It is characterized by strong vertical mixing due to instabilities that arise from a temperature decrease with height. It is convective with an average lapse rate of -6.5 Kkm^{-1} [Marshall, Plumb and Alan (1965)].

The tropopause is the top of the troposphere. Above that, the stratosphere and mesosphere comprise what is called middle atmosphere. The two regions are separated by stratopause. The stratosphere is poorly mixed, the temperature increases with height resulting in strong stability. The mesosphere is a region in which the temperature decreases again. The top of the mesosphere is the mesopause. Above these is the upper atmosphere composed of the thermosphere, where the temperature increases rapidly with height [for more details see, for example, Marshall, Plumb and Alan (1965), Lodders and Fegley (1998) and Holton (2004)]. It is known that convection in the troposphere (the lower atmosphere) generates gravity waves that propagate into the stratosphere, mesosphere and thermosphere (the middle and upper atmosphere).

The lapse rate of temperature in the Earth's atmosphere is approximately constant in each layer, changing abruptly at the tropopause, stratopause and mesopause. The abrupt change in lapse rate is coincident with an abrupt change in the buoyancy frequency N [McHugh (2009)]. The buoyancy frequency is also called the Brunt-Väisälä frequency after David Brunt and Vilho Väisälä and it is the most convenient measure of stability. The square of the buoyancy frequency is given by

$$N^2 = -\frac{g}{\rho} \frac{\partial \rho}{\partial z},$$

where ρ is the density of the fluid flow and g is the gravitational acceleration. Stability occurs when $N^2 > 0$ or equivalently when the density decreases with height i.e., $\frac{\partial \rho}{\partial z} < 0$ for all z . This means that an upwardly-displaced particle will be heavier than its surroundings and so will experience a restoring force. This force causes the particle to oscillate around its starting position with frequency N . In the stratosphere $N^2 \approx 5 \times 10^{-4} \text{ s}^{-2}$ [see, for example, Sutherland (2010)]. Conversely for unstable stratification the density increases with height i.e., $\frac{\partial \rho}{\partial z} > 0$, as heavier fluid particles are resting on top of lighter ones. In this case, an upwardly-displaced particle will be surrounded by heavier particles and so it will continue to rise and convection then ensues.

Studying convectively-generated gravity waves is important because they transport energy and momentum from the troposphere into the middle atmosphere. They thus contribute towards turbulence and mixing [e.g., Lindzen (1981)] and modify the global circulation of the atmosphere [e.g., Holton (1982) and (1983), Vincent and Ried (1983), Chun and Baik (1998)]. They influence the wind speed and consequently the thermal structure of the atmosphere and thus have important implications for both weather and climate.

1.2 Previous Work and Motivation

There have been a number of studies aiming to reveal more information on the structure of gravity waves and their properties such as propagation speed and direction, gravity wave interactions with each other, e.g., linear superposition of the waves or nonlinear wave interactions which is formulated mathematically by products of wave terms in the equations and interactions with the background fluid flow. There have been analytical studies [e.g., Bretherton (1966), Booker and Bretherton (1967) and Brown and Stewartson (1982)] as well as numerical simulations [e.g., Fritts (1982) and Campbell and Maslowe (2003)]. Some mechanisms by which gravity waves affect the mean flow are described by the models of Lindzen (1981) and Holton (1982).

It has been known for decades that convection is an important source of the atmospheric waves. In more recent years, observations and numerical models have begun to study the characteristic of the gravity waves and the mechanism of their generation by convection. Observations show that gravity wave propagation is related to convection as a source in the middle atmosphere [e.g., Pfister et al. (1986), Tsuda et al. (1994) and Vincent and Alexander (2000)] and in the upper atmosphere [e.g., Tsuda et al. (1990) and Venkateswara Rao, Wiryosumarto and Harijono (2011)]. Relations between convection and gravity waves have been observed in the troposphere and the lower atmosphere [e.g., Alexander et al. (2008) and Dutta et al. (2009)] and

in the upper atmosphere [e.g., Kovalam, Vincent and Love (2006) and Taylor et al. (2009)].

Although the observations suggest some relations between gravity waves and convection, the results are insufficient to give a complete description of the global circulation of the atmosphere due to the limitation of radar observations to only a few locations [Sato (1992)] and inadequate information given by the aircrafts observations about the vertical structure of the waves [e.g., Alexander and Pfister (1995) and Alexander, Beres and Pfister (2000)]. A better understanding of the atmospheric behaviour can be obtained by modeling the atmospheric circulation using large-scale atmospheric models.

Large-scale atmospheric models or global circulation models (GCMs) are complex computer programs based on the governing equations of motion which are solved using numerical methods to accurately predict weather (short term) and climate (long term). In order for these predictions to be reliable, the models need to accurately simulate phenomena such as convection, internal gravity waves and other atmospheric waves. However, because gravity waves are relatively small-scale phenomena with wavelengths smaller than a few hundred kilometers, very high spatial resolution is needed in order for GCMs to represent them accurately. The order of magnitude of the GCM grid spacing is typically of 100 km, and hence they are not always able to resolve small-scale gravity waves.

Atmospheric scientists and modelers parameterize the effects of the gravity waves in GCMs by adding to the equations in the model a term called the gravity wave drag to represent the drag force that would have resulted from the gravity waves if they had been correctly resolved by the model. Some examples of gravity wave drag parameterization schemes are those developed by Lindzen (1981), Hines (1997) and Alexander and Dunkerton (1999). In order to develop accurate parameterizations of the gravity wave drag, it is important to understand the mechanisms that generate them. The mechanisms responsible for generating gravity waves by convection are not well un-

derstood due to insufficient information that relates gravity waves and convection. This lack of information results in difficulty in obtaining accurate parameterizations of the convectively-generated gravity waves.

There have been several studies in the past few years that involved simulations using general circulation models to test out various hypotheses for gravity wave generation mechanisms [for example Alexander, Holton and Durran (1995), Pandya and Alexander (1999), Piani et al. (2000) and Song and Chun (2003)]. Numerical simulations and modeling proposed three simplified mechanisms that generate gravity waves by convection. These are summarized by Beres, Alexander and Holton (2002) and Fritts and Alexander (2003) as follows:

(1) Deep heating:

In this mechanism, the gravity waves are mainly excited by a thermal forcing with no shear. Some studies based on numerical simulations and models showed that in this mechanism, the wavelength of the dominant vertical wavelength of the excited tropospheric gravity waves is approximately twice the depth of the heating. Since the buoyancy frequency in the stratosphere is approximately double that in the troposphere then in the absence of shear the vertical wavenumber is proportional to the buoyancy frequency [see, for example, Salby and Garcia (1987), Bergman and Salby (1994), Alexander, Holton and Durran (1995) and Piani et al. (2000)].

(2) Obstacle effect:

In this mechanism, the gravity waves are generated in a similar way to how topographic gravity waves are generated, in which the topography or mountain acts as an obstacle [e.g., Clark, Hauf and Kuettner (1986) and Pfister et al. (1993)].

(3) Mechanical oscillator:

In this mechanism, updraft and downdraft oscillations produce vertically propagating

gravity waves similar to that of a mechanical oscillator in a stratified fluid [e.g., Fovell, Durran and Holton (1992) and Lane, Reeder and Clark (2001)].

In reality the three mechanisms are not distinct but are coupled. Clark, Hauf and Kuettner (1986) found that gravity waves in a stable layer overlying convection can be generated by both the deep heating and the obstacle effect mechanisms.

A number of studies have been carried out to develop parameterizations for convective gravity waves [e.g., Chun et al. (2004), Beres (2005), Beres et al. (2005) and Richter, Sassi and Garcia (2010)]. Various parameterizations that are based on theoretical studies provide realistic estimates of the convectively-generated gravity waves activity, for example, the parameterization given in Chun and Baik (2002) and Chun et al. (2004), which is based on the linear theory [Chun and Baik (1998)], and the parameterization of Beres (2005) and Beres et al. (2005), which is based on the theory described by Beres, Alexander and Holton (2004).

Although simulations and modeling have been able to examine and interpret various hypotheses for the convective generation of gravity waves, the high level of complexity of GCMs and the large number of degrees of freedom involved makes it complicated to identify and quantify relationships between the gravity waves and the convection. A mathematical study based on relatively simple equations that can be solved either analytically or numerically would allow us to investigate these relationships in a more straightforward way.

Most theoretical studies of gravity waves have been based on linear theory. If the wave amplitude is sufficiently small relative to the magnitude of the background flow, we can linearize the equations. The linearization process expresses each of the total quantities in the governing equations as the sum of a background steady part and a perturbation. The waves are considered to be perturbations to a basic background state. Further simplifications can be made to the equations based on the particular problem we wish to study. Linearity simplifies the governing equations and allows the use of Fourier transforms (and other transform methods such as Laplace transforms),

but means there can be no interaction between waves (different Fourier modes) or between the waves and the background flow. In our study we first consider the linear problem and then carry out a weakly-nonlinear analysis.

In a nonlinear problem, there are interactions between waves and between the waves and the mean flow. The mean flow changes as a result of these interactions. Although wave-mean flow interaction is in general a fully nonlinear phenomenon, some insight can be obtained by studying relatively simple weakly-nonlinear models. The process of the weakly-nonlinear analysis introduces a small parameter ε such that $\varepsilon \ll 1$ and seeks a solution of the form of a series of powers of ε , i.e., $w = w^{(0)} + \varepsilon w^{(1)} + \mathcal{O}(\varepsilon^2)$. The leading-order term $w^{(0)}$ is the linear solution which can be obtained by taking the limit as ε approaches zero. The first correction $w^{(1)}$ can be obtained from the substitution of w into the nonlinear model. The mean flow evolves with time due to the wave interactions and the change in the mean flow $\bar{u}$ can be described by the following equation

$$\frac{\partial \bar{u}}{\partial t} = \varepsilon^2 \frac{\partial \bar{F}}{\partial z}, \quad (1.1)$$

where $\bar{F}$ is the horizontal average of the vertical flux of horizontal momentum of the $\mathcal{O}(\varepsilon)$ perturbation (see Appendix C).

It is known that the interactions between gravity waves and the mean flow produce important oscillations in the middle atmosphere. Examples of these oscillations are the quasi-biennial oscillation (QBO) and the semi-annual oscillation (SAO). The QBO is a variation in mean flow $\bar{u}$ resulting from the divergence of wave momentum flux according to mechanism shown in equation (1.1). It is understood that the QBO is driven by large-scale atmospheric waves [Holton and Lindzen (1972)] as well as smaller-scale gravity waves [Lindzen (1981)] that are convectively-generated in the troposphere, although there has been considerable debate in the past few decades on the relative contributions of the two types of waves. A realistic representation of the QBO in a climate model is one of the major issues in global climate modeling. In

order to correctly model the QBO and other phenomena in the middle atmosphere, it is important to correctly represent the generation of the gravity waves in the lower atmosphere. The QBO is now simulated in a growing number of climate models [e.g., Takahashi (1996) and Giorgetta, Manzini and Roeckner (2006)] either by using very high resolution or by using a gravity wave drag parameterization. There are many simulations that dealt with parameterization of the convectively-generated gravity waves in QBO models [e.g., Choi and Chun (2011) and Kim et al. (2013)].

In summary, an important goal of studying gravity waves is to develop good physical and theoretically based gravity wave parameterizations for use in GCMs. There have been some theoretical studies that dealt with the generation of the gravity waves by convection and demonstrate connections between the characteristics of the gravity waves and the convection. One such study was carried out by Sang (1991). This study considers a two-layer model comprising an unstable lower layer and a stable upper layer with a thermal forcing centered at the interface. The equations are linear steady and approximated by the Boussinesq approximation. This study focused on the case where the amplitude of the disturbances in the upper layer decay exponentially with height. These are trapped disturbances which would not reach higher levels of the atmosphere or have any significant effect on the background flow. However, there are some unclear assumptions and omissions. The model that is developed and described in this thesis is similar to that of Sang (1991), but with inconsistencies corrected and with more details added.

Another theoretical study of gravity waves generated by convection is that of Chun and Baik (1998). Once again, the equations are linear two-dimensional Boussinesq with constant background velocity. The buoyancy frequency is assumed to be constant with height. The gravity wave drag parameterizations of Chun and Baik (2002) and Chun et al. (2004) are based on the results of this study.

An interesting series of contributions to this topic was started by Holton, Beres and Zhuo (2002). Their analysis involves a two-dimensional linear Boussinesq model

with zero background velocity (i.e., a motionless atmosphere) and with a thermal forcing added as a nonhomogeneous term in the equation for conservation of energy. The configuration is based on that of Hayashi (1975) in which the thermal forcing has a half sine-wave structure in the underlying heating region and is specified in the form of a Gaussian distribution horizontally and periodic in time. The buoyancy frequency is assumed constant and does not change with height. This study was extended by Beres, Alexander and Holton (2004) to a multifrequency thermal forcing with constant mean velocity and further extended by Beres (2004) to a three-dimensional model. The model described in these papers form the basis for a parameterization in Beres (2005) and Beres et al. (2005) that can be used to represent gravity wave effects in GCMs.

In this study, we develop a two-dimensional atmospheric model for gravity waves generated by convection in a two-dimensional region defined by cartesian coordinates x in the horizontal direction and z in the vertical direction. The model comprises two layers: an upper layer with a density profile with stable stratification and an unstable lower layer. The two layers are connected according to specified interface conditions. The lower boundary of the lower layer represents the surface of the earth and the upper layer extends up to infinity. We add a nonhomogeneous term to the governing equations to represent a thermal forcing (deep heating). This forcing term generates a solution in the lower layer comprising closed convective cells connected by the interface conditions to a solution in the upper layer consisting of oscillations in the vertical direction with upward (positive) group velocity. The horizontal profile for the thermal forcing term is chosen to be either a periodic function of x i.e., e^{ikx} which generates waves with a single wavenumber k or a function in the form of Gaussian distribution, the latter configuration is similar to that of Sang (1991) and Chun and Baik (1998).

We consider a background density that varies with height, denoted by $\bar{\rho}(z)$. We assume that the variation of $\bar{\rho}$ with height is small and so the derivative $\frac{d\bar{\rho}}{dz}$ is small

compared with $\bar{\rho}$ and terms involving $\frac{d\bar{\rho}}{dz}$ can be neglected. This simplification gives a form of the Boussinesq approximation. Previous related studies [e.g., Chun and Baik (1998) and Holton, Beres and Zhuo (2002), Beres (2004) and Beres, Alexander and Holton (2004)] used the standard form of the Boussinesq approximation where $\bar{\rho}$ is set to a constant everywhere in the governing equations except in the buoyancy term. By allowing $\bar{\rho}$ to vary with height, we obtain a more realistic model but can still find exact solutions under certain conditions. The density in the upper layer decreases with height and the gravity waves propagate upwards. In the lower layer the density increases with height and unstable stratification occurs and convection ensues.

By changing the vertical structure, the depth and the vertical location of the thermal forcing, our model can be set in to represent any of the three convective gravity wave generation mechanisms described before. The “obstacle effect” mechanism can be constructed in the model by centering the thermal forcing in the lower layer such that it affects only the lower layer. In this case the thermal forcing generates convection in the lower layer which then generates gravity waves in the upper layer through the interface conditions in a similar manner to how topographic gravity waves are generated by flow over an obstacle such as a mountain. The “deep heating” mechanism can also be obtained by considering a model with a single stable layer with the thermal forcing applied to generate waves. Both mechanisms can be achieved by centering the thermal forcing at the interface such that it affects both layers. The “mechanical oscillator” mechanism can be obtained by applying an oscillatory lower boundary condition to a single stable layer to generate gravity waves without the nonhomogeneous thermal forcing term. This would be similar to the configuration used in earlier studies [see Booker and Bretherton (1967), Figure 1].

In our configuration, the effects of the gravity waves result in a drag force on the mean flow. The vertical profile of the drag depends on the location and the vertical profile of the thermal forcing. The appearance of the convective cells in the unstable lower layer also depends on the vertical and horizontal structure of the

thermal forcing. By means of numerical solutions, we find that the time evolution of the convection depends on the Rayleigh number which is a measure of the relative importance of the effects of the unstable stratification compared with the effects of viscosity and heat conduction. In an inviscid and non-heat-conducting flow the Rayleigh number is infinite and the amplitude of the convective cells increases rapidly with time. For a certain finite value of the Rayleigh number, the convection has a finite and approximately steady amplitude.

1.3 Overview of the Thesis

An overview of the thesis is as follows. Chapter 2 is devoted to deriving the governing equations of our study. It begins with a statement of the basic equations of fluid dynamics which describe the conservation of mass, momentum and energy of a fluid. We then describe the Boussinesq approximation which simplifies these equations. It is useful to rewrite the governing equations in terms of nondimensional quantities, so we nondimensionalize the equations then move on to linearize the dimensionless equations by using the perturbation method. In this method the flow variables are divided into two parts, a basic state portion and a perturbation portion. The perturbation quantities are assumed to be small enough that all the product of the perturbations can be neglected, so that the nonlinear governing equations are reduced to linear differential equations in the perturbation variables. The linear equations are then combined into a single perturbed equation which describes the vertical velocity perturbation of the gravity waves and convection. This equation is a second-order differential equation that also includes a first derivative term. The first derivative term can be eliminated by assuming $\frac{d\bar{\rho}}{dz} \ll \bar{\rho}$. We finish the chapter by presenting a chart to summarize the derivation process.

We then move on, in chapter 3, to establish our model. We formulate a two-layer model for gravity waves and convection and introduce the thermal forcing as a

nonhomogeneous term and describe the boundary and the interface conditions. Next we derive analytical solutions. We focus first on finding normal mode solutions for the configuration where the thermal forcing is periodic in the horizontal x direction with a single horizontal wavenumber (monochromatic forcing). We also consider a configuration where the thermal forcing is in the form of a wave packet comprising a continuous spectrum of horizontal wavenumbers (Fourier modes) and thus the solution is obtained by evaluating a Fourier transform. Two configurations are considered. In the first configuration the thermal forcing is centered in the lower layer and in the second configuration it is centered at the interface ($z = 0$). We derive the general solution in terms of arbitrary constants, but do not apply the boundary and interface conditions to determine the values of the constants. We then consider the case with $\frac{d\bar{\rho}}{dz} \ll \bar{\rho}$ (which is one of the forms of the Boussinesq approximation), the solutions are simpler in this case so we apply the boundary and interface conditions and evaluate the constants. The exact solution satisfying the specified conditions is thus obtained and we present the solution in the form of contour plots. From these plots we observe that the steady solution that we have obtained takes the form of closed convective cells in the lower layer and gravity waves in the upper layer.

In Chapter 4, the time-dependent problem is discussed. The waves are still assumed to be periodic in the horizontal direction, but the wave amplitude now varies with time and in the vertical direction. The solution is obtained by taking a Laplace transform in time, for $t > 0$. We observe that it is complicated to get an analytical time-dependent solution for the full two-layer model, so we consider some simplifications. We take the long-wave limit in which the vertical scale is assumed to be much smaller than the horizontal scale. We also consider each layer separately, first the case where there is only one layer (upper layer only) with gravity waves excited by a thermal forcing and then the case with a convective lower layer only before examining the two-layer model.

Chapter 5 is dedicated to the nonlinear analysis. In the nonlinear problem, the

interaction between waves with positive and negative wavenumbers leads to waves with higher harmonics and zero-wavenumber terms. The mean flow varies with time due to such interactions. We first formulate the nonlinear model and then derive the governing equations in terms of the streamfunction. Next, we carry out a weakly-nonlinear analysis to examine the wave-mean flow interactions in the configuration in which the gravity waves are generated by a thermal forcing only (upper layer only). We derive expressions for the momentum flux and the gravity wave drag. These theoretically based expressions can be used for the parameterization of convectively-generated gravity waves in global circulation models.

In Chapter 6, we carry out numerical simulations for the linear time-dependent evolution of gravity waves and convection generated by a thermal forcing. The simulations are carried out by taking a Fourier transform in the horizontal direction and in the vertical direction making use of the finite difference method to approximate the derivatives. For the gravity waves, the upward and downward propagation of the waves can be determined analytically according to the sign of the group velocity (see Appendix A). In the numerical simulations, the domain is truncated to a finite domain. To allow the gravity waves to propagate upward and to avoid any instability due to being reflected from the upper boundary, a radiation condition is needed to guarantee that the waves propagate out of the computational domain reaching the boundary. Campbell and Maslowe (2003) developed a time-dependent radiation condition for numerical simulations involving gravity waves generated by a sinusoidal lower boundary condition. In our configuration where the governing equations have a nonhomogeneous term (the thermal forcing) the radiation condition of Campbell and Maslowe does not apply. We derive the appropriate time-dependent nonhomogeneous radiation condition for our problem. For the numerical simulations of convection generated by a thermal forcing, we add viscosity and heat conduction in order to obtain convection with finite-amplitude and we are able to identify the critical Rayleigh number needed to maintain a constant amplitude.

To round off the thesis, Chapter 7 contains conclusions of our study of the generation of the gravity waves by convection in the atmosphere. Then we finish the thesis with some appendices which contain some basic concepts, background information and some numerical tests to explain some special circumstances.

The discussion of the basic equations of fluid dynamics and the Boussinesq approximation in Sections 2.2 and 2.3 is based on Kundu and Cohen (2008). Appendices A-D describe background material needed in various parts of the thesis. The rest of the thesis describes my original work (unless otherwise stated): the formulation of the model, the solutions and the discussion of the results.

Chapter 2

Governing Equations of a Two-dimensional Model for Gravity Waves and Convection

2.1 Introduction

The motion of the atmosphere is governed by the fundamental laws of fluid dynamics and thermodynamics. These laws can be expressed in terms of partial differential equations involving physical quantities (such as density, pressure, temperature) as dependent variables and space and time as independent variables. The basic equations of fluid dynamics are based on the laws of conservations of mass, momentum and energy. The derivation of these equations of motion can be found in any textbook on fluid dynamics such as Landau and Lifshitz (1953), Batchelor (1967) or Kundu and Cohen (2008). In this chapter, we use these equations to develop a two-dimensional two-layer model for gravity waves and convection in the lower atmosphere.

This chapter begins by presenting the governing equations of fluid dynamics. These equations have no known general solutions. In order to derive analytical so-

lutions, it is necessary to consider some simplifications and approximations. In certain contexts the Boussinesq approximation, named after Joseph Valentin Boussinesq (1842-1929), is often used to simplify the equations that govern the fluid motion in order to facilitate both theoretical analysis and numerical computation. In Section 2.3 we show how the Boussinesq approximation simplifies the governing equations of fluid dynamics and discuss the conditions under which the approximation can be made. We present two levels of approximation: Boussinesq approximation I in which the background density varies with height such that $\frac{d\bar{\rho}}{dz} \ll \bar{\rho}$ and Boussinesq approximation II where the background density $\bar{\rho}$ is replaced by a constant value ρ_0 in the equations except in the gravitational terms. After that, we write the governing equations in terms of dimensionless quantities. In Section 2.5, we linearize the equations by using the perturbation method. The linearized equations can be combined into a single partial differential equation (2.51) that describes gravity waves and convection. This equation can be simplified by using the Boussinesq approximation I.

2.2 The Basic Equations of Fluid Dynamics

The equations of motion of fluid dynamics are based on the laws of conservation of mass, momentum and energy. In this section, we describe these equations following Kundu and Cohen (2008). The law of conservation of mass states that the rate of increase mass within a fixed volume must equal the rate of inflow through the boundaries. The principle of conservation of mass can be expressed by the following differential equation

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0, \quad (2.1)$$

where the scalar function ρ is the fluid density, the vector function $\mathbf{u}$ is the velocity of the fluid motion and both are functions of space represented by the position vector $\mathbf{x}$ and the time variable t . $\frac{D\rho}{Dt}$ represents the rate of change of the density with respect

to time t following the motion of the fluid and is given by

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \mathbf{u} \cdot \nabla\rho.$$

Equation (2.1) is called the continuity equation.

A fluid is usually called incompressible if its density does not change with pressure. Most liquids are almost incompressible. Gases are compressible, but for speed $\lesssim 100 \text{ ms}^{-1}$ changes in pressure are small and the density changes in the flow are also small [Kundu and Cohen (2008)]. In this case, we can assume that $\frac{1}{\rho} \frac{D\rho}{Dt}$ is small relative to the partial derivatives of the components of the vector $\mathbf{u}$ and we can thus neglect the first term of (2.1) and write the continuity equation as

$$\nabla \cdot \mathbf{u} = 0. \quad (2.2)$$

The principle of conservation of momentum or Newton's second law states that the net forces acting on fluid element equal to its mass times its acceleration. Applied to incompressible fluid flows this leads to the Navier-Stokes equation which is given by

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}. \quad (2.3)$$

By dividing by ρ , we get

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad (2.4)$$

where the vector $\mathbf{g}$ is the acceleration due to gravity, p is the pressure of the fluid, μ is the dynamic viscosity coefficient and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient. If viscous effects are negligible, then μ can be set to zero and we obtain Euler's equation.

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g}. \quad (2.5)$$

If the fluid is at rest with $\mathbf{u} = 0$, the first and the second terms on the right-hand side of equation (2.5) are balanced and then we get

$$\nabla p = \rho \mathbf{g}. \quad (2.6)$$

This is called hydrostatic balance. In cartesian coordinates (x, y, z) , $\mathbf{g}$ acts only in the vertical direction and can be written as $(0, 0, -g)$ where g is scalar ($g = 9.8 \text{ ms}^{-2}$) and equation (2.6) can be written as

$$\frac{\partial p}{\partial z} = -\rho g. \quad (2.7)$$

The thermal energy equation based on the principle of conservation of energy is given by

$$\rho \frac{De}{Dt} = -p(\nabla \cdot \mathbf{u}) + \varphi - \nabla \cdot \mathbf{q}, \quad (2.8)$$

where e is the internal energy of the fluid, $\mathbf{q}$ is the heat flux vector and φ is the viscous dissipation.

2.3 The Boussinesq Approximation

In 1903, Boussinesq suggested that density changes under certain conditions in the fluid can be neglected except in the terms involving gravity where ρ is multiplied by g . A formal discussion and the conditions under which the Boussinesq approximation holds are given in Spiegel and Vernois (1960). The discussion given here follows Kundu and Cohen (2008).

The following assumptions are made under the Boussinesq approximation:

1. The variation of the density ρ in space and time is small enough that we can set $\frac{1}{\rho} \frac{D\rho}{Dt}$ to zero in the continuity equation (2.1).
2. The variation of the density in the vertical direction $\frac{\partial \rho}{\partial z}$ is small.
3. The variation of the density with pressure $\frac{\partial \rho}{\partial p}$ is small, i.e. the fluid is almost incompressible.
4. The other properties of the fluid such as the viscosity coefficient μ , the thermal conductivity coefficient k and the specific heat capacity at constant volume or

pressure C_V and C_p , respectively, are assumed to be constant and the viscous dissipation can be neglected.

Under the Boussinesq approximation (by assumption 1), the continuity equation (2.1) becomes

$$\nabla \cdot \mathbf{u} = 0. \quad (2.9)$$

In three-dimensional cartesian coordinates (x, y, z) with a velocity vector $\mathbf{u} = (u, v, w)$, the continuity equation can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

In two-dimensional cartesian coordinates, for example x and z , with a velocity vector $\mathbf{u} = (u, w)$ the continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

and we can then define a streamfunction Ψ by

$$\frac{\partial \Psi}{\partial x} = w \quad \text{and} \quad \frac{\partial \Psi}{\partial z} = -u. \quad (2.10)$$

The streamfunction along with the vorticity can be used to simplify many partial differential equations that govern fluid motion.

The vorticity of a fluid with velocity $\mathbf{u}$ is defined as

$$\xi = \nabla \times \mathbf{u}.$$

It can be written in three dimensions as

$$\begin{aligned} \xi_x &= w_y - v_z, \\ \xi_y &= u_z - w_x, \\ \xi_z &= v_x - u_y. \end{aligned}$$

In two-dimensional space (x and z), the vorticity has only a y -component ξ_y , which we shall simply denote as ξ . It is related to the streamfunction by

$$\xi = u_z - w_x = \Psi_{zz} + \Psi_{xx} = \nabla^2 \Psi. \quad (2.11)$$

We now examine the thermal energy equation (2.8) under Boussinesq approximation. Although the continuity equation is approximated by $\nabla \cdot \mathbf{u} = 0$, the term $p(\nabla \cdot \mathbf{u})$ in equation (2.8) is not negligible compared with the other terms in the equation; it is negligible only for incompressible fluids. But we have from the continuity equation (2.1) that

$$-p(\nabla \cdot \mathbf{u}) = \frac{p}{\rho} \frac{D\rho}{Dt}.$$

Assuming that the density depends on both the temperature and pressure, then

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial p} \frac{Dp}{Dt} + \frac{\partial \rho}{\partial T} \frac{DT}{Dt}.$$

Using the Boussinesq approximation (assumption 3), the term $\frac{\partial \rho}{\partial p} \frac{Dp}{Dt}$ is considered to be small relative to the second term in the equation, so it can be neglected giving the approximation

$$-p(\nabla \cdot \mathbf{u}) \approx \frac{p}{\rho} \frac{\partial \rho}{\partial T} \frac{DT}{Dt},$$

where in the partial derivative of ρ with respect to T , the pressure p is fixed and does not depend on T . Assuming a perfect gas, for which $p = \rho RT$ and $C_p - C_V = R$, where R here denotes the gas constant. The derivative of ρ with respect to T for fixed p is $\frac{\partial \rho}{\partial T} = \frac{-p}{RT^2}$ and so

$$\begin{aligned} -p(\nabla \cdot \mathbf{u}) &\approx \frac{-p}{T} \frac{DT}{Dt} \\ &= -\rho R \frac{DT}{Dt} \\ &= -(C_p - C_V) \rho \frac{DT}{Dt}. \end{aligned}$$

The thermal energy equation (2.8) thus becomes

$$\rho \frac{De}{Dt} = -(C_p - C_V) \rho \frac{DT}{Dt} + \varphi - \nabla \cdot \mathbf{q}. \quad (2.12)$$

For a perfect gas, the internal energy e is related to the temperature T by the linear relation $e = C_V T$. Substituting this into equation (2.12), we obtain

$$\rho C_V \frac{DT}{Dt} = -\rho C_p \frac{DT}{Dt} + \rho C_V \frac{DT}{Dt} + \varphi - \nabla \cdot \mathbf{q}.$$

Then

$$\frac{DT}{Dt} = \frac{1}{\rho C_p} (\varphi - \nabla \cdot \mathbf{q}). \quad (2.13)$$

The expression on the right-hand side is the diabatic heating Q . We can thus write

$$\frac{DT}{Dt} = Q. \quad (2.14)$$

The viscous dissipation of energy φ is negligible under the Boussinesq approximation. Neglecting φ and assuming Fourier's law of heat conduction

$$\mathbf{q} = -k \nabla T,$$

we have

$$Q = \frac{1}{\rho C_p} (k \nabla^2 T) = \kappa \nabla^2 T, \quad (2.15)$$

where $\kappa = \frac{k}{\rho C_p}$ is the thermal diffusivity. Equation (2.14) thus becomes

$$\frac{DT}{Dt} = \kappa \nabla^2 T. \quad (2.16)$$

Equation (2.16) can also be written in terms of the density. We write ρ and T in terms of their reference states ρ_0 and T_0 as

$$\rho = \rho_0 + \rho' \text{ and } T = T_0 + T',$$

where ρ' and T' are departures from the reference quantities, such that $\rho' \ll \rho_0$ and $T' \ll T_0$. Under the Boussinesq approximation, the pressure is considered to be almost constant. So from the equation of state $p = \rho R T$ we can say that ρT is approximately constant and equal to its reference state value and then we write

$$\rho T = \rho_0 T_0.$$

This means

$$\begin{aligned}(\rho_0 + \rho') (T_0 + T') &= \rho_0 T_0, \\ \rho_0 T_0 + \rho_0 T' + \rho' T_0 + \rho' T' &= \rho_0 T_0.\end{aligned}$$

The term $\rho' T'$ is negligible compared with others terms since it is a product of perturbation quantities which are considered to be small. Thus the above equation is approximated as

$$\rho' T_0 = -\rho_0 T',$$

which implies that

$$\rho' = -\rho_0 \frac{T'}{T_0},$$

and so

$$\rho = \rho_0 \left(1 - \frac{T'}{T_0} \right), \quad (2.17)$$

or

$$\rho = \rho_0 \left(1 - \frac{1}{T_0} (T - T_0) \right). \quad (2.18)$$

By using this linear relation (2.18) between the density and the temperature, equation (2.16) can also be written in terms of ρ as

$$\frac{D\rho}{Dt} = \kappa \nabla^2 \rho. \quad (2.19)$$

To study gravity waves in the atmosphere it is reasonable to consider $\kappa = 0$ and we will assume that in our gravity wave model. However we shall see in Section 6.4 that to obtain finite amplitude convection in a time-dependent configuration we need to have κ nonzero. If $\kappa = 0$, equation (2.19) becomes

$$\frac{D\rho}{Dt} = 0. \quad (2.20)$$

The set of equations (2.9), (2.20) and (2.3) represents a fluid in which incompressibility (Boussinesq effects) has been assumed in the continuity and the energy

equations but not in the momentum equation. Since the full Boussinesq approximation has not been made, we will refer to this system of equations as non-Boussinesq.

We can make further simplifications by approximating the momentum equation (2.3) as well. There are different forms of the Boussinesq equations for the momentum equation. At one level of approximation, the background density is a function depending on z , denoted by $\bar{\rho}(z)$. In this approximation the vertical change in the density is small relative to the background density i.e., $\frac{d\bar{\rho}}{dz} \ll \bar{\rho}$. We will refer to this version as the Boussinesq approximation I. In this approximation we decompose the density and pressure into background ($\bar{\rho}(z)$ and $\bar{p}(z)$) and perturbation (ρ' and p') terms as

$$\rho(x, y, z, t) = \bar{\rho}(z) + \rho'(x, y, z, t), \quad (2.21)$$

$$p(x, y, z, t) = \bar{p}(z) + p'(x, y, z, t), \quad (2.22)$$

where $\rho' \ll \bar{\rho}$ and $p' \ll \bar{p}$. Then we substitute these expressions into the momentum equations (2.3). We shall consider this level of approximation in our model and it is derived in Section 2.5.

The other version of approximation take the background density $\bar{\rho}(z)$ to be a constant ρ_0 . This approximation is valid over a vertical distance where the vertical change in the density is so small such that it can be considered to be negligible. We will refer to this version by the Boussinesq approximation II. In this approximation the density is decomposed as

$$\rho(x, y, z, t) = \rho_0 + \rho'(x, y, z, t), \quad (2.23)$$

where $\rho' \ll \rho_0$. By substituting (2.23) and (2.22) into the momentum equation (2.3), we get

$$(\rho_0 + \rho') \frac{D\mathbf{u}}{Dt} = -\nabla(\bar{p} + p') + (\rho_0 + \rho') \mathbf{g} + \mu \nabla^2 \mathbf{u}.$$

The gradient of the background pressure is given by

$$\nabla \bar{p} = \left(\frac{\partial \bar{p}}{\partial x}, \frac{\partial \bar{p}}{\partial y}, \frac{\partial \bar{p}}{\partial z} \right) = (0, 0, -\rho_0 g) = \rho_0 \mathbf{g}, \quad (2.24)$$

where $\frac{\partial \bar{p}}{\partial z} = -\rho_0 g$ is the statement (2.6) of background hydrostatic balance in which the pressure is independent of the horizontal variable x and varies vertically as a linear function of z . The momentum equation becomes

$$\left(1 + \frac{\rho'}{\rho_0}\right) \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0} \nabla p' + \frac{\rho'}{\rho_0} \mathbf{g} + \frac{\mu}{\rho_0} \nabla^2 \mathbf{u}. \quad (2.25)$$

Once again, we can neglect the ratio $\frac{\rho'}{\rho_0}$ in the inertia term $\frac{D\mathbf{u}}{Dt}$ since $\frac{\rho'}{\rho_0} \ll 1$. However, we cannot neglect it in the gravity term $\frac{\rho'}{\rho_0} \mathbf{g}$ because the effect of gravity still needs to be represented. Thus, equation (2.25) becomes

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0} \nabla p' + \frac{\rho'}{\rho_0} \mathbf{g} + \nu \nabla^2 \mathbf{u}. \quad (2.26)$$

We now have a set of equations (2.9), (2.19) and (2.26) in which all three equations have been simplified by the Boussinesq approximation:

$$\left. \begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \frac{D\mathbf{u}}{Dt} &= -\frac{1}{\rho_0} \nabla p' + \frac{\rho'}{\rho_0} \mathbf{g} + \nu \nabla^2 \mathbf{u}, \\ \frac{D\rho}{Dt} &= \kappa \nabla^2 \rho. \end{aligned} \right\} \quad (2.27)$$

We will refer to this approximation as the Boussinesq approximation II. We will not use it in our investigation but we present it here for comparison purposes since it is used in many previous related studies.

2.4 Nondimensionalization

In the derivation of the equations so far, we have been assuming that all fluid properties (independent and dependent variables) are measured in terms of pre-defined units such as m, s, K. This means that the equations are dimensional and all the terms in a given equation must have the same dimensions. It is useful to rewrite the governing equations in terms of nondimensional quantities. This allows us to compare the magnitude of the different quantities and assess which terms are more important

than others and determine whether any terms can be neglected under any conditions. This technique is known as nondimensionalization. When we nondimensionalize an equation, we remove units by scaling all the variables and parameters in the equation with respect to some fixed reference values. We define

$$\text{Nondimensional quantity} = \frac{\text{Dimensional quantity}}{\text{Reference value}}.$$

The starting point is the two-dimensional (x and z) system of equations of motion: the momentum equation (2.3), the continuity equation (2.9) and the energy equation (2.16). In the generation of the gravity waves by convection in the atmosphere, the convection occurs much more efficiently than conduction and the viscosity and the conduction terms in the governing equations are assumed to be small relative to other terms in the equation and can be neglected, at least as a first approximation. We shall however reintroduce these terms to describe time-dependent convection in Chapter 6.

