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**$f_0(1370)$ Resonance Contribution
to Final State Interactions
in the Reaction $\gamma\gamma \rightarrow \pi\pi$**

by

Genevieve Hay

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Master of Science

Ottawa-Carleton Institute for Physics
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May, 1998

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acceptance of this thesis

$f_0(1370)$ Resonance Contribution to Final State
Interactions in the Reaction $\gamma\gamma \rightarrow \pi\pi$

submitted by
Genevieve Hay, B.Sc., MBA

in partial fulfillment of the requirements for
the degree of Master of Science

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Date 14th May '98

Abstract

We studied the reaction $\gamma\gamma \rightarrow \pi\pi$. This reaction is not expected to produce any chargeless pions since at low energies the process is described by scalar QED where the photon only couples to the charge of the particle. However, some experiments have observed a small, but significant, $\pi^0\pi^0$ cross-section. Using the quark model a previous attempt was made to describe the experimental observations by considering final state interactions of the $\pi^+\pi^-$ through t-channel and s-channel gluon exchange. This produced a $\pi^0\pi^0$ cross-section too small in magnitude. The $f_0(1370)$ resonance, which has a very broad width, may be contributing to the cross-section. A multichannel model was developed to include the $f_0(1370)$. An enhancement was found in the $\pi^0\pi^0$ cross-section, however, the prediction is still smaller in magnitude than the experimental data.

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Chapter 1

Introduction

In these fast-paced days of ‘high’ technology we have yet to unlock the answer to the simplest question that we have inherited from the dawn of man, “What is matter?” It is the shared dream of many physicists that by smashing particles together and observing the results, they will be able to determine the constituents of matter. This approach began in the early days of this century with the success of the cathode ray tube and has emerged into huge multi-billion dollar particle accelerators of today. Throughout this century, the results of these accelerators have astounded physicists because as they approach higher and higher energies, more and more new particles emerge and it has been a major pursuit in the latter half of this century to try to put all the pieces together into one coherent picture. The first obvious thing to do was to group particles into common lifetimes and mass ranges and thereby define different types of reactions. For example, the strong reaction is characterized by very short reaction times and has a much stronger coupling strength than the electromagnetic reaction. Soon empirical laws emerged which provided quantum numbers that further differentiated the types of particles observed in the scattering experiments. These laws deal with conservation principles and are so basic that they provide the cornerstone of most of today’s analysis. To mention a few we have conservation of angular momentum, isospin, charge, particle type (strangeness, for example), and energy in any given type of reaction.

In the 1950's and 1960's a proliferation of resonances were observed in hadron-hadron scattering and patterns emerged. It was found that these states (particles) could be arranged in multiplets of the group $SU(3)$ on the basis of isospin and strangeness. In the early 1960's Gell-Mann and Zweig postulated the existence of unobserved particle constituents that helped to explain both the decuplet patterns of the baryons and types of reactions observed: quarks were born. In particular, it was postulated that mesons (integer spin hadrons) were composed of a quark-antiquark ($q\bar{q}$) bound state and baryons (half-odd integer spin hadrons) were composed of three quarks. All particles comprised of quarks are known as hadrons. For example, a proton is a baryon of three quarks, uud , where u is the up quark and d is the down quark. Further, pions, discovered in cosmic rays in 1947, are perhaps the most common meson and are comprised of up and down quarks and their antiquarks. The production of these mesons from photons is the focus of this thesis.

Quarks are bound together via the exchange of gluons in analogy to electron and positron bound together in positronium via the exchange of photons. The theory of strong interactions describing quarks and gluons is known as quantum chromodynamics (QCD). However, it is a very complicated theory to solve and it is useful to use phenomenological models to describe properties of hadrons. The quark model is one such model.

1.1 Mesons and The Quark Model

At this point it is useful to introduce the quantum numbers of mesons and some notation. We consider the parity of the spatial wave function and charge conjugation (in which we reverse the charges of the quarks). The antiquark has the opposite intrinsic parity of the quark and since the parity of the wavefunction with angular momentum L is given by $(-1)^L$, the parity operator acting on a meson has eigenstates given by $(-1)^{L+1}$. Similarly, for a neutral $q\bar{q}$ system, the eigenstates of the charge conjugation

operator are $(-1)^{L+S}$ where S is the spin of the system. For total angular momentum J , parity P and charge conjugation parity C , we label all mesons by J^{PC} (even when the meson is not neutral) and since each one represents a unique combination of L and S , we also use a spectroscopic notation given by $^{2s+1}L_J$ where we use S for $L = 0$, P for $L = 1$, D for $L = 2$, etc. as shown in Table 1.1

L	S	J^{PC}	Notation
0	0	0^{-+}	1S_0
	1	1^{--}	3S_1
1	0	1^{+-}	1P_1
	1	0^{++}	3P_0
		1^{++}	3P_1
		2^{++}	3P_2
2	0	2^{-+}	1D_2
	1	1^{--}	3D_1
		2^{--}	3D_2
		3^{--}	3D_3

Table 1.1: The angular momentum and charge-parity quantum numbers and spectroscopic notation for mesons based on the spin and orbital angular momentum of the $q\bar{q}$ system.

In the quark potential model, mesons are described as $q\bar{q}$ pairs bound together by a phenomenological potential. The QCD motivated effective potential between the quark and antiquark is dominated by one-gluon exchange at short distance and a linear confinement at long range. In the simplest form, meson energy levels and wavefunctions are found by solving the Schrödinger equation

$$H|\psi\rangle = [H_0 + V_{q\bar{q}}]|\psi\rangle = E|\psi\rangle \quad (1.1)$$

where H_0 is the kinetic energy term given by

$$H_0 = \sum_{i=1}^2 \left(m_i + \frac{p^2}{2m_i} \right). \quad (1.2)$$

and $V_{q\bar{q}}$ is the effective quark-antiquark potential which in addition to the spin-independent Coulomb and linear pieces will in general have spin dependent terms which give rise to mass splittings such as the $^3S_1 - ^1S_0$ ($\rho - \pi$) hyperfine splitting from the $\vec{S}_q \cdot \vec{S}_{\bar{q}}$ interactions and 3L_J splitting ($f_2 - f_1 - f_0$) from spin-orbit and tensor interactions. The constituent quark model has been successful in predicting meson spectroscopy and properties (Godfrey [1]).

Most scattering reactions are a black box of events which we are unable to probe directly. We can only hypothesize as to the mechanisms involved that may produce the particles observed. In those areas of physics where the main theories fall short of immediate answers to observed phenomena, we have the realm of phenomenology, which leans on models like the quark model to explain the data. The more applications we can find to exercise the quark model, the more widely useful it will become and may give us some physical intuition of the processes involved. The work in this thesis is one such application.

1.2 The Reaction $\gamma\gamma \rightarrow \pi\pi$

This thesis was concerned with the reaction $\gamma\gamma \rightarrow \pi\pi$ and the cross-sections for both $\pi^+\pi^-$ and $\pi^0\pi^0$ as a further test of the quark model. Since at low energies the process is described by scalar QED in which the photon only couples to the charge of the pion and not its quark constituents, this reaction is not expected to produce the $\pi^0\pi^0$ final state which was observed. Since scalar QED fails to account for the $\pi^0\pi^0$ cross-section, we used the quark model in a phenomenological approach to attempt to explain the data. We assumed there are final state interactions occurring between the charged pions to produce the chargeless pions. A preliminary calculation of the process [2] only partially explained the data so the purpose of this thesis is to perform a more refined calculation.

A final state interaction (FSI) is an interaction that takes place between final state

particles beyond the basic interaction. An example of one such possibility is given in Fig. 1.1

Figure 1.1: This diagram depicts the basic reaction $\gamma\gamma \rightarrow \pi^+\pi^-$ followed by a possible final state interaction producing $\pi^0\pi^0$.

In 1986, when measuring the cross-section for the reaction $\gamma\gamma \rightarrow \pi^+\pi^-$ A. Courau *et al.* [3] found a significant increase over what was expected from the basic interaction which is given by scalar QED. In 1987, Barnes, Dooley and Isgur [4] (BDI) attributed the excess to strong interactions between the final state pions. To calculate the effects of the FSI, they used the Fermi approximation which assumes that the interaction potential is very short range. Because of this, only the amplitude of the S-wave at the origin, $|u_0(k,0)|^2$, contributes to the enhancement of the cross-section while the other angular momentum states remain unaffected by the FSI since for higher partial waves the amplitudes at the origin are zero. They found that the cross section thus calculated was enhanced three-fold below 500 MeV, even after taking into account the limited detector polar acceptance, and it fit the initial data. However, the data was in error and subsequent measurements in 1990 by Boyer *et al.* [5] of the cross-section gave results closer to the scalar QED predictions as shown in Figure 1.2

In 1996, Blundell [2] performed a more complete calculation than the work done by Barnes *et al.* without resorting to the Fermi approximation and by including FSI's using different $\pi\pi$ potentials for the different isospin channels. His work was successful

Figure 1.2: Total cross-section vs. $\sqrt{s}$, the centre of mass energy, for $\gamma\gamma \rightarrow \pi^+\pi^-$ showing the BDI calculations and scalar QED predictions with data from Behrend *et al.* [6], Boyer *et al.* [5] and Aihara *et al.* [7]. The curves have all been corrected for limited detector acceptance.

in that he obtained only a small enhancement for the reaction $\gamma\gamma \rightarrow \pi^+\pi^-$ over the scalar QED cross-sections that fit the new data well, shown in Figure 1.3. Moreover, his calculations predicted a small $\gamma\gamma \rightarrow \pi^0\pi^0$ cross-section. At low energies, the photons have long wavelengths and do not couple to the quarks in the pions so the pions can be treated as point particles and scalar QED is a sufficient treatment for calculating the cross-section. However, in scalar QED there is no mechanism for the production of $\pi^0\pi^0$ since the photons only couple to charged particles. However, a small, but significant,

Figure 1.3: Total cross-section vs. $\sqrt{s}$ for $\gamma\gamma \rightarrow \pi^+\pi^-$ showing FSI predictions from Blundell [2] and scalar QED calculations with data from Behrend *et al.* [6], Boyer *et al.* [5] and Aihara *et al.* [7]. The curves have all been corrected for limited detector acceptance.

