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A Moment Method for Statistical Analysis of High Speed VLSI Interconnects

by

Lena Le Li

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Engineering

Ottawa-Carleton Institute for Electrical Engineering

Department of Electronics

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Ottawa, Ontario K1S 5B6

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**A Moment Method for Statistical Analysis of High Speed VLSI
Interconnects**

submitted by Lena Le Li,
in partial fulfillment of the requirements for
the degree of Master of Engineering

Q.J. Zhang, Thesis Supervisor
M.S. Nakhla, Thesis Co-Supervisor
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Abstract

Statistical analysis of VLSI interconnects is important in the manufacturability-driven design of printed circuit boards (PCBs) and multichip modules (MCMs). The conventional Monte-Carlo method for statistical analysis requires highly repetitive circuit simulations. Since interconnects are usually modeled by a network of transmission lines, such statistical analysis is highly CPU intensive.

An efficient approach for statistical analysis based on the statistical moment theory and family of Pearson distributions is presented in this thesis. A high-order sensitivity analysis technique is developed to extend the classical first-order adjoint sensitivity analysis. The technique is used to obtain a high-order response model for the extraction of statistical moment information. Based on statistical moment theory, an explicit analytical relationship between the statistics of the circuit output and that of circuit parameters is derived. The final probability density function of the network responses is then obtained from moment information through a Pearson family of distributions.

This approach is suitable for fast estimation of network statistical analysis; it does not require the Monte-Carlo analysis. The efficiency of the proposed method and the result comparisons between the proposed method and the Monte-Carlo method are validated by several high-speed VLSI interconnect examples.

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List of Symbols

$\mathbf{X}$	variable vector in the lower level
$\mathbf{Y}$	output performance vector in the higher level
$f(x)$	function of variable x
$E[x]$	expected value of x
s	sample value of a function
n	number of variables in $\mathbf{X}$
r	order of the sensitivity
N	highest order in the polynomial
q	order of moments
$\text{pdf}(y)$	probability density function of y
h_q	q th-order moment
μ_q	q th-order moment about mean
σ	standard deviation
$\text{var}(y)$	variance of y
β_1	test of skewness
β_2	test of kurtosis
a_i	coefficients of polynomial
α_i	scaling factor for variable x_i

$\mathbf{Z}$	variable vector scaled from $\mathbf{X}$ space
ρ	correlation coefficient
$\mathbf{C}$	symmetric matrix representing the capacitance per unit length of the transmission line
$\mathbf{L}$	symmetric matrix representing the inductor per unit length of the transmission line
l	length of the transmission line
$\mathbf{R}$	symmetric matrix representing the resistance per unit length of the transmission line
$\mathbf{G}$	symmetric matrix representing the conductance per unit length of the transmission line
$\mathbf{A}_k$	modified nodal admittance matrix for the linear subnetwork k
$\mathbf{b}$	source vector
$\mathbf{Y}_\pi$	modified nodal admittance matrix for the network π
$\mathbf{V}$	a vector of node voltages appended by branch currents for inductors and voltage sources in the s-domain
$\mathbf{D}_k$	selector matrix that maps the terminal currents of subnetwork into the N_π space of network π
$\mathbf{C}_\pi$	constant matrix with entries determined by the lumped linear components in the network π
$\mathbf{G}_\pi$	constant matrix with entries determined by the lumped linear components in the network π
$\mathbf{\Gamma}$	diagonal matrix containing the square-roots of the eigenvalues of the transmission line matrix $\mathbf{Z}_L \mathbf{Y}_L$
γ_i^2	eigenvalues of the transmission line matrix $\mathbf{Z}_L \mathbf{Y}_L$
$\mathbf{S}_v$	a matrix whose columns contain the eigenvectors of the transmission line matrix $\mathbf{Z}_L \mathbf{Y}_L$
N_π	number of elements in the vector $\mathbf{V}$
$\mathbf{U}_\phi$	incidence matrix

Chapter 1

Introduction

1.1 Motivation of Statistical Design in High-Speed VLSI Interconnects

As the signal speed and interconnection density increase, transmission line effects such as signal delay, ringing, ground noise, crosstalk, reflection, false switching, among others [1], of VLSI interconnects become a limiting factor in the design of an overall VLSI system. These interconnect effects exist at several levels: at the printed circuit boards (PCBs), the multichip modules (MCMs), and the system level. Therefore, such signal integrity issues must be dealt with in the design stage because improperly designed interconnects can result in increasing these unwanted interconnect effects.

In a realistic manufacturing process of printed circuit boards and multichip modules, tolerance in interconnect dimensions and material properties leads to variations in critical interconnect performances which in turn reduce the production yield. Therefore, there is an increasing need for a statistical design approach which would address the large number of interconnects and the tolerance from on-chip, interchip, and intermodule connections.

Traditional or deterministic design focuses on finding a set of circuit parameter values to meet all circuit performance specifications. Here, a single point in the space of variables is sought for optimization which best meets a given set of performance specifications (in design) or best matches a given set of response measurements (in modeling). It is usually assumed that this set of values can be accurately realized when the circuit is built. Unfortunately, this assumption is not realistic for mass-produced integrated circuits.

In the manufacturing process of PCBs and MCMs, tolerances in VLSI interconnects and material properties lead to variations in signal delay, crosstalk, ground noise, etc. Such signal integrity issues become critical at high signal speeds where random variations in interconnect performances can have a serious effect on the overall manufacturing yield [1]-[3][6]. Yield is defined as the ratio of the number of circuits that meet specifications to the number of circuits that are manufactured. For example, if an amplifier is designed for a dc gain of $50 \text{ dB} \pm 5 \text{ dB}$, we could expect to see a distribution of actual gain for a population of manufactured circuits — a distribution similar to the one shown in Figure 1.1. The yield is less than 50%.

As we see, the unsatisfactory yield could be caused by tolerances in the circuit manufacturing process. Because of the close correlation between high yield and high profits, IC manufacturers strive to maximize yield. To increase the manufacturing yield from the point of view of a designer in initial stages of circuit design, it is necessary to predict as accurately as possible: (1) the performance results of products; (2) how such a circuit may be redesigned to reduce unwanted effects. Such redesign is called statistical design. The goal of statistical design is to manage the performance uncertainty in terms of the parameter-value uncertainty. Statistical design methods for maximizing the circuit yield have been an active area of research for more than

Figure 1.1: Distribution of performances of manufactured circuits.

two decades. Recently, as the complexity and speed of VLSI circuits continues to increase, statistical design of the interconnects becomes an important part of VLSI system design [1]-[3].

1.2 Computational Issues in Statistical Design

Statistical design methods, such as statistical modeling, analysis and yield optimization, have been studied for many years [7]-[13] and have recently been applied to high-speed VLSI interconnect design [2]-[5]. A statistical design problem can be formulated as an optimization problem such as cost minimization, performance variance minimization, and yield optimization. Solving an optimization problem is an iterative procedure. In each iterative step, statistical analysis is needed to evaluate the statistical properties of the circuit outputs from the statistical properties of the circuit parameters. Techniques involved in this iterative step, e.g., Monte-Carlo anal-

ysis, require highly repetitive circuit simulations and are, therefore, computationally expensive. In general, to determine whether a circuit satisfies a given specification, a circuit simulation must be performed. A circuit simulation, especially of VLSI interconnect networks with multiconductor transmission line models, is computationally expensive.

To address the computational problems in statistical analysis, control variate (CV) Monte-Carlo analysis was developed and has been extensively used. By using CV Monte-Carlo analysis, expensive circuit simulations are replaced by model building and model evaluation. Based on the CV technique, various modeling approaches have been previously proposed: e.g., statistical modeling at the device level [10][14]-[17] and performance modeling [12][18]-[22]. Based on these modeling approaches, several techniques and algorithms for further reducing the computational effort such as parameter screening [17], Maximally Flat Quadratic Interpolation (MFQI) algorithms [20], Group Method of Data Handling (GMDH) algorithms [22], and some combining algorithms [23] have been proposed.

1.3 Thesis Objective

Recently, in the circuit simulation area, moment matching techniques such as Asymptotic Waveform Evaluation [24] [25] were developed and substantially improved network simulation time. Motivated by this idea, we exploit the concept of moments in a different environment: i.e., statistical moments [26]. Although statistical moment is a classical theory, only low order moment information has been previously considered [27]. The potential of full moment information similar to that in [24] [25] has not been exploited.

In this thesis, an analysis method — a moment method instead of the Monte-Carlo analysis — is developed for statistical analysis. First, a more accurate model with high-order derivative terms is used for the output performance modeling. Unlike the response surface and interpolation schemes to determine the coefficients in quadratic approximation model, the proposed method uses the high-order sensitivity information to determine the coefficients of the output performance polynomial expression.

This method having been developed, an analytical and explicit relationship between the statistics of the circuit output and that of circuit parameters is derived to evaluate the statistical moments of the circuit output performance.

Finally, the probability distribution function of circuit responses is obtained from moment information through a Pearson family [26] of distributions by the moment matching technique.

The method is intended not only to be suitable for fast estimation of network statistical performances with arbitrary distributions, but also to provide the probability density function which is very important in statistical design problems.

1.4 Outline of the Thesis

This thesis is organized as follows. Chapter 2 reviews the literature and the existing techniques for CV strategy. Chapter 2 also reviews the basic statistical analysis methods: Monte-Carlo, CV Monte-Carlo, and shadow modeling. Basic moment concepts are also presented.

Chapter 3 describes the principles of the moment method based on the statistical moment theory. The output performance is expressed in a general polynomial

in terms of circuit parameters. An analytical and explicit relationship between the moments of the output and the moments of the design variables is derived. Because of the physical/geometric oriented modeling, a simplified formula is obtained for independent Gaussian distribution parameters.

In Chapter 4, the high-order sensitivity information is applied to determine the coefficients of the polynomial expression by the Taylor series. Two approaches — one based on the exact adjoint sensitivity analysis and the other on perturbation analysis — are used to calculate the high-order sensitivity of both lumped and distributed networks. The concept of first-order exact adjoint sensitivity technique is expanded into high-order sensitivity formulas.

Chapter 5 briefly discusses two methods for distribution determination: i.e., the Gram-Charlier series of type A and the family of Pearson distributions. An analytical method to determine the parameters of the distributions is presented. As an alternative, an optimization problem based on moment matching is formulated to determine the parameters of the probability density function from moments information.

In Chapter 6, five examples are used to illustrate the usefulness and efficiency of the moment method. For each example, the Monte-Carlo method and the proposed method are applied to illustrate that the results (moment values and probability density functions) from both methods are in good agreement and that the moment method is faster than the Monte-Carlo method.

Finally, Chapter 7 describes the conclusions and offers future work and research suggestions.

Chapter 2

Basic Concepts and Literature Review

2.1 Methods for Statistical Analysis

Because of parameter-value uncertainty in the manufacturing process of PCBs and MCMs, statistical design is needed to minimize the variations in expected product performance in the presence of uncertainties and variations in functional parameters.

The basic step in statistical design is the evaluation of statistical properties of circuit output from the statistical properties of the circuit parameters. Starting with some knowledge of the statistical variation of component values resulting from random variations in a manufacturing process, statistical analysis seeks to predict certain aspects of the resulting variation in a circuit performance: for example, to estimate the manufacturing yield. In general, the yield calculation can be accomplished in three ways: the analysis method, the geometry method, and the Monte-Carlo method.

It is important to choose an optimization strategy that requires a minimum number of objective function evaluations. Because the Monte-Carlo method is conceptually simple, relatively easily programmed, and independent of the dimensions of de-

sign parameters, it is employed extensively for statistical analysis. But this statistical method requires highly repetitive circuit simulations and is therefore computationally expensive.

One of the solutions to the CPU intensive problem is to accelerate the convergence of a Monte-Carlo computation or to reduce the required sample size in Monte-Carlo analysis. Any technique with this purpose is classified as a variance-reduction technique. Some research has been reported in the general literature on the Monte-Carlo analysis. J.M. Hammersley and D.C. Handscomb [28] described techniques such as control variate, importance sampling, antithetic variates, and regression method. The control variate (CV) technique is a general statistical method that helps reduce the computational burden of the Monte-Carlo analysis[13].

The CV technique was first applied to electrical circuits by Neill [29]. It was used for circuit yield estimation by Hocevar, Lightner, and Trick [30], and by Soin and Rankin [31]. Soin and Rankin expand on a novel application of CV technique to circuit tolerance analysis problems [32]. The efficiency of these techniques is determined by the cost of building a sufficiently accurate approximation model, referred to as the shadow model [13]. Since the shadow model is simply a surrogate simulator which is computationally more economical than the original circuit simulator, the main challenge with the shadow model method is in identifying a shadow model which follows the original circuit "closely" and yet has negligible computational effort as compared to the original circuit simulator. Approaches — like macro modeling or model simplification and approximation expression of circuit performance — have been developed for the construction of shadow models.

Techniques involving the approximation expression of circuit performance have been developed since the early 1980s. The basic idea of the method is to use a

polynomial as the approximation expression of circuit performance. The coefficients of the polynomial expression can be determined by either response surface, interpolation, or sensitivity analysis.

Response surface models are constructed by computing the coefficients of linear or quadratic polynomials through linear regression from the simulation results [14][33]. This requires the number of simulations to be larger than the number of coefficients to be computed, with the extra degrees of freedom being used to estimate the modeling error. For this method, techniques such as parameter screening [17] or stepwise regression [34] have been used to reduce the number of coefficients.

In contrast to response surface, interpolation models pass exactly through all the sample points, but provide no means for estimating the resulting modeling error. A simple linear interpolation is used in [35] and [36]. A more complex model such as the quadratic model was proposed by Hocevar, Lightner and Trick [30]. To reduce the large number of circuit analyses required for determining the coefficients in quadratic interpolation, the Maximally Flat Quadratic Interpolation (MFQI) [20] and Group Method of Data Handling (GMDH) were proposed. The GMDH is an approximation technique based on the perception principle; it has a neural network architecture. It was first proposed by Ivakhnenko in [22]; since then several modifications to the standard GMDH algorithm have been proposed [37]-[39]. The MFQI technique was proposed by Biernacki and Styblinski in [20] and was modified [21] in 1989. The MFQI technique is described more specifically in Subsection 2.2.4. To overcome the limitations of both algorithms, the method of combining the MFQI algorithm and the GMDH algorithm [23] was proposed by Styblinski and Aftab.

