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PROJECTION BASED TECHNIQUES FOR THE SIMULATION OF RF
CIRCUITS AND HIGH SPEED INTERCONNECTS

by

Roni Khazaka

A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

Ottawa-Carleton Institute for Electrical Engineering,
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submitted by Roni Khazaka,
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Abstract

As operating frequencies increase and device sizes decrease, the complexity of current designs is rising in both the digital VLSI area, and the analog RF area. In the digital area, interconnect effects are becoming dominant contributors to signal integrity degradation, and are therefore an important bottleneck in high-end, high frequency VLSI design. In the analog RF area, the increasingly complex topologies, and device models, coupled with specific analysis requirements such as finding the steady-state solution, are pushing the limits of current analysis tools. Design automation tools are an integral part of the design process, and an important factor in the current rush to reduce the design cycle and improve the time-to-market. The performance of current commercial systems is increasingly limited by the short time to market requirements, and by the capabilities of current design tools, rather than by what the technology can optimally achieve.

In this thesis, novel model order reduction methods based on congruent transformation are suggested in order to deal with the increasing complexity of modern designs. The proposed methods significantly improve the CPU and memory requirements, by transforming a large circuit into a much smaller one that is more easily simulated. Different reduction methods for linear distributed interconnect networks, and for nonlinear steady-state analysis are proposed. A new approach is proposed for finding reduced order time domain macromodels of interconnect networks, which significantly reduces the size of the macromodel compared with previous reduction methods. A new sensitivity analysis method based on model order reduction is proposed. Finally, the new concept of congruent transformation based circuit reduction for nonlinear steady-state analysis of radio frequency circuits is introduced.

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Contents

1	Introduction	1
1.1	Background and Motivation	1
1.2	Contributions	5
1.3	Organization of the Thesis	7
2	Unified Circuit Formulation	8
2.1	Formulation of Lumped Circuits	8
2.1.1	Example of Linear Elements	9
2.1.2	Example of Nonlinear Elements	10
2.2	Models for High-Speed Interconnects	12
2.2.1	Lumped Models	15
2.2.2	Distributed Model	15
2.2.3	Full-wave Models	16
2.3	Frequency Domain Interconnect Stencil	17
2.3.1	Transmission Line Equations	17
2.3.2	Eigenvalue Based Transmission Line Stencil	19
2.3.3	Matrix Exponential Based Transmission Line Stencil	24
2.4	Time Domain Interconnect Stamp	27
2.4.1	Method of Characteristics	27
2.4.2	Discretized Time Domain Interconnect Models	30

2.5	MNA Stencil of a Linear Subsection	33
2.6	Conclusion	36
3	Background on Circuit Simulation	37
3.1	Linear Frequency Domain Analysis	37
3.2	DC Analysis	38
3.3	Transient Time Domain Analysis	40
3.4	Concept of Steady-State Analysis	41
3.5	Time Domain Steady-State Simulation Methods	42
3.5.1	Conventional Transient Analysis	44
3.5.2	Shooting Method	45
3.6	Harmonic Balance Technique	47
3.7	Recent Difficulties in Circuit Simulation	49
3.7.1	Simulation of High-Speed Interconnects and Packages	50
3.7.2	Simulation of RF Circuits	52
4	Review of Model Order Reduction	55
4.1	Direct Moment Matching	55
4.1.1	Padé Approximants	56
4.1.2	Single Expansion Moment Matching	57
4.1.3	Multi-point Moment Matching	59
4.2	Krylov-Subspace Based Reduction	60
4.3	Passivity of the Macromodel	63
5	Macromodeling of Interconnect Networks	66
5.1	Introduction	66
5.2	Problem formulation	68
5.3	Proposed Reduction Algorithm	70
5.3.1	Preliminaries	71

5.3.2	First step of reduction	72
5.3.3	Second step of reduction	74
5.4	Properties of the Reduced-Order Model	75
5.4.1	Conservation of poles	76
5.4.2	Preservation of passivity	77
5.5	Numerical Examples	82
5.5.1	Example 1	82
5.5.2	Example 2	85
5.5.3	Example 3	87
5.5.4	Example 4	89
5.6	Conclusion	92
6	Sensitivity Analysis of Interconnect Networks	93
6.1	Introduction	93
6.2	Formulation of Network Equations	94
6.3	Sensitivity Analysis	96
6.4	Sensitivity analysis using model reduction	98
6.4.1	Reduction of the Circuit Equations	98
6.4.2	Reduction of the Adjoint Network	100
6.4.3	Calculation of the Sensitivity of the MNA Martrix	102
6.4.4	Sensitivity with Respect to Physical Parameters of Transmission Lines .	103
6.5	Numerical Results	104
6.6	Conclusion	111
7	Circuit Reduction Method for Steady-State Analysis	112
7.1	Introduction	112
7.2	Harmonic Balance Simulation	114
7.2.1	Formulation of the HB Equations	114

7.2.2	Handling Nonlinear Components Using FFT	118
7.2.3	Solution of the HB Equations	122
7.2.4	The Harmonic Balance Jacobian	124
7.3	Continuation Methods	126
7.4	Overview of the Circuit Reduction Algorithm	127
7.4.1	Augmenting the HB Equations with Continuation Parameter	128
7.4.2	Circuit Reduction	129
7.4.3	Solution of the Reduced Equations	130
7.5	Calculation of the Subspace Q_h	132
7.6	Proof of Preservation of Derivatives	135
7.7	Circuits with Multi-Tone Excitation	141
7.7.1	Multi-Tone HB Equations	142
7.7.2	FFT for Almost Periodic Signals	144
7.7.3	Reduction Strategy for Circuits with Multi-Tone Inputs	147
7.8	Numerical Examples	149
7.8.1	Example 1: Amplifier Circuit	149
7.8.2	Example 2: Mixer Circuit	153
7.9	Conclusion	155
8	Summary and Future Work	156
8.1	Summary	156
8.2	Future Work	157
	Bibliography	159

List of Tables

5.1	Summary of computational results (Examples 1&2)	87
5.2	CPU comparison for example 3	88
7.1	Formulas for the derivatives of some simple functions	136
7.2	Amplitudes of the harmonics	145
7.3	Mapping of the harmonics	146

List of Figures

1.1	High speed interconnects	2
1.2	Concept of model reduction	4
2.1	Example circuit	9
2.2	Component stamps in MNA formulation	11
2.3	Rectifier circuit with nonlinear capacitor	12
2.4	High-Speed interconnects	13
2.5	Example of a physical interconnect	14
2.6	Electrical interconnect models	14
2.7	Interconnect cross-section/Full-wave analysis	16
2.8	Multi-Conductor Transmission Line	17
2.9	Linear circuit with interconnects	23
2.10	Single lossless line	27
2.11	Method of characteristics interconnect model	29
2.12	Lumped model example	32
2.13	Example of a a linear subcircuit	35
3.1	Simple second order circuit	41
3.2	Steady-state response with a step input	43
3.3	Steady-state response with a periodic input	43
3.4	Mixed Frequency/Time problem	51

3.5	Interconnect discretization	52
4.1	Poles captured using AWE	59
4.2	Poles captured using CFH	60
4.3	Tunnel diode	64
4.4	Biased tunnel diode	64
5.1	Example of a linear subcircuit	69
5.2	Circuit for example 1	82
5.3	Example 1 - Y_{13}	83
5.4	Voltage on signal line (example1)	83
5.5	Voltage on victim line (example 1)	84
5.6	Nine transmission line circuit	85
5.7	Voltage on signal line (example 2)	86
5.8	Voltage on victim line (example 2)	86
5.9	Large interconnect network	87
5.10	Transient response of the network	88
5.11	Y_{11} of the network in example 4	89
5.12	Y_{12} of the network in example 4	90
5.13	Large interconnect network with nonlinear terminations	90
5.14	Transient response of the network at node V_{n2}	91
5.15	Transient response of the network at node V_{out}	91
6.1	A microstrip line from the numerical example	104
6.2	Time domain response	105
6.3	Frequency domain response	105
6.4	Frequency domain relative sensitivity w.r.t resistance per unit length	106
6.5	Time domain relative sensitivity w.r.t resistance per unit length	106
6.6	Frequency domain relative sensitivity w.r.t inductance per unit length	107

6.7	Time domain relative sensitivity w.r.t inductance per unit length	107
6.8	Frequency domain relative sensitivity w.r.t capacitance per unit length	108
6.9	Time domain relative sensitivity w.r.t capacitance per unit length	108
6.10	Frequency domain relative sensitivity w.r.t conductance per unit length	109
6.11	Time domain relative sensitivity w.r.t conductance per unit length	109
6.12	Frequency domain relative sensitivity w.r.t interconnect width	110
6.13	Time domain relative sensitivity w.r.t interconnect width	110
7.1	Example circuit for HB formulation	116
7.2	Newton-Raphson Iteration	123
7.3	Continuation example (DC solution)	126
7.4	Overview of the nonlinear circuit reduction algorithm	127
7.5	Amplifier circuit for Example 1	150
7.6	Example 1: Response to single tone input at 900MHz	151
7.7	Example 1: Output power at 2700MHz (single tone input)	151
7.8	Example 1: Response to a two-tone input	152
7.9	Mixer circuit for Example 2	154
7.10	Example 2: Response to single tone local oscillator input	154
7.11	Example 2: IF output power at 100MHz with respect to RF input power	155

Notation

$\mathbf{A}$	Matrix $\mathbf{A}$
$\mathbf{A}^T$	Matrix $\mathbf{A}$ transposed
$\mathbf{A}^{-T}$	Matrix $\mathbf{A}$ inverted and transposed
$\mathbf{A}^*$	Complex conjugate of matrix $\mathbf{A}$
$\mathbb{R}^N$	The set of real vectors of size N
$\mathbb{R}^{N \times N}$	The set of real matrices of size $N \times N$
$\mathbb{C}^N$	The set of complex vectors of size N
$\mathbb{C}^{N \times N}$	The set of complex matrices of size $N \times N$
$\mathbf{b}_\phi$	Source vector $\mathbf{b}$ for circuit ϕ in MNA equations, as defined in equation (2.1)
$d_{i,j}$	Matrix composed of elements $d_{i,j}$ in i^{th} row and j^{th} column
$\mathbf{C}_\phi$	MNA matrix $\mathbf{C}$ for circuit ϕ , as defined in equation (2.1)
$\mathbf{C}_\pi$	Per-unit-length capacitance matrix of transmission line set π as defined in (2.5)
$\mathbf{f}_\phi(\mathbf{x}_\phi)$	Vector of nonlinear elements in MNA equations of circuit ϕ , as defined in equation (2.1)
$\mathbf{G}_\phi$	MNA matrix $\mathbf{G}$ for circuit ϕ , as defined in equation (2.1)
$\mathbf{G}_\pi$	Per-unit-length conductance matrix of transmission line set π as defined in (2.5)
$\mathbf{J}_s$	The jacobian used in the shooting method as defined in (3.23)
$\mathbf{L}_\pi$	Per-unit-length inductance matrix of transmission line set π as defined in (2.5)
N_ϕ	Total number of variables in MNA formulation of circuit ϕ as defined in (2.1)
$\mathbf{R}_\pi$	Per-unit-length resistance matrix of transmission line set π as defined in (2.5)
$\mathbf{S}_v$	Matrix of transmission line eigenvectors as defined in (2.12)

U	The identity matrix
$\mathbf{x}_\phi(t)$	Vector of unknowns in MNA formulation of circuit ϕ , as defined in (2.1)
$\mathbf{Y}_\pi$	Y -parameter matrix of linear network π
$\mathbf{Y}_L$	Transmission line per-unit-length admittance matrix as defined in (2.10)
$\mathbf{Z}_L$	Transmission line per-unit-length impedance matrix as defined in (2.10)

Abbreviations

AWE	Asymptotic waveform evaluation
CFH	Complex frequency hopping
CPU	Central processing unit
EDA	Electronic design automation
HB	Harmonic balance
FFT	Fast Fourier transform
IFFT	Inverse fast Fourier transform
IC	Integrated circuit
KCL	Kirchhoff's current law
LO	Local oscillator
LU	Lower/upper triangular decomposition.
MMTs	Moment matching techniques
MNA	Modified nodal analysis
MOR	Model order reduction
MVP	Matrix vector product
NR	Newton Raphson iteration
RF	Radio frequency
RC	Resistor-capacitor network
RLC	Resistor-inductor-capacitor network
TEM	Transverse eletromagnetic

Chapter 1

Introduction

1.1 Background and Motivation

In recent years, Electronic Design Automation (EDA) tools have been gaining more importance due to increasing design complexity, and a greater emphasis on shortening the design cycle in order to reduce development costs, and shrink the time-to-market. One of the main reasons for using EDA tools, is to be able to handle more complex designs. In fact, due to recent advances in device and fabrication technologies, the limiting factors to achieving better performance are increasingly related to the allowable time-to-market, and available design tools rather than to what can be optimally achieved with current technologies. This trend can be observed in both the digital VLSI area, and the analog area, where the complexity of designs have significantly increased, and is pushing the limits of current EDA tools.

One example of increased complexity in the digital VLSI area is high speed interconnects. The recent trend in the VLSI industry toward smaller feature sizes, low power consumption, and the use of mixed analog/digital circuits has made signal integrity analysis a challenging task [1]–[4]. In addition, operating frequencies have been rapidly increasing. For example, the central processor clock frequency is well into the GHz range, and communication switches are being designed to transmit data with bit rates faster than 1 Gb/s. For such designs, the perfor-

Figure 1.1: High speed interconnects

mance of interconnects and packages is critical to that of the overall design. Due to these higher operating frequencies and sharper rise times, interconnects can no longer be represented by a lumped model or as a simple short circuit, but must be treated as distributed multiconductor transmission line elements [2]. Therefore, at high frequencies, interconnects start exhibiting effects such as delays, dispersion, reflection, attenuation, ringing, and crosstalk due to coupling between lines, thus causing a significant degradation in signal quality. Furthermore, as devices are optimized and reduced in size, the proportion of the delay and signal degradation attributed to interconnects is becoming dominant [1], [5]. If not detected early during the design stage, interconnect effects can result in a faulty circuit and severely increase the design cycle time by adding more design iterations. Extra design iteration are very costly, and a longer design cycle can be critical for profitability, and for capturing market share. Hence it is very important for designer to be able to accurately simulate interconnect effects, and correct signal integrity problems as early as possible in the design stage. Furthermore, interconnect issues are not confined to any one part of the overall system. In fact, interconnects are present at all

levels of the design hierarchy, from the die to the package, printed circuit board, backplane and cables (Fig. 1.1). Consequently efficient simulation tools for interconnect networks are a necessary requirement in recent designs. One of the objectives of this thesis is to develop efficient interconnect analysis tools, for both frequency and time domain interconnect simulation.

Another example of increased complexity is in the analog area, where the design of Radio Frequency Integrated Circuits (RFICs) has recently gained much prominence. Due to the recent boom in wireless communications, RF circuits such as Low Noise Amplifiers (LNAs), mixers, and power amplifiers which are typically found in RF front ends have gained more and more importance. In this area, design complexity has mainly increased due to three main factors. The first is related to using more complicated circuit topologies, in order to handle more aggressive design criteria such as a lower operating voltage, or lower power consumption for example. Another source of complexity is the drive towards higher integration. One example of this, is including passive elements such as on chip spiral inductors with complicated models [6], [7]. Another example is integrating more circuit blocks on the same chip in order to achieve a fully integrated front end [8]. Finally, another source of increased complexity is the active device models. RF circuits typically stretch the limits of available technologies, and even second order effects must be accounted for in device models. Furthermore, as device sizes decrease and RF designers explore other technologies such as CMOS [9], the complexity of device models is on the rise, thus adding more internal nodes to the circuit. The increased problem size, coupled with the difficult analysis requirement of finding the nonlinear steady-state solution is pushing the limits of current RF design tools. In this thesis, more efficient steady-state analysis methods will be developed for the analysis of complex RF circuits.

One of the objectives of this thesis is to develop solutions that address the increasing complexity of current simulation problems. It is important to note that, even with the ever increasing power of modern computers, acquiring faster workstations is not a sufficient solution to this problem. This is because the CPU cost of the simulation is a superlinear function of circuit size. Therefore a relatively small increase in circuit size can quickly erase any gains

Figure 1.2: Concept of model reduction

made by purchasing a more powerful workstation. In this thesis, new efficient analysis methods are introduced in order to improve the simulation speed of complex circuits. The proposed techniques can be classified as projection based model order reduction techniques. The main concept behind order reduction is to transform a large problem that is difficult to solve, into a much smaller one that conserves the key characteristics of the original one. The analysis is done in the reduced problem space and the results are then mapped back to the original space (Fig. 1.2). Since the reduced system is much smaller than the original system, significant CPU savings can be achieved.

In recent years, model order reduction (MOR) has been the subject of a significant amount of research in the circuit simulation area [10]–[35]. The methods available in the literature are focused on the reduction of linear networks such as high speed-interconnect networks. These techniques can be classified under two general classes. The first class is based on direct Padé approximation [11]–[16], [27]–[35], and the second includes indirect methods based Krylov subspace techniques [17]–[25], that are based on the approximation of the *leading* eigenvalues. The first approach has the advantage of matching the network with a fewer number of poles; however, the resulting macromodel is not guaranteed passive by construction. On the other hand, macromodels developed using the second approach (i.e. Krylov methods) can be made passive by construction [20], [24]; however, such techniques result in a large number of redun-

dant poles [25], [26]. Some attempts have been made to remove the redundant poles of Krylov techniques, but the resulting macromodels are not passive by construction [25], [26]. In this thesis, a new Krylov based reduction method, that removes the redundant poles obtained using conventional reduction while at the same time preserving the passivity of the macromodel by construction.

Model order reduction has also been suggested in the literature for improving the efficiency of sensitivity analysis of high speed interconnects [16], [17], [36]. However, these approaches require the computation of the sensitivity of the transformation used in the reduction, which is a very cumbersome process. In this thesis, a reduction method for sensitivity analysis of distributed interconnects is developed. Unlike previous methods this approach does not require the cumbersome calculation of the sensitivity of transformation matrix used in the reduction. Instead a projection based reduction is applied directly to the adjoint network equations.

Finally, the main focus of MOR research in the circuit area has been on the analysis of linear interconnect networks. In this thesis, a new nonlinear projection based reduction method is proposed for the nonlinear steady-state analysis of RF circuit. This approach uses continuation methods as a vehicle to enable the nonlinear circuit reduction, and also benefits from the improved convergence provided by the continuation techniques.

1.2 Contributions

In this thesis, projection based model order reduction methods are developed for the analysis of both analog RF circuits, and digital VLSI interconnect networks. Specifically, the main contributions are,

1. A new nonlinear circuit reduction method was developed for finding the nonlinear steady-state solution of Radio Frequency Integrated Circuits [37], [38]. Model Order Reduction (MOR) algorithms have been traditionally confined to linear systems. The new order

reduction method for nonlinear steady-state analysis contributes to developing a new concept in circuit analysis, namely nonlinear reduction methods. This concept has many other potential applications, and it is expected that the field of nonlinear model reduction will be the subject of significant research in the future. Details of this technique can be found in Chapter 7.

2. A derivative calculation method for obtaining the moments of the Harmonic Balance equations was developed [38]. The new method is used to obtain the basis of the subspace used for nonlinear order reduction, but also has other potential applications in the field of intermodulation analysis. The details of this technique can be found in Section 7.5.
3. A new order reduction method for time domain macromodelling of linear circuits was developed [39]–[44]. The new method is passive by construction, and removes the redundant poles obtained by previous passive MOR techniques. The size of the macromodel obtained using the new method is typically less than half the size obtained by previous MOR methods. The details of this order reduction method are given in chapter 5.
4. An MOR method for frequency domain sensitivity analysis of transmission line networks was developed [45], [46]. Previous approaches to using MOR for sensitivity analysis required the computation of the sensitivity of the transformation matrix, which is an expensive process. The proposed method efficiently applies MOR techniques directly to the adjoint equations, without requiring the sensitivity of the transformation matrix. The work has also contributed to more developments in the area of multi-dimensional MOR methods for large scale sensitivity analysis of interconnect networks [47], [48]. The MOR based sensitivity evaluation method is outlined in Chapter 6.

1.3 Organization of the Thesis

This thesis is organized as follows. In Chapter 2, a unified circuit formulation for RF circuits and VLSI interconnects including frequency and time domain transmission line stamps is presented. Chapter 3 provides the background on current simulation methods for RF and VLSI circuits, and is followed by a review of order reduction techniques in Chapter 4. In Chapter 5 a new reduction method for interconnect macromodeling is presented. This method is based on congruent transformation, and is shown to preserve the dominant poles while maintaining the passivity of the reduced macromodel. A new reduction technique for the sensitivity analysis of high speed distributed interconnects is presented in Chapter 6. In Chapter 7 a nonlinear circuit reduction method for finding the nonlinear steady-state solution of Radio Frequency circuit is proposed. Finally a summary and a list of future work is given in Chapter 8.

Chapter 2

Unified Circuit Formulation

For clarity, completeness and consistency throughout the thesis, a unified formulation of circuit equations is presented in this chapter, with special attention given to difficult circuit elements such as multi-conductor transmission lines. The formulation of the network equations, also provides some insight into the difficulties faced in the analysis of interconnect networks and radio frequency circuits. The formulation presented in this chapter is general, and will be used throughout the thesis for both analog RF circuits, and high-speed digital circuits.

2.1 Formulation of Lumped Circuits

Consider a circuit ϕ containing both linear and nonlinear elements. Using the Modified Nodal Admittance (MNA) formulation [49], [50], the circuit equations can be written in matrix form as

$$\mathbf{G}_\phi \mathbf{x}_\phi(t) + \mathbf{C}_\phi \dot{\mathbf{x}}_\phi(t) + \mathbf{f}_\phi(\mathbf{x}_\phi(t)) = \mathbf{b}_\phi(t) \quad (2.1)$$

where

- $\mathbf{x}_\phi(t) \in \mathbb{R}^{N_\phi}$ is a vector of node voltage waveforms, appended by independent voltage source current, linear inductor current, nonlinear capacitor charge and nonlinear inductor flux waveforms,

Figure 2.1: Example circuit

- $C_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$ and $G_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$ are constant matrices describing the linear lumped memory and memoryless elements respectively,
- $b_\phi(t) \in \mathbb{R}^{N_\phi}$ is a vector with entries determined by the independent voltage and current sources,
- $f_\phi(x_\phi(t))$ is a vector containing function describing the nonlinear elements of the circuit,
- N_ϕ is the total number of variables in the circuit, including node voltages, voltage source currents, linear inductor currents, nonlinear capacitor charges and nonlinear inductor flux densities.

The matrices C_ϕ and G_ϕ are typically referred to as the MNA matrices. One important advantage of this formulation is that the construction of these matrices can be automated using component stamps or *stencils*. The derivation of an MNA stamp is best shown by an example.

2.1.1 Example of Linear Elements

Consider the simple linear circuit in Fig. 2.1, with the circuit nodes numbered from 1 to 3. Kirchhoff's current law at each node in the circuit gives

$$\begin{aligned}
g_1 (v_1 - v_2) &= J \\
g_1 (v_2 - v_1) + g_2 v_2 + sC (v_2 - v_3) &= 0 \\
g_3 v_3 + sC (v_3 - v_2) &= 0
\end{aligned} \tag{2.2}$$

Re-writing equation (2.2) in matrix form results in

$$\begin{bmatrix} \mathbf{g_1} & -\mathbf{g_1} & 0 \\ -\mathbf{g_1} & \mathbf{g_1} + g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + s \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & -c \\ 0 & c & -c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix} \tag{2.3}$$

Looking at the contribution of the resistor g_1 connected between nodes 1 and 2, which is shown in bold in the MNA equations in (2.3), one can easily deduce the general resistor stamp shown in Fig. 2.2. Fig. 2.2 also shows the stamps for some other common components. These can be easily derived in a similar fashion. It is to be noted that, since each node in the circuit is typically connected to 2 or 3 other nodes only, the resulting circuit equations from this formulation are very sparse. A detailed explanation of the MNA formulation can be found in [50].

2.1.2 Example of Nonlinear Elements

The nonlinear equations that model nonlinear elements are added to the vector $\mathbf{f}_\phi(\mathbf{x}(t))$ in equation (2.1). Additional variables may be added in order ensure that the nonlinear functions in $\mathbf{f}(\mathbf{x})$ are algebraic. For a nonlinear capacitor for example, a new variable representing the charge on the capacitor is introduced. In addition to ensuring that the nonlinear vector is algebraic, this capacitor formulation also helps in ensuring charge conservation at each time step of the transient analysis [51]. This nonlinear capacitor formulation is best illustrated by an example. Consider the rectifier circuit in Fig. 2.3. A nonlinear capacitor was used, and the capacitor charge was therefore taken to be a nonlinear function of the voltage. For this circuit,

Figure 2.2: Component stamps in MNA formulation

Figure 2.3: Rectifier circuit with nonlinear capacitor

the MNA equations are

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & g & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ I_v \\ Q_c \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{I}_v \\ \dot{Q}_c \end{bmatrix} + \begin{bmatrix} I_s(e^{(v_1-v_2)/v_t} - 1) \\ -I_s(e^{(v_1-v_2)/v_t} - 1) \\ 0 \\ -kv_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E(t) \\ 0 \end{bmatrix} \quad (2.4)$$

where I_v is the voltage source current, and Q_c is the charge on the nonlinear capacitor. The above system of equations simply represents Kirchhoff's current law (KCL) at each node of the circuit, in addition to other equations such as the one introduced by the voltage source for example [50]. One important point to note in equation (2.4) is that the nonlinear vector contains only algebraic equations.

2.2 Models for High-Speed Interconnects

Interconnects are present at various levels of the design hierarchy, such as packaging structures, multichip modules (MCM), printed circuit boards (PCB), backplanes and on-chip dies

Figure 2.4: High-Speed interconnects

(Fig. 2.4), and are becoming a common element in circuit design. Many different types of interconnections are used in current designs. One example of a physical interconnect is the microstrip line as shown in Fig. 2.5, other examples include striplines and cables [1], [3]. In order to perform a circuit analysis or simulation, an electrical model that can be included in a circuit simulator is required. Naturally, such a model would be a function of the physical properties of the interconnect, such as the width of the line or the thickness of the dielectric. However, the nature and complexity of electrical models also varies with the operating frequency of interest. For example, at very low frequencies, most interconnects can be accurately treated as a simple short circuit. As the frequency increases the required models become more complicated, and at high frequencies (as the electrical length of the line becomes a significant fraction of the wavelength [2]) interconnects must be treated as distributed transmission lines (Fig. 2.6). For even higher frequencies, as the wavelength becomes of the same order of the cross-sectional dimensions of the line, a full electromagnetic analysis may be required. In this section, the different electrical models for interconnects will be presented.

Figure 2.5: Example of a physical interconnect

Figure 2.6: Electrical interconnect models

2.2.1 Lumped Models

At low frequencies, a lumped model such as a resistor/capacitor (RC) model may be used to represent the interconnect structure with sufficient accuracy. In fact, some of the early work on interconnect timing simulation was based on analysis RC trees (no floating capacitors, and no resistors to ground) using the Elmore delay [52], which provided a dominant time constant approximation for monotonic step responses. However, with increasing operating frequencies, the RC representation is no longer accurate, and an RLC model is used to capture the ringing in the signal waveforms. At even higher frequencies, lumped models are no longer adequate, and interconnects must be considered as distributed lossy coupled multi-conductor transmission lines [2].

2.2.2 Distributed Model

At relatively higher signal-speeds, the electrical length of interconnects becomes a significant fraction of the wavelength giving rise to signal distorting effects that do not exist at lower frequencies. Consequently, the conventional lumped impedance interconnect models become inadequate and distributed transmission line models with per-unit-length parameters are needed. The basic distributed model is the simple delay-line (or lossless line). More complicated models include per-unit-length loss (either in the direction of the travelling wave or due to dielectric substrate loss), or coupling between adjacent transmission lines, where the coupling may be resistive, inductive, capacitive or a combination of these. The models may even require nonuniform per-unit-length parameters to accurately model different geometries [53]. Under the quasi-TEM assumption, a lossy coupled multi-conductor transmission line set π containing n coupled signal lines in addition to one reference conductor, can be modelled using the Telegrapher's equations [2]

$$\frac{\partial v_{\pi}(z, t)}{\partial z} = -R_{\pi} i_{\pi}(z, t) - L_{\pi} \frac{\partial i_{\pi}(z, t)}{\partial t} \quad (2.5a)$$

$$\frac{\partial i_{\pi}(z, t)}{\partial z} = -G_{\pi} v_{\pi}(z, t) - C_{\pi} \frac{\partial v_{\pi}(z, t)}{\partial t} \quad (2.5b)$$

Figure 2.7: Interconnect cross-section/Full-wave analysis

where $v_\pi(z, t) \in \mathbb{R}^n$ and $i_\pi(z, t) \in \mathbb{R}^n$ represent the voltages and currents as a function of the displacement z along the line. R_π , L_π , G_π , and $C_\pi \in \mathbb{R}^{n \times n}$ are the per-unit-length resistance, inductance, capacitance and conductance matrices respectively. The Telegrapher's equations are partial differential equations in space and time, and are difficult to solve in the time domain. Furthermore at high frequencies, the frequency dependence of the per-unit-length parameters must be accounted for [3], and depending on the geometries those parameters may vary along the the length of the line [53]. For most practical cases, one of theses variations of the distributed transmission line models is sufficiently accurate for signal integrity simulation [4].

2.2.3 Full-wave Models

At further sub-nanosecond rise times the cross-sectional dimensions of the interconnects are of the same order as the wave length (Fig. 2.7). In this case, an interconnect block must be treated as a dispersive structure and field components in the direction of propagation can no longer be neglected [54]. Consequently, even the distributed models based on quasi-TEM approximations become inaccurate in describing the interconnect performance. In such situations full-wave models which take into account all possible field components, and satisfy all boundary conditions, are required in order to give an accurate estimation of the high-frequency effects.

Naturally, a full EM analysis is very CPU expensive. Methods such as the partial element equivalent circuit (PEEC) models [55]–[58] have been developed for cases when the TEM approximation no longer holds, but such models result in very large lumped equivalent circuits

Figure 2.8: Multi-Conductor Transmission Line

and require a very large CPU time when coupled with nonlinear circuit simulators such as SPICE.

2.3 Frequency Domain Interconnect Stencil

As is evident from section. 2.1, a stamp or *stencil* is required for each element in order to form the MNA equations in (2.1). The stamp simply represents the relationship between the terminal voltages and currents of the corresponding circuit element. In this section, frequency domain stamps for interconnects modelled as distributed transmission lines are derived.

2.3.1 Transmission Line Equations

Consider the lossy coupled multi-conductor transmission lines containing n coupled signal conductors and one reference conductor as shown in Fig. 2.8. The voltages and currents as functions of the distance z along the line are $v_{\pi 1}(t, z), \dots, v_{\pi n}(t, z)$ and $i_{\pi 1}(t, z), \dots, i_{\pi n}(t, z)$

respectively. The voltage and current vectors $\mathbf{v}_\pi(t, z)$ and $\mathbf{i}_\pi(t, z) \in \mathbb{R}^n$ are defined as

$$\mathbf{v}_\pi(t, z) = \begin{bmatrix} v_{\pi 1}(t, z) \\ \vdots \\ v_{\pi n}(t, z) \end{bmatrix}; \quad \mathbf{i}_\pi(t, z) = \begin{bmatrix} i_{\pi 1}(t, z) \\ \vdots \\ i_{\pi n}(t, z) \end{bmatrix} \quad (2.6)$$

As discussed in Section 2.2.2, under the quasi-TEM assumption, these voltages and currents are governed by the Telegrapher's equations [2]

$$\frac{\partial \mathbf{v}_\pi(z, t)}{\partial z} = -\mathbf{R}_\pi \mathbf{i}_\pi(z, t) - \mathbf{L}_\pi \frac{\partial \mathbf{i}_\pi(z, t)}{\partial t} \quad (2.7a)$$

$$\frac{\partial \mathbf{i}_\pi(z, t)}{\partial z} = -\mathbf{G}_\pi \mathbf{v}_\pi(z, t) - \mathbf{C}_\pi \frac{\partial \mathbf{v}_\pi(z, t)}{\partial t} \quad (2.7b)$$

where $\mathbf{v}_\pi(z, t) \in \mathbb{R}^n$ and $\mathbf{i}_\pi(z, t) \in \mathbb{R}^n$ represent the voltages and currents as a function of the displacement z along the line. $\mathbf{R}_\pi$, $\mathbf{L}_\pi$, $\mathbf{G}_\pi$, and $\mathbf{C}_\pi \in \mathbb{R}^{n \times n}$ are the per-unit-length resistance, inductance, capacitance and conductance matrices respectively. In the Laplace domain, equation (2.7) becomes

$$\frac{\partial \mathbf{V}_\pi(z, s)}{\partial z} = -\mathbf{R}_\pi \mathbf{I}_\pi(z, s) - s\mathbf{L}_\pi \mathbf{I}_\pi(z, s) \quad (2.8a)$$

$$\frac{\partial \mathbf{I}_\pi(z, s)}{\partial z} = -\mathbf{G}_\pi \mathbf{V}_\pi(z, s) - s\mathbf{C}_\pi \mathbf{V}_\pi(z, s) \quad (2.8b)$$

Equation (2.8) can be re-written as

$$\frac{\partial \mathbf{V}_\pi(z, s)}{\partial z} = -\mathbf{Z}_p \mathbf{I}_\pi(z, s) \quad (2.9a)$$

$$\frac{\partial \mathbf{I}_\pi(z, s)}{\partial z} = -\mathbf{Y}_p \mathbf{V}_\pi(z, s) \quad (2.9b)$$

where $\mathbf{Z}_p$ and $\mathbf{Y}_p$ represent the transmission line per-unit-length *impedance* and *admittance* matrices, defined as

$$\mathbf{Z}_p = \mathbf{R} + s\mathbf{L}; \quad \mathbf{Y}_p = \mathbf{G} + s\mathbf{C} \quad (2.10)$$

The above matrices are convenient definitions used in transmission line literature, and are not related to the general concepts of impedance and admittance used in circuit theory.