$\pi^0\pi^0$ cross-section is observed. Blundell's calculation, although a start, does not account for the entire production as shown in Figure 1.4

The $f_0(1370)$ resonance is considered to be a $q\bar{q}$ meson with a very broad width decaying into $\pi^+\pi^-$ and $\pi^0\pi^0$. It is expected that because of its large width at the lower energies this resonance may contribute to the cross-section. The purpose of this thesis is to include the effects of the $f_0(1370)$ resonance into Blundell's calculations to study its effects on the $\pi^0\pi^0$ cross-section. One method of including this resonance is to introduce

Figure 1.4: Total cross-section vs. $\sqrt{s}$ for $\gamma\gamma \rightarrow \pi^0\pi^0$ showing FSI predictions from Blundell [2] with data from Bienlein *et al.* [8], Marsiske *et al.* [9] and Edwards *et al.* [10] corrected to full polar detection acceptance. The various cases include different assumptions about the potentials, but we are interested in Case A' with a maximum of 6.5 nb.

the Schrödinger equation for the $q\bar{q}$ resonance and then couple it to the Schrödinger equation for the $\pi\pi$ eigenstate by considering an energy transfer (or potential) representing a mixing between the two states. This system of coupled differential equations is known as the ‘multichannel’ model. This model, developed by Weinstein [11], is a numerical matrix method for solving coupled Schrödinger equations. Each ‘channel’ represents a simultaneous eigenstate for a given center of mass energy, hence a possible final state for a reaction. In his paper Weinstein gives the example of S-wave $K\pi$ scattering through an unstable K_0^* scalar meson resonance. This model was applied to

our situation of S-wave $\pi\pi$ scattering through the $f_0(1370)$ resonance. The $\pi\pi$ wavefunction generated numerically was used to calculate phase shifts and cross-sections. It was found that there was an enhancement of the $\pi^0\pi^0$ cross-section but mostly in the lower energy range (below 500 MeV).

This thesis is presented in three main sections: the FSI model of Blundell, the multichannel model as presented by Weinstein and finally, the combined model. Initially Chapter 2 we give an overview of the basic interaction as calculated in scalar QED. We then show how Blundell introduced FSI's into the basic reaction and generated the corrected amplitudes for the cross-section calculations. Subsequently in Chapter 3 we develop Weinstein's multichannel model for the $\pi\pi$ scattering through the $f_0(1370)$ resonance to generate wavefunctions and phase shift data. Putting it all together, in Chapter 4 we present the results of using Weinstein's wavefunctions and Blundell's FSI-corrected cross-section calculations. Finally, in Chapter 5 we present our conclusions.

Chapter 2

The Scalar QED Cross-Section for $\gamma\gamma \rightarrow \pi^+\pi^-$ and Final State Interaction Calculations

The FSI calculations are based on non-relativistic scattering theory [12] and follow the work done by Blundell [2]. These interactions occur between final state particles beyond the initial reaction vertex. In this chapter we consider the basic reaction $\gamma\gamma \rightarrow \pi^+\pi^-$, and then we include FSIs between the charged pions. In the next chapter we include the $\pi\pi \leftrightarrow f_0(1370)$ interaction through a multichannel model.

Here we begin with the scalar QED cross section calculation for the basic reaction and express it in terms of its angular momentum partial wave amplitudes. We show how to derive the spherical wave solution for FSIs using $\pi\pi$ potentials (in this case, describing t and s-channel gluon exchange), and how to apply it in calculations of the partial wave amplitudes. Since the potentials are different for specific isospin states, the calculations simplify by rotating into the isospin basis, computing the partial wave amplitudes and then rotating back into the final state basis. Finally, we apply corrections for limited polar detector acceptance before computing the net cross-sections.

2.1 Scalar QED Cross-Section for $\gamma\gamma \rightarrow \pi\pi$ and Partial Wave Amplitudes

At low energies, the long wavelength photons do not ‘see’ the quark constituents of the pions, so the pions are treated as point particles and scalar QED is used to compute the cross section.

Figure 2.1: The three scalar QED vertices contributing to the $\gamma\gamma \rightarrow \pi^+\pi^-$ cross-section. In general p_i is the four-momentum of the particle and ϵ_i is the polarization of the photon.

There are three Feynman diagrams, shown in Figure 2.1, contributing to the basic interaction amplitude

$$\mathcal{M} = 4\pi\alpha \left[\frac{\epsilon_1 \cdot (2p_3 - p_1)\epsilon_2 \cdot (2p_4 - p_2)}{(p_2 - p_4)^2 - m^2} + \frac{\epsilon_1 \cdot (2p_4 - p_1)\epsilon_2 \cdot (2p_3 - p_2)}{(p_1 - p_4)^2 - m^2} + 2\epsilon_1 \cdot \epsilon_2 \right] \quad (2.1)$$

which yields upon squaring, summing over the photon polarizations and integrating over the final state phase space, the total cross section

$$\sigma = \frac{\pi\alpha^2}{4m^2} \left[2x(1+x)\sqrt{1-x} - x^2(2-x)\ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right) \right] \quad (2.2)$$

where $x \equiv 4m^2/s$, α is the fine structure constant and m is the mass of the pion.

We want to write the cross-section in terms of its angular momentum partial waves and to do this we need to calculate the helicity amplitudes. In the center of mass frame,

define the incident photon momentum $\vec{p}_1$ to be along the positive $\hat{z}$ direction, the angle θ to be the scattering angle between the outgoing pion momentum $\vec{p}_3$ and the $\hat{z}$ axis and the angle ϕ to be taken with respect to the plane of scattering. We choose circular polarizations for the photons:

$$\begin{aligned}\epsilon_1(+) = \epsilon_2(-) &= \frac{1}{\sqrt{2}}(0, -1, -i, 0), \\ \epsilon_1(-) = \epsilon_2(+) &= \frac{1}{\sqrt{2}}(0, 1, -i, 0).\end{aligned}\tag{2.3}$$

Substituting these directly into the Feynman amplitude above, Eq. 2.1, we obtain the helicity amplitudes

$$\begin{aligned}\mathcal{M}_{++} = \mathcal{M}_{--} &= \frac{8\pi\alpha x}{1 - (1-x)\cos^2\theta} \\ \mathcal{M}_{\pm\mp} &= \frac{8\pi\alpha(1-x)\sin^2\theta e^{\pm 2i\phi}}{1 - (1-x)\cos^2\theta}\end{aligned}\tag{2.4}$$

which can be expanded in terms of the partial wave amplitudes. For example,

$$\mathcal{M}_{--} = \sum_{L=0}^{\infty} \sum_{M_L=-L}^L f_{LM_L}^{--} Y_{LM_L}(\theta, \phi)\tag{2.5}$$

where $f_{LM_L}^{--}$ is the partial wave amplitude and Y_{LM_L} is the spherical harmonic. We can use orthogonality conditions of the spherical harmonics to invert the helicity amplitudes to obtain the partial wave amplitudes

$$f_{LM_L}^{--} = \int d\Omega Y_{LM_L}^*(\theta, \phi) \mathcal{M}_{--}\tag{2.6}$$

where it can be shown that $f_{LM_L}^{++} = f_{LM_L}^{--} = 0$ except when $M_L = 0$ and likewise, $f_{LM_L}^{+-} = f_{LM_L}^{-+} = 0$ except when $M_L = 2$. Thus we can write the total cross-section in terms of the angular momentum partial waves (as given by Blundell [2]) as

$$\sigma = \frac{\sqrt{1-x}}{128\pi^2 s} \left[(f_{00})^2 + \sum_{L \geq 2, \text{even}}^{\infty} [(f_{L0})^2 + (f_{L2})^2] \right].\tag{2.7}$$

Figure 2.2: Scalar QED partial wave cross-sections for $\gamma\gamma \rightarrow \pi^+\pi^-$. Here the total represents the sum of $L=0,2,4$ partial waves. Note that these predictions are not corrected for limited detector acceptance while the prediction in Fig. 1.2 has been corrected.

The non-zero partial wave amplitudes are given explicitly below:

$$\begin{aligned}
 f_{00} &= 8\pi^{\frac{3}{2}}\alpha \frac{x}{\sqrt{1-x}} \ln \left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} \right) \\
 f_{20} &= 4\sqrt{5}\pi^{\frac{3}{2}}\alpha \frac{x}{(1-x)} \left[-6 + \frac{(2+x)}{\sqrt{1-x}} \ln \left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} \right) \right] \\
 f_{22} &= 4\sqrt{\frac{15}{2}}\pi^{\frac{3}{2}}\alpha \left[\frac{10}{3} - \frac{2}{(1-x)} + \frac{x^2}{(1-x)^{\frac{3}{2}}} \ln \left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} \right) \right] \\
 f_{40} &= 3\pi^{\frac{3}{2}}\alpha \frac{x}{(1-x)} \left[\frac{110}{3} - \frac{70}{(1-x)} + \frac{(3x^2+24x+8)}{(1-x)^{\frac{3}{2}}} \ln \left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
f_{42} &= 3\sqrt{10}\pi^{\frac{3}{2}}\alpha \left[\frac{-54}{5} + \frac{76}{3(1-x)} - \frac{14}{(1-x)^2} + \frac{x^2(6+x)}{(1-x)^{\frac{5}{2}}} \ln \left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} \right) \right] \\
f_{60} &= 8\sqrt{13}\pi^{\frac{3}{2}}\alpha \frac{x}{(1-x)} \left[-\frac{231}{40} + \frac{7(4-15x)}{8(1-x)} - \frac{21(1+5x+5x^2)}{8(1-x)^2} \right. \\
&\quad \left. + \frac{(16+120x+90x^2+5x^3)}{16(1-x)^{\frac{5}{2}}} \ln \left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} \right) \right]
\end{aligned} \tag{2.8}$$

These represent the amplitudes from the scalar QED treatment without polar acceptance or FSI corrections (and appear as $f_{LM_L}(s(k_f))$ in Eqs. 2.5 and 2.27 in Section 2.3 below). They are used to calculate the uncorrected cross-section contributions for each of the $L = 0, 2, 4$ partial waves as in Figure 2.2

2.2 Final State Interaction Wavefunction

To include final state interactions into the basic reaction described in the previous section, we derive a final state wavefunction which includes the interaction and then include it directly into the Born approximation for the amplitude calculation. From this we obtain partial wave amplitudes that have been corrected for FSIs in terms of the uncorrected partial wave amplitudes given in the previous section.

