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**A Universal Closed-Form High-Speed Interconnect
Model for General Purpose Circuit Simulators**

by

Xin Li

**A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Master of Engineering**

**Ottawa-Carleton Institute for Electrical Engineering,
Department of Electronics,
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General Purpose Circuit Simulators ”

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Abstract

The continuing advances in circuit density and speed, at both the chip and package level, are placing increasing demands on the performance of interconnection technologies, thus also increasing the need for efficient and accurate interconnect analysis. In this thesis, a passive closed-form model for multiconductor, lossy transmission line analysis is presented. The proposed model is suitable for inclusion in general purpose circuit simulators and overcomes the mixed frequency/time simulation difficulties encountered during the transient analysis.

The method offers an efficient means to discretize transmission lines compared to the conventional lumped discretization while preserving the passivity of the discrete model. Coefficients of the approximation are computed using closed-loop *Padé* approximates of the exponential matrix. The algorithm also provides an error criterion for automatically selecting the order of the approximation for the frequency-bandwidth of interest. Numerical examples are presented to demonstrate the validity of the proposed model and to illustrate its application to a variety of interconnect structures.

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Chapter 1 Introduction

The problem of transmission line simulation arises in numerous applications. It gained special importance in connection with the contemporary development of digital electronics to the speed level when transmission-line behavior of on-board and even on-chip interconnections become important in the design [1]-[3].

Due to the advances in scaling technology over the past 20 years, device performance and operating speeds have skyrocketed. Minimum interconnect line width and spacing have been decreased along with the minimum feature size of integrated circuits scaled down to submicron range.

As a result, interconnect performance has become a limiting factor impinging on circuit performance. Not only is the signal delay due to interconnect often larger than the transistor delay, but also the interconnect effects, such as reflection, dispersion and crosstalk, may cause false action of the circuit. In general, the interconnect delay and the coupled signal noise must be considered together as they impact the overall system delay. There can be considerable delays due to transmission line effects, and transmission line noise can have an impact on the settling time of network, thereby determining the maximum rate at which the interconnect can be reliably clocked[3].

Thus, designers must observe the timing behavior of component blocks as early as possible in the design cycle. One of the major goals in the timing-level simulation of circuit blocks is to allow simulation of the entire signal path including nonlinear devices and interconnections as efficiently as possible without compromising accuracy to any great extent.

1.1 Transmission Line Simulation Problems

The problem of transmission line simulation consists of two parts. First, the distributed line parameters *RLGC* must be found. The parameters can be obtained from measurements or electromagnetic simulations. Next, the distributed *RLGC* lines must be simulated.

Although the theory of distributed lines is well-developed and the mathematical solution for multiconductor lines was formulated long ago, the problems of distributed line modeling and transient simulation still appear to be open since the problem involves several major areas of electronics engineering.

First, a solution of the problem requires knowledge of transmission line physics, a part of electromagnetics. An analysis and simulation of distributed line structures involves system theory and computational mathematics for a proper choice of transient simulation and linearization techniques. In addition, the development of device models for a transmission line that could be used for a circuit simulation involves circuit theory and theory of electronic devices.

The modeling and simulation of interconnects has been the focus of many recent papers [1]-[15] and can essentially be divided into two categories: *lumped models* (*RC*-class and *RLGC*-class) and *distributed* models.

The lumped models are the equivalent-circuit models. A distributed line can be represented as a chain of lumped *RLGC* segments. As interconnection densities and switching speeds increase, the electrical length of interconnects becomes a significant fraction of a wavelength and lumped models can no longer be used for accurate simulation of delay and crosstalk. Instead, a *distributed* model for the interconnect should be used.

After a transmission line model has been chosen, a transient simulation technique has to be applied to the model to simulate a transient response of a transmission line network. Lossy transmission lines are traditionally modeled and analyzed in the frequency domain, while the nonlinear terminations can only be described in the time domain. The major difficulty usually encountered

while linking the distributed transmission line models and the nonlinear simulators is the problem of mixed frequency/time.

There are several frequency- and time-domain transient simulation techniques available as seen in [5]-[15]: the frequency-domain techniques are based on numerical *Laplace* and *Fourier* transformations, while the time-domain transient simulation methods include numerical integration and convolution.

A great deal of work has been devoted to the development of mixed approaches. These methods exploit the natural formulation of both linear and nonlinear electric systems. They adopt a frequency domain characterization for the line, the linear element of the system, and the time domain characteristic equations for the loads, which contains the nonlinearities. The two are combined by means of a convolution operation [5],[7]. In order to overcome the computational complexity, a recursive convolution method has been developed [6].

Moment matching techniques such as Asymptotic waveform evaluation (*AWE*) and its variants, complex frequency hopping (*CFH*) and Krylov subspace techniques, have been successfully used to analyze interconnect networks. These techniques are based on *Padé* synthesis. Typical efficiency gains of several magnitudes over traditional methods have been reported.

In this thesis, an alternative approach is proposed to resolve the issue of mixed frequency / time simulation. The proposed approach is based on the idea of “macromodelling”. Using the macromodel approach, the transmission line stamp is converted to set of ordinary differential equations which can be linked easily to conventional circuit simulators.

1.2 Objectives and Contributions

The objective of this thesis is to develop an efficient, stable and passive macromodel for transient simulation of transmission lines. The model should be accurate, and able to be stenciled into general nonlinear simulators.

To achieve this objective, a novel model for discretization of distributed transmission lines is presented. The method uses closed-loop *Padé* expressions of the exponential matrix to efficiently approximate Telegrapher's equations. The proposed technique keeps the order of the approximation very small compared to the conventional discretization. The algorithm also provides an error criterion for automatically selecting the order of the approximation needed for capturing the interconnect behavior in the frequency bandwidth of interest.

It's always possible to come up with stable reduced order macromodels by utilizing a number of heuristics, but it is not guaranteed to obtain a provably passive approximation. In the absence of such a guarantee the simulation and analysis of the circuit may become problematic. A passive reduce-order model eliminates this concern.

In this thesis, the proof of passivity of the new model is given. This enables the development of a time-domain macromodel that can be stenciled in the general nonlinear simulators. Simulation examples in frequency and time domain are presented to demonstrate the validity of the proposed model and to illustrate its application to a variety of interconnect structures.

1.3 Organization

The material in this work consists of three major parts: introduction of transmission line modeling and simulation, presentation of a new macromodel for the transmission line, and macromodel implementation. These material is organized in four chapters:

Chapter 2 discusses the interconnecting modeling. Different interconnect models and their applications are described. Transmission line theory is reviewed, and various simulation methods are discussed.

Chapter 3 presents the new transmission line model. *Padé* theory is discussed, and a new macromodel of transmission line is presented using closed-loop *Padé* expressions of the exponential matrix. A proof of the stability of the model is presented.

Chapter 4 is concerned with the passivity issue. Passivity and its importance for interconnect modeling is discussed. Also proof of passivity of the proposed model is given.

Chapter 5 presents the macromodel implementation and simulation results in frequency and time domain for single, coupled lines are presented. The results are compared with results obtained with other simulators.

Chapter 2. Interconnect Modeling

2.1 Interconnect Modeling

The modeling and simulation of interconnects has been the focus of many recent papers [1]-[15],[16]-[27],and can essentially be divided into two categories: *lumped models* (*RC*-class and *RLGC*-class) and *distributed* models.

RC network models include most of the on-chip interconnects in middle to high frequency MOS integrated circuits, where the inductance effects of the lines are dominated by the resistance effects. These on-chip interconnections typically have high values of line resistance due to their small cross-sections, and they can be modeled to a fair degree of accuracy by linear *RC* circuits.

RLGC network models, which include most of the off-chip interconnects, have been analyzed mostly to correctly estimating the crosstalk and delay effect. These networks are typically generated by automatic parasitic extraction programs that often produce vast quantities of data, even for modest interconnects.

As interconnection densities and switching speeds increase, the electrical length of interconnects becomes a significant fraction of a wavelength and lumped models can no longer be used for accurate simulation of delay and crosstalk. Instead a *distributed* model for the interconnect should be used in this case.

The basic model is the simple delay line or lossless line. More complicated models include per-unit-length loss (either in the direction of travelling wave or due to dielectric substrate loss) or coupling between adjacent transmission lines, where the coupling may be resistive,inductive,capacitive or a combination of three.

The models may even require frequency dependent parameters (for addressing high-frequency effects such as skin effect, edge effect or proximity effect) or non-uniform per-unit-length param-

eters to accurately model different geometries. The main problem in simulating these high-frequency models is that the distributed model is derived from the solutions of partial differential equations, which is best described in the frequency-domain. However, these high-frequency models are difficult to interface with nonlinear drivers/loads.

The transmission line models can be considered to be multi-ports, and are usually characterized in terms of the sets of interchangeable descriptors, namely, $\mathbf{Y}$ (admittance), $\mathbf{Z}$ (impedance), $\mathbf{H}$ (hybrid) matrices, or transmission parameters $\mathbf{ABCD}$. To facilitate the interface with general-purpose circuit simulators such as *SPICE*, which employs *Modified Nodal Analysis*(*MNA*), $\mathbf{Y}$ -parameters are generally used to characterize the multi-port networks.

The ultimate goal of the modeling is to generate an element that can be coupled together in a circuit simulator which can account for the devices and interconnects in one simulation. As such, circuit elements rather than field equations, $\mathbf{S}$ -parameters, or other models are found to be useful.

The circuit models for a transmission line relate the termination voltages and currents directly. The implementation of the device model for a single lossless line was named the method of characteristics, and some further implementations of the device models for a transmission line are described in [8]. For modeling of a multiconductor line, the model decomposition technique [10],[17] has found many applications. This technique decomposes n coupled lines into n single model lines and model transformation networks. Accordingly, a multiconductor line can be represented in a circuit simulator using single line models and controlled sources.

To efficiently analyze transmission lines, methods based on reduced-order modeling techniques are used. To address the complexity issues, intermediate processing is required to approximate the generated linear networks by smaller circuits which exhibit similar electrical properties. This technique is generally referred to as *network reduction*. The approximation order is determined by the interesting frequency range.

The first step in building the reduced-order model of an interconnect is to determine a finite, approximate bandwidth of interest. In most digital applications, the minimum wavelength of

interest of the propagating signal can be approximated.

For instance, while a ramp has an energy spectrum spread over an infinite frequency range, most of the signal energy is low-frequency in nature, and decreases rapidly and monotonically, with increasing frequency, as shown in Fig.2-1.

The ramp signal has 1ns rise-time, so its expression in time-domain is:

$$v(t) = 10^9 t - 10^9 (t - 10^{-9}) U(t - 10^{-9}) , \text{ unit in seconds } (s), \quad (2-1)$$

and its *Laplace* form is:

$$V(s) = \frac{10^9}{s^2} - \frac{10^9}{s^2} e^{(-10^{-9}s)} \quad (2-2)$$

The following graph is the signal spectrum.

Fig. 2-1. Ramp signal (rise-time=1ns) and associated amplitude spectrum

ω_{max} is a chosen value, and if we ignore the high-frequency above it, this will introduce a minimum distortion in the overall signal shape. ω_{max} can be defined as the 3-dB bandwidth point, i.e.,

$$\omega_{max} \approx \frac{2\pi}{t_r} \quad (2-3)$$

The reduction of Eq.(2-3) is shown in appendix 1. This approximate bandwidth of interest can be employed in the selection of the order of approximation.

2.2 Transmission Line Theory

Multiconductor transmission line structures form a basic block of microwave circuits, and due to the increasing bit-rate in emerging digital systems, the behavior of multiconductor buses on board and backplanes becomes increasingly important. Hence, the prediction of their behavior is essential to the correct functioning of these devices.

The prediction of digital signal behavior includes not only transmission, reflection, and crosstalk in buses but also the behavior of the active circuits, packages, connectors, pads, and via-holes and other discontinuities on the signal propagation path. This is possible only if powerful simulation programs are used (such as the many variants of the *SPICE* program), so that a circuit description of multiconductor lines is necessary.

Multiconductor lines are structures with a constant cross-section which propagate a number of electromagnetic modes. In the direction perpendicular to the cross-section, that is, the propagation direction x , the complex field amplitudes behave as voltage and current

on a transmission line.

The multiconductor transmission line (*MTL*) equations for frequency-domain analysis (sinusoidal, steady-state excitation of the line) are [8]:

$$\frac{d}{dx}\hat{\mathbf{V}}(x) = -\hat{\mathbf{Z}}\hat{\mathbf{I}}(x) \quad (2-4a)$$

$$\frac{d}{dx}\hat{\mathbf{I}}(x) = -\hat{\mathbf{Y}}\hat{\mathbf{V}}(x) \quad (2-4b)$$

where $\hat{\mathbf{V}}(x)$, $\hat{\mathbf{I}}(x)$ are $n \times 1$ vectors containing the phase line voltages (with respect to the reference conductor) and phase line currents, respectively. x is the distance variable.

We use carets to denote complex valued quantities. The $n \times n$ complex matrices of per-unit-length impedance, $\hat{\mathbf{Z}}$, and admittance, $\hat{\mathbf{Y}}$, are symmetric and contain the $n \times n$ matrices of per-unit-length resistance $\mathbf{R}$, inductance $\mathbf{L}$, conductance $\mathbf{G}$, and capacitance $\mathbf{C}$ as:

$$\hat{\mathbf{Z}} = \mathbf{R} + j\omega\mathbf{L} \quad (2-5a)$$

$$\hat{\mathbf{Y}} = \mathbf{G} + j\omega\mathbf{C} \quad (2-5b)$$

The matrices $\mathbf{R}$, $\mathbf{L}$, $\mathbf{C}$, and $\mathbf{G}$ are also positive definite as may be shown from energy considerations[8].

The fundamental assumption in modeling a *MTL* with the transmission line equations is that the electric and magnetic fields lie in a plane transverse to the x axis which is called the transverse electromagnetic field structure or mode (*TEM*) of propagation. This field structure is identical to that of the static field in the transverse plane.

It has been shown [18] that the *TEM* mode can exist in a lossy medium provided that the conductors are perfect and the medium is homogeneous. For the case of imperfect conductors and/or inhomogeneous surrounding media, the *TEM* mode cannot exist. In this case it is assumed that these fields remain approximately *TEM* which is referred to as the quasi-*TEM* approximation.