2.3.2 Eigenvalue Based Transmission Line Stencil

In this section, a transmission line stamp based on the eigndecomposition of the transmission line parameters will be presented. Equations (2.9a) and (2.9b) can be combined to form a set of wave equations

$$\frac{\partial^2 \mathbf{V}_\pi(z, s)}{\partial z^2} = \mathbf{Z}_p \mathbf{Y}_p \mathbf{V}_\pi(z, s) \quad (2.11a)$$

$$\frac{\partial^2 \mathbf{I}_\pi(z, s)}{\partial z^2} = \mathbf{Y}_p \mathbf{Z}_p \mathbf{I}_\pi(z, s) \quad (2.11b)$$

The eigenvalue based transmission line stamp is based on decoupling the different propagation modes on the coupled transmission line set [2]. Let $\mathbf{S}_v$ be the matrix containing the right eigenvectors of $\mathbf{Z}_p \mathbf{Y}_p$. Therefore, by definition

$$\mathbf{Z}_p \mathbf{Y}_p \mathbf{S}_v = \mathbf{S}_v \mathbf{\Gamma}_m \quad (2.12)$$

where $\mathbf{\Gamma}_m$ is a diagonal matrix containing the eigenvalues γ_i^2 of $\mathbf{Z}_p \mathbf{Y}_p$.

$$\mathbf{\Gamma}_m = \text{diag} [\gamma_1^2, \gamma_2^2, \dots, \gamma_n^2] \quad (2.13)$$

Note that for the general case of transmission lines in an inhomogeneous dielectric medium, $\mathbf{Z}_L \mathbf{Y}_L$ typically has n distinct eigenvalues, where n is the number of coupled signal conductors. Each eigenvalue is related to the propagation constant of the corresponding propagation mode as will be evident from the remainder of this analysis. The matrix $\mathbf{Z}_p \mathbf{Y}_p$ can now be diagonalized as follows

$$\mathbf{Z}_p \mathbf{Y}_p = \mathbf{S}_v \mathbf{\Gamma}_m \mathbf{S}_v^{-1} \quad (2.14)$$

Substituting from (2.14) into (2.11a) results in

$$\frac{\partial^2 \mathbf{V}_\pi(z, s)}{\partial z^2} = \mathbf{S}_v \mathbf{\Gamma}_m \mathbf{S}_v^{-1} \mathbf{V}_\pi(z, s) \quad (2.15)$$

Define the variable substitution

$$\mathbf{V}_p(z, s) \leftarrow \mathbf{S}_v^{-1} \mathbf{V}_\pi(z, s) \quad (2.16)$$

Applying the above substitution to equation (2.15) results in

$$\frac{\partial^2 \mathbf{V}_p(z, s)}{\partial z^2} = \mathbf{\Gamma}_m \mathbf{V}_p(z, s) \quad (2.17)$$

Since $\mathbf{\Gamma}_m$ is a diagonal matrix, equation (2.17) represents a set of decoupled second order differential equations in z . The solution of (2.17) can therefore be written as a sum of forward travelling waves and a backward travelling waves as follows

$$\mathbf{V}_p(z, s) = \begin{bmatrix} e^{-\gamma_1 z} & & & \\ & e^{-\gamma_2 z} & & \\ & & \ddots & \\ & & & e^{-\gamma_n z} \end{bmatrix} \mathbf{V}_p^+ + \begin{bmatrix} e^{\gamma_1 z} & & & \\ & e^{\gamma_2 z} & & \\ & & \ddots & \\ & & & e^{\gamma_n z} \end{bmatrix} \mathbf{V}_p^- \quad (2.18)$$

where $\mathbf{V}_p^+$ and $\mathbf{V}_p^-$ are constant vectors determined from the terminal conditions at both ends of the line. Define the diagonal matrix $\mathbf{E}_m$ as

$$\mathbf{E}_m(z) = \begin{bmatrix} e^{-\gamma_1 z} & & & \\ & e^{-\gamma_2 z} & & \\ & & \ddots & \\ & & & e^{-\gamma_n z} \end{bmatrix} \quad (2.19)$$

Substituting from (2.19) and (2.16) into (2.18) results in

$$\mathbf{V}_\pi(z, s) = \mathbf{S}_v \mathbf{E}_m(z) \mathbf{V}_p^+ + \mathbf{S}_v [\mathbf{E}_m(z)]^{-1} \mathbf{V}_p^- \quad (2.20)$$

From equation (2.20) the near and far end voltages at $z = 0$ and $z = d$ respectively, can be expressed in matrix form as

$$\begin{bmatrix} \mathbf{V}_\pi(0) \\ \mathbf{V}_\pi(d) \end{bmatrix} = \begin{bmatrix} \mathbf{S}_v & \mathbf{S}_v \\ \mathbf{S}_v \mathbf{E}_m(d) & \mathbf{S}_v [\mathbf{E}_m(d)]^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{V}_p^+ \\ \mathbf{V}_p^- \end{bmatrix} \quad (2.21)$$

In order to obtain expressions for the currents on the lines, substitute from (2.20) into (2.9a) to obtain

$$-\mathbf{S}_v \mathbf{\Gamma}_s \mathbf{E}_m(z) \mathbf{V}_p^+ + \mathbf{S}_v \mathbf{\Gamma}_s [\mathbf{E}_m(z)]^{-1} \mathbf{V}_p^- = -\mathbf{Z}_p \mathbf{I}_\pi(z, s) \quad (2.22)$$

with Γ_s defined as

$$\Gamma_s = \text{diag}[\gamma_1, \gamma_2, \dots, \gamma_n] \quad (2.23)$$

By pre-multiplying by Z_p^{-1} , equation (2.22) can be simplified to

$$I_\pi(z, s) = S_i E_m(z) V_p^+ - S_i [E_m(z)]^{-1} V_p^- \quad (2.24)$$

with S_i defined as

$$S_i = Z_p^{-1} S_v \Gamma_s \quad (2.25)$$

From equation (2.24) the near and far end currents at $z = 0$ and $z = d$ respectively can be expressed as

$$\begin{bmatrix} I_\pi(0) \\ I_\pi(d) \end{bmatrix} = \begin{bmatrix} S_i & -S_i \\ S_i E_m(d) & -S_i [E_m(d)]^{-1} \end{bmatrix} \begin{bmatrix} V_p^+ \\ V_p^- \end{bmatrix} \quad (2.26)$$

Eliminating the constants V_p^+ and V_p^- between equations (2.26) and (2.21) results in the following relation between the terminal voltages and currents

$$\begin{bmatrix} I_\pi(0) \\ I_\pi(d) \end{bmatrix} = \begin{bmatrix} S_i & -S_i \\ S_i E_m(d) & -S_i [E_m(d)]^{-1} \end{bmatrix} \begin{bmatrix} S_v & S_v \\ S_v E_m(d) & S_v [E_m(d)]^{-1} \end{bmatrix}^{-1} \begin{bmatrix} V_\pi(0) \\ V_\pi(d) \end{bmatrix} \quad (2.27)$$

The Y parameters of the transmission line can be easily deduced from equation (2.27), as equation (2.27) can be simplified to

$$\begin{bmatrix} I_\pi(0) \\ -I_\pi(d) \end{bmatrix} = \begin{bmatrix} S_i E_1 S_v^{-1} & S_i E_2 S_v^{-1} \\ S_i E_2 S_v^{-1} & S_i E_1 S_v^{-1} \end{bmatrix} \begin{bmatrix} V_\pi(0) \\ V_\pi(d) \end{bmatrix} \quad (2.28)$$

where E_1 and E_2 are defined as

$$E_1 = \text{diag} \left\{ \frac{1 + e^{-2\gamma_i d}}{1 - e^{-2\gamma_i d}}, \quad i = 1, \dots, n \right\} \quad (2.29a)$$

$$E_2 = \text{diag} \left\{ \frac{-2e^{-\gamma_i d}}{1 - e^{-2\gamma_i d}}, \quad i = 1, \dots, n \right\} \quad (2.29b)$$

It is important to note the negative sign in front of the far end current $I_\pi(d)$ in equation (2.28). This is due to the fact that $I_\pi(d)$ is defined as leaving the interconnect (see Fig. 2.8), which is inconsistent with the standard definition of the Y parameters where all currents are defined as entering the subsystem. From (2.28) we can simply write the Y parameters of the interconnects as

$$\mathbf{Y}_\pi = \begin{bmatrix} \mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1} \\ \mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1} \end{bmatrix} \quad (2.30)$$

With the Y parameters of the multi-conductor transmission line known, a stamp for the line can now be constructed. Consider a linear network ϕ containing linear lumped components and N_t lossy coupled transmission line sets, with n_k coupled conductors in transmission line set k . Assume the network ϕ has N_ϕ nodal variables. The Modified Nodal Analysis (MNA) matrix equations in (2.1) can be written in the frequency domain, and appended with the transmission line stamp as follows

$$\mathbf{G}_\phi \mathbf{X}_\phi(s) + s\mathbf{C}_\phi \mathbf{X}_\phi(s) + \mathbf{Y}_\phi(s) \mathbf{X}_\phi(s) = \mathbf{B}_\phi(s) \quad (2.31)$$

Where the matrix $\mathbf{Y}_\phi$ contains the mapped Y parameters of all the transmission line sets, and is expressed as

$$\mathbf{Y}_\phi(s) = \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{Y}_k(s) \mathbf{D}_k^T \quad (2.32)$$

where $\mathbf{D}_k = [d_{i,j}]$ is a selector matrix with elements $d_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, N_\phi\}$, $j \in \{1, 2, \dots, 2n_k\}$ with a maximum of one non-zero in each row or column, that maps $\mathbf{I}_k(s) \in \mathbb{R}^{2n_k}$, the vector of currents entering the interconnect subnetwork k , into the node space $\mathbb{R}^{N_\phi}$ of the network.

The above formulation is best illustrated by an example. Consider the linear circuit in Fig. 2.9. There is only one transmission line set in this case, which we will refer to as transmission line set 1. The Y -parameters of a transmission line set are expressed in (2.30), and can

Figure 2.9: Linear circuit with interconnects

be summarized for transmission line set 1 as,

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \mathbf{Y}_1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \quad (2.33)$$

The MNA equations of this circuit are,

$$\mathbf{X}_\phi = [v_1, v_2, v_3, v_4, v_5, I_E]^T \quad (2.34)$$

$$\mathbf{B}_\phi = [0, 0, 0, 0, 0, E_i]^T \quad (2.35)$$

$$\mathbf{G}_\phi = \begin{bmatrix} g_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_1 & 0 & -g_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g_1 & 0 & g_1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{C}_\phi = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.36)$$

$$Y_\phi = D_1 Y_1 D_1^T = \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} & 0 & 0 \\ y_{21} & y_{22} & y_{23} & y_{24} & 0 & 0 \\ y_{31} & y_{32} & y_{33} & y_{34} & 0 & 0 \\ y_{41} & y_{42} & y_{43} & y_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.37)$$

2.3.3 Matrix Exponential Based Transmission Line Stencil

In this section, a transmission line stamp based on the calculation of the matrix exponential is presented. Equation (2.8) can be written in matrix form as,

$$\frac{\partial X_\pi(z)}{\partial z} = -(D_\pi + sE_\pi)X_\pi(z) \quad (2.38)$$

where

$$D_\pi = \begin{bmatrix} 0 & R_\pi \\ G_\pi & 0 \end{bmatrix}; \quad E_\pi = \begin{bmatrix} 0 & L_\pi \\ C_\pi & 0 \end{bmatrix}; \quad X_\pi(z) = \begin{bmatrix} V_\pi(z) \\ I_\pi(z) \end{bmatrix} \quad (2.39)$$

Equation (2.38) is simply a set of ordinary differential equations in z . Writing the solution of (2.38) and applying the terminal conditions provides a relation between the terminal voltages and currents at the near and far ends of the multi-conductor transmission line structure, in the form

$$\begin{bmatrix} V_\pi(d, s) \\ I_\pi(d, s) \end{bmatrix} = e^{-(D_\pi + sE_\pi)d} \begin{bmatrix} V_\pi(0, s) \\ I_\pi(0, s) \end{bmatrix} \quad (2.40)$$

where d is the length of the line. Equation (2.40) provides an expression for the ABCD or *hybrid* parameters of the transmission line in the form of a matrix exponential

$$T_\pi(s) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = e^{-(D_\pi + sE_\pi)d} \quad (2.41)$$

It is possible to obtain an expression for the Y -parameters from the ABCD parameters and proceed to insert the transmission line stamp into the MNA equations as outlined in Sec-

tion 2.3.2. However, an alternate stamp based on the ABCD parameters will be presented in the remainder of this section.

Using equations (2.40) and (2.41), the relationship between terminal voltages and currents of transmission line set k can be written as

$$\mathbf{A}_k(s)\mathbf{V}_k(s) + \mathbf{B}_k(s)\mathbf{I}_k(s) = 0 \quad (2.42)$$

The terminal voltages in (2.42) are related to those in (2.40) by

$$\mathbf{V}_k(s) = \begin{bmatrix} \mathbf{V}_\pi(0, s) \\ \mathbf{V}_\pi(d, s) \end{bmatrix}; \quad \mathbf{I}_k(s) = \begin{bmatrix} \mathbf{I}_\pi(0, s) \\ -\mathbf{I}_\pi(d, s) \end{bmatrix} \quad (2.43)$$

Note the negative sign before the far end current in (2.43). This was added in order to redefine the direction of the current at the far end of the line as flowing into the line as opposed to what is shown in Fig. 2.8. Noting this, and using some algebraic manipulations, simple expressions can be found for $\mathbf{A}_k$ and $\mathbf{B}_k$ from (2.40) and (2.42) as follows

$$\mathbf{A}_k(s) = \begin{bmatrix} \mathbf{T}_{11} & -\mathbf{U} \\ \mathbf{T}_{21} & 0 \end{bmatrix}; \quad \mathbf{B}_k(s) = \begin{bmatrix} \mathbf{T}_{12} & 0 \\ \mathbf{T}_{22} & \mathbf{U} \end{bmatrix} \quad (2.44)$$

With the terminal relationships of the transmission lines expressed in the form shown in (2.44), it is now simple to construct a stamp for the lines. Consider a linear network ϕ containing linear lumped components and N_t lossy coupled transmission line sets, with n_k coupled conductors in transmission line set k . Assume the network ϕ has N_ϕ nodal variables. The Modified Nodal Analysis (MNA) matrix equations in (2.1) can be written in the frequency domain, and appended with the transmission line stamp as follows

$$\begin{bmatrix} \mathbf{G}_\phi + s\mathbf{C}_\phi & \mathbf{D}_1 & \cdots & \mathbf{D}_{N_t} \\ \mathbf{A}_1\mathbf{D}_1^T & \mathbf{B}_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \mathbf{A}_{N_t}\mathbf{D}_{N_t}^T & 0 & 0 & \mathbf{B}_{N_t} \end{bmatrix} \begin{bmatrix} \mathbf{X}_\phi \\ \mathbf{I}_1 \\ \vdots \\ \mathbf{I}_{N_t} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_\phi \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2.45)$$

where $\mathbf{D}_k = [d_{i,j}]$ is a selector matrix with elements $d_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, N_\phi\}$, $j \in \{1, 2, \dots, 2n_k\}$ with a maximum of one non-zero in each row or column, that maps $\mathbf{I}_k(s) \in$

$\mathbb{R}^{2n_k}$, the vector of currents entering the interconnect subnetwork k , into the node space $\mathbb{R}^{N_\phi}$ of the network. Note that in this formulation, the terminal currents of the transmission lines are appended to the set of unknowns in the MNA equations.

This formulation is illustrated by an example. Consider the linear circuit in Fig. 2.9. This circuit has only one transmission line set, which will be referred to as transmission line set 1. For this circuit the MNA equations in (2.45) become

$$\begin{bmatrix} \mathbf{G}_\phi + s\mathbf{C}_\phi & \mathbf{D}_1 \\ \mathbf{A}_1\mathbf{D}_1^T & \mathbf{B}_1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_\phi \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_\phi \\ 0 \end{bmatrix} \quad (2.46)$$

where $\mathbf{I}_1$ represents the terminal currents of transmission line set 1

$$\mathbf{I}_1 = [i_1, i_2, i_3, i_4]^T \quad (2.47)$$

$\mathbf{X}_\phi$ is the vector of the unknowns in the standard MNA formulation as shown in (2.1)

$$\mathbf{X}_\phi = [v_1, v_2, v_3, v_4, v_5, I_E]^T \quad (2.48)$$

$$\mathbf{B}_\phi = [0, 0, 0, 0, 0, E_i]^T \quad (2.49)$$

$$\mathbf{G}_\phi = \begin{bmatrix} g_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_1 & 0 & -g_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g_1 & 0 & g_1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{C}_\phi = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.50)$$

$\mathbf{A}_1(s)$ and $\mathbf{B}_1(s)$ are defined in (2.44), and the selector matrix $\mathbf{D}_1$ is given by

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.51)$$

Figure 2.10: Single lossless line

2.4 Time Domain Interconnect Stamp

As is evident from their derivations, the frequency domain stamps of Section 2.3 are complex functions of frequency and cannot be readily expressed in the time domain. This section is devoted to possible time domain representations of interconnects.

2.4.1 Method of Characteristics

The method of characteristics was initially developed by Branin [59] in 1967 for single lossless lines, and later extended to coupled lines [60]. Using this approach, interconnects are essentially treated as delay elements. Naturally, dispersive effects in lossy lines would cause difficulties in such an approach. An extension of this method to handle lossy lines was later proposed [61], however, the method of characteristics remains most practical for lossless lines. In this section, the general concept behind the method of characteristics will be outlined. This approach is best illustrated using the simple example of a single lossless line. The original derivation of the method of characteristics was developed in the time domain [59] using what was referred to as *characteristic curves* (hence the name). A simpler alternative derivation in the frequency domain will be presented here. This frequency domain approach has the added advantage of clearly illustrating the difficulties encountered when using the method of characteristics for lossy lines.

Consider the single lossless line in Fig. 2.10. The Y -parameters of this line can be ex-

pressed as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{Z_o(1 - e^{-2\gamma d})} \begin{bmatrix} 1 + e^{-2\gamma d} & -2e^{-\gamma d} \\ -2e^{-\gamma d} & 1 + e^{-2\gamma d} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad (2.52)$$

As is evident from equation (2.52), the Y -parameters of the transmission line are complex functions of frequency and are not readily transformed into a time domain representation. The method of characteristics succeeds in doing such a transformation, but only for lossless lines. By re-arranging the terms in equation (2.52) the following expressions are obtained

$$V_1 = Z_o I_1 + e^{-\gamma d} [2V_2 - e^{-\gamma d} (Z_o I_1 + V_1)] \quad (2.53a)$$

$$V_2 = Z_o I_2 + e^{-\gamma d} [2V_1 - e^{-\gamma d} (Z_o I_2 + V_2)] \quad (2.53b)$$

where Z_o is the characteristic impedance of the line. Now we define W_{c1} and W_{c2} as

$$W_{c1} = e^{-\gamma d} [2V_2 - e^{-\gamma d} (Z_o I_1 + V_1)] \quad (2.54a)$$

$$W_{c2} = e^{-\gamma d} [2V_1 - e^{-\gamma d} (Z_o I_2 + V_2)] \quad (2.54b)$$

Substituting from (2.54) into (2.53) results in

$$V_1 = Z_o I_1 + W_{c1} \quad (2.55)$$

$$V_2 = Z_o I_2 + W_{c2}$$

The terms $Z_o I_1$ and $Z_o I_2$ in (2.54) are now substituted with their values obtained from (2.54), thus resulting in

$$W_{c1} = e^{-\gamma d} [2V_2 - e^{-\gamma d} (2V_1 - W_{c1})] \quad (2.56a)$$

$$W_{c2} = e^{-\gamma d} [2V_1 - e^{-\gamma d} (2V_2 - W_{c2})] \quad (2.56b)$$

Noting the symmetry in (2.56a) and (2.56b), allows for convenient recursive expressions for W_{c1} and W_{c2}

$$W_{c1} = e^{-\gamma d} [2V_2 - W_{c2}] \quad (2.57)$$

$$W_{c2} = e^{-\gamma d} [2V_1 - W_{c1}]$$

Figure 2.11: Method of characteristics interconnect model

A lumped representation of the transmission line based on controlled sources can now be deduced from equations (2.55) and (2.57) as shown in Fig. 2.11. However, the expressions for W_{c1} and W_{c2} in 2.57 are still in the frequency domain. A time domain expression is needed in order to mix this interconnect stamp with time domain nonlinear differential equations. This is where the lossless line assumption is imposed. For the case of lossless lines, the propagation constant is simply $\gamma = s\sqrt{LC}$. In this case, the term $e^{-\gamma d}$ in (2.57) represents a simple delay in the time domain. Therefore, by analytically taking the inverse Laplace transform of (2.57) for the lossless line case we obtain

$$\begin{aligned} w_{c1}(t + \tau) &= 2v_2(t) - w_{c2}(t) \\ w_{c2}(t + \tau) &= 2v_1(t) - w_{c1}(t) \end{aligned} \tag{2.58}$$

where $\tau = d\sqrt{LC}$. With the aid of delayed sources, a time domain model for the lossless interconnects has been derived. In addition to its limited application to lossless lines, this approach presents additional difficulties for time domain simulators because the time shift affects the stability of the integration formula and typically results in more time steps being required for the integration. It remains, however, advantageous for lines with long flat delays [62].

For lossy lines, the propagation constant is not purely imaginary and, hence, cannot be replaced by a pure delay. In that case, analytical expressions such as (2.58) cannot be found in the time domain. Extensions of this method through Padé synthesis were proposed in order to handle the lossy lines case [61]. In the case of coupled lines, the method of characteristics is

applied through decoupling of the transmission line equations [2], [63].

2.4.2 Discretized Time Domain Interconnect Models

As was seen from the previous sections, one of the main difficulties in interconnect simulation is the fact that the Multi-conductor Transmission Line (MTL) equations or the *Telegrapher's equation* are partial differential equations and best solved in the frequency domain. However, a time domain interconnect stamp is required for nonlinear transient simulation. Delay extraction methods, based on the techniques presented in Section 2.4.1, are suitable for addressing this problem in the case of low loss lines with very long flat delays [62]. However, for the general case, some form of discretization of the Telegrapher's equations is required in order to obtain a time domain interconnect stamp in the form of ordinary differential equations. A number of such techniques have been proposed in the literature [2], [64]–[69]. Some of the main interconnect discretization methods are outlined below.

1. *Lumped Segmentation:* This approach is the brute force way of discretizing the lines. It is based on cutting the line into small sections, which are small enough to be modelled with lumped RLCG elements [2]. The resulting time domain model typically contains a large number of nodes.
2. *Chebyshev Polynomial Based Discretization:* This approach is based on expressing the voltage and current waveforms along the line in terms of Chebyshev polynomials [65]. This allows for the elimination of the spacial derivative in the Telegrapher's equations. A system of ordinary time domain differential equations can then be constructed in terms of the unknown Chebyshev coefficients. This approach is relatively more efficient than lumped segmentation, however the passivity of the macromodel cannot be guaranteed.
3. *Compact Difference Based Discretization:* This approach [66] is based on discretizing the voltages and currents along the line, and using compact difference [70] in order to

approximate the spacial derivatives along the line. It results in a passive lumped model that is relatively smaller and than that obtained from simple lumped segmentation.

4. *Interpolation Based Methods:* Interpolation techniques such as the one suggested in [71], can be used to obtain a frequency domain rational approximation of the multi-port scattering parameters of the interconnect. These rational approximations are then be converted to the time domain by finding the poles and zeros. The main difficulty with this approach is that it does not guarantee passivity by construction.
5. *Matrix Rational Approximation:* The Matrix Rational Approximation (MRA) approach is based on approximating the matrix exponential stamp in equation (2.40) by a matrix rational function. This is possible through the use of pre-determined scalar coefficients that can be obtained based on a Padé approximation [67], or on other more suitable approximations [68], [69]. Note that in the case of Padé these coefficients are obtained using closed form formulas and therefore do not suffer from the ill-conditioning of the Padé approximation. Expressing the matrix exponential in rational form, allows the construction of an efficient macromodel in the form of ordinary differential equations. This algorithm guarantees the passivity of the macromodel by construction, and is suitable for use with order reduction techniques.

What the above methods have in common, is that the resulting macromodel typically contains a large number of nodes. Such large circuits require a high CPU cost for time domain transient simulation. Regardless of the discretization method used, the final objective is to express the transmission line stamp in terms of time domain ordinary differential equations, which can be linked with nonlinear circuit simulators. For example, in the case of an interconnect network π , the stamp relating the port voltages $v_{\pi p}$ and current $i_{\pi p}$ is expressed in the form

$$G_{\pi} x_{\pi}(t) + C_{\pi} \frac{\partial x_{\pi}(t)}{\partial t} = B_{\pi} v_{\pi p}(t); \quad i_{\pi p} = B_{\pi}^T x_{\pi}(t) \quad (2.59)$$

Figure 2.12: Lumped model example

The above formulation is general regardless of how the discretized model is obtained. To illustrate this approach, a simple example is used. Consider the transmission line in Fig. 2.12. This line was discretized using brute force lumped segmentation into one RLCG section. For this simple circuit, the formulation in (2.59) is constructed as

$$\mathbf{v}_\pi = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}; \quad \mathbf{i}_\pi = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (2.60)$$

$$\mathbf{x}_\pi(t) = [V_1, V_x, V_2, I_L, I_1, I_2]^T \quad (2.61)$$

$$\mathbf{G}_\pi = \begin{bmatrix} \frac{1}{r} & -\frac{1}{r} & 0 & 0 & -1 & 0 \\ -\frac{1}{r} & \frac{1}{r} & 0 & 1 & 0 & 0 \\ 0 & 0 & g & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{C}_\pi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.62)$$

$$\mathbf{B}_\pi^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.63)$$

The above example was used only to illustrate the form of the equations. In general, a large number of lumped sections is required, resulting in large matrices. However, the form of the equations remains unchanged. The following section shows how a linear network described in the form of equation (2.59) can be combined with circuits containing nonlinear elements.

2.5 MNA Stencil of a Linear Subsection

In Section 2.4.2 interconnects were treated as a linear subsection, and modelled using a set of ordinary differential equations. This section describes how these subsections are linked with nonlinear elements during circuit simulation. Consider a general linear subsection k that may contain a single set of coupled interconnects as shown in the previous section, or many sets of coupled transmission lines as well as many lumped elements. Assume the linear subnetwork k contains n_k ports. After discretization, the linear subnetwork is represented by the following set of ordinary differential equations of order N_k

$$\mathbf{G}_k \mathbf{x}_k(t) + \mathbf{C}_k \frac{\partial \mathbf{x}_k(t)}{\partial t} = \mathbf{B}_k \mathbf{v}_k(t); \quad \mathbf{i}_k = \mathbf{B}_k^T \mathbf{x}_k(t) \quad (2.64)$$

where $\mathbf{G}_k$ and $\mathbf{C}_k \in \mathbb{R}^{N_k \times N_k}$ are the MNA matrices of the discretized network, $\mathbf{v}_k$ and $\mathbf{i}_k \in \mathbb{R}^{n_k}$ are the vectors containing the port voltages and currents, and $\mathbf{B}_k \in \mathbb{R}^{N_k \times n_k}$ is a selector matrix mapping the port voltages and currents to the node space of the ordinary differential equations describing subnetwork k . In the remainder of this section, we shall describe the stencil used to include linear subnetworks of the form in (2.64), into nonlinear time domain circuit equations.

Consider a circuit ϕ containing linear and nonlinear lumped components and N_t linear subnetworks. Linear subnetwork k is modelled using ordinary differential equations as described

in equation (2.64). Assume the network ϕ has N_ϕ nodal variables. The Modified Nodal Analysis (MNA) matrix equations in (2.1) can be written in the time domain, and appended with the linear sub-circuit stamp as follows

$$\begin{bmatrix} \mathbf{G}_\phi & \mathbf{D}_1 \mathbf{B}_1^T & \cdots & \mathbf{D}_{N_t} \mathbf{B}_{N_t}^T \\ -\mathbf{B}_1 \mathbf{D}_1^T & \mathbf{G}_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\mathbf{B}_{N_t} \mathbf{D}_{N_t}^T & 0 & 0 & \mathbf{G}_{N_t} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\phi(t) \\ \mathbf{x}_1(t) \\ \vdots \\ \mathbf{x}_{N_t}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{C}_\phi & 0 & \cdots & 0 \\ 0 & \mathbf{C}_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{C}_{N_t} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_\phi(t) \\ \dot{\mathbf{x}}_1(t) \\ \vdots \\ \dot{\mathbf{x}}_{N_t}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{f}_\phi(\mathbf{x}_\phi(t)) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_\phi(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2.65)$$

where $\mathbf{G}_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$, $\mathbf{C}_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$, $\mathbf{f}_\phi(\mathbf{x}_\phi) \in \mathbb{R}^{N_\phi}$ and $\mathbf{b}_\phi(t) \in \mathbb{R}^{N_\phi}$ represent the MNA equations of the lumped elements without the sub-circuits as defined in equation (2.1), $\mathbf{D}_k = [d_{i,j}]$ is a selector matrix with elements $d_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, N_\phi\}$, $j \in \{1, 2, \dots, n_k\}$ with a maximum of one non-zero in each row or column, that maps $\mathbf{v}_k(t) \in \mathbb{R}^{n_k}$ and $\mathbf{i}_k(t) \in \mathbb{R}^{n_k}$, the vectors of port voltages and currents entering subnetwork k , into the node space $\mathbb{R}^{N_\phi}$ of network ϕ . Note that in this formulation, the set of unknowns of each linear subnetwork are appended to the set of unknowns in the general MNA equations.

The above formulation will be illustrated by a simple example. Consider the circuit in Fig. 2.13. This example contains only one sub-circuit which we shall refer to as sub-circuit 1. This two-port sub-circuit is described by the following ordinary differential equations

$$\mathbf{G}_1 \mathbf{x}_1(t) + \mathbf{C}_1 \frac{\partial \mathbf{x}_1(t)}{\partial t} = \mathbf{B}_1 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad (2.66a)$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \mathbf{B}_1^T \mathbf{x}_1(t) \quad (2.66b)$$

Figure 2.13: Example of a linear subcircuit

where V_1 , V_2 , I_1 and I_2 are the port voltages and currents as shown in Fig. 2.13. Now we proceed to writing the circuit equations for the network in Fig. 2.13. The MNA equations of the overall circuit can be written according to the formulation in (2.65) as

$$\begin{bmatrix} \mathbf{G}_\phi & \mathbf{D}_1 \mathbf{B}_1^T \\ -\mathbf{B}_1 \mathbf{D}_1^T & \mathbf{G}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_\phi(t) \\ \mathbf{x}_1(t) \end{bmatrix} + \begin{bmatrix} \mathbf{C}_\phi & 0 \\ 0 & \mathbf{C}_1 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_\phi(t) \\ \dot{\mathbf{x}}_1(t) \end{bmatrix} + \begin{bmatrix} \mathbf{f}_\phi(\mathbf{x}_\phi(t)) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_\phi(t) \\ 0 \end{bmatrix} \quad (2.67)$$

Where the MNA matrices $\mathbf{G}_\phi$, $\mathbf{C}_\phi$, $\mathbf{b}_\phi$ and $\mathbf{f}(\mathbf{x}_\phi)$ describing the lumped elements without the sub-network are given by

$$\mathbf{x}_\phi = [V_1, V_2, V_3, I_E]^T \quad (2.68)$$

$$\mathbf{f}(\mathbf{x}_\phi) = [f(V_1), 0, 0, 0]^T \quad (2.69)$$

$$\mathbf{b}_\phi = [0, 0, 0, E_i]^T \quad (2.70)$$

$$\mathbf{G}_\phi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_1} & -\frac{1}{R_1} & 0 \\ 0 & -\frac{1}{R_1} & \frac{1}{R_1} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{C}_\phi = \begin{bmatrix} C_2 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.71)$$

The matrix D_1 that maps the port voltages and currents of the sub-circuit into the node space $\mathbb{R}^{N_\phi}$ of the network ϕ is given by

$$D_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (2.72)$$

2.6 Conclusion

In this chapter, a unified circuit formulation based on the modified nodal analysis (MNA) approach was presented. This approach is general, and consistently used throughout the thesis for both interconnect analysis and nonlinear steady-state analysis. Furthermore, special attention was given to difficult elements such as high speed interconnects, and a review of interconnect models and stencils was provided. These models provide the basis for understanding the difficulties of interconnect simulation. The formulations provided in this chapter form the basis on which the model order reduction methods discussed in later chapters are applied.