We denote all the variables in the dimensional equations by a star $\star$. Note that for convenience we did not include the star in Sections 2.2 and 2.3 and note also that this symbol $\star$ is not an asterisk $*$. (We will use the asterisk throughout the thesis to denote the complex conjugate). The governing dimensional equations (2.5), (2.9) and (2.20) now become

$$\rho^\star \left[\frac{\partial u^\star}{\partial t^\star} + u^\star \frac{\partial u^\star}{\partial x^\star} + w^\star \frac{\partial u^\star}{\partial z^\star} \right] = -\frac{\partial p^\star}{\partial x^\star}, \quad (2.28)$$

$$\rho^\star \left[\frac{\partial w^\star}{\partial t^\star} + u^\star \frac{\partial w^\star}{\partial x^\star} + w^\star \frac{\partial w^\star}{\partial z^\star} \right] = -\frac{\partial p^\star}{\partial z^\star} - \rho^\star g^\star, \quad (2.29)$$

$$\frac{\partial u^\star}{\partial x^\star} + \frac{\partial w^\star}{\partial z^\star} = 0, \quad (2.30)$$

$$\frac{\partial \rho^\star}{\partial t^\star} + u^\star \frac{\partial \rho^\star}{\partial x^\star} + w^\star \frac{\partial \rho^\star}{\partial z^\star} = 0. \quad (2.31)$$

Let L_x and L_z be the typical length scales in the x and z directions, respectively, on the order of magnitude of the horizontal and vertical wavelengths of the gravity waves. U and W be the typical velocities scales in the x and z directions, respectively,

and R be the typical density scale. If T is the time scale then the velocity and length scales must satisfy $U = \frac{L_x}{T}$ and $W = \frac{L_z}{T}$. Therefore the time scale is $T = \frac{L_x}{U} = \frac{L_z}{W}$, the gravity acceleration scale is $\frac{U^2}{L_z}$ and the pressure scale is RU^2 . We define the nondimensional variables in terms of the corresponding dimensional ones as follows

$$\begin{aligned} x &= \frac{x^*}{L_x}, & z &= \frac{z^*}{L_z}, & t &= \frac{t^*}{T} = \frac{Ut^*}{L_x}, & \mathcal{U} &= \frac{u^*}{U}, \\ \mathcal{W} &= \frac{w^*}{W}, & \varrho &= \frac{\rho^*}{R}, & g &= \frac{g^* L_z}{U^2}, & \mathcal{P} &= \frac{p^*}{RU^2}. \end{aligned} \quad (2.32)$$

Then the derivatives with respect to the nondimensional variables are related to the dimensional variables by

$$\frac{\partial}{\partial x^*} = \frac{1}{L_x} \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial z^*} = \frac{1}{L_z} \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t^*} = \frac{U}{L_x} \frac{\partial}{\partial t}.$$

The continuity equation (2.30) becomes

$$\frac{U}{L_x} \frac{\partial \mathcal{U}}{\partial x} + \frac{W}{L_z} \frac{\partial \mathcal{W}}{\partial z} = 0, \quad (2.33)$$

and since

$$\frac{1}{T} = \frac{U}{L_x} = \frac{W}{L_z}, \quad (2.34)$$

equation (2.33) becomes

$$\frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{W}}{\partial z} = 0. \quad (2.35)$$

The momentum equation in the x direction (2.28) becomes

$$\rho \left[\frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{U}}{\partial z} \right] = - \frac{\partial \mathcal{P}}{\partial x}. \quad (2.36)$$

The momentum equation in the z direction (2.29) becomes

$$R\varrho \left[\frac{UW}{L_x} \frac{\partial \mathcal{W}}{\partial t} + \frac{UW}{L_x} \mathcal{U} \frac{\partial \mathcal{W}}{\partial x} + \frac{W^2}{L_z} \mathcal{W} \frac{\partial \mathcal{W}}{\partial z} \right] = - \frac{RU^2}{L_z} \frac{\partial \mathcal{P}}{\partial z} - \frac{RU^2}{L_z} \varrho g.$$

Once again by using equation (2.34), we get

$$\delta \varrho \left[\frac{\partial \mathcal{W}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{W}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{W}}{\partial z} \right] = - \frac{\partial \mathcal{P}}{\partial z} - \varrho g, \quad (2.37)$$

where $\delta = L_z^2/L_x^2$ is the square of the aspect ratio which gives a measure of the magnitude of the vertical scale to the horizontal. In the geophysical flow configuration

studied here, $L_z \ll L_x$ and so δ can be considered a small parameter for the purpose of asymptotics. The limit $\delta \rightarrow 0$ is called the long-wave limit.

The energy equation (2.31) becomes

$$\frac{UR}{L_x} \frac{\partial \varrho}{\partial t} + \frac{UR}{L_x} \mathcal{U} \frac{\partial \varrho}{\partial x} + \frac{WR}{L_z} \mathcal{W} \frac{\partial \varrho}{\partial z} = 0,$$

which can be written by using equation (2.34) as

$$\frac{\partial \varrho}{\partial t} + \mathcal{U} \frac{\partial \varrho}{\partial x} + \mathcal{W} \frac{\partial \varrho}{\partial z} = 0. \quad (2.38)$$

The governing equations in terms of nondimensional variables are equations (2.36), (2.37), (2.35) and (2.38). In the configuration that we study, we include an external thermal forcing which appears as a specified nonhomogeneous term in equation for the energy or temperature. In our formulation where the energy equation is written in terms of the density, we add the thermal forcing as a nonhomogeneous term in the energy equation. The governing equations thus become

$$\left. \begin{aligned} \varrho \left[\frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{U}}{\partial z} \right] &= -\frac{\partial \mathcal{P}}{\partial x}, \\ \delta \varrho \left[\frac{\partial \mathcal{W}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{W}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{W}}{\partial z} \right] &= -\frac{\partial \mathcal{P}}{\partial z} - \varrho g, \\ \frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{W}}{\partial z} &= 0, \\ \frac{\partial \varrho}{\partial t} + \mathcal{U} \frac{\partial \varrho}{\partial x} + \mathcal{W} \frac{\partial \varrho}{\partial z} &= \mathcal{F}, \end{aligned} \right\} \quad (2.39)$$

where the nonhomogeneous term $\mathcal{F}(x, z, t)$ represents a thermal forcing and is needed to generate the convection and gravity waves. The exact form of this function will be specified in the next chapter. In the long-wave limit $\delta \rightarrow 0$, the vertical momentum gives

$$\frac{\partial \mathcal{P}}{\partial z} = -\varrho g.$$

Thus, the long-wave limit is a condition for the flow to be hydrostatic.

2.5 The Perturbed Equations

The dependent variables in equations (2.39) are total quantities. Each of these total quantities is expressed as a sum of a background flow quantity (the basic state) and a perturbation (the deviation field). The background velocity takes the form $(\bar{u}, 0)$ with $\bar{u} \neq 0$ and the background flow is a shear flow, its velocity $\bar{u}$ depends on height z . We consider the case where the background density is a function of z , also we write

$$\begin{aligned}\mathcal{U}(x, z, t) &= \bar{u}(z) + \varepsilon u(x, z, t), \\ \mathcal{W}(x, z, t) &= \varepsilon w(x, z, t), \\ \varrho(x, z, t) &= \bar{\rho}(z) + \varepsilon \rho(x, z, t), \\ \mathcal{P}(x, z, t) &= \bar{p}(z) + \varepsilon p(x, z, t), \\ \mathcal{F}(x, z, t) &= \varepsilon F(x, z, t).\end{aligned}$$

The parameter ε characterizes the extent of nonlinearity of the equations. We substitute these expressions into the system (2.39) and make use of the following background hydrostatic balance equation for the background flow

$$\frac{d\bar{p}}{dz} = -\bar{\rho}g, \quad (2.40)$$

we get

$$\bar{\rho} \left(\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + \frac{d\bar{u}}{dz} w \right) + \varepsilon \left[\bar{\rho} \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \rho \left(\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + \frac{d\bar{u}}{dz} w \right) \right] = -\frac{\partial p}{\partial x}, \quad (2.41)$$

$$\delta \bar{\rho} \left(\frac{\partial w}{\partial t} + \bar{u} \frac{\partial w}{\partial x} \right) + \varepsilon \left[\delta \bar{\rho} \left(u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \delta \rho \left(\frac{\partial w}{\partial t} + \bar{u} \frac{\partial w}{\partial x} \right) \right] = -\frac{\partial p}{\partial z} - \rho g, \quad (2.42)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (2.43)$$

$$\frac{\partial \rho}{\partial t} + \bar{u} \frac{\partial \rho}{\partial x} + \frac{d\bar{\rho}}{dz} w + \varepsilon \left[u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} \right] = F. \quad (2.44)$$

The two momentum equations (2.41) and (2.42) can be combined into a single equation, by eliminating the pressure, as follows:

First, we differentiate equation (2.41) with respect to z and by making use of the continuity equation (2.43), we get

$$\begin{aligned} & \bar{\rho} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial u}{\partial z} + \frac{d^2 \bar{u}}{dz^2} w \right] + \frac{d\bar{\rho}}{dz} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u + \frac{d\bar{u}}{dz} w \right] \\ & + \varepsilon \left\{ \bar{\rho} \left[-u \frac{\partial^2 w}{\partial z^2} + w \frac{\partial^2 u}{\partial z^2} \right] + \frac{d\bar{\rho}}{dz} \left[-u \frac{\partial w}{\partial z} + w \frac{\partial u}{\partial z} \right] + \rho \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial u}{\partial z} + \frac{d^2 \bar{u}}{dz^2} w \right] \right. \\ & \left. + \frac{\partial \rho}{\partial z} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u + \frac{d\bar{u}}{dz} w \right] \right\} = -\frac{\partial^2 p}{\partial x \partial z}. \end{aligned}$$

Next, we differentiate equation (2.42) with respect to x and once again by using (2.43), we obtain

$$\begin{aligned} & \bar{\rho} \delta \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} \right] + \varepsilon \left\{ \bar{\rho} \delta \left[u \frac{\partial^2 w}{\partial x^2} - w \frac{\partial^2 u}{\partial x^2} \right] + \rho \left[\delta \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} \right] \right. \\ & \left. + \frac{\partial \rho}{\partial x} \left[\delta \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) w \right] \right\} = -\frac{\partial^2 p}{\partial x \partial z} - \frac{\partial \rho}{\partial x} g. \end{aligned}$$

Now by subtraction, we get

$$\begin{aligned} & \bar{\rho} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\frac{\partial u}{\partial z} - \delta \frac{\partial w}{\partial x} \right) + \frac{d^2 \bar{u}}{dz^2} w \right] + \frac{d\bar{\rho}}{dz} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u + \frac{d\bar{u}}{dz} w \right] \\ & + \varepsilon \left\{ \bar{\rho} \left[-u \left(\frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2} \right) w + w \left(\frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2} \right) u \right] + \frac{d\bar{\rho}}{dz} \left[-u \frac{\partial w}{\partial z} + w \frac{\partial u}{\partial z} \right] \right. \\ & + \rho \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\frac{\partial u}{\partial z} - \delta \frac{\partial w}{\partial x} \right) + \frac{d^2 \bar{u}}{dz^2} w \right] + \frac{\partial \rho}{\partial z} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u + \frac{d\bar{u}}{dz} w \right] \\ & \left. - \frac{\partial \rho}{\partial x} \left[\delta \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) w \right] \right\} = g \frac{\partial \rho}{\partial x}. \end{aligned} \tag{2.45}$$

Now, the nonlinear model for the gravity waves generated by convection consists of the equations of motion (2.43), (2.44) and (2.45). For small-amplitude waves ($\varepsilon \ll 1$) these nonlinear differential equations can be reduced to linear differential equations for the perturbations in which the basic state variables are specified.

2.6 The Linear Equation for the Gravity Waves and Convection

The Linear Non-Boussinesq Equations

Setting $\varepsilon = 0$ in equations (2.44) and (2.45) gives

$$\bar{\rho} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\frac{\partial u}{\partial z} - \delta \frac{\partial w}{\partial x} \right) + \frac{d^2 \bar{u}}{dz^2} w \right] + \frac{d\bar{\rho}}{dz} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u + \frac{d\bar{u}}{dz} w \right] = g \frac{\partial \rho}{\partial x}, \quad (2.46)$$

and

$$\frac{\partial \rho}{\partial t} + \bar{u} \frac{\partial \rho}{\partial x} + \frac{d\bar{\rho}}{dz} w = F. \quad (2.47)$$

The linear problem comprises (2.46) and (2.47) along with the continuity equation (2.43). These equations can be combined into a single time-dependent linear equation describes the vertical velocity perturbation of the convectively-generated gravity waves.

In order to combine these equations, we apply the linear operator $\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial}{\partial x}$ to equation (2.46) and make use of equation (2.43) to obtain

$$\begin{aligned} & \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \left(\frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2} \right) w - \frac{d^2 \bar{u}}{dz^2} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} \right] \\ & + \frac{1}{\bar{\rho}} \frac{d\bar{\rho}}{dz} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \frac{\partial w}{\partial z} - \frac{d\bar{u}}{dz} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} \right] + \chi \frac{\partial^2 w}{\partial x^2} = -g \frac{1}{\bar{\rho}} \frac{\partial^2 F}{\partial x^2}, \end{aligned} \quad (2.48)$$

where $\chi = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}$. This quantity provides a useful description of atmospheric stability. If it is positive, there is stable stratification and $\chi = N^2$, where N is called the buoyancy frequency or Brunt-Väisälä frequency. If it is negative, there is unstable stratification.

Now we have a linear differential equation in the perturbations which can be solved by standard methods to determine the characteristic structure of the perturbations in terms of the known basic state.

Gravity waves in the atmosphere may be approximated by periodic oscillations in the horizontal direction. We will represent them, at least initially, as sinusoidal

oscillations. To generate solutions of this form, we consider a thermal forcing of the form

$$F(x, z) = \hat{F}(z) e^{ikx} + \text{c.c.}, \quad (2.49)$$

where c.c. denotes the complex conjugate of the preceding term, k is a constant and $\hat{F}(z)$ is the specified amplitude of the thermal forcing. The nonhomogeneous term $F(x, z)$ is a time-independent thermal disturbance that is present over a finite range of altitudes as might be observed when there is deep convective heating. The horizontal structure of F determines the structure of the velocity perturbation. The form (2.49) leads to a solution of equation (2.48) in normal mode form which can represent either gravity waves or convection

$$w(x, z, t) = \hat{w}(z, t) e^{ikx} + \text{c.c.} \quad (2.50)$$

Substituting by (2.49) and (2.50) into equation (2.48), we get

$$\begin{aligned} \left(\frac{\partial}{\partial t} + ik\bar{u} \right)^2 \frac{\partial^2 \hat{w}}{\partial z^2} - \frac{\chi}{g} \left(\frac{\partial}{\partial t} + ik\bar{u} \right)^2 \frac{\partial \hat{w}}{\partial z} - \left[k^2 \chi + \delta k^2 \left(\frac{\partial}{\partial t} + ik\bar{u} \right)^2 \right. \\ \left. + ik \frac{d^2 \bar{u}}{dz^2} \left(\frac{\partial}{\partial t} + ik\bar{u} \right) - ik \frac{\chi}{g} \left(\frac{\partial}{\partial t} + ik\bar{u} \right) \frac{d\bar{u}}{dz} \right] \hat{w} = k^2 \frac{g}{\bar{\rho}} \hat{F}. \end{aligned} \quad (2.51)$$

This is the linear time-dependent equation for our non-Boussinesq configuration. In Chapter 3, we consider some configurations for which we can derive exact analytical solutions to equation (2.51).

If we consider waves with time-independent amplitude

$$w(x, z) = \hat{w}(z) e^{ikx} + \text{c.c.}, \quad (2.52)$$

equation (2.51) becomes

$$\frac{d^2 \hat{w}}{dz^2} - \frac{\chi}{g} \frac{d\hat{w}}{dz} + \left[\frac{\chi}{\bar{u}^2} - \frac{1}{\bar{u}} \frac{d^2 \bar{u}}{dz^2} - \delta k^2 + \frac{\chi}{g} \frac{1}{\bar{u}} \frac{d\bar{u}}{dz} \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\bar{\rho}}. \quad (2.53)$$

For a basic flow with no shear, $\bar{u}$ is constant and equation (2.53) becomes

$$\frac{d^2 \hat{w}}{dz^2} - \frac{\chi}{g} \frac{d\hat{w}}{dz} + \left[\frac{\chi}{\bar{u}^2} - \delta k^2 \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\bar{\rho}}. \quad (2.54)$$

The Linear Equation under the Boussinesq Approximation I

In order to simplify the governing equation by applying the Boussinesq approximation I, we consider the time-dependent linear equations (2.46), (2.47) and (2.43). According to the Boussinesq approximation I, the derivative $\frac{d\bar{\rho}}{dz}$ is small compared with $\bar{\rho}$ and we can thus omit the term evolving $\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dz}$ from equation (2.46). This gives

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial z} - \delta\frac{\partial w}{\partial x}\right) + \frac{d^2\bar{u}}{dz^2}w = g\frac{1}{\bar{\rho}}\frac{\partial\rho}{\partial x}. \quad (2.55)$$

In a similar manner to the technique used to combine the three equations into a single equation in the non-Boussinesq configuration, we combine the three equations into the following equation

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2}{\partial z^2} + \delta\frac{\partial^2}{\partial x^2}\right)w - \frac{d^2\bar{u}}{dz^2}\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial w}{\partial x} + \chi\frac{\partial^2 w}{\partial x^2} = -g\frac{1}{\bar{\rho}}\frac{\partial^2 F}{\partial x^2}. \quad (2.56)$$

This equation describes the vertical velocity perturbation under the Boussinesq approximation I. Note that in the case of the Boussinesq approximation II, $\bar{\rho}(z)$ on the right-hand side of equation (2.56) is replaced by a constant ρ_0 and χ becomes $\chi = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}$. This is analogous to the model used in most previous studies. In some cases χ is replaced by a constant which is unrelated to the value of the constant ρ_0 .

To solve equation (2.56), we seek a solution of the form of (2.50). Substituting (2.49) and (2.50) into equation (2.56) gives

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right)^2\frac{\partial^2\hat{w}}{\partial z^2} - \left[k^2\chi + \delta k^2\left(\frac{\partial}{\partial t} + ik\bar{u}\right)^2 + ik\frac{d^2\bar{u}}{dz^2}\left(\frac{\partial}{\partial t} + ik\bar{u}\right)\right]\hat{w} = k^2g\frac{\hat{F}}{\bar{\rho}(z)}. \quad (2.57)$$

This is the linear time-dependent equation for the Boussinesq approximation I. It is of the same form as the corresponding equation for the non-Boussinesq case (2.51) by omitting the terms involving χ/g .

Our goal is to solve this time-dependent equation, and we shall do this in Chapter 4. As a first approximation, we consider the corresponding steady equation

$$\frac{d^2\hat{w}}{dz^2} + \left[\frac{\chi}{\bar{u}^2} - \frac{1}{\bar{u}}\frac{d^2\bar{u}}{dz^2} - \delta k^2\right]\hat{w} = -\frac{g}{\bar{u}^2}\frac{\hat{F}}{\bar{\rho}(z)}. \quad (2.58)$$

For constant $\bar{u}$, equation (2.58) becomes

$$\frac{d^2 \hat{w}}{dz^2} + \left[\frac{\chi}{\bar{u}^2} - \delta k^2 \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\bar{\rho}(z)}. \quad (2.59)$$

This is equation (2.54) without the first derivative term. It represents the linear steady vertical velocity perturbation under Boussinesq approximation I. Once again, under the Boussinesq approximation II, equation (2.59) would be

$$\frac{d^2 \hat{w}}{dz^2} + \left[\frac{\chi}{\bar{u}^2} - \delta k^2 \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\rho_0}, \quad (2.60)$$

where $\chi = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}$.

Table 2.1 shows a summary of the steps taken in deriving the governing equations of the gravity waves and convection generated by a thermal forcing.

Table 2.2 shows a comparison between the governing equations under the Boussinesq approximation I and the governing equations under the Boussinesq approximation II.

Summary in a Chart

↓ Equations (2.5), (2.9) and (2.20) ↓

$$\boxed{\begin{aligned}\rho^* \frac{D^* \mathbf{u}^*}{D^* t^*} &= -\nabla^* p^* + \rho^* \mathbf{g}^*, \\ \nabla^* \cdot \mathbf{u}^* &= 0, \\ \frac{D^* \rho^*}{D^* t^*} &= 0.\end{aligned}}$$

↓ 2-D, Nondimensionalization ↓

$$\boxed{\begin{aligned}\varrho \left[\frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{U}}{\partial z} \right] &= -\frac{\partial \mathcal{P}}{\partial x}, \\ \delta \varrho \left[\frac{\partial \mathcal{W}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{W}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{W}}{\partial z} \right] &= -\frac{\partial \mathcal{P}}{\partial z} - \varrho g, \\ \frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{W}}{\partial z} &= 0, \\ \frac{\partial \varrho}{\partial t} + \mathcal{U} \frac{\partial \varrho}{\partial x} + \mathcal{W} \frac{\partial \varrho}{\partial z} &= \mathcal{F}.\end{aligned}}$$

↓ In terms of perturbations ↓

$$\boxed{\begin{aligned}\bar{\rho} \left(\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + \frac{d\bar{u}}{dz} w \right) + \varepsilon \left[\bar{\rho} \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \rho \left(\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + \frac{d\bar{u}}{dz} w \right) \right] &= -\frac{\partial p}{\partial x}, \\ \delta \bar{\rho} \left(\frac{\partial w}{\partial t} + \bar{u} \frac{\partial w}{\partial x} \right) + \varepsilon \left[\delta \bar{\rho} \left(u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \delta \rho \left(\frac{\partial w}{\partial t} + \bar{u} \frac{\partial w}{\partial x} \right) \right] &= -\frac{\partial p}{\partial z} - g\rho, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\partial \rho}{\partial t} + \bar{u} \frac{\partial \rho}{\partial x} + \frac{d\bar{\rho}}{dz} w + \varepsilon \left[u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} \right] &= F.\end{aligned}}$$

↓ Combining the 2 momentum equations ↓

$$\boxed{\begin{aligned}\bar{\rho} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\frac{\partial u}{\partial z} - \delta \frac{\partial w}{\partial x} \right) + \frac{d^2 \bar{u}}{dz^2} w \right] + \frac{d\bar{\rho}}{dz} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u + \frac{d\bar{u}}{dz} w \right] \\ + \varepsilon \left\{ \bar{\rho} \left[-u \left(\frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2} \right) w + w \left(\frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2} \right) u \right] + \frac{d\bar{\rho}}{dz} \left[-u \frac{\partial w}{\partial z} + w \frac{\partial u}{\partial z} \right] \right. \\ \left. + \rho \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\frac{\partial u}{\partial z} - \delta \frac{\partial w}{\partial x} \right) + \frac{d^2 \bar{u}}{dz^2} w \right] + \frac{\partial \rho}{\partial z} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u + \frac{d\bar{u}}{dz} w \right] \right. \\ \left. - \frac{\partial \rho}{\partial x} \left[\delta \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) w \right] \right\} = g \frac{\partial \rho}{\partial x}, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \\ \frac{\partial \rho}{\partial t} + \bar{u} \frac{\partial \rho}{\partial x} + \frac{d\bar{\rho}}{dz} w + \varepsilon \left[u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} \right] = F.\end{aligned}}$$

↓ Linearize and combine the equations ↓

$$\boxed{\left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \left(\frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2} \right) w - \frac{d^2 \bar{u}}{dz^2} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} \right] + \frac{1}{\bar{\rho}} \frac{d\bar{\rho}}{dz} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \frac{\partial w}{\partial z} - \frac{d\bar{u}}{dz} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} \right] + \chi \frac{\partial^2 w}{\partial x^2} = -g \frac{1}{\bar{\rho}} \frac{\partial^2 F}{\partial x^2}}.$$

↓ Normal mode ↓

$$\boxed{\left(\frac{\partial}{\partial t} + ik\bar{u} \right)^2 \frac{\partial^2 \hat{w}}{\partial z^2} - \frac{\chi}{g} \left(\frac{\partial}{\partial t} + ik\bar{u} \right)^2 \frac{\partial \hat{w}}{\partial z} - \left[k^2 \chi + \delta k^2 \left(\frac{\partial}{\partial t} + ik\bar{u} \right)^2 + ik \frac{d^2 \bar{u}}{dz^2} \left(\frac{\partial}{\partial t} + ik\bar{u} \right) - ik \left(\frac{\partial}{\partial t} + ik\bar{u} \right) \frac{\chi}{g} \frac{d\bar{u}}{dz} \right] \hat{w} = k^2 g \frac{\hat{F}}{\bar{\rho}}}.$$

↓ Steady state and constant $\bar{u}$ ↓

$$\boxed{\frac{d^2 \hat{w}}{dz^2} - \frac{\chi}{g} \frac{d\hat{w}}{dz} + \left[\frac{\chi}{\bar{u}^2} - \delta k^2 \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\bar{\rho}}}.$$

↓ Boussinesq approximation I ($\frac{1}{\bar{\rho}} \frac{d\bar{\rho}}{dz} \ll 1$ and $\chi = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}$) ↓

$$\boxed{\frac{d^2 \hat{w}}{dz^2} + \left[\frac{\chi}{\bar{u}^2} - \delta k^2 \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\bar{\rho}(z)}}.$$

↓ Boussinesq approximation II ($\bar{\rho} = \rho_0$ but $\chi = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}$) ↓

$$\boxed{\frac{d^2 \hat{w}}{dz^2} + \left[\frac{\chi}{\bar{u}^2} - \delta k^2 \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\rho_0}}}.$$

Table 2.1: Summary of the steps taken in deriving the governing equations of the gravity waves and convection generated by a thermal forcing.

↓ Boussinesq approximation I ↓	↓ Boussinesq approximation II ↓
$\rho(x, z, t) = \bar{\rho}(z) + \rho(x, z, t)$ and $\frac{d\bar{\rho}}{dz} \ll \bar{\rho}$ ↓ Linearize equations (2.40)-(2.43) ↓ $\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + \frac{d\bar{u}}{dz} w = -\frac{1}{\bar{\rho}(z)} \frac{\partial p}{\partial x},$ $\delta \left(\frac{\partial w}{\partial t} + \bar{u} \frac{\partial w}{\partial x} \right) = -\frac{1}{\bar{\rho}(z)} \frac{\partial p}{\partial z} - \frac{g}{\bar{\rho}(z)} \rho,$ $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$ $\frac{\partial \rho}{\partial t} + \bar{u} \frac{\partial \rho}{\partial x} + \frac{d\bar{\rho}}{dz} w = F.$	$\bar{\rho}(z) \longrightarrow \rho_0$ ↓ Linearize equations (2.27) ↓ $\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + \frac{d\bar{u}}{dz} w = -\frac{1}{\rho_0} \frac{\partial p}{\partial x},$ $\delta \left(\frac{\partial w}{\partial t} + \bar{u} \frac{\partial w}{\partial x} \right) = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{g}{\rho_0} \rho,$ $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$ $\frac{\partial \rho}{\partial t} + \bar{u} \frac{\partial \rho}{\partial x} + \frac{d\bar{\rho}}{dz} w = F.$
↓ Combine ↓ $\left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \left(\frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2} \right) w - \frac{d^2 \bar{u}}{dz^2} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} \right] + \chi \frac{\partial^2 w}{\partial x^2} = -\frac{g}{\bar{\rho}(z)} \frac{\partial^2 F}{\partial x^2},$ where $\chi = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}$. It is constant since $\bar{\rho} = \rho_0 e^{\pm \frac{z}{H}}$.	↓ Combine ↓ $\left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \left(\frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2} \right) w - \frac{d^2 \bar{u}}{dz^2} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} \right] + \chi \frac{\partial^2 w}{\partial x^2} = -\frac{g}{\rho_0} \frac{\partial^2 F}{\partial x^2},$ where $\chi = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}$. It is generally assumed to be constant.

Table 2.2: The governing equations under the Boussinesq approximation I against the governing equations under the Boussinesq approximation II.

Chapter 3

Linear Steady Two-Layer Model of Gravity Waves Generated by Convection

In this chapter, we present a two-layer formulation for internal gravity waves and convection generated by a thermal forcing and give analytical solutions for the linear steady motion where the basic flow is assumed to have constant shear. In Section 3.1 we formulate the model and describe the boundary and interface conditions. In Section 3.2 we derive the solution for a wave comprising a single wavenumber k . In Sections 3.2.1 we find a normal mode solution where the thermal disturbance F is centered at $z = -h_1$ (obstacle effect) and in Sections 3.2.2 we find a normal mode solution where F is centered at $z = 0$ (obstacle effect and deep heating). Next, we generalize the solutions to a spectrum of wavenumbers by using a Fourier transform for both configurations and present contour plots for all the cases considered.

3.1 The Model Formulation

Consider a two-layer atmospheric model represented by the vertical domain $-h \leq z < \infty$ comprising an unstable convective layer in the domain $-h \leq z \leq 0$ in which $\chi = -N_0^2 < 0$ and a stable upper layer in the domain $0 < z < \infty$ in which $\chi = N^2 > 0$. The interface between the two layers is set at $z = 0$. In both layers, we consider a steady perturbation with vertical velocity $w(x, z)$ given in normal mode form according to

$$w(x, z) = \hat{w}(z)e^{ikx} + \text{c.c.}, \quad (3.1)$$

where k is the positive horizontal wavenumber and $\hat{w}$ satisfies equation (2.54). A schematic of the two-layer model is shown in Figure 3.1.

A reasonable approximation to the mean atmospheric density is given by [see, for example, Baines (1995)]

$$\bar{\rho}(z) = \rho_0 e^{-\frac{z}{H}}, \quad (3.2)$$

where ρ_0 is a constant and specifies the magnitude of the basic state density at $z = 0$ and H is the scale height. In the upper layer the density has positive gradient $\frac{d\bar{\rho}}{dz} > 0$ while it is negative in the lower layer. We use this profile in the upper layer so $\chi = N^2 = \frac{g}{H} > 0$. In the lower layer, we specify unstable stratification by setting

$$\bar{\rho}(z) = \rho_0 e^{\frac{z}{H}}. \quad (3.3)$$

This density profile implies that $\chi = -N_0^2 = -\frac{g}{H} < 0$.

A thermal forcing in the atmosphere generally takes the form of a function of the vertical variable which of finite depth. It attains a maximum value at some central level and decays to zero as a function of height above and below that central level. So a reasonable representation of the vertical profile of the thermal forcing is a function of the form $e^{-b|z+h_1|}$ where h_1 is the central level of the forcing. We assume a monochromatic thermal forcing that oscillates in the horizontal direction and is

centered vertically at some level in the lower layer $z = -h_1$ where $-h < -h_1 < 0$:

$$F(x, z) = F_0 e^{-b|z+h_1|} e^{ikx} + \text{c.c.}, \quad (3.4)$$

where $b > 0$ and F_0 are constants, b determines the depth of the forcing. In Section 3.2.2 we consider the case where F is centered at the interface $z = 0$. We will use the subscript 0 for the solution in the interval $-h \leq z < -h_1$, the subscript 1 for the solution in the interval $-h_1 \leq z \leq 0$ and the subscript 2 for the solution in the upper layer where $0 < z < \infty$. We then denote the profile background density by $\bar{\rho}_2(z) = \rho_{02} e^{-\frac{z}{H}}$ in the upper layer and $\bar{\rho}_1(z) = \rho_{01} e^{\frac{z}{H}}$ in the lower layer.

We solve equation (2.54) in the two layers to obtain $\hat{w}_2$ in the upper layer and $\hat{w}_0$ and $\hat{w}_1$ in the lower layer. We impose a zero boundary condition at the bottom

$$\hat{w}_0 = 0 \quad \text{at } z = -h, \quad (3.5)$$

and at the upper boundary we assume that there is no incoming wave energy from above the domain. This means that only waves with upward group velocity are considered.

The continuity of the vertical velocity and the density everywhere within the lower layer (including the level $z = -h_1$) implies that at $z = -h_1$

$$\hat{w}_0 = \hat{w}_1, \quad (3.6)$$

$$\frac{d\hat{w}_0}{dz} = \frac{d\hat{w}_1}{dz}. \quad (3.7)$$

The continuity of the vertical velocity and pressure and the discontinuity of the density at $z = 0$ imply the following interface conditions at $z = 0$

$$\hat{w}_1 = \hat{w}_2, \quad (3.8)$$

$$\frac{d\hat{w}_1}{dz} - \gamma \hat{w}_1 = \frac{\bar{\rho}_2}{\bar{\rho}_1} \frac{d\hat{w}_2}{dz}, \quad (3.9)$$

where $\gamma = \frac{\bar{\rho}_1 - \bar{\rho}_2}{\bar{\rho}_1} \frac{g}{\bar{u}^2}$ is constant. The interface condition (3.9) is commonly used in this kind of model, [see, for example, Sutherland (2010)]. The following is a derivation of this condition.

Figure 3.1: A schematic diagram of the two-layer model in the domain $-h \leq z < \infty$ with gravity waves in the upper layer and convection in the lower layer. The thermal disturbance is centered at $z = -h_1$.

The total pressure is assumed to be continuous across the interface and to satisfy

$$\frac{D\mathcal{P}}{Dt} = \frac{\partial \mathcal{P}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{P}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{P}}{\partial z} = 0. \quad (3.10)$$

By expressing each total quantity in the above equation as a sum of a background flow and perturbation quantities as done in Section 2.5, we write

$$\mathcal{U}(x, z, t) = \bar{u}(z) + \varepsilon u(x, z, t),$$

$$\mathcal{W}(x, z, t) = \varepsilon w(x, z, t),$$

$$\mathcal{P}(x, z, t) = \bar{p}(z) + \varepsilon p(x, z, t).$$

Considering only the linear steady terms, (3.10) becomes

$$\bar{u} \frac{\partial p}{\partial x} + w \frac{d\bar{p}}{dz} = 0.$$

Using equation (2.40), this can be written as

$$\frac{\partial p}{\partial x} = \frac{\bar{\rho}g}{\bar{u}}w. \quad (3.11)$$

By substituting the continuity equation $\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z}$ into the linear steady form of the horizontal momentum equation (2.41), we obtain

$$\bar{\rho} \left(-\bar{u} \frac{\partial w}{\partial z} + \frac{d\bar{u}}{dz} w \right) = -\frac{\partial p}{\partial x}. \quad (3.12)$$

We write

$$f(x, z) = \hat{f}(z) e^{ikx} + c.c.,$$

where f represents w or p . Equations (3.11) and (3.12) become, respectively,

$$\hat{p} = \frac{\bar{\rho}g}{ik\bar{u}}\hat{w}, \quad (3.13)$$

$$\hat{p} = -\frac{\bar{\rho}}{ik} \left(\frac{d\bar{u}}{dz}\hat{w} - \bar{u} \frac{d\hat{w}}{dz} \right). \quad (3.14)$$

So we must have continuity of the following quantity

$$\bar{\rho} \left(\bar{u} \frac{d\hat{w}}{dz} - \frac{d\bar{u}}{dz}\hat{w} - \frac{g}{\bar{u}}\hat{w} \right).$$

This implies that we have the following condition at the interface

$$\frac{d\hat{w}_1}{dz} - \frac{\bar{\rho}_1 - \bar{\rho}_2}{\bar{\rho}_1} \frac{g}{\bar{u}^2} \hat{w}_1 - \frac{\bar{\rho}_1 - \bar{\rho}_2}{\bar{\rho}_1} \frac{1}{\bar{u}} \frac{d\bar{u}}{dz} \hat{w}_1 = \frac{\bar{\rho}_2}{\bar{\rho}_1} \frac{d\hat{w}_2}{dz}. \quad (3.15)$$

For constant $\bar{u}$, equation (3.15) becomes

$$\frac{d\hat{w}_1}{dz} - \gamma \hat{w}_1 = \frac{\bar{\rho}_2}{\bar{\rho}_1} \frac{d\hat{w}_2}{dz}.$$

Under the Boussinesq approximation I, the change of the density in the vertical direction is small and can be neglected except in the terms involving gravity. This implies that the ratio $\frac{\bar{\rho}_2}{\bar{\rho}_1} = 1 - \frac{\bar{\rho}_1 - \bar{\rho}_2}{\bar{\rho}_1}$ can be approximated by 1. Thus the Boussinesq approximation I version of the interface condition (3.9) is

$$\frac{d\hat{w}_1}{dz} - \gamma \hat{w}_1 = \frac{d\hat{w}_2}{dz}. \quad (3.16)$$

In a configuration where the density is continuous at the interface then $\gamma = 0$ and the interface condition becomes

$$\frac{d\hat{w}_1}{dz} = \frac{d\hat{w}_2}{dz}. \quad (3.17)$$

Under the Boussinesq approximation II $\bar{\rho} = \rho_0$, a constant, and so $\gamma = 0$ and $\frac{\bar{\rho}_2}{\bar{\rho}_1} = 1$. This also gives the simplified condition (3.17).

Since from now on we will use only the Boussinesq approximation I, we will refer it, for simplicity, as the Boussinesq approximation.

3.2 General Solutions

3.2.1 The Configuration with a Thermal Forcing Centered in the Lower Layer

In this section, we derive a solution of equation (2.54) in the two-layer model $-h \leq z < \infty$ where the thermal forcing (3.4) is applied. This configuration describes the obstacle effect mechanism.

Equation (2.54) is a second-order nonhomogeneous ordinary differential equation with constant coefficients. Its characteristic equation has the roots

$$r_{1,2} = \frac{\chi}{2g} \pm \sqrt{\left(\frac{\chi}{2g}\right)^2 - \left(\frac{\chi}{\bar{u}^2} - \delta k^2\right)}.$$

In the lower layer, $\chi = -N_0^2 = -\frac{g}{H} < 0$ so $\frac{\chi}{\bar{u}^2} - \delta k^2 < 0$.

Let

$$\lambda^2 = \left(\frac{N_0^2}{\bar{u}^2} + \delta k^2\right) = \frac{g}{H\bar{u}^2} + \delta k^2.$$

The discriminant can be written as

$$4 \left[\left(\frac{1}{2H}\right)^2 + \lambda^2 \right] > 0,$$

so the characteristic equation has two real roots

$$r_{1,2} = -\frac{1}{2H} \pm \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}.$$

Thus two linearly independent solutions of the homogeneous equation corresponding to (2.54) are $e^{(r_{1,2})z}$, an exponentially-decaying and an exponentially-growing function. A particular solution of the nonhomogeneous equation (2.54) can be obtained in each of the layers by using the method of undetermined coefficients.

The interval $-h \leq z < -h_1$:

In the lower part of the lower layer, the amplitude of the thermal forcing takes the form $\hat{F} = F_0 e^{b(z+h_1)}$ and so equation (2.54) becomes

$$\frac{d^2 \hat{w}_0}{dz^2} + \frac{1}{H} \frac{d\hat{w}_0}{dz} - \lambda^2 \hat{w}_0 = -E_{01} e^{(b-\frac{1}{H})z} e^{bh_1}, \quad (3.18)$$

where $E_{01} = \frac{gF_0}{\rho_{01}\bar{u}^2}$. Thus the general solution of equation (3.18) is

$$\begin{aligned} \hat{w}_0 = & A_0 e^{\left(-\frac{1}{2H} + \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}\right)z} + B_0 e^{\left(-\frac{1}{2H} - \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}\right)z} \\ & - \frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H}\left(b - \frac{1}{H}\right) - \lambda^2} e^{(b-\frac{1}{H})z} e^{bh_1}, \end{aligned} \quad (3.19)$$

where A_0 and B_0 are constants which can be determined from the specified boundary and interface conditions.

The interval $-h_1 \leq z \leq 0$:

In the upper part of the lower layer, the amplitude of the thermal forcing takes the form $\hat{F} = F_0 e^{-b(z+h_1)}$ and so equation (2.54) becomes

$$\frac{d^2 \hat{w}_1}{dz^2} + \frac{1}{H} \frac{d\hat{w}_1}{dz} - \lambda^2 \hat{w}_1 = -E_{01} e^{-(b+\frac{1}{H})z} e^{-bh_1}. \quad (3.20)$$

The solution of this equation is given by

$$\begin{aligned} \hat{w}_1(z) = & A_1 e^{\left(-\frac{1}{2H} + \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}\right)z} + B_1 e^{\left(-\frac{1}{2H} - \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}\right)z} \\ & - \frac{E_{01}}{\left(b + \frac{1}{H}\right)^2 - \frac{1}{H}\left(b + \frac{1}{H}\right) - \lambda^2} e^{-(b + \frac{1}{H})z} e^{-bh_1}, \end{aligned} \quad (3.21)$$

where A_1 and B_1 are constants which can be determined from the specified boundary and interface conditions.

The interval $0 < z < \infty$:

In the stable upper layer, $0 < z < \infty$, the background density is given by $\bar{\rho}_2 = \rho_{02} e^{-\frac{z}{H}}$ so $\frac{d\bar{\rho}_2}{dz} < 0$ and $\chi = N^2 = \frac{g}{H} > 0$. The amplitude of the thermal forcing takes the form $\hat{F} = F_0 e^{-b(z+h_1)}$, so equation (2.54) becomes

$$\frac{d^2 \hat{w}_2}{dz^2} - \frac{1}{H} \frac{d\hat{w}_2}{dz} + \left[\frac{N^2}{\bar{u}^2} - \delta k^2 \right] \hat{w}_2 = -E_{02} e^{-(b - \frac{1}{H})z} e^{-bh_1}, \quad (3.22)$$

where $E_{02} = \frac{gF_0}{\rho_{02}\bar{u}^2}$.

There are two possibilities to consider: **Case a** in which $\frac{N^2}{\bar{u}^2} - \delta k^2 > 0$ and **Case b** in which $\frac{N^2}{\bar{u}^2} - \delta k^2 < 0$.

Consider first **Case a**:

Let

$$\mu^2 = \frac{N^2}{\bar{u}^2} - \delta k^2.$$

Equation (3.22) becomes

$$\frac{d^2 \hat{w}_2}{dz^2} - \frac{1}{H} \frac{d\hat{w}_2}{dz} + \mu^2 \hat{w}_2 = -E_{02} e^{-(b - \frac{1}{H})z} e^{-bh_1}, \quad (3.23)$$

with discriminant $4 \left[\left(\frac{1}{2H}\right)^2 - \mu^2 \right]$. Then we have three cases:

Case a I in which $\left(\frac{1}{2H}\right)^2 - \mu^2 < 0$, **Case a II** in which $\left(\frac{1}{2H}\right)^2 - \mu^2 = 0$ and **Case a III** in which $\left(\frac{1}{2H}\right)^2 - \mu^2 > 0$.