The quasi-*TEM* approximation remains as the dominant trend in the electronics industry for analyzing lossy multiconductor, multielectric transmission line systems. The reasons are that the quasi-*TEM* approximation remains valid for most practical transmission line structures and offers relative ease and low cost when compared with a full-wave approach in obtaining the time-domain response to arbitrary excitations. The equivalence between the *TEM* mode field equations and the transmission line equations allows for the description of propagation in terms of the line circuit parameters or the medium characteristics. The validity of the quasi-*TEM* approximation was investigated in [19],[20].

A solution for Eq.(2-4) in terms of x is a matrix exponential,

$$\Phi(x, s) = e^{(A + sB)x} \Phi(0, s) \quad (2-6)$$

where, $\Phi(x, s) = \begin{bmatrix} V(x, s) \\ I(x, s) \end{bmatrix}$, $A = \begin{bmatrix} \mathbf{0} & -\mathbf{R} \\ -\mathbf{G} & \mathbf{0} \end{bmatrix}$, and $B = \begin{bmatrix} \mathbf{0} & -L \\ -C & \mathbf{0} \end{bmatrix}$, with $\mathbf{0}$ as a $n \times n$ matrix

of zeros. The far end voltages and currents can be described as $\Phi(d, s)$ where d is the length of the line. Therefore, in order to solve the far end voltages and currents in terms of the near end voltages and currents, we get:

$$\Phi(d, s) = e^{(A + sB)d} \Phi(0, s) \quad (2-7)$$

The model for multiconductor lines is reciprocal if no non-reciprocal materials exist in the multiconductor structures.

The propagation of voltage and current in a distributed two-conductor line is described in time-domain by the line telegrapher's equations:

$$\begin{aligned} -\frac{\partial}{\partial x}v(x, t) &= L \frac{\partial}{\partial t}i(x, t) + R i(x, t) \\ -\frac{\partial}{\partial x}i(x, t) &= C \frac{\partial}{\partial t}v(x, t) + G v(x, t) \end{aligned} \quad (2-8)$$

The transient simulation of transmission lines is complicated by the fact that since a distributed line is described by the line telegrapher's equations, which are partial differential equations of time and space, the transfer functions of any transmission line model modules are transcendental functions of the *Laplace* operator.

An approximate finite-order differential equation representation for the modules of the line model has to be found. It can be performed either in the frequency-domain, by approximating the transfer functions with transfer functions of a finite-order system (finite-order rational polynomial functions), or in the time-domain, by approximating any time-response with the corresponding response of a finite-order system (for instance, the step-response can be approximated with a finite-order exponential series).

2.3 Transient Simulation Methods

The fundamental difficulty encountered in integrating transmission line simulation into a transient circuit simulator is that circuits containing nonlinear devices must be characterized in the time-domain while transmission lines with loss, dispersion, or discontinuity are best characterized in the frequency domain. To cope with this difficulty, several types of approaches have been proposed in previous works.

One uses a network of lumped elements and segments of ideal transmission line to approximate the frequency response of each lossy transmission line or multiconductor line [16]. The approximated circuit models are suitable for existing general-purpose circuit simulators. However, the drawback of this type of approach is that the amount of computation increases for the simulation because a large number of extra nodes and elements are introduced [17]. Increasing the number of lumps also results in larger pole values which may force the integration time-step to become smaller to avoid unwanted numerical oscillations.

The second type of approach adopts convolution technique. For each integration, the outputs of linear lossy multiconductor lines are the convolutions of the inputs with the impulse responses (Green's function) of the multiconductor lines. The multiconductor lines are treated as a linear n -port system. To determine the pulse responses of an arbitrary multiconductor line system, people used the inverse fast *Fourier* transformation technique (*IFFT*) [17], the numerical inverse *Laplace* transformation technique [18] to determine the impulse response.

However, the convolution operation needs to extend over the entire past history. The computation time required at any time t is then proportional to t ; therefore, the convolution at large time points will be very time-consuming. In addition, the handling of nonlinear dynamic systems with the above techniques is extremely difficult and usually invokes iterative methods that additionally degrade the overall efficiency.

In the case of the numerical Fourier transform, the computational accuracy depends not only on the computational step (discretization error), but also on the computational interval (truncation error). For an accurate simulation, the transient simulation interval has to exceed the duration of the transient process in the system. As a result, a minimum of about a thousand time-points has to be computed to achieve a satisfactory accuracy for a

line with a pulse excitation. Furthermore, the impulse responses associated with a transmission line model are very slowly decreasing functions of time, and the transfer functions are very slowly decreasing functions of frequency. These factors make the applications of numerical *Laplace* transform, *Fourier* transform to a transmission line model even more inefficient.

To avoid the time-consuming convolution integration, the state-based approach [20] and the waveform relaxation based approach [21] have been proposed. The state-based approach utilizes information about the internal states of a transmission line at a given time to solve the states for the next time point. The voltage and the current at the sample points are kept as the states of the line, and are assumed to be piecewise-linear between adjacent sample points. Based on this assumption, the state variables can be determined by using integration on space, hence avoiding convolution.

The waveform relaxation based approach solves the line's equations in the frequency domain and uses the *FFT* to transform the results back and forth between the frequency and time domains for each iteration. Hence, time-domain convolutions are avoided by performing multiplication on the frequency domain.

A fast convolution integration algorithm is presented in [6]. From *Padé* approximation (or the reduced moment method), the impulse response of an arbitrary transfer function can be determined, and by employing the particular form of the impulse response, a recursive convolution formulation is presented. But there is no guarantee of the stability of the model, and it is difficult to predict the order of matching for getting enough accuracy. The multi-point moment-matching technique such as *CFH* [22] can be used to improve the accuracy and stability of the moment-matching technique. The other approaches, *PVL* [23], *Arnoldi* [24], are more suitable for network-reduction of linear lumped networks, but not efficient for handling distributed sub-networks. For example, in order to use the *PVL*

technique to analyze circuits containing transmission lines, numerical preprocessing is needed to approximate the frequency characteristics of Telegrapher's equations in terms of a set of ordinary differential equations [14].

Transient analysis of the entire circuitry requires solution of differential equations in time-domain due to the existing nonlinear blocks. Derivation of differential equations from reduced-order circuit models is known as *macromodel synthesis*.

A good circuit model for the transmission line is critical to accurately and efficiently analyze the transient propagation of signals. In next chapter, we present a new transmission line model, and also prove its stability and passivity. Through this reduced-model, we can directly write out the time-domain expression of transmission lines in differential equations.

Chapter 3. *Padé* Approximate Model of Transmission Lines

The analysis, design and simulation of electronic circuits containing transmission lines require some kinds of transmission line models to be utilized that would describe the physical processes of propagation, reflection and coupling that take place in transmission line structures. It is important to remember that a simulation is only as good as its models. In this chapter, a new macromodel of transmission line is presented.

The idea for the new model is from *Padé* approximation. Techniques from *Padé* theory have been quite successful in the area of rational approximation due to the linearity of the problem and the ease of the computation[30],[31],[32]. The problem of partial minimal realization has been recognized to be equivalent with *Padé Approximation*. Next, we discuss *Padé* theory, and employ the theory to realize a rational approximation of transmission line model.

3.1 Theory of *Padé* Approximates

The *Padé* approximates to a function are essentially the same as the set of approximates derived from an infinite set of continued fractions; One advantage of *Padé* approximates is that a given power series determines a set of approximates which are usually unique, whereas there are many ways of writing an associated continued fraction[31].

Considering the evaluation of a function $f(s)$, where s is complex variable, by forming a power series:

$$K(s) = \sum_{r=0}^{\infty} c_r s^r \quad (3-1)$$

When s lies within the circle of convergence R , we can write:

$$f(s) = K(s) \quad (3-2)$$

and approximate $f(s)$ by calculating a power series approximate:

$$k_l(s) = \sum_{r=0}^l c_r s^r \quad (3-3)$$

By taking l large enough we can in principle approximate $f(s)$ as closely as possible for s lying with the circle of convergence R . In many applications, it is very difficult to evaluate more than a few of the coefficients c_r , so that the accuracy to which $f(s)$ can be approximated by Eq.(3-3) is limited.

Padé approximates provide us with a practical method of calculating $f(s)$, for a large class of divergent series $K(s)$; in addition, the method frequently accelerates convergence when $K(s)$ is convergent [33],[34],[35].

The (N,M) *Padé* approximate to $f(s)$ defined from a series $K(s)$ is a rational function:

$$f_{N,M}(s) = \frac{\sum_{m=0}^M a_m s^m}{\sum_{n=0}^N b_n s^n} \quad (3-4)$$

and the coefficients in the numerator and the denominator are defined by the formal identity:

$$\left(\sum_{n=0}^N b_n s^n \right) \left(\sum_{r=0}^{\infty} c_r s^r \right) = \sum_{m=0}^M a_m s^m + o(s^{M+N}) \quad (3-5)$$

By the phrase ‘formal identity’ and the notation $o(s^{M+N})$, the coefficients of $1, s, s^2, \dots, s^{M+N}$ on the two sides of Eq.(3-5) are equated. Then, it may be proved that the coefficients a_m, b_n always exist and the rational fraction $f_{N,M}$ is unique [32].

$f(s)=e^s$ is the foundation of *Padé’s* original work, and coefficients of the Maclaurin expansion of $f(s)$ are sufficiently simple so that explicit forms of numerator and denominator of the approximates can be found.

For a given accuracy it is preferable to use *diagonal Padé approximations* ($M=N$). In fact, suppose that $M>N$, the evaluation of the approximation $f_{N,M}$ which is of order $M+N$ requires approximately Mn^3 flops. However, the same work is necessary to compute the approximation $f_{M,M}$ which is of order $2M>M+N$. Similar arguments hold for the case $M<N$ [37].

Since e^{-s} is a regular all-pass function, ideally it would be desirable for its rational approximations also retaining the all-pass property. However, since all regular rational all-pass functions are of the form:

$$H(s) = \frac{A(-s)}{A(s)} = \frac{a_0 - a_1 s + \dots + (-1)^n a_n s^n}{a_0 + a_1 s + \dots + a_n s^n} \quad (3-6)$$

where $A(s)$ is a strict Hurwitz polynomial, it follows that only diagonal *Padé* approximation can retain the all-pass nature of e^{-s} [38].

If we denote the unique *Padé* approximation to the matrix exponential $e^{\mathbf{Z}}$ by $f_{N,M}$, where $\mathbf{Z}$ is a matrix, with numerator $\mathbf{Q}_{N,M}$ of degree M and denominator $\mathbf{P}_{N,M}$ of degree N , then it is known [35] that:

$$\mathbf{Q}_{N,M}(\mathbf{Z}) = \sum_{j=0}^M \frac{(M+N-j)!M!}{(M+N)!j!(M-j)!} \mathbf{Z}^j \quad (3-7a)$$

$$\mathbf{P}_{N,M}(\mathbf{Z}) = \sum_{j=0}^N \frac{(M+N-j)!N!}{(M+N)!j!(N-j)!} (-\mathbf{Z})^j \quad (3-7b)$$

3.2 Padé Approximates of Transmission Lines

Next, we use Padé theory to approximate Eq.(2-7) by matrix rational functions.

Eq.(2-7) is written in the s -domain as:

$$\Phi(x, s) = e^{(\mathbf{A} + s\mathbf{B})x} \Phi(0, s)$$

where, $\Phi(x, s) = \begin{bmatrix} \mathbf{V}(x, s) \\ \mathbf{I}(x, s) \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} \mathbf{0} & -\mathbf{R} \\ -\mathbf{G} & \mathbf{0} \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} \mathbf{0} & -\mathbf{L} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix}$, with $\mathbf{0}$ an $n \times n$ matrix

of zeros. And we can rewrite it as:

$$\begin{bmatrix} \mathbf{V}(d, s) \\ -\mathbf{I}(d, s) \end{bmatrix} = e^{\mathbf{Z}d} \begin{bmatrix} \mathbf{V}(0, s) \\ \mathbf{I}(0, s) \end{bmatrix} \quad (3-8)$$

where $\mathbf{Z}=(\mathbf{A}+s\mathbf{B})d$, d is the length of transmission line. The negative sign before $\mathbf{I}(d,s)$ means that we take the direction in which current flows into the circuit as positive.

Approximating the exponential matrix e^Z using *Padé* theory from Eq.(3-6a),(3-6b), we get:

$$\begin{bmatrix} V(d, s) \\ -I(d, s) \end{bmatrix} = e^Z \begin{bmatrix} V(0, s) \\ I(0, s) \end{bmatrix} = Q_{N, M}(Z)(P^{-1}_{N, M}(Z)) \begin{bmatrix} V(0, s) \\ I(0, s) \end{bmatrix} \quad (3-9)$$

where $Q_{N, M}(Z)$, $P_{N, M}(Z)$ are polynomial matrices, and there is a closed-form relation which can be derived to generate these matrices. From the equation:

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} e^Z \equiv \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3-10)$$

$Q_{N, M}(Z)$, $P_{N, M}(Z)$ are computed using the following relations:

$$Q_{11} = \sum_{i=0}^N (M+N-i)! \binom{N}{i} \left[\frac{1}{2} (1 + (-1)^i) (ab)^{i/2} \right] \quad (3-11a)$$

$$Q_{12} = \sum_{i=0}^N (M+N-i)! \binom{N}{i} \left[\frac{1}{2} (1 - (-1)^i) (ab)^{\frac{i-1}{2}} a \right] \quad (3-11b)$$

$$Q_{21} = \sum_{i=0}^N (M+N-i)! \binom{N}{i} \left[\frac{1}{2} (1 - (-1)^i) (ba)^{\frac{i-1}{2}} b \right] \quad (3-11c)$$

$$Q_{22} = \sum_{i=0}^N (M+N-i)! \binom{N}{i} \left[\frac{1}{2} (1 + (-1)^i) (ba)^{i/2} \right] \quad (3-11d)$$

$$P_{11} = \sum_{i=0}^M (M+N-i)! \binom{M}{i} (-1)^i \left[\frac{1}{2} (1 + (-1)^i) (ab)^{i/2} \right] \quad (3-11e)$$

$$\mathbf{P}_{12} = \sum_{i=0}^M (M+N-i)! \binom{M}{i} (-1)^i \left[\frac{1}{2} (1 - (-1)^i) (\mathbf{a}\mathbf{b})^{\frac{i-1}{2}} \mathbf{a} \right] \quad (3-11f)$$

$$\mathbf{P}_{21} = \sum_{i=0}^M (M+N-i)! \binom{M}{i} (-1)^i \left[\frac{1}{2} (1 - (-1)^i) (\mathbf{b}\mathbf{a})^{\frac{i-1}{2}} \mathbf{b} \right] \quad (3-11g)$$

$$\mathbf{P}_{22} = \sum_{i=0}^M (M+N-i)! \binom{M}{i} (-1)^i \left[\frac{1}{2} (1 + (-1)^i) (\mathbf{b}\mathbf{a})^{i/2} \right] \quad (3-11h)$$

where $\mathbf{a}$ and $\mathbf{b}$ are matrices or scalars a, b (if R, L, G, C are scalars when line is single) defined as:

$$\mathbf{a} = -(\mathbf{R} + s\mathbf{L})d; \quad \mathbf{b} = -(\mathbf{G} + s\mathbf{C})d; \quad (3-12)$$

$$\text{and } \binom{N}{i} = \frac{N!}{i!(N-i)!}.$$

Using above expressions for $\mathbf{Q}_{N,M}(\mathbf{Z})$, $\mathbf{P}_{N,M}(\mathbf{Z})$, Eq.(3-8) can be written in a closed-loop form as:

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}(d, s) \\ -\mathbf{I}(d, s) \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}(0, s) \\ \mathbf{I}(0, s) \end{bmatrix} \quad (3-13)$$

Closed-loop relationships (3-11a) - (3-11h) makes the computation of approximates represented by Eq.(3-13) very efficient. Since all the coefficients in Eq.(3-13) are known, it can be easily stenciled into a circuit simulator. In other words the transmission line can now be simply viewed as another “element” in the circuit simulator, without the mask of its complex distributed form.

