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TWO GENERATOR SUBGROUPS OF $SL(2, \mathbb{C})$

by

ABDUL MAJEED

A thesis presented to

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STATEMENT

Except where it is indicated, this thesis is my own work.

TWO GENERATOR SUBGROUPS OF $SL(2,C)$

by

Abdul Majeed

ABSTRACT

This thesis is mainly concerned with the determination of structure of two generator subgroups of $SL(2,C)$. However, there are also results of general nature and to some extent independent of the topics concerning two generator subgroups of $SL(2,C)$. After determining all the complex 2×2 faithful matrix representation of a free abelian group of rank 2, we discuss the question of linearity over the ring of integers of the free product of two infinite dihedral groups and of two symmetric groups of degree 3. One of our theorems reduces the study of free subgroups of $GL(n,F)$ to the quest for free subgroups of $SL(n,F)$.

We discuss also the question whether, in the case when $\langle x,y \rangle < SL(2,F)$ is not solvable-by-finite, free subgroups of finite index always exist. We give a counter example to this and show that there exist torsion-free non-free irreducible subgroups of $GL(2,Q)$ which are not solvable-by-finite. Consequently, if a subgroup of $GL(2,F)$ is torsion free but not solvable-by-finite, free subgroups of finite index need not exist in general.

Let $\alpha, \beta, \gamma \in C$ be such that

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0$$

It is shown that there exists $a, b \in SL(2,C)$ such that $\text{tr } a = \alpha$, $\text{tr } b = \beta$,

$\text{tr } ab = \gamma$ and $\langle a, b \rangle$ is an irreducible subgroup. Moreover, a, b are completely determined up to conjugacy; that is, if for $a', b' \in \text{SL}(2, \mathbb{C})$, $\text{tr } a' = \alpha$, $\text{tr } b' = \beta$, $\text{tr } a'b' = \gamma$, then there exists a $c \in \text{GL}(2, \mathbb{C})$ such that

$$a' = cac^{-1}, b' = cbc^{-1}$$

Denote by $F_{\alpha, \beta, \gamma}^{\infty}$ the group generated by $a, b \in \text{SL}(2, \mathbb{C})$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. It is shown that the set

$$S(\alpha, \beta) : \{ \gamma : F_{\alpha, \beta, \gamma}^{\infty} \text{ is not the free product of } \langle a \rangle, \langle b \rangle \}$$

is at most countable. In fact, we show that $F_{\alpha, \beta, \gamma}^{\infty}$ is the free product of $\langle a \rangle, \langle b \rangle$ provided that γ is algebraically independent over $\mathbb{Q}(\alpha, \beta)$, the field of quotients of the polynomial ring $\mathbb{Z}[\alpha, \beta]$ in α, β with integer coefficients.

A recent result of J. Tits states that a linear group is either solvable-by-finite or else contains a non-cyclic free subgroup. It is therefore worth looking at the irreducible solvable-by-finite subgroups of $\text{SL}(2, \mathbb{C})$. In one of the theorems given in the last chapter of this thesis, we obtain a necessary and sufficient condition for an infinite subgroup $\langle a, b \rangle$ of $\text{SL}(2, \mathbb{C})$ to be solvable-by-finite. It is also shown that a subgroup G of $\text{SL}(2, \mathbb{C})$ is reducible if and only if every two generator subgroup of G is reducible. This result is used to prove that an irreducible subgroup G of $\text{SL}(2, \mathbb{C})$ has an irreducible solvable-by-finite subgroup if and only if G contains an element x such that $x^2 = -I$.

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NOTATIONS

Most of the notations used in this thesis are standard in the modern books in the theory of linear groups and of groups. The main sources of my information on linear groups have been the books by Dixon [4], Wehrfritz [27] and Curtis and Reiner [3]. For the purposes of clarity and convenience, we list a few of the notations used in the thesis. A and B shall denote the groups unless stated otherwise.

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ the set of integers, the set of rationals, the set of real numbers and the set of complex numbers, respectively

I the identity matrix

$\text{tr } a$ the trace, that is, the sum of diagonal elements of the matrix a

$\langle a, b, \dots; r_i=1, i=1, 2, \dots \rangle$ the group generated by the elements $a, b, \dots$, satisfying the relations $r_i=1, i=1, 2, \dots$

$\{x, y, \dots\}$ the set consisting of elements $x, y, \dots$ such that ...

$H \leq A$ H is a subgroup of A or a subset of A

$[x, y]$ the commutator $xyx^{-1}y^{-1}$ of x, y in A

$C_A(S)$ the centralizer of the subset S of A

$\langle A, B \rangle$ the group generated by A, B

$\pi_{\alpha \in \Lambda}^* G_\alpha$ the free product of groups $G_\alpha, \alpha \in \Lambda$

$\langle A * B; H \rangle$ the generalized free product of A and B amalgamating a subgroup H

$GL(n, F)$

the general linear group consisting of all $n \times n$ invertible matrices with entries from a field F

$SL(n, F)$

the special linear group consisting of all $n \times n$ matrices with entries from F and of determinant 1

CHAPTER 0

INTRODUCTION

Let a, b be a pair of matrices in $SL(2, \mathbb{C})$ and $\langle a, b \rangle$ the group generated by a, b . Various authors, notably among them are Frasc [7], Sanov [22], Doniakhi [6], Chang, Jennings and Ree [2], Specht [23], Newman [20], Lehner and Newman [13], Goldberg and Newman [9] and Lyndon and Ullman [14], [15], have discussed the question about the freeness of $\langle a, b \rangle$ at one time or other in their respective papers. Purzitsky [30], [31] and Herzberger [32], simultaneously but working independently, have used the concept of fundamental domains to determine certain necessary and sufficient conditions for which $\langle a, b \rangle \subset SL(2, \mathbb{R})$ is a discrete free group of rank 2. Here we give an alternative proof of the sufficiency condition of one of their theorems. The ideas contained in the proof of this result are used later to prove one of our theorems. Some new sufficient conditions for $\langle a, b \rangle$ to be free of rank 2 are also obtained.

A recent theorem of J. Tits [25] states that a linear group is either solvable-by-finite or else contains a non-cyclic free group. It is, therefore, worth looking at the irreducible solvable-by-finite subgroups of $SL(2, \mathbb{C})$. In one of the theorems given in the last chapter of this thesis, we obtain a necessary and sufficient condition for an infinite subgroup $\langle a, b \rangle$ of $SL(2, \mathbb{C})$ to be solvable-by-finite.

It is also shown that a subgroup G of $SL(2, \mathbb{C})$ is reducible if and only if any two generator subgroups of G is reducible. This result is used to prove that an irreducible subgroup G of $SL(2, \mathbb{C})$ has an irreducible solvable-by-finite subgroup if and only if G contains an element x such that $x^2 = -I$.

Let V be a vector space of dimension n over a field F . The set $\text{Hom}(V, V)$ of all linear transformations from V to V form a ring with identity. The units (that is, the invertible linear transformations of V) of $\text{Hom}(V, V)$ form a group called the general linear group. The ring $\text{Hom}(V, V)$ is isomorphic to the ring $M_n(F)$ of all $n \times n$ matrices with entries from the field F . A matrix $x \in M_n(F)$ is invertible if and only if $\det x$ (the determinant of x) is a non-zero element of F . The subgroup of $M_n(F)$, consisting of those matrices in $M_n(F)$ which have inverses, is called the general linear group of degree n over F and is denoted by $GL(n, F)$. Clearly $GL_n(V)$ and $GL(n, F)$ are isomorphic groups. A subgroup of $GL(n, F)$ is called a linear group. The collection of those matrices in $GL(n, F)$ whose determinant is 1, the identity of F , is a subgroup of $GL(n, F)$, called the special linear group and is denoted by $SL(n, F)$. More generally, if R is a commutative ring with unity, then $GL(n, R)$ is the group of units in the ring $M_n(R)$ of all $n \times n$ matrices over R , and $SL(n, R)$ is the subgroup of those matrices with determinant 1, the identity in R .

By a representation of degree n over a field F of a group G , we mean any homomorphism of G into the group $GL(n, F)$. A representation σ is called faithful if and only if σ is a monomorphism. If σ and ρ are two representations, both of degree n , of a group G , then σ and ρ are called equivalent representations provided that there exists an $x \in GL(n, F)$ such that

$$\rho(g) = x\sigma(g)x^{-1} \quad \text{for all } g \in G.$$

A matrix $a = (a_{ij}) \in M_n(F)$ is said to be (lower) triangular if $a_{ij} = 0$ for all $j > i$. The set of all triangular matrices in $GL(n, F)$ form a subgroup, called the triangular subgroup of $GL(n, F)$ and is denoted by $TL(n, F)$. Those matrices in $TL(n, F)$ whose determinant is 1, form a subgroup $STL(n, F)$ of $TL(n, F)$. A matrix $a = (a_{ij}) \in GL(n, F)$ is called a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$. The set $D(n, F)$ of all diagonal matrices of $GL(n, F)$ form a subgroup of $GL(n, F)$. If F^* denotes the multiplicative group of F , then $D(n, F)$ is isomorphic to the direct product of n copies of F^* . A scalar matrix $a = (a_{ij})$ is a diagonal matrix with $a_{ii} = \alpha$ for $i = 1, 2, \dots, n$. The set of scalar matrices in $GL(n, F)$ is a subgroup of $GL(n, F)$. In fact, the scalar matrices in $GL(n, F)$ constitute the centre of $GL(n, F)$.

Let $a \in GL(n, F)$ and $\lambda \in F$. Then, the equation

$$\det(a - \lambda I) = 0 \tag{1}$$

is called the characteristic equation of a and the roots of this equation

are called the characteristic roots of a . In the special case when $a \in \text{SL}(2, F)$ the characteristic equation has the form $\lambda^2 - \alpha\lambda + 1 = 0$, $\alpha = \text{tr } a$, and so by the Cayley-Hamilton theorem

$$a^2 - \alpha a + I = 0 \quad (2)$$

If λ_1, λ_2 are the characteristic roots of a , then by induction on n , we have for $\lambda_1 \neq \lambda_2$,

$$a^n = s_{n-1}a - s_{n-2}I \quad (3)$$

where $s_n = (\lambda_1^{n+1} - \lambda_2^{n+1}) / (\lambda_1 - \lambda_2)$. Use of these equations will be made in our latter discussions.

Let G be a subgroup of $\text{GL}_n(V)$. A subspace W of V is called invariant under G if $xW \subseteq W$ for all $x \in G$. G is called irreducible if and only if the only subspaces of V which are invariant under G are the null space $\{0\}$ and V itself. Otherwise, G is called reducible. A subgroup G of $\text{GL}_n(V)$ is called completely reducible if there exist minimal subspaces $W_1, \dots, W_s$, $s \geq 1$ of V which are invariant under G such that V is the direct sum of $W_1, \dots, W_s$. It is then clear that every irreducible group is completely reducible. For various properties of reducible and irreducible groups and conditions characterizing these groups, see Dixon's books [4], [5] and Wehrfritz' book [27].

We now define the free product of groups as follows. Let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of groups. A group G , which shall be written as $\pi_{\alpha \in \Lambda}^* G_\alpha$, is said to be the ordinary free product of its subgroups G_α , $\alpha \in \Lambda$ if

each element $g \neq 1$ of G is expressible uniquely as a product of a finite number of elements from the G_α , $\alpha \in \Lambda$:

$$g = g_1 g_2 \cdots g_k \quad (A)$$

$g_i \in G_{\alpha_i}$, $i=1, 2, \dots, k$; $g_i \neq 1$, $\alpha_i \neq \alpha_{i+1}$, $1 \leq i \leq k-1$ and the expression (A) is unique for all $g \in G$.

The free product of a family of groups G_α , $\alpha \in \Lambda$, with $G_\alpha \cap G_\beta = H$ for all $\alpha, \beta \in \Lambda$ is called the generalized free product of G_α with the subgroup H amalgamated (cf. Magnus, Karass and Solitar [16a] or Kurosh [11a]).

Let $\mathcal{L}$ consist of all functions $a: \mathbb{C}U\{\infty\} \rightarrow \mathbb{C}U\{\infty\}$, of the form

$$a(z) = (\alpha z + \beta) / (\gamma z + \delta), \quad \alpha\delta - \beta\gamma = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}.$$

The elements of $\mathcal{L}$ are the conformal self mappings of the extended complex plane $\mathbb{C}U\{\infty\}$. $\mathcal{L}$ is a group under the usual product of mappings. There is a relationship between the group $\mathcal{L}$ and the special linear group $SL(2, \mathbb{C})$ consisting of the matrices:

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C} \quad (1)$$

In fact, $SL(2, \mathbb{C}) / \{\pm I\}$ is isomorphic to $\mathcal{L}$. Here I denotes the identity matrix in $SL(2, \mathbb{C})$. With each a of $SL(2, \mathbb{C})$ given in (1), we can associate a uniquely defined complex number $\text{tr } a = \alpha + \delta$, called the trace of a .

The mappings in $\mathcal{L}$ are classified in various ways. One of these is as follows.

For each $\underline{a} \in \mathcal{L}$

$$\underline{a}(z) = (\alpha z + \beta) / (\gamma z + \delta), \quad \alpha\delta - \beta\gamma = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}$$

Let us identify $\underline{a}$ with the matrix a given in (1). Then, we have the following well defined real number:

$$|\text{tra}|^2 = |\alpha + \delta|^2$$

If $\underline{a}$ is not the identity linear transformation, we call $\underline{a}$

elliptic if and only if $0 \leq |\text{tra}|^2 < 4$,

parabolic if and only if $|\text{tra}|^2 = 4$,

loxodromic if and only if $|\text{tra}|^2 \notin [0, 4]$.

Those loxodromic elements $\underline{a}$ with $|\text{tra}|^2 > 4$ are called hyperbolic. The element $\underline{a}$ is parabolic if and only if $\underline{a}$ has only one fixed point. It is quite easy to see that:

$\underline{a}$ is loxodromic if and only if $\underline{a}$ is conjugate (in $\mathcal{L}$) to $z \mapsto \lambda z$,

$$|\lambda| > 1$$

$\underline{a}$ is hyperbolic if and only if $\underline{a}$ is conjugate to $z \mapsto \lambda z$, $\lambda > 1$

$\underline{a}$ is elliptic if and only if $\underline{a}$ is conjugate to $z \mapsto \lambda z$, $|\lambda| = 1$

$\underline{a}$ is parabolic if and only if $\underline{a}$ is conjugate to $z \mapsto z + 1$.

(See, for example, Shimura [22a]).

Let Γ denote the proper unimodular group consisting of all 2×2 matrices with rational integral elements and determinant 1. For m , a positive integer, define the principal congruence group $\Gamma(m)$ by:

$$\Gamma(m) = \left\{ a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma : \alpha, \delta \equiv 1 \pmod{m}, \beta, \gamma \equiv 0 \pmod{m} \right\}$$

It was shown by Fraşch [7] that for each prime p , $\Gamma(p)$ is a finitely generated free group. In particular, it follows that the matrices

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

for $p=2$ in $\Gamma(p)$ generate a free group of rank 2. This was later proved also by Sanov [22], Brenner [1] and Specht [23]. The search for free subgroups of Γ and in general for those of $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$ was continued by Chang, Jennings and Ree in [2], who proved that the matrices

$$a = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

generate a free group of rank 2 for all $\alpha, \beta \in \mathbb{C}$ provided that $|\alpha\beta| \geq 4$.

Morris Newman in [20] showed that the matrices

$$a_1 = \begin{pmatrix} -\alpha & -\beta \\ \gamma & \delta \end{pmatrix}, \quad b_1 = \begin{pmatrix} -\alpha' & \beta' \\ -\gamma' & \delta' \end{pmatrix}$$

where $\alpha, \beta, \dots, \delta, \alpha', \dots, \delta' \geq 0$ are real numbers such that $\text{tr } a_1 = \delta - \alpha \geq 2$,

$\text{tr } b_1 = \delta' - \alpha' \geq 2$, generate a free group of rank 2. R.C. Lyndon and J.L. Ullman in [14], using a result of A.M. Macbeath, gave an alternative proof for Newman's result. In [14] and in another subsequent paper [15], Lyndon and Ullman examined pairs of elliptic and parabolic linear transformations which generate non-cyclic free groups. They also gave a few rational values of $\mu < 2$ for which

$$G_\mu = \langle a = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \rangle$$

is not free of rank 2. It is still unknown whether or not, for example, G_μ is free for $\mu = 7/4$.

