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Dynamic Particle Systems And Multilevel Measure Branching Processes

by

YADONG WU

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University
Ottawa, Ontario, Canada
September 9th, 1991

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Dr. John Dixon, Chairman,
Department of Mathematics and Statistics

Dr. D.A. Dawson, Thesis Supervisor

Dr. K.J. Hochberg, External Examiner

Carleton University

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TO MY FATHER AND MY MOTHER

Abstract

In this thesis we study the multilevel $M(R^d)$ -valued and $M(M(R^d))$ -valued measure branching processes respectively. They are natural extensions of measure-valued branching processes introduced by Dawson and Watanabe which arise as weak limits of branching diffusions when we change the scales of mass and lifetime in an appropriate way.

In the first part of the thesis, we introduce a multilevel birth and death system and consider the continuous limit to the system. We show that the limiting $M(R^d)$ -valued process is the unique solution of the $M(R^d)$ -valued martingale problem for the limiting generator. We also derive the moment structures of the limiting diffusion process.

In the second part of the thesis we consider a multilevel branching diffusion particle system and its diffusion approximation which can be characterized as a $M(M(R^d))$ -valued process. For the local behaviour of this process, we prove that the Hausdorff dimension of the support of the aggregated process of the $M(M(R^d))$ -valued process is equal to or greater than 4 if $d > 4$. For the long time behaviour, we show that if $d \leq 4$ then the $M(M(R^d))$ -valued process suffers local extinction, and if $d = 4$, then the process has a self-similarity property.

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Introduction

The theory of measure-valued branching processes or superprocesses has been developing for over twenty years. This class of processes first arose from the study of continuous state branching in the work of Jirina (1958, 1964) and Watanabe (1968). It was also linked to the study of stochastic evolution equations in Dawson (1975).

Until recently the measure-valued processes studied were "one level" measure branching processes which arise as approximations to branching diffusion particle systems in which particles perform one level branching independently. During the last few years, considerable interest has developed in dynamic multilevel particle models. For example such models can be used to model the replication, updating and transfer of digitized data sets as they are processed through information networks. A dynamic multilevel system consists of different levels of objects. At any given level, it is assumed that each object (particle) of that level can be removed or copied. Collections of objects (particles) at one level comprise objects at the next higher level. Once an object is copied, it is assumed that subsequent alterations may cause it to differ from the object from which it was copied.

In this thesis, we study two new multilevel stochastic models i.e., multilevel birth and death model and multilevel branching diffusion model, which are motivated by both biological and information system problems. Although these are related to and grow out of the theory of branching process and superprocesses they involve systems of strongly interacting particles (in contrast to ordinary branching processes in which no further interaction between particles occurs after their birth). Ordinary superprocesses which can be viewed as generalized stochastic partial differential equations are one of the most active areas of current research in stochastic processes. One of the main tools in the study of superprocesses is a class of nonlinear partial differential equations. This is also true in the case of multilevel measure branching processes, but instead of nonlinear partial differential equations, we have to work with nonlinear equations for functions on an infinite dimensional space involving Gateaux-type derivatives - more precisely a class of infinite dimensional elliptic operators. This is a mathematical subject in which little prior work existed and we have to develop completely new ideas to handle them.

We divide the thesis into two parts. In the first part we introduce a multilevel birth and death particle model, which is a generalization of the multilevel branching model introduced by D. Dawson and K. Hochberg in 1991 [5] and then consider the measure valued diffusion approximation to the multilevel birth and death particle system.

For the sake of simplicity, we restrict our attention to the two level birth and death case. Each level two particle or superparticle consists of a number (called “size”) of level one particles and undergoes a level two birth and death process which depends on the size of the superparticle. Each level one particle or particle undergoes a level one birth and death process which depends on number of the superparticles which have the same size as the one that the particle belongs to. In the first chapter, we describe the two level birth and death particle system as a $N(Z^+)$ -valued process and show the existence of the process.

To consider the measure-valued diffusion approximation to the two level birth and death particle system, we show the tightness of sequence of the $N(Z^+)$ -valued processes using the criterion for tightness (see L. Gorostiza and J. Lopez-Mimbela in 1989 [29]) in section 2 of the second chapter. We prove the existence and uniqueness of the continuous $M(R^+)$ -valued process by the martingale problem method in the last section of the second chapter. We discuss the moment structure of the limiting process in the third chapter.

In the second part of the thesis, we investigate the two level branching diffusion model which was formulated by Dawson and Hochberg in 1991. We describe the continuous diffusion limit $Y(t)$ of the two level branching diffusion system as a $M(M(R^d))$ -valued process in section 1 of the fourth chapter. We discuss the moment measures for the “aggregated” measure-valued process $\bar{Y}(t) \equiv \int \mu Y(t, d\mu)$

in the following section. A lower bound of the Hausdorff dimension of the support of the process $\bar{Y}(t)$ is given in section 3 of the fourth chapter, i.e., we show that if $d > 4$ and the initial value of the process $Y(t)$ is the atomic measure δ_λ , where λ denotes the d -dimensional Lebesgue measure on R^d , then the Hausdorff dimension of support of the process $\bar{Y}(t)$ is equal to or greater than 4.

In Chapter 5, we investigate the long time behaviour of the $M(M(R^d))$ -valued process $Y(t)$, which is an analogue of Dawson's results in the one level case (see, Dawson (1977) [18]), where he showed that if the initial value is given by Lebesgue measure on R^d then existence or non-existence of a non-trivial limiting distribution of the one level classical critical branching Brownian motion is equivalent to transience or recurrence of the spatial motion. The motivation for the work in this chapter is a conjecture of Dawson:

Conjecture 1 (*D. Dawson*). *The $M(M(R^d))$ -valued process $Y(t)$ suffers local extinction if and only if $d \leq 4$.*

Specifically we show that if $d \leq 4$, then the process $Y(t)$ suffers local extinction, and if $d = 4$ and the initial value of the process is given by the canonical equilibrium measure R_∞ of the one level $M(R^d)$ -valued branching process, then $Y(t)$ possesses a self-similarity property. The method of the proof for the local extinction we use here was suggested by Dawson.

Part I

MULTILEVEL BIRTH AND DEATH PROCESS

Chapter 1

Branching Processes and Multilevel Birth and Death Particle System

The study of dynamic particle systems has played an important role in the development of stochastic processes both from the theoretical as well as the applied points of view. Some branching particle systems have been used to describe animal, biology, or plant populations, information transfer processes through networks and reaction and diffusion models of particles in physics or chemistry.

In the first section of this chapter, we review the basic concepts and state some recent developments and applications of branching processes. In section two we introduce a multilevel birth and death particle system which is a generalization to the multilevel branching model which was introduced by Dawson and Hochberg in 1991 ([5]). We prove the existence of the process in the last section of this chapter.

1.1 Basic Concepts and Preliminary Facts

The study of branching processes has a long history. The original problem was introduced by Galton (1873) and Watson (1874). The subject has been extensively studied in the last twenty years, and there are many natural directions for generalization.

(a) Galton – Watson process:

This process is a Markov chain $\{Z_n, n = 0, 1, 2, \dots\}$ on the non-negative integers, which can be defined recursively by

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_k^n,$$

where $Z_0, \xi_k^n, k = 1, 2, \dots$ are given independent, identically distributed nonnegative integer-valued random variables. We can consider Z_n as the number of particles in the generation of a population in which individual particles reproduce independently and in the same manner.

Let

$$\{p_k, k = 0, 1, 2, \dots\}, \quad p_k \geq 0, \quad \sum p_k = 1$$

be the distribution of ξ , called the offspring distribution, and that of Z_0 the initial distribution of the process.

The transition functions of Galton-Watson processes are defined in terms of a

given offspring distribution $\{p_k : k = 0, 1, 2, \dots\}$ by

$$p(i, j) = p(Z_{n+1} = j | Z_n = i) \\ = \begin{cases} p_j^{*i} & \text{if } i \geq 1, \quad j \geq 0 \\ \delta_{i,j} & \text{if } i = 0, \quad j \geq 0, \end{cases}$$

where $\{p_k^{*i}, k = 0, 1, 2, \dots\}$ is the i -fold convolution of $\{p_k : k = 0, 1, 2, \dots\}$.

Remark 1.1.1 *Let*

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \neq 1; \quad f_0(s) = s,$$

$$f_1(s) = f(s), \quad f_{n+1}(s) = f([f_n(s)]),$$

$$u(s) = a[f(s) - s]$$

and

$$m = EZ_1 = \sum_j p(1, j)j = f'(1), \quad \lambda = u'(1) = a(m - 1).$$

Then when $m < 1$, $= 1$, or > 1 , we shall refer to the Galton-Watson process as subcritical, critical or supercritical, respectively.

Now we are going to state some results which are required later.

Theorem 1.1.1 $\lim_{n \rightarrow \infty} P\{Z_n = k\} = 0$ for $k \geq 1$. Furthermore,

$$(1.1.1) \quad P\{\lim_n Z_n = 0\} = 1 - P\{\lim_n Z_n = \infty\} = q,$$

where q is the smallest nonnegative root of the equation $t = f(t)$.

Proof. cf. Athreya and Ney (1977) [3], p8.

□

Next is the exponential limit law for the critical process:

Theorem 1.1.2 *If $m = 1$ and $\sigma^2 < \infty$ then as $n \rightarrow \infty$*

$$(1.1.2) \quad [1 - f_n(0)] = P\{Z_n > 0\} \sim \frac{2}{n\sigma^2}.$$

Proof. See Athreya and Ney (1977) [3], p19.

□

Theorem 1.1.3 *If $m = 1$ and $\sigma^2 < \infty$ then*

$$(1.1.3) \quad \lim_{n \rightarrow \infty} P\left\{\frac{Z_n}{n} > z | Z_n > 0\right\} = \exp\left\{-\frac{2z}{\sigma^2}\right\}, z \geq 0.$$

Proof. See Athreya and Ney (1977) [3], p20.

□

Of particular interest is the Feller diffusion approximation which was formulated by Feller (1951) and proved by Jirina (1969) and Lindvall (1972). These results have been extended to the age-dependent case by Jagers (1971).

Theorem 1.1.4 *Let $E\xi_k^n = 1$ (the critical case) and let $\text{Var}(\xi_k^n) = \sigma^2 < \infty$. Suppose that $\frac{Z_0^{(m)}}{m}$ converges in distribution, where $Z_k^{(m)}$ is a sequence of Galton-Watson processes.*

If $W_m(t) = \frac{Z_{[mt]}^{(m)}}{m}$, then $W_m(t)$ converges in distribution in $D_{[0,\infty)}([0,\infty))$ to a diffusion process with generator as $m \rightarrow \infty$:

$$Ag(x) = \frac{1}{2}\sigma^2 x g''(x)$$

for $g \in C_c^\infty([0,\infty))$. where $D_{[0,\infty)}([0,\infty))$ denotes the space of positive right continuous functions with left limits on $[0,\infty)$ and $C_c^\infty([0,\infty))$ is the space of infinitely differentiable functions with compact support on $[0,\infty)$.

Proof. Note that $\frac{Z_n^{(m)}}{m}$ is a Markov chain with values in $E_m := \{\frac{j}{m} : j = 0, 1, 2, \dots\}$, and define

$$T_m g(x) = E[g(\frac{1}{m} \sum_{k=1}^{mx} \xi_k)],$$

where the ξ_k are independent and distributed as the ξ_k^n . To prove the theorem, it is sufficient to show

$$(1.1.4) \quad \lim_{m \rightarrow \infty} \sup_{x \in E_m} |m(T_m g(x) - g(x)) - \frac{1}{2}\sigma^2 x g''(x)| = 0, \quad g \in C_c^\infty([0,\infty)).$$

It is easy to see that, for $x \in E_m$,

$$\begin{aligned} (1.1.5) \quad & |m(T_m g(x) - g(x)) - \frac{1}{2}\sigma^2 x g''(x)| \\ &= |mE[g(\frac{1}{m} \sum_{k=1}^{mx} \xi_k) - g(x)] - \frac{1}{2}E[(\frac{1}{\sqrt{m}} \sum_{k=1}^{mx} (\xi_k - 1))^2]g''(x)| \end{aligned}$$

$$\begin{aligned}
 &= |E[\int_0^1 S_{mx}^2 x(1-v)(g''(x + v\sqrt{\frac{x}{m}} S_{mz}) - g''(x))dv]| \\
 &\leq E[\int_{0 \vee (1-\frac{c}{x})}^1 S_{mx}^2 x(1-v)2||g''||dv] \\
 &= E[x||g''||((\frac{c}{x}) \wedge 1)^2 S_{mx}^2],
 \end{aligned}$$

where $S_n = n^{-\frac{1}{2}} \sum_{k=1}^n (\xi_k - 1)$ and suppose the support of g is contained in the interval $[0, c]$. Since

$$(1.1.6) \quad \lim_{m \rightarrow \infty} E[x_m ||g''||((\frac{c}{x_m}) \wedge 1)^2 S_{mx_m}^2] = E[x ||g''||((\frac{c}{x}) \wedge 1)^2 V^2],$$

where $x_m \rightarrow x$, for $0 < x < \infty$ and $S_{mx_m} \Rightarrow V \sim N(0, \sigma^2)$, the dominated convergence theorem implies that

$$\lim_{m \rightarrow \infty} [m(T_m g(x) - g(x)) - \frac{1}{2} \sigma^2 x g''(x)] = 0.$$

Thus the proof of the theorem is complete. (cf. S.Ethier and T.Kurtz (1985) [26]).

(b) Markov branching process

In the Galton - Watson process, the lifetime of each particle is one unit of time. A natural generalization is to allow these lifetimes to be random variables. Instead of the discrete time Markov chain $\{Z_n, n = 0, 1, 2, \dots\}$, we must consider a process $\{Z(t) : t \geq 0\}$, where $Z(t)$ denotes the number of particles at time t . We only consider the case in which the lifetimes are independent exponentially distributed random variables.

Definition 1.1.1 *A stochastic process $\{Z(t) : t \geq 0\}$ on a probability space $(\Omega, \mathfrak{F}, P)$ is called a one dimensional continuous time Markov branching process if*

(i) its state space is the set of non-negative integers:

(ii) it is a stationary Markov chain with respect to $\mathfrak{F}_t = \sigma\{Z(s, \omega), s \leq t\}$;

(iii) the transition probabilities $p_{i,j}(t)$ satisfy

$$(1.1.7) \quad \sum_{j=0}^{\infty} p_{i,j}(t) s^j = \left[\sum_{j=0}^{\infty} p_{1,j}(t) s^j \right]^i$$

for all $i \geq 0$ and $|s| \leq 1$.

Remark 1.1.2 (i) and (ii) imply that $Z(t)$ is a continuous time Markov process on the integers; (iii) characterizes the basic branching property.

The transition functions of the Markov branching process can be determined by the infinitesimal rate $0 < a < \infty$, and the offspring distribution $0 \leq p_i, \sum_{i=0}^{\infty} p_i = 1$ as the solution of the Kolmogorov forward and backward equations:

$$(1.1.8) \quad \frac{d}{dt} p_{i,j}(t) = -j a p_{i,j}(t) + \sum_{k=1}^{j+1} k a p_{i,k}(t) p_{j-k+1},$$

(forward equation)

$$(1.1.9) \quad \frac{d}{dt} p_{i,j}(t) = -i a p_{i,j}(t) + i a \sum_{k=i-1}^{\infty} p_{k-i+1} p_{k,j}(t)$$

(backward equation)

with boundary conditions

$$p_{i,j}(0) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Note that the Markov branching process $\{Z(t), t \geq 0\}$ has generator (on an appropriate domain):

$$(1.1.10) \quad Ag(k) = \sum_j akp_j(g(k-1+j) - g(k)).$$

The infinitesimal probabilities have probabilistic interpretation as follows: The process starts with a specified number of particles. A given particle lives an exponentially distributed lifetime with parameter a and at death is replaced by a random number of offspring with probabilities $\{p_k : k = 0, 1, \dots\}$. As usual, particles act independently of other particles. Consequently

$$p_{i,j}(\varepsilon) = iap_{j-i+1}\varepsilon + 0(\varepsilon)$$

as $\varepsilon \longrightarrow 0$ if $j \geq i-1, j \neq i$;

$$p_{i,i}(\varepsilon) = 1 - ia\varepsilon + 0(\varepsilon)$$

as $\varepsilon \longrightarrow 0$ and

$$p_{i,j}(\varepsilon) = 0(\varepsilon)$$

as $\varepsilon \longrightarrow 0$ if $j < i-1$.

Remark 1.1.3 *When particles have general (i.e., not exponential) lifetime distribution, then the process $Z(t)$ which gives to the number of particles existing at t is called an age-dependent process. (It is not Markovian).*

Remark 1.1.4 *The Kolmogorov forward and backward equations always have solutions satisfying $\sum_j p_{i,j}(t) \leq 1$. We allow the possibility that $\sum_j p_{i,j}(t) < 1$, corresponding to a case in which $Z(t)$ may reach ∞ in a finite period of time. In such cases, the backward equations have solutions that do not satisfy the forward equations. It can be shown that the forward equations are satisfied by only one set of transition probabilities. We shall see that under an additional hypothesis to be given below, Kolmogorov equations have a unique solution $p_{i,j}(t)$ satisfying $\sum_j p_{i,j}(t) = 1$.*

Non-Explosion Hypothesis: For every $\varepsilon > 0$,

$$(1.1.11) \quad \int_{1-\varepsilon}^1 \frac{1}{f(s) - s} ds = \infty.$$

Now we have the following basic results:

Theorem 1.1.5 *The non-explosion hypothesis is necessary and sufficient for $P\{Z(t) < \infty\} = 1$, where $Z(t)$ is the minimal process.*

See T.Harris (1963) [30], p119 for the proof.

□

Remark 1.1.5 *According to Feller (1940), the minimal process means that the process with the transition probabilities $\{p_{i,j}(t)\}$ which are solutions of Kolmogorov equations and any other solutions $p_{i,j}^*(t)$ satisfy $p_{i,j}^*(t) \geq p_{i,j}(t)$ for $i, j, = 0, 1, 2, \dots$.*

The asymptotic behaviour of $Z(t)$ for large t is very similar to that of the discrete time Galton-Watson process:

Theorem 1.1.6 *The family $\{Z(t) \exp\{-\lambda t\}, \mathfrak{F}_t; t \geq 0\}$ is a nonnegative martingale and hence*

$$(1.1.12) \quad \lim_{t \rightarrow \infty} Z(t) \exp\{-\lambda t\} = W \quad \text{exists in probability 1.}$$

Proof. cf. Athreya and Ney (1977) [3], p111.

□

Next result is an analogue of Theorem 1.1.2:

Theorem 1.1.7 *If $\lambda = 0$ and $\sigma^2 < \infty$, then*

$$(1.1.13) \quad \lim_{t \rightarrow \infty} P\{Z(t) > \frac{\sigma^2}{2}tx | Z(t) > 0\} = \exp\{-r\}$$

The proof may be found in Athreya and Ney (1977) [3], p113.

□

In the subcritical case, we have the following theorem:

Theorem 1.1.8 *If $\lambda < 0$, then*

$$(1.1.14) \quad \lim_{t \rightarrow \infty} P\{Z(t) = j | Z(t) > 0\} = \pi_j, j \geq 0$$

exists, where $\pi_j \geq 0$ and $\sum_{j=0}^{\infty} \pi_j = 1$.

The idea of the proof is the same as in the discrete time.

□

(c) Multi-type Markov Branching Process

Another natural direction for generalization is to consider a number of types of particles having different probabilistic behaviour, and it is to such processes that we now turn to. In this section, we only consider a finite number N of particle types. Such processes arise in a variety of biological and physical applications, where for example, they could represent genetic types in an animal population, mutant types in a bacterial population, electrons, photons, nucleons, etc. in a cosmic ray cascade, and so on. It is also possible to construct processes with a countable number or a continuum of types, or even with an abstract space as a "type-space". We shall see later that branching diffusion process provides an example of "continuous type" processes. However we prefer to consider them as measure-valued processes, because this setting can be extended to the more general case.

We assume that there are N types of particles with each particle capable, at its death, of producing particles not only of its type but of other types as well. We assume that each particle lives a random length of time, which has an exponential distribution.

To define the particle production of an N -type process, we need N generating functions, each in N variables. The i th generating function f_i will determine the distribution of the number of offspring of various types to be produced by a type i

particle:

$$f^{(i)}(s_1, s_2, \dots, s_N) = \sum_{j_1, \dots, j_N \geq 0} p^{(i)}(j_1, \dots, j_N) s_1^{j_1} \dots s_N^{j_N}$$

$$0 \leq s_\alpha \leq 1, \alpha = 1, \dots, N.$$

where $p^{(i)}(j_1, \dots, j_N)$ equals to the probability that a type i parent produces j_1 particle of type 1, j_2 of type 2, $\dots$, j_N of type N . (see, e.g., Athreya, K.B and Ney, P.E. 1977 [3] for further details).

In order to define multi-type branching processes, we adopt the following conventions.

Let

Z^N be the N -dimensional lattice space,

$$Z_+^N = \{(x_1, \dots, x_N) \in Z^N; x_i \geq 0, i = 1, \dots, N\},$$

R^N be the N -dimensional Euclidean space,

$$R_+^N = \{(x_1, \dots, x_N); x_i \geq 0, i = 1, \dots, N\},$$

$$B^N = \{(x_1, \dots, x_N) \in R^N; 0 \leq x_i < 1, i = 1, \dots, N\}.$$

We write

$$\mathbf{p}(\mathbf{j}) = (p^{(1)}(\mathbf{j}), \dots, p^{(N)}(\mathbf{j})),$$

$$\mathbf{f}(\mathbf{s}) = (f^{(1)}(\mathbf{s}), \dots, f^{(N)}(\mathbf{s})),$$

where $\mathbf{j} = (j_1, \dots, j_N) \in Z_+^N$, and $\mathbf{s} = (s_1, \dots, s_N) \in B^N$.

Definition 1.1.2 *A multi(N)-type Galton-Watson process is a Markov chain*

$\{Z_n, n = 0, 1, \dots\}$ on R_+^N with transition function:

$$(1.1.15) \quad P(i, j) = P\{Z_{n+1} = j | Z_n = i\} = \text{coefficient of } \mathbf{s}^j \text{ in } [f(\mathbf{s})]^i,$$

where $i, j \in Z_+^N$, $\mathbf{s} \in B^N$.

If particles have random lifetimes which are independent exponentially distributed random variables, then we have the following definition:

Definition 1.1.3 A stochastic process $\{Z(t, \omega); t \geq 0\}$ on a probability space $(\Omega, \mathfrak{F}, P)$ is called an N -dimensional continuous time Markov branching process if

(i). its state space is Z_+^N ;

(ii). it is a stationary strong Markov process with respect to the fields

$$\mathfrak{F}_t = \sigma\{Z(s, \omega); s \leq t\};$$

(iii). the transition functions $P(t; i, j)$ satisfy

$$(1.1.16) \quad \sum_{j \in Z_+^N} P(t; i, j) \mathbf{s}^j = \prod_{k=1}^N \left[\sum_{j \in Z_+^N} P(t; \mathbf{e}_k, j) \mathbf{s}^j \right]^{i_k}$$

for all $i = (i_1, i_2, \dots, i_N) \in Z_+^N$ and $\mathbf{s} \in B^N$, where $\mathbf{e}_k = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in the k th component. $Z(t) = (Z_1(t), \dots, Z_N(t))$, $t \geq 0$, and $Z_i(t)$ can be considered as the number of type i particles existing at time t .

The basic construction and limit theory of multi-type branching processes can be found in Athreya and Ney (1977) [3]. For further developments in this direction,

for example, we refer to B.G. Ivanoff (1981) [34], A.Joffe and M. Metivier (1986) [35] and L. Gorostiza and J. Lopez-Mimbela (1989) [29].

For instance, in the last paper, the authors extend the multi-type branching processes to the measure-valued case. They consider the multi-type measure branching process as a small particle limit of a system of particles of N types in R^d with immigration undergoing migration, branching and mutation. They characterize the limit process as a solution of a martingale problem.

d) Branching Diffusion Processes and Dawson-Watanabe Processes

A very interesting model for generalization is the branching diffusion process or branching random field in R^d , which is described as follows.

If $X(t)$ denote the random distribution of particles in R^d , then at each instant the process $X(t)$ is described by an infinitely divisible random field of particles in R^d . At $t = 0$, $X(0)$ is given by a Poisson random field. Each particle has a random lifetime which is exponentially distributed. At the end of its lifetime, each particle branches into n identical particles with probability $p_n, n = 0, 1, 2, \dots$. In addition, during its lifetime each particle moves in R^d according to a Markov motion process. Note that if B is a Borel subset of R^d , then $X(t, B)$ denotes the number of particles in the set B at time t .

If let $Z_t =$ number of particles at time t , then the branching diffusion process

X_t can be expressed as

$$X_t = \{X_t^1, X_t^2, \dots, X_t^{Z_t}\},$$

where $X_t^j, j = 1, 2, \dots, Z_t$ denotes the location of j th particle.

Let

$$f(s) = \sum_{n=0}^{\infty} p_n s^n$$

be the offspring generating function and lifetime parameter be c . Then $\{Z_t\}$ has its infinitesimal generating function as follows:

$$u(s) = c[f(s) - s],$$

$$m = f'(1) \begin{cases} < 1 & \text{(subcritical case)} \\ = 1 & \text{(critical case)} \\ > 1 & \text{(supercritical case)} . \end{cases}$$

A branching diffusion process can be considered as a $N(R^d)$ -valued process (cf. Dawson 1975 [17]).

Now we assume that the motion process is a d -dimensional Brownian motion .

We have the following theorems.

Theorem 1.1.9 *Let $f''(1-) < \infty$, $f'(1) > 1$, $f(0) = 0$. Then for any bounded domain $B \subset R^d$.*

$$(1.1.17) \quad P_x \left\{ \lim_{t \rightarrow \infty} (2\pi t)^{\frac{d}{2}} \frac{X(t, B)}{Z_t} = |B| \right\} = 1,$$

where $|B|$ is the Lebesgue measure of B and P_x denotes the probability measure corresponding to starting with one particle at x .

Proof. See Athreya and Ney (1977) [3], p243.

□

Theorem 1.1.10 (*G.Ivanoff*) Assume that all moments of the offspring distribution are finite.

(i). For fixed t , $X(t)$ satisfies the Brillinger mixing condition, that is, if $q_n(x_1, \dots, x_n)$ denotes the n th order factorial cumulant density function, then

$$(1.1.18) \quad \int \cdots \int |q_n(x_1, \dots, x_n)| dx_1 \cdots dx_{n-1} < \infty.$$

(ii). Given a Borel set A , let $kA = \{x : x = kA, y \in A\}$. If A is compact, then as $k \rightarrow \infty$

$$(1.1.19) \quad \frac{X(t, kA) - EX(t, kA)}{\sqrt{\text{Var}(X(t, kA))}} \rightarrow N(0, 1).$$

That is, $X(t, kA)$ is asymptotically normally distributed.

Branching diffusion processes (or branching random fields) in a random environment have been studied by several authors, and many important results in this direction have been obtained. For example, the persistence for a model of branching random walk in a random environment is considered by D.Dawson and K.Fleischmann (1985) [7], and they show that there exists a critical dimension d_c , such that the system is persistent if and only if $d > d_c$ and fails to be persistent if $d < d_c$.

Dawson-Watanabe Process:

The Dawson-Watanabe Process as the measure diffusion approximation to a branching diffusion process was given by Watanabe (1968) [43] and Dawson (1975) [17]. This process possesses interesting properties which have been studied by Dawson (1975 [17], 1977 [18], 1979 [8]), Dawson-Hochberg (1979) [11], P.J.Fitzsimmons (1988) [27], I.Iscoe (1988) [33], and others.

Watanabe (1968) [43] treats the case of finite-measure states in which the spatial part of the process is diffusion in R^d . The construction of the measure-valued branching diffusion processes which take their values among finite measures on standard Borel spaces and whose spatial part is not necessarily a diffusion process is due to Dynkin (1989a) [23].

In Dawson (1975) [17], infinite-measure states are considered in the case in which the spatial motion is Brownian motion or a symmetric stable process on R^d and in which the branching mechanism has finite variance. These are extended to the case in which the branching mechanism has infinite variance by I.Iscoe (1988) [33], and are extended to the case in which the spatial part of the measure-valued process is any regular Markov process by A.M. Etheridge (1990) [25].

To complete the introduction about Dawson-Watanabe processes, we provide a list of notations. Let

$M(R^d)$ denote the all measures on R^d ,

$N(Z^+)$ denote the collection of the integer-valued measures on Z^+ .

Remarks 1.1.1 1) A $M(R^d)$ -valued process is said to be multiplicative if for all $\mu_1, \mu_2 \in M(R^d)$, $P_{\mu_1 + \mu_2} = P_{\mu_1} * P_{\mu_2}$, where “*” denotes convolution. In other words, the distribution of the process with initial condition $\mu_1 + \mu_2$ is equal to the sum of independent versions of the processes with initial conditions μ_1 and μ_2 respectively.

2) It is known that the symmetric stable process of index α in R^d is transient if and only if $d > \alpha$.

Dawson-Watanabe processes have been extensively studied during the last fifteen years. Now we proceed to state some important results about local and long time behaviour of the measure-valued branching processes.

Let A be a subset of R^d . The Hausdorff α -measure is defined by

$$(1.1.20) \quad H^\alpha(A) = \lim_{\delta \downarrow 0} \inf_{\substack{\cup B_i \supset A \\ d(B_i) \leq \delta}} \sum (d(B_i))^\alpha,$$

where $d(B_i)$ denotes the diameter of B_i and each B_i is a closed sphere. There exists a unique $0 \leq \beta \leq d$ such that $\alpha < \beta \implies H^\alpha(A) = \infty$, $\alpha > \beta \implies H^\alpha(A) = 0$. β is called the Hausdorff-Besicovitch dimension of A , $\dim(A)$.