Unfortunately, all the approaches mentioned above need the Monte-Carlo analysis. The goal of these techniques is on the number of simulations in the Monte-Carlo

analysis. Can a different statistical method be used as surrogate for the Monte-Carlo method in statistical analysis? The method of moments mentioned in [27] could be developed for this goal. A very simple case is considered by [27], but a more general case still does not exist and is the subject of this thesis.

2.2 Monte-Carlo Analysis

The Monte-Carlo method is the most reliable technique for the statistical design of electric circuits. It is applicable to any type of circuit (linear or nonlinear), any type of circuit analysis (time domain or frequency domain), and any form of probability distribution. In this section, we outline the Monte-Carlo method, the control variate Monte-Carlo method, and the quadratic modeling.

2.2.1 Basic Monte-Carlo Analysis

The basic Monte-Carlo analysis for estimating statistics, e.g., the output performance y which should be a random variable, would be as follows. Generate M sets of representative values for the component parameters $\mathbf{X}$ by a pseudorandom selection according to the probability density function (pdf) of parameter variables; analyze the circuits with a circuit simulator; obtain M samples of m output performance functions. The quantities which describe the properties of the random variable are determined approximately by means of observations of the random process and the computation of statistical characteristics (see Figure 2.1).

Figure 2.2 illustrates the distribution of the sample points generated in the parameter space in the course of a basic Monte-Carlo analysis for the case in which x_1 is uniformly distributed over its tolerance range, and x_2 is described by a trun-

Figure 2.1: Flow chart of the basic Monte-Carlo procedure [32]

cated Gaussian distribution function. M circuit samples generated in this way are then simulated. By checking whether the specifications are met for the M samples, the manufacturing yield can be calculated from the M samples. Thus, a Monte-Carlo analysis is akin to measurements made on M actually manufactured circuits. Usually, when M is sufficiently large, this yield provides a reasonable estimate of yield.

Figure 2.2: Generation of M circuit samples (M points in parameter space) within the tolerance region. For each circuit, simulation determines whether the circuit passes (●) or fails (○) the customer's specifications [27].

Because the frequency distribution is mathematically analogous to a probability distribution function for a discrete distribution and because it shares the mathematical properties of such a function, the histogram Figure 2.3 of a circuit performance can be generated and be useful to the circuit designer.

Figure 2.3: The histograms of circuit properties of interest from the results of the simulations illustrated in Figure 2.2

2.2.2 Control Variate Monte-Carlo Analysis

Because Monte-Carlo analysis is extremely expensive, many researchers have addressed the computational burden associated with Monte-Carlo method. Techniques that accelerate Monte-Carlo computation are classified as variance-reduction techniques. The control-variate (CV) technique is a general statistical method for variance reduction that helps to reduce the computational burden of Monte-Carlo.

The basic technique of CV Monte-Carlo analysis comprises of two Monte-Carlo analyses. First, analyses of output performances between the main circuit and its approximate model are performed. The Monte-Carlo analysis involves a small number (size m) of samples when the same randomly selected component values are used for both models. Second, a large sample (size M) auxiliary run is performed, when a more accurate estimate of the output performance (e.g., yield) of the shadow

Figure 2.4: Flow chart of the CV Monte-Carlo procedure for yield estimation [32]

model is obtained. Finally, by adding the estimate of the difference in yield of the two models to the accurate estimate of the yield of the shadow model, we obtain the estimate of yield of the main circuit. The overall procedure is illustrated in Figure 2.4.

In [32], the details of the CV technique are described, the expressions for the efficiency of the CV yield estimation over the primitive Monte-Carlo method are presented, and the yield sensitivities and moments of output performances can be estimated by employing the CV method.

2.2.3 Construction of Shadow Models

The CV technique assumes the existence of a model which approximates the functional behavior of the circuit in question. This model may be a circuit, a mathematical model, or a mixture. General suggestions for the construction of shadow models are given in [32]. One approach is to harness the designer's insight into his/her circuit's behavior; building up a functional equivalent as the circuit itself is designed. In some cases, the (desired) behavior may be summarized in a simple circuit model whose elements can be expressed as functions of the statistical variables of the actual circuit. An illustration provided in Figure 2.5, shows the circuit under evaluation (a bipolar integrated band-pass filter) and its shadow model comprising a discrete component circuit. In this case, the integrated circuit was intended as a replacement for the bulkier discrete circuit, and so the choice of shadow model was obvious.

Another approach is to express the performance of each circuit performance as a polynomial. Applying the CV method both in the yield estimation and design centering, Hocevar *et al.* [18] [19] have successfully used a quadratic approximation to the circuit responses. In the next subsection, the quadratic modeling is discussed.

Figure 2.5: (a) Bipolar integrated band-pass filter: a large circuit. (b) Discrete counterpart of bipolar integrated circuit: the shadow model

2.2.4 Quadratic Modeling

Let $\mathbf{X}$ be the vector of circuit elements or parameters, and $\mathbf{Y}$ the vector of circuit responses (performance functions of interest) such as time delay, DC voltages etc.. Let $\mathbf{Y} = \mathbf{f}(\mathbf{X})$ be a circuit function mapping the parameter space into the response space. For each output performance, $y_i = f_i(\mathbf{X})$ can be represented by using the quadratic polynomial

$$f_i(\Phi, \mathbf{X}) = a_0 + \sum_{i=1}^n a_i x_i + \sum_{i=1}^n \sum_{j=1, j \geq i}^n a_{ij} x_i x_j \quad (2.1)$$

where n is the dimensionality of the parameter space. Let

$$\Phi^T = [a_0, a_1, \dots, a_n, a_{11}, a_{12}, \dots, a_{nn}]. \quad (2.2)$$

The total number of elements in this vector is

$$M = 1 + n + \frac{n^2 - n}{2} + n \quad (2.3)$$

To determine Φ , the m base points $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(m)}$ are sampled in the x -space, and the corresponding responses $s(\mathbf{X}^{(1)}), s(\mathbf{X}^{(2)}), \dots, s(\mathbf{X}^{(m)})$ are found by simulation or measurement. Now, the problem is to find Φ such that

$$Q(\Phi, \mathbf{X}^{(k)}) - s(\mathbf{X}^{(k)}) = 0 \quad k = 1, 2, \dots, m \quad (2.4)$$

Clearly, M (i.e., $m=M$) samples are needed to uniquely determine Φ . A linear system

$$\mathbf{Q}\Phi = \mathbf{b} \quad (2.5)$$

can be set and solved to obtain the coefficients $a_0, a_1, \dots, a_n, a_{11}, a_{12}, \dots, a_{nn}$, where $\mathbf{b}^T = [s(\mathbf{X}^{(1)}), s(\mathbf{X}^{(2)}), \dots, s(\mathbf{X}^{(M)})]$. Obviously, this unique quadratic interpolation

technique requires a large number of samples: i.e., the number of circuit simulations equal to $m = \binom{n+2}{2}$. For a circuit with 50 elements, 1326 circuit simulations are needed to create 1326 samples, while the matrix of the linear system requires storage of a 1326×1326 matrix. Clearly, for a large-scale problem, the unique quadratic models do not reduce computational costs effectively. The Maximally Flat Quadratic Interpolation technique (MFQI) first proposed in [20] by Styblinski and Biernacki overcomes this limitation of the traditional approach. MFQI uses $m \leq M$ samples. Thus, the solution of Φ could not be unique: it follows that the optimal one could be selected as a solution. The basic philosophy of the MFQI approach is to search for the "least prejudiced" solution in the sense that the coefficient of the second-order and mixed terms are as small as possible in the least square sense.

Suppose that m ($n+1 < m < M$) samples of $f(\mathbf{X}^{(k)})$ are simulated at some point $\mathbf{X}^{(k)}$, $k = 1, 2, \dots, m$, by some simulator such as HSPICE. These points are called the base points. The MFQI programming problem is

$$\min_{\Phi_2} \|\Phi_2\|^2 \quad (2.6)$$

subject to

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad (2.7)$$

Where

$$\Phi_1 = [a_0, a_1, \dots, a_n]^T, \quad (2.8)$$

and Φ_2 contains all mixed and quadratic term coefficients of Equation (2.1)

$$\Phi_2 = [a_{11}, a_{12}, \dots, a_{nn}]^T. \quad (2.9)$$

The vectors f_1 and f_2 are of dimensions $n + 1$ and $m - (n + 1)$ respectively. Q_{ij} , ($i = 1, 2; j = 1, 2$), are constant matrix determined by samples $X^{(k)}$. The reduced system with variables Φ_2 can be written in the form

$$C\Phi_2 = e \quad (2.10)$$

where

$$C = Q_{22} - Q_{21}Q_{11}^{-1}Q_{12} \quad (2.11)$$

and

$$e = f_2 - Q_{21}Q_{11}^{-1}f_1. \quad (2.12)$$

Then, the problem presented by Equations (2.6)-(2.9) can be represented as:

$$\min_{\Phi_2} \Phi_2^T \Phi_2 \quad (2.13)$$

$$s.t. \quad C\Phi_2 = e. \quad (2.14)$$

The unique solution to the MFQI problem can be obtained as

$$\Phi_2 = C^T(CC^T)^{-1}e \quad (2.15)$$

and

$$\Phi_1 = Q_{11}^{-1}f_1 - Q_{11}^{-1}Q_{12}\Phi_2. \quad (2.16)$$

Both Φ_1 and Φ_2 give the maximally flat quadratic interpolations in the form of Equation (2.1) to $f_i(X)$. The term "maximally flat quadratic approximation" comes from the mechanism of the least squares constraint that forces the second order derivatives to be as small as possible. Once $f_i(X)$ is available, the output performance can be determined from the maximally flat quadratic expression (2.1) without repeated circuit simulations.

2.3 Estimation of Statistical Moments from Sampled Data

The probability distribution of random variables which could be design variables or output performances can be described by a probability density function, histograms, or moment information. The first and second order moments have been used for statistical analysis in [27]. In this section, the estimation of moments from the sample points is described.

Let us suppose that M samples of output performance $y_1, y_2, \dots, y_M$ generated by a circuit simulator are based on the samples of parameters; then the first-order moment (mean value) of random variable y can be estimated by:

$$h_1 = \frac{\sum_{i=1}^M y_i}{M} \quad (2.17)$$

Generally, the q -order moment can be calculated by:

$$h_q = \frac{\sum_{i=1}^M y_i^q}{M} \quad (2.18)$$

and the q -order moments about mean μ_q is:

$$\mu_q = \frac{\sum_{i=1}^M (y_i - h_1)^q}{M}. \quad (2.19)$$

The first-order moment is the mean value of the random variable and the second-order moment gives the variance or standard deviation of the random variable. The third-order and fourth-order moments give the measure of skewness and kurtosis. Skewness, which refers to the symmetry of the distribution, can be denoted by [42]:

$$\beta_1 = \frac{\mu_3}{\mu_2 \sqrt{\mu_2}} \quad (2.20)$$

Kurtosis refers to the flatness of one distribution in relation to the Gaussian distribution. It can be presented in terms of moments μ_4 and μ_2 [42]:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \quad (2.21)$$

Figure 2.6 and Figure 2.7 give the pictorial explanation of skewness and kurtosis. Therefore, the third moment measures the degree of symmetry and the fourth moment measures the degree of flatness for a random variable. Usually, the first to 4th-order moments (i.e., mean value, variance, skewness, and kurtosis) give the major properties of the distributions.

Figure 2.6: The test of skewness gives the degree of symmetry

Figure 2.7: The test of kurtosis gives the degree of flatness

Chapter 3

Moment-Based Statistical Analysis

In this chapter, the statistical analysis problem is formulated and the principle of the moment method is described. Based on the N th-order multidimensional polynomial, an analytical and explicit relationship between the moments of circuit output and that of the circuit variables is derived. In this way, the distribution of the output performance can be obtained by converting the moments to pdf.

3.1 Problem Formulation

Let $x_i, i = 1, 2, \dots, n$, represent the parameters of the VLSI interconnect network: i.e., the transmission line physical/geometrical parameters and the interconnect terminations. Let y represent various signal integrity responses of the network: e.g., the signal propagation delay, crosstalk, ground noise, etc. The relationship between x_i and response y is multidimensional and nonlinear: i.e.,

$$y = f(x_1, x_2, \dots, x_n). \quad (3.1)$$

In statistical analysis, the interconnect parameters $x_1, x_2, \dots, x_n$, are subject to random variations; it is required to determine the statistical properties of the signal integrity response y from that of the x 's.

A complete representation of the statistical properties of y can be its probability density function (pdf). Therefore a complete statistical analysis is to determine the pdf of y from the pdf of the x 's. However, from the nonlinear relationship in (3.1), such a pdf is extremely difficult to obtain unless it is indirectly approximated through the Monte-Carlo analysis.

Thus it is desirable to seek an alternative form to represent statistical information which would make it possible to derive a more direct relationship between the statistical properties of x 's and y 's. In statistical theory, a variety of functions that generate certain aspects of a probability distribution can be defined for a given distribution. These functions are considered as transforms of the probability density function. *Characteristic Function* is one of them. The characteristic function of the distribution of a random variable y is defined by a Fourier transform as [40][41]:

$$\phi(t) = \int_{-\infty}^{\infty} e^{jvt} \text{pdf}(y) dy \quad (3.2)$$

where $\text{pdf}(y)$ is the probability density function of y . From the Fourier transform inversion formula, the pdf of y is given by

$$\text{pdf}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-jty} dt \quad (3.3)$$

It then follows that every pdf and its characteristic function form a unique Fourier transform pair.

The characteristic function can be represented by a series [40]:

$$\phi(t) = 1 + E[y](jt) + E[y^2] \frac{(jt)^2}{2!} + \dots + E[y^k] \frac{(jt)^k}{k!} + \dots \quad (3.4)$$

where $E[y^q]$ is the q th-order moment of random variable y . The q th moment of y is defined as [26, 41, 42]:

$$h_q = E[y^q] = \int_{-\infty}^{\infty} \text{pdf}(y) y^q dy \quad (3.5)$$

The q th moment about the mean is

$$\mu_q = E[(y - E[y])^q] \quad (3.6)$$

A set of moments can determine a distribution uniquely under certain conditions which are obeyed by distributions commonly occurring in statistical practice [40]. For practical purposes, therefore, a knowledge of the moments is equivalent to the knowledge of the distribution functions. The first two moments give the information about the mean $\bar{y} = h_1$ and standard deviation $\sigma = \sqrt{\mu_2}$ or variance $\text{var}(y) = \mu_2$. The third and fourth moments further give the test of skewness $\beta_1 = \mu_3/(\mu_2)^{3/2}$ and test of kurtosis $\beta_2 = \mu_4/\mu_2^2$ [26].