Recently, Purzitsky [30], [31] and Herzberger [32] have obtained necessary and sufficient conditions on the trace values of certain elements of $\langle a, b \rangle \text{SL}(2, \mathbb{R})$ in order that $\langle a, b \rangle$ should be a discrete free group on two generators. This thesis also contains some results in this direction. We give a few other sufficient conditions on the trace values of a, b and ab for $\langle a, b \rangle \text{SL}(2, \mathbb{C})$ to be free on two generators. Some theorems concerning the structure of two generator subgroups of $\text{SL}(2, \mathbb{C})$ are also proved in the thesis.

Chapter I consists of first a discussion on the determination of all non-scalar faithful representations of free abelian groups of rank 2 in $\text{SL}(2, \mathbb{C})$ followed by faithful representations over $\mathbb{Z}$ of the free product of two infinite dihedral groups and of two symmetric groups of degree 3. A theorem in Section 3 of this chapter reduces the study of free subgroups of $\text{GL}(2, \mathbb{C})$ to the free subgroups of $\text{SL}(2, \mathbb{C})$. In Section 4, we discuss the question whether, in case $\langle x, y \rangle$, $x, y \in \text{GL}(n, F)$ is not solvable-by-finite, non-abelian free subgroups of finite index always exist. By a theorem of

John Stallings [24], any torsion-free group with a free subgroup of finite index is itself free. We give an example to show that there exists a torsion-free, non-free subgroup of $GL(2, \mathbb{Q})$ having no solvable subgroup of finite index. Although there are easily constructable examples of torsion-free non-free irreducible subgroups of $SL(2, \mathbb{C})$ which are generated by more than two elements (we give one in that section), it is still an open question whether or not there exist 2-generator non-free torsion-free subgroups in $SL(2, \mathbb{C})$.

In Chapter II, we initiate our discussion on free subgroups of $SL(2, \mathbb{C})$. A necessary and sufficient condition on the trace values of a, b and ab for the group $\langle a, b \rangle \subset SL(2, \mathbb{C})$ to be irreducible is obtained in section 1 of this chapter. Section II.2 contains an alternative proof of the sufficiency condition of one of the theorems of Purzitsky and Herzberger quoted above. Section II.3 gives some new sufficient conditions for $\langle a, b \rangle \subset SL(2, \mathbb{C})$ to be free of rank 2.

For any triplet α, β, γ of real or complex numbers, let $F_{\alpha, \beta, \gamma}$ denote a subgroup of $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$ which is generated by a pair of matrices a, b with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. We shall show that, provided $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0$, $F_{\alpha, \beta, \gamma}$ is an irreducible subgroup of $SL(2, \mathbb{C})$ determined up to conjugacy. Moreover, the set

$$S(\alpha, \beta) = \{ \gamma : F_{\alpha, \beta, \gamma} \text{ is a free product of } \langle a \rangle \text{ and } \langle b \rangle \}$$

is at most countable. As a consequence of this, we obtain that for fixed α, β, γ if a and b have infinite order, then $F_{\alpha, \beta, \gamma}$ is free of rank 2, provided that γ is algebraically independent over $\mathbb{Q}(\alpha, \beta)$. We give an

example to show that if γ is not algebraically independent over $\mathbb{Q}(\alpha, \beta)$ then $F_{\alpha, \beta, \gamma}$ need not be free of rank 2.

Suppose that $\langle a, b \rangle$ is not a free subgroup of $SL(2, \mathbb{R})$. What can we say about the freeness of the group $\langle a^n, b^n \rangle$ for sufficiently large n ? In Chapter III, we show that if the trace of at least one of the matrices a and b is 2 while the trace of the other is ≥ 2 , then there exists a positive integer n such that $\langle a^n, b^n \rangle$ is free of rank 2. We also show that for $\text{tr} a = \alpha$, $\text{tr} b = \beta$, $\alpha, \beta \in \mathbb{C}$, $|\alpha| > 2$, $|\beta| > 2$, $\langle a^n, b^n \rangle$ is free for sufficiently large n .

By Tits' theorem [25], every linear group is either solvable-by-finite or else contains a non-abelian free subgroup. Section 2 of Chapter III contains some results about the nature of irreducible infinite solvable-by-finite subgroups of $SL(2, \mathbb{C})$. It is shown that an irreducible infinite subgroup $\langle a, b \rangle$ of $SL(2, \mathbb{C})$ is solvable-by-finite if and only if $\langle a, b \rangle$ is isomorphic, modulo $\pm I$, to a dihedral group. A necessary and sufficient condition for a subgroup of $SL(2, \mathbb{C})$ to be reducible is also given in this section. This result is used to prove that an irreducible subgroup G of $SL(2, \mathbb{C})$ has an irreducible solvable-by-finite subgroup if and only if G contains an element x such that $x^2 = -I$.

There are still some unresolved questions concerning the topics discussed in the thesis. For example, we have been unable to decide

whether or not for a pair a, b of matrices in $SL(2, \mathbb{C})$ with $\text{tr } a = \alpha \in \mathbb{C} \setminus \mathbb{R}$, $\text{tr } b = \beta \in \mathbb{C} \setminus \mathbb{R}$, the conditions $|\alpha| \geq 2$, $|\beta| \geq 2$, $|\gamma| \geq |\alpha\beta| + 2$ are sufficient to imply $\langle a, b \rangle$ is a free group of rank 2. (Compare with the theorem II.2.1 where $\alpha, \beta \in \mathbb{R}$). If a, b correspond to elliptic transformations with $\text{tr } x = 2 \cos \bar{\lambda}/n$, ($n \geq 2$), $x = a, b$, then Purzitsky [30], [31], proved that for $a \neq -b$, $\langle a, b \rangle$ is the discrete free product of $\langle a \rangle$ and $\langle b \rangle$ if and only if either $\text{tr } a^{-1}b \leq -2$ or $\text{tr } ab \leq -2$. The corollary II.3.4 also gives a sufficient condition for certain pairs a, b of transformations (elliptic or otherwise) to generate a free product of $\langle a \rangle$, $\langle b \rangle$. However, to my knowledge, nothing seems to be known about when an arbitrary pair of elliptic transformations a, b generate the free product of $\langle a \rangle$, $\langle b \rangle$.

Lastly, we state the following.

Conjecture (compare with theorem III.2.5).

A subgroup G of $SL(n, F)$, ($n \geq 2$) is reducible if and only if every two generator subgroup of G is reducible.

CHAPTER I

I.0 INTRODUCTION

This chapter consists of results which are, in a sense, of general nature and, to some extent, independent of the topics discussed in later chapters. In Section 1 of this chapter, we determine all complex faithful 2×2 non-scalar matrix representations of free abelian groups of rank 2. In Section 2, we discuss the linearity of the free product of two infinite dihedral groups and of two symmetric groups of degree 3. According to a result of Nisnevich [17], the free product

$\pi_{\alpha \in \Lambda}^* G_\alpha$ of linear groups G_α , $\alpha \in \Lambda$, each of degree n over a field F of characteristic $p \geq 0$, has a faithful representation of degree $n+1$ over some field F' of characteristic p ; and if the centre of each of the G_α is non-trivial, then $\pi_{\alpha \in \Lambda}^* G_\alpha$ even has a faithful representation of degree n . In general, the field F' contains elements which are transcendental over F . However, we show that the free product of two infinite dihedral groups and of two symmetric groups of degree 3 has a faithful representation of degree 2 over $\mathbb{Z}$.

Section I.3 contains a result which reduces the study of free subgroups of $GL(n, F)$, F an arbitrary field, to the quest for free subgroups of $SL(n, F)$. In Section I.4, we discuss the question whether, in the case when $\langle x, y \rangle \leq CGL(2, F)$ is not solvable-by-finite, free subgroups of finite index always exist. We give a counter example to this and show that there exist torsion-free, non-free subgroups of $GL(2, \mathbb{Q})$ which

are not solvable-by-finite. Consequently, if a subgroup of $GL(n, F)$ is not solvable-by-finite, free subgroups of finite index need not exist in general.

I.1 FAITHFUL NON-SCALAR 2x2 REPRESENTATIONS OF A FREE ABELIAN GROUP OF RANK 2

An $n \times n$ matrix a is said to lower triangular if it can be written as:

$$\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ * & & \alpha_n \end{pmatrix}$$

where 0 stands for all zero entries above the main diagonal. It is known [24a] that for any subset M of commutative matrices in $GL(n, F)$, there exists a finite extension F^* of F and a similarity transformation over F^* which reduces all matrices in M simultaneously to the triangular form. In particular, if all the roots of the characteristic polynomial of each matrix in M belongs to F , then we may take F as the field F^* . We now determine all (faithful) complex 2x2 matrix representations of free abelian groups of rank 2.

Let $a, b \in GL(2, C)$ where C is the field of complex numbers. Because of the remarks above, if $ab=ba$, then there exists a similarity transformation such that a and b can be brought simultaneously into the lower triangular form:

$$\begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & \beta_3 \end{pmatrix} \quad (1)$$

In fact, since C is algebraically closed, we can suppose that $\alpha_i, \beta_i, 1 \leq i \leq 3$ are in C .

We now determine all complex 2×2 faithful non-scalar representations of a free abelian group of rank 2.

I.1.1. Theorem

Let

$$a = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & \alpha_3 \end{pmatrix}, b = \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & \beta_3 \end{pmatrix}$$

be two commutative non-scalar matrices of infinite order.

(i) If a has two distinct eigenvalues α_1, α_3 , then b has two distinct eigenvalues β_1, β_3 and $\langle a, b \rangle$ is a free abelian group of rank 2 if and only if either the multiplicative group generated by $\langle \alpha_1/\alpha_3, \beta_1/\beta_3 \rangle$ is a free abelian group of rank 2 or both the multiplicative groups generated by α_1, β_1 and by α_3, β_3 are free abelian of rank 2.

(ii) If a has a repeated eigenvalue α_1 , then b has a repeated eigenvalue β_1 with $\beta_2 \neq 0$ and $\langle a, b \rangle$ is a free abelian group of rank 2 if and only if either $\langle \alpha_1, \beta_1/\beta_2 \rangle$ are linearly independent over $\mathbb{Z}$ or the multiplicative group generated by α_1, β_1 is free abelian of rank 2.

Proof

Case (i). If a has two distinct eigenvalues α_1, α_3 then, by a suitable conjugation, we can suppose that, up to equivalence,

$$a = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_3 \end{pmatrix}, \quad \alpha_1 \neq \alpha_3$$

Then, $ab=ba$ implies

$$b = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_3 \end{pmatrix}$$

so that, being a non-scalar matrix, b has two distinct eigenvalues β_1, β_3 . Also,

$$a^m b^n = \begin{pmatrix} \alpha_1^m \beta_1^n & 0 \\ 0 & \alpha_3^m \beta_3^n \end{pmatrix}$$

so that $a^m b^n = I$ if and only if $\alpha_1^m \beta_1^n = 1 = \alpha_3^m \beta_3^n$. Hence, $\langle a, b \rangle$ is a free abelian group of rank 2 if and only if either the multiplicative group $\langle \alpha_1 / \alpha_3, \beta_1 / \beta_3 \rangle$ is a free abelian group of rank 2, or both the multiplicative groups generated by α_1, β_1 and by α_3, β_3 are free abelian of rank 2.

Case (ii). If a has a repeated eigenvalue α_1 , then conjugate a to the Jordan form

$$a = \begin{pmatrix} \alpha_1 & 0 \\ 1 & \alpha_1 \end{pmatrix}$$

Now, $ab=ba$ implies

$$b = \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & \beta_1 \end{pmatrix}$$

with $\beta_2 \neq 0$ (otherwise b is a scalar), so that b has a repeated eigenvalue β_1 . Also,

$$a^m = \begin{pmatrix} \alpha_1^m & 0 \\ m\alpha_1^{m-1} & \alpha_1^m \end{pmatrix} \quad b^n = \begin{pmatrix} \beta_1^n & 0 \\ n\beta_1^{n-1}\beta_2 & \beta_1^n \end{pmatrix}$$

so that

$$a^m b^n = \begin{pmatrix} \alpha_1^m \beta_1^n & 0 \\ \alpha_1^{m-1} \beta_1^{n-1} (m\beta_1 + n\alpha_1 \beta_2) & \alpha_1^m \beta_1^n \end{pmatrix}$$

Hence, $a^m b^n = I$ implies $\alpha_1^m \beta_1^n = 1$, $m\beta_1 + n\alpha_1 \beta_2 = 0$. Therefore, $\langle a, b \rangle$ is a free abelian group of rank 2 if and only if either $\alpha_1, \beta_1/\beta_2$ are linearly independent over $\mathbb{Z}$ or $\langle \alpha_1, \beta_1 \rangle$ is a free abelian group of rank 2.

Hence, the theorem.

I.2 FAITHFUL 2x2 REPRESENTATIONS OVER Z OF FREE PRODUCT OF CERTAIN DIHEDRAL GROUPS

Let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of linear groups of degree n and the same characteristic $p \geq 0$. The theorem of Nisnevič [17] quoted at the beginning of I.0 shows that the free product $\pi_{\alpha \in \Lambda}^* G_\alpha$ has a faithful representation of degree $n+1$ over some field F' of characteristic p but usually F' contains transcendental elements. In what follows, we show that the free product of two infinite dihedral groups has a faithful representation of degree 2 over Z . We also show that the free product of two symmetric groups each of degree 3 has a faithful representation of degree 2 over Z . In the proof of these results, we make use of a theorem of A.M. Macbeath [16]. For the sake of completeness, we state and prove Macbeath's theorem. We shall be using it again in later sections. The proof given here is due to Lyndon and Ullman [14].

I.2.1 Theorem (The Ping Pong Lemma of Macbeath)

Let A and B be groups of permutations of a set Ω and let G be the group generated by A and B together. Suppose that Ω contains two disjoint non-empty sets Γ and Δ such that each non-trivial element of A maps Γ into Δ and each non-trivial element of B maps Δ into Γ . Then, either G is the free product of its subgroups A and B or else both A and B have order 2 and G is a dihedral group.

Proof (Lyndon and Ullman [14])

Suppose that A has order greater than 2 so that A has more than one non-trivial element. Let a_1, a_2 be two distinct non-trivial elements of A . Then $a_1(\Gamma) \subseteq \Delta$, $a_2(\Gamma) \subseteq \Delta$. We show that $a_1(\Gamma) \cap a_2(\Gamma) = \emptyset$. Suppose otherwise and let $u \in a_1(\Gamma) \cap a_2(\Gamma)$. Then, there exist $v_1, v_2 \in \Gamma$ such that $a_1(v_1) = u = a_2(v_2)$ so that $a_2^{-1}a_1(v_1) = v_2 \in \Delta \cap \Gamma = \emptyset$, a contradiction. Hence, the images $a(\Gamma)$, of Γ under the non-trivial elements a of A are disjoint non-empty subsets of Δ . Thus, it follows that each $a(\Gamma)$ is properly contained in Δ , that is, $a(\Gamma) \subset \Delta$.

Now let

$$w = b_k a_k \dots b_1 a_1$$

where $k \geq 1$, $1 \neq a_i \in A$, $1 \neq b_i \in B$, $i = 1, 2, \dots, k$. Then, $a_1(\Gamma) \subset \Delta$, $b_1 a_1(\Gamma) \subset b_1(\Delta) \subset \Gamma$. By a continuation of this argument, $w(\Gamma) \subset \Gamma$. Thus, $w(\Gamma) \neq \Gamma$ and so $w \neq 1$. This shows that G is a free product of A and B .