3.3 Criteria for selecting order of approximation

Accuracy of the proposed model depends on the order of the approximation (M, N) . Following error criterion can be used to efficiently decide the order of the approximation:

$$\left\| e^Z - P_M^{-1}(Z) Q_N(Z) \right\| < \varepsilon \quad (3-14)$$

where ε is a pre-defined error tolerance in the frequency range of interest. Since all the elements in Eq.(3-11) are computed in a recursive and closed-loop form, the required order (M, N) can be easily estimated. Alternatively,

$$\left\| P_{M+1}^{-1}(Z) Q_{N+1}(Z) - P_M^{-1}(Z) Q_N(Z) \right\| < \varepsilon \quad (3-15)$$

can also be used to efficiently estimate the order of the approximation.

3.4 Stability of *Padé* Model

Before converting Eq.(3-13) from frequency domain to time domain using *Laplace* transformation, we must consider the characteristic of the model. Next, we will discuss the stability of our model.

The essential requirement of a closed-loop dynamic system is stability: the ability of the system to operate under a variety of conditions without self destructing.

To linear, time-invariant systems, the basic stability criterion is directly determined by the locations of the system poles, i.e., the roots of the characteristic equation of the system. In terms of the so-called characteristic equation of the system, which has the form:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$$

with real coefficients $a_0 \dots a_n$, the stability question is answered in the affirmative if it can be established that all the real roots and all the real parts of the complex roots are negative. Stated in another way, the stability of the physical system is assured if it can be shown that all zeros of the polynomial:

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

lie in the left half of the s -plane. A polynomial having this property is called a *Hurwitz polynomial* [46].

The methods for determining whether the roots of a polynomial are located in the stable region, without finding the roots, are provided by *Routh-Hurwitz* criterion for continuous system, and by *Jurrry-Blanchard* criterion for discrete systems[46]. In addition, Nyquist criterion provides a method of determine *BIBO* stability of a closed-loop feed-back system from the transfer-function of the open-loop system. These methods can be found in standard textbooks on control systems. There are also some approximate but simpler stability conditions proposed for using in practice[41]. By approximate conditions, it means either only sufficient conditions or only necessary conditions (sufficient conditions for instability) for stability.

Consider our model, we derive Y matrix of the transmission line from Eq. (3-13). As explained in the above, we prefer *N/N Padé* approximate to the exponential matrix. So that, from Eq.(3-11a) to Eq.(3-11h), we know that Q_{11} equals to P_{11} , Q_{22} equals to P_{22} , Q_{12} equals to $-P_{12}$, and Q_{21} equals to $-P_{21}$.

Eq.(3-13) can be written into:

$$\begin{bmatrix} Q_{11} & -Q_{12} \\ -Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} V(d, s) \\ -I(d, s) \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} V(0, s) \\ I(0, s) \end{bmatrix} \quad (3-16)$$

and

$$Q_{11}V_2 + Q_{12}I_2 = Q_{11}V_1 + Q_{12}I_1 \quad (3-17a)$$

$$-Q_{21}V_2 - Q_{22}I_2 = Q_{21}V_1 + Q_{22}I_1 \quad (3-17b)$$

where V_2 is for $V(d, s)$, I_2 for $I(d, s)$, and V_1 for $V(0, s)$, I_1 for $I(0, s)$.

From Eq.(3-17a),(3-17b), we can get the Y parameters as:

$$Y_{11} = \frac{-Q_{12}^{-1}Q_{11} - Q_{22}^{-1}Q_{21}}{2}, \quad \text{and} \quad Y_{12} = \frac{Q_{12}^{-1}Q_{11} - Q_{22}^{-1}Q_{21}}{2} \quad (3-18a)$$

$$Y_{21} = \frac{Q_{12}^{-1}Q_{11} - Q_{22}^{-1}Q_{21}}{2}, \quad \text{and} \quad Y_{22} = \frac{-Q_{12}^{-1}Q_{11} - Q_{22}^{-1}Q_{21}}{2} \quad (3-18b)$$

From $Y_{12} = Y_{21}$, we know the model is reciprocal.

First, consider the transmission line as a single, lossless line. L, C in Eq.(3-12) are now scalars. The Y parameters are:

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{-Q_{11}Q_{11} - Q_{12}Q_{21}}{2Q_{22}Q_{12}} & \frac{Q_{11}Q_{11} - Q_{12}Q_{21}}{2Q_{22}Q_{12}} \\ \frac{Q_{11}Q_{11} - Q_{12}Q_{21}}{2Q_{22}Q_{12}} & \frac{-Q_{11}Q_{11} - Q_{12}Q_{21}}{2Q_{22}Q_{12}} \end{bmatrix} \quad (3-19)$$

where Q_{ij} are polynomials of s .

If the expansion order N is even, and we assume line length d is 1, Q_{ij} are simplified as

$$Q_{22} = \sum_{i=0}^{N/2} \frac{(2N-2i)!N!(LC)^i s^{2i}}{(2i)!(N-2i)!} \quad (3-20a)$$

$$Q_{12} = \sum_{i=0}^{\frac{N-2}{2}} \frac{(2N-2i-1)!N!(-1)L^{i+1}C^i s^{2i+1}}{(2i+1)!(N-2i-1)!} \quad (3-20b)$$

$$Q_{21} = \sum_{i=0}^{\frac{N-2}{2}} \frac{(2N-2i-1)!N!(-1)L^i C^{i+1} s^{2i+1}}{(2i+1)!(N-2i-1)!} \quad (3-20c)$$

and Q_{22} equals to Q_{11} .

If the expansion order N is odd, Q_{ij} are simplified as

$$Q_{11} = \sum_{i=0}^{(N-1)/2} \frac{(2N-2i)!N!(LC)^i s^{2i}}{(2i)!(N-2i)!} \quad (3-21a)$$

$$Q_{12} = \sum_{i=0}^{\frac{N-1}{2}} \frac{(2N-2i-1)!N!(-1)L^{i+1}C^i s^{2i+1}}{(2i+1)!(N-2i-1)!} \quad (3-21b)$$

$$Q_{21} = \sum_{i=0}^{\frac{N-1}{2}} \frac{(2N-2i-1)!N!(-1)L^i C^{i+1} s^{2i+1}}{(2i+1)!(N-2i-1)!} \quad (3-21c)$$

and Q_{22} equals to Q_{11} .

The stability question is to check if all roots of $Q_{22}Q_{12}$ are in left region of s -plane, including $j\omega$ axis.

Lemmas:

1. Polynomials with real coefficients whose zeros all lie in the region ($Re(s) < 0$) will be called *strict Hurwitz* polynomials. If there are simple zeros on the $j\omega$ axis ($Re(s) = 0$), we call such a polynomial a *modified Hurwitz* polynomial.

2. The zeros of odd and even parts of a *Hurwitz* polynomial are restricted to lie on the $j\omega$ axis where they occur in conjugate pairs and alternate with each other. It means that the odd and even parts of a *Hurwitz* polynomial are modified *Hurwitz* polynomials.

3. The diagonal *Padé* approximates of e^{-s} is:

$$H(s) = \frac{A(-s)}{A(s)} = \frac{a_0 - a_1 s + \dots + (-1)^n a_n s^n}{a_0 + a_1 s + \dots + a_n s^n}$$

where $A(s)$ is a strict Hurwitz polynomial[35].

Because the *Padé* approximates of e^{-x} is:

$$e^{-x} = \frac{\sum_{i=0}^N \frac{(2N-i)! N! (-x)^i}{i! (N-i)!}}{\sum_{i=0}^N \frac{(2N-i)! N! x^i}{i! (N-i)!}} = \frac{A(-x)}{A(x)}$$

when N is an even number, replacing $\sqrt{LC}s$ with x , Q_{22} and Q_{12} can be written as:

$$Q_{22} = \sum_{i=0}^{N/2} \frac{(2N-2i)! N! x^{2i}}{(2i)! (N-2i)!}$$

$$Q_{12} = \left(-\sqrt{\frac{L}{C}}\right) \sum_{i=0}^{\frac{N-2}{2}} \frac{(2N-2i-1)! N! x^{2i+1}}{(2i+1)! (N-2i-1)!}$$

We find that Q_{22} is the even part of the denominator of *Padé* approximates of e^{-x} , and Q_{12} is the odd part of the denominator of *Padé* approximates of e^{-x} multiplied by $-\sqrt{\frac{L}{C}}$. So that Q_{22} and Q_{12} are *modified Hurwitz* polynomials, and all roots of $Q_{22}Q_{12}$ are on the $j\omega$ axis. This is the character of a lossless network.

When N is a odd number, Q_{22} is still the even part of the denominator of *Padé* approximates of e^{-x} , and Q_{12} is the odd part of the denominator of *Padé* approximates of e^{-x} multiplied by $-\sqrt{\frac{L}{C}}$. All roots of $Q_{22}Q_{12}$ are on the $j\omega$ axis.

For single, lossy line, we know $R \neq 0$, and $G \neq 0$. When N is even, and assume line length d is 1, Q_{ij} are simplified as

$$Q_{22} = \sum_{i=0}^{N/2} \frac{(2N-2i)!N!((R+sL)(G+sC))^i}{(2i)!(N-2i)!}$$

$$Q_{12} = \sum_{i=0}^{\frac{N-2}{2}} \frac{(2N-2i-1)!N!(-1)(R+sL)^{i+1}(G+sC)^i}{(2i+1)!(N-2i-1)!}$$

$$Q_{21} = \sum_{i=0}^{\frac{N-2}{2}} \frac{(2N-2i-1)!N!(-1)(R+sL)^i(G+sC)^{i+1}}{(2i+1)!(N-2i-1)!}$$

and Q_{22} equals to Q_{11} .

If the expansion order N is odd, Q_{ij} are simplified as

$$Q_{11} = \sum_{i=0}^{(N-1)/2} \frac{(2N-2i)!N!((R+sL)(G+sC))^i}{(2i)!(N-2i)!}$$

$$Q_{12} = \sum_{i=0}^{\frac{N-1}{2}} \frac{(2N-2i-1)!N!(-1)(R+sL)^{i+1}(G+sC)^i}{(2i+1)!(N-2i-1)!}$$

$$Q_{21} = \sum_{i=0}^{\frac{N-1}{2}} \frac{(2N-2i-1)!N!(-1)(R+sL)^i(G+sC)^{i+1}}{(2i+1)!(N-2i-1)!}$$

Replacing $\sqrt{(R+sL)(G+sC)}$ with x , we can prove that when $Q_{22}Q_{12}$ equal to zero, x is on the $j\omega$ axis, and s has negative real parts.

For coupled and lossy lines, R, L, G, C are matrices. To prove its stability, we need to check if roots of the polynomial made by multiplication of determinants of Q_{22} and Q_{12} are all in the left region of s -plane, including $j\omega$ axis.

When N is an even number,

$$Q_{22} = \sum_{i=0}^{N/2} \frac{(2N-2i)!N![(G+sC)(R+sL)]^i}{(2i)!(N-2i)!}$$

and

$$Q_{12} = \sum_{i=0}^{\frac{N-2}{2}} \frac{(2N-2i-1)!N!(-1)[(R+sL)(G+sC)]^i}{(2i+1)!(N-2i-1)!}(R+sL)$$

We rewrite Q_{22} as:

$$Q_{22} = \sum_{i=0}^{N/2} f(N, i) [(G + sC)(R + sL)]^i$$

and Q_{12} as:

$$Q_{12} = \left(\sum_{i=0}^{\frac{N-2}{2}} (-1)^i g(N, i) [(R + sL)(G + sC)]^i \right) (R + sL) = q_{12}(R + sL)$$

Lemmas[49]:

1. $Det(AB) = Det(A)Det(B)$, where A, B are matrices and Det is for determinant.

2. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of matrix C , then the matrix polynomial

$$b[C] = b_0 C^m + b_1 C^{m-1} + \dots + b_m, \text{ with eigenvalues of } b(\lambda_1), b(\lambda_2), \dots, b(\lambda_n).$$

3. The determinant of a matrix is equal to the product of its eigenvalues. It follows that

$$Det(b[C]) = b(\lambda_1) \cdot b(\lambda_2) \dots b(\lambda_n)$$

Suppose eigenvalues of $(G + sC)(R + sL)$ are $\lambda_1, \lambda_2, \dots, \lambda_k$, and λ_i is polynomial of s .

we have:

$$Det[Q_{22}] = \sum_{i=0}^{N/2} f(N, i) \lambda_1^i \sum_{i=0}^{N/2} f(N, i) \lambda_2^i \dots \sum_{i=0}^{N/2} f(N, i) \lambda_k^i$$

Let $\lambda_i = x_i^2$, we find that when $Det[Q_{22}] = 0$, the variables x_i equal to $\pm j\omega_i$. Because

$\sum_{i=0}^{N/2} f(N, i)x_k^{2i}$ is even part of a *Hurwitz* polynomial, their roots are on the $j\omega$ axis.