Chapter 3

Background on Circuit Simulation

In the previous chapter, a general circuit formulation including RF circuits and high-speed interconnects was introduced. This chapter introduces conventional simulation methods that are applied to the above mentioned formulation for different types of analysis. Some of the difficulties facing these conventional techniques are outlined. These difficulties are addressed in later chapters using projection based MOR methods.

3.1 Linear Frequency Domain Analysis

Frequency domain or AC simulation is based on phasor analysis [72]. It is generally performed on linear circuits, or on nonlinear circuits which are linearized around an operating point (i.e. small signal analysis). This approach is very useful for certain types of applications, such as finding the frequency response of filters and amplifiers. However, because all nonlinearities are linearized around the operating point, this type of analysis cannot be used for finding the nonlinear distortion effects, or to analyze circuits such as mixers where the nonlinear behavior is an integral part of the circuit function. The main CPU cost of this analysis is one LU decomposition of the MNA matrices at each frequency point. This can be easily deduced by observing the analysis steps. The MNA equations for a linear circuit ϕ are easily obtained from

equation (2.1) and can be written as

$$\mathbf{G}_\phi \mathbf{x}_\phi(t) + \mathbf{C}_\phi \dot{\mathbf{x}}_\phi(t) = \mathbf{b}_\phi(t) \quad (3.1)$$

For a sinusoidal input at a frequency ω , the phasor form of vector $\mathbf{b}_\phi(t)$ can be expressed as

$$\mathbf{b}_\phi(t) \rightarrow \mathbf{B}_{ph} e^{j\omega t} \quad (3.2)$$

where the phasor $\mathbf{B}_{ph} \in \mathbb{C}^{N_\phi}$ is a vector of complex numbers containing the phase and amplitude information of the input. Since the circuit is linear, the output $\mathbf{x}_\phi(t)$ would also contain a single frequency ω with a phasor given by

$$\mathbf{x}_\phi(t) \rightarrow \mathbf{X}_{ph} e^{j\omega t} \quad (3.3)$$

In phasor form, the MNA equations in (3.1) become

$$\mathbf{G}_\phi \mathbf{X}_{ph} e^{j\omega t} + \mathbf{C}_\phi \frac{\partial}{\partial t} \mathbf{X}_{ph} e^{j\omega t} = \mathbf{B}_{ph} e^{j\omega t} \quad (3.4)$$

After performing the time domain differentiation, and eliminating the $e^{j\omega t}$ equation (3.4) becomes

$$\mathbf{G}_\phi \mathbf{X}_{ph} + j\omega \mathbf{C}_\phi \mathbf{X}_{ph} = \mathbf{B}_{ph} \quad (3.5)$$

The phasor $\mathbf{X}_{ph}$ can therefore be found by solving the above system of linear equations, i.e.

$$\mathbf{X}_{ph}(j\omega) = (\mathbf{G}_\phi + j\omega \mathbf{C}_\phi)^{-1} \mathbf{B}_{ph} \quad (3.6)$$

As can be seen from (3.6) the main cost of the frequency domain analysis is one LU decomposition at each frequency point. The most common type of frequency analysis is to set the input to 1 in order to obtain the frequency domain transfer function.

3.2 DC Analysis

DC analysis is the most common type of analysis because it is typically required as a starting point for other types of analysis such as transient simulation or small signal AC analysis. The

DC analysis computes the DC solution or the operating point of a circuit with respect to the DC sources. Consider the MNA equations of a DC circuit based on the formulation in (2.1)

$$\mathbf{G}_\phi \mathbf{x}_\phi(t) + \mathbf{C}_\phi \dot{\mathbf{x}}_\phi(t) + \mathbf{f}_\phi(\mathbf{x}_\phi(t)) = \mathbf{b}_{\phi dc} \quad (3.7)$$

Note that in the above MNA equations, the source vector $\mathbf{b}_{\phi dc}$ is constant with respect to time. Therefore, at equilibrium, the solution vector $\mathbf{x}_\phi(t)$ is also constant, and its derivative with respect to time is zero. Equation (3.7) is therefore reduced to

$$\mathbf{G}_\phi \mathbf{x}_\phi + \mathbf{f}_\phi(\mathbf{x}_\phi) = \mathbf{b}_{\phi dc} \quad (3.8)$$

Removing the $\mathbf{C}_\phi$ term from (3.7) essentially amounts to short circuiting all inductors and open circuiting all capacitors. The resulting equations in (3.8) are simply a set of nonlinear algebraic equations which can be solved using iterative methods such as Newton Iteration. Newton iteration methods essentially convert the problem of solving a set of nonlinear algebraic equations, into a sequence of solutions of linear equations as follows. The problem in (3.8) is reformulated as finding the root of the following equation, with the subscript ϕ dropped for simplicity

$$\psi(\mathbf{x}) = \mathbf{G}\mathbf{x} + \mathbf{f}(\mathbf{x}) - \mathbf{b}_{dc} \quad (3.9)$$

Starting with an initial guess $\mathbf{x}^{(0)}$, the solution is updated at each iteration as follows

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} + \Delta\mathbf{x} \quad (3.10)$$

where

$$\Delta\mathbf{x} = \mathbf{J}_{dc}^{-1} \psi(\mathbf{x}^{(i)}) \quad (3.11)$$

and the jacobian matrix $\mathbf{J}_{dc}$ is given by

$$\mathbf{J}_{dc} = \left. \frac{\partial}{\partial \mathbf{x}} \psi(\mathbf{x}) \right|_{\mathbf{x}=\mathbf{x}^{(i)}} = \mathbf{G} + \left. \frac{\partial}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) \right|_{\mathbf{x}=\mathbf{x}^{(i)}} \quad (3.12)$$

Convergence is reached when two conditions are satisfied. The first is $\Delta\mathbf{x}$ in (3.11) must be less than an error tolerance, and the second is that equation (3.9) or in more precisely Kirchhoff's Current Law (KCL) must be satisfied. Note that the main CPU cost in each iteration is the LU factorization of the jacobian matrix in (3.12).

3.3 Transient Time Domain Analysis

Transient simulation is another important form of analysis common in nonlinear simulators such as SPICE. Transient simulation is an initial value problem, where the initial conditions are known. The response is obtained by starting with these initial conditions, and stepping forward in time using numerical time-domain integration. Typical integration methods are based on Linear Multi-Step (LMS) [50] formulas such as the Trapezoidal Rule and the Backward Euler rule. As an example, consider the Trapezoidal rule which is a second order method defined as

$$\mathbf{x}(t+h) = \mathbf{x}(t) + \frac{h}{2}(\dot{\mathbf{x}}(t) + \dot{\mathbf{x}}(t+h)) \quad (3.13)$$

The Trapezoidal Rule can be applied to the MNA equations in (2.1) in order to derive the following difference equations

$$\begin{aligned} (\mathbf{G}_\phi + \frac{2}{h}\mathbf{C}_\phi)\mathbf{x}_\phi(t+h) + \mathbf{f}_\phi(\mathbf{x}_\phi(t+h)) &= \\ &= (\frac{2}{h}\mathbf{C}_\phi - \mathbf{G}_\phi)\mathbf{x}_\phi(t) - \mathbf{f}_\phi(\mathbf{x}_\phi(t)) + \mathbf{b}_\phi(t) + \mathbf{b}_\phi(t+h) \end{aligned} \quad (3.14)$$

Equation (3.14) can be written in the form

$$\mathbf{A}_\phi \mathbf{x}_\phi(t+h) + \mathbf{f}_\phi(\mathbf{x}_\phi(t+h)) = \mathbf{B}_\phi \quad (3.15)$$

with

$$\begin{aligned} \mathbf{A}_\phi &= \mathbf{G}_\phi + \frac{2}{h}\mathbf{C}_\phi \\ \mathbf{B}_\phi &= (\frac{2}{h}\mathbf{C}_\phi - \mathbf{G}_\phi)\mathbf{x}_\phi(t) - \mathbf{f}_\phi(\mathbf{x}_\phi(t)) + \mathbf{b}_\phi(t) + \mathbf{b}_\phi(t+h) \end{aligned} \quad (3.16)$$

By applying the Trapezoidal rule in (3.13), the nonlinear differential algebraic equations (DAEs) in (2.1) were converted into the difference equations in (3.15) which are a set of nonlinear algebraic equations that must be solved at each time step. This solution can be found using iterative methods such as Newton iteration in a similar way to what was described for the nonlinear equations resulting from the DC analysis as shown in Section 3.2.

Note that the trapezoidal rule is a Linear Multi-Step (LMS) [50] method of order 2, which is the highest order that is absolutely stable [73]. While the use of such a low order method

Figure 3.1: Simple second order circuit

is necessary in order to ensure stability, maintaining accuracy for low order methods results in a very small step size. This in turn leads to a large number of time steps required to perform the transient analysis. At each time step, Newton Iteration is used to find the solution of the nonlinear equation in (3.14), thus requiring many costly LU decompositions of the Jacobian matrix. Indeed nonlinear transient analysis is one of the most expensive types of analysis in conventional simulators such as SPICE, especially for the case of interconnect networks with large MNA matrices.

3.4 Concept of Steady-State Analysis

The types of analysis covered in the previous sections are the most common in general circuit simulation and are available in conventional simulators such as SPICE. In the case of nonlinear RF circuits, design requirements based on nonlinear distortion are imposed, and metrics such as intercept points (IP3) and 1-dB gain compression points [74] must be simulated in the design process. However, such figures of merit are not readily obtained using the analysis methods discussed so far, and another class of techniques, namely nonlinear steady-state analysis methods must be used.

The steady-state solution of a circuit is most commonly defined as the solution after all

the transients have died out. In order to illustrate the general concept of finding the steady-state response, the simple second order circuit shown in Fig. 3.1 is considered. The transient response of the circuit with a step input of $I(t) = u(t)A$ is shown in Fig. 3.2. In this case, the steady-state after all transients have died out corresponds to the DC solution. The DC solution is a common example of a steady-state response. The focus in this thesis is, however, on the steady-state response due to periodic inputs. In Fig. 3.3, the response of the circuit due to a 120MHz sinusoidal input $I(t) = \sin(2\pi 120 \times 10^6 t)$ is shown. In this case as well, the state state response is the signal after all transients have died out. However, as is evident from the plots, the signal goes through a number of cycles before reaching steady-state even for this simple circuit. Practical circuits with widely varying time constants, and circuits, such as mixers, with a wide range of frequencies may require thousands of cycles before reaching steady-state. The calculation of the steady-state response of a linear circuit such as the example in Fig. 3.1, can be easily done using phasor analysis [72] (see Section 3.2). This method is in fact available in most conventional simulators. However, phasor analysis is not appropriate for the simulation of circuits, such as mixers, with inherent nonlinear behavior, and for distortion and intermodulation analysis, where the nonlinearities in the circuit cannot be ignored. In the following two sections, analysis methods for finding the nonlinear steady-state solution will be presented.

3.5 Time Domain Steady-State Simulation Methods

Due to the presence of nonlinear elements, the circuit equations are most naturally expressed in the time domain. In this section some approaches for finding the steady-state solution directly in the time domain are described. It is important to note however, that such techniques would face difficulties dealing with distributed elements which are typically modelled in the frequency domain.

Figure 3.2: Steady-state response with a step input

Figure 3.3: Steady-state response with a periodic input

3.5.1 Conventional Transient Analysis

The most obvious way to find the steady-state solution, is to perform a transient simulation, as described in Section 3.3, which is long enough for all the transients to disappear. FFT is then applied to the steady-state signal in order to obtain the tones. In fact this is the basis of the *four* analysis in SPICE [51]. This approach presents a number of difficulties in terms of accuracy and CPU cost.

As was seen in Section 3.3, transient analysis is one of the most CPU intensive types of analysis in SPICE. This is due to the fact that at each iteration, a large system of nonlinear equations must be solved at a cost of several LU decompositions of a large jacobian. In the case of steady-state analysis long simulation time intervals requiring a large number of time steps are often required before steady-state is reached. This large number of time steps directly translates into a prohibitive CPU cost. This problem is encountered in circuits with a high Q , where the circuit goes through many cycles before reaching the steady-state. This is also an issue for circuits with non-commensurate input tones, which is typically the case when doing intermodulation analysis. For example, consider a circuit with input tones at $f_1 = 900$ MHz and $f_2 = 901$ MHz. Fourier analysis requires the tones to be commensurate (i.e. all frequencies must be multiples of a common frequency). In this case the common period will correspond to the frequency $f_2 - f_1 = 1$ MHz. Assuming that the highest frequency is at 9010 MHz (although higher harmonics may be present), the time step must be less than half the period at this high frequency. With the ratio of the highest to lowest frequency at 9010:1, a large number of time steps and considerable CPU cost would be required before reaching the steady-state.

In addition to the CPU cost, this approach of finding the steady-state response faces a number of accuracy difficulties related to Fourier analysis. There are a number of mechanisms that lead to such errors [51], such as an incorrect period, aliasing, interpolation, and simulation noise. The condition of reaching steady-state is the periodicity of the signal, or in other words $x(t + T) = x(t)$. Any error in meeting this condition results in a small transient at the end of the signal in order to insure periodicity. This small transient has a very broad spectrum

and add to the error of the FFT analysis. Aliasing errors arise if the sampling period of the FFT is too large (i.e. not enough tones are considered). In this case, the power of higher frequency tones is aliased into the lower frequency tones causing errors. Interpolation errors arise because FFT requires equally spaced points which are found using interpolation from the simulated time points. Finally when using transient analysis to obtain steady-state results, tight error tolerances must be imposed on the nonlinear simulator in order to capture frequency domain tones with a wide dynamic range.

3.5.2 Shooting Method

The shooting method [75]–[81] is a time domain method specifically constructed to find the steady-state solution. When steady-state is reached, the response is periodic with $\mathbf{x}_\phi(t) = \mathbf{x}_\phi(t+T)$, where the vector $\mathbf{x}_\phi(t)$ is defined in (2.1), and T is the period of the input. Using the shooting method, the periodic response is formulated as the solution of a two point boundary value problem [82]. The basic idea is to find an initial condition vector $\mathbf{x}_\phi(0)$ such that when the system equations are integrated over one period T we obtain

$$\mathbf{x}_\phi(T) - \mathbf{x}_\phi(0) = 0 \quad (3.17)$$

Using the shooting method, the problem is therefore reduced to finding an initial value $\mathbf{x}_\phi(0) = \mathbf{x}_0$, such that equation (3.17) is satisfied. This would, by definition, result in the periodic steady-state solution. In order to solve the boundary value problem in (3.17), the state transition function $\Phi(\mathbf{x}_\phi(0), t)$ is defined. This function is evaluated by performing transient analysis from 0 to t with $\mathbf{x}_\phi(0)$ as the initial condition. Therefore, assuming an initial condition of $\mathbf{x}_\phi(0) = \mathbf{x}_0$ we can write

$$\mathbf{x}_\phi(T) = \Phi(\mathbf{x}_0, T) \quad (3.18)$$

In order to satisfy the boundary condition in (3.17) we must have

$$\mathbf{x}_0 = \Phi(\mathbf{x}_0, T) \quad (3.19)$$

The solution to such a set of nonlinear equations is found using iterative methods. One possible method is the fixed point iteration. That is, starting with an initial guess of $\mathbf{x}_0 = \mathbf{x}^{(0)}$ the following iteration is performed

$$\mathbf{x}^{(i+1)} = \Phi(\mathbf{x}^{(i)}, T) \quad (3.20)$$

Convergence is achieved when $\mathbf{x}^{(i+1)} \approx \mathbf{x}^{(i)}$. This method has the advantage of being very simple but it is inefficient. In fact, if examined carefully, it is equivalent to performing transient analysis until transients die out. One can therefore think of the transient based approach to finding the steady-state solution as described in Section 3.5.1, as a special case of the shooting method. Other more efficient methods for solving the shooting equations, such as gradient and extrapolation methods [78], [81], have been proposed in the literature. In this section the solution by Newton iterations will be described as an example. In order to simplify the notation, the problem will be defined as finding the root of the nonlinear function $\Psi(\mathbf{x}_0)$, which is written as

$$\Psi(\mathbf{x}_0) = \Phi(\mathbf{x}_0, T) - \mathbf{x}_0 \quad (3.21)$$

Starting with an initial guess of $\mathbf{x}_0 = \mathbf{x}_0^{(0)}$, and applying the Newton-Raphson algorithm, the following iteration can be derived

$$\mathbf{x}_0^{(i+1)} = \mathbf{x}_0^{(i)} - \mathbf{J}_s^{-1} \Psi(\mathbf{x}_0^{(i)}) \quad (3.22)$$

where the Jacobian $\mathbf{J}_s$ of the shooting equations is defined as

$$\mathbf{J}_s = \frac{\partial \Psi(\mathbf{x}_0)}{\partial \mathbf{x}_0} = \frac{\partial \Phi(\mathbf{x}_0, T)}{\partial \mathbf{x}_0} - \mathbf{U} \quad (3.23)$$

where $\mathbf{U}$ is the identity matrix and $\partial \Phi(\mathbf{x}_0, T)/\partial \mathbf{x}_0$ is the sensitivity of the state transition function with respect to the initial condition. In order to calculate $\partial \Phi(\mathbf{x}_0, T)/\partial \mathbf{x}_0$, the first variational equation is derived by differentiating the MNA equations in (2.1) with respect to $\mathbf{x}_0$ to obtain

$$\mathbf{G}_\phi \frac{\partial \mathbf{x}_\phi(t)}{\partial \mathbf{x}_0} + \mathbf{C}_\phi \frac{\partial \dot{\mathbf{x}}_\phi(t)}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{f}(\mathbf{x}_\phi(t))}{\partial \mathbf{x}_\phi(t)} \frac{\partial \mathbf{x}_\phi(t)}{\partial \mathbf{x}_0} = 0 \quad (3.24)$$

or

$$\mathbf{G}_\phi \mathbf{z}(t) + \mathbf{C}_\phi \dot{\mathbf{z}}(t) + \mathbf{A}_\phi(t) \mathbf{z}(t) = 0 \quad (3.25)$$

where

$$\mathbf{z}(t) = \frac{\partial \mathbf{x}_\phi(t)}{\partial \mathbf{x}_0}; \quad \mathbf{A}_\phi(t) = \frac{\partial \mathbf{f}_\phi(\mathbf{x}_\phi(t))}{\partial \mathbf{x}_\phi(t)} \quad (3.26)$$

Note that $\mathbf{z}(t) \in \mathbb{R}^{N_\phi \times N_\phi}$ is a matrix, with the initial value equal to the identity matrix

$$\mathbf{z}(0) = \frac{\partial \mathbf{x}(0)}{\partial \mathbf{x}_0} = \frac{\partial \mathbf{x}(0)}{\partial \mathbf{x}(0)} = \mathbf{U} \quad (3.27)$$

and

$$\mathbf{z}(T) = \frac{\partial \mathbf{x}_\phi(T)}{\partial \mathbf{x}_0} = \frac{\partial \Phi(\mathbf{x}_0, T)}{\partial \mathbf{x}_0} \quad (3.28)$$

The desired sensitivity matrix $\partial \Phi(\mathbf{x}_0, T)/\partial \mathbf{x}_0$ can therefore be obtained by integrating the first variational equation in (3.25) over the shooting interval $[0, T]$. Since $\mathbf{z}(t)$ is a matrix, this integration has to be done N_ϕ times. Computational savings can be achieved if all the simulations were done simultaneously with the transient simulation over the shooting interval which is also required to compute $\Psi(\mathbf{x}_0^{(i)})$ in (3.22). However, this is still a costly process especially for large circuits. Note that this simulation must be done for each Newton iteration. Furthermore, at each Newton iteration, the resulting dense jacobian $\mathbf{J}_s$ must be factorized.

3.6 Harmonic Balance Technique

The Harmonic Balance (HB) technique is a method for finding the steady-state solution of nonlinear circuits in the frequency domain. This method relies on the fact that, for periodic inputs, the steady-state response is known to consist of frequencies at the harmonics of the input tone. Consider a circuit ϕ with the MNA equation as defined in equation (2.1) as

$$\mathbf{G}_\phi \mathbf{x}_\phi(t) + \mathbf{C}_\phi \dot{\mathbf{x}}_\phi(t) + \mathbf{f}_\phi(\mathbf{x}_\phi(t)) = \mathbf{b}_\phi(t) \quad (3.29)$$

For a periodic input $\mathbf{b}_\phi(t)$ the response is also known to be periodic in the steady-state. In the Harmonic Balance approach, the steady space response is found by assuming a periodic

solution for equation (3.29) in the form of a finite Fourier series

$$\mathbf{x}_\phi(t) = \mathbf{A}_0 + \sum_{k=1}^H (\mathbf{A}_k \cos(k\omega t) + \mathbf{B}_k \sin(k\omega t)) \quad (3.30a)$$

$$\dot{\mathbf{x}}_\phi(t) = \sum_{k=1}^H (\mathbf{B}_k k\omega \cos(k\omega t) - \mathbf{A}_k k\omega \sin(k\omega t)) \quad (3.30b)$$

The value of the nonlinear functions in $\mathbf{f}_\phi(\mathbf{x}_\phi(t))$ can also be expressed as a truncated Fourier series

$$\mathbf{f}_\phi(\mathbf{x}_\phi(t)) = \mathbf{F}_{co} + \sum_{k=1}^H (\mathbf{F}_{ci} \cos(k\omega t) + \mathbf{F}_{si} \sin(k\omega t)) \quad (3.31)$$

The number of coefficients in the above series is given by

$$N_h = 2H + 1 \quad (3.32)$$

Expressing the vector of unknowns $\mathbf{x}_\phi(t)$ in (3.29) as a Fourier series as shown in equation (3.30), and equating the coefficients of the sines and cosines, leads to a system of $N_{hb} = N_\phi N_h$ nonlinear algebraic equations with the Fourier coefficients as the unknowns. This set of nonlinear equations is referred to as the Harmonic Balance equations and can be expressed in the form

$$\bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) = \mathbf{B}_h \quad (3.33)$$

where

- $\mathbf{X}_h \in \mathbb{R}^{N_{hb}}$ is a vector containing the unknown Fourier coefficients of $\mathbf{x}_\phi(t)$,
- $\mathbf{B}_h \in \mathbb{R}^{N_{hb}}$ is a vector of the Fourier coefficients of the input sources,
- $\bar{\mathbf{G}}$ is a block matrix $\bar{\mathbf{G}} = [\mathbf{G}_{ij}]$ whose blocks $\mathbf{G}_{ij} \in \mathbb{R}^{N_h \times N_h}$ are diagonal matrices given by

$$\mathbf{G}_{ij} = \text{diag}(g_{ij}, \dots, g_{ij}) \quad (3.34)$$

where g_{ij} is the corresponding element in the $\mathbf{G}_\phi$ matrix in (7.1),

- $\bar{C}$ is also a block matrix $\bar{C} = [C_{ij}]$ whose blocks $C_{ij} \in \mathbb{R}^{N_h \times N_h}$ are given by

$$C_{ij} = c_{ij} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega & \cdots & 0 & 0 \\ 0 & -\omega & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & H\omega \\ 0 & 0 & 0 & \cdots & -H\omega & 0 \end{bmatrix} \quad (3.35)$$

where c_{ij} is the corresponding element in the C_ϕ matrix in (7.1),

- $F(X_h)$ contains the Fourier coefficients of the nonlinear vector $f_\phi(x_\phi(t))$ defined in (7.1)
- $N_h = 2H + 1$ is the number of unknown Fourier coefficients at each node
- $N_{hb} = N_\phi N_h$ is the total number of unknowns in the harmonic balance equations.

The harmonic balance technique succeeds in converting the nonlinear differential algebraic MNA equations into a set of nonlinear algebraic equations. The steady-state solution is therefore found by solving this set of nonlinear algebraic equations, instead of having to deal with the original set of differential algebraic equations. However, as can be seen from the above formulation, the resulting set of equations is quite large and is therefore computationally expensive to solve.

3.7 Recent Difficulties in Circuit Simulation

In this section, some of the main difficulties facing circuit simulation, namely the issues of high-speed interconnects and the nonlinear steady-state simulation of RF circuits are discussed. These tend to lead to increasingly large sets of circuit equations to be solved. These issues are addressed in later chapters using projection based model order reduction techniques.

3.7.1 Simulation of High-Speed Interconnects and Packages

In recent years, signal integrity has become an important performance limiting factor in high speed VLSI systems [1], [3]. Furthermore, signal integrity simulations are presenting significant challenges to current circuit simulators. These challenges are twofold. The first is related to circuit size. Due to the large number of parasitics at high frequencies, circuit and package extraction typically results in a very large number of elements. This significantly increases the size of the MNA matrices, and could lead to a prohibitive CPU cost. The second issue is related to high-speed interconnects, which are becoming dominant contributors to signal integrity degradation [1], [3], [5]. Accurate interconnect simulation is therefore a necessary step in the design process. Interconnects, however, represent a significant challenge for conventional circuit simulators. The main reason interconnects cause simulation difficulties is the fact that, at high frequencies, they are modelled as distributed multi-conductor transmission lines. The interconnect simulation problem has been extensively studied in the literature [2], [10]–[35], [64]–[69], [83], [84] and a number of solutions have been proposed. Each approach addresses the problem from a different perspective; however, from a conceptional point of view, all these methods are attempting to efficiently include a distributed element, in a nonlinear time domain simulation. The main issue here is that distributed elements are best characterized in the frequency domain, while nonlinear elements are represented in the time domain. In other words, the fundamental problem is the so called mixed frequency/time domain nature.

The mixed frequency/time domain problem is illustrated in Fig. 3.4. The fundamental issue is that, as shown in Section 2.3, multi-conductor transmission lines are modelled by the Telegrapher's equations which are partial differential equations in space and time

$$\frac{\partial \mathbf{v}_\pi(z, t)}{\partial z} = -\mathbf{R}_\pi \mathbf{i}_\pi(z, t) - \mathbf{L}_\pi \frac{\partial \mathbf{i}_\pi(z, t)}{\partial t} \quad (3.36a)$$

$$\frac{\partial \mathbf{i}_\pi(z, t)}{\partial z} = -\mathbf{G}_\pi \mathbf{v}_\pi(z, t) - \mathbf{C}_\pi \frac{\partial \mathbf{v}_\pi(z, t)}{\partial t} \quad (3.36b)$$

Such partial differential equations are best solved in the frequency domain, thus resulting in

Figure 3.4: Mixed Frequency/Time problem

frequency domain interconnect stamps as was described in sections 2.3.2 and 2.3.3. These Transmission line stamps are complex functions of frequency, and cannot be readily expressed as ordinary differential equations in the time domain. Furthermore, the per-unit-length parameters of the interconnects, may also be functions of frequency, which makes the transmission line stamp an even more complicated function of frequency. Obtaining a time domain representation of the TL stamp is, therefore, not a simple task. As illustrated in Fig. 3.4, this mixed frequency/time domain representation would cause difficulties when simulating circuits containing nonlinear elements. In this case, parts of the circuit equations (those describing the interconnects) are expressed in the frequency domain, while the nonlinear elements are best characterized in the time domain. Convolution methods can be used in this case, but their CPU and memory requirements are very high, as the impulse response has to be convolved with the entire computed waveform at each time step.

A time domain interconnect model in the form of ordinary differential equations is therefore required for nonlinear transient simulations. Such models are typically obtained using some

Figure 3.5: Interconnect discretization

form of discretization of the Telegrapher's equations [4], [64]–[67], [83], [85] as was discussed in Section 2.4. The resulting time domain models can now be included in the simulation as illustrated in Fig. 3.5. However, the difficulty with such models is that they are typically very large, and require a very high CPU cost when simulated using conventional nonlinear simulators such as SPICE.

Whether the large size of the MNA equations results from a large number of parasitics emanating from packages, vias and ground planes, or whether it is due to the large lumped models representing high speed interconnects, the main issue facing the simulator is to efficiently solve these large systems of equations. This has led to the use of model reduction techniques in order to improve the efficiency of signal integrity simulation.

3.7.2 Simulation of RF Circuits

In this chapter, the problem of finding the nonlinear steady-state solution was introduced, and the various general approaches to obtaining the steady-state solution were discussed. The harmonic balance method reformulates the steady-state problems by setting the Fourier coeffi-

icients of the solution as the unknowns. This approach has the advantage of transforming the set of nonlinear Differential Algebraic Equations (DAEs) into a set of nonlinear algebraic equations which can be solved using iterative methods. These equation can be solved using any of the iterative techniques available in the literature, such Newton iteration for example. However, there are a number of difficulties associated with this approach. The first is a general problem related to the convergence of iterative methods. For example, the Newton Iteration does not guarantee convergence for any initial guess. The second issues is due to the dense structure of the jacobian matrix. This matrix is composed of a large number of dense blocks, which significantly affects the memory and CPU cost requirements of the Newton Iteration. Finally, the size of the harmonic balance equations is significantly larger than that of the original set of differential algebraic equations. For example, while each node voltage in the original set of equations corresponded to on time domain variable in the original DAEs, in the harmonic balance equations each node voltage is associated with a large number of unknowns corresponding to the Fourier series coefficients of the steady-state waveform at that node. The size of the original equations is therefore multiplied by the number of Fourier coefficients. This could be a significant problem in circuit with highly nonlinear behavior where a large number of harmonics must be considered. Another example, is circuits with two or three tone inputs where not only the harmonics of all tones must be included, but also the intermodulation products. In such cases the size of the equations could quickly overwhelm existing circuit simulators. As outlined in Section 7.2.3 a number of improvements have been proposed in the literature such as [86]–[91]. However, even the most efficient of these approaches still require a good pre-conditioner in order to converge, and finding such a pre-conditioner is a difficult problem for the case of the harmonic balance equations [90]–[92]. All these techniques, essentially try to apply the latest and most efficient iterative method for the case of solving the HB equations. In this thesis, the problem is addressed from a different perspective. Instead of solving the original HB equations, a reduction method is proposed for reducing the size of the original circuit and resulting in a much smaller set of equations. The new method uses continuation techniques

as a vehicle to enable the reduction process, thus, simultaneously addressing both convergence and numerical complexity issues. The details of this method are given in Chapter 7.

Chapter 4

Review of Model Order Reduction

Model order reduction techniques have been proposed in the circuit simulation literature as a way to deal with the large lumped models that result from packaging and interconnect structures [4], [10]–[35]. The basic concept is that while these lumped models contain a large number of poles (an infinite number of poles in the case of networks with distributed components), only a fraction of these poles has a significant impact on the circuit response in the frequency range of interest. Model reduction takes advantage of this fact by approximating the original large system with reduced order model that capture the essential characteristics, or the dominant poles of the original system. The reduction methods in the circuit simulation literature can be classified under two main classes: 1) Direct moment matching based on Padé approximation, and 2) Indirect approximations based on finding the leading eigenvalues (those with the largest magnitude).

4.1 Direct Moment Matching

Direct moment matching techniques (MMTs) [4], [10]–[16], [27]–[35] are based on directly finding a rational approximation of the impulse response in the frequency domain, typically through the use of Padé approximation. This rational approximation is constructed so that it captures the dominant poles of the system, and matches the behavior of the original impulse

response from dc up to a frequency which depends on the order of the approximation. Once a rational function approximation of the circuit response is found, it can simply be evaluated in order to find the frequency domain response, or it can be used to find the poles and residues of the system and construct a time domain macromodel. There are two main types of direct moment matching techniques. The first, also referred to in the literature as asymptotic waveform evaluation (AWE), is based on a single moment expansion (typically at the origin of the complex plane), and the second is based on performing multiple expansions along the complex frequency axis. This section starts with a review of Padé approximation, which is the basis for direct moment matching methods. It is then followed by brief descriptions of single and multi-point expansion MMTs.