Let $\{X(t) : t \geq 0\}$ be the multiplicative critical stochastic measure diffusion process in R^d defined by the Laplace functional

$$(1.1.21) \quad \begin{aligned} E[\exp\{-\int g(x)X(t, dx)\} | X(0) = v] \\ = \exp\{-\int u(t, x)v(dx)\}, \end{aligned}$$

where $u(t, x)$ satisfies the nonlinear initial value problem

$$(1.1.22) \quad \begin{aligned} \frac{\partial u(t, x)}{\partial t} &= G_\alpha u(t, x) - ru^2(t, x), t > 0, \\ u(0, x) &= g(x), \end{aligned}$$

where r is a given positive constant, and G_α denotes the infinitesimal generator of the Markov semigroup associated with the symmetric stable process of index α , $0 < \alpha \leq 2$ on R^d . Then we have

Theorem 1.1.11 (*Dawson and Hochberg 1979 [11]*) *For fixed $t > 0$, there exists a random set A such that*

$$(1.1.23) \quad X(t, \omega, C \cap A(\omega)) = X(t, \omega, C)$$

for every compact set C and almost every ω , and

$$(1.1.24) \quad \dim A(\omega) \leq \alpha$$

for every ω .

Theorem 1.1.12 *For $d = \alpha = 1$ or 2 , $X(t, \cdot)$ is almost surely singular for $t > 0$. For $d < \alpha$, then $X(t, \cdot)$ has an L^2 - density.*

Proof. This is proved in Dawson and Hochberg (1979) [11] for the case $\alpha = 2$ and in the case $\alpha < 2$ in Roelly-Coppoletta (1984) . The existence of an L^2 density is established in Roelly- Coppoletta (1984).

□

U.Zähle (1988) [46], [47], [48] considered the carrying Hausdorff dimension of the more general processes, so-called localizable measure-valued Markov processes. He obtained not only an upper bound of the Hausdorff dimension, but also a lower bound as well.

Theorem 1.1.13 (*Zähle 1988*). *Let*

$$(1.1.25) \quad D = \inf \{ \varepsilon > 0 : \liminf_{s \rightarrow 0+} Q_s B(0; s^{1/\varepsilon})^c = 0 \}$$

where $B(x, r)$ denotes the closed ball in R^d with center x and radius r and $Q_t, t \geq 0$, is the transition probability of a homogeneous space-homogeneous Markov process on R^d , i.e.,

$$Q_{s+t} = Q_s * Q_t, \quad S_t g(x) = \int g(x+z) Q_t(dz),$$

$$S_t(g((\cdot) - x))(z) = (S_t g)(z - x), \quad x \in R^d, \quad g \in \mathcal{F},$$

where $\mathcal{F}$ is the set of all non-negative measurable functions on R^d . Then there exists a sequence Ξ_n of random closed sets on R^d , given over $(\mathcal{M}[(R^d)^{[0,t]}],$

$\sigma(\mathcal{M}[(R^d)^{[0,t]}]), \mathfrak{R}_{t,\mu}$), such that $\dim \Xi_n \leq D, \mathfrak{R}_{t,\mu}$ -a.s.,

$$(1.1.26) \quad \xi \circ \pi_t^{-1} \left(\bigcap_{n \in N} \Xi_n \right)^c = 0$$

$\mathfrak{R}_{t,\mu}$ -a.s., where $\mathcal{M}(E)$ denotes the family of all random measures on E , $\sigma(E)$ denotes the usual σ -algebra in $\mathcal{M}(E)$, $\mathfrak{R}_{t,\mu}$ denotes the backward tree distribution $\mathfrak{R}_\mu^{[0,t]}$ on $\mathcal{M}[(R^d)^{[0,t]}]$ and the π_t denotes the projection of $(R^d)^{[0,t]}$ onto the last factor.

Proof. See Zähle (1988) [48].

□

Theorem 1.1.14 *If $Q_s, s > 0$, has density q_s with respect to $\mathcal{L}^d$, if there exists some $\varepsilon > 0$ such that*

$$(1.1.27) \quad \sup_{x \in \mathbb{R}^d} |x|^{-d-\varepsilon} q_t(x) < \infty$$

and if there exists $D > 0$ and a bounded and $\mathcal{L}^d$ - integrable function g on $\mathbb{R}^d$ satisfying

$$(1.1.28) \quad \delta^{d/D} q_\delta(\delta^{1/D} x) \leq g(x), \delta > 0, x \in \mathbb{R}^d,$$

then

$$(1.1.29) \quad \dim B \geq \min\{D, d\}$$

whenever $\nu(B) > 0$, $B \subset \mathbb{R}^d$ holds for $K_{t,\mu}$ - a.e. ν . Here $\mathcal{L}^d$ denotes Lebesgue measure on $\mathbb{R}^d$ and $K_{t,\mu}$ denotes the distribution of the random measure which is the result of the evolution over the time span t when starting with μ .

Proof. See U.Zähle (1988) [48].

□

The question of asymptotic behaviour is also interesting. Dawson has shown that if the spatial part of the measure-valued process is Brownian motion or a symmetric stable process on $\mathbb{R}^d$, and if we take the initial value of the process to be Lebesgue

measure, then existence or non-existence of a non-trivial limiting distribution is equivalent to transience or recurrence of the spatial motion. That is we have the following theorem:

Theorem 1.1.15 (*Dawson 1977 [18]*). *Let $\{X(t) : t \geq 0\}$ be critical measure diffusion process, whose spatial part is a symmetric stable process of index $0 < \alpha \leq 2$ on R^d . Then*

(i) *if spatial diffusion is transient, then there is a parameter family $\{P_\rho : 0 < \rho < \infty\}$ of probability measures on $M(R^d)$, which are both steady state random measures for the process and also invariant under spatial translation. The parameter ρ corresponds to the mean density of the random measure and $P_{\rho_1 + \rho_2} = P_{\rho_1} * P_{\rho_2}$ where “ $*$ ” denotes the convolution operation.*

(ii). *If the spatial diffusion is recurrent, then there is no non-trivial steady state random measure and for every compact set $K \subset R^d$, $\varepsilon > 0$, and $\nu \in M_1(R^d)$,*

$$(1.1.30) \quad \lim_{t \rightarrow \infty} P_\nu(X(t, K) > \varepsilon) = 0,$$

where $M_1(R^d) = \{\mu \in M(R^d) : 0 < \lim_{k \rightarrow \infty} \frac{\lambda(A_k)}{\mu(A_k)} < \infty\}$ and A_k denotes the cube centred at the origin in R^d of side k .

Dawson's results above have been extended to a more general class of processes by Dynkin (1989b) [22] and Etheridge (1990) [25]. If we consider the spatial motion as a regular Markov process, then Etheridge (1990) [25] shows that under some

mild conditions, recurrence of the spatial motion implies "local extinction" of the measure-valued process, and the transience of the spatial motion implies existence of a non-trivial limiting distribution is a result of Dynkin (1989b) [22].

The global size of the supports of the measure-valued critical branching Brownian motion $\{X(t) : t \geq 0\}$ was investigated in I. Iscoe (1988) [33], where the hitting (i.e., charging) probabilities of distant balls are evaluated asymptotically and are shown to depend strongly on the dimension, d , of the underlying Euclidean space R^d .

Theorem 1.1.16 (*I. Iscoe*). *Let $\mu(R^d) < +\infty$ with $\text{supp } \mu \subset B(0, R_0)$ and let $R \geq R_0$. Then*

$$(1.1.31) \quad \begin{aligned} P_\mu(X \text{ ever charges } [\overline{B}(0; R)]^c) \\ = 1 - \exp(-R^{-2} \langle u(\frac{\cdot}{R}), \mu \rangle), \end{aligned}$$

where u is the unique positive solution of the singular elliptic boundary value problem

$$(1.1.32) \quad \begin{aligned} \Delta u(x) &= u^2(x), x \in B(0, 1), \\ u(x) &\longrightarrow +\infty, \text{ as } x \rightarrow \partial B(0, 1). \end{aligned}$$

In the case $d = 1$, the value of $u(0)$ is given by

$$(1.1.33) \quad \begin{aligned} u(0) &= \frac{1}{6} \left\{ \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} \right\}^2 \in (8.372, 8.388). \\ u(0) &\simeq 8.38 \end{aligned}$$

Theorem 1.1.17 (*I. Iscoe 1988 [33]*). For each $\varepsilon > 0$,

$$P_{\delta_0}(X \text{ ever charges } B(x, \varepsilon)) = \begin{cases} \frac{6}{x^2} & , \quad d = 1 \\ \frac{4}{x^2} & , \quad d = 2 \\ \frac{2}{x^2} & , \quad d = 3 \\ \frac{2}{[x^2 \log |x|]} & , \quad d = 4 \\ \frac{ce^{d-4}}{|x|^{d-2}} & , \quad d \geq 5, \end{cases}$$

where c is some constant (depending only on the dimension d).

Finally, we are going to state Perkins' three much sharper results on the Hausdorff measure of the support of the measure-valued branching diffusions. In the remainder of this section we assume that X_t is the coordinate process on $(C([0, \infty), M_f(R^d)), \mathcal{B}(M_f), Q^\mu)$, where $M_f(R^d)$ is the set of finite measures on R^d equipped with the topology of weak convergence of measures, $C([0, \infty), M_f)$ denotes the space of continuous functions from $[0, \infty)$ to M_f with the compact-open topology, and $\mathcal{B}(M_f)$ is the class of Borel subsets. We define

$$\mathcal{H} = \{ \phi : [0, \delta_0] \longrightarrow [0, \infty) : \delta_0 > 0, \phi \text{ strictly increasing and continuous} \}.$$

If $\phi \in \mathcal{H}$ and $A \subset R^d$, $\phi - m(A)$ denotes the Hausdorff ϕ -measure of A , i.e.,

$$\phi - m(A) = \lim_{\delta \rightarrow 0} \inf \{ \sum_{i=1}^{\infty} \phi(d(B_i)) : B_i \text{ are ball of diameter } d(B_i) \leq \delta$$

such that $A \subset \bigcup_{i=1}^{\infty} B_i \}$.

If $x > 0$, let

$$\log^+ x = (\log x) \vee 0, \quad \phi_\alpha(x) = x^\alpha \log^+ \log^+ \left(\frac{1}{x} \right),$$

$$\phi_\alpha^{(1)}(x) = x^\alpha \log^+(\frac{1}{x}), \quad \phi_\alpha^{(2)}(x) = x^\alpha (\log^+(\frac{1}{x}))(\log^+ \log^+(\frac{1}{x})),$$

$$\phi_\alpha^{(3)}(x) = x^\alpha (\log^+(\frac{1}{x}))^2.$$

Theorem 1.1.18 (Perkins 1986, [38], p5) *Let $d > \alpha$. There are constants*

$0 < C_1(\alpha, d) \leq C_2(\alpha, d) < \infty$ such that for any $\mu \in M_f$ and $Q^\mu - a.a.\omega$:

$\forall t > 0$ there is a random Borel set, $\Lambda_t(\omega)$, which supports X_t and satisfies

$$(1.1.34) \quad C_1(\alpha, d)\phi_2 - m(A \cap \Lambda_t) \leq X_t(A) \leq C_2(\alpha, d)\phi_\alpha - m(A \cap \Lambda_t)$$

$$\forall A \in \mathcal{B}(R^d).$$

The theorem say that X_t spreads its mass over its support in a very uniform manner and does so for all $t > 0$ simultaneously.

Theorem 1.1.19 (Perkins 1986) [38] *Let $d = \alpha$. There are constants $0 < C_1(\alpha, d) \leq$*

$C_2(\alpha, d) < \infty$ such that for any $\mu \in M_f$ and $Q^\mu - a.a.\omega$:

$\forall t > 0$ there is a random Borel support for X_t , Λ_t , which satisfies

$$(1.1.35) \quad C_1(\alpha, d)\phi_\alpha^{(1)} - m(A \cap \Lambda_t) \leq X_t(A) \leq C_2(\alpha, d)\phi_\alpha^{(3)} - m(A \cap \Lambda_t)$$

$$\forall A \in \mathcal{B}(R^d).$$

Theorem 1.1.20 (Dawson and Perkins 1990) [15] *If X_t is super Brownian motion in R^d , $d \geq 3$, starting at $\mu \in M_f(R^d)$ under Q^μ , then there is a $C(d) > 0$, such that*

$$(1.1.36) \quad X_t(A) = C(d)\phi_2 - m(A \cap \Lambda_t)$$

$$\forall A \in \mathcal{B}(R^d) \quad Q^\mu - a.s. \quad \forall t > 0, \quad \mu \in M_f(R^d).$$

All of the processes we have studied up to now are one level branching models. Next section we are going to consider a multilevel branching system.

1.2 Multilevel Birth and Death Particle System

Multilevel branching systems were first introduced by D.Dawson and K.Hochberg in 1991 [5]. In this section, we consider a more general dynamic multilevel information structure. We suppose that a collection of information units at a given level comprises an information unit at the next higher level . At each level, individual information units undergo a Galton-Watson type branching process in which they can be copied or removed. Dynamic multilevel information structures have been used to describe the replication, updating and transfer of digitized data collections as they are processed through information networks. Once duplicated, copies of such data undergo alteration independent of other extant copies, and then each in turn is duplicated and transferred to others, again to undergo independent alterations. Analogously, colonies of individuals, be they allelic variations at some genetic locus or members of animal or plant populations, may undergo similar multilevel restructuring, with individuals evolving both at the local level and at the colony level, in the sense that genetic structures of individuals may be altered through births and deaths within a given colony, while the colony itself may disappear at random times due to natural calamities or it may replicate, say to establish a copy of itself at a new habitat. For example, such dynamic multilevel information system can be used

to describe new mutations on mitochondrial DNA.

For simplicity, we restrict our attention to the two level case. The dynamic two level population system can be studied from several different points of view. The superparticle process can be considered as a multi-type branching process in which the type corresponds to the number of particles it contains; thus the space Z^+ of non-negative integers serves as the type space. Another possibility is to view the superparticle process as a branching random walk on Z^+ , where again, a superparticle consisting of k particles will be located at site k . However, we will describe it as a two level birth and death process, since this is the framework that most naturally extends to higher level situations.

We consider the two level birth and death process $\{X(t), t \geq 0\}$ as a $N(Z^+)$ -valued pure jump Markov process. We assume that each level 1 particle can be removed or copied according to a level one birth and death process. Each level 2 particle which is a collection of level one particles performs a level two birth and death process and different level two particles (i.e. superparticles) maybe perform different level two birth and death processes which depend on the size of the superparticles (We say that a superparticle is of size i if it consists of exactly i level 1 particles). A measure in $N(Z^+)$ can be represented by a vector $((n_i), i \in Z^+)$ where n_i denotes the number of level two particles (superparticles) of size i . The possible transitions of state in this process can be described as follows:

(1) Birth of a level two particle of size i :

$$\mu \longrightarrow \mu + \delta_i \text{ at rate } \lambda_2(n_i, i),$$

i.e.
$$n_i \longrightarrow n_i + 1$$

at the rate which depends on the size i and the number of the level two particles of size i .

Figure 1.1: Birth of a Superparticle of Size i

(2) Death of a level two particle of size i :

$$\mu \longrightarrow \mu - \delta_i \text{ at rate } \mu_2(n_i, i),$$

i.e.
$$n_i \longrightarrow n_i - 1$$

at the rate which depends on the size i and the number of the level two particles of size i .

Figure 1.2: Death of a Superparticle of Size i

(3) The birth of a level one particle in a level two particle of size i :

$$\mu \longrightarrow \mu - \delta_i + \delta_{i+1} \text{ at rate } \lambda_1(n_i, i),$$

i.e.

$$(n_i, n_{i+1}) \longrightarrow (n_i - 1, n_{i+1} + 1)$$

at the rate which depends on the size i and the number of the level two particles of size i .

Figure 1.3: Birth of a Particle In a Superparticle of Size i

(4) Death of a level one particle in a level two particle of size i :

$$\mu \longrightarrow \mu - \delta_i + \delta_{i-1} \text{ at rate } \mu_i(n_i, i),$$

i.e. $(n_{i-1}, n_i) \longrightarrow (n_{i-1} + 1, n_i - 1)$

at the rate which depends on the size i and the number of the level two particles of size i .

Figure 1.4: Death of a Particle In a Superparticle of Size i

To describe the generator G of this $N(Z^+)$ -valued Markov process, we consider the class of test functions of the form:

$$(1.2.1) \quad F(\nu) = f\left(\sum_i \phi(i)n_i\right) = f(\langle \phi, \nu \rangle),$$

where $\nu = \sum_i n_i \delta_i$, and $f \in C_b(R^+)$.

By 1) - 4), we obtain

$$(1.2.2) \quad Gf(\nu) = \lim_{t \downarrow 0} \frac{E\{F(X(t)) \mid X(0) = \nu\} - F(X(0))}{t},$$

which is given by

$$(1.2.3) \quad \begin{aligned} GF(\nu) = & \sum_k \lambda_1(\nu(k), k)[f(\langle \phi, \nu \rangle - \phi(k) + \phi(k+1)) - f(\langle \phi, \nu \rangle)] \\ & + \sum_k \mu_1(\nu(k), k)[f(\langle \phi, \nu \rangle - \phi(k) + \phi(k-1)) - f(\langle \phi, \nu \rangle)] \\ & + \sum_k \lambda_2(\nu(k), k)[f(\langle \phi, \nu \rangle + \phi(k)) - f(\langle \phi, \nu \rangle)] \\ & + \sum_k \mu_2(\nu(k), k)[f(\langle \phi, \nu \rangle - \phi(k)) - f(\langle \phi, \nu \rangle)]. \end{aligned}$$

1.3 Existence

In this section, we consider the existence of the multilevel birth and death process. We can use the martingale method to show that there exists a Markov process $\{X(t)\}$, which is the unique solution of the $N(Z^+)$ -valued martingale problem associated with the generator G given in the previous section. Here we would like to prove the existence of the process via the Laplace transform. By Theorem 1.1.5, to

prove the existence, it suffices to demonstrate the absence of explosions. Without loss of generality , we only consider the two level pure birth process.

Theorem 1.3.1 *Suppose that*

(i) *there exists function $\lambda_2(.)$, such that for each $n \in Z^+$*

$$(1.3.1) \quad \sup_i \lambda_2(n, i) \leq \lambda_2(n)$$

and

$$(1.3.2) \quad n\lambda_2(1) \leq \lambda_2(n) \leq nl\lambda_2(1),$$

where $1 \leq l < \infty$;

(ii) *for each $m \in Z^+$*

$$(1.3.3) \quad \sum_i \frac{1}{\lambda_1(m, i)} = \infty.$$

Then the two level pure birth process $\{X(t), t \geq 0\}$ has no explosion.

Proof of the theorem. Let us define the random times between successive births as following:

$$\tau_1 = \inf\{t : X(t) \neq X(0)\},$$

$$\tau_n = \inf\{t : X(t + \tau_1 + \cdots + \tau_{n-1}) \neq X(\tau_1 + \cdots + \tau_{n-1})\},$$

$$T_n = \tau_1 + \cdots + \tau_n.$$

If we let $X(0) = \mu = \sum_i n_i \delta_i \in N(Z^+)$ and $N_0 = \sum_i n_i$, then it is easy to see that

(a). the rate of a level 1 birth is $\sum_i \lambda_1(n_i, i)$;

(b). the rate of a level 2 birth is $\sum_i \lambda_2(n_i, i)$.

Because all jumps of the two level pure birth process are level one or level two births, we can suppose that the first m jumps consist of $k \leq m$ jumps of level one and $m - k$ jumps of level two. Therefore the Laplace transform of T_n is given by

$$(1.3.4) \quad \phi_m(\theta) = \prod_{u=1}^k \left(1 + \frac{\theta}{\sum_i \lambda_1(m_i^{(u)}, i)}\right)^{-1} \prod_{v=1}^{m-k} \left(1 + \frac{\theta}{\sum_i \lambda_2(n_i^{(v)}, i)}\right)^{-1},$$

where we suppose that when the u th level 1 birth occurs, the state of the process $X(t)$ is $\sum_i m_i^{(u)} \delta_i$, and when the v th level 2 birth occurs, the state of the process is $\sum_i n_i^{(v)} \delta_i$.

When m goes to infinity, if the number of jumps of level two goes to infinity, then by (1.3.1), (1.3.2) and (1.3.3), we have

$$\begin{aligned} \phi_m(\theta) &\leq \prod_{u=1}^k \left(1 + \frac{\theta}{\sum_i \lambda_1(m_i^{(u)}, i)}\right)^{-1} \prod_{v=1}^{m-k} \left(1 + \frac{\theta}{\sum_i \lambda_2(n_i^{(v)}, i)}\right)^{-1} \\ &= \prod_{u=1}^k \left(1 + \frac{\theta}{\sum_i \lambda_1(m_i^{(u)}, i)}\right)^{-1} \frac{1}{\prod_{v=1}^{m-k} \left(1 + \frac{\theta}{\sum_i \lambda_2(n_i^{(v)}, i)}\right)} \\ &\leq \prod_{u=1}^k \left(1 + \frac{\theta}{\sum_i \lambda_1(m_i^{(u)}, i)}\right)^{-1} \frac{1}{\prod_{v=1}^{m-k} \left(1 + \frac{\theta}{\sum_i l \lambda_2(1) n_i^{(v)}}\right)} \\ &\leq \prod_{u=1}^k \left(1 + \frac{\theta}{\sum_i \lambda_1(m_i^{(u)}, i)}\right)^{-1} \frac{1}{\prod_{v=1}^{m-k} \left(1 + \frac{\theta}{(N_0 + v) l \lambda_2(1)}\right)} \end{aligned}$$

and

$$\sum_v \frac{1}{(N_0 + v) l \lambda_2(1)} = \infty$$

implies that

$$\prod_{v=1}^{\infty} \left(1 + \frac{\theta}{(N_0 + v)l\lambda_2(1)}\right) = \infty.$$

Thus we obtain that

$$\phi_m(\theta) \longrightarrow 0, \text{ as } m \rightarrow \infty.$$

If the number of jumps of level two is finite, i.e., there exists a $N_1 > 0$, such that $m - k \leq N_1$ for any m , then $\{m_i^{(u)}\}$ is bounded (in particular, $\max_i m_i^{(u)} \leq N_0 + N_1$), and (1.3.3) implies that

$$\begin{aligned} \phi_m(\theta) &\leq \max_{1 \leq j \leq N_0 + N_1} \prod_{u=1}^k \left(1 + \frac{\theta}{(N_0 + N_1)\lambda_1(j, u)}\right)^{-1} \prod_{v=1}^{m-k} \left(1 + \frac{\theta}{\sum_i \lambda_2(n_i^{(v)})}\right)^{-1} \\ &\longrightarrow 0 \end{aligned}$$

as $m \longrightarrow \infty$ in this case.

Therefore we come to the conclusion that $T_m \longrightarrow \infty$ as $m \longrightarrow \infty$ a.e.. That is, the two level pure birth process has at most finite number of jumps in any finite time interval. This means that the process $\{X(t); t \geq 0\}$ has no explosion. The proof of the theorem 1.3.1 is complete.

□

Remark 1.3.1 . *In many practical examples of two level birth and death processes, the conditions (1.3.1), (1.3.2) and (1.3.3) of the Theorem 1.3.1 are met. The multilevel branching process introduced by Dawson and Hochberg in 1991 [5] is such an*

example. However, the conditions (1.3.1), (1.3.2) and (1.3.3) are sufficient but not necessary for the existence of the processes. Let us look at the following example:

Example. We suppose that the birth rate $\lambda_2(n, i)$ of the level two particles satisfy (1.3.1) and (1.3.2). Let $\lambda_1(m, i) = \mu_1(m, i) = i^2$ for $m \in \mathbb{Z}^+$, $i = 1, 2, \dots$ and $\lambda_1(m, 0) = 1$, $\mu_1(m, 0) = 0$.

Since for each m ,

$$\sum_i \frac{1}{\lambda_1(m, i)} = \sum_i \frac{1}{i^2} < \infty,$$

the condition (1.3.3) fails to hold.

It is clear that in any finite time interval, the two level birth and death process given here has at most finite number of level two jumps but it could have an infinite number of level one jumps. However, using Karlin's (1969, [36]) notations, we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} \pi_n \sum_{k=0}^n \frac{1}{\lambda_k \pi_k} \\ &= \sum_{n=0}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \sum_{k=0}^n \frac{1}{k^2} \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n^2} \sum_{k=0}^n 1 = \sum_{n=0}^{\infty} \frac{1}{n^2} (n+1) = \infty, \end{aligned}$$

where λ_k, μ_k can be explained as level one birth and death rates, respectively, in our case. This implies that there exists a unique level one birth and death process, say, $X_1(t)$, such that $P(X_1(t) < \infty) = 1$ for any $t > 0$. Therefore, the two level birth and death process given in this example exists.

Chapter 2

Measure Diffusion Approximation to the Multilevel Birth and Death Particle System

The diffusion approximation to a stochastic population process was originally introduced by W. Feller. It has been extended to branching random fields by S. Watanabe (1968) [43], D. Dawson (1975 [17], 1977 [18]), D. Dawson and G. Ivanoff (1978) [6] and many others.

The basic idea of the diffusion approximation is to assign masses to each particle and each superparticle and to consider the situation in which there are a large number of particles and superparticles in such a way as to obtain non-degenerate means and variances.

The organization of this chapter is as follows. In section 1, we introduce some notations, remarks and definitions. In section 2, we prove the tightness of the rescaled processes. Finally, the limit process is characterized in section 3.

2.1 Notations, Definitions and Rescaled Multi-level Birth and Death Processes

In this section, we will introduce a group of scaling transformations to the two level birth and death process obtained in the previous chapter. Firstly, we provide a list of some basic definitions and notations:

Let

$C_c^\infty(R)$ be the class of infinitely differentiable functions with compact support on R ,

$C^\infty(R)$ be the family of infinitely differentiable functions on R ,

$C_0(R^+) = \{f : R^+ \longrightarrow R : f \text{ continuous and } \lim_{|x| \longrightarrow +\infty} f(x) = 0\}$,

$(\Omega, \mathfrak{F}, P)$ denote a probability space,

$M(R^+)$ be collection of measures on R^+ ,

$M_f(R^+)$ be collection of finite measures on R^+ ,

$N(\varepsilon Z^+)$ denote family of integer-valued measures on εZ^+ , where

$\varepsilon Z^+ = \{\varepsilon n : n \in Z^+\}$,

$\langle \phi, \mu \rangle = \int \phi(x) \mu(dx)$,

$D_{[0,\infty)}([0,\infty))$ denote the set of all mapping from $[0,\infty)$ into $[0,\infty)$ which are continuous from right and possess limits from the left at every point in $(0,\infty)$,

$(C(R^+), \|\cdot\|)$ denote the Banach space of all real valued continuous functions on R^+ with the norm $\|f\| = \sup_{x \in R^+} |f(x)|$,

$C_b(R^+)$ denote the set of bounded continuous functions on R^+ ,

$C_c(R^+) \subset C_b(R^+)$ denote the set of all continuous functions on R^+ with compact support,

$C_c^2(R^+)$ is the set of twice differentiable functions on R^+ with compact support.

We also define:

$\dot{R} = R \cup \{\infty\}$ is the one point compactification of R ,

$\phi_p(x) = \frac{1}{1+|x|^p}$ for $p > 0$, and

$M_p(R) = \{\mu \in M(R) : \langle \phi_p, \mu \rangle < \infty\}$. Iscoe has shown that $M_p(R)$ when furnished with the weakest topology making the maps $\mu \longrightarrow \langle \phi, \mu \rangle$ continuous for every $\phi \in C_c(R)$ and in addition for $\phi = \phi_p$ is locally compact. This topology is a little bit stronger than the vague topology.

In order to consider the diffusion approximation to the multilevel birth and death process, we assume that

$$(2.1.1) \quad \lambda_1(k, j) = [a_1(j) + b_1(j)]k,$$

$$\mu_1(k, j) = [a_1(j) - b_1(j)]k,$$

$$\lambda_2(k, j) = [a_2(j) + b_2(j)]k,$$

$$\mu_2(k, j) = [a_2(j) - b_2(j)]k,$$

where $a_2(\cdot), b_2(\cdot) \in C_b(R^+)$, $a_1(\cdot), b_1(\cdot) \in C(R^+)$ and for $\forall 0 < c < 1$, $a_1(cx) \leq c a_1(x)$, $b_1(cx) \leq c b_1(x)$, and

$$(2.1.2) \quad \lambda_1^e(j) = a_1^e(j) + b_1^e(j),$$

$$\mu_1^\varepsilon(j) = a_1^\varepsilon(j) - b_1^\varepsilon(j),$$

$$\lambda_2^\varepsilon(j) = a_2^\varepsilon(j) + b_2^\varepsilon(j),$$

$$\mu_2^\varepsilon(j) = a_2^\varepsilon(j) - b_2^\varepsilon(j),$$

where $a_1^\varepsilon(j) = a_1(j)/\varepsilon_1^2\varepsilon_2$, $b_1^\varepsilon(j) = b_1(j)/\varepsilon_1\varepsilon_2$, $a_2^\varepsilon(j) = a_2(j)/\varepsilon_2^2$, $b_2^\varepsilon(j) = b_2(j)/\varepsilon_2$. Let $X^\varepsilon(t) = X^{\varepsilon_1, \varepsilon_2}(t)$ be a two level birth and death process with the two level birth and death rates $k\lambda_i^\varepsilon(j)$, $k\mu_i^\varepsilon(j)$, $i = 1, 2$ respectively.

Let

$$(2.1.3) \quad Y^\varepsilon(t, A) = \varepsilon_2 X^\varepsilon(t, A/\varepsilon_1).$$

If we denote by G^ε the generator of the rescaled process $Y^\varepsilon(t)$, then for any $f \in C^\infty(R)$, $\phi \in C_b^2(R^+)$, $F(\mu) := f(\langle \phi, \mu \rangle)$,

$$(2.1.4) \quad G^\varepsilon F(\mu)$$

$$\begin{aligned} &= \sum_j [f(\langle \phi, \mu \rangle + \varepsilon_2 \phi(j\varepsilon_1)) - f(\langle \phi, \mu \rangle)] \mu(j\varepsilon_1) \lambda_2^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [f(\langle \phi, \mu \rangle - \varepsilon_2 \phi(j\varepsilon_1)) - f(\langle \phi, \mu \rangle)] \mu(j\varepsilon_1) \mu_2^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [f(\langle \phi, \mu \rangle - \varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j+1)\varepsilon_1)) - f(\langle \phi, \mu \rangle)] \mu(j\varepsilon_1) \lambda_1^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [f(\langle \phi, \mu \rangle - \varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j-1)\varepsilon_1)) - f(\langle \phi, \mu \rangle)] \mu(j\varepsilon_1) \mu_1^\varepsilon(j\varepsilon_1). \end{aligned}$$

2.2 The Tightness of the Rescaled Multilevel Birth and Death Processes

In order to study the continuous diffusion approximation to the multilevel birth and death particle system, we need to consider the tightness of the rescaled pro-

cesses. The main result in this section is that we use the criterion for tightness (see L. Gorostiza and J. Lopez-Mimbela [29]) to prove that the family of the rescaled multilevel birth and death processes $\{Y^\epsilon(t)\}_\epsilon$ is tight in $D_{[0,\infty)}(M_f(R^+))$.