The same conclusion holds if B has order greater than 2 and also trivially if either A or B has order 1. The case remains where A is generated by a , and B is generated by b with $a^2 = b^2 = 1$. Any further relation between a and b can be reduced to the form $(ab)^n = 1$, $n > 0$, and indeed to at most one such relation. If such a relation holds, then G is a dihedral group of order $2n$. If no such relation holds, then G is the infinite dihedral group, a free product of A and B .

1.2.2 Theorem. The free product of two infinite dihedral groups has a faithful representation of degree 2 over $\mathbb{Z}$.

Proof

Consider the Matrices:

$$a = \begin{pmatrix} -\alpha & \alpha-1 \\ -(\alpha+1) & \alpha \end{pmatrix}, \quad b = \begin{pmatrix} -\beta & \beta-1 \\ -(\beta+1) & \beta \end{pmatrix}$$

$$\alpha, \beta \in \mathbb{Z}, \quad \alpha \neq \beta.$$

Then

$$a^2 = b^2 = I$$

and

$$ab = \begin{pmatrix} 1-(\alpha-\beta) & \alpha-\beta \\ -(\alpha-\beta) & 1+\alpha-\beta \end{pmatrix}$$

$$= \begin{pmatrix} 1-\xi & \xi \\ -\xi & 1+\xi \end{pmatrix}, \quad \xi = \alpha - \beta$$

Hence

$$(ab)^m = \begin{pmatrix} 1-m\xi & m\xi \\ -m\xi & 1+m\xi \end{pmatrix}$$

so that ab has infinite order since $\xi \neq 0$. Therefore, $\langle a, b \rangle$ is the infinite dihedral group. Also,

$$\begin{aligned}
 a(ab)^m &= \begin{pmatrix} -\alpha & \alpha-1 \\ -(\alpha+1) & \alpha \end{pmatrix} \begin{pmatrix} 1-m(\alpha-\beta) & m(\alpha-\beta) \\ -m(\alpha-\beta) & 1+m(\alpha-\beta) \end{pmatrix} \\
 &= \begin{pmatrix} -\alpha+m\xi & -1+\alpha-m\xi \\ -1-(\alpha-m\xi) & \alpha-m\xi \end{pmatrix} \quad m \in \mathbb{Z}
 \end{aligned}$$

Let $\alpha=5$, $\beta=3$. Then, $\xi=\alpha-\beta=2$. So

$1-m\xi = 1-2m < 0$ for all $m \geq 1$, and $-2m < 0$ for all $m \geq 1$.

Hence, $(ab)^m$ is a matrix of the form

$$\begin{pmatrix} -\delta_1 & \delta_2 \\ -\delta_3 & \delta_4 \end{pmatrix}, \quad \delta_i \in \mathbb{Z}^+, \quad i=1,2,3,4.$$

Similarly, $(ab)^{-m}$ is of the form:

$$\begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_3 & -\delta_4 \end{pmatrix}, \quad \delta_i \in \mathbb{Z}^+, \quad i=1,2,3,4.$$

Also,

$$a(ab)^m = \begin{pmatrix} -5+2m & -1+5-2m \\ -1-(5-2m) & 5-2m \end{pmatrix} = \begin{pmatrix} -5+2m & 4-2m \\ -6+2m & 5-2m \end{pmatrix}$$

for $m > 0$.

$5-2m=3$ when $m=1$, and $5-2m=1$ when $m=2$

and

$5-2m < 0$ for all $m=3,4,\dots$

so that $a(ab)$, $a(ab)^2$ are of the form

$$\begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$

and $a(ab)^3$, $a(ab)^4$, etc. are of the form

$$\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}, \text{ etc.}$$

Also, $a(ab)^{-m}$, $m > 0$ is of the form

$$\begin{pmatrix} -(5+2m) & -1+(5+2m) \\ -1-(5+2m) & 5+2m \end{pmatrix}$$

Hence, each element of the group $A = \langle a, b \rangle$ is of the form

$$\begin{pmatrix} -\delta_1 & \delta_2 \\ -\delta_3 & \delta_4 \end{pmatrix} \text{ or } \begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_3 & -\delta_4 \end{pmatrix}, \delta_i \in \mathbb{Z}^+, i=1,2,3,4.$$

Consider another pair of matrices:

$$a' = \begin{pmatrix} -\alpha & -(1+\alpha) \\ \alpha-1 & \alpha \end{pmatrix}, b' = \begin{pmatrix} -\beta & -(1+\beta) \\ \beta-1 & \beta \end{pmatrix}$$

Then, $a'^2 = b'^2 = I$. For $\alpha=5, \beta=3$, it follows, as above, that every element of the group $A' = \langle a', b' \rangle$ can be expressed as:

$$\begin{pmatrix} -\delta'_1 & -\delta'_2 \\ \delta'_3 & \delta'_4 \end{pmatrix} \text{ or } \begin{pmatrix} \delta'_1 & \delta'_2 \\ -\delta'_3 & -\delta'_4 \end{pmatrix}, \delta'_i \in \mathbb{Z}^+, i=1,2,3,4$$

We show that A and A' generate their free product. For this, we make use of the Ping Pong lemma 1.2.1 of Macbeath. As our disjoint sets, we take

$$\Gamma = (-\infty, 0), \Delta = (0, \infty).$$

Then, for each $z \in \Gamma$ and any $g \in A$, we have

$$g(z) = \frac{-\delta'_1 z + \delta'_2}{-\delta'_3 z + \delta'_4} \text{ or } g(z) = \frac{\delta_1 z - \delta_2}{\delta_3 z - \delta_4}$$

according as g is of the form:

$$\begin{pmatrix} -\delta_1 & \delta_2 \\ -\delta_3 & \delta_4 \end{pmatrix} \text{ or } \begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_3 & -\delta_4 \end{pmatrix}$$

Now, $z \in \Gamma$ implies $z = -x$ for some $x > 0$. Hence,

$$g(z) = \frac{\delta_1 x + \delta_2}{\delta_3 x + \delta_4} > 0 \text{ for all } z = -x \in \Gamma \text{ and } g \in A.$$

so that every non-trivial element of A maps Γ into Δ . Also, $z' \in \Delta$ and $g' \in A'$ implies

$$g'(z') = \frac{-\delta'_1 z' - \delta'_2}{\delta'_3 z' + \delta'_4} \text{ or } g'(z') = \frac{\delta'_1 z' + \delta'_2}{-\delta'_3 z' - \delta'_4}$$

according as

$$g' = \begin{pmatrix} -\delta'_1 & -\delta'_2 \\ \delta'_3 & \delta'_4 \end{pmatrix} \text{ or } \begin{pmatrix} \delta'_1 & \delta'_2 \\ -\delta'_3 & -\delta'_4 \end{pmatrix}$$

Hence, $g'(z') < 0$ for all $z' \in \Delta$. Therefore, every non-trivial element of A' maps Δ into Γ . Thus, by Macbeath's lemma, A and A' generate their free product. Hence, the free product of two infinite dihedral groups has a faithful representation of degree 2 over Z .

I.2.3 Theorem. The free product of two symmetric groups of degree 3 has a faithful matrix representation of degree 2 over Z .

Proof

Consider the matrices:

$$a = \begin{pmatrix} 2 & 1 \\ -7 & -3 \end{pmatrix}, \quad b = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$$

Then, $a^3 = b^2 = I$ and

$$ab = \begin{pmatrix} 1 & 0 \\ -5 & -1 \end{pmatrix}, (ab)^2 = I$$

$$a^2b = \begin{pmatrix} -3 & -1 \\ 7 & 3 \end{pmatrix}$$

Hence, every non-trivial element of $\langle a, b \rangle$ is of the form:

$$\begin{pmatrix} \delta_1 & \delta_2 \\ -\delta_3 & -\delta_4 \end{pmatrix} \text{ or } \begin{pmatrix} -\delta_1 & -\delta_2 \\ \delta_3 & \delta_4 \end{pmatrix} \quad \delta_i \in \mathbb{Z}^+, i=1,2,3,4$$

Consider another pair of matrices:

$$a' = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix} \quad b' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$$

Then, $a'^3 = b'^2 = (a'b')^2 = I$. As above, it is easy to see that every non-trivial element of $\langle a', b' \rangle$ can be expressed as

$$\begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_3 & -\delta_4 \end{pmatrix} \text{ or } \begin{pmatrix} -\delta_1 & \delta_2 \\ -\delta_3 & \delta_4 \end{pmatrix}$$

Take $\Gamma = (-\infty, 0)$, $\Delta = (0, \infty)$. Then, each non-trivial element of $\langle a, b \rangle$ maps Δ into Γ , and each non-trivial element of $\langle a', b' \rangle$ maps Γ into Δ . Hence, by Macbeath's lemma 1.2.1, $\langle a, b \rangle$ and $\langle a', b' \rangle$ generate their free product.

I.3 RELATION BETWEEN FREE SUBGROUPS OF $GL(n,F)$ and of $SL(n,F)$

In this section, we establish a certain relationship between free subgroups of $SL(n,F)$ and the free subgroups of $GL(n,F)$. The significance of this result lies in the fact that it reduces the study of free subgroups of $GL(n,F)$ to the investigation of free subgroups of $SL(n,F)$. Our first theorem deals with the free product of some special types of subgroups of $GL(n,F)$.

I.3.1 Theorem. Let

$$A = \langle a_1, \dots, a_m \rangle, B = \langle b_1, \dots, b_n \rangle$$

be subgroups of $GL(n,F)$. For $0 \neq \alpha, \beta \in F$, let

$$A_\alpha = \langle \alpha a_1, \dots, \alpha a_m \rangle, B_\beta = \langle \beta b_1, \dots, \beta b_n \rangle$$

and $\langle \alpha I, \beta I \rangle \cap \langle A, B \rangle = \langle \alpha I, \beta I \rangle \cap \langle A_\alpha, B_\beta \rangle = I$.

Then, $\langle A, B \rangle$ is the free product of A, B if and only if $\langle A_\alpha, B_\beta \rangle$ is the free product of A_α, B_β .

Proof

Suppose that $\langle A, B \rangle$ is the free product of A, B and let $0 \neq \alpha, 0 \neq \beta$ in F satisfy the conditions in the statement of the theorem. Let

$$w' = x'_1 y'_1 \dots x'_k y'_k = I, (k \geq 1) \tag{1}$$

$x'_i \in A_\alpha, y'_i \in B_\beta, 1 \leq i \leq k$, be a non-trivial relation in $\langle A_\alpha, B_\beta \rangle$. Then, Eqn. (1) can be written as

$$w' = \alpha^{p_\beta q} x_1 y_1 \dots x_k y_k = I$$

where $x_i \in A$, $y_i \in B$, $1 \leq i \leq k$. Therefore

$$w = x_1 y_1 \dots x_k y_k = \alpha^{-p} \beta^{-q} I$$

is an element of $\langle A, B \rangle$. Since $\langle \alpha I, \beta I \rangle \cap \langle A, B \rangle = \{I\}$, we have $\alpha^p \beta^q = I$, the identity element of F . But then

$$w = x_1 y_1 \dots x_k y_k = I, \quad (k \geq 1)$$

is a non-trivial relation in $\langle A, B \rangle$, a contradiction. Hence, $\langle A_\alpha, B_\beta \rangle$ is the free product of A_α , B_β .

For the converse replace α by α^{-1} and β by β^{-1} . Then, the above argument shows that

$$\langle \alpha^{-1} A_\alpha, \beta^{-1} B_\beta \rangle = \langle A, B \rangle$$

is the free product of A, B . Hence, the theorem.

I.3.2. Theorem. Let $A = \langle a_i : i \in \Gamma \rangle$ be a subgroup of $GL(n, F)$, F any arbitrary field. Then A is a free group on free generators a_i , $i \in \Gamma$, if and only if for any family $\{\alpha_i : 0 \neq \alpha_i \in F, i \in \Gamma\}$ the group $A' = \langle \alpha_i a_i : i \in \Gamma \rangle$ is free on free generators $\alpha_i a_i$.

Proof

Suppose that A is free on free generators a_i , $i \in \Gamma$ and let

$$w = x_1^{\xi_1} \dots x_k^{\xi_k} = I, \quad k \geq 1$$

$x_i \in \langle \alpha_i a_i : i \in \Gamma \rangle$, $\xi_i \in \mathbb{Z}$, be a non-trivial relation in A' . Then

$$w = \alpha_i^{q_i} a_{i_1}^{\xi_1} \dots a_{i_k}^{\xi_k} = I$$

so that

$$w' = a_{i_1}^{\xi_1} \dots a_{i_k}^{\xi_k} = \alpha^{-q} I, \alpha = \alpha_{i_1}^{\xi_1} \dots \alpha_{i_k}^{\xi_k},$$

is an element in the centre of A . Since A is free on free generators $a_i, i \in \Gamma$, A has trivial centre. So $\alpha^q = 1$. But then $w' = I$ is a non-trivial relation in A , contradicting the freeness of A .

The converse follows by replacing α_i by $\alpha_i^{-1}, i \in \Gamma$ in A' so that

$$A = \langle \alpha_i^{-1} a_i : i \in \Gamma \rangle = \langle a_i : i \in \Gamma \rangle$$

is free on free generators $a_i, i \in \Gamma$. Hence the theorem.

I.3.3 Corollary

Let $A = \langle a_i : i \in \Gamma \rangle$ be a free subgroup of $GL(n, F)$. Then, assuming each

$\det a_i$ has an n -th root in F , $A' = \langle \alpha_i^{-1} a_i : 0 \neq \alpha_i^n = \det a_i, i \in \Gamma \rangle$

is a free subgroup of $SL(n, F)$.

Proof

Follows directly from theorem I.3.3.

I.4 AN EXAMPLE OF A TORSION-FREE NON-FREE SUBGROUP OF $GL(2, Q)$ WHICH IS NOT SOLVABLE-BY-FINITE.

Let x, y be elements of $GL(2, C)$. Then x, y have a common eigenvector if and only if $\langle x, y \rangle$ is reducible because in case of degree 2 reducibility is equivalent to the existence of an invariant 1-dimensional subspace (spanned by the common eigenvector). One may ask the question:

Is it true that every $\langle x, y \rangle \subseteq GL(2, C)$ has a reducible subgroup of finite index or has a non-abelian free subgroup of finite index?

J. Tits [25], says that either $\langle x, y \rangle$ has a free subgroup (possibly of finite index) or $\langle x, y \rangle$ is solvable-by-finite. However, a theorem of Mal'cev (cf. Structure of Linear Groups by Dixon [4] or Infinite Linear Groups by Wehrfritz [27]) says that in the latter case there is a subgroup of finite index which is triangularizable (and hence reducible). Thus the only additional unknown in the above question is whether, in the case where $\langle x, y \rangle$ is not solvable-by-finite, non-abelian free subgroups of finite index always exist. It may be pointed out that, by a theorem of John Stallings [24], any torsion-free group with a free subgroup of finite index is itself free. We give a counter example to show that there exists a torsion-free, non-free subgroup $\langle x, y \rangle$ of $GL(2, Q)$ with no solvable subgroup of finite index.

Before giving our counter example, we prove a theorem about free subgroups of $GL(n, F)$, F an arbitrary field which may be of independent interest.

Let

$$A = \langle a_1, \dots, a_m \rangle \quad (1)$$

$$B = \langle b_1, \dots, b_n \rangle \quad (2)$$

be free subgroups of $GL(n, F)$ freely generated by the given generators.