Now, the stable problem is to prove that when eigenvalues λ_i of $(G+sC)(R+sL)$ are $-\omega_i^2$, what are the s values? Before further proof, here is another theorem, and its proof is in Appendix 3.

Theorem 3.1 If A , B , and C are symmetric non-negative definite, and either A or C positive definite, then

$$|\lambda^2 A + 2\lambda B + C| = 0$$

has no roots with positive real parts.

If A , B , and C are positive definite, then the roots of

$$|\lambda^2 A + 2\lambda B + C| = 0$$

has negative real parts. [40]

Expand $(G+sC)(R+sL)$ to:

$$(G+sC)(R+sL) = s^2 CL + s(GL + CR) + GR = As^2 + Bs + C = M$$

We can prove later that A, B and C are positive definite matrices.

Let $\text{Det}[(G+sC)(R+sL)] = \text{Det}[M] = \lambda_1 \cdot \lambda_2 \dots \lambda_k$

We know that when $\text{Det}[M]=0$, s has negative real parts, that means when $\lambda_i=0$, s has negative real parts. How about when $\lambda_i = -\omega_i^2$?

A scalar λ is said to be an *eigenvalue* of an $n \times n$ matrix A if there exists an $n \times 1$ non-null vector x such that $Ax = \lambda x$ or, equivalently, $(A - \lambda I)x = 0$, where I is unit matrix. Clearly, λ is an eigenvalue of A if and only if the matrix $A - \lambda I$ is singular. So that the

above question is to check the s value when $Det[\mathbf{M}-\lambda \mathbf{I}]=Det[\mathbf{M}+\omega_i^2 \mathbf{I}]=0$.

Because of $\mathbf{M}+\omega_i^2 \mathbf{I}=\mathbf{A}s^2+\mathbf{B}s+\mathbf{C}+\omega_i^2 \mathbf{I}=\mathbf{A}s^2+\mathbf{B}s+\tilde{\mathbf{C}}$, where $\mathbf{A}, \mathbf{B}$, and $\tilde{\mathbf{C}}$ are positive definite matrices, we conclude that s has negative real parts when $Det[\mathbf{Q}_{22}]=0$.

Next, we prove that s has negative real parts when $Det[\mathbf{Q}_{12}]=0$. Suppose eigenvalues of $(\mathbf{R}+s\mathbf{L})(\mathbf{G}+s\mathbf{C})$ are $\mu_1, \mu_2, \dots, \mu_k$, and μ_i is polynomial of s , we have:

$$Det[\mathbf{Q}_{12}]=(-1)^{\sum_{i=0}^{\frac{N-2}{2}} g(N, i) \mu_1^i} \sum_{i=0}^{\frac{N-2}{2}} g(N, i) \mu_2^i \dots \sum_{i=0}^{\frac{N-2}{2}} g(N, i) \mu_k^i Det(\mathbf{R}+s\mathbf{L})$$

The determinate of $\mathbf{Q}_{12}$ equals to $Det[\mathbf{q}_{12}]Det(\mathbf{R}+s\mathbf{L})$, where

$$Det[\mathbf{q}_{12}]=(-1)^{\sum_{i=0}^{\frac{N-2}{2}} g(N, i) \mu_1^i} \sum_{i=0}^{\frac{N-2}{2}} g(N, i) \mu_2^i \dots \sum_{i=0}^{\frac{N-2}{2}} g(N, i) \mu_k^i$$

Because $\sum_{i=0}^{(N-2)/2} g(N, i) x_k^{2i}$ is odd part of a *Hurwitz* polynomial, their roots are on the

$j\omega$ axis. Following the proof procedure when $Det[\mathbf{Q}_{22}]=0$, we can find that s has non-positive real parts when $Det[\mathbf{q}_{12}]=0$, and $Det(\mathbf{R}+s\mathbf{L})=0$.

When N is a odd number,

$$Q_{22} = \sum_{i=0}^{\frac{N-1}{2}} \frac{(2N-2i)!N![(G+sC)(R+sL)]^i}{(2i)!(N-2i)!}$$

$$Q_{12} = \left(\sum_{i=0}^{\frac{N-1}{2}} \frac{(2N-2i-1)!N!(-1)[(R+sL)(G+sC)]^i}{(2i+1)!(N-2i-1)!} \right) (R+sL)$$

in the same way, we can prove that when N is odd ,the proposed model is still stable.

To prove A, B and C above are positive definite matrices, we do *similarity transformation* of these matrices [49]. For matrices A, B, C , the transformation defined by $B = C^{-1}AC$ is termed *similarity* or *collineatory* transformation, where A, B, C are real matrices. The two matrices related by the *similarity transformation* are termed *similar* or are said to be *transforms* of each other. The *similarity transformation* does not change either the characteristic equation or the characteristic values of a matrix.

Because R, G, L, C are positive definite matrices, consider the *similarity transformation* on GL below:

$$G^{-\frac{1}{2}}(GL)G^{\frac{1}{2}} = G^{\frac{1}{2}}LG^{\frac{1}{2}}$$

Since $G^{\frac{1}{2}}LG^{\frac{1}{2}}$ is a positive definite matrix according to the definition of positive-real in appendix 5, we prove that GL is also a positive definite matrix.

Chapter 4. Passivity of Proposed Model

4.1 Introduction

It's always possible to come up with stable reduced order macromodels by utilizing a number of heuristics, but it is not guaranteed to obtain a provably passive approximation.

It is known that passive networks are necessarily stable, but the converse is not true. Passivity and stability differ in the following fundamental way: while the connection of two stable networks is not necessarily stable, any multiport connection of passive networks is guaranteed to be passive [46].

This closure property is of paramount importance from a practical point of view for the following reason. The reduced-order model is intended to replace the original interconnect in the global netlist. It will have the same drivers and loads as the original model. the output impedances of the drivers and the input impedances of the loads are represented by passive elements. If the reduced-order model is only stable but not passive, there is no guarantee that the network composed of output impedances, reduce-order model, and input impedance is stable. In the absence of such a guarantee the simulation and analysis of the circuit may become problematic. A passive reduce-order model eliminates this concern.

Fig.4-1. is a numerical example generated in [48] that demonstrates the passivity problem. It is demonstrated that, in addition to requiring macromodel stability, macromodel passivity is required to guarantee the overall circuit stability once the active and passive drive/load models are connected.

$Y_1(s)$ represents a reduced order transfer function which has all poles and zeros in the left half plane, and internally stable. $Y_2(s)$ represents a capacitor and resistor in parallel. If we drive this circuit with a current source, the output will oscillate at $2.5/\pi \text{Hz}$. (Poles at $s = \pm 5j$, so that $f = \omega/(2\pi) = \frac{2.5}{\pi} \text{Hz}$).

It is clear from the above example that the stability of the combined system cannot be assured. So that for the class of *LLF* network, a multi-port is unconditionally stable if, and only if it is passive.

Fig 4-1: A system example demonstrating potential instability.

4.2 Passivity of Linear Circuit

Problems in analysis are that a physical network and an input are given, the mathematical function (e.g., a current, voltage, or an impedance) is to be found. A solution always exists and is unique. In the converse problem, that of synthesis, a rational function of a complex-frequency variable, or a set of these rational functions, is given as a speci-

cation of a desired network, and a network characterized by the given functions is to be found. A solution may not exist, in which case the given function cannot be represented by a physical network, or alternatively, is not physically realizable.

As a first step, a check is made to determine whether the given function satisfies the necessary and sufficient conditions for physical realizability; these conditions guarantee that a network represented by the function can be devised. Satisfying this condition ensures that, the synthesis procedure is guaranteed to work. Can the specification be met by the given function? We require necessary and sufficient conditions.

A network is said to be *stable* if all of its natural frequencies are restricted to the *LHS* (left half of the complex frequency s -plane). For a two-port network, its stability cannot be determined by the two-port itself; it also depends on the terminations. Thus, a two-port network is said to be *potentially unstable* if it is possible to select passive one-port impedances that, when terminated at the ports, produce an unstable network. If no such passive termination can be found, the two-port network is called *absolutely stable*. A property in network theory states that: *interconnections of stable systems may not necessarily be stable, but (strictly) passive networks are (asymptotically) stable; and arbitrary interconnections of (strictly) passive networks are (strictly) passive, and therefore, (asymptotically) stable* [43].

Consider any network for which the application of Kirchoff's laws leads to a set of equations. The network is termed *linear*, if these network equations are *linear*. The network is said to be *lumped* (or to consist of *lumped elements*), if in these network equations, time is the only independent variable, i.e., voltage or current variation with space coordinates has not entered the problem. The network is also finite, since the finite number of equations, each with a finite number of terms, implies a finite number of elements. For the class of reciprocal *LLF* (*linear, lumped, finite*) networks, a multiport is uncondi-

tionally stable if, and only if it is passive[44].

Definition: Passivity An n -port network is said to be passive if, for all admissible signal pairs $\mathbf{v}(t)$ and $\mathbf{i}(t)$, the total energy stored in and delivered to the n -port is nonnegative for all time t ; that is:

$$\varepsilon(t) = \varepsilon(t_0) + \int_{t_0}^t \mathbf{v}^T(x) \mathbf{i}(x) dx \geq 0 \quad (4-1)$$

for all initial time t_0 and all time $t \geq t_0$, where $\varepsilon(t_0)$ is the finite energy stored in the n -port at the initial time t_0 .

The concept of passivity introduced above is a useful qualitative measure of the time-domain behavior of networks. The definition, although very general, is difficult to apply.

An important result will be established concerning the physical realizability and synthesis of an arbitrary passive network. If any arbitrary matrix array satisfies these constraints, it is possible to find a passive network with this matrix as its circuit matrix. We translate the time-domain passivity criterion into frequency-domain passivity conditions for linear time-invariant n -port networks, and show that the linear time-invariant n -port networks are passive if and only if their general hybrid matrices are positive-real matrices. The general hybrid matrix includes the impedance matrix $\mathbf{Z}$, the admittance matrix $\mathbf{Y}$, and the hybrid matrix $\mathbf{H}$ [45].

First consider the necessary conditions that the impedance matrix $\mathbf{Z}$ (if it exists) of any physical passive network must satisfy. At any accessible terminal pairs with voltage and current v_k, i_k , the complex power W_k is given by:

$$W_k = P_k + jQ_k = v_k i_k^* \quad (4-2)$$

where P_k is the real power in watts, Q_k is the reactive power, and the asterisk indicates

the complex conjugate. Complex power may be defined as $v i^*$ or $v^* i$. Either definition can be used since the average power is $P = \text{Re}(v^* i) = \text{Re}(v i^*)$, and the sign of the reactive power depends on the choice of the definition.

The total complex power supplied to a network with n -port is:

$$W = \sum_{k=1}^n W_k = \sum_{k=1}^n v_k i_k^* = P + jQ \quad (4-3)$$

This may be written in matrix notation as:

$$W = [i_1^*, i_2^* \dots i_n^*] \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \bar{I} V \quad (4-4)$$

The sign (-) indicates the conjugate transpose. The current and voltage matrices are related by the impedance matrix for the accessible ports:

$$V = Z I \quad (4-5)$$

Substituting this for V in Eq. (4-4):

$$W = \bar{I} Z I \quad (4-6)$$

The real power is of importance, since physical realizability in a passive network demands that:

$$P = \text{Re}(W) \geq 0 \quad (4-7)$$

To obtain the power in convenient form requires further consideration of the matrix Z in the following section.

Any given matrix may be written as the sum of a hermitian and skew-hermitian matrix:

$$\mathbf{Z} = \mathbf{Z}_H + \mathbf{Z}_s \quad (4-8)$$

A matrix which satisfies $\mathbf{Z}' = -\mathbf{Z}^*$ is said to be *skew-hermitian matrix*, and a matrix which satisfies $\mathbf{Z}' = \mathbf{Z}^*$ is said to be *hermitian matrix*, where $\mathbf{Z}'$ is the transpose of $\mathbf{Z}$, and $\mathbf{Z}^*$ is the conjugate of $\mathbf{Z}$. Thus let:

$$\mathbf{Z}_H = \frac{1}{2}(\mathbf{Z} + \bar{\mathbf{Z}}) \quad (4-9)$$

$$\text{and} \quad \mathbf{Z}_s = \frac{1}{2}(\mathbf{Z} - \bar{\mathbf{Z}}) \quad (4-10)$$

We observe that $\mathbf{Z}_H$ is a hermitian matrix, and $\mathbf{Z}_s$ is a skew-hermitian matrix.

If Eq. (4-8) is substituted into Eq. (4-6):

$$\begin{aligned} \mathbf{W} &= \bar{\mathbf{I}}(\mathbf{Z}_H + \mathbf{Z}_s)\mathbf{I} \\ &= \bar{\mathbf{I}}\mathbf{Z}_H\mathbf{I} + \bar{\mathbf{I}}\mathbf{Z}_s\mathbf{I} \end{aligned} \quad (4-11)$$

It is shown in the following that the first term of Eq. (4-11) is a single real number, i.e., the real power P . Observe that the conjugate of the first term is:

$$(\bar{\mathbf{I}}\mathbf{Z}_H\mathbf{I})^* = \mathbf{I}'\mathbf{Z}_H^*\mathbf{I}^* \quad (4-12)$$

and since the expansion of the matrices result in a scalar element, which is equal to its transpose, so that:

$$(\bar{\mathbf{I}}\mathbf{Z}_H\mathbf{I})^* = (\mathbf{I}'\mathbf{Z}_H^*\mathbf{I}^*)' = \bar{\mathbf{I}}\bar{\mathbf{Z}}_H\mathbf{I} = \bar{\mathbf{I}}\mathbf{Z}_H\mathbf{I} \quad (4-13)$$

since $\mathbf{Z}_H$ is a hermitian matrix, and the sign $(-)$ indicates conjugate transpose, and $(\quad)'$ is the sign for transpose. $\bar{\mathbf{I}}\mathbf{Z}_H\mathbf{I}$ is equal to its own conjugate, hence it must be real. In the

same way, $\bar{I}Z_s I$ is equal to its negative conjugate, and hence must be purely imaginary.