4.1.1 Padé Approximants

Cauchy provided the foundation of Padé approximants in 1821, and many others have studied them since. Baker [93] is credited with introducing this concept to the circuit simulation field in 1975. Padé approximants employ a rational function approximation to fit the desired response. This type of approximation has proven to be very suitable for network transfer functions. MMTs use Padé to fit a rational function to the moments of the system transfer function. For example, for the following transfer function with moments m_i

$$h(s) = m_0 + m_1s + m_2s^2 + \dots \quad (4.1)$$

a set of equations could be derived to obtain a rational approximation of order L/M for $h(s)$

$$\begin{aligned} h(s) &= m_0 + m_1s + m_2s^2 + \dots \\ &= \frac{a_0 + a_1s + a_2s^2 + \dots + a_Ls^L}{1 + b_1s + \dots + b_Ms^M} \end{aligned} \quad (4.2)$$

By cross multiplying equation (4.2) and equating the coefficients of equal powers of s , expressions for the coefficients a_i and b_i are obtained as

$$\begin{bmatrix} m_{L-M+1} & m_{L-M+2} & \cdots & m_L \\ m_{L-M+2} & \ddots & \ddots & m_{L+1} \\ \vdots & \ddots & \ddots & \vdots \\ m_L & m_{L+1} & \cdots & m_{L+M-1} \end{bmatrix} \begin{bmatrix} b_M \\ b_{M-1} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_{L+1} \\ m_{L+2} \\ \vdots \\ m_{L+M} \end{bmatrix} \quad (4.3)$$

$$a_0 = m_0$$

$$a_1 = m_1 + b_1 m_0$$

$$a_2 = m_2 + b_1 m_1 + b_2 m_0 \quad (4.4)$$

$$\vdots$$

$$a_L = m_L + \sum_{i=1}^{\min(L,M)} b_i m_{L-i}$$

Obtaining the Padé coefficients is therefore a very simple procedure once the moments of the transfer function are found. Determining the optimum order of the approximation is, however, a less obvious problem to solve. A discussion of Padé orders can be found in [13]. Padé approximations based on a single expansion point MMTs, and multi-expansion point MMTs were proposed in the literature, and are described in the following sections.

4.1.2 Single Expansion Moment Matching

Moment matching based on a single expansion point (normally at the origin of the complex plane) [11], [13], [14] is referred to as asymptotic waveform evaluation (AWE). As seen in the previous section Padé approximation as applied to the moments (i.e. the derivatives with respect to s) of the original system. As an example of moment calculation, consider a linear circuit ϕ , containing lumped elements only. For such a circuit, the MNA equations in (2.1) can be written in the frequency domain as

$$\mathbf{G}_\phi \mathbf{X}_\phi(s) + s \mathbf{C}_\phi \mathbf{X}_\phi(s) = \mathbf{B}_\phi(s) \quad (4.5)$$

The moments of this circuit are simply the Taylor coefficients of $X_\phi(s)$ such that

$$X_\phi(s) = M_0 + M_1s + M_2s^2 + \dots \quad (4.6)$$

Note that for impulse input excitation, the laplace transform B_ϕ of the input vector is simply a constant. Substituting from (4.6) into (4.5) and equating the coefficients of equal powers of s , yields the following expressions for the moments

$$\begin{aligned} G_\phi M_0 &= B_\phi \\ G_\phi M_{n+1} &= -C_\phi M_n; \quad n = 0, 1, 2, \dots \end{aligned} \quad (4.7)$$

Note from the above equation that only one LU decomposition of G_ϕ is required in order to calculate all the moments. The moments corresponding to the output node of interest are then used to obtain a rational Padé approximation as outlined in the previous section. Such methods obtain a low order approximation of the transfer function by capturing the poles that are close to the origin. However these methods are limited by some fundamental properties of Padé approximations. Following is a list of those properties which have the most impact on MMTs:

- Experience shows that the matrix in (4.3) is ill-conditioned if its size is large. This implies that one can only expect to detect 6 to 8 accurate poles from a single expansion.
- Padé often produces unstable poles on the right hand side of the complex plane.
- Like other approximants, Padé accuracy deteriorates as we move away from the expansion point.
- Padé provides no estimates for error bounds

Due to the above limitations, single expansion MMTs are very limited in their scope. While a single expansion may be sufficient for some RC networks, it is not adequate for RLC circuits. Circuits with both capacitors and inductors can have poles with similar damping constants but different resonant frequencies. A single expansion would capture the low frequency poles near

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Figure 4.2: Poles captured using CFH

ensures that all dominant poles are captured. In addition CFH provides clear bounds on the region of accuracy. This approach therefore addresses most of the difficulties encountered when using a single Padé expansion, and was shown to be suitable for interconnect simulation.

The most important remaining issue facing direct moment matching methods (both single and multi-point expansions) is that of passivity. Since interconnects are passive elements, it is expected that the reduced macromodel would also be passive. However, direct MMTs do not guarantee the passivity of the reduced model. This is a significant problem because a non-passive network could become unstable when connected to certain terminations (even passive terminations). Some passivity guarantees were provided in [26], but in general direct MMTs are not passive by construction.

4.2 Krylov-Subspace Based Reduction

The Krylov subspace was introduced by Nikolai Krylov in 1931, and later studied and developed by many others such as Arnoldi [94] and Lanczos [95]. Recently, these methods were introduced to circuit simulation, as a way to perform model order reduction on high-speed in-

terconnect networks [17], [20], [23], [24], [96]. Unlike the direct moment matching methods which were described earlier, Krylov subspace methods are based on the computation of the *leading* eigenvalues (those with the largest magnitude). By making use of numerically well conditioned algorithms such as the Arnoldi process [97], Krylov methods are much better behaved than direct Padé approximation. For example, while Padé is limited to 6 to 8 poles from a single expansion, Krylov methods can capture hundreds of poles at a single expansion. Furthermore, while Padé approximations do not even guarantee stability [13], reduced order models obtained using Krylov subspace methods were proven to be passive by construction [20], [24]. The passivity of the macromodel is an essential property in order to guarantee the stability of the simulation. This property of passivity by construction is one of the main reasons that has lead to the recent popularity of Krylov reduction methods in circuit simulation.

Krylov subspace based model order reduction, can be expressed in terms of a congruent transformation. Consider a linear multi-port network π . As shown in sections 2.4.2 and 2.5, the MNA equations of this network can be written as [24], [50]

$$\mathbf{G}_\pi \mathbf{x}_\pi + \mathbf{C}_\pi \dot{\mathbf{x}}_\pi = \mathbf{B}_\pi \mathbf{v}_\pi; \quad \mathbf{i}_\pi = \mathbf{B}_\pi^T \mathbf{x}_\pi \quad (4.8)$$

where $\mathbf{v}_\pi \in \mathbb{R}^{N_p}$ and $\mathbf{i}_\pi \in \mathbb{R}^{N_p}$ are the port voltages and currents, $\mathbf{G}_\pi$ and $\mathbf{C}_\pi \in \mathbb{R}^{N_\pi \times N_\pi}$ represent the memoryless and memory elements in the network respectively, and $\mathbf{B}_\pi \in \mathbb{R}^{N_\pi \times N_p}$ is a selector matrix. The number of ports in the network is N_p , and N_π is the number of states in the vector of unknowns $\mathbf{x}_\pi$. In the Laplace domain, equation (4.8) becomes

$$\mathbf{G}_\pi \mathbf{X}_\pi + s \mathbf{C}_\pi \mathbf{X}_\pi = \mathbf{B}_\pi \mathbf{V}_\pi; \quad \mathbf{I}_\pi = \mathbf{B}_\pi^T \mathbf{X}_\pi \quad (4.9)$$

The block moments $\mathbf{M}_i$ (or the derivatives w.r.t. s) of the multi-port system in (4.9) can be expressed as

$$\begin{aligned} \mathbf{M}_0 &= \mathbf{G}_\pi^{-1} \mathbf{B} \\ \mathbf{M}_i &= \mathbf{G}_\pi^{-1} \mathbf{C}_\pi \mathbf{M}_{i-1} = \mathbf{A}_\pi \mathbf{M}_{i-1}; \quad i > 0 \end{aligned} \quad (4.10)$$

where

$$\mathbf{A}_\pi = \mathbf{G}_\pi^{-1} \mathbf{C}_\pi \quad (4.11)$$

The k^{th} moment M_k can therefore be related to the zeroth moment M_0 by

$$M_k = A_\pi^k M_0 \quad (4.12)$$

In direct moment matching, the Padé approximation is applied directly to the moments calculated in (4.10). However, due to the ill-conditioned nature of the Padé matrix, a number of different expansions must be performed at different frequencies in order to capture all the dominant poles. In Krylov subspace based reduction, this process is done indirectly through congruent transformation. Consider the matrix $K_\pi \in \mathbb{R}^{N_\pi \times N_k}$ defined as

$$\begin{aligned} K_\pi &= [M_0, M_1, M_2, \dots, M_k] \\ &= [M_0, A_\pi M_0, A_\pi^2 M_0, \dots, A_\pi^k M_0] \end{aligned} \quad (4.13)$$

Note that K_π is a thin and tall matrix. The number of columns N_k in K_π is given by,

$$N_k = k \times N_p \quad (4.14)$$

where k is the number of block moments used in (4.13), and N_p is the number of ports. As can be seen from (4.13), due to the relationship between the moments as stated in (4.11), the column space spanned by the moments is simply the Krylov subspace of the matrix A_π . The difficulty here is that although the matrix K_π spans the Krylov subspace, it is very ill-conditioned. This is because the higher order moments are almost parallel to each other. It is therefore difficult to explicitly construct a high order Krylov subspace. A well known solution to this problem is the Arnoldi process [97], which is used to construct an orthonormal basis $Q_\pi \in \mathbb{R}^{N_\pi \times q}$ of the Krylov subspace

$$\text{colsp}[Q_\pi] = \text{colsp}[K_\pi] \quad (4.15)$$

The Arnoldi process does not explicitly calculate the moments, and can construct an orthonormal basis for a very high order Krylov subspace without ill-conditioning. In addition, further improvements to the application of this process to circuit model order reduction through the use of double orthogonalization are suggested in [98]. The order reduction is done by congruent transformation using the orthonormal basis Q_π . The resulting reduced system is expressed

as

$$\hat{G}_\pi \hat{X}_\pi + s \hat{C}_\pi \hat{X}_\pi = \hat{B}_\pi V_\pi; \quad I_\pi = \hat{B}_\pi^T \hat{X}_\pi \quad (4.16)$$

where $\hat{G}_\pi \in \mathbb{R}^{N_k \times N_k}$, $\hat{C}_\pi \in \mathbb{R}^{N_k \times N_k}$, $\hat{B}_\pi \in \mathbb{R}^{N_k \times N_p}$ and $\hat{X}_\pi \in \mathbb{R}^{N_k}$ are given by

$$\begin{aligned} \hat{G}_\pi &= Q_\pi^T G_\pi Q_\pi; & \hat{C}_\pi &= Q_\pi^T C_\pi Q_\pi \\ \hat{B}_\pi &= Q_\pi^T B_\pi; & \hat{X}_\pi &= Q_\pi^T X_\pi \end{aligned} \quad (4.17)$$

Note that the order N_k of the reduced system is much smaller than that of the original system. Model order reduction using Krylov subspace methods has two main advantages. The first is that the reduced order models can be guaranteed passive by construction [20], [24]. The second advantage is that, unlike direct moment matching, a large number of poles up to very high frequencies can be captured from a single expansion. However, while Krylov methods can capture all the dominant poles, the resulting reduced order models contain many redundant poles making them unnecessarily large compared to multi-point direct expansions such as complex frequency hopping (CFH) [25], [26].

4.3 Passivity of the Macromodel

An important issue in model order reduction is preserving the passivity of the macromodel. This is important for guaranteeing the stability of the over all circuit after terminations are connected. The original interconnect network is passive, and is therefore guaranteed stable for *any* passive termination. On the other hand, a non passive (i.e. active) network cannot be guaranteed stable for all possible passive terminations [99]. It is therefore important to ensure that the reduced order macromodel is passive. In this section the passivity property shall be defined and clarified for general networks.

By definition, a passive network is a network that is incapable of delivering energy at any time. In other words, for a passive network, the power consumption for all allowable voltage/current pairs at the ports can never be negative. Mathematically this can be expressed

Figure 4.3: Tunnel diode

Figure 4.4: Biased tunnel diode

as

$$\int_{t_o}^t v^T(\tau) i(\tau) d\tau \geq 0; \quad \forall t_o, \quad \forall t \in [t_o, \infty) \quad (4.18)$$

Note that the above equation assumes that the network is initially relaxed (i.e. contains no initial stored energy). As an example of a passive circuit consider the tunnel diode in Figure 4.3. As can be seen from the i, v curve, this network is passive because the curve lies in the first and third quadrants only and equation (4.18) is therefore satisfied. On the other hand, the biased tunnel diode in Figure 4.4 is active as can be seen from its i, v curve which partially lies in the second and fourth quadrants.

The passivity condition for a multi-port network can be derived in the frequency domain by considering the power consumed in the network. For an allowable port voltage and current

pair $\mathbf{V}$ and $\mathbf{I}$ the power is given by

$$P = \text{real}[\mathbf{V}^{*T} \mathbf{I}] \quad (4.19)$$

where the superscript $*T$ denotes the conjugate transpose. Noting that $\mathbf{I} = \mathbf{Y}(s)\mathbf{V}$, the power delivered to the network can be written as

$$P = \text{real}[\mathbf{V}^{*T} \mathbf{Y}(s) \mathbf{V}] \quad (4.20)$$

Therefore, for the system to be passive we must have

$$P = \text{real}[\mathbf{V}^{*T} \mathbf{Y}(s) \mathbf{V}] \geq 0 \quad (4.21)$$

or alternatively

$$\mathbf{V}^{*T} \left[\frac{1}{2} (\mathbf{Y}(s) + \mathbf{Y}^{*T}(s)) \right] \mathbf{V} \geq 0; \quad \forall \text{real}(s) > 0 \quad (4.22)$$

and for all vectors $\mathbf{V}$. In other words, the matrix $\mathbf{Y}(s)$ must be a *positive* matrix. A detailed explanation of this topic can be found in [99].

Chapter 5

Macromodeling of Interconnect Networks

5.1 Introduction

Due to the recent developments in VLSI technology at both the chip and package levels, the central processor clock frequency is well into the GHz range, and communication switches are being designed to transmit data with bit rates faster than 1 Gb/s. At such high frequencies the performance of interconnects and packages becomes critical to that of the overall design. In particular, interconnects in high end packages are becoming a critical bottleneck in current designs. This has highlighted the previously negligible transmission line effects such as delay, reflection, ringing, crosstalk, and signal distortion, which can in fact be present at all levels of the design hierarchy, such as board, package, multi-chip module, packages and chip-level interconnects. Furthermore, due to the trend towards higher operating frequencies, high bandwidth element models are required for accurate simulation. At these frequencies, interconnects can no longer be treated as lumped elements, and distributed transmission line models based on the Telegrapher's equations must be used instead [2], [100]. Furthermore, skin and proximity effects also become prominent at high frequencies and distributed models with frequency dependent parameters may be needed [3].

The main difficulty encountered in the simulation of interconnect networks arises while

linking distributed interconnect models to nonlinear elements. Distributed transmission line models are best characterized in the frequency domain, while typical circuits containing nonlinear elements such as drivers and receivers are described in the time domain. This mixed frequency/time domain representation of the circuit equations poses a fundamental challenge for the transient time domain simulation of interconnect networks. A brute force solution to this problem is the lumped segmentation of transmission lines [2]. In addition, as discussed in Section 2.4.2, other more efficient techniques have been proposed in the literature to obtain a discretized time domain transmission line model [4], [64]–[67], [83]. However, even with the most efficient discretization methods, the resulting circuits are very large, and require a prohibitively high computation cost.

To solve the above difficulty order reduction techniques have been proposed in the literature. The concept of model reduction relies on the fact that, although distributed elements introduce an infinite number of poles, only a fraction of these poles have a significant contribution to the circuit response. A large circuit with a large number of poles can therefore be replaced by a much smaller equivalent circuit that matches the dominant poles of the original circuit. Several model reduction techniques based on direct Padé approximation have been proposed in the literature [12], [14], [15], [101], [102]. Indirect Padé approximation was later introduced in order to solve the ill-conditioning problem of Padé [18], [84]. These techniques were later modified in order to ensure the passivity of the reduced order model [20], [24]. A passive macromodel is necessary, because a stable but non-passive model could result in an unstable circuit when connected to passive termination. However, a generally found difficulty with such Krylov-space based techniques is the issue of redundant poles [25], [26]. When the order of the approximation is increased to enhance the accuracy range of a Krylov-space based approximation, it tends to capture a large number of redundant poles which have a negligible contribution to the circuit response. This results in a much larger macromodel than is necessary, and significantly reduces the efficiency of the simulation. Some attempts have been made to remove the unnecessary poles [12], [25], but these approaches do not guarantee the passivity

of the reduce order model by construction.

In this chapter a method for obtaining reduced order passive macromodels for lossy coupled transmission line networks is presented. Using the proposed approach, redundant poles are eliminated while preserving the passivity of the transmission line model. The reduction is done in two steps. The first step of reduction is done by projecting the original system onto an orthonormal basis of the Krylov subspace. The reduced system is guaranteed passive but still contains many unnecessary poles. This severely reduces the efficiency of standard time domain simulators such as SPICE. This problem is addressed using a second step of reduction. In this step, the unnecessary poles are removed by projection onto an orthonormal basis of the eigenspace of the dominant poles. Pole selection is done based on their contribution to the response in the frequency range of interest. The resulting interconnect macromodel has two main properties: 1) it is passive by construction, and 2) it conserves the dominant poles of the original system. Several practical interconnects were considered and significant reduction in the size of the macromodel and CPU cost were obtained.

This chapter is organized as follows. The formulation of a transmission line network is presented in Section 5.2. In Section 5.3, the proposed macromodelling algorithm is presented. The properties of the proposed macromodel, such as the preservation of passivity and of the dominant poles are discussed in section 5.4. This is followed by numerical examples and conclusions in sections 5.5 and 5.6 respectively.

5.2 Problem formulation

Consider a linear network π (Fig. 5.1), which may contain a single set of multi-conductor transmission lines, or a large number of lines along with other lumped elements. The distributed transmission lines in this network are modelled using the Telegrapher's equations [2]. In order to perform nonlinear transient simulation a lumped time domain representation of the network π is required. Such a lumped model is obtained using some form of discretization of the Tele-

Figure 5.1: Example of a linear subcircuit

grapher's equation (see Section 2.4.2 for more details). After discretization, the network π is represented using ordinary differential equations as was defined for a general linear subnetwork in Section 2.5. Assume that the network π has n_p ports. The relation between the port voltages and currents is represented as

$$\mathbf{G}_\pi \mathbf{x}_\pi(t) + \mathbf{C}_\pi \frac{\partial \mathbf{x}_\pi(t)}{\partial t} = \mathbf{B}_\pi \mathbf{v}_{\pi p}(t) \quad (5.1a)$$

$$\mathbf{i}_{\pi p} = \mathbf{B}_\pi^T \mathbf{x}_\pi(t) \quad (5.1b)$$

where $\mathbf{G}_\pi, \mathbf{C}_\pi \in \mathbb{R}^{N_\pi \times N_\pi}$ and $\mathbf{B}_\pi \in \mathbb{R}^{N_\pi \times n_p}$ are the MNA matrices of the discretized network, $\mathbf{v}_{\pi k}$ and $\mathbf{i}_k \in \mathbb{R}^{n_p}$ are the vectors containing the port voltages and currents defined as

$$\begin{aligned} \mathbf{v}_{\pi k} &= [V_1, V_2, \dots, V_{n_p}] \\ \mathbf{i}_{\pi k} &= [I_1, I_2, \dots, I_{n_p}] \end{aligned} \quad (5.2)$$

The matrix $\mathbf{B}_k \in \mathbb{R}^{N_\pi \times n_p}$ is a selector matrix mapping the port voltages and currents to the node space of the ordinary differential equations describing subnetwork π . Time domain representations in the form of (5.1) and can be combined with nonlinear terminations as shown in

Section 2.5 and simulated using conventional simulators such as SPICE. However, the resulting circuits are typically large and require a prohibitively high CPU cost. A new model reduction algorithm is proposed in this chapter in order to reduce the size of the lumped equivalent model in (5.1) the proposed method removes the redundant poles obtained using previous reduction methods while, at the same time, maintaining the passivity of the reduced macromodel by construction.

5.3 Proposed Reduction Algorithm

Model order reduction has been proposed in the literature as a solution to large discretized linear interconnect models [12], [14], [15], [18], [20], [24]–[26], [84], [101], [102]. The main idea behind order reduction techniques is that while linear networks may contain a large number of poles (an infinite number of poles in the case of distributed interconnects), typically their response is governed by a small number of dominant poles. The objective of model order reduction is to generate a small network that captures the dominant poles and therefore the response of the original large network. In [20], [24] passive order reduction methods were introduced for the analysis on interconnect networks. The major contribution of these methods was the ability to preserve the passivity of the macromodel by construction. The reduced order models are however not optimal, and contain a large number of redundant poles [25], [26]. This leads to an unnecessary increase in computational cost and reduces the effectiveness of model reduction techniques.

The reduction algorithm used in this chapter is a multi-level technique that eliminates the unnecessary poles obtained using Krylov methods, while maintaining the passivity of the macromodel by construction. The algorithm consists of two main steps. First a passive reduced order model is obtained using conventional Krylov methods [24]. A second level of reduction is then performed in order to remove the unnecessary poles. This is done through projection onto an orthonormal basis specifically constructed to preserve the dominant poles. This approach

removes the unnecessary poles while exploiting the properties of congruent transformation in order to ensure passivity.

5.3.1 Preliminaries

Indirect order reduction methods such as Krylov subspace methods are based on the approximation of the leading eigenvalues. In this section we examine the frequency domain characterization of a multi-port linear network, and the relation between the poles of the network and the eigenvalues. The equations describing subsystem π in (5.1) can be written in the frequency domain as

$$\mathbf{G}_\pi \mathbf{X}_\pi + s \mathbf{C}_\pi \mathbf{X}_\pi = \mathbf{B}_\pi \mathbf{V}_{\pi p} \quad (5.3a)$$

$$\mathbf{I}_{\pi p} = \mathbf{B}_\pi^T \mathbf{X}_\pi \quad (5.3b)$$

The Y -parameter matrix $\mathbf{Y}_\pi(s)$ of the network relates the port voltages and currents by

$$\mathbf{I}_{\pi p} = \mathbf{Y}_\pi(s) \mathbf{V}_{\pi p} \quad (5.4)$$

An expression for the Y -parameters can be derived from (5.3a) and (5.3b) as

$$\mathbf{Y}_\pi(s) = \mathbf{B}_\pi^T (\mathbf{U} + s \mathbf{G}_\pi^{-1} \mathbf{C}_\pi) \mathbf{G}_\pi^{-1} \mathbf{B}_\pi \quad (5.5)$$

or alternatively

$$\mathbf{Y}_\pi(s) = \mathbf{B}_\pi^T (\mathbf{U} + s \mathbf{A}_\pi) \mathbf{R}_\pi \quad (5.6)$$

with

$$\mathbf{A}_\pi = \mathbf{G}_\pi^{-1} \mathbf{C}_\pi; \quad \mathbf{R}_\pi = \mathbf{G}_\pi^{-1} \mathbf{B}_\pi \quad (5.7)$$

Assume that the matrix $\mathbf{A}_\pi$ can be diagonalized in the form

$$\mathbf{A}_\pi = \mathbf{S}_\pi \mathbf{\Gamma}_\pi \mathbf{S}_\pi^{-1} \quad (5.8)$$

where $\mathbf{\Gamma}_\pi = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_{N_\pi}]$ is a diagonal matrix containing the eigenvalues of the matrix $\mathbf{A}_\pi$, and $\mathbf{S}_\pi$ is a matrix formed by the corresponding eigenvectors. Using (5.8), equation (5.6)

can be simplified to

$$\begin{aligned} Y_\pi(s) &= B_\pi^T S_\pi (U + s\Gamma_\pi)^{-1} S_\pi^{-1} R_\pi \\ &= B_\pi^T S_\pi \begin{bmatrix} \frac{1}{1+s\lambda_1} & & \\ & \ddots & \\ & & 0 \\ & & & \frac{1}{1+s\lambda_{N_\pi}} \end{bmatrix} S_\pi^{-1} R_\pi \end{aligned} \quad (5.9)$$

It can be deduced from (5.9) that the poles of the network are simply the reciprocal of the eigenvalues of $A_\pi = G_\pi^{-1} C_\pi$.

5.3.2 First step of reduction

The first step of reduction is done using conventional Krylov reduction methods. This reduction step is done through orthogonal projection on the the Krylov subspace of the system. Considering the equations of the system in (5.3), the Krylov subspace is defined as

$$K = [R_\pi, A_\pi R_\pi, A_\pi^2 R_\pi, \dots, A_\pi^k R_\pi] \quad (5.10)$$

where the columns of the matrix $K \in \mathbb{R}^{N_\pi \times N_{kr}}$ span the Krylov subspace, with $N_{kr} = N_\pi n_p$, and R_π and A_π defined as

$$A_\pi = G_\pi^{-1} C_\pi; \quad R_\pi = G_\pi^{-1} B_\pi \quad (5.11)$$

Note that the terms $R_\pi, A_\pi R_\pi, A_\pi^2 R_\pi$ etc. in (5.10) are simply the block moments of the system (i.e. the derivatives of X_π w.r.t s). Furthermore, it is important to note that the matrix K is typically ill-conditioned. This can be deduced by examining the properties of a matrix vector product of the form $A^k v$. For large values of k , this product converges to the eigenvector of A corresponding to the eigenvalue with the largest magnitude. Therefore, for large values of k the last terms in (5.10) are almost parallel and do not add any more dimensions to the subspace. Therefore, finding an orthonormal basis Q_k of the Krylov subspace directly from the moments in (5.10) is an ill-conditioned process. The basis Q_k can be accurately evaluated using the block Arnoldi process (See Algorithm 5.1) [97], [103]. This approach analytically

eliminates troublesome cancellations and results in a numerically accurate orthonormal basis for K .

Algorithm 5.1 Block Arnoldi Process

$[q_0, R] \leftarrow \text{qr}(R_\pi); \quad \text{qr factorization of } R_\pi$

for $j = 1$ to k **do**

$z = A_\pi q_{j-1}$

for $i = 0$ to $j - 1$ **do**

$h = q_i^T z$

$z = z - q_i h$

end for

$[q_j, R] \leftarrow \text{qr}(z); \quad \text{qr factorization of } z$

end for

$Q_k \leftarrow [q_0, q_1, \dots, q_k]$

The reduced macromodel is obtained by projecting the original system in (5.1) onto the orthonormal basis $Q_k \in \mathbb{R}^{N_\pi \times N_{kr}}$ of the Krylov subspace. The resulting system is given by

$$\hat{G}_\pi \hat{x}_\pi(t) + \hat{C}_\pi \dot{\hat{x}}_\pi(t) = \hat{B}_\pi v_{\pi p}(t); \quad i_{\pi p}(t) = \hat{B}_\pi^T \hat{x}_\pi(t) \quad (5.12)$$

where $\hat{G}_\pi$ and $\hat{C}_\pi \in \mathbb{R}^{N_{kr} \times N_{kr}}$ and $\hat{B}_\pi \in \mathbb{R}^{N_{kr} \times n_p}$ are given by

$$\hat{G}_\pi = Q_k^T G_\pi Q_k; \quad \hat{C}_\pi = Q_k^T C_\pi Q_k; \quad \hat{B}_\pi = Q_k^T B_\pi \quad (5.13)$$

It can be shown that the reduced system in (5.12) is guaranteed passive and that it preserves the first k moments of the original system [24]. Furthermore, since $N_{kr} \ll N_\pi$ the reduced system is much smaller in size than the original system. However, macromodels obtained using this method were shown to contain a large number of unnecessary poles [25], [26]. The large size of the macromodel severely increases the CPU cost of the time domain transient simulation. This problem is addressed using a second level of reduction as discussed in the following section.

5.3.3 Second step of reduction

The objective of this reduction step is to remove the redundant poles left in the reduced macro-model obtained using conventional Krylov methods, while at the same time conserving passivity by construction. The unnecessary poles in (5.12) are removed by projection onto an orthonormal basis $Q_v \in \mathbb{R}^{N_{kr} \times N_v}$ of the eigenspace X_v of the dominant poles. Pole selection is done based on their contribution to the response in the frequency range of interest. For example if a pole's contribution to one of the Y parameters is given by

$$y(s) = \frac{r}{s - p}; \quad p = \alpha + j\beta \quad (5.14)$$

The area under the curve for $\|y(s)\|^2$ up to the highest frequency ω_h of interest is a good indication of this poles contribution to the response with respect to the other poles. This can be computed analytically as

$$\int_0^{\omega_h} \|y(s)\|^2 d\omega = \frac{\|r\|^2}{\alpha} \left(\arctan\left(\frac{\omega_h - \beta}{\alpha}\right) - \arctan\left(\frac{-\beta}{\alpha}\right) \right) \quad (5.15)$$

This second reduction step is performed using orthonormal projection onto subspace K_v , which spans the eigenspace of the dominant eigenvalues. This reduction step preserves the dominant poles while at the same time guaranteeing the passivity of the reduced system by construction. Proofs for pole and passivity conservation are given in Section 5.4. In order to obtain a real basis of the eigenspace of the dominant poles, the real and imaginary parts of complex eigenvectors are separated and the matrix K_v is constructed as

$$K_v = \begin{bmatrix} \text{real}(v_1), \text{imag}(v_1), \text{real}(v_2), \text{imag}(v_2), \\ \dots, \text{real}(v_m), \text{imag}(v_m) \end{bmatrix} \quad (5.16)$$

where v_i are eigenvectors corresponding to the dominant eigenvalues of the matrix $\hat{A}_\pi$ defined as

$$\hat{A}_\pi = \hat{G}_\pi^{-1} \hat{C}_\pi \quad (5.17)$$

The transformation matrix $Q_v \in \mathbb{R}^{N_{kr} \times N_v}$ is an orthonormal basis of K_v and is obtained from the eigenvectors using a standard orthogonalization process [97],

$$\text{colsp}[Q_v] = \text{colsp}[K_v] \quad (5.18)$$

It is to be noted that while the eigenvectors are in general complex, Q_v is made real by splitting the real and imaginary parts of complex eigenvectors. Furthermore, while the eigenvectors are split into real and imaginary parts in order to obtain a real basis, the size of the basis remains unchanged because two columns would still span the space of a pair of complex conjugate eigenvectors. A real transformation matrix is necessary for the conservation of passivity.

The reduced order system is now obtained by congruent transformation with the orthonormal basis Q_v

$$\tilde{G}_\pi \tilde{x}_\pi(t) + \tilde{C}_\pi \dot{\tilde{x}}_\pi(t) = \tilde{B}_\pi v_{\pi p}(t); \quad i_{\pi p}(t) = \tilde{B}_\pi^T \tilde{x}_\pi(t) \quad (5.19)$$

where $\tilde{G}_\pi, \tilde{C}_\pi \in \mathbb{R}^{N_v \times N_v}$ and $\tilde{B}_\pi \in \mathbb{R}^{N_v \times n_p}$ are given by

$$\tilde{G}_\pi = Q_v^T \hat{G}_\pi Q_v; \quad \tilde{C}_\pi = Q_v^T \hat{C}_\pi Q_v; \quad \tilde{B}_\pi = Q_v^T \hat{B}_\pi \quad (5.20)$$

The final reduced macromodel given in (5.19) is guaranteed passive as will be shown in the next section, and conserves the dominant poles of the original systems. This time domain macromodel can be efficiently used in conventional nonlinear simulators such as SPICE in order to perform time domain simulations. The approach presented in this section is general, and can be applied to a single set of transmission lines, or to large networks containing many interconnects. In the following section, the proofs of passivity and conservation of dominant poles are presented.

5.4 Properties of the Reduced-Order Model

The interconnect macromodel in (5.19) has two main properties: 1) it is passive by construction, and 2) it conserves the dominant poles of the original system. Proofs for both properties

are provided in this section.