We assume there is a FSI potential, $V(r)$, between the two outgoing pions and that the related Schrödinger equation will yield the appropriate final state wavefunction. In the centre of mass frame, the radial Schrödinger equation for relative motion is given by

$$\left[\nabla_{\vec{r}}^2 - 2\mu V(r) + k^2 \right] \psi(\vec{r}) = 0 \tag{2.9}$$

where μ is the reduced mass of the system and k is the relative momentum of the particles in natural units where $c \equiv \hbar \equiv 1$. We separate out the spherical harmonics by taking $\psi(\vec{r}) = R_l(r)Y_{lm}(\Omega)$ and obtain the radial equation

$$\left[\frac{1}{r} \frac{d^2}{dr^2} r - \frac{l(l+1)}{r^2} - 2\mu V(r) + k^2 \right] R_l(r) = 0 \tag{2.10}$$

whose solution in the absence of a potential is a spherical Bessel function $R_l(r) = j_l(kr)$ with asymptotic behaviour

$$j_l(kr) \xrightarrow{kr \rightarrow \infty} \frac{1}{kr} \sin(kr - \frac{1}{2}l\pi) \quad (2.11)$$

This, then, characterizes the behaviour of the general solution to Eq. 2.9 whose solution for $V(r) = 0$ is given by the free wave equation

$$\psi_{\vec{k}}^o(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} \quad (2.12)$$

which can be expanded in terms of the angular momentum partial waves as

$$e^{i\vec{k} \cdot \vec{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m_l=-l}^l i^l j_l(kr) Y_{lm}^*(\Omega_{\vec{k}}) Y_{lm}(\Omega_{\vec{r}}). \quad (2.13)$$

When we include a scattering potential, we assume in the final state a spherical disturbance to the free wave solution so that the solution to Eq. 2.9 above has the asymptotic form

$$\psi_{\vec{k}}^{\pm}(\vec{r}) \xrightarrow{r \rightarrow \infty} \psi_{\vec{k}}^o(\vec{r}) + f_{\vec{k}}^{\pm}(\Omega_r) \frac{e^{-ikr}}{r} \quad (2.14)$$

where $f_{\vec{k}}^{\pm}(\Omega_r)$ is the scattering amplitude dependent on the potential with $+$ denoting an outgoing and $-$ denoting an incoming spherical wave. Furthermore, we can assume the amplitude is symmetric about the incident axis of propagation and so has only a θ dependence given by Legendre polynomials

$$f_{\vec{k}}^{\pm}(\Omega_r) = f_{\vec{k}}^{\pm}(\theta_r) = \sum_{l=0}^{\infty} f_l^{\pm} P_l(\cos \theta_r). \quad (2.15)$$

Altogether, Eq. 2.14 then has the asymptotic form

$$\psi_{\vec{k}}^{\pm}(\vec{r}) \xrightarrow{r \rightarrow \infty} 4\pi \sum_{l=0}^{\infty} \sum_{m_l=-l}^l i^l \frac{1}{kr} \sin(kr - \frac{1}{2}l\pi) Y_{lm}^*(\Omega_{\vec{k}}) Y_{lm}(\Omega_{\vec{r}}) + \sum_{l=0}^{\infty} f_l^{\pm} P_l(\cos \theta_r) \frac{e^{-ikr}}{r} \quad (2.16)$$

where the spherical harmonics can be rewritten interchangeably with the Legendre polynomials.

Alternatively, we can propagate the spherical wave back in time to interact with the incident wave and so choose the incoming spherical wave solution $\psi_{\vec{k}}^-(\vec{r})$ which must also be symmetric about the axis of propagation of the incident wave. In terms of the Legendre polynomials this can be written as

$$\psi_{\vec{k}}^-(r, \theta_r) = \sum_{l=0}^{\infty} \frac{y_l(r)}{r} P_l(\cos \theta_r) \quad (2.17)$$

where $y_l(r)/r = R_l(r)$ is a solution to Eq. 2.10 above. Therefore, we have the asymptotic behaviour

$$y_l(r) \xrightarrow{r \rightarrow \infty} a_l \sin(kr - \frac{1}{2}l\pi + \delta_l). \quad (2.18)$$

Matching this solution behaviour to the asymptotic solution of the right hand side of Eq. 2.16 for the incoming wavefunction $\psi_{\vec{k}}^-$, permits us to solve for a_l and f_l . Thus the incoming spherical wave solution of equation Eq. 2.9 becomes

$$\psi_{\vec{k}}^-(\vec{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m_l=-l}^l i^l e^{-i\delta_l} u_l(k, r) Y_{lm}^*(\Omega_{\vec{k}}) Y_{lm}(\Omega_{\vec{r}}) \quad (2.19)$$

where $u_l(k, r)$ is the radial wavefunction we seek, a solution to Eq. 2.10 with asymptotic behaviour

$$u_l(k, r) \xrightarrow{r \rightarrow \infty} \frac{1}{kr} \sin(kr - \frac{1}{2}l\pi + \delta_l). \quad (2.20)$$

2.3 Modifying QED Amplitudes to Include FSI Potentials

Consider the Hamiltonian $H_o + W + V$ where H_o is the kinetic energy term, W is the basic interaction and V is the FSI potential. In the absence of any interaction

potentials, we define the plane waves $\phi_{\vec{k}_f}$ as solutions to

$$(H_o - E)\phi_{\vec{k}_f} = 0 \quad (2.21)$$

The transition amplitude from initial to final states, restricted to the basic interaction, is given by $\langle \phi_{\vec{k}_f}^- | W | \phi_{\vec{k}_i}^- \rangle$ where we have used the Born approximation assuming that the potential W is sufficiently small that we approximate the final state as a plane wave. Blundell made the assumption

$$|\mathcal{M}|^2 = s |\langle \phi_{\vec{k}_f}^- | W | \phi_{\vec{k}_i}^- \rangle|^2 \quad (2.22)$$

where s is the square of the centre of mass energy and is introduced to match the relativistic normalization of the matrix elements used in the QED calculation to the non-relativistic normalization used in the FSI calculation.

We can include the FSI directly into the final wavefunction by using our solution in Eq. 2.19 in momentum space representation

$$\mathcal{M}^{FSI}(\vec{k}_f) = \langle \phi_{\vec{k}_f}^- | W + V | \phi_{\vec{k}_i}^- \rangle = \langle \psi_{\vec{k}_f}^- | W | \phi_{\vec{k}_i}^- \rangle \quad (2.23)$$

Using the closure relation, this becomes

$$\begin{aligned} \mathcal{M}^{FSI}(\vec{k}_f) &= \frac{1}{(2\pi)^3} \int d^3\vec{k}' \langle \psi_{\vec{k}_f}^- | \phi_{\vec{k}'}^- \rangle \langle \phi_{\vec{k}'}^- | W | \phi_{\vec{k}_i}^- \rangle \\ &= \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}' \psi_{\vec{k}_f}^{-*} \langle \phi_{\vec{k}'}^- | W | \phi_{\vec{k}_i}^- \rangle \\ &= \frac{2}{\pi} \sqrt{s(k_f)} \sum_{l=0}^{\infty} \sum_{m_l=-l}^l e^{i\delta_l} Y_{lm_l}(\Omega_{k_f}) \int d^3\vec{k}' \langle \phi_{\vec{k}_f}^- | W | \phi_{\vec{k}_i}^- \rangle Y_{lm_l}^*(\Omega_{k'}) \\ &\quad \times \int_0^{\infty} dr r^2 j_l(k' r) u_l(k_f, r) \end{aligned} \quad (2.24)$$

In QED, we obtain transition amplitudes

$$\mathcal{M}(\vec{k}) = \langle \phi_{\vec{k}_f}^- | S | \phi_{\vec{k}_i}^- \rangle \quad (2.25)$$

which can be expanded in terms of the angular momentum partial waves as in Eq. 2.5

$$\mathcal{M}(\vec{k}) = \sum_{L=0}^{\infty} \sum_{M_L=-L}^L f_{LM_L}(s(k)) Y_{LM_L}(\Omega_k). \quad (2.26)$$

We can substitute this into Eq. 2.24 using Blundell's approximation, and obtain for a single partial wave corrected for FSI, the expression

$$f_{LM_L}^{FSI}(s(k_f)) = \frac{2}{\pi} \sqrt{s(k_f)} e^{i\delta_L} \int_0^{\infty} dk k^2 \int_0^{\infty} dr r^2 \frac{f_{LM_L}(s(k_f))}{\sqrt{s(k)}} j_L(kr) u_L(k_f, r) \quad (2.27)$$

2.4 Cross-Section Calculations for $\gamma\gamma \rightarrow \pi\pi$ Using the FSI Amplitudes

2.4.1 Isospin Dependence

Since we are considering a strong interaction, we have charge conjugation invariance between the pions and this implies they must be in an $L = \text{even}$ state. For two identical bosons of spin 0, the total wavefunction must be symmetric which means $L + I$ must be even, hence the isospin I states must be even. Since the FSI potentials used are dependent on the isospin states of the pions, Blundell decided to rotate the final states $\{|\pi^+\pi^-\rangle, |\pi^0\pi^0\rangle\}$ into the isospin basis $|II_z\rangle = \{|00\rangle, |20\rangle\}$

$$\begin{aligned} |\pi^+\pi^-\rangle &= \sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{3}}|20\rangle \\ |\pi^0\pi^0\rangle &= -\frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|20\rangle \end{aligned} \quad (2.28)$$

because this allowed him to uncouple the Schroedinger equations and solve the wavefunctions for each isospin state directly. To do this, we repeat the computations in

Eqs. 2.23 through 2.27 to obtain the FSI amplitudes in terms of the isospin basis wavefunctions, starting with

$$\begin{aligned}\mathcal{M}^{FSI}(\vec{k}) &= \frac{1}{(2\pi)^3} \int d^3\vec{k} \langle \psi_{\vec{k}_f}^- | [|\phi_{\vec{k}}^{\pi^+\pi^-} \rangle \langle \phi_{\vec{k}}^{\pi^+\pi^-} | + |\phi_{\vec{k}}^{\pi^0\pi^0} \rangle \langle \phi_{\vec{k}}^{\pi^0\pi^0} |] W | \phi_{\vec{k}_i} \rangle \\ &= \frac{1}{(2\pi)^3} \int d^3\vec{k} \langle \psi_{\vec{k}_f}^- | \phi_{\vec{k}}^{\pi^+\pi^-} \rangle \langle \phi_{\vec{k}}^{\pi^+\pi^-} | W | \phi_{\vec{k}_i} \rangle\end{aligned}\quad (2.29)$$

where the $\pi^0\pi^0$ term drops out since this is not a final state of the scalar QED process. We already have expressions for the right hand bra-ket given using Eq. 2.22 and we can write

$$\begin{aligned}\langle \psi_{\vec{k}_f}^{-\pi^+\pi^-} | \phi_{\vec{k}}^{\pi^+\pi^-} \rangle &= \left[\sqrt{\frac{2}{3}} \langle \psi_{\vec{k}_f}^{-0} | + \frac{1}{\sqrt{3}} \langle \psi_{\vec{k}_f}^{-2} | \right] \left[\sqrt{\frac{2}{3}} | \phi_{\vec{k}}^0 \rangle + \frac{1}{\sqrt{3}} | \phi_{\vec{k}}^2 \rangle \right] \\ &= (2\pi)^{\frac{3}{2}} \left[\frac{2}{3} \psi_{\vec{k}_f}^{-0*} + \frac{1}{3} \psi_{\vec{k}_f}^{-2*} \right]\end{aligned}\quad (2.30)$$

$$\begin{aligned}\langle \psi_{\vec{k}_f}^{-\pi^0\pi^0} | \phi_{\vec{k}}^{\pi^+\pi^-} \rangle &= \left[-\frac{1}{\sqrt{3}} \langle \psi_{\vec{k}_f}^{-0} | + \sqrt{\frac{2}{3}} \langle \psi_{\vec{k}_f}^{-2} | \right] \left[\sqrt{\frac{2}{3}} | \phi_{\vec{k}}^0 \rangle + \frac{1}{\sqrt{3}} | \phi_{\vec{k}}^2 \rangle \right] \\ &= (2\pi)^{\frac{3}{2}} \left[-\frac{\sqrt{2}}{3} \psi_{\vec{k}_f}^{-0*} + \frac{\sqrt{2}}{3} \psi_{\vec{k}_f}^{-2*} \right]\end{aligned}\quad (2.31)$$