Accordingly passivity requires:

$$P = \bar{I}Z_H I \geq 0 \quad (4-14)$$

$$jQ = \bar{I}Z_s I \quad (4-15)$$

The result of the past few paragraphs may be summarized in the following statement:

Theorem 4.1 Any physical passive network (not necessarily reciprocal) at any real frequency must have an impedance matrix (if it exists) whose hermitian part is the matrix of positive semidefinite or positive-definite hermitian form. [46]

When Z is symmetric, $Z_H = \text{Re}(Z)$, and

$$P = \bar{I}(\text{Re}(Z))I \geq 0 \quad (4-16)$$

The converse problem will be considered next. Given an arbitrary impedance matrix with complex numbers as elements, under what conditions can a passive network be constructed having this impedance matrix? It will be shown that if Z has a positive definite or positive semidefinite hermitian part, a passive network (i.e., one which has only positive R, L, C elements and gyrators) exists.

To show this, a method of synthesizing the actual network will be demonstrated. Suppose, that Z is given as Eq.(4-8):

$$Z = Z_H + Z_s$$

Methods for synthesizing Z_s , Z_H individually will be given. The final matrix Z is synthesized by connecting the networks of Z_s , Z_H .

Now

$$\mathbf{Z}_s = \text{Re}(\mathbf{Z}_s) + j\text{Im}(\mathbf{Z}_s)$$

and since $\mathbf{Z}_s$ is skew-hermitian, its real part, $\text{Re}(\mathbf{Z}_s)$ is skew-symmetric, and $\text{Im}(\mathbf{Z}_s)$ is symmetric.

The construction of a network corresponding to the $\text{Re}(\mathbf{Z}_s)$ will be discussed next. The general impedance matrix can be written as the sum of two matrices:

$$\mathbf{Z} = \mathbf{Z}_R + \mathbf{Z}_A \quad (4-17)$$

The matrix $\mathbf{Z}_R$ is symmetric (though not generally real), the matrix $\mathbf{Z}_A = -\mathbf{Z}_A'$ is skew-symmetric (real), and $\mathbf{Z}'$ is the transpose of $\mathbf{Z}$.

An important point with this process is that, if the mesh impedance matrix $\mathbf{Z}$ with all meshes exposed is written as the sum of its real and imaginary parts, then the imaginary part will be symmetric, and it is only the real part which must be broken down into its symmetric and skew-symmetric parts. Thus:

$$\begin{aligned} \mathbf{Z} &= \text{Re}(\mathbf{Z}) + j\text{Im}(\mathbf{Z}) \\ &= \{[\text{Re}(\mathbf{Z})]_R + j\text{Im}(\mathbf{Z})\} + [\text{Re}(\mathbf{Z})]_A \end{aligned} \quad (4-18)$$

The first bracket of Eq.(4-18) is $\mathbf{Z}_R$ of Eq.(4-17), the second bracket is $\mathbf{Z}_A$ of Eq.(4-17).

As an illustration of the synthesis process, consider the representation of the matrix for a 2-port network:

$$\mathbf{Z} = \begin{bmatrix} r_{11} + jx_{11} & r_{12} + jx_{12} \\ r_{21} + jx_{12} & r_{22} + jx_{22} \end{bmatrix}$$

The symmetric part of $\mathbf{Z}$ is:

$$\mathbf{Z}_R = \begin{bmatrix} r_{11} + jx_{11} & \frac{r_{12} + r_{21}}{2} + jx_{12} \\ \frac{r_{12} + r_{21}}{2} + jx_{12} & r_{22} + jx_{22} \end{bmatrix}$$

and the skew-symmetric part of $\mathbf{Z}$ is:

$$\mathbf{Z}_A = \begin{bmatrix} 0 & \frac{r_{12} - r_{21}}{2} \\ \frac{-r_{12} + r_{21}}{2} & 0 \end{bmatrix}$$

$\mathbf{Z}_R$ can be represented by an equivalent T-network, and the real skew-symmetric matrix $\mathbf{Z}_A$ is a gyrator with $a = \frac{(r_{21} - r_{12})}{2}$, which is the same for $\text{Re}(\mathbf{Z}_s)$. The number of gyrators M for a real skew-symmetric matrix of n rows and columns is in this representation $M = \frac{n(n-1)}{2}$.

Symmetric portion of $\mathbf{Z}$

Gyrator portion of $\mathbf{Z}$

The gyrator is defined from a circuit point of view in terms of the voltage-current relations which exist at its terminals, and the impedance matrix is $\mathbf{Z} = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$. The number a

is real and somewhat analogous to the turns ratio of a transformer.

The realization of the symmetric reactance matrix $Im(\mathbf{Z}_s)$ is also straightforward. It can be processed in several ways. These methods can be found in standard textbooks on circuit synthesis[46]. So far, we have solved for the synthesis of $\mathbf{Z}_s$.

The resolution of $\mathbf{Z}_H$ into real and imaginary parts as was done for $\mathbf{Z}_s$ does not lead to physical networks. The basis of the method to synthesize $\mathbf{Z}_H$ depends on first transforming the matrix into diagonal form with all elements off the main diagonal equal to zero[46]. This diagonal matrix is symbolized by $\tilde{\mathbf{Z}}_H$. The transformation is performed to keep the resulting matrix $\tilde{\mathbf{Z}}_H$ as hermitian matrix. This means that the diagonal elements are all real, and the transformed matrix can be synthesized as a group of resistors. The final step in the synthesis is to interconnect the network corresponding to the diagonal matrix so as to get a new network whose impedance matrix is the original $\mathbf{Z}_H$. The interconnection is specified by the form of the transformations used to obtain the diagonal form $\tilde{\mathbf{Z}}_H$. If $\mathbf{Z}_H$ is positive-definite or semidefinite, then the network for $\tilde{\mathbf{Z}}_H$ consists of a number of real positive resistors. The interconnection of these resistors, requiring only conventional network elements and gyrators, can be performed once the transformation is known.

The final network is obtained by interconnecting the network for $\mathbf{Z}_s$, $\mathbf{Z}_H$ to give a structure which realizes $\mathbf{Z}$. The following theory may, therefore, be stated as:

Theorem 4.2 *The necessary and sufficient condition for a linear, passive network, which posses an impedance matrix $\mathbf{Z}$, to be physical realizable as a lumped passive network is that the hermitian part of the impedance matrix be passive definite or semidefinite.*[46]

The statement of the definition of passivity at a point is given in terms of the impedance matrix $\mathbf{Z}(s)$. But if the admittance matrix $\mathbf{Y}(s)$ exists, or if the hybrid matrix $\mathbf{H}(s)$ exists (whether $\mathbf{Z}(s)$ exists or not), it may be substituted in the definition[45].

There is no special limitation on the circuit matrix of the general *LLF* networks besides that its element must be a rational function of the frequency variable s with real coefficients. So that the $n \times n$ circuit matrix characterizing a *LLF* n -port network must be a real matrix.

Definition: Positive-real matrix An $n \times n$ matrix function $\mathbf{A}(s)$ of the complex variable s is said to be a *positive-real matrix* if it satisfies the following three conditions:

1. $\mathbf{A}(s)$ is analytic in the open right half of the complex s -plane (*RHS*).
2. $\mathbf{A}^*(s) = \mathbf{A}(s^*)$ for all s in the open *RHS*, and sign $*$ is for the conjugate.
3. Its hermitian part $\mathbf{A}_h(s)$ is nonnegative definite for all s in the open *RHS*.

where a hermitian matrix $\mathbf{A}_h(s)$ is said to be nonnegative definite if its associated hermitian form $(\mathbf{X})^* \mathbf{A}_h \mathbf{X}$ is nonnegative; that is

$$(\mathbf{X})^* \mathbf{A}_h \mathbf{X} \geq 0$$

for all complex n -vector $\mathbf{x}$. [45]

To use the definition to test for positive realness of a matrix is usually complicated. The first two conditions are relatively easy to check. The third constraint is somewhat difficult to verify, because the nonnegative definiteness of the hermitian part must be investigated for all s in the open *RHS*. In the case where a matrix is rational, each element being

the ratio of two polynomials, an equivalent set of conditions can be stated that reduces the test for the points in the vast open *RHS* to the points on the boundary, the real-frequency axis($j\omega$ – axis).

Theorem 4.3 The necessary and sufficient conditions for a symmetric n th-order matrix of rational functions $[w_{ik}(s)]$ to be a positive real matrix are:

1. Each element of the matrix has real coefficients.
2. $[w_{ik}(s)]$ is analytic in the open right half of the complex s -plane (*RHS*).
3. If any element has a simple pole $s=j\omega_p$ (no multiple-order poles) on the j -axis, and K_{ik}^v is the residue of $[w_{ik}(s)]$ at this pole, the matrix of the residues at this pole,

$$\begin{bmatrix} k_{11}^v & k_{12}^v & \dots & k_{1n}^v \\ k_{21}^v & k_{22}^v & \dots & k_{2n}^v \\ \dots & \dots & \dots & \dots \\ k_{n1}^v & k_{n2}^v & \dots & k_{nn}^v \end{bmatrix}$$

is positive semidefinite.

4. The real part of the matrix for $s=j\omega$ must be nonnegative for all ω ; in other words, if for each i and k , $w_{ik}(j\omega)$ is separated into its rectangular components $w_{ik}(j\omega) = r_{ik}(\omega) + jx_{ik}(\omega)$, the matrix

$$\begin{bmatrix} r_{11}(\omega) & r_{12}(\omega) & \dots & r_{1n}(\omega) \\ r_{21}(\omega) & r_{22}(\omega) & \dots & r_{2n}(\omega) \\ \dots & \dots & \dots & \dots \\ r_{n1}(\omega) & r_{n2}(\omega) & \dots & r_{nn}(\omega) \end{bmatrix}$$

is nonnegative. [46]

The theorem applies for the $\mathbf{Z}$ (impedance matrix), the $\mathbf{Y}$ (admittance matrix), and the $\mathbf{H}$ (hybrid matrix). Although all the permissible characterization would eventually result in the same conclusion, the amount of computation involved may be quite different. The availability of so many different characterizations permits one to choose a particular

approach or a combination of approaches, so that the problem at hand can be solved in the simplest and the most satisfying manner.

4.3 Test of Nonnegative definite

Following the above discussion, to check passivity, we need to check if $Re(Z(j\omega))$ is nonnegative definite. A matrix $[a_{ik}]$ is nonnegative definite if and only if each principal minor is non-negative. We may choose the sequence of numbers in natural order; that is, we remove the first row and column from the determinant to form the first minor, then remove the first row and column of the first minor, and so on [46].

For example, the matrix $[A]$ of order 4 is nonnegative definite if and only if:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \geq 0 \quad \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} \geq 0 \quad \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \geq 0 \quad a_{44} \geq 0$$

As we know, evaluation of determinants is tedious. We need more simple methods to test the passivity.

Theorem 4.4. Let:

$$A = \begin{bmatrix} T & U \\ U' & W \end{bmatrix}$$

where T is of dimension $m \times m$, U of dimension $m \times n$, and W of dimension $n \times n$, U' is the transposition of U .

If T is symmetric positive definite and the Schur complement $W - U'T^{-1}U$ of T is positive definite, or if W is symmetric positive definite and the Schur complement $T - UW^{-1}U'$ of W is positive definite, then A is positive definite.

If T is symmetric nonnegative definite and the Schur complement $W - U'T^{-1}U$ of T is nonnegative definite, then A is nonnegative definite. [49]

Proof of the above is included in Appendix 4.

If we employ the above theorem to Z matrix and check if it is nonnegative definite for all s in the open *RHS*, that is ($Re(Z(j\omega)) \geq 0$), we need to test if and only if:

$$Re(Z_{11}(j\omega)) \geq 0 \quad (4-19)$$

$$Re(Z_{22}(j\omega)) - Re(Z_{21}(j\omega))[Re(Z_{11}(j\omega))]^{-1}Re(Z_{12}(j\omega)) \geq 0 \quad (4-20)$$

assuming that Z is analytic in the open *RHS*.

4.4 Passivity of Proposed Model

From Eq.(3-6), we know that diagonal *Padé* approximation of e^{-s} are of the form:

$$e^{-s} = \frac{A(-s)}{A(s)} = \frac{a_0 - a_1s + \dots + (-1)^n a_n s^n}{a_0 + a_1s + \dots + a_n s^n}$$

where $A(s)$ is a strict Hurwitz polynomial, which means that the roots of $A(s)$ are on the negative axis, or a pair of complex values with negative real parts. Because a strict Hurwitz polynomial can result only from the product of the following factors,

$$(s+a) \text{ for } a \text{ real and positive} \\ (s^2 + 2as + a^2 + b^2) \text{ for } b \text{ real, and } a \text{ real and positive}$$

The first represents a zero on the negative real axis. The second represents the pair of conjugate complex zeros: $s = -a \pm jb$.

Approximating the exponential matrix e^Z using *Padé* theory, we change Eq.(3-9) into

$$\begin{bmatrix} V(d, s) \\ -I(d, s) \end{bmatrix} = e^Z \begin{bmatrix} V(0, s) \\ I(0, s) \end{bmatrix} = (Q_{N, M}(Z)) [P^{-1}_{N, M}(Z)] \begin{bmatrix} V(0, s) \\ I(0, s) \end{bmatrix}$$

and we can rewrite diagonal *Padé* approximation as:

$$(Q_{N, N}(Z)) [P^{-1}_{N, N}(Z)] = \prod_{i=1}^{\frac{N}{2}} [(a_i I + Z)(a_i^* I + Z)] [(a_i I - Z)(a_i^* I - Z)]^{-1} \quad \text{for even value of } N \quad (4-21a)$$

$$(Q_{N, N}(Z)) [P^{-1}_{N, N}(Z)] = [a_0 I + Z] [a_0 I - Z]^{-1} \prod_{i=1}^{\frac{N-1}{2}} [(a_i I + Z)(a_i^* I + Z)] [(a_i I - Z)(a_i^* I - Z)]^{-1} \quad \text{for odd value of } N \quad (4-21b)$$

where $a_i = x_i + jy_i$, and x_i are positive constants. * is the sign for conjugate, I is unit matrix. If there is only one factor $a_0 I + Z$, a_0 is a real positive constant.

So that, we decompose the network into a series cascaded small sections, and each of them can be described by,

$$[(a_i I + Z)(a_i^* I + Z)][(a_i I - Z)(a_i^* I - Z)]^{-1}$$

and

$$[a_0 I + Z][a_0 I - Z]^{-1}$$

We know that the series or parallel connections of two passive networks are again passive. If we can prove each small section is a passive network, then the whole system is a passive network. The passivity question is simplified to test the passivity of these second-order and first-order systems.