Notice that the moments of interconnect parameters x_i can be computed from their pdfs by using Equation (3.5). The problem now is to compute the statistical moments of signal integrity responses y given the moments of interconnect parameters $x_i, i = 1, 2, \dots, n$. The final pdf of y can then be obtained from the moments $E[y^q]$.

3.2 Moment Computation

The above problem is still generally difficult to solve because the relationship between x 's and y is nonlinear, multidimensional, and generally very complex. In practical problems such as high speed VLSI interconnect analysis this relationship is not in explicit form. To provide an efficient solution, we consider the N th order multidimensional polynomial of y ,

$$y = f(x_1, x_2, \dots, x_n) \\ \approx a_0 + \sum_{r=1}^N \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_r=1}^n a_{i_1, i_2, \dots, i_r} (x_{i_1} - x_{i_1}^0)(x_{i_2} - x_{i_2}^0) \cdots (x_{i_r} - x_{i_r}^0) \quad (3.7)$$

where $x_i^0, i = 1, 2, \dots, n$, is an expansion point of x_i and is equal to the mean value of x_i . Special cases of such a polynomial are quadratic models [21] and Taylor series expansion.

We first scale all variables such that

$$z_i = \alpha_i(x_i - x_i^0), \quad i = 1, 2, \dots, n \quad (3.8)$$

where α_i is a scaling factor. For all $x_1, x_2, \dots, x_n$, the distributions are given; thus all the moments for $x_1, x_2, \dots, x_n$, and $z_1, z_2, \dots, z_n$, can be found. The moments of $z_i, i = 1, 2, \dots, n$, can be found by

$$E[z_i^q] = \alpha_i^q E[(x_i - x_i^0)^q], \quad (3.9)$$

where the q is the order of the moments.

The moments of y can now be computed from that of the z 's. Taking the expected value on both sides of Equation (3.7), the first moment of y can be found as

$$E[y] = a_0 + \sum_{r=1}^N \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_r=1}^n a_{i_1, i_2, \dots, i_r} \frac{E[z_{i_1} z_{i_2} \cdots z_{i_r}]}{\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_r}} \quad (3.10)$$

In order to calculate the higher order moments $E(y^q)$, $q > 1$, we first express the y^q in term of $z_i, i = 1, 2, \dots, n$. For y^2 , we multiply y by y and the expression of y^2 is

$$y^2 = a_0^2 + \sum_{r=1}^N \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_r=1}^n \sum_{j=0}^r a_{i_1, \dots, i_j} a_{i_{j+1}, \dots, i_r} \frac{z_{i_1} z_{i_2} \cdots z_{i_r}}{\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_r}} \quad (3.11)$$

In general, by multiplying the polynomial expression (3.7), the q th power of function y is

$$y^q = a_0^q + \sum_{r=1}^N \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_r=1}^n \sum_{j_1=0}^r \sum_{j_2=0}^{r-j_1} \cdots \sum_{j_{q-1}=0}^{r-\alpha} A_0 \quad (3.12)$$

$$\times a_{i_1, \dots, i_{j_1}} a_{i_{j_1+1}, \dots, i_{j_1+j_2}} \cdots a_{i_{\beta+1}, \dots, i_r} z_{i_1} z_{i_2} \cdots z_{i_r}$$

where A_0 symbolically represents $(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_r})^{-1}$. The α and β symbolically represent $\sum_{k=1}^{q-2} j_k$ and $\sum_{k=1}^{q-1} j_k$ respectively. Taking the expected value on both sides of Equation (3.12), the q th-order moments of function y is

$$E[y^q] = a_0^q + \sum_{r=1}^N \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_r=1}^n \sum_{j_1=0}^r \sum_{j_2=0}^{r-j_1} \cdots \sum_{j_{q-1}=0}^{r-\alpha} A_0 \quad (3.13)$$

$$\times a_{i_1, \dots, i_{j_1}} a_{i_{j_1+1}, \dots, i_{j_1+j_2}} \cdots a_{i_{\beta+1}, \dots, i_r} E[z_{i_1} z_{i_2} \cdots z_{i_r}]$$

Thus an explicit relationship between the moments of signal integrity response y and interconnect network parameters x_i is found. To illustrate Formula (3.13) with an example, we consider $y = f(x_1, x_2)$ and $\alpha_1 = \alpha_2 = 1$. The second-order moment ($q = 2$) of y with $N = 2$ is

$$\begin{aligned} E[y^2] = & a_0^2 + a_0 a_1 E[x_1] + a_1 a_0 E[x_1] + a_0 a_2 E[x_2] + a_2 a_0 E[x_2] \\ & + a_0 a_{11} E[x_1^2] + a_{11} a_0 E[x_1^2] + a_{11} a_0 E[x_1^2] + a_0 a_{12} E[x_1 x_2] \\ & + a_1 a_2 E[x_1 x_2] + a_{12} a_0 E[x_1 x_2] + a_0 a_{21} E[x_2 x_1] + a_2 a_1 E[x_2 x_1] \\ & + a_{21} a_0 E[x_2 x_1] + a_0 a_{22} E[x_2^2] + a_2 a_2 E[x_2^2] + a_{22} a_0 E[x_2^2] \end{aligned}$$

3.3 Physical/Geometrical Oriented Modeling

High-speed interconnect models can be classified into two categories: equivalent circuit and physical/geometrical based models. In physical/geometrical models, the variables $x_1, \dots, x_n$ are design variables such as the conductor length, conductor width, distance between two adjacent conductors, PCB height and dielectric constant. Figure 3.1 shows the parameters of a 2-conductor coupled interconnect configuration. The statistical distributions of these model parameters are typically Gaussian and independent [4][8][43]. In this case, the variables $z_1, z_2, \dots, z_n$, become independent Gaussian random variables with zero mean values. The odd moments for $z_i, \dots, z_n$ are zero: i.e.,

$$E[z_{i_1}^{q_1} z_{i_2}^{q_2} \dots z_{i_k}^{q_k}] = 0 \quad (3.14)$$

if any one of the q 's is an odd number. The computation with Equation (3.10) may be much simplified: e.g. for $N = 4$ and $q = 1$,

$$\begin{aligned} E[y] = & a_0 + \sum_{i=1}^n (a_{ii} + a_i^2) \frac{E[z_i^2]}{\alpha_i^2} + \sum_{i=1}^n \sum_{j=1, j \neq i}^n (a_{ij}^2 + a_{iijj} + a_{ii}a_{jj}) \frac{E[z_i^2]E[z_j^2]}{\alpha_i^2 \alpha_j^2} \\ & + \sum_{i=1}^n (a_{iiii} + a_{ii}^2 + a_i^4) \frac{E[z_i^4]}{\alpha_i^4} \end{aligned} \quad (3.15)$$

Similar simplification can also be achieved even when some variables are correlated because z_i has symmetrical Gaussian distribution. In this case, factor analysis [44] can be used to find a set of independent factors with respect to which the proposed moment computation is simplified.

Figure 3.1: Physical/geometrical parameters of a 2-conductor transmission line.

3.4 High-Order Models Used

Polynomial models, especially quadratic models to represent the nonlinear relationship between y and x 's, have frequently been used in the literature for yield optimization: e.g., [8]-[11][21]. It is noted that our proposed moment computation method can be readily applied to these existing models. However, our method is not constrained to quadratic models. To cover a larger range in the parameter space, we used higher order moments. Therefore, models with orders higher than the quadratic are necessary. Also needed is an efficient technique to determine the coefficients $a_{i_1, i_2, \dots, i_r}$. Because the model building method in [21] is suitable only for up to quadratic models, we use an alternative formulation here. The N th-order multidimensional Taylor expansion of y is

$$\begin{aligned}
 y &= f(x_1, x_2, \dots, x_n) \\
 &= \sum_{r=0}^N \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_r=1}^n \frac{1}{r!} \frac{\partial^r f(x_1^0, x_2^0, \dots, x_n^0)}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_r}} \\
 &\quad \times (x_{i_1} - x_{i_1}^0)(x_{i_2} - x_{i_2}^0) \dots (x_{i_r} - x_{i_r}^0)
 \end{aligned} \tag{3.16}$$

The coefficients in Equation (3.7) can be obtained from the high-order sensitivity information: i.e.,

$$a_{i_1, i_2, \dots, i_r} = \frac{1}{r!} \frac{\partial^r f(x_1^0, x_2^0, \dots, x_n^0)}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_r}} \tag{3.17}$$

Substituting the sensitivities into the Equation (3.13), we have the following formula:

$$E[y^q] = \sum_{r=0}^N \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_r=1}^n \sum_{j_1=0}^r \sum_{j_2=0}^{r-j_1} \cdots \sum_{j_{q-1}=0}^{r-\alpha} \frac{A_0}{j_1!} \frac{1}{j_2!} \cdots \frac{1}{j_{q-1}!} \frac{1}{(r-\beta)!} \\ \frac{\partial^{j_1} f}{\partial x_{i_1} \cdots \partial x_{i_{j_1}}} \frac{\partial^{j_2} f}{\partial x_{i_{j_1+1}} \cdots \partial x_{i_{j_1+j_2}}} \cdots \frac{\partial^{r-\beta} f}{\partial x_{i_{\beta+1}} \cdots \partial x_{i_r}} E[z_{i_1} z_{i_2} \cdots z_{i_r}] \quad (3.18)$$

Sensitivity information coupled with high-order Taylor expansion (3.16) makes our method more general and more efficient. Efficiency of the method is due to sensitivity analysis which can be computationally less expensive than repeated simulations needed for creating regular polynomial models.

Chapter 4

High-Order Sensitivity Analysis

4.1 Introduction

The high order sensitivities $\frac{\partial^r y}{\partial x_i^r}$ required in (3.18) can be obtained by using either exact or approximate sensitivity analysis. In this chapter, we describe higher order sensitivity analysis approaches which are extensions of existing first-order sensitivity techniques.

For linear networks the main computational effort is one LU factorization of the network modified nodal admittance matrix, and several forward/backward substitutions. The LU factorization is for solving the circuit at its nominal point. Different forward/backward substitutions allow the computation of high-order adjoint sensitivities for different variables. In the approximate sensitivity analysis approach, high-order numerical differentiation formulas [53] can be used.

4.2 Formulation of Network Equations

Consider a network π consisting of lumped elements and N_s multiconductor transmission lines. The s -domain equation is

$$\mathbf{Y}_\pi \mathbf{V} = \mathbf{b} \quad (4.1)$$

where $\mathbf{V}$ is the vector of node voltages appended by branch currents for inductors and voltage sources in the s -domain; $\mathbf{b}$ is a source vector; $\mathbf{Y}_\pi$ is the modified nodal admittance matrix [45][46] of the network expressed as

$$\mathbf{Y}_\pi = s\mathbf{C}_\pi + \mathbf{G}_\pi + \sum_{k=1}^{N_s} \mathbf{D}_k \mathbf{A}_k \mathbf{D}_k^T \quad (4.2)$$

Where $\mathbf{C}_\pi$ and $\mathbf{G}_\pi$ are N_π by N_π matrices determined by lumped elements in the network, $\mathbf{A}_k$ is the nodal admittance matrix of the k th distributed transmission line subnetwork and $\mathbf{D}_k$ is an incidence matrix containing 1's and 0's which maps the terminals of the k th transmission line onto the node space of the overall network π . The lossy multiconductor transmission line is assumed to be uniform along its length with an arbitrary cross-section. The cross-section of an N_k -conductor transmission line can be represented by the following $[N_k \times N_k]$ dimensional matrices of line parameter: the resistance per unit length $\mathbf{R}$, the inductance per unit length $\mathbf{L}$, the capacitance per unit length $\mathbf{C}$, and the conductance per unit length $\mathbf{G}$. The matrices can be computed from the physical/geometrical parameters of the transmission line through the quasi-static analysis [47],[48] or empirical formulas [49]. Let γ_j^2 be an eigenvalue of the matrix $\mathbf{Z}_L \mathbf{Y}_L$ with an associated eigenvector $\mathbf{S}_{\nu_j}$, where $\mathbf{Z}_L = \mathbf{R} + s\mathbf{L}$ and $\mathbf{Y}_L = \mathbf{G} + s\mathbf{C}$. The admittance matrix for the multiconductor transmission line is [46]:

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1} \\ \mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1} \end{bmatrix} \quad (4.3)$$

where

$$\mathbf{E}_1 = \text{diag} \left\{ \frac{(1 + e^{-2\gamma_j l})}{(1 - e^{-2\gamma_j l})} \quad j = 1, 2, \dots, N_k \right\}, \quad (4.4)$$

$$\mathbf{E}_2 = \text{diag} \left\{ \frac{2}{(e^{-\gamma_j l} - e^{\gamma_j l})} \quad j = 1, 2, \dots, N_k \right\}, \quad (4.5)$$

and l is the length of the transmission line. $\mathbf{S}_v$ is a matrix containing all eigenvectors $\mathbf{S}_{v,j}$, $j = 1, 2, \dots, N_k$, $\mathbf{\Gamma}$ is a diagonal matrix with $\Gamma_{i,i} = \gamma_i$, and $\mathbf{S}_i = \mathbf{Z}_L^{-1} \mathbf{S}_v \mathbf{\Gamma}$.

4.3 High-Order Exact Adjoint Sensitivity Analysis

The exact adjoint sensitivity method developed in [50] for first-order sensitivity analysis was an important breakthrough in circuit simulation and optimization. Recently, it has been extended to transmission line networks [51]. However, these methods are not adequate in our case because high order sensitivities are needed. The expansion from first order sensitivity to higher order sensitivity for distributed networks is now described.