Before proceeding with the matter of the tightness of the processes $\{Y^\epsilon(t)\}_\epsilon$, we take up a preliminary lemma:

Lemma 2.2.1 *Suppose that $X(t), t \geq 0$ is a linear birth and death process on $\{0, 1, 2, \dots\}$ with birth and death rates $\lambda_n = \lambda n$ and $\mu_n = \mu n$, $\lambda > 0$, $\mu > 0$, respectively. Then*

$$(2.2.1) \quad \begin{aligned} E_i X(t) &:= \sum_j j p_{i,j}(t) \\ &= \begin{cases} i & \text{if } \lambda = \mu \\ i \exp\{(\lambda - \mu)t\} & \text{if } \lambda \neq \mu. \end{cases} \end{aligned}$$

Moreover, we have

$$(2.2.2) \quad \begin{aligned} E_i X^2(t) &= \sum_j j^2 p_{i,j}(t) \\ &= \begin{cases} i^2 + (\lambda + \mu)it & \text{if } \lambda = \mu \\ i^2 \exp\{2(\lambda - \mu)t\} - \frac{\lambda + \mu}{\lambda - \mu} i \exp\{(\lambda - \mu)t\} + \frac{(\lambda + \mu)i}{\lambda - \mu} \exp\{2(\lambda - \mu)t\} & \text{if } \lambda \neq \mu \end{cases} \end{aligned}$$

and

$$(2.2.3) \quad \text{Var} X(t) = \begin{cases} i(\lambda + \mu)t & \text{if } \lambda = \mu \\ i \frac{\lambda + \mu}{\lambda - \mu} (\exp\{2(\lambda - \mu)t\} - \exp\{(\lambda - \mu)t\}) & \text{if } \lambda \neq \mu, \end{cases}$$

where we assume that $P\{X(0) = i\} = 1$.

Proof. Let

$$M(t) = EX(t) = \sum_{j=1}^{\infty} j p_{i,j}(t).$$

Since

$$p'_{i,0}(t) = \mu p_{i,1}(t)$$

and for $j \geq 1$, $p_{i,j}(t)$ satisfies the following forward Kolmogorov differential equation:

$$(2.2.4) \quad p'_{i,j}(t) = \lambda(j-1)p_{i,j-1}(t) - (\lambda + \mu)j p_{i,j}(t) + \mu(j+1)p_{i,j+1}(t).$$

It is easy to see that

$$\begin{aligned} (2.2.5) \quad M'(t) &= \sum_j j p'_{i,j}(t) \\ &= \sum_j j [\lambda(j-1)p_{i,j-1}(t) - (\lambda + \mu)j p_{i,j}(t) + \mu(j+1)p_{i,j+1}(t)] \\ &= \sum_j (j-1)^2 \lambda p_{i,j-1}(t) + \lambda \sum_j (j-1)p_{i,j-1}(t) \\ &\quad - (\lambda + \mu) \sum_j j^2 p_{i,j}(t) + \mu \sum_j (j+1)^2 p_{i,j+1}(t) - \mu \sum_j (j+1)p_{i,j+1}(t) \\ &= \lambda \sum_j j^2 p_{i,j}(t) + \lambda M(t) - (\lambda + \mu) \sum_j j^2 p_{i,j}(t) \\ &\quad + \mu \sum_j j^2 p_{i,j}(t) - \mu p_{i,1}(t) - \mu M(t) + \mu p_{i,1}(t) \\ &= (\lambda - \mu)M(t). \end{aligned}$$

Notice that $X(0) = i$, $M(0) = i$.

If $\lambda = \mu$, then by calculation

$$M'(t) = 0,$$

$$\int_0^t M'(s)ds = M(s)|_0^t = M(t) - M(0) = 0,$$

$$M(t) = M(0) = i.$$

If $\lambda \neq \mu$, then we have relation

$$\frac{M'(t)}{M(t)} = \lambda - \mu.$$

Computing the integral of both sides over the interval $[0, t]$, we get

$$\ln \frac{M(t)}{M(0)} = (\lambda - \mu)t,$$

that is

$$M(t) = M(0) \exp\{(\lambda - \mu)t\} = i \exp\{(\lambda - \mu)t\}.$$

In conclusion we obtain

$$(2.2.6) \quad M(t) = \begin{cases} i & \text{if } \lambda = \mu \\ i \exp\{(\lambda - \mu)t\} & \text{if } \lambda \neq \mu. \end{cases}$$

Now we let

$$M_2(t) = EX^2(t) = \sum_j j^2 p_{i,j}(t).$$

Then by (2.2.4) we have

$$\begin{aligned} (2.2.7) \quad M_2'(t) &= \sum_j j^2 p'_{i,j}(t) \\ &= \sum_j j^2 [\lambda(j-1)p_{i,j-1}(t) - (\lambda + \mu)jp_{i,j}(t) + \mu(j+1)p_{i,j+1}(t)] \\ &= \sum_j (j-1)^2 \lambda(j-1)p_{i,j-1}(t) + \sum_j (2j-1)\lambda(j-1)p_{i,j-1}(t) \\ &\quad - (\lambda + \mu) \sum_j j^3 p_{i,j}(t) \end{aligned}$$

$$\begin{aligned}
& + \sum_j (j+1)^2 \mu (j+1) p_{i,j+1}(t) - \sum_j (2j+1) \mu (j+1) p_{i,j+1}(t) \\
= & \lambda \sum_j j^3 p_{i,j}(t) + 2\lambda \sum_j j^2 p_{i,j}(t) + \sum_j \lambda (j-1) p_{i,j-1}(t) \\
& - (\lambda + \mu) \sum_j j^3 p_{i,j}(t) + \mu \sum_j j^3 p_{i,j}(t) - \mu p_{i,1}(t) - 2\mu \sum_j j^2 p_{i,j}(t) \\
& + \mu \sum_j (j+1) p_{i,j+1}(t) + 2\mu p_{i,1}(t) + \mu \sum_j j p_{i,j}(t) - \mu p_{i,1}(t) \\
= & 2(\lambda - \mu) \sum_j j^2 p_{i,j}(t) + (\lambda + \mu) M(t) \\
= & 2(\lambda - \mu) M_2(t) + (\lambda + \mu) M(t).
\end{aligned}$$

So we get a linear differential equation:

$$(2.2.8) \quad M_2'(t) + 2(\mu - \lambda) M_2(t) = (\lambda + \mu) M(t).$$

Now we consider two cases separately.

If $\lambda \neq \mu$, we let ρ denote the integrating factor and we have

$$\begin{aligned}
\rho(t) &= \exp\left\{\int 2(\mu - \lambda) dt\right\} = \exp\{-2(\lambda - \mu)t\}, \\
\frac{d}{dt} \exp\{-2(\lambda - \mu)t\} M_2(t) &= \exp\{-2(\lambda - \mu)t\} (\lambda + \mu) i \exp\{(\lambda - \mu)t\}, \\
\int_0^t d \exp\{-2(\lambda - \mu)s\} M_2(s) &= \int_0^t \exp\{-(\lambda - \mu)s\} (\lambda + \mu) i ds, \\
\exp\{-2(\lambda - \mu)t\} M_2(t) - M_2(0) &= -\frac{\lambda + \mu}{\lambda - \mu} i \exp\{-(\lambda - \mu)t\} + \frac{(\lambda + \mu)i}{\lambda + \mu}.
\end{aligned}$$

Thus we obtain that

$$M_2(t) = i^2 \exp\{2(\lambda - \mu)t\} - \frac{\lambda + \mu}{\lambda - \mu} i \exp\{(\lambda - \mu)t\} + \frac{(\lambda + \mu)i}{\lambda - \mu} \exp\{2(\lambda - \mu)t\}$$

as claimed. Note that $X(0) = i$ implies that $M_2(0) = i^2$.

If $\lambda = \mu$, then by calculation

$$\begin{aligned}
 (2.2.9) \quad M_2'(t) &= (\lambda + \mu)M(t), \\
 \int_0^t M_2'(s)ds &= \int_0^t (\lambda + \mu)ids, \\
 M_2(s) &= M_2(0) + (\lambda + \mu)it = i^2 + (\lambda + \mu)it.
 \end{aligned}$$

In conclusion, we obtain

$$\begin{aligned}
 (2.2.10) \quad EX^2(t) &= M_2(t) \\
 &= \begin{cases} i^2 + (\lambda + \mu)it & \text{if } \lambda = \mu \\ i^2 \exp\{2(\lambda - \mu)t\} - \frac{(\lambda + \mu)i}{\lambda - \mu} \exp\{(\lambda - \mu)t\} + \frac{(\lambda + \mu)i}{\lambda - \mu} \exp\{2(\lambda - \mu)t\} & \text{if } \lambda \neq \mu, \end{cases}
 \end{aligned}$$

and so

$$(2.2.11) \quad Var.X(t) = \begin{cases} i(\lambda + \mu)t & \text{if } \lambda = \mu \\ \frac{(\lambda + \mu)i}{\lambda - \mu} (\exp\{2(\lambda - \mu)t\} - \exp\{(\lambda - \mu)t\}) & \text{if } \lambda \neq \mu. \end{cases}$$

The proof of the Lemma 2.2.1 is complete.

□

Let $X_{2,j}^\varepsilon(t)$ be a level two birth and death process with birth and death rates $k\lambda_2^\varepsilon(j)$ and $k\mu_2^\varepsilon(j)$ at state k .

If $X_{2,j}^\varepsilon(0) = k$, then the lemma 2.2.1 implies that

$$(2.2.12) \quad Var(X_{2,j}^\varepsilon(t)) = \begin{cases} 2k\varepsilon_2^\varepsilon(j)t & \text{if } b_2(j) = 0 \\ \frac{2a_2^\varepsilon(j)k}{2b_2^\varepsilon(j)} (\exp\{4b_2^\varepsilon(j)t\} - \exp\{2b_2^\varepsilon(j)t\}) & \text{if } b_2(j) \neq 0. \end{cases}$$

$$= \begin{cases} \frac{2ka_2(j)}{\varepsilon_2^2} t & \text{if } b_2(j) = 0 \\ \frac{a_2(j)k}{\varepsilon_2 b_2(j)} (\exp\{\frac{4b_2(j)}{\varepsilon_2} t\} - \exp\{\frac{2b_2(j)}{\varepsilon_2} t\}) & \text{if } b_2(j) \neq 0. \end{cases}$$

Now we are going to our main theorem in this section:

Theorem 2.2.1 *The sequence of the processes $\{Y^\varepsilon(t)\}$ constructed in the last section is tight in $D_{[0,\infty)}(M_f(R^+))$, under the assumptions (2.1.1) and (2.1.2).*

Proof. To prove the tightness of the processes $\{Y^\varepsilon\}$ in $D_{[0,\infty)}(M_f(R^+))$, by the criterion in [29], it suffices to show that for each $\phi \in C_c^2(R^+)$, the sequence of real-valued processes $\{\langle Y^\varepsilon(t), \phi \rangle\}_\varepsilon$ is tight in R for each $t \geq 0$, and for each $\varepsilon > 0$, $T > 0$, and $\delta > 0$, there is a random variable $\xi(\varepsilon, T, \delta, \phi)$, such that

$$(2.2.13) \quad \xi(\varepsilon, T, \delta, \phi) \geq 0$$

and

$$(2.2.14) \quad \begin{aligned} E[(\langle Y^\varepsilon(T + \delta), \phi \rangle - \langle Y^\varepsilon(T), \phi \rangle)^2 \mid \langle Y^\varepsilon(t), \phi \rangle, 0 \leq t \leq T] \\ \leq E[\xi(\varepsilon, T, \delta, \phi) \mid \langle Y^\varepsilon(t), \phi \rangle, 0 \leq t \leq T], \end{aligned}$$

where $\{\langle Y^\varepsilon(t), \phi \rangle, 0 \leq t \leq T\}$ denotes the σ - algebra generated by

$\{\langle Y^\varepsilon(t), \phi \rangle, 0 \leq t \leq T\}$, and

$$(2.2.15) \quad \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} E\xi(\varepsilon, T, \delta, \phi) = 0.$$

We note that

$$X^\varepsilon(t, R^+) = \frac{1}{\varepsilon_2} Y^\varepsilon(t, R^+) = \frac{1}{\varepsilon_2} \langle Y^\varepsilon(t), 1 \rangle$$

as the number of superparticles at time t is a branching process, but the offspring distribution of each superparticle, which depends on the size of the superparticle, may be different. For example, the probability that a superparticle of size j produces a new copy is

$$p_2(j) = \frac{\lambda_2^\varepsilon(j)}{\lambda_2^\varepsilon(j) + \mu_2^\varepsilon(j)}$$

and the probability that a superparticle of size j dies is

$$p_0(j) = \frac{\mu_2^\varepsilon(j)}{\lambda_2^\varepsilon(j) + \mu_2^\varepsilon(j)}.$$

If for all j , $p_2(j) = p_2(1)$, $p_0(j) = p_0(1)$, then by the Lemma 2.2.1, we have

$$\begin{aligned} (2.2.16) \quad EY^\varepsilon(t, R^+) &= \varepsilon_2 EX^\varepsilon(t, R^+) \\ &= \begin{cases} \varepsilon_2 \left[\frac{Y^\varepsilon(0, R^+)}{\varepsilon_2} \right] & \text{if } b_2^\varepsilon(1) = 0 \\ \varepsilon_2 \frac{Y^\varepsilon(0, R^+)}{\varepsilon_2} \exp\{2b_2^\varepsilon(1)t\} & \text{if } b_2^\varepsilon(1) \neq 0 \end{cases} \\ &= \begin{cases} Y^\varepsilon(0, R^+) & \text{if } b_2(1) = 0 \\ Y^\varepsilon(0, R^+) \exp\left\{\frac{2b_2(1)}{\varepsilon_2}t\right\} & \text{if } b_2(1) \neq 0. \end{cases} \end{aligned}$$

Now we turn to the general case, in which $\lambda_2^\varepsilon(j)$ and $\mu_2^\varepsilon(j)$ depend on the size j .

We let

$$A_0 = \sup_x a_2(x), \quad a_0 = \inf_x a_2(x),$$

and consider two cases for $b_2(x)$:

(i) $b_2(x) \equiv 0$ or

(ii) $b_2(x) < 0$ and $b_0 \equiv \inf_x |b_2(x)| > 0$, $B_0 \equiv \sup_x |b_2(x)| < \infty$.

In the remainder of this section, when we use the term " $b_2(x) < 0$ " we shall always mean (ii) holds.

Using Lemma 2.2.1, we have

$$(2.2.17) \quad EY^\varepsilon(t, R^+) \leq \begin{cases} Y^\varepsilon(0, R^+) & \text{if } b_2(x) \equiv 0 \\ Y^\varepsilon(0, R^+) \exp\{-\frac{2b_0}{\varepsilon_2} t\} & \text{if } b_2(x) < 0, \end{cases}$$

where $\varepsilon = \varepsilon_1 = \varepsilon_2$ and

$$(2.2.18) \quad \begin{aligned} E(Y^\varepsilon(t, R^+))^2 &= \varepsilon_2^2 E(X^\varepsilon(t, R^+))^2 \\ &\leq \begin{cases} \varepsilon_2^2 \left[\left(\frac{Y^\varepsilon(0, R^+)}{\varepsilon_2} \right)^2 + 2A_0 \frac{1}{\varepsilon_2^2} \frac{Y^\varepsilon(0, R^+)}{\varepsilon_2} t \right] & \text{if } b_2(x) \equiv 0 \\ \varepsilon_2^2 \left[\left(\frac{Y^\varepsilon(0, R^+)}{\varepsilon_2} \right)^2 \exp\left\{-\frac{4b_0}{\varepsilon_2} t\right\} + \frac{A_0}{b_0} \frac{1}{\varepsilon_2^2} Y^\varepsilon(0, R^+) \exp\left\{-\frac{2b_0}{\varepsilon_2} t\right\} \right. \\ \quad \left. - \frac{a_0}{B_0} \frac{1}{\varepsilon_2^2} Y^\varepsilon(0, R^+) \exp\left\{-\frac{4B_0}{\varepsilon_2} t\right\} \right] & \text{if } b_2(x) < 0. \end{cases} \end{aligned}$$

Letting now $Y^\varepsilon(0) \rightarrow Y(0) \in M_f(R^+)$, we obtain that

$$\sup_\varepsilon Y^\varepsilon(0, R^+) < \infty.$$

Hence for $\phi \in C_c^2(R^+)$, $t \geq 0$

$$(2.2.19) \quad \sup_\varepsilon E \langle Y^\varepsilon(t), \phi \rangle \leq \|\phi\| \sup_\varepsilon EY^\varepsilon(t, R^+) < \infty,$$

and Markov's inequality implies that

$$\begin{aligned} P[\langle Y^\varepsilon(t), \phi \rangle \geq \alpha] &\leq \frac{1}{\alpha} E \langle Y^\varepsilon(t), \phi \rangle \\ &< \infty. \end{aligned}$$

Therefore we obtain that the sequence of the real-valued random variables.

$$\{ \langle Y^\varepsilon(t), \phi \rangle \}_\varepsilon$$

is tight in R for each $t \geq 0$.

Now we are going to show that there exists a random variable $\xi(\varepsilon, T, \delta, \phi)$, such that (2.2.13) and (2.2.14) are satisfied in the two cases:

(i). If $b_2(x) \equiv 0$, then by (2.1.4)

$$\begin{aligned}
 (2.2.20) \quad G^\varepsilon &< Y^\varepsilon(t), \phi \rangle^2 \\
 &= \sum_j [(\langle Y^\varepsilon(t), \phi \rangle + \varepsilon_2 \phi(j\varepsilon_1))^2 - \langle Y^\varepsilon(t), \phi \rangle^2] Y^\varepsilon(t, j\varepsilon_1) \lambda_2^\varepsilon(j\varepsilon_1) \\
 &\quad + \sum_j [(\langle Y^\varepsilon(t), \phi \rangle - \varepsilon_2 \phi(j\varepsilon_1))^2 - \langle Y^\varepsilon(t), \phi \rangle^2] Y^\varepsilon(t, j\varepsilon_1) \mu_2^\varepsilon(j\varepsilon_1) \\
 &\quad + \sum_j [(\langle Y^\varepsilon(t), \phi \rangle - \varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j+1)\varepsilon_1))^2 - \langle Y^\varepsilon(t), \phi \rangle^2] Y^\varepsilon(t, j\varepsilon_1) \lambda_1^\varepsilon(j\varepsilon_1) \\
 &\quad + \sum_j [(\langle Y^\varepsilon(t), \phi \rangle - \varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j-1)\varepsilon_1))^2 - \langle Y^\varepsilon(t), \phi \rangle^2] Y^\varepsilon(t, j\varepsilon_1) \mu_1^\varepsilon(j\varepsilon_1) \\
 &= \sum_j 2\varepsilon_2^2 \phi^2(j\varepsilon_1) a_2^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) + \sum_j 2 \langle Y^\varepsilon(t), \phi \rangle \varepsilon_2 \phi''(j\varepsilon_1) \varepsilon_1^2 a_1^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) \\
 &\quad + \sum_j 2 \langle Y^\varepsilon(t), \phi \rangle 2\varepsilon_2 \phi'(j\varepsilon_1) \varepsilon_1 b_1^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) + 0(\varepsilon^2) \\
 &\leq \sum_j 2\varepsilon_2^2 \phi^2(j\varepsilon_1) a_2^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) + \sum_j 2 \langle Y^\varepsilon(t), \phi \rangle \varepsilon_2 \phi''(j\varepsilon_1) \varepsilon_1^2 \varepsilon_1 a_1^\varepsilon(j) Y^\varepsilon(t, j\varepsilon_1) \\
 &\quad + \sum_j 2 \langle Y^\varepsilon(t), \phi \rangle 2\varepsilon_2 \phi'(j\varepsilon_1) \varepsilon_1 \varepsilon_1 b_1^\varepsilon(j) Y^\varepsilon(t, j\varepsilon_1) + 0(\varepsilon^2) \\
 &\leq 2\|\phi^2\| \cdot \|a_2\| Y^\varepsilon(t, R^+) + 2\|a_1 \phi''\| \cdot \|\phi\| \varepsilon_1 (Y^\varepsilon(t, R^+))^2 \\
 &\quad + 4\|b_1 \phi'\| \cdot \|\phi\| \varepsilon_1 (Y^\varepsilon(t, R^+))^2 + 0(\varepsilon^2).
 \end{aligned}$$

Since for $\phi \in C_c^2(R^+)$, $f \in C^\infty(R)$,

$$(2.2.21) \quad f(\langle Y^\varepsilon(t), \phi \rangle) - \int_0^t G^\varepsilon f(\langle Y^\varepsilon(s), \phi \rangle) ds$$

is a martingale, we have

$$(2.2.22) \quad \begin{aligned} & E[\langle Y^\varepsilon(T + \delta), \phi \rangle^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ &= \langle Y^\varepsilon(T), \phi \rangle^2 + E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle^2 ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \end{aligned}$$

and

$$(2.2.23) \quad \begin{aligned} & E[\langle Y^\varepsilon(T + \delta), \phi \rangle \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ &= \langle Y^\varepsilon(T), \phi \rangle + E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle ds \mid Y^\varepsilon(t), 0 \leq t \leq T]. \end{aligned}$$

We wish to estimate $G^\varepsilon \langle Y^\varepsilon(t), \phi \rangle$. Now by (2.1.4)

$$(2.2.24) \quad \begin{aligned} & G_\varepsilon \langle Y^\varepsilon(t), \phi \rangle \\ &= \sum_j [\langle Y^\varepsilon(t), \phi \rangle + \varepsilon_2 \phi(j\varepsilon_1) - \langle Y^\varepsilon(t), \phi \rangle] Y^\varepsilon(t, j\varepsilon_1) \lambda_2^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [\langle Y^\varepsilon(t), \phi \rangle - \varepsilon_2 \phi(j\varepsilon_1) - \langle Y^\varepsilon(t), \phi \rangle] Y^\varepsilon(t, j\varepsilon_1) \mu_2^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [\langle Y^\varepsilon(t), \phi \rangle - \varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j+1)\varepsilon_1) - \langle Y^\varepsilon(t), \phi \rangle] Y^\varepsilon(t, j\varepsilon_1) \lambda_1^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [\langle Y^\varepsilon(t), \phi \rangle - \varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j-1)\varepsilon_1) - \langle Y^\varepsilon(t), \phi \rangle] Y^\varepsilon(t, j\varepsilon_1) \mu_1^\varepsilon(j\varepsilon_1) \\ &= \sum_j \varepsilon_2 \phi(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) \lambda_2^\varepsilon(j\varepsilon_1) + \sum_j [-\varepsilon_2 \phi(j\varepsilon_1)] \mu_2^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) \\ &\quad + \sum_j [\phi((j+1)\varepsilon_1) - \phi(j\varepsilon_1)] \varepsilon_2 Y^\varepsilon(t, j\varepsilon_1) \lambda_1^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [\phi((j-1)\varepsilon_1) - \phi(j\varepsilon_1)] \varepsilon_2 Y^\varepsilon(t, j\varepsilon_1) \mu_1^\varepsilon(j\varepsilon_1) \end{aligned}$$

$$\begin{aligned}
&= \sum_j \varepsilon_2 \phi''(j\varepsilon_1) \varepsilon_1^2 a_1^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) + \sum_j 2\varepsilon_2 \phi'(j\varepsilon_1) \varepsilon_1 b_1^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) + 0(\varepsilon^2) \\
&\leq \sum_j \varepsilon_2 \phi''(j\varepsilon_1) \varepsilon_1^2 \varepsilon_1 a_1^\varepsilon(j) Y^\varepsilon(t, j\varepsilon_1) + \sum_j 2\varepsilon_2 \phi'(j\varepsilon_1) \varepsilon_1^2 b_1^\varepsilon(j) Y^\varepsilon(t, j\varepsilon_1) + 0(\varepsilon^2) \\
&\leq \varepsilon_1 \|\phi'' a_1\| Y^\varepsilon(t, R^+) + 2\|b_1 \phi'\| \varepsilon_1 Y^\varepsilon(t, R^+) + 0(\varepsilon^2).
\end{aligned}$$

Putting the results above together, we have

$$\begin{aligned}
(2.2.25) \quad &E[(\langle Y^\varepsilon(T+\delta), \phi \rangle - \langle Y^\varepsilon(T), \phi \rangle)^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] \\
&= E[\langle Y^\varepsilon(T+\delta), \phi \rangle^2 - 2\langle Y^\varepsilon(T+\delta), \phi \rangle \langle Y^\varepsilon(T), \phi \rangle \\
&\quad + \langle Y^\varepsilon(T), \phi \rangle^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] \\
&= E[\langle Y^\varepsilon(T+\delta), \phi \rangle^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] + \langle Y^\varepsilon(T), \phi \rangle^2 \\
&\quad - 2\langle Y^\varepsilon(T), \phi \rangle (\langle Y^\varepsilon(T), \phi \rangle + E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle ds \\
&\quad \mid Y^\varepsilon(t), 0 \leq t \leq T]) \\
&= E[\langle Y^\varepsilon(T+\delta), \phi \rangle^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] - \langle Y^\varepsilon(T), \phi \rangle^2 \\
&\quad - 2\langle Y^\varepsilon(T), \phi \rangle E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\
&\leq E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle^2 ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\
&\quad + 2|\langle Y^\varepsilon(T), \phi \rangle| E[\int_T^{T+\delta} (\varepsilon_1 \|\phi'' a_1\| + 2\|b_1 \phi'\| \varepsilon_1 + 0(\varepsilon^2)) \\
&\quad Y^\varepsilon(s, R^+) ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\
&\leq E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle^2 ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\
&\quad + 2\|\phi\| Y^\varepsilon(T, R^+) (\varepsilon_1 \|\phi'' a_1\| + 2\|b_1 \phi'\| \varepsilon_1 + 0(\varepsilon^2)) Y^\varepsilon(T, R^+) \delta \\
&\leq E[\int_T^{T+\delta} (2\|\phi^2\| \cdot \|a_2\| Y^\varepsilon(s, R^+) + 2\|a_1 \phi''\| \cdot \|\phi\| \varepsilon_1 (Y^\varepsilon(s, R^+))^2
\end{aligned}$$

$$\begin{aligned}
& +4\|b_1\phi'\| \cdot \|\phi\|\varepsilon_1(Y^\varepsilon(s, R^+))^2 + 0(\varepsilon^2))ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\
& +2\|\phi\|\varepsilon_1(\|\phi''a_1\| + 2\|b_1\phi'\| + 0(\varepsilon))(Y^\varepsilon(T, R^+))^2\delta \\
\leq & \int_T^{T+\delta} 2\|\phi^2\| \cdot \|a_2\|Y^\varepsilon(T, R^+)ds \\
& + \int_T^{T+\delta} (2\|a_1\phi''\| \cdot \|\phi\|\varepsilon_1 + 4\|b_1\phi'\| \cdot \|\phi\|\varepsilon_1 + 0(\varepsilon^2)) \\
& \quad [(Y^\varepsilon(T, R^+))^2 + 2A_0\frac{Y^\varepsilon(T, R^+)}{\varepsilon_2}(s - T)]ds \\
& +2\|\phi\|\varepsilon_1(\|\phi''a_1\| + 2\|b_1\phi'\| + 0(\varepsilon))(Y^\varepsilon(T, R^+))^2\delta \\
= & 2\|\phi^2\| \cdot \|a_2\|\delta Y^\varepsilon(T, R^+) + (2\|a_1\phi''\| \cdot \|\phi\|\varepsilon_1 \\
& +4\|b_1\phi'\| \cdot \|\phi\|\varepsilon_1 + 0(\varepsilon^2))[\delta(Y^\varepsilon(T, R^+))^2 + A_0\frac{Y^\varepsilon(T, R^+)}{\varepsilon_2}\delta^2] \\
& +2\|\phi\|\varepsilon_1(\|\phi''a_1\| + 2\|b_1\phi'\| + 0(\varepsilon))(Y^\varepsilon(T, R^+))^2\delta,
\end{aligned}$$

where $\|\cdot\|$ denotes the supremum norm and $a(x) \in C(R^+)$, $\phi(x) \in C_c(R)$ imply that $\|a\phi\| < \infty$.

Put

$$\begin{aligned}
C_1 &= 2\|\phi^2\| \cdot \|a_2\|, \quad C_2 = 2\|a_1\phi''\| \cdot \|\phi\| + 4\|b_1\phi'\| \cdot \|\phi\| + 0(\varepsilon), \\
C_3 &= 2\|\phi\|(\|\phi''a_1\| + 2\|b_1\phi'\| + 0(\varepsilon)).
\end{aligned}$$

Then

$$\begin{aligned}
(2.2.26) \quad E[(\langle Y^\varepsilon(T + \delta), \phi \rangle - \langle Y^\varepsilon(T), \phi \rangle)^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] \\
\leq C_1\delta Y^\varepsilon(T, R^+) + C_2\varepsilon_1\frac{A_0}{\varepsilon_2}\delta^2 Y^\varepsilon(T, R^+) \\
+ C_2\delta\varepsilon_1(Y^\varepsilon(T, R^+))^2 + C_3\varepsilon_1\delta(Y^\varepsilon(T, R^+))^2
\end{aligned}$$

$$= (C_1\delta + C_2A_0\delta^2)Y^\varepsilon(T, R^+) + (C_2 + C_3)\varepsilon_1\delta(Y^\varepsilon(T, R^+))^2.$$

Let

$$\xi(\varepsilon, T, \delta, \phi) = (C_1\delta + C_2A_0\delta^2)Y^\varepsilon(T, R^+) + (C_2 + C_3)\varepsilon_1\delta(Y^\varepsilon(T, R^+))^2.$$

Then it is easy to see that

$$\begin{aligned} (2.2.27) \quad & E[(\langle Y^\varepsilon(T + \delta), \phi \rangle - \langle Y^\varepsilon(T), \phi \rangle)^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ & \leq E[\xi(\varepsilon, T, \delta, \phi) \mid Y^\varepsilon(t), 0 \leq t \leq T] \end{aligned}$$

and

$$\begin{aligned} (2.2.28) \quad & \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} E\xi(\varepsilon, T, \delta, \phi) \\ & = \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} [(C_1\delta + C_2A_0\delta^2)Y^\varepsilon(0, R^+) \\ & \quad + (C_2 + C_3)\varepsilon_1\delta((Y^\varepsilon(0, R^+))^2 + 2A_0\frac{Y^\varepsilon(0, R^+)}{\varepsilon_2}T)] \\ & = 0. \end{aligned}$$

Remark 2.2.1 We also can use the following lemma of B. Fernández (1986) (cf. Luis G. Gorostiza (1988) [28]) to show that there exists a random variable $\xi(\varepsilon, T, \delta, \phi)$, such that (2.2.13), (2.2.14) and (2.2.15) are satisfied.