For any second-order section, it is written as:

$$Q(Z)(P^{-1}(Z)) = (a_i I + Z)(a_i^* I + Z)[(a_i I - Z)(a_i^* I - Z)]^{-1} \quad (4-22)$$

and the input-output relation of the network is:

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} V(d, s) \\ -I(d, s) \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} V(0, s) \\ I(0, s) \end{bmatrix} \quad (4-23)$$

which equals to

$$[(x_i^2 + y_i^2)I - 2Zx_i I + Z^2] \begin{bmatrix} V(d, s) \\ -I(d, s) \end{bmatrix} = [(x_i^2 + y_i^2)I + 2Zx_i I + Z^2] \begin{bmatrix} V(0, s) \\ I(0, s) \end{bmatrix}$$

With

$$\begin{aligned} Z^2 &= \begin{bmatrix} \mathbf{0} & -\mathbf{R} - s\mathbf{L} \\ -\mathbf{G} - s\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{R} - s\mathbf{L} \\ -\mathbf{G} - s\mathbf{C} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{R} + s\mathbf{L})(\mathbf{G} + s\mathbf{C}) & \mathbf{0} \\ \mathbf{0} & (\mathbf{G} + s\mathbf{C})(\mathbf{R} + s\mathbf{L}) \end{bmatrix} \end{aligned}$$

and

$$2\mathbf{Z}x_i\mathbf{I} = \begin{bmatrix} \mathbf{0} & -\mathbf{R} - s\mathbf{L} \\ -\mathbf{G} - s\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} 2x_i\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 2x_i\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -2x_i\mathbf{I}(\mathbf{R} + s\mathbf{L}) \\ -2x_i\mathbf{I}(\mathbf{G} + s\mathbf{C}) & \mathbf{0} \end{bmatrix}$$

We have:

$$\begin{aligned} &\begin{bmatrix} (x_i^2 + y_i^2)\mathbf{I} + (\mathbf{R} + s\mathbf{L})(\mathbf{G} + s\mathbf{C}) & 2x_i\mathbf{I}(\mathbf{R} + s\mathbf{L}) \\ 2x_i\mathbf{I}(\mathbf{G} + s\mathbf{C}) & (x_i^2 + y_i^2)\mathbf{I} + (\mathbf{G} + s\mathbf{C})(\mathbf{R} + s\mathbf{L}) \end{bmatrix} \begin{bmatrix} \mathbf{V}(d, s) \\ -\mathbf{I}(d, s) \end{bmatrix} \\ &= \begin{bmatrix} (x_i^2 + y_i^2)\mathbf{I} + (\mathbf{R} + s\mathbf{L})(\mathbf{G} + s\mathbf{C}) & (-2)x_i\mathbf{I}(\mathbf{R} + s\mathbf{L}) \\ (-2)x_i\mathbf{I}(\mathbf{G} + s\mathbf{C}) & (x_i^2 + y_i^2)\mathbf{I} + (\mathbf{G} + s\mathbf{C})(\mathbf{R} + s\mathbf{L}) \end{bmatrix} \begin{bmatrix} \mathbf{V}(0, s) \\ \mathbf{I}(0, s) \end{bmatrix} \end{aligned}$$

and the $\mathbf{Y}$ matrix is:

$$\begin{aligned} Y_{11} &= \frac{-\mathbf{Q}_{12}^{-1}\mathbf{Q}_{11} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}}{2}, & Y_{12} &= \frac{\mathbf{Q}_{12}^{-1}\mathbf{Q}_{11} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}}{2} \\ Y_{21} &= \frac{\mathbf{Q}_{12}^{-1}\mathbf{Q}_{11} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}}{2}, & Y_{22} &= \frac{-\mathbf{Q}_{12}^{-1}\mathbf{Q}_{11} - \mathbf{Q}_{22}^{-1}\mathbf{Q}_{21}}{2} \end{aligned}$$

So that, for the second-order section,

$$Y_{11} = \frac{[2x_i\mathbf{I}(\mathbf{R} + s\mathbf{L})]^{-1}[(x_i^2 + y_i^2)\mathbf{I} + (\mathbf{R} + s\mathbf{L})(\mathbf{G} + s\mathbf{C})]}{2} +$$

$$\frac{[(x_i^2 + y_i^2)I + (G + sC)(R + sL)]^{-1}(2x_i I(G + sC))}{2}$$

$$= m_2 + m_1$$

and

$$Y_{12} = (-1) \frac{[2x_i I(R + sL)]^{-1}[(x_i^2 + y_i^2)I + (R + sL)(G + sC)]}{2} +$$

$$\frac{[(x_i^2 + y_i^2)I + (G + sC)(R + sL)]^{-1}(2x_i I(G + sC))}{2}$$

$$= -m_2 + m_1$$

then, Y matrix of the second-order section can be written as:

$$Y = \begin{bmatrix} m_1 & m_1 \\ m_1 & m_1 \end{bmatrix} + \begin{bmatrix} m_2 & -m_2 \\ -m_2 & m_2 \end{bmatrix} \quad (4-24)$$

For any first-order section, it is written as

$$Q(Z)P^{-1}(Z) = (a_i I + Z)(a_i I - Z)^{-1} \quad (4-25)$$

and the input-output relation of the network is

$$\begin{bmatrix} a_i I & R + sL \\ G + sC & a_i I \end{bmatrix} \begin{bmatrix} V(d, s) \\ -I(d, s) \end{bmatrix} = \begin{bmatrix} a_i I & -(R + sL) \\ -(G + sC) & a_i I \end{bmatrix} \begin{bmatrix} V(0, s) \\ I(0, s) \end{bmatrix}$$

The Y parameters of this section are:

$$Y_{11} = \frac{-Q_{12}^{-1}Q_{11} - Q_{22}^{-1}Q_{21}}{2} = \frac{(R + sL)^{-1}a_i I + (a_i I)^{-1}(G + sC)}{2} = n_1 + n_2$$

$$Y_{12} = \frac{Q_{12}^{-1}Q_{11} - Q_{22}^{-1}Q_{21}}{2} = \frac{-(R + sL)^{-1}(a_i I) + (a_i I)^{-1}(G + sC)}{2} = -n_1 + n_2$$

So that, Y matrix of the 1st order section can be written as:

$$Y = \begin{bmatrix} n_1 & -n_1 \\ -n_1 & n_1 \end{bmatrix} + \begin{bmatrix} n_2 & n_2 \\ n_2 & n_2 \end{bmatrix} \quad (4-26)$$

There are several important operations on nonnegative definite matrices that keep its nonnegative definite character invariant; that is, they lead to new nonnegative definite matrices.

Let W and V are n th-order nonnegative definite matrices whose elements are rational functions. Then the operations leading to new nonnegative definite matrices are [46]:

1. *Addition*; that is, $W+V$ is a nonnegative definite matrix.

2 *Forming the inverse, if it exists*; Thus, if W is nonsingular, W^{-1} is a nonnegative definite matrix.

3. If $W_H > 0$ for all s in $\sigma > 0$, then W^{-1} exists. W_H is the hermitian form of W , and $s = \sigma + j\omega$.

Note that R , L , C , and G are positive definite matrices as shown from energy considerations[8],and from the similarity transformation on the multiplication of them, we can prove that the multiplication result is also a positive definite matrix.

The passivity proof is as following,

$$m_1 = \frac{[(x_i^2 + y_i^2)I + (G + sC)(R + sL)]^{-1} (2x_i I (G + sC))}{2}$$

If we can prove m_1^{-1} is positive real matrix, then m_1 is also positive-real matrix.

$$\begin{aligned}
\text{Because } \frac{\mathbf{m}_1^{-1}}{2} &= (2x_i \mathbf{I}(\mathbf{G} + s\mathbf{C}))^{-1} [(x_i^2 + y_i^2)\mathbf{I} + (\mathbf{G} + s\mathbf{C})(\mathbf{R} + s\mathbf{L})] \\
&= (2x_i \mathbf{I}(\mathbf{G} + s\mathbf{C}))^{-1} (x_i^2 + y_i^2)\mathbf{I} + (2x_i \mathbf{I}(\mathbf{G} + s\mathbf{C}))^{-1} (\mathbf{G} + s\mathbf{C})(\mathbf{R} + s\mathbf{L}) \\
&= (2x_i \mathbf{I}(\mathbf{G} + s\mathbf{C}))^{-1} (x_i^2 + y_i^2)\mathbf{I} + (2x_i \mathbf{I})^{-1} (\mathbf{R} + s\mathbf{L})
\end{aligned}$$

It is two positive real matrices added together, so that $\mathbf{m}_1^{-1}$ is positive real matrix, and then $\mathbf{m}_1$ is also a positive real matrix.

$$\begin{aligned}
\mathbf{m}_2 &= \frac{[2x_i \mathbf{I}(\mathbf{R} + s\mathbf{L})]^{-1} [(x_i^2 + y_i^2)\mathbf{I} + (\mathbf{R} + s\mathbf{L})(\mathbf{G} + s\mathbf{C})]}{2} \\
&= \frac{[2x_i \mathbf{I}(\mathbf{R} + s\mathbf{L})]^{-1} (x_i^2 + y_i^2)\mathbf{I} + [2x_i \mathbf{I}(\mathbf{R} + s\mathbf{L})]^{-1} (\mathbf{R} + s\mathbf{L})(\mathbf{G} + s\mathbf{C})}{2} \\
&= \frac{[2x_i \mathbf{I}(\mathbf{R} + s\mathbf{L})]^{-1} (x_i^2 + y_i^2)\mathbf{I} + 2x_i \mathbf{I}(\mathbf{G} + s\mathbf{C})}{2}
\end{aligned}$$

It is two positive real matrices added together, so that $\mathbf{m}_2$ is positive real matrix.

$$\mathbf{n}_1 = \frac{(\mathbf{R} + s\mathbf{L})^{-1} a_i \mathbf{I}}{2}, \text{ and } \mathbf{n}_2 = \frac{(a_i \mathbf{I})^{-1} (\mathbf{G} + s\mathbf{C})}{2}.$$

According to the above lemma, $\mathbf{n}_1, \mathbf{n}_2$ are also positive real matrices.

We have proven that the proposed model is analytic in the open *RHS*. If we employ theorem (4.4) to check $\mathbf{Y}$ matrix if it is nonnegative definite in the open *RHS*, that is ($\text{Re}(\mathbf{Y}(j\omega)) \geq 0$), we need to test if and only if:

$$\text{Re}(\mathbf{Y}_{11}(j\omega)) \geq 0$$

$$Re(Y_{22}(j\omega)) - Re(Y_{21}(j\omega))[Re(Y_{11}(j\omega))]^{-1}Re(Y_{12}(j\omega)) \geq 0$$

From above, we know for the 1st order section, Y matrix can be written as

$$Y = \begin{bmatrix} n_1 & -n_1 \\ -n_1 & n_1 \end{bmatrix} + \begin{bmatrix} n_2 & n_2 \\ n_2 & n_2 \end{bmatrix}$$

and second order section, Y matrix can be written as

$$Y = \begin{bmatrix} m_1 & m_1 \\ m_1 & m_1 \end{bmatrix} + \begin{bmatrix} m_2 & -m_2 \\ -m_2 & m_2 \end{bmatrix}$$

When $s = j\omega$, $Re(m_1), Re(m_2), Re(n_1)$ and $Re(n_2)$ are all ≥ 0 , and

$$Re(n_1(j\omega)) - Re((-1)n_1(j\omega))[Re(n_1(j\omega))]^{-1}Re((-1)n_1(j\omega)) = 0$$

$$Re(n_2(j\omega)) - Re((-1)n_2(j\omega))[Re(n_2(j\omega))]^{-1}Re((-1)n_2(j\omega)) = 0$$

$$Re(m_1(j\omega)) - Re((-1)m_1(j\omega))[Re(m_1(j\omega))]^{-1}Re((-1)m_1(j\omega)) = 0$$

$$Re(m_2(j\omega)) - Re((-1)m_2(j\omega))[Re(m_2(j\omega))]^{-1}Re((-1)m_2(j\omega)) = 0$$

We find that Y matrices of first and second order sections are made up by two positive real matrices, and they are all passive networks. The high order approximation of the exponential matrix e^Z using *Padé* theory can be expressed by several multiplied second order and first order functions. The networks described by these first order and second order functions are proved to be passive. Then, we can conclude that the *Padé* approximation of the exponential matrix e^Z is a passive model.

Chapter 5 Transmission Line Macromodel

Complex high-speed electronic circuits may contain thousands or tens of thousands of on-chip and on-board interconnections that are, in effect, microstrip transmission lines terminated with nonlinear active devices. To apply circuit methods of analysis and simulation to an electronic circuit, a circuit model(i.e., the model relating termination voltages and currents) has to be available for every device in the circuit.

Thus, the circuit models for a transmission line are of the most practical importance. Combined with an efficient transient simulation method, the device models can form the basis of an efficient and practically applicable technique for the transient simulation of transmission line networks.

In this chapter, time-domain implementation of the proposed model is discussed, and numerical examples are presented to demonstrate the validity of the proposed model and to illustrate its application to a variety of interconnect structures. The simulation results are compared with those obtained by other simulation methods, and proved to be reliable.

5.1 Model Implementation

Before simulating a circuit, we must formulate a mathematical equation for it. Three practical methods for formulating network equations have been developed: 1) state variables; 2) tableau; 3) modified nodal. Each method has its advantages and disadvantages with regard to ease of formulation, their use in theoretical studies, solution methods, and applications.

The oldest method for computer applications is the state variable method [52]. It reduces the system of equations to the smallest possible set of first-order differential equations, and gets a system of first-order ordinary differential equations(ODE's). Many results in mathematics are developed specially for ODE's, and their properties are well understood.

The tableau method [50] is in some way the opposite of the state variables. It preserves all equations of the network including the topological ones. This increases the size of the system matrix, but with one advantage: the equations are very sparse.

Modified nodal formulation [51], is now widely used in practically all CAD programs. It is obtained from the tableau by eliminating many branch currents and all branch voltages, since these can be easily found from known nodal voltages. The system matrix is much smaller than the tableau matrix, and sparsity is preserved.