Case a I $\left(\frac{1}{2H}\right)^2 - \mu^2 < 0$:

The roots of the characteristic equation are a complex conjugate pair with positive

real parts

$$r_{1,2} = \frac{1}{2H} \pm i \sqrt{\mu^2 - \left(\frac{1}{2H}\right)^2}.$$

The homogeneous equation corresponding to equation (3.23) has the solution

$$\hat{w}_2(z) = e^{\left(\frac{1}{2H}\right)z} \left[C_1 e^{-i\left(\sqrt{\mu^2 - \left(\frac{1}{2H}\right)^2}\right)z} + C_2 e^{i\left(\sqrt{\mu^2 - \left(\frac{1}{2H}\right)^2}\right)z} \right], \quad (3.24)$$

where C_1 and C_2 are constants.

Let

$$m = \sqrt{\mu^2 - \left(\frac{1}{2H}\right)^2},$$

equation (3.24) becomes

$$\hat{w}_2 = e^{\left(\frac{1}{2H}\right)z} [C_1 e^{-imz} + C_2 e^{imz}].$$

Following Booker and Bretherton (1967), the group velocity argument given in Appendix A tells us that with $\bar{u} > 0$ and $k > 0$ the solution e^{imz} corresponds to an upward-propagating wave with positive group velocity, while the other solution e^{-imz} corresponds to a downward-propagating wave with negative group velocity. So we set $C_1 = 0$ in order to have an upward-propagating wave, and the general solution of the equation (3.23) is

$$\hat{w}_2 = C_2 e^{\left(\frac{1}{2H}\right)z} e^{i\sqrt{\mu^2 - \left(\frac{1}{2H}\right)^2}z} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H}\left(b - \frac{1}{H}\right) + \mu^2} e^{-(b - \frac{1}{H})z} e^{-bh_1}, \quad (3.25)$$

where C_2 is a constant which can be determined from the specified boundary and interface conditions. Recall that $\hat{w}_2$ is complex and the real solution $w(x, z)$ is given by (3.1). In the complex conjugate the horizontal wavenumber is $-k < 0$ and we take the solution e^{-imz} in order to have an upward-propagating wave. If $\bar{u} < 0$ the signs of the vertical wavenumbers are the opposite of those given here.

Case a II $\left(\frac{1}{2H}\right)^2 = \mu^2$:

The roots of the characteristic equation for (3.23) are two equal positive real numbers

$$r_{1,2} = \frac{1}{2H}.$$

Thus the general solution of equation (3.23) in this case is

$$\hat{w}_2 = (D_1 + D_2 z) e^{\left(\frac{1}{2H}\right)z} - E_{02} \frac{1}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H} \left(b - \frac{1}{H}\right) + \mu^2} e^{-\left(b - \frac{1}{H}\right)z} e^{-bh_1}, \quad (3.26)$$

where D_1 and D_2 are constants which can be determined from the specified boundary and interface conditions. There is no oscillation in z and if D_1 and D_2 are nonzero, the amplitude of the solution increases exponentially with z .

Case a III $\left(\frac{1}{2H}\right)^2 - \mu^2 > 0$:

The roots of the characteristic equation for (3.23) are two distinct real numbers

$$r_{1,2} = \frac{1}{2H} \pm \sqrt{\left(\frac{1}{2H}\right)^2 - \mu^2}.$$

Thus the general solution of equation (3.23) is

$$\begin{aligned} \hat{w}_2 = & E_1 e^{\left(\frac{1}{2H} + \sqrt{\left(\frac{1}{2H}\right)^2 - \mu^2}\right)z} + E_2 e^{\left(\frac{1}{2H} - \sqrt{\left(\frac{1}{2H}\right)^2 - \mu^2}\right)z} \\ & - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H} \left(b - \frac{1}{H}\right) + \mu^2} e^{-\left(b - \frac{1}{H}\right)z} e^{-bh_1}, \end{aligned} \quad (3.27)$$

where E_1 and E_2 are constants which can be determined from the specified boundary and interface conditions. Once again there is no oscillation in z and if E_1 and E_2 are nonzero, the amplitude of the solution increases exponentially as $z \rightarrow \infty$.

Consider now **Case b** in which $\frac{N^2}{\bar{u}^2} - \delta k^2 < 0$:

Let

$$\nu^2 = -\left(\frac{N^2}{\bar{u}^2} - \delta k^2\right).$$

In this case, the discriminant is $4 \left[\left(\frac{1}{2H}\right)^2 + \nu^2\right] > 0$ and equation (3.22) becomes

$$\frac{d^2 \hat{w}_2}{dz^2} - \frac{1}{H} \frac{d \hat{w}_2}{dz} - \nu^2 \hat{w}_2 = -E_{02} e^{-\left(b - \frac{1}{H}\right)z} e^{-bh_1}. \quad (3.28)$$

The roots of the characteristic equation are two distinct positive real numbers

$$r_{1,2} = \frac{1}{2H} \pm \sqrt{\left(\frac{1}{2H}\right)^2 + \nu^2}.$$

The general solution of equation (3.28) is

$$\begin{aligned} \hat{w}_2 = & F_1 e^{\left(\frac{1}{2H} + \sqrt{\left(\frac{1}{2H}\right)^2 + \nu^2}\right)z} + F_2 e^{\left(\frac{1}{2H} - \sqrt{\left(\frac{1}{2H}\right)^2 + \nu^2}\right)z} \\ & - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H}\left(b - \frac{1}{H}\right) - \nu^2} e^{-\left(b - \frac{1}{H}\right)z} e^{-bh_1}, \end{aligned} \quad (3.29)$$

where F_1 and F_2 are constants which can be determined from the specified boundary and interface conditions. There is no oscillation in z and if F_1 is nonzero, the amplitude of the solution increases exponentially with z .

We note that the solutions in the lower layer are exponential functions of the vertical variable. In the upper layer, we are interested in Case a I which is the only configuration for which the solution oscillates vertically representing upward-propagating waves. In each case, the constants can be determined by applying the lower boundary condition and the interface condition.

Summary in a Chart

$\frac{d^2 \hat{w}}{dz^2} - \frac{\chi}{g} \frac{d\hat{w}}{dz} + \left[\frac{\chi}{\bar{u}^2} - \delta k^2 \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\rho(z)}.$	
$\downarrow \quad F(x, z) = F_0 e^{-b z+h_1 } e^{ikx}. \quad \downarrow$	
$\downarrow \text{Equations} \downarrow$	
$\text{Upper layer } (\chi = N^2 > 0)$	
Case a: $\mu^2 = \frac{N^2}{\bar{u}^2} - \delta k^2 > 0.$ $\frac{d^2 \hat{w}_2}{dz^2} - \frac{1}{H} \frac{d\hat{w}_2}{dz} + \mu^2 \hat{w}_2 = -E_{02} e^{-(b-\frac{1}{H})z} e^{-bh_1}.$	Case b: $-\nu^2 = \frac{N^2}{\bar{u}^2} - \delta k^2 < 0.$ $\frac{d^2 \hat{w}_2}{dz^2} - \frac{1}{H} \frac{d\hat{w}_2}{dz} - \nu^2 \hat{w}_2 = -E_{02} e^{-(b-\frac{1}{H})z} e^{-bh_1}.$
$\text{Lower layer } (\chi = -N_0^2 < 0)$	
$-\lambda^2 = \frac{-N_0^2}{\bar{u}^2} - \delta k^2 < 0.$	
$\frac{d^2 \hat{w}_1}{dz^2} + \frac{1}{H} \frac{d\hat{w}_1}{dz} - \lambda^2 \hat{w}_1 = -E_{01} e^{-(b+\frac{1}{H})z} e^{-bh_1}.$	
$\frac{d^2 \hat{w}_0}{dz^2} + \frac{1}{H} \frac{d\hat{w}_0}{dz} - \lambda^2 \hat{w}_0 = -E_{01} e^{(b-\frac{1}{H})z} e^{bh_1}.$	
$\downarrow \text{Solutions} \downarrow$	
Upper layer	
Case a I: $\left(\frac{1}{2H}\right)^2 - \mu^2 < 0.$ $\hat{w}_2 = C e^{(\frac{1}{2H})z} e^{i\sqrt{\mu^2 - \left(\frac{1}{2H}\right)^2} z} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H}\left(b - \frac{1}{H}\right) + \mu^2} e^{-(b-\frac{1}{H})z} e^{-bh_1}.$	
Case a II: $\left(\frac{1}{2H}\right)^2 - \mu^2 = 0.$ $\hat{w}_2 = (D_1 + D_2 z) e^{(\frac{1}{2H})z} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H}\left(b - \frac{1}{H}\right) + \mu^2} e^{-(b-\frac{1}{H})z} e^{-bh_1}.$	
Case a III: $\left(\frac{1}{2H}\right)^2 - \mu^2 > 0.$ $\hat{w}_2 = E_1 e^{\left(\frac{1}{2H} + \sqrt{\left(\frac{1}{2H}\right)^2 - \mu^2}\right)z} + E_2 e^{\left(\frac{1}{2H} - \sqrt{\left(\frac{1}{2H}\right)^2 - \mu^2}\right)z} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H}\left(b - \frac{1}{H}\right) + \mu^2} e^{-(b-\frac{1}{H})z} e^{-bh_1}.$	
Case b: $\hat{w}_2 = F_1 e^{\left(\frac{1}{2H} + \sqrt{\left(\frac{1}{2H}\right)^2 + \nu^2}\right)z} + F_2 e^{\left(\frac{1}{2H} - \sqrt{\left(\frac{1}{2H}\right)^2 + \nu^2}\right)z} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H}\left(b - \frac{1}{H}\right) - \nu^2} e^{-(b-\frac{1}{H})z} e^{-bh_1}.$	
Lower layer	
$\hat{w}_1 = A_1 e^{\left(-\frac{1}{2H} + \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}\right)z} + B_1 e^{\left(-\frac{1}{2H} - \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}\right)z} - \frac{E_{01}}{\left(b + \frac{1}{H}\right)^2 - \frac{1}{H}\left(b + \frac{1}{H}\right) - \lambda^2} e^{-(b+\frac{1}{H})z} e^{-bh_1}.$	
$\hat{w}_0 = A_0 e^{\left(-\frac{1}{2H} + \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}\right)z} + B_0 e^{\left(-\frac{1}{2H} - \sqrt{\left(\frac{1}{2H}\right)^2 + \lambda^2}\right)z} - \frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 + \frac{1}{H}\left(b - \frac{1}{H}\right) - \lambda^2} e^{(b-\frac{1}{H})z} e^{bh_1}.$	

Table 3.1: Summary of the steps taken in deriving the solutions of the non-Boussinesq equation in the two-layer model when the thermal forcing is centered in the lower layer.

3.2.2 The Configuration with a Thermal Forcing Centered at the Interface

In this subsection, we consider the configuration when the thermal forcing is centered vertically at the interface, $z = 0$. We set $h_1 = 0$ in the forcing function (3.4):

$$F(x, z) = F_0 e^{-b|z|} e^{ikx} + \text{c.c.} \quad (3.30)$$

In this configuration, we no longer have w_1 . We will use the subscript 0 for the solution in the lower layer and the subscript 2 for the solution in the upper layer. We again consider upward-propagating waves and the boundary and interface conditions become

$$\hat{w}_0 = 0 \quad \text{at } z = -h, \quad (3.31)$$

$$\hat{w}_0 = \hat{w}_2 \quad \text{at } z = 0, \quad (3.32)$$

$$\frac{d\hat{w}_0}{dz} - \gamma\hat{w}_0 = \frac{\bar{\rho}_2}{\bar{\rho}_1} \frac{d\hat{w}_2}{dz} \quad \text{at } z = 0. \quad (3.33)$$

The thermal forcing influences both upper and lower layers. In the upper layer it generates gravity waves (the deep heating mechanism) and in the lower layer it generates convection which influence the gravity waves via the interface conditions (the obstacle effect mechanism). Thus, this configuration describes a combination of the two gravity wave generation mechanisms. A schematic of our two-layer model in this configuration is shown in Figure 3.2.

We derive the solutions in the case when the thermal forcing is centered at the interface following the procedure we used in Section 3.2.1 for the problem when the thermal forcing is centered in the lower layer. Table 3.2 shows the solutions in this configuration.

Figure 3.2: A schematic diagram of the two-layer model in the domain $-h \leq z < \infty$ with gravity waves in the upper layer and convection in the lower layer. The thermal disturbance is centered at $z = 0$.

$\frac{d^2 \hat{w}}{dz^2} - \frac{\chi}{g} \frac{d\hat{w}}{dz} + \left[\frac{\chi}{\bar{u}^2} - \delta k^2 \right] \hat{w} = -\frac{g}{\bar{u}^2} \frac{\hat{F}}{\rho}.$		
$\downarrow \quad F(x, z) = F_0 e^{-b z } e^{ikx}. \quad \downarrow$		
$\downarrow \text{Equations} \downarrow$		
$\text{Upper layer } (\chi = N^2 > 0)$		
Case a: $\mu^2 = \frac{N^2}{\bar{u}^2} - \delta k^2 > 0.$ $\frac{d^2 \hat{w}_2}{dz^2} - \frac{1}{H} \frac{d\hat{w}_2}{dz} + \mu^2 \hat{w}_2 = -E_{02} e^{-(b-\frac{1}{H})z}.$	Case b: $-\nu^2 = \frac{N^2}{\bar{u}^2} - \delta k^2 < 0.$ $\frac{d^2 \hat{w}_2}{dz^2} - \frac{1}{H} \frac{d\hat{w}_2}{dz} - \nu^2 \hat{w}_2 = -E_{02} e^{-(b-\frac{1}{H})z}.$	
$\text{Lower layer } (\chi = -N_0^2 < 0)$		
$-\lambda^2 = \frac{-N_0^2}{\bar{u}^2} - \delta k^2 < 0.$		
$\frac{d^2 \hat{w}_0}{dz^2} + \frac{1}{H} \frac{d\hat{w}_0}{dz} - \lambda^2 \hat{w}_0 = -E_{01} e^{(b-\frac{1}{H})z}.$		
$\downarrow \text{Solutions} \downarrow$		
Upper layer		
Case a I: $(\frac{1}{2H})^2 - \mu^2 < 0.$ $\hat{w}_2 = C e^{(\frac{1}{2H})z} e^{i\sqrt{\mu^2 - (\frac{1}{2H})^2}z} - \frac{E_{02} e^{-(b-\frac{1}{H})z}}{(b-\frac{1}{H})^2 + \frac{1}{H}(b-\frac{1}{H}) + \mu^2}.$		
Case a II: $(\frac{1}{2H})^2 - \mu^2 = 0.$ $\hat{w}_2 = (D_1 + D_2 z) e^{(\frac{1}{2H})z} - \frac{E_{02} e^{-(b-\frac{1}{H})z}}{(b-\frac{1}{H})^2 + \frac{1}{H}(b-\frac{1}{H}) + \mu^2}.$		
Case a III: $(\frac{1}{2H})^2 - \mu^2 > 0.$ $\hat{w}_2 = E_1 e^{\left(\frac{1}{2H} + \sqrt{(\frac{1}{2H})^2 - \mu^2}\right)z} + E_2 e^{\left(\frac{1}{2H} - \sqrt{(\frac{1}{2H})^2 - \mu^2}\right)z} - \frac{E_{02} e^{-(b-\frac{1}{H})z}}{(b-\frac{1}{H})^2 + \frac{1}{H}(b-\frac{1}{H}) + \mu^2}.$		
Case b: $\hat{w}_2 = F_1 e^{\left(\frac{1}{2H} + \sqrt{(\frac{1}{2H})^2 + \nu^2}\right)z} + F_2 e^{\left(\frac{1}{2H} - \sqrt{(\frac{1}{2H})^2 + \nu^2}\right)z} - \frac{E_{02} e^{-(b-\frac{1}{H})z}}{(b-\frac{1}{H})^2 + \frac{1}{H}(b-\frac{1}{H}) - \nu^2}.$		
Lower layer		
$\hat{w}_0 = A_0 e^{\left(-\frac{1}{2H} + \sqrt{(\frac{1}{2H})^2 + \lambda^2}\right)z} + B_0 e^{\left(-\frac{1}{2H} - \sqrt{(\frac{1}{2H})^2 + \lambda^2}\right)z} - \frac{E_{01} e^{(b-\frac{1}{H})z}}{(b-\frac{1}{H})^2 + \frac{1}{H}(b-\frac{1}{H}) - \lambda^2}.$		

Table 3.2: Summary of the steps taken in deriving the solutions of the non-Boussinesq equation in the two-layer model when the thermal forcing is centered at the interface.

Note that setting $h_1 = 0$ in the solutions given in Section 3.2.1 does not recover the above solutions. This is obvious since $h_1 = 0$ does not imply that the two differential equations in regions “0” and “1” [i.e., (3.18) and (3.20), respectively] are identical. In summary, as before, Cases a II and a III give solutions that increase

exponentially with z according to the magnitude of the scale height. We are interested in the solution where there are upward-propagating oscillatory gravity waves in the upper layer, so we consider Case a I. In all cases, the solution in the lower layer is a linear combination of exponential functions of z . In the next section we will see that when the boundary and interface conditions are applied and the values of the constants in the general solution are determined, then the solution in the lower layer represents convective cells coupled with upward-propagating gravity waves in the upper layer.

3.3 Application of the Boundary and Interface Conditions

In the previous section we obtained the general solution of equation (2.54). The application of the boundary and interface conditions gives expressions that are very complicated. In order to reduce the complexity of the expressions, we will simplify the solutions by applying the Boussinesq approximation I before applying the boundary and interface conditions. In the Boussinesq approximation I, the first derivative term $(-\frac{\chi}{g} \frac{d\hat{w}}{dz})$ in equation (2.54) is eliminated giving equation (2.59). In this section, we solve equation (2.59) in each layer then we apply the boundary and interface conditions to determine the constants.

3.3.1 Thermal Disturbance Centered in the Lower Layer with the Boussinesq Approximation

Consider first the lower layer $-h \leq z < 0$. We have $-\frac{N_0^2}{\bar{u}^2} - \delta k^2 < 0$.

Let

$$\lambda^2 = - \left(\frac{-N_0^2}{\bar{u}^2} - \delta k^2 \right) = \frac{g}{H\bar{u}^2} + \delta k^2.$$

The interval $-h \leq z < -h_1$:

In the lower part of the lower layer, equation (2.59) becomes

$$\frac{d^2 \hat{w}_0}{dz^2} - \lambda^2 \hat{w}_0 = -E_{01} e^{(b-\frac{1}{H})z} e^{bh_1}. \quad (3.34)$$

The solution of equation (3.34) is

$$\hat{w}_0(z) = A_0 \cosh \lambda z + B_0 \sinh \lambda z - \frac{E_{01}}{(b - \frac{1}{H})^2 - \lambda^2} e^{(b-\frac{1}{H})z} e^{bh_1}. \quad (3.35)$$

The interval $-h_1 \leq z \leq 0$:

In the upper part of the lower layer, equation (2.59) becomes

$$\frac{d^2 \hat{w}_1}{dz^2} - \lambda^2 \hat{w}_1 = -E_{01} e^{-(b+\frac{1}{H})z} e^{-bh_1}. \quad (3.36)$$

The solution of this equation is given by

$$\hat{w}_1(z) = A_1 \cosh \lambda z + B_1 \sinh \lambda z - \frac{E_{01}}{(b + \frac{1}{H})^2 - \lambda^2} e^{-(b+\frac{1}{H})z} e^{-bh_1}. \quad (3.37)$$

The interval $0 < z < \infty$:

In the stable upper layer, $0 < z < \infty$, equation (2.59) becomes

$$\frac{d^2 \hat{w}_2}{dz^2} + \left[\frac{N^2}{\bar{u}^2} - \delta k^2 \right] \hat{w}_2 = -E_{02} e^{-(b-\frac{1}{H})z} e^{-bh_1}. \quad (3.38)$$

We have two cases for the upper layer: **Case a** in which $\frac{N^2}{\bar{u}^2} - \delta k^2 > 0$ and **Case b** in which $\frac{N^2}{\bar{u}^2} - \delta k^2 < 0$.

Consider first **Case a** $\frac{N^2}{\bar{u}^2} - \delta k^2 > 0$:

Let

$$m^2 = \frac{N^2}{\bar{u}^2} - \delta k^2 = \frac{g}{H\bar{u}^2} - \delta k^2.$$

Equation (3.38) becomes

$$\frac{d^2 \hat{w}_2}{dz^2} + m^2 \hat{w}_2 = -E_{02} e^{-(b-\frac{1}{H})z} e^{-bh_1}. \quad (3.39)$$

Using the group velocity argument in Appendix A, we can show that with $\bar{u} > 0$ and $k > 0$ the solution e^{imz} corresponds to an upward-propagating wave with positive group velocity, while the other solution e^{-imz} corresponds to a downward-propagating wave with negative group velocity. Thus the solution of equation (3.39) is

$$\hat{w}_2(z) = C_2 e^{imz} - \frac{E_{02}}{(b - \frac{1}{H})^2 + m^2} e^{-(b - \frac{1}{H})z} e^{-bh_1}. \quad (3.40)$$

Consider now **Case b** $\frac{N^2}{\bar{u}^2} - \delta k^2 < 0$:

Let

$$\nu^2 = - \left(\frac{N^2}{\bar{u}^2} - \delta k^2 \right).$$

Equation (2.59) becomes

$$\frac{d^2 \hat{w}_2}{dz^2} - \nu^2 \hat{w}_2 = -E_{02} e^{-(b - \frac{1}{H})z} e^{-bh_1}. \quad (3.41)$$

The solutions of the corresponding homogeneous equation either grow or decay with height

$$\hat{w}_2(z) = D_1 e^{\nu z} + D_2 e^{-\nu z}.$$

To have a solution that is bounded as $z \rightarrow \infty$, we set $D_1 = 0$. We obtain a particular solution by using the method of undetermined coefficients. The solution of equation (3.41) thus is given by

$$\hat{w}_2(z) = D_2 e^{-\nu z} - \frac{E_{02}}{(b - \frac{1}{H})^2 - \nu^2} e^{-(b - \frac{1}{H})z} e^{-bh_1}. \quad (3.42)$$

The solution in the upper layer gives either upward-propagating oscillatory waves according to Case a or trapped waves according to Case b. Since the trapped waves obtained in Case b decay to zero exponentially, they cannot affect the development of the flow at the higher levels and they are of less interest to us. We focus on the oscillatory gravity waves obtained in Case a.

The constants A_0 , B_0 , A_1 , B_1 , C_2 and D_2 are determined by applying the interface and bottom boundary conditions: (3.5), (3.6), (3.7), (3.8) and (3.16). We

show the application of the conditions for Case a. Applying condition (3.5), we get

$$A_0 \cosh \lambda h - B_0 \sinh \lambda h = \frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} e^{-(b - \frac{1}{H})h} e^{bh_1}. \quad (3.43)$$

Conditions (3.6) and (3.7) give the following equations, respectively,

$$\begin{aligned} A_0 \cosh \lambda h_1 - B_0 \sinh \lambda h_1 - A_1 \cosh \lambda h_1 + B_1 \sinh \lambda h_1 \\ = E_{01} e^{\frac{1}{H}h_1} \left(\frac{1}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} - \frac{1}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} \right), \end{aligned} \quad (3.44)$$

and

$$\begin{aligned} -A_0 \lambda \sinh \lambda h_1 + B_0 \lambda \cosh \lambda h_1 + A_1 \lambda \sinh \lambda h_1 - B_1 \lambda \cosh \lambda h_1 \\ = E_{01} e^{\frac{1}{H}h_1} \left(\frac{b - \frac{1}{H}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} + \frac{b + \frac{1}{H}}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} \right). \end{aligned} \quad (3.45)$$

Conditions (3.8) and (3.16) give the following equations, respectively,

$$A_1 - C_2 = e^{-bh_1} \left(\frac{E_{01}}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + m^2} \right), \quad (3.46)$$

$$-\gamma A_1 + \lambda B_1 - im C_2 = e^{-bh_1} \left(\frac{E_{02} \left(b - \frac{1}{H}\right)}{\left(b - \frac{1}{H}\right)^2 + m^2} - \frac{E_{01} \left(b + \frac{1}{H} + \gamma\right)}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} \right). \quad (3.47)$$

Equations (3.43)-(3.47) can be written in matrix form as

$$\begin{pmatrix} \cosh \lambda h & -\sinh \lambda h & 0 & 0 & 0 \\ \cosh \lambda h_1 & -\sinh \lambda h_1 & -\cosh \lambda h_1 & \sinh \lambda h_1 & 0 \\ -\lambda \sinh \lambda h_1 & \lambda \cosh \lambda h_1 & \lambda \sinh \lambda h_1 & -\lambda \cosh \lambda h_1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -\gamma & \lambda & -im \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} P \\ Q \\ R \\ S \\ T \end{pmatrix},$$

where

$$\begin{pmatrix} P \\ Q \\ R \\ S \\ T \end{pmatrix} = \begin{pmatrix} E_{01} e^{bh_1} e^{-(b-\frac{1}{H})h} \frac{1}{(b-\frac{1}{H})^2 - \lambda^2} \\ E_{01} e^{\frac{1}{H}h_1} \left(\frac{1}{(b-\frac{1}{H})^2 - \lambda^2} - \frac{1}{(b+\frac{1}{H})^2 - \lambda^2} \right) \\ E_{01} e^{\frac{1}{H}h_1} \left(\frac{b-\frac{1}{H}}{(b-\frac{1}{H})^2 - \lambda^2} + \frac{b+\frac{1}{H}}{(b+\frac{1}{H})^2 - \lambda^2} \right) \\ e^{-bh_1} \left(\frac{E_{01}}{(b+\frac{1}{H})^2 - \lambda^2} - \frac{E_{02}}{(b-\frac{1}{H})^2 + m^2} \right) \\ e^{-bh_1} \left(\frac{E_{02}(b-\frac{1}{H})}{(b-\frac{1}{H})^2 + m^2} - \frac{E_{01}(b+\frac{1}{H}+\gamma)}{(b+\frac{1}{H})^2 - \lambda^2} \right) \end{pmatrix}.$$

By solving the above system of equations, we get

$$B_0 = \frac{\cosh \lambda h}{\lambda \cosh \lambda h - (\gamma + im) \sinh \lambda h} \left\{ (\gamma + im) \operatorname{sech} \lambda h P - [(\gamma + im) \cosh \lambda h_1 - \lambda \sinh \lambda h_1] Q \right. \\ \left. - \left[\frac{\gamma + im}{\lambda} \sinh \lambda h_1 - \cosh \lambda h_1 \right] R - imS + T \right\},$$

$$A_0 = \frac{\sinh \lambda h}{\cosh \lambda h} B_0 + \frac{1}{\cosh \lambda h} P,$$

$$B_1 = B_0 - \sinh \lambda h_1 Q - \frac{1}{\lambda} \cosh \lambda h_1 R,$$

$$A_1 = \frac{1}{\gamma + im} [\lambda B_1 + im S - T],$$

$$C_2 = A_1 - S.$$

By using the following fact

$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b, \quad (3.48)$$

the solutions can be written as follows:

In the upper layer we have for Case a (gravity waves)

$$\hat{w}_2(z) = \left[\frac{1}{\gamma + im} (\lambda \alpha \cosh \lambda h - \lambda \sinh \lambda h_1 Q - \cosh \lambda h_1 R + imS - T) - S \right] e^{imz} \\ - \frac{E_{02} e^{-bh_1}}{(b-\frac{1}{H})^2 + m^2} e^{-(b-\frac{1}{H})z}, \quad (3.49)$$

and for Case b (trapped disturbance)

$$\hat{w}_2(z) = \left[\frac{1}{\gamma - \nu} (\lambda \alpha \cosh \lambda h - \lambda \sinh \lambda h_1 Q - \cosh \lambda h_1 R - \nu S - T) - S \right] e^{-\nu z} \\ - \frac{E_{02} e^{-bh_1}}{(b-\frac{1}{H})^2 - \nu^2} e^{-(b-\frac{1}{H})z}. \quad (3.50)$$

In the lower layer, the constants are different depending on whether the upper layer is in Case a or Case b.

In the upper part of the lower layer, if the upper layer is in Case a we have

$$\begin{aligned}\hat{w}_1(z) = & \left(\alpha \cosh \lambda h - \sinh \lambda h_1 Q - \frac{\cosh \lambda h_1}{\lambda} R \right) \sinh \lambda z \\ & + \frac{1}{\gamma + im} \left(\lambda \alpha \cosh \lambda h - \lambda \sinh \lambda h_1 Q - \cosh \lambda h_1 R + imS - T \right) \cosh \lambda z \\ & - \frac{E_{01} e^{-bh_1}}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} e^{-(b + \frac{1}{H})z},\end{aligned}\quad (3.51)$$

and if the upper layer is in Case b, we have

$$\begin{aligned}\hat{w}_1(z) = & \left(\alpha \cosh \lambda h - \sinh \lambda h_1 Q - \frac{\cosh \lambda h_1}{\lambda} R \right) \sinh \lambda z \\ & + \frac{1}{\gamma - \nu} \left(\lambda \alpha \cosh \lambda h - \lambda \sinh \lambda h_1 Q - \cosh \lambda h_1 R - \nu S - T \right) \cosh \lambda z \\ & - \frac{E_{01} e^{-bh_1}}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} e^{-(b + \frac{1}{H})z}.\end{aligned}\quad (3.52)$$

In the lower part of the lower layer, we have

$$\hat{w}_0(z) = \alpha \sinh \lambda(z + h) + P \frac{\cosh \lambda z}{\cosh \lambda h} - \frac{E_{01} e^{bh_1}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} e^{(b - \frac{1}{H})z}, \quad (3.53)$$

where α has two values depending on whether the upper layer in Case a or Case b.

The constants are

$$\begin{aligned}E_{01} &= \frac{gF_0}{\rho_{01}\bar{u}^2}, \\ E_{02} &= \frac{gF_0}{\rho_{02}\bar{u}^2}, \\ P &= E_{01} e^{bh_1} e^{-(b - \frac{1}{H})h} \frac{1}{\left(b - \frac{1}{H}\right)^2 - \lambda^2}, \\ Q &= E_{01} e^{\frac{1}{H}h_1} \left(\frac{1}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} - \frac{1}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} \right), \\ R &= E_{01} e^{\frac{1}{H}h_1} \left(\frac{b - \frac{1}{H}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} + \frac{b + \frac{1}{H}}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} \right),\end{aligned}$$

Case a

$$\begin{aligned}
S &= e^{-bh_1} \left(\frac{E_{01}}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + m^2} \right), \\
T &= e^{-bh_1} \left(\frac{E_{02} \left(b - \frac{1}{H}\right)}{\left(b - \frac{1}{H}\right)^2 + m^2} - \frac{E_{01} \left(b + \frac{1}{H} + \gamma\right)}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} \right), \\
\alpha &= \frac{1}{\lambda \cosh \lambda h - (\gamma + im) \sinh \lambda h} \left\{ (\gamma + im) \operatorname{sech} \lambda h P \right. \\
&\quad \left. - [(\gamma + im) \cosh \lambda h_1 - \lambda \sinh \lambda h_1] Q - \left[\frac{\gamma + im}{\lambda} \sinh \lambda h_1 - \cosh \lambda h_1 \right] R \right. \\
&\quad \left. - imS + T \right\},
\end{aligned}$$

Case b

$$\begin{aligned}
S &= e^{-bh_1} \left(\frac{E_{01}}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 - \nu^2} \right), \\
T &= e^{-bh_1} \left(\frac{E_{02} \left(b - \frac{1}{H}\right)}{\left(b - \frac{1}{H}\right)^2 - \nu^2} - \frac{E_{01} \left(b + \frac{1}{H} + \gamma\right)}{\left(b + \frac{1}{H}\right)^2 - \lambda^2} \right), \\
\alpha &= \frac{1}{\lambda \cosh \lambda h - (\gamma - \nu) \sinh \lambda h} \left\{ (\gamma - \nu) \operatorname{sech} \lambda h P \right. \\
&\quad \left. - [(\gamma - \nu) \cosh \lambda h_1 - \lambda \sinh \lambda h_1] Q - \left[\frac{\gamma - \nu}{\lambda} \sinh \lambda h_1 - \cosh \lambda h_1 \right] R \right. \\
&\quad \left. + \nu S + T \right\}.
\end{aligned}$$

The solutions obtained in this subsection are summarized in Table 3.3.

Summary in a Chart

Upper layer	
Case a $\hat{w}_2(z) = \left[\frac{1}{\gamma+im} (\lambda \alpha \cosh \lambda h - \lambda \sinh \lambda h_1 Q - \cosh \lambda h_1 R + imS - T) - S \right] e^{imz} - \frac{E_{02} e^{-bh_1}}{(b-\frac{1}{H})^2+m^2} e^{-(b-\frac{1}{H})z}.$	Case b $\hat{w}_2(z) = \left[\frac{1}{\gamma-\mu} (\lambda \alpha \cosh \lambda h - \lambda \sinh \lambda h_1 Q - \cosh \lambda h_1 R - \nu S - T) - S \right] e^{-\nu z} - \frac{E_{02} e^{-bh_1}}{(b-\frac{1}{H})^2-\nu^2} e^{-(b-\frac{1}{H})z}.$
Lower layer	
Case a $\hat{w}_1(z) = (\alpha \cosh \lambda h - \sinh \lambda h_1 Q - \frac{\cosh \lambda h_1}{\lambda} R) \sinh \lambda z + \frac{1}{\gamma+im} \left(\lambda \alpha \cosh \lambda h - \lambda \sinh \lambda h_1 Q - \cosh \lambda h_1 R + imS - T \right) \cosh \lambda z - \frac{E_{01} e^{-bh_1}}{(b+\frac{1}{H})^2-\lambda^2} e^{-(b+\frac{1}{H})z}.$	
Case b $\hat{w}_1(z) = (\alpha \cosh \lambda h - \sinh \lambda h_1 Q - \frac{\cosh \lambda h_1}{\lambda} R) \sinh \lambda z + \frac{1}{\gamma-\nu} \left(\lambda \alpha \cosh \lambda h - \lambda \sinh \lambda h_1 Q - \cosh \lambda h_1 R - \nu S - T \right) \cosh \lambda z - \frac{E_{01} e^{-bh_1}}{(b+\frac{1}{H})^2-\lambda^2} e^{-(b+\frac{1}{H})z}.$	
Case a or b $\hat{w}_0 = \alpha \sinh \lambda(z+h) + P \frac{\cosh \lambda z}{\cosh \lambda h} - \frac{E_{01} e^{-bh_1}}{(b-\frac{1}{H})^2-\lambda^2} e^{(b-\frac{1}{H})z}.$	
↓	↓
Constants	
$E_{01} = \frac{gF_0}{\rho_{01} \bar{u}^2}, E_{02} = \frac{gF_0}{\rho_{02} \bar{u}^2},$ $P = E_{01} e^{bh_1} e^{-(b-\frac{1}{H})h} \frac{1}{(b-\frac{1}{H})^2-\lambda^2},$ $Q = E_{01} e^{\frac{1}{H}h_1} \left(\frac{1}{(b-\frac{1}{H})^2-\lambda^2} - \frac{1}{(b+\frac{1}{H})^2-\lambda^2} \right),$ $R = E_{01} e^{\frac{1}{H}h_1} \left(\frac{b-\frac{1}{H}}{(b-\frac{1}{H})^2-\lambda^2} + \frac{b+\frac{1}{H}}{(b+\frac{1}{H})^2-\lambda^2} \right),$	
Case a	
$S = e^{-bh_1} \left(\frac{E_{01}}{(b+\frac{1}{H})^2-\lambda^2} - \frac{E_{02}}{(b-\frac{1}{H})^2+m^2} \right), T = e^{-bh_1} \left(\frac{E_{02}(b-\frac{1}{H})}{(b-\frac{1}{H})^2+m^2} - \frac{E_{01}(b+\frac{1}{H}+\gamma)}{(b+\frac{1}{H})^2-\lambda^2} \right),$ $\alpha = \frac{1}{\lambda \cosh \lambda h - (\gamma+im) \sinh \lambda h} \left\{ (\gamma+im) \operatorname{sech} \lambda h P - [(\gamma+im) \cosh \lambda h_1 - \lambda \sinh \lambda h_1] Q - \left[\frac{\gamma+im}{\lambda} \sinh \lambda h_1 - \cosh \lambda h_1 \right] R - imS + T \right\},$	
Case b	
$S = e^{-bh_1} \left(\frac{E_{01}}{(b+\frac{1}{H})^2-\lambda^2} - \frac{E_{02}}{(b-\frac{1}{H})^2-\nu^2} \right), T = e^{-bh_1} \left(\frac{E_{02}(b-\frac{1}{H})}{(b-\frac{1}{H})^2-\nu^2} - \frac{E_{01}(b+\frac{1}{H}+\gamma)}{(b+\frac{1}{H})^2-\lambda^2} \right),$ $\alpha = \frac{1}{\lambda \cosh \lambda h - (\gamma-\nu) \sinh \lambda h} \left\{ (\gamma-\nu) \operatorname{sech} \lambda h P - [(\gamma-\nu) \cosh \lambda h_1 - \lambda \sinh \lambda h_1] Q - \left[\frac{\gamma-\nu}{\lambda} \sinh \lambda h_1 - \cosh \lambda h_1 \right] R + \nu S + T \right\}.$	

Table 3.3: Summary of the steps taken in deriving the solutions of the Boussinesq equation in the two-layer model when the thermal forcing is centered in the lower layer.

We note that we derived steady solutions by assuming the thermal forcing and the disturbance take the forms (2.49) and (2.50), respectively. In order to obtain solutions that oscillate in time as well as x , we would assume the thermal forcing and the disturbance to take the forms

$$F(x, z, t) = \hat{F}(z) e^{ik(x-ct)} + \text{c.c.},$$

$$w(x, z, t) = \hat{w}(z) e^{ik(x-ct)} + \text{c.c.},$$

where c is the wave phase speed. Or equivalently

$$F(x, z, t) = \hat{F}(z) e^{i(kx-\omega t)} + \text{c.c.},$$

$$w(x, z, t) = \hat{w}(z) e^{i(kx-\omega t)} + \text{c.c.},$$

where ω is the wave frequency. In this case equation (2.60) would become

$$\frac{d^2 \hat{w}}{dz^2} + \left[\frac{N^2}{(\bar{u} - c)^2} - \delta k^2 \right] \hat{w} = -\frac{g}{(\bar{u} - c)^2} \frac{\hat{F}}{\bar{\rho}},$$

or

$$\frac{d^2 \hat{w}}{dz^2} + \left[\frac{k^2 N^2}{(\omega - k\bar{u})^2} - \delta k^2 \right] \hat{w} = -\frac{k^2 g}{(\omega - k\bar{u})^2} \frac{\hat{F}}{\bar{\rho}}.$$

This would give the same steady solutions we obtained but with $\bar{u}$ replaced by $\bar{u} - c$ or $\bar{u} - \frac{\omega}{k}$.

Plots of the Exact Steady Solutions

We plot the Boussinesq linear steady normal mode solutions according to (3.49)-(3.53). The figures are given in a rectangular domain $0 \leq x \leq 2\pi$ and $-10 \leq z \leq 20$ with the following parameters: $h_1 = 5$, $F_0 = 10$, the wavenumber $k = 1$, the background velocity $\bar{u} = \sqrt{10}$, the scale height $H = 10$, the gravitational acceleration $g = 9.8$, $N_0^2 = 0.98$ and $N^2 = 0.98$. We keep N^2 , $\bar{u}$, k fixed and then the square of the aspect ratio δ determines whether the solution is in Case a or Case b. We set

$\delta = 0.01$ to get $\frac{N^2}{\bar{u}^2} - \delta k^2 > 0$ (Case a) and $\delta = 0.1$ to get $\frac{N^2}{\bar{u}^2} - \delta k^2 < 0$ (Case b). We match the densities $\bar{\rho}_1$ and $\bar{\rho}_2$ at the interface by setting $\rho_{01} = \rho_{02} = 2$, in this case we have $E_{01} = E_{02}$ and $\gamma = 0$. We consider two configurations: with $b = 0.7$ and $b = 0.07$.