Lemma 2.2.2 Let $X \equiv \{X(t), t \geq 0\}$ be a process in D_R adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$. If there exists adapted processes $\theta^{(1)}$ and $\theta^{(2)}$ in D_R , such that

$$(2.2.29) \quad M_t \equiv X_t - \int_0^t \theta_s^{(1)} ds, \quad t \geq 0$$

is a square integrable martingale with increasing process:

$$(2.2.30) \quad \int_0^t \theta_s^{(2)} ds, \quad t \geq 0,$$

then for each $T > 0$ and $\delta > 0$, there exists a random variable $\xi_{T,\delta} \geq 0$,

$$(2.2.31) \quad E[(X_{t+\delta} - X_t)^2 | \mathcal{F}_t] \leq E[\xi_{T,\delta} | \mathcal{F}_t]$$

for $0 \leq t \leq T$. Such a random variable is

$$(2.2.32) \quad \xi_{T,\delta} = 2[\delta \sup_{0 \leq t \leq T+\delta} \theta_t^{(2)} + \delta^2 \sup_{0 \leq t \leq T+\delta} (\theta_t^{(1)})^2].$$

(ii). If $b_2(x) < 0$ for each x , then from (2.2.18)

$$\begin{aligned} (2.2.33) \quad & E[(\langle Y^\varepsilon(T+\delta), \phi \rangle - \langle Y^\varepsilon(T), \phi \rangle)^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ &= E[\langle Y^\varepsilon(T+\delta), \phi \rangle^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] - \langle Y^\varepsilon(T), \phi \rangle^2 \\ &\quad - 2 \langle Y^\varepsilon(T), \phi \rangle E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ &\leq E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle^2 ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ &\quad + 2Y^\varepsilon(T, R^+) \|\phi\| E[\int_T^{T+\delta} G^\varepsilon \langle Y^\varepsilon(s), \phi \rangle ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ &\leq E[\int_T^{T+\delta} (2\|\phi^2\| \cdot \|b_2\| + 2\|a_1 \phi''\| \cdot \|\phi\| + 4\|b_1 \phi'\| \cdot \|\phi\| + 0(\varepsilon^2)) \\ &\quad (Y^\varepsilon(s, R^+))^2 ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ &\quad + E[\int_T^{T+\delta} 2\|\phi^2\| \cdot \|a_2\| Y^\varepsilon(s, R^+) ds \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ &\quad + 2Y^\varepsilon(T, R^+) \|\phi\| E[\int_T^{T+\delta} (\sum_j 2\varepsilon_2 \phi(j\varepsilon_1) b_2^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) \end{aligned}$$

$$\begin{aligned}
 & + \sum_j \varepsilon_2 \phi''(j\varepsilon_1) \varepsilon_1^2 a_1^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) \\
 & + \sum_j 2\varepsilon_2 \phi'(j\varepsilon_1) \varepsilon_1 b_1^\varepsilon(j\varepsilon_1) Y^\varepsilon(t, j\varepsilon_1) + 0(\varepsilon^2) ds \quad | \quad Y^\varepsilon(t), 0 \leq t \leq T \\
 \leq & (2\|\phi^2\| \cdot \|b_2\| + 2\|a_1 \phi''\| \cdot \|\phi\| + 4\|b_1 \phi'\| \cdot \|\phi\| + 0(\varepsilon^2)) \int_T^{T+\delta} [(Y^\varepsilon(T, R^+))^2 \\
 & \exp\{-\frac{4b_0}{\varepsilon_2}(s-T)\} + \frac{A_0}{b_0} Y^\varepsilon(T, R^+) \exp\{-\frac{2b_0}{\varepsilon_2}(s-T)\} \\
 & - \frac{a_0}{B_0} Y^\varepsilon(T, R^+) \exp\{-\frac{4B_0}{\varepsilon_2}(s-T)\}] ds \\
 & + 2\|\phi^2\| \cdot \|a_2\| \int_T^{T+\delta} Y^\varepsilon(T, R^+) \exp\{-\frac{2b_0}{\varepsilon_2}(s-T)\} ds \\
 & + 2Y^\varepsilon(T, R^+) \|\phi\| (2\|\phi\| \cdot \|b_2\| + \|\phi'' a_1\| + 2\|\phi' b_1\| + 0(\varepsilon^2)) \\
 & \int_T^{T+\delta} Y^\varepsilon(T, R^+) \exp\{-\frac{2b_0}{\varepsilon_2}(s-T)\} ds.
 \end{aligned}$$

If we let

$$C_4 = 2\|\phi^2\| \cdot \|b_2\| + 2\|a_1 \phi''\| \cdot \|\phi\| + 4\|b_1 \phi'\| \cdot \|\phi\| + 0(\varepsilon^2)$$

$$C_5 = 2\|\phi\| \cdot \|b_2\| + \|\phi'' a_1\| + 2\|\phi' b_1\| + 0(\varepsilon^2),$$

then

$$\begin{aligned}
 & E[(\langle Y^\varepsilon(T+\delta), \phi \rangle - \langle Y^\varepsilon(T), \phi \rangle)^2 \quad | \quad Y^\varepsilon(t), 0 \leq t \leq T] \\
 \leq & C_4 [(Y^\varepsilon(T, R^+))^2 (-\frac{\varepsilon_2}{4b_0}) \exp\{-\frac{4b_0}{\varepsilon_2}s\} \Big|_0^\delta + \frac{A_0}{b_0} Y^\varepsilon(T, R^+) (-\frac{\varepsilon_2}{2b_0}) \exp\{-\frac{2b_0}{\varepsilon_2}s\} \Big|_0^\delta \\
 & - \frac{a_0}{B_0} Y^\varepsilon(T, R^+) (-\frac{\varepsilon_2}{4B_0}) \exp\{-\frac{4B_0}{\varepsilon_2}s\} \Big|_0^\delta] \\
 & + 2\|\phi^2\| \cdot \|a_2\| [Y^\varepsilon(T, R^+) (-\frac{\varepsilon_2}{2b_0}) \exp\{-\frac{2b_0}{\varepsilon_2}s\} \Big|_0^\delta] \\
 & + 2C_5 \|\phi\| (Y^\varepsilon(T, R^+))^2 (-\frac{\varepsilon_2}{2b_0}) \exp\{-\frac{2b_0}{\varepsilon_2}s\} \Big|_0^\delta \\
 = & C_4 [(Y^\varepsilon(T, R^+))^2 \frac{\varepsilon_2}{4b_0} (1 - \exp\{-\frac{4b_0}{\varepsilon_2}\delta\}) + \frac{A_0}{b_0} Y^\varepsilon(T, R^+) \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2}\delta\}) \\
 & - \frac{a_0}{B_0} Y^\varepsilon(T, R^+) \frac{\varepsilon_2}{4B_0} (1 - \exp\{-\frac{4B_0}{\varepsilon_2}\delta\})] \\
 & + 2\|\phi^2\| \cdot \|a_2\| Y^\varepsilon(T, R^+) \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2}\delta\}) \\
 & + 2C_5 \|\phi\| (Y^\varepsilon(T, R^+))^2 \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2}\delta\})
 \end{aligned}$$

$$\begin{aligned} & -\frac{a_0}{B_0} Y^\varepsilon(T, R^+) \frac{\varepsilon_2}{4B_0} (1 - \exp\{-\frac{4B_0}{\varepsilon_2} \delta\})] \\ & + 2\|\phi^2\| \cdot \|a_2\| Y^\varepsilon(T, R^+) \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2} \delta\}) \\ & + 2C_5 \|\phi\| (Y^\varepsilon(T, R^+))^2 \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2} \delta\}). \end{aligned}$$

If we let

$$\begin{aligned} (2.2.34) \quad & \xi(\varepsilon, T, \delta, \phi) \\ & = C_4 [(Y^\varepsilon(T, R^+))^2 \frac{\varepsilon_2}{4b_0} (1 - \exp\{-\frac{4b_0}{\varepsilon_2} \delta\}) + \frac{A_0}{b_0} Y^\varepsilon(T, R^+) \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2} \delta\}) \\ & \quad - \frac{a_0}{B_0} Y^\varepsilon(T, R^+) \frac{\varepsilon_2}{4B_0} (1 - \exp\{-\frac{4B_0}{\varepsilon_2} \delta\})] \\ & \quad + 2\|\phi^2\| \cdot \|a_2\| Y^\varepsilon(T, R^+) \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2} \delta\}) \\ & \quad + 2C_5 \|\phi\| (Y^\varepsilon(T, R^+))^2 \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2} \delta\}), \end{aligned}$$

in this case, then it is easy to see that

$$\begin{aligned} (2.2.35) \quad & E[(\langle Y^\varepsilon(T + \delta), \phi \rangle - \langle Y^\varepsilon(T), \phi \rangle)^2 \mid Y^\varepsilon(t), 0 \leq t \leq T] \\ & \leq E[\xi(\varepsilon, T, \delta, \phi) \mid Y^\varepsilon(t), 0 \leq t \leq T] \end{aligned}$$

and

$$\begin{aligned} (2.2.36) \quad & \lim_{\delta \rightarrow 0} \limsup_{0 < \varepsilon < 1} E\xi(\varepsilon, T, \delta, \phi) \\ & = \lim_{\delta \rightarrow 0} \limsup_{0 < \varepsilon < 1} \{ [C_4 \frac{\varepsilon_2}{4b_0} (1 - \exp\{-\frac{4b_0}{\varepsilon_2} \delta\}) + C_5 \|\phi\| \frac{\varepsilon_2}{b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2} \delta\})] \\ & \quad [(Y^\varepsilon(0, R^+))^2 \exp\{-\frac{4b_0}{\varepsilon_2} T\} + Y^\varepsilon(0, R^+) \frac{A_0}{b_0} \exp\{-\frac{2b_0}{\varepsilon_2} T\} \\ & \quad - \frac{a_0}{B_0} Y^\varepsilon(0, R^+) \exp\{-\frac{4B_0}{\varepsilon_2} T\}] \} \end{aligned}$$

$$\begin{aligned}
& + [C_4 \frac{A_0}{b_0} \frac{\varepsilon_2}{2b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2} \delta\}) - C_4 \frac{a_0}{B_0} \frac{\varepsilon_2}{4B_0} (1 - \exp\{-\frac{4B_0}{\varepsilon_2} \delta\}) \\
& \quad + \|\phi^2\| \cdot \|a_2\| \frac{\varepsilon_2}{b_0} (1 - \exp\{-\frac{2b_0}{\varepsilon_2} \delta\})] [Y^\varepsilon(0, R^+) \exp\{-\frac{2b_0}{\varepsilon_2} T\}] \\
& = 0.
\end{aligned}$$

Therefore, we conclude that $\{Y^\varepsilon(t), t \geq 0\}$ is tight in $D_{[0, \infty)}(M_f(R^+))$.

□

2.3 The Limiting Diffusion Process

In this section, we are going to investigate the existence and uniqueness of the continuous diffusion limit of the multilevel birth and death processes. The main result is that we show that the limiting process is a measure-valued Markov process which is characterized as the unique solution of the measure-valued martingale problem for the limiting generator. Specifically, we have the following theorem:

Theorem 2.3.1 *Assume that $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $Y_0^- \longrightarrow Y_0 \in M_f(R^+)$ as $\varepsilon \longrightarrow 0$. Then the $N(\varepsilon Z^+)$ - valued processes $Y^\varepsilon(t)$ converge weakly to a $M_f(R^+)$ - valued Markov process $Y(t)$ which is characterized as the unique solution to the martingale problem associated with*

$$\begin{aligned}
(2.3.1) \quad G_\varepsilon F(\mu) &= f''(\langle \phi, \mu \rangle) \langle a_2 \phi^2, \mu \rangle \\
&+ 2f'(\langle \phi, \mu \rangle) \langle b_2 \phi, \mu \rangle \\
&+ f'(\langle \phi, \mu \rangle) \langle a_1 \phi'', \mu \rangle
\end{aligned}$$

$$+2f'(\langle \phi, \mu \rangle) \langle b_1 \phi', \mu \rangle,$$

where $F(\mu) = f(\langle \phi, \mu \rangle)$, $f \in C^\infty(R)$ and $\phi \in C_c^2(R^+)$.

Morcover, the transition probabilities on $M_f(R^+)$ are determined by the Laplace transition functional

$$(2.3.2) \quad E[\exp\{-\langle \phi, Y(t) \rangle\} | Y(0) = \mu] = \exp\{-\langle u(t), \mu \rangle\}.$$

where the cumulant generating function $u(.,.)$ satisfies the following nonlinear differential equations:

$$(2.3.3) \quad \begin{aligned} \frac{\partial u(t, x)}{\partial t} &= -a_2(x)u^2(t, x) + 2b_2(x)u(t, x) \\ &\quad + a_1(x)\frac{\partial^2 u(t, x)}{\partial x^2} + 2b_1(x)\frac{\partial u(t, x)}{\partial x}, \\ u(0, x) &= \phi(x). \end{aligned}$$

Proof. At first, we are going to consider the convergence of the generators. Using a two term Taylor expansion we know that for each $f \in C^\infty(R)$ and $\phi \in C_c^2(R^+)$

$$(2.3.4) \quad \begin{aligned} G^\varepsilon F(\mu) &= \sum_j [f(\langle \phi, \mu \rangle + \varepsilon_2 \phi(j\varepsilon_1)) - f(\langle \phi, \mu \rangle)] \mu(j\varepsilon_1) \lambda_2^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [f(\langle \phi, \mu \rangle - \varepsilon_2 \phi(j\varepsilon_1)) - f(\langle \phi, \mu \rangle)] \mu(j\varepsilon_1) \mu_2^\varepsilon(j\varepsilon_1) \\ &\quad + \sum_j [f(\langle \phi, \mu \rangle - \varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j+1)\varepsilon_1)) - f(\langle \phi, \mu \rangle)] \mu(j\varepsilon_1) \lambda_1^\varepsilon(j\varepsilon_1) \end{aligned}$$

$$\begin{aligned}
& + \sum_j [f'(\langle \phi, \mu \rangle - \varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j-1)\varepsilon_1)) - f'(\langle \phi, \mu \rangle)] \mu(j\varepsilon_1) \mu_1^\varepsilon(j\varepsilon_1) \\
= & \sum_j [f'(\langle \phi, \mu \rangle) \varepsilon_2 \phi(j\varepsilon_1) + \frac{f''(\langle \phi, \mu \rangle)}{2} \varepsilon_2^2 \phi^2(j\varepsilon_1)] (a_2^\varepsilon(j\varepsilon_1) + b_2^\varepsilon(j\varepsilon_1)) \mu(j\varepsilon_1) \\
& + \sum_j [f'(\langle \phi, \mu \rangle) (-\varepsilon_2) \phi(j\varepsilon_1) + \frac{f''(\langle \phi, \mu \rangle)}{2} \varepsilon_2^2 \phi^2(j\varepsilon_1)] \\
& \quad (a_2^\varepsilon(j\varepsilon_1) - b_2^\varepsilon(j\varepsilon_1)) \mu(j\varepsilon_1) \\
& + \sum_j [f'(\langle \phi, \mu \rangle) (-\varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j+1)\varepsilon_1)) \\
& + \frac{f''(\langle \phi, \mu \rangle)}{2} (-\varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j+1)\varepsilon_1))^2] (a_1^\varepsilon(j\varepsilon_1) + b_1^\varepsilon(j\varepsilon_1)) \mu(j\varepsilon_1) \\
& + \sum_j [f'(\langle \phi, \mu \rangle) (-\varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j-1)\varepsilon_1)) \\
& + \frac{f''(\langle \phi, \mu \rangle)}{2} (-\varepsilon_2 \phi(j\varepsilon_1) + \varepsilon_2 \phi((j-1)\varepsilon_1))^2] (a_1^\varepsilon(j\varepsilon_1) - b_1^\varepsilon(j\varepsilon_1)) \mu(j\varepsilon_1) + O(\varepsilon^2) \\
= & \sum_j f''(\langle \phi, \mu \rangle) \varepsilon_2^2 \phi^2(j\varepsilon_1) a_2^\varepsilon(j\varepsilon_1) \mu(j\varepsilon_1) \\
& + \sum_j 2f'(\langle \phi, \mu \rangle) \varepsilon_2 \phi(j\varepsilon_1) b_2^\varepsilon(j\varepsilon_1) \mu(j\varepsilon_1) \\
& + \sum_j [f'(\langle \phi, \mu \rangle) \varepsilon_2 (-2\phi(j\varepsilon_1) + \phi((j-1)\varepsilon_1) + \phi((j+1)\varepsilon_1)) \\
& \quad + \frac{f''(\langle \phi, \mu \rangle)}{2} \varepsilon_2^2 (\phi'^2(j\varepsilon_1) \varepsilon_1^2 + \phi'(j\varepsilon_1) \phi''(j\varepsilon_1) \varepsilon_1^3 \\
& \quad + \phi'^2(j\varepsilon_1) \varepsilon_1^2 - \phi'(j\varepsilon_1) \phi''(j\varepsilon_1) \varepsilon_1^3)] a_1^\varepsilon(j\varepsilon_1) \mu(j\varepsilon_1) \\
& + \sum_j [f'(\langle \phi, \mu \rangle) \varepsilon_2 (\phi((j+1)\varepsilon_1) - \phi((j-1)\varepsilon_1)) \\
& \quad + \frac{f''(\langle \phi, \mu \rangle)}{2} \varepsilon_2^2 (\phi'^2(j\varepsilon_1) \varepsilon_1^2 + \phi'(j\varepsilon_1) \phi''(j\varepsilon_1) \varepsilon_1^3 \\
& \quad - \phi'^2(j\varepsilon_1) \varepsilon_1^2 + \phi'(j\varepsilon_1) \phi''(j\varepsilon_1) \varepsilon_1^3)] b_1^\varepsilon(j\varepsilon_1) \mu(j\varepsilon_1) + O(\varepsilon^2) \\
= & \sum_j f''(\langle \phi, \mu \rangle) \varepsilon_2^2 \phi^2(j\varepsilon_1) a_2^\varepsilon(j\varepsilon_1) \mu(j\varepsilon_1)
\end{aligned}$$

$$\begin{aligned}
& + \sum_j 2f'(\langle \phi, \mu \rangle) \varepsilon_2 \phi(j\varepsilon_1) b_2^\varepsilon(j\varepsilon_1) \mu(j\varepsilon_1) + \sum_j [f'(\langle \phi, \mu \rangle) \varepsilon_2 \phi''(j\varepsilon_1) \varepsilon_1^2 \\
& + \frac{f''(\langle \phi, \mu \rangle)}{2} \varepsilon_2^2 2\phi'^2(j\varepsilon_1) \varepsilon_1^2] a_1^\varepsilon(j\varepsilon_1) \mu(j\varepsilon_1) + \sum_j [f'(\langle \phi, \mu \rangle) \varepsilon_2 2\phi'(j\varepsilon_1) \varepsilon_1 \\
& + \frac{f''(\langle \phi, \mu \rangle)}{2} \varepsilon_2^2 2\phi'(j\varepsilon_1) \phi''(j\varepsilon_1) \varepsilon_1^3] b_1^\varepsilon(j\varepsilon_1) \mu(j\varepsilon_1) + 0(\varepsilon^2) \\
\longrightarrow & f''(\langle \phi, \mu \rangle) \langle a_2 \phi^2, \mu \rangle + 2f'(\langle \phi, \mu \rangle) \langle b_2 \phi, \mu \rangle \\
& + f'(\langle \phi, \mu \rangle) \langle a_1 \phi'', \mu \rangle + 2f'(\langle \phi, \mu \rangle) \langle b_1 \phi', \mu \rangle \\
= & G^c F(\mu).
\end{aligned}$$

By the tightness of $\{Y^\varepsilon(t), t \geq 0\}$ with a new indexation if necessary, we know that there exists a $M_f(R^+)$ -valued process $\{Y(t), t \geq 0\}$ such that Y^ε converge weakly to Y .

Next, we proceed to show that $\{Y(t), t \geq 0\}$ is a solution of the martingale problem for G_c . That is, for each bounded stopping time τ , we need to show that

$$(2.3.5) \quad E[F(Y(\tau)) - F(Y(0)) - \int_0^\tau G_c F(Y(s)) ds] = 0.$$

We already know that for each $\varepsilon > 0$,

$$(2.3.6) \quad E[F(Y^\varepsilon(\tau)) - F(Y^\varepsilon(0)) - \int_0^\tau G^\varepsilon F(Y^\varepsilon(s)) ds] = 0$$

and we have

$$\begin{aligned}
(2.3.7) \quad & \lim_{\varepsilon \rightarrow 0} E[F(Y^\varepsilon(\tau)) - F(Y^\varepsilon(0))] \\
& = E[F(Y(\tau)) - F(Y(0))].
\end{aligned}$$

Observe that

$$\begin{aligned}
 (2.3.8) \quad E\left[\int_0^\tau G_c F(Y(s))ds\right] &= E\left[\int_0^\tau G^\varepsilon F(Y^\varepsilon(s))ds\right] \\
 &= E\left[\int_0^\tau G_c F(Y(s))ds\right] - E\left[\int_0^\tau G_c F(Y^\varepsilon(s))ds\right] \\
 &\quad + E\left[\int_0^\tau (G_c - G^\varepsilon)F(Y^\varepsilon(s))ds\right].
 \end{aligned}$$

It is clear that

$$(2.3.9) \quad E\left[\int_0^\tau G_c F(Y^\varepsilon(s))ds\right] \longrightarrow E\left[\int_0^\tau G_c F(Y(s))ds\right]$$

as $\varepsilon \longrightarrow 0$, and the last term of the right side of (2.3.8) satisfies

$$\begin{aligned}
 (2.3.10) \quad |E\left[\int_0^\tau (G_c - G^\varepsilon)F(Y^\varepsilon(s))ds\right]| \\
 = |E\left[\int_0^\tau 0(\varepsilon)Y^\varepsilon(s, R^+)ds\right]| \\
 \leq 0(\varepsilon) \int_0^\tau EY^\varepsilon(s, R^+)ds,
 \end{aligned}$$

where τ is bounded, say by T , and $0(\varepsilon)$ depends only on F .

Using (2.2.17) we can continue to calculate:

$$\begin{aligned}
 (2.3.11) \quad |E\left[\int_0^\tau (G_c - G^\varepsilon)F(Y^\varepsilon(s))ds\right]| \\
 \leq \begin{cases} 0(\varepsilon)TY^\varepsilon(0, R^+) & \text{if } b_2(x) \equiv 0 \\ 0(\varepsilon)TY^\varepsilon(0, R^+) \exp\{-\frac{2b_0}{\varepsilon_2}0\} & \text{if } b_2(x) < 0 \end{cases} \\
 \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.
 \end{aligned}$$

In conclusion, from (2.3.7), (2.3.8), (2.3.9) and (2.3.11) we see that

$$E[F(Y(\tau)) - F(Y(0)) - \int_0^\tau G_c F(Y(s))ds]$$

$$\begin{aligned}
&= \lim_{\epsilon \rightarrow 0} E[F(Y^\epsilon(\tau)) - F(Y^\epsilon(0)) - \int_0^\tau G^\epsilon F(Y^\epsilon(s)) ds] \\
&= 0,
\end{aligned}$$

namely, the limiting process $Y(t)$ is a solution of the martingale problem for G_c .

To complete the proof of the Theorem 2.3.1, by the theorem of D.W. Stroock and S.R.S. Varadhan it suffices to show that the distribution of $Y(t)$ is uniquely determined. Now we suppose that the Equation (2.3.3) has a unique solution.

By Itô's formula and the fact that the process $Y(t)$ is a solution to the martingale problem for the generator G_c , it is easy to see that

$$(2.3.12) \quad W(s) \equiv f(s, Y(s)) - \int_0^s \left[\frac{\partial}{\partial v} f(v, Y(v)) + G_c f(v, Y(v)) \right] dv$$

is a martingale for $0 \leq s \leq t$, $f \in C^{1,2}([0, \infty) \times M_f(R^+))$ and $f(s, \cdot) \in \mathcal{D}(G_c)$, where $C^{1,2}([0, \infty) \times M_f(R^+))$ denotes the collection of twice differentiable functions on the space $[0, \infty) \times M_f(R^+)$ and $\mathcal{D}(G_c)$ is the domain of the generator G_c . In particular if we assume that

$$f(s, \mu) = \exp\{-\langle u(t-s), \mu \rangle\}$$

for $0 \leq s \leq t$ and $\mu \in M_f(R^+)$ where $u(t, x)$ is the solution to the Equation (2.3.3), then (2.3.12) shows that for $0 \leq s \leq t$

$$\frac{\partial}{\partial s} E_\mu f(s, Y(s)) + G_c E_\mu f(s, Y(s)) = 0$$

and therefore $\exp\{-\langle u(t-s), Y(s) \rangle\}$ is a martingale for $0 \leq s \leq t$. Hence we obtain the relation

$$E_\mu \exp\{-\langle u(0), Y(t) \rangle\} = \exp\{-\langle u(t), Y(0) \rangle\}.$$

Thus to complete the proof of the Theorem 2.3.1, it remains to check that the Equation (2.3.3) does have a unique solution, but this is true from the following remark. Hence the proof of the theorem 2.3.1 is complete.

□

Remark 2.3.1 *The existence and uniqueness of the partial differential equation (2.3.3) follows from S. Roelly and A. Rouault (1990) [2] or N. El-Karoui and S. Roelly (1991) [1].*

Remark 2.3.2 *That the martingale problem for the generator G_c has a unique solution can be easily checked by the following lemma:*

Condition C: Let $p > 0$ be fixed and set $\phi_p(x) = (1 + |x|^2)^{-\frac{p}{2}}$, $\phi_p^{-1} = \frac{1}{\phi_p}$. Then

(C.1) $C_c^\infty(R) \subset \mathcal{D}(A)$, $\phi_p \in \mathcal{D}(A)$ and $\phi_p^{-1} A \phi_p \in C_0(R)$ for every $\phi \in C_c^\infty(R)$;

(C.2) $\sup_{0 \leq t \leq T} \|\phi_p^{-1} T_t \phi_p\| < \infty$ for every $T > 0$,

where T_t denotes the strongly continuous Markov semigroup on $C_0(R)$ with the generator A .

Lemma 2.3.1 *Suppose that for $a \in C^+(\dot{R})$, $b \in C(\dot{R})$ and $\nu \in M_p(R)$, where $\dot{R}$ is the one point compactification of R and $C^+(\dot{R})$ denotes the collection of non-negative continuous functions defined on $\dot{R}$, the generator G is given by*

$$(2.3.13) \quad GF(\mu) = \frac{1}{2} \int_R \mu(dx) a(x) \frac{\partial^2 F}{\partial \mu(dx)^2} + \int_R \mu(dx) (A + b) \left(\frac{\partial F}{\partial \mu} \right)(x) + \int_R \nu(dx) \frac{\partial F}{\partial \mu}(x).$$

Let $p > 0$ be fixed. Suppose that the Condition C is satisfied for p , then for every $\mu \in M_p(R)$, the $(M_p(R), G, \mu)$ -martingale problem has a unique solution. Accordingly, G defines a diffusion process $(\Omega, \mathcal{F}, P_\mu, X_t)$ taking values in $M_p(R)$ uniquely, which is called a measure-valued branching diffusion with immigration.

(See K. Konno and T. Shiga (1988) [37] for the proof of the Lemma 2.3.1.)

Remark 2.3.3 *Our present results include as a special case the multilevel branching case in Dawson and Hochberg (1991) [5], where*

$$\begin{aligned} a_1(x) &= \gamma_1 x \quad , \quad b_1(x) = \frac{1}{2} \gamma_1 c_1 x, \\ a_2(x) &= \gamma_2 \quad , \quad b_2(x) = \frac{1}{2} \gamma_2 c_2 \end{aligned}$$

and (2.3.3) becomes

$$(2.3.14) \quad \frac{\partial u}{\partial t} = -\gamma_2 u^2 + \gamma_2 c_2 u + \gamma_1 x \frac{\partial^2 u}{\partial x^2} + \gamma_1 c_1 x \frac{\partial u}{\partial x}.$$

Chapter 3

The Moment Structure for the Limiting Process

In the previous chapter we studied the diffusion approximation to the multi-level birth and death particle system and characterized the limiting $M(R^+)$ -valued diffusion process as a unique solution of the martingale problem for the limiting generator. In this chapter, we proceed to consider the moment structures of this diffusion process.

3.1 Notations and Definitions

The purpose of this section is to establish some definitions and notations that will be employed in the next section.

Definition 3.1.1 *Suppose $Y(t)$ to be the limiting $M(R^+)$ -valued diffusion process obtained in the last chapter. We define the moments of the process as following:*

$$(3.1.1) \quad m_k(t) = E(\langle x^k, Y(t) \rangle),$$

$$(3.1.2) \quad M_k(t) = E(\langle x, Y(t) \rangle^k).$$

Throughout this chapter, we assume that

$$(3.1.3) \quad \begin{aligned} b_1(x) &= c_1 x, & b_2(x) &\equiv c_2, \\ a_1(x) &= c_3 x + c_4 x^2, \end{aligned}$$

where c_1, c_2, c_3 and c_4 are constants.

Definition 3.1.2 *If we let $X(t)$ denote the two level birth and death particle system introduced in the Chapter 1, then we define its n -th moment measure by*

$$(3.1.4) \quad m_n(t, \mu; j_1, \dots, j_n) = E\left[\prod_{i=1}^n X_t(j_i) \mid X_0 = \mu\right],$$

where $j_1, \dots, j_n \in Z^+$ (not necessarily distinct).