Then the equivalent expressions for Eq. 2.27 are given below where $u_L^I(k_f, r)$ are the two isospin radial wavefunction solutions to Eq. 2.10 with corresponding asymptotic phase shifts δ_L^I . The superscript I corresponds to the isospin channel and the subscript L to the relative angular momentum of the outgoing pions.

$$\begin{aligned}f_{LM_L}^{FSI\pi^+\pi^-}(s(k_f)) &= \frac{2}{\pi} \sqrt{s(k_f)} \int_0^\infty dk k^2 \int_0^\infty dr r^2 \frac{f_{LM_L}(s(k_f))}{\sqrt{s(k)}} j_L(kr) \\ &\times \left[\frac{2}{3} e^{i\delta_L^0} u_L^0(k_f, r) + \frac{1}{3} e^{i\delta_L^2} u_L^2(k_f, r) \right]\end{aligned}\quad (2.32)$$

$$\begin{aligned}f_{LM_L}^{FSI\pi^0\pi^0}(s(k_f)) &= \frac{2}{\pi} \sqrt{s(k_f)} \int_0^\infty dk k^2 \int_0^\infty dr r^2 \frac{f_{LM_L}(s(k_f))}{\sqrt{s(k)}} j_L(kr) \\ &\times \left[-\frac{\sqrt{2}}{3} e^{i\delta_L^0} u_L^0(k_f, r) + \frac{\sqrt{2}}{3} e^{i\delta_L^2} u_L^2(k_f, r) \right]\end{aligned}\quad (2.33)$$

To simplify the expression, we define isospin amplitudes

$$g_{LM_L}^I(s(k_f)) = \frac{2}{\pi} \sqrt{s(k_f)} \int_0^\infty dk k^2 \int_0^\infty dr r^2 \frac{f_{LM_L}(s(k_f))}{\sqrt{s(k)}} j_L(kr) u_L^I(k_f, r) \quad (2.34)$$

then using these amplitudes, the angular momentum partial wave amplitudes, after converting back into the final state basis, become

$$|f_{LM_L}^{FSI\pi^+\pi^-}(s(k_f))|^2 = \frac{4}{9}(g_{LM_L}^0)^2 + \frac{1}{9}(g_{LM_L}^2)^2 + \frac{4}{9}g_{LM_L}^0 g_{LM_L}^2 \cos(\delta_L^0 - \delta_L^2) \quad (2.35)$$

$$|f_{LM_L}^{FSI\pi^0\pi^0}(s(k_f))|^2 = \frac{2}{9}(g_{LM_L}^0)^2 + \frac{2}{9}(g_{LM_L}^2)^2 - \frac{4}{9}g_{LM_L}^0 g_{LM_L}^2 \cos(\delta_L^0 - \delta_L^2) \quad (2.36)$$

These are the corrected amplitudes to be substituted into the cross-section of Eq. 2.7 above. The integrations were done using Simpson's method where step sizes and ranges were varied to ensure convergence.

2.4.2 FSI Potentials

To compute the wavefunction in $g_{LM_L}^I$ above, we use the standard Numerov technique. The FSI potentials included in the Schrödinger equation were calculated by Swanson and collaborators [13, 14] using the quark model. They included the colour-hyperfine and quark confinement contributions in the t-channel gluon exchange and s-channel gluon exchange. They parametrized the resulting meson-meson potentials in the form

$$V(r) = V_o e^{-\frac{1}{2}(\frac{r}{r_o})^2}. \quad (2.37)$$

The parameters V_o and r_o are given by Swanson *et al.* [13, 14], and are tabulated by Blundell [2] in Table 2.1

In Swanson *et al.*'s notation, the potential is divided into a contribution arising from a gluon exchange in the t-channel from the colour hyperfine interaction and the confinement contribution in Fig. 2.3 (a) and a contribution from gluon exchange in the s-channel in Fig. 2.3 (b).

Pion State		$\pi\text{-}\pi$ Potentials (V_0 in GeV, r_0 in GeV^{-1})					
		t-channel gluon exchange				s-channel	
L	I	hyperfine		confinement		gluon exchange	
		V_0	r_0	V_0	r_0	V_0	r_0
0	0	-0.392	1.36	-0.024	2.29	-1.75	1.48
0	2	0.786	1.36	0.047	2.29		
2	0	-0.044	1.40	0.0175	1.49		
2	2	0.088	1.40	-0.035	1.49		

Table 2.1: These are the $\pi\pi \leftrightarrow \pi\pi$ potential parameters used in the functional form given by Eq. 2.37

Figure 2.3: Representations of (a) t-channel gluon exchange and (b) s-channel gluon exchange used to find the FSI potentials.

2.4.3 Polar Acceptance

Due to experimental limitations we cannot integrate over the entire polar range. Selecting $|\cos(\theta)| \leq 0.6$ as typical, we use this in all of the following corrections. First, with this angular acceptance included, the scalar QED cross-section in Eq. 2.2 becomes

$$\begin{aligned}
\sigma_{acc} = & \frac{\pi\alpha^2}{4m^2} \left[2 \cos \theta_{acc} x \sqrt{1-x} \left(\frac{x^2}{1 - (1-x) \cos^2 \theta_{acc}} + 1 \right) \right. \\
& \left. - x^2(2-x) \ln \left(\frac{1 + \sqrt{1-x} \cos \theta_{acc}}{1 - \sqrt{1-x} \cos \theta_{acc}} \right) \right] \quad (2.38)
\end{aligned}$$

To write this in terms of partial waves, we note that the spherical harmonics are no longer orthogonal to each other and the equivalent expression for Eq. 2.7 becomes an

infinite sum (Blundell [2])

$$\begin{aligned} \sigma_{acc} = & \frac{\sqrt{1-x}}{128\pi^2 s} \int_0^{2\pi} d\phi \int_{-\cos\theta_{acc}}^{\cos\theta_{acc}} d(\cos\theta) \left[2 \left| \sum_{L \geq 0, even}^{\infty} (f_{L0}) Y_{L0}(\theta, \phi) \right|^2 \right. \\ & \left. + \left| \sum_{L \geq 2, even}^{\infty} (f_{L2}) Y_{L2}(\theta, \phi) \right|^2 + \left| \sum_{L \geq 2, even}^{\infty} (f_{L-2}) Y_{L-2}(\theta, \phi) \right|^2 \right] \end{aligned} \quad (2.39)$$

To simplify calculations, define the harmonic integrals

$$h_{l_1 l_2}^m \equiv \int_0^{2\pi} d\phi \int_{-\cos\theta_{acc}}^{\cos\theta_{acc}} d(\cos\theta) Y_{l_1 m}^*(\theta, \phi) Y_{l_2 m}(\theta, \phi) \quad (2.40)$$

Numerically it was found that the FSI's only contribute significantly to the L=0 state. We construct the total cross section by taking σ_{acc} and subtracting the terms with the uncorrected QED amplitudes in Eq. 2.8 in the L=0 state and add them back using the FSI corrected amplitudes. Eq. 2.7 becomes

$$\begin{aligned} \sigma = & \sigma_{acc} + \frac{\sqrt{1-x}}{128\pi^2 s} \left[h_{00}^0 \left\{ |f_{00}^{FSI\pi^+\pi^-}|^2 - (f_{00})^2 \right\} \right. \\ & \left. + \sum_{L \geq 2, even}^{\infty} h_{L0}^0 \left\{ f_{00}^{FSI\pi^+\pi^-} f_{L0} + (f_{00}^{FSI\pi^+\pi^-})^* f_{L0} - 2f_{00} f_{L0} \right\} \right] \end{aligned} \quad (2.41)$$

Making substitutions for the partial waves in terms of the isospin amplitudes defined above, we finally obtain for the $\pi^+\pi^-$ cross-section

$$\begin{aligned} \sigma = & \sigma_{acc} + \frac{\sqrt{1-x}}{128\pi^2 s} \left[h_{00}^0 \left\{ \frac{4}{9}(g_{00}^0)^2 + \frac{1}{9}(g_{00}^2)^2 + \frac{4}{9}g_{00}^0 g_{00}^2 \cos(\delta_0^0 - \delta_0^2) - (f_{00})^2 \right\} \right. \\ & \left. + \sum_{L \geq 2, even}^6 2h_{L0}^0 f_{L0} \left\{ \frac{2}{3}(g_{00}^0) \cos \delta_0^0 + \frac{1}{3}(g_{00}^2) \cos \delta_0^2 - f_{00} \right\} \right] \end{aligned} \quad (2.42)$$

Note the sum over angular momentum states was terminated at L=6 since numerically, the cross section converged within the first few angular momentum states.

For the $\pi^o\pi^o$ cross-section, we use the sum over corrected partial wave amplitudes given in Eq. 2.7. Since only the L=0 state contributes significantly to the FSI's, only the $f_{00}^{FSI\pi^o\pi^o}$ partial wave enters into the sum. Since the pions are identical particles, we need to apply a statistical factor of $\frac{1}{2}$ to the cross-section in Eq. 2.7.

Chapter 3

The Multichannel Model Calculations

As described in the last chapter, we began with a model for final state interactions between the pions. The model generates good results for the $\pi^+\pi^-$ cross-section but the corresponding $\gamma\gamma \rightarrow \pi^0\pi^0$ cross-section magnitude is too small. We want to include the $f_0(1370)$ resonance in the calculations because the $f_0(1370)$ resonance is very broad and it participates in low energy $\pi\pi$ scattering. Being part of a triplet state, the $f_0(1370)$ has spin $S = 1$ and this is assumed throughout the calculations. There is a possibility the resonance will enhance the $\pi^0\pi^0$ channel due to the change in the wavefunction and phase shifts. The method we use assumes that asymptotic meson states scatter through resonances using transition potentials. As mentioned before, we introduced the Schrödinger equation for the $q\bar{q}$ resonance and then coupled it to the Schrödinger equation for the $\pi\pi$ eigenstate through a transition potential. This system of coupled differential equations is known as the ‘Multichannel’ model. This model is a non-relativistic method of using matrices to solve coupled Schrödinger equations within the context of the quark model and is based on work presented by Weinstein [11]. We now have a numerical matrix method (based on the standard Numerov method explained later) for solving coupled Schrödinger equations.