5.4.1 Conservation of poles

Consider the interconnect macromodel in (5.12) obtained using Krylov reduction. The relation between the port voltages and currents can be written as

$$\mathbf{I}_{\pi p} = \hat{\mathbf{B}}_{\pi}^T \left(\mathbf{U} + s \hat{\mathbf{G}}_{\pi}^{-1} \hat{\mathbf{C}}_{\pi} \right) \hat{\mathbf{G}}_{\pi}^{-1} \hat{\mathbf{B}}_{\pi} \mathbf{V}_{\pi p} \quad (5.21)$$

In a similar analysis to the one in Section 5.3.1, it can shown from (5.21) that the poles $\hat{p}_i$ of this network are related to the eigenvalues $\hat{\lambda}_i$ of the matrix $\hat{\mathbf{A}}_{\pi} = \hat{\mathbf{G}}_{\pi}^{-1} \hat{\mathbf{C}}_{\pi}$

$$\hat{p}_i = -\frac{1}{\hat{\lambda}_i} \quad (5.22)$$

Similarly a pole $\tilde{p}_i$ of the final reduced macromodel in (5.19) is related to the corresponding eigenvalue of the matrix $\tilde{\mathbf{A}}_{\pi} = \tilde{\mathbf{G}}_{\pi}^{-1} \tilde{\mathbf{C}}_{\pi}$ by

$$\tilde{p}_i = -\frac{1}{\tilde{\lambda}_i} \quad (5.23)$$

Consider a dominant pole $\hat{p}_i = -1/\hat{\lambda}_i$ of the Krylov reduced system in (5.21). $\hat{\lambda}_i$ is an eigenvalue of $\hat{\mathbf{A}}_{\pi}$ with an associated eigenvector $\hat{\mathbf{v}}_i \in \mathbb{C}^{N_{kr}}$ such that

$$\hat{\mathbf{A}}_{\pi} \hat{\mathbf{v}}_i = \hat{\lambda}_i \hat{\mathbf{v}}_i; \quad or \quad (5.24)$$

Replacing $\hat{\mathbf{A}}_{\pi}$ with its definition $\hat{\mathbf{A}}_{\pi} = \hat{\mathbf{G}}_{\pi}^{-1} \hat{\mathbf{C}}_{\pi}$ equation (5.24) can be written as

$$\hat{\mathbf{C}}_{\pi} \hat{\mathbf{v}}_i = \hat{\lambda}_i \hat{\mathbf{G}}_{\pi} \hat{\mathbf{v}}_i \quad (5.25)$$

From the definition of $\mathbf{Q}_k \in \mathbb{R}^{N_{kr} \times N_v}$ in (5.18), $\mathbf{Q}_v$ spans the subspace formed by the eigenvectors of the dominant eigenvalues of $\hat{\mathbf{A}}_{\pi}$. Therefore the eigenvector corresponding to $\hat{\lambda}_i$ is in the subspace $\mathbf{Q}_v$ (i.e $\hat{\mathbf{v}}_i \in \mathbf{Q}$), and $\hat{\mathbf{v}}_i$ can be written as

$$\hat{\mathbf{v}}_i = \mathbf{Q}_v \tilde{\mathbf{v}}_i \quad (5.26)$$

where $\tilde{\mathbf{v}}_i \in \mathbb{C}^{N_v}$. Substituting from (5.26) into (5.25) yields

$$\hat{C}_\pi Q_v \tilde{\mathbf{v}}_i = \hat{\lambda}_i \hat{G}_\pi Q_v \tilde{\mathbf{v}}_i \quad (5.27)$$

Pre-multiplying by Q_v^T results in

$$Q_v^T \hat{C}_\pi Q_v \tilde{\mathbf{v}}_i = \hat{\lambda}_i Q_v^T \hat{G}_\pi Q_v \tilde{\mathbf{v}}_i \quad (5.28)$$

or alternatively

$$\tilde{C}_\pi \tilde{\mathbf{v}}_i = \hat{\lambda}_i \tilde{G}_\pi \tilde{\mathbf{v}}_i \quad (5.29)$$

Equation (5.29) can be easily reduced to

$$\tilde{G}_\pi^{-1} \tilde{C}_\pi \tilde{\mathbf{v}}_i = \hat{\lambda}_i \tilde{\mathbf{v}}_i \quad (5.30)$$

Therefore $\hat{\lambda}_i$ is also an eigenvalue of $\tilde{A}_\pi = \tilde{G}_\pi^{-1} \tilde{C}_\pi$ with corresponding eigenvector $\tilde{\mathbf{v}}_i$. A similar argument can easily be made for all the other eigenvalues whose eigenvectors were included in (5.16).

5.4.2 Preservation of passivity

In this section the reduced interconnect macromodel in equation (5.19) is proven passive by construction. Passivity is an important property of the reduced order system, because a non-passive circuit (even if it is stable) may lead to instability when connected to other passive circuits. Physically, passivity implies that the macromodel cannot generate energy. Mathematically, there are a number of equivalent conditions that the circuit must satisfy in order to be passive. In [99] a necessary and sufficient condition for passivity is imposed on the Y -parameters of the system as follows. A network is passive if and only if its Y -parameters $\tilde{\mathbf{Y}}(s)$ satisfy the following two conditions:

1. $\tilde{\mathbf{Y}}(s^*) = \tilde{\mathbf{Y}}^*(s)$ for all $s \in \mathbb{C}$. The symbol $*$ denotes the complex conjugate operator.
2. $\tilde{\mathbf{Y}}(s)$ is a positive real matrix. In other words

$$\mathbf{z}^{*T} \left(\tilde{\mathbf{Y}}(s) + \tilde{\mathbf{Y}}^T(s^*) \right) \mathbf{z} \geq 0; \quad (5.31)$$

for all vectors $z \in \mathbb{C}^{n_p}$ and for all $s \in \mathbb{C}$ such that $\text{real}(s) \geq 0$.

The proof in this section begins along the same line as [24]. First we prove that a reduction of a system described by equations in the form of the original MNA equations, by congruent transformation with a real matrix results in a passive reduced order model. Then we proceed to prove that the second level of reduction in the proposed algorithm is equivalent to a reduction of the original MNA equations through a congruent transformation.

Consider the set of MNA equations representing the linear subnetwork

$$\mathbf{G}_\pi \mathbf{x}_\pi(t) + \mathbf{C}_\pi \frac{\partial \mathbf{x}_\pi(t)}{\partial t} = \mathbf{B}_\pi \mathbf{v}_{\pi p}(t) \quad (5.32a)$$

$$\mathbf{i}_{\pi p} = \mathbf{B}_\pi^T \mathbf{x}_\pi(t) \quad (5.32b)$$

As can be seen from the example in Section 2.4.2, the MNA matrices can be written in a way such that $\mathbf{C}_\pi$ is symmetric non-negative definite and $\mathbf{G}_\pi$ in the form

$$\mathbf{G}_\pi = \begin{pmatrix} \mathbf{N} & \mathbf{E} \\ -\mathbf{E}^T & \mathbf{0} \end{pmatrix} \quad (5.33)$$

where $\mathbf{N}$ is symmetric non-negative definite. Let the system in (5.32) be reduced using a real congruent transformation $\mathbf{Q}_{kv} \in \mathbb{R}^{N_\pi \times N_v}$. The resulting reduced system is given by

$$\tilde{\mathbf{G}}_\pi \tilde{\mathbf{x}}_\pi(t) + \tilde{\mathbf{C}}_\pi \dot{\tilde{\mathbf{x}}}_\pi(t) = \tilde{\mathbf{B}}_\pi \mathbf{v}_{\pi p}(t); \quad \mathbf{i}_{\pi p}(t) = \tilde{\mathbf{B}}_\pi^T \tilde{\mathbf{x}}_\pi(t) \quad (5.34)$$

where $\tilde{\mathbf{G}}_\pi, \tilde{\mathbf{C}}_\pi \in \mathbb{R}^{N_v \times N_v}$ and $\tilde{\mathbf{B}}_\pi \in \mathbb{R}^{N_v \times n_p}$ are given by

$$\tilde{\mathbf{G}}_\pi = \mathbf{Q}_{kv}^T \hat{\mathbf{G}}_\pi \mathbf{Q}_{kv}; \quad \tilde{\mathbf{C}}_\pi = \mathbf{Q}_{kv}^T \hat{\mathbf{C}}_\pi \mathbf{Q}_{kv}; \quad \tilde{\mathbf{B}}_\pi = \mathbf{Q}_{kv}^T \hat{\mathbf{B}}_\pi \quad (5.35)$$

The Y -parameters of the network in (5.34) can be written as

$$\tilde{\mathbf{Y}}_\pi(s) = \tilde{\mathbf{B}}_\pi^T \left(\tilde{\mathbf{G}}_\pi + s \tilde{\mathbf{C}}_\pi \right)^{-1} \tilde{\mathbf{B}}_\pi \quad (5.36)$$

One important property of the reduced order system is that the reduced matrices $\tilde{\mathbf{G}}_\pi, \tilde{\mathbf{C}}_\pi$ and $\tilde{\mathbf{B}}_\pi$ are all real. This implies that the first condition of passivity, namely $\tilde{\mathbf{Y}}(s^*) = \tilde{\mathbf{Y}}^*(s)$ is automatically satisfied. We now turn our attention to the second condition of passivity, namely

$$\mathbf{z}^{*T} \left(\tilde{\mathbf{Y}}(s) + \tilde{\mathbf{Y}}^T(s^*) \right) \mathbf{z} \geq 0; \quad \forall \mathbf{z} \in \mathbb{C}^{n_p}; \quad \forall s \in \mathbb{C} \text{ such that } \text{real}(s) > 0 \quad (5.37)$$

Define the matrix $\tilde{\mathbf{E}}_h(s)$ as

$$\tilde{\mathbf{E}}_h(s) = \mathbf{z}^{*T} \left(\tilde{\mathbf{Y}}(s) + \tilde{\mathbf{Y}}^T(s^*) \right) \mathbf{z} \quad (5.38)$$

The condition in (5.37) therefore becomes

$$\tilde{\mathbf{E}}_h(s) \geq 0; \quad \forall s \in \mathbb{C} \text{ such that } \text{real}(s) > 0 \quad (5.39)$$

With some algebraic manipulation $\tilde{\mathbf{E}}_h(s)$ can be written as

$$\begin{aligned} \tilde{\mathbf{E}}_h(s) &= \mathbf{z}^{*T} \left[\tilde{\mathbf{B}}_\pi^T \left(\tilde{\mathbf{G}}_\pi + s \tilde{\mathbf{C}}_\pi \right)^{-1} \tilde{\mathbf{B}}_\pi + \tilde{\mathbf{B}}_\pi^T \left(\tilde{\mathbf{G}}_\pi + s^* \tilde{\mathbf{C}}_\pi \right)^{-T} \tilde{\mathbf{B}}_\pi \right] \mathbf{z} \\ &= \mathbf{z}^{*T} \tilde{\mathbf{B}}_\pi^T \left[\left(\tilde{\mathbf{G}}_\pi + s \tilde{\mathbf{C}}_\pi \right)^{-1} + \left(\tilde{\mathbf{G}}_\pi + s^* \tilde{\mathbf{C}}_\pi \right)^{-T} \right] \tilde{\mathbf{B}}_\pi \mathbf{z} \\ &= \mathbf{z}^{*T} \tilde{\mathbf{B}}_\pi^T \left(\tilde{\mathbf{G}}_\pi + s \tilde{\mathbf{C}}_\pi \right)^{-1} \left[\left(\tilde{\mathbf{G}}_\pi + s \tilde{\mathbf{C}}_\pi \right) + \left(\tilde{\mathbf{G}}_\pi + s^* \tilde{\mathbf{C}}_\pi \right)^T \right] \\ &\quad \times \left(\tilde{\mathbf{G}}_\pi + s^* \tilde{\mathbf{C}}_\pi \right)^{-T} \tilde{\mathbf{B}}_\pi \mathbf{z} \end{aligned} \quad (5.40)$$

Where the superscript $-T$ designates the inverse transpose. Define $\mathbf{w}$ and s as

$$\mathbf{w} = \left(\tilde{\mathbf{G}}_\pi + s^* \tilde{\mathbf{C}}_\pi \right)^{-T} \tilde{\mathbf{B}}_\pi \mathbf{z} \quad (5.41)$$

$$s = \sigma + j\omega; \quad \sigma > 0 \quad (5.42)$$

Substituting from (5.41) and (5.42) into (5.40) results in

$$\begin{aligned} \tilde{\mathbf{E}}_h(s) &= \mathbf{w}^{*T} \left[\left(\tilde{\mathbf{G}}_\pi + (\sigma + j\omega) \tilde{\mathbf{C}}_\pi \right) + \left(\tilde{\mathbf{G}}_\pi^T + (\sigma - j\omega) \tilde{\mathbf{C}}_\pi^T \right) \right] \mathbf{w} \\ &= \mathbf{w}^{*T} \mathbf{Q}_{kv}^T \left[\mathbf{G}_\pi + \mathbf{G}_\pi^T + \sigma (\mathbf{C}_\pi + \mathbf{C}_\pi^T) \right] \mathbf{Q}_{kv} \mathbf{w} \end{aligned} \quad (5.43)$$

Setting $\mathbf{u} = \mathbf{Q}_{kv} \mathbf{w}$ results in

$$\tilde{\mathbf{E}}_h(s) = \mathbf{u}^{*T} \left[\mathbf{G}_\pi + \mathbf{G}_\pi^T + \sigma (\mathbf{C}_\pi + \mathbf{C}_\pi^T) \right] \mathbf{u} \quad (5.44)$$

Since $\mathbf{C}_\pi$ is symmetric nonnegative definite we have

$$\mathbf{u}^{*T} \sigma (\mathbf{C}_\pi + \mathbf{C}_\pi^T) \mathbf{u} = 2\sigma \mathbf{u}^{*T} \mathbf{C}_\pi \mathbf{u} \geq 0; \quad \forall \mathbf{u} \in \mathbb{C}^{N_\pi} \quad (5.45)$$

Also, since $\mathbf{G}_\pi$ has the form shown in (5.33), and noting that $\mathbf{N}$ in (5.33) is symmetric non-negative definite we have

$$\begin{aligned} \mathbf{u}^{*T} (\mathbf{G}_\pi + \mathbf{G}_\pi^T) \mathbf{u} &= \mathbf{u}^{*T} \left(\begin{bmatrix} \mathbf{N} & \mathbf{E} \\ -\mathbf{E}^T & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{N} & \mathbf{E} \\ -\mathbf{E}^T & \mathbf{0} \end{bmatrix}^T \right) \mathbf{u} \\ &= \mathbf{u}^{*T} \begin{bmatrix} 2\mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{u} \geq 0; \quad \forall \mathbf{u} \in \mathbb{C}^{N_\pi} \end{aligned} \quad (5.46)$$

From (5.45) and (5.46) we can deduce that $\tilde{\mathbf{E}}_h(s) \geq 0$, and that the second passivity condition as stated in (5.31) is satisfied. Therefor, a reduced system obtained from the original system π through congruent transformation with a real matrix $\mathbf{Q}_{kv}$ is passive by construction. In the remainder of this section, we shall prove that the reduced order system obtained in the second level of reduction in Section 5.3.3 using the congruent transformation $\mathbf{Q}_v \in \mathbb{R}^{N_{kr} \times N_v}$, is equivalent to reducing the original network π using a congruent transformation $\mathbf{Q}_{kv} \in \mathbb{R}^{N_\pi \times N_v}$.

The final reduced model obtained after the second level of reduction in Section 5.3.3 was defined in equation (5.19) as

$$\tilde{\mathbf{G}}_\pi \tilde{\mathbf{x}}_\pi(t) + \tilde{\mathbf{C}}_\pi \dot{\tilde{\mathbf{x}}}_\pi(t) = \tilde{\mathbf{B}}_\pi \mathbf{v}_{\pi p}(t); \quad \mathbf{i}_{\pi p}(t) = \tilde{\mathbf{B}}_\pi^T \tilde{\mathbf{x}}_\pi(t) \quad (5.47)$$

where $\tilde{\mathbf{G}}_\pi, \tilde{\mathbf{C}}_\pi \in \mathbb{R}^{N_v \times N_v}$ and $\tilde{\mathbf{B}}_\pi \in \mathbb{R}^{N_v \times n_p}$ are given by

$$\tilde{\mathbf{G}}_\pi = \mathbf{Q}_v^T \hat{\mathbf{G}}_\pi \mathbf{Q}_v; \quad \tilde{\mathbf{C}}_\pi = \mathbf{Q}_v^T \hat{\mathbf{C}}_\pi \mathbf{Q}_v; \quad \tilde{\mathbf{B}}_\pi = \mathbf{Q}_v^T \hat{\mathbf{B}}_\pi \quad (5.48)$$

The matrix $\mathbf{Q}_v \in \mathbb{R}^{N_{kr} \times N_v}$ is the congruent transformation used in the second level of reduction. The matrices $\hat{\mathbf{G}}_\pi, \hat{\mathbf{C}}_\pi \in \mathbb{R}^{N_{kr} \times N_{kr}}$ and $\hat{\mathbf{B}}_\pi \in \mathbb{R}^{N_{kr} \times n_p}$ were obtained from the first level of reduction and are defined in equation (5.13) as

$$\hat{\mathbf{G}}_\pi = \mathbf{Q}_k^T \mathbf{G}_\pi \mathbf{Q}_k; \quad \hat{\mathbf{C}}_\pi = \mathbf{Q}_k^T \mathbf{C}_\pi \mathbf{Q}_k; \quad \hat{\mathbf{B}}_\pi = \mathbf{Q}_k^T \mathbf{B}_\pi \quad (5.49)$$

The matrix $Q_k \in \mathbb{R}^{N_\pi \times N_{kr}}$ is the congruent transformation used in the first level of reduction.

Substituting from (5.49) into (5.48) yields

$$\begin{aligned}\tilde{G}_\pi &= Q_v^T Q_k^T G_\pi Q_k Q_v \\ \tilde{C}_\pi &= Q_v^T Q_k^T C_\pi Q_k Q_v \\ \tilde{B}_\pi &= Q_v^T Q_k^T B_\pi\end{aligned}\tag{5.50}$$

Setting $Q_{kv} = Q_k Q_v$ results in

$$\begin{aligned}\tilde{G}_\pi &= Q_{kv}^T G_\pi Q_{kv} \\ \tilde{C}_\pi &= Q_{kv}^T C_\pi Q_{kv} \\ \tilde{B}_\pi &= Q_{kv}^T B_\pi\end{aligned}\tag{5.51}$$

From (5.51) we can conclude that the reduced macromodel is equivalent to a macromodel obtained through congruent transformation with a real matrix Q_{kv} . Therefore the reduced model is passive by construction.

Figure 5.2: Circuit for example 1

5.5 Numerical Examples

5.5.1 Example 1

For the first example an interconnect consisting of two coupled lines suggested in [22] was considered. The per-unit-length parameters of the lines are given by, $L_{11} = L_{22} = 403.043$ nH/m, $L_{12} = L_{21} = 76.96$ nH/m, $C_{11} = C_{22} = 45.65$ pF/m, $C_{12} = C_{21} = -12.319$ pF/m, $R_{11} = R_{22} = 10$ Ω/m, $R_{12} = R_{21} = 0$ Ω/m. The line was taken to be 10cm long. For this example the PRIMA [24] based macromodel was found to be of size 80 in order to match up to 10GHz [22]. Using the proposed approach the size of the macromodel was reduced to 34 state variables. A comparison between the Y_{13} parameter of the PRIMA macromodel and the proposed macromodel is shown in fig 5.3. As can be seen, the match is very good. The macromodel was also tested in time domain simulation. The lines were terminated with 30 Ω resistors at the near end, and 1.5 pF capacitors at the far end as shown in Fig. 5.2. The input signal was a 2nsec pulse with 0.1nsec rise and fall time. The simulation results at the signal line and the victim line are shown in Fig. 5.4 and Fig. 5.5. The results matched those obtained using the conventional PRIMA macromodel with a CPU speed-up of 3.7 as shown in table 5.1.

Figure 5.3: Example 1 - Y_{13}

Figure 5.4: Voltage on signal line (example1)

Figure 5.5: Voltage on victim line (example 1)

Figure 5.6: Nine transmission line circuit

5.5.2 Example 2

In this example a practical interconnect network containing nine lossy coupled lines was considered. The per-unit-length parameters of the lines are given in [104]. The length of the lines was taken to be 10cm. The PRIMA [22], [24] based macromodel for this interconnect structure had 398 states, matching the Y-parameters up to 5.5 GHz. The proposed macromodel was of size 211, and matched up the same upper frequency. The nine coupled lines were terminated with 30Ω resistors at the near end and 1.5pF capacitors at the far end as shown in Fig. 5.6. A 2nsec input pulse with rise and fall time of 0.1nsec was applied, and the time domain responses at the signal line and one of the victim lines are shown in Fig. 5.7 and Fig. 5.8. As can be seen, the results matched very well with those obtained from the conventional PRIMA [24] macromodel. The CPU cost was 3.74 times less that the macromodel obtained through conventional reduction as shown in table 5.1.

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	PRIMA Macromodel [22], [24]		Proposed Macromodel		Speed-up
	Size	CPU Cost (sec)	Size	CPU Cost (sec)	
Example 1	80	11.98	34	3.21	3.7
Example 2	398	1268	211	339	3.74

Table 5.1: Summary of computational results (Examples 1&2)

Figure 5.9: Large interconnect network

5.5.3 Example 3

A distributed interconnect network with three sets of nine coupled lines with frequency independent parameters, and six sets of two coupled lines with frequency dependent parameters is considered for this example (Fig. 5.9). The network was discretized using the matrix-rational Padé approximation [67]. The size of the MNA matrices was 1288×1288 . A 2-port macromodel of the network was found using a single level of reduction and then a second level of reduction was applied. The macromodel obtained using conventional Krylov methods was of size 420, while the one obtained using the proposed approach was of size 235. A comparison of the time domain simulation using the macromodel, is shown in Fig. 5.10. A CPU comparison showed that the proposed method gave a CPU speedup of 3 times over the conventional Krylov reduction as shown in Table 5.2.

Table 5.2: CPU comparison for example 3

	Krylov	Krylov + Pole selection
Size	420×420	235×235
CPU time (excluding overhead)	5.2s	1.7s
Simulation Speed-up	1	3

Figure 5.10: Transient response of the network

Figure 5.11: Y_{11} of the network in example 4

5.5.4 Example 4

In this example, an interconnect network containing 7 transmission lines and other lumped elements was used. After lumped discretization, the network contained 1448 nodes. The proposed algorithm was used to obtain a reduced order macromodel of order 50 that matches the response of the original system up to 4.5GHz as shown in Figs. 5.11 and 5.12. Conventional Krylov reduction required an order of 150 in order to match the original system up to the same frequency. The reduced system obtained using the proposed algorithm was connected to non-linear terminations as shown in Fig. 5.13. The transient response of the reduced order model matched that of the original system as shown in figs. 5.14 and 5.15. The CPU speedup of the time domain simulation of the reduced system compared to that of the original system was 77 times.

Figure 5.12: Y_{12} of the network in example 4

Figure 5.13: Large interconnect network with nonlinear terminations

Figure 5.14: Transient response of the network at node Vn2

Figure 5.15: Transient response of the network at node Vout

5.6 Conclusion

In this chapter an order reduction method for transmission line networks is proposed. The proposed method can be classified as an indirect reduction method, which reduces the original network through congruent transformation. This approach consists of two reduction steps. The first step is performed through orthogonal projection onto the Krylov subspace, resulting in the first reduced model. However, a well known difficulty with this model is that it contains a large number of or redundant poles. This issue is addressed through the second step of reduction which is done through orthogonal projection onto the eigenspace of the dominant eigenvalues. The new method is passive, and eliminated the unnecessary poles that are present in previous order reduction techniques, thus resulting in a much smaller model. Reduced models obtained using the proposed method were used in conventional time domain simulators such as SPICE, and significant CPU speed up was achieved.

Chapter 6

Sensitivity Analysis of Interconnect Networks

6.1 Introduction

Advances in fabrication methods and decreasing device sizes have significantly increased the relative contribution of high-speed interconnects to overall signal degradation [10]. At higher frequencies interconnects are modeled by transmission lines. Transmission line analysis has therefore become imperative at all levels of the design hierarchy from backplanes to printed circuit boards, packages, and integrated circuits [3]. The analysis of large systems containing transmission lines have presented additional problems to circuit simulation, stemming from the fact that such distributed elements are modeled by partial differential equations as opposed to ordinary differential equations as in the case of lumped elements. A significant amount of research has been done to overcome these difficulties and reduce the CPU cost of transmission line analysis [13].

Recently, techniques based on congruent transformation have emerged as an efficient tool to generate reduced order passive models of very large linear networks. These techniques have been applied successfully to lumped [24] and distributed networks [23], [105], resulting

in significant savings in CPU time. Such methods allow the circuit designer to determine the response of a given circuit. However, these techniques are limited to finding the circuit response. One type of analysis that is often required during the design process (for optimization or design centering for example) is sensitivity analysis. In this chapter a reduction method for sensitivity analysis is presented.

Several techniques can be found in the literature that use model reduction based on Padé approximation to perform sensitivity analysis of linear lumped networks [16], [17], [36]. However, these approaches require finding the derivatives of the poles and zeros of the reduced model, or in other words finding the sensitivity of transformation used for reduction. In this chapter, an efficient algorithm is proposed for sensitivity analysis of transmission line networks based on congruent transformation techniques. The proposed technique handles distributed transmission line circuits without the need for discretizing the Telegrapher's equations. The algorithm is based on projecting the adjoint network equations onto a reduced order subspace that preserves the circuit moments. The sensitivity can then be evaluated over any number of frequency points. The additional cost of the sensitivity analysis at all frequencies of interest is just one QR factorization of the transformation matrix (equivalent to one LU decomposition of the MNA). In contrast, the cost of sensitivity analysis without the proposed model reduction methods is one LU decomposition at each frequency point of interest. The rest of the chapter is organized as follows. Section 6.2 presents the formulation of the circuit equations. This is followed in Section 6.3 by a brief description of adjoint sensitivity analysis. The proposed model reduction algorithm is then discussed in Section 6.4. Finally the results and conclusion are given in sections 6.5 and 6.6 respectively.

6.2 Formulation of Network Equations

The formulation in this section is based on the eigenvalue based transmission line stamp described in Section 2.3.2. Consider a linear network ϕ containing linear lumped components and

N_t lossy coupled transmission line sets, with n_k coupled conductors in transmission line set k . Assume the network ϕ has N_ϕ nodal variables. The Modified Nodal Analysis (MNA) matrix equations in (2.1) can be written in the frequency domain, and appended with the transmission line stamp as follows

$$\mathbf{G}_\phi \mathbf{X}_\phi(s) + s\mathbf{C}_\phi \mathbf{X}(s) + \mathbf{Y}_\phi(s) \mathbf{X}_\phi(s) = \mathbf{B}_\phi(s) \quad (6.1)$$

or alternatively

$$\begin{aligned} \mathbf{A}_\phi \mathbf{X}_\phi(s, \lambda) &= \mathbf{B}_\phi(s) \\ \mathbf{A}_\phi(s, \lambda) &= \mathbf{G}_\phi + s\mathbf{C}_\phi + \mathbf{Y}_\phi(s) \end{aligned} \quad (6.2)$$

- $\mathbf{X}_\phi(s, \lambda) \in \mathbb{C}_\phi^N$ is the vector of the complex variables describing the subnetwork which includes node voltage waveforms appended by independent and dependent voltage source currents and inductor currents,
- $\mathbf{G}_\phi, \mathbf{C}_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$ are constant matrices describing the lumped memory and memory-less elements of network ϕ respectively.
- $\mathbf{B}_\phi \in \mathbb{C}^N$ is a vector with entries determined by the independent voltage and current sources,
- $\mathbf{Y}_\phi$ contains the mapped Y -parameters of all the transmission line sets, and is expressed as

$$\mathbf{Y}_\phi(s) = \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{Y}_k(s) \mathbf{D}_k^T \quad (6.3)$$

where $\mathbf{D}_k = [d_{i,j}]$ is a selector matrix with elements $d_{i,j} \in \{0, 1\}$ where $i \in \{1, 2, \dots, N_\phi\}$, $j \in \{1, 2, \dots, 2n_k\}$ with a maximum of one non-zero in each row or column, that maps $\mathbf{I}_k(t) \in \mathbb{R}^{2n_k}$, the vector of currents entering the interconnect subnetwork k , into the node space $\mathbb{R}^{N_\phi}$ of the network. The Y -parameters $\mathbf{Y}_k(s)$ of transmission line set k are given by

$$\mathbf{Y}_k \mathbf{V}_k = \mathbf{I}_k \quad (6.4)$$

where $V_k \in \mathbb{R}^{2n_k}$ is the vector of terminal voltages of the interconnect subnetwork k and

$$Y_k = \begin{bmatrix} S_i E_1 S_v^{-1} & S_i E_2 S_v^{-1} \\ S_i E_2 S_v^{-1} & S_i E_1 S_v^{-1} \end{bmatrix} \quad (6.5)$$

where E_1 and E_2 are the diagonal matrices,

$$E_1 = \text{diag} \left\{ \frac{1 + e^{-2\gamma_i d}}{1 - e^{-2\gamma_i d}}, i = 1 \dots n_k \right\} \quad (6.6a)$$

$$E_2 = \text{diag} \left\{ \frac{2}{e^{-\gamma_i d} - e^{\gamma_i d}}, i = 1 \dots n_k \right\} \quad (6.6b)$$

where d is the length of the transmission line set, and γ_i^2 are the eigenvalues of the matrix $Z_L Y_L$ corresponding to transmission line set k with associated eigenvectors x_i .

The impedance per-unit-length Z_L and admittance per-unit-length Y_L are defined as

$$Z_L = R_k + sL_k \quad (6.7a)$$

$$Y_L = G_k + sC_k \quad (6.7b)$$

where R_k, L_k, G_k and $C_k \in \mathbb{R}^{n_k \times n_k}$ are the per-unit-length resistance, inductance, capacitance and conductance matrices of the transmission line set k . S_v is a matrix with the eigenvectors x_i in the columns, and

$$S_i = Z_L^{-1} S_v \Gamma_s \quad (6.8)$$

where $\Gamma_s = \text{diag}\{\gamma_i, i = 1 \dots n_k\}$.

6.3 Sensitivity Analysis

An often required step in the design process is the calculation of the sensitivity of an output node of interest with respect to a circuit parameter λ . The parameter λ may be any parameter of interest in the circuit formulation such as the value of a circuit component like as a resistor or a capacitor or it may be a transmission line parameter such as resistance per-unit-length or width for example. Several techniques can be used to determine the sensitivity of the output

w.r.t. any parameter λ in the network. Amongst these, the adjoint technique has been shown to be the most efficient method [50]. Recalling the MNA formulation in (6.2)

$$\mathbf{A}_\phi \mathbf{X}_\phi(s, \lambda) = \mathbf{B}_\phi(s) \quad (6.9)$$

and setting V_{out} to be the output node of interest, results in

$$V_{out} = \mathbf{d}^T \mathbf{X}_\phi \quad (6.10)$$

where $\mathbf{d}$ is a selector vector with a non-zero entry corresponding to the output variable. The sensitivity of V_{out} w.r.t. λ can therefore be written as

$$\frac{\partial V_{out}}{\partial \lambda} = \mathbf{d}^T \frac{\partial \mathbf{X}_\phi}{\partial \lambda} \quad (6.11)$$

Differentiating (6.9) with respect to lambda results in

$$\mathbf{A}_\phi \frac{\partial \mathbf{X}_\phi(s, \lambda)}{\partial \lambda} = -\frac{\partial \mathbf{A}_\phi}{\partial \lambda} \mathbf{X}_\phi(s, \lambda) \quad (6.12)$$

or alternatively

$$\frac{\partial \mathbf{X}_\phi(s, \lambda)}{\partial \lambda} = -\mathbf{A}_\phi^{-1} \frac{\partial \mathbf{A}_\phi}{\partial \lambda} \mathbf{X}_\phi(s, \lambda) \quad (6.13)$$

Substituting from (6.13) into (6.11) yields

$$\frac{\partial V_{out}}{\partial \lambda} = -\mathbf{d}^T \mathbf{A}_\phi^{-1} \frac{\partial \mathbf{A}_\phi}{\partial \lambda} \mathbf{X}_\phi \quad (6.14)$$

Defining the adjoint vector $\mathbf{X}_a$ as

$$\mathbf{X}_a^T = \mathbf{d}^T \mathbf{A}_\phi^{-1} \quad (6.15)$$

results in the following expression for the sensitivity

$$\frac{\partial V_{out}}{\partial \lambda} = -\mathbf{X}_a^T \frac{\partial \mathbf{A}_\phi}{\partial \lambda} \mathbf{X}_\phi \quad (6.16)$$

From equation (6.15) the following expression can be derived for $\mathbf{X}_a$

$$\mathbf{A}_\phi^T(s, \lambda) \mathbf{X}_a = \mathbf{d} \quad (6.17)$$

The sensitivity of V_{out} w.r.t. λ is therefore found by solving the adjoint equations in (6.17) and using (6.16) to write the sensitivity as

$$\frac{\partial V_{out}}{\partial \lambda} = -\mathbf{X}_a^T \frac{\partial \mathbf{A}_\phi}{\partial \lambda} \mathbf{X}_\phi \quad (6.18)$$

where $\mathbf{X}_\phi$ is the solution of the network equations, and $\mathbf{X}_a$ is the solution of the adjoint equations. The relative sensitivity is defined as

$$S_\lambda = \lambda \frac{\partial V_{out}}{\partial \lambda} \quad (6.19)$$

The definition in (6.19) is directly related to the partial derivatives, however it is more meaningful when comparing sensitivity w.r.t. components of different magnitudes.

The computational cost of this method is one L/U factorization of $\mathbf{A}_\phi(s, \lambda)$ at each frequency point. It is to be noted that for large circuits this method can be computationally expensive. To address this problem, in the next section we propose an algorithm to perform sensitivity analysis on large networks using model reduction techniques.

6.4 Sensitivity analysis using model reduction

Model order reduction methods have been proposed in the literature for the analysis of high speed interconnect networks. Recently, a Krylov subspace based MOR technique was proposed for the analysis distributed networks [19]. In this section, an MOR method for sensitivity analysis of high speed systems including distributed interconnects is presented. This method is based on the adjoint sensitivity technique and does not require the calculation of the sensitivity of the transformation matrix used in the order reduction.