3.2 The Moment Structure

At first we are going to consider the moment measures for the two level birth and death particle system $\{X(t)\}$. Specifically we have

Proposition 3.2.1 *Under the assumption of (3.1.3), we have*

$$m_1(t, \delta_i, j) = p^{(1)}(t, i, j) \exp\{c_2 t\},$$

where $p^{(1)}(t, i, j)$ denotes the probability transition function for the level 1 birth and death process with the birth rate $a_1(\cdot) + b_1(\cdot)$ and death rate $a_1(\cdot) - b_1(\cdot)$ respectively.

Our proposition requires the following formula:

Feynman Kac formula. If let A denote the infinitesimal generator of a Markov process $X(t)$, i.e.

$$Af(x) = \lim_{t \rightarrow 0} \frac{E^x(f(X_t)) - f(x)}{t},$$

then

$$u(t, x) \equiv E^x[f(X_t) \exp\{-\int_0^t g(X_s) ds\}]$$

is a solution of the initial-value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= Au - gu, \\ u(0, x) &= f(x), \end{aligned}$$

where $g(x)$ is bounded and continuous.

(See, e.g., Øksendal (1985): "Stochastic Differential Equations", Springer-Verlag, pp95 – 96.)

Proof of proposition 3.2.1. Let $F(\mu) = \langle \delta_j, \mu \rangle$, then from (1.2.3) we see that

$$\begin{aligned} (3.2.1) \quad \frac{dn_1(t, \mu; j)}{dt} &= \frac{dE[X_t(j)|X_0 = \mu]}{dt} \\ &= \frac{dE[F(X_t)|X_0 = \mu]}{dt} = E_\mu GF(X_t) \\ &= E_\mu \left[\sum_k \lambda_1(X_t(k), k)(\delta_j(k+1) - \delta_j(k)) \right. \\ &\quad \left. + \sum_k \mu_1(X_t(k), k)(\delta_j(k-1) - \delta_j(k)) \right. \\ &\quad \left. + \sum_k \lambda_2(X_t(k), k)\delta_j(k) - \sum_k \mu_2(X_t(k), k)\delta_j(k) \right] \end{aligned}$$

$$\begin{aligned}
&= E_\mu \left[\sum_k X_t(k)(a_1(k) + b_1(k))(\delta_j(k+1) - \delta_j(k)) \right. \\
&\quad \left. + \sum_k X_t(k)(a_1(k) - b_1(k))(\delta_j(k-1) - \delta_j(k)) \right. \\
&\quad \left. + \sum_k X_t(k)(a_2(k) + b_2(k))\delta_j(k) - \sum_k X_t(k)(a_2(k) - b_2(k))\delta_j(k) \right] \\
&= E_\mu [X_t(j-1)(a_1(j-1) + b_1(j-1)) - X_t(j)(a_1(j) + b_1(j)) \\
&\quad + X_t(j+1)(a_1(j+1) - b_1(j+1)) \\
&\quad - X_t(j)(a_1(j) - b_1(j)) + X_t(j)(a_2(j) + b_2(j)) - X_t(j)(a_2(j) - b_2(j))] \\
&= E_\mu [X_t(j-1)(a_1(j-1) + b_1(j-1)) \\
&\quad - X_t(j)2a_1(j) + X_t(j+1)(a_1(j+1) - b_1(j+1)) + X_t(j)2b_2(j)] \\
&= (a_1(j-1) + b_1(j-1))m_1(t, j-1) - 2a_1(j)m_1(t, j) \\
&\quad + (a_1(j+1) - b_1(j+1))m_1(t, j+1) + 2b_2(j)m_1(t, j),
\end{aligned}$$

where $m_1(t, j) \equiv m_1(t, \mu; j)$.

Applying the Kolmogorov forward equation we have

$$\begin{aligned}
(3.2.2) \quad \frac{dp^{(1)}(t, i, j)}{dt} &= (a_1(j-1) + b_1(j-1))p^{(1)}(t, i, j-1) \\
&\quad - 2a_1(j)p^{(1)}(t, i, j) \\
&\quad + (a_1(j+1) - b_1(j+1))p^{(1)}(t, i, j+1) \\
&\equiv Ap^{(1)}(t, i, j),
\end{aligned}$$

where A denotes the infinitesimal generator of the level one birth and death process with the probability transition function $p^{(1)}(t, i, j)$.

If we choose $g(x) = -2b_2(x)$ in the Feynman-Kac formula and compare with (3.2.2), then we know that

$$(3.2.3) \quad \frac{dm_1(t, \delta_i, j)}{dt} = Am_1(t, \delta_i, j) - g(j)m_1(t, \delta_i, j)$$

has the solution:

$$m_1(t, \delta_i, j) = p^{(1)}(t, i, j) \exp\{c_2 t\}.$$

Thus the proof is complete.

□

Theorem 3.2.1 (i) $m_k(t)$ satisfies the following differential equation:

$$(3.2.4) \quad \begin{aligned} \frac{dm_k(t)}{dt} = & (2c_2 + k(k-1)c_4 + 2kc_1)m_k(t) \\ & + k(k-1)c_3m_{k-1}(t). \end{aligned}$$

(ii) The first two mean empirical moments $m_1(t)$ and $m_2(t)$ are given by

(a)

$$(3.2.5) \quad m_1(t) = m_0 \exp\{2(c_1 + c_2)t\};$$

(b)

$$(3.2.6) \quad \begin{aligned} m_2(t) = & m_2(0) \exp\{2(2c_1 + c_2 + c_4)t\} \\ & + \frac{c_3 m_1(0)}{c_1 + c_4} \exp\{2(c_1 + c_2)t\} [\exp\{2(c_1 + c_4)t\} - 1]. \end{aligned}$$

Proof. Since

$$\begin{aligned} m_k(t) &= E \langle x^k, Y(t) \rangle \\ &= E \langle x^k, Y(0) \rangle + E \int_0^t G_c \langle x^k, Y(s) \rangle ds, \end{aligned}$$

by taking derivative of both sides with respect to t , we have

$$\begin{aligned} (3.2.7) \quad \frac{dm_k(t)}{dt} &= \frac{dE \langle x^k, Y(t) \rangle}{dt} = EG_c \langle x^k, Y(t) \rangle \\ &= E[2 \langle b_2 x^k, Y(t) \rangle + \langle a_1 k(k-1)x^{k-2}, Y(t) \rangle \\ &\quad + 2 \langle b_1 k x^{k-1}, Y(t) \rangle] \\ &= 2E \langle b_2 x^k, Y(t) \rangle + k(k-1)E \langle a_1 x^{k-2}, Y(t) \rangle \\ &\quad + 2kE \langle b_1 x^{k-1}, Y(t) \rangle \\ &= 2E \langle c_2 x^k, Y(t) \rangle + k(k-1)E \langle c_3 x^{k-1}, Y(t) \rangle \\ &\quad + k(k-1)E \langle c_4 x^k, Y(t) \rangle + 2kE \langle c_1 x^k, Y(t) \rangle \\ &= (2c_2 + k(k-1)c_4 + 2kc_1)m_k(t) + k(k-1)c_3 m_{k-1}(t). \end{aligned}$$

In particular, we obtain

$$\begin{aligned} \frac{dm_1(t)}{dt} &= 2(c_1 + c_2)m_1(t), \\ \int_0^t \frac{dm_1(s)}{m_1(s)} &= \int_0^t (2c_1 + 2c_2)ds, \\ \ln m_1(s)|_0^t &= 2(c_1 + c_2)t; \end{aligned}$$

that is,

$$(3.2.8) \quad m_1(t) = m_1(0) \exp\{2(c_1 + c_2)t\}.$$

Substituting 2 for k in (3.2.7), we have

$$(3.2.9) \quad \frac{dm_2(t)}{dt} = 2(2c_1 + c_2 + c_4)m_2(t) + 2c_3m_1(t).$$

Putting together with (3.2.8), we obtain

$$(3.2.10) \quad \begin{aligned} \frac{dm_2(t)}{dt} + (-4c_1 - 2c_2 - 2c_4)m_2(t) \\ = 2c_3m_1(0) \exp\{2(c_1 + c_2)t\}. \end{aligned}$$

To solve the differential equation (3.2.10), if we let ρ be the integrating factor, then

$$\begin{aligned} \rho &= \exp\left\{\int(-4c_1 - 2c_2 - 2c_4)dt\right\} \\ &= \exp\{-2(2c_1 + c_2 + c_4)t\}, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}[\exp\{-2(2c_1 + c_2 + c_4)t\}m_2(t)] \\ = \exp\{-2(2c_1 + c_2 + c_4)t\}2c_3m_1(0) \exp\{2(c_1 + c_2)t\}, \end{aligned}$$

and so

$$\begin{aligned} \exp\{-2(2c_1 + c_2 + c_4)t\}m_2(t) - m_2(0) \\ = \int_0^t \exp\{(-2c_1 - 2c_4)s\}2c_3m_1(0)ds \\ = 2c_3m_1(0) \frac{1}{-2c_1 - 2c_4} [\exp\{(-2c_1 - 2c_4)t\} - 1], \end{aligned}$$

$$\begin{aligned} \exp\{-(4c_1 + 2c_2 + 2c_4)t\}m_2(t) \\ = m_2(0) + \frac{c_3m_1(0)}{c_1 + c_4} [1 - \exp\{-(2c_1 + 2c_4)t\}]. \end{aligned}$$

Therefore

$$\begin{aligned}
 m_2(t) &= m_2(0) \exp\{2(2c_1 + c_2 + c_4)t\} \\
 &\quad + \frac{c_3 m_1(0)}{c_1 + c_4} [\exp\{2(2c_1 + c_2 + c_4)t\} - \exp\{(2c_1 + 2c_2)t\}] \\
 &= m_2(0) \exp\{2(2c_1 + c_2 + c_4)t\} \\
 &\quad + \frac{c_3 m_1(0)}{c_1 + c_4} \exp\{2(c_1 + c_2)t\} [\exp\{2(c_1 + c_4)t\} - 1].
 \end{aligned}$$

The proof of the Theorem 3.2.1 is complete.

□

Theorem 3.2.2 *If $\bar{Y}(t) = Y(t, R^+)$ and $b_2(x) \equiv 0$, then $\bar{Y}(t)$ is a martingale and there exists a non-negative random variable ξ , such that*

$$\bar{Y}(t) \longrightarrow \xi \text{ a.e.}$$

as $t \rightarrow \infty$ and

$$E\xi \leq E\bar{Y}(0).$$

Proof. Let $\mathfrak{F}_s$ denote the σ -algebra generated by $\{Y(u) : 0 \leq u \leq s\}$, then for each $t > 0$ and $s > 0$,

$$\begin{aligned}
 (3.2.11) \quad E[\bar{Y}(t) \mid \mathfrak{F}_s] &= E[\langle Y(t), 1 \rangle \mid \mathfrak{F}_s] \\
 &= E[\langle 1, Y(s) \rangle + \int_s^t G_\cdot \langle 1, Y(u) \rangle du \mid \mathfrak{F}_s] \\
 &= \langle 1, Y(s) \rangle + E[\int_s^t 2 \langle b_2, Y(u) \rangle du \mid \mathfrak{F}_s] \\
 &= \bar{Y}(s)
 \end{aligned}$$

implies that $\bar{Y}(t)$ is a martingale. In addition, it is easy to see that for $b_2(x) \equiv 0$,

$$\sup_t E|\bar{Y}(t)| = E\bar{Y}(0) < \infty.$$

Therefore the Doob's martingale convergence theorem implies that there exists a random variable ξ , such that

$$\bar{Y}(t) \longrightarrow \xi$$

as $t \longrightarrow \infty$ with probability 1, where ξ satisfies

$$E|\xi| \leq E\bar{Y}(0).$$

Thus the proof is complete. □

Theorem 3.2.3 (i). If $c_1 + c_2 \leq 0$, then $M_2(t)$ satisfies the following inequality:

$$\begin{aligned} (3.2.12) \quad M_2(t) &\leq \exp\{4(c_1 + c_2)t\} M_2(0) + 2\|a_2\| \int_0^t m_2(s) ds \\ &\leq M_2(0) + 2\|a_2\| \sup_{0 \leq s \leq t} m_2(s)t \end{aligned}$$

for each $t \geq 0$.

(ii). Let $a_2(x) \equiv r$ (constant). Then if $c_1 + c_2 = 0$, we have

$$\begin{aligned} (3.2.13) \quad M_2(t) &= M_2(0) + \frac{rm_2(0)}{c_1 + c_4} (\exp\{2(c_1 + c_4)t\} - 1) \\ &\quad + \frac{rc_3m_1(0)}{(c_1 + c_4)^2} (\exp\{2(c_1 + c_4)t\} - 1) - \frac{2rc_3m_1(0)}{c_1 + c_4} t. \end{aligned}$$

and if $c_1 + c_2 \neq 0$, we have

$$\begin{aligned}
 (3.2.14) \quad M_2(t) = & M_2(0) \exp\{4(c_1 + c_2)t\} \\
 & + \frac{rm_2(0)}{c_4 - c_2} (\exp\{2(2c_1 + c_2 + c_4)t\} - \exp\{4(c_1 + c_2)t\}) \\
 & - \frac{rc_3m_1(0)}{(c_4 - c_2)(c_1 + c_2)} \exp\{4(c_1 + c_2)t\} \\
 & + \frac{rc_3m_1(0)}{c_4 + c_1} \left(\frac{1}{c_4 - c_2} \exp\{2(2c_1 + c_2 + c_4)t\} \right. \\
 & \quad \left. + \frac{1}{c_1 + c_2} \exp\{2(c_1 + c_2)t\} \right).
 \end{aligned}$$

Proof. Since

$$\begin{aligned}
 (3.2.15) \quad M_k(t) &= E \langle x, Y(t) \rangle^k \\
 &= E \langle x, Y(0) \rangle^k + E \int_0^t G_c \langle x, Y(s) \rangle^k ds,
 \end{aligned}$$

we obtain

$$\begin{aligned}
 (3.2.16) \quad \frac{dM_k(t)}{dt} &= EG_c \langle x, Y(t) \rangle^k \\
 &= E[k(k-1) \langle x, Y(t) \rangle^{k-2} \langle a_2 x^2, Y(t) \rangle \\
 &\quad + 2k \langle x, Y(t) \rangle^{k-1} \langle b_2 x, Y(t) \rangle \\
 &\quad + 2k \langle x, Y(t) \rangle^{k-1} \langle b_1, Y(t) \rangle] \\
 &= E[k(k-1) \langle x, Y(t) \rangle^{k-2} \langle a_2 x^2, Y(t) \rangle \\
 &\quad + 2kc_2 \langle x, Y(t) \rangle^k + 2kc_1 \langle x, Y(t) \rangle^k]
 \end{aligned}$$

$$= E[k(k-1) < x, Y(t) >^{k-2} < a_2 x^2, Y(t) > \\ + 2k(c_2 + c_1) < x, Y(t) >^k].$$

In particular we have

$$(3.2.17) \quad \frac{dM_2(t)}{dt} = 2E < a_2 x^2, Y(t) > + 4(c_2 + c_1)E < x, Y(t) >^2 \\ = 2E < a_2 x^2, Y(t) > + 4(c_2 + c_1)M_2(t).$$

(i). If $c_1 + c_2 \leq 0$, then

$$(3.2.18) \quad \frac{dM_2(t)}{dt} - 4(c_2 + c_1)M_2(t) = 2E < a_2 x^2, Y(t) >$$

is a linear differential equation and the integrating factor ρ is obtained by

$$(3.2.19) \quad \rho = \exp\left\{-\int 4(c_1 + c_2)t\right\} \\ = \exp\{-4(c_1 + c_2)t\}.$$

Hence

$$(3.2.20) \quad \frac{d}{dt}(M_2(t)\rho(t)) = \rho(t)2E < a_2 x^2, Y(t) > ,$$

$$(3.2.21) \quad M_2(t)\exp\{-4(c_1 + c_2)t\} - M_2(0) \\ = \int_0^t \exp\{-4(c_1 + c_2)s\} 2E < a_2 x^2, Y(s) > ds,$$

$$M_2(t) = \exp\{4(c_1 + c_2)t\}M_2(0) \\ + 2\exp\{4(c_1 + c_2)t\} \int_0^t \exp\{-4(c_1 + c_2)s\} E < a_2 x^2, Y(s) > ds \\ \leq \exp\{4(c_1 + c_2)t\}M_2(0) + 2\|a_2\| \int_0^t m_2(s)ds \\ \leq M_2(0) + 2\|a_2\| \sup_{0 \leq s \leq t} m_2(s)t.$$

(ii). Now we suppose that $a_2 \equiv r$. If $c_1 + c_2 = 0$, then

$$\begin{aligned}
 \frac{dM_2(t)}{dt} &= 2rm_2(t), \\
 M_2(t) - M_2(0) &= 2r \int_0^t m_2(s) ds \\
 &= 2r \int_0^t [m_2(0) \exp\{2(c_1 + c_4)s\} \\
 &\quad + \frac{c_3 m_1(0)}{c_1 + c_4} (\exp\{2(c_1 + c_4)s\} - 1)] ds \\
 &= 2rm_2(0) \frac{1}{2(c_1 + c_4)} (\exp\{2(c_1 + c_4)t\} - 1) \\
 &\quad + \frac{2rc_3 m_1(0)}{2(c_1 + c_4)^2} (\exp\{2(c_1 + c_4)t\} - 1) - \frac{2rc_3 m_1(0)}{c_1 + c_4} t.
 \end{aligned}$$

This implies that (3.2.13) holds.

If $c_1 + c_2 \neq 0$, then

$$(3.2.22) \quad \frac{dM_2(t)}{dt} - 4(c_2 + c_1)M_2(t) = 2rm_2(t),$$

and the integrating factor ρ is given by

$$\begin{aligned}
 (3.2.23) \quad \rho &= \exp\left\{-\int 4(c_1 + c_2)dt\right\} \\
 &= \exp\{-4(c_1 + c_2)t\}.
 \end{aligned}$$

in this case. Therefore

$$\begin{aligned}
 (3.2.24) \quad M_2(t) \exp\{-4(c_1 + c_2)t\} - M_2(0) \\
 &= 2r \int_0^t [m_2(0) \exp\{2(c_4 - c_2)s\} \\
 &\quad + \frac{c_3 m_1(0)}{c_1 + c_4} \exp\{-2(c_1 + c_2)s\} (\exp\{2(c_1 + c_4)s\} - 1)] ds
 \end{aligned}$$

$$\begin{aligned}
&= 2rm_2(0)\frac{1}{2(c_4 - c_2)}(\exp\{2(c_4 - c_2)t\} - 1) \\
&\quad + \frac{2rc_3m_1(0)}{c_1 + c_4}\frac{1}{2(c_4 - c_2)}(\exp\{2(c_4 - c_2)t\} - 1) \\
&\quad + \frac{2rc_3m_1(0)}{c_1 + c_4}\frac{1}{2(c_1 + c_2)}(\exp\{-2(c_1 + c_2)t\} - 1)
\end{aligned}$$

and

$$\begin{aligned}
M_2(t) &= M_2(0)\exp\{4(c_1 + c_2)t\} \\
&\quad + \frac{rm_2(0)}{c_4 - c_2}(\exp\{2(2c_1 + c_2 + c_4)t\} - \exp\{4(c_1 + c_2)t\}) \\
&\quad + \frac{rc_3m_1(0)}{(c_1 + c_4)(c_4 - c_2)}(\exp\{2(2c_1 + c_2 + c_4)t\} - \exp\{4(c_1 + c_2)t\}) \\
&\quad + \frac{rc_3m_1(0)}{(c_1 + c_4)(c_1 + c_2)}(\exp\{2(c_1 + c_2)t\} - \exp\{4(c_1 + c_2)t\}) \\
&= M_2(0)\exp\{4(c_1 + c_2)t\} \\
&\quad + \frac{rm_2(0)}{c_4 - c_2}(\exp\{2(2c_1 + c_2 + c_4)t\} - \exp\{4(c_1 + c_2)t\}) \\
&\quad - \exp\{4(c_1 + c_2)t\}\frac{rc_3m_1(0)}{c_1 + c_4}\left(\frac{1}{c_4 - c_2} + \frac{1}{c_1 + c_2}\right) \\
&\quad + \frac{rc_3m_1(0)}{c_1 + c_4}\left(\frac{1}{c_4 - c_2}\exp\{2(2c_1 + c_2 + c_4)t\} \right. \\
&\quad \quad \left. + \frac{1}{c_1 + c_2}\exp\{2(c_1 + c_2)t\}\right) \\
&= M_2(0)\exp\{4(c_1 + c_2)t\} \\
&\quad + \frac{rm_2(0)}{c_4 - c_2}(\exp\{2(2c_1 + c_2 + c_4)t\} - \exp\{4(c_1 + c_2)t\}) \\
&\quad - \exp\{4(c_1 + c_2)t\}\frac{rc_3m_1(0)}{(c_1 + c_2)(c_4 - c_2)} \\
&\quad + \frac{rc_3m_1(0)}{c_1 + c_4}\left(\frac{1}{c_4 - c_2}\exp\{2(2c_1 + c_2 + c_4)t\} + \frac{1}{c_1 + c_2}\exp\{2(c_1 + c_2)t\}\right).
\end{aligned}$$

The proof of the theorem 3.2.3 is complete.

Part II

MULTILEVEL
 $M(M(R^d))$ –VALUED
BRANCHING DIFFUSION
PROCESS

Chapter 4

Multilevel Branching Diffusion System

In the previous chapters, we investigate a dynamic multilevel particle system the multilevel birth and death system and its continuous diffusion limit. The all processes studied there them can be characterized as $M(R^+)$ -valued processes. From now on we study a more complicated stochastic model—the multilevel dynamic branching diffusion particle system in R^d and its diffusion approximation. In this chapter, we first introduce the multilevel branching and diffusion particle model which was formulated by D. Dawson and K. Hochberg in 1991. We consider its continuous analogue as a $M(M(R^d))$ -valued process $Y(t)$. Then we study the “aggregated” $M(R^d)$ -valued process $\bar{Y}(t)$, which is defined in terms of $Y(t)$. Finally, we discuss the moment measures and the Hausdorff dimension of the support of the “aggregated” process $\bar{Y}(t)$. Specifically, a lower bound of the Hausdorff dimension is obtained.

4.1 Introduction

We define $\dot{R}^d = R^d \cup \{\tau\}$, τ being an isolated adjoined point. Let $M_\rho(\dot{R}^d)$ be the space of ρ -tempered measures introduced by Iscoe (1983) with the ρ -vague topology, that is, the smallest topology making the maps $\mu \longrightarrow \langle \phi, \mu \rangle$ continuous for $\phi \in C_c(R^d) \cup \{\phi_\rho\}$, where $d < \rho \leq d+2$ and for $x \in R^d$

$$(4.1.1) \quad \phi_\rho(x) = \frac{1}{1 + |x|^\rho}$$

and

$$\phi_\rho(\tau) = 1.$$

Iscoe has shown that M_ρ is locally compact with the ρ -vague topology (see, Iscoe 1986, [31]). In the remainder of this thesis, we also need the following notations:

$N(R^d)$ denotes all integer-valued measures on R^d ,

$M(M(R^d))$ is the collection of all positive measures on $M(R^d)$,

$C_b(R)$ is the Banach space of bounded continuous functions on R with $\|f\| = \sup_{x \in R} |f(x)|$,

$C_b^2(R) \subset C_b(R)$ is the subspace of bounded twice differentiable functions on R ,

$C_c(R^d)$ denotes the collection of continuous functions on R^d with compact support,

$C_c^2(R^d)$ denotes the collection of twice differentiable functions on R^d with compact support,

$C_c^+(R^d)$ is the collection of positive continuous functions on R^d with compact support.

Let

$$M_\rho^2(\dot{R}^d) \equiv \{\nu \in M(M_\rho(\dot{R}^d)) : \int \int \phi_\rho(x) \mu(dx) \nu(d\mu) < \infty\}.$$

We endow $M_\rho^2(\dot{R}^d)$ with the smallest topology which make continuous the maps

$$\nu \longrightarrow \int \int \phi(x) \mu(dx) \nu(d\mu)$$

for all $\phi \in C_c(R^d) \cup \{\phi_\rho\}$.

The basic process in this part is a $M_\rho^2(\dot{R}^d)$ -valued process $Y(t)$. The reason that we consider $M_\rho^2(\dot{R}^d)$ as the state space of the process is that in order to study the long time behaviour of the process we must use a class of infinite measures as initial measures of the process $Y(t)$. It is easy to see that the state space $M_\rho^2(\dot{R}^d)$ is rich enough for our needs.

We will not discuss in detail the construction of the process in this thesis but explain briefly how it can be characterized as the continuous limit of a two level branching diffusion particle system.

Now we consider a multilevel branching diffusion system or multilevel branching random field which was introduced by D. Dawson and K. Hochberg in 1991. The multilevel branching diffusion particle system is a natural extension of the multilevel branching model which can be described as following.

We consider a two level branching random field. By "two level" we mean that the system consists of particles at two different levels. Each level two particle (i.e., superparticle) is a collection of level one particles (i.e., particle) in R^d . We say that a superparticle is of size i if it consists of exactly i particles. We suppose that after an exponentially distributed time, each particle undergoes a level one binary branching process with the branching rate γ_1 in which case we note that the total number of superparticles is unchanged. We also assume that each superparticle performs a level two binary branching process with the branching rate γ_2 , i.e., it either dies or produces a copy, each with probability one-half, after an exponentially distributed holding time with the parameter γ_2 . During the holding times of both level one and level two, each particle moves in R^d according to the d -dimensional Brownian motion. The entire system can be represented as a random atomic measure on $N(R^d)$:

$$(4.1.2) \quad Y_a(t) \equiv \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \delta_{\sum_{r=1}^i \delta_{x_{i,k,r}(t)}} + n_0(t) \delta_{\phi}$$

where $x_{i,k,r}(t)$ denotes the location in R^d of the r th particle in the k th superparticle $X_{i,k}(t)$ of size i at time t , n_i is the number of the superparticles of size i at time t and $n_0(t)$ denotes the number of the null superparticles at time t where the term "null superparticle" means that it does not contain any particles.

2

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2

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The possible transitions of state of the two level branching diffusion particle system can be described by the following pictures:

(a). Copy of the superparticle X_{i,k_1} :

Figure 4.1: Level Two Birth

with the transition of state

$$\sum_{k=1}^{n_i} \delta_{\sum_{r=1}^i \delta_{x_{i,k},r}} \longrightarrow \sum_{k \neq k_1} \delta_{\sum_{r=1}^i \delta_{x_{i,k},r}} + 2\delta_{\sum_{r=1}^i \delta_{x_{i,k_1},r}}$$

at rate γ_2 .

(b). Death of the superparticle X_{i,k_1} :

Figure 4.2: **Level Two Death**

with the transition of state

$$\sum_{k=1}^{n_i} \delta_{\sum_{r=1}^i \delta_{x_{i,k},r}} \longrightarrow \sum_{k \neq k_1} \delta_{\sum_{r=1}^i \delta_{x_{i,k},r}}$$

at the rate γ_2 .

(c). Birth of a particle in the superparticle X_{i,k_1} :

Figure 4.3: **Level One Birth**

with the transition of state

$$\begin{aligned} & \sum_{k=1}^{n_i} \delta_{\sum_{r=1}^i \delta_{x_{i,k},r}} + \sum_{k=1}^{n_{i+1}} \delta_{\sum_{r=1}^{i+1} \delta_{x_{i+1,k},r}} \\ \longrightarrow & \sum_{k \neq k_1} \delta_{\sum_{r=1}^i \delta_{x_{i,k},r}} + \sum_{k=1}^{n_{i+1}+1} \delta_{\sum_{r=1}^{i+1} \delta_{x_{i+1,k},r}} \end{aligned}$$

at rate $i\gamma_1$, in which case the number of the superparticles of size $i+1$ has changed from n_{i+1} to $n_{i+1} + 1$ and the number of superparticles of size i has changed from n_i to $n_i - 1$ due to the birth of a particle in a superparticle of size i .

(d). Death of a particle in the superparticle X_{i,k_1} :

Figure 4.4: Level One Death

with the transition of state

$$\begin{aligned} & \sum_{k=1}^{n_{i-1}} \delta_{\sum_{r=1}^{i-1} \delta_{x_{i-1,k},r}} + \sum_{k=1}^{n_i} \delta_{\sum_{r=1}^i \delta_{x_{i,k},r}} \\ \longrightarrow & \sum_{k=1}^{n_{i-1}+1} \delta_{\sum_{r=1}^{i-1} \delta_{x_{i-1,k},r}} + \sum_{k \neq k_1} \delta_{\sum_{r=1}^i \delta_{x_{i,k},r}} \end{aligned}$$

at rate $i\gamma_1$, in which case the number of the superparticles of size $i - 1$ has changed from n_{i-1} to $n_{i-1} - 1$ and the number of superparticles of size i has changed from n_i to $n_i - 1$ due to the death of a particle in a superparticle of size i .

(e). Each particle at $x_{i,k,r}$ can change its location in R^d to some new location $x'_{i,k,r}$ according to the d -dimensional Brownian motion:

Figure 4.5: Spatial Diffusion of Particles

where the transition of the state can be described as

$$\delta_{\sum_{r=1}^i \delta_{x_{i,k,r}}} \longrightarrow \delta_{\sum_{r=1}^i \delta_{x'_{i,k,r}}}.$$

In this case, the total number of the particles has no change.

Remark 4.1.1 By “different” superparticles X_{i_1,k_1} and X_{i_2,k_2} we mean that either the number of particles or locations of particles or both numbers and locations of particles of the superparticles are different, that is

$$\delta_{\sum_{r=1}^{m_1} \delta_{x_{i_1,k_1,r}}} \neq \delta_{\sum_{r=1}^{m_2} \delta_{x_{i_2,k_2,r}}}$$

Consider the test function $F(\nu)$ on M_ρ^2 of the following form:

$$(4.1.3) \quad F(\nu) = f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu \rangle\rangle),$$

where $\nu \in M_\rho^2$, $h_1, f \in C_b^2(R)$ and $h_2 \in C_c^2(R^d)$ and $\langle\langle g(\mu), \nu \rangle\rangle \equiv \int g(\mu) \nu(d\mu)$.