Contour plots of the vertical velocities perturbations for both cases for the first configuration ($b = 0.7$) are shown in Figure 3.3. Figure 3.4 shows the amplitudes of the vertical velocities perturbations as functions of z in the first configuration. With $b = 0.7$ the thermal forcing spans a narrow range of altitudes. This means that the thermal forcing generates convection in the lower layer which in turn generates gravity waves in the upper layer via the interface conditions, this corresponds to the obstacle effect mechanism. The second term in (3.49) includes a factor of $e^{-bh_1} = e^{-3.5}$ and decreases with height according to $e^{-(b-\frac{1}{H})z} = e^{-0.6z}$ and is thus negligible in the upper layer. So the upper layer solution takes the form of oscillatory gravity waves with a constant amplitude, as expected with the Boussinesq approximation and for large z the real part of w in the upper layer is proportional to $\cos\left(kx + \sqrt{\frac{N^2}{\bar{u}^2} - \delta k^2} z\right)$. Note that in the non-Boussinesq solution obtained in the previous section, the waves increase in amplitude according to the scale height $e^{\frac{z}{2H}}$ and the convection decreases in amplitude according to the scale height $e^{-\frac{z}{2H}}$.

The solutions in the second configuration ($b = 0.07$) are plotted in Figure 3.5 and the amplitudes of the vertical velocities perturbations are shown in Figure 3.6. In the second configuration we set a smaller value $b = 0.07$. This corresponds to a thermal forcing spanning a large range of altitudes in both the lower and upper layers. This means that the thermal forcing directly generates gravity waves as well as convection. This corresponds to a combination of the obstacle effect and deep heating mechanisms. In this configuration, $b - \frac{1}{H} < 0$ so the second term in (3.49) is not small and then the real part of w in the upper layer is proportional to $\cos\left(kx + \sqrt{\frac{N^2}{\bar{u}^2} - \delta k^2} z\right) + \cos(kx)$ which is a linear superposition of waves with different phases.

All the graphs show are in terms of nondimensional quantities. Our input pa-

rameters are the nondimensional scale height H , the nondimensional acceleration due to gravity g , the nondimensional parameter b which defines the depth of the thermal forcing, the square of the aspect ratio δ , the nondimensional mean flow velocity $\bar{u}$ and the nondimensional horizontal wavenumber k . The results shown in the graphs can be transformed into dimensional configurations by using the relations (2.32). The dimensional scale height H^* in the troposphere is approximately 7 km [Gill (1982)]. For our graphs we used a nondimensional scale height $H = 10$ so the vertical length scale $L_z = H^*/H = 0.7$ km and the horizontal length scale is $L_x = L_z/\sqrt{\delta} = 7$ km. In the real atmosphere, the acceleration due to gravity is approximately $g^* = 9.8 \text{ ms}^{-2}$. We nondimensionalized g^* by defining $g = L_z g^*/U^2$ in (2.32), where U is the horizontal velocity scale. For our choice of g and L_z , we can find U and then use the relation (2.34) to find the vertical velocity scale W .

For example, considering the nondimensional quantities used in plotting the Case a solution in Figure 3.3: $H = 10$, $\delta = 0.01$, $\bar{u} = \sqrt{10}$ and $g = 9.8$, we have $L_z = 0.7$ km, $L_x = 7$ km and $U = 0.8 \text{ ms}^{-1}$, so $\bar{u} = \sqrt{10}$ corresponds to $\bar{u}^* = 8 \text{ ms}^{-1}$. In Case a the nondimensional vertical wavenumber is $m = \sqrt{N^2/\bar{u}^2 - \delta k^2}$, so in Figure 3.3(a) $m \approx 0.29$ and the nondimensional vertical wavelength is $\lambda = 2\pi/m \approx 21.2$. This corresponds to a dimensional vertical wavelength of $\lambda^* = L_z \lambda \approx 14.7$ km. The depth of the thermal forcing is determined by the parameter b . Let D denote the nondimensional heating depth and D^* the dimensional heating depth. With our choice of b in Figure 3.3(a), $D \approx 10$ and so $D^* = L_z D \approx 7$ km. Studies and observations [e.g., Salby and Garcia (1987), Pandya and Alexander (1999) and McLandress, Alexander and Wu (2000)] indicate that the vertical wavelength of convectively-generated gravity waves is generally about twice the depth of the heating. So our choice of input nondimensional parameters used to illustrate our solution is realistic compared with the real atmosphere.

It is important to note, however, that the gravity wave vertical wavelength does not depend of the heating depth according to our solutions (3.49). Clearly m depends

on $N^2 = g/H$ and on $\bar{u}$, δ and k , but not on b . Changing the heating depth (by changing b) does not affect the vertical wavelength $2\pi/m$ but it does change the overall structure of the waves. If the heating is shallow (large b), then the solution takes the form $\cos(kx + mz) + \cos(kx)$ over the range of low altitudes corresponding to the heating depth and takes the form $\cos(kx + mz)$ at higher altitudes. If the heating is deep, then the solution takes the form $\cos(kx + mz)$ over a wider range of altitudes corresponding to the heating depth. In particular, if $b = 1/H$, then the solution takes the form $\cos(kx + mz) + \cos(kx)$ at all levels.

Figure 3.3: Contour plots of the vertical velocity perturbation $w(x, z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = -5$ with $b = 0.7$.

Figure 3.4: The amplitude of the vertical velocity perturbation $\hat{w}(z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = -5$ with $b = 0.7$.

Figure 3.5: Contour plots of the vertical velocity perturbation $w(x, z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = -5$ with $b = 0.07$.

Figure 3.6: The amplitude of the vertical velocity perturbation $\hat{w}(z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = -5$ with $b = 0.07$.

3.3.2 Thermal Disturbance Centered at the Interface with the Boussinesq Approximation

In this section, we consider the case where the thermal disturbance F is centered at $z = 0$ as given in (3.30). We get the following solutions.

For the lower layer

$$\hat{w}_0(z) = A_0 \cosh \lambda z + B_0 \sinh \lambda z - \frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} e^{\left(b - \frac{1}{H}\right)z}. \quad (3.54)$$

For the upper layer **Case a:**

$$\hat{w}_2(z) = C_2 e^{imz} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + m^2} e^{-\left(b - \frac{1}{H}\right)z}. \quad (3.55)$$

For the upper layer **Case b:**

$$\hat{w}_2(z) = D_2 e^{-\nu z} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 - \nu^2} e^{-\left(b - \frac{1}{H}\right)z}. \quad (3.56)$$

By applying the boundary and interface conditions (3.31)-(3.33) we get the following system of equations

$$\begin{pmatrix} \cosh \lambda h & -\sinh \lambda h & 0 \\ 1 & 0 & -1 \\ -\gamma & \lambda & -im \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \\ C_2 \end{pmatrix} = \begin{pmatrix} P_0 \\ S_0 \\ T_0 \end{pmatrix},$$

where

$$\begin{pmatrix} P_0 \\ S_0 \\ T_0 \end{pmatrix} = \begin{pmatrix} E_{01} e^{-\left(b - \frac{1}{H}\right)h} \frac{1}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} \\ \frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + m^2} \\ \frac{E_{02}\left(b - \frac{1}{H}\right)}{\left(b - \frac{1}{H}\right)^2 + m^2} + \frac{E_{01}\left(b - \frac{1}{H} - \gamma\right)}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} \end{pmatrix}.$$

By solving the above system of equations, we get the constants

$$\begin{aligned} B_0 &= \frac{\cosh \lambda h}{\lambda \cosh \lambda h - (\gamma + im) \sinh \lambda h} \left\{ (\gamma + im) \operatorname{sech} \lambda h P_0 - im S_0 + T_0 \right\}, \\ A_0 &= \frac{\sinh \lambda h}{\cosh \lambda h} B_0 + \frac{1}{\cosh \lambda h} P_0, \\ C_2 &= A_0 - S_0. \end{aligned}$$

Thus the solution in the upper layer is

Case a:

$$\hat{w}_2(z) = [\alpha_0 \sinh \lambda h + P_0 \operatorname{sech} \lambda h - S_0] e^{imz} - \frac{E_{02}}{(b - \frac{1}{H})^2 + m^2} e^{-(b - \frac{1}{H})z}, \quad (3.57)$$

Case b:

$$\hat{w}_2(z) = [\alpha_0 \sinh \lambda h + P_0 \operatorname{sech} \lambda h - S_0] e^{-\nu z} - \frac{E_{02}}{(b - \frac{1}{H})^2 - \nu^2} e^{-(b - \frac{1}{H})z}, \quad (3.58)$$

and the solution in the lower layer is

$$\hat{w}_0(z) = \alpha_0 \sinh \lambda(z + h) + P_0 \frac{\cosh \lambda z}{\cosh \lambda h} - \frac{E_{01}}{(b - \frac{1}{H})^2 - \lambda^2} e^{(b - \frac{1}{H})z}, \quad (3.59)$$

where the constants are

$$E_{01} = \frac{gF_0}{\rho_{01}\bar{u}^2},$$

$$E_{02} = \frac{gF_0}{\rho_{02}\bar{u}^2},$$

$$P_0 = e^{-(b - \frac{1}{H})h} \frac{E_{01}}{(b - \frac{1}{H})^2 - \lambda^2},$$

Case a

$$S_0 = \frac{E_{01}}{(b - \frac{1}{H})^2 - \lambda^2} - \frac{E_{02}}{(b - \frac{1}{H})^2 + m^2},$$

$$T_0 = \frac{E_{02} (b - \frac{1}{H})}{(b - \frac{1}{H})^2 + m^2} + \frac{E_{01} (b - \frac{1}{H} - \gamma)}{(b - \frac{1}{H})^2 - \lambda^2},$$

$$\alpha_0 = \frac{1}{\lambda \cosh \lambda h - (\gamma + im) \sinh \lambda h} \left\{ (\gamma + im) \operatorname{sech} \lambda h P_0 - im S_0 + T_0 \right\},$$

Case b

$$S_0 = \frac{E_{01}}{(b - \frac{1}{H})^2 - \lambda^2} - \frac{E_{02}}{(b - \frac{1}{H})^2 - \nu^2},$$

$$T_0 = \frac{E_{02} (b - \frac{1}{H})}{(b - \frac{1}{H})^2 - \nu^2} + \frac{E_{01} (b - \frac{1}{H} - \gamma)}{(b - \frac{1}{H})^2 - \lambda^2},$$

$$\alpha_0 = \frac{1}{\lambda \cosh \lambda h - (\gamma - \nu) \sinh \lambda h} \left\{ (\gamma - \nu) \operatorname{sech} \lambda h P_0 + \nu S_0 + T_0 \right\}.$$

The solutions obtained in this subsection are summarized in Table 3.4.

Summary in a Chart

Upper layer	
Case a $\hat{w}_2 = \left\{ \left[\alpha_0 \sinh \lambda h + P_0 \operatorname{sech} \lambda h - S_0 \right] e^{imz} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + m^2} e^{-\left(b - \frac{1}{H}\right)z} \right\}.$	Case b $\hat{w}_2 = \left\{ \left[\alpha_0 \sinh \lambda h + P_0 \operatorname{sech} \lambda h - S_0 \right] e^{-\nu z} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 - \nu^2} e^{-\left(b - \frac{1}{H}\right)z} \right\}.$
Lower layer	
$\hat{w}_0 = \left\{ \alpha_0 \sinh \lambda(z + h) + P_0 \frac{\cosh \lambda z}{\cosh \lambda h} - \frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} e^{\left(b - \frac{1}{H}\right)z} \right\}.$	
↓	↓
Constants	
$E_{01} = \frac{gF_0}{\rho_{01}\bar{u}^2},$ $E_{02} = \frac{gF_0}{\rho_{02}\bar{u}^2},$ $P_0 = e^{-\left(b - \frac{1}{H}\right)h} \frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2},$	
Case a	
$\alpha_0 = \frac{1}{\lambda \cosh \lambda h - (\gamma + im) \sinh \lambda h} \left\{ (\gamma + im) \operatorname{sech} \lambda h P_0 - im S_0 + T_0 \right\},$ $S_0 = \left(\frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 + m^2} \right),$ $T_0 = \left(\frac{E_{02} \left(b - \frac{1}{H}\right)}{\left(b - \frac{1}{H}\right)^2 + m^2} + \frac{E_{01} \left(b - \frac{1}{H} - \gamma\right)}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} \right),$	
Case b	
$\alpha_0 = \frac{1}{\lambda \cosh \lambda h - (\gamma - \nu) \sinh \lambda h} \left\{ (\gamma - \nu) \operatorname{sech} \lambda h P_0 + \nu S_0 + T_0 \right\},$ $S_0 = \left(\frac{E_{01}}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} - \frac{E_{02}}{\left(b - \frac{1}{H}\right)^2 - \nu^2} \right),$ $T_0 = \left(\frac{E_{02} \left(b - \frac{1}{H}\right)}{\left(b - \frac{1}{H}\right)^2 - \nu^2} + \frac{E_{01} \left(b - \frac{1}{H} - \gamma\right)}{\left(b - \frac{1}{H}\right)^2 - \lambda^2} \right).$	

Table 3.4: Summary of the steps taken in deriving the solutions of the Boussinesq equation in the two-layer model when the thermal forcing is centered at the interface.

We note, as we mentioned at the end of Section 3.2, that setting $h_1 = 0$ in the solutions that we got in Section 3.3.1 does not recover the above solutions. That is because in Section 3.3.1, w_0 and w_1 are matched according to the conditions at

$z = -h_1$ and w_1 and w_2 are matched according to the conditions at $z = 0$. In this section there is no w_1 and so w_0 and w_2 are matched according to the conditions at $z = 0$. So the constants in Section 3.3.1 are different from the constants in this section (in particular, $Q = 0$, $R = 0$ and there are slight differences in both S and T and so α_0 is slightly different from α).

Plots of the Exact Solutions

We plot the Boussinesq linear steady solutions (3.57)-(3.59) by using the same parameter values that were used in Section 3.3.1 with $b = 0.7$ and $b = 0.07$. Both cases give gravity waves in the upper layer and partial convection in the lower layer, weaker than what was obtained when the thermal forcing was centered below the interface within the lower layer (Figures 3.3 and 3.5).

Contour plots of the vertical velocities perturbations for Cases a and b with $b = 0.7$ are shown in Figure 3.7. Figure 3.8 shows the amplitudes of the vertical velocities perturbations as functions of z . As noted before, changing the depth of the thermal forcing by changing b changes the structure of the waves. For shallow heating the waves take the form $\cos\left(kx + \sqrt{\frac{N^2}{\bar{u}^2} - \delta k^2} z\right)$ and for deep heating they take the form $\cos\left(kx + \sqrt{\frac{N^2}{\bar{u}^2} - \delta k^2} z\right) + \cos(kx)$.

Figure 3.7: Contour plots of the vertical velocity perturbation $w(x, z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = 0$ with $b = 0.7$.

Figure 3.8: The amplitude of the vertical velocity perturbation $\hat{w}(z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = 0$ with $b = 0.7$.

Figure 3.9: Contour plots of the vertical velocity perturbation $w(x, z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = 0$ with $b = 0.07$.

Figure 3.10: The amplitude of the vertical velocity perturbation $\hat{w}(z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = 0$ with $b = 0.07$.

Note that in all the above plots, we set $\gamma = 0$ in the interface condition (3.9) meaning that the density is continuous at the interface i.e., $\rho_{01} = \rho_{02} = 2$. Figure 3.11 shows contour plots of the vertical velocities perturbation for Cases a and b in which the density is not continuous at the interface i.e., $\gamma \neq 0$. We set $\rho_{01} = 10$ and $\rho_{02} = 2$ and show the solutions with $b = 0.7$ and $h_1 = 5$.

Figure 3.11: Contour plots of the vertical velocity perturbation $w(x, z)$ in the two-layer model with a monochromatic forcing. The thermal disturbance is centered at $z = -5$ with $b = 0.7$ for the case where the density is discontinuous at the interface $z = 0$.

The main difference between these plots and those obtained before is the obvious discontinuity in the vertical velocity across the interface $z = 0$. This physically unrealistic result suggests that the density should be continuous and we shall assume that in the rest of the thesis.

3.4 Horizontally-localized Wave Packet Solutions

In this section, we explore the solutions by considering a spectrum of horizontal wavenumbers k so that the solution can be written as a Fourier transform with respect to the horizontal variable x . We use a wave packet thermal disturbance F of the form:

$$F(x, z) = F_0 e^{-b|z+h_1|} \frac{a^2}{a^2 + x^2} e^{ik_0 x} + \text{c.c.}, \quad (3.60)$$

where $a > 0$, $b > 0$ and k_0 are constants and $F \rightarrow 0$ as $x \rightarrow \pm\infty$.

Let

$$w(x, z) = \int_{-\infty}^{\infty} \hat{w}(k, z) e^{ikx} dk,$$

where k is the wavenumber in the x direction. The complex Fourier coefficient $\hat{w}(k, z)$ is given by

$$\hat{w}(k, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w(x, z) e^{-ikx} dx.$$

The Fourier transform of the thermal forcing function (3.60) is

$$\hat{F}(k, z) = \frac{a}{2} F_0 e^{-b|z+h_1|} e^{-a|k-k_0|}.$$

By taking the Fourier transform of the equations with respect to x , we get for each Fourier mode k a nonhomogeneous second-order ordinary differential equation with constant coefficients. The general solutions of those equations are the same as those obtained in Sections 3.2 and 3.3 except that each solution is multiplied by $\frac{a}{2} e^{-a|k-k_0|}$ to give $\hat{w}(k, z)$. We then invert the transform to get $w(x, z)$.

Over the spectrum of the wavenumbers, we obtain the solution in Case a when the condition $\frac{N^2}{\bar{u}^2} - \delta k^2 > 0$ is satisfied and the solution in Case b when the condition $\frac{N^2}{\bar{u}^2} - \delta k^2 < 0$ is satisfied. This means that we have the Case a solution representing oscillatory gravity waves valid in the interval $|k| < k_1$ and Case b solution representing trapped waves valid in the interval $|k| > k_1$, where $k_1 = \sqrt{\frac{N^2}{\delta \bar{u}^2}}$ is the high wavenumber cutoff point for oscillatory gravity waves. To invert the transform, we

sum up over the range of Fourier modes and the solution thus comprises a combination of oscillatory waves and trapped waves. At high altitudes only the oscillatory modes are present. With the thermal forcing (3.60), the solution takes the form of a horizontally-localized wave packet with horizontal profile proportional to $\frac{a^2}{a^2+x^2}e^{ik_0x}$. The constant a determines the breadth of the wave packet and k_0 is the wavenumber that determining the wavelength and the number of oscillations within the wave packet. If $k_0 = 0$, we get a single convective cell in the lower layer and a gravity wave that resembles a topographic wave generated by an isolated mountain. If $k_0 \neq 0$ and the wavelength is short, there are multiple oscillations within the wave packet.

We note that the gravity waves are generally assumed to take the form of a continuous spectrum of vertical wavenumbers or phase speeds representing a wave packet localized in time. To make such a representation using our solutions we could re-derive our solution assuming that the disturbance is a periodic function of t as well as x . To obtain a solution as a spectrum of wavenumbers and a spectrum of phase speeds, the thermal forcing can be written as an integral of its spectral components in both space and time

$$F(x, z, t) = \hat{F}(z) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{F}_x(k) \hat{F}_t(c) e^{ik(x-ct)} dk dc,$$

where $\hat{F}_x(k)$ is the Fourier transform of the horizontal distribution and $\hat{F}_t(c)$ is the Fourier transform of the temporal distribution. The disturbance can then be written as

$$w(x, z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{w}(k, z, c) e^{ik(x-ct)} dk dc.$$

The solution for $\hat{w}(k, z, c)$ would be in the form of the same steady solutions we obtained but with $\bar{u}$ replaced by $\bar{u} - c$. Then we would invert the transform to find $w(x, z, t)$.

Plots of the Horizontally-localized Solutions: Thermal Disturbance Centered in the Lower Layer

We plot the horizontally-localized solutions when the thermal wave packet forcing is centered at $z = -h_1 = -5$. We consider again the two configurations that we considered before, in the first configuration $b = 0.7$ and in the second configuration $b = 0.07$. The figures are given in a rectangular domain $-2\pi \leq x \leq 2\pi$ and $-10 \leq z \leq 20$ with the following parameter values: $a = 0.7$, $F_0 = 10$, the background velocity $\bar{u} = \sqrt{2}$, the scale height $H = 10$, the gravitational acceleration $g = 9.8$, the square of the buoyancy frequency $N^2 = 0.98$, $N_0^2 = 0.98$, the square of the aspect ratio $\delta = 0.001$ and we set $k_0 = 1$ in the thermal forcing (3.60). With these values of a and k_0 the horizontal wavelength is the same order as the width of the wave packet so we have one convective cell only. We also set $\rho_{01} = \rho_{02} = 2$, so $\gamma = 0$ and the density is continuous across the interface. Contour plots of the vertical velocities perturbations for the case where the thermal forcing is centered at $z = -h_1 = -5$ are shown in Figure 3.12. In Figure 3.12(a) $b = 0.7$ and in Figure 3.12(b) $b = 0.07$. The amplitudes of the vertical velocities perturbations as functions of z corresponding to the wavenumber $k = 1$ for both configurations are shown in Figure 3.13. Figure 3.14 shows the amplitude of the Fourier transform of the vertical velocity perturbation $\hat{w}(k, z)$ as a function of the wavenumber k at $z = 10$ in the upper layer. With $k_0 = 1$, the horizontal wavelength is 2π so one half of a wavelength is approximately the same as the width of the packet, so there is only one convective cell. For the values of the input parameters used, the cutoff wavenumber for oscillatory waves $k_1 = \sqrt{\frac{N^2}{\delta \bar{u}^2}} \approx 22$. We see that the dominant contribution to the solution comes from the wavenumbers $|k| < k_1$ which correspond to Case a. The wavenumbers $|k| > k_1$ which correspond to Case b have small amplitudes and decay exponentially with height. In Figure 3.14(b) the Case b modes from the high wavenumber part of the spectrum where $|k| > k_1$ are shown on a different scale. The amplitude of these modes is order 10^{-7} while the

Case a modes have amplitudes of order 1.

Figure 3.12: Contour plots of the vertical velocity perturbation $w(x, z)$ in the two-layer model with a horizontally-localized forcing. The thermal disturbance is centered at $z = -5$ with (a) $b = 0.7$ and (b) $b = 0.07$.

Figure 3.13: The amplitude of the vertical velocity perturbation $\hat{w}(z, k)$ in the two-layer model with a horizontally-localized forcing as a function of z with $k = 1$. The thermal disturbance is centered at $z = -5$ with (a) $b = 0.7$ and (b) $b = 0.07$.

Figure 3.14: (a) The amplitude of the vertical velocity perturbation $\hat{w}(z, k)$ in the two-layer model with a horizontally-localized forcing as a function of k at $z = 10$. The thermal disturbance is centered at $z = -5$ with $b = 0.7$. (b) The Case b solutions shown on a different scale.

Plots of the Horizontally-localized Solutions: Thermal Disturbance Centered at the Interface

We use the same parameters that we used in plotting the solutions when the thermal disturbance centered in the lower layer to plot the solutions when the thermal forcing is centered at the interface $z = 0$. Figure 3.15 shows contour plots of the vertical velocities perturbations for the two configurations. The amplitudes of the vertical velocities perturbations as functions of z with wavenumber $k = 1$ are shown in Figure 3.16.

Figure 3.15: Contour plots of the vertical velocity perturbation $w(x, z)$ with a horizontally-localized forcing. The thermal disturbance is centered at $z = 0$ with (a) $b = 0.7$ and (b) $b = 0.07$.

Figure 3.16: The amplitude of the vertical velocity perturbation $\hat{w}(z, k)$ in the two-layer model with a horizontally-localized forcing as a function of z with $k = 1$. The thermal disturbance centered at $z = 0$ with (a) $b = 0.7$ and (b) $b = 0.07$.

Comparing Figures 3.15 and 3.16 with the corresponding Figures 3.12 and 3.13 where the thermal forcing was centered at $z = -h_1$, we observe that now the convection is weaker and the gravity wave amplitude is larger. By examining the spectrum of the solution (not shown) we note again that the dominant contribution to the solution comes from the Case a modes ($|k| < k_1$).

Chapter 4

The Linear Time-dependent Problem

A more realistic representation of gravity waves and convection is a model in which the amplitude of the perturbations is allowed to vary with time as well with the vertical variable. We look for time-dependent solutions first analytically and then numerically. We find that it is quite complicated to obtain an analytical solution (even an approximate one) for the general time-dependent problem so we consider some special configurations for which it is possible to derive time-dependent analytical solutions. In particular, we consider the case where the thermal forcing is centered at $z = 0$, so that we have w_0 and w_2 only, no w_1 , and we take the long-wave limit of zero aspect ratio ($\delta = 0$). This limit corresponds to the case where the background flow is hydrostatic. With $\delta = 0$ the expression $N^2/\bar{u}^2 - \delta k^2$ in the solution (see Table 3.4) becomes $N^2/\bar{u}^2$ which is always positive. In this case for the solution in the upper layer there is only Case a and no Case b. This does not change the qualitative behaviour of the solution. Recall from Figure 3.14 that in the spectrum of wavenumbers, the Case a modes (gravity waves) give the dominant contribution to the solution and the Case b modes (trapped disturbances) modes decay exponentially

to zero with height so they can be neglected. It is reasonable therefore to consider the long-wave limit which only has Case a modes.

Before examining the two-layer model we consider simple configurations where there is only one layer, either a single stable layer with gravity waves or a single unstable layer with convection. In each of these configurations we note that the corresponding steady solution is a special case of the more general configuration studied in Chapter 3.

4.1 Time-dependent Model

The governing equation for the linear time-dependent problem was derived in Chapter 2 and is given by equation (2.57) defined for $t > 0$. We impose the initial conditions that the vorticity and the first time derivative of the vorticity are zero at $t = 0$. As before we apply a steady thermal forcing of the form $F(x, z) = \hat{F}(z)e^{ikx} + \text{c.c.} = e^{-b|z|}e^{ikx} + \text{c.c.}$ In the long-wave limit ($\delta = 0$) and with constant $\bar{u}$, the linear time-dependent equation for the wave amplitude $\hat{w}(z, t)$ is

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right)^2 \frac{\partial^2 \hat{w}(z, t)}{\partial z^2} - \chi k^2 \hat{w}(z, t) = k^2 \frac{g}{\rho} \hat{F}, \quad (4.1)$$

and the initial conditions are $\hat{w}_{zz}(z, 0) = \hat{w}_{zzt}(z, 0) = 0$. We solve this equation on the domain $t > 0$ and on an appropriate interval for z depending on whether we are studying gravity waves layer or the convective layer. The solution can be obtained by taking a Laplace transform in time. We define $\tilde{w}(z, s)$ to be the Laplace transform of $\hat{w}(z, t)$ as follows

$$\tilde{w}(z, s) = \int_0^\infty e^{-st} \hat{w}(z, t) dt = \mathcal{L}[\hat{w}(z, t)],$$

where s is a complex variable. The function $\hat{w}(z, t)$ can be found by inverting the Laplace transform

$$\hat{w}(z, t) = \mathcal{L}^{-1}[\tilde{w}(z, s)] = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} e^{st} \tilde{w}(z, s) ds, \quad (4.2)$$

where τ is a real number chosen so that the contour of integration lies to the right of all the singularities of the integrand. The transformed equation is

$$\frac{\partial^2 \tilde{w}}{\partial z^2} - \frac{\chi k^2}{(s + ik\bar{u})^2} \tilde{w} = \frac{k^2 g}{s (s + ik\bar{u})^2} \frac{\hat{F}}{\bar{\rho}}. \quad (4.3)$$

As before, the sign of χ is positive or negative depending on the layer that we consider. First, we examine a one-layer model with time-dependent upward-propagating gravity waves only on the vertical domain $0 \leq z < \infty$ with $\chi = N^2 > 0$. Next, we consider a one-layer model with convection only on the vertical domain $-h \leq z \leq 0$ with $\chi = -N_0^2 < 0$ and zero upper and lower boundary conditions, then we extend the vertical domain to $-\infty < z \leq 0$. After that, we consider a two-layer model with both convection and gravity waves on the domain $-\infty < z < \infty$ with interface conditions at $z = 0$.

4.2 One-layer Model with Gravity Waves Generated by a Thermal Forcing

4.2.1 Solution for the Vertical Velocity Perturbation

In this section, we solve the linear time-dependent equation (4.1) in the vertical domain $0 \leq z < \infty$ with stable stratification given by the density profile $\bar{\rho}_2(z) = \rho_{02} e^{-\frac{z}{H}}$. This corresponds to the upper layer in the two-layer model in the previous chapter. Since there is only one layer in the model now, we omit the subscripts of 2 and simply write $w = w_2$, $\bar{\rho} = \bar{\rho}_2$ and so on. The thermal forcing takes the form

$F(x, z) = F_0 e^{-bz} e^{ikx} + \text{c.c.}$ We apply a zero boundary condition at $z = 0$ and require upward-propagating waves as before. This configuration corresponds to the deep heating mechanism, analogous to the models used by Chun and Baik (1998) and Holton, Beres and Zhuo (2002), and is shown in Figure 4.1.

Figure 4.1: A schematic diagram of the one-layer model in the domain $0 \leq z < \infty$ with gravity waves generated by a thermal disturbance centered at $z = 0$.

We first note that the steady equation for this configuration is

$$\frac{d^2 \hat{w}}{dz^2} + \frac{N^2}{\bar{u}^2} \hat{w} = -\frac{E_0}{\bar{u}^2} e^{-(b - \frac{1}{H})z}, \quad (4.4)$$

where $E_0 = \frac{gF_0}{\rho_0}$.

The solution of equation (4.4) is given by

$$\hat{w}(z) = \frac{E_0}{N^2 + (b - \frac{1}{H})^2 \bar{u}^2} \left(e^{\frac{iNz}{\bar{u}}} - e^{-(b - \frac{1}{H})z} \right). \quad (4.5)$$

In the time-dependent problem, the Laplace transform $\tilde{w}(z, s)$ satisfies

$$\frac{\partial^2 \tilde{w}}{\partial z^2} - \frac{N^2 k^2}{(s + ik\bar{u})^2} \tilde{w} = E_0 \frac{k^2}{s (s + ik\bar{u})^2} e^{-(b - \frac{1}{H})z}. \quad (4.6)$$

There are two solutions for the homogeneous equation proportional to $e^{\pm \frac{Nk}{(s + ik\bar{u})} z}$, and we consider the solution which corresponds to upward-propagating waves and impose

a zero boundary condition at $z = 0$. A particular solution can be obtained by using the method of undetermined coefficients. We obtain the following solution

$$\tilde{w}(z, s) = \frac{-E_0 k^2}{\left(b - \frac{1}{H}\right)^2} \frac{1}{s \left((s + ik\bar{u})^2 - \frac{N^2 k^2}{\left(b - \frac{1}{H}\right)^2} \right)} \left[e^{\frac{-Nkz}{s+ik\bar{u}}} - e^{-(b-\frac{1}{H})z} \right]. \quad (4.7)$$

Making use of the first shifting theorem we obtain

$$\hat{w}(z, t) = \frac{-E_0 k^2}{\left(b - \frac{1}{H}\right)^2} e^{-ik\bar{u}t} \mathcal{L}^{-1} \left\{ \frac{1}{(s - ik\bar{u}) \left(s^2 - \frac{N^2 k^2}{\left(b - \frac{1}{H}\right)^2} \right)} \left(e^{\frac{-Nkz}{s}} - e^{-(b-\frac{1}{H})z} \right) \right\}. \quad (4.8)$$

The singularities of this expression in the complex s -plane are shown in Table 4.1.

Singular points	Classification
$s_1 = 0$	Essential singularity
$s_2 = ik\bar{u}$	Simple pole
$s_{3,4} = \pm \frac{Nk}{b - \frac{1}{H}}$	Simple poles

Table 4.1: The singularities of the inverse Laplace transform for gravity waves generated by a thermal forcing.

In order to evaluate the inverse Laplace transform, we write (4.8) in terms of partial fractions as follows

$$\begin{aligned} \hat{w}(z, t) = \frac{-E_0 k^2}{\left(b - \frac{1}{H}\right)^2} e^{-ik\bar{u}t} \mathcal{L}^{-1} \left\{ C_1 \left[\frac{e^{-\frac{Nkz}{s}}}{s - ik\bar{u}} - \frac{e^{-(b-\frac{1}{H})z}}{s - ik\bar{u}} \right] \right. \\ + C_2 \left[\frac{e^{-\frac{Nkz}{s}}}{s - \frac{Nk}{\left(b - \frac{1}{H}\right)}} - \frac{e^{-(b-\frac{1}{H})z}}{s - \frac{Nk}{\left(b - \frac{1}{H}\right)}} \right] \\ \left. + C_3 \left[\frac{e^{-\frac{Nkz}{s}}}{s + \frac{Nk}{\left(b - \frac{1}{H}\right)}} - \frac{e^{-(b-\frac{1}{H})z}}{s + \frac{Nk}{\left(b - \frac{1}{H}\right)}} \right] \right\}, \quad (4.9) \end{aligned}$$

where $C_1 = \frac{-(b-\frac{1}{H})^2}{N^2 k^2 + k^2 \bar{u}^2 (b-\frac{1}{H})^2}$, $C_2 = \frac{(b-\frac{1}{H})^2}{2N^2 k^2 - 2ik^2 N \bar{u} (b-\frac{1}{H})}$ and $C_3 = \frac{(b-\frac{1}{H})^2}{2N^2 k^2 + 2ik^2 N \bar{u} (b-\frac{1}{H})}$.

In each line of equation (4.9) we have a sum of two terms: the first term takes the form $\frac{e^{\frac{a}{s}}}{s-c}$ and the second term takes the form $\frac{e^{-(b-\frac{1}{H})z}}{s-c}$ where $a = -Nkz$ and c is

one of the singularities s_2 , s_3 or s_4 . The evaluation of the inverse Laplace transform of the second term is straightforward. The first term has two singularities, a simple pole $s = c$ and an essential singularity $s = 0$. The evaluation of the inverse Laplace transform of a function of this form was carried out by Nadon and Campbell (2007) for a different problem involving time-dependent gravity waves. This inverse Laplace transform consists of a sum of two terms: a term arising from the residue at the pole $s = c$ and a series that arising from the residue of the essential singularity $s = 0$. Calculating the inverse transforms following Nadon and Campbell (2007), we obtain

$$\begin{aligned} \hat{w}(z, t) = & \frac{E_0}{N^2 + (b - \frac{1}{H})^2 \bar{u}^2} \left[e^{\frac{iN}{\bar{u}}z} - e^{-(b - \frac{1}{H})z} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{Nkz}}{k\bar{u}\sqrt{t}} \right)^n J_n(2\sqrt{Nkzt}) \right] \\ & - \frac{E_0}{2N^2 - 2iN(b - \frac{1}{H})\bar{u}} \left[\left(e^{-(b - \frac{1}{H})z} - e^{-(b - \frac{1}{H})z} \right) e^{-ik\bar{u}t} e^{\frac{Nk}{(b - \frac{1}{H})}t} \right. \\ & \quad \left. - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-(b - \frac{1}{H})\sqrt{Nkz}}{Nk\sqrt{t}} \right)^n J_n(2\sqrt{Nkzt}) \right] \quad (4.10) \\ & - \frac{E_0}{2N^2 + 2iN(b - \frac{1}{H})\bar{u}} \left[\left(e^{(b - \frac{1}{H})z} - e^{-(b - \frac{1}{H})z} \right) e^{-ik\bar{u}t} e^{-\frac{Nk}{(b - \frac{1}{H})}t} \right. \\ & \quad \left. - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{(b - \frac{1}{H})\sqrt{Nkz}}{Nk\sqrt{t}} \right)^n J_n(2\sqrt{Nkzt}) \right], \end{aligned}$$

where J_n is the Bessel function of the first type of order n .

The above solution involves a sum of steady terms and time-dependent terms. In order to study the behaviour of the solution at large t , we use the behaviour of the Bessel function that is given in formula 9.2.1 of Abramowitz and Stegun (1972): For $|\varsigma| \gg 1$, the Bessel function has the asymptotic form

$$J_n(\varsigma) \sim \sqrt{\frac{2}{\pi\varsigma}} \left\{ \cos \left(\varsigma - \frac{1}{2}n\pi - \frac{1}{4}\pi \right) + \mathcal{O}(|\varsigma|^{-1}) \right\}. \quad (4.11)$$

This means that all the time-dependent terms that involve Bessel functions are $\mathcal{O}(t^{-\frac{3}{4}})$ and go to zero as $t \rightarrow \infty$. We note also that the sign of $b - \frac{1}{H}$ affects the behaviour of the solution. In addition to the terms involving Bessel functions, there are two more time-dependent terms, both of which are exponential functions that include the

factor $e^{\pm \frac{Nk}{b-\frac{1}{H}}t}$. The term with the plus sign in the exponent is multiplied by zero, and the term with the negative sign gives a solution that is bounded as $t \rightarrow \infty$ for finite z provided that $b - \frac{1}{H} \geq 0$. This means that the thermal forcing must decay at the same rate as or faster than the background density. Under this condition all the time-dependent terms go to zero as $t \rightarrow \infty$ and the time-dependent solution (4.10) approaches the steady-state solution (4.5). Figure 4.2 shows the solution for the case $b - \frac{1}{H} > 0$ and Figure 4.3 shows the solution for the case $b - \frac{1}{H} = 0$. If $b - \frac{1}{H} < 0$ then the term with the negative factor in the exponent becomes infinite as $t \rightarrow \infty$ although the other time-dependent terms go to zero. Thus $b \geq \frac{1}{H}$ is the condition for the solution to be stable: the rate of decay of the thermal forcing with altitude must not exceed the rate of decay of the background density.

Another useful form of the solution can be obtained by rewriting the solution in a different way in which the coefficients of Bessel functions are expressed in terms of $\frac{\sqrt{t}}{\sqrt{Nkz}}$ instead of $\frac{\sqrt{Nkz}}{\sqrt{t}}$. To do this, we follow Nijimbere and Campbell (preprint 2014). We use the following generating function for Bessel functions that is given in formula 9.1.41 of Abramowitz and Stegun (1972):

$$e^{\frac{1}{2}Z(T-\frac{1}{T})} = \sum_{n=-\infty}^{\infty} T^n J_n(Z). \quad (T \neq 0)$$

For negative n we use the fact that $J_{-n}(Z) = (-1)^n J_n(Z)$ and we get

$$\sum_{n=1}^{\infty} T^n J_n(Z) = e^{\frac{1}{2}Z(T-\frac{1}{T})} - \sum_{n=0}^{\infty} \left(\frac{-1}{T}\right)^n J_n(Z). \quad (4.12)$$

The solution (4.10) can now be rewritten by using formula (4.12) as

$$\begin{aligned}
\hat{w}(z, t) = & \frac{E_0}{N^2 + \left(b - \frac{1}{H}\right)^2 \bar{u}^2} \left[-e^{-(b - \frac{1}{H})z} + e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{ik\bar{u}\sqrt{t}}{\sqrt{Nkz}} \right)^n J_n(2\sqrt{Nkzt}) \right] \\
& - \frac{E_0}{2N^2 - 2iN \left(b - \frac{1}{H}\right) \bar{u}} \left[-e^{-(b - \frac{1}{H})z} e^{-ik\bar{u}t} e^{\frac{Nk}{(b - \frac{1}{H})}t} \right. \\
& \quad \left. + e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{Nk\sqrt{t}}{(b - \frac{1}{H})\sqrt{Nkz}} \right)^n J_n(2\sqrt{Nkzt}) \right] \\
& - \frac{E_0}{2N^2 + 2iN \left(b - \frac{1}{H}\right) \bar{u}} \left[-e^{-(b - \frac{1}{H})z} e^{-ik\bar{u}t} e^{\frac{-Nk}{(b - \frac{1}{H})}t} \right. \\
& \quad \left. + e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-Nk\sqrt{t}}{(b - \frac{1}{H})\sqrt{Nkz}} \right)^n J_n(2\sqrt{Nkzt}) \right].
\end{aligned} \tag{4.13}$$

This form of the solution is useful in obtaining insight into the behaviour of the solution as $z \rightarrow \infty$, which is not so evident from the version (4.10). Based on the asymptotic behaviour (4.11) of the Bessel function, we observe that the terms involving Bessel functions approach zero as $z \rightarrow \infty$ at finite time and thus $\hat{w} \rightarrow 0$ as $z \rightarrow \infty$ at finite time. This is to be expected since the waves must propagate at finite speed and cannot reach infinite height in finite time.