Let $c_1, c_2, \dots, c_p; d_1, d_2, \dots, d_p$ be distinct elements of A and B respectively

such that the length ℓ of c_1 (as a word in the generators $a_1, \dots, a_m$ but not

their inverses) is equal to the length of c_j , and also equal to the length

of each d_k (as a word in $b_1, \dots, b_n$ but not their inverses). Further let

$$H = A \cap B = \langle c_1, \dots, c_p : c_i = d_i, 1 \leq i \leq p \rangle \quad (3)$$

be a subgroup common to both A and B. For any $0 \neq \alpha \in F$, let

$$A_\alpha = \langle \alpha a_1, \dots, \alpha a_m \rangle \quad (1')$$

$$B_\alpha = \langle \alpha b_1, \dots, \alpha b_n \rangle \quad (2')$$

$$H_\alpha = \langle \alpha^\ell c_1, \dots, \alpha^\ell c_p : c_i = d_i, 1 \leq i \leq p \rangle \quad (3')$$

Then

$$H_\alpha = A_\alpha \cap B_\alpha$$

1.4.1 Theorem. For A, B, H, A_α , B_α , H_α as in (1), (2), (3), (1'), (2'), (3') respectively, let

$$\langle A, B \rangle \cap \langle \alpha I \rangle = \langle A_\alpha, B_\alpha \rangle \cap \langle \alpha I \rangle = \{I\}$$

Then $\langle A, B \rangle$ is the generalized free product of A, B amalgamating H if and only if $\langle A_\alpha, B_\alpha \rangle$ is the generalized free product of A_α , B_α amalgamating H_α .

Proof

We denote the free product of the groups A, B amalgamating a subgroup H by $\langle A * B : H \rangle$. Suppose that $\langle A, B \rangle = \langle A * B : H \rangle$. Let

$$w = x_1 y_1 x_2 y_2 \dots x_k y_k = I$$

$x_i \in A_\alpha$, $y_j \in B_\alpha$, $1 \leq i, j \leq k$, be a non-trivial relation in $\langle A_\alpha, B_\alpha \rangle$. Since each x_i is a word in $\alpha a_1, \dots, \alpha a_m$ and each y_i is a word in $\alpha b_1, \dots, \alpha b_n$, we can write w as:

$$w = \alpha^q x'_1 y'_1 \dots x'_k y'_k = I$$

where now each x'_i is a word in $a_1, \dots, a_m$ and each y'_j is a word in $b_1, \dots, b_n$. Hence

$$w' = x'_1 y'_1 \dots x'_k y'_k = \alpha^{-q} I$$

is an element of $\langle A, B \rangle$. Since $\langle \alpha I \rangle \cap \langle A, B \rangle = \{I\}$, we must have $\alpha^q = 1$, the identity element of F . But, then

$$w' = x'_1 y'_1 \dots x'_k y'_k = I$$

is a non-trivial relation in $\langle A, B \rangle = \langle A * B : H \rangle$, a contradiction. Hence,

$$\langle A_\alpha, B_\alpha \rangle = \langle A * B : H_\alpha \rangle.$$

Conversely, suppose that $\langle A_\alpha, B_\alpha \rangle = \langle A * B : H_\alpha \rangle$ and let

$$w = x_1 y_1 \dots x_k y_k = I$$

$x_i \in A, y_j \in B, 1 \leq i \leq k, 1 \leq j \leq k$, be a non-trivial relation in $\langle A, B \rangle$. If m_i, n_j are the lengths of x_i and y_j respectively, $1 \leq i, j \leq k$, then we have

$$w' = \alpha^{m_1 + \dots + m_k + n_1 + \dots + n_k} x_1 y_1 \dots x_k y_k$$

$$= \alpha^{m_1 + \dots + m_k + n_1 + \dots + n_k} I$$

$$= \alpha^q I$$

as an element of $\langle A_\alpha, B_\alpha \rangle$. As $\langle A_\alpha, B_\alpha \rangle \cap \langle I \rangle = \{I\}$, we get $\alpha^q = 1$ and so $w' = I$

is a non-trivial relation in $\langle A_\alpha, B_\alpha \rangle$, a contradiction. Hence,

$$\langle A, B \rangle = \langle A * B : H \rangle.$$

As a very special case of the above theorem, we have the following:

1.4.2 Corollary

Let $a, b \in \text{SL}(2, \mathbb{C})$ be elements of the same finite or infinite order such that $A = \langle a \rangle$, $B = \langle b \rangle$ generate their generalized free product amalgamating a subgroup $H = \langle a^k = b^k \rangle$. Then, for any $0 \neq \alpha \in \mathbb{C}$ which is not a $2n$ -th primitive root of unity for any n , $A_\alpha = \langle \alpha a \rangle$, $B_\alpha = \langle \alpha b \rangle$ generate the generalized free product of A_α, B_α amalgamating $H_\alpha = \langle \alpha^k a^k = \alpha^k b^k \rangle$.

Proof

For each $x \in \text{SL}(2, \mathbb{C})$, $\det x = 1$. Thus, if $x \in \langle a, b \rangle \langle \alpha I \rangle$, then $x = \alpha^q I$ for some integer q and so

$$1 = \det x = \alpha^{2q}$$

Since α is not a primitive $2n$ -th root of unity for any n , $q=0$ so that $x=I$. Hence, $\langle a, b \rangle \langle \alpha I \rangle = I$ and the corollary follows from the above theorem.

1.4.3 Counter Example

Consider the matrices

$$a = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

These matrices satisfy the relation

$$a^3 = b^3 = -I.$$

Morris Newman [19] has shown that $\langle a \rangle, \langle b \rangle$ generate their free product amalgamating the subgroup $H = \{\pm I\}$. Now,

$$2a = \begin{pmatrix} 2 & -2 \\ 2 & 0 \end{pmatrix}, \quad 2b = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}$$

so, by corollary 1.4.2, we see that $\langle 2a, 2b \rangle$ is the generalized free product of $A_2 = \langle 2a \rangle$, $B_2 = \langle 2b \rangle$ amalgamating the subgroup $H_2 = \langle -8I \rangle$. Since

$$(2a)^3 = 8a^3 = -8I = (2b)^3$$

both $2a$ and $2b$ generate infinite cyclic groups A_2 and B_2 respectively. B.H. Neumann (cf. [18], corollary 5.1) has shown that the free product of two torsion-free groups amalgamating a subgroup is torsion free.

Thus,

$$G = \langle 2a, 2b \rangle = \langle A_2 * B_2 : H_2 \rangle$$

is torsion-free. G is obviously non-free because $(2a)^3 = (2b)^3$ is a non-trivial relation in G . Also, G cannot be solvable-by-finite because G has a factor group isomorphic to

$$G_1 = \langle a_1, b_1 : a_1^3 = b_1^3 = 1 \rangle$$

which contains non-abelian free subgroups. (For instance, the derived group of G_1 is free of rank 4.) Thus, the group G furnishes an example of a torsion-free, non-free group in $GL(2, Q)$ which is not solvable-by-finite.

We now prove a theorem about the torsion-free subgroups of the free product of periodic groups.

I.4.4 Theorem

Let G be the free product of periodic groups $A_\alpha, \alpha \in \Lambda$, and H , a subgroup of G . Then, H is free if and only if H is torsion-free.

Proof

Since every free group is torsion-free, the condition is necessary. To prove the sufficiency of the condition, suppose that H is a torsion free subgroup of the free product of periodic groups $A_\alpha, \alpha \in \Lambda$. By Kurosh's subgroup theorem (cf. [11a]) for free products

$$H = F * \pi_{x_\alpha \in G}^* (x_\alpha A_\alpha x_\alpha^{-1} \cap H)$$

where F is a free group and $\pi_{x_\alpha \in G}^* (x_\alpha A_\alpha x_\alpha^{-1} \cap H)$ is the free product of $x_\alpha A_\alpha x_\alpha^{-1} \cap H, \alpha \in \Lambda, x_\alpha \in G$. Since H is torsion-free and each $x_\alpha A_\alpha x_\alpha^{-1}$ is periodic, $x_\alpha A_\alpha x_\alpha^{-1} \cap H = \{1\}$ for all $\alpha \in \Lambda$. Hence, $\pi_{x_\alpha \in G}^* (x_\alpha A_\alpha x_\alpha^{-1} \cap H) = \{1\}$, so that $H = F$ is free.

The situation is not very hopeful if the condition that the free factors be periodic is removed. In such a case, a torsion-free subgroup of a free product need not be free. (See example I.4.6 below.)

I.4.5 Corollary (Newman [19a]).

Every torsion-free subgroup of $SL(2, \mathbb{Z})$ is free.

Proof

Every torsion-free subgroup G of $SL(2, \mathbb{Z})$ is isomorphic to a subgroup $\bar{G} = G\mathbb{Z}^*/\mathbb{Z}^*$ of $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}^*$ where $\mathbb{Z}^* = \{\pm 1\}$. However, $PSL(2, \mathbb{Z})$ is the free product of a cyclic group of order 2 and a cyclic group of order 3, [19a].

One may ask the question whether or not there exist torsion-free non-free subgroups in $SL(2, \mathbb{C})$. The following example shows that, in general, an irreducible torsion-free subgroup of $SL(2, \mathbb{C})$ may not be free.

I.4.6 Example

Let

$$G = \langle a_1 = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 & 0 \\ xi & 1 \end{pmatrix}, b_1 = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 & yi \\ 0 & 1 \end{pmatrix} \rangle$$

where $|x| \geq 2$, $|y| \geq 2$, $x, y \in \mathbb{R}$, $i = \sqrt{-1}$. Chang, Jennings and Ree have shown that G is the free product of two free abelian groups $A = \langle a_1, a_2 \rangle$, $B = \langle b_1, b_2 \rangle$ each of rank 2. By the result of B.H. Neumann, about the free product of torsion-free groups already mentioned above, G is torsion-free. Now, every subgroup of a torsion-free group is itself torsion-free. However, the subgroup $H = \langle a_1, b_1, b_2 \rangle$ of G is torsion-free but not free. H is, of course, irreducible because $\langle a_1, b_1 \rangle$ is free (cf. [2]).

However, I do not know whether there exists a two generator irreducible torsion-free non-free group in $SL(2, \mathbb{C})$. In this connection, it is worth mentioning that Alice Whitmore [29] has given a representation of the group of Listing's knot as a subgroup of $SL(2, \mathbb{C})$. The group L of Listing's knot has the following presentation:

$$L = \langle a, b : b^{-1} a^{-1} b a b^{-1} a b a^{-1} b^{-1} a = 1 \rangle$$

L is torsion-free and non-free. The faithfulness of the representation of L as a subgroup of $SL(2, \mathbb{C})$ is still an open question. A positive answer to this will lead to the solution of the question about the existence of torsion-free non-free 2-generator irreducible subgroups of $SL(2, \mathbb{C})$.

Another related question, still, unsolved, is the following:
Let $a, b \in SL(2, \mathbb{C})$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$ and let $\alpha, \beta, \gamma \in \mathbb{Z}$. Suppose that $\langle a, b \rangle$ is torsion-free. Is $\langle a, b \rangle$ free? (See corollary I.4.5.)

CHAPTER II

II.0 INTRODUCTION

Sufficient conditions of different types for a pair of matrices a, b in $SL(2, \mathbb{C})$ to generate a free group of rank 2 were given by various authors, e.g. Frasc [7], Sanov [22], Doniakhi [6], Brenner [1], Chang, Jennings and Ree [2], Newman [20], Goldberg and Newman [9] and Lyndon and Ullman [14], [15]. Recently, Purzitsky [30], [31] and Herzberger [32], working independently, gave necessary and sufficient conditions for the group $\langle a, b \rangle \leq SL(2, \mathbb{R})$ to be the discrete free product of the cyclic groups $\langle a \rangle, \langle b \rangle$. We give an alternative proof of the sufficiency condition of one of their theorems. This result is inserted here since the ideas contained in its proof will be used to establish one of our later results.

Section I of this chapter contains a necessary and sufficient condition for $\langle a, b \rangle$ to be an irreducible subgroup of $SL(2, \mathbb{C})$. In this section, we reduce by conjugation an arbitrary pair of matrices $a, b \in SL(2, \mathbb{C})$ into a special form involving the traces of a, b and ab only. By a result of Purzitsky [30], [31] and Herzberger [32], $\langle a, b \rangle \leq SL(2, \mathbb{R})$ is a discrete free group of rank 2 if $\text{tr } a = \alpha \geq 2$, $\text{tr } b = \beta \geq 2$, $\text{tr } ab^{-1} \leq -2$ (or equivalently $\text{tr } ab = \gamma \geq \alpha\beta + 2$). Suppose that α, β, γ are integers and denote $\langle a, b \rangle$ by $F_{\alpha, \beta, \gamma}$. Then, by the above result, $F_{3, 2, \gamma}$ is a discrete free group iff $\gamma \geq 8$ or $\gamma \leq -2$. In Theorem II.2.7, we examine $F_{3, 2, \gamma}$ in the non-discrete case to see whether or not $F_{3, 2, \gamma}$ is free for integral values of γ such that $-2 < \gamma < 8$.

In Section II.3, we discuss the relationship of the transcendence degree of $Q(\alpha, \beta, \gamma)$ over Q , $\alpha = \text{tr } a$, $\beta = \text{tr } b$, $\gamma = \text{tr } ab$, and the freeness of the non-cyclic group $\langle a, b \rangle$. We show that while for the case when the transcendence degree of $Q(\alpha, \beta, \gamma)$ over Q is 3, $\langle a, b \rangle$ is necessarily free.

of rank 2, the same may not be true when the transcendence degree of $Q(\alpha, \beta, \gamma)$ over Q is less than 3, as example II.3.2 shows. In this section, we also show that the set $S(\alpha, \beta)$ consisting of all $\gamma \in C$ such that $\langle a, b \rangle$ is not the free product of $\langle a \rangle$ and $\langle b \rangle$ is at most countable; and deduce that $\langle a, b \rangle$ is free of rank 2 whenever γ is not algebraic over $Q(\alpha, \beta)$.

In Section II.4, we determine some values of α, β, γ $\alpha = \text{tr } a$, $\beta = \text{tr } b$, $\gamma = \text{tr } ab$, for which $\langle a, b \rangle$ is not free of rank 2.

II.1 REDUCIBLE AND IRREDUCIBLE 2-GENERATOR SUBGROUPS OF $SL(2, C)$

Let $a, b \in SL(2, C)$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. In this section, we determine a necessary and sufficient condition involving only the traces of a , b and ab for the group $G = \langle a, b \rangle$ to be irreducible. For that, we shall bring the matrices a and b , by a suitable conjugation, into a special form. Such a representation will help us give a criterion for the freeness of the group $G = \langle a, b \rangle$ as well.