If we let $\nu = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \delta_{\sum_{r=1}^i \delta_{x_{i,k,r}}}$, where $x_{i,k,r} \in R^d$ then the generator $G^{(2)}$ of the two level branching diffusion particle system can be described formally by the following expression:

$$G^{(2)}F(\nu) = B_1F(\nu) + D_1F(\nu) + B_2F(\nu) + D_2F(\nu) + WF(\nu),$$

where

(i) B_2 is the term of the birth of the superparticle, that is

$$B_2F(\nu) = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \gamma_2 [f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu + \delta_{\sum_{r=1}^i \delta_{x_{i,k,r}}} \rangle\rangle) - f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu \rangle\rangle)];$$

(ii) D_2 is the term of the death of the superparticle, that is

$$D_2F(\nu) = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \gamma_2 [f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu - \delta_{\sum_{r=1}^i \delta_{x_{i,k,r}}} \rangle\rangle) - f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu \rangle\rangle)];$$

(iii) B_1 is the level one birth term, that is

$$B_1F(\nu) = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \sum_{r_1=1}^i \gamma_1 [f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu - \delta_{\sum_{r=1}^i \delta_{x_{i,k,r}}} + \delta_{\sum_{r=1}^i \delta_{x_{i,k,r}} + \delta_{x_{i,k,r_1}}} \rangle\rangle) - f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu \rangle\rangle)];$$

(iv) D_1 is the level one death term, that is

$$D_1 F(\nu) = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \sum_{r_1=1}^i \gamma_1 [f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu - \delta_{\sum_{r=1}^i \delta_{x_i, k, r}} \rangle\rangle + \delta_{\sum_{r=1}^i \delta_{x_i, k, r} - \delta_{x_i, k, r_1}} \rangle\rangle) - f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu \rangle\rangle)];$$

(v) W is the diffusion term resulting from the spatial motion of particles in R^d .

To study the continuous limit of the two level branching diffusion particle systems, we rescale the system as follows: If $\mathcal{B}(M_\rho(\dot{R}^d))$ denotes the *Borel σ -algebra of $M_\rho(\dot{R}^d)$* (ref. [31], p90), then for $A \in \mathcal{B}(M_\rho(\dot{R}^d))$ and $t > 0$. we define

$$Y_n(t, A) = \frac{1}{n} Y_n(nt, A_n),$$

where

$$A_n = \{\mu : \frac{1}{n}\mu \in A\}.$$

Following the procedure of Chapter 2 and using the same methods we can prove the tightness of the processes $\{Y_n(t)\}$. We can also show that when $n \rightarrow \infty$, $Y_n(t)$ converges weakly in $D_{[0, \infty)}(M_\rho^2(R^d))$ to a M_ρ^2 -valued process $Y(t)$ which can be characterized as the unique solution of the martingale problem for the limiting generator $G_c^{(2)}$ which is given by

$$(4.1.4) \quad G_c^{(2)} F(\nu) = \langle\langle \mathcal{L}F'(\nu, \cdot), \nu \rangle\rangle + \gamma_2 \langle\langle F''(\nu), \delta_{\mu_1}(d\mu_2)\nu(d\mu_1) \rangle\rangle.$$

In (4.1.4) $\mathcal{L}$ denotes the generator of the $M_\rho(R^d)$ -valued branching process, i.e.

$$(4.1.5) \quad \mathcal{L}F'(\nu, \mu) = \mathcal{L}_1 F'(\nu, \mu) + \mathcal{L}_2 F'(\nu, \mu)$$

$$\begin{aligned}
 &= f'(<< h_1(< h_2, \cdot >), \nu >>) h_1'(< h_2, \mu >) < \Delta h_2, \mu > \\
 &\quad + \gamma_1 f'(<< h_1(< h_2, \cdot >), \nu >>) h_1''(< h_2, \mu >) < h_2^2, \mu >,
 \end{aligned}$$

where Δ is the d -dimensional Laplacian,

$$(4.1.6) \quad \mathcal{L}_1 F'(\nu, \mu) = \int \Delta \frac{\delta F'(\nu, \mu)}{\delta \mu(x)} \mu(dx)$$

and

$$(4.1.7) \quad \mathcal{L}_2 F'(\nu, \mu) = \gamma_1 \int \int \frac{\delta^2 F'(\nu, \mu)}{\delta \mu(x) \delta \mu(y)} \delta_x(dy) \mu(dx),$$

where

$$\begin{aligned}
 (4.1.8) \quad F'(\nu, \mu) &= \frac{\partial F(\nu)}{\partial \nu(\mu)} \\
 &= \frac{d}{d\varepsilon} [F(\nu + \varepsilon \delta_\mu)]_{\varepsilon=0} \\
 &= f'(<< h_1(< h_2, \cdot >), \nu >>) h_1(< h_2, \mu >).
 \end{aligned}$$

The Laplace transition functional of the M_ρ^2 -valued process $Y(t)$ is given by

$$\begin{aligned}
 (4.1.9) \quad L_{t,\nu}(H) &= E[\exp(-\int_{M_\rho(R^d)} H(\mu) Y(t, d\mu)) \mid Y(0) = \nu] \\
 &= \exp\{-\int u(t, \mu) \nu(d\mu)\},
 \end{aligned}$$

where $u(t, \mu)$ is a solution of the weak form of the following integral equations :

$$\begin{aligned}
 (4.1.10) \quad u(t, \mu) &= T_t u(0, \mu) - \gamma_2 \int_0^t [T_{t-s} u^2(s, \cdot)](\mu) ds \\
 u(0, \mu) &= H(\mu)
 \end{aligned}$$

and $H(\mu) = h(< \phi, \mu >)$, $\phi \in C_c(R^d)$ and $h \in C_b(R)$.

Remarks 4.1.1 *In order to have a better understanding of the two level branching diffusion particle system $Y_a(t)$ and its continuous limit process $Y(t)$, we consider a special case in which there is no level two branching, i.e. $\gamma_2 \equiv 0$.*

Firstly, let us look at the particle system $Y_a(t)$. It is clear that if $\gamma_2 \equiv 0$ and $Y_a(0) = \sum_{i=1}^N \delta_{\sum_{j=1}^{n_i} \delta_{z_j}}$, then for any $t > 0$, $Y_a(t)$ must have the form

$$Y_a(t) = \sum_{i=1}^N \delta_{\sum_{j=1}^{m_i} \delta_{v_j}},$$

that is, the number N of the superparticles is nonincreasing.

Next, we consider the continuous analogue $Y(t)$ of the infinite particle branching system. It is not hard to see that if $\gamma_2 \equiv 0$, then there is no superparticle branching in the particle system, so as the continuous limit of this particle system, $Y(t)$ must be a process with atomic measure values on $M_\rho(R^d)$. In particular, if $Y(0) = \delta_{\mu_1} + \delta_{\mu_2} + \cdots + \delta_{\mu_n}$, then for any $t > 0$ we have

$$Y(t) = \delta_{X_1(t)} + \delta_{X_2(t)} + \cdots + \delta_{X_n(t)},$$

where $X_i(t)$ is a one level $M_\rho(R^d)$ -valued branching process with initial distribution $X_i(0) = \mu_i$, $i = 1, \dots, n$.

Given the M_ρ^2 -valued process $Y(t)$, we define the "aggregated" measure-valued process $\bar{Y}(t)$ by

$$(4.1.11) \quad \bar{Y}(t) = \int \mu Y(t, d\mu).$$

It follows from the expression for the Laplace functional of $\bar{Y}(t)$ that $\bar{Y}(t)$ is infinitely divisible but not multiplicative.

4.2 Moment Measures for $\bar{Y}(t)$

In this section, we are going to consider the moment measures for the $M_p(R^d)$ valued aggregated process $\bar{Y}(t)$ defined before. We describe the moment measures in terms of the semigroups associated with the generator $\mathcal{L}$ and the d dimensional Brownian motion respectively. Finally we investigate the long time behaviour for the second moment measure.

Now we are going to consider the moment measures for $\bar{Y}(t)$. The n th moment measure is defined by

$$(4.2.1) \quad \begin{aligned} m_n(t, \mu_0; dx_1, \dots, dx_n) \\ \equiv E\left[\prod_{i=1}^n \bar{Y}(t, dx_i) \mid \bar{Y}(0) = \int \mu \nu_0(d\mu) \equiv \mu_0\right], \end{aligned}$$

where $\nu_0 \in M_p^2(R^d)$ and hence $\mu_0 \in M_p(R^d)$.

Before coming to the main theorem of this section, we take up a following preliminary lemma:

Lemma 4.2.1 *Let $\{T_t\}$ be the semigroup of the $M_p(R^d)$ - valued branching process with the generator $\mathcal{L}$ and $\{S_t\}$ be the Brownian semigroup. Then we have*

$$(4.2.2) \quad T_t < \phi, \mu > = < S_t \phi, \mu >$$

and

$$\begin{aligned}
 (4.2.3) \quad T_t < \phi, \cdot >^2 (\mu) \\
 &= < S_t \phi, \mu >^2 + 2\gamma_1 \int \int_0^t S_{t-s} (S_s \phi)^2 ds \mu(dx)
 \end{aligned}$$

for $t > 0$, $\phi \in C_c(R^d)$ and $\mu \in M_p(R^d)$.

Proof. Let $\mathcal{P}(t, \mu_0, d\mu)$ be the supertransition function over $\{S_t\}$ with generator $\mathcal{L}$. Then it is known that (See Dynkin [24])

$$\begin{aligned}
 (4.2.4) \quad T_t \exp\{- < \phi \theta, \mu >\} \\
 &= \int \exp\{- < \phi \theta, m >\} \mathcal{P}(t, \mu, dm) \\
 &= \exp\{- < v(t), \mu >\},
 \end{aligned}$$

where $v(t, x)$ is the solution of the following equations:

$$\begin{aligned}
 (4.2.5) \quad v(t, x) &= S_t v(0, x) - \gamma_1 \int_0^t S_{t-s} (v(s, x))^2 ds, \\
 v(0, x) &= \theta \phi(x).
 \end{aligned}$$

By an argument similar to that of (4.2.14), we have

$$\begin{aligned}
 (4.2.6) \quad v(t, x) &= S_t \theta \phi(x) - \gamma_1 \int_0^t S_{t-s} [S_s v(0)(x) \\
 &\quad - \gamma_1 \int_0^s S_{s-u} (v(u))^2(x) du]^2 ds \\
 &= S_t \theta \phi(x) - \gamma_1 \theta^2 \int_0^t S_{t-s} (S_s \phi)^2(x) ds \\
 &\quad + \text{higher order terms in } \theta.
 \end{aligned}$$

Using the Taylor expansion it is easy to see that

$$\begin{aligned}
 (4.2.7) \quad \exp\{-\langle r(t), \mu \rangle\} &= 1 - \langle r(t), \mu \rangle + \frac{1}{2} \langle r(t), \mu \rangle^2 + \dots \\
 &= 1 - \theta \langle S_t \phi, \mu \rangle \\
 &\quad + \theta^2 \left[\langle \gamma_1 \int_0^t S_{t-s}(S_s \phi)^2 ds, \mu \rangle \right. \\
 &\quad \left. + \frac{1}{2} \langle S_t \phi, \mu \rangle^2 \right] + \text{higher order terms in } \theta.
 \end{aligned}$$

Hence we obtain the relation

$$\begin{aligned}
 (4.2.8) \quad &\int \exp\{-\langle \theta \phi, m \rangle\} \mathcal{P}(t, \mu, dm) \\
 &= \int \mathcal{P}(t, \mu, dm) \left[1 - \langle \theta \phi, m \rangle + \frac{\theta^2}{2} \langle \phi, m \rangle^2 + \dots \right] \\
 &= 1 - \theta \int \mathcal{P}(t, \mu, dm) \langle \phi, m \rangle \\
 &\quad + \frac{\theta^2}{2} \int \mathcal{P}(t, \mu, dm) \langle \phi, m \rangle^2 + \dots \\
 &= 1 - \theta T_t \langle \phi, \mu \rangle + \frac{\theta^2}{2} T_t \langle \phi, \mu \rangle^2 + \dots.
 \end{aligned}$$

Therefore the result follows by equating the coefficients of θ and θ^2 in (4.2.7) and (4.2.8) respectively. The proof of the Lemma 4.2.1 is complete. □

Now we turn to our main theorem of this section:

Theorem 4.2.1 *The first and second moment measures of the process $\bar{Y}(t)$ are given by*

$$(4.2.9) \quad \int \phi(x) m_1(t, \mu_0; dx) = \int \int S_t \phi(x) \mu(dx) \nu_0(d\mu)$$

and

$$\begin{aligned}
 (4.2.10) \quad & \int \int \phi(x_1)\phi(x_2)m_2(t, \mu_0; dx_1, dx_2) \\
 &= 2\gamma_2 \int_0^t \int_0^s T_{t-s} [\int \int S_s \phi(x_1) S_s \phi(x_2) \mu(dx_1) \mu(dx_2)] ds \nu_0(d\mu) \\
 &\quad + [\int \int S_t \phi(x) \mu(dx) \nu_0(d\mu)]^2
 \end{aligned}$$

respectively for $\phi \in C_c(R^d)$.

Proof. Note that the Laplace functional of $\bar{Y}(t)$ is given by

$$\begin{aligned}
 (4.2.11) \quad & E[\exp\{-\int \theta \phi(x) \bar{Y}(t, dx)\} \mid \bar{Y}(0) = \mu_0] \\
 &= E[\exp\{-\int \int \theta \phi(x) \mu(dx) Y(t, dx)\} \mid Y(0) = \nu_0] \\
 &= \exp\{-\langle\langle u(t), \nu_0 \rangle\rangle\},
 \end{aligned}$$

where $u(t, \mu)$ is the solution of

$$\begin{aligned}
 (4.2.12) \quad & \frac{\partial u(t, \mu)}{\partial t} = \mathcal{L}u(t, \mu) - \gamma_2(u(t, \mu))^2, \\
 & u(0, \mu) = \int \theta \phi(x) \mu(dx),
 \end{aligned}$$

so we can determine the moment measures of $\bar{Y}(t)$ by finding the solution of the equation (4.2.12).

Observe that $u(t, \mu)$ is also a solution of the equation:

$$(4.2.13) \quad u(t) = T_t u(0) - \gamma_2 \int_0^t T_{t-s} (u(s))^2 ds.$$

We solve this integral equation iteratively:

$$\begin{aligned}
(4.2.14) u(t) &= T_t u(0) - \gamma_2 \int_0^t T_{t-s} (u(s))^2 ds \\
&= T_t u(0) - \gamma_2 \int_0^t T_{t-s} [T_s u(0) - \gamma_2 \int_0^s T_{s-v} (u(v))^2 dv]^2 ds \\
&= T_t u(0) - \gamma_2 \int_0^t T_{t-s} [(T_s u(0))^2 - 2T_s u(0) \gamma_2 \int_0^s T_{s-v} (u(v))^2 dv \\
&\quad + (\gamma_2 \int_0^s T_{s-v} (u(v))^2 dv)^2] ds \\
&= T_t u(0) - \gamma_2 \int_0^t T_{t-s} (T_s u(0))^2 ds \\
&\quad + 2\gamma_2^2 \int_0^t T_{t-s} T_s u(0) \int_0^s T_{s-v} (u(v))^2 dv ds \\
&\quad - \gamma_2^3 \int_0^t T_{t-s} (\int_0^s T_{s-v} (u(v))^2 dv)^2 ds \\
&= T_t u(0) - \gamma_2 \int_0^t T_{t-s} (T_s u(0))^2 ds \\
&\quad + 2\gamma_2^2 \int_0^t T_t u(0) \int_0^s T_{s-v} [T_v u(0) \\
&\quad - \gamma_2 \int_0^v T_{v-w} (u(w))^2 dw]^2 dv ds \\
&\quad - \gamma_2^3 \int_0^t T_{t-s} (\int_0^s T_{s-v} (u(v))^2 dv)^2 ds \\
&= \theta T_t < \phi, . > - \gamma_2 \theta^2 \int_0^t T_{t-s} (T_s < \phi, . >)^2 ds \\
&\quad + 2\gamma_2^2 \theta^3 \int_0^t T_t < \phi, . > \int_0^s T_{s-v} (T_v < \phi, . >)^2 dv ds \\
&\quad + \text{higher order terms in } \theta,
\end{aligned}$$

where $u(0, \mu) = < \theta \phi, \mu > = \theta < \phi, \mu >$.

Together with the Lemma 4.2.1 we have

$$(4.2.15) \mu(t, \mu) = \theta \int S_t \phi(x) \mu(dx)$$

$$-\gamma_2\theta^2 \int_0^t T_{t-s} [\int \int S_s\phi(x_1)S_s\phi(x_2)\mu(dx_1)\mu(dx_2)]ds + \dots.$$

We can continue to calculate

$$(4.2.16) \quad \begin{aligned} \langle\langle u(t), \nu_0 \rangle\rangle &= \theta \int \int S_t\phi(x)\mu(dx)\nu_0(d\mu) \\ &\quad -\gamma_2\theta^2 \int \int_0^t T_{t-s} [\int \int S_s\phi(x_1)S_s\phi(x_2) \\ &\quad \mu(dx_1)\mu(dx_2)]ds\nu_0(d\mu) + \dots \end{aligned}$$

and

$$(4.2.17) \quad \begin{aligned} &\exp\{-\langle\langle u(t), \nu_0 \rangle\rangle\} \\ &= 1 - \langle\langle u(t), \nu_0 \rangle\rangle + \frac{1}{2} \langle\langle u(t), \nu_0 \rangle\rangle^2 \\ &\quad - \frac{1}{3!} \langle\langle u(t), \nu_0 \rangle\rangle^3 + \dots \\ &= 1 - [\theta \int \int S_t\phi(x)\mu(dx)\nu_0(d\mu) \\ &\quad -\gamma_2\theta^2 \int \int_0^t T_{t-s} (\int \int S_s\phi(x_1)S_s\phi(x_2)\mu(dx_1)\mu(dx_2))ds\nu_0(d\mu) + \dots] \\ &\quad + \frac{1}{2} [\theta \int \int S_t\phi(x)\mu(dx)\nu_0(d\mu) \\ &\quad -\gamma_2\theta^2 \int \int_0^t T_{t-s} (\int \int S_s\phi(x_1)S_s\phi(x_2)\mu(dx_1)\mu(dx_2))ds\nu_0(d\mu) \\ &\quad + \dots]^2 + \dots. \end{aligned}$$

Together with (4.2.11), we obtain

$$(4.2.18) \quad \begin{aligned} &E[\exp\{-\int \theta\phi(x)\bar{Y}(t, dx)\} \mid \bar{Y}(0) = \mu_0] \\ &= 1 - [\theta \int \int S_t\phi(x)\mu(dx)\nu_0(d\mu) \end{aligned}$$

$$\begin{aligned}
& -\gamma_2 \theta^2 \int \int_0^t T_{t-s} \left(\int \int S_s \phi(x_1) S_s \phi(x_2) \mu(dx_1) \mu(dx_2) \right) ds \nu_0(d\mu) + \dots] \\
& \quad + \frac{1}{2} [\theta \int \int S_t \phi(x) \mu(dx) \nu_0(d\mu) \\
& -\gamma_2 \theta^2 \int \int_0^t T_{t-s} \left(\int \int S_s \phi(x_1) S_s \phi(x_2) \mu(dx_1) \mu(dx_2) \right) ds \nu_0(d\mu) \\
& \quad + \dots]^2 + \dots.
\end{aligned}$$

Taking the first and second derivatives in both sides of the equation (4.2.18) with respect to θ at $\theta = 0$ respectively, we have

$$E\left[\int \phi(x) \bar{Y}(t, dx) \mid \bar{Y}(0) = \mu_0\right] = \int \int S_t \phi(x) \mu(dx) \nu_0(d\mu)$$

and

$$\begin{aligned}
(4.2.19) \quad & E\left[\left(\int \phi(x) \bar{Y}(t, dx)\right) \left(\int \phi(x) \bar{Y}(t, dx)\right) \mid \bar{Y}(0) = \mu_0\right] \\
& = 2\gamma_2 \int \int_0^t T_{t-s} \left[\int \int S_s \phi(x_1) S_s \phi(x_2) \mu(dx_1) \mu(dx_2) \right] ds \nu_0(d\mu) \\
& \quad + \left[\int \int S_t \phi(x) \mu(dx) \nu_0(d\mu) \right]^2.
\end{aligned}$$

Thus we obtain the result as claimed.

□

Now we proceed to discuss the long-time behaviour of $m_2(t, \mu_0; dx_1, dx_2)$ in the case of $Y(0) = \delta_{\delta_x} dx$. That is

$$d\mu_0 = \int \mu \delta_{\delta_x} dx = \delta_x dx.$$

By (4.2.10), for any $\phi \in C_c(R^d)$,

$$\begin{aligned}
 (4.2.20) \quad & \int \int \phi(x_1)\phi(x_2)m_2(t, \mu_0; dx_1 dx_2) \\
 = & 2\gamma_2 \int \int_0^t T_{t-s} \left[\int \int S_s \phi(x_1) S_s \phi(x_2) \delta_x(dx_1) \delta_x(dx_2) \right] ds \delta_x dx \\
 & + \left[\int \int S_t \phi(x) \delta_x(dx_1) \delta_x dx \right]^2 \\
 = & 2\gamma_2 \int \int_0^t T_{t-s} \left[\int \int S_s \phi(x_1) S_s \phi(x_2) \delta_x(dx_1) \delta_x(dx_2) \right] ds dx \\
 & + \left[\int \int S_t \phi(x) \delta_x(dx_1) dx \right]^2 \\
 = & 2\gamma_2 \int \int_0^t T_{t-s} \langle S_s \phi, \cdot \rangle^2 (\delta_x) ds dx + \left[\int S_t \phi(x) dx \right]^2 \\
 = & \left[\int S_t \phi(x) dx \right]^2 + 2\gamma_2 \int \int_0^t \left[\langle S_{t-s} S_s \phi, \delta_x \rangle^2 \right. \\
 & \left. + 2\gamma_1 \int \int_0^{t-s} S_{t-s-v} (S_v S_s \phi)^2 dv \delta_x(dx_1) \right] ds dx \\
 = & \left[\int S_t \phi(x) dx \right]^2 + 2\gamma_2 \int \int_0^t [(S_t \phi(x))^2 \\
 & + 2\gamma_1 \int_0^{t-s} S_{t-s-v} (S_{v+s} \phi)^2(x) dv] ds dx \\
 = & \left[\int \phi(x) dx \right]^2 + 2\gamma_2 t \int (S_t \phi(x))^2 dx \\
 & + 4\gamma_1 \gamma_2 \int \int_0^t \int_0^{t-s} S_{t-s-v} (S_{v+s} \phi)^2(x) dv ds dx \\
 = & 2\gamma_2 t \int \int \int p(t, x_1 - x) p(t, x_2 - x) \phi(x_1) \phi(x_2) dx_1 dx_2 dx \\
 & + 4\gamma_1 \gamma_2 \int \int_0^t \int_s^t S_{t-u} (S_u \phi)^2(x) du ds dx + \left[\int \phi(x) dx \right]^2 \\
 = & 2\gamma_2 t \int \int p(2t, x_1 - x_2) \phi(x_1) \phi(x_2) dx_1 dx_2 \\
 & + 4\gamma_1 \gamma_2 \int_0^t \int_s^t (S_u \phi)^2(x) dx du ds + \left[\int \phi(x) dx \right]^2 \\
 = & 2\gamma_2 t \int \int (2\pi 2t)^{-\frac{d}{2}} \exp\left\{-\frac{|x_1 - x_2|^2}{4t}\right\} \phi(x_1) \phi(x_2) dx_1 dx_2
 \end{aligned}$$

$$\begin{aligned}
& +4\gamma_1\gamma_2 \int_0^t \int_s^t \int \int \int p(u, x; x_1)p(u, x; x_2) \\
& \quad \phi(x_1)\phi(x_2)dx_1dx_2dxdu ds + \left[\int \phi(x)dx \right]^2 \\
= & 2\gamma_2 \int \int (4\pi)^{-\frac{d}{2}} t^{-\frac{d}{2}+1} \exp\left\{-\frac{|x_1-x_2|^2}{4t}\right\} \phi(x_1)\phi(x_2)dx_1dx_2 \\
& +4\gamma_1\gamma_2 \int_0^t \int_s^t \int \int p(2u, x_2-x_1)\phi(x_1)\phi(x_2)dx_1dx_2du ds + \left[\int \phi(x)dx \right]^2,
\end{aligned}$$

where

$$\begin{aligned}
(4.2.21) \quad p(u, x_2-x_1) &= p(u, x_1; x_2) \\
&= (2\pi u)^{-\frac{d}{2}} \exp\left\{-\frac{|x_1-x_2|^2}{2u}\right\}.
\end{aligned}$$

Note that the first term of the right hand side above

$$\begin{aligned}
(4.2.22) \quad & 2\gamma_2 \int \int (4\pi)^{-\frac{d}{2}} t^{-\frac{d}{2}+1} \exp\left\{-\frac{|x_1-x_2|^2}{4t}\right\} \phi(x_1)\phi(x_2)dx_1dx_2 \\
& \longrightarrow 0 \text{ as } t \longrightarrow \infty
\end{aligned}$$

if $d > 2$ and ϕ has compact support. In addition, if $d > 4$, then the second term of the right hand side becomes

$$\begin{aligned}
(4.2.23) \quad & 4\gamma_1\gamma_2 \int_0^t \int_s^t \int \int p(2u, x_2-x_1)\phi(x_1)\phi(x_2)dx_1dx_2du ds \\
&= 4\gamma_1\gamma_2 \int \int \left[\int_0^t \int_s^t p(2u, x_2-x_1)du ds \right] \phi(x_1)\phi(x_2)dx_1dx_2 \\
&= 4\gamma_1\gamma_2 \int \int \left[\int_0^t \int_0^u p(2u, x_2-x_1)ds du \right] \phi(x_1)\phi(x_2)dx_1dx_2 \\
&= 4\gamma_1\gamma_2 \int \int \left[\int_0^t up(2u, x_2-x_1)du \right] \phi(x_1)\phi(x_2)dx_1dx_2 \\
&= 4\gamma_1\gamma_2 \int \int \left[(4\pi)^{\frac{d}{2}} 2^{d-4} |x_2-x_1|^{-d+4} \right]
\end{aligned}$$

$$\begin{aligned}
& \int_{\frac{|x_2-x_1|^2}{4t}}^{\infty} v^{(\frac{d}{2}-2)-1} \exp\{-v\} dv] \phi(x_1) \phi(x_2) dx_1 dx_2 \\
& \longrightarrow 4\gamma_1 \gamma_2 \int \int (4\pi)^{\frac{d}{2}} 2^{d-4} |x_2 - x_1|^{-d+4} \\
& \qquad \qquad \qquad \Gamma\left(\frac{d}{2} - 2\right) \phi(x_1) \phi(x_2) dx_1 dx_2 \\
& < \infty \text{ as } t \longrightarrow \infty.
\end{aligned}$$

In conclusion, we can summarize the result above as the following theorem:

Theorem 4.2.2 *If $d > 4$ and $Y(0) = \delta_{\delta_x} dx$, then the second moment measure of $\bar{Y}(t)$ converges weakly in $M_\rho(R^d) \times M_\rho(R^d)$ to the measure with the density function*

$$4\gamma_1 \gamma_2 (4\pi)^{\frac{d}{2}} 2^{d-4} |x_2 - x_1|^{-d+4} \Gamma\left(\frac{d}{2} - 2\right)$$

as $t \longrightarrow \infty$.

4.3 Hausdorff Dimension of Support of $\bar{Y}(t)$

D. Dawson and K. Hochberg (1979) [11] investigated the carrying dimension of the one level stochastic measure diffusion. They showed that there is a random carrying set for the process and bounded the Hausdorff dimension from above. In this section, we study the support of the aggregated process $\bar{Y}(t)$. The main result is that a corresponding lower bound of Hausdorff dimension is given by using the Zähle lemma.