4.2.2 Solutions for the Streamfunction and Density

In this section, we derive expressions for the streamfunction and density perturbations corresponding to the solution for the vertical velocity obtained in Section 4.2.1. We will need these expressions for the weakly-nonlinear analysis in Chapter 5.

The streamfunction perturbation ψ is defined using (2.10) as

$$u = -\frac{\partial\psi}{\partial z} \quad \text{and} \quad w = \frac{\partial\psi}{\partial x}, \tag{4.14}$$

and so the vorticity perturbation is given by

$$\zeta = \frac{\partial^2\psi}{\partial z^2} + \delta \frac{\partial^2\psi}{\partial x^2} = \nabla^2\psi. \tag{4.15}$$

The Boussinesq version of equations (2.46) and (2.47) in the long-wave limit and with

constant $\bar{u}$ can be written in terms of the streamfunction and the density as

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) \frac{\partial^2 \psi}{\partial z^2} = -\frac{g}{\bar{\rho}} \frac{\partial \rho}{\partial x} \quad (4.16)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) \rho + \frac{d\bar{\rho}}{dz} \frac{\partial \psi}{\partial x} = F. \quad (4.17)$$

We write the streamfunction and density perturbations in normal mode form as

$$\psi(x, z, t) = e^{ikx} \hat{\psi}(z, t) + \text{c.c.},$$

$$\rho(x, z, t) = e^{ikx} \hat{\rho}(z, t) + \text{c.c.}$$

Equations (4.16) and (4.17) become, respectively

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right) \frac{\partial^2 \hat{\psi}}{\partial z^2} = -ik \frac{g}{\bar{\rho}} \hat{\rho} \quad (4.18)$$

and

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right) \hat{\rho} + ik \frac{d\bar{\rho}}{dz} \hat{\psi} = \hat{F}. \quad (4.19)$$

The amplitude of the streamfunction perturbation $\hat{\psi}$ can be obtained by dividing the amplitude of the vertical velocity perturbation by ik :

$$\begin{aligned} \hat{\psi}(z, t) &= \frac{\hat{w}(z, t)}{ik} \\ &= \frac{E_0}{ik \left(N^2 + \left(b - \frac{1}{H}\right)^2 \bar{u}^2\right)} \left[e^{\frac{iN}{\bar{u}}z} - e^{-(b-\frac{1}{H})z} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{Nkz}}{k\bar{u}\sqrt{t}}\right)^n J_n(2\sqrt{Nkzt}) \right] \\ &\quad - \frac{E_0}{ik \left(2N^2 - 2iN \left(b - \frac{1}{H}\right) \bar{u}\right)} \left[\left(e^{-(b-\frac{1}{H})z} - e^{-(b-\frac{1}{H})z}\right) e^{-ik\bar{u}t} e^{\frac{Nk}{(b-\frac{1}{H})}t} \right. \\ &\quad \left. - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-(b-\frac{1}{H})\sqrt{Nkz}}{Nk\sqrt{t}}\right)^n J_n(2\sqrt{Nkzt}) \right] \\ &\quad - \frac{E_0}{ik \left(2N^2 + 2iN \left(b - \frac{1}{H}\right) \bar{u}\right)} \left[\left(e^{(b-\frac{1}{H})z} - e^{-(b-\frac{1}{H})z}\right) e^{-ik\bar{u}t} e^{-\frac{Nk}{(b-\frac{1}{H})}t} \right. \\ &\quad \left. - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{(b-\frac{1}{H})\sqrt{Nkz}}{Nk\sqrt{t}}\right)^n J_n(2\sqrt{Nkzt}) \right]. \end{aligned} \quad (4.20)$$

The density can be obtained by solving equation (4.19). This is a first-order linear differential equation in $\hat{\rho}$ and can be solved, for example by using an integrating factor, to obtain

$$\hat{\rho}(z, t) = e^{-ik\bar{u}t} \int_0^t e^{ik\bar{u}\tau} \left(\hat{F} - ik \frac{d\bar{\rho}}{dz} \hat{\psi}(z, \tau) \right) d\tau. \quad (4.21)$$

To evaluate the integral, we need to integrate the terms that contain Bessel functions given in (4.20). In doing so, we make use of formula 9.1.30 of Abramowitz and Stegun (1972):

$$\frac{d}{d\tau} \{ \tau^{-n} J_n(\tau) \} = -\tau^{-n} J_{n+1}(\tau).$$

We can write this formula as

$$\frac{d}{dt} \left\{ t^{-\frac{n}{2}+\frac{1}{2}} J_{n-1}(2\sqrt{Nkzt}) \right\} = -\sqrt{Nk} z^{\frac{1}{2}} t^{-\frac{n}{2}} J_n(2\sqrt{Nkzt}).$$

After performing the integrations, we get

$$\begin{aligned} \hat{\rho} = & e^{-\frac{1}{H}z} \left\{ \frac{F_0}{ik\bar{u}} e^{-(b-\frac{1}{H})z} - \frac{F_0}{ik\bar{u}} e^{-ik\bar{u}t} e^{-(b-\frac{1}{H})z} \right. \\ & + \frac{\rho_0 E_0}{H \left(N^2 + \left(b - \frac{1}{H} \right)^2 \bar{u}^2 \right)} \left[\frac{1}{ik\bar{u}} \left(e^{\frac{iN}{\bar{u}}z} - e^{-(b-\frac{1}{H})z} \right) - \frac{e^{-ik\bar{u}t}}{ik\bar{u}} \left(e^{\frac{iN}{\bar{u}}z} - e^{-(b-\frac{1}{H})z} \right) \right. \\ & \quad \left. + \frac{1}{\sqrt{Nk}} e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{Nk}}{k\bar{u}} \right)^n z^{\frac{n}{2}-\frac{1}{2}} t^{-\frac{n}{2}+\frac{1}{2}} J_{n-1} \left(2\sqrt{Nkzt} \right) \right] \\ & - \frac{\rho_0 E_0}{H \left(2N^2 - 2iN \left(b - \frac{1}{H} \right) \bar{u} \right)} \left[\frac{1}{\sqrt{Nk}} e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-(b-\frac{1}{H})}{\sqrt{Nk}} \right)^n z^{\frac{n}{2}-\frac{1}{2}} t^{-\frac{n}{2}+\frac{1}{2}} J_{n-1} \left(2\sqrt{Nkzt} \right) \right] \\ & - \frac{\rho_0 E_0}{H \left(2N^2 + 2iN \left(b - \frac{1}{H} \right) \bar{u} \right)} \left[\frac{\left(b - \frac{1}{H} \right)}{Nk} \left(e^{(b-\frac{1}{H})z} - e^{-(b-\frac{1}{H})z} \right) e^{-ik\bar{u}t} e^{-\frac{Nk}{(b-\frac{1}{H})}t} \right. \\ & \quad \left. + \frac{1}{\sqrt{Nk}} e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{(b-\frac{1}{H})}{\sqrt{Nk}} \right)^n z^{\frac{n}{2}-\frac{1}{2}} t^{-\frac{n}{2}+\frac{1}{2}} J_{n-1} \left(2\sqrt{Nkzt} \right) \right] \left. \right\}. \end{aligned} \quad (4.22)$$

The linear solutions for the streamfunction and density perturbations that are given in equations (4.20) and (4.22), respectively, will be considered as a first approximation for the nonlinear solutions that we will study in the next chapter.

Plots of the Solutions of the One-layer Model with Gravity Waves Generated by a Thermal Forcing

We plot the vertical velocity perturbation according to equation (4.13) at different time levels in a rectangular domain $0 \leq x \leq 2\pi$ and $0 \leq z \leq 20$ with the following parameters values: $\bar{u} = \sqrt{10}$, $H = 5$, $F_0 = 1$, $\rho_0 = 1$, $k = 1$ and $g = 9.8$. We consider two configurations: the first configuration with $b = 0.7$ for which $(b - \frac{1}{H}) > 0$ and the second configuration with $b = 0.2$ for which $(b - \frac{1}{H}) = 0$. Figure 4.2 shows contour plots of the vertical velocity perturbation for the first configuration and Figure 4.3 shows the second configuration. In each case the solution approaches the steady-state solution (4.5).

Comparing Figures 4.2 and 4.3, we observe as before that the vertical structure of the gravity waves once they reach their steady state is determined by the vertical profile of the thermal forcing. As before with $b = 0.7$ in Figure 4.2, w is proportional to $\cos(kx + \frac{N}{u}z)$, but with $b = 0.2$ in Figure 4.3, w is proportional to $\cos(kx + \frac{N}{u}z) + \cos(kx)$, which is a linear superposition of waves with different phases.

Configuration (I) $b - \frac{1}{H} > 0$:

(a) $t = 2$

(b) $t = 5$

(c) $t = 10$

(d) $t = 20$

Figure 4.2: Contour plots of the vertical velocity perturbation $w(x, z, t)$ of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.7$. Plots are obtained at (a) $t = 2$, (b) $t = 5$, (c) $t = 10$ and (d) $t = 20$.

Configuration (II) $b - \frac{1}{H} = 0$:

(a) $t = 2$

(b) $t = 5$

(c) $t = 10$

(d) $t = 20$

Figure 4.3: Contour plots of the vertical velocity perturbation $w(x, z, t)$ of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.2$. Plots are obtained at (a) $t = 2$, (b) $t = 5$, (c) $t = 10$ and (d) $t = 20$.

4.3 One-layer Model with Convection Generated by a Thermal Forcing

In this section, we consider a single layer with unstable stratification given by the density profile $\bar{\rho}_1 = \rho_{01} e^{\frac{z}{H}}$ in a vertical domain $-h \leq z \leq 0$. A thermal forcing given by $F(x, z) = F_0 e^{bz} e^{ikx} + \text{c.c.}$ generates convection only with no gravity waves. We solve equation (4.1) in this domain for $t > 0$ with zero boundary conditions at $z = 0$ and $z = -h$ and with initial conditions $\hat{w}_{zz}(z, 0) = \hat{w}_{zzt}(z, 0) = 0$. The configuration for this layer is shown in Figure 4.4. Once again since there is only one layer, we drop the subscript of zero on the vertical velocity perturbation and other quantities and write $w = w_0$, $\hat{w} = \hat{w}_0$, $\bar{\rho} = \bar{\rho}_0$ and so on.

Figure 4.4: A schematic diagram of the one-layer model in the domain $-h \leq z < 0$ with convection generated by a thermal disturbance centered at $z = 0$.

Due to the unstable stratification in this layer, χ is negative and we represent it as $\chi = -N_o^2$. We first consider the steady equation for this configuration that is given by

$$\frac{d^2 \hat{w}}{dz^2} - \frac{N_o^2}{\bar{u}^2} \hat{w} = -\frac{E_0}{\bar{u}^2} e^{(b - \frac{1}{H})z}, \quad (4.23)$$

where $E_0 = \frac{gF_0}{\rho_0}$.

The solution of equation (4.23) subject to the zero boundary conditions is given by

$$\hat{w} = \frac{-E_0}{N_0^2 - \left(b - \frac{1}{H}\right)^2 \bar{u}^2} \left(e^{\frac{N_0 z}{\bar{u}}} - e^{\left(b - \frac{1}{H}\right)z} + \frac{-e^{-(b - \frac{1}{H})h} e^{\frac{N_0(z+h)}{\bar{u}}} + e^{\frac{N_0 z}{\bar{u}}} + e^{-(b - \frac{1}{H})h} e^{-\frac{N_0(z-h)}{\bar{u}}} - e^{-\frac{N_0 z}{\bar{u}}}}{e^{\frac{2N_0 h}{\bar{u}}} - 1} \right). \quad (4.24)$$

This solution can also be written as

$$\hat{w} = \frac{-E_0}{N_0^2 - \left(b - \frac{1}{H}\right)^2 \bar{u}^2} \left(e^{\frac{N_0 z}{\bar{u}}} - e^{\left(b - \frac{1}{H}\right)z} + \left(e^{-\frac{N_0 h}{\bar{u}}} - e^{-(b - \frac{1}{H})h} \right) \frac{\sinh \frac{N_0 z}{\bar{u}}}{\sinh \frac{N_0 h}{\bar{u}}} \right). \quad (4.25)$$

To simplify the mathematics in the discussion of the time-dependent solution later in this section, it is helpful to consider an unbounded domain. This would correspond to a physical situation in which the heating and convection are relatively shallow in vertical scale and occur at a relatively high altitude above the ground so that we can neglect the ground and consider the ground to be at $-\infty$. If we assume the vertical domain to be $-\infty < z \leq 0$ and solve equation (4.23) with the boundary and limiting conditions $\hat{w}(z) = 0$ at $z = 0$ and $\hat{w}(z) \rightarrow 0$ as $z \rightarrow -\infty$. We obtain the following steady solution

$$\hat{w} = \frac{-E_0}{N_0^2 - \left(b - \frac{1}{H}\right)^2 \bar{u}^2} \left(e^{\frac{N_0 z}{\bar{u}}} - e^{\left(b - \frac{1}{H}\right)z} \right). \quad (4.26)$$

In the solution (4.25) in the bounded domain, there is no restriction on the sign of $\left(b - \frac{1}{H}\right)$. However the solution (4.26) in the unbounded domain requires that $\left(b - \frac{1}{H}\right) > 0$ in order to obtain a solution that goes to zero as $z \rightarrow -\infty$. Note that by assuming $\left(b - \frac{1}{H}\right) > 0$ in (4.25) and taking the limit as $h \rightarrow \infty$ to (4.25) we obtain the solution (4.26). The condition $\left(b - \frac{1}{H}\right) > 0$ means that the thermal forcing increases more rapidly with height than the background density.

We plot the solutions with $\left(b - \frac{1}{H}\right) > 0$ using the same parameter values that we used in the previous section in which $b = 0.7$. Figure 4.5 shows the solution in the vertical domain $-5 \leq z \leq 0$ that corresponds the bounded domain. For an approximation to an unbounded domain, we plot the solution on the interval $-20 \leq z \leq 0$. The solution in this case is shown in Figure 4.6. In both cases the thermal forcing,

coupled with unstable stratification and the zero boundary conditions, gives closed contours below $z = 0$ that resemble convective cells in the atmosphere.

Figure 4.5: (a) The amplitude of the vertical velocity perturbation $\hat{w}(z)$ of the convection generated by a thermal forcing centered $z = 0$ with $b = 0.7$ in the vertical domain $-5 \leq z \leq 0$. (b) Contour plots of the vertical velocity perturbation $w(x, z)$.

Figure 4.6: (a) The amplitude of the vertical velocity perturbation $\hat{w}(z)$ of the convection generated by a thermal forcing centered $z = 0$ with $b = 0.7$ in the vertical domain $-20 \leq z \leq 0$. (b) Contour plots of the vertical velocity perturbation $w(x, z)$.

To examine the time-dependent problem, we take the Laplace transform of (4.1) and obtain

$$\frac{\partial^2 \tilde{w}}{\partial z^2} + \frac{N_0^2 k^2}{(s + ik\bar{u})^2} \tilde{w} = E_0 \frac{k^2}{s(s + ik\bar{u})^2} e^{(b - \frac{1}{H})z}, \quad (4.27)$$

with boundary conditions $\tilde{w}(-h, s) = 0$ and $\tilde{w}(0, s) = 0$. The solution is

$$\begin{aligned} \tilde{w} = & \frac{-E_0 k^2}{(b - \frac{1}{H})^2} \frac{1}{s \left((s + ik\bar{u})^2 + \frac{N_0^2 k^2}{(b - \frac{1}{H})^2} \right)} \left[e^{\frac{iN_0 k}{s + ik\bar{u}} z} - e^{(b - \frac{1}{H})z} \right. \\ & \left. + \frac{-e^{-(b - \frac{1}{H})h} e^{\frac{iN_0 k(z+h)}{s + ik\bar{u}}} + e^{\frac{iN_0 kz}{s + ik\bar{u}}} + e^{-(b - \frac{1}{H})h} e^{-\frac{iN_0 k(z-h)}{s + ik\bar{u}}} - e^{-\frac{iN_0 kz}{s + ik\bar{u}}}}{e^{\frac{2iN_0 kh}{s + ik\bar{u}}} - 1} \right]. \end{aligned} \quad (4.28)$$

By making use of the first shifting theorem, we obtain

$$\begin{aligned} \hat{w}(z, t) = & \frac{-E_0 k^2}{(b - \frac{1}{H})^2} e^{-ik\bar{u}t} \mathcal{L}^{-1} \left\{ \frac{1}{(s - ik\bar{u}) \left(s^2 + \frac{N_0^2 k^2}{(b - \frac{1}{H})^2} \right)} \left[e^{\frac{iN_0 k}{s} z} - e^{(b - \frac{1}{H})z} \right. \right. \\ & \left. \left. + \frac{-e^{-(b - \frac{1}{H})h} e^{\frac{iN_0 k(z+h)}{s}} + e^{\frac{iN_0 kz}{s}} + e^{-(b - \frac{1}{H})h} e^{-\frac{iN_0 k(z-h)}{s}} - e^{-\frac{iN_0 kz}{s}}}{e^{\frac{2iN_0 kh}{s}} - 1} \right] \right\}. \end{aligned} \quad (4.29)$$

The singularities of this expression are shown in Table 4.2.

Singular points	Classification
$s_1 = 0$	Essential singularity
$s_2 = ik\bar{u}$	Simple pole
$s_{3,4} = \pm i \frac{N_0 k}{b - \frac{1}{H}}$	Simple poles
$s_n^* = \frac{N_0 kh}{n\pi} \quad n = 0, \pm 1, \pm 2, \dots$	Simple poles

Table 4.2: The singularities of the inverse Laplace transform for the convection generated by a thermal forcing.

Evaluating the inverse Laplace transform of equation (4.29) is complicated because of the infinite number of singularities s_n^* extending along the real line in the complex s -plane from the origin to $\pm\infty$ and the fact that there is a singularity at

infinity, $s_0^* \rightarrow \pm\infty$. We can simplify the solution by extending the vertical domain to the semi-infinite domain $-\infty < z \leq 0$. In this case we have only the first two terms on the right-hand side of equation (4.29) and we no longer have the singularities s_n^* . Solving equation (4.27) with the specified boundary conditions, we obtain

$$\tilde{w} = \frac{-E_0 k^2}{\left(b - \frac{1}{H}\right)^2} \frac{1}{s \left((s + ik\bar{u})^2 + \frac{N_0^2 k^2}{\left(b - \frac{1}{H}\right)^2} \right)} \left[e^{\frac{iN_0 k}{s + ik\bar{u}} z} - e^{\left(b - \frac{1}{H}\right) z} \right]. \quad (4.30)$$

We invert the transform as we did in Section 4.2 and get the following solution

$$\begin{aligned} \hat{w}(z, t) = & \frac{-E_0}{N_0^2 - \left(b - \frac{1}{H}\right)^2 \bar{u}^2} \left[e^{\frac{N_0}{\bar{u}} z} - e^{\left(b - \frac{1}{H}\right) z} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{-iN_0 k z}}{k\bar{u}\sqrt{t}} \right)^n J_n \left(2\sqrt{-iN_0 k z t} \right) \right] \\ & + \frac{E_0}{2N_0^2 - 2N_0 \left(b - \frac{1}{H}\right) \bar{u}} \left[\left(e^{\left(b - \frac{1}{H}\right) z} - e^{\left(b - \frac{1}{H}\right) z} \right) e^{-ik\bar{u}t} e^{\frac{iN_0 k}{\left(b - \frac{1}{H}\right)} t} \right. \\ & \quad \left. - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{i \left(b - \frac{1}{H}\right) \sqrt{-iN_0 k z}}{N_0 k \sqrt{t}} \right)^n J_n \left(2\sqrt{-iN_0 k z t} \right) \right] \\ & + \frac{E_0}{2N_0^2 + 2N_0 \left(b - \frac{1}{H}\right) \bar{u}} \left[\left(e^{-\left(b - \frac{1}{H}\right) z} - e^{\left(b - \frac{1}{H}\right) z} \right) e^{-ik\bar{u}t} e^{\frac{-iN_0 k}{\left(b - \frac{1}{H}\right)} t} \right. \\ & \quad \left. - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-i \left(b - \frac{1}{H}\right) \sqrt{-iN_0 k z}}{N_0 k \sqrt{t}} \right)^n J_n \left(2\sqrt{-iN_0 k z t} \right) \right]. \end{aligned} \quad (4.31)$$

$J_n(2\sqrt{-iN_0 k z t})$ is a solution of the modified Bessel's equation of order n . See Appendix B for more details.

The asymptotic behaviour of the Bessel function $J_n(\varsigma)$ with complex argument $|\varsigma| \gg 1$ is given by (4.11). This means that for $t \gg 1$

$$J_n(2\sqrt{-iN_0 k z t}) \sim t^{-1/4} \cos \left(2\sqrt{-iN_0 k z t} - \frac{1}{2}n\pi - \frac{1}{4}\pi \right).$$

Thus, all the series in (4.31) are divergent and approach infinity as $t \rightarrow \infty$ and the amplitude of the perturbation becomes infinite as $t \rightarrow \infty$.

The solution (4.31) can also be rewritten in an alternative form by using (4.12) as

$$\begin{aligned}
\hat{w}(z, t) = & \frac{-E_0}{N_0^2 - (b - \frac{1}{H})^2 \bar{u}^2} \left[-e^{(b - \frac{1}{H})z} + e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{ik\bar{u}\sqrt{t}}{\sqrt{-iN_0kz}} \right)^n J_n \left(2\sqrt{-iN_0kzt} \right) \right] \\
& + \frac{E_0}{2N_0^2 - 2N_0(b - \frac{1}{H})\bar{u}} \left[-e^{(b - \frac{1}{H})z} e^{-ik\bar{u}t} e^{\frac{iN_0k}{(b - \frac{1}{H})}t} \right. \\
& \quad \left. + e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{iN_0k\sqrt{t}}{(b - \frac{1}{H})\sqrt{-iN_0kz}} \right)^n J_n \left(2\sqrt{-iN_0kzt} \right) \right] \\
& + \frac{E_0}{2N_0^2 + 2N_0(b - \frac{1}{H})\bar{u}} \left[-e^{(b - \frac{1}{H})z} e^{-ik\bar{u}t} e^{\frac{-iN_0k}{(b - \frac{1}{H})}t} \right. \\
& \quad \left. + e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-iN_0k\sqrt{t}}{(b - \frac{1}{H})\sqrt{-iN_0kz}} \right)^n J_n \left(2\sqrt{-iN_0kzt} \right) \right].
\end{aligned} \tag{4.32}$$

Once again the asymptotic behaviour of the Bessel function (4.11) tells us that

$$J_n(2\sqrt{-iN_0kzt}) \sim z^{-1/4} \cos \left(2\sqrt{-iN_0kzt} - \frac{1}{2}n\pi - \frac{1}{4}\pi \right) \quad \text{for } |z| \gg 1,$$

and so the solution approaches infinity as $z \rightarrow -\infty$ for finite t . Recall that in the steady problem we imposed the limiting condition that $\hat{w} \rightarrow 0$ as $z \rightarrow -\infty$. However it is not possible to find a time-dependent solution that satisfies $\tilde{w}(z, s) \rightarrow 0$ as $z \rightarrow -\infty$ for all s .

4.4 Time-dependent Linear Two-layer Model

We return to consideration of a two-layer model with unstable stratification in the lower layer and stable stratification in the upper layer with a thermal forcing that generates both gravity waves and convection (both obstacle effect and deep heating mechanisms). We assume that the background densities match at the interface and the stable density stratification takes the form $\bar{\rho}_2 = \rho_0 e^{-\frac{z}{H}}$ in the upper layer and $\bar{\rho}_1 = \rho_0 e^{\frac{z}{H}}$ in the lower layer. The thermal forcing is specified as

$F(x, z) = F_0 e^{-b|z|} e^{ikx} + \text{c.c.}$ A schematic of this configuration is shown in Figure 4.7. We again apply the assumption of upward-propagating waves and the following interface conditions at $z = 0$:

$$\begin{aligned}\hat{w}_0 &= \hat{w}_2, \\ \frac{d\hat{w}_0}{dz} &= \frac{d\hat{w}_2}{dz}.\end{aligned}$$

The solution is also required to go to zero as $z \rightarrow -\infty$.

Figure 4.7: A schematic diagram of the two-layer model in the domain $-\infty < z < \infty$ with gravity waves in the upper layer and convection in the lower layer. The thermal disturbance is centered at $z = 0$.

The steady solutions in the two layers are

$$\hat{w}_0 = \frac{-E_0}{N^2 - (b - \frac{1}{H})^2 \bar{u}^2} \left\{ \frac{(1 - i) \left(N^2 + iN \left(b - \frac{1}{H} \right) \bar{u} - \left(b - \frac{1}{H} \right)^2 \bar{u}^2 \right)}{N \left(N + i \left(b - \frac{1}{H} \right) \bar{u} \right)} e^{\frac{Nz}{\bar{u}}} - e^{(b - \frac{1}{H})z} \right\} \quad (4.33)$$

and

$$\hat{w}_2 = \frac{E_0}{N^2 + \left(b - \frac{1}{H}\right)^2 \bar{u}^2} \left\{ \frac{(1+i) \left(N^2 + N \left(b - \frac{1}{H}\right) \bar{u} + \left(b - \frac{1}{H}\right)^2 \bar{u}^2 \right)}{N \left(N + \left(b - \frac{1}{H}\right) \bar{u} \right)} e^{\frac{iNz}{\bar{u}}} - e^{-(b - \frac{1}{H})z} \right\}, \quad (4.34)$$

where $E_0 = \frac{gF_0}{\rho_0}$ and N is the buoyancy frequency of the upper layer.

Once again in order to obtain a bounded solution as $z \rightarrow \pm\infty$, we consider the case that $\left(b - \frac{1}{H}\right) > 0$.

We plot the solutions with $\left(b - \frac{1}{H}\right) > 0$ using the same parameter values that we used in plotting the one-layer models with $b = 0.7$ in Sections 4.2 and 4.3. Figure 4.8 shows contour plots of the vertical velocity perturbation for the model in the domain $-20 \leq z \leq 10$. The amplitude of the vertical velocity perturbation under this configuration is plotted in Figure 4.9. Since the thermal forcing is centered at the interface ($z = 0$), the convection in the lower layer is relatively weak compared with that in the cases shown in Chapter 3 where the thermal forcing was centered in the lower layer.

Figure 4.8: Contour plots of the vertical velocity perturbation $w(x, z)$ in the two-layer model. The thermal disturbance is centered at $z = 0$ with $b = 0.7$. For the case where the vertical domain is unbounded.

Figure 4.9: The amplitude of the vertical velocity perturbation $\hat{w}(z)$ in the two-layer model. The thermal disturbance centered at $z = 0$ when $b = 0.7$. For the case where the vertical domain is unbounded.

Now, we consider the time-dependent version of our model. So we solve equation (4.3) in the domain $-h \leq z < \infty$. We take Laplace transform of equation (4.3). In the lower layer there is an infinite number of singularities in the complex s -plane as shown in Table 4.2. In order to invert the transform and as done in Section 4.3, we first simplify the problem by extending the vertical domain to be $-\infty < z < \infty$ and assuming that the solution is bounded as $z \rightarrow -\infty$. We obtain the following solutions

$$\hat{w}_0(z, t) = -k^2 E_0 e^{-ik\bar{u}t} \mathcal{L}^{-1} \left\{ \frac{1}{s - ik\bar{u}} \left[\frac{(1-i) \left[\left(b - \frac{1}{H}\right)^2 s^2 + Nk \left(b - \frac{1}{H}\right) s + N^2 k^2 \right]}{Nk \left[\left(b - \frac{1}{H}\right)^2 s^2 + N^2 k^2 \right] \left[\left(b - \frac{1}{H}\right) s + Nk \right]} e^{\frac{iNk}{s}z} - \frac{e^{(b - \frac{1}{H})z}}{\left(b - \frac{1}{H}\right)^2 s^2 + N^2 k^2} \right] \right\} \quad (4.35)$$

and

$$\hat{w}_2(z, t) = -k^2 E_0 e^{-ik\bar{u}t} \mathcal{L}^{-1} \left\{ \frac{1}{s - ik\bar{u}} \left[\frac{(1-i) \left[\left(b - \frac{1}{H}\right)^3 s^3 + k^3 N^3 + ik^3 N^3 \right]}{Nk \left[\left(b - \frac{1}{H}\right)^4 s^4 - k^4 N^4 \right]} e^{\frac{-Nk}{s}z} - \frac{e^{-(b - \frac{1}{H})z}}{\left(b - \frac{1}{H}\right)^2 s^2 - N^2 k^2} \right] \right\}, \quad (4.36)$$

where N is the buoyancy frequency of the gravity waves in the upper layer.

Inverting the transforms gives the following solutions

$$\begin{aligned} \hat{w}_0(z, t) = & \frac{(-1+i)E_0 \left(N^2 + iN \left(b - \frac{1}{H}\right) \bar{u} - \left(b - \frac{1}{H}\right)^2 \bar{u}^2 \right)}{N \left(N + i \left(b - \frac{1}{H}\right) \bar{u} \right) \left(N^2 - \left(b - \frac{1}{H}\right)^2 \bar{u}^2 \right)} \left[e^{\frac{N}{\bar{u}}z} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{-iNkz}}{k\bar{u}\sqrt{t}} \right)^n J_n \left(2\sqrt{-iNkzt} \right) \right] \\ & - \frac{E_0}{2N^2 - 2N \left(b - \frac{1}{H}\right) \bar{u}} \left[e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{i \left(b - \frac{1}{H}\right) \sqrt{-iNkz}}{Nk\sqrt{t}} \right)^n J_n \left(2\sqrt{-iNkzt} \right) \right] \\ & - \frac{iE_0}{2N^2 + 2N \left(b - \frac{1}{H}\right) \bar{u}} \left[e^{-(b - \frac{1}{H})z} e^{-ik \left(\frac{N}{(b - \frac{1}{H})} + \bar{u} \right) t} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-i \left(b - \frac{1}{H}\right) \sqrt{-iNkz}}{Nk\sqrt{t}} \right)^n J_n \left(2\sqrt{-iNkzt} \right) \right] \\ & - \frac{(1-i)E_0}{2N^2 + 2iN \left(b - \frac{1}{H}\right) \bar{u}} \left[e^{-i(b - \frac{1}{H})z} e^{-k \left(\frac{N}{(b - \frac{1}{H})} + i\bar{u} \right) t} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{\left(b - \frac{1}{H}\right) \sqrt{-iNkz}}{Nk\sqrt{t}} \right)^n J_n \left(2\sqrt{-iNkzt} \right) \right] \\ & + \frac{E_0}{N^2 - \left(b - \frac{1}{H}\right)^2 \bar{u}^2} e^{(b - \frac{1}{H})z} - \frac{E_0}{2N^2 + 2N \left(b - \frac{1}{H}\right) \bar{u}} e^{(b - \frac{1}{H})z} e^{-ik \left(\frac{N}{(b - \frac{1}{H})} + \bar{u} \right) t} \end{aligned} \quad (4.37)$$

and

$$\begin{aligned}
\hat{w}_2(z, t) = & \frac{(1+i)E_0 \left(N^2 + N \left(b - \frac{1}{H} \right) \bar{u} + \left(b - \frac{1}{H} \right)^2 \bar{u}^2 \right)}{N \left(N + \left(b - \frac{1}{H} \right) \bar{u} \right) \left(N^2 + \left(b - \frac{1}{H} \right)^2 \bar{u}^2 \right)} \left[e^{\frac{iN}{\bar{u}}z} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{Nkz}}{k\bar{u}\sqrt{t}} \right)^n J_n \left(2\sqrt{Nkzt} \right) \right] \\
& + \frac{E_0}{2N^2 - 2iN \left(b - \frac{1}{H} \right) \bar{u}} \left[e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-\left(b - \frac{1}{H} \right) \sqrt{Nkz}}{Nk\sqrt{t}} \right)^n J_n \left(2\sqrt{Nkzt} \right) \right] \\
& - \frac{iE_0}{2N^2 + 2iN \left(b - \frac{1}{H} \right) \bar{u}} \left[e^{(b-\frac{1}{H})z} e^{-k \left(\frac{N}{(b-\frac{1}{H})} + i\bar{u} \right) t} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{(b-\frac{1}{H}) \sqrt{Nkz}}{Nk\sqrt{t}} \right)^n J_n \left(2\sqrt{Nkzt} \right) \right] \\
& - \frac{(1+i)E_0}{2N^2 + 2N \left(b - \frac{1}{H} \right) \bar{u}} \left[e^{-i(b-\frac{1}{H})z} e^{-ik \left(\frac{N}{(b-\frac{1}{H})} + \bar{u} \right) t} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left(\frac{-i(b-\frac{1}{H}) \sqrt{Nkz}}{Nk\sqrt{t}} \right)^n J_n \left(2\sqrt{Nkzt} \right) \right] \\
& - \frac{E_0}{N^2 + \left(b - \frac{1}{H} \right)^2 \bar{u}^2} e^{-(b-\frac{1}{H})z} + \frac{E_0}{2N^2 + 2iN \left(b - \frac{1}{H} \right) \bar{u}} e^{-(b-\frac{1}{H})z} e^{-k \left(\frac{N}{(b-\frac{1}{H})} + i\bar{u} \right) t} \Big].
\end{aligned} \tag{4.38}$$

The terms that include Bessel functions in the solution of the upper layer (4.38) go to zero as $t \rightarrow \infty$ and so the solution for the gravity waves in the upper layer approaches the steady-state solution (4.34). But in the solution for the lower layer (4.37) the terms involving Bessel functions become infinite as $t \rightarrow \infty$ and as $z \rightarrow -\infty$ as in Section 4.3.

We thus conclude that the time-dependent model for convection generated by a thermal forcing in a region of unstable stratification does not give a bounded solution, at least not under the conditions that we imposed on our model. These solutions were obtained based on the simplifications $\delta = 0$ (the long-wave limit) and the lower boundary being at $z \rightarrow -\infty$ rather than at $z = -h$. The obvious question is whether a stable time-dependent solution could be obtained without these simplifications. We explore this question using numerical simulations in Chapter 6. A more general question is under what circumstances we can derive a stable time-dependent solution for convection in a layer with unstable stratification.

A well-known model for time-dependent convection is the Lorenz model [Lorenz (1963)] which is derived and discussed in Appendix D. In this model, the spatial structure of the convection is specified according to equations (D.16) and (D.17) and only the time evolution is investigated. The nonlinear terms are retained in the

equations. This leads to an initial-value problem comprising a system of nonlinear ordinary differential equations (ODEs) for the amplitude of the convection as a function of time only. The equations in this system can be linearized and the stability of both the linear and nonlinear systems can be analyzed. Another important difference between the Lorenz model and ours is that the Lorenz equations include viscosity and heat conduction which we have so far neglected.

The behaviour of the solution of the Lorenz equations is found to depend on the relative magnitude of the effects of the unstable stratification to the effects of the viscosity and heat conduction. The ratio of these effects is measured by the Rayleigh number R_a . For values of R_a below a certain critical value R_c the ODE system has only one critical point or equilibrium solution corresponding to the zero solution and this critical point is found to asymptotically stable for both the linear and the nonlinear systems (Appendix D). This means that for any initial condition, the solution approaches the equilibrium state of zero as $t \rightarrow \infty$. For values of R_a larger than R_c , i.e., when the viscous and heat conduction effects are weaker or the unstable stratification is stronger, the system of nonlinear ODEs has two additional critical points or equilibrium solutions. Near these critical points the solution evolves into a state of aperiodic motion resembling chaotic motion. For $R_a > R_c$ the linear system has the one critical point corresponding to the zero solution but it found to be a saddle point. This means that every solution of the linear system is a linear combination of an exponentially-growing and an exponentially-decaying mode. If the initial condition is such that only the decaying mode is present, then the solution approaches the zero equilibrium solution as $t \rightarrow \infty$ but for other initial conditions the solution diverges to infinity as $t \rightarrow \infty$. For $R_a = R_c$ every solution is a linear combination of a constant amplitude mode and an exponentially-decaying mode and so the solution is stable.

The Lorenz model assumes that R_a is finite, this means that the viscous and heat conduction coefficients are nonzero. Letting $R_a \rightarrow \infty$ in the nonlinear or linear

ODEs would make the solution go to infinity.

Examining the Lorenz model suggests that the inclusion of viscosity and heat conduction may be needed to model time-dependent convection. We explore this numerically in Section 6.4. Before describing the numerical investigation we carry out a weakly-nonlinear analysis of the evolution of the gravity waves in the one-layer stably-stratified layer described in Section 4.2.

Chapter 5

Nonlinear Time-dependent Analysis of the Gravity Waves Evolution

The linear solution (4.10) can be considered as a first approximation to the solution of the nonlinear equations (2.39) for a situation where the perturbation amplitude is very small. However, in general, the effects of the nonlinearity must be taken into account. We begin this chapter by deriving the nonlinear governing equations then we carry out a weakly-nonlinear analysis for $\varepsilon \ll 1$ in which the solution is written as a perturbation series in powers of the nonlinear parameter ε . The leading order term in the series is the linear solution derived in Chapter 4. Taking the first few terms of the perturbation series gives us an approximate solution valid for small ε . We examine the problem in the stable layer and investigate the nonlinear interactions of the gravity waves with the mean flow.

5.1 The Nonlinear Model

The starting point of our analysis of the nonlinear problem is the system of equations (2.39) which are written in terms of total quantities. We write each of these total quantities as the sum of a basic state quantity and a perturbation as done before

$$\begin{aligned} (\bar{\rho} + \varepsilon\rho) \left[\left(\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + \frac{d\bar{u}}{dz} w \right) + \varepsilon \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) \right] &= -\frac{\partial p}{\partial x}, \\ \delta(\bar{\rho} + \varepsilon\rho) \left[\left(\frac{\partial w}{\partial t} + \bar{u} \frac{\partial w}{\partial x} \right) + \varepsilon \left(u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) \right] &= -\frac{\partial p}{\partial z} - \rho g, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\partial \rho}{\partial t} + \bar{u} \frac{\partial \rho}{\partial x} + \frac{d\bar{\rho}}{dz} w + \varepsilon \left[u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} \right] &= F. \end{aligned}$$

We consider small-amplitude perturbations for which $\varepsilon \ll 1$. We assume that $\varepsilon\rho \ll \bar{\rho}$ and then combine the zonal and the vertical momentum equations as we did in Section 2.5. We get the following equation

$$\begin{aligned} \bar{\rho} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left(\frac{\partial u}{\partial z} - \delta \frac{\partial w}{\partial x} \right) + \frac{d^2 \bar{u}}{dz^2} w \right] + \frac{d\bar{\rho}}{dz} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u + \frac{d\bar{u}}{dz} w \right] \\ + \varepsilon \left\{ \bar{\rho} [u \nabla^2 w - w \nabla^2 u] + \frac{d\bar{\rho}}{dz} \left[u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right] \right\} = g \frac{\partial \rho}{\partial x}, \end{aligned} \quad (5.1)$$

where $\nabla^2 = \frac{\partial^2}{\partial z^2} + \delta \frac{\partial^2}{\partial x^2}$ is the nondimensional Laplacian operator.