In the Multichannel model, each ‘channel’ represents a simultaneous eigenstate for a given center of mass energy, hence a possible final state for a reaction. The model assumes that there is an energy transfer (transition potential) due to the presence of other possible simultaneous eigenstates. It further assumes that this can be represented by adding a potential term in the Hamiltonian resulting in a ‘mixing’ between possible eigenstates, or channels. This gives rise to a coupled system of differential equations. Since reactions can be quite complex, the model can accomodate many channels. For the purposes of this thesis, however, our emphasis will be restricted to two channels. In Weinstein’s paper there is an example of s-wave $K\pi$ scattering through an unstable K_0^* scalar meson resonance. In this chapter, this model is applied to our situation of s-wave $\pi\pi$ scattering through the unstable $f_0(1370)$ resonance. Essentially, the process generates wavefunctions numerically from the coupled equations which can then be used to generate phase shifts. Indeed, the phase shifts generated are in excellent agreement with the non-relativistic Breit-Wigner predictions [11]. Finally, the wavefunctions themselves can be used in generating the corrected cross-sections, showing the effects of adding the resonance into the FSI model.

3.1 Quark Model Approach

In the quark model, we solve for the spectra of mesons using the eigenvalue equation

$$H_R(r)\psi_{R_{q\bar{q}}}(r) = m_R\psi_{R_{q\bar{q}}}(r) \quad (3.1)$$

for each resonance R of the $q\bar{q}$ meson where m_R is the mass of the resonance and

$$H_R(r) = \left[-\frac{\hbar^2}{2\mu} \nabla \cdot \nabla \right] + m_q + m_{\bar{q}} + V_{R_{q\bar{q}}}(r) \quad (3.2)$$

and $V_{R_{q\bar{q}}}(r)$ is the $q\bar{q}$ potential. Using spherical harmonics we can rewrite the wavefunction

$$\psi_{R_{q\bar{q}}}(r) = \frac{u_{R_{q\bar{q}}}}{r} Y_{LM}(\theta, \phi) \quad (3.3)$$

and the radial wavefunction $u_{R_{q\bar{q}}}$ then solves the equivalent eigenvalue equation as Eq. 3.1 above for the radial part of the Hamiltonian

$$\begin{aligned} H_R(r) &= -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + m_q + m_{\bar{q}} + V_{R_{q\bar{q}}}(r) \\ &= -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \mathcal{V}_{eff} \end{aligned} \quad (3.4)$$

The problem with this approach is that it assumes the resonances are stable (with infinite lifetimes) so that by the Heisenberg uncertainty principle the mass resonance, the mass eigenvalue, is a fixed value, *i.e.* it has no width. Clearly, since many mesons have a measurable resonance width around 200 MeV, this simple approach cannot be expected to make accurate predictions regarding the spectra of all unstable mesons. To solve the problem we need to consider a process that allows for the transition between meson-meson and unstable resonance states. Weinstein's multichannel model describes this process.

3.2 The Two-Channel Model: Simultaneous Eigenfunction Equations

Consider the reaction $AB \rightarrow R \rightarrow AB$ for two meson scattering through a resonance R back into the original products. We obtain the relative wavefunction of the AB system by solving the Schrödinger equation

$$H_{AB}(r_{AB})u_{AB}(r_{AB}) = E_{AB}u_{AB}(r_{AB}) \quad (3.5)$$

and likewise for the R resonance, a $q\bar{q}$ system,

$$H_{R_{q\bar{q}}}(r_{q\bar{q}})u_{R_{q\bar{q}}}(r_{q\bar{q}}) = E_{R_{q\bar{q}}}u_{R_{q\bar{q}}}(r_{q\bar{q}}) \quad (3.6)$$

Since these are states transforming one to the other, we assume that some of the energy available to an AB eigenstate is transferred to the R eigenstates through a potential $V_{AB \rightarrow R}$ so that there is a net loss of energy in the Schrödinger equation; denoted as

$$H_{AB}(r_{AB})u_{AB}(r_{AB}) = E_{AB}u_{AB}(r_{AB}) - V_{AB \rightarrow R}u_{R_{q\bar{q}}}(r_{q\bar{q}}) \quad (3.7)$$

Similarly, for the resonance structure,

$$H_{R_{q\bar{q}}}(r_{q\bar{q}})u_{R_{q\bar{q}}}(r_{q\bar{q}}) = E_{AB}u_{R_{q\bar{q}}}(r_{q\bar{q}}) - V_{R \rightarrow AB}u_{AB}(r_{AB}) \quad (3.8)$$

where $E_{R_{q\bar{q}}} \equiv E_{AB}$ is the centre of mass energy available in the system (which must exceed m_A and m_B) and by time reversal invariance, we require $V_{R \rightarrow AB} \equiv V_{AB \rightarrow R}$. These coupled equations can be written in matrix form

$$\begin{bmatrix} H_{AB}(r_{AB}) & V_{AB \rightarrow R}(r_{AB}) \\ V_{AB \rightarrow R}(r_{AB}) & H_{R_{q\bar{q}}}(r_{q\bar{q}}) \end{bmatrix} \begin{bmatrix} u_{AB}(r_{AB}) \\ u_{R_{q\bar{q}}}(r_{q\bar{q}}) \end{bmatrix} = E_{AB} \begin{bmatrix} u_{AB}(r_{AB}) \\ u_{R_{q\bar{q}}}(r_{q\bar{q}}) \end{bmatrix} \quad (3.9)$$

It is a straightforward matter to solve for the eigenfunctions $u_{AB}(r_{AB})$ and $u_{R_{q\bar{q}}}(r_{q\bar{q}})$ numerically using standard Numerov techniques recast in matrix form and this will give us information needed to compute the phase shifts and cross sections. From the phase shift information we obtain the spectra of the mesons and we will show they have non-zero resonant widths which, in essence, is a prediction of their instability. Hence, the coupled channel model solves the problem of fixed mass eigenvalues.

3.3 Numerov Techniques: Solving for Simultaneous Eigenfunctions

To solve Eq. 3.9 using standard Numerov techniques, we start by simplifying our notation: take the geometric relation $2r_{AB} \equiv r_{q\bar{q}}$ as shown in Fig. 3.1, rescale and drop the $q\bar{q}$ subscript $u_{R_{q\bar{q}}}(2r_{AB}) \equiv u_R(r_{AB})$ and further drop the subscript AB so that $r_{AB} = r$ and $E_{AB} = E$. This gives

$$\begin{bmatrix} H_{AB}(r) & V_{AB \rightarrow R}(r) \\ V_{AB \rightarrow R}(r) & H_{R_{q\bar{q}}}(2r) \end{bmatrix} \begin{bmatrix} u_{AB}(r) \\ u_R(r) \end{bmatrix} = E \begin{bmatrix} u_{AB}(r) \\ u_{R_{q\bar{q}}}(r) \end{bmatrix} \quad (3.10)$$

Figure 3.1: The resonance R has interquark separation $r_{q\bar{q}}$. In the decay of the R , a $q\bar{q}$ pair emerges from the vacuum. These quarks bind with their corresponding antiparticle constituents to form mesons A and B at a separation of r_{AB} . If the masses of the quarks in the resonance are equal (as they are in the $f_0(1370)$ resonance) then the mesons A and B are positioned at the centers of the lines joining the quarks which correspond to the centers of mass. Hence $r_{q\bar{q}} = 2r_{AB}$.

To write this as a matrix equation, we define the 2×2 matrix

$$\mathcal{H}(r) = \begin{bmatrix} H_{AB}(r) & V_{AB \rightarrow R}(r) \\ V_{AB \rightarrow R}(r) & H_{R_{q\bar{q}}}(2r) \end{bmatrix} \quad (3.11)$$

and a radial wavefunction vector

$$\mathbf{u}(r) \equiv \begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix} \equiv \begin{bmatrix} u_{AB}(r) \\ u_{R_{q\bar{q}}}(2r) \end{bmatrix} \equiv \begin{bmatrix} u_{AB}(r) \\ u_R(r) \end{bmatrix} \quad (3.12)$$

We then obtain for Eq. 3.10

$$\mathcal{H}(r)\mathbf{u}(r) = E\mathbf{u}(r) \quad (3.13)$$

Note that the $\mathcal{H}(r)$ term has the kinetic energy terms on the diagonal so we can separate these terms from $\mathcal{H}(r)$ and obtain, after rearranging,

$$\mathbf{u}''(r) = \frac{2\mu}{\hbar^2}(\mathcal{V}_{eff} - E)\mathbf{u}(r) \quad (3.14)$$

where $u''(r)$ is the second derivative with respect to r . The operator on the right hand side of Eq. 3.14 which includes the potential and energy terms is defined by Weinstein as the ‘potential’ function $\mathcal{W}(r)$:

$$u''(r) = \mathcal{W}(r)u(r) \quad (3.15)$$

Now, to solve numerically, we label the radial steps by r_i and define our initial increment $\epsilon = r_i - r_{i-1}$. A Taylor expansion of the wavefunctions $u(r_{i+1})$ and $u(r_{i-1})$ about r_i yields

$$u(r_{i+1}) = u(r_i) + \epsilon u'(r_i) + \frac{1}{2}\epsilon^2 u''(r_i) + \frac{1}{6}\epsilon^3 u'''(r_i) + \frac{1}{24}\epsilon^4 u''''(r_i) + \dots \quad (3.16)$$

$$u(r_{i-1}) = u(r_i) - \epsilon u'(r_i) + \frac{1}{2}\epsilon^2 u''(r_i) - \frac{1}{6}\epsilon^3 u'''(r_i) + \frac{1}{24}\epsilon^4 u''''(r_i) - \dots \quad (3.17)$$

and the sum yields

$$u(r_{i+1}) + u(r_{i-1}) = 2u(r_i) + \epsilon^2 u''(r_i) + \frac{1}{12}\epsilon^4 u''''(r_i) + \mathcal{O}(\epsilon^6) \quad (3.18)$$