If the initial value of the two level measure branching process $Y(t)$ is the atomic measure on $M(R^d)$ with the mass at λ , where $\lambda(dx) = dx$ denotes Lebesgue measure

on R^d , then by (4.2.10) we have

$$\begin{aligned}
(4.3.1) \quad & \int \int \phi(x_1)\phi(x_2)m_2(t, \lambda; dx_1, dx_2) \\
&= 2\gamma_2 \int_0^t \int_0^s T_{t-s} [\int \int S_s \phi(x_1) S_s \phi(x_2) \mu(dx_1) \mu(dx_2)] ds \delta_\lambda(d\mu) \\
&\quad + [\int \int S_t \phi(x) \mu(dx) \delta_\lambda(d\mu)]^2 \\
&= 2\gamma_2 \int_0^t T_{t-s} [\int \int S_s \phi(x_1) S_s \phi(x_2) \lambda(dx_1) \lambda(dx_2)] ds + [\int S_t \phi(x) \lambda(dx)]^2 \\
&= 2\gamma_2 \int_0^t T_{t-s} \langle S_s \phi, \cdot \rangle^2(\lambda) ds + \langle S_s \phi, \lambda \rangle^2 \\
&= 2\gamma_2 \int_0^t [\langle S_{t-s} S_s \phi, \lambda \rangle^2 \\
&\quad + 2\gamma_1 \langle \int_0^{t-s} S_{t-s-v} (S_v S_s \phi)^2 dv, \lambda \rangle] ds + \langle S_t \phi, \lambda \rangle^2 \\
&= 2\gamma_2 \int_0^t [\langle S_t \phi, \lambda \rangle^2 + 2\gamma_1 \langle \int_0^{t-s} S_{t-s-v} (S_{v+s} \phi)^2 dv, \lambda \rangle] ds \\
&\quad + \langle S_t \phi, \lambda \rangle^2 \\
&= 2\gamma_2 \int_0^t [\int S_t \phi(x) dx]^2 ds + [\int S_t \phi(x) dx]^2 \\
&\quad + 4\gamma_1 \gamma_2 \int_0^t \int \int_0^{t-s} S_{t-s-v} (S_{v+s} \phi)^2(x) dv dx ds \\
&= 2\gamma_2 \int_0^t [\int \phi(x) dx]^2 ds + [\int \phi(x) dx]^2 \\
&\quad + 4\gamma_1 \gamma_2 \int_0^t \int \int_0^{t-s} (S_{v+s} \phi)^2(x) dv dx ds \\
&= 2\gamma_2 t [\int \phi(x) dx]^2 + [\int \phi(x) dx]^2 + 4\gamma_1 \gamma_2 \int_0^t \int \int_s^t (S_w \phi)^2(x) dw dx ds \\
&= (2\gamma_2 t + 1) [\int \phi(x) dx]^2 \\
&\quad + 4\gamma_1 \gamma_2 \int_0^t \int \int_s^t [\int \int p(w, x_1 - x) p(w, x_2 - x) \\
&\quad \phi(x_1) \phi(x_2) dx_1 dx_2] dw dx ds
\end{aligned}$$

$$\begin{aligned}
&= (2\gamma_2 t + 1) \left[\int \phi(x) dx \right]^2 \\
&\quad + 4\gamma_1 \gamma_2 \int_0^t \int_s^t \int \int \left[\int p(w, x_1 - x) p(w, x_2 - x) dx \right] \\
&\quad \quad \quad \phi(x_1) \phi(x_2) dx_1 dx_2 dw ds \\
&= (2\gamma_2 t + 1) \left[\int \phi(x) dx \right]^2 \\
&\quad + 4\gamma_1 \gamma_2 \int_0^t \int_s^t \int \int p(2w, x_1 - x_2) \phi(x_1) \phi(x_2) dx_1 dx_2 dw ds \\
&= (2\gamma_2 t + 1) \left[\int \phi(x) dx \right]^2 \\
&\quad + 4\gamma_1 \gamma_2 \int \int \left[\int_0^t w p(2w, x_1 - x_2) dw \right] \phi(x_1) \phi(x_2) dx_1 dx_2.
\end{aligned}$$

If $g(t, \lambda; x_1, x_2)$ denotes the covariance density of $m_2(t, \lambda; dx_1, dx_2)$, then

$$\begin{aligned}
(4.3.2) \quad & \int \int g(t, \lambda; x_1, x_2) \phi(x_1) \phi(x_2) dx_1 dx_2 \\
&= (2\gamma_2 t + 1) \left[\int \phi(x) dx \right]^2 \\
&\quad + 4\gamma_1 \gamma_2 \int \int \left[\int_0^t w p(2w, x_1 - x_2) dw \right] \phi(x_1) \phi(x_2) dx_1 dx_2,
\end{aligned}$$

where

$$\begin{aligned}
(4.3.3) \quad & \int_0^t w p(2w, x_1 - x_2) dw \\
&= \int_0^t w (2\pi 2w)^{-\frac{d}{2}} \exp\left\{-\frac{|x_2 - x_1|^2}{4w}\right\} dw \\
&= \int_0^t (4\pi)^{-\frac{d}{2}} w^{-\frac{d}{2}+1} \exp\left\{-\frac{|x_2 - x_1|^2}{4w}\right\} dw \\
&= (4\pi)^{-\frac{d}{2}} \int_{\frac{|x_2 - x_1|^2}{4t}}^{\frac{|x_2 - x_1|^2}{4t}} \left(\frac{|x_2 - x_1|^2}{4v}\right)^{-\frac{d}{2}+1} \exp\{-v\} \left(-\frac{|x_2 - x_1|^2}{4v^2}\right) dv \\
&= (4\pi)^{-\frac{d}{2}} \int_{\frac{|x_2 - x_1|^2}{4t}}^{\infty} |x_2 - x_1|^{-d+4} v^{\frac{d}{2}-3} 2^{d-4} \exp\{-v\} dv.
\end{aligned}$$

Hence if $d > 4$, then for small $|x_2 - x_1|$ we have

$$\begin{aligned}
 (4.3.4) \quad & \int_0^t wp(2w, x_2 - x_1) dw \\
 & \approx (4\pi)^{-\frac{d}{2}} |x_2 - x_1|^{-d+4} \Gamma\left(\frac{d}{2} - 2\right) 2^{d-4} \\
 & = \text{const.} \cdot |x_2 - x_1|^{-d+4}
 \end{aligned}$$

and so, in this case we obtain

$$(4.3.5) \quad g(t, \lambda; x_1, x_2) \sim \text{const.} \cdot |x_2 - x_1|^{-d+4}.$$

Now we are going to state the main theorem of this section.

Theorem 4.3.1 *Assume that $d > 4$ and $Y(0) = \delta_\lambda$, then the Hausdorff dimension of support of the aggregated process $\bar{Y}(t)$ is equal to or greater than 4.*

The proof of the Theorem 4.3.1 requires the following Lemma:

Lemma 4.3.1 (Zähle 1984) *Let X be a random measure with second moment measure $k(x, dy)$ and assume that for $EX - a.e. x \in R^d$*

$$(4.3.6) \quad \int_{S(x, \varepsilon)} |x - y|^{-D} k(x, dy) < \infty$$

for some $\varepsilon > 0$, where $S(x, \varepsilon)$ denotes a sphere with center x and radius ε . Then

$$P\{(X(B) > 0) \setminus (\dim(\Xi \cap B) \geq D)\} = 0,$$

where Ξ denotes the closed support of X .

Proof of Theorem 4.3.1. By (4.3.5) and Zähle's Lemma, we have

$$\begin{aligned}
 (4.3.7) \quad & \int_{S(x,\epsilon)} |x-y|^{-D} k(x,y) dy \\
 & \approx \int_{S(x,\epsilon)} |x-y|^{-D} \text{const.} \cdot |x-y|^{-(d-4)} dy \\
 & = \text{const.} \cdot \int_0^\epsilon \frac{1}{r^{d-4+D}} r^{d-1} dr \\
 & = \text{const.} \cdot \int_0^\epsilon \frac{1}{r^{D-3}} dr.
 \end{aligned}$$

Since the latter converges if and only if $D - 3 < 1$, i.e., $D < 4$. We conclude that $\dim \Xi \geq D$ for $D < 4$. This implies that $\dim \Xi \geq 4$.

The proof of the Theorem 4.3.1 is complete.

□

Chapter 5

Long Time Behaviour of Two Level Measure Branching Processes

The question of asymptotic behaviour of measure-valued processes is of particular interest. In the case of one level measure-valued branching processes, D. Dawson (1977) [18] showed that if the spatial part of the measure-valued process is Brownian motion or a symmetric stable process on R^d , and if the initial value of the process is Lebesgue measure, then existence or non-existence of a non trivial limiting distribution is equivalent to transience or recurrence of the spatial motion. Alison M. Etheridge (1990) [25] obtained a similar result in a more general case. In this chapter, we consider the asymptotic behaviour of the two level $M(M(R^d))$ valued branching processes defined in previous chapter. The main results prove Dawson's conjecture about extinction of the two level measure branching process. Specifically, we show in section 1 that if $d \leq 4$, then the $M(M(R^d))$ valued process

suffers local extinction. We obtain the self-similarity property of the process when $d = 4$ in section 2.

5.1 The Extinction of The Two Level $M(M(R^d))$ -valued Branching Processes

In this section we will study the asymptotic behaviour of the two level measure-valued process. Since the process always goes to extinction if the initial measure is finite we need introduce a class of infinite measures. In particular, the measure $\delta_{\delta_x} dx \in M(M(R^d))$ will be used in this section. We know that δ_{δ_x} is an atomic measure on $M(R^d)$ with the mass at the atomic measure δ_x . So the measure $\delta_{\delta_x} dx$ can be considered as δ_{δ_x} in which the location x of the atomic measure δ_{δ_x} is given by d -dimensional Lebesgue measure in R^d .

If we let $\nu_0 = \delta_{\delta_x} dx$, then it is easy to see that

$$\begin{aligned}
 (5.1.1) \quad & \int \int \phi_\rho(y) \mu(dy) \nu_0(d\mu) \\
 &= \int \int \phi_\rho(y) \delta_x(dy) dx = \int \phi_\rho(x) dx \\
 &< \infty
 \end{aligned}$$

implies that $\delta_{\delta_x} dx \in M_\rho^2(R^d)$.

As stated in the introduction, the one level measure branching process in R^d with spatial Brownian motion and with initial value given by d -dimensional Lebesgue measure converges to a equilibrium invariant probability measure in dimension $d \geq$

3, and suffers local extinction in dimension $d \leq 2$. The main result of this section is Theorem 5.1.1 in which we show that if $d \leq 4$ then the two level $M(M(R^d))$ valued process goes to extinction. Our proof in dimension $d = 3, 4$ in fact uses the invariant probability measure of the one level process. If we let R_∞ denote the canonical equilibrium measure of the one level $M_\rho(R^d)$ -valued branching process, then by

$$\begin{aligned}
 (5.1.2) \quad & \int \int \phi_\rho(x) \mu(dx) R_\infty(d\mu) \\
 &= \lim_{t \rightarrow \infty} \int \int \phi_\rho(x) \mu(dx) T_t^* \nu_0(d\mu) = \lim_{t \rightarrow \infty} \int T_t \langle \phi_\rho, \cdot \rangle (\delta_x) dx \\
 &= \lim_{t \rightarrow \infty} \int \langle S_t \phi_\rho, \delta_x \rangle dx = \lim_{t \rightarrow \infty} \int S_t \phi_\rho(x) dx \\
 &= \int \phi_\rho(x) dx < \infty,
 \end{aligned}$$

we know that $R_\infty \in M_\rho^2(R^d)$, where T_t^* and T_t will be defined later.

The method of proof for local extinction we use here was suggested by Dawson which is different from any approach used before in the one level case. The motivation result of this section is to prove the conjecture of Dawson that the two level $M(M(R^d))$ -valued process suffers local extinction if and only if $d \leq 4$.

Let $Y(t)$ denote the two level measure branching process with the Laplace functional given by

$$\begin{aligned}
 (5.1.3) \quad L_{t,\nu}(H) &= E[\exp\{-\langle H, Y(t) \rangle\} | Y(0) = \nu] \\
 &= \exp\{-\langle u(t), \nu \rangle\},
 \end{aligned}$$

where $u(t, \mu)$ satisfies the following nonlinear differential equations:

$$(5.1.4) \quad \begin{aligned} \frac{\partial u(t, \mu)}{\partial t} &= \mathcal{L}u(t, \mu) - \gamma_2 u^2(t, \mu). \\ u(0, \mu) &= H(\mu). \end{aligned}$$

$u(t, \mu)$ is also a solution of the following integral equation:

$$(5.1.5) \quad u(t, \mu) = T_t u(0, \mu) - \gamma_2 \int_0^t T_{t-s} [u(s)]^2(\mu) ds,$$

where T_t denotes the semigroup of operators of the one level $M_\rho(R^d)$ -valued branching process with the generator $\mathcal{L}$.

Definition 5.1.1 *We shall say that the two level $M_\rho^2(R^d)$ -valued process $Y(t)$ suffers local extinction if for every compact set B in R^d , $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$,*

$$(5.1.6) \quad \lim_{t \rightarrow \infty} P(\langle \langle 1_{K_{B, \varepsilon_2}}, Y(t) \rangle \rangle > \varepsilon_1) = 0$$

where K_{B, ε_2} is given by

$$K_{B, \varepsilon_2} := \{\mu \in M_\rho(R^d) : \mu(B) > \varepsilon_2\}$$

and $1_{K_{B, \varepsilon_2}}$ is the indicator function of the set of K_{B, ε_2} .

Before statement of our main theorem, we are going to give a lower bound on the second moment measure.

Let $u(t)$ denote the solution of the equation (5.1.4) with initial values $u(0, \mu) = \langle \phi, \mu \rangle$ and put $\nu_0 = \delta_{\delta_x} dx$ as before. then

$$(5.1.7) \quad \int [1 - \exp\{-\langle \phi, \cdot \rangle\}](\mu) T_t^* \nu_0(d\mu)$$

$$\begin{aligned}
 &\equiv \int T_t[1 - \exp\{-\langle \phi, \cdot \rangle\}](\mu) \nu_0(d\mu) \\
 &= \int T_t[1 - \exp\{-\langle \phi, \cdot \rangle\}](\delta_x) dx \\
 &= \int [1 - \exp\{-\langle v(t), \delta_x \rangle\}] dx \\
 &= \int [1 - \exp\{-v(t, x)\}] dx,
 \end{aligned}$$

where $v(t, x)$ is the unique solution of the following nonlinear differential equation:

$$\begin{aligned}
 \frac{\partial v(t, x)}{\partial t} &= \Delta v(t, x) - v^2(t, x), \\
 v(0, x) &= \phi(x).
 \end{aligned}$$

Since

$$\lim_{t \rightarrow \infty} \sup_x v(t, x) = 0,$$

$$1 - \exp\{-v(t, x)\} \approx v(t, x) \text{ as } t \rightarrow \infty,$$

we have

$$\begin{aligned}
 (5.1.8) \quad &\int [1 - \exp\{-\langle \phi, \cdot \rangle\}](\mu) T_t^* \nu_0(d\mu) \\
 &\approx \int v(t, x) dx \\
 &\rightarrow \int [1 - \exp\{-\langle \phi, \mu \rangle\}] R_\infty(d\mu).
 \end{aligned}$$

This implies that

$$(5.1.9) \quad T_t^* \nu_0 \Rightarrow R_\infty.$$

Proposition 5.1.1 *If $d \leq 4$, then*

$$(5.1.10) \quad \gamma \int_0^t \int [T_s \langle \phi, \cdot \rangle(\mu)]^2 R_\infty(d\mu) ds \rightarrow \infty$$

as $t \rightarrow \infty$.

Proof. In Chapter 4, we have shown that

$$(5.1.11) \quad T_t \langle \phi, \cdot \rangle (\mu) = \langle S_t \phi, \mu \rangle$$

$$(5.1.12) \quad T_t \langle \phi, \mu \rangle^2 = \langle S_t \phi, \mu \rangle^2 + 2\gamma_1 \int \int_0^t S_{t-s} [S_s \phi]^2 ds d\mu,$$

where $\{S_t\}$ is the semigroup of operators of Brownian motion on R^d .

Therefore

$$\begin{aligned}
 (5.1.13) \quad & \gamma_2 \int_0^t \int [T_s \langle \phi, \cdot \rangle (\mu)]^2 R_\infty(d\mu) ds \\
 &= \gamma_2 \int_0^t \int \langle S_s \phi, \mu \rangle^2 R_\infty(d\mu) ds \\
 &= \lim_{t_1 \rightarrow \infty} \gamma_2 \int_0^t \int \langle S_s \phi, \mu \rangle^2 T_{t_1-s}^* \nu_0(d\mu) ds \\
 &= \lim_{t_1 \rightarrow \infty} \gamma_2 \int_0^t \int T_{t_1-s} \langle S_s \phi, \cdot \rangle^2 \nu_0(d\mu) ds \\
 &= \lim_{t_1 \rightarrow \infty} \gamma_2 \int_0^t \int T_{t_1-s} \langle S_s \phi, \cdot \rangle^2 \delta_{\delta_x}(dx) ds \\
 &= \lim_{t_1 \rightarrow \infty} \gamma_2 \int_0^t \int T_{t_1-s} \langle S_s \phi, \cdot \rangle^2 (\delta_x) dx ds \\
 &= \lim_{t_1 \rightarrow \infty} \gamma_2 \int \int_0^t [\langle S_{t_1} \phi, \delta_x \rangle^2 \\
 &\quad + 2\gamma_1 \int_0^{t_1-s} S_{t_1-s-u} [S_{u+s} \phi]^2(x) du] ds dx \\
 &= \lim_{t_1 \rightarrow \infty} \gamma_2 \int \int_0^t (S_{t_1} \phi(x))^2 ds dx \\
 &\quad + \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int \int_0^t \int_0^{t_1-s} [S_{u+s} \phi]^2(x) du ds dx \\
 &= \lim_{t_1 \rightarrow \infty} \gamma_2 t \int (S_{t_1} \phi(x))^2 dx \\
 &\quad + \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int \int_0^t \int_s^{t_1} [S_w \phi]^2(x) dw ds dx
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int \int_0^t \int_s^{t_1} [S_w \phi]^2(x) dw ds dx \\
 &= \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int \int_0^t \int_0^w [S_w \phi]^2(x) ds dw dx \\
 &\quad + \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int \int_t^{t_1} \int_0^t [S_w \phi]^2(x) ds dw dx \\
 &\geq \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int \int_0^t w [S_w \phi]^2(x) dw dx \\
 &= \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int \int_0^t w \int \int p(w, x; y_1) p(w, x; y_2) \\
 &\quad \phi(y_1) \phi(y_2) dy_1 dy_2 dw dx.
 \end{aligned}$$

By the relation

$$(5.1.14) \quad |y_1 - x|^2 + |y_2 - x|^2 = \frac{1}{2}|y_1 - y_2|^2 + 2|x - \frac{1}{2}(y_1 + y_2)|^2$$

we have

$$(5.1.15) \quad p(w, x; y_1) p(w, x; y_2) = p(2w, y_1; y_2) p\left(\frac{1}{2}w, x; \frac{y_1 + y_2}{2}\right).$$

Thus we obtain the relation

$$\begin{aligned}
 (5.1.16) \quad &\gamma_2 \int_0^t \int < S_s \phi, \mu >^2 R_\infty(d\mu) ds \\
 &\geq \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int \int_0^t w \int \int p(2w, y_1; y_2) p\left(\frac{1}{2}w, x; \frac{y_1 + y_2}{2}\right) \\
 &\quad \phi(y_1) \phi(y_2) dy_1 dy_2 dw dx \\
 &= \lim_{t_1 \rightarrow \infty} 2\gamma_1 \gamma_2 \int_0^t w \int \int p(2w, y_1; y_2) \phi(y_1) \phi(y_2) dy_1 dy_2 dw \\
 &= 2\gamma_1 \gamma_2 \int_0^t w \int \int \frac{1}{(2\pi 2w)^{\frac{d}{2}}} \exp\left\{-\frac{|y_1 - y_2|^2}{4w}\right\} \phi(y_1) \phi(y_2) dy_1 dy_2 dw \\
 &= 2\gamma_1 \gamma_2 \int \int \int_{\infty}^{\frac{|y_1 - y_2|^2}{4t_1}} (4\pi)^{-\frac{d}{2}} \left(\frac{4v}{|y_1 - y_2|^2}\right)^{\frac{d}{2}-1} \exp\{-v\}
 \end{aligned}$$

$$\begin{aligned}
 & \left(-\frac{|y_1 - y_2|^2}{4v^2}\right) \phi(y_1) \phi(y_2) dv dy_1 dy_2 \\
 = & 2\gamma_1 \gamma_2 \int \int \frac{1}{16|y_1 - y_2|^{d-4} \pi^{\frac{d}{2}}} \\
 & \left[\int_{\frac{|y_1 - y_2|^2}{4t_1}}^{\infty} v^{\frac{d}{2}-3} \exp\{-v\} dv \right] \phi(y_1) \phi(y_2) dy_1 dy_2 \\
 = & \frac{\gamma_1 \gamma_2}{8\pi^{\frac{d}{2}}} \int \int |y_1 - y_2|^{4-d} \left[\int_{\frac{|y_1 - y_2|^2}{4t_1}}^{\infty} \exp\{-v\} v^{\frac{d}{2}-3} dv \right] \phi(y_1) \phi(y_2) dy_1 dy_2 \\
 \longrightarrow & \frac{\gamma_1 \gamma_2}{8\pi^{\frac{d}{2}}} \int \int |y_1 - y_2|^{4-d} \int_0^{\infty} v^{\frac{d}{2}-3} \exp\{-v\} \phi(y_1) \phi(y_2) dy_1 dy_2
 \end{aligned}$$

as $t \rightarrow \infty$. It is easy to see that the right hand side of (5.1.16) is infinity if $d \leq 4$.

This completes the proof of the Proposition 5.1.1.

□

Our main theorem requires the following lemma which was given by Dawson and Perkins (1991).

Lemma 5.1.1 *Let $\mathfrak{R}_d$ denote the Borel σ -algebra in R^d . Then for each $B \in \mathfrak{R}_d$ and $d \geq 3$, we have*

$$\begin{aligned}
 (5.1.17) \quad & R_{\infty}(\{\mu : \frac{1}{t} \mu(\sqrt{t}B) > a\}) \\
 & = t^{\frac{d-2}{2}} R_{\infty}(\{\mu : \mu(B) > a\}),
 \end{aligned}$$

where R_{∞} denotes the canonical equilibrium measure of one level measure branching process.

Proof. See D. Dawson and E. Perkins (1991) [15].

□

Now let us turn to our main result about the asymptotic behaviour.

Theorem 5.1.1 *If $d \leq 4$ and $Y_0 = \delta_{x_0} dx$, then the two level process $Y(t)$ suffers local extinction.*

Proof. Since the one level $M_\rho(R^d)$ -valued process with the generator $\mathcal{L}$ suffers local extinction for $d \leq 2$, it is clear that the theorem is true in this case. So we only need to show the theorem for $d = 3$ and $d = 4$. In the remainder of this section, we always assume that $d \geq 3$. Noting that for each $t > 0$, and $\mu \in M(R^d)$,

$$(5.1.18) \quad \begin{aligned} u(t, \mu) &= T_t u(0, \mu) - \gamma_2 \int_0^t T_{t-s} [u(s)]^2(\mu) ds \\ &\geq 0, \end{aligned}$$

we have

$$(5.1.19) \quad \gamma_2 \int_0^t T_{t-s} [u(s)]^2(\mu) ds \leq T_t u(0)(\mu)$$

and so

$$(5.1.20) \quad \begin{aligned} &\gamma_2 \int \int_0^t T_{t-s} [u(s)]^2(\mu) ds R_\infty(d\mu) \\ &\leq \int T_t u(0)(\mu) R_\infty(d\mu) = \int u(0, \mu) R_\infty(d\mu) \\ &= \int \langle \phi, \mu \rangle R_\infty(d\mu) < \infty. \end{aligned}$$

Therefore we obtain that

$$(5.1.21) \quad \lim_{t \rightarrow \infty} \int \int_0^t u^2(s, \mu) ds R_\infty(d\mu)$$

$$\begin{aligned}
 &= \lim_{t \rightarrow \infty} \int_0^t \int_0^s T_{t-s}[u(s)]^2(\mu) ds R_\infty(d\mu) \\
 &< \infty.
 \end{aligned}$$

Next we proceed to show that for $s \geq 1$

$$(5.1.22) \quad \int_{\{\mu: \langle S_s \phi, \mu \rangle \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}}\}} \langle S_s \phi, \mu \rangle R_\infty(d\mu) \longrightarrow 0$$

as $\varepsilon \longrightarrow 0$. Observe that for $\phi \in C_c^+(R^d)$, $\phi \neq 0$,

$$\begin{aligned}
 (5.1.23) \quad S_s \phi(x) &= \int (2\pi s)^{-\frac{d}{2}} \exp\left\{-\frac{|y-x|^2}{2s}\right\} \phi(y) dy \\
 &= \int (2\pi s)^{-\frac{d}{2}} \exp\left\{-\frac{x^2}{2s}\right\} \exp\left\{-\frac{y^2 - 2xy}{2s}\right\} \phi(y) dy,
 \end{aligned}$$

therefore there exist two positive constants $0 < C_1 \leq C_2 < \infty$, such that

$$C_1 s^{-\frac{d}{2}} \exp\left\{-\frac{x^2}{2s}\right\} \leq S_s \phi(x) \leq C_2 s^{-\frac{d}{2}} \exp\left\{-\frac{x^2}{2s}\right\}$$

uniformly in x for fixed non-trivial ϕ .

Now we define a new measure $\mu_s \in M_\rho(R^d)$ in terms of $\mu \in M_\rho(R^d)$ by

$$(5.1.24) \quad \mu_s(B) = \frac{1}{s} \mu\left(\left\{x : \frac{x}{\sqrt{s}} \in B\right\}\right),$$

where $B \in \mathfrak{R}_d$.

Since (5.1.23) implies that

$$C_1 s^{-\frac{d}{2}} \langle \exp\left\{-\frac{x^2}{2s}\right\}, \mu \rangle \leq \langle S_s \phi, \mu \rangle = C_2 s^{-\frac{d}{2}} \langle \exp\left\{-\frac{x^2}{2s}\right\}, \mu \rangle$$

and

$$C_1 s^{-\frac{d-2}{2}} \langle \exp\left\{-\frac{x^2}{2}\right\}, \mu_s \rangle \leq \langle S_s \phi, \mu \rangle \leq C_2 s^{-\frac{d-2}{2}} \langle \exp\left\{-\frac{x^2}{2}\right\}, \mu_s \rangle,$$

thus together with the lemma 5.1.1, we have

$$\begin{aligned}
 (5.1.25) \quad & R_\infty(\{\mu : \langle S_s \phi, \mu \rangle > a\}) \\
 & \leq R_\infty(\{\mu : C_2 s^{-\frac{d-2}{2}} < \exp\{-\frac{x^2}{2}\}, \mu_s > a\}) \\
 & = s^{\frac{d-2}{2}} R_\infty(\{\mu : C_2 s^{-\frac{d-2}{2}} < \exp\{-\frac{x^2}{2}\}, \mu > a\}) \\
 & = s^{\frac{d-2}{2}} R_\infty(\{\mu : \langle \exp\{-\frac{x^2}{2}\}, \mu \rangle > \frac{1}{C_2} s^{\frac{d-2}{2}} a\}).
 \end{aligned}$$

For $a \geq 0$, we define

$$\nu((a, \infty)) = R_\infty(\{\mu : C_2 s^{-\frac{d-2}{2}} < \exp\{-\frac{x^2}{2}\}, \mu_s > a\})$$

and

$$(5.1.26) \quad \tilde{\nu}((\frac{1}{C_2} s^{\frac{d-2}{2}} a, \infty)) = R_\infty(\{\mu : \langle \exp\{-\frac{x^2}{2}\}, \mu \rangle > \frac{1}{C_2} s^{\frac{d-2}{2}} a\}).$$

Then it is easy to see that

$$(5.1.27) \quad d\nu(a) = s^{\frac{d-2}{2}} d\tilde{\nu}(\frac{1}{C_2} s^{\frac{d-2}{2}} a)$$

and

$$\begin{aligned}
 (5.1.28) \quad & \int_{\{\mu : \langle S_s \phi, \mu \rangle \leq \frac{\epsilon}{s^{\frac{d-2}{2}}}\}} \langle S_s \phi, \mu \rangle R_\infty(d\mu) \\
 & \leq \int_{\{\mu : C_1 s^{-\frac{d-2}{2}} < \exp\{-\frac{x^2}{2}\}, \mu_s \leq \frac{\epsilon}{s^{\frac{d-2}{2}}}\}} C_2 s^{-\frac{d-2}{2}} < \exp\{-\frac{x^2}{2}\}, \mu_s > R_\infty(d\mu) \\
 & = \int_{\{\mu : C_2 s^{-\frac{d-2}{2}} < \exp\{-\frac{x^2}{2}\}, \mu_s \leq \frac{C_2 \epsilon}{C_1 s^{\frac{d-2}{2}}}\}} C_2 s^{-\frac{d-2}{2}} < \exp\{-\frac{x^2}{2}\}, \mu_s > R_\infty(d\mu) \\
 & = \int_0^{\frac{C_2 \epsilon}{C_1 s^{\frac{d-2}{2}}}} a d\nu(a) \\
 & = \int_0^{\frac{C_2 \epsilon}{C_1 s^{\frac{d-2}{2}}}} a s^{\frac{d-2}{2}} d\tilde{\nu}(\frac{1}{C_2} s^{\frac{d-2}{2}} a).
 \end{aligned}$$

If let $b = \frac{1}{C_2} s^{\frac{d-2}{2}} a$, then

$$(5.1.29) \quad \int_{\{\mu: \langle S_s \phi, \mu \rangle \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}}\}} \langle S_s \phi, \mu \rangle R_\infty(d\mu) \leq C_2 \int_0^{\frac{\varepsilon}{C_1}} b d\tilde{\nu}(b).$$

Since

$$\begin{aligned} (5.1.30) \quad & \int \langle S_s \phi, \mu \rangle R_\infty(d\mu) \\ &= \int T_s \langle \phi, \cdot \rangle (\mu) R_\infty(d\mu) \\ &= \int \langle \phi, \mu \rangle R_\infty(d\mu) \equiv C_0 < \infty \end{aligned}$$

and for $\frac{\varepsilon}{C_1} < 1$, we have

$$\begin{aligned} (5.1.31) \quad & \int_0^{\frac{\varepsilon}{C_1}} b d\tilde{\nu}(b) \leq \int_0^1 b d\tilde{\nu}(b) \\ &= \int_{\{\mu: \langle \exp\{-\frac{x^2}{2}\}, \mu \rangle \leq 1\}} \langle \exp\{-\frac{x^2}{2}\}, \mu \rangle R_\infty(d\mu) \\ &= \frac{1}{C_1} \int_{\{\mu: C_2 \langle \exp\{-\frac{x^2}{2}\}, \mu \rangle \leq C_2\}} C_1 \langle \exp\{-\frac{x^2}{2}\}, \mu \rangle R_\infty(d\mu) \\ &\leq \frac{1}{C_1} \int_{\{\mu: \langle S_1 \phi, \mu \rangle \leq C_2\}} \langle S_1 \phi, \mu \rangle R_\infty(d\mu) \\ &\leq \frac{1}{C_1} \int \langle S_1 \phi, \mu \rangle R_\infty(d\mu) \\ &< \infty. \end{aligned}$$

Hence

$$\begin{aligned} (5.1.32) \quad & \int_{\{\mu: \langle S_s \phi, \mu \rangle \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}}\}} \langle S_s \phi, \mu \rangle R_\infty(d\mu) \\ &\leq C_2 \int_0^{\frac{\varepsilon}{C_1}} b d\tilde{\nu}(b) \\ &\longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0. \end{aligned}$$

Next we show that

$$(5.1.33) \quad \lim_{t \rightarrow \infty} \int u(t, \mu) R_{\infty}(d\mu) = 0$$

by contradiction. Suppose that

$$(5.1.34) \quad \lim_{t \rightarrow \infty} \int u(t, \mu) R_{\infty}(d\mu) = l > 0.$$

By (5.1.32), for any fixed $s \geq 1$, there exists $\varepsilon_0 > 0$ such that

$$(5.1.35) \quad \int_{\{\mu: \langle S_s \phi, \mu \rangle \leq \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} \langle S_s \phi, \mu \rangle R_{\infty}(d\mu) < \frac{l}{2}.$$

From (5.1.18) we know that for each $s > 0$

$$(5.1.36) \quad u(s, \mu) \leq \langle S_s \phi, \mu \rangle,$$

so we have

$$(5.1.37) \quad \int_{\{\mu: \langle S_s \phi, \mu \rangle \leq \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_{\infty}(d\mu) < \frac{l}{2}.$$