It is difficult to write the nonlinear equations in terms of the vertical velocity perturbation only (as we did in the linear equations). We find it more convenient to write the equations in terms of the streamfunction which is defined by (2.10). Equation (5.1) can be written in terms of ψ and ρ as

$$\begin{aligned} \bar{\rho} \left[\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi - \frac{d^2 \bar{u}}{dz^2} \psi_x \right] + \frac{d\bar{\rho}}{dz} \left[\frac{\partial}{\partial t} \psi_z + \bar{u} \frac{\partial}{\partial x} \psi_z - \frac{d\bar{u}}{dz} \psi_x \right] \\ + \varepsilon \left\{ \bar{\rho} (\psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x) + \frac{d\bar{\rho}}{dz} (\psi_x \psi_{zz} - \psi_z \psi_{zx}) \right\} = -g\rho_x. \end{aligned}$$

By applying the Boussinesq approximation I ($\frac{d\bar{\rho}}{dz} \ll \bar{\rho}$), we get

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi - \frac{d^2 \bar{u}}{dz^2} \psi_x + \varepsilon (\psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x) = -\frac{g}{\bar{\rho}} \rho_x. \quad (5.2)$$

Once again, we consider the case where there is no vertical shear in the mean flow. This means that $\bar{u}$ is constant and we thus obtain

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi + \varepsilon (\psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x) = -\frac{g}{\bar{\rho}} \rho_x.$$

Now our nonlinear model is

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi + \frac{g}{\bar{\rho}} \rho_x = \varepsilon (\psi_z \nabla^2 \psi_x - \psi_x \nabla^2 \psi_z) \quad (5.3)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \rho + \frac{d\bar{\rho}}{dz} \psi_x - F = \varepsilon (\psi_z \rho_x - \psi_x \rho_z). \quad (5.4)$$

In the long-wave limit, the Laplacian operator is replaced by the second derivative with respect to z and then the model can be written as

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \psi_{zz} + \frac{g}{\bar{\rho}} \rho_x = \varepsilon (\psi_z \psi_{zzx} - \psi_x \psi_{zzz}) \quad (5.5)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \rho + \frac{d\bar{\rho}}{dz} \psi_x - F = \varepsilon (\psi_z \rho_x - \psi_x \rho_z). \quad (5.6)$$

5.2 Weakly-nonlinear Analysis

5.2.1 Equations for the Zero-wavnumber Components

In order to obtain approximate solutions to the nonlinear equations (5.5) and (5.6), we carry out a weakly-nonlinear analysis for $\varepsilon \ll 1$. The solutions are written as perturbation series in powers of the nonlinear parameter ε :

$$\psi(x, z, t) \sim \psi^{(0)}(x, z, t) + \varepsilon \psi^{(1)}(x, z, t) + \mathcal{O}(\varepsilon^2) \quad (5.7)$$

and

$$\rho(x, z, t) \sim \rho^{(0)}(x, z, t) + \varepsilon \rho^{(1)}(x, z, t) + \mathcal{O}(\varepsilon^2). \quad (5.8)$$

We substitute the series (5.7) and (5.8) into the nonlinear equations (5.5) and (5.6) and group terms of $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$.

At $\mathcal{O}(1)$ the leading-order terms in the series, $\psi^{(0)}$ and $\rho^{(0)}$, satisfy the linear equations (4.16) and (4.17) and so they can be written as

$$\psi^{(0)}(x, z, t) = e^{ikx} \phi^{(0)}(z, t) + \text{c.c.}$$

and

$$\rho^{(0)}(x, z, t) = e^{ikx} \mathcal{R}^{(0)}(z, t) + \text{c.c.},$$

where $\phi^{(0)}(z, t)$ and $\mathcal{R}^{(0)}(z, t)$ are found according to the linear solutions (4.20) and (4.22), respectively.

At $\mathcal{O}(\varepsilon)$, we find that $\psi^{(1)}$ and $\rho^{(1)}$ satisfy the equations

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \psi_{zz}^{(1)} + \frac{g}{\bar{\rho}} \rho_x^{(1)} = \psi_z^{(0)} \psi_{zzx}^{(0)} - \psi_x^{(0)} \psi_{zzz}^{(0)} \quad (5.9)$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \rho^{(1)} + \frac{d\bar{\rho}}{dz} \psi_x^{(1)} = \psi_z^{(0)} \rho_x^{(0)} - \psi_x^{(0)} \rho_z^{(0)}. \quad (5.10)$$

This implies that $\psi^{(1)}$ and $\rho^{(1)}$ can be written in the form

$$\psi^{(1)}(x, z, t) = e^{2ikx} \phi_2^{(1)}(z, t) + \phi_0^{(1)}(z, t) + \text{c.c.} \quad (5.11)$$

and

$$\rho^{(1)}(x, z, t) = e^{2ikx} \mathcal{R}_2^{(1)}(z, t) + \mathcal{R}_0^{(1)}(z, t) + \text{c.c.}, \quad (5.12)$$

where $\phi_0^{(1)}$ and $\mathcal{R}_0^{(1)}$ are the leading-order contributions to the zero-wavenumber component of the disturbances, and $\phi_2^{(1)}$ and $\mathcal{R}_2^{(1)}$ are the leading-order contributions to the amplitude of the second harmonic.

By substituting equations (5.11) and (5.12) into equations (5.9) and (5.10), we find that the functions $\phi_0^{(1)}$ and $\mathcal{R}_0^{(1)}$ satisfy the following uncoupled partial differential equations

$$\frac{\partial}{\partial t} \left(\frac{\partial^2 \phi_0^{(1)}}{\partial z^2} \right) = ik \left(\frac{\partial \phi^{(0)*}}{\partial z} \frac{\partial^2 \phi^{(0)}}{\partial z^2} - \phi^{(0)} \frac{\partial^3 \phi^{(0)*}}{\partial z^3} \right) \quad (5.13)$$

and

$$\frac{\partial}{\partial t} \left(\mathcal{R}_0^{(1)} \right) = ik \left(\frac{\partial \phi^{(0)*}}{\partial z} \mathcal{R}^{(0)} - \phi^{(0)} \frac{\partial \mathcal{R}^{(0)*}}{\partial z} \right), \quad (5.14)$$

where the asterisks denote complex conjugates. The functions $\phi_2^{(1)}$ and $\mathcal{R}_2^{(1)}$ satisfy the following coupled partial differential equations

$$\left(\frac{\partial}{\partial t} + 2ik\bar{u}\right) \frac{\partial^2 \phi_2^{(1)}}{\partial z^2} + \frac{2ikg}{\bar{\rho}} \mathcal{R}_2^{(1)} = ik \left(\frac{\partial \phi^{(0)}}{\partial z} \frac{\partial^2 \phi^{(0)}}{\partial z^2} - \phi^{(0)} \frac{\partial^3 \phi^{(0)}}{\partial z^3} \right) \quad (5.15)$$

and

$$\left(\frac{\partial}{\partial t} + 2ik\bar{u}\right) \mathcal{R}_2^{(1)} + 2ik \frac{d\bar{\rho}}{dz} \phi_2^{(1)} = ik \left(\frac{\partial \phi^{(0)}}{\partial z} \mathcal{R}^{(0)} - \phi^{(0)} \frac{\partial \mathcal{R}^{(0)}}{\partial z} \right). \quad (5.16)$$

We wish to determine the behaviour of the solution for large t . So although we cannot solve these equations exactly we seek to determine the order of each of the functions $\phi_0^{(1)}$, $\mathcal{R}_0^{(1)}$, $\phi_2^{(1)}$ and $\mathcal{R}_2^{(1)}$. The order of the right-hand side of equations (5.13)-(5.16) at large t can be determined as follows. The right-hand side of these equations comprise products of partial derivatives of the linear solutions $\phi^{(0)}$ and $\mathcal{R}^{(0)}$ and their complex conjugates. According to the linear solution that is given in equation (4.20), we can write $\phi^{(0)}$ as

$$\begin{aligned} \phi^{(0)} = & AG(z) + e^{-ik\bar{u}t} \sum_{n=1}^{\infty} B_n z^{\frac{n}{2}} J_n \left(2\sqrt{Nkzt} \right) t^{-\frac{n}{2}} \\ & + C \left(e^{(b-\frac{1}{H})z} - e^{-(b-\frac{1}{H})z} \right) e^{-ik\bar{u}t} e^{-\frac{Nk}{(b-\frac{1}{H})}t}, \end{aligned} \quad (5.17)$$

where

$$G(z) = e^{\frac{iN}{\bar{u}}z} - e^{-(b-\frac{1}{H})z}, \quad (5.18)$$

and the constants

$$\begin{aligned} B_n &= -A \left(\frac{i\sqrt{Nk}}{k\bar{u}} \right)^n - C \left(\frac{b-\frac{1}{H}}{\sqrt{Nk}} \right)^n + D \left(\frac{-(b-\frac{1}{H})}{\sqrt{Nk}} \right)^n, \\ A &= \frac{E_{02}}{ik \left(N^2 + \left(b - \frac{1}{H} \right)^2 \bar{u}^2 \right)}, \\ C &= \frac{-E_{02}}{ik \left(2N^2 + 2iN \left(b - \frac{1}{H} \right) \bar{u} \right)}, \\ D &= \frac{E_{02}}{ik \left(2N^2 - 2iN \left(b - \frac{1}{H} \right) \bar{u} \right)}. \end{aligned}$$

Similarly, according to the linear solution (4.22) we can write $\mathcal{R}^{(0)}$ as

$$\begin{aligned}\mathcal{R}^{(0)} = & H(z) + e^{-\frac{1}{H}z} e^{-ik\bar{u}t} \sum_{n=1}^{\infty} E_n z^{\frac{n}{2}-\frac{1}{2}} J_{n-1} \left(2\sqrt{Nkzt} \right) t^{-\frac{n}{2}+\frac{1}{2}} \\ & + R e^{-\frac{1}{H}z} e^{-ik\bar{u}t} \left(e^{\frac{iN}{\bar{u}}z} - e^{-(b-\frac{1}{H})z} \right) + P e^{-bz} e^{-ik\bar{u}t} \\ & + Q e^{-\frac{1}{H}z} \left(e^{(b-\frac{1}{H})z} - e^{-(b-\frac{1}{H})z} \right) e^{-ik\bar{u}t} e^{-\frac{Nk}{(b-\frac{1}{H})}t},\end{aligned}$$

where

$$H(z) = -P e^{-bz} + R e^{-\frac{1}{H}z} \left(e^{\frac{iN}{\bar{u}}z} - e^{-(b-\frac{1}{H})z} \right), \quad (5.19)$$

and the constants

$$E_n = \frac{-ik\rho_0}{H\sqrt{Nk}} B_n, \quad P = -\frac{F_0}{ik\bar{u}}, \quad Q = \frac{ik\rho_0 \left(b - \frac{1}{H} \right)}{HNk} C \text{ and } R = \frac{\rho_0}{H\bar{u}} A,$$

where A , C and B_n are given in equation (5.17).

On computing the partial derivatives of $\phi^{(0)}$ and $\mathcal{R}^{(0)}$, the derivatives of Bessel functions can be computed by using formula 9.1.27 of Abramowitz and Stegun (1972):

$$\frac{d}{d\varsigma} J_n(\varsigma) = \frac{1}{2} [J_{n-1}(\varsigma) - J_{n+1}(\varsigma)].$$

The steady term and the largest time-dependent term of $\phi^{(0)}$ and its partial derivatives with respect to z are:

$$\begin{aligned}\phi^{(0)} & \sim AG(z) + B e^{-ik\bar{u}t} z^{\frac{1}{2}} J_1(\varsigma) t^{-\frac{1}{2}}, \\ \frac{\partial \phi^{(0)}}{\partial z} & \sim A \frac{dG}{dz} + \frac{\sqrt{Nk}}{2} B e^{-ik\bar{u}t} [J_0(\varsigma) - J_2(\varsigma)], \\ \frac{\partial^2 \phi^{(0)}}{\partial z^2} & \sim A \frac{d^2 G}{dz^2} + \frac{Nk}{4} B e^{-ik\bar{u}t} z^{-\frac{1}{2}} [J_{-1}(\varsigma) - 2J_1(\varsigma) + J_3(\varsigma)] t^{\frac{1}{2}}, \\ \frac{\partial^3 \phi^{(0)}}{\partial z^3} & \sim A \frac{d^3 G}{dz^3} + \frac{\sqrt{N^3 k^3}}{8} B e^{-ik\bar{u}t} z^{-1} [J_{-2}(\varsigma) - 3J_0(\varsigma) + 3J_2(\varsigma) - J_4(\varsigma)] t,\end{aligned}$$

where $\varsigma = 2\sqrt{Nkzt}$.

The steady term and the largest time-dependent term of $\mathcal{R}^{(0)}$ and its first partial derivative with respect to z are:

$$\mathcal{R}^{(0)} = H(z) + E e^{-\frac{1}{H}z} e^{-ik\bar{u}t} J_0(\varsigma),$$

$$\frac{\partial \mathcal{R}^{(0)}}{\partial z} \sim \frac{dH}{dz} + \frac{\sqrt{Nk}}{2} E e^{-\frac{1}{H}z} e^{-ik\bar{u}t} z^{-\frac{1}{2}} (J_{-1}(\varsigma) - J_1(\varsigma)) t^{-\frac{3}{2}}.$$

We first consider equation (5.13). By substituting the partial derivatives into its right-hand side, the product of the steady terms gives

$$-ikA^2 \left(\frac{dG^*}{dz} \frac{d^2G}{dz^2} + G \frac{d^3G^*}{dz^3} \right). \quad (5.20)$$

The largest term arising from the product of the steady terms by the time-dependent terms is

$$-ik \frac{\sqrt{N^3 k^3}}{8} AB^* G(z) z^{-1} e^{ik\bar{u}t} \left(J_{-2}(\varsigma) - 3J_0(\varsigma) + 3J_2(\varsigma) - J_4(\varsigma) \right) t, \quad (5.21)$$

and the largest term arising from the product of the time-dependent terms is

$$ik \frac{\sqrt{N^3 k^3}}{8} BB^* z^{-\frac{1}{2}} \left([J_0(\varsigma) - J_2(\varsigma)][J_{-1}(\varsigma) - 2J_1(\varsigma) + J_3(\varsigma)] \right. \\ \left. - J_1(\varsigma)[J_{-2}(\varsigma) - 3J_0(\varsigma) + 3J_2(\varsigma) - J_4(\varsigma)] \right) t^{\frac{1}{2}}. \quad (5.22)$$

By using the asymptotic behaviour of the Bessel function (4.11), we can see that the largest term on the right-hand side of equation (5.13) is given by (5.21) and it is $\mathcal{O}(t^{\frac{3}{4}})$.

Similarly for equation (5.14), the product of the steady terms on the right-hand side gives

$$-ikA \left(\frac{dG^*}{dz} H(z) + G(z) \frac{dH^*}{dz} \right). \quad (5.23)$$

The largest term arising from the product of the steady terms and the time-dependent terms is

$$-ik \frac{\sqrt{Nk}}{2} AE^* G(z) e^{-\frac{1}{H}z} z^{-\frac{1}{2}} e^{ik\bar{u}t} [J_{-1}(\varsigma) - J_1(\varsigma)] t^{\frac{1}{2}}, \quad (5.24)$$

and the largest term arising from the product of the time-dependent terms is

$$ik \frac{\sqrt{Nk}}{2} e^{-\frac{1}{H}z} \left(B^* E J_0(\varsigma) [J_0(\varsigma) - J_2(\varsigma)] - BE^* J_1(\varsigma) [J_{-1}(\varsigma) - J_1(\varsigma)] \right). \quad (5.25)$$

This implies that the largest term on the right-hand side of equation (5.14) is given by (5.24) and is $\mathcal{O}(t^{\frac{1}{4}})$. So in equation (5.3) the nonlinear terms are $\mathcal{O}(t^{\frac{3}{4}})$ and in equation (5.4) the nonlinear terms are $\mathcal{O}(t^{\frac{1}{4}})$. So for large t , the nonlinear terms in equation (5.5) become $\mathcal{O}(1)$ on a time scale of $t \sim \mathcal{O}(\varepsilon^{-\frac{4}{3}})$.

5.2.2 Order of $\phi_0^{(1)}$ and $\mathcal{R}_0^{(1)}$ for Large t

In this subsection, we examine the order of magnitude of the zero-wavenumber terms $\phi_0^{(1)}$ and $\mathcal{R}_0^{(1)}$ for large t . In doing so, we solve equations (5.13) and (5.14) by performing direct integrations.

Consider first equation (5.13), we need to integrate both sides once with respect to t and twice with respect to z . The right-hand side comprises a steady part and time-dependent terms. the steady part is given in (5.20) and the largest time-dependent terms are given in (5.21) and (5.22).

The term given in (5.21) takes the form of sum of terms having the following form (for some constants a , b , n , m and l)

$$e^{az} z^n e^{bt} t^m J_l(\varsigma).$$

In order to integrate this term with respect to t , we perform integration by parts by integrating e^{bt} and differentiating $t^m J_l(\varsigma)$. This choice gives us well-ordered terms:

$$\begin{aligned} & \sim \left[\frac{1}{b} e^{az} z^n e^{bt} J_l(\varsigma) \right] t^m + \left[-\frac{\sqrt{Nk}}{2b^2} e^{az} z^{n+\frac{1}{2}} e^{bt} \left(J_{l-1}(\varsigma) - J_{l+1}(\varsigma) \right) \right] t^{m-\frac{1}{2}} \\ & + \left[-\frac{m}{b^2} e^{az} z^n e^{bt} J_l(\varsigma) + \frac{Nk}{4b^3} e^{az} z^{n+1} e^{bt} \left(J_{l-2}(\varsigma) - 2J_l(\varsigma) + J_{l+2}(\varsigma) \right) \right] t^{m-1} + \dots \end{aligned} \quad (5.26)$$

We need also to get well-ordered terms after integrating with respect to z . So we consider the derivative formula, 9.1.30 of Abramowitz and Stegun (1972), of the Bessel function:

$$\frac{d}{d\varsigma} \varsigma^l J_l(\varsigma) = \varsigma^l J_{l-1}(\varsigma).$$

This formula can be written as

$$\frac{\partial}{\partial z} z^{\frac{l}{2}+\frac{1}{2}} J_{l+1}(\varsigma) = \sqrt{Nk} z^{\frac{l}{2}} t^{\frac{1}{2}} J_l(\varsigma) \quad (5.27)$$

or

$$\frac{\partial}{\partial t} t^{\frac{l}{2}+\frac{1}{2}} J_{l+1}(\varsigma) = \sqrt{Nk} z^{\frac{l}{2}} t^{\frac{l}{2}} J_l(\varsigma), \quad (5.28)$$

where $\varsigma = 2\sqrt{Nkzt}$. Now, we integrate the leading-order term in equation (5.26) with respect to z . We perform integration by parts by integrating $z^{\frac{l}{2}} J_l(\varsigma)$ [by using the anti-derivate form of the formula (5.27)] and differentiating $e^{az} z^{n-\frac{l}{2}}$. We get the following well-ordered terms:

$$\begin{aligned} & \sim \left[\frac{1}{b\sqrt{Nk}} e^{az} z^{n+\frac{1}{2}} e^{bt} J_{l+1}(\varsigma) \right] t^{m-\frac{1}{2}} + \left[-\frac{1}{bNk} e^{az} \left(\left(n - \frac{l}{2} \right) z^n + a z^{n+1} \right) e^{bt} J_{l+2}(\varsigma) \right] t^{m-1} \\ & + \left[\frac{1}{b\sqrt{N^3 k^3}} e^{az} \left(\left(n - \frac{l}{2} - \frac{1}{2} \right) z^{n-\frac{1}{2}} + a \left(n - \frac{l}{2} + \frac{3}{2} \right) z^{n+\frac{1}{2}} + a^2 z^{n+\frac{3}{2}} \right) e^{bt} J_{l+3}(\varsigma) \right] t^{m-\frac{3}{2}} + \dots \end{aligned} \quad (5.29)$$

After carrying out one more integration by parts with respect to z , we get to the leading-order term

$$\left[\frac{1}{bNk} e^{az} z^{n+1} e^{bt} J_{l+2}(\varsigma) \right] t^{m-1}.$$

This implies that after performing the integrations and according to the asymptotic behaviour of Bessel functions (4.11), the largest term arising from (5.21) is $\mathcal{O}(t^{-\frac{1}{4}})$.

Analogous to what we did for integrating (5.21), we do similar processes for (5.22). We find that the largest term arising after integrating (5.22) is $\sim \mathcal{O}(t^{-\frac{1}{2}})$. This means that the dominant contribution comes from integrating the steady part, then

$$\phi_0^{(1)} \sim I(z)t \sim \mathcal{O}(t),$$

where the function $I(z)$ is a function of z only and can be obtained by integrating the steady part twice with respect to z .

Note that the constants of the integrations do not change the order of the solution. After integrating with respect to t the constant of integration is an arbitrary

function of z , the initial conditions imply that this constant must be zero. Integrating with respect to z twice gives terms of the form $f(t)z + g(t)$, where $f(t)$ and $g(t)$ are arbitrary functions of t . The requirement of a bounded solution as $z \rightarrow \infty$ yields that $f(t) = 0$. and the zero boundary condition at $z = 0$ implies that $g(t) \sim I(0)t$.

In a similar manner to the technique used to integrate equation (5.13), we integrate both sides of equation (5.14) with respect to t . We find that

$$\mathcal{R}_0^{(1)} \sim L(z)t \sim \mathcal{O}(t),$$

where $L(z) = -ikA \left(\frac{dG^*}{dz} H(z) + G(z) \frac{dH^*}{dz} \right)$. The constant of integration is a function of z only, and does not change the order of the solution.

We conclude that for large t , both of the zero-wavenumber components $\phi_0^{(1)}$ and $\mathcal{R}_0^{(1)}$ are $\mathcal{O}(t)$. We have not been able to obtain the order of magnitude of the terms $\phi_2^{(1)}$ and $R_2^{(1)}$ because that would involve solving equations (5.15) and (5.16). The order of magnitude of these terms determines when the expansions (5.11) and (5.12) become invalid. Suppose $\phi_2^{(1)} \sim t^{p_1}$, $R_2^{(1)} \sim t^{p_2}$ and p is the largest of p_1 , p_2 and 1. Then the expansions becomes invalid when $t \sim \mathcal{O}(\varepsilon^{-1/p})$. To examine the late-time (fully nonlinear) evolution of the disturbance using multiple scaling, we would have to define a slow time scale $\tau = \varepsilon^{\frac{1}{p}}t$ where $p \geq 1$ and could possibly obtain a late-time solution in terms of this variable.

5.2.3 Gravity Wave Drag in Relevance to Parameterizations

In Appendix C, we see that the mean flow changes as a result of the nonlinear wave interactions, according to equation (C.4):

$$\frac{\partial \bar{u}_0}{\partial t} = -\varepsilon^2 \frac{\partial \bar{F}}{\partial z},$$

where $\bar{F}$ is the horizontal average of the vertical flux of horizontal momentum, $\frac{\partial \bar{F}}{\partial z}$ is the momentum flux divergence or the gravity wave drag and $\frac{\partial \bar{u}_0}{\partial t}$ is the rate of change

of the mean flow with time. In relevance to gravity wave drag parameterization, in order to have a good representation of the convectively-generated gravity waves in the global circulation models, it is useful to know the leading-order term in the momentum flux and the leading-order term of the drag. According to equation (C.9), the drag in our notation is given by

$$\frac{\partial \bar{F}}{\partial z} = -2k \text{Im} \left(\phi^{(0)*} \frac{\partial^2 \phi^{(0)}}{\partial z^2} \right). \quad (5.30)$$

In a linear problem, the mean flow velocity is $\bar{u}$ which we have chosen to be a constant. In a nonlinear problem, the mean flow evolves with time. The total mean velocity at time t is

$$\bar{u}_{\text{total}}(z, t) = \bar{u} + \bar{u}_0(z, t),$$

where $\bar{u}$ is the initial background flow velocity and

$$\bar{u}_0(z, t) = -\varepsilon^2 \frac{\partial}{\partial z} \left(\phi_0^{(1)}(z, t) + \phi_0^{(1)*}(z, t) \right).$$

Equation (5.13) can be written as

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial \bar{u}_0}{\partial z} &= -\varepsilon^2 \frac{\partial}{\partial t} \frac{\partial^2}{\partial z^2} \left(\phi_0^{(1)}(z, t) + \phi_0^{(1)*}(z, t) \right) \\ &= -ik\varepsilon^2 \left(\frac{\partial \phi^{(0)*}}{\partial z} \frac{\partial^2 \phi^{(0)}}{\partial z^2} - \phi^{(0)} \frac{\partial^3 \phi^{(0)*}}{\partial z^3} \right) + \text{c.c.} \\ &= -ik\varepsilon^2 \frac{\partial}{\partial z} \left(\phi^{(0)*} \frac{\partial^2 \phi^{(0)}}{\partial z^2} - \phi^{(0)} \frac{\partial^2 \phi^{(0)*}}{\partial z^2} \right) \\ &= -ik\varepsilon^2 \frac{\partial^2}{\partial z^2} \left(\phi^{(0)*} \frac{\partial \phi^{(0)}}{\partial z} - \phi^{(0)} \frac{\partial \phi^{(0)*}}{\partial z} \right) \\ &= 2k\varepsilon^2 \frac{\partial^2}{\partial z^2} \text{Im} \left(\phi^{(0)*} \frac{\partial \phi^{(0)}}{\partial z} \right). \end{aligned}$$

This can be written in terms of $\bar{F}$, according to equation (C.8), as

$$\frac{\partial}{\partial t} \frac{\partial \bar{u}_0}{\partial z} = -\varepsilon^2 \frac{\partial^2 \bar{F}}{\partial z^2}. \quad (5.31)$$

We note that by integrating equation (5.31) with respect to z , we obtain equation (C.4) which describes the change in the mean flow velocity with time due to the nonlinear waves interactions.

$$\frac{\partial \bar{u}_0}{\partial t} = -\varepsilon^2 \frac{\partial \bar{F}}{\partial z}. \quad (5.32)$$

This means that the wave drag can also be obtained by integrating the right-hand side of equation (5.13) once with respect to z .

In a similar way, we now examine the changes of the density. The background density is a function of z and denoted by $\bar{\rho}(z)$. In the linear problem the density does not change with time, but it does so in the nonlinear problem. The mean total density can be written as

$$\bar{\rho}_{\text{total}} = \bar{\rho}(z) + \bar{\rho}_0(z, t), \quad (5.33)$$

where

$$\bar{\rho}_0(z, t) = \varepsilon^2 \left(\mathcal{R}_0^{(1)}(z, t) + \mathcal{R}_0^{(1)*}(z, t) \right).$$

In order to obtain an equation describing the changes in the density, we write equation (5.14) as

$$\frac{\partial \bar{\rho}_0}{\partial t} = ik\varepsilon^2 \left(\frac{\partial \phi^{(0)*}}{\partial z} \mathcal{R}^{(0)} - \phi^{(0)} \frac{\partial \mathcal{R}^{(0)*}}{\partial z} \right) + \text{c.c.} = -2k\varepsilon^2 \frac{\partial}{\partial z} \text{Im} \left(\phi^{(0)*} \mathcal{R}^{(0)} \right). \quad (5.34)$$

Equation (5.34) describes the rate of change of the density.

Now, we find expression for the momentum flux and the drag. According to equation (C.8), the momentum flux is given by

$$\bar{F} \approx -2k|A|^2 \text{Im} \left(G^*(z) \frac{dG}{dz} \right),$$

so

$$\bar{F} \approx \frac{-2E_0^2}{k \left(N^2 + \left(b - \frac{1}{H} \right)^2 \bar{u}^2 \right)^2} \left[\frac{N}{\bar{u}} - e^{-(b-\frac{1}{H})z} \left(\frac{N}{\bar{u}} \cos \frac{Nz}{\bar{u}} + \left(b - \frac{1}{H} \right) \sin \frac{Nz}{\bar{u}} \right) \right],$$

and the drag is given by

$$\frac{\partial \bar{F}}{\partial z} \approx -2kA^2 \text{Im} \left(G^*(z) \frac{d^2 G}{dz^2} \right),$$

so

$$\frac{\partial \bar{F}}{\partial z} \approx \frac{-2E_0^2}{k \left(N^2 + \left(b - \frac{1}{H} \right)^2 \bar{u}^2 \right)^2} e^{-(b - \frac{1}{H})z} \left[\frac{N^2}{\bar{u}^2} \sin \frac{Nz}{\bar{u}} + \left(b - \frac{1}{H} \right)^2 \sin \frac{Nz}{\bar{u}} \right].$$

These expressions for the momentum flux and drag could be used as the basis for a gravity wave parameterization.

Figure 5.1 shows the momentum flux and the drag with parameters $g = 9.8$, $F_0 = 1$, $\rho_0 = 1$, (then $E_0 = \frac{gF_0}{\rho_0} = 9.8$), $k = 1$, $\bar{u} = \sqrt{10}$ and $H = 5$, and in the computational vertical domain $0 \leq z \leq 20$. There are two configurations according to whether $b - \frac{1}{H}$ is positive or zero. In Figure 5.1(a) we show the configuration with $b - \frac{1}{H} > 0$ by setting $b = 0.7$, which means that the heating depth $\approx 1/b$ is less than the scale height H . In Figure 5.1(b) we show the configuration where $b - \frac{1}{H} = 0$ by setting $b = 0.2$, which means that the heating depth equals the scale height.

We note that for the configuration where $b - \frac{1}{H} > 0$, the values of the nondimensional parameters correspond to a dimensional configuration where $L_z = \frac{H^*}{H} = \frac{7}{5} = 1.4$ km. The dimensional heating depth $D^* \approx 1/b^* = 2$ km and the depth of the region of nonzero drag is approximately 8.4 km.

(a) Configuration $b - \frac{1}{H} > 0$:

(b) Configuration $b - \frac{1}{H} = 0$:

Figure 5.1: The momentum flux and the drag for gravity waves generated by a thermal forcing. The thermal disturbance is centered at $z = 0$ with (a) $b = 0.7$ for which $b - \frac{1}{H} > 0$ and (b) $b = 0.2$ for which $b - \frac{1}{H} = 0$.

We observe that the momentum flux changes with height due to the existence of the thermal forcing. In the case where $b - \frac{1}{H} > 0$ the momentum flux is the sum of a constant and a function that decays exponentially with height so as $z \rightarrow \infty$ it approaches a constant value. So there is a drag force on the mean flow at low altitudes but at higher altitudes the drag is zero. In the case where $b - \frac{1}{H} = 0$ the momentum flux oscillates with the vertical wavelength of the gravity waves. We observe that in both cases there is nonzero drag even in absence of vertical shear and critical levels, so the Eliassen-Palm theorem (see Appendix C) does not apply to our configuration.

Chapter 6

Numerical Simulations of Gravity Waves and Convection Generated by a Thermal Forcing

In this chapter, we carry out numerical simulations for linear time-dependent gravity waves and convection generated by a thermal forcing. In the gravity wave simulations, the linear version of equations (5.3) and (5.4) are solved numerically with a zero lower boundary condition. The infinite vertical domain is truncated and a time-dependent non-reflecting non-homogeneous boundary condition is imposed at the upper boundary. The derivation of this non-reflecting boundary condition is given in Section 6.2 and in Appendix E we give an example to illustrate the instability that results when a zero boundary condition is applied instead of the non-reflecting condition at the upper boundary. In the simulations of convection, we solve the linear version of equations (5.3) and (5.4) with the addition of viscosity and heat conductions as suggested in Chapter 4.

6.1 Numerical Model

The starting point of our numerical simulations of gravity waves is the perturbed equations (5.3) and (5.4) that we derived in chapter 5:

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi + \varepsilon (\psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x) = -\frac{g}{\bar{\rho}} \rho_x$$

and

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \rho + \frac{d\bar{\rho}}{dz} \psi_x + \varepsilon (\psi_x \rho_z - \psi_z \rho_x) = F.$$

In the numerical simulations, it is more convenient to use the vorticity. The perturbation of the vorticity is given by

$$\zeta = \psi_{zz} + \delta \psi_{xx} = \nabla^2 \psi. \quad (6.1)$$

The numerical model can be written in terms of the perturbed streamfunction, density and vorticity as

$$\zeta_t + \bar{u} \zeta_x + \varepsilon (\psi_x \zeta_z - \psi_z \zeta_x) = -\frac{g}{\bar{\rho}} \rho_x, \quad (6.2)$$

$$\zeta = \nabla^2 \psi, \quad (6.3)$$

$$\rho_t + \bar{u} \rho_x + \frac{d\bar{\rho}}{dz} \psi_x + \varepsilon (\psi_x \rho_z - \psi_z \rho_x) = F. \quad (6.4)$$

The linear model is defined by setting $\varepsilon = 0$ in the governing equations and is given by

$$\zeta_t + \bar{u} \zeta_x = -\frac{g}{\bar{\rho}} \rho_x, \quad (6.5)$$

$$\zeta = \nabla^2 \psi, \quad (6.6)$$

$$\rho_t + \bar{u} \rho_x + \frac{d\bar{\rho}}{dz} \psi_x = F. \quad (6.7)$$

We take a Fourier transform in the x direction and use finite differences to approximate the derivatives of z and t . The numerical simulation is carried out on a rectangular domain in the xz -plane. The computational domain is defined by $x_1 < x < x_2$ and $0 < z < z_2$ with zero initial conditions

$$\psi(x, z, 0) = \zeta(x, z, 0) = \rho(x, z, 0) = 0, \quad (6.8)$$

and zero lower boundary conditions

$$\psi(x, 0, t) = \zeta(x, 0, t) = \rho(x, 0, t) = 0. \quad (6.9)$$

In order to allow the waves to propagate out of the computational domain on reaching the upper boundary, we need to apply a non-reflecting boundary condition or radiation condition at the upper boundary. If a zero boundary condition is applied instead once the waves get at the upper boundary they are not allowed to propagate further, so instabilities develop and the results are inaccurate. This case is shown in Appendix E.

6.2 Time-dependent Radiation Condition

In this section, we derive a nonhomogeneous time-dependent radiation condition. A time-dependent radiation condition for a homogeneous problem was derived in Campbell and Maslowe (2003); they studied the evolution of a forced gravity wave packet where the wave packet, localized in the horizontal direction, is forced at the lower boundary of a two-dimensional domain and propagates vertically towards a critical layer. In our model the gravity waves are generated by a thermal forcing that is expressed in the equations as a nonhomogeneous term and therefore the radiation condition of Campbell and Maslowe cannot be applied. To derive the radiation condition we consider the streamfunction written in normal mode form

$$\psi(x, z, t) = \hat{\psi}(z, t)e^{ikx} + \text{c.c.}$$

The amplitude $\hat{\psi}$ is given by $ik\hat{\psi} = \hat{w}$ so it satisfies almost the same equation as $\hat{w}$ (equation (2.57) divided by ik):

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right)^2 \frac{\partial^2 \hat{\psi}}{\partial z^2} - \left[k^2 N^2 + \delta k^2 \left(\frac{\partial}{\partial t} + ik\bar{u}\right)^2 + ik \frac{d^2 \bar{u}}{dz^2} \left(\frac{\partial}{\partial t} + ik\bar{u}\right) \right] \hat{\psi} = -ikg \frac{\hat{F}}{\bar{\rho}(z)}. \quad (6.10)$$

With constant $\bar{u}$, we have

$$\left(\frac{\partial}{\partial t} + ik\bar{u}\right)^2 \frac{\partial^2 \hat{\psi}}{\partial z^2} - \left[k^2 N^2 + \delta k^2 \left(\frac{\partial}{\partial t} + ik\bar{u}\right)^2\right] \hat{\psi} = -ikg \frac{\hat{F}}{\bar{\rho}(z)}. \quad (6.11)$$

We define $\tilde{\psi}(z, s)$ to be the Laplace transform of $\hat{\psi}(z, t)$ and take the Laplace transform in t of equation (6.11). This gives

$$\frac{\partial^2 \tilde{\psi}}{\partial z^2} - \mathcal{H}^2 \tilde{\psi} = \frac{-ikg}{\bar{\rho}s(s + ik\bar{u})^2} \hat{F}(z), \quad (6.12)$$

where $\tilde{\psi}(z, s)$ is the Fourier-Laplace transform of $\psi(z, t)$ and

$$\mathcal{H}(k, s) = \delta^{1/2} k \left[\frac{(s + ik\bar{u})^2 + N^2/\delta}{(s + ik\bar{u})^2} \right]^{1/2}. \quad (6.13)$$

By using the integrating factor method and integrating from some level z_0 to z , equation (6.12) can be written as

$$\frac{\partial \tilde{\psi}}{\partial z} + \mathcal{H} \tilde{\psi} = \frac{-ikg}{s(s + ik\bar{u})^2} e^{\mathcal{H}z} \int_{z_0}^z e^{-\mathcal{H}\varsigma} \frac{\hat{F}(\varsigma)}{\bar{\rho}(\varsigma)} d\varsigma. \quad (6.14)$$

We use the mean density profile and the form of the thermal forcing that are given in equations (3.2) and (3.30), respectively:

$$\bar{\rho}(z) = \rho_0 e^{-\frac{z}{H}}$$

and

$$F(x, z) = F_0 e^{-bz} e^{ikx} + \text{c.c.}$$

Equation (6.14) becomes

$$\frac{\partial \tilde{\psi}}{\partial z} + \mathcal{H} \tilde{\psi} = \frac{-E_0}{s(s + ik\bar{u})^2} e^{\mathcal{H}z} \int_{z_0}^z e^{-\mathcal{H}\varsigma} e^{-(b - \frac{1}{H})\varsigma} d\varsigma, \quad (6.15)$$

where $E_0 = \frac{ikgF_0}{\rho_0}$. By evaluating the integral and applying the requirement of upward-propagating waves only [see Section 3.2.1], we obtain

$$\frac{\partial \tilde{\psi}}{\partial z} + \mathcal{H} \tilde{\psi} = \frac{E_0}{s(s + ik\bar{u})^2} \frac{e^{-(b - \frac{1}{H})z}}{[\mathcal{H}(k, s) + (b - \frac{1}{H})]}. \quad (6.16)$$

The inverse Laplace transform of the function $\mathcal{H}(k, s)$ is derived in Campbell and Maslowe (2003) and is given by

$$h(k, t) = \frac{N^2 k}{\delta^{1/2}} \int_0^t \frac{a}{\tau} J_1(a\tau) d\tau e^{-ik\bar{u}t} + \delta^{1/2} k e^{-ik\bar{u}t} \delta(t - 0),$$

where J_1 is the Bessel function of order 1, $\delta(t - 0)$ is the delta function and the constant $a = \frac{N^2 k}{\delta}$. If we consider the case where $\delta \ll 1$, then $\mathcal{H}$ can be approximated by

$$\mathcal{H}(k, s) = \frac{Nk}{s + ik\bar{u}}, \quad (6.17)$$

so that

$$h(k, t) = Nk e^{-ik\bar{u}t}, \quad (6.18)$$

and equation (6.16) becomes

$$\frac{\partial \tilde{\psi}}{\partial z} + \mathcal{H} \tilde{\psi} = \frac{E_0}{\left(b - \frac{1}{H}\right)} \frac{e^{-(b - \frac{1}{H})z}}{s(s + ik\bar{u}) \left(s + ik\bar{u} + \frac{Nk}{(b - \frac{1}{H})}\right)}. \quad (6.19)$$

The inverse Laplace transform of the right-hand side is straightforward and by using a convolution integral of $\mathcal{H}$ and $\tilde{\psi}$, the inverse Laplace transform of equation (6.19) gives the following radiation condition

$$\begin{aligned} \frac{\partial}{\partial z} \hat{\psi}(z, t) + \int_0^t \hat{\psi}(z, \tau) h(k, t - \tau) d\tau &= E_0 e^{-(b - \frac{1}{H})z} \left[\frac{-i}{k\bar{u} (Nk + ik\bar{u} (b - \frac{1}{H}))} \right. \\ &\quad \left. + \frac{i}{Nk^2 \bar{u}} e^{-ik\bar{u}t} + \frac{(b - \frac{1}{H})}{Nk (Nk + ik\bar{u} (b - \frac{1}{H}))} e^{-\left(\frac{Nk}{(b - \frac{1}{H})} + ik\bar{u}\right)t} \right]. \end{aligned} \quad (6.20)$$

By using equation (6.18), the radiation condition becomes

$$\begin{aligned} \frac{\partial}{\partial z} \hat{\psi}(z, t) + Nk e^{-ik\bar{u}t} \int_0^t \hat{\psi}(z, \tau) e^{ik\bar{u}\tau} d\tau &= E_0 e^{-(b - \frac{1}{H})z} \left[\frac{-i}{k\bar{u} (Nk + ik\bar{u} (b - \frac{1}{H}))} \right. \\ &\quad \left. + \frac{i}{Nk^2 \bar{u}} e^{-ik\bar{u}t} + \frac{(b - \frac{1}{H})}{Nk (Nk + ik\bar{u} (b - \frac{1}{H}))} e^{-\left(\frac{Nk}{(b - \frac{1}{H})} + ik\bar{u}\right)t} \right]. \end{aligned} \quad (6.21)$$

In the absence of the thermal forcing, i.e. $E_0 = 0$, we would obtain the homogeneous radiation condition which is given in Campbell and Maslowe (2003)

$$\frac{\partial}{\partial z} \hat{\psi}(z, t) + Nk e^{-ik\bar{u}t} \int_0^t \hat{\psi}(z, \tau) e^{ik\bar{u}\tau} d\tau = 0. \quad (6.22)$$

Note that the homogeneous radiation condition (6.22) could be applied when the thermal forcing is zero or approximately zero at the outflow boundary. In that case, condition (6.22) would be considered as an approximation of (6.21). So, and according to the definition of the thermal forcing (3.30), when the value of b is large enough that the thermal forcing is approximately zero at the upper boundary, equation (6.22) can be applied as an approximation. However, if the thermal forcing is non-zero at the upper boundary, then we must apply the nonhomogeneous radiation condition (6.21) and we cannot approximate it by equation (6.22).