We now define the Numerov vector $z(r_i)$ as

$$z(r_i) \equiv u(r_i) - \frac{1}{12}\epsilon^2 u''(r_i) \quad (3.19)$$

then using Eq. 3.15 it can be rewritten as

$$z(r_i) = (1 - \frac{1}{12}\epsilon^2 \mathcal{W}(r_i))u(r_i) \quad (3.20)$$

To obtain an iterative method, we take the sum

$$z(r_{i+1}) + z(r_{i-1}) = (u(r_{i+1}) + u(r_{i-1})) - \frac{1}{12}\epsilon^2 (u''(r_{i+1}) + u''(r_{i-1})) \quad (3.21)$$

then make the substitution from Eq. 3.18

$$\begin{aligned}
z(r_{i+1}) + z(r_{i-1}) &= 2u(r_i) + \epsilon^2 u''(r_i) + \frac{1}{12} \epsilon^4 u''''(r_i) + \mathcal{O}(\epsilon^6) \\
&\quad - \frac{1}{12} \epsilon^2 \left(2u''(r_i) + \epsilon^2 u''''(r_i) + \frac{1}{12} \epsilon^4 u''''''(r_i) + \mathcal{O}(\epsilon^6) \right) \\
&= 2 \left(u(r_i) - \frac{1}{12} \epsilon^2 u''(r_i) \right) + \epsilon^2 u''(r_i) + \mathcal{O}(\epsilon^6) \\
&= 2z(r_i) + \epsilon^2 u''(r_i) + \mathcal{O}(\epsilon^6) \\
&= 2z(r_i) + \epsilon^2 \mathcal{W}(r_i)(u(r_i) + \mathcal{O}(\epsilon^6))
\end{aligned} \tag{3.22}$$

where, in the last line, we have substituted Eq. 3.15 for $u''(r_i)$. This gives us an iterative form for the Numerov function up to order $\mathcal{O}(\epsilon^6)$ which is very accurate for small radial step sizes. Finally,

$$z(r_{i+1}) = 2z(r_i) + \epsilon^2 \mathcal{W}(r_i)u(r_i) - z(r_{i-1}) \tag{3.23}$$

This relation can then be substituted into the inverted form of Eq. 3.20

$$u(r_{i+1}) = (\mathcal{I}_2 - \frac{1}{12} \epsilon^2 \mathcal{W}(r_{i+1}))^{-1} z(r_{i+1}) \tag{3.24}$$

and we have an iterative method for obtaining the wavefunctions given by the vector $u(r_{i+1})$.

3.4 An Application to the Reaction $\pi\pi \leftrightarrow f_0(1370)$

The $f_0(1370)$ resonance has a very broad width which we wish to include in the FSI model of the previous chapter. We will consider the 2-channel model applied to $\pi\pi \leftrightarrow f_0(1370)$. In this reaction, the $\pi\pi$ system has total angular momentum $J = 0$ and so the resulting resonance must have $J = 0$. Since the $q\bar{q}$ system has spin one because it is a member of a triplet state, the $q\bar{q}$ orbital angular momentum must also be one to couple with the spin to give $J = 0$. The $\pi\pi$ system, by charge conjugation invariance must have an even orbital angular momentum, which implies by Bose-Einstein symmetry it must have an even isospin state. The isospin of the u, d quarks is $\frac{1}{2}$ which can only combine into an isospin 0 or 1 state. By isospin conservation,

then, it can only combine into the isospin 0 state and so the isospin 2 state cannot be coupled to the f_0 resonance. These quantum numbers must be taken into account when evaluating the Hamiltonians.

In this chapter, we take the $\pi\pi$ system as one channel and the $f_0(1370)$ as the other. We use transition potentials derived from the 3P_0 model with harmonic oscillator wavefunctions to compute the ‘potential’ matrix and then solve for the physical solutions (those that converge asymptotically to either trigonometric functions or exponentially decaying functions); see Swanson [15]. Subsequently, we substitute the physical isospin dependent wavefunctions generated by this two-channel model into the isospin amplitude computation in Eq. 2.34 to obtain the cross-section in Eq. 2.42.

3.4.1 The ‘Potential’ Function $\mathcal{W}(r)$ and Potentials

We need the explicit forms of the Hamiltonian as defined in Eq. 3.11:

$$\mathcal{H}(r) = \begin{bmatrix} H_{\pi\pi}(r) & V_{\pi\pi \rightarrow R}(r) \\ V_{\pi\pi \rightarrow R}(r) & H_{R_{q\bar{q}}}(2r) \end{bmatrix} \quad (3.26)$$

Using the quark model we have for the diagonal Hamiltonians

$$H_{\pi\pi}(r) = -\frac{\hbar^2}{2\mu_{\pi\pi}} \nabla \cdot \nabla + V_{\pi\pi \leftrightarrow \pi\pi}(r) + m_\pi + m_\pi \quad (3.27)$$

where $m_\pi = 133$ MeV, the $V_{\pi\pi \leftrightarrow \pi\pi}$ interaction potentials are given by the FSI’s defined in the previous chapter in Table 2.1, and

$$H_{R_{q\bar{q}}}(2r) = -\frac{\hbar^2}{2\mu_{q\bar{q}}} \nabla \cdot \nabla + V_{R_{q\bar{q}}}(2r) + m_q + m_{\bar{q}} \quad (3.28)$$

where the quark-antiquark potential, as given by Weinstein [11], includes the Coulomb potential and the colour magnetic hyperfine interaction from one-gluon exchange

$$\begin{aligned} V_{R_{q\bar{q}}}(r_{q\bar{q}} = 2r) = & -\left[\frac{3}{4}br_{q\bar{q}} - \frac{\alpha_c}{r_{q\bar{q}}} + \frac{3}{4}C \right] \mathbf{F}_q \cdot \mathbf{F}_{\bar{q}} \\ & - \frac{8\pi}{3} \frac{\alpha_h}{m_q m_{\bar{q}}} \frac{\sigma_h^3}{\pi^{3/2}} e^{-\sigma_h^2 r_{q\bar{q}}^2} \mathbf{S}_q \cdot \mathbf{S}_{\bar{q}} \mathbf{F}_q \cdot \mathbf{F}_{\bar{q}} \end{aligned} \quad (3.29)$$

The expectation value of the colour operator interaction $\mathbf{F}_q \cdot \mathbf{F}_{\bar{q}}$ is $-4/3$ for mesons and that of the spin-spin interaction $\mathbf{S}_q \cdot \mathbf{S}_{\bar{q}}$ is $1/4$ for the spin one $q\bar{q}$ system. The parameters used in the potentials, given by Weinstein [11] from fitting the spectra to the known masses of mesons, are $b = 0.154 \text{ GeV}^2$, $\alpha_c = 0.857$, $\alpha_h = 0.840$, $\sigma_h = 700 \text{ MeV}$, $m_q = m_{\bar{q}} = 375 \text{ MeV}$. The constant C is determined by adjusting the resonance peak to the accepted value 1370 MeV ; we found the value at $C = -431 \text{ MeV}$.

In spherical polar coordinates, the radial part of the Laplacian is

$$\nabla \cdot \nabla \equiv \left(\frac{d^2}{dr^2} - \frac{l_{\pi\pi}(l_{\pi\pi} + 1)}{r^2} \right) \quad (3.30)$$

and note that with the change of variable $r_{q\bar{q}} = 2r$ there is a factor of $1/4$ to be carried with the $\hbar^2/2\mu_{q\bar{q}}$ coefficient of $\frac{d^2}{dr^2}$.

Next, we require the off-diagonal component, $V_{\pi\pi \leftrightarrow R}(r)$. This was given by Swanson [15] as

$$V_{\pi\pi \leftrightarrow R}(r) = A\beta \left(\frac{1}{\beta r} - 8\beta r \right) e^{-(\beta r)^2} \quad (3.31)$$

where $A = 0.5$ is a parameter adjusted to give the experimental resonance width of 500 MeV and $\beta = 400 \text{ MeV}$ is the standard quark model oscillator parameter.

In setting up the ‘potential’ function $\mathcal{W}(r)$, we obtain from Eqs. 3.14 and 3.15

$$\begin{aligned} \mathcal{W}(r) &= \frac{2\mu}{\hbar^2} (\mathcal{V}_{eff} - E) \mathcal{I}_2 \\ &= \begin{bmatrix} W_{11}(r) & W_{12}(r) \\ W_{21}(r) & W_{22}(r) \end{bmatrix} \end{aligned} \quad (3.32)$$

where channel 1 is the $\pi\pi$ system and channel 2 is for the $q\bar{q}$ resonance R . Combining Eqs. 3.27, 3.28 and 3.30 the explicit elements of $\mathcal{W}(r)$ are:

$$W_{11}(r) = \frac{2\mu_{\pi\pi}}{\hbar^2} \left(\frac{\hbar^2}{2\mu_{\pi\pi}} \frac{l_{\pi\pi}(l_{\pi\pi} + 1)}{r^2} + V_{\pi\pi \rightarrow \pi\pi} \right) - k_f^2 \quad (3.33)$$

$$W_{12}(r) = \frac{2\mu_{\pi\pi}}{\hbar^2} V_{\pi\pi \rightarrow R} \quad (3.34)$$

$$W_{21}(r) = \frac{2\mu_{q\bar{q}}}{\hbar^2} 4V_{\pi\pi \rightarrow R} \quad (3.35)$$

$$W_{22}(r) = \frac{2\mu_{q\bar{q}}}{\hbar^2} 4 \left(\frac{\hbar^2}{2\mu_{q\bar{q}}} \frac{1}{4} \frac{l_{\pi\pi}(l_{\pi\pi} + 1)}{r^2} + V_{R_{q\bar{q}}} \right) - 4 \frac{\mu_{q\bar{q}}}{\mu_{\pi\pi}} k_f^2 - 8\mu_{q\bar{q}}(2m_\pi - m_q - m_{\bar{q}}) \quad (3.36)$$

The potentials used in $\mathcal{W}(r)$ are shown in Fig. 3.2 which is blown up near the origin to capture the behaviour of the $\pi\pi$ potentials. Note that at the origin, $V_{\pi\pi \rightarrow R}$ is close to 30,000 MeV and $V_{R_{q\bar{q}}}$ is close to -30,000 MeV. Also note that beyond 0.005 MeV^{-1} the

Figure 3.2: These are the potentials used in the potential function $\mathcal{W}(r)$, where $V_{\pi\pi \rightarrow \pi\pi}$ is the $\pi\pi$ potential, $V_{\pi\pi \rightarrow R}$ is the $\pi\pi - f_0$ potential, and $V_{R_{q\bar{q}}}$ is the $q - q$ potential given in the text.

quark-antiquark potential is essentially the only non-zero potential.