6.4.1 Reduction of the Circuit Equations

As can be seen from (6.18) the solution of both the MNA equations and of the adjoint equations are required for sensitivity analysis. The focus of this section is on using MOR for finding the

solution of the MNA equations. The original system ϕ in (6.2) with N_ϕ nodal variables can be reduced by a congruent transformation into a smaller system of size N_r such that $N_r \ll N_\phi$. The congruent transformation matrix $Q_m \in \mathbb{R}^{N_\phi \times N_r}$ used in the reduction is an orthonormal basis, constructed such that

$$\text{colsp}[Q_m] = \text{colsp}[M_0, M_1, \dots, M_q] \quad (6.20)$$

where M_i is the i^{th} moment of the circuit response. The moments M_i are evaluated using the recursive relation given by [13]

$$[A_\phi]_{s=0} M_i = \sum_{r=1}^n \frac{\left. \frac{\partial^r A}{\partial s^r} \right|_{s=0}}{r!} M_{i-r} \quad (6.21)$$

with

$$[A_\phi]_{s=0} M_0 = B \quad (6.22)$$

Order reduction is performed on the original system in (6.2) through congruent transformation with Q_m . The resulting reduced system is given by

$$\hat{A}_\phi(s) \hat{X}_\phi(s) = \hat{B}_\phi \quad (6.23)$$

where

$$\hat{A}_\phi = Q_m^T A_\phi Q_m; \quad \hat{B}_\phi = Q_m^T B_\phi; \quad X_\phi = Q_m \hat{X}_\phi \quad (6.24)$$

and

$$X_\phi = Q_m \hat{X}_\phi \quad (6.25)$$

Such a congruent transformation was shown to produce a passive reduced system and to preserve the moments of the original system [19]. The analysis is done in the reduced system space, and the solution is mapped back to the original space using (6.25). The response is therefore found without the need to factorize the original large MNA matrix at each frequency point. The main CPU cost of the reduction is one LU decomposition of the original MNA equations used to find the moments in (6.21) and (6.22), and one QR decomposition to orthogonalize the matrix Q_m . The moments shown in (6.21) are for an expansion at the origin of

the complex plane. It is often advantageous to collect moments from more than one expansion point, and algorithms for choosing the expansion points were presented in [12] and [19]. In this case, an LU decomposition is required at each expansion point.

Applying the MOR approach to the original system, allows the computation of the response without having to perform an LU decomposition of the large equations at each frequency point. However, obtaining sensitivity information from the reduced system is a difficult problem. From (6.25) it can be seen that

$$\frac{\partial \mathbf{X}_\phi}{\partial \lambda} = \frac{\partial \mathbf{Q}_m}{\partial \lambda} \hat{\mathbf{X}}_\phi + \mathbf{Q}_m \frac{\partial \hat{\mathbf{X}}_\phi}{\partial \lambda} \quad (6.26)$$

Equation (6.26) demonstrates that finding the sensitivity of $\mathbf{X}_\phi$ w.r.t λ involves finding the derivative of the congruent transformation w.r.t λ which is difficult and cumbersome to evaluate. In the following section an MOR method is proposed for finding the sensitivity of the output without having to evaluate the sensitivity of the transformation matrix.

6.4.2 Reduction of the Adjoint Network

In this section a reduction method for the adjoint network described in (6.17) is described. This approach eliminates the need to evaluate the sensitivity $\partial \mathbf{Q}_m / \partial \lambda$ of the transformation matrix $\mathbf{Q}_m$ used in the reduction of the MNA equation as described in the previous section. Instead, model reduction is applied directly on the adjoint equations

$$\mathbf{A}_\phi^T(s, \lambda) \mathbf{X}_a = \mathbf{d} \quad (6.27)$$

The reduction of the adjoint network above, is done using a congruent transformation with $\mathbf{Q}_a$. The resulting reduced adjoint network is given by

$$\hat{\mathbf{A}}_a \hat{\mathbf{X}}_a(s) = \hat{\mathbf{d}}_a \quad (6.28)$$

where

$$\hat{\mathbf{A}}_a = \mathbf{Q}_a^T \mathbf{A}_\phi^T \mathbf{Q}_a; \quad \hat{\mathbf{d}}_a = \mathbf{Q}_a^T \mathbf{d} \quad (6.29)$$

where the congruent transformation matrix Q_a is an orthonormal matrix that spans the same subspace as the moments of the adjoint network

$$\text{colsp}[Q_a] = \text{colsp}[M_0^a, M_1^a, \dots, M_q^a] \quad (6.30)$$

where M_i^a represents the i^{th} moment of the adjoint network. These moments are evaluated using the recursive relationship given by

$$[A_\phi^T]_{s=0} M_i^a = \sum_{r=1}^i \frac{\left. \frac{\partial^r A_\phi^T}{\partial s^r} \right|_{s=0}}{r!} M_{i-r}^a \quad (6.31)$$

with

$$[A_\phi^T]_{s=0} M_0^a = d \quad (6.32)$$

It should be noted that the factorization of A_ϕ^T can be easily deduced from that of A_ϕ which was evaluated when calculating the moments of network ϕ in (6.22). Therefore, the main additional CPU expense of finding the transformation matrix Q_a is the cost of the QR decomposition required to orthonormalize Q_a in (6.30).

In order to find the solution of the adjoint network, the analysis is done on the reduced adjoint in (6.28), and the results are mapped back to the original adjoint space using

$$X_a(s) = Q_a \hat{X}_a(s) \quad (6.33)$$

The solution of the adjoint network is therefore found without having to perform an LU decomposition of the original large MNA at each frequency point. Hence the solution of the reduced system of equations is much less computationally expensive when compared to that of the original system. It is to be noted that the system moments and the adjoint moments are computed using the same L/U decomposition in (6.21) and (6.31). However, there is no obvious relationship between Q and Q_a . Therefore two separate QR decompositions are required to find the two transformation matrices. Furthermore, the algorithm does not require Q and Q_a to be of the same order, although using the same order for both transformations is recommended.

Once the solution of the adjoint network is obtained, the sensitivity is found using (6.16) as

$$\frac{\partial V_{out}}{\partial \lambda} = -\mathbf{X}_a^T \frac{\partial \mathbf{A}_\phi}{\partial \lambda} \mathbf{X}_\phi \quad (6.34)$$

Note that the sensitivity of the MNA matrix $\frac{\partial \mathbf{A}_\phi}{\partial \lambda}$ is required in the above expression. In the following section the calculation of this sensitivity matrix for the case of transmission line networks is discussed.

6.4.3 Calculation of the Sensitivity of the MNA Martrix

From (6.34) it can be seen that the derivative of the MNA equations is required to calculate the output sensitivities. The method of calculating this derivative however depends on the type of the parameter λ . The following subsections briefly describe the computation of $\frac{\partial \mathbf{A}_\phi}{\partial \lambda}$ when λ represents lumped and distributed elements.

Sensitivity with Respect to Parameters of Lumped Components

When λ represents a parameter of a lumped element it can be seen from (6.2) that

$$\frac{\partial \mathbf{A}_\phi}{\partial \lambda} = \frac{\partial \mathbf{G}_\phi}{\partial \lambda} + s \frac{\partial \mathbf{C}_\phi}{\partial \lambda} \quad (6.35)$$

which can be easily evaluated by differentiating the corresponding component stamp in Fig. 2.2 [50].

Sensitivity with Respect to Electrical Parameters of Transmission Lines

If λ represents an electrical parameter of a distributed element, it can be seen from (6.2) that

$$\frac{\partial \mathbf{A}}{\partial \lambda} = \frac{\partial \mathbf{Y}_\phi(s)}{\partial \lambda} = \frac{\partial}{\partial \lambda} \sum_{k=1}^{N_t} \mathbf{D}_k \mathbf{Y}_k(s) \mathbf{D}_k^T \quad (6.36)$$

In order to compute $\frac{\partial \mathbf{A}_\phi}{\partial \lambda}$ using (6.36) the derivatives of $\mathbf{Y}_k(s)$ w.r.t. to λ are needed, where $\mathbf{Y}_k(s)$ represents the Y -parameters of transmission line set k . It can be shown that $\mathbf{Y}_k(s)$

satisfies the relation [16]

$$\begin{aligned} \frac{\partial \mathbf{Y}_k}{\partial \lambda} \begin{bmatrix} \mathbf{S}_v & 0 \\ 0 & \mathbf{S}_v \end{bmatrix} &= \begin{bmatrix} \frac{\partial \mathbf{S}_i}{\partial \lambda} & 0 \\ 0 & \frac{\partial \mathbf{S}_i}{\partial \lambda} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_2 & \mathbf{E}_1 \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{S}_i & 0 \\ 0 & \mathbf{S}_i \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{E}_1}{\partial \lambda} & \frac{\partial \mathbf{E}_2}{\partial \lambda} \\ \frac{\partial \mathbf{E}_2}{\partial \lambda} & \frac{\partial \mathbf{E}_1}{\partial \lambda} \end{bmatrix} - \mathbf{Y}_k \begin{bmatrix} \frac{\partial \mathbf{S}_v}{\partial \lambda} & 0 \\ 0 & \frac{\partial \mathbf{S}_v}{\partial \lambda} \end{bmatrix} \end{aligned} \quad (6.37)$$

From (6.37) it can be seen that the sensitivity of $\mathbf{Y}_k$ depends on the partial derivatives of the eigenvalues γ_i^2 and eigenvectors $\mathbf{x}_i$ of the matrix $\mathbf{Z}_L \mathbf{Y}_L$ which can be obtained using

$$\begin{bmatrix} \gamma_i \mathbf{U} - \mathbf{Z}_L \mathbf{Y}_L & \mathbf{x}_i \\ \mathbf{x}_i^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}_i}{\partial \lambda} \\ \frac{\partial \gamma_i^2}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial}{\partial \lambda} (\mathbf{Z}_L \mathbf{Y}_L) \right) \mathbf{x}_i \\ 0 \end{bmatrix} \quad (6.38)$$

where $\mathbf{U}$ is the identity matrix.

6.4.4 Sensitivity with Respect to Physical Parameters of Transmission Lines

While sensitivity with respect to electrical parameters of transmission lines is a useful measure, it is often an intermediate step in the calculation of sensitivity w.r.t. to physical parameters such as width because ultimately physical parameters are the design parameters. Suppose λ is a physical parameter of a transmission line system with N conductors. A simple expression for the output sensitivity can be written as

$$\frac{\partial V}{\partial \lambda} = \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\partial V}{\partial \mathbf{R}_{i,j}} \frac{\partial \mathbf{R}_{i,j}}{\partial \lambda} + \frac{\partial V}{\partial \mathbf{L}_{i,j}} \frac{\partial \mathbf{L}_{i,j}}{\partial \lambda} + \frac{\partial V}{\partial \mathbf{G}_{i,j}} \frac{\partial \mathbf{G}_{i,j}}{\partial \lambda} + \frac{\partial V}{\partial \mathbf{C}_{i,j}} \frac{\partial \mathbf{C}_{i,j}}{\partial \lambda} \right) \quad (6.39)$$

The above formulation requires the computation of the sensitivity of the output with respect to all $4 \times N^2$ electrical parameters of the transmission lines. For example an interconnect containing 4 coupled lines will have 65 electrical parameters (including the length). Since all these electrical parameters are functions of the same physical parameter of interest (e.g. the width of the line) it is much more efficient to include the physical parameters directly in (6.38).

The derivatives of the electrical parameters with respect to the physical parameters can be found using numerical methods [106]–[108], but for many cases simple analytical expressions for the electrical parameters (such as those in [109]) can be used.

6.5 Numerical Results

To demonstrate the efficiency of the method, a relatively large interconnect circuit consisting of 458 resistors, inductors and capacitors and 12 transmission lines is considered. Fig. 6.1, shows the dimensions of the microstrip transmission line of interest. An input pulse of a $4ns$ and a rise and fall time of $0.3ns$ was applied at the input. The model order reduction method proposed in this chapter was applied to the circuit and to the adjoint network. The original network contained 403 nodes. The frequency response was matched up to 4GHz using a reduced system of order 45 with two expansion points. A comparison between the time and frequency domain responses obtained from the original and reduced systems is shown in figures. 6.2 and 6.3. The time and frequency domain sensitivities of the output voltage with respect to the R, L, C, and G parameters of one of the interconnects, and with respect to its width are shown in figs. 6.4–6.13. In order to obtain the time domain results, 2048 frequency point were used in the FFT algorithm. This requires 2048 L/U decompositions of the original large equations when conventional adjoint method is used. However, the proposed reduction methods required only 2 L/U decomposition of the original MNA equations.

Figure 6.1: A microstrip line from the numerical example

Figure 6.2: Time domain response

Figure 6.3: Frequency domain response

Figure 6.4: Frequency domain relative sensitivity w.r.t resistance per unit length

Figure 6.5: Time domain relative sensitivity w.r.t resistance per unit length

Figure 6.6: Frequency domain relative sensitivity w.r.t inductance per unit length

Figure 6.7: Time domain relative sensitivity w.r.t inductance per unit length

Figure 6.8: Frequency domain relative sensitivity w.r.t capacitance per unit length

Figure 6.9: Time domain relative sensitivity w.r.t capacitance per unit length

Figure 6.10: Frequency domain relative sensitivity w.r.t conductance per unit length

Figure 6.11: Time domain relative sensitivity w.r.t conductance per unit length

Figure 6.12: Frequency domain relative sensitivity w.r.t interconnect width

Figure 6.13: Time domain relative sensitivity w.r.t interconnect width

6.6 Conclusion

An efficient algorithm, based on congruent transformation, for evaluating the frequency and time domain sensitivity of large lossy coupled transmission line networks was presented in this chapter. The algorithm is based on projecting the adjoint network equations on a reduced order subspace that preserves the circuit moments. Using this algorithm, output sensitivities of large linear networks can be calculated accurately with respect to lumped components and parameters of distributed elements. The proposed algorithm provides a significant decrease in the computational complexity for sensitivity analysis.

Chapter 7

Circuit Reduction Method for Steady-State Analysis

7.1 Introduction

The rapid growth in Radio Frequency (RF) silicon Integrated Circuits (ICs) for mobile communications systems has placed new demands on simulation tools. Quantities such as intermodulation and harmonic distortion are of interest to circuit designers. Such figures of merit are obtained using nonlinear steady-state analysis. However, finding the steady-state response for large RF and microwave circuits represents an increasing challenge. The reason for this, is that such circuits exhibit characteristics that make most traditional transient-based time domain simulators perform poorly. For example, if there are widely separated time constants in the circuit, a traditional transient analysis would require a very large number of time steps before the steady-state solution is reached. In addition to the high computation cost of this type of analysis, the accuracy of the results may be poor because of accumulated errors [110]. Another problem faced by this approach, is the difficulty of handling distributed elements such as transmission lines in the time domain [4].

The harmonic balance (HB) technique [75], [88], [111], [112] has been introduced to ad-

dress the above difficulties. The main philosophy behind HB is that waveforms in a periodically excited nonlinear circuit are periodic in the steady-state and, therefore, can be represented as a finite Fourier series. The nonlinear differential equations are thus transformed into a set of nonlinear algebraic equations that can be solved using iterative techniques such as the Newton-Raphson (NR) method. The difficulty with the HB technique is that each of the Fourier coefficients becomes an unknown, which results in a very large system of equations. Solving this set of equations involves storing and factorizing the Jacobian matrix at each NR iteration. For large RF circuits with strong nonlinearities, this Jacobian matrix is large and dense [113] and its factorization is very expensive even for medium sized circuits [91]. This problem is particularly difficult for simulations involving multi-tone inputs, such as IP3 analysis, or simulation of mixer circuits, because the number of variables is very large due to the large number of intermodulation tones. In addition, like all other locally convergent methods, NR has convergence problems. Convergence can only be achieved by finding a sufficiently close initial guess to the solution. This is particularly difficult for circuits with high excitation levels such as power amplifiers. Continuation or *homotopy* methods have been proposed in the literature in order to improve the convergence properties of iterative methods such as the NR iteration [114]–[116]. However, using these methods requires finding a number of intermediate solutions while sweeping the continuation parameter, and results in a further increase in the computation cost.

In this chapter, a circuit reduction method is proposed in order to improve the efficiency of harmonic balance simulation. The proposed algorithm poses the problem from a different perspective. The idea is to reduce the size of the circuit and, therefore, of the Jacobian matrix needed for the HB solution, through the introduction of the new concept of nonlinear circuit reduction for steady-state analysis. Continuation methods are used as a vehicle in order to enable the circuit reduction. Using the new approach, the original set of HB equations is replaced with a much smaller set of nonlinear equations. The reduced set, shares with the original one the first few derivatives with respect to the continuation parameter. A continuation

scheme is then used to sweep the reduced system of equations, and the solution is mapped back to the space of the original circuit equations.

7.2 Harmonic Balance Simulation

In this section, the harmonic balance technique for finding the steady-state solution is described. The methods presented, are then used as a foundation in the latter section when introducing the proposed circuit reduction method for steady-state simulation.

7.2.1 Formulation of the HB Equations

Consider a circuit ϕ containing linear and nonlinear elements. Assuming that the circuit ϕ has N_ϕ nodal variables, its MNA equations as defined in equation (2.1) can be written as

$$\mathbf{G}_\phi \mathbf{x}_\phi(t) + \mathbf{C}_\phi \dot{\mathbf{x}}_\phi(t) + \mathbf{f}_\phi(\mathbf{x}_\phi(t)) = \mathbf{b}_\phi(t) \quad (7.1)$$

where $\mathbf{G}_\phi$ and $\mathbf{C}_\phi \in \mathbb{R}^{N_\phi \times N_\phi}$ are the MNA matrices as was outlined in section. 2.1. Assuming a periodic input $\mathbf{b}_\phi(t)$ for circuit ϕ , the response is also known to be periodic in the steady-state. The solution can therefore be written in terms of a truncated Fourier series including the first H harmonics as

$$\mathbf{x}_\phi(t) = \mathbf{A}_0 + \sum_{k=1}^H (\mathbf{A}_k \cos(k\omega t) + \mathbf{B}_k \sin(k\omega t)) \quad (7.2)$$

Substituting from (7.2) into (7.1) and equating the coefficients of the sines and cosines, results in the following set of nonlinear algebraic equations

$$\bar{\mathbf{G}} \mathbf{X}_h + \bar{\mathbf{C}} \mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) = \mathbf{B}_h \quad (7.3)$$

where

- $\mathbf{X}_h \in \mathbb{R}^{N_{hb}}$ is a vector containing the unknown Fourier coefficients of $\mathbf{x}_\phi(t)$,
- $\mathbf{B}_h \in \mathbb{R}^{N_{hb}}$ is a vector of the Fourier coefficients of the input sources,

- $\bar{G}$ is a block matrix $\bar{G} = [\bar{G}_{ij}]$ whose blocks $\bar{G}_{ij} \in \mathbb{R}^{N_h \times N_h}$ are diagonal matrices given by

$$\bar{G}_{ij} = \text{diag}(g_{ij}, \dots, g_{ij}) \quad (7.4)$$

where g_{ij} is the corresponding element in the G_ϕ matrix in (7.1),

- $\bar{C}$ is also a block matrix $\bar{C} = [\bar{C}_{ij}]$ whose blocks $\bar{C}_{ij} \in \mathbb{R}^{N_h \times N_h}$ are given by

$$\bar{C}_{ij} = c_{ij} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega & \cdots & 0 & 0 \\ 0 & -\omega & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & H\omega \\ 0 & 0 & 0 & \cdots & -H\omega & 0 \end{bmatrix} \quad (7.5)$$

where c_{ij} is the corresponding element in the C_ϕ matrix in (7.1),

- $F(X_h)$ contains the Fourier coefficients of the nonlinear vector $\mathbf{f}_\phi(\mathbf{x}_\phi(t))$ defined in (7.1)
- $N_h = 2H + 1$ is the number of unknown Fourier coefficients at each node
- $N_{hb} = N_\phi N_h$ is the total number of unknowns in the harmonic balance equations.

The harmonic balance technique succeeds in converting the nonlinear differential algebraic MNA equations into a set of nonlinear algebraic equations. However, as can be seen from the above formulation, the resulting set of equations is quite large and is therefore computationally expensive to solve. In this chapter, a circuit reduction technique for steady-state analysis is proposed in order to address this problem.

In order to illustrate how the HB equations presented above are constructed, the simple one node circuit in Fig. 7.1 is considered. The MNA equations for this circuits are

$$\dot{x}(t) + x(t) + f(x(t)) = A \cos(\omega t) \quad (7.6)$$

Figure 7.1: Example circuit for HB formulation

where $f(x(t)) = I_s(e^{\frac{x(t)}{v_t}} - 1)$ is the nonlinear equation representing the current in the diode. Note that for this circuit, $x(t)$ is a scalar. After the steady-state is reached, $x(t)$ can be expressed as a Fourier series

$$x(t) = a_0 + \sum_{k=1}^H (a_k \cos(k\omega t) + b_k \sin(k\omega t)) \quad (7.7)$$

The derivative of $x(t)$ can therefore be written as

$$\dot{x}(t) = \sum_{k=1}^H (b_k k\omega \cos(k\omega t) - a_k k\omega \sin(k\omega t)) \quad (7.8)$$

and the nonlinear diode current can also be represented as

$$f(x(t)) = f_{co} + \sum_{k=1}^H (f_{ck} \cos(k\omega t) + f_{sk} \sin(k\omega t)) \quad (7.9)$$

By plugging (7.7), (7.8) and (7.9) into equation (7.6) we obtain the relation

$$a_0 + \sum_{k=1}^H (a_k \cos(k\omega t) + b_k \sin(k\omega t)) + \sum_{k=1}^H (b_k k\omega \cos(k\omega t) - a_k k\omega \sin(k\omega t)) \\ f_{co} + \sum_{k=1}^H (f_{ck} \cos(k\omega t) + f_{sk} \sin(k\omega t)) = A \cos(\omega t) \quad (7.10)$$

Equation (7.10) is satisfied if and only if the coefficients of $\cos(k\omega t)$ and $\sin(k\omega t)$ are matched on the left hand side and the right hand side of the equation. The following equations must therefore be satisfied

$$a_0 + f_{co} = 0 \quad (7.11a)$$

$$a_1 + b_1\omega + f_{c1} = A \quad (7.11b)$$

$$b_1 - a_1\omega + f_{s1} = 0 \quad (7.11c)$$

$$a_2 + 2b_2\omega + f_{c2} = 0 \quad (7.11d)$$

$$b_2 - 2a_2\omega + f_{s2} = 0 \quad (7.11e)$$

Relations in the form of equation (7.11) can be found up to the highest harmonic H of interest.

Assembling these equations in matrix form yields

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_H \\ b_H \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega & \cdots & 0 & 0 \\ 0 & -\omega & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & H\omega \\ 0 & 0 & 0 & \cdots & -H\omega & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_H \\ b_H \end{bmatrix} + \begin{bmatrix} f_{c0} \\ f_{c1} \\ f_{s1} \\ \vdots \\ f_{cH} \\ f_{sH} \end{bmatrix} = \begin{bmatrix} 0 \\ A \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (7.12)$$

The above set of equations is a system of nonlinear algebraic equations, where the unknowns are the Fourier coefficients of $x(t)$. The coefficients f_{ci} and f_{si} of the in the nonlinear vector are directly related to the unknown variables through the use of the Fast Fourier Transform (FFT) as shown in Section 7.2.2. The above formulation can be generalized for a network with N_ϕ nodes. In this case the HB matrix equations are written as

$$\bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) = \mathbf{B}_h \quad (7.13)$$

where $\mathbf{X}_h \in \mathbb{R}^{N_{hb}}$ is a vector containing the unknown Fourier coefficients of $\mathbf{x}_\phi(t)$, $\mathbf{B}_h \in \mathbb{R}^{N_{hb}}$ is a vector of the Fourier coefficients of the input sources, $\bar{\mathbf{G}}$ is a block matrix $\bar{\mathbf{G}} = [\mathbf{G}_{ij}]$ whose blocks $\mathbf{G}_{ij} \in \mathbb{R}^{N_h \times N_h}$ are diagonal matrices given by

$$\mathbf{G}_{ij} = \text{diag}(g_{ij}, \dots, g_{ij}) \quad (7.14)$$

where g_{ij} is the corresponding element in the $\mathbf{G}_\phi$ matrix in (3.29). The matrix $\bar{\mathbf{C}}$ is also a block matrix $\bar{\mathbf{C}} = [\mathbf{C}_{ij}]$ whose blocks $\mathbf{C}_{ij} \in \mathbb{R}^{N_h \times N_h}$ are given by

$$\mathbf{C}_{ij} = c_{ij} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega & \cdots & 0 & 0 \\ 0 & -\omega & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & H\omega \\ 0 & 0 & 0 & \cdots & -H\omega & 0 \end{bmatrix} \quad (7.15)$$

where c_{ij} is the corresponding element in the $\mathbf{C}_\phi$ matrix in (3.29). The vector $\mathbf{F}(\mathbf{X}_h)$ contains the Fourier coefficients of the nonlinear vector $\mathbf{f}_\phi(\mathbf{x}_\phi(t))$ defined in (3.29). More precisely, the vector $\mathbf{f}_\phi(\mathbf{x}_\phi(t))$ is a vector of nonlinear scalar functions

$$\mathbf{f}_\phi(\mathbf{x}_\phi(t)) = [f_1(\mathbf{x}_\phi), f_2(\mathbf{x}_\phi), \dots, f_{N_\phi}(\mathbf{x}_\phi)]^T \quad (7.16)$$

and the vector $\mathbf{F}(\mathbf{X}_h)$ is given by

$$\mathbf{F}(\mathbf{X}_h) = [F_1(\mathbf{X}_h), F_2(\mathbf{X}_h), \dots, F_{N_\phi}(\mathbf{X}_h)]^T \quad (7.17)$$

where $F_1(\mathbf{X}_h)$ is a vector containing the Fourier coefficients of $f_1(\mathbf{x}_\phi)$, $F_2(\mathbf{X}_h)$ contains the coefficients of $f_2(\mathbf{x}_\phi)$ etc. An example with only one scalar nonlinear function is shown in equation (7.12). The relationship between $\mathbf{X}_h$ and $\mathbf{F}(\mathbf{X}_h)$ is best evaluated through the use of the Fast Fourier Transform (FFT) as explained in the following section.

7.2.2 Handling Nonlinear Components Using FFT

In this section, the evaluation of the nonlinear vector $\mathbf{F}(\mathbf{X}_h)$ through the use of the Fast Fourier Transform (FFT) is explained. Without loss of generality, the discussion in this section is limited to one of the scalar functions in (7.16). The same procedure is applied to each scalar element of (7.16) in order to obtain (7.17). First a brief review the Direct Fourier Transform (DFT) algorithm, and its relationship to the Fast Fourier Transform is presented. The use of FFT for the nonlinear vector in the HB equations is then outlined.

The Direct Fourier Transform

The Fast Fourier Transform is simply an efficient method to evaluate the direct Fourier transform, for the case when the number of time points is a power of 2. While the FFT algorithm exploits some repetitive computations in order to reduce the computation complexity, the two approaches (FFT and DFT) are mathematically equivalent. In this section the DFT approach will be followed because it provides more insight, and will be useful in explaining the evaluation of the Jacobian matrix in later sections. One must however keep in mind that in the software implementation, the FFT approach is followed and leads to identical numerical results with less CPU cost.

Let $x(t)$ be a periodic signal with a period $T = \frac{2\pi}{\omega}$ be expressed as a Fourier series.

$$x(t) = a_0 + \sum_{k=1}^H (a_k \cos(k\omega t) + b_k \sin(k\omega t)) \quad (7.18)$$

The function $x(t)$ is sampled at N_h equally spaced time points $[t_0, t_1, \dots, t_{N_h-1}]$ in the interval $[0, T]$ such that

$$t_n = n \frac{T}{N_h}; \quad n = 0, 1, \dots, N_h - 1 \quad (7.19)$$

Writing the Fourier series in (7.18) at each time point yields

$$x(t_0) = a_0 + \sum_{k=1}^H (a_k \cos(k\omega t_0) + b_k \sin(k\omega t_0)) \quad (7.20a)$$

$$x(t_1) = a_0 + \sum_{k=1}^H (a_k \cos(k\omega t_1) + b_k \sin(k\omega t_1)) \quad (7.20b)$$

...

$$x(t_{N-1}) = a_0 + \sum_{k=1}^H (a_k \cos(k\omega t_{N_h-1}) + b_k \sin(k\omega t_{N_h-1})) \quad (7.20c)$$

Equation (7.20) can be written in matrix form as

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N_h-1} \end{bmatrix} = \begin{bmatrix} 1 & \cos(\omega t_0) & \sin(\omega t_0) & \cdots & \cos(H\omega t_0) & \sin(H\omega t_0) \\ 1 & \cos(\omega t_1) & \sin(\omega t_1) & \cdots & \cos(H\omega t_1) & \sin(H\omega t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega t_{N_h-1}) & \sin(\omega t_{N_h-1}) & \cdots & \cos(H\omega t_{N_h-1}) & \sin(H\omega t_{N_h-1}) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_H \\ b_H \end{bmatrix} \quad (7.21)$$

where $x_n = x(t_n)$ represents the time sample of $x(t)$ at time t_n . Equation (7.21) represents the inverse direct Fourier transform (DFT), where the time samples are obtained by multiplication with the inverse DFT matrix. The inverse DFT can be represented more compactly as,

$$X_s = \Gamma X_h \quad (7.22)$$

where $X_s = [x_0, x_1, \dots, x_{N_h-1}]^T$ represents the time samples, and the Fourier coefficients are assembled in the vector $X_h = [a_0, a_1, b_1, \dots, a_H, b_H]$. Note that the DFT matrix is independent of frequency. The only requirement is that the time domain signal is sampled at equally spaced time points along one period. This can be deduced by noting the arguments of the sine and cosine in the inverse DFT matrix in (7.21), which are not functions of frequency. These arguments of the form $k\omega t_n$, and can be written as

$$k\omega t_n = k \left(\frac{2\pi}{T} \right) n \left(\frac{T}{N_h} \right) = kn \left(\frac{2\pi}{N_h} \right) \quad (7.23)$$

the matrix Γ can be written independently of the frequency as

$$\Gamma = \begin{bmatrix} 1 & \cos(0) & \sin(0) & \cdots & \cos(0) & \sin(0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos\left(\left(\frac{2\pi n}{N_h}\right)\right) & \sin\left(\left(\frac{2\pi n}{N_h}\right)\right) & \cdots & \cos\left(\left(\frac{2\pi Hn}{N_h}\right)\right) & \sin\left(\left(\frac{2\pi Hn}{N_h}\right)\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos\left(\frac{2\pi(N_h-1)}{N_h}\right) & \sin\left(\frac{2\pi(N_h-1)}{N_h}\right) & \cdots & \cos\left(\frac{2\pi H(N_h-1)}{N_h}\right) & \sin\left(\frac{2\pi H(N_h-1)}{N_h}\right) \end{bmatrix} \quad (7.24)$$

or alternatively

$$\Gamma = \begin{bmatrix} 1 & \cos(\Theta_{0,1}) & \sin(\Theta_{0,1}) & \cdots & \cos(\Theta_{0,H}) & \sin(\Theta_{0,H}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\Theta_{n,1}) & \sin(\Theta_{n,1}) & \cdots & \cos(\Theta_{n,H}) & \sin(\Theta_{n,H}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\Theta_{N_h-1,1}) & \sin(\Theta_{N_h-1,1}) & \cdots & \cos(\Theta_{N_h-1,H}) & \sin(\Theta_{N_h-1,H}) \end{bmatrix} \quad (7.25)$$

with

$$\Theta_{n,k} = kn \left(\frac{2\pi}{N_h} \right) \quad (7.26)$$

The only requirement for the above formulation is that the time points be equally spaced along the period $[0, T]$. The direct Fourier transform (DFT) can be deduced from (7.22) as

$$X_h = \Gamma^{-1} X_s \quad (7.27)$$

The numerical complexity of the direct Fourier transform is $O(N_h^2)$, or that of a matrix vector multiplication. The FFT algorithm essentially achieves the same result, but with a numerical complexity of $O(N_h \ln(N_h))$. In the remainder of the chapter, the FFT process will be represented as a matrix vector multiplication as defined in (7.22) and (7.27), while keeping in mind that the actual implementation is different in order to take advantage of the CPU speed-up.