6.3 Time-dependent Gravity Wave Simulations

We consider the linear equations (6.5)-(6.7). Since there is no nonlinear interaction in this case, we consider a wave comprising a single wavenumber k and write the equation in a normal mode by defining each flow variable $f(x, z, t)$ as

$$f(x, z, t) = e^{ikx} \hat{f}(z, t) + \text{c.c.}$$

The equations for the amplitude functions $\hat{\zeta}$, $\hat{\psi}$ and $\hat{\rho}$ are

$$\hat{\zeta}_t + ik\bar{u}\hat{\zeta} = -ikg \frac{\hat{\rho}}{\bar{\rho}}, \quad (6.23)$$

$$\hat{\psi}_{zz} - \delta k^2 \hat{\psi} = \hat{\zeta}, \quad (6.24)$$

$$\hat{\rho}_t + ik\bar{u}\hat{\rho} + ik \frac{d\bar{\rho}}{dz} \hat{\psi} = \hat{F}. \quad (6.25)$$

We solve the above equations with zero initial conditions at $t = 0$ and a zero boundary condition at $z = 0$. At the upper boundary, the radiation condition (6.21) is applied. We observe that the radiation condition is needed to obtain the correct solution.

To show this, we carried out same test computations with a zero upper boundary condition. The result of these tests are shown in Appendix E and they are clearly not accurate.

Discretization

The numerical simulations are carried out on the rectangular domain with the following spatial discretization of the independent variables

$$x_i = x_1 + (i - 1)\Delta x \text{ and } z_j = z_1 + (j - 1)\Delta z, \quad i = 1, 2, \dots, I, \quad j = 1, 2, \dots, J$$

where $\Delta x = (x_I - x_1)/(I - 1)$ and $\Delta z = (z_J - z_1)/(J - 1)$ are the increments in x and z , respectively. The time is discretized as $t_n = n\Delta t$, $n = 0, 1, 2, \dots, N$, with time increment $\Delta t = t_N/N$.

The approximations for the functions $\hat{\psi}(z, t)$, $\hat{\zeta}(z, t)$ and $\hat{\rho}(z, t)$ are denoted, respectively, by

$$\hat{\psi}(z_j, t_n) = \hat{\psi}_j^n,$$

$$\hat{\zeta}(z_j, t_n) = \hat{\zeta}_j^n,$$

$$\hat{\rho}(z_j, t_n) = \hat{\rho}_j^n.$$

The time derivative is approximated by using the forward Euler method, which is defined by

$$\begin{aligned} \hat{\zeta}_t(z_j, t_n) &= \frac{\hat{\zeta}_j^{n+1} - \hat{\zeta}_j^n}{\Delta t}, \\ \hat{\rho}_t(z_j, t_n) &= \frac{\hat{\rho}_j^{n+1} - \hat{\rho}_j^n}{\Delta t}. \end{aligned}$$

The second z -derivative is approximated by using the second-order central finite difference approximation:

$$\hat{\psi}_{zz}(z_j, t_n) = \frac{\hat{\psi}_{j+1}^n - 2\hat{\psi}_j^n + \hat{\psi}_{j-1}^n}{(\Delta z)^2}.$$

The discretized equations corresponding to (6.23)-(6.25) are given by

$$\hat{\zeta}_j^{n+1} = \hat{\zeta}_j^n - \Delta t \left[ik\bar{u}_j \hat{\zeta}_j^n + \frac{ikg}{\bar{\rho}_j} \hat{\rho}_j^n \right], \quad (6.26)$$

$$\hat{\zeta}_j^n = \frac{\hat{\psi}_{j+1}^n - (\delta k^2 (\Delta z)^2 + 2) \hat{\psi}_j^n + \hat{\psi}_{j-1}^n}{(\Delta z)^2}, \quad (6.27)$$

$$\hat{\rho}_j^{n+1} = \hat{\rho}_j^n - \Delta t \left[ik\bar{u}_j \hat{\rho}_j^n - \frac{ik}{H} \bar{\rho}_j \hat{\psi}_j^n - \hat{F}_j \right], \quad (6.28)$$

The simulations are carried out in a rectangular domain $0 < x < 2\pi$, $0 < z < 20$ with the following parameters; $k = 1$, $H = 5$, $g = 9.8$, $\rho_0 = 1$, $F_0 = 1$, $\bar{u} = 3$ and in the long-wave limit, $\delta = 0$. According to the discussion in Chapter 4 about the choice of the sign of $b - \frac{1}{H}$ in the analytical solution (4.10), we have shown that in order to get a bounded solution we must have $b - \frac{1}{H} \geq 0$. We consider the two configurations: $b = 0.7$, so that $b - \frac{1}{H} > 0$, and $b = 0.2$, so that $b - \frac{1}{H} = 0$.

We also carried out a test with $b - \frac{1}{H} < 0$ which is described in Appendix E. As expected with this choice, the amplitude of the solution increases rapidly with height and with time.

Solution Plots of the Configuration $b - \frac{1}{H} > 0$:

Contour plots of the linear simulation of the vertical velocity perturbation $w(x, z, t)$ for the gravity waves propagation obtained at $t = 2$, $t = 10$, $t = 30$ and $t = 50$ are shown in Figure 6.1. Figure 6.2 shows the absolute value of the amplitude of the vertical velocity perturbation $|\hat{w}(z, t)|$ as a function of t obtained at $z = 5$ and $z = 10$. In Figure 6.3 we plot the numerical simulations for the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ as a function of z at $t = 50$ against the analytical solution of the steady vertical velocity perturbation $\hat{w}(z)$ that is given by (4.10).

Solution Plots of the Configuration $b - \frac{1}{H} = 0$:

Contour plots of the linear simulation of the vertical velocity perturbation $w(x, z, t)$ for the gravity waves propagation obtained at $t = 2$, $t = 10$, $t = 30$ and $t = 50$ are shown in Figure 6.4. Figure 6.5 shows the absolute value of the amplitude of the vertical velocity perturbation $|\hat{w}(z, t)|$ as a function of t obtained at $z = 5$ and $z = 10$. In Figure 6.6 we plot the numerical simulation for the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ as a function of z at $t = 50$ against the analytical solution of the steady vertical velocity perturbation $\hat{w}(z)$ that is given in (4.10). There is very close agreement between the numerical simulations and the exact solutions.

First configuration, $b - \frac{1}{H} > 0$:

Figure 6.1: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.7$. Plots are obtained at (a) $t = 2$, (b) $t = 10$, (c) $t = 30$ and (d) $t = 50$. A radiation condition is applied at the upper boundary.

Figure 6.2: Numerical simulations: the absolute value of the amplitude of the vertical velocity perturbation $|\hat{w}(z, t)|$ as a function of t of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.7$. Plots are obtained at (a) $z = 5$ and (b) $z = 10$. A radiation condition is applied at the upper boundary.

Figure 6.3: Numerical simulations of the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ as a function of z at $t = 50$, dashed line, against the analytical solution of the steady vertical velocity perturbation $\hat{w}(z)$, solid line. The thermal forcing is centered at $z = 0$ with $b = 0.7$. A radiation condition is applied at the upper boundary.

Second configuration, $b - \frac{1}{H} = 0$:

Figure 6.4: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.2$. Plots are obtained at (a) $t = 2$, (b) $t = 10$, (c) $t = 30$ and (d) $t = 50$. A radiation condition is applied at the upper boundary.

Figure 6.5: Numerical simulations: the absolute value of the amplitude of the vertical velocity perturbation $|\hat{w}(z, t)|$ of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.2$ as a function of t . Plots are obtained at (a) $z = 5$ and (b) $z = 10$. A radiation condition is applied at the upper boundary.

Figure 6.6: Numerical simulations of the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.2$ as a function of z at $t = 50$, dashed line, against the analytical solution of the steady vertical velocity perturbation $\hat{w}(z)$, solid line. A radiation condition is applied at the upper boundary.

6.4 Time-dependent Simulations of Convection

In Section 4.3 we examined a linear inviscid time-dependent model for convection generated by a thermal forcing in a single layer of unstable stratification. We derived an analytical solution and found that the amplitude becomes infinite as $t \rightarrow \infty$. On the other hand, the Lorenz model for convection in a viscous heat-conducting fluid indicates that the evolution of the convection depends on the magnitude of the viscosity and heat conduction. This is discussed in detail in Appendix D.

In the Lorenz model the spatial structure of the convective cells is specified and kept fixed and only the amplitude is assumed to change as a function of time. We define the Rayleigh number R_a which is a nondimensional parameter that specifies the relative magnitude of the effects of the unstable stratification to the effects of viscosity and heat conduction. For values of R_a less than a certain critical value R_c , the amplitude of the convection decays to zero as a function of time. For $R_a = R_c$ or $r = R_a/R_c = 1$, the convection maintains a steady amplitude. For $R_a > R_c$ or $r > 1$, the amplitude may decay to zero or grow exponentially depending on the specified initial conditions. This suggests that in our time-dependent model viscosity and heat conduction may be needed to maintain a finite-amplitude as $t \rightarrow \infty$.

In this section we investigate this possibility using numerical simulations. The numerical model is similar to that used for the gravity wave simulation in Section 6.3, but with unstable stratification $\bar{\rho} = \rho_0 e^{\frac{z}{H}}$ instead of stable stratification and with the addition of viscosity and heat conduction. The governing equations are

$$\zeta_t + \bar{u}\zeta_x = -\frac{g}{\bar{\rho}}\rho_x + \nu\nabla^2\zeta, \quad (6.29)$$

$$\zeta = \nabla^2\psi, \quad (6.30)$$

$$\rho_t + \bar{u}\rho_x + \frac{d\bar{\rho}}{dz}\psi_x = F + \kappa\nabla^2\rho. \quad (6.31)$$

As before the thermal forcing takes the form $F = F_0 e^{bz} e^{ikx} + \text{c.c.}$ and we write the

solution as

$$\psi(x, z, t) = \hat{\psi}(z, t)e^{ikx} + \text{c.c.},$$

$$\rho(x, z, t) = \hat{\rho}(z, t)e^{ikx} + \text{c.c.},$$

and solve for $\hat{\psi}$ and $\hat{\rho}$ using finite difference discretization in z and t . The results shown here were obtained with $\delta = 0.1$. Changing to $\delta = 0$, the value in the analytical solutions, does not affect the qualitative behaviour of the numerical results. We also fix the structure of the thermal forcing by setting $b = 0.7$.

In the Lorenz configuration [Saltzman (1962)], the Rayleigh number is defined as (see Appendix D)

$$R_a = \frac{g\alpha\Delta Th^3}{\nu\kappa} = \frac{g\Delta Th^3}{T_0\nu\kappa},$$

where T_0 is the background temperature and ΔT is the temperature between the lower boundary and the upper boundary of the layer. The analogue of this for our configuration is

$$R_a = \frac{\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz} h^4}{\nu\kappa} = \frac{N_0^2 h^4}{\nu\kappa}.$$

We first carry out the computation with $N_0^2 = 2$ and $\nu = \kappa = 0$. This corresponds to infinite Rayleigh number. Contour plots of the vertical velocity perturbation $w(x, z, t)$ are shown in Figure 6.7 at different values of t . Figure 6.8 shows the amplitude of the vertical velocity as a function of z at fixed time $t = 20$ and the evolution of the amplitude at fixed height $z = -5$ as a function of t . The amplitude increases rapidly with time and by $t = 20$ it has reached a value of 10^{27} . However, the convective cells maintain their basic shape all through this process. This is in good agreement with the exponentially-growing behaviour predicted by the analytical solution (for the case $\delta = 0$) that was derived in Section 4.3.

Figure 6.7: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ of the convection generated by a thermal forcing centered at $z = -5$ with $b = 0.7$. Plots are obtained at (a) $t = 2$, (b) $t = 5$, (c) $t = 10$ and (d) $t = 20$. For the case with $N_0^2 = 2$ and $\nu = \kappa = 0$.

Figure 6.8: Numerical simulations: the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ of the convection generated by a thermal forcing centered at $z = -5$ with $b = 0.7$ (a) as a function of z obtained at $t = 20$ and (b) as a function of t obtained at $z = -5$. For the case with $N_0^2 = 2$ and $\nu = \kappa = 0$.

Next, we introduce viscosity and heat conduction by setting $\nu = \kappa = 0.5$ and also reduce the strength of the unstable stratification by reducing the rate of increase of the background density with height setting $N_0^2 = 10^{-3}$. This corresponds to a Rayleigh number of $R_a = 39.2$. Again, the basic structure of the convective cells is maintained but now the amplitude increases initially and then remains approximately constant from about $t = 12$. This behaviour suggests that $R_a = 39.2$ is close to the critical Rayleigh number for this configuration. Contour plots of the vertical velocity perturbation in this case are shown in Figure 6.9 at different values of t . Figure 6.10 shows the amplitude of the vertical velocity as a function of z at fixed time $t = 20$ and the evolution of the amplitude at fixed height $z = -5$ as a function of t .

Figure 6.9: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ of the convection generated by a thermal forcing centered at $z = -5$ with $b = 0.7$. Plots are obtained at (a) $t = 2$, (b) $t = 5$, (c) $t = 10$ and (d) $t = 20$. For the case with $N_0^2 = 10^{-3}$ and $\nu = \kappa = 0.5$.

Figure 6.10: Numerical simulations: the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ of the convection generated by a thermal forcing centered at $z = -5$ with $b = 0.7$ (a) as a function of z obtained at $t = 20$ and (b) as a function of t obtained at $z = -5$. For the case with $N_0^2 = 10^{-3}$ and $\nu = \kappa = 0.5$.

Finally, we increase the viscosity and heat conduction to $\nu = \kappa = 1$ while keeping $N_0^2 = 10^{-3}$. This has the effect of reducing the Rayleigh number by a factor 4 to $R_a = 9.8$. With this value, the amplitude of the convection decays to zero as a function of time consistent with what occurs in the Lorenz model when $R_a < R_c$. Contour plots of the vertical velocity perturbation in this case are shown in Figure 6.11 at different values of t . Figure 6.12 shows the amplitude of the vertical velocity as a function of z at fixed time $t = 20$ and the evolution of the amplitude at fixed height $z = -5$ as a function of t .

Figure 6.11: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ of the convection generated by a thermal forcing centered at $z = -5$ with $b = 0.7$. Plots are obtained at (a) $t = 2$, (b) $t = 5$, (c) $t = 10$ and (d) $t = 20$. For the case with $N_0^2 = 10^{-3}$ and $\nu = \kappa = 1$.

Figure 6.12: Numerical simulations: the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ of the convection generated by a thermal forcing centered at $z = -5$ with $b = 0.7$ (a) as a function of z obtained at $t = 20$ and (b) as a function of t obtained at $z = -5$. For the case with $N_0^2 = 10^{-3}$ and $\nu = \kappa = 1$.

Other experiments with a range of different values of R_a were carried out as well (not shown here). The results support our conclusions that for $R_a = R_c \approx 40$ the amplitude of the convection is approximately constant, for $R_a < R_c$ it decays to zero with t and for $R_a > R_c$ it increases with t . It is clear that viscosity and heat conduction are necessary to maintain finite-amplitude convection.

6.5 Time-dependent Simulations of the Two-layer Model

In this section we carry out time-dependent numerical simulations for the two-layer model comprising both gravity waves and convection. We use the same numerical methods described in Sections 6.3 and 6.4. We impose a zero condition at the lower boundary and apply the radiation condition (6.21) at the upper boundary. We consider the case when the thermal forcing is centered in the lower layer (at $z = -h_1$) and apply the interface conditions (3.6)-(3.9) which we discretize using finite differences.

The simulations are carried out in a rectangular domain $0 < x < 2\pi$ and $-10 < z < 40$ with the following parameters $k = 1$, $\rho_0 = 1$, $F_0 = 1$, $g = 10$, $h_1 = 5$, $\delta = 0.1$ and $\bar{u} = 3$. In the upper layer $H = 5$ which implies that $N^2 = 2$ and in the lower layer $H = 10^4$ which implies that $N_0^2 = 10^{-3}$. First, we consider the configuration where the two layers have viscosity and heat conduction by setting $\nu = \kappa = 0.5$ in both layers. Contour plots of the vertical velocity perturbation for this configuration at different time levels are shown in Figure 6.13(a), the upper layer is shown in Figure 6.13(b) on a different scale. This amount of viscosity and heat conduction allows the convection amplitude to remain finite and damps the gravity waves so their amplitude is small. To prevent damping the gravity waves, we carry out another simulation in which we apply viscosity and heat conduction in the lower layer only i.e. $\nu = \kappa = 0.5$ in the lower layer and $\nu = \kappa = 0$ in the upper layer. In this case we get stronger gravity waves in the upper layer. Contour plots of the vertical velocity perturbation for this configuration at different time levels are shown in Figure 6.14(a) and the upper layer is shown in Figure 6.14(b) on a different scale.

Figure 6.13: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ for the two-layer model generated by a thermal forcing centered at $z = -5$ with $b = 0.7$. Plots are obtained at different time levels for the case with $\nu = \kappa = 0.5$ in the two layers. (b) the upper layer solutions shown on a different scale.

Figure 6.14: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ for the two-layer model generated by a thermal forcing centered at $z = -5$ with $b = 0.7$. Plots are obtained at different time levels for the case with $\nu = \kappa = .5$ only in the lower layer. (b) the upper layer solutions shown on a different scale.

Chapter 7

Conclusions

7.1 Summary

This concludes our study of the generation of the gravity waves by a thermal forcing and convection in the atmosphere. We considered a two-dimensional atmospheric two-layer model. The model comprised an upper stable layer in which the gravity waves propagated and a lower unstable layer in which convection occurred. The two layers were connected according to interface conditions. In the upper layer, we required upward-propagating waves and we set the lower boundary to zero. We started our analysis by deriving the steady linear solutions for the model and steadily increased the complexity of the problem by considering time-dependence and then adding nonlinear terms. In the time-dependent problem, we considered each layer separately and then we examined the two-layer model. After that, we examined the nonlinear problem for the gravity waves generated by a thermal forcing (upper layer only). The nonlinear analysis was based on weakly-nonlinear perturbation theory. We finished this study by carrying out numerical simulations for the linear time-dependent problem for a single layer with gravity waves generated by a thermal forcing and a single layer with convection generated by a thermal forcing then for the two-layer

model.

The thesis began by deriving the governing equations for the gravity waves and convection. We considered two-dimensional (x and z) equations and wrote these equations in terms of dimensionless quantities. This enabled us to compare the magnitude of the terms of any equations regardless their units. After that, we linearized the governing equations by using the perturbation method in which each total quantity was written as the sum of a background state and a perturbation. Waves were considered as perturbations to a basic state flow and assumed to be small relative to the background state. At this stage the background velocity was assumed to have only a horizontal component and varied in the vertical direction (shear flow). The perturbations were assumed to be periodic in the horizontal direction and dependent on z and t . The linear equations were combined into a single linear time-dependent second-order nonhomogeneous partial differential equation in the vertical velocity perturbation $w(x, z, t)$. The coefficients of this governing equation became constant since the buoyancy frequency was constant and we also assumed that the background velocity was constant.

We began our analysis by deriving steady linear solutions for the model. In this case, the perturbation was periodic in the x direction and the wave amplitude was dependent on z only. We examined two configurations: The first configuration described the obstacle effect mechanism where the thermal forcing generated convection in the lower layer and then the convection generated the gravity waves in the upper layer. The second configuration described both the obstacle effect and the deep heating mechanisms where the thermal forcing influenced both layers. In this configuration, the gravity waves were generated by thermal forcing and convection. For the two configurations, we derived the solutions for the non-Boussinesq configuration (the Boussinesq effects have been taken into account in the continuity and energy equations only) then for the Boussinesq approximation I. In the non-Boussinesq configuration, the governing equation comprised second and first order z -derivatives of

the amplitude of the perturbation and there were many cases according to the various possibilities of the roots of the characteristic equation. In the lower layer, the solution in the form of a combination of exponentially-decaying and growing functions of z . With the zero boundary condition at the lower boundary, this solution gave closed convective cells. In the upper layer there was one case that gave gravity waves (Case aI) while other cases gave solutions in the form of a combination of decaying and growing exponential function of z . By applying the Boussinesq approximation I, the first z -derivative was eliminated. This reduced the number of possibilities of the roots of the characteristic equation to only two cases. We denoted these cases by Case a and Case b, and only Case a gave gravity waves. For simplicity we applied the boundary and the interface conditions to the solutions in the Boussinesq approximation I only. We finished the analysis of the steady solutions by deriving Fourier solutions over a spectrum of wavenumbers. In a normal mode solution, Case a and Case b were determined according to the sign of $\frac{N^2}{u^2} - \delta k^2$ whether it was positive or negative. In the Fourier solution we summed up the solutions of the two cases. We found that the dominant contribution came from the wavenumbers $|k| < k_1 = \sqrt{\frac{N^2}{\delta u^2}}$ which correspond to Case a (see Figure 3.14).

We noted that the structure of the gravity waves and convection depends on the profile of the thermal forcing. The parameter h_1 determined the center of the thermal forcing in the z direction. If the thermal forcing was at low altitude acting within the lower layer ($h_1 = 5$), it generated strong convection in the lower layer (Figure 3.3) which in turn generated gravity waves in the upper layer via the interface conditions (the obstacle effect mechanism). When it was centered at the interface the convection was weaker (Figure 3.7) and the thermal forcing generated both gravity waves and convection (obstacle effect and deep heating mechanisms). If the thermal forcing is in the upper layer or if there is no convection, then the thermal forcing directly generates gravity waves with no obstacle effect. This is a deep heating mechanism analogous to what was assumed in most previous studies [e.g., Chun and Baik (1998) and Holton,

Beres and Zhuo (2002)]. The depth of the thermal forcing was determined by the parameter b . For $b = 0.7$ the thermal forcing was shallow (spanning a narrow range of altitudes) and for $h_1 = 5$ or $h_1 = 0$ the gravity waves solution was given by (3.49) or (3.57), respectively. In this configuration the particular solution was relatively small and the real part of w was proportional to $\cos\left(kx + \sqrt{\frac{N^2}{\bar{u}^2} - \delta k^2} z\right)$ and gave the solution in Figures 3.3 and 3.7. For $b = 0.07$ the thermal forcing was deep (spanning wide range of altitude) and for $h_1 = 5$ or $h_1 = 0$ the gravity waves solution was again given by (3.49) and (3.57). In this configuration the particular solution was not small and the real part of w was proportional to $\cos\left(kx + \sqrt{\frac{N^2}{\bar{u}^2} - \delta k^2} z\right) + \cos(kx)$ which was a linear superposition of waves with different phases and gave the solution in Figures 3.5 or 3.9, respectively.

Next, we progressed to consider the time-dependent problem in which the perturbation was periodic in the x direction and the wave amplitude varied in time and in the vertical direction. In order to be able to find an analytical solution for the time-dependent problem, we considered the long-wave limit and also considered each layer separately, then we went on to solve the two-layer model under this approximation. The corresponding steady solutions in this configuration were shown before we derived the time-dependent solutions.

We started with a one-layer model where the gravity waves were excited by a thermal forcing (the upper layer only). This configuration describes the deep heating mechanism. We derived a solution for $t > 0$ using a Laplace transform. The solution comprised steady terms corresponding to the steady solution and time-dependent terms. The time-dependent terms involved Bessel functions which arose from the essential singularity. As $t \rightarrow \infty$ the time-dependent terms went to zero and the solution then approached the steady solution. The condition $b - \frac{1}{H} \geq 0$ was required to obtain a bounded solution for large z and large t . This means that the rate of decay of the thermal forcing with altitude must not exceed the rate of decay of the background density.

In the one-layer model with convection generated by a thermal forcing, we found that the transformed equation produced an infinite number of singularities including a singularity at infinity. In order to invert the transform, we extended the vertical domain to be $-\infty < z \leq 0$. By doing so, we were able to have only four finite singularities. Inverting the transform produced steady terms and time-dependent terms. The time-dependent terms involved modified Bessel functions which again arose from the essential singularity. These terms made the solution grow exponentially and became infinite as $t \rightarrow \infty$ and as $z \rightarrow -\infty$. We then went on to derive the solutions for the two-layer model in the vertical domain $-\infty < z < \infty$ which also showed that the convection amplitude became infinite as $t \rightarrow \infty$ or $z \rightarrow \infty$. By comparing our model with the well-known Lorenz model (Appendix D), we concluded that viscosity and heat conduction are needed in order to generate finite-amplitude convection.

After that, we examined the nonlinear time-dependent problem for the case where the gravity waves were excited only by a thermal forcing. We derived the nonlinear equations and wrote them in terms of the streamfunction ψ . In the linear problem, we were able to write the equations in terms of the vertical velocity perturbation. In the nonlinear problem it was complicated to combine the equations in terms of one dependent variable. However it was convenient to write the equations in terms of the streamfunction and the density. We carried out a weakly-nonlinear analysis for $\varepsilon \ll 1$. In doing so, we searched for a solution written as a perturbation series in powers of the nonlinear parameter ε .

$$\psi(x, z, t) \sim \psi^{(0)}(x, z, t) + \varepsilon \psi^{(1)}(x, z, t) + \mathcal{O}(\varepsilon^2)$$

and

$$\rho(x, z, t) \sim \rho^{(0)}(x, z, t) + \varepsilon \rho^{(1)}(x, z, t) + \mathcal{O}(\varepsilon^2).$$

At leading-order, we found the same linear solutions that were derived in Chapter 4. At $\mathcal{O}(\varepsilon)$ the nonlinear interactions between the waves with positive and negative

wavenumbers produced zero-wavenumber components ($\phi_0^{(1)}$ and $\mathcal{R}_0^{(1)}$) and second harmonic terms ($\phi_2^{(1)}$ and $\mathcal{R}_2^{(1)}$).

The zero-wavenumber components satisfied two uncoupled partial differential equations (5.13) and (5.14), which could be solved by direct integration. These equations described the changes in the mean flow with time owing to the nonlinear interactions between the waves and the mean flow. We were able to write equation (5.13) in terms of the mean momentum flux $\bar{F}$

$$\frac{\partial \bar{u}_0}{\partial t} = -\varepsilon^2 \frac{\partial \bar{F}}{\partial z},$$

where $\frac{\partial \bar{u}_0}{\partial t}$ gave the rate of change of the mean flow with time. We also obtained expressions for the momentum flux and the wave drag. Having analytical expressions for the momentum flux and the drag is of importance because they can be used to give better representations of convectively-generated gravity waves in global circulation models. We observed that the momentum flux was a function of z given in equation (5.35). The drag on the mean flow as a result of the divergence of the momentum flux is $X(t) = \frac{\partial \bar{F}}{\partial z}$; it was given in equation (5.35). If nonlinearity is included, there is a nonzero divergence of the momentum flux which acts as a drag force on the mean flow. Thus, the mean flow evolved with time. We were able to determine the order of the zero-wavenumber components for large t by performing integration by parts to the leading-order terms of the right-hand side of equations (5.13) and (5.14). We found that both of the zero-wavenumber components were $\mathcal{O}(t)$.

Lastly, we carried out numerical simulations for the linear time-dependent problems involving simulations of gravity waves only, convection only and then the two-layer model. In the numerical simulations of gravity waves, a time-dependent nonhomogeneous radiation condition was needed at the upper boundary to guarantee that the waves propagated out of the computational domain reaching the boundary and to avoid any downward-propagating reflected waves. The simulations were carried out by taking a Fourier transform in the horizontal direction and approximating the

z and t derivatives using the finite difference method. The results obtained from the numerical simulations confirmed the analytical solutions. In Section 6.4 we presented the results of numerical simulations of convection generated by a thermal forcing in a layer with viscosity and heat conduction as well as unstable stratification. With an infinite Rayleigh number (no viscosity and heat conduction), we found that the amplitude of the convection grow exponentially with time as predicted by the analytical solution. For a Rayleigh number of about $R_a = R_c \approx 40$ the amplitude remained approximately constant for large t and for $R_a < R_c$ the amplitude decreased to zero with time. Then we carried out numerical simulations for the two-layer model including both gravity waves and convection. The two layers have different magnitudes of viscosity and heat conduction, weaker in the upper layer to avoid damping the gravity waves to zero and larger in the lower layer to keep the convection finite.

7.2 Discussion

A number of previous analytical studies by other researchers have dealt with analytical models for gravity waves and convection or gravity waves with a thermal forcing. In this section, we highlight the differences between our configuration and theirs and then discuss our results relative to the conclusions of these related studies.

In the derivation of the equations, we considered the general two-dimensional inviscid equations of fluid dynamics written in terms of the velocity components, the density and the pressure. We then nondimensionalized these equations and used two different typical length scales: L_x and L_z which led to the introduction of the square of the aspect ratio $\delta = L_z^2/L_x^2$. We linearized and combined the zonal and the vertical momentum equations. We then combined the resulting linear equation with the continuity and the linearized energy equations to give the single linear time-dependent equation (2.48) in terms of the vertical velocity perturbation. Up to this stage the only approximation made was linearization. In the analytical solutions, we

required upward-propagating waves and concluded that for $\bar{u} > 0$ and $k > 0$, we had to choose the solution with positive vertical wavenumber since that is the one with positive group velocity (see Appendix A).

To derive solutions, we considered a constant background velocity $\bar{u}$ and a background density in the form of exponential function of z (3.2) and (3.3) so that the buoyancy frequency would be constant. We also looked at the special cases where $\frac{d\bar{\rho}}{dz} \ll \bar{\rho}$ (Boussinesq approximation I) and where $\delta = 0$. Other studies such as Sang (1991), Chun and Baik (1998), Holton, Beres and Zhou (2002), Beres (2004) and Beres, Alexander and Holton (2004) considered alternative forms of the equations in which they were written in terms of other dependent variables such as the potential temperature and kinematic pressure. They also linearized the equations but in addition they made the Boussinesq approximation II in which the density is set to a constant ρ_0 everywhere except in the buoyancy term and N^2 is set to a constant. They combined the linearized equation into a single equation analogous to our equation (2.48). This means that the combined equation in each of these studies can be obtained from equation (2.48), but written in different dependent variables, by applying some simplifications. For example, equation (5) in Holton, Beres and Zhou (2002) can be obtained from equation (2.48) if we set $\bar{u} = 0$, apply the Boussinesq approximation II with $\bar{\rho} = \rho_0$ being a constant and set $\delta = 1$.

We also considered a more realistic background density, this density profile gave stable stratification with $N^2 = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}$ being positive in the upper layer and unstable stratification in the lower layer. Other studies replaced $\bar{\rho}(z)$ by a constant state ρ_0 in the equations except the gravity terms and then considered $N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}$ being positive in the whole domain which meant there was no actual convection. This configuration is similar to the steady version of the one-layer model for the gravity waves generated by convection studied in Section 4.2. We derived the steady solution for this configuration and then examined the time-dependent and the nonlinear solutions and carried out numerical simulations.

Sang (1991) used a two-layer model with convection in the lower layer like our model, however he focused on the case where the amplitude of the disturbances in the upper layer decay exponentially with height (as in our Case b in Section 3.3.2). These are trapped disturbances which would not reach higher levels of the atmosphere or have any significant effect on the background flow. In our configuration, we obtained both trapped disturbances (Case b in Section 3.3.2) and oscillatory gravity waves (Case a in Section 3.3.2), but it is the latter that is of interest.

We also studied a one-layer configuration with stable stratification and gravity waves generated by a thermal forcing. This is the configuration studied by Holton, Beres and Zhou (2002), Beres (2004) and Beres, Alexander and Holton (2004) which can be used to represent the deep heating mechanisms for gravity wave generation. Since the one-layer configuration is simpler than the two-layer version, we extended the one-layer investigation to include time-dependence and nonlinearity and examined the interaction of the waves and the background flow.

The gravity wave analytical solutions obtained here could be used as the basis for a gravity wave drag parameterization scheme for use in general circulation models. In parameterizations schemes [e.g., Beres (2005) and Beres et al. (2005)] the gravity waves are generally assumed to take the form of a spectrum of vertical wavenumbers or phase speeds representing a wave packet localized in time. To make such a representation using our solutions we could re-derive our solution assuming that the disturbance is a periodic function of t as well as x ; i.e. using a normal mode form $w = \hat{w}e^{ik(x-ct)}$. This would be given the same steady solutions we obtained but with $\bar{u}$ replaced by $\bar{u} - c$. We could then specify a spectrum of phase speeds c and sum the solution over c . To obtain a more realistic representation we could instead use our time-dependent solution.

Possible further extensions of the present study would be to continue the weakly-nonlinear analysis for gravity waves into the late time regime or to carry out a nonlinear numerical simulations with either gravity waves or convection or both. We could

also examine cases with vertical shear; i.e. with $\bar{u}$ being a function of z .

Appendix A

Group Velocity and Upward Propagation of Gravity Waves

Consider a wave of the form

$$\psi(\mathbf{x}, t) = Ae^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad (\text{A.1})$$

where A is the amplitude of the wave, $\mathbf{k} = (k, l, m)$ is the wavenumber vector (k, l and m are the wavenumbers in the x, y and z directions, respectively) and ω is the frequency. By substituting (A.1) into the governing equation, the partial differential equation which describes wave propagation, we get a relation between the frequency and the wavenumber. This relation is called dispersion relation:

$$G(\omega, \mathbf{k}) = 0.$$

In the case of two dimensions, for example x and z , equation (A.1) can be written as

$$\psi(x, z, t) = Ae^{i(kx + mz - \omega t)}, \quad (\text{A.2})$$

and the dispersion relation takes the form

$$G(\omega, k, m) = 0.$$

If we solve for ω , we obtain

$$\omega = W(k, m).$$

ω may have different solutions, we refer to those solutions as different modes.

The quantity $\theta = kx + mz - \omega t$ is called the phase. The phase velocity $\mathbf{c}$ is given by

$$\mathbf{c} = \left(\frac{\omega k}{k^2 + m^2}, \frac{\omega m}{k^2 + m^2} \right).$$

The phase speeds in the x and z directions, respectively, are given by

$$c_x = \frac{\omega}{k}, \quad c_z = \frac{\omega}{m}.$$

Note that c_x and c_z are not the components of the vector $\mathbf{c}$ (see, for example, Kundu and Cohen (2008): Figure 7.3, Page 218).

The speed and direction of propagation of the wave group is given by the group velocity. In two dimensions (x and z), the group velocity is defined as

$$\mathbf{C}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m} \right). \quad (\text{A.3})$$

For the case of two-dimensional gravity waves, the frequency is given by

$$\omega = \bar{u}k \pm \frac{Nk}{\sqrt{k^2 + m^2}}, \quad (\text{A.4})$$

and so the group velocity is

$$\mathbf{C}_g = (C_{gx}, C_{gz}) = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m} \right), \quad (\text{A.5})$$

where

$$C_{gx} = \bar{u} \pm \frac{m^2 N}{(k^2 + m^2)^{3/2}}, \quad (\text{A.6})$$

$$C_{gz} = \mp \frac{kmN}{(k^2 + m^2)^{3/2}}. \quad (\text{A.7})$$

The phase speeds are

$$c_x = \bar{u} \pm \frac{N}{\sqrt{k^2 + m^2}}, \quad (\text{A.8})$$

$$c_z = \bar{u} \frac{k}{m} \pm \frac{Nk}{m\sqrt{k^2 + m^2}}. \quad (\text{A.9})$$

Consider a solution in the form

$$\hat{w} = A e^{imz} + B e^{-imz},$$

where A and B are constants. We can determine the sign of the group velocity for each of the solutions e^{imz} and e^{-imz} as follows. We combine the z -component of the group velocity C_{gz} that is given in (A.7) and the phase speed in the horizontal direction $c_x = c$ that is given in (A.8) as

$$(\bar{u} - c)^3 = \mp \frac{N^3}{(k^2 + m^2)^{3/2}} = \frac{N^2 C_{gz}}{km}.$$

Then

$$C_{gz} = \frac{km(\bar{u} - c)^3}{N^2}.$$

Now, consider the case where the horizontal wavenumber $k > 0$ and define m so that $\text{sgn}(m) = \text{sgn}(\bar{u} - c)$. Then regardless of the sign of m and $(\bar{u} - c)$, C_{gz} is positive and the solution e^{imz} corresponds to an upward-propagating wave.

On the other hand, the group velocity corresponding to the wavenumber $-m$ is given by

$$C_{gz} = \frac{\partial \omega}{\partial(-m)} = -\frac{km(\bar{u} - c)^3}{N^2}.$$

As before, $m(\bar{u} - c)^3$ is always positive, and so C_{gz} is negative which means that the solution e^{-imz} corresponds to a downward-propagating wave. Thus for $\bar{u} - c > 0$, $k > 0$ and $m > 0$, the solution corresponding to an upward-propagating wave is $w = A e^{ikx} e^{imz} + A^* e^{-ikx} e^{-imz}$ and the solution corresponding to a downward-propagating wave is $w = A e^{ikx} e^{-imz} + A^* e^{-ikx} e^{imz}$.

Appendix B

The Modified Bessel Function

Bessel's equation of order ν and parameter λ takes the form

$$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2)y = 0.$$

The solution of Bessel's equation is given by

$$y = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x),$$

where $J_\nu(\lambda x)$ and $Y_\nu(\lambda x)$ are known as Bessel functions of the first and second kind, respectively, of order ν and parameter λ . They are given by

$$J_\nu(\lambda x) = \sum_{l=0}^{\infty} \frac{(-1)^l (\lambda x)^{\nu+2l}}{2^{\nu+2l} (l)! \Gamma(\nu + l + 1)} \quad (\text{B.1})$$

and

$$Y_\nu(\lambda x) = \frac{\cos \nu\pi J_\nu(\lambda x) - J_{-\nu}(\lambda x)}{\sin \nu\pi}.$$

Modified Bessel's equation of order ν and parameter λ takes the form

$$x^2 y'' + xy' - (\lambda^2 x^2 + \nu^2)y = 0.$$

The solution is

$$y = c_1 I_\nu(\lambda x) + c_2 K_\nu(\lambda x),$$

where

$$I_\nu(\lambda x) = i^{-\nu} J_\nu(i\lambda x)$$

and

$$K_\nu(\lambda x) = \frac{\pi}{2} \frac{I_{-\nu}(\lambda x) - I_\nu(\lambda x)}{\sin \nu \pi}.$$

Now consider the following equation

$$x^2 y'' + xy' - (ix^2 + n^2)y = 0.$$

This equation is the modified Bessel's equation of order n and parameter $\sqrt{i}$ and it has a solution $J_n(\sqrt{-i}x)$ or customarily denoted by $I_n(\sqrt{i}x)$. According to the series expansion (B.1), $J_n(\sqrt{-i}x)$ takes the form

$$\begin{aligned} J_n(\sqrt{-i}x) &= J_n(i^{3/2}x) = \sum_{l=0}^{\infty} \frac{(-1)^l (i^{3/2}x)^{n+2l}}{2^{n+2l}(l)!\Gamma(n+l+1)} \\ &= i^{3n/2} \sum_{l=0}^{\infty} \frac{(-1)^l i^{3l} x^{n+2l}}{2^{n+2l}(l)!\Gamma(n+l+1)}. \end{aligned} \quad (\text{B.2})$$

By separating this series into real and imaginary parts, using the fact that i^{3l} is real for even $l = 2j$ and imaginary for odd $l = 2j + 1$, we get

$$\begin{aligned} J_n(\sqrt{-i}x) &= i^{3n/2} \left[\sum_{j=0}^{\infty} \frac{(-1)^j x^{n+4j}}{2^{n+4j}(2j)!\Gamma(n+2j+1)} + i \sum_{j=0}^{\infty} \frac{(-1)^j x^{n+2+4j}}{2^{n+2+4j}(2j+1)!\Gamma(n+2j+2)} \right] \\ &= i^{3n/2} (\Sigma_R + i\Sigma_I). \end{aligned}$$

Since $i^{3n/2} = (e^{i\pi/2})^{3n/2} = \cos \frac{3n\pi}{4} + i \sin \frac{3n\pi}{4}$, we get

$$J_n(\sqrt{-i}x) = \left(\cos \frac{3n\pi}{4} \Sigma_R - \sin \frac{3n\pi}{4} \Sigma_I \right) + i \left(\cos \frac{3n\pi}{4} \Sigma_I + \sin \frac{3n\pi}{4} \Sigma_R \right). \quad (\text{B.3})$$

$J_n(\sqrt{-i}x)$ thus consists of one real series plus i times a second real series. The series forming the real part of this expression defines the function represented as **ber** _{n} x . The series forming the imaginary part defines the function represented as **bei** _{n} x . The letter *be* suggest the relation between these new functions and Bessel functions

themselves. The terminal letters r and i suggest the real and imaginary. For more details see, for example, Wylie and Barrett (1995). Plots of $|J_1(\sqrt{-ix})|$, $ber_1 x$ and $bei_1 x$ are shown in Figure B.1.