Suppose a random variable ϕ is any lumped element parameter or a transmission line parameter in the k th distributed transmission line subnetwork. Suppose the circuit output V_{out} is obtained from the solution vector $\mathbf{V}$ as

$$V_{out} = \mathbf{U}_{out}^T \mathbf{V} \quad (4.6)$$

where U_{out} is a N_π -vector. The first order sensitivity is

$$\frac{\partial V_{out}}{\partial \phi} = -\hat{V}^T \frac{\partial Y_\pi}{\partial \phi} V \quad (4.7)$$

where $\hat{V}$ is the solution of the adjoint linear equation:

$$Y_\pi^T \hat{V} = U_{out} \quad (4.8)$$

In our approach, the higher order (i.e., the r th-order) sensitivities to $\phi_1, \phi_2, \dots, \phi_r$ are needed. Therefore, it is necessary to expand the first order sensitivity techniques in [51]. For the general case, the r th-order sensitivity formula is derived as follows. First, differentiating both sides in Equation (4.1) with respect to $\phi_1, \phi_2, \dots, \phi_r$, we have

$$Y_\pi \frac{\partial^r V}{\partial \phi_1 \partial \phi_2 \dots \partial \phi_r} + \sum_{k_1=1}^r \frac{\partial Y_\pi}{\partial \phi_{k_1}} \frac{\partial^{r-1} V}{\partial \phi_{k_2} \partial \phi_{k_3} \dots \partial \phi_{k_r}} = 0 \quad (4.9)$$

where $k_1 \neq k_2 \neq \dots \neq k_r$. The r th-order sensitivity of the vector V to these variables is

$$\frac{\partial^r V}{\partial \phi_1 \partial \phi_2 \dots \partial \phi_r} = -Y_\pi^{-1} \sum_{k_1=1}^r \frac{\partial Y_\pi}{\partial \phi_{k_1}} \frac{\partial^{r-1} V}{\partial \phi_{k_2} \partial \phi_{k_3} \dots \partial \phi_{k_r}} \quad (4.10)$$

In (4.9), we have used a simplifying assumption that if the variables are different, i.e., if ϕ_i and ϕ_j represent different variables for $i \neq j$, then

$$\frac{\partial^r Y_\pi}{\partial \phi_1 \partial \phi_2 \dots \partial \phi_r} = 0, \quad r > 1 \quad (4.11)$$

This condition is true for all lumped components. The high-order sensitivity of V_{out} is

$$\frac{\partial^r V_{out}}{\partial \phi_1 \partial \phi_2 \cdots \partial \phi_r} = -\dot{V}^T \sum_{k_1=1}^r \frac{\partial Y_\pi}{\partial \phi_{k_1}} \frac{\partial^{r-1} V}{\partial \phi_{k_2} \partial \phi_{k_3} \cdots \partial \phi_{k_r}} \quad (4.12)$$

By substituting Equation (4.10) for $\frac{\partial^{r-1} V}{\partial \phi_{k_2} \partial \phi_{k_3} \cdots \partial \phi_{k_r}}$ in Equation (4.12) and by repeatedly replacing the $(r-1)$ th, $(r-2)$ th, ..., 1st derivatives of V , the r th-order sensitivity formula becomes

$$\begin{aligned} \frac{\partial^r V_{out}}{\partial \phi_1 \partial \phi_2 \cdots \partial \phi_r} = (-1)^r & \left[\sum_{k_1=1}^r \sum_{k_2=1, k_2 \neq k_1}^r \cdots \sum_{k_r=1, k_r \neq k_1, \dots, k_{r-1}}^r (\dot{V}^T \frac{\partial Y_\pi}{\partial \phi_{k_1}} \right. \\ & \left. Y_\pi^{-1} \frac{\partial Y_\pi}{\partial \phi_{k_2}} \cdots Y_\pi^{-1} \frac{\partial Y_\pi}{\partial \phi_{k_r}} V) \right] \quad (4.13) \end{aligned}$$

If in the case where $\phi_1 = \phi_2 = \cdots \phi_r = \phi$, the r th-order sensitivity to parameter ϕ is directly derived in the following part. By differentiating both sides of the Equation (4.1) with respect to variable ϕ , for the first-order sensitivity, we have

$$\frac{\partial Y_\pi}{\partial \phi} V + Y_\pi \frac{\partial V}{\partial \phi} = 0 \quad (4.14)$$

Differentiating Equation (4.14) again, for the second sensitivity, we have

$$\frac{\partial^2 Y_\pi}{\partial \phi^2} V + 2 \frac{\partial Y_\pi}{\partial \phi} \frac{\partial V}{\partial \phi} + Y_\pi \frac{\partial^2 V}{\partial \phi^2} = 0 \quad (4.15)$$

For r th-order sensitivity, the corresponding equation is

$$\sum_{k_1=0}^r \binom{r}{k_1} \frac{\partial^{k_1} Y_\pi}{\partial \phi^{k_1}} \frac{\partial^{r-k_1} V}{\partial \phi_{r-k_1}} = 0 \quad (4.16)$$

or

$$\mathbf{Y}_\pi \frac{\partial^r \mathbf{V}}{\partial \phi^r} + \sum_{k_1=1}^r \binom{r}{k_1} \frac{\partial^{k_1} \mathbf{Y}_\pi}{\partial \phi^{k_1}} \frac{\partial^{r-k_1} \mathbf{V}}{\partial \phi_{r-k_1}} = 0 \quad (4.17)$$

The r th-order sensitivity of vector $\mathbf{V}$ can be solved from the above equation,

$$\frac{\partial^r \mathbf{V}}{\partial \phi^r} = -\mathbf{Y}_\pi^{-1} \sum_{k_1=1}^r \binom{r}{k_1} \frac{\partial^{k_1} \mathbf{Y}_\pi}{\partial \phi^{k_1}} \frac{\partial^{r-k_1} \mathbf{V}}{\partial \phi_{r-k_1}} \quad (4.18)$$

Recursively substituting the high-order sensitivity in the right side of Equation (4.18) by lower sensitivity of vector $\mathbf{V}$, the r th-order sensitivity to parameter ϕ becomes

$$\begin{aligned} \frac{\partial^r V_{out}}{\partial \phi^r} &= \sum_{j_1=1}^r \binom{r}{j_1} \sum_{j_2=1}^{r-j_1} \binom{r-j_1}{j_2} \dots \sum_{j_p=1}^1 \binom{r-\sum_{i=1}^{p-1} j_i}{j_p} \\ &\quad (-1)^p \hat{\mathbf{V}}^T \frac{\partial^{j_1} \mathbf{Y}_\pi}{\partial \phi^{j_1}} \mathbf{Y}_\pi^{-1} \frac{\partial^{j_2} \mathbf{Y}_\pi}{\partial \phi^{j_2}} \dots \mathbf{Y}_\pi^{-1} \frac{\partial^{j_p} \mathbf{Y}_\pi}{\partial \phi^{j_p}} \mathbf{V} \end{aligned} \quad (4.19)$$

where $1 \leq p \leq r$.

4.4 Special Cases

The computation of high-order sensitivities can be further simplified for VLSI interconnect analysis where the variable ϕ is either a transmission line parameter or a termination parameter in the form of lumped element R , L or C .

We first define incidence matrix $\mathbf{U}_\phi$ as

$$\mathbf{U}_\phi = \begin{cases} (\mathbf{u}_i - \mathbf{u}_j) & \text{if } \phi \text{ is a lumped parameter between nodes } i \text{ and } j \\ \mathbf{D}_k & \text{if } \phi \text{ is a transmission line parameter in the } k\text{th subnetwork} \end{cases} \quad (4.20)$$

and element level sensitivity $\frac{\partial Y_\phi}{\partial \phi}$ as

$$\frac{\partial Y_\phi}{\partial \phi} = \begin{cases} \frac{\partial Y_b}{\partial \phi} & \text{if } \phi \text{ is a lumped parameter represented by admittance } Y_b \\ \frac{\partial A_k}{\partial \phi} & \text{if } \phi \text{ is a transmission line parameter in the } k\text{th subnetwork} \\ & \text{represented by admittance matrix } A_k \end{cases} \quad (4.21)$$

where u_i is a unit vector with 1 in i th location and 0's everywhere else.

The first order sensitivity can be expressed by:

$$\frac{\partial V_{out}}{\partial \phi} = -\hat{V}^T U_\phi \frac{\partial Y_\phi}{\partial \phi} U_\phi^T V = -\hat{V}_\phi^T \frac{\partial Y_\phi}{\partial \phi} V_\phi \quad (4.22)$$

where $\hat{V}_\phi$ and V_ϕ are interpreted as branch voltages $\hat{V}_\phi = U_\phi^T \hat{V}$ and $V_\phi = U_\phi^T V$.

To use the formula (4.13) for calculating the r th-order sensitivities to $\phi_i, i = 1, 2, \dots, r$, we first solve a set of adjoint equations:

$$\left. \begin{aligned} Y_\pi^T \hat{V}^{\phi_1} &= U_{\phi_1} \\ Y_\pi^T \hat{V}^{\phi_2} &= U_{\phi_2} \\ &\vdots \\ Y_\pi^T \hat{V}^{\phi_r} &= U_{\phi_r} \end{aligned} \right\} \quad (4.23)$$

Because LU factors of Y_π are already available from solving the original circuit of (4.1), only forward/backward substitutions are needed to obtain the solutions $\hat{V}^{\phi_1}, \hat{V}^{\phi_2}, \dots, \hat{V}^{\phi_r}$ from (4.23). Using these adjoint solutions, the formula (4.13) becomes:

$$\frac{\partial^r V_{out}}{\partial \phi_1 \partial \phi_2 \dots \partial \phi_r} = (-1)^r \left[\sum_{k_1=1}^r \sum_{k_2=1, k_2 \neq k_1}^r \dots \sum_{k_r=1, k_r \neq k_1, \dots, k_{r-1}}^r (\hat{V}^T U_{\phi_{k_1}} \frac{\partial Y_{\phi_{k_1}}}{\partial \phi_{k_1}} U_{\phi_{k_1}}^T \right]$$

$$\mathbf{Y}_\pi^{-1} \mathbf{U}_{\phi_{k_2}} \frac{\partial \mathbf{Y}_{\phi_{k_2}}}{\partial \phi_{k_2}} \mathbf{U}_{\phi_{k_2}}^T \cdots \mathbf{Y}_\pi^{-1} \mathbf{U}_{\phi_{k_r}} \frac{\partial \mathbf{Y}_{\phi_{k_r}}}{\partial \phi_{k_r}} \mathbf{U}_{\phi_{k_r}}^T \mathbf{V})] \quad (4.24)$$

which can be further simplified as:

$$\begin{aligned} \frac{\partial^r V_{out}}{\partial \phi_1 \partial \phi_2 \cdots \partial \phi_r} = (-1)^r & \left[\sum_{k_1=1}^r \sum_{k_2=1, k_2 \neq k_1}^r \cdots \sum_{k_r=1, k_r \neq k_1, \dots, k_{r-1}}^r ((\hat{\mathbf{V}}_{\phi_{k_1}})^T \frac{\partial \mathbf{Y}_{\phi_{k_1}}}{\partial \phi_{k_1}} \right. \\ & \left. (\hat{\mathbf{V}}_{\phi_{k_2}})^T \frac{\partial \mathbf{Y}_{\phi_{k_2}}}{\partial \phi_{k_2}} \cdots (\hat{\mathbf{V}}_{\phi_{k_r}})^T \frac{\partial \mathbf{Y}_{\phi_{k_r}}}{\partial \phi_{k_r}} \mathbf{V}_{\phi_{k_r}}) \right] \quad (4.25) \end{aligned}$$

where $\hat{\mathbf{V}}_{\phi_{k_1}}$ and $\mathbf{V}_{\phi_{k_r}}$ are defined as:

$$\begin{aligned} \hat{\mathbf{V}}_{\phi_{k_2}}^{\phi_{k_1}} &= \mathbf{U}_{\phi_{k_2}}^T \hat{\mathbf{V}}^{\phi_{k_1}}; \\ \mathbf{V}_{\phi_{k_r}} &= \mathbf{U}_{\phi_{k_r}}^T \mathbf{V} \end{aligned}$$

In the special case where $\phi_1 = \phi_2 = \cdots \phi_r = \phi$ and ϕ is a lumped parameter, all the $r!$ terms in (4.25) are identical and the formula (4.25) becomes:

$$\frac{\partial^r V_{out}}{\partial \phi^r} = (-1)^r r! (\hat{\mathbf{V}}_\phi (\hat{\mathbf{V}}_\phi^\phi)^{r-1} \mathbf{V}_\phi) \left(\frac{\partial Y_\phi}{\partial \phi} \right)^r \quad (4.26)$$

In (4.26), Y_ϕ is the admittance across element ϕ . $\hat{\mathbf{V}}_\phi^\phi$ is the voltage across element ϕ when a unit independent current excitation is applied across ϕ in the adjoint system: i.e., $\hat{\mathbf{V}}_\phi^\phi$ is the effective adjoint input impedance across the port of ϕ .

In another special case, if ϕ is a transmission line parameter in the k th subnetwork, then to obtain the high-order sensitivity, the formula (4.18) could be changed to:

$$\begin{aligned}
\frac{\partial^r V_{out}}{\partial \phi^r} &= -\hat{\mathbf{V}}^T \sum_{k=1}^r \binom{r}{k} \mathbf{D}_k \frac{\partial^k \mathbf{A}_k}{\partial \phi^k} \mathbf{D}_k^T \frac{\partial^{r-k} \mathbf{V}}{\partial \phi^{r-k}} \\
&= \sum_{j_1=1}^r \binom{r}{j_1} \sum_{j_2=1}^{r-j_1} \binom{r-j_1}{j_2} \dots \sum_{j_p=1}^1 \binom{r-\sum_{i=1}^{p-1} j_i}{j_p} \\
&\quad (-1)^p (\hat{\mathbf{V}}_\phi)^T \frac{\partial^{j_1} \mathbf{A}_k}{\partial \phi^{j_1}} (\hat{\mathbf{V}}_\phi)^T \frac{\partial^{j_2} \mathbf{A}_k}{\partial \phi^{j_2}} \dots (\hat{\mathbf{V}}_\phi)^T \frac{\partial^{j_p} \mathbf{A}_k}{\partial \phi^{j_p}} \mathbf{V}_\phi \quad (4.27)
\end{aligned}$$

Where $1 \leq p \leq r$, for example, using the formula above, the third-order sensitivity is

$$\begin{aligned}
\frac{\partial^3 V_{out}}{\partial \phi^3} &= 6(-1)^3 (\hat{\mathbf{V}}_\phi)^T \frac{\partial \mathbf{A}_k}{\partial \phi} (\hat{\mathbf{V}}_\phi)^T \frac{\partial \mathbf{A}_k}{\partial \phi} (\hat{\mathbf{V}}_\phi)^T \frac{\partial \mathbf{A}_k}{\partial \phi} \mathbf{V}_\phi \\
&\quad + 3(-1)^2 (\hat{\mathbf{V}}_\phi)^T \frac{\partial \mathbf{A}_k}{\partial \phi} (\hat{\mathbf{V}}_\phi)^T \frac{\partial^2 \mathbf{A}_k}{\partial \phi^2} \mathbf{V}_\phi \\
&\quad + 3(-1)^2 (\hat{\mathbf{V}}_\phi)^T \frac{\partial^2 \mathbf{A}_k}{\partial \phi^2} (\hat{\mathbf{V}}_\phi)^T \frac{\partial \mathbf{A}_k}{\partial \phi} \mathbf{V}_\phi + (-1)^1 (\hat{\mathbf{V}}_\phi)^T \frac{\partial^3 \mathbf{A}_k}{\partial \phi^3} \mathbf{V}_\phi
\end{aligned}$$

By using the formula (4.27) to obtain the high-order sensitivity with respect to transmission line parameters, the evaluation of $\frac{\partial^r \mathbf{A}_k}{\partial \phi^r}$ is required. The technique for first-order derivatives for transmission lines [51] is expanded into higher orders in Appendix A.