II.1.1 Theorem

Let $a, b \in SL(2, C)$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. Then, $G = \langle a, b \rangle$ is irreducible if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0 \quad (*)$$

Proof

Suppose that $\langle a, b \rangle$ is irreducible. Then, $a \neq I$, $b \neq I$. By a similarity transformation, both a and b can be brought into the form:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} u_1 & u_2' \\ u_3' & \beta - u_1 \end{pmatrix} \quad (1)$$

where $\lambda + \lambda^{-1} = \alpha = \text{tr } a \neq \pm 2$, $\mu_1(\beta - \mu_1) - \mu_2'\mu_3' = 1$, $\beta = \text{tr } b$, $\mu_2' \neq 0$, $\mu_3' \neq 0$, because of irreducibility; or

$$\pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \beta - \mu_1 \end{pmatrix} \quad (2)$$

when $\text{tr } a = \pm 2$. In the latter case, $\mu_2' \neq 0$ because of irreducibility of $\langle a, b \rangle$: In the case of (1), we conjugate the matrices by

$$c = \begin{pmatrix} \delta & 0 \\ 0 & \delta' \end{pmatrix}$$

and obtain

$$a' = c \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} c^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

$$b' = c \begin{pmatrix} \mu_1 & \mu_2' \\ \mu_3' & \beta - \mu_1 \end{pmatrix} c^{-1} = \begin{pmatrix} \mu_1 & \delta\delta'^{-1}\mu_2' \\ \delta^{-1}\delta'\mu_3' & \beta - \mu_1 \end{pmatrix}$$

Since $\mu_3' \neq 0$, we choose δ, δ' such that $\mu_3' = \delta\delta'^{-1}$. Then

$$a' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad b' = \begin{pmatrix} \mu_1 & \mu_2' \\ 1 & \beta - \mu_1 \end{pmatrix} \quad (3)$$

where $\mu_2' = \mu_2'\mu_3' = -1 + \mu_1(\beta - \mu_1)$

In case (2), that is, when $\alpha = \pm 2$, we conjugate both matrices in

(2) by

$$c = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$$

so that

$$a'' = cac^{-1} = \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$b'' = cbc^{-1} = \begin{pmatrix} \mu_1 - \delta\mu_2 & \mu_2 \\ \delta\mu_1 + \mu_3 - \delta^2\mu_2 - \delta(\beta - \mu_1) & \delta\mu_2 + \beta - \mu_1 \end{pmatrix}$$

Take

$$\delta = \mu_1/\mu_2, \mu_2 \neq 0.$$

Then

$$a'' = \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, b'' = \begin{pmatrix} 0 & \mu_2 \\ -1/\mu_2 & \beta \end{pmatrix}, \mu_2 \neq 0 \quad (4)$$

Now

$$a'b' = \begin{pmatrix} \lambda\mu_1 & \lambda\mu_2 \\ \lambda^{-1} & \lambda^{-1}(\beta - \mu_1) \end{pmatrix}, a''b'' = \begin{pmatrix} 0 & \mu_2 \\ -1/\mu_2 & \beta + \mu_2 \end{pmatrix}$$

in cases (3) and (4), respectively. Since $\text{tr } a'b' = \text{tr } ab$ and $\text{tr } a''b'' = \text{tr } ab$ in the respective cases, and $\text{tr } ab = \gamma$, we have:

$$\mu_1 = (\gamma - \beta/\lambda)/(\lambda - \lambda^{-1}), \mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4)/(\alpha^2 - 4), \alpha \neq \pm 2 \quad (5)$$

in case (3) and

$$\mu_2 = \gamma' - \beta, \gamma' = \pm \gamma \quad (6)$$

in case of (4). Hence, a, b can be transformed by a suitable conjugation into the form:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix}, \quad \lambda + \lambda^{-1} = \alpha \neq \pm 2$$

with μ_1, μ_2 being given by (5); and into the form:

$$\pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \gamma' - \beta \\ -1/\gamma' - \beta & \beta \end{pmatrix}, \quad \gamma' = \pm \gamma$$

when $\alpha \neq \pm 2$. By irreducibility of $\langle a, b \rangle$

$$\mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4)/(\alpha^2 - 4) \neq 0, \quad \alpha \neq \pm 2$$

that is

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0 \quad (7)$$

which conforms to the condition

$$\gamma' - \beta \neq 0, \quad \gamma' = \pm \gamma \quad (8)$$

for the case (2) when $\alpha = \pm 2$.

Conversely, if conditions (7) and (8) are satisfied, then $\mu_2 \neq 0$ in both cases and so $\langle a, b \rangle$ is irreducible. Hence, $\langle a, b \rangle$ is irreducible if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0.$$

The following corollary is immediate.

II.1.2 Corollary

Let $a, b \in SL(2, C)$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. Then, $\langle a, b \rangle$ is reducible if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 = 0$$

II.1.3 Corollary

Suppose that $\langle a, b \rangle$ is a subgroup of $SL(2, C)$ and let $\lambda, \lambda^{-1}, \mu, \mu^{-1}$ be the characteristic roots of a, b respectively. Then, $\langle a, b \rangle$ is reducible if and only if

$$\text{tr } ab = \lambda/\mu + \mu/\lambda \text{ or } \lambda\mu + \lambda^{-1}\mu^{-1}$$

Proof

By Corollary II.1.2 $\langle a, b \rangle$ is reducible if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 = 0, \alpha = \text{tr } a, \beta = \text{tr } b, \gamma = \text{tr } ab$$

Now, $\alpha = \lambda + \lambda^{-1}$, $\beta = \mu + \mu^{-1}$. Putting the values of α, β in terms of the characteristic roots of a and b , and solving the quadratic equation in γ given above, we see that:

$$\begin{aligned} \gamma &= [\alpha\beta \pm \{(\alpha^2 - 4)(\beta^2 - 4)\}^{1/2}] / 2 \\ &= \lambda/\mu + \mu/\lambda \text{ or } \lambda\mu + \lambda^{-1}\mu^{-1} \end{aligned}$$

Hence the corollary:

II.1.4 Corollary

If α , β , and γ satisfy the inequality (*), then there exist $a, b \in \text{SL}(2, \mathbb{C})$ such that $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$ and $\langle a, b \rangle$ is an irreducible subgroup. Moreover, a and b are completely determined up to conjugacy; that is, if $\text{tr } a' = \alpha$, $\text{tr } b' = \beta$ and $\text{tr } a'b' = \gamma$ then there exists $c \in \text{GL}(2, \mathbb{C})$ such that $a' = cac^{-1}$, $b' = cbc^{-1}$.

Proof

From Equations (3) and (4) of theorem I.1.1, we see that any two matrices $a, b \in \text{SL}(2, \mathbb{C})$ can be brought, by conjugation, to the form:

$$a_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad b_1 = \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix}$$

when $\text{tr } a = \alpha \neq \pm 2$, $\mu_1 = (\gamma - \beta/\lambda)/(\lambda - \lambda^{-1})$, $\mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4)/(\alpha^2 - 4) \neq 0$; or into the form:

$$a_1 = \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & \gamma' - \beta \\ -1/(\gamma' - \beta) & \beta \end{pmatrix}, \quad \gamma' = \pm \gamma$$

when $\text{tr } a_1 = \pm 2$. If a' and b' are any two other elements in $\text{SL}(2, \mathbb{C})$ with $\text{tr } a' = \alpha = \text{tr } a$, $\text{tr } b' = \beta = \text{tr } b$, $\text{tr } a'b' = \gamma = \text{tr } ab$, then a', b' also are conjugate to a_1, b_1 respectively. Hence, there exist $c_1, c_2 \in \text{GL}(2, \mathbb{C})$ such that

$$a_1 = c_1 a c_1^{-1}, \quad b_1 = c_1 b c_1^{-1}$$

and

$$a_1 = c_2 a' c_2^{-1}, \quad b_1 = c_2 b' c_2^{-1}$$

and so

$$a' = cac^{-1}, b' = cbc^{-1}$$

where

$$c = c_2^{-1} c_1.$$

Hence, the corollary.

II.1.5 Theorem

Let $G = \langle a, b \rangle$ be an irreducible subgroup of $SL(2, C)$.

- (i) If $\text{tr } a = \alpha = \pm 2$, $\text{tr } b = \beta = \pm 2$, $\text{tr } ab = \gamma$, then a, b can be put, by a similarity transformation, simultaneously into the form:

$$\pm \begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & \gamma' - 2 \\ 0 & 1 \end{pmatrix}, \gamma' = \pm \gamma \quad (1)$$

respectively, the signs being taken appropriately.

- (ii) If the trace of at least one of the a and b , say of a , is different from ± 2 , then a, b can be put, by a similarity transformation, simultaneously into the form:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix} \quad (2)$$

respectively, where $\lambda + \lambda^{-1} = \alpha = \text{tr } a$, $\beta = \text{tr } b$, $\gamma = \text{tr } ab$,

$$\mu_1 = (\gamma - \beta/\lambda)/(\lambda - \lambda^{-1}), \mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4)/(\alpha^2 - 4), \alpha \neq \pm 2$$

Proof

We have already shown that if $\langle a, b \rangle$ is irreducible, a and b can be put in the form:

$$a = \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \gamma' - \beta \\ -1/\gamma' - \beta & \beta \end{pmatrix} \quad (1')$$

where $\text{tr } a = \pm 2$, the signs being taken appropriately, and

$$a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix} \quad (2')$$

otherwise, $\lambda, \beta, \gamma, \mu_1, \mu_2$ are as given above.

In (1') if $\beta = \pm 2$, then conjugate b by

$$c = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$$

We obtain

$$cbc^{-1} = \begin{pmatrix} -\delta(\gamma' \mp 2) & \gamma' \mp 2 \\ -1/(\gamma' \mp 2) - \delta^2(\gamma' \mp 2) \mp 2\delta & \delta(\gamma' \mp 2) \pm 2 \end{pmatrix}$$

Choose

$$\delta = \mp 1/(\gamma' \mp 2)$$

Then

$$cbc^{-1} = \begin{pmatrix} \pm 1 & \gamma' \mp 2 \\ 0 & \pm 1 \end{pmatrix}, \quad cac^{-1} = a$$

If, however, $\text{tr } a = \pm 2$, then as shown before, we have:

$$a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix}$$

for all β where $\lambda, \beta, \mu_1, \mu_2$ are as given in the theorem. Hence, the result.

II.1.6 Theorem

Let $G = \langle a, b \rangle$ be an irreducible subgroup of $SL(2, C)$, λ a characteristic root of a and μ a characteristic root of b . Then a and b can be brought, by conjugation, simultaneously into the form:

$$a = \begin{pmatrix} \lambda & 0 \\ \epsilon & \lambda^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}, \quad \epsilon \neq 0, \eta \neq 0$$

Proof

We can suppose that the trace of at least one of the a and b is different from ± 2 , for otherwise, the fact that a and b can be put in the form mentioned in the statement of the theorem is part (1) of theorem II.1.5. Without any loss of generality, we can assume that $\text{tr } a \neq \pm 2$. Then, by theorem II.1.5, a and b can be brought into the form:

$$a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix}$$

respectively, $\lambda + \lambda^{-1} = \alpha = \text{tr } a$, $\beta = \text{tr } b$, $\gamma = \text{tr } ab$, $\mu_1 = (\gamma - \beta/\lambda)/(\lambda - \lambda^{-1})$,
 $\mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4)/(\alpha^2 - 4)$.

To prove the theorem, we require an invertible c such that

$$c^{-1}ac = \begin{pmatrix} \lambda & 0 \\ * & \lambda^{-1} \end{pmatrix}, \quad c^{-1}bc = \begin{pmatrix} \mu & * \\ 0 & \mu^{-1} \end{pmatrix}$$

Take

$$c = \begin{pmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_4 \end{pmatrix}$$

Then

$$ac = c \begin{pmatrix} \lambda & 0 \\ * & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} * & \gamma_3 \lambda^{-1} \\ * & \gamma_4 \lambda^{-1} \end{pmatrix}$$

$$bc = c \begin{pmatrix} \mu & * \\ 0 & \mu^{-1} \end{pmatrix} = \begin{pmatrix} \mu \gamma_1 & * \\ \mu \gamma_2 & * \end{pmatrix}$$

Thus, if we choose $\begin{pmatrix} \gamma_3 \\ \gamma_4 \end{pmatrix}$ as eigenvector for λ^{-1} for a and $\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ as eigenvector for μ for b , then $c^{-1}ac$ and $c^{-1}bc$ are in the required form.

Here, of course, the eigenvectors $\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ and $\begin{pmatrix} \gamma_3 \\ \gamma_4 \end{pmatrix}$ are linearly independent (otherwise, they would span a 1-dimensional space invariant under $\langle a, b \rangle$ contrary to the irreducibility of $\langle a, b \rangle$).

II.1.7 Corollary

Suppose that $\langle a, b \rangle$ is an irreducible subgroup of $SL(2, R)$ with $\text{tr } a = \alpha \geq 2$, $\text{tr } b = \beta \geq 2$. Then a, b can be brought simultaneously into the form:

$$\begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}, \xi \neq 0, \eta \neq 0$$

where $\lambda \geq 1, \mu \geq 1$.

Proof

Just invert the matrices, if necessary.

II.1.8 Theorem

Suppose that

$$a = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

Then $\langle a, b \rangle$ is reducible if and only if either $\xi\eta=0$ or $\xi\eta=-(\lambda-\lambda^{-1})(\mu-\mu^{-1})$.

Proof

Here, $\text{tr } ab = \xi\eta + \lambda\mu + \lambda^{-1}\mu^{-1}$. Hence the condition

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 = 0$$

of reducibility of $\langle a, b \rangle$, after substituting $\lambda + \lambda^{-1}$ for $\alpha, \mu + \mu^{-1}$ for β , $\xi\eta + \lambda\mu + \lambda^{-1}\mu^{-1}$ for γ , becomes

$$\xi\eta(\xi\eta + (\lambda - \lambda^{-1})(\mu - \mu^{-1})) = 0$$

i.e. $\xi\eta=0$ or $\xi\eta=-(\lambda-\lambda^{-1})(\mu-\mu^{-1})$.

II.1.9 Corollary

Let

$$a = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

Then, $\langle a, b \rangle$ is not free if $\xi\eta=0$ or $\xi\eta=-(\lambda-\lambda^{-1})(\mu-\mu^{-1})$.

II.1.10 Corollary

The subgroups $\langle a, b \rangle$ and $\langle a', b' \rangle$ where

$$a = \begin{pmatrix} e^{\theta} & 0 \\ \xi & e^{-\theta} \end{pmatrix}, \quad b = \begin{pmatrix} e^{\phi} & \eta \\ 0 & e^{-\phi} \end{pmatrix}$$

and

$$a' = \begin{pmatrix} e^{i\theta} & 0 \\ \xi & e^{-i\theta} \end{pmatrix}, \quad b' = \begin{pmatrix} e^{i\phi} & \eta \\ 0 & e^{-i\phi} \end{pmatrix}$$

are not free if $\xi\eta=0$ or $\xi\eta=-4\sinh\theta\sinh\phi$ in the case of $\langle a, b \rangle$; and $\xi\eta=0$ or $\xi\eta=4\sin\theta\sin\phi$ in case of $\langle a', b' \rangle$.

Proof

Here both $\langle a, b \rangle$ and $\langle a', b' \rangle$ are reducible.

II.2 FREE SUBGROUPS IN $SL(2, R)$

Let a, b be elements of $SL(2, R)$ where R is the field of real numbers. Then, a and b correspond to linear fractional transformations of the form:

$$\underline{a}(z) = (\lambda_1 z + \lambda_2) / (\lambda_3 z + \lambda_4), \quad \underline{b}(z) = (\mu_1 z + \mu_2) / (\mu_3 z + \mu_4)$$

$z \in \mathbb{C}$ with $\lambda_1\lambda_4 - \lambda_2\lambda_3 = 1$, $\mu_1\mu_4 - \mu_2\mu_3 = 1$, $\lambda_i, \mu_i \in R$, $i=1, 2, 3, 4$. We then write a and b as:

$$a = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}, \quad b = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$$

Various authors have given different sufficient conditions for certain pairs of hyperbolic, parabolic or elliptic linear transformations to generate free groups of rank 2. (A linear fractional transformation a is called hyperbolic, parabolic or elliptic according as $(\text{tr } a)^2 = (\lambda_1 + \lambda_4)^2 > 4$, $(\text{tr } a)^2 = 4$ or $(\text{tr } a)^2 < 4$ respectively).

Newman [20] has given sufficient conditions for two hyperbolic or parabolic linear fractional transformations to generate a free group of rank 2. Chang, Jennings, and Ree [2] have discussed certain pairs of parabolic linear transformations which generate non-cyclic free groups (cf. also Brenner [1]). Lyndon and Ullman ([14],[15]) also have examined pairs of parabolic transformations that generate free groups.

Purzitsky [30], [31] and Herzberger [32] have given necessary and sufficient conditions for the group generated by a pair a, b in $SL(2, R)$ to be the discrete free product of $\langle a \rangle$, $\langle b \rangle$. In what follows, we give an alternative proof of the sufficiency condition of one of their results. The particular form of the matrices obtained in the proof of our result is used in a later result (theorem II.2.7).

II.2.1 Theorem

Let $G = \langle a, b \rangle$ be a subgroup of $SL(2, R)$ with $\text{tr } a = \alpha \geq 2$, $\text{tr } b = \beta \geq 2$, $\text{tr } ab = \gamma \geq \alpha\beta + 2$. Then, G is free of rank 2.