3.4.2 Physical Solutions

We must now solve for the physically allowed solutions which satisfy asymptotic conditions, namely that the resonance wavefunction must vanish for large $r_{q\bar{q}}$. Define r_c as the classical turning point at which the $q\bar{q}$ kinetic energy, given by Eq. 3.38 below, vanishes. For $r > r_c$, in the classically forbidden region, the general form of the solution for the vector $u(r)$ is given by

$$u(r) = \begin{bmatrix} D_1 \sin(k_1 r) + G_1 \cos(k_1 r) \\ D_2 e^{-y_2(r)/\sqrt{\kappa_2(r)}} + G_2 e^{+y_2(r)/\sqrt{\kappa_2(r)}} \end{bmatrix} \quad (3.36)$$

where we define the kinetic energy terms

$$\frac{\hbar^2 k_1^2}{2\mu_{\pi\pi}} = E - m_\pi - m_\pi \Rightarrow k_1 = \sqrt{\frac{2\mu_{\pi\pi}}{\hbar^2} (E - m_\pi - m_\pi)} \quad (3.37)$$

$$-\frac{\hbar^2 \kappa_2^2}{2\mu_{q\bar{q}}} = E - m_q - m_{\bar{q}} - C - br \Rightarrow \kappa_2 = \sqrt{-\frac{2\mu_{q\bar{q}}}{\hbar^2} (E - m_q - m_{\bar{q}} - C - br)} \quad (3.38)$$

and the exponent

$$y_2(r) = \frac{2}{3b} \sqrt{-\frac{2\mu_{q\bar{q}}}{\hbar^2} (E - m_q - m_{\bar{q}} - C - br)^3} \quad (3.39)$$

The top solution represents the two free pions and the bottom solution represents the WKB approximation to the Schrödinger equation with a pure linear confining potential between the two quarks:

$$-\frac{\hbar^2}{2\mu} u''(r) + bru(r) = (E - m_q - m_{\bar{q}} - C)u(r). \quad (3.40)$$

Clearly, in the forbidden region, the G_2 term of Eq. 3.36 cannot contribute to the physical solution since it is exponentially increasing and has no physical meaning. We must develop a method in which this term is not included in the physical solution.

We need to solve for the coefficients given by D and G since they will not only give us the physical solution, they also carry the phase shift information. To do this we evaluate the wavefunction at two points beyond r_c , say $r_b > r_a > r_c$, then

$$u(r_a) = Dd(r_a) + Gg(r_a) \quad (3.41)$$

$$u(r_b) = Dd(r_b) + Gg(r_b) \quad (3.42)$$

where we can substitute either the sine and cosine in the top solution of Eq. 3.36 for $d(r)$ and $g(r)$ or the exponential functions in the bottom solution. In either case, we solve the set of linear equations to obtain

$$\begin{aligned} G &= \frac{u(r_a)d(r_b) - u(r_b)d(r_a)}{g(r_a)d(r_b) - g(r_b)d(r_a)} \\ D &= -\frac{u(r_a)g(r_b) - u(r_b)g(r_a)}{g(r_a)d(r_b) - g(r_b)d(r_a)} \end{aligned} \quad (3.43)$$

Since these depend on the values of the wavefunctions, we use the Numerov technique described above to obtain the wavefunction. All that is required are suitable initial conditions to begin the iteration. If we choose two different initial conditions, then we can solve for the desired solutions as well as the phase shift.

Consider

$$u_1(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u_1(\epsilon) = \begin{bmatrix} \epsilon^{(l_1+1)} \\ 0 \end{bmatrix} \quad (3.44)$$

and

$$u_2(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u_2(\epsilon) = \begin{bmatrix} 0 \\ \epsilon^{(l_2+1)} \end{bmatrix} \quad (3.45)$$

where $l_1 = 0$ and $l_2 = 1$ are the orbital angular momenta of the $\pi\pi$ and $q\bar{q}$ states respectively. The asymptotic solutions for each case are given by Eq. 3.36 and can be written explicitly as

$$u(r) = \begin{bmatrix} D_{11} \sin(k_1 r) + G_{11} \cos(k_1 r) \\ D_{21} e^{-y_2(r)/\sqrt{\kappa_2(r)}} + G_{21} e^{+y_2(r)/\sqrt{\kappa_2(r)}} \end{bmatrix} \quad (3.46)$$

for the first solution (note the second index is the solution number and the first refers to the channel) and for the second solution,

$$\mathbf{u}(r) = \begin{bmatrix} D_{12} \sin(k_1 r) + G_{12} \cos(k_1 r) \\ D_{22} e^{-y_2(r)/\sqrt{\kappa_2(r)}} + G_{22} e^{+y_2(r)/\sqrt{\kappa_2(r)}} \end{bmatrix} \quad (3.47)$$

The indexing above accommodates the definition of a solution matrix

$$\mathcal{U}(r) \equiv [\mathbf{u}_1(r), \mathbf{u}_2(r)] \quad (3.48)$$

remembering that the left column as opposed to the top row refers to solution 1. Explicitly,

$$\begin{aligned} \mathcal{U}(r) &= \begin{bmatrix} \sin(k_1 r) & 0 \\ 0 & e^{-y_2(r)/\sqrt{\kappa_2(r)}} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \\ &+ \begin{bmatrix} \cos(k_1 r) & 0 \\ 0 & e^{+y_2(r)/\sqrt{\kappa_2(r)}} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \end{aligned} \quad (3.49)$$

We define the matrices $\mathcal{D} = [D_{ij}]$, $\mathcal{G} = [G_{ij}]$ and $\mathcal{C} \equiv \mathcal{G}^{-1}$ where i refers to the channel and j refers to the solution of the j^{th} initial condition. By Eq. 3.13 we have

$$(\mathcal{H}(r) - E\mathcal{I}_2)\mathcal{U} = 0. \quad (3.50)$$

To pick out the physical solutions, we want to take a linear combination that excludes the exponentially growing terms. To do this, we right multiply the above equation by $\mathcal{C}$ and note that $\mathcal{U}\mathcal{C}$ is still a solution to the equation whose first column is, up to some normalization, the solution we want. To ease notation, define

$$\mathcal{F} = \mathcal{D}\mathcal{C} = [F_{ij}], \quad (3.51)$$

where

$$F_{ij} = \sum_{k=1}^2 D_{ik} C_{kj}. \quad (3.52)$$

Then the physical solution is

$$u(r)|_{r>r_c} = [\mathcal{U}(r)|_{r>r_c} C]_1 = N \begin{bmatrix} F_{11} \sin(k_1 r) + \cos(k_1 r) \\ F_{21} e^{-y_2(r)} / \sqrt{\kappa_2(r)} \end{bmatrix} \quad (3.53)$$

up to a normalization constant which is now necessarily dependent on $\mathcal{C}$. The top solution representing the $\pi\pi$ wavefunction is the one used in computing the isospin amplitudes for the cross-section.

3.4.3 Obtaining Phase Shifts: Two Channel Model and Breit-Wigner

The phase shift due to $q\bar{q}$ resonance formation and decay is obtained from Eq. 3.53 by normalizing the pion system to unit amplitude per unit length such that

$$u(r)|_{r>r_c} = \begin{bmatrix} \sin(k_1 r + \delta) \\ N_R e^{-y_2(r)} / \sqrt{\kappa_2(r)} \end{bmatrix} \quad (3.54)$$

where

$$N_R = \frac{F_{21}}{\sqrt{F_{11}^2 + 1}} \quad (3.55)$$

and

$$\delta = \arctan \frac{1}{F_{11}}. \quad (3.56)$$

The mass of the resonance is obtained when the phase shift passes through 90° and the width is taken as the energy interval between 45° and 135° .

The non-relativistic Breit-Wigner phase shifts for the resonance is given by

$$\delta^{BW} = \arctan \left(\frac{\frac{1}{2}\Gamma_R}{M_R - E} \right) \quad (3.57)$$

where Γ_R is the width and M_R is the mass of the resonance taken from experimental measures. The phase shifts generated by Weinstein's model and from the Breit-Wigner are numerically equivalent.

3.4.4 Phase Shifts Input to the Model

The final step in computing the corrected cross-section is to obtain the phase shifts that go into Eq. 2.42 and there are a number of legitimate choices. First of all, we have the phase shifts generated from Weinstein's model. The phase shifts when the resonance width is zero, given by $A = 0$, are different from those when the width is set to 500 MeV given by $A = 0.5$. Secondly, since our theory is based on the quark model, we can also use the phase shifts calculated from the quark model. For $I = 0$ states we use those given by Li *et al.* [14],

$$\delta_{l=even}^{f=0} = \frac{\alpha_s}{9m_q^2} k E_\pi \left[e^{-x} i_l(x) + \frac{8}{\sqrt{27}} e^{-4x/3} \delta_{l,0} \right] - \frac{\alpha_s}{2m_q^2} k E_\pi e^{-x} i_l(x) \quad (3.58)$$

with $\alpha_s = 0.6$, $m_q = 0.33$ GeV, $x = k^2/4\beta_\pi^2$, $\beta_\pi = 0.337$ GeV, $i_0 = \sinh(x)/x$ and $\delta_{l,0}$ is the Kronecker δ (not to be confused with any phase shift). The first term comes from t-channel gluon exchange and the second from s-channel gluon exchange. For $I = 2$ states, we use those from Barnes & Swanson [16]

$$\sin \delta_{l=0}^{f=2} = - \left\{ \frac{\alpha_s}{9\lambda m_q^2} \sqrt{\frac{s}{\xi}} \left(1 - e^{-\lambda\xi/2} + \frac{4\lambda\xi}{\sqrt{27}} e^{-\lambda\xi/3} \right) \right\} \quad (3.59)$$

$$\sin \delta_{l \geq 2, even}^{f=2} = - \left\{ \frac{\alpha_s}{18m_q^2} \sqrt{s\xi} e^{-\lambda\xi/4} i_l(\lambda\xi/4) \right\} \quad (3.60)$$

with parameters defined similarly as above and $\lambda = 2.20$ GeV⁻², $\xi = s - 4M^2$, $M = 0.138$ GeV and s is the center of mass energy. The last option available for generating phase shifts to be used in the cross-section calculations is to take a linear fit to the experimental data. The equations used are given by Blundell [2]:

$$I = 0 : \delta_0^0 = 0.0027 \text{rad/MeV} \times (\sqrt{s} - 250 \text{MeV}) \quad (3.61)$$

$$I = 2 : \delta_0^2 = -0.00062 \text{rad/MeV} \times (\sqrt{s} - 250 \text{MeV}) \quad (3.62)$$

So, altogether we identify four cases for generating the isospin 0 phase shifts to be used in the cross-section calculations:

- Case 1: from the FSI-Multichannel model with zero resonance width, given by $A = 0$
- Case 2: from the FSI-Multichannel model with resonance width 500 MeV, given by $A = 0.5$
- Case 3: from the quark model
- Case 4: from the straight-line fit to experimental data

Chapter 4

Results

There were two essential calculations completed in this thesis. The first was the 2-channel model for $\pi\pi$ scattering through the $f_0(1370)$ resonance which is presented in the first section of this chapter. The second was the combined FSI-Multichannel program to determine the contribution to the $\pi^0\pi^0$ cross section due to this resonance which is presented in the second section.