4.5 A High-Order Approximate Sensitivity Analysis Approach

A second approach to high-order sensitivity analysis is based on perturbation [53]. Approximate first order derivatives (or sensitivity) of an output function may be found in the formula:

$$\frac{\partial V_{out}}{\partial \phi_i} \approx \frac{V_{out}(\Phi + \Delta \Phi_i) - V_{out}(\Phi)}{\Delta \phi_i} \quad (4.28)$$

where $\Delta \Phi_i = \Delta \phi_i u_i$. To perform high order sensitivity analysis, we define several operators.

Let operator ξ be defined by

$$\xi^q y_k = y_{k+q} \quad (4.29)$$

where y_k and y_{k+q} are 2 discrete values of function y . For example, $\xi^{-1/2} y_1 = y_{1/2}$.

The central difference operator δ is defined by

$$\delta = \xi^{1/2} - \xi^{-1/2} \quad (4.30)$$

As an example, $\delta y_{1/2} = y_1 - y_0$.

The averaging operator μ is defined as

$$\mu = \frac{1}{2}(\xi^{1/2} + \xi^{-1/2}) \quad (4.31)$$

For example, $\mu y_{1/2} = \frac{1}{2}(y_1 + y_0)$.

The r th order sensitivity $\frac{\partial^r y}{\partial x^r}$ can be expressed in terms of the central difference operator δ by [53]:

$$\left. \begin{aligned}
 \frac{dy}{dx} &= \frac{\mu}{\Delta x} \left(\delta - \frac{1^2}{3!} \delta^3 + \frac{1^2 \cdot 2^2}{5!} \delta^5 - \frac{1^2 \cdot 2^2 \cdot 3^2}{7!} \delta^7 + \dots \right) y_k \\
 \frac{d^2 y}{dx^2} &= \frac{1}{\Delta x^2} \left(\delta^2 - \frac{1}{12} \delta^4 + \frac{1}{90} \delta^6 - \frac{1}{560} \delta^8 + \frac{1}{3150} \delta^{10} + \dots \right) y_k \\
 \frac{d^3 y}{dx^3} &= \frac{\mu}{\Delta x^3} \left(\delta^3 - \frac{1}{4} \delta^5 + \frac{7}{120} \delta^7 - \dots \right) y_k \\
 \frac{d^4 y}{dx^4} &= \frac{1}{\Delta x^4} \left(\delta^4 - \frac{1}{6} \delta^6 + \frac{7}{240} \delta^8 - \dots \right) y_k
 \end{aligned} \right\} \quad (4.32)$$

where Δx is an interval of variable x .

Chapter 5

Conversion of Moments to Statistical Distributions

Having computed the moments of circuit output from Equation (3.18), the pdf of the output should now be determined. Although a pdf-moment relationship is available from the definition of characteristic functions in (3.2)-(3.4), such a definition is not suitable for computational purposes because it requires the Fourier transform operation and the need of a large number of moments. Two main approaches are described in this chapter. The first, due to Gram and Charlier [26], seeks to represent a given frequency function as a series in the derivatives of the normal frequency function, and the second, due to Karl Pearson [26], seeks to ascertain a family of distributions that will satisfactorily represent observed data.

5.1 The Gram-Charlier Series of Type A

Let $\alpha(y)$ be a normal function:

$$\alpha(y) = \frac{1}{\sqrt{2\pi}} e^{(-\frac{1}{2}y^2)} \quad (5.1)$$

Let $H_k(y)$ be a Chebyshev-Hermite polynomial defined as

$$H_k(y) = \sum_{i=0}^{k/2} (-1)^i \frac{k!}{2^i i! (k-2i)!} y^{k-2i} \quad (5.2)$$

The pdf can be expanded formally in a series as

$$\text{pdf}(y) = \sum_{k=0}^{\infty} c_k H_k(y) \alpha(y) \quad (5.3)$$

where the coefficients c_k can be solved from

$$\begin{aligned} c_k &= \frac{1}{k!} \int_{-\infty}^{\infty} \text{pdf}(y) H_k(y) dy \\ &= \frac{1}{k!} \sum_{i=0}^{k/2} (-1)^i \frac{k!}{2^i i! (k-2i)!} h_{k-2i} \end{aligned} \quad (5.4)$$

For example, 4th order expansion is

$$\begin{aligned} \text{pdf}(y) &= \alpha(y) \left[1 + h_1 H_1(y) + \frac{1}{2} (h_2 - 1) H_2(y) \right. \\ &\quad \left. + \frac{1}{6} (h_3 - 3h_1) H_3(y) + \frac{1}{24} (h_4 - 6h_2 + 3) H_4(y) + \dots \right] \end{aligned} \quad (5.5)$$

where the Chebyshev Hermit polynomials are

$$\left. \begin{aligned} H_0 &= 1 \\ H_1 &= y \\ H_2 &= y^2 - 1 \\ H_3 &= y^3 - 3y \\ H_4 &= y^4 - 6y^2 + 3 \end{aligned} \right\} \quad (5.6)$$

and h_i is the i th order moment defined in (3.5). This is the so-called Gram-Charlier series of Type A. In this way, one can obtain pdf from moments.

5.2 Family of Pearson Distributions

The Gram-Charlier approach provides a general and direct relation between pdf and moments and is suitable when many moments are available. In practice, when the order of moments is very high, the accuracy in the moments deteriorates. We consider a more realistic case where up to 4 moments are needed. In this case a more specific family of distributions (i.e., the family of Pearson distributions provides better accuracy [54]). With this family, the special distributions — such as chi-square distribution, the t or student distribution, the F-distribution, and a number of the practically important distributions — satisfy the following differential equation:

$$\frac{d(\text{pdf})}{dy} = \frac{(y-a)\text{pdf}}{b_0 + b_1y + b_2y^2} \quad (5.7)$$

where b 's and a are constant. By putting $Y = y - a$, the equation becomes

$$\frac{\partial \ln(\text{pdf})}{\partial Y} = \frac{Y}{B_0 + B_1Y + B_2Y^2} \quad (5.8)$$

The coefficients b 's, B 's and a in (5.7) and (5.8) are related to the three central moments μ_2 to μ_4 by using the following equations.

$$B_0 = b_0 + a^2(1 + b_2) \quad (5.9)$$

$$B_1 = a(1 + 2b_2) \quad (5.10)$$

$$B_2 = b_2 \quad (5.11)$$

$$b_1 = a = -\frac{\mu_3(\mu_4 + 3\mu_2^2)}{A} \quad (5.12)$$

$$b_0 = -\frac{\mu_2(4\mu_2\mu_4 - 3\mu_3^2)}{A} \quad (5.13)$$

$$b_2 = -\frac{\mu_2(2\mu_2\mu_4 - 3\mu_3^2 - 6\mu_2^3)}{A} \quad (5.14)$$

$$A = 10\mu_4\mu_2 - 18\mu_2^3 - 12\mu_3^2 \quad (5.15)$$

The form of pdf solved from (5.7) or (5.8) is dependent on the roots of the denominator in (5.8). Let the two roots of the denominator be $r_1 = v_1 + jw_1$, $r_2 = v_2 + jw_2$ and let K be defined as:

$$K = \frac{B_1^2}{4B_0B_2} \quad (5.16)$$

If the roots are real and of opposite signs, and $K < 0$, the pdf is a Beta distribution (Type I distribution)

$$\text{pdf} = \frac{a_1^{p-1} a_2^{q-1}}{(a_1 + a_2)^{p+q-1} B(p, q)} \left(1 + \frac{y}{a_1}\right)^{p-1} \left(1 - \frac{y}{a_2}\right)^{q-1} \quad (5.17)$$

where $B(p, q)$ is a Beta function with the parameters p and q . The parameters in the distribution are

$$\Phi = \begin{bmatrix} p \\ q \\ a_1 \\ a_2 \end{bmatrix} \quad (5.18)$$

which need to be determined from the moments.

As another example, if the roots of $(B_0 + B_1y + B_2y^2)$ are complex and $0 \leq K \leq 1$, we have the so-called Type IV distribution:

$$\text{pdf} = k \left(1 + \frac{y^2}{a_1^2}\right)^{-m} \exp(-\nu \cdot \tan^{-1}(y/a_1)), \quad m > \frac{1}{2} \quad (5.19)$$

The parameters in the distribution of Type IV are:

$$\Phi = \begin{bmatrix} \nu \\ m \\ a_1 \\ k \end{bmatrix} \quad (5.20)$$

The complete set all types of distributions are provided in Appendix B.

5.3 Determination of Parameters in Pearson Distributions

The parameters of the distributions (e.g., Φ in (5.18) and (5.20)) need to be determined from the moment information. There are two possible ways to find Φ . The first method is through analytical solution. In our experiment, the analytical approach is sometimes ill-conditioned. The second approach, namely, optimization, can be used.

To fit a Pearson distribution, we equate the calculated moments to the moments of this type of distribution expressed in terms of its parameters and then solve the resulting equations for those parameters. According to the definition of q th-order moments in Equation (3.5), a set of nonlinear equations are formulated as

$$\left. \begin{aligned} F_1(\Phi) &= h_1 - \int_{-\infty}^{\infty} \text{pdf}(y, \Phi) y dy = 0 \\ F_2(\Phi) &= h_2 - \int_{-\infty}^{\infty} \text{pdf}(y, \Phi) y^2 dy = 0 \\ F_3(\Phi) &= h_3 - \int_{-\infty}^{\infty} \text{pdf}(y, \Phi) y^3 dy = 0 \\ F_4(\Phi) &= h_4 - \int_{-\infty}^{\infty} \text{pdf}(y, \Phi) y^4 dy = 0 \end{aligned} \right\} \quad (5.21)$$

These nonlinear equations can be solved by the Newton's method or in our work by

optimization. In this section, solutions to the two example types (i.e., Type I and Type IV) are described.

Beta Distribution

In Type I (Beta distribution), $B(p, q)$ is a Beta function and the unknown parameters are p , q , a_1 , and a_2 , which are to be found from the moments. The moments h_i , ($i = 1, 2, 3, 4$) of the circuit output have been obtained by the moments method described in Chapter 3. By substituting the pdf of the Beta distribution (5.17) into the Equations (5.21), a set of nonlinear equations is obtained

$$\left. \begin{aligned} F_1(\Phi) &= y_1 - a_1 - A = 0 \\ F_2(\Phi) &= y_1 y_2 - 2y_1 a_1 + a_1^2 - B = 0 \\ F_3(\Phi) &= y_1 y_2 y_3 - 3y_1 y_2 a_1 + 3y_1 a_1^2 - a_1^3 - C = 0 \\ F_4(\Phi) &= y_1 y_2 y_3 y_4 - 4y_1 y_2 y_3 a_1 + 6y_1 y_2 a_1^2 - 4y_1 a_1^3 + a_1^4 - D = 0 \end{aligned} \right\} \quad (5.22)$$

where

$$\left. \begin{aligned} y_1 &= \frac{p}{p+q}(a_1 + a_2) \\ y_2 &= \frac{p+1}{p+q+1}(a_1 + a_2) \\ y_3 &= \frac{p+2}{p+q+2}(a_1 + a_2) \\ y_4 &= \frac{p+3}{p+q+3}(a_1 + a_2) \end{aligned} \right\} \quad (5.23)$$

and A , B , C , and D are constants determined by the moments h_i .

$$\left. \begin{aligned}
 A &= h_1 - d \\
 B &= h_2 - 2h_1d + d^2 \\
 C &= h_3 - 3h_2d + 3h_1d^2 - d^3 \\
 D &= h_4 - 4h_3d + 6h_2d^2 - 4h_1d^3 + d^4
 \end{aligned} \right\} \quad (5.24)$$

where $d = h_1 + a$ and a can be evaluated from moments in Equation (5.12).

From these 4 equations the 4 unknowns p , q , a_1 , and a_2 can be obtained. These equations are not well-conditioned for normal nonlinear equation solvers. To solve this set of nonlinear equations, we formulate it as an unconstrained least-squares optimization problem.

$$\min_{\Phi_2} \sum_{i=1}^4 F_i^2(\Phi) \quad (5.25)$$

The Jacobian matrix $\mathbf{J}$ for this optimization is

$$\mathbf{J} = \mathbf{J}_1\mathbf{J}_2 + \mathbf{J}_3 \quad (5.26)$$

where $\mathbf{J}$, $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ symbolically represent the following matrices:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial p} & \frac{\partial F_1}{\partial q} & \frac{\partial F_1}{\partial a_1} & \frac{\partial F_1}{\partial a_2} \\ \frac{\partial F_2}{\partial p} & \frac{\partial F_2}{\partial q} & \frac{\partial F_2}{\partial a_1} & \frac{\partial F_2}{\partial a_2} \\ \frac{\partial F_3}{\partial p} & \frac{\partial F_3}{\partial q} & \frac{\partial F_3}{\partial a_1} & \frac{\partial F_3}{\partial a_2} \\ \frac{\partial F_4}{\partial p} & \frac{\partial F_4}{\partial q} & \frac{\partial F_4}{\partial a_1} & \frac{\partial F_4}{\partial a_2} \end{bmatrix} \quad (5.27)$$

$$\mathbf{J}_1 = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & 0 & 0 & 0 \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & 0 & 0 \\ \frac{\partial F_3}{\partial y_1} & \frac{\partial F_3}{\partial y_2} & \frac{\partial F_3}{\partial y_3} & 0 \\ \frac{\partial F_4}{\partial y_1} & \frac{\partial F_4}{\partial y_2} & \frac{\partial F_4}{\partial y_3} & \frac{\partial F_4}{\partial y_4} \end{bmatrix} \quad (5.28)$$

$$\mathbf{J}_2 = \begin{bmatrix} \frac{\partial y_1}{\partial p} & \frac{\partial y_1}{\partial q} & \frac{\partial y_1}{\partial a_1} & \frac{\partial y_1}{\partial a_2} \\ \frac{\partial y_2}{\partial p} & \frac{\partial y_2}{\partial q} & \frac{\partial y_2}{\partial a_1} & \frac{\partial y_2}{\partial a_2} \\ \frac{\partial y_3}{\partial p} & \frac{\partial y_3}{\partial q} & \frac{\partial y_3}{\partial a_1} & \frac{\partial y_3}{\partial a_2} \\ \frac{\partial y_4}{\partial p} & \frac{\partial y_4}{\partial q} & \frac{\partial y_4}{\partial a_1} & \frac{\partial y_4}{\partial a_2} \end{bmatrix} \quad (5.29)$$

$$\mathbf{J}_3 = \begin{bmatrix} 0 & 0 & \frac{\partial F_1}{\partial a_1} & 0 \\ 0 & 0 & \frac{\partial F_2}{\partial a_1} & 0 \\ 0 & 0 & \frac{\partial F_3}{\partial a_1} & 0 \\ 0 & 0 & \frac{\partial F_4}{\partial a_1} & 0 \end{bmatrix} \quad (5.30)$$

This optimization problem with the Jacobian matrix can be efficiently solved by [55] which uses a gradient based on L2 optimization technique. Numerical examples are presented in the next chapter.