Since

$$\begin{aligned} (5.1.38) \quad & \int \frac{\partial u(s, \mu)}{\partial s} R_{\infty}(d\mu) \\ &= \int \mathcal{L}u(s, \mu) R_{\infty}(d\mu) - \gamma_2 \int u^2(s, \mu) R_{\infty}(d\mu) \\ &= -\gamma_2 \int u^2(s, \mu) R_{\infty}(d\mu) < 0 \end{aligned}$$

we obtain that $\int u(s, \mu) R_{\infty}(d\mu)$ is monotone decreasing. Thus from the assumption in (5.1.34), for any $s > 0$ we have

$$(5.1.39) \quad \int u(s, \mu) R_{\infty}(d\mu) \geq \lim_{t \rightarrow \infty} \int u(t, \mu) R_{\infty}(d\mu) = l.$$

Therefore

$$(5.1.40) \quad \int_{\{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_\infty(d\mu) > \frac{l}{2}$$

and

$$(5.1.41) \quad \begin{aligned} & \int_{\{\mu: u(s, \mu) \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}}\} \cap \{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_\infty(d\mu) \\ & \leq \int_{\{\mu: u(s, \mu) \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}}\} \cap \{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} \frac{\varepsilon}{s^{\frac{d-2}{2}}} R_\infty(d\mu) \\ & \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}} \int_{\{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} R_\infty(d\mu) \\ & \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}} \int_{\{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} \frac{\langle S_s \phi, \mu \rangle}{\frac{\varepsilon_0}{s^{\frac{d-2}{2}}}} R_\infty(d\mu) \\ & = \frac{\varepsilon}{\varepsilon_0} \int_{\{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} \langle S_s \phi, \mu \rangle R_\infty(d\mu) \\ & \leq \frac{\varepsilon}{\varepsilon_0} C_0 < \infty. \end{aligned}$$

We choose ε small enough, such that

$$(5.1.42) \quad \frac{\varepsilon}{\varepsilon_0} C_0 < \frac{l}{4};$$

then

$$(5.1.43) \quad \int_{\{\mu: u(s, \mu) \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}}\} \cap \{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_\infty(d\mu) < \frac{l}{4}$$

and so

$$(5.1.44) \quad \begin{aligned} & \int_{\{\mu: u(s, \mu) > \frac{\varepsilon}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_\infty(d\mu) \\ & \geq \int_{\{\mu: u(s, \mu) > \frac{\varepsilon}{s^{\frac{d-2}{2}}}\} \cap \{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_\infty(d\mu) \end{aligned}$$

$$\begin{aligned}
 &= \int_{\{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_\infty(d\mu) \\
 &\quad - \int_{\{\mu: u(s, \mu) \leq \frac{\varepsilon}{s^{\frac{d-2}{2}}\}} \cap \{\mu: \langle S_s \phi, \mu \rangle > \frac{\varepsilon_0}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_\infty(d\mu) \\
 &\geq \frac{l}{2} - \frac{l}{4} = \frac{l}{4}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 (5.1.45) \quad &\int u^2(s, \mu) R_\infty(d\mu) \\
 &\geq \int_{\{\mu: u(s, \mu) > \frac{\varepsilon}{s^{\frac{d-2}{2}}}\}} u^2(s, \mu) R_\infty(d\mu) \\
 &\geq \frac{\varepsilon}{s^{\frac{d-2}{2}}} \int_{\{\mu: u(s, \mu) > \frac{\varepsilon}{s^{\frac{d-2}{2}}}\}} u(s, \mu) R_\infty(d\mu) \\
 &\geq \frac{\varepsilon}{s^{\frac{d-2}{2}}} \frac{l}{4}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 (5.1.46) \quad &\lim_{t \rightarrow \infty} \int_0^t \int u^2(s, \mu) R_\infty(d\mu)(ds) \\
 &\geq \lim_{t \rightarrow \infty} \int_1^t \frac{\varepsilon}{s^{\frac{d-2}{2}}} \frac{l}{4} ds \\
 &= \begin{cases} \frac{\varepsilon l}{4} \lim_{t \rightarrow \infty} \ln s|_1^t & \text{if } d = 4 \\ \frac{\varepsilon l}{4} \lim_{t \rightarrow \infty} \frac{1}{-\frac{d-2}{2}+1} s^{-\frac{d-2}{2}+1}|_1^t & \text{if } d \neq 4 \end{cases} \\
 &= \begin{cases} \infty & \text{if } d = 4 \\ \frac{\varepsilon l}{4} \frac{2}{4-d} \lim_{t \rightarrow \infty} [t^{\frac{4-d}{2}} - 1] & \text{if } d \neq 4 \end{cases}
 \end{aligned}$$

$$= \begin{cases} \infty & \text{if } d = 4 \\ \infty & \text{if } d = 3 \\ \text{finite} & \text{if } d > 4. \end{cases}$$

This is contradiction to the (5.1.34) if $d = 3$ and $d = 4$. Hence we conclude that for $d = 3$ and $d = 4$

$$(5.1.47) \quad \lim_{t \rightarrow \infty} \int u(t, \mu) R_{\infty}(d\mu) = 0.$$

On the other hand, for each $t > 0$

$$\begin{aligned} (5.1.48) \quad & \int u(t_1 + t, \mu) \nu_0(d\mu) \\ &= \int T_{t_1+t} u(0, \mu) \nu_0(d\mu) - \gamma_2 \int \int_0^{t+t_1} T_{t_1+t-s} [u(s)]^2(\mu) ds \nu_0(d\mu) \\ &= \int T_{t_1} T_t u(0, \mu) \nu_0(d\mu) - \gamma_2 \int \int_0^t T_{t_1} T_{t-s} [u(s)]^2(\mu) ds \nu_0(d\mu) \\ &\quad - \gamma_2 \int \int_t^{t+t_1} T_{t_1+t-s} [u(s)]^2(\mu) ds \nu_0(d\mu) \\ &\leq \int T_t u(0, \mu) T_{t_1}^* \nu_0(d\mu) - \gamma_2 \int \int_0^t T_{t-s} [u(s)]^2(\mu) ds T_{t_1}^* \nu_0(d\mu). \end{aligned}$$

This implies that for each $t > 0$,

$$\begin{aligned} (5.1.49) \quad & \limsup_{t_1 \rightarrow \infty} \int u(t_1, \mu) \nu_0(d\mu) \\ &= \limsup_{t_1 \rightarrow \infty} \int u(t_1 + t, \mu) \nu_0(d\mu) \\ &\leq \lim_{t_1 \rightarrow \infty} \left[\int T_t u(0, \mu) T_{t_1}^* \nu_0(d\mu) \right. \\ &\quad \left. - \gamma_2 \int \int_0^t T_{t-s} [u(s)]^2(\mu) ds T_{t_1}^* \nu_0(d\mu) \right] \\ &= \int T_t u(0, \mu) R_{\infty}(d\mu) - \gamma_2 \int \int_0^t T_{t-s} [u(s)]^2(\mu) ds R_{\infty}(d\mu) \end{aligned}$$

$$= \int u(t, \mu) R_{\infty}(d\mu).$$

Thus together with (5.1.47), we obtain

$$(5.1.50) \quad \limsup_{t_1 \rightarrow \infty} \int u(t_1, \mu) \nu_0(d\mu) = 0.$$

Hence the theorem 5.1.1 is proved.

□

5.2 The Rescaled Process and the Self-Similarity Property

In this section, we are going to consider the rescaled processes for the process $Y(t)$ and study the self-similarity property. To do so, we introduce the following group of scaling transformations: Let

$$(5.2.1) \quad Y_k(t, A) = Y(k^2 t, k^{-\frac{d}{2}} \{ \mu : \exists \tilde{\mu} \in A, s.t. \mu(x : \frac{x}{k} \in \cdot) = \tilde{\mu}(\cdot) \}) k^{-\frac{d}{2}}$$

for each $k \geq 1$, where $A \in \mathcal{B}(M_p(R^d))$ and $\mathcal{B}(M_p(R^d))$ denotes the family of the all Borel subsets of $M_p(R^d)$. Before coming to the main theorem of the section, we are going to consider the generators $G_k^{(2)}$ of $Y_k(t)$.

If

$$F(\nu) = f(\langle \langle h_1(\langle h_2, \cdot \rangle), \nu \rangle \rangle),$$

$$\tilde{f}(a) = f(k^{-\frac{d}{2}} a), \quad \tilde{h}_1(b) = h_1(k^{-\frac{d}{2}} b),$$

$$\tilde{h}_2(x) = h_2(\frac{x}{k}),$$

then

$$\begin{aligned}
 (5.2.2) \quad \frac{d}{dt} E F(Y_k(t)) &= \frac{d}{dt} E[f(\langle\langle h_1(\langle h_2, \cdot \rangle), Y_k(t) \rangle\rangle)] \\
 &= \frac{d}{dt} E[f(k^{-\frac{d}{2}} \int h_1(k^{-\frac{d}{2}} \int h_2(\frac{x}{k}) \mu(dx)) Y(k^2 t, d\mu))] \\
 &\approx \frac{d}{dt} E[\tilde{f}(\int \tilde{h}_1(\int \tilde{h}_2(x) \mu(dx)) Y(k^2 t, d\mu))] \\
 &= \frac{d}{dk^2 t} E[\tilde{f}(\langle\langle \tilde{h}_1(\langle \tilde{h}_2, \cdot \rangle), Y(k^2 t) \rangle\rangle)] \frac{d(k^2 t)}{dt} \\
 &= \{ \int [\tilde{f}'(\langle\langle \tilde{h}_1(\langle \tilde{h}_2, \cdot \rangle), Y(k^2 t) \rangle\rangle) \tilde{h}_1'(\langle \tilde{h}_2, \mu \rangle) < \Delta \tilde{h}_2, \mu > \\
 &\quad + \gamma_1 \tilde{f}'(\langle\langle \tilde{h}_1(\langle \tilde{h}_2, \cdot \rangle), Y(k^2 t) \rangle\rangle) \\
 &\quad \tilde{h}_1''(\langle \tilde{h}_2, \mu \rangle) < \tilde{h}_2^2, \mu >] Y(k^2 t, d\mu) \\
 &\quad + \gamma_2 \int \int \tilde{f}''(\langle\langle \tilde{h}_1(\langle \tilde{h}_2, \cdot \rangle), Y(k^2 t) \rangle\rangle) \tilde{h}_1(\langle \tilde{h}_2, \mu_1 \rangle) \\
 &\quad \tilde{h}_1(\langle \tilde{h}_2, \mu_2 \rangle) \delta_{\mu_1}(d\mu_2) Y(k^2 t, d\mu_1)] k^2.
 \end{aligned}$$

Since

$$\begin{aligned}
 (5.2.3) \quad \tilde{f}'(a) &= f'(k^{-\frac{d}{2}} a) k^{-\frac{d}{2}} \\
 \tilde{f}''(a) &= f''(k^{-\frac{d}{2}} a) k^{-d} \\
 \tilde{h}_1'(b) &= h_1'(k^{-\frac{d}{2}} b) k^{-\frac{d}{2}} \\
 \tilde{h}_2''(b) &= h_2''(\frac{x}{k}) k^{-2} \\
 \Delta \tilde{h}_2(x) &= \frac{\partial^2}{\partial x^2} \tilde{h}_2(x) = \frac{\partial^2}{\partial x^2} h_2(\frac{x}{k}) \\
 &= \frac{1}{k^2} h_2''(\frac{x}{k}) = \frac{1}{k^2} \Delta h_2(\frac{x}{k}),
 \end{aligned}$$

we have

$$\begin{aligned}
 (5.2.4) \quad G_k^{(2)} F(\nu) &= k^2 \tilde{f}'(\langle \langle \tilde{h}_1(\langle \tilde{h}_2, \cdot \rangle), \nu \rangle \rangle) \\
 &\quad [\int \tilde{h}_1'(\langle \tilde{h}_2, \mu \rangle) \langle \Delta \tilde{h}_2, \mu \rangle \nu(d\mu) \\
 &\quad + \gamma_1 \int \tilde{h}_1''(\langle \tilde{h}_2, \mu \rangle) \langle \tilde{h}_2^2, \mu \rangle \nu(d\mu)] \\
 &\quad + \gamma_2 k^2 \tilde{f}''(\langle \langle \tilde{h}_1(\langle \tilde{h}_2, \cdot \rangle), \nu \rangle \rangle) \int [\tilde{h}_1(\langle \tilde{h}_2, \mu \rangle)]^2 \nu(d\mu) \\
 &= k^2 k^{-\frac{d}{2}} f'(k^{-\frac{d}{2}} \langle \langle h_1(k^{-\frac{d}{2}} \langle h_2(\frac{\cdot}{k}), \cdot \rangle), \nu \rangle \rangle) \\
 &\quad [\int k^{-\frac{d}{2}} h_1'(k^{-\frac{d}{2}} \langle h_2(\frac{\cdot}{k}), \mu \rangle) \langle \frac{1}{k^2} \Delta h_2(\frac{\cdot}{k}), \mu \rangle \nu(d\mu) \\
 &\quad + \gamma_1 \int k^{-d} h_1''(k^{-\frac{d}{2}} \langle h_2(\frac{\cdot}{k}), \mu \rangle) \langle h_2^2(\frac{\cdot}{k}), \mu \rangle \nu(d\mu)] \\
 &\quad + \gamma_2 k^{2-d} f''(k^{-\frac{d}{2}} \langle \langle h_1(k^{-\frac{d}{2}} \langle h_2(\frac{\cdot}{k}), \cdot \rangle), \nu \rangle \rangle) \\
 &\quad \int [h_1(k^{-\frac{d}{2}} \langle h_2(\frac{\cdot}{k}), \mu \rangle)]^2 \nu(d\mu).
 \end{aligned}$$

Let

$$\begin{aligned}
 &\int k^{-\frac{d}{2}} h_1(\langle \cdot \rangle^{-\frac{d}{2}} \int h_2(\frac{x}{k}) \mu(dx)) \nu(d\mu) \\
 &= \int h_1(\int h_2(x) \tilde{\mu}(dx)) \tilde{\nu}(d\tilde{\mu})
 \end{aligned}$$

and

$$\begin{aligned}
 F(\tilde{\nu}) &= f(\int h_1(\int h_2(x) \tilde{\mu}(dx)) \tilde{\nu}(d\tilde{\mu})), \\
 \tilde{F}_k(\nu) &= f(k^{-\frac{d}{2}} \int h_1(k^{-\frac{d}{2}} \int h_2(\frac{x}{k}) \mu(dx)) \nu(d\mu);
 \end{aligned}$$

then it is easy to see that

$$(5.2.5) \quad G_k^{(2)} F(\nu)$$

$$\begin{aligned}
&= f'(\langle\langle h_1(\langle h_2, \cdot \rangle), \tilde{\nu} \rangle\rangle) \int h'_1(\langle h_2, \tilde{\mu} \rangle) \langle \Delta h_2, \tilde{\mu} \rangle \tilde{\nu}(d\tilde{\mu}) \\
&\quad + \gamma_1 k^{2-\frac{d}{2}} f'(\langle\langle h_1(\langle h_2, \cdot \rangle), \tilde{\nu} \rangle\rangle) \\
&\quad \int h''_1(\langle h_2, \tilde{\mu} \rangle) \langle h_2^2, \tilde{\mu} \rangle \tilde{\nu}(d\tilde{\mu}) \\
&\quad + \gamma_2 k^{2-\frac{d}{2}} f''(\langle\langle h_1(\langle h_2, \cdot \rangle), \tilde{\nu} \rangle\rangle) \int [h_1(\langle h_2, \tilde{\mu} \rangle)]^2 \tilde{\nu}(d\tilde{\mu}) \\
&= G^{(2)} F(\tilde{\nu}) = G^{(2)} \tilde{F}_k(\nu).
\end{aligned}$$

Now we turn to our main theorem about the self-similarity property.

Theorem 5.2.1 (Self-Similarity Property) *If $d = 4$ and the initial value of the process Y is the canonical equilibrium measure of the one level measure branching process, then the rescaled process $Y_k(t)$ has the same distribution as $Y(t)$.*

Proof. At first, we are going to calculate the Laplace transition functional of the rescaled process $Y_k(t)$.

Let

$$\begin{aligned}
\tilde{H}(\mu) &= h_1(\langle h_2, \mu \rangle), \\
H(\mu) &= h_1(k^{-\frac{d}{2}} \langle h_2(\frac{\cdot}{k}), \mu \rangle) k^{-\frac{d}{2}}
\end{aligned}$$

and

$$\begin{aligned}
(5.2.6) \quad L_{Y_k(t)}(\tilde{H}) &= E[\exp\{-\int \tilde{H}(\mu) Y_k(t, d\mu)\} \mid Y_k(0)] \\
&= \exp\{-\int u_k(t, \mu) Y_k(0, d\mu)\}.
\end{aligned}$$

Since we already know that the Laplace functional of process $Y(t)$ is given by

$$(5.2.7) \quad \begin{aligned} L_{Y(t)}(H) &= E[\exp\{-\int H(\mu)Y(t, d\mu)\} \mid Y(0)] \\ &= \exp\{-\int u(t, \mu)Y(0, d\mu)\}, \end{aligned}$$

where $u(t, \mu)$ is the solution of the following nonlinear differential equations

$$(5.2.8) \quad \begin{aligned} \frac{\partial u(t, \mu)}{\partial t} &= \mathcal{L}u(t, \mu) - \gamma_2 u^2(t, \mu), \\ u(0, \mu) &= H(\mu), \end{aligned}$$

we have

$$(5.2.9) \quad \begin{aligned} L_{Y_k(t)}(\tilde{H}) &= E[\exp\{-\int h_1(< h_2, \mu >)Y_k(t, d\mu)\} \mid Y_k(0)] \\ &= E[\exp\{-\int k^{-\frac{d}{2}}h_1(k^{-\frac{d}{2}}\int h_2(\frac{x}{k})\mu(dx))Y(k^2t, d\mu)\} \mid Y_k(0)] \\ &= E[\exp\{-\int H(\mu)Y(k^2t, d\mu)\} \mid Y(0)] \\ &= \exp\{-\int u(k^2t, \mu)Y(0, d\mu)\}. \end{aligned}$$

where we use the fact that $Y(0) = Y_k(0)$ for any $k \geq 1$ if $Y(0) = R_\infty$ which will be shown later. On the other hand, the rescaling transformation (5.2.1) implies that

$$(5.2.10) \quad \begin{aligned} &\exp\{-\int u_k(t, \mu)Y_k(0, d\mu)\} \\ &= \exp\{-\int k^{-\frac{d}{2}}u_k(t, k^{-\frac{d}{2}}\mu(k\cdot))Y(0, d\mu)\}. \end{aligned}$$

Equating (5.2.9) to (5.2.10) we obtain

$$(5.2.11) \quad u(k^2t, \mu(\cdot)) = k^{-\frac{d}{2}}u_k(t, k^{-\frac{d}{2}}\mu(k\cdot)).$$

Therefore

$$(5.2.12) \quad u_k(0, k^{-\frac{d}{2}}\mu(k\cdot)) = k^{\frac{d}{2}}u(0, \mu(\cdot)) = h_1(k^{-\frac{d}{2}} < h_2(\frac{\cdot}{k}), \mu >),$$

and for $t > 0$, by taking derivative with respect to t in (5.2.11) we have

$$\begin{aligned} (5.2.13) \quad & \frac{\partial u_k(t, k^{-\frac{d}{2}}\mu(k\cdot))}{\partial t} \\ &= k^{\frac{d}{2}} \frac{\partial u(k^2t, \mu)}{\partial t} = k^{\frac{d}{2}} \frac{\partial u(k^2t, \mu)}{\partial k^2t} \frac{\partial k^2t}{\partial t} \\ &= k^{\frac{d}{2}+2} [\mathcal{L}u(k^2t, \mu) - \gamma_2 u^2(k^2t, \mu)] \\ &= k^{\frac{d}{2}+2} \mathcal{L}_1 u(k^2t, \mu) + k^{\frac{d}{2}+2} \mathcal{L}_2 u(k^2t, \mu) - k^{\frac{d}{2}+2} \gamma_2 u^2(k^2t, \mu). \end{aligned}$$

Now let us discuss the three terms of the right hand side of (5.2.13) respectively. At first, since

$$\begin{aligned} \frac{\delta u(k^2t, \mu)}{\delta \mu(x)} &= \lim_{\varepsilon \rightarrow 0} \frac{u(k^2t, \mu + \varepsilon \delta_x) - u(k^2t, \mu)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{k^{-\frac{d}{2}} [u_k(t, k^{-\frac{d}{2}}(\mu(k\cdot) + \varepsilon \delta_x(k\cdot))) - u_k(t, k^{-\frac{d}{2}}\mu(k\cdot))]}{\varepsilon} \\ &= k^{-\frac{d}{2}} \lim_{\varepsilon \rightarrow 0} \frac{u_k(t, k^{-\frac{d}{2}}\mu(k\cdot) + k^{-\frac{d}{2}}\varepsilon \delta_{\frac{x}{k}}(\cdot)) - u_k(t, k^{-\frac{d}{2}}\mu(k\cdot))}{\varepsilon} \\ &= k^{-d} \lim_{\varepsilon \rightarrow 0} \frac{u_k(t, \tilde{\mu} + k^{-\frac{d}{2}}\varepsilon \delta_{\frac{x}{k}}) - u_k(t, \tilde{\mu})}{k^{-\frac{d}{2}}\varepsilon} \\ &= k^{-d} \frac{\delta u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(\frac{x}{k})}, \end{aligned}$$

where $\tilde{\mu}(\cdot) = k^{-\frac{d}{2}}\mu(k\cdot)$ and

$$\begin{aligned} \Delta \frac{\delta u(k^2t, \mu)}{\delta \mu(x)} &= \frac{\partial^2}{\partial x^2} [k^{-d} \frac{\delta u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(\frac{x}{k})}] \\ &= k^{-d} \frac{\partial^2}{\partial (\frac{x}{k})^2} \frac{\delta u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(\frac{x}{k})} \frac{d^2(\frac{x}{k})}{dx^2} \end{aligned}$$

$$= k^{-d-2} \frac{\partial^2}{\partial(\frac{x}{k})^2} \frac{\delta u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(\frac{x}{k})},$$

imply that

$$\begin{aligned} (5.2.14) \quad \mathcal{L}_1 u(k^2 t, \mu) &= \int \Delta \frac{\delta u(k^2 t, \mu)}{\delta \mu(x)} \mu(dx) \\ &= k^{-d-2} \int \frac{\partial^2}{\partial(\frac{x}{k})^2} \frac{\delta u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(\frac{x}{k})} \mu(dx) \\ &= k^{-\frac{d}{2}-2} \int \frac{\partial^2}{\partial x^2} \frac{\delta u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(x)} \tilde{\mu}(dx) \\ &= k^{-\frac{d}{2}-2} \int \Delta \frac{\delta u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(x)} \tilde{\mu}(dx) \\ &= k^{-\frac{d}{2}-2} \mathcal{L}_1 u_k(t, k^{-\frac{d}{2}} \mu(k \cdot)). \end{aligned}$$

Next by calculation

$$\begin{aligned} (5.2.15) \quad & \frac{\delta^2 u(k^2 t, \mu)}{\delta \mu(x) \delta \mu(y)} \\ &= \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \left\{ \frac{u(k^2 t, \mu + \varepsilon_1 \delta_x + \varepsilon_2 \delta_y) - u(k^2 t, \mu + \varepsilon_2 \delta_y)}{\varepsilon_1 \varepsilon_2} \right. \\ & \quad \left. - \frac{u(k^2 t, \mu + \varepsilon_1 \delta_x) - u(k^2 t, \mu)}{\varepsilon_1 \varepsilon_2} \right\} \\ &= \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} k^{-\frac{d}{2}} \left\{ \frac{u_k(t, k^{-\frac{d}{2}}(\mu(k \cdot) + \varepsilon_1 \delta_x(k \cdot) + \varepsilon_2 \delta_y(k \cdot)))}{\varepsilon_1 \varepsilon_2} \right. \\ & \quad \left. - \frac{u_k(t, k^{-\frac{d}{2}}(\mu(k \cdot) - \varepsilon_2 \delta_y(k \cdot)))}{\varepsilon_1 \varepsilon_2} \right. \\ & \quad \left. - \frac{u_k(t, k^{-\frac{d}{2}}(\mu(k \cdot) + \varepsilon_1 \delta_x(k \cdot))) - u_k(t, k^{-\frac{d}{2}} \mu(k \cdot))}{\varepsilon_1 \varepsilon_2} \right\} \\ &= k^{-\frac{d}{2}} \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \left\{ \frac{u_k(t, k^{-\frac{d}{2}} \mu(k \cdot) + k^{-\frac{d}{2}} \varepsilon_1 \delta_{\frac{x}{k}}(\cdot) + k^{-\frac{d}{2}} \varepsilon_2 \delta_{\frac{y}{k}}(\cdot))}{\varepsilon_1 \varepsilon_2} \right. \\ & \quad \left. - \frac{u_k(t, k^{-\frac{d}{2}} \mu(k \cdot) - k^{-\frac{d}{2}} \varepsilon_2 \delta_{\frac{y}{k}}(\cdot))}{\varepsilon_1 \varepsilon_2} \right. \\ & \quad \left. - \frac{u_k(t, k^{-\frac{d}{2}} \mu(k \cdot) + k^{-\frac{d}{2}} \varepsilon_1 \delta_{\frac{x}{k}}(\cdot)) - u_k(t, k^{-\frac{d}{2}} \mu(k \cdot))}{\varepsilon_1 \varepsilon_2} \right\} \end{aligned}$$

$$\begin{aligned}
 &= k^{-\frac{d}{2}-d} \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \left\{ \frac{u_k(t, \tilde{\mu} + k^{-\frac{d}{2}}\varepsilon_1\delta_{\frac{x}{k}} + k^{-\frac{d}{2}}\varepsilon_2\delta_{\frac{y}{k}}) - u_k(t, \tilde{\mu} + k^{-\frac{d}{2}}\varepsilon_2\delta_{\frac{y}{k}})}{k^{-\frac{d}{2}}\varepsilon_1 k^{-\frac{d}{2}}\varepsilon_2} \right. \\
 &\quad \left. - \frac{u_k(t, \tilde{\mu} + k^{-\frac{d}{2}}\varepsilon_1\delta_{\frac{x}{k}}) - u_k(t, \tilde{\mu})}{k^{-\frac{d}{2}}\varepsilon_1 k^{-\frac{d}{2}}\varepsilon_2} \right\} \\
 &= k^{-\frac{d}{2}-d} \frac{\delta^2 u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(\frac{x}{k}) \delta \tilde{\mu}(\frac{y}{k})}
 \end{aligned}$$

we obtain the relation

$$\begin{aligned}
 (5.2.16) \quad \mathcal{L}_2 u(k^2 t, \mu(\cdot)) &= \gamma_1 \int \int \frac{\delta^2 u(k^2 t, \mu(\cdot))}{\delta \mu(x) \delta \mu(y)} \delta_x(dy) \mu(dx) \\
 &= k^{-\frac{d}{2}-d} \gamma_1 \int \int \frac{\delta^2 u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(\frac{x}{k}) \delta \tilde{\mu}(\frac{y}{k})} \delta_x(dy) \mu(dx) \\
 &= k^{-d} \gamma_1 \int \int \frac{\delta^2 u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(x) \delta \tilde{\mu}(y)} \delta_{kx}(dky) \mu(dkx) \\
 &= k^{-d} \gamma_1 \int \int \frac{\delta^2 u_k(t, \tilde{\mu})}{\delta \tilde{\mu}(x) \delta \tilde{\mu}(y)} \delta_x(dy) \tilde{\mu}(dx) \\
 &= k^{-d} \mathcal{L}_2 u_k(t, \tilde{\mu}) \\
 &= k^{-d} \mathcal{L}_2 u_k(t, k^{-\frac{d}{2}} \mu(k \cdot)).
 \end{aligned}$$

Thus using (5.2.13) and the relations (5.2.14) and (5.2.16), we can continue to calculate

$$\begin{aligned}
 (5.2.17) \quad \frac{\partial u_k(t, k^{-\frac{d}{2}} \mu(k \cdot))}{\partial t} &= k^{\frac{d}{2}+2} k^{-\frac{d}{2}-2} \mathcal{L}_1 u_k(t, k^{-\frac{d}{2}} \mu(k \cdot)) \\
 &\quad + k^{\frac{d}{2}+2} k^{-d} \mathcal{L}_2 u_k(t, k^{-\frac{d}{2}} \mu(k \cdot)) - k^{\frac{d}{2}+2} \gamma_2 k^{-d} u_k^2(t, k^{-\frac{d}{2}} \mu(k \cdot)) \\
 &= \mathcal{L}_1 u_k(t, k^{-\frac{d}{2}} \mu(k \cdot)) + k^{2-\frac{d}{2}} \mathcal{L}_2 u_k(t, k^{-\frac{d}{2}} \mu(k \cdot)) \\
 &\quad - k^{-\frac{d-4}{2}} \gamma_2 u_k^2(t, k^{-\frac{d}{2}} \mu(k \cdot)).
 \end{aligned}$$

That is

$$(5.2.18) \quad \frac{\partial u_k(t, \mu)}{\partial t} = \mathcal{L}_1 u_k(t, \mu) + k^{2-\frac{d}{2}} \mathcal{L}_2 u_k(t, \mu) - \gamma_2 k^{-\frac{d-4}{2}} u_k^2(t, \mu).$$

In particular, if we take $d = 4$, then (5.2.18) becomes

$$(5.2.19) \quad \begin{aligned} \frac{\partial u_k(t, \mu)}{\partial t} &= \mathcal{L}_1 u_k(t, \mu) + \mathcal{L}_2 u_k(t, \mu) - \gamma_2 u_k^2(t, \mu) \\ &= \mathcal{L} u_k(t, \mu) - \gamma_2 u_k^2(t, \mu). \end{aligned}$$

Moreover, if we let $Y(0) = R_\infty$, where R_∞ denotes the canonical equilibrium measure of the one level measure branching process, and apply the Lemma 5.1.1 to the case of $d = 4$ and $t = k^2$, then

$$R_\infty(\{\mu : k^2 \mu(kB) > a\}) = k^2 R_\infty(\{\mu : \mu(B) > a\});$$

that is

$$R_\infty(A) = k^{-2} R_\infty(k^{-2} \{\mu : \exists \tilde{\mu} \in A, \text{ s.t. } \mu(\frac{\cdot}{k}) = \tilde{\mu}(\cdot)\}).$$

Therefore we have $Y_k(0) = Y(0) = R_\infty$. Thus if the initial value of process $Y(t)$ is R_∞ , then for each $k \geq 1$, $Y_k(t)$ and $Y(t)$ have the same Laplace functional. Hence the proof of the Theorem 5.2.1 is complete.

□

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