Proof

If $\text{tr } a = \alpha \geq 2$, $\text{tr } b = \beta \geq 2$, the matrices a and b can be transformed, by theorem II.1.5, into the form:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \gamma - 2 \\ 0 & 1 \end{pmatrix} \quad (1)$$

when $\alpha = 2$, $\beta = 2$; and into the form:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix} \quad (2)$$

when $\alpha \neq 2$. In case (1), $\langle a, b \rangle$ is free by results of Chang, Jennings and Ree [2] (see also Lyndon and Ullman [14] §3) provided that:

$$\gamma - 2 \geq 4, \text{ that is } \gamma \geq 6 = 2 \times 2 + 2$$

When $\alpha = \lambda + \lambda^{-1} > 2$, then a and b are of the form given in (2), with $\mu_1 = (\gamma - \beta / \lambda) / (\lambda - \lambda^{-1})$ and $\mu_2 = -(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4) / (\lambda - \lambda^{-1})^2$. Let

$$c = \begin{pmatrix} 1 & -\mu_1 \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} a_1 = cac^{-1} &= \begin{pmatrix} 1 & -\mu_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \mu_1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & \mu_1(\lambda - \lambda^{-1}) \\ 0 & \lambda^{-1} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} b_0 = cbc^{-1} &= \begin{pmatrix} 1 & -\mu_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 \\ 1 & \beta - \mu_1 \end{pmatrix} \begin{pmatrix} 1 & \mu_1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mu_2 - \mu_1(\beta - \mu_1) \\ 1 & \beta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & \beta \end{pmatrix} \end{aligned}$$

because $\mu_1(\beta - \mu_1) - \mu_2 = 1$. Hence

$$\begin{aligned} a_0 &= b_0 a_1^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} \lambda^{-1} & -\mu_1(\lambda - \lambda^{-1}) \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & \beta\lambda - \mu_1(\lambda - \lambda^{-1}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & -\gamma + \alpha\beta \end{pmatrix} \end{aligned}$$

by using $\mu_1 = (\gamma - \beta/\lambda)/(\lambda - \lambda^{-1})$. Now

$$G = \langle a, b \rangle \approx \langle a_1, b_0 \rangle = \langle b_0 a_1^{-1}, b_0 \rangle \approx \langle a_0, b_0 \rangle$$

so that we have only to show that $\langle a_0, b_0 \rangle$ is free of rank 2 provided that α, β, γ satisfy the conditions stated in the theorem. Now, it is shown in [3a] that for any matrix

$$x = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, C)$$

$$x^n = \frac{\lambda_2 \lambda_1^n - \lambda_1 \lambda_2^n}{(\lambda_2 - \lambda_1)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

when the characteristic roots λ_1, λ_2 of x are distinct; and

$$x^n = (1-n)\lambda_1^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + n\lambda_1^{n-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

when the characteristic roots of x are identical. For $x \in \text{SL}(2, \mathbb{R})$, $\lambda_1 \lambda_2 = 1$, that is, $\lambda_2 = \lambda_1^{-1}$, in the case when the characteristic roots of x are distinct and $\lambda_1 = \lambda_2 = \pm 1$ when the characteristic roots of x are identical. Hence, if we put $s_n = (\lambda_1^{n+1} - \lambda_1^{-(n+1)}) / (\lambda_1 - \lambda_1^{-1})$, we have

$$a_o^n = \begin{pmatrix} -s_{n-2} & -\lambda s_{n-1} \\ \lambda^{-1} s_{n-1} & s_n \end{pmatrix}$$

where $\alpha' = \gamma - \alpha\beta \neq 2$; and

$$a_o^n = (-1)^n \begin{pmatrix} -(n-1) & n\lambda \\ -n\lambda^{-1} & n+1 \end{pmatrix}$$

where $\alpha' = 2$. In both cases, a_o^n is either of the form:

$$\begin{pmatrix} -\alpha & \beta \\ -\gamma & \delta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & -\beta \\ \gamma & -\delta \end{pmatrix}$$

$\alpha, \beta, \gamma, \delta \geq 0$, according as n is even or odd. Similarly, each power (positive or negative) of b_o is of the form:

$$\begin{pmatrix} -\alpha' & -\beta' \\ \gamma' & \delta' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha' & \beta' \\ -\gamma' & -\delta' \end{pmatrix}$$

Let $\mathbb{R}^*$ denote the extended real line. Then, $G = \langle a, b \rangle = \langle a_o, b_o \rangle$ acts as a group of linear transformations on $\mathbb{R}^*$. We take

$$\Gamma = (-\infty, 0) \text{ and } \Delta = (0, \infty)$$

as the two disjoint subsets of R^* . It is easy to see that each element of $\langle a_0 \rangle$ transforms Γ into Δ and each element of $\langle b_0 \rangle$ transforms Δ into Γ . By Macbeath's lemma I.2.1, $\langle a_0, b_0 \rangle$ is a free product of $\langle a_0 \rangle$ and $\langle b_0 \rangle$. Since a_0, b_0 both are non-trivial parabolic or hyperbolic linear transformations, they have infinite order. Hence, $\langle a_0, b_0 \rangle$ is a free group of rank 2. Consequently, $G = \langle a, b \rangle$ is a free group of rank 2.

II.2.2 Corollary

Let $G = \langle a, b \rangle$ be an irreducible subgroup of $SL(2, R)$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. Then, in all the following cases, G is free of rank 2.

- (i) $\alpha \geq 2, \beta \geq 2, \gamma \leq -2$
- (ii) $\beta \geq 2, \gamma \geq 2, \alpha \geq \beta\gamma + 2$ or $\alpha \leq -2$
- (iii) $\alpha \geq 2, \gamma \geq 2, \beta \geq \alpha\gamma + 2$ or $\beta \leq -2$
- (iv) $\alpha \leq -2, \beta \geq 2, \gamma \geq 2$ or $\gamma \leq \alpha\beta - 2$
- (v) $\beta \leq -2, \alpha \geq 2, \gamma \geq 2$ or $\gamma \leq \alpha\beta - 2$
- (vi) $\alpha \leq -2, \beta \leq -2, \gamma \geq \alpha\beta + 2$ or $\gamma \leq -2$

Proof

- (i) Since $\langle a, b \rangle$ is free of rank 2 if and only if $\langle a^{-1}, b \rangle$ is free of rank 2, we apply theorem II.2.1 to the group $\langle a^{-1}, b \rangle$ observing that $\text{tr } a^{-1} = \alpha$, $\text{tr } b = \beta$, $\text{tr } a^{-1}b = \text{tr}(\alpha - a)b = \alpha\beta - \gamma$. Hence, $\langle a^{-1}, b \rangle$ is free if $\alpha\beta - \gamma \geq \alpha\beta + 2$, that is, if $\gamma \leq -2$ which is so. Hence, $\langle a, b \rangle$ is free.
- (ii) In this case, we consider the group $\langle ab, b^{-1} \rangle$ which is free if and only if $\langle a, b \rangle$ is. We apply theorem II.2.1 to the group $\langle ab, b^{-1} \rangle$. Here $\text{tr } ab = \gamma$, $\text{tr } b^{-1} = \beta$, and $\text{tr } ab.b^{-1} = \text{tr } a = \alpha$. Hence,

$\langle ab, b^{-1} \rangle$ is free if $\alpha \geq \beta\gamma + 2$ or $\alpha \leq -2$, using the theorem and part (1) of this corollary.

(iii) The proof of (iii) is similar to that of (ii).

(iv) The case when $\beta \geq 2$, $\gamma \geq 2$ and $\alpha \leq -2$ has been dealt with in (ii). So we examine the case when $\gamma \leq \alpha\beta - 2$. Here we consider the group $\langle -a, b \rangle$ which is free if and only if $\langle a, b \rangle$ is free. But in the case of $\langle -a, b \rangle$, $\text{tr}(-a) = -\alpha \geq 2$, $\text{tr } b = \beta \geq 2$ and $\text{tr}(-ab) = -\text{tr } ab = -\gamma \geq -\alpha\beta + 2$, that is $\gamma \leq \alpha\beta - 2$ for $\langle -a, b \rangle$ and hence also $\langle a, b \rangle$ to be free of rank 2.

(v) This is similar to the proof of (iv).

(vi) If $\text{tr } a = \alpha \leq -2$, $\text{tr } b = \beta \leq -2$, then $\text{tr}(-a)(-b) = \text{tr } ab = \gamma$. As $\langle a, b \rangle$ is free on two generators if and only if $\langle -a, -b \rangle$ is free on two generators, we have, for the group $\langle -a, -b \rangle$

$$\text{tr}(-a) = -\alpha \geq 2, \text{tr}(-b) = -\beta \geq 2, \text{tr}(-a)(-b) = \text{tr } ab = \gamma$$

where $\gamma \geq (-\alpha)(-\beta) + 2 = \alpha\beta + 2$ or $\gamma \leq -2$. Hence, by theorem II.2.1 and part (1) of this corollary, $\langle -a, -b \rangle$ and hence also $\langle a, b \rangle$ is free of rank 2 whenever $\text{tr } a = \alpha \leq -2$, $\text{tr } b = \beta \leq -2$, $\text{tr } ab = \gamma \geq \alpha\beta + 2$ or $\gamma \leq -2$.

II.2.3 Corollary (Newman [20]).

Let

$$A = \begin{pmatrix} -a & b \\ -c & d \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & -\beta \\ \gamma & \delta \end{pmatrix}$$

be elements of $SL(2, R)$ with $a, \dots, d, \alpha, \dots, \delta \geq 0$.

If $\text{tr } A = d - a \geq 2$, $\text{tr } B = \delta - \alpha \geq 2$, then $\langle A, B \rangle$ is free of rank 2.

Proof

Here

$$AB = \begin{pmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{pmatrix}$$

so

$$\text{tr } AB = a\alpha+b\gamma+c\beta+d\delta$$

Now $\text{tr } A \cdot \text{tr } B = (d-a)(\delta-\alpha)$. Since $\det A = bc - ad = 1$, $\det B = \beta\gamma - \alpha\delta = 1$, $bc = 1 + ad \geq 1$, $\beta\gamma = 1 + \alpha\delta \geq 1$. Hence, $bc\beta\gamma \geq 1$. Moreover, since b, c, β, γ are real numbers ≥ 0 , the arithmetic mean $(c\beta + b\gamma)/2$ of $c\beta$ and $b\gamma$ is greater than or equal to their geometric mean and so ≥ 1 . That is, $c\beta + b\gamma \geq 2$. Hence,

$$\begin{aligned} \text{tr } AB &= a\alpha + b\gamma + c\beta + d\delta = (d-a)(\delta-\alpha) + d\alpha + a\delta + c\beta + b\gamma \\ &\geq (d-a)(\delta-\alpha) + 2 + d\alpha + a\delta \\ &\geq \text{tr } A \cdot \text{tr } B + 2 \end{aligned}$$

Hence, by theorem II.2.1, $\langle A, B \rangle$ is free of rank 2.

II.2.4 Corollary

Let

$$a = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

where $\text{tr } a = \alpha = \lambda + \lambda^{-1}$, $\text{tr } b = \beta = \mu + \mu^{-1}$. Then,

- (i) if $\alpha \geq 2$, $\beta \geq 2$, $\langle a, b \rangle$ is free of rank 2 whenever either $\xi\eta \geq \lambda/\mu + \mu/\lambda + 2$ or $\xi\eta \leq -2 - \lambda\mu - \lambda^{-1}\mu^{-1}$

- (ii) if $\alpha \leq -2$, $\beta \geq 2$; $\langle a, b \rangle$ is free of rank 2 whenever either $\xi\eta \geq 2 - \lambda\mu - \lambda^{-1}\mu^{-1}$ or $\xi\eta \leq \lambda/\mu + \mu/\lambda - 2$
- (iii) if $\alpha \geq 2$, $\beta \leq -2$; $\langle a, b \rangle$ is free of rank 2 when either $\xi\eta \geq 2 - \lambda\mu - \lambda^{-1}\mu^{-1}$ or $\xi\eta \leq \lambda/\mu + \mu/\lambda - 2$
- (iv) if $\alpha \leq -2$, $\beta \leq -2$; $\langle a, b \rangle$ is free of rank 2 whenever either $\xi\eta \geq \lambda/\mu + \mu/\lambda + 2$ or $\xi\eta \leq -2 - \lambda\mu - \lambda^{-1}\mu^{-1}$.

Proof

- (i) Here, $\text{tr } a = \lambda + \lambda^{-1} = \alpha \geq 2$, $\text{tr } b = \mu + \mu^{-1} = \beta \geq 2$, $\text{tr } ab = \xi\eta + \lambda\mu + \lambda^{-1}\mu^{-1} = \gamma$.

Hence, by theorem II.2.1, $\langle a, b \rangle$ is free on two generators if

$$\xi\eta + \lambda\mu + \lambda^{-1}\mu^{-1} \geq \lambda\mu + \lambda^{-1}\mu^{-1} + \lambda/\mu + \mu/\lambda + 2$$

or

$$\xi\eta + \lambda\mu + \lambda^{-1}\mu^{-1} \leq -2$$

that is, if

$$\xi\eta \geq \lambda/\mu + \mu/\lambda + 2 \text{ or } \xi\eta \leq -2 - \lambda\mu - \lambda^{-1}\mu^{-1}.$$

The parts (ii) - (iv) of the corollary follow from corollary II.2.2.

It is known that if $\lambda = \mu = 1$ and $\xi = 1$ with η transcendental over Q , then $\langle a, b \rangle$ is free of rank 2. (cf. Sanov [22], Brenner [1] or Chang, Jennings and Ree [2]). One may expect that $\langle a, b \rangle$ may be free in general if one of ξ or η is transcendental over Q . However, the following example shows that $\langle a, b \rangle$ need not be free of rank 2 even in the case when $\alpha = 2$ and β, η are transcendental over Q .

II.2.5 Example

Consider the matrices

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

where η is transcendental over $\mathbb{Q}$. One can easily see that a, b are obtained from the matrices

$$a_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 & \eta \\ -1/\eta & \beta \end{pmatrix}, \quad \eta \neq 0$$

after conjugation by

$$c = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$$

when

$$\delta = \{-\beta \pm (\beta^2 - 4)^{1/2}\} / 2\eta.$$

Hence, $\langle a, b \rangle$ is free if and only if $\langle a_0, b_0 \rangle$ is free. However, if we let $\beta = \eta \pm 2 \cos 2k\pi/n$, $n > 2$, $(k, n) = 1$, then

$$a_0^{-1} b_0 = \begin{pmatrix} 0 & \eta \\ -1/\eta & \beta - \eta \end{pmatrix}$$

so that $a_0^{-1} b_0$ has characteristic polynomial $\lambda^2 - 2\cos 2k\pi/n + 1$, and eigenvalues $\pm e^{\pm 2k\pi i/n}$. Hence,

$$(a_0^{-1} b_0)^n = \pm I.$$

Therefore, $\langle a, b \rangle$ is not free even though β and η are transcendental over $\mathbb{Q}$.

Suppose $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0$ and $a, b \in \text{SL}(2, \mathbb{R})$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$ and let $F_{\alpha, \beta, \gamma}$ denote the group generated by a and b . By corollary II.1.4, this is uniquely determined, up to conjugacy, by α, β, γ . Since

$$\langle a, b \rangle = \langle b, (ab)^{-1} \rangle = \langle ba, a^{-1} \rangle$$

and

$$\text{tr } x = -\text{tr } x^{-1}$$

(for $x \in \text{SL}(2, \mathbb{C})$), we have

$$F_{\alpha, \beta, \gamma} \cong F_{\beta, \gamma, \alpha} \cong F_{\gamma, \alpha, \beta}$$

For some specific values of α, β, γ we have the following.

II.2.6 Corollary

- (i) $F_{\alpha, \alpha, \alpha}$ is free for all $\alpha \leq -2$, $\alpha \in \mathbb{R}$
- (ii) $F_{\alpha, \alpha, -\alpha}$ is free for all $\alpha \geq 2$, $\alpha \in \mathbb{R}$

Proof

This is obtained by a straight forward application of theorem II.2.1.