Type IV Distribution

As an example of analytically solving the pdf from moments, we consider the Type IV distribution function in Equation (5.19). From the first to the fourth moments,

we can find the parameters by:

$$r = \frac{6(\beta_2 - \beta_1 - 1)}{2\beta_2 - 3\beta_1 - 6} \quad (5.31)$$

$$\nu = \frac{r(r-2)\sqrt{\beta_1}}{\sqrt{16(r-1) - \beta_1(r-2)^2}} \quad (5.32)$$

$$a_1 = \sqrt{\frac{\mu_2}{16}[16(r-1) - \beta_1(r-2)^2]} \quad (5.33)$$

$$m = \frac{r+2}{2} \quad (5.34)$$

where $\beta_1 = \mu_3/(\mu_2)^{3/2}$ is skewness and $\beta_2 = \mu_4/\mu_2^2$ is kurtosis. Substituting the values of the parameters into the function of the distribution, k can be determined by:

$$k = \frac{1}{\int_{-\infty}^{\infty} (1 + \frac{y^2}{a_1^2})^{-m} \exp(-\nu \cdot \tan^{-1}(y/a_1)) dy} \quad (5.35)$$

Chapter 6

Numerical Examples

6.1 Example 1: A Simple Illustrative Example

Consider a nonlinear function

$$y = 3 + x_1^2 + x_2^3 \quad (6.1)$$

This simple example is used to illustrate and verify the proposed moment computation method. Suppose x_1 and x_2 are independent variables and follow Gaussian distributions with 5% standard deviation. The nominal values x_i^0 for variables x_1 and x_2 are both 1. Equation (3.8) becomes

$$z_1 = x_1 - 1;$$

$$z_2 = x_2 - 1;$$

the moments of z 's are

$$E[z_i^q] = \begin{cases} 0 & \text{if } q \text{ is odd} \\ (\alpha_i \sigma_i)^q \cdot (1 \cdot 3 \cdot 5 \cdot \dots \cdot (q-1)) & \text{if } q \text{ is even} \end{cases} \quad (6.2)$$

where q is the order of the moment and the scaling factors are chosen as 1. All the odd order moments vanish, because z_i are symmetrical Gaussian distributions with

zero mean. The covariance between z_1 and z_2 is zero because they are independent variables.

The effects of the 6th and higher moments of x_1 and x_2 are deliberately truncated.

The moments of y computed by the proposed methods are listed in Table 6.1. To compare moments of the moment method and the Monte-Carlo method, the OSA90 [56] software package was used to create 2000 samples by the Monte-Carlo method and the moments are calculated from the 2000 samples. The results calculated by the proposed method as listed in Table 6.1 are in good agreement with that from the Monte-Carlo method.

Table 6.1: Moments of y for Example 1

		Monte-Carlo	moment method
moments	h_1	5.0117	5.0100
	h_2	25.1487	25.1328
	h_3	126.3539	126.2447
	h_4	635.6324	634.9713
mean	$\bar{y}$	5.0117	5.0100
stand. dev.	σ	0.1770	0.1809
test of skewness	β_1	0.2092	0.1941
test of kurtosis	β_2	3.0126	2.8910

According to these moments, the pdf can be determined by the Pearson distribution system. First, we find the roots and parameter K from which the type of Pearson distribution can be determined. The roots of the polynomial in (5.8) for the proposed method are $r_1 = 1.441$, $r_2 = -0.777$, and $K = -0.0986$. According to the criterion in Appendix B, the probability density functions should belong to Type I (Beta distribution). The parameters p , q , a_1 and a_2 are solved by least squares

optimization program L2OS [55] and the optimal solution is

$$\Phi = \begin{bmatrix} p \\ q \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 12.0009 \\ 21.3832 \\ 0.7750 \\ 1.4360 \end{bmatrix}$$

To compare the solutions, the Monte-Carlo analysis is also performed. The results are illustrated in the forms of histograms in Figure 6.1 and by the pdfs in Figure 6.2. To obtain this pdf, we use Equations (2.18) and (2.19) and thus compute moments from the samples of y 's obtained from Monte-Carlo analysis. These moments are also used to find the pdf by Pearson distribution. This pdf is shown in Figure 6.2, in comparison with that of the proposed moment method. In Figure 6.1, the pdf computed by the proposed approach is compared to the raw histogram from the Monte-Carlo simulation. For both figures, the results from the proposed moment method agree with that from the Monte-Carlo analysis.

Figure 6.1: The pdf of y by moment method and the bar chart from the Monte-Carlo method for Example 1.

Figure 6.2: The pdf of y for Example 1. The moments of y were computed from the proposed method and also from the statistics of Monte-Carlo samples. The two sets of moments are converted to pdf and are compared.

6.2 Example 2: A 3-Transmission Line Network

High speed VLSI interconnects are modeled by multiconductor transmission lines. The circuit in Figure 6.3 contains three coupled transmission lines. Five variables — including each transmission line length l_1, l_2, l_3 , the distance between coupled conductors of each transmission line d , and the width of the conductors w — were chosen as random variables.

All the variables follow independent Gaussian distributions with 5% standard deviation. The nominal values x_i^0 for l_1, l_2, l_3, d, w (see Figure 3.1) are 0.05m, 0.04m, 0.03m, 0.0024m, 0.00058m, respectively. The distribution of the signal propagation delay of V_{out1} and V_{out2} (i.e., τ_1 and τ_2) are dependent on these 5 variables. Following the same procedure as in Example 1, the moments of the time delay τ_1 and τ_2 can be found by the formula (3.18). We transform the variables x 's to the z 's by using Equation (3.8). The scaling factors are chosen as:

$$\alpha_i = \frac{0.1}{x_i^0} \cdot 100$$

The moments of z_i are $E[z_i] = E[z_i^3] = 0$; $E[z_i^2] = 0.25$; and $E[z_i^4] = 0.1875$.

Approximate sensitivity analysis was conducted. The sensitivities of τ_1 from first to fourth order are listed in Table 6.2. The second sensitivity is presented in the following matrix:

$$\begin{bmatrix} 3.51597e-4 & 1.86313e-4 & 8.00789e-5 & -3.46574e-4 & 2.83366e-4 \\ 1.86313e-4 & -3.97673e-4 & -1.69984e-4 & 5.44732e-4 & -1.78502e-4 \\ 8.00789e-5 & -1.69984e-4 & 1.05661e-4 & 9.96910e-5 & -7.098830e-6 \\ -3.46574e-4 & 5.44732e-4 & 9.96910e-5 & 1.56765e-4 & -4.74543e-4 \\ 2.83366e-4 & -1.78502e-4 & -7.09883e-6 & -4.74543e-4 & 7.01997e-4 \end{bmatrix}$$

Table 6.2: High-order sensitivities of τ_1

Variables	$f'(z_i)$	$f^{(2)}(z_i)$	$f^{(3)}(z_i)$	$f^{(4)}(z_i)$
z_1	3.045754e-02	3.515974e-04	-7.050484e-05	6.455746e-05
z_2	2.160552e-02	-3.976725e-04	1.423514e-05	6.591023e-05
z_3	9.976228e-03	1.056607e-04	2.743118e-05	-8.239121e-06
z_4	1.284672e-02	1.567645e-04	6.963350e-05	-1.732768e-03
z_5	-3.445640e-03	7.019971e-04	-1.401994e-04	3.058302e-04

The entry on row i and column j is the sensitivity $\frac{\partial^2 f}{\partial z_i \partial z_j}$. The moments for the time delay of τ_1 and τ_2 are computed and listed in Tables 6.3 and 6.4, respectively. From the Table 6.3, it can be seen that the results by the proposed and the Monte-Carlo methods are in good agreement with each other.

Table 6.3: Moments of τ_1 for the 3-transmission line example

		Monte-Carlo	proposed method
moments	h_1	1.857233	1.857337
	h_2	3.449708	3.450117
	h_3	6.408371	6.409580
	h_4	11.905907	11.909073
mean	$\bar{y}$	1.857233	1.857337
stand. dev.	σ	1.9817e-02	2.0440e-02
test of skewness	β_1	0.0656	0.0119
test of kurtosis	β_2	2.9779	2.9998

From these moments, the Pearson distribution system is used to find the pdf of time delays τ_1 and τ_2 . For τ_1 , the roots of (5.8) — real and of opposite sign — indicate that Beta distributions should be used for the pdf of τ_1 . But for τ_2 , because the roots are complex and $0 \leq K < 1$, the pdf of τ_2 should be of Type IV. The parameters for distribution of τ_1 are solved by optimization described in Chapter 5

Table 6.4: Moments of τ_2 for the 3-transmission line example

		Monte-Carlo	proposed method
moments	h_1	1.8159	1.8159
	h_2	3.2978	3.2980
	h_3	5.9896	5.9904
	h_4	10.8798	10.8819
mean	$\bar{y}$	1.8159	1.8159
stand. dev.	σ	1.8708e-02	1.9378e-02
test of skewness	β_1	0.0658	0.0182
test of kurtosis	β_2	3.0299	3.0018

and are

$$\Phi = \begin{bmatrix} p \\ q \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10.0432 \\ 10.0426 \\ 0.0937 \\ 0.0939 \end{bmatrix}$$

and the parameters for distribution of τ_2 are solved by the analytical approach, also described in Chapter 5. The solutions are $r = 5365$ and

$$\Psi = \begin{bmatrix} \nu \\ m \\ a_1 \\ k \end{bmatrix} = \begin{bmatrix} -1.3146 \\ 2.6835e + 03 \\ 2.1826e + 03 \\ 3.5 \end{bmatrix}$$

The pdf's computed from the proposed method are compared with those from the Monte-Carlo method (1000 samples generated) in Figures 6.4 and 6.5. The results are in good agreement with each other.

Table 6.5 compares the run-time of the moment method and the Monte-Carlo method. It shows that the moment method provides significant speed improvement when compared with the Monte-Carlo method. The comparison was made on a Sun SPARC 2 workstation.

Table 6.5: CPU comparison between two methods for the 3-transmission line example.

Number of Variables	CPU Time (seconds)		Speed-Up Ratio
	Moment Method	Monte-Carlo	
5	5 mins	1 hrs 24 mins	17

Figure 6.3: Example 2. A 3-transmission line example

Figure 6.4: The pdf of τ_1 for the 3-transmission line example. The two sets of pdf were computed in a way similar to that case in Figure 6.2. The bar chart is from the Monte-Carlo analysis.

Figure 6.5: The pdf of τ_2 for the 3-transmission line example. The bar chart is from the Monte-Carlo analysis.

6.3 Example 3: Ground Noise Example

Ground noise due to simultaneous switching is a serious problem in signal integrity design of multichip modules and printed circuit boards. This saturation is due to the inductive or transmission line effects of the ground/power pins or interconnects. Figure 6.6 represents a ground interconnection from 4 circuit blocks which switch simultaneously. We choose 10 variables in this example including lengths of the interconnects $l_1 - l_8$, termination R and the time-rise t_r . The applied input voltage is step with t_r rise time, and 5V amplitude. The nominal values x_i^0 of the variables are

$$\begin{aligned}
 x_1^0 &= l_1 = 0.02 \text{ m} \\
 x_2^0 &= l_2 = 0.04 \text{ m} \\
 x_3^0 &= l_3 = 0.02 \text{ m} \\
 x_4^0 &= l_4 = 0.04 \text{ m} \\
 x_5^0 &= l_5 = 0.04 \text{ m} \\
 x_6^0 &= l_6 = 0.04 \text{ m} \\
 x_7^0 &= l_7 = 0.02 \text{ m} \\
 x_8^0 &= l_8 = 0.05 \text{ m} \\
 x_9^0 &= R = 50.0 \text{ } \Omega \\
 x_{10}^0 &= t_r = 0.1 \text{ (ns)}.
 \end{aligned}$$

All the variables are independent random variables following Gaussian distributions with $\sigma_i = 2\%$ standard deviation. The output is the peak value of the ground noises V_4 . The scaling factor is chosen in the same way as in Example 2, and the moments of variables z_i can be found from moments of x_i . Sensitivity analysis is carried out. Table 6.6 shows the sensitivities of the ground noise V_4 for all variables.