4.1 $\pi\pi$ Scattering Through the $f_0(1370)$ Resonance

We studied a 2-channel model for $\pi\pi$ scattering via the $f_0(1370)$ resonance using the model as described in Chapter 3. We found that the line shape of the resonance was dependent on the $\pi\pi$ potential. Briefly, the resonance with $V_{\pi\pi\leftrightarrow\pi\pi} = 0$ was narrower than when the potential was non-zero as shown in Fig. 4.1. Note specifically the broad resonance behaviour, when the $\pi\pi$ potential is turned on, at lower energies where the $\pi\pi$ cross-sections are calculated. There is also a shift in the resonant peak to lower energy when the $\pi\pi$ potential is included. We also show the effects of varying the width parameter A when the $\pi\pi$ potential is non-zero. As A increases from zero, the resonance broadens and the peak shifts downwards as shown in Fig. 4.2.

Figure 4.1: The $f_0(1370)$ resonance with and without the $V_{\pi\pi\leftrightarrow\pi\pi}$ potential. When the $\pi\pi$ potential is on, $A = 0.5$, otherwise $A = 0$

4.2 Contribution to the $\pi^0\pi^0$ Cross-Section Due to the $f_0(1370)$ Resonance

We used the 2-channel model from the previous section to generate the wavefunction for the FSI model described in Chapter 2. We refer to this combined program as the FSI-Multichannel model. The results are presented in two stages. We first present the phase shift data generated by the coupled channel calculations, the quark model and the linear fit to experimental data. Then we present the resulting cross-sections using these phase shifts.

Figure 4.2: The $f_0(1370)$ resonance for the values of the $f_0 \leftrightarrow \pi\pi$ coupling strength given by $A = 0, 0.25, 0.5, 0.75$

4.2.1 Phase Shift Data

The phase shifts from the 2-channel $\pi\pi$ scattering found using Weinstein's procedure and those from an asymptotic fit to the $\pi\pi$ wavefunctions were numerically identical and this was used as a measure of the stability of the program. To obtain such agreement, it was found necessary to take $r_b > r_a \gg r_c$, close to the maximum r over which we integrated, a minor modification from Weinstein's procedure. Thus we established that Weinstein's model correctly predicts the asymptotic phase shifts.

The isospin 0 phase shifts for the four cases outlined at the end of the previous chapter are:

Figure 4.3: Isospin = 0 phase shifts from the radial wavefunction when the resonance width is zero (Case 1) and set to 500 MeV (Case 2), from the quark model (Case 3) and from a linear fit (Case 4) to experimental data from Mukhin [17], Rosselet [18], Estabrooks [19] and Protopopescu [20].

Case 1: from the FSI-Multichannel model with zero resonance width, given by $A = 0$

Case 2: from the FSI-Multichannel model with resonance width 500 MeV, given by $A = 0.5$

Case 3: from the quark model

Case 4: from the straight-line fit to experimental data

They are shown in Fig. 4.3. Note that when the resonance width is zero, we reproduce, as we should, the phase shift data that was obtained in the original FSI model developed by Blundell [2]. Since the isospin 2 calculations were entirely decoupled from the f_0

Figure 4.4: Isospin = 2 phase shifts from the radial wavefunction and the quark model with a linear fit to experimental data from Hoogland [21] and Prukop [22].

resonance, the phase shift data produced by the new model again reproduces that of the original FSI model as seen in Fig 4.4.

It is interesting to note that the experimental phase shift data hints at a resonance near to 850 MeV as it passes through 90 degrees. Clearly there is no such simple resonance as the data reflects the contributions from many resonances and the $\pi\pi$ potentials. It implies, however, that on this empirical evidence the $f_0(980)$ may contribute to the cross-section. Since numerically it matters little how the resonance is generated, we attempted to find a more linear representation of the phase shifts being generated by the model by simply playing with the parameters C , which adjusts the location

of the $f_0(1370)$ resonance peak, and A, which gives the $f_0 \leftrightarrow \pi\pi$ coupling strength. Unfortunately, we were not successful in obtaining a lift in the central dip of the phase shift data even as we shifted the resonance peak as low as 850 MeV. Clearly, what is required is a more sophisticated model including other resonance channels such as the $f_0(980)$ and other open channels such as the $K\bar{K}$, $\eta\eta$, etc. (For examples see Weinstein [23, 24].) We anticipate that this would eventually give better agreement of the phase shift data between the model and experiment.

4.2.2 Cross-section Data

Consistent with Blundell's original FSI model, we used a non-relativistic energy-momentum relationship for the potentials in the 'potential function $\mathcal{W}$ ' but used a relativistic relationship globally throughout the rest of the program. This reproduced Blundell's results when the resonance width is set to zero. It was simply a matter of setting the resonance width to a non-zero value to observe the effects of the $f_0(1370)$ resonance on the cross-sections. We illustrate this in Fig. 4.5. Here we use the case with the straight-line fit to experimental data for the $\pi^0\pi^0$ cross-section. As the coupling strength increases, a peak emerges in the low energy range at the expense of the contribution in the middle range. There is clearly a correspondence between these results and the behaviour of the resonance shown in Fig. 4.2. The low energy arm of the resonance below 350 MeV rises quickly with the width. Since the cross-section is dependent on the phase shift, there is a corresponding rise in the low energy peak of the $\pi^0\pi^0$ cross-section as seen in Fig. 4.5.

Using the amplitudes from Eq. 2.34 and the phase shifts described in the previous section, for $\pi^+\pi^-$ we computed the cross-section from Eq. 2.42 and for $\pi^0\pi^0$ we computed the cross-section from Eq. 2.7 directly (as there was no correction for polar detector acceptance). These are shown for the four phase shift cases in Fig. 4.6 and Fig. 4.7 respectively. Whereas we found little dependence in the $\pi^+\pi^-$ cross-section on the phase shift data used to generate it, the $\pi^0\pi^0$ cross-section was highly sensitive to

Figure 4.5: Total cross-section vs. $\sqrt{s}$ for $\gamma\gamma \rightarrow \pi^0\pi^0$ showing FSI-Multichannel predictions as the $f_0 \leftrightarrow \pi\pi$ coupling strength increases from zero. Here for the phase shifts we use the straight-line fit to experimental data to generate the cross-section.

the phase shift data.

The insensitivity of the $\pi^+\pi^-$ cross-section is illustrated in Fig. 4.6 where Cases 2, 3, and 4 all overlap. Case 1, since it had no $\pi\pi - f_0$ coupling, has a lower peak amplitude and corresponds to Blundell's original curves. These predictions show that the presence of the coupling produces a significant enhancement in the S-wave contribution to the $\pi^+\pi^-$ cross-section.

The $\pi^0\pi^0$ cross-section data is extremely sensitive to the phase shift data used because it depends strongly on the $\delta^0 - \delta^2$ phase shift difference. The dip below the

Figure 4.6: Total cross-section vs. $\sqrt{s}$ for the reaction $\gamma\gamma \rightarrow \pi^+\pi^-$ showing scalar QED calculations, FSI-Multichannel predictions corrected for limited polar detector acceptance for various phase shift inputs with data from Behrend *et al.* [6], Boyer *et al.* [5] and Aihara *et al.* [7].

experimental data in the mid energy region (450 - 700 MeV) of the phase shift data generated by the multichannel model, shown in Cases 1 and 2 of Fig. 4.3, is directly related to the low amplitude of the cross-section predictions in that region as given in Cases 1 and 2 in Fig. 4.7. Even though the $\pi^0\pi^0$ cross-section in Case 2, where the $f_0 \leftrightarrow \pi\pi$ coupling is turned on, shows a strong enhancement over Case 1 where the coupling is turned off, the prediction is still too low in the critical region 450 - 700 MeV. Using quark model phase shifts, we obtain no improvement in the $\pi^0\pi^0$ cross-section

Figure 4.7: Total cross-section vs. $\sqrt{s}$ for the reaction $\gamma\gamma \rightarrow \pi^0\pi^0$ showing FSI-Multichannel predictions with data from Bienlein *et al.* [8], Marsiske *et al.* [9] and Edwards *et al.* [10] for full polar detection acceptance. The various cases represent the different phase shift data used in the calculations.

over using the ones generated from the combined FSI-Multichannel program. However, as with Blundell's FSI model, the best results come from the straight line fit to the experimental data. We exceed Blundell's peak of 6.5 nb around 420 MeV by only 1 nb. An undesirable effect of the multichannel program is that we also experience a peak in the very low energy range below 350 MeV which is not supported by experimental data.

Overall, the results of this work are promising but inconclusive. The presence of

alternative channels does enhance the $\pi^0\pi^0$ cross-section although it is still not in agreement with the experimental data. It is anticipated that the model is too simplistic and other resonances must be included before a match between phase shift data generated by the model and that obtained from experiment, will occur. The obvious next step is to include the $f_0(980)$ resonance and other open channels such as $K\overline{K}$ and $\eta\eta$ into the multichannel part of the combined model.

Chapter 5

Conclusions

In this thesis we considered the reaction $\gamma\gamma \rightarrow \pi\pi$ and the final state interaction contributions to the cross-sections with particular attention to the $\pi^0\pi^0$ prediction. We modified the FSI calculation of Blundell to include the contribution of the $f_0(1370)$ resonance. Using the multichannel method of including the $f_0(1370)$ resonance into the final state interactions we found an enhancement only in the low energy range, below 400 MeV for the $\pi^0\pi^0$ cross-section. However, there are two pieces of promising information obtained from the FSI-Multichannel model. Firstly, the phase shift data produced by the combined model is closer to the experimental data than Blundell's previous FSI model. Secondly, just the presence of alternative channels creates a small enhancement of the $\pi^0\pi^0$ cross-section.

We noted that the experimental phase shift data hints at a resonance around 850 MeV as it passes through 90 degrees. Since this does not represent any known resonance, it must be a compilation of many underlying resonances and open channels. Clearly, what is required is a more sophisticated model including channels such as the $f_0(980)$. We anticipate that this would eventually give better agreement with the phase shift data from the model and more accurate cross-section predictions.

Overall, the results of this work are promising but inconclusive. The presence of alternative channels does enhance cross-sections but does not, at this point, give quan-

titative agreement with experiment. We believe that the model is too simplistic and other resonances must be included to give good agreement between phase shift data generated by the model and that obtained from experiment. The obvious next step is to include the $f_0(980)$, $K\overline{K}$, $\eta\eta$ and other channels into the multichannel part of the combined model.

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