In part (ii), of the corollary II.2.6, it is easy to see, in view of the remarks preceding corollary II.2.6, that

$$F_{\alpha, \alpha, -\alpha} \cong F_{\alpha, -\alpha, \alpha} \cong F_{-\alpha, \alpha, \alpha}$$

If any one of the α, β, γ is 0 or ± 1 or $2 \cos 2k\pi/n$, $n \geq 2$, $(k, n) = 1$, then $F_{\alpha, \beta, \gamma}$ is not free because the corresponding elements have finite

order. Also $F_{2,2,2}$ is not free because in this case one of a, b or ab is the identity matrix. Suppose that $\alpha=3, \beta=2$. Theorem II.2.1 shows that $F_{3,2,\gamma}$ is discrete free group of rank 2 if and only if $\gamma \leq -2$ or $\gamma \geq 8$. We examine integral values of γ such that $-2 < \gamma < 8$ to see whether or not $F_{3,2,\gamma}$ is, in the non-discrete case, free for these values.

II.2.7 Theorem

$F_{3,2,\gamma}$ is not free (both in the discrete and non-discrete case) on two generator for any $\gamma \in \mathbb{Z}$ such that $-2 < \gamma < 8$.

Proof

We have only to examine the non-discrete case. As remarked above, for $\gamma=0, \pm 1$, $F_{3,2,\gamma}$ is not free on two generators. Let $\gamma=2$. Then, by the remarks preceding Corollary II.2.6, $F_{3,2,2}$ is isomorphic to $F_{2,2,3}$ which, by Corollary II.1.4 is isomorphic to

$$\left\langle \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \cong \text{SL}(2, \mathbb{Z})$$

(cf. [15]), and hence not free on two generators. For $\gamma=3$, $F_{3,2,3}$ is reducible by Corollary II.1.2 because

$$3^2 + 2^2 + 3^2 - 2 \cdot 3 \cdot 3 - 4 = 0.$$

For $\gamma=4$, $F_{3,2,4}$ is isomorphic to $F_{3,4,2}$ and, by Corollary II.1.4, also isomorphic to

$$G = \langle a' = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}, b' = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix} \rangle$$

But in G ,

$$a'b' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, a'^{-1}b'^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so that G and hence also $F_{3,2,4}$ is not free. To discuss the cases for $\gamma=5,6,7$, we observe that from the proof of Theorem II.2.1, the subgroup

$\langle a, b \rangle$ of $SL(2, \mathbb{C})$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$, is isomorphic to the group generated by the matrices

$$a_0 = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & -\gamma + \alpha\beta \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 & -1 \\ 1 & \beta \end{pmatrix}, \quad \lambda + \lambda^{-1} = \alpha\beta + 2.$$

Putting the values of α, β, γ in a_0, b_0 , we have

$$F_{3,2,5} = \langle a_0 = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & +1 \end{pmatrix}, b_0 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \rangle$$

$$F_{3,2,6} = \langle a_0 = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}, b_0 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \rangle$$

$$F_{3,2,7} = \langle a_0 = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & -1 \end{pmatrix}, b_0 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \rangle$$

so that $a_0^3 = -I$, $a_0^2 = -I$, $a_0^3 = -I$ are the non-trivial relations in

$F_{3,2,5}$, $F_{3,2,6}$ and $F_{3,2,7}$ respectively. Hence for all γ such that $-2 < \gamma < \beta$, $F_{3,2,\gamma}$ is not free.

From the above result, one may suspect that for all integral values of γ such that $-2 < \gamma < \alpha\beta + 2$, $F_{\alpha,\beta,\gamma}$ is not free. However, this is not the case. For example, $F_{3,3,3}$ is isomorphic to the group generated by the matrices:

$$a = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

and $\langle a, b \rangle$ being the derived group of the unimodular group, is free of rank 2 (cf. Newman [19]).

In fact, we here give an infinite family of groups $F_{\alpha,\beta,\gamma}$:

$\alpha, \beta, \gamma \in \mathbb{Z}$, $-2 < \gamma < \alpha\beta + 2$, $\alpha \geq 2$, $\beta \geq 2$, for which $F_{\alpha,\beta,\gamma}$ is free and simultaneously

an infinite family of $F_{\alpha,\beta,\gamma}$, $\alpha, \beta, \gamma \in \mathbb{Z}$, $\alpha \geq 2$, $\beta \geq 2$, $-2 < \gamma < \alpha\beta + 2$ for which

$F_{\alpha,\beta,\gamma}$ is not free of rank 2.

II.2.8 Theorem

Let $t \in \mathbb{R}$.

- (i) If $t \leq 0$ or $t \geq 1$, then the transformations:

$$\underline{a}(z) = \frac{tz-1}{(t^2+1)z-t}, \quad \underline{b}(z) = \frac{(-t+1)z-1}{(t^2-t+1)z+t}$$

generate a group isomorphic to $SL(2, \mathbb{Z})/\{ \pm I \}$, that is, a free product of a cyclic group of order 2 and a cyclic group of order 3.

- (ii) If $0 < t < 1$, then the linear transformation

$$\underline{a}(z) = \frac{-tz-1}{(t^2+1)z+t}, \quad \underline{b}(z) = \frac{(t+1)z-1}{(t^2+t+1)z-t}$$

generate a group isomorphic to $SL(2, \mathbb{Z})/\{ \pm I \}$.

Proof

- (i) Here $\underline{a}^2(z)=z$, $\underline{b}^3(z)=z$ for all $z \in \mathbb{C}$. Hence $\underline{a}^2 = \underline{b}^3 = I$. Choose disjoint sets $I = (-\infty, 0)$ and $\Delta = (0, \infty)$ and observe that $\underline{a}$ maps I into Δ and $\underline{b}, \underline{b}^2$ map Δ into I . Hence, by Macbeath's lemma I.2.1, $\langle \underline{a}, \underline{b} \rangle$ generate the free product of a cyclic group $\langle \underline{a} \rangle$ of order 2 and a cyclic group $\langle \underline{b} \rangle$ of order 3. Consequently, $\langle \underline{a}, \underline{b} \rangle$ is isomorphic to $SL(2, \mathbb{Z})/\{ \pm I \}$.

- (ii) Proof of (ii) is similar to that of (i).

Consider now the matrices:

$$a_1 = aba^{-1}b^{-1}, \quad b_1 = ab^2a^{-1}b^{-2}$$

where

$$a = \begin{pmatrix} t & -1 \\ t^2+1 & -t \end{pmatrix}, \quad b = \begin{pmatrix} -t+1 & -1 \\ t^2-t+1 & t \end{pmatrix}$$

are the matrices corresponding to the linear transformations $\underline{a}, \underline{b}$ in part (i) of theorem II.2.8. By straight forward computation, we have:

$$\text{tr } a_1 = 16t^4 - 16t^3 + 20t^2 - 8t + 3 = \alpha_t, \text{ say} \quad (1)$$

$$= \text{tr } b_1$$

(because b_1 is conjugate to a_1), and

$$\text{tr } a_1 b_1 = \alpha_t^2 - \alpha_t, \quad \text{tr } a_1 b_1^{-1} = \alpha_t.$$

Since $\langle a_1, b_1 \rangle$ is contained in the derived group of $\langle a, b \rangle$ and contains no scalars, for all $t \geq 1$, $\langle a_1, b_1 \rangle$ which is isomorphic to $\langle a_1, b_1^{-1} \rangle$ and hence to $F_{\alpha_t, \alpha_t, \alpha_t}$, is free. Thus, we have an infinite family of free groups $F_{\alpha_t, \alpha_t, \alpha_t}$ where α_t is given by (1), for all $t \geq 1$.

For the second part of our statement that there exists an infinite family of groups $F_{\alpha, \beta, \gamma}$, $\alpha \geq 2$, $\beta \geq 2$, $-2 < \gamma < \alpha\beta + 2$, $\alpha, \beta, \gamma \in \mathbb{Z}$ which is not free, we first make note of the following result.

II.2.9 Theorem

Let

$$a = \begin{pmatrix} 0 & -1 \\ 1 & m \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & m+1 \end{pmatrix}.$$

Then, for all $m \in \mathbb{C}$, the group $\langle a, b \rangle$ contains the unimodular group.

Proof

Here we have only to observe that:

$$ab^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad a^{-1}b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\langle ab^{-1}, a^{-1}b \rangle$ generate the unimodular group (cf. [15], p. 1398).

II.2.10 Corollary

For all $m \in \mathbb{Z}$,

$$a = \begin{pmatrix} 0 & -1 \\ 1 & m \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & m+1 \end{pmatrix}$$

generate the unimodular group.

Proof

Here, $SL(2, \mathbb{Z}) \subseteq \langle a, b \rangle$ by II.2.9. Also, both a and b are in $SL(2, \mathbb{Z})$.

Hence, $\langle a, b \rangle \subseteq SL(2, \mathbb{Z})$. Consequently,

$$\langle a, b \rangle = SL(2, \mathbb{Z}).$$

II.2.11 Corollary

For any $m \in \mathbb{Z}$,

$$F_{m, m+1, (m-1)(m+2)}$$

is not free of rank 2.

Proof

By corollary II.1.4, $F_{m, m+1, (m-1)(m+2)}$ is isomorphic to the group $\langle a, b \rangle$ generated by the matrices:

$$a = \begin{pmatrix} 0 & -1 \\ 1 & m \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & m+1 \end{pmatrix}$$

and $\langle a, b \rangle$ is, by corollary II.2.10, isomorphic to $SL(2, \mathbb{Z})$, which is not free on two generators.

Hence, the corollary.

11.3 RELATION WITH TRANSCENDENCE DEGREE

Let F' be an extension field of a field F . Let S be a subset of F' . We say that S (or the elements of S) is algebraically independent over F whenever $x_1, \dots, x_k$, ($k \geq 1$) are distinct elements of S and $p(x_1, \dots, x_k)$ a non-zero polynomial over F , then $p(x_1, \dots, x_k) \neq 0$. We can introduce ordering in the algebraically independent subsets of F' by ascending inclusion. The subsets are obviously inductively ordered so that, by Zorn's lemma, there exist maximal subsets and it can be shown that these maximal algebraically independent subsets all have the same cardinality (see Lang [12], p. 253). A subset S of F' which is algebraically independent over F and is maximal with respect to inclusion ordering will be called a transcendence base of F' over F and its cardinality the transcendence degree of F' over F . From the maximality of S , it is clear that if S is a transcendental base of F' over F , then F' is algebraic over $F(S)$, the field generated by F and S . For further information on transcendence base, transcendence degree and related topics, see Lang [12].

For any $v \in \text{SL}(2, F)$ we have

$$v^2 - (\text{tr } v)v + I = 0$$

by the Cayley-Hamilton theorem. Hence, $uv - (\text{tr } v)u + uv^{-1} = 0$ which gives

$$\text{tr } uv = \text{tr } u \text{tr } v - \text{tr } u v^{-1} \quad (1)$$

Using (1), we find that, for any integer m , the trace of w^n , $w \in \text{SL}(2, F)$ can be written as:

$$\text{tr } w^n = \text{tr } w \cdot \text{tr } w^{n-1} - \text{tr } w^{n-2}$$

from which it follows that

$$\text{tr } w^n = J_n(\text{tr } w) \quad (2)$$

where $J_m(x)$ is the Chebyshev polynomial of second kind defined by the recursive relation

$$J_m(x) = x J_{m-1}(x) - J_{m-2}(x) \quad (3)$$

with initial conditions $J_0(x)=2$, $J_1(x)=x$ (see Cheney [2a]). Also, from the relation $\text{tr } u = \text{tr } u^{-1}$, we have $\text{tr } w^{-n} = \text{tr } w^n$. Thus, we conclude that for any arbitrary integer m ,

$$\text{tr } w^m = J_{|m|}(\text{tr } w) \quad (4)$$

where $J_m(x)$, given by the recursive relation (3), is a polynomial in x of degree m with integer coefficients.

Horowitz [10] has shown that the trace of any element w of a free group F_n of rank n (can be expressed as a polynomial with integer coefficients of the traces of 2^n-1 elements of F_n). From this, he concludes that the trace of any element w of a finitely generated subgroup G of $SL(2, F)$ can be expressed as a polynomial over Z in the traces of elements in a certain finite subset of G . We shall use his results to prove some of our theorems.

Let a, b be a pair of matrices in $SL(2, C)$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. Suppose that α, β, γ are algebraically independent over Q . Then, the transcendence degree of $Q(\alpha, \beta, \gamma)$ over Q is 3.

II.3.1 Theorem

Let $a, b \in \text{SL}(2, \mathbb{C})$, $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. Suppose that the transcendence degree of $\mathbb{Q}(\alpha, \beta, \gamma)$ over $\mathbb{Q}$ is 3. Then, $\langle a, b \rangle$ is free of rank 2.

Proof

By Horowitz' theorem 3.1 [10], the trace of an arbitrary element w of $\langle a, b \rangle$ can be expressed as a polynomial in α, β, γ with coefficients from $\mathbb{Z}$. If $w = a^m$ or b^n , then since $\text{tr } w$ is a polynomial in α or β respectively and α and β are algebraically independent over $\mathbb{Q}$ and hence over $\mathbb{Z}$, $\text{tr } w \neq 2$ so that $w \neq I$ for any $m, n \in \mathbb{Z}$. And if $w = a^{m_1} b^{n_1} \dots a^{m_k} b^{n_k}$, $k \geq 1$, then $\text{tr } w$ is a polynomial $p_w(\alpha, \beta, \gamma)$ in α, β, γ . Since the transcendence degree of $\mathbb{Q}(\alpha, \beta, \gamma)$ over $\mathbb{Q}$ is 3, $p_w(\alpha, \beta, \gamma) \neq 2$ for any such $w \in \langle a, b \rangle$. Consequently, $w \neq I$ and so $\langle a, b \rangle$ is a free group on two generators.

However, it should be noted that if the transcendence degree of $\mathbb{Q}(\alpha, \beta, \gamma)$ over $\mathbb{Q}$ is less than 3, say 2, then $\langle a, b \rangle$ need not be free on two generators.

II.3.2 Example

For $a, b \in \text{SL}(2, \mathbb{C})$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$, suppose that $\mathbb{Q}(\beta, \gamma)$ has transcendence degree 2 over $\mathbb{Q}$. Choose $\alpha \in \mathbb{C} \setminus \mathbb{Q}$ such that

$$\alpha = \beta\gamma/\gamma^2 - 1 \quad (1)$$

Then, using the Cayley-Hamilton theorem for a, b and ab , we have:

$$\begin{aligned}
 \text{tr } a^2 bab &= \text{tr}(\alpha a - I)bab \\
 &= \alpha \text{tr}(ab)^2 - \text{tr } bab \\
 &= \alpha(\gamma^2 - 2) - \text{tr } a b^2 \\
 &= \alpha(\gamma^2 - 2) - \text{tr } a(\beta b - I) \\
 &= \alpha(\gamma^2 - 2) - \beta\gamma + \alpha \\
 &= \alpha\gamma^2 - \alpha - \beta\gamma \\
 &= 0
 \end{aligned}$$

by (i). Hence

$$(a^2 bab)^2 = -I.$$

Thus, $\langle a, b \rangle$ is not free of rank 2.

For $a, b \in \text{SL}(2, \mathbb{C})$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$, let $F_{\alpha, \beta, \gamma} = \langle a, b \rangle$.

If any one of the α and β , say α , is fixed then the number of subgroups $F_{\alpha, \beta, \gamma}$ of $\text{SL}(2, \mathbb{C})$ which are not the free product of the groups $\langle a \rangle$ and $\langle b \rangle$ is uncountable. This is because taking γ to be $2 \cos 2k\pi/n$, $(k, n)=1$, $n > 2$ and letting β assume any value in $\mathbb{C}$, $F_{\alpha, \beta, \gamma}$ has a non-trivial relation $(ab)^n = -I$ for all β , so that $F_{\alpha, \beta, \gamma}$ cannot be the free product of $\langle a \rangle$ and $\langle b \rangle$. However, if we fix both α, β , then we prove that there are not 'many' values of γ for which $\langle a, b \rangle$ is not a free product of $\langle a \rangle, \langle b \rangle$.