The physical meaning of system state $x(t)$ is used to describe completely the energy distribution of the system at time t , so that the number of system states also indicates the number of the system's independent energy storage devices, such as inductors, capacitors in electrical circuit systems.

In large-scale systems, with many inputs and outputs, the state-variable approach can have distinct advantages over conventional methods. A feature of the state-variable method is that it decomposes a complex system into a set of smaller systems that can be normalized to have a minimum interaction and that can be solved individually [53].

The *Laplace* transform is a convenient method of calculating the fundamental matrix of a system whose differential equations are in normal form. Consider the equations:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (5-1)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are constant matrices. Taking the *Laplace* transform of both sides of Eq.(5-1), we obtain:

$$s\mathbf{X}(s) - \mathbf{X}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \quad (5-2)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \quad (5-3)$$

where:

$$\mathbf{X}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \dots \\ X_k(s) \end{bmatrix}$$

The definition of $\mathbf{U}(s)$ and $\mathbf{Y}(s)$ are similar, and $\mathbf{X}(s)$, $\mathbf{U}(s)$, and $\mathbf{Y}(s)$ are the *Laplace* transform of $\mathbf{x}(t)$, $\mathbf{U}(t)$, and $\mathbf{y}(t)$, respectively.

From Eq.(5-2):

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{X}(0) + \mathbf{B}\mathbf{U}(s)$$

where $\mathbf{I}$ is the $k \times k$ unit matrix. Hence:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{X}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \quad (5-4)$$

Substitution of Eq.(5-4) into Eq.(5-3) results in:

$$Y(s) = C(sI - A)^{-1}X(0) + [C(sI - A)^{-1}B + D]U(s) \quad (5-5)$$

With zero initial conditions $X(0)=0$, this becomes:

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) = H(s)U(s) \quad (5-6)$$

where:

$$H(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \dots & H_{1n}(s) \\ H_{21}(s) & H_{22}(s) & \dots & H_{2n}(s) \\ \dots & \dots & \dots & \dots \\ H_{m1}(s) & H_{m2}(s) & \dots & H_{mn}(s) \end{bmatrix} \quad (5-7)$$

Thus the i th component $Y_i(s)$ of the transform $Y(s)$ of the output vector may be written as:

$$Y_i(s) = H_{i1}U_1(s) + H_{i2}U_2(s) + \dots + H_{in}U_n(s)$$

Hence it is apparent that $H_{ij}(s)$, which is the ij th element of

$H(s) = C(sI - A)^{-1}B + D$ is the transfer function between $u_j(t)$ and $y_i(t)$.

We have then found a general expression for the transfer functions of an arbitrary lumped system in terms of the matrices appearing in the normal form characterization Eq.(5-1).

Many techniques are available for obtaining state-space representations of transfer function systems. Consider a system defined by:

$$y^n + a_1 y^{n-1} + \dots + a_{n-1} y^1 + a_n y = b_0 u^n + b_1 u^{n-1} + \dots + b_{n-1} u^1 + b_n u \quad (5-8)$$

where u is the input and y is the output. This equation can also be written as:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n} \quad (5-9)$$

In what following we will present state-space representation of the system defined by the above equation in diagonal canonical form.

Consider the transfer function system defined by Eq. (5-8), and the case where the denominator polynomial involves only distinct roots, the Eq.(5-9) can be written as:

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{(s + p_1)(s + p_2) \dots (s + p_n)} \\ &= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \dots + \frac{c_n}{s + p_n} \end{aligned} \quad (5-10)$$

The diagonal canonical form of the state-space representation of this system is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & \dots & \dots & 0 \\ 0 & -p_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix} u \quad (5-11a)$$

$$y = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} + b_0 u \quad (5-11b)$$

If the denominator polynomial involves multiple roots, the preceding diagonal canonical form must be modified into the Jordan canonical form. Suppose the transfer function is:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{(s + p_1)^2 (s + p_2) \dots (s + p_n)}$$

The partial fraction expansion of this system becomes:

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s + p_1)^2} + \frac{c_2}{s + p_1} + \frac{c_3}{s + p_2} + \dots + \frac{c_n}{s + p_n}$$

A state-space representation of this system in the Jordan canonical form is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & \dots & 0 \\ 0 & -p_1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \dots \\ 1 \end{bmatrix} u \quad (5-12a)$$

$$y = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} + b_0 u \quad (5-12b)$$

The block-diagram representation of the system can be turned into hardware only if all the poles $p_1, p_2, \dots, p_n$ are real. If they are complex, the feedback gains and the gains corresponding to the residues are complex. The representation must be considered as being purely conceptual: valid for theoretical studies, but not physically realizable. If a physically realizable representation is desired, it is possible to combine a pair of complex poles and residues into a single second-order transfer function with real coefficients. The resulting second-order transfer function of the subsystem is then realized in one of the companion forms.

Suppose, for example, that p_1, p_2 are a complex conjugate pair. For a transfer function having real coefficients (as it must in a real system), the residues at a pair of complex conjugate poles must be themselves complex conjugates, as λ_1, λ_2 .

Hence in the state-space realization of the transfer function, we have the following equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_1^* \end{bmatrix} u \quad (5-13a)$$

$$y = \begin{bmatrix} \lambda_1 & \lambda_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (5-13b)$$

where A_1 is the Jordan block associated with p_1 , and A_1^* is the complex conjugate of A_1 .

Introduce the equivalence transformation $\bar{x} = P x$, where

$$P = \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{2} \begin{bmatrix} I & -iI \\ I & iI \end{bmatrix} \quad (i = \sqrt{-1})$$

Then it can be verified that the Eq. (5-13) can be transformed into:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} Re(A_1) & Im(A_1) \\ -Im(A_1) & Re(A_1) \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 2Re(b_1) \\ -2Im(b_1) \end{bmatrix} u \quad (5-14)$$

$$y = \begin{bmatrix} Re(\lambda_1) & Im(\lambda_1) \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

where $Re(A)$ and $Im(A)$ denote the real and the imaginary parts of A , respectively. Since all the coefficients in the above equation are real, this equation can be simulated in an analog computer. With the above formulation, we write the circuit equations for the proposed transmission line model into a state-space formula, and calculate the time-domain responses.

5.2 Computational Results

Three examples are presented in this section to demonstrate the validity and efficiency of the proposed model. The network configurations and simulation results are shown in the following figures

Fig.5.2 Frequency Response for example 1.

Fig. 5.3. Frequency responses for example 1

Example 2: In this example, a circuit with a coupled distributed line is simulated using the proposed method (Fig. 5.4).

Fig.5.4. Circuit for example 2

Fig. 5.5. Frequency responses for example 2

Comparison of the frequency responses from both the proposed method and the conventional discretization (up to 8GHz) is given in Fig.5.5.

The proposed method required an approximation of order (16,16) to match the exact solutions accurately. However, as seen from the graph, conventional lumped approximation using 16 sections deviates significantly from the exact solution. Also, the waveform

from conventional approximation suffers from spurious oscillations. It was found during the experiment that a discretization with 70 lumped sections is required to match the exact solution for this example.

From frequency-domain results, we can conclude that the new model has a good accuracy, with less approximation order. Because the proposed model is passive, we can directly change the frequency-domain model to time-domain and still keep the stability of the model. The following is transient simulation of a interconnect network.

Example 3: In Fig.5.6, a small network is shown. Two transmission lines(T_1, T_2) are cascaded. The parameters of T_1 are $Rd = 25\Omega$, $Gd = 0.005\Omega^{-1}$, $Ld = 0.1\mu H$, $Cd = 40pF$, where d is the length of the transmission line. The parameters of T_2 are the same except for $Rd = 3.125\Omega$. A 1V step signal of 0.5 ns rise time is placed at the input, and the output waveforms are compared with result from *CFH* method in Fig.5.8. As we explain in chapter 3, we approximate the circuit up to the desired frequency, $\omega_{max} \approx \frac{2\pi}{t_r}$, that is $f_{max} \approx \frac{1}{t_r} \approx 2\text{ GHz}$, and the frequency response is in Fig. 5.7.

Fig.5.6 Example Circuit

Fig.5.7 The output frequency response of the example circuit

Fig.5.8 Transient response of above circuit

Chapter 6. Summary and Conclusions

In this thesis, the problem of transmission line modeling and simulation was analyzed. Existing transmission line simulation techniques as well as models were reviewed and compared. A new macromodel for transmission line was proposed for transient simulation of distributed networks. The stability and passivity of the new model were proven, and numerical examples were presented to demonstrate the validity of the model.

The material presented in this work consists of three parts: introduction of transmission line modeling and simulation, presentation of a new macromodel for the transmission line, and macromodel implementation.

The new model was presented in Chapter 3. The idea for this new model is based on *Padé* approximation. Reciprocal Y matrices of single and coupled lines were derived from diagonal *Padé* approximation, and coefficients of the approximation were computed using closed-loop *Padé* approximates of exponential matrix. The algorithm also provided an error criterion for automatically selecting the order of the approximation for the frequency-bandwidth of interest. A proof of the stability of the approximated model was presented. The transmission line was now viewed as an element in the circuit simulator, without the mask of its distributed form.

Chapter 4 was concerned with passivity issue. It is always possible to come up with stable reduced order macromodels by utilizing a number of heuristics, but it is not guaranteed to obtain provably passive approximations. In the absence of such a guarantee the simulation and analysis of the circuit may become problematic. A passive reduce-order model eliminates this concern.

The diagonal *Padé* approximation retains the all-pass nature of e^{-s} , and all regular rational all-pass functions can be divided into first and second order functions [36]. Physically states, the series or parallel connections of two passive networks are also a passive network.

The new macromodel was decomposed into a series cascaded sections, and the passivity question was simplified to test the passivity of these second-order and first-order systems. After each first and second order section was proven to be passive, it was concluded that the new model is a guaranteed passive approximation.

In chapter 5, the macromodel implementation was discussed. The circuit equations of the proposed model were written into state-space formulae for time domain simulation. Numerical examples were presented to demonstrate the validity of the proposed model and to illustrate its application to a variety of interconnect structures. Simulation results show that the new method offers a more efficient and accurate means to discretize transmission lines compared to the conventional lumped discretization.

The research work in this thesis was published in [54]-[55], and the result of further study on transmission line with frequency-dependent parameters was published in [56].

Appendix 1

Risetime and Frequency Content of a Transition

The formula $f = 0.35/T_r$ underscores the inverse relationship between the risetime T_r of a transition and the highest frequency content f of the transition. The risetime is the waveform transition from LOW to HIGH, typically 10% to 90%, or 20% to 80% as specified. We intuitively feel that the sharper the edge is, the more Fourier component it contains. It is not obvious, however, whether the relationship is a simple inverse and even if it is, what the proportionality constant is. The constant 0.35 we adopt is approximate; numbers in the range 0.3 to 0.4 have been variously used elsewhere.

One justification of the formula lies in amplifier design. When a unit step is passed through a filter with Gaussian response, the output signal has degraded risetime and is shown below to be related to the Gaussian curve half-width by the formula. As shown in Fig. A1.1, suppose the amplifier transfer function has the magnitude

$$|H(\omega)| = e^{-\omega^2 T^2 / 2} \quad (1)$$

where $1/T$ is the standard deviation point on the curve. The bandwidth at half-power is given by

$$|H(\omega)| = e^{-(\omega_h T)^2} = \frac{1}{2}, \quad \omega_h = \frac{0.8326}{T} \quad (2)$$

The impulse response of a Gaussian filter is also Gaussian, a well-known result in Fourier transform theory:

$$h(t) = F^{-1}[H(\omega)] = \frac{1}{T\sqrt{2\pi}} e^{-t^2/(2T^2)} \quad (3)$$

The unit step response is the integral of the impulse response:

$$g(t) = \int_{-\infty}^t h(t)dt = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{t}{2T} \right) \right] \quad (4)$$

where erf is the error function. The 10% to 90% risetime of the error function is numerically found to be $T_r = 2.563T$. Hence

$$T_r \frac{\omega_h}{2\pi} = 2.563T \frac{0.8326}{2\pi T} = 0.34 \quad (5)$$

and can be taken as the basis for the inverse relationship between the risetime and the highest frequency content of a signal transition. The more conservative formula is

$$\omega_{max} = \frac{2\pi}{T_r}$$

Fig A1.1 The response of a unit step through a Gaussian filter is an error function.

The above result has been derived for a very specific signal, that of the step response of a Gaussian amplifier, but it is widely applied to general signal transitions. In any such applications the formula should be viewed as a rule of thumb.

Appendix 2

Maximum and Minimum Modulus Theorems and Extensions

A function $f(z)$ which is one-valued and differentiable at every point of z domains is said to be analytic in the domain z .

If $f(z)$ is analytic throughout the domain D and z_0 lies in D , then:

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \quad (1)$$

if C is any closed path lying entirely within D and z is in the interior of C . This is known as the *Cauchy integral formula*.

The *Cauchy integral formula* shows us that an analytic function is defined throughout the interior of a region by its values on the boundary of that region. Since the real and imaginary parts of an analytic function are conjugate harmonic functions, they will be determined throughout the interior by their values on the boundary and they will be inter-related by the *Cauchy-Riemann differential* equations, which further restrict the totality of information required to define the function in the interior. The consequences of these facts are of great importance in applications of complex variable theory.

Maximum and minimum modulus theorems: If $f(z)$ is analytic in a domain D , and a closed region R , with boundary curve C , lies entirely within D , then the maximum value of $|f(z)|$ in R will be attained on its boundary. Further, if the maximum is attained at any interior point, the $f(z)$ is a constant.

Proof: Suppose there is an interior point z_0 (Fig. A2.1) at which $f(z)$ attains its maxi-

imum value M , that is, $|f(z_0)| = M$. Taking z_0 as center, draw a circle about it lying entirely interior to C , calling its radius r_0 . If $m(r_0)$ is the maximum value of $f(z)$ on the periphery of the circle, integrating around the circle we have:

$$\left| \int_c f(z) ds \right| \leq 2\pi r_0 m(r_0) \quad (2)$$

Fig. A2.1

From the mean value theorem,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_c \frac{f(z) r_0 e^{i\phi}}{r_0 e^{i\phi}} d\phi = \frac{1}{2\pi r_0} \int_c f(z) r_0 d\phi \\ &= \frac{1}{2\pi r_0} \int_c f(z) ds \end{aligned} \quad (3)$$

We conclude that:

$$|f(z_0)| = \frac{1}{2\pi r_0} \left| \int_c f(z) ds \right| \leq m(r_0) \quad (4)$$

This is consistent with the assumption that $|f(z)|=M$, the maximum of $f(z)$ throughout R , only if

$$|f(z_0)| = M = m(r_0) \quad (5)$$

This is to say that $f(z)$ must be a constant through the region R , or else that it cannot achieve its maximum value in the interior of R , which proves the theorem.