Evaluation of the Nonlinear Vector Using FFT

In this section we limit our analysis to one of the scalar functions in the nonlinear $\mathbf{f}_\phi(\mathbf{x}_\phi(t))$. The other elements are treated in a similar fashion. Our task is to evaluate the vector $F_1(\mathbf{X}_h)$ containing the Fourier coefficients of $f_1(\mathbf{x}_\phi(t))$, as a function of $\mathbf{X}_h$. Without loss of generality we can assume that $f(\mathbf{x}_\phi)$ is a function of one variable x_1 . The objective is therefore to find $F(X_1)$, as a function of X_1 which is the vector containing the Fourier coefficients of $x_1(t)$. Knowing X_1 , the time samples of $x(t)$ over one period can be obtained by applying the inverse DFT in equation (7.22)

$$X_s = \Gamma X_1 \quad (7.28)$$

where $X_s = [x_0, x_1, \dots, x_{N_h}]$ is a vector containing the time samples of $x(t)$. The vector F_s containing the time samples of $f(x(t))$ is therefore given by

$$F_s = [f(x_0), f(x_1), \dots, f(x_{N_h})] \quad (7.29)$$

Knowing F_s , $F(X_1)$ can now be easily obtained using the DFT defined in equation (7.27)

$$F(X_h) = \Gamma^{-1} F_s \quad (7.30)$$

7.2.3 Solution of the HB Equations

The Harmonic Balance equations defined in (7.13) are a set of nonlinear algebraic equations where the unknowns are the Fourier coefficients of the steady-state solution. The solution of these equations is found by applying iterative methods. Newton iteration is an obvious choice [117], and benefits from quadratic convergence near the solution, however each iteration is costly in terms of CPU requirements and convergence is not guaranteed. Relaxation based techniques [86], [87] improve the CPU cost, but have difficulties in convergence. A number of variations of the above techniques such as Gauss-Jacobi Newton harmonic relaxation [88] and inexact Newton iteration [89], were later proposed in order to improve convergence, and CPU cost. The HB method is still however limited by the large size and dense structure of the jacobian. More recently iterative linear techniques such as the quasi-minimal residual (QMR) [90] and generalized minimal residual (GMRES) [91] were used for solving the HB equations. These techniques, however, rely on the availability of a good pre-conditioner, which is not an obvious problem to solve [90]–[92]. In this section, the basic Newton iteration for solving the HB equation is presented. This serves to illustrate the solution procedure, and also forms a basis for the solution done using circuit reduction in the latter sections of this chapter. The HB equations can be written as

$$\Phi(X_h) = \bar{G}X_h + \bar{C}X_h + F(X_h) - B_h = 0 \quad (7.31)$$

The target solution is the vector X_h that satisfies equation (7.31). The solution is found iteratively using Newton Raphson (NR) iteration, by starting with an initial guess, and updating

Figure 7.2: Newton-Raphson Iteration

the solution at each iteration as shown in fig 7.2. In the NR algorithm the solution vector $\mathbf{X}_h$ is updated, as follows

$$\mathbf{X}_h^{(i+1)} = \mathbf{X}_h^i - \mathbf{J}_{hb}(\mathbf{X}_h^{(i)})^{-1} \Phi(\mathbf{X}_h^{(i)}) \quad (7.32)$$

where i is the iteration number, $\mathbf{X}_h^{(i)}$ is the previous guess, and $\mathbf{J}_{hb}(\mathbf{X}_h^{(i)})$ is the Jacobian matrix

$$\mathbf{J}_{hb}(\mathbf{X}_h^{(i)}) = \frac{\partial \Phi(\mathbf{X}_h)}{\partial \mathbf{X}_h} \Big|_{\mathbf{X}_h^{(i)}} = \bar{\mathbf{G}} + \bar{\mathbf{C}} + \frac{\partial \mathbf{F}(\mathbf{X}_h)}{\partial \mathbf{X}_h} \Big|_{\mathbf{X}_h^{(i)}} \quad (7.33)$$

The main difficulty with the Harmonic Balance simulation is the factorization of the Jacobian matrix due to its large size and dense structure. This problem is particularly important in circuits with multi-tone inputs due to the large number of harmonics and intermodulation tones present. In the following section, the evaluation of the harmonic balance jacobian is explained.

7.2.4 The Harmonic Balance Jacobian

As mentioned in the previous section, the evaluation and inversion of the Harmonic Balance jacobian represents the bulk of the computation complexity of finding the steady-state response. This section is devoted to explaining the structure of this jacobian, and how it is evaluated. The HB jacobian was defined in Section 7.2.3 as

$$\mathbf{J}_{hb}(\mathbf{X}_h^{(i)}) = \left. \frac{\partial \Phi(\mathbf{X}_h)}{\partial \mathbf{X}_h} \right|_{\mathbf{X}_h^{(i)}} = \bar{\mathbf{G}} + \bar{\mathbf{C}} + \left. \frac{\partial \mathbf{F}(\mathbf{X}_h)}{\partial \mathbf{X}_h} \right|_{\mathbf{X}_h^{(i)}}$$

The first two terms ($\bar{\mathbf{G}}$ and $\bar{\mathbf{C}}$) are defined in equations (7.4) and (7.5). The evaluation of the remaining term $\partial \mathbf{F}(\mathbf{X}_h)/\partial \mathbf{X}_h$ is the subject of this section. In order to find $\partial \mathbf{F}(\mathbf{X}_h)/\partial \mathbf{X}_h$, the derivative of the nonlinear vector $\mathbf{f}_\phi$ are first considered in the time domain

$$\frac{\partial \mathbf{f}_\phi(\mathbf{x}_\phi)}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (7.34)$$

where $f_1, \dots, f_n$ are the scalar nonlinear functions that form the vector $\mathbf{f}(\mathbf{x})$, and $x_1, \dots, x_n$ are the scalar variables in the vector $\mathbf{x}_\phi(t)$. Now we can write $\partial \mathbf{F}(\mathbf{X}_h)/\partial \mathbf{X}_h$ as

$$\frac{\partial \mathbf{F}(\mathbf{X}_h)}{\partial \mathbf{X}_h} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial X_1} & \cdots & \frac{\partial F_n}{\partial X_n} \end{bmatrix} \quad (7.35)$$

where F_1 , for example, is a vector containing the Fourier coefficient of f_1 and X_1 contains the Fourier coefficients of x_1 . Note that the terms in (7.35) ($\frac{\partial F_1}{\partial X_1}$ for example) are matrices. Each block is computed from the corresponding term in (7.34). In order to simplify the notation we will consider the case of one of the blocks, and the same approach is used to compute the remaining blocks. Consider the function $\frac{\partial f_1}{\partial x_1}$. Let $\mathbf{X}_s$ be the vector containing the time domain

samples of $x_1(t)$

$$X_s = \begin{bmatrix} x_1(t_0) \\ x_1(t_1) \\ \vdots \\ x_1(t_{N_h-1}) \end{bmatrix} \quad (7.36)$$

from (7.22) we can write

$$X_s = \Gamma X_1 \quad (7.37)$$

The time samples of $f_1(x_1(t))$ are assembled in the vector F_s such that

$$F_s = \begin{bmatrix} f_1(x_1(t_0)) \\ f_1(x_1(t_1)) \\ \vdots \\ f_1(x_1(t_{N_h-1})) \end{bmatrix} \quad (7.38)$$

From (7.27), the following expression can be written

$$F_1 = \Gamma^{-1} F_s \quad (7.39)$$

Now we can evaluate the derivatives of the Fourier coefficients of $f_1(x_1)$ with respect to the Fourier coefficients of x_1

$$\frac{\partial F_1}{\partial X_1} = \Gamma^{-1} \frac{\partial F_s}{\partial X_1} = \Gamma^{-1} \frac{\partial X_s}{\partial F_s} \frac{\partial F_s}{\partial X_1} = \Gamma^{-1} \frac{\partial F_s}{\partial X_s} \Gamma \quad (7.40)$$

where $\frac{\partial F_s}{\partial X_s}$ is a diagonal matrix given by

$$\frac{\partial F_s}{\partial X_s} = \text{diag} \left[\left. \frac{\partial f_1(x_1)}{\partial x_1} \right|_{x_1(t_0)}, \left. \frac{\partial f_1(x_1)}{\partial x_1} \right|_{x_1(t_1)}, \dots, \left. \frac{\partial f_1(x_1)}{\partial x_1} \right|_{x_1(t_{N-1})} \right] \quad (7.41)$$

Each block term in equation (7.35) is found by using (7.40). Note that the multiplication by Γ and Γ^{-1} is not done in practice, and the FFT and IFFT algorithms are used instead. The resulting jacobian contains many dense blocks and is very expensive to store, and factorize.

Figure 7.3: Continuation example (DC solution)

7.3 Continuation Methods

As explained in Section 7.2.3, iterative methods such as the Newton Iteration are used to solve the Harmonic Balance equations. Such methods start with an initial guess in order to start the iteration. A good initial guess is important for two main reasons. The first is related to simulation speed. A good initial guess would result in a significantly fewer iterations before convergence is achieved. The second reason is equally important, and is related to the convergence of iterative methods. Newton's method for example is not globally convergent (i.e. convergence is not guaranteed for all initial guesses). A good initial guess is often necessary simply to achieve convergence. The difficulty with this problem is that, typically, there are no obvious ways of finding a good initial guess. Continuation or *homotopy* methods [114]–[116], which are rooted in the mathematical field of *Topology* where introduced to circuit simulation in order to address the convergence issues of locally convergent iterative methods such as the Newton iteration. The basic concept behind continuation methods is to augment the original equations $\Phi(\mathbf{X}) = 0$ with a new variable α in order to obtain a new system of equations $\Psi(\mathbf{X}, \alpha) = 0$. The new system Ψ is constructed such that it has an easy solution at $\alpha = 0$, and that it is equivalent to the original system Φ at $\alpha = 1$, that is $\Psi(\mathbf{X}_h, 1) = \Phi(\mathbf{X}_h)$. The parameter α is then swept from 0 to 1, while tracking the solution. At $\alpha = 0$, the solution is of course trivial, and as the value of α is incremented between 0 and 1, the solution at the previous point is used as an initial guess.

The continuation method is best illustrated by an example. Consider the circuit in Fig. 7.3

Figure 7.4: Overview of the nonlinear circuit reduction algorithm

with a 5 volts dc input. Finding the dc solution of this circuit involves solving a set of nonlinear equations where convergence is often a problem. Continuation is applied to this problem by ramping the value of the input voltage from 0 to 5 volts. When the value is at 0 volts, the solution is trivial with all values set to 0. When the value is 5 volts we are of course back the original problem. For the intermediate steps between 0 and 5, the initial guess is taken to be the solution at the previous value of α (in this case α being the dc input voltage). This is an example where the parameter α is a recognizable physical parameter of the circuit. However, continuation methods are not limited to physical parameters and it is often advantageous to introduce artificial parameters [114].

7.4 Overview of the Circuit Reduction Algorithm

In this section a new algorithm that reduces the size of the Harmonic Balance equations is described. The new approach is based on orthogonal projection methods through congruent transformation. An overview of the proposed algorithm is shown in Fig. 7.4. The first step

of the algorithm consists of embedding a continuation parameter α in the original Harmonic Balance equations $\Phi(\mathbf{X}_h) = 0$. The parameter α is chosen such that the *embedded* system $\Psi(\mathbf{X}_h, \alpha) = 0$ has a trivial solution when $\alpha = 0$, and is equivalent to the original system Φ when $\alpha = 1$. The system Ψ is then reduced into a much smaller reduced system $\hat{\Psi}$. The parameter α is tracked from 0 to 1 in the space of the reduced system, and the solution at $\alpha = 1$ is mapped back to the space of the original system. The solution is therefore found entirely in the reduced space without the high cost of solving the large system of nonlinear Harmonic Balance equations. The details of the circuit reduction are discussed in the remainder of this section.

7.4.1 Augmenting the HB Equations with Continuation Parameter

The first step in the proposed algorithm is to augment the HB equations with a continuation parameter α as shown in Fig. 7.4. In this case, the parameter α multiplies the input ac power, resulting in a power ramping homotopy. The augmented HB equations can therefore be written as follows

$$\begin{aligned} \Psi(\mathbf{X}_h, \alpha) = & \bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) \\ & - \mathbf{B}_{dc} - \alpha\mathbf{B}_{ac} \end{aligned} \quad (7.42)$$

where $\mathbf{B}_{dc}$ represents the dc input sources and $\mathbf{B}_{ac}$ contains the ac sources. When $\alpha = 0$ the problem is reduced to finding the dc solution (a relatively simple problem), and when $\alpha = 1$ the problem is equivalent to the original Harmonic Balance problem. This is a classical power ramping continuation parameter (in this case for the ac power). As will be seen in the following sections the continuation parameter is used as a vehicle in order to enable the circuit reduction. However, the reduced circuit will also benefit from the convergence improving properties of continuation methods.

7.4.2 Circuit Reduction

In this section, the second step of the reduction scheme shown in Fig. 7.4 is presented, where the large system of equations $\Psi(\mathbf{X}_h, \alpha)$ is reduced into a much smaller system $\hat{\Psi}(\hat{\mathbf{X}}_h, \alpha)$. The main objective of the reduction algorithm is to significantly reduce the size of the original circuit and, consequently, the size of the jacobian matrix. The analysis is done in the reduced space, thus resulting in CPU cost savings. The reduced set of equations is obtained by effecting the following change of variables in the original set of equations (7.42)

$$\mathbf{X}_h \rightarrow \mathbf{Q}_h \hat{\mathbf{X}}_h \quad (7.43)$$

With the change of variables in (7.43) the augmented Harmonic Balance equations in (7.42) become

$$\begin{aligned} \bar{\mathbf{G}}\mathbf{Q}_h \hat{\mathbf{X}}_h + \bar{\mathbf{C}}\mathbf{Q}_h \hat{\mathbf{X}}_h + \mathbf{F}(\mathbf{Q}_h \hat{\mathbf{X}}_h) = \\ = \mathbf{B}_{dc} + \alpha \mathbf{B}_{ac} \end{aligned} \quad (7.44)$$

Equation (7.44) is then pre-multiplied by $\mathbf{Q}_h^T$, and the following reduced system is obtained

$$\hat{\mathbf{G}}\hat{\mathbf{X}}_h + \hat{\mathbf{C}}\hat{\mathbf{X}}_h + \hat{\mathbf{F}}(\hat{\mathbf{X}}_h) - \hat{\mathbf{B}}_{dc} - \alpha \hat{\mathbf{B}}_{ac} = 0 \quad (7.45)$$

where

$$\begin{aligned} \hat{\mathbf{G}} &= \mathbf{Q}_h^T \bar{\mathbf{G}} \mathbf{Q}_h & \hat{\mathbf{C}} &= \mathbf{Q}_h^T \bar{\mathbf{C}} \mathbf{Q}_h \\ \hat{\mathbf{F}}(\hat{\mathbf{X}}_h) &= \mathbf{Q}_h^T \mathbf{F}(\mathbf{Q}_h \hat{\mathbf{X}}_h) & \hat{\mathbf{B}}_{dc} &= \mathbf{Q}_h^T \mathbf{B}_{dc} \dots \end{aligned} \quad (7.46)$$

In the above equations, $\mathbf{Q}_h \in \mathbb{R}^{N_{hb} \times N_q}$ is an orthonormal basis of the subspace spanned by the first q derivatives of $\mathbf{X}_h$ with respect to α , with $N_q \ll N_{hb}$. In other words, if the Taylor expansion of $\mathbf{X}_h$ with respect to α is

$$\mathbf{X}_h = \sum_{k=0}^{\infty} \mathbf{A}_k (\alpha - \alpha_0)^k \quad (7.47)$$

then

$$\text{colsp}[\mathbf{Q}_h] = \text{colsp}[\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{q-1}] \quad (7.48)$$

The basic idea behind this change of variables is that the reduced system obtained by this transformation preserves the first N_q derivatives of $\mathbf{X}_h$ with respect to α under the mapping $\mathbf{Q}_h$. More precisely, if $\hat{\mathbf{X}}_h$ is expanded as the Taylor series

$$\hat{\mathbf{X}}_h = \sum_{k=0}^{\infty} \hat{\mathbf{A}}_k (\alpha - \alpha_0)^k \quad (7.49)$$

then

$$\mathbf{A}_k = \mathbf{Q}_h \hat{\mathbf{A}}_k, \quad k = 0, \dots, q-1 \quad (7.50)$$

The proof of conservation of derivatives is shown in Section 7.6. The solution of the reduced system is therefore expected to accurately approximate that of the original system as α is tracked from 0 to 1.

7.4.3 Solution of the Reduced Equations

After the augmented system of equations is reduced, the next step is to solve the reduced order system. As mentioned in the previous section, the CPU cost savings result from the fact that the analysis is done in the reduced system space, instead of solving the original large system. In this section the steps for solving the reduced system are presented. Consider the system, of equations in (7.45)

$$\hat{\Psi}(\hat{\mathbf{X}}_h, \alpha) = \hat{\mathbf{G}}\hat{\mathbf{X}}_h + \hat{\mathbf{C}}\hat{\mathbf{X}}_h + \hat{\mathbf{F}}(\hat{\mathbf{X}}_h) - \hat{\mathbf{B}}_{dc} - \alpha\hat{\mathbf{B}}_{ac}$$

In order to simplify the notation, the above system will be denoted as $\hat{\Psi}_i(\hat{\mathbf{X}}_h) = \hat{\Psi}(\hat{\mathbf{X}}_h, \alpha_i)$ when $\alpha = \alpha_i$. The solution of the reduced system of equations is found by applying the Newton iteration to

$$\hat{\Psi}_i(\hat{\mathbf{X}}_h) = \hat{\mathbf{G}}\hat{\mathbf{X}}_h + \hat{\mathbf{C}}\hat{\mathbf{X}}_h + \hat{\mathbf{F}}(\hat{\mathbf{X}}_h) - \hat{\mathbf{B}}_t \quad (7.51)$$

with

$$\hat{\mathbf{B}}_t = \hat{\mathbf{B}}_{dc} - \alpha_i \hat{\mathbf{B}}_{ac} \quad (7.52)$$

Starting with an initial guess $\hat{\mathbf{X}}_{guess}$, the solution is updated, at each Newton iteration as follows,

$$\hat{\mathbf{X}}_{new} = \hat{\mathbf{X}}_{old} - \hat{\mathbf{J}}_{hb}(\hat{\mathbf{X}}_{old})^{(-1)} \hat{\Psi}_i(\hat{\mathbf{X}}_{old}) \quad (7.53)$$

where $\hat{\mathbf{X}}_{old}$ is the old guess, $\hat{\mathbf{X}}_{new}$ is the new updated solution, and $\hat{\mathbf{J}}_{hb}(\hat{\mathbf{X}}_{old})$ is the jacobian of the reduced equations, which can be written as

$$\begin{aligned} \hat{\mathbf{J}}_{hb}(\hat{\mathbf{X}}_{old}) &= \left. \frac{\partial \hat{\Psi}_i(\hat{\mathbf{X}})}{\partial \hat{\mathbf{X}}} \right|_{\hat{\mathbf{X}}_{old}} = \hat{\mathbf{G}} + \hat{\mathbf{C}} + \left. \frac{\partial \hat{\mathbf{F}}(\hat{\mathbf{X}})}{\partial \hat{\mathbf{X}}} \right|_{\hat{\mathbf{X}}_{old}} \\ &= \hat{\mathbf{G}} + \hat{\mathbf{C}} + \left. \frac{\partial \left(\mathbf{Q}_h^T \mathbf{F}(\mathbf{Q}_h \hat{\mathbf{X}}) \right)}{\partial \hat{\mathbf{X}}} \right|_{\hat{\mathbf{X}}_{old}} = \hat{\mathbf{G}} + \hat{\mathbf{C}} + \mathbf{Q}_h^T \left(\left. \frac{\partial \mathbf{F}(\mathbf{X}_h)}{\partial \mathbf{X}_h} \right|_{\mathbf{Q}_h \hat{\mathbf{X}}_{old}} \right) \mathbf{Q}_h \end{aligned} \quad (7.54)$$

The jacobian of the reduced circuit is therefore related to the original jacobian defined in Section 7.2.4 by a congruent transformation with the matrix $\mathbf{Q}_h$

$$\hat{\mathbf{J}}_{hb} = \mathbf{Q}_h \mathbf{J}_{hb} \mathbf{Q}_h \quad (7.55)$$

where $\hat{\mathbf{J}}_{hb} \in \mathbb{R}^{N_q \times N_q}$ is much smaller than the original jacobian. Note that the structure of the blocks of the original jacobian as shown in (7.40) can be exploited to obtain the reduced jacobian through a series of Matrix Vector Products (MVPs) efficiently computed using the FFT algorithm [103]. The reduced jacobian therefore be efficiently computed without explicitly evaluating the original large jacobian.

The iteration in (7.53) is repeated until the difference $\|\Delta \hat{\mathbf{X}}\|$ between the old guess and the updated values is less than the error tolerance ϵ . The analysis is therefore done completely in the domain of the reduced system. Once convergence is achieved the solution is mapped back to the original system space

$$\mathbf{Q}_h \hat{\mathbf{X}} \rightarrow \mathbf{X} \quad (7.56)$$

An estimate of the error can be obtained by substituting the solution $\mathbf{X}_h$ into the Harmonic Balance equations in (7.13), and checking if Kirchhoff's current law is satisfied. If the solution is found to be unsatisfactory, the order of the reduced system can be increased or an expansion at an intermediate value of α can be performed.

7.5 Calculation of the Subspace Q_h

In this section, the algorithm for finding the subspace Q_h is described. The circuit reduction approach which was described in the previous sections is based on an orthogonal projection onto this subspace. Q_h was defined in (7.48) as

$$\text{colsp}[Q_h] = \text{colsp}[M_0, M_1, \dots, M_{N_q-1}]$$

In order to obtain Q_h , the moments $M_0, \dots, M_{N_q-1}$ are evaluated, and a standard orthonormalization method [103] is applied in order to obtain an orthonormal basis Q_h that spans the space of these moments. The focus of this section is therefore on the evaluation of the moments A_k , which correspond to the Taylor coefficients of X_h with respect to the continuation parameter α as defined below

$$X_h(\alpha) = \sum_{k=0}^{N_q-1} A_k(\alpha)^k \quad (7.57)$$

Now recall the Harmonic Balance equations appended with the continuation parameter α as defined in (7.42),

$$\Psi(X_h, \alpha) = \bar{G}X_h + \bar{C}X_h + F(X_h) \quad (7.58)$$

$$-B_{dc} - \alpha B_{ac} \quad (7.59)$$

The nonlinear vector $F(X_h)$ can also be expanded in a Taylor series as follows

$$F(X_h(\alpha)) = \sum_{k=0}^{N_q-1} D_k(\alpha)^k \quad (7.60)$$

At $\alpha = 0$, we have $F(X_h(0)) = D_0 = F(A_0)$. Substituting from (7.57) and (7.60) into (7.58), we get,

$$\bar{G} \sum_{k=0}^{N_q-1} A_k(\alpha)^k + \bar{C} \sum_{k=0}^{N_q-1} A_k(\alpha)^k + \sum_{k=0}^{N_q-1} D_k(\alpha)^k = B_{dc} + \alpha B_{ac} \quad (7.61)$$

Setting $\alpha = 0$ we get

$$\bar{G}A_0 + \bar{C}A_0 + F(A_0) = B_{dc} \quad (7.62)$$

Note that A_0 in (7.62) is simply the dc solution. Also, equating the first power of α in (7.61) yields

$$\bar{G}A_1 + \bar{C}A_1 + D_1 = B_{ac} \quad (7.63)$$

Now define the matrix $T(\alpha)$ to be

$$T(\alpha) = \frac{\partial F(X)}{\partial X}; \quad T_0 = T(\alpha)|_{\alpha=0} \quad (7.64)$$

Since $D_1 = (\partial F/\partial \alpha)|_{\alpha=0} = (\partial F/\partial \alpha) A_1$, then substituting in (7.63) gives the following relation for A_1

$$(\bar{G} + \bar{C} + T_0)A_1 = B_{ac} \quad (7.65)$$

The first moment is therefore found by simply solving the above set of linear equations. Note that the left hand side matrix is simply the jacobian of the harmonic balance equations. The CPU cost of this operation is one LU decomposition of the jacobian matrix. In order to compute $A_2, \dots, A_q$, we expand $T(\alpha)$ as a Taylor series in α

$$T(\alpha) = \sum_{k=0} T_k \alpha^k \quad (7.66)$$

and then use this expansion to write $\partial F/\partial \alpha$ in the following form:

$$\frac{\partial F}{\partial \alpha} = T \frac{\partial X}{\partial \alpha} \quad (7.67)$$

Plugging equations (7.57), (7.60) and (7.66) into (7.67) we get

$$\sum_{i=1} i D_i \alpha^{i-1} = \sum_{i=0} T_i \alpha^i \sum_{i=1} i A_i \alpha^{i-1} \quad (7.68)$$

Taking the n^{th} derivative in (7.68) with respect to α and putting $\alpha = 0$, we get the following recursive relationship

$$D_n = T_0 A_n + \frac{1}{n} \sum_{j=1}^{n-1} (n-j) T_j A_{n-j} \quad (7.69)$$

Equating the n^{th} power of α in (7.61) we get

$$\bar{G}A_n + \bar{C}A_n + D_n = 0 \quad (7.70)$$

Now using equation (7.69) we can find the following recursive relationship for the derivatives A_n

$$(\bar{G} + \bar{C} + T_0)A_n = -\frac{1}{n} \sum_{j=1}^{n-1} (n-j)T_j A_{n-j} \quad (7.71)$$

From (7.65) and (7.71) we can conclude that computing $A_1, \dots, A_q$ requires only one LU decomposition of the Jacobian matrix. Each additional derivatives can be obtained by forward/backward substitution. Next we proceed to show how to compute the right hand side of equation (7.71). The term A_{n-j} represents the $(n-j)^{th}$ derivative of X with respect to α , with A_0 and A_1 evaluated using (7.65) and (7.71). The higher order derivatives are found recursively using (7.71). The remaining term required in (7.71) is T_j . First, recall the definition of $T(\alpha)$ from equations (7.64), (7.66) and (7.35)

$$T(\alpha) = \sum_{j=0} T_j \alpha^j = \frac{\partial F(X)}{\partial X} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial X_1} & \cdots & \frac{\partial F_n}{\partial X_n} \end{bmatrix} \quad (7.72)$$

For simplicity of presentation, we will consider one of the terms, $\frac{\partial F_1}{\partial X_1}$ for example, in equation (7.72). The remaining terms are treated in a similar fashion. We define the matrix P as

$$P = T_{11} = \frac{\partial F_1}{\partial X_1} = \sum_{j=0} P_j \alpha^j \quad (7.73)$$

The Taylor coefficient P_j is entered in T_j at the location corresponding to $\frac{\partial F_1}{\partial X_1}$. As shown in Section 7.2.4, P can be written as

$$P = \Gamma^{-1} G \Gamma \quad (7.74)$$

with

$$G = \text{diag} [g(x_1(t_1)), \dots, g(x_1(t_s))] \quad (7.75)$$

where

$$g(x_1) = \frac{\partial f_1(x_1)}{\partial x_1} \quad (7.76)$$

The time points $t_1, \dots, t_s$ are spaced over the fundamental period. Noting the Taylor expansion of $g(x_1)$

$$g(x_1) = \frac{\partial f_1(x_1)}{\partial x_1} = \sum_{j=0} g_j \alpha^j \quad (7.77)$$

we can write the expression for P_j as

$$P_i = \Gamma^{-1} G_j \Gamma \quad (7.78)$$

where G_j is a diagonal matrix whose elements are the time samples of g_i . As an example of computing g_i consider the case where

$$f_1(x_1) = I_s (e^{x_1/v_t} - 1); \quad g(x_1) = \frac{I_s}{v_t} e^{x_1/v_t} \quad (7.79)$$

Noting the Taylor series

$$x_1(t) = \sum_{i=0} a_i(t) \alpha^i \quad (7.80)$$

where the coefficients a_i are the time domain versions of the corresponding entry in (7.57), it can be shown that [118]

$$g_n = \frac{1}{nv_t} \sum_{i=0}^{n-1} g_i a_{n-i}(n-i) \quad (7.81)$$

Similar expressions can be found for other functions as shown in Table 7.1 . It is to be noted that, while the implementation of the above algorithm is quite complex, the computational cost of finding the the coefficients A_i is relatively low. The bulk of the CPU cost is due to the LU factorization of the jacobian matrix, which is required only once to find the first moment, and is then re-used to find the remaining moments. The remaining operations are relatively inexpensive, and are made even more efficient by using the FFT algorithm which is used to replace the Matrix Vector Products (MVPs) [103].

7.6 Proof of Preservation of Derivatives

One of the main features of the reduced order system presented in this chapter is that it shares the first q derivatives with respect to α with the original system. In this section we prove that

Equation	$y = \sum_{i=0} a_i \alpha^i, z = \sum_{i=0} c_i \alpha^i, f = \sum_{i=0} d_i \alpha^i$
$f = \exp(y)$	$d_0 = \exp(a_0)$ $d_n = (1/n) \sum_{i=0}^{n-1} d_i a_{n-i} (n-i)$
$f = \log(y)$	$d_0 = \log(a_0)$ $d_n = (1/a_0) \left(a_n - \sum_{i=1}^{n-1} a_i d_{n-i} \frac{(n-i)}{n} \right)$
$f = y^p$	$d_0 = a_0^p$ $d_n = (1/na_0) (npd_0 a_n + \sum_{i=1}^{n-1} d_i a_{n-i} (pn - i(p+1)))$
$f = y + z$	$d_n = a_n + c_n$
$f = y - z$	$d_n = a_n - c_n$
$f = yz$	$d_n = \sum_{i=0}^n a_i c_{n-i}$
$f = y/z$	$d_n = (1/c_0) \left(a_n - \sum_{i=0}^{n-1} a_i c_{n-i} \right)$

Table 7.1: Formulas for the derivatives of some simple functions [118]

the reduced system in (7.45) shares the first N_q derivatives with the original system in (7.42) as stated in (7.50). The proof is given by *mathematical induction*. First, the zeroth derivative $\mathbf{A}_0$ is proved to be preserved. Then we show that the first derivative $\hat{\mathbf{A}}_1$ of $\hat{\mathbf{X}}_h$ with respect to α as obtained from the reduced equations in (7.45) is equivalent to the one obtained from the original HB equations in (7.42) through mapping by $\mathbf{Q}_h$. Next we proceed to show that the same relation applies to the k^{th} derivative of $\hat{\mathbf{X}}_h$ provided that the previous $(k-1)$ derivatives are preserved under the mapping $\mathbf{Q}_h$. In other words, for $k = 0, \dots, N_q - 1$, if $\mathbf{A}_k$ is the k^{th} derivative of the original system, and $\hat{\mathbf{A}}_k$ is that of the reduced system then we have

$$\mathbf{A}_k = \mathbf{Q}_h \hat{\mathbf{A}}_k \quad (7.82)$$

First $\hat{\mathbf{X}}_h(\alpha)$ and $\mathbf{F}(\mathbf{Q}_h \hat{\mathbf{X}}_h)$ in (7.44) are expanded as Taylor series in the form

$$\hat{\mathbf{X}}_h(\alpha) = \sum_{k=0} \hat{\mathbf{A}}_k \alpha^k \quad (7.83a)$$

$$\mathbf{F}(\mathbf{Q}_h \hat{\mathbf{X}}_h) = \sum_{k=0} \hat{\mathbf{D}}_k \alpha^k \quad (7.83b)$$

Substituting from (7.83) into (7.45) results in the following

$$\left(\hat{G} + \hat{C}\right) \sum_{k=0} \hat{A}_k \alpha^k + Q_h^T \sum_{k=0} \hat{D}_k \alpha^k - Q_h^T B_{dc} - \alpha Q_h^T B_{ac} = 0 \quad (7.84)$$

Equating similar powers of α in equation (7.84) results in

$$\left(\hat{G} + \hat{C}\right) \hat{A}_0 + Q_h^T \hat{D}_0 = Q_h^T B_{dc} \quad (7.85a)$$

$$\left(\hat{G} + \hat{C}\right) \hat{A}_1 + Q_h^T \hat{D}_1 = Q_h^T B_{ac} \quad (7.85b)$$

$$\left(\hat{G} + \hat{C}\right) \hat{A}_k + Q_h^T \hat{D}_k = 0 \quad (7.85c)$$

Proof of Preservation of A_0

In this section it shall be proven that the zeroth moment A_0 is preserved in the reduced system.

Noting that

$$\hat{D}_0 = F(Q_h \hat{A}_0) \quad (7.86)$$

Equation (7.85a) can be written as

$$Q_h^T \left((\bar{G} + \bar{C}) Q_h \hat{A}_0 + F(Q_h \hat{A}_0) - B_{dc} \right) = 0 \quad (7.87)$$

Since Q_h is an orthonormal basis for the subspace spanned by the columns of the matrix $[A_0, \dots, A_{N_q-1}]$, it can be represented in the form [119]

$$Q_h = [A_0, \dots, A_{N_q-1}] R^{-1} \quad (7.88)$$

where R is an upper triangular matrix. Substituting from (7.88) into (7.87) results in

$$Q_h^T \left((\bar{G} + \bar{C}) [A_0, \dots, A_{N_q-1}] R^{-1} \hat{A}_0 + F \left([A_0, \dots, A_{N_q-1}] R^{-1} \hat{A}_0 \right) - B_{dc} \right) = 0 \quad (7.89)$$

Now we verify that $e_1 = R^{-1} \hat{A}_0$ satisfies equation (7.89), where e_r represents the r^{th} column of the identity matrix. Substituting into (7.89) yields

$$Q_h^T \left((\bar{G} + \bar{C}) [A_0, \dots, A_{N_q-1}] e_1 + F([A_0, \dots, A_{N_q-1}] e_1) - B_{dc} \right) = 0 \quad (7.90)$$

Simplifying equation (7.90) results in

$$\mathbf{Q}_h^T ((\bar{\mathbf{G}} + \bar{\mathbf{C}}) \mathbf{A}_0 + \mathbf{F}(\mathbf{A}_0) - \mathbf{B}_{dc}) = 0 \quad (7.91)$$

Comparing with (7.62) we can determine that equation (7.91) is satisfied. Therefore we can conclude that $\mathbf{e}_1 = \mathbf{R}^{-1} \hat{\mathbf{A}}_0$ is indeed a solution of equation (7.89). $\hat{\mathbf{A}}_0$ can then be written as

$$\hat{\mathbf{A}}_0 = \mathbf{R} \mathbf{e}_1 \quad (7.92)$$

Pre-multiplying by $\mathbf{Q}_h$ results in

$$\mathbf{Q}_h \hat{\mathbf{A}}_0 = \mathbf{Q}_h \mathbf{R} \mathbf{e}_1 = [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{R}^{-1} \mathbf{R} \mathbf{e}_1 = [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{e}_1 = \mathbf{A}_0 \quad (7.93)$$

Therefore the zeroth moment $\mathbf{A}_0$ is preserved under the mapping $\mathbf{Q}_h$.