Table 6.6: The high-order sensitivities of V_4

Variables	$f'(z_i)$	$f^{(2)}(z_i)$	$f^{(3)}(z_i)$	$f^{(4)}(z_i)$
z_1	4.715238e-04	1.200000e-03	-8.958333e-04	-1.666667e-02
z_2	-4.150357e-04	4.666667e-04	1.458333e-05	-6.666666e-02
z_3	-5.008744e-02	-4.883333e-03	1.464583e-03	1.666667e-03
z_4	1.941071e-03	4.491666e-03	-3.654167e-03	-4.500000e-02
z_5	1.658333e-05	-1.632500e-02	-6.943750e-03	-2.666667e-02
z_6	-1.009445e-01	-6.091666e-03	5.662500e-03	-2.666667e-02
z_7	3.296583e-02	-1.189167e-02	-6.500000e-03	-1.500000e-02
z_8	8.278039e-02	-8.984166e-02	-3.229854e-01	-5.605000e+00
z_9	-1.622311e-01	1.470000e-02	-2.322917e-03	-3.000000e-02
z_{10}	-7.518678e-03	-6.616666e-03	-8.020833e-04	-3.166667e-02

Substituting those sensitivities and moments of the variables into Equation (3.18) gives the moments of the ground noises V_4 . Table 6.7 shows the results of the moments calculated by the proposed method and those from a Monte-Carlo analysis for V_4 . Again good agreement between the two methods is obtained. For both methods, the roots of (5.8) are found to be complex and $0 \leq K < 1$. Therefore, the Type IV should be used for the pdf functions. The parameters for the distribution are found to be $r = 63.1884$ and

$$\Phi = \begin{bmatrix} \nu \\ m \\ a_1 \\ k \end{bmatrix} = \begin{bmatrix} 9.4097 \\ 32.5942 \\ -0.3348 \\ 4.5 \end{bmatrix} \quad (6.3)$$

Figure 6.7 compares the pdf of V_4 from the proposed method and the Monte-Carlo analysis. Good agreement is achieved.

To create 2000 samples in the Monte-Carlo method, 2000 circuit simulations are required and to calculate the high-order sensitivities only 190 circuit simulations are needed. The major CPU requirement of both methods is tabulated in Table 6.8.

Table 6.7: Moments of V_4 for the ground noise example

		Monte-Carlo	proposed method
moments	h_1	3.2291	3.2265
	h_2	10.4287	10.4121
	h_3	33.6866	33.6064
	h_4	108.8322	108.4882
mean	$\bar{y}$	3.2291	3.2265
stand. dev.	σ	4.2029e-02	4.2917e-02
test of skewness	β_1	0.03926	0.07593
test of kurtosis	β_2	3.0188	3.1085

Table 6.8: CPU comparison between two methods for the ground noise example.

Number of Variables	CPU Time (seconds)		Speed-Up Ratio
	Moment Method	Monte-Carlo	
10	1 hrs 7 mins	9 hrs	6.6

Figure 6.6: Example 3. A ground noise example

Figure 6.7: The pdf of V_4 for the ground noise example. The bar chart is from the Monte-Carlo analysis.

6.4 Example 4: Op Amp Circuit

In this example, the 741 operational amplifier circuit [57] shown in Figure 6.8 is considered. Five parameters are chosen as random variables. They are transistor parameters g_m , r_π , resistors R_3 and R_{out} , and capacitor C_c . The function y used is the gain $G = |V_{out}/V_{in}|$ of the operational amplifier. The nominal values for the variables are

$$x_1^0 = g_m = 0.01 \text{ } 1/\Omega$$

$$x_2^0 = r_\pi = 10000 \text{ } \Omega$$

$$x_3^0 = R_3 = 39000 \text{ } \Omega$$

$$x_4^0 = C_c = 30 \text{ } pf$$

$$x_5^0 = R_{out} = 8000 \text{ } \Omega$$

The scaling factor is $\alpha_i = 1/x_i^0$. The moments of z_i can be obtained from the moments of x_i . The exact high-order adjoint sensitivity technique is used to calculate the sensitivities. The sensitivities of gain are computed from the voltage sensitivities following the formulas in Appendix C. The first to 4th-order sensitivities of the gain are listed in Table 6.9. Let the standard deviation σ_i be 5% of the nominal value of each variable and then the solution of moments are listed in Table 6.10.

Table 6.9: Sensitivities of z_i for the op amp circuit example.

Variables	$f'(z_i)$	$f^{(2)}(z_i)$	$f^{(3)}(z_i)$	$f^{(4)}(z_i)$
z_1	-0.167443	-0.101850	0.003390	0.000104
z_2	264.883728	-0.140745	0.412915	-1.614155
z_3	0.000000	0.000000	0.000000	0.000000
z_4	-263.051208	526.105835	-1578.315918	6313.261230
z_5	-2.095883	0.033398	-0.000798	0.000025

Table 6.10: Moments of Gain in the op amp circuit example

		Monte-Carlo	proposed method
moments	h_1	264.0413	263.7157
	h_2	70080.7651	69898.2452
	h_3	18697426.9467	18620437.9265
	h_4	5014420382.8954	4985441233.3500
mean	$\bar{y}$	264.0413	263.7157
stand. dev.	σ	1.9051	1.8770
test of skewness	β_1	0.2234	0.205871
test of kurtosis	β_2	2.9801	2.897577

Figure 6.9 gives the pdfs for both the moment and Monte-Carlo methods and the bar chart for Monte-Carlo analysis. For the Monte-Carlo method 2000 samples are used. Both pdfs are almost the same and match perfectly with the frequency distribution which is presented by the bar chart. The pdfs are nonsymmetrical, a condition which is also indicated by the skewness parameter. Because the test of skewness $\beta_1 > 0$ for both methods, the function is more steep on the left side.

Figure 6.8: Example 4. The 741 op amp circuit [57]

Figure 6.9: The pdf of the gain for the 741 op amp circuit. The bar chart is from Monte-Carlo analysis.

6.5 Example 5: The 7-Transmission Line Network

An interconnect circuit consisting of a terminal network and 7 transmission lines is shown in Figure 6.10. The applied voltage is a pulse with 10ps rise/fall time, 5ns pulse width and 5V amplitude. The lengths of the 7 transmission lines are chosen as random variables, and the time delay of V_{out} using threshold of 0.7V is the output y . The nominal values for the variables x_i , ($i = 1, 2, \dots, 7$) are

$$x_1^0 = l_1 = 0.03 \text{ m}$$

$$x_2^0 = l_2 = 0.03 \text{ m}$$

$$x_3^0 = l_3 = 0.03 \text{ m}$$

$$x_4^0 = l_4 = 0.05 \text{ m}$$

$$x_5^0 = l_5 = 0.03 \text{ m}$$

$$x_6^0 = l_6 = 0.04 \text{ m}$$

$$x_7^0 = l_7 = 0.02 \text{ m}$$

and the scaling factors are chosen as

$$\alpha_i = \frac{0.1}{x_i^0} \cdot 100$$

The moments of z_i are $E[z_i] = E[z_i^3] = 0$; $E[z_i^2] = 0.25$; and $E[z_i^4] = 0.1875$. For the moment method, the approximate sensitivity analysis was conducted. The Monte-Carlo analysis with 5000 samples is used to verify the solutions from proposed method. The moments from both methods for time delay of V_{out} are computed and listed in Table 6.11.

Figure 6.11 gives the pdf from the both methods and also the bar chart from the Monte-Carlo analysis. From Figure 6.11 and the Table 6.11, the results, from both the moment and Monte-Carlo methods, are in quite good agreement. The example with 10% standard deviation was also tested and the results are shown in Figure 6.12.

Table 6.11: Moments of time delay for the 7-transmission line example

		Monte-Carlo	proposed method
moments	h_1	1.7945	1.7945
	h_2	3.22034	3.2204
	h_3	5.7794	5.7796
	h_4	10.3722	10.3727
mean	$\bar{y}$	1.7945	1.7945
stand. dev.	σ	0.01095	0.01113
test of skewness	β_1	0.03118	0.05675
test of kurtosis	β_2	2.9601	3.0082

Table 6.12: CPU comparison between two methods for the 7-transmission line example.

Number of Variables	CPU Time (seconds)		Speed-Up Ratio
	Moment Method	Monte-Carlo	
7	3 mins	2 hrs 10 mins	43

In this example, 112 circuit simulations are needed for the moment method, and 5000 circuit simulations are required to create 5000 samples for the Monte-Carlo method. To compare the efficiency of the proposed method and Monte-Carlo method, we represent the major computation in Table 6.12. It is noted that the speed-up ratio will become higher when the number of circuit simulations in the Monte-Carlo analysis is increased and lower when the number of circuit variables is increased.

Figure 6.10: Example 5. The 7-transmission line network

Figure 6.11: The pdf of the time delay of V_{out} in the 7-transmission line example with 5% standard deviation in component values. The bar chart is from the Monte-Carlo analysis.

Figure 6.12: The pdf of the time delay of V_{out} in the 7-transmission line example with 10% standard deviation in component values. The bar chart is from the Monte-Carlo analysis.

Chapter 7

Conclusions

7.1 Summary

This research was motivated by the need for an efficient statistical analysis approach for speeding up the optimization process in the statistical design of VLSI interconnects. In this thesis, an efficient method has been developed to obtain the probability density function of circuit output performance (high level) from the distribution of lower level parameters. Several contributions have been presented in this thesis to address the statistical design and modeling of VLSI circuits.

- A set of analytical and explicit formulas have been derived to calculate the moments of circuit output performance from the moments of the circuit parameters. The formulas are derived for the general case where the high level output y is a multidimensional and nonlinear function of n random variables x_i with arbitrary distributions. As special cases the formulas also can be applied to existing response surface modeling methods, e.g., quadratic modeling.
- Formulas have been derived for the computation of high-order sensitivities of circuit output with respect to circuit parameters. The exact adjoint sensitivity

technique has been extended from the first order sensitivity analysis to the high-order sensitivity analysis for linear frequency domain analysis with distributed transmission-line subnetworks. The parameters in sensitivity analysis could be lumped parameters or transmission-line parameters.

- Because a set of moments gives certain aspects of a probability distribution, the moments are converted to a probability density function. The relationship between moments and pdf described in statistical theory is applied to circuit analysis. The family of Pearson distribution is applied to obtain the pdf of the circuit output performance. By moment-matching from the first to 4th-order moments, the parameters in the Pearson distribution function can be determined through an optimization problem.
- The moment method has been successfully applied to statistical analysis of high-speed VLSI interconnect networks. Five examples demonstrate that the pdfs of output performances from moment method are in good agreement with those of the Monte-Carlo method and the CPU speed-up is confirmed in these circuits.

Generally speaking, in theory, the proposed moment method can work with nonlinear functions and arbitrary distributions. But in reality, the efficiency of the technique will be lower if the nonlinearity is too strong and the distributions of its parameters are too complicated. In this case, the order of sensitivity information and order of moments needed will be high resulting in reduced accuracy and efficiency.

Another situation where the efficiency of the proposed technique will be affected is when the number of statistical variables is large. In this case a factor analysis may be used in order to reduce the number of variables. If after factor analysis the number of

statistical variables is still large, the moment method will not be efficient, and other techniques, e.g., the Monte-Carlo method could be used instead.

7.2 Suggestions for Further Research

1. The moment method should be extended to more complex case where the parameter variables are correlated and have arbitrary distributions. Formula (3.18) is suitable for both independent and multivariable correlated distributions. In this thesis, the independent case has been considered in the examples. In the correlation case, the high-order mixed sensitivities and covariance are needed to calculate the moments of output performance. Because formulas for the high-order mixed sensitivity have been derived, the implementation of the correlation case can be further developed and tested by circuit examples.
2. By using the explicit relationship between the moments of output performance and the moment of the design variables, the explicit relationship between the moments of output performance and the nominal values of design variables can be derived. The moment values such as the mean value of delay and standard deviation of gain can be controlled by adjusting the design variables. This research topic will be an interesting one in optimization problems.
3. The moment method can be used in hierarchical statistical modeling as from device level to MCM level, and from the MCM level to the PCB level where the output y of a higher level is a function of some random variables of a lower level. For example, the higher level output function y could be the cost of circuits which is a function of all output performances or specifications. The moments method could be used to determine the pdf of cost which could be

very useful in microwave circuit and system design.

4. The moment method, based on a high-order sensitivity analysis, is more efficient when the number of variables is smaller. AWE techniques [24] [25] can be used to do sensitivity analysis, and the multilevel optimization by decomposition can be used to reduce the variables. In such a way, the moment method becomes more efficient. Combining these efficient methods in the optimization problems can be the topic of additional, interesting, and challenging research.

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Appendix A

High-Order Sensitivities of the Transmission Lines

The r th-order derivative of admittance matrix $\mathbf{A}_k$ is represented by:

$$\mathbf{A}_k^{(r)} = \frac{\partial^r \mathbf{A}_k}{\partial \phi^r} = \begin{bmatrix} (\mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1})^{(r)} & (\mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1})^{(r)} \\ (\mathbf{S}_i \mathbf{E}_2 \mathbf{S}_v^{-1})^{(r)} & (\mathbf{S}_i \mathbf{E}_1 \mathbf{S}_v^{-1})^{(r)} \end{bmatrix} \quad (\text{A.1})$$

Obviously, for the first-order derivative of $\mathbf{A}_k$, the evaluation of $\frac{\partial \mathbf{S}_v}{\partial \phi}$, $\frac{\partial \mathbf{S}_i}{\partial \phi}$, $\frac{\partial \mathbf{E}_1}{\partial \phi}$, and $\frac{\partial \mathbf{E}_2}{\partial \phi}$ is required. First, the equation [52]

$$\begin{bmatrix} \gamma_j^2 \mathbf{U} - \mathbf{Z}_p \mathbf{Y}_p & 2\gamma_j \mathbf{S}_v \\ \mathbf{S}_v^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{S}_v}{\partial \phi} \\ \frac{\partial \gamma_j}{\partial \phi} \end{bmatrix} = \begin{bmatrix} (\mathbf{Z}_p \frac{\partial \mathbf{Y}_p}{\partial \phi} + \frac{\partial \mathbf{Z}_p}{\partial \phi} \mathbf{Y}_p) \mathbf{S}_v \\ 0 \end{bmatrix} \quad (\text{A.2})$$

is used to evaluate the derivative of the eigenvalues and eigenvectors γ_j and $\mathbf{S}_v$, for $j = 1, 2, \dots, N_k$. The derivatives of $\mathbf{S}_i$ can be evaluated from

$$\mathbf{Z}_p \frac{\partial \mathbf{S}_i}{\partial \phi} = \frac{\partial \mathbf{S}_v}{\partial \phi} \Gamma + \mathbf{S}_v \frac{\partial \Gamma}{\partial \phi} - \frac{\partial \mathbf{Z}_p}{\partial \phi} \mathbf{S}_i \quad (\text{A.3})$$

For the derivatives of $\mathbf{E}_1$ and $\mathbf{E}_2$, partial differentiation gives

$$\frac{\partial \mathbf{E}_1}{\partial \phi} = \text{diag} \left\{ \frac{-\frac{\partial \gamma_l}{\partial \phi} l e^{-\gamma_l l} + \frac{\partial \gamma_l}{\partial \phi} l e^{\gamma_l l}}{2} \right\} = -l \frac{\partial \Gamma}{\partial \phi} \mathbf{E}_2 \quad (\text{A.4})$$

and

$$\frac{\partial \mathbf{E}_2}{\partial \phi} = \text{diag} \left\{ \frac{-\frac{\partial \gamma_1}{\partial \phi} l e^{-\gamma_1 l} - \frac{\partial \gamma_2}{\partial \phi} l e^{\gamma_2 l}}{2} \right\} = -l \frac{\partial \Gamma}{\partial \phi} \mathbf{E}_1 \quad (\text{A.5})$$

For the r th-order derivative of $\mathbf{A}_k$, the r th-order derivatives $\mathbf{S}_v^{(r)}$, $\mathbf{S}_i^{(r)}$, $\mathbf{E}_1^{(r)}$ and $\mathbf{E}_2^{(r)}$ are required.