The minimum modulus theorem is more restricted, since it is possible for the modulus to go to zero in the interior of an analytic region. If, however, we restrict our function so that $|f(z)|>0$ in the interior of D , then the minimum value of $|f(z)|$ in any closed region interior to D will be attained on the boundary of that region. This is readily proved by introducing the function $g(z)=1/f(z)$ and noting that $|g(z)|$ attains its maximum on the boundary by the maximum modulus theorem.

These results for the maximum and minimum values for the modulus of an analytic function permit us to prove an important theorem for harmonic functions. A *Harmonic function* is a two-variable function which has two-order partial differential and meets *Laplace equation*.

Theorem: If $u(x,y)$ is a harmonic function which is finite in and on the closed path C , then $u(x,y)$ will take its maximum and minimum values on C and never in the interior of C unless it is a constant.

To prove this, construct the function:

$$F(z) = e^{u+iv} = e^u e^{iv} \quad (6)$$

where v is the conjugate harmonic function to u . The function $F(z)$ will be an analytic function which will take on its maximum absolute value on the boundary unless it is a constant. Furthermore

$$|F(z)| = e^u \quad (7)$$

does not vanish in or on C , so the conditions for the minimum value theorem are satisfied for $F(z)$, and we can conclude that e^u , and hence u , will take on its minimum value on C (note that u can be negative) unless u is a constant.

Theorem: Any real and image part of analytic function meets Laplace equation, and has two -order partial differentials, so that the real and image part of analytic function are harmonic functions.

Proof: Because an analytic function $f(z) = u + iv$ satisfies the *Cauchy-Riemann* relations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $z=x+jy$, so that the real part u and image part v meet the *Laplace* differential equations,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Another thing is that a differential of an analytic function is still an analytic function, so that the real and image parts of an analytic function have any order of continuously partial differential.

This clearly means that both u , and v are so-called harmonic functions. Thus, we can apply *Maximum and Minimum Modulus Theorems* to any real and image parts of analytic functions.

Appendix 3

Proof of Theorem 3.1

Theorem. If A , B , and C are symmetric non-negative definite, and either A or C positive definite, then

$$|\lambda^2 A + 2\lambda B + C| = 0 \quad (1)$$

has no roots with positive real parts.

If A and C are symmetric non-negative definite and B is symmetric positive definite, then the only root with zero real part is $\lambda = 0$

Proof: Let us give a proof which makes use of the physical background of the statement. In this case, we shall use energy considerations.

Consider the differential equation

$$Ax'' + 2Bx' + Cx = 0 \quad x(0) = c^1 \quad x'(0) = c^2 \quad (2)$$

which, if considered to arise from the study of electric circuits possessing capacitances, inductances, and resistances, is such that A, B , and C are non-negative definite.

Let us write

$$(x', Ax'') + 2(x', Bx') + (x', Cx) = 0 \quad (3)$$

Thus, for any $s > 0$,

$$\int_0^s [(x', Ax'') + 2(x', Bx') + (x', Cx)] dt = 0 \quad (4)$$

or

$$(x', Ax')|_0^s + 4 \int_0^s (x', Bx') dt + (x, Cx)|_0^s = 0 \quad (5)$$

This is equivalent to the equation

$$(x'(s), Ax'(s)) + 4 \int_0^s (x', Bx') dt + (x(s), Cx(s)) = c_3 \quad (6)$$

where $c_3 = (c^2, Ac^2) + (c^1, Cc^1)$.

From equations (4) to (5), we use the equation $\int u dv = uv - \int v du$, and because A, C are symmetric, so that $(Ax, x') = (x, Ax')$, and $(Cx, x') = (x, Cx')$, where x is vector, and x' is the differential of x .

If λ is the root of (1), then (2) has a solution of the form $e^{\lambda t} c$. If λ is real, c is real.

If λ is complex, $\lambda = r_1 + ir_2$, then the real part of $e^{\lambda t} c$, which has the form $e^{r_1 t} (a^1 \cos(r_2 t) + a^2 \sin(r_2 t))$, is also a solution. Substituting in (6), we see that

$$e^{2r_1 s} (b^1, Ab^1) + 4 \int_0^s e^{2r_1 t} (b^2(t), Bb^2(t)) dt + e^{2r_1 s} (b^3, Cb^3) = c_3 \quad (7)$$

where b^1 and b^3 are constant vectors, and b^2 is a variable vector given by

$$(a^1 r_1 + a^2 r_2) \cos(r_2 t) + (a^2 r_1 - a^1 r_2) \sin(r_2 t)$$

If A or C is positive definite, with $B \geq 0$, we see that $r_1 > 0$ leads to a contradiction as $s \rightarrow \infty$.

If $A, C \geq 0$, then B positive definite requires that $r_1 \leq 0$. Furthermore, since

$a^1 \cos(r_2 t) + a^2 \sin(r_2 t)$ is periodic, the integral $\int_0^s e^{2r_1 t} (b^2(t), Bb^2(t)) dt$ diverges to

plus infinity as $s \rightarrow \infty$, unless $r_2 = 0$, if $r_1 = 0$.

Another proof is as follows,

Theorem. *If A , B , and C are positive definite, then the roots of*

$$|\lambda^2 A + 2\lambda B + C| = 0 \quad (8)$$

has negative real parts.

Proof: Equation (8) is the determinant of matrix polynomial

$$L(\lambda) = A\lambda^2 + B\lambda + C,$$

which is governed by the system of equations

$$L\left(\frac{d}{dt}\right)x = A\frac{d^2 x}{dt^2} + B\frac{dx}{dt} + Cx = f \quad (9)$$

Let

$$(L(\lambda)f, f) = \lambda^2(Af, f) + \lambda(Bf, f) + (Cf, f) = 0 \quad (10)$$

for some λ and $f \neq 0$. Then

$$\lambda = \frac{-(Bf, f) \pm \sqrt{(Bf, f)^2 - 4(Af, f)(Cf, f)}}{2(Af, f)} \quad (11)$$

and since $(Af, f) > 0$, $(Bf, f) > 0$, and $(Cf, f) > 0$, the real part of λ is negative. This fact reflects dissipation of energy in damped oscillatory systems.

These argument can be extended to treat similar questions for equations in which the coefficients A, B , and C are variable.

Appendix 4

Nonnegative Definiteness of Schur Complement

Sufficient conditions for the positive definiteness or positive semidefiniteness of a Schur complement are given by the following theorem.

Theorem 1. Let

$$A = \begin{bmatrix} T & U \\ V & W \end{bmatrix}$$

where T is of dimensions $m \times m$, U of dimensions $m \times n$, V of dimensions $n \times m$, and W of dimensions $n \times n$.

If A is positive definite, then the Schur complement $W - VT^{-1}U$ of T and the Schur complement $T - UW^{-1}V$ of W are positive definite.

Proof: Define:

$$P = \begin{bmatrix} I_m & -T^{-1}U \\ 0 & I_n \end{bmatrix} \quad (1)$$

where I_m , and I_n are unitary matrices, and P is nonsingular. Further,

$$P'AP = \begin{bmatrix} T & (I - TT^{-1})U \\ V - (T^{-1}U)'T & Q \end{bmatrix} \quad (2)$$

where $Q = W - VT^{-1}U - (T^{-1}U)'(I - TT^{-1})U$

Suppose that A is positive definite, then $P'AP$ is positive definite. Because any principal submatrix of a positive definite matrix is positive definite, so that Q, T are positive def-

inite, and $Q = W - VT^{-1}U$. We conclude that if A is positive definite, then the Schur complement of T is positive definite.

When T is nonsingular, the matrix $Q = W - VT^{-1}U$ is called the *Schur complement* of T in $\begin{bmatrix} T & U \\ V & W \end{bmatrix}$ or $\begin{bmatrix} W & V \\ U & T \end{bmatrix}$.

Theorem 2. *Let*

$$A = \begin{bmatrix} T & U \\ U' & W \end{bmatrix}$$

where T is of dimensions $m \times m$, U of dimensions $m \times n$, and W of dimensions $n \times n$, U' is the transposition of U .

If T is symmetric positive definite and the Schur complement $W - U'T^{-1}U$ of T is positive definite, or if W is symmetric positive definite and the Schur complement $T - UW^{-1}U'$ of W is positive definite, then A is positive definite.

If T is symmetric nonnegative definite and that the Schur complement $W - U'T^{-1}U$ of T is nonnegative definite, then A is nonnegative definite.

Proof: Suppose that T is symmetric positive definite (in which case T is nonsingular) or more generally that T is symmetric nonnegative definite.

Define:

$$P = \begin{bmatrix} I_m & -T^{-1}U \\ 0 & I_n \end{bmatrix}$$

P is nonsingular. Moreover,

$$P'AP = \text{diag}(T, W - U'T^{-1}U)$$

or

$$A = (P^{-1})' \text{diag}(T, W - U'T^{-1}U) P^{-1}$$

If $W - U'T^{-1}U$ is nonnegative definite, then $\text{diag}(T, W - U'T^{-1}U)$ is nonnegative definite, implying that A is nonnegative definite. Similarly, if T is positive definite (and hence invertible) and $W - U'T^{-1}U$ is positive definite, then $\text{diag}(T, W - U'T^{-1}U)$ is positive definite, implying that A is positive definite.

In the special case of a symmetric partitioned matrix, we have the following corollary of the above theorem.

Corollary. Suppose that a symmetric matrix A is partitioned as

$$A = \begin{bmatrix} T & U \\ U' & W \end{bmatrix}$$

where T and W are square. Then, A is positive definite if and only if T and Schur complement $W - U'T^{-1}U$ of T are both positive definite. Similarly, A is positive definite if and only if W and the Schur complement $T - UW^{-1}U'$ of W are both positive definite.

Appendix 5.

Properties of Nonnegative Definite Matrix

In this section, we first consider some definitions and properties of nonnegative definite matrices.

Let $\bar{x}Ax$ represent an arbitrary quadratic form (in an n -dimensional vector $x = (x_1, \dots, x_n)'$, where the sign $(-)$ indicates the conjugate transpose, $(\ ')$ is the sign for the transpose. The quadratic form $\bar{x}Ax$ is said to be *nonnegative definite* if $\bar{x}Ax \geq 0$ for every x vector.

Note that $\bar{x}Ax = 0$ for at least one value of x , namely, $x=0$. If $\bar{x}Ax$ is nonnegative definite and if, the null vector 0 is the only value of x for which $\bar{x}Ax = 0$, then $\bar{x}Ax$ is said to be *positive definite*. A quadratic form that is nonnegative definite, but not positive definite, is called *positive semidefinite*.

The term positive definite is applied to matrices as well as to quadratic forms. An $n \times n$ matrix A is said to be positive definite if the quadratic form $x'Ax$ is positive definite for all non-zero complex n -vector x .

The terms of *nonnegative definite*, *positive definite*, and *positive semidefinite* are applied to matrices as well as to quadratic forms. An $n \times n$ matrix A is said to be *nonnegative definite*, *positive definite*, and *positive semidefinite* if the quadratic form $\bar{x}Ax$ is *nonnegative definite*, *positive definite*, and *positive semidefinite* respectively.

The following two lemmas give some basic results on scale multiples and sums of nonnegative definite matrices.

Lemma 1. let $k > 0$ represent a positive scale, and $\mathbf{A}$ an $n \times n$ matrix. If $\mathbf{A}$ is positive definite, then $k\mathbf{A}$ is also positive definite; if $\mathbf{A}$ is positive semidefinite, then $k\mathbf{A}$ is also positive semidefinite.

Lemma 2. Let $\mathbf{A}$ and $\mathbf{B}$ represent $n \times n$ matrices. If $\mathbf{A}$ and $\mathbf{B}$ are both nonnegative definite, then $\mathbf{A} + \mathbf{B}$ is nonnegative definite. Moreover, if either $\mathbf{A}$ or $\mathbf{B}$ is positive definite, and the other is nonnegative definite, then $\mathbf{A} + \mathbf{B}$ is positive definite.

Some basic properties of positive definite matrices are described in the following lemmas.

Lemma 3. Any positive definite matrix is nonsingular.

Lemma 4. Let $\mathbf{A}$ represent an $n \times n$ matrix and $\mathbf{P}$ an $n \times n$ nonsingular matrix. (1) If $\mathbf{A}$ is positive definite, then $\bar{\mathbf{P}}\mathbf{A}\mathbf{P}$ is positive definite. (2) If $\mathbf{A}$ is positive semidefinite, then $\bar{\mathbf{P}}\mathbf{A}\mathbf{P}$ is positive semidefinite, where $(-)$ is conjugate transpose.

Lemma 5. Any principal submatrix of a positive definite matrix is positive definite; any principal submatrix of a positive semidefinite matrix is nonnegative definite.

When we have a problem in matrix polynomial, we need some properties and rules of manipulation of matrices.

Let $\mathbf{A}$ be a square matrix of order n and $\mathbf{I}$ the unit matrix of order n . Since the distributive law holds for the multiplication of matrices if they are kept in the same order, it follows that any polynomial identity in a scalar x will remain an identity when x is replaced by $\mathbf{A}$, and every term independent of x is multiplied by $\mathbf{I}$. In short, the algebra of polynomials in one square matrix $\mathbf{A}$, with scalar coefficients, is the same as the algebra of ordinary polynomials.

It is easy to show that if $f(\mathbf{A})$ and $g(\mathbf{A})$ are two polynomials in the n th-order square matrix $\mathbf{A}$ with scalar coefficients, then:

$$f(\mathbf{A})g(\mathbf{A}) = g(\mathbf{A})f(\mathbf{A})$$

Now let the polynomial $g(\mathbf{A})$ have the property that $\det(\mathbf{A}) \neq 0$. In such case, the polynomial $g(\mathbf{A})$ has the inverse $[g(\mathbf{A})]^{-1}$. Hence, premultiplying and postmultiplying above equation by $[g(\mathbf{A})]^{-1}$, we obtain:

$$[g(\mathbf{A})]^{-1}f(\mathbf{A}) = f(\mathbf{A})[g(\mathbf{A})]^{-1}$$

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