Figure B.1: The modified Bessel function ($J_1(\sqrt{-ix}) = I(\sqrt{ix})$) of order 1 and parameter $\sqrt{i}$.

Appendix C

The Eliassen-Palm Theorem and the Mean Flow Evolution

C.1 Momentum Flux and the Mean Flow

In the nonlinear problem, the mean flow varies with time due to nonlinear interactions with the waves and so the changes in the wave amplitude affect the mean flow. The interaction between the waves with positive and negative wavenumbers ($\pm k$) generates waves with higher wavenumbers ($\pm 2k, \pm 3k, \dots$) and a zero wavenumber component which represents changes in the mean flow. The mean is defined by taking the average over a horizontal wavelength $2\pi/k$:

$$\overline{(\dots)} = \frac{k}{2\pi} \int_0^{\frac{2\pi}{k}} (\dots) dx.$$

In order to obtain an equation that describes the mean flow, we consider the horizontal momentum equation in terms of the total dependent variables (2.36) and we approximate the total density $\varrho = \bar{\rho}(z) + \varepsilon \rho(x, z, t)$ by $\bar{\rho}$:

$$\frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{U}}{\partial z} = -\frac{1}{\bar{\rho}} \frac{\partial \mathcal{P}}{\partial x}. \quad (\text{C.1})$$

Combining this equation with the continuity equation $\frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{W}}{\partial z} = 0$ yields

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial}{\partial x} \mathcal{U}^2 + \frac{\partial}{\partial z} \mathcal{U} \mathcal{W} = -\frac{1}{\bar{\rho}} \frac{\partial \mathcal{P}}{\partial x}. \quad (\text{C.2})$$

The term $\mathcal{U} \mathcal{W} = -\Psi_x \Psi_z$ is the vertical component of the flux of horizontal momentum, where Ψ is the total streamfunction and the subscripts denote the partial derivatives.

Let

$$\begin{aligned} \mathcal{U} &= \bar{u}(z) + \varepsilon u(x, z, t), \\ \mathcal{W} &= \varepsilon w(x, z, t). \end{aligned}$$

By substituting the total velocities into equation (C.2) and taking the horizontal average over a wavelength $2\pi/k$, we get

$$\frac{\partial \bar{u}_0}{\partial t} + \varepsilon^2 \frac{\partial}{\partial z} \overline{uw} = 0. \quad (\text{C.3})$$

The first term is the rate of change of the mean flow with time resulting from the nonlinear interactions with the initial mean flow. Equation (C.3) can be written in terms of the momentum flux as

$$\frac{\partial \bar{u}_0}{\partial t} = -\varepsilon^2 \frac{\partial \bar{F}}{\partial z}, \quad (\text{C.4})$$

where $\bar{F} = \overline{uw} = -\overline{\psi_x \psi_z}$ is the horizontal average of the vertical flux of horizontal momentum and is given by

$$\bar{F} = \frac{k}{2\pi} \int_0^{\frac{2\pi}{k}} uw \, dx = -\frac{k}{2\pi} \int_0^{\frac{2\pi}{k}} \psi_x \psi_z \, dx. \quad (\text{C.5})$$

The divergence of the mean momentum flux $\frac{\partial \bar{F}}{\partial z}$ is called the wave drag.

For the time-dependent problem where the amplitude of the waves depends on z and t , the streamfunction can be written as

$$\psi(x, z, t) = \phi(z, t)e^{ikx} + \phi^*(z, t)e^{-ikx}.$$

Thus

$$\begin{aligned}\psi_x \psi_z &= ik(\phi e^{ikx} - \phi^* e^{-ikx})(\phi_z e^{ikx} + \phi_z^* e^{-ikx}) \\ &= ik(\phi \phi_z e^{2ikx} - \phi^* \phi_z^* e^{-2ikx} + \phi \phi_z^* - \phi^* \phi_z).\end{aligned}\tag{C.6}$$

Taking the average over a horizontal wavelength $2\pi/k$, we get the mean momentum flux

$$\bar{F} = -\overline{\psi_x \psi_z} = -ik(\phi \phi_z^* - \phi^* \phi_z),\tag{C.7}$$

or

$$\bar{F} = -2k\text{Im}(\phi^* \phi_z).\tag{C.8}$$

The momentum flux divergence (the drag) is given by

$$\frac{\partial \bar{F}}{\partial z} = -ik(\phi \phi_{zz}^* - \phi^* \phi_{zz}) = -2k\text{Im}(\phi^* \phi_{zz}).\tag{C.9}$$

C.2 The Eliassen-Palm Theorem

In the special case where the wave amplitude is steady and an oscillatory function of z ($\phi = e^{imz}$) and the background velocity is constant, the following theorem holds [Eliassen and Palm (1961)].

The Eliassen-Palm theorem:

For linear waves with a constant background velocity ($\bar{u} = \text{constant}$), the momentum flux is independent of height.

Proof:

For constant $\bar{u}$, we can represent the waves as periodic functions of x , z and t :

$$\psi(x, z, t) = A e^{i(kx + mz - \omega t)} + A^* e^{-i(kx + mz - \omega t)}.$$

Then

$$\psi_x \psi_z = -km \left(A^2 e^{2i(kx + mz - \omega t)} + A^{*2} e^{-2i(kx + mz - \omega t)} - 2AA^* \right).$$

Taking the average over a horizontal wavelength $2\pi/k$, we get the mean momentum flux

$$\bar{F} = -\overline{\psi_x \psi_z} = -2km|A|^2. \quad (\text{C.10})$$

This is called the Eliassen-Palm flux. Equation (C.10) means that the mean momentum flux is constant and the mean momentum flux divergence is zero at all levels.

The momentum flux would vary with height in a more general configuration, for example if $\bar{u}$ is a function of z . We also see (Section 5.2.3) that the theorem does not hold in the case we consider in this thesis, where $\bar{u}$ is constant but the waves are generated by a thermal forcing and do not take the simple form $\phi = e^{imz}$. Our configuration thus describes a situation where there is a nonzero momentum flux divergence although $\bar{u}$ is constant.

Appendix D

The Lorenz Model for Atmospheric Convection

D.1 Governing Equations

In this appendix, we describe the Lorenz model for atmospheric convection. Lorenz (1963) introduced a system of three ordinary differential equations for a two-dimensional configuration governing convection in a layer of unstable stratification resulting from a constant temperature difference between the upper and lower boundary. The solution of these equations affords a simple example of deterministic non-periodic flow. We will use this solution to interpret our time-dependent convection solution in Section 6.4 and, in particular, to comment on the role that viscosity and heat conduction play in keeping the convection solution stable.

In 1916 Rayleigh studied the flow occurring in a layer of fluid of uniform depth h , when the temperature difference between the upper and lower surfaces is maintained at a constant value ΔT . Such a system possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth.

The equations governing convection in a viscous, heat-conducting fluid are

$$\nabla \cdot \mathbf{u} = 0, \quad (\text{D.1})$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0} \nabla p + \frac{\rho}{\rho_0} \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad (\text{D.2})$$

$$\frac{DT}{Dt} = \kappa \nabla^2 T, \quad (\text{D.3})$$

where ν is the viscosity coefficient and κ is the heat conduction coefficient. Lorenz used the streamfunction-vorticity formulation of these equations derived in Saltzman (1962) to study finite-amplitude convection. In this appendix we present these equations using Saltzman's notation. We follow the Boussinesq formulation with reference density, temperature and pressure ρ_0 , T_0 and p_0 , as done in Section 2.3, and define

$$P = \frac{1}{\rho_0}(p - p_0).$$

Making use of equation (2.17), we rewrite the momentum equation (D.2) as

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} &= -\frac{1}{\rho_0} \nabla(\rho_0 P + p_0) + \left(1 - \frac{T'}{T_0}\right) \mathbf{g} + \nu \nabla^2 \mathbf{u} \\ &= -\nabla P - \frac{1}{\rho_0} \nabla p_0 + \mathbf{g} - \frac{T'}{T_0} \mathbf{g} + \nu \nabla^2 \mathbf{u} \\ &= -\nabla P - \mathbf{g} + \mathbf{g} - \frac{T'}{T_0} \mathbf{g} + \nu \nabla^2 \mathbf{u}. \end{aligned}$$

Thus the momentum equation becomes

$$\frac{D\mathbf{u}}{Dt} = -\nabla P - \alpha T' \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad (\text{D.4})$$

where $\alpha = \frac{1}{T_0}$ is the thermal expansion coefficient.

In a two-dimensional configuration in terms of x and z , equations (D.1)-(D.3) can be written as

$$\frac{\partial \mathcal{U}}{\partial x} + \frac{\partial \mathcal{W}}{\partial z} = 0, \quad (\text{D.5})$$

$$\frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{U}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{U}}{\partial z} = -\frac{\partial P}{\partial x} + \nu \nabla^2 \mathcal{U}, \quad (\text{D.6})$$

$$\frac{\partial \mathcal{W}}{\partial t} + \mathcal{U} \frac{\partial \mathcal{W}}{\partial x} + \mathcal{W} \frac{\partial \mathcal{W}}{\partial z} = -\frac{\partial P}{\partial z} + \alpha T g + \nu \nabla^2 \mathcal{W}, \quad (\text{D.7})$$

$$\frac{\partial T}{\partial t} + \mathcal{U} \frac{\partial T}{\partial x} + \mathcal{W} \frac{\partial T}{\partial z} = \kappa \nabla^2 T. \quad (\text{D.8})$$

As before, we define a steamfunction Ψ by

$$\mathcal{U} = -\frac{\partial \Psi}{\partial z} \quad \text{and} \quad \mathcal{W} = \frac{\partial \Psi}{\partial x}, \quad (\text{D.9})$$

and the two momentum equations can then be combined to give

$$\frac{\partial}{\partial t}(\nabla^2 \Psi) = -\frac{\partial \Psi}{\partial x} \frac{\partial}{\partial z}(\nabla^2 \Psi) + \frac{\partial \Psi}{\partial z} \frac{\partial}{\partial x}(\nabla^2 \Psi) + \nu \nabla^4 \Psi + \alpha g \frac{\partial T}{\partial x}. \quad (\text{D.10})$$

We consider a layer of fluid in a two-dimensional region $0 < z < h$ with zero background flow velocity. This means that the total quantities Ψ , $\mathcal{U}$ and $\mathcal{W}$ are equal to the corresponding perturbation quantities. It is assumed however that there is a background temperature profile so the total temperature T is equal to the sum of the background temperature and a perturbation. Following the notation of Saltzman (1962), we write

$$T(x, z, t) = \bar{T}(z, t) + T'(x, z, t), \quad (\text{D.11})$$

where $\bar{T}(z, t)$ is the average in the x direction and $T'(x, z, t)$ is the departure from $\bar{T}$. In addition, $\bar{T}(z, t)$ is expressed as the sum of a linear variation between the upper and lower boundaries and a departure from this linear variation

$$\bar{T}(z, t) = \left[\bar{T}(0, t) - \frac{\Delta T}{h} z \right] + \bar{T}''(z, t), \quad (\text{D.12})$$

where h is the height of the fluid, $\Delta T = \bar{T}(0) - \bar{T}(h)$ and $\bar{T}''(z, t)$ is the departure from the linear variation. It is assumed that $\Delta T > 0$ so that there is unstable stratification. The reason for writing $\bar{T}(z, t)$ in this form is so that the linear terms in the temperature equation will have constant coefficients which will make it easier to express the solution in terms of Fourier modes.

We now define

$$\theta(x, z, t) = \bar{T}''(z, t) + T'(x, z, t),$$

so the total temperature is written as

$$T(x, z, t) = \left[\bar{T}(0, t) - \frac{\Delta T}{h} z \right] + \theta(x, z, t). \quad (\text{D.13})$$

Substituting equation (D.13) into equation (D.10) gives

$$\frac{\partial}{\partial t}(\nabla^2 \Psi) + \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial z}(\nabla^2 \Psi) - \frac{\partial \Psi}{\partial z} \frac{\partial}{\partial x}(\nabla^2 \Psi) - \nu \nabla^4 \Psi = \alpha g \frac{\partial \theta}{\partial x}, \quad (\text{D.14})$$

and substituting equation (D.13) into equation (D.8), and making use of (D.9) gives

$$\frac{\partial}{\partial t} \bar{T}(0, t) + \frac{\partial \theta}{\partial t} - \frac{\partial \Psi}{\partial z} \frac{\partial \theta}{\partial x} + \frac{\partial \Psi}{\partial x} \left[-\frac{\Delta T}{h} + \frac{\partial \theta}{\partial z} \right] = \kappa \nabla^2 \theta.$$

Assume that the temperature is kept constant at the upper and lower boundaries i.e.,

$\frac{\partial}{\partial t} \bar{T}(0, t) = \frac{\partial}{\partial t} \bar{T}(h, t) = 0$, gives

$$\frac{\partial \theta}{\partial t} - \frac{\partial \Psi}{\partial z} \frac{\partial \theta}{\partial x} + \frac{\partial \Psi}{\partial x} \frac{\partial \theta}{\partial z} - \frac{\partial \Psi}{\partial x} \frac{\Delta T}{h} = \kappa \nabla^2 \theta. \quad (\text{D.15})$$

Equations (D.14) and (D.15) describe the evolution of the streamfunction and temperature perturbations in the $x - z$ plane. Both equations are nonlinear because of the products in the second and third terms. Note however that, the coefficients of the linear terms (αg and $\frac{\Delta T}{h}$) are constant which makes it easier to express the solutions in terms of Fourier modes. We note that $\frac{g \alpha \Delta T}{h}$ in this model is analogous to $\frac{g}{\rho} \frac{d\bar{\rho}}{dz} = \frac{g}{H} = N_0^2$ in our model.

D.2 Derivation of the Lorenz Equations

We now derive the Lorenz system of ordinary differential equations using equations (D.14) and (D.15) showing the details which are omitted in Lorenz's paper. Rayleigh found that fields of motion of the form [Lorenz (1963)]

$$\begin{aligned} \Psi &= \Psi_0 \sin\left(\frac{\pi a x}{h}\right) \sin\left(\frac{\pi z}{h}\right), \\ \theta &= \theta_0 \cos\left(\frac{\pi a x}{h}\right) \sin\left(\frac{\pi z}{h}\right), \end{aligned}$$

develop if the Rayleigh number

$$R_a = \frac{g\alpha h^3 \Delta T}{\nu \kappa}$$

exceeds a critical value

$$R_c = \frac{\pi^4(1+a^2)^3}{a^2}.$$

The minimum value of R_c , namely $\frac{27\pi^4}{4}$, occurs when $a^2 = \frac{1}{2}$. The constant a is the vertical to horizontal aspect ratio of the convective cells.

In order to obtain a system of ordinary differential equations, Lorenz used the following expressions

$$\Psi = \frac{\kappa(1+a^2)\sqrt{2}}{a} X \sin\left(\frac{\pi ax}{h}\right) \sin\left(\frac{\pi z}{h}\right), \quad (\text{D.16})$$

$$\theta = \frac{R_c \Delta T}{\pi R_a} \left[\sqrt{2} Y \cos\left(\frac{\pi ax}{h}\right) \sin\left(\frac{\pi z}{h}\right) - Z \sin\left(\frac{2\pi z}{h}\right) \right], \quad (\text{D.17})$$

where X , Y and Z are functions of time only.

We then set

$$L = \frac{\kappa(1+a^2)\sqrt{2}}{a}, \quad M = \frac{\pi a}{h}, \quad N = \frac{\pi}{h} \quad \text{and} \quad P = \frac{R_c \Delta T}{\pi R_a}.$$

It is then easy to verify the following relations

$$M^2 + N^2 = \frac{\pi^2}{h^2}(1+a^2), \quad (\text{D.18})$$

$$\frac{\sqrt{2} M g P \alpha}{L(M^2 + N^2)^2} = \nu, \quad (\text{D.19})$$

$$\frac{LMN}{\sqrt{2}\kappa(M^2 + N^2)} = 1, \quad (\text{D.20})$$

$$\frac{\Delta T L M}{\sqrt{2}\kappa(M^2 + N^2) P h} = \frac{R_a}{R_c}, \quad (\text{D.21})$$

$$\frac{4N^2}{M^2 + N^2} = \frac{4}{1+a^2}. \quad (\text{D.22})$$

In terms of L , M , N and P , equations (D.16) and (D.17) become

$$\Psi = LX \sin(Mx) \sin(Nz), \quad (\text{D.23})$$

$$\theta = P[\sqrt{2}Y \cos Mx \sin Nz - Z \sin 2Nz]. \quad (\text{D.24})$$

We differentiate these and substitute into equations (D.14) and (D.15) and make use of the trigonometric identities

$$\begin{aligned} \sin(2Nz) &= 2 \sin(Nz) \cos(Nz), \\ \cos^2(Mx) + \sin^2(Mx) &= 1, \\ \sin(Nz) \cos(2Nz) &= \frac{1}{2}[\sin 3Nz - \sin Nz]. \end{aligned}$$

Equation (D.14) becomes

$$\frac{dX}{dt} = -\nu(M^2 + N^2)X + \frac{\sqrt{2}MPg\alpha}{L(M^2 + N^2)}Y, \quad (\text{D.25})$$

and equation (D.15) becomes

$$\begin{aligned} P \left[\sqrt{2} \frac{dY}{dt} \cos(Mx) \sin(Nz) - \frac{dZ}{dt} \sin(2Nz) \right] = \\ - (LMX \cos(Mx) \sin(Nz))(P[\sqrt{2}NY \cos(Mx) \cos(Nz) - 2NZ \cos(2Nz)]) \\ + (LN X \sin(Mx) \cos(Nz))(P[-\sqrt{2}Y M \sin(Mx) \sin(Nz)]) \\ + \frac{\Delta T}{h} [LMX \cos(Mx) \sin(Nz)] \\ + \kappa [-P\sqrt{2}(M^2 + N^2)Y \cos(Mx) \sin(Nz) + 4N^2 PZ \sin(2Nz)], \end{aligned}$$

which can be simplified to

$$\begin{aligned} P \left[\sqrt{2} \frac{dY}{dt} \cos(Mx) \sin(Nz) - \frac{dZ}{dt} \sin(2Nz) \right] = \\ - \frac{\sqrt{2}}{2} PLMNXY \sin(2Nz) + PLMN XZ \cos(Mx) [\sin(3Nz) - \sin(Nz)] \\ + \frac{\Delta T}{h} [LMX \cos(Mx) \sin(Nz)] \\ + \kappa [-P\sqrt{2}(M^2 + N^2)Y \cos(Mx) \sin(Nz) + 4N^2 PZ \sin(2Nz)]. \end{aligned}$$

Now equating the coefficients of $\cos(Mx) \sin(Nz)$ and $\sin(2Nz)$ for both sides and neglecting higher modes, we get

$$\sqrt{2}P \frac{dY}{dt} = -PLMN XZ + \frac{\Delta T}{h} LMX - \sqrt{2}\kappa P(M^2 + N^2)Y, \quad (\text{D.26})$$

$$-P \frac{dZ}{dt} = -\frac{\sqrt{2}}{2} PLMNXY + 4\kappa N^2 PZ. \quad (\text{D.27})$$

We then define a new time variable by $\tau = \frac{\pi^2(1+a^2)\kappa}{h^2}t$. Then

$$\frac{d}{dt} = \frac{\pi^2(1+a^2)\kappa}{h^2} \frac{d}{d\tau} = \kappa(M^2 + N^2) \frac{d}{d\tau}.$$

Equation (D.25) then becomes

$$\frac{dX}{d\tau} = -\frac{\nu}{\kappa}X + \frac{\sqrt{2}MPg\alpha}{L(M^2 + N^2)^2\kappa}Y,$$

which can be simplified by using equation (D.19) to

$$\frac{dX}{d\tau} = -\frac{\nu}{\kappa}X + \frac{\nu}{\kappa}Y.$$

The ratio $\sigma = \nu/\kappa$ is called the Prandtl number. The first equation in the Lorenz system is

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y. \quad (\text{D.28})$$

Similarly, equation (D.26) becomes

$$\frac{dY}{d\tau} = -\frac{LMN}{\sqrt{2}\kappa(M^2 + N^2)}XZ + \frac{\Delta TLM}{h\sqrt{2}P\kappa(M^2 + N^2)}X - Y,$$

which can be simplified by using equations (D.20) and (D.21) to give

$$\frac{dY}{d\tau} = -XZ + \frac{R_a}{R_c}X - Y.$$

The ratio $r = R_a/R_c$ is the Rayleigh number scaled with respect to the critical Rayleigh number. The second equation in the Lorenz system is

$$\frac{dY}{d\tau} = -XZ + rX - Y. \quad (\text{D.29})$$

Lastly, equation (D.27) becomes

$$\frac{dZ}{d\tau} = \frac{LMN}{\sqrt{2}\kappa(M^2 + N^2)}XY - \frac{4N^2}{(M^2 + N^2)}Z.$$

Equations (D.20) and (D.22) simplify the above equation to give the third equation in Lorenz system

$$\frac{dZ}{d\tau} = XY - bZ, \quad (\text{D.30})$$

where $b = \frac{4}{1+a^2}$.

D.3 Fixed Points and Stability

We have derived the Lorenz system of nonlinear differential equations (D.28), (D.29) and (D.30):

$$\begin{aligned}\dot{X} &= \sigma(Y - X), \\ \dot{Y} &= rX - Y - XZ, \\ \dot{Z} &= XY - bZ,\end{aligned}\tag{D.31}$$

where X represents the amplitude of the streamfunction, Y and Z are the amplitudes of the two components of the temperature perturbation and σ , r , $b > 0$ are parameters. σ is the Prandtl number and r is the Rayleigh number scaled with respect to the critical Rayleigh number. The dots represent derivatives with respect to the scaled time variable τ .

In this section we determine the fixed points of the Lorenz model and study their stability. A fixed point or a critical point of a system of ordinary differential equations such as (D.31) occurs where the first derivative $\dot{X}$, $\dot{Y}$ and $\dot{Z}$ are zero. This corresponds to a rest or equilibrium state [see, for example, Perko (2001)]. In the nonlinear Lorenz system (D.31) $\dot{X} = 0$ implies that $Y = X$, $\dot{Y} = 0$ implies that either $X = 0$ or $Z = r - 1$, and $\dot{Z} = 0$ implies that $X^2 = bZ$. If $X^* = 0$, then $Y^* = 0$ and $Z^* = 0$, so the origin is a fixed point. If $Z^* = r - 1$, then $X^* = Y^* = \pm\sqrt{b(r - 1)}$. If $r > 1$ there is a symmetric pair of fixed points: $C^+ = (\sqrt{b(r - 1)}, \sqrt{b(r - 1)}, r - 1)$ and $C^- = (-\sqrt{b(r - 1)}, -\sqrt{b(r - 1)}, r - 1)$ in addition to the origin. If $r \leq 1$ the origin is the only fixed point.

Stability of the Fixed Points of the Linear System

By omitting the nonlinear terms XY and XZ in (D.31), we obtain the linear equations

$$\begin{aligned}\dot{X} &= \sigma(Y - X), \\ \dot{Y} &= rX - Y, \\ \dot{Z} &= -bZ.\end{aligned}\tag{D.32}$$

The origin is a fixed point of this linear system. If $r \neq 1$, then the origin is the only fixed point so the zero solution is the only equilibrium solution. If $r = 1$, then every point that satisfies $X = Y$ and $Z = 0$ is a fixed point, so there are infinitely many fixed points lying on the line $X = Y$ in $X - Y$ phase space.

In the linear system (D.32), the equation for Z is decoupled from the other two equations. The solution is

$$Z(t) = e^{-bt} Z_0,$$

and so $Z(t) \rightarrow 0$ as $t \rightarrow \infty$.

The other two directions are governed by the system

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.\tag{D.33}$$

The eigenvalues of the matrix are the roots of

$$\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r) = 0,$$

which gives

$$\lambda^\pm = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}}{2}.$$

Let

$$\mu = \sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}.$$

Then

$$\lambda^\pm = \frac{-(\sigma + 1) \pm \mu}{2}.$$

If $r < 1$, then $(\sigma + 1)^2 + 4\sigma(r - 1) = (\sigma - 1)^2 + 4\sigma r > 0$ and $\lambda^\pm$ are both negative real numbers and so the zero solution is asymptotically stable. This means that for all solutions which start near the origin remain near the origin for all time and furthermore tend towards it as $t \rightarrow \infty$. Since the origin is the only fixed point all solutions approach the origin. The general solution is a linear combination of exponentially-decaying functions $e^{\lambda^\pm t}$.

If $r = 1$, then the eigenvalues are $\lambda^+ = 0$ and $\lambda^- = -(\sigma + 1)$. The eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\sigma \\ 1 \end{pmatrix}$. Thus a fundamental matrix for the system is

$$\Phi(t) = \begin{pmatrix} 1 & -\sigma e^{-(\sigma+1)t} \\ 1 & e^{-(\sigma+1)t} \end{pmatrix},$$

and the general solution is

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \vec{\Phi} \vec{c} = \begin{pmatrix} c_1 - \sigma c_2 e^{-(\sigma+1)t} \\ c_1 + c_2 e^{-(\sigma+1)t} \end{pmatrix},$$

where $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and the constants c_1 and c_2 are determined by the initial conditions. We observe that $X(t) \rightarrow c_1$ and $Y(t) \rightarrow c_1$ as $t \rightarrow \infty$, so every solution approaches an equilibrium solution $X = Y = c_1$ as $t \rightarrow \infty$. These equilibrium solutions correspond to stable fixed points $(X, Y, 0)$ and $X = Y$. So with $r = 1$, we obtain convective cells with finite amplitude.

If $r > 1$, then the eigenvalues are both real numbers and $\lambda^+ > 0 > \lambda^-$ and the origin is a saddle point. This means that some solutions approach the zero equilibrium solution as $t \rightarrow \infty$ and others diverge to infinity as $t \rightarrow \infty$. The behaviour of a solution depends on its initial condition. We observe that the eigenvectors of the matrix in (D.33) corresponding to the eigenvalues $\lambda^\pm$ are

$$\vec{V}^\pm = \begin{pmatrix} \sigma \\ \frac{\sigma-1 \pm \mu}{2} \end{pmatrix}.$$

Thus a fundamental matrix for the system is

$$\Phi(t) = \begin{pmatrix} e^{\lambda^+ t} \vec{V}^+ & e^{\lambda^- t} \vec{V}^- \end{pmatrix},$$

and the general solution is

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \vec{\Phi} \vec{d} = \begin{pmatrix} d_1 \sigma e^{\frac{-(\sigma+1)+\mu}{2}t} + d_2 \sigma e^{\frac{-(\sigma+1)-\mu}{2}t} \\ d_1 \frac{\sigma-1+\mu}{2} e^{\frac{-(\sigma+1)+\mu}{2}t} + d_2 \frac{\sigma-1-\mu}{2} e^{\frac{-(\sigma+1)-\mu}{2}t} \end{pmatrix},$$

where $\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ and the constants d_1 and d_2 are determined by the initial conditions. We note that the solution is bounded as $t \rightarrow \infty$ if $d_1 = 0$. In that case $X(t) = d_2 \sigma e^{\frac{-(\sigma+1)-\mu}{2}t}$ and $Y(t) = d_2 \frac{\sigma-1-\mu}{2} e^{\frac{-(\sigma+1)-\mu}{2}t}$. So solutions that start with initial conditions given by $Y(0) = \frac{\sigma-1-\mu}{2\sigma} X(0)$ approach the zero equilibrium solution. In the two-dimensional phase portrait for the system (D.33) in $X - Y$ space, trajectories that start along the line $Y = \frac{\sigma-1-\mu}{2\sigma} X$ approach the origin along that line. All other trajectories are repelled from the origin.

We note also that as $r \rightarrow \infty$ the solution becomes infinite. This corresponds to setting the viscosity and heat conduction to zero in the governing linear equations.

Nonlinear Stability of the Origin

The following method due to Liapunov can be used for determining the stability of the origin for the nonlinear Lorenz system [Strogatz (1994)].

Consider a system $\dot{\mathbf{X}} = f(\mathbf{X})$ with a fixed point at $\mathbf{X}^*$. Suppose that we can find a Liapunov function, i.e., a continuously differentiable, real valued function $V(\mathbf{X})$ with the following properties:

1. $V(\mathbf{X}) > 0$ for all $\mathbf{X} \neq \mathbf{X}^*$, and $V(\mathbf{X}^*) = 0$ (we say that V is positive definite).
2. $\dot{V} = (\nabla f) \cdot f < 0$ for all $\mathbf{X} \neq \mathbf{X}^*$.

Then $\mathbf{X}^*$ is globally asymptotically stable; for all initial conditions, $\mathbf{X}(t) \rightarrow \mathbf{X}^*$ as $t \rightarrow \infty$.

Now, we apply the above method to the Lorenz system (D.31). Consider the Liapunov function

$$V(X, Y, Z) = \frac{1}{\sigma} X^2 + Y^2 + Z^2.$$

This is a smooth, positive definite function, $V(X, Y, Z) > 0$ if $(X, Y, Z) \neq (0, 0, 0)$ and

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial X} \frac{dX}{dt} + \frac{\partial V}{\partial Y} \frac{dY}{dt} + \frac{\partial V}{\partial Z} \frac{dZ}{dt} \\ &= \frac{2}{\sigma} X(\sigma(Y - X)) + 2Y(rX - Y - XZ) + 2Z(XY - bZ) \\ &= 2X(Y - X) + 2Y(rX - Y - XZ) + 2Z(XY - bZ). \end{aligned}$$

So

$$\begin{aligned} \frac{1}{2} \dot{V} &= (1 + r)XY - X^2 - Y^2 - bZ^2 \\ &= -[X^2 - (1 + r)XY] - Y^2 - bZ^2. \end{aligned}$$

By completing the square in the first two terms, we get

$$\begin{aligned} \frac{1}{2} \dot{V} &= - \left[\left(X - \frac{(1 + r)Y}{2} \right)^2 - \left(\frac{1 + r}{2} \right)^2 Y^2 \right] - Y^2 - bZ^2 \\ &= - \left[X - \frac{1 + r}{2} Y \right]^2 - \left[1 - \left(\frac{1 + r}{2} \right)^2 \right] Y^2 - bZ^2. \end{aligned}$$

If $r < 1$, then $\dot{V} < 0$ for all $(X, Y, Z) \neq (0, 0, 0)$. Note that $\dot{V} = 0$ iff $(X, Y, Z) = (0, 0, 0)$ (since $b \neq 0$ and $1 - \left(\frac{1+r}{2}\right)^2 \neq 0$ as $r < 1$).

Thus the origin is globally asymptotically stable for $r < 1$. This means that for $r < 1$ every solution $(X(t), Y(t), Z(t))$ of the nonlinear system (D.31) for a given set of initial conditions approaches the zero equilibrium solution.

Stability of the Fixed Points C^+ and C^-

Before studying the stability of the fixed points $C^\pm$, we define the stability of a hyperbolic fixed point [Strogatz (1994)]. Consider the nonlinear system $\dot{\mathbf{X}} = f(\mathbf{X})$, the fixed point $\mathbf{X}^*$ of this system ($f(\mathbf{X}^*) = 0$) is called a hyperbolic point if none of the eigenvalues of the Jacobian matrix $Df(\mathbf{X}^*)$ has zero real part. The stability of the hyperbolic fixed point $\mathbf{X}^*$ is determined by the sign of the real part of the eigenvalues of $Df(\mathbf{X}^*)$; If the eigenvalues of the Jacobian $Df(\mathbf{X}^*)$ all have negative real part, then $\mathbf{X}^*$ is asymptotically stable. If some eigenvalues of $Df(\mathbf{X}^*)$ have positive real part, then $\mathbf{X}^*$ is an unstable fixed point.

The Jacobian matrix for the Lorenz system is given by

$$Df = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z & -1 & -X \\ Y & X & -b \end{pmatrix}.$$

Let $s = \pm\sqrt{b(r-1)}$, then $C^\pm = (s, s, r-1)$ and the Jacobian matrix becomes

$$Df(C^\pm) = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -s \\ s & s & -b \end{pmatrix}.$$

The eigenvalues of this matrix are given by the characteristic equation

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (b + \sigma b + s^2)\lambda + 2s^2\sigma = 0.$$

But $s^2 = b(r-1)$, then

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2b\sigma(r-1) = 0. \quad (\text{D.34})$$

Let

$$A = \sigma + b + 1, \quad (\text{D.35})$$

$$B = b(\sigma + r), \quad (\text{D.36})$$

$$C = 2b\sigma(r-1). \quad (\text{D.37})$$

Equation (D.34) becomes

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \quad (\text{D.38})$$

where A, B and C are positive real numbers (as C^+ and C^- exist only if $r > 1$).

Such an equation has either:

- (a) three real roots or
- (b) one real root and two complex conjugate roots.

Rearranging equation (D.38) gives

$$\lambda(\lambda^2 + B) = -A\lambda^2 - C < 0.$$

Then if λ is real, then it is negative.

Case(a): all λ are real, then they are negative and in this case the origin is stable.

Case(b): let λ_1 be the real root, then it is negative. Let $\lambda_{2,3} = \alpha \pm i\beta$ be the complex conjugate roots (so instability occurs if $\alpha > 0$).

Now the characteristic equation of the Jacobian matrix is

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0,$$

$$\lambda^3 + (-\lambda_1 - \lambda_2 - \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0.$$

Comparing with equation (D.38), we get

$$A = -\lambda_1 - \lambda_2 - \lambda_3,$$

$$B = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$

$$C = -\lambda_1\lambda_2\lambda_3.$$

Then

$$A = -\lambda_1 - 2\alpha,$$

$$\begin{aligned}
B &= \lambda_1(\lambda_2 + \lambda_3) + \lambda_2\lambda_3 = 2\lambda_1\alpha + \alpha^2 + \beta^2, \\
C &= -\lambda_1(\alpha^2 + \beta^2).
\end{aligned}$$

We observe that

$$\begin{aligned}
C - AB &= -\lambda_1(\alpha^2 + \beta^2) + [\lambda_1 + 2\alpha][2\lambda_1\alpha + \alpha^2 + \beta^2] \\
&= 2\lambda_1^2\alpha + 4\alpha^2\lambda_1 + 2\alpha(\alpha^2 + \beta^2) \\
&= 2\alpha[(\lambda_1 + \alpha)^2 + \beta^2].
\end{aligned}$$

The fixed points C^+ and C^- are stable if $\alpha < 0$. The stability condition is $C - AB < 0$, this can be written using (D.35)-(D.37) as

$$2b\sigma(r - 1) - (\sigma + b + 1)(b(\sigma + r)) < 0.$$

This is equivalent to

$$(\sigma - b - 1)r < \sigma^2 + 3\sigma + \sigma b.$$

If $\sigma - b - 1 < 0$, the fixed points $C^\pm$ are stable for $1 < r < \infty$. If $\sigma - b - 1 > 0$, the fixed points $C^\pm$ are stable for $1 < r < r_h = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$. With the choice of values of the constants used by Saltzman, $\sigma = 10$, $a^2 = \frac{1}{2}$ and $b = \frac{8}{3}$, the value of $r_h = 24.74$.

The stability for both the nonlinear and the linear systems indicates that for relatively small values of the Rayleigh number ($r < 1$) the solutions approach zero but for large values of Rayleigh number ($r > 1$) the behaviour of the solution depends on the the initial conditions. Numerical solutions of the nonlinear system show that the solution settles into an irregular oscillation that persists as $t \rightarrow \infty$ but never repeats exactly. The motion is called aperiodic. This system is a relatively simple nonlinear time-dependent model that is sometimes used for studying convection in a viscous heat-conducting atmosphere.

Appendix E

Some Numerical Tests

E.1 Gravity Wave Solution with Zero Upper Boundary Condition

In the gravity wave simulations described in Section 6.3 it was necessary to apply a nonreflecting boundary condition or radiation condition in order to allow the waves to propagate out of the computational domain. In this subsection, we demonstrate the importance of applying the radiation condition. We present some test simulations with zero boundary conditions

$$\hat{\psi}(k, z_2, t) = \hat{\zeta}(k, z_2, t) = \hat{\rho}(k, z_2, t) = 0, \quad (\text{E.1})$$

instead of the radiation condition. The simulations are carried out using the same parameters that we used in Section 6.3 and we show the solution only for the case with $b = 0.7$. Figure E.1 shows contour plots of the linear simulation of the vertical velocity perturbation $w(x, z, t)$ of the gravity waves propagation obtained at $t = 2$, $t = 10$, $t = 30$ and $t = 50$.

We observe that the results are not accurate. The closed circular contours seen in Figure E.1 (b),(c) and (d) represent waves reflected from the upper boundary and

superimposed on the upward propagating waves. Hence, the radiation condition is needed in order to obtain accurate results.

Figure E.1: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.7$. Plots are obtained at (a) $t = 2$, (b) $t = 10$, (c) $t = 30$ and (d) $t = 50$. Zero condition is applied at the upper boundary.

E.2 Gravity Wave Solution obtained with $b - \frac{1}{H} < 0$

In order to get a bounded solution for the gravity wave simulations described in Section 6.3, we must have $b - \frac{1}{H} \geq 0$. This means that the thermal forcing must decay at the same rate as or faster than the background density. Breaking this condition yields a solution that is unbounded as $t \rightarrow \infty$. We show the result of a numerical simulation carried out using the same parameters that we used in Section 6.3 in the vertical domain $0 < z < 40$ with $b = 0.1$ and $H = 5$. Contour plots of the linear simulation of the vertical velocity perturbation $w(x, z, t)$ obtained at $t = 2$ and $t = 10$ are shown in Figure E.2. Figure E.3 shows the absolute value of the amplitude of the vertical velocity perturbation $|\hat{w}(z, t)|$ as a function of t obtained at $z = 5$ and $z = 10$. In Figure E.4, we plot the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ as a function of z at early time $t = 0.1$. We see that the amplitude increases rapidly with t and with z and hence we conclude that in order to have a bounded solution, we must have $b - \frac{1}{H} \geq 0$.

Figure E.2: Numerical simulations: contour plots of the vertical velocity perturbation $w(x, z, t)$ of the gravity waves generated by a thermal forcing centered at $z = 0$ with $b = 0.1$ and $H = 5$. Plots are obtained at (a) $t = 2$ and (b) $t = 10$. A radiation condition is applied at the upper boundary.

Figure E.3: Numerical simulations: the absolute value of the amplitude of the vertical velocity perturbation $|\hat{w}(z, t)|$ of the gravity waves as a function of t generated by a thermal forcing centered at $z = 0$ with $b = 0.1$ and $H = 5$. Plots are obtained at (a) $z = 5$ and (b) $z = 10$. A radiation condition is applied at the upper boundary.

Figure E.4: Numerical simulations: the amplitude of the vertical velocity perturbation $\hat{w}(z, t)$ of the gravity waves as a function of z at $t = 0.1$ generated by a thermal forcing centered at $z = 0$ with $b = 0.1$ and $H = 5$. A radiation condition is applied at the upper boundary.

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