With $\frac{\partial \mathbf{S}_v}{\partial \phi}$ and $\frac{\partial \Gamma}{\partial \phi}$ from Equations (A.2)-(A.5) as a starting point, $\mathbf{S}_v^{(r)}$ and $\Gamma^{(r)}$ may be evaluated recursively by the following equations:

$$\mathbf{S}_v^{(r)} = \begin{bmatrix} \mathbf{S}_{v_1}^{(r)} & \mathbf{S}_{v_2}^{(r)} & \dots & \mathbf{S}_{v_{N_s}}^{(r)} \end{bmatrix} \quad (\text{A.6})$$

$$\Gamma^{(r)} = \text{diag} \left[\gamma_1^{(r)} \quad \gamma_2^{(r)} \quad \dots \quad \gamma_{N_s}^{(r)} \right] \quad (\text{A.7})$$

$$\mathbf{T} \mathbf{D}^{(r)} = \begin{bmatrix} \sum_{k=0}^{r-1} \binom{r-1}{k} [\mathbf{Z}_L \mathbf{Y}_L]^{(r-k)} \mathbf{S}_{v_j}^{(k)} \\ 0 \end{bmatrix} - \sum_{k=0}^{r-2} \binom{r-1}{k} \mathbf{T}^{(r-k-1)} \mathbf{D}^{(k+1)} \quad (\text{A.8})$$

where $\mathbf{T}$ and $\mathbf{D}^{(r)}$ symbolically express the two matrices as:

$$\mathbf{T} = \begin{bmatrix} \gamma_j^2 \mathbf{U} - \mathbf{Z}_p \mathbf{Y}_p & 2\gamma_j \mathbf{S}_{v_j} \\ \mathbf{S}_{v_j}^T & 0 \end{bmatrix} \quad (\text{A.9})$$

and

$$\mathbf{D}^{(r)} = \begin{bmatrix} \mathbf{S}_{v_j}^{(r)} \\ \gamma_j^{(r)} \end{bmatrix} \quad (\text{A.10})$$

Once $\mathbf{S}_v^{(r)}$ and $\Gamma^{(r)}$ have been evaluated by formula (A.8), the derivative $\mathbf{S}_i^{(r)}$ can be obtained by

$$\mathbf{Z}_L \mathbf{S}_i^{(r)} = \sum_{k=0}^r \binom{r}{k} \mathbf{S}_v^{(r-k)} \Gamma^{(k)} - \sum_{k=0}^{r-1} \binom{r}{k} \mathbf{Z}_L^{(r-k)} \mathbf{S}^{(k)} \quad (\text{A.11})$$

The sensitivities $\mathbf{E}_1^{(r)}$ and $\mathbf{E}_2^{(r)}$ can be found by

$$\mathbf{E}_1^{(r)} = -l \sum_{k=0}^{r-1} \binom{r-1}{k} \Gamma^{(r-k)} \mathbf{E}_2^{(k)} \quad (\text{A.12})$$

and

$$\mathbf{E}_2^{(r)} = -l \sum_{k=0}^{r-1} \binom{r-1}{k} \Gamma^{(r-k)} \mathbf{E}_1^{(k)} \quad (\text{A.13})$$

Although the above equations may look tedious to derive and evaluate, only minimal CPU cost would be required in the actual calculation because the matrix $\mathbf{T}$ has been decomposed into LU factors during the original circuit analysis. Therefore, only forward/backward substitutions would be required. For the special case where ϕ is the length of a transmission line, the sensitivity in Equation (A.1) can be simplified as

$$\mathbf{A}_k^{(r)} = \frac{\partial^r \mathbf{A}_k}{\partial \phi^r} = \begin{bmatrix} \mathbf{S}_i \mathbf{E}_1^{(r)} \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_2^{(r)} \mathbf{S}_v^{-1} \\ \mathbf{S}_i \mathbf{E}_2^{(r)} \mathbf{S}_v^{-1} & \mathbf{S}_i \mathbf{E}_1^{(r)} \mathbf{S}_v^{-1} \end{bmatrix} \quad (\text{A.14})$$

Appendix B

The Family of Pearson Distributions [26]

TYPE I. (Beta distribution)

If the roots are real and of opposite sign, and $K < 0$, the pdf is a Beta distribution.

$$f = \frac{a_1^{p-1} a_2^{q-1}}{(a_1 + a_2)^{p+q-1} B(p, q)} \left(1 + \frac{y}{a_1}\right)^{p-1} \left(1 - \frac{y}{a_2}\right)^{q-1} \quad (\text{B.1})$$

where $B(p, q)$ is a Beta function with the parameters p, q . The parameters of the distribution are p, q, a_1 , and a_2 .

TYPE II.

If $K = 0, B_2 > 0, B_1 = 0, B_0 < 0$ and if $\beta_1 = 0, \beta_2 < 3$, the distribution of the function is

$$f = \frac{1}{a B(\frac{1}{2}, m + 1)} \left(1 - \frac{y^2}{a^2}\right)^m, \quad -a \leq y \leq a \quad (\text{B.2})$$

where $B(\frac{1}{2}, m + 1)$ is a Beta function, the parameters are a, m and σ .

TYPE III. (Gamma distribution)

If $B_2 = 0$, the distribution becomes

$$f = k(1 + \frac{y}{a})^p e^{-\frac{y}{a}}, \quad -a \leq y \leq \infty, \quad (\text{B.3})$$

where a , p , and k are the parameters.

TYPE IV.

If the roots of $(B_0 + B_1y + B_2y^2)$ are complex and $0 \leq K \leq 1$, we have the distribution

$$f = k(1 + \frac{y^2}{a^2})^{-m} \exp(-\nu \cdot \tan^{-1}(\frac{y}{a})), \quad m > \frac{1}{2}, \quad (\text{B.4})$$

In this function, the parameters are m , ν , and a .

TYPE V.

If the roots of $(B_0 + B_1y + B_2y^2)$ are equal and the criterion $K = 1$, the distribution becomes

$$f = \frac{\gamma^{p-1}}{\Gamma(p-1)} y^{-p} e^{-\gamma/y}, \quad 0 \leq y \leq \infty, \quad (\text{B.5})$$

where $\Gamma(p-1)$ is a gamma function, and γ and p are parameters.

TYPE VI.

If the real roots of the quadratic have the same sign and are not equal and $K > 1$, we may write the distribution function in the form

$$f = \frac{1}{B(p, q)} \frac{y^{p-1}}{(1+y)^{p+q}}, \quad 0 \leq y \leq \infty; \quad p, q > 0, \quad (\text{B.6})$$

where $B(p, q)$ is a Beta function and the parameters are p and q .

TYPE VII.

If $K = 0$, $B_2 > 0$, $B_1 = 0$, and $B_0 > 0$, $\beta_1 = 0$, $\beta_2 > 3$, the function is symmetrical and is defined as

$$f = \frac{1}{aB(\frac{1}{2}, m - \frac{1}{2})} (1 + \frac{y^2}{a^2})^{-m}, \quad -\infty \leq y \leq \infty. \quad (\text{B.7})$$

The parameters are a and m .

In all types of distributions listed above, the parameters can be determined from the moment information.

Appendix C

High-Order Sensitivities of the Magnitude of an Output

Here, the formulas for the high-order sensitivities of a magnitude value of an output (e.g., V_{out}) are derived. The magnitude value of V_{out} can be presented by

$$|V_{out}| = \sqrt{V_{out} V_{out}^*} \quad (C.1)$$

where V_{out}^* is the conjugate of V_{out} . The formulas for calculating the sensitivities from first to 4th-order are

$$\frac{\partial |V_{out}|}{\partial \phi} = \frac{1}{2} (V_{out} V_{out}^*)^{-\frac{1}{2}} \left(\frac{\partial V_{out}}{\partial \phi} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi} V_{out} \right) \quad (C.2)$$

$$\begin{aligned} \frac{\partial^2 |V_{out}|}{\partial \phi^2} = & -\frac{1}{4} (V_{out} V_{out}^*)^{-\frac{3}{2}} \left(\frac{\partial V_{out}}{\partial \phi} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi} V_{out} \right)^2 \\ & + \frac{1}{2} (V_{out} V_{out}^*)^{-\frac{1}{2}} \left(\frac{\partial^2 V_{out}}{\partial \phi^2} V_{out}^* + 2 \frac{\partial V_{out}^*}{\partial \phi} \frac{\partial V_{out}}{\partial \phi} + \frac{\partial^2 V_{out}^*}{\partial \phi^2} V_{out} \right) \end{aligned} \quad (C.3)$$

$$\frac{\partial^3 |V_{out}|}{\partial \phi^3} = \frac{3}{8} (V_{out} V_{out}^*)^{-\frac{5}{2}} \left(\frac{\partial V_{out}}{\partial \phi} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi} V_{out} \right)^3$$

$$\begin{aligned}
& -\frac{3}{4}(V_{out}V_{out}^*)^{-\frac{3}{2}} \left(\frac{\partial V_{out}}{\partial \phi} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi} V_{out} \right) \\
& \times \left(\frac{\partial^2 V_{out}}{\partial \phi^2} V_{out}^* + 2 \frac{\partial V_{out}^*}{\partial \phi} \frac{\partial V_{out}}{\partial \phi} + \frac{\partial^2 V_{out}^*}{\partial \phi^2} V_{out} \right) + \frac{1}{2}(V_{out}V_{out}^*)^{-\frac{1}{2}} \\
& \times \left(\frac{\partial^3 V_{out}}{\partial \phi^3} V_{out}^* + 3 \frac{\partial^2 V_{out}}{\partial \phi^2} \frac{\partial V_{out}^*}{\partial \phi} + 3 \frac{\partial^2 V_{out}^*}{\partial \phi^2} \frac{\partial V_{out}}{\partial \phi} + \frac{\partial^3 V_{out}^*}{\partial \phi^3} V_{out} \right) \quad (C.4)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^4 |V_{out}|}{\partial \phi^4} = & -\frac{15}{16}(V_{out}V_{out}^*)^{-\frac{7}{2}} \left(\frac{\partial V_{out}}{\partial \phi} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi} V_{out} \right)^4 \\
& + \frac{18}{8}(V_{out}V_{out}^*)^{-\frac{5}{2}} \left(\frac{\partial V_{out}}{\partial \phi} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi} V_{out} \right)^2 \\
& \times \left(\frac{\partial^2 V_{out}}{\partial \phi^2} V_{out}^* + 2 \frac{\partial V_{out}^*}{\partial \phi} \frac{\partial V_{out}}{\partial \phi} + \frac{\partial^2 V_{out}^*}{\partial \phi^2} V_{out} \right) \\
& - \frac{3}{4}(V_{out}V_{out}^*)^{-\frac{3}{2}} \left(\frac{\partial^2 V_{out}}{\partial \phi^2} V_{out}^* + 2 \frac{\partial V_{out}^*}{\partial \phi} \frac{\partial V_{out}}{\partial \phi} + \frac{\partial^2 V_{out}^*}{\partial \phi^2} V_{out} \right)^2 \\
& - (V_{out}V_{out}^*)^{-\frac{1}{2}} \left(\frac{\partial V_{out}}{\partial \phi} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi} V_{out} \right) \\
& \times \left(\frac{\partial^3 V_{out}}{\partial \phi^3} V_{out}^* + 3 \frac{\partial^2 V_{out}}{\partial \phi^2} \frac{\partial V_{out}^*}{\partial \phi} + 3 \frac{\partial^2 V_{out}^*}{\partial \phi^2} \frac{\partial V_{out}}{\partial \phi} + \frac{\partial^3 V_{out}^*}{\partial \phi^3} V_{out} \right) \\
& + \frac{1}{2}(V_{out}V_{out}^*)^{-\frac{1}{2}} \left(\frac{\partial^4 V_{out}}{\partial \phi^4} V_{out}^* + 4 \frac{\partial^3 V_{out}}{\partial \phi^3} \frac{\partial V_{out}^*}{\partial \phi} \right. \\
& \left. + 6 \frac{\partial^2 V_{out}}{\partial \phi^2} \frac{\partial^2 V_{out}^*}{\partial \phi^2} + 4 \frac{\partial^3 V_{out}^*}{\partial \phi^3} \frac{\partial V_{out}}{\partial \phi} + \frac{\partial^4 V_{out}^*}{\partial \phi^4} V_{out} \right) \quad (C.5)
\end{aligned}$$

The second order cross sensitivity formula is

$$\begin{aligned}
 \frac{\partial^2 |V_{out}|}{\partial \phi_1 \partial \phi_2} = & -\frac{1}{4} (V_{out} V_{out}^*)^{-3/2} \left(\frac{\partial V_{out}}{\partial \phi_2} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi_2} V_{out} \right) \\
 & \times \left(\frac{\partial V_{out}}{\partial \phi_1} V_{out}^* + \frac{\partial V_{out}^*}{\partial \phi_1} V_{out} \right) + (V_{out} V_{out}^*)^{-1/2} \\
 & \times \left(\frac{\partial^2 V_{out}}{\partial \phi_1 \partial \phi_2} V_{out}^* + \frac{\partial V_{out}}{\partial \phi_2} \frac{\partial V_{out}}{\partial \phi_1} + \frac{\partial V_{out}^*}{\partial \phi_1} \frac{\partial V_{out}}{\partial \phi_2} + \frac{\partial^2 V_{out}^*}{\partial \phi_1 \partial \phi_2} V_{out} \right) \quad (C.6)
 \end{aligned}$$

END

0 5-0 6-9 6

FIN