Proof of Preservation of $\mathbf{A}_1$

From equation (7.83b) $\hat{\mathbf{D}}_1$ can be written as

$$\hat{\mathbf{D}}_1 = \left. \frac{\partial F(\mathbf{Q}_h \hat{\mathbf{X}}_h)}{\partial \alpha} \right|_{\alpha=0} = \left. \frac{\partial F(\mathbf{Q}_h \hat{\mathbf{X}}_h)}{\partial (\mathbf{Q}_h \hat{\mathbf{X}}_h)} \frac{\partial \mathbf{Q}_h \hat{\mathbf{X}}_h}{\partial \hat{\mathbf{X}}_h} \frac{\partial \hat{\mathbf{X}}_h}{\partial \alpha} \right|_{\alpha=0} \quad (7.94)$$

Using the definition in (7.64), equation (7.94) results in

$$\hat{\mathbf{D}}_1 = \mathbf{T}_0 \mathbf{Q}_h \hat{\mathbf{A}}_1 \quad (7.95)$$

Substituting (7.95) into (7.85b), we get

$$\mathbf{Q}_h^T ((\bar{\mathbf{G}} + \bar{\mathbf{C}}) \mathbf{Q}_h \hat{\mathbf{A}}_1 + \mathbf{T}_0 \mathbf{Q}_h \hat{\mathbf{A}}_1 - \mathbf{B}_{ac}) = 0 \quad (7.96)$$

Next we substitute the value for $\mathbf{Q}_h$ from (7.88) to get

$$\begin{aligned} \mathbf{Q}_h^T ((\bar{\mathbf{G}} + \bar{\mathbf{C}}) [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{R}^{-1} \hat{\mathbf{A}}_1 + \\ \mathbf{T}_0 [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{R}^{-1} \hat{\mathbf{A}}_1 - \mathbf{B}_{ac}) = 0 \end{aligned} \quad (7.97)$$

Now we verify that $\mathbf{e}_2 = \mathbf{R}^{-1} \hat{\mathbf{A}}_1$ satisfies equation (7.89), where $\mathbf{e}_r$ represents the r^{th} column of the identity matrix. Substituting into (7.97) results in

$$\mathbf{Q}_h^T ((\bar{\mathbf{G}} + \bar{\mathbf{C}}) [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{e}_2 + \mathbf{T}_0 [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{e}_2 - \mathbf{B}_{ac}) = 0 \quad (7.98)$$

Simplifying (7.98) we obtain the following

$$\mathbf{Q}_h^T ((\bar{\mathbf{G}} + \bar{\mathbf{C}})\mathbf{A}_1 + \mathbf{T}_0\mathbf{A}_1 - \mathbf{B}_{ac}) = 0 \quad (7.99)$$

Comparing with (7.65), we can determine that equation (7.99) is satisfied. The first moment $\hat{\mathbf{A}}_1$ of the reduced system can therefore be written as

$$\hat{\mathbf{A}}_1 = \mathbf{R}\mathbf{e}_2 \quad (7.100)$$

Pre-multiplying by $\mathbf{Q}_h$ results in

$$\mathbf{Q}_h\hat{\mathbf{A}}_1 = \mathbf{Q}_h\mathbf{R}\mathbf{e}_2 = [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{R}^{-1}\mathbf{R}\mathbf{e}_2 = [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{e}_2 = \mathbf{A}_1 \quad (7.101)$$

Therefore the zeroth moment $\mathbf{A}_1$ is preserved under the mapping $\mathbf{Q}_h$.

Proof of Preservation of $\mathbf{A}_n$

In this section it shall be proven that the n^{th} moment $\mathbf{A}_n$ is preserved in the reduced system if the previous moments $\mathbf{A}_0, \dots, \mathbf{A}_{n-1}$ are also preserved. First the matrix $\hat{\mathbf{T}}$ is defined as

$$\hat{\mathbf{T}}(\alpha) = \frac{\partial \mathbf{F}(\mathbf{Q}_h\hat{\mathbf{X}}_h)}{\partial \hat{\mathbf{X}}_h} \quad (7.102)$$

The Taylor expansion of $\hat{\mathbf{T}}$ with respect to α is

$$\hat{\mathbf{T}}(\alpha) = \sum_{k=0} \hat{\mathbf{T}}_k \alpha^k \quad (7.103)$$

Recalling that $\hat{\mathbf{A}}_k$ and $\hat{\mathbf{D}}_k$ are the Taylor coefficients of $\hat{\mathbf{X}}_h$ and $\mathbf{F}(\mathbf{Q}_h\hat{\mathbf{X}}_h)$ respectively as defined in equations (7.83a) and (7.83b), and using the chain rule results in

$$\frac{\partial \mathbf{F}(\mathbf{Q}_h\hat{\mathbf{X}}_h)}{\partial \alpha} = \frac{\partial \mathbf{F}(\mathbf{Q}_h\hat{\mathbf{X}}_h)}{\partial \hat{\mathbf{X}}_h} \frac{\partial \hat{\mathbf{X}}_h}{\partial \alpha} = \hat{\mathbf{T}}(\alpha) \frac{\partial \hat{\mathbf{X}}_h}{\partial \alpha} \quad (7.104)$$

or

$$\sum_{k=1} k \hat{\mathbf{D}}_k \alpha^{k-1} = \sum_{k=0} \hat{\mathbf{T}}_k \alpha^k \sum_{k=1} k \hat{\mathbf{A}}_k \alpha^{k-1} \quad (7.105)$$

Equating the coefficients of equal powers of α in (7.104) results in the following recursive relationship

$$\hat{D}_k = \hat{T}_0 \hat{A}_k + \frac{1}{k} \sum_{j=1}^{k-1} (k-j) \hat{T}_j \hat{A}_{k-j} \quad (7.106)$$

Using the chain rule, $\hat{T}$ can be related to the matrix T which was defined in (7.64)

$$\hat{T} = \frac{\partial F(Q_h \hat{X}_h)}{\partial \hat{X}_h} = \frac{\partial F(Q_h \hat{X}_h)}{\partial (Q_h \hat{X}_h)} \frac{\partial (Q_h \hat{X}_h)}{\partial \hat{X}_h} = T Q_h = \sum_{k=0} T_k Q_h \alpha^k \quad (7.107)$$

Substituting (7.107) into (7.106) results in

$$\hat{D}_k = T_0 Q_h \hat{A}_k + \frac{1}{k} \sum_{j=1}^{k-1} (k-j) T_j Q_h \hat{A}_{k-j} \quad (7.108)$$

The recursive equation in (7.85c) for finding the n^{th} moment of the reduced system can therefore be written as

$$Q_h^T \left((\bar{G} + \bar{C}) Q_h \hat{A}_k + T_0 Q_h \hat{A}_k + \frac{1}{k} \sum_{j=1}^{k-1} (k-j) T_j Q_h \hat{A}_{k-j} \right) = 0 \quad (7.109)$$

Substituting for Q_h from (7.88) results in

$$Q_h^T \left((\bar{G} + \bar{C}) [A_0, \dots, A_{N_q-1}] R^{-1} \hat{A}_k + T_0 [A_0, \dots, A_{N_q-1}] R^{-1} \hat{A}_k + \frac{1}{k} \sum_{j=1}^{k-1} (k-j) T_j Q_h \hat{A}_{k-j} \right) = 0 \quad (7.110)$$

Substituting $e_{k+1} = R^{-1} \hat{A}_k$ as a solution for the above equation results in

$$Q_h^T \left((\bar{G} + \bar{C}) [A_0, \dots, A_{N_q-1}] e_{k+1} + T_0 [A_0, \dots, A_{N_q-1}] e_{k+1} + \frac{1}{k} \sum_{j=1}^{k-1} (k-j) T_j Q_h \hat{A}_{k-j} \right) = 0 \quad (7.111)$$

Assuming that the previous moments $A_0, \dots, A_{k-1}$ are preserved under the mapping Q_h or more precisely

$$A_i = Q_h \hat{A}_i; \quad i = 0, \dots, k-1 \quad (7.112)$$

Equation (7.111) can be written as

$$\mathbf{Q}_h^T \left((\bar{\mathbf{G}} + \bar{\mathbf{C}}) \mathbf{A}_k + \mathbf{T}_0 \mathbf{A}_k + \frac{1}{k} \sum_{j=1}^{k-1} (k-j) \mathbf{T}_j \mathbf{A}_{k-j} \right) = 0 \quad (7.113)$$

Comparing with (7.71), we can determine that equation (7.113) is satisfied. The k^{th} moment $\hat{\mathbf{A}}_k$ of the reduced system can therefore be written as

$$\hat{\mathbf{A}}_k = \mathbf{R} \mathbf{e}_{k+1} \quad (7.114)$$

Pre-multiplying by $\mathbf{Q}_h$ results in

$$\mathbf{Q}_h \hat{\mathbf{A}}_k = \mathbf{Q}_h \mathbf{R} \mathbf{e}_{k+1} = [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{R}^{-1} \mathbf{R} \mathbf{e}_{k+1} = [\mathbf{A}_0, \dots, \mathbf{A}_{N_q-1}] \mathbf{e}_{k+1} = \mathbf{A}_k \quad (7.115)$$

Therefore the zeroth moment $\mathbf{A}_k$ is preserved under the mapping $\mathbf{Q}_h$ if the previous moments $\mathbf{A}_0, \dots, \mathbf{A}_{k-1}$ are preserved. It can be therefore deduced by induction that the reduced system preserves the first N_q moments of the original system under the mapping $\mathbf{Q}_h$, where N_q is the number of column moments forming the subspace spanned by $\mathbf{Q}_h$.

7.7 Circuits with Multi-Tone Excitation

The analysis methods described so far have assumed a periodic input that can be expressed in terms of a truncated Fourier series. However, steady-state analysis of circuits with multi-tone inputs is often required. This may be due to the function of the circuit, such as the case of mixer circuits. It may also be due to the type of analysis required. For example in the case of IP3 analysis, two input tones are used in order to measure the 3rd order intermodulation product. When more than one input tone are used, the tones are in general non-commensurate (i.e. not multiples of each other) and the input cannot be written as a Fourier series, and is referred to as quasi-periodic. In this section, the circuit reduction approach for case of multi-tone inputs is described.

7.7.1 Multi-Tone HB Equations

For the case of non-commensurate multi-tone inputs. The steady-state response consists of frequencies at the harmonics of the input tones, as well as their intermodulation products. For example, if the two input tones are at ω_1 and ω_2 with $\omega_2 > \omega_1$, the steady-state response will have tones at dc, $\omega_1, 2\omega_1, \dots, \omega_2, 2\omega_2, \dots, 2\omega_1 - \omega_2, 2\omega_2 - \omega_2, \dots$ etc. In general, the steady-state response can therefore be represented as

$$\mathbf{x}_\phi(t) = \mathbf{A}_0 + \sum_{k=1}^H (\mathbf{A}_k \cos(\omega_k t) + \mathbf{B}_k \sin(\omega_k t)) \quad (7.116)$$

where the frequencies ω_k represent the harmonics and intermodulation products of the input tones. For the remainder of this section, the coefficients $\mathbf{A}_k$ and $\mathbf{B}_k$ of the sine and cosine will be referred to as the Fourier coefficients. Substituting the above solution into the MNA equations of circuit ϕ in (7.1) and equating the coefficients of the sines and cosines at the same frequencies results in the Harmonic Balance of the form

$$\bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) = \mathbf{B}_{dc} + \mathbf{B}_{h1} + \mathbf{B}_{h2} + \dots \quad (7.117)$$

where $\mathbf{X}_h \in \mathbb{R}^{N_{hb}}$ is a vector containing the unknown coefficients of the sines and cosines representing $\mathbf{x}_\phi(t)$, $\mathbf{B}_{dc} \in \mathbb{R}^{N_{hb}}$ is a vector of the Fourier coefficients of the dc input sources, $\mathbf{B}_{dc} \in \mathbb{R}^{N_{h1}}, \mathbf{B}_{dc} \in \mathbb{R}^{N_{h2}} \dots$ are vectors representing the various input tones, and $\bar{\mathbf{G}}$ is a block matrix $\bar{\mathbf{G}} = [\mathbf{G}_{ij}]$ whose blocks $\mathbf{G}_{ij} \in \mathbb{R}^{N_h \times N_h}$ are diagonal matrices given by

$$\mathbf{G}_{ij} = \text{diag}(g_{ij}, \dots, g_{ij}) \quad (7.118)$$

where g_{ij} is the corresponding element in the $\mathbf{G}_\phi$ matrix in (7.1). The matrix $\bar{\mathbf{C}}$ is also a block matrix $\bar{\mathbf{C}} = [\mathbf{C}_{ij}]$ whose blocks $\mathbf{C}_{ij} \in \mathbb{R}^{N_h \times N_h}$ are given by

$$\mathbf{C}_{ij} = c_{ij} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega & \cdots & 0 & 0 \\ 0 & -\omega & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \omega_H \\ 0 & 0 & 0 & \cdots & -\omega_H & 0 \end{bmatrix} \quad (7.119)$$

where c_{ij} is the corresponding element in the $\mathbf{C}_\phi$ matrix in (7.1). The vector $\mathbf{F}(\mathbf{X}_h)$ contains the Fourier coefficients of the nonlinear vector $\mathbf{f}_\phi(\mathbf{x}_\phi(t))$ defined in (7.1). More precisely, the vector $\mathbf{f}_\phi(\mathbf{x}_\phi(t))$ is a vector of nonlinear scalar functions

$$\mathbf{f}_\phi(\mathbf{x}_\phi(t)) = [f_1(\mathbf{x}_\phi), f_2(\mathbf{x}_\phi), \dots, f_{N_\phi}(\mathbf{x}_\phi)]^T \quad (7.120)$$

and the vector $\mathbf{F}(\mathbf{X}_h)$ is given by

$$\mathbf{F}(\mathbf{X}_h) = [F_1(\mathbf{X}_h), F_2(\mathbf{X}_h), \dots, F_{N_\phi}(\mathbf{X}_h)]^T \quad (7.121)$$

where $F_1(\mathbf{X}_h)$ for example, is a vector containing the Fourier coefficients of $f_1(\mathbf{x}_\phi)$.

The above formulation is similar to the one shown in Section 7.2.1, except that in this case, the frequencies corresponding to the coefficients in $\mathbf{X}_h$ and $\mathbf{F}(\mathbf{X}_h)$ and in equation (7.119) are non-commensurate. This causes difficulties when attempting to use the FFT algorithm to evaluate the Fourier coefficients of the nonlinear vector $\mathbf{F}(\mathbf{X}_h)$ as was done for the case of single tone inputs. The same difficulty also applies to the calculation of the jacobian. In the following section, a modification of the FFT is shown, that would make the above equations appear as if they represent a periodic signal as opposed to a quasi-periodic signal from the perspective of the FFT algorithm.

7.7.2 FFT for Almost Periodic Signals

The Fourier transform as presented in Section 7.2.2 is limited to periodic signals composed of one fundamental frequency along with its harmonics. In this case the period T of the signal is simply that of the fundamental harmonic, and applying the FFT algorithm is a straight forward procedure. However, the input of RF circuits is in general composed of multi-tone signals with non-commensurate frequencies. For example, one can have inputs at 900MHz and 910MHz. In such cases the period of the combined signal is very large compared to that of the input tones. Such signals are referred to as "almost periodic." In such cases, applying the FFT algorithm in a brute force way is both inaccurate and inefficient (the signal would have to be sampled over the very long combined period). There are a number of methods in the literature that can be used to solve this problem [120]–[123]. The approach used here is the one presented in [75], and is simple, efficient, and suitable for Harmonic Balance analysis.

The approach used for handling circuits with almost periodic inputs relies on two main properties of the Harmonic Balance problem. The first property is that the nonlinear vector $f_\phi(\mathbf{x}_\phi(t))$ in the MNA equations in (7.1) is memoryless. The second property is that, when evaluating $F(\mathbf{X})$ in the harmonic balance algorithm, the time domain waveforms are of no importance, and only the frequency domain Fourier coefficients are of interest. This approach is best explained using an example. Consider the memoryless nonlinear function

$$f(x(t)) = x(t)^2 \quad (7.122)$$

For this example we assume

$$x(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) \quad (7.123)$$

with $\omega_2 > \omega_1$. This simple problem is identical to what is encountered in the Harmonic Balance algorithm. The Fourier coefficients of $x(t)$ are known, and the task is to find the Fourier coefficients of $f(x)$. In this case, the solution can be found analytically using the trigonometric

Frequency	Amplitude
DC	$\frac{a_1}{2} + \frac{a_2}{2}$
$\omega_2 - \omega_1$	$\frac{a_1 a_2}{2}$
ω_1	0
ω_2	0
$2\omega_1$	$\frac{a_1}{2}$
$\omega_1 + \omega_2$	$\frac{a_1 a_2}{2}$
$2\omega_2$	$\frac{a_2}{2}$

Table 7.2: Amplitudes of the harmonics

identities.

$$\begin{aligned}
 f(x(t)) &= (a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t))^2 \\
 &= a_1^2 \cos^2(\omega_1 t) + a_2^2 \cos^2(\omega_2 t) + a_1 a_2 \cos(\omega_1 t) \cos(\omega_2 t) \\
 &= \frac{a_1}{2} + \frac{a_1}{2} \cos(2\omega_1 t) + \frac{a_2}{2} + \frac{a_2}{2} \cos(2\omega_2 t) \\
 &\quad + \frac{a_1 a_2}{2} \cos((\omega_1 + \omega_2)t) + \frac{a_1 a_2}{2} \cos((\omega_2 - \omega_1)t) \quad (7.124)
 \end{aligned}$$

By examining equation (7.124) we can determine that $f(x)$ has 5 different harmonics as listed in Table 7.2. The key conclusion to be drawn from this exercise is that the amplitude of the harmonics is independent of the frequencies ω_1 and ω_2 . Of course these frequencies will affect the time domain waveforms, but if our interest is simply to determine Fourier coefficients, we can choose any frequency and still obtain the same result. Let us consider an example where the values of ω_1 and ω_2 are given as

$$\omega_1 = 900 \text{ MHz}; \quad \omega_2 = 910 \text{ MHz} \quad (7.125)$$

These two frequencies are non-commensurate and the resulting waveform is quasi-periodic. However, as was seen earlier, modifying the value of the frequencies does not affect the Fourier

Frequency	Actual Frequency	Mapped Frequency	Amplitude
DC	DC	DC	$\frac{a_1}{2} + \frac{a_2}{2}$
$\omega_2 - \omega_1$	10 MHz	300 MHz	$\frac{a_1 a_2}{2}$
ω_1	900 MHz	600 MHz	0
ω_2	910 MHz	900 MHz	0
$2\omega_1$	1800 MHz	1200 MHz	$\frac{a_1}{2}$
$\omega_1 + \omega_2$	1810 MHz	1500 MHz	$\frac{a_1 a_2}{2}$
$2\omega_2$	1820 MHz	1800 MHz	$\frac{a_2}{2}$

Table 7.3: Mapping of the harmonics

coefficients of the results. So let us consider the following mapping

$$\omega_1 \leftarrow 600 \text{ MHz}; \quad \omega_2 \leftarrow 900 \text{ MHz} \quad (7.126)$$

The harmonics of $f(x)$ obtained from the original definition of ω_1 and ω_2 in (7.125), and from the mapping in (7.126) are shown in Table 7.3. As can be seen from the table, using the mapped frequencies, $f(x)$ becomes a periodic signal with a fundamental frequency at 300 MHz. The FFT algorithm can therefore be applied to the mapped frequencies, and the resulting amplitudes are then mapped back to the original frequencies. Practically, this approach simply results in a reordering of the harmonics and requires no additional CPU expense. A systematic way of finding the appropriate frequency mapping is given in [75].

Using this approach multi-tone circuits can be handled without any significant modification to the harmonic balance analysis. The circuit reduction method described in Section 7.4 is therefore applicable to the case of multi-tone inputs. In this case, however, more than one ac input is present, and more possibilities are available for augmenting the HB equations with a continuation parameter α as will be explained in the following section.

7.7.3 Reduction Strategy for Circuits with Multi-Tone Inputs

As seen from the previous sections, with a relatively simple modification to the FFT process, the proposed circuit reduction algorithm is applicable to circuits with multi-tone inputs. However, due to the presence of more than one input tones, more possibilities are available for augmenting the harmonic balance equations with continuation parameters. Before discussing the details of the continuation strategies we shall express the harmonic balance equations for circuits with multi-tone input as

$$\Psi(\mathbf{X}_h, \alpha) = \bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) - \mathbf{B}_{dc} - \mathbf{B}_{h1} + \mathbf{B}_{h2} + \dots \quad (7.127)$$

where the vectors $\mathbf{B}_{h1}, \mathbf{B}_{h2}, \dots$ represent the contribution of each of the input tones.

Single Parameter Sweep

The first strategy, which we shall refer to as a single parameter sweep, is used in simulation problems where the input tones are swept simultaneously. This is commonly encountered in simulations designed to determine the third order intercept point (IP3). Typically, for this type of analysis, two input tones of similar power, and that fall within the bandwidth of the circuit, are applied at the input. The power of the input tones is increased simultaneously and the output power is tracked at one of the fundamentals and at one of the third order intermodulation products ($2f_2 - f_1$ for example). This allows the designers to determine the third order intercept point (IP3). An example of such an analysis is given in Section 7.8.1. In this case, since the power of the two input tone is swept simultaneously, the most appropriate reduction strategy is as follows. The HB equations in (7.127) are appended with a single continuation parameter α as follows

$$\Psi(\mathbf{X}_h, \alpha) = \bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) - \mathbf{B}_{dc} - \alpha(\mathbf{B}_{h1} + \mathbf{B}_{h2}) \quad (7.128)$$

or alternatively

$$\Psi(\mathbf{X}_h, \alpha) = \bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) - \mathbf{B}_{dc} - \alpha\mathbf{B}_{ac} \quad (7.129)$$

where

$$\mathbf{B}_{ac} = \mathbf{B}_{h1} + \mathbf{B}_{h2} \quad (7.130)$$

The equations in (7.129) are now similar to (7.42). The system in (7.129) is then reduced using congruent transformation with a subspace spanning the moments with respect to the parameter α as was outlined in Section 7.4. Note that in this case the moments with respect to α are matched in the reduced system. Therefore, the reduced system response will match that of the original as α is tracked from 0 to 1, or in other words as in the input power of both input tones is increased simultaneously. An example of such a reduction is given in Section 7.8.1. Note that in this type of reduction does not allow for the powers of the input tones to be changed independently. Cases where this is required will be discussed in the following section.

Multi-parameter Sweep

In the previous section, a reduction strategy was given for the case when the input tones are simultaneously increased. However, this is not appropriate for all types of analysis. Mixer circuits for example, are often analyzed with a constant power at the LO input, while the input RF power is swept in order to find the 1dB compression point. An example of such a simulation is shown in Section 7.8.2. In such cases, the input powers are not simultaneously changed, and the reduction strategy outlined in the previous section is not appropriate. In this case, what we shall call a multi-parameter sweep will be used. Consider for example a down converter mixer circuit. The HB equations in (7.127) are augmented as follows

$$\Psi(\mathbf{X}_h, \alpha_1, \alpha_2) = \bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) - \mathbf{B}_{dc} - \alpha_1\mathbf{B}_{LO} + \alpha_2\mathbf{B}_{RF} \quad (7.131)$$

where $\mathbf{B}_{LO}$ contains the tone at the LO input, and $\mathbf{B}_{RF}$ contains the tone at the RF input. In this case the following strategy is employed. First, α_2 is set to zero thus making the input as single tone input at the LO frequency. The circuit is therefore treated as a single tone circuit of the form

$$\Psi(\mathbf{X}_h, \alpha_1) = \bar{\mathbf{G}}\mathbf{X}_h + \bar{\mathbf{C}}\mathbf{X}_h + \mathbf{F}(\mathbf{X}_h) - \mathbf{B}_{dc} - \alpha_1\mathbf{B}_{LO} \quad (7.132)$$

and a reduction is done on a subspace spanning the moments with respect to α_1 . The parameter α_1 is now tracked from 0 to 1 (or in other words, the LO power is increased up to the desired value). Once the operating solution at the desired LO input power is obtained (with no RF input), the LO input power is set to a constant. The augmented HB equations now become

$$\Psi(X_h, \alpha_2) = \bar{G}X_h + \bar{C}X_h + F(X_h) - B_{dc} - B_{LO} + \alpha_2 B_{RF} \quad (7.133)$$

or alternatively

$$\Psi(X_h, \alpha_2) = \bar{G}X_h + \bar{C}X_h + F(X_h) - B_t + \alpha_2 B_{RF} \quad (7.134)$$

where $B_t = B_{dc} + B_{LO}$. The above equations are in a similar form to (7.42). The system in (7.133) is now reduced using a subspace spanning the moments with respect to α_2 . Those moments are conserved in the reduced system, and the reduced circuit will match the response of the original one as α_2 is tracked from 0 to 1 (or in other words as the input RF power is tracked with the LO power kept constant). An example of such an analysis is given in Section 7.8.2.

7.8 Numerical Examples

7.8.1 Example 1: Amplifier Circuit

The proposed circuit reduction method was applied to the amplifier circuit shown in Figure 7.5. The transistor models included base emitter and base collector junction capacitances given as, $C_{be} = 2pF$ and $C_{bc} = 0.3pF$. With a single tone input at 900MHz the original size of the Harmonic Balance equations was 792. Using the circuit reduction method described in this chapter, the size of the equations was reduced to 25. The output power delivered to the load at the first harmonic (900MHz) and the third harmonic at (2.7GHz) is plotted against the input power in Figures 7.6 and 7.7. As can be seen from the plots the results are in good agreement.

A typical analysis requirement in the design of RF circuits is to apply a two tone input in order to determine the third order intermodulation product (IP3). A two tone input was applied

Figure 7.5: Amplifier circuit for Example 1

to the circuit in Figure 7.5, with the tones at 900MHz and 910MHz. The size of the resulting Harmonic Balance equations was 7512. The reduction algorithm was applied to both tones simultaneously resulting in a reduced order system of size 40. The output power at one of the main tones (910MHz) as well as two of the intermodulation products (920MHz and 930MHz) are plotted in Figure 7.8 against the input power.

Figure 7.6: Example 1: Response to single tone input at 900MHz

Figure 7.7: Example 1: Output power at 2700MHz (single tone input)

Figure 7.8: Example 1: Response to a two-tone input

7.8.2 Example 2: Mixer Circuit

In this example the differential Gilbert cell mixer shown in Figure 7.9 was considered. The transistor models included base emitter and base collector junction capacitances given as, $C_{be} = 2pF$ and $C_{bc} = 0.3pF$. The mixer was used in a down-converter configuration with the local oscillator (LO) input at 1GHz, the RF input at 900MHz, and the IF output matched at 100MHz. For this type of circuits, a common analysis requirement is to sweep the RF input power, while keeping the LO input power constant in order to determine the conversion gain and 1dB compression point for example. In order to achieve this result, a two step continuation scheme was employed as was discussed in Section 7.7.3. First the RF input was set to zero, and the circuit was reduced using a subspace that spans the moments with respect to the LO input power. The response was then tracked as the LO input power is increased up to 1dBm. For this single tone input configuration (RF input set to zero) the size of the original Harmonic Balance equations was 1406, and the size of the reduced system was 25. Figure 7.10 shows the amplitude of the 2GHz harmonic at node $N1$ for the original and reduced system with respect to input LO power. In the second step, the mixer is considered with a two tone input (one at the LO and one at the RF input). With the LO power kept constant at 1dBm, the circuit is reduced using a subspace spanning the moments with respect to the RF input power. In this case, the size of the original system of equations was 13394, and the size of the reduced system was 40. In Figure 7.11 the IF output power at 100MHz is plotted against the RF input power for both the original and reduced system. As can be seen, the results match with no noticeable difference.

Figure 7.9: Mixer circuit for Example 2

Figure 7.10: Example 2: Response to single tone local oscillator input

Figure 7.11: Example 2: IF output power at 100MHz with respect to RF input power

7.9 Conclusion

In this chapter, a projection based reduction method for nonlinear steady-state analysis was proposed. The new method builds on the advances in model order reduction methods for linear circuits, and extends the concept of model reduction to the steady state analysis of nonlinear circuits. Continuation methods were used as a vehicle in order to enable the reduction of the circuit equations, allowing the continuation parameter to be used as the basis for the reduction technique. Model reduction is particularly suitable for steady state analysis due to the very large size of the equations resulting from the harmonic balance approach. The new approach was implemented and tested and shown to be both accurate and efficient.

Chapter 8

Summary and Future Work

8.1 Summary

This thesis provides new algorithms designed for managing the increasing complexity of high-speed interconnect networks and Radio Frequency circuits. Several projection based model order reduction methods were presented in order to improve the efficiency of the analysis of high-speed interconnect networks, as well as the steady-state analysis of nonlinear circuits. First, a new technique is presented for obtaining reduced order macromodels of interconnect networks. Previous methods for obtaining reduced order macromodels produced models with a large number of redundant poles. This significantly increases the size of the macromodel and reduces the efficiency of the simulation. Previous attempts to rectify this problem resulted in macromodels that are not passive by construction. However, a non-passive model cannot be guaranteed stable with all possible passive terminations, and could lead to oscillation in the simulation. The new projection based reduction method presented in this thesis, removes the redundant poles, while at the same time preserving the passivity of the macromodel. The macromodel is half the size of that obtained using conventional reduction methods, and can be efficiently simulated using nonlinear simulators. Another contribution in this thesis, is a new technique for sensitivity analysis of interconnect networks using projection based model order

reduction. The new approach reduces the adjoint circuit equations directly through congruent transformation, and does not require the cumbersome computations of the sensitivity of the transformation matrix as is the case in previous methods.

Another contribution in this thesis, as a new algorithm for efficient steady-state analysis of nonlinear radio frequency circuits. Most of the current work in model order reduction in the circuit simulation area has been targeted towards macromodeling and analyzing linear circuits such as interconnect networks. In this thesis, a projection based reduction technique for nonlinear steady-state analysis is presented. The new approach uses continuation methods as a vehicle for enabling the circuit reduction, and at the same time benefits from the convergence improving properties of continuation techniques. The reduction is done by congruent transformation onto a subspace spanning the moments with respect to the continuation parameters, and these moments are proven to be preserved in the reduced system. The proposed nonlinear circuit reduction technique was shown to be very efficient for steady-state analysis of nonlinear RF circuits. The underlying concept is, however, very general and is expected to lead to many new results in nonlinear circuit reduction.

8.2 Future Work

1. The proposed nonlinear circuit reduction method uses continuation methods as vehicle to enable the reduction. In this case, the RF input power was used as the continuation parameter. An initial investigation shows that for some types of circuits it may be beneficial to use other parameters, perhaps even artificially constructed parameters [114] in order to do the reduction. Further research into choosing the continuation parameters used for reduction would provide a significant contribution to this area.
2. The proposed reduction method for nonlinear steady state analysis was used for analysis and simulation purposes. However, sensitivity information is often required. An important application of this work would be to develop a scheme for applying the concept

of nonlinear circuit reduction to the problem of sensitivity analysis of the steady-state solution of nonlinear circuits.

3. The concept of order reduction for sensitivity analysis presented in this thesis was recently extended multi-dimensional large scale sensitivity analysis for high speed interconnects networks [47], [48]. This approach allows for parameter sweeping for more than one parameters without having to reduce the system again. This approach is of interest in optimization loops, and for designers simply wishing to sweep a number of parameters.
4. The continuation based reduction for nonlinear circuit reduction is especially suitable for the multi-dimensional reduction methods described above. A useful contribution to the current work would be extending multi-dimensional reduction methods to the nonlinear domain and applying them to the steady-state solution of RF circuits.
5. The interconnect order reduction methods presented in this thesis are limited to cases where the transmission line parameters are known, or at least an (albeit large) passive lumped representation of the interconnect is given. However it is often the case when only measured or calculated frequency domain data is available. A number of methods are available for obtaining a lumped time representation of networks characterized by frequency domain data. However, ensuring the passivity of the macromodel is still an open problem. Addressing this issue is an important challenge currently facing the interconnect and VLSI design and simulation community.

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