II.3.3 Theorem

Let α, β be any fixed real or complex numbers. Let $F_{\alpha, \beta, \gamma}$ be the subgroup of $\text{SL}(2, \mathbb{C})$ generated by the matrices a, b with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. Then, the set

$$S(\alpha, \beta) = \{ \gamma : F_{\alpha, \beta, \gamma} \text{ is not a free product of } \langle a \rangle, \langle b \rangle. \}$$

is at most countable.

Proof

Let

$$w = (a^{m_1} b^{n_1} \dots a^{m_k} b^{n_k}), k \geq 1,$$

be an arbitrary element of $\langle a, b \rangle$. We can suppose that $a^{m_i} \neq I, b^{n_j} \neq I$, for otherwise, two of the adjacent factors amalgamate and we again have an expression of the above form. By Cayley-Hamilton theorem, we have:

$$a^2 - \alpha a + I = 0, b^2 - \beta b + I = 0$$

so that

$$a^{m_i} = \lambda_i a + \lambda'_i I, b^{n_j} = \mu_j b + \mu'_j I, i, j = 1, 2, \dots, k.$$

Hence

$$w = (\lambda_1 a + \lambda'_1 I)(\mu_1 b + \mu'_1 I) \dots (\lambda_k a + \lambda'_k I)(\mu_k b + \mu'_k I)$$

$$= \lambda_1 \mu_1 \dots \lambda_k \mu_k (ab)^k + \text{terms of the form } a^\epsilon (ab)^i b^\delta \text{ or}$$

$$b^\delta (ab)^i a^\epsilon, \text{ where } \epsilon = 0 \text{ or } 1, \delta = 0 \text{ or } 1, 0 \leq i \leq k-1, \text{ and for } i = k-1,$$

at least one of the ϵ or δ is 0

$$= I_1 + I_2$$

where $I_1 = \lambda_1 \mu_1 \dots \lambda_k \mu_k (ab)^k$ and $I_2 = \text{remaining terms}$. So

$$\text{tr } w = \text{tr } I_1 + \text{tr } I_2$$

Now

$$\lambda_1 \mu_1 \dots \lambda_k \mu_k = 0$$

implies there exist integers i or j such that $\lambda_i = 0$ or $\mu_j = 0$, whence $a^{m_i} = \lambda_i^{\pm 1} 1$ or $b^{n_j} = \mu_j^{\pm 1} 1$. Since $a, b \in \text{SL}(2, \mathbb{C})$, $\lambda_i^{\pm 1} = \pm 1$ or $\mu_j^{\pm 1} = \pm 1$. Consequently, $a^{m_i} = \pm 1$ or $b^{n_j} = \pm 1$, contradicting our choice of m_i and n_j . Hence, $\lambda_1 \mu_1 \neq 0$.

Since each of the λ_i and μ_j is a polynomial in α and β respectively, $1 \leq i \leq k$, $1 \leq j \leq k$, we have

$$\lambda = \lambda(\alpha, \beta) = \lambda_1 \mu_1 \dots \lambda_k \mu_k$$

as a polynomial in α, β . Thus,

$$\text{tr } l_1 = \lambda \text{tr}(ab)^k = \lambda J_k(\text{tr}(ab)) = \lambda J_k(\gamma)$$

by Equation (2) preceding theorem II.3.1, where $\text{tr } ab = \gamma$ and J_k is Chebyshev polynomial of second kind and of degree k with integer coefficients.

Also, since each term in l_2 is of the form $a^\epsilon (ab)^i b^\delta$ or $b^\delta (ab)^i a^\epsilon$ for $0 \leq i \leq k-1$ and for $i=k-1$, at least one of the ϵ or δ is zero, the trace of each such term is a polynomial in γ of degree $< k$ with coefficients from $\mathbb{Z}[\alpha, \beta] \subset \mathbb{Q}(\alpha, \beta)$. Thus, $\text{tr } l_2$ is a polynomial in γ of degree at most $k-1$, with coefficients from $\mathbb{Q}(\alpha, \beta)$. Hence, $\text{tr } w$ is a polynomial in γ of degree k with coefficients from $\mathbb{Q}(\alpha, \beta)$.

If now $w=1$ in $\langle a, b \rangle$, then

$$\text{tr } w = a_0 \gamma^k + \dots + a_k = 2, \quad a_0 \neq 0, \quad a_i \in \mathbb{Q}(\alpha, \beta), \quad 0 \leq i \leq k,$$

so that γ is algebraic over $\mathbb{Q}(\alpha, \beta)$. Thus, if $F_{\alpha, \beta, \gamma}$ is not a free product of the groups $\langle a \rangle$, $\langle b \rangle$, then $w=1$ is a non-trivial relation in

$\langle a, b \rangle$ for some w and this implies γ is algebraic over $Q(\alpha, \beta)$. Therefore, a necessary condition for $F_{\alpha, \beta, \gamma}$ not to be a free product of $\langle a \rangle$, $\langle b \rangle$ is that γ be algebraic over $Q(\alpha, \beta)$. However, there are only countably many γ algebraic over $Q(\alpha, \beta)$, and therefore the set

$$S(\alpha, \beta) = \{\gamma \in \mathbb{C} : F_{\alpha, \beta, \gamma} \text{ is not a free product of } \langle a \rangle, \langle b \rangle\}$$

is at most countable. Hence, the theorem.

II.3.4 Corollary

Let $G = \langle a, b : \text{tr } a = \alpha, \text{tr } b = \beta, \text{tr } ab = \gamma \rangle$. Then, for given values of α, β , the group G is the free product of its subgroups $\langle a \rangle$, $\langle b \rangle$ if γ is algebraically independent over $Q(\alpha, \beta)$.

Proof

A part of the proof of theorem II.3.3.

An element $x \neq 1$ in $SL(2, \mathbb{C})$ has finite order if and only if x has a characteristic root which is a primitive n th root of unity for some $n \geq 2$. This remark, combined with corollary II.3.4, gives us the following.

II.3.5 Corollary

Suppose that the characteristic roots of a and b are not primitive n th roots of unity for any $n \geq 2$, and suppose that $\text{tr } ab = \gamma$ is algebraically independent over $Q(\alpha, \beta)$, $\text{tr } a = \alpha$, $\text{tr } b = \beta$. Then, $\langle a, b \rangle$ is a free group of rank 2.

Remark

If γ is not algebraically independent over $Q(\alpha, \beta)$, then $F_{\alpha, \beta, \gamma}$ need not be the free product of $\langle a \rangle$ and $\langle b \rangle$ as the example II.3.2 shows.

II.3.6 Corollary

Suppose that $\alpha=2$ and β, γ algebraically independent over Q . Then, $F_{2, \beta, \gamma}$ is free of rank 2.

Proof

The fact that β, γ are algebraically over Q implies γ is algebraically independent over $Q(2, \beta) = Q(\beta)$. The corollary then follows from theorem II.3.3.

II.3.7 Corollary

Let $\alpha=\beta$ and β, γ be algebraically independent over Q . Then, $F_{\alpha, \beta, \gamma} = F_{\beta, \beta, \gamma}$ is free of rank 2.

Proof

Here γ is algebraically independent over $Q(\alpha, \beta) = Q(\beta)$.

Once again, it is worth noticing that in the above corollary if β, γ are not algebraically independent over Q (and hence over Z), then $F_{2, \beta, \gamma}$ and $F_{\beta, \beta, \gamma}$ may not be free of rank 2. Example II.3.2 suffices to prove this point.

In the following theorem, we establish a necessary and sufficient condition for an arbitrary element of $SL(2, C)$ to be the identity element.

II.3.8 Theorem

Let $\langle a, b \rangle$ be an irreducible subgroup of $SL(2, \mathbb{C})$ and w be an arbitrary element of $SL(2, \mathbb{C})$. Then, $w=I$ if and only if

$$\text{tr } w = 2, \text{tr } wa = \text{tr } a, \text{tr } wb = \text{tr } b$$

Proof

It is trivial that $w=I$ implies $\text{tr } w = 2, \text{tr } wa = \text{tr } a, \text{tr } wb = \text{tr } b$, for all $a, b \in SL(2, \mathbb{C})$.

Conversely, suppose that $w \neq I$. Then, by the Cayley-Hamilton theorem, we have

$$w^2 - 2w + I = 0, (\text{tr } w = 2)$$

so that

$$(w^2 - w)a = wa - a$$

$$(w^2 - w)b = wb - b$$

Hence

$$\text{tr}(w^2 - w)a = \text{tr}(w^2 - w)b = 0 \quad (1)$$

Now, $\text{tr } w = 2$ implies w has the Jordan form

$$w = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } w^2 - w = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

If

$$a = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \quad b = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$$

then

$$\text{tr}(w^2 - w)a = \alpha_2, \text{tr}(w^2 - w)b = \beta_2$$

are both zero by (1). But, this contradicts the irreducibility of $\langle a, b \rangle$. Hence, $w=1$.

II.3.9 Corollary (Whitemore [29])

Let $a, b \in SL(2, C)$ with $\text{tr } a = \text{tr } b = \alpha$ and $\text{tr } ab \neq 2$, $\text{tr } ab \neq \alpha^2 - 2$. If w satisfies $\text{tr } w = 2$, $\text{tr } wa = \text{tr } a$, $\text{tr } wb = \text{tr } b$, then $w=1$.

Proof

If $\text{tr } a = \text{tr } b = \alpha$, then $\langle a, b \rangle$ is reducible if and only if

$$\mu_2 = -(2\alpha^2 + \gamma^2 - \alpha^2 \gamma - 4) / (\alpha^2 - 4) = 0$$

when $\alpha \neq \pm 2$; and

$$\mu_2 = \gamma' - 2 = 0, \quad \gamma' = \pm \gamma = \pm \text{tr } ab$$

when $\alpha = \pm 2$. Hence, $\langle a, b \rangle$ is irreducible if and only if $\gamma \neq \pm 2$ and $\gamma \neq \alpha^2 - 2$, the signs in the first of the equations being taken appropriately. The corollary now follows from the above theorem.

II.4 SOME EXAMPLES OF NON-FREE GROUPS

In this section, we give a few examples of non-free subgroups of $SL(2, C)$ which are generated by pairs of matrices. These examples also provide answers to some of the more or less plausible questions related with the largeness of the traces of a, b and ab . We shall derive some relations among the traces of a, b and ab for which $\langle a, b \rangle$ is not free of rank 2.

Let $a, b \in \text{SL}(2, \mathbb{C})$ with $\text{tr } a = \alpha, \text{tr } b = \beta$. In the theorem given below, we find values of γ for which $\langle a, b \rangle$ is not free on two generators.

For any $\alpha, \beta \in \mathbb{C}$, we define the sequences $s_n, t_n, (n=0, 1, 2, \dots)$ by:

$$s_0 = t_0 = 1, s_1 = \alpha, t_1 = \beta$$

and

$$s_n = \alpha s_{n-1} - s_{n-2}, t_n = \beta t_{n-1} - t_{n-2}, (n \geq 2)$$

Now, for any fixed integers k, ℓ, m, n (not all zero), we define:

$$u = s_{n-1} t_{\ell-1} s_{m-1} t_{n-1}$$

$$v = s_k s_{m-1} t_{\ell-1} t_{n-2} + s_{k-1} s_{m-2} t_{\ell-1} t_n + s_{k-1} s_m t_{\ell-2} t_{n-1} \\ + s_{k-2} s_{m-1} t_{\ell} t_{n-1}$$

$$w = s_{k-1} s_{m-1} (t_{\ell} t_{n-2} - t_{\ell-2} t_n) + t_{\ell-1} t_{n-1} (s_k s_{m-2} + s_{k-2} s_m) \\ + s_k s_m t_{\ell-2} t_{n-1} + s_{k-2} s_{m-2} t_{\ell} t_n - 2 s_{k-1} t_{\ell-1} s_{m-1} t_{n-1} \\ - 2 \cos 2\pi/q$$

II.4.1 Theorem

With the notation above, if $\gamma \in \mathbb{C}$ satisfies the equation

$$u\gamma^2 - v\gamma + w = 0$$

then $F_{\alpha, \beta, \gamma}$ is not free.

Proof

By the Cayley-Hamilton theorem,

$$a^2 - \alpha a + 1 = 0, \quad b^2 - \beta b + 1 = 0, \quad (ab)^2 - \gamma(ab) + 1 = 0 \quad (1)$$

From the first two equations in (1), we have

$$a^m = s_{m-1}a - s_{m-2}I, \quad b^n = t_{n-1}b - t_{n-2}I \quad (2)$$

where s_n, t_n satisfy the equations

$$s_m = \alpha s_{m-1} - s_{m-2}, \quad t_n = \beta t_{n-1} - t_{n-2}$$

with initial conditions $s_0 = 1, s_1 = \alpha, t_0 = 1, t_1 = \beta$. Calculations which we omit and which use equations (1) and (2) show that:

$$\begin{aligned} a^k b^\ell a^m b^n &= s_{k-1} t_{\ell-1} [s_{m-1} t_{n-1} (ab)^2 - s_{m-1} t_{n-2} aba - s_{m-2} t_{n-1} ab^2 + s_{m-2} t_{n-2} ab] \\ &\quad - s_{k-1} t_{\ell-2} [s_{m-1} t_{n-1} ab^2 - s_{m-1} t_{n-2} a^2 - s_{m-2} t_{n-1} ab + s_{m-2} t_{n-2} a] \\ &\quad - s_{k-2} t_{\ell-1} [s_{m-1} t_{n-1} bab - s_{m-1} t_{n-2} ba - s_{m-2} t_{n-1} b^2 + s_{m-2} t_{n-2} b] \\ &\quad + s_{k-2} t_{\ell-2} [s_{m-1} t_{n-1} ab - s_{m-1} t_{n-2} a - s_{m-2} t_{n-1} b + s_{m-2} t_{n-2} I] \end{aligned}$$

and so

$$\text{tr } a^k b^\ell a^m b^n = \alpha \gamma^2 + \beta \gamma + w,$$

where $w' = w + 2 \cos 2p\pi/q$ and u, v and w are as defined above. Thus, if γ satisfies the equation

$$u\gamma^2 - v\gamma + w = 0,$$

then

$$\text{tr } a^k b^\ell a^m b^n = (e^{2p\pi i/q} + e^{-2p\pi i/q}) = c_1 + c_2$$

where

$$c_1 = e^{2p\pi i/q}$$

$$c_2 = e^{-2p\pi i/q}$$

are the characteristic roots of $a^k b^\ell a^m b^n$. Since $q > 2$, $(p, q) = 1$, $c_1 \neq c_2$ so that $a^k b^\ell a^m b^n$ can be diagonalized with diagonal elements c_1 and c_2 .

Hence

$$(a^k b^\ell a^m b^n)^q = I$$

is a non-trivial relation in $\langle a, b \rangle$. Thus, $\langle a, b \rangle$ is not free on two generators.

Chang, Jennings and Ree [2] have discussed the freeness of the groups

$$F_{\alpha, \beta} = \langle a, b : a = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \rangle$$

They have shown that the group $F_{\alpha, \beta}$ is free on two generators if and only if the group

$$F_\lambda = \langle a, b : a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix}, 2\lambda = \alpha\beta \rangle$$

is free. In such a case, they have called the number λ as a free number, otherwise non-free. They have also shown (cf. Theorem 4, [23]) that for integers k, ℓ, m, n, p and q such that $q > 2, (p, q) = 1$, the number λ is given by

$$\lambda = [-(k+m)(\ell+n) \pm \{(k+m)^2(\ell+n)^2 - 16k\ell mn \sin^2 p\pi/q\}^{1/2}] / 4k\ell mn \quad (3)$$

is not free. We show that their theorem is a special case of theorem II.4.1 with $\alpha=2, \beta=2$ and $\gamma=2\lambda+2$. Indeed

$$F_{\lambda} \approx F_{2,2,\gamma} \text{ where } \gamma=2\lambda+2.$$

Thus, we have only to show that for $\alpha=2, \beta=2, \gamma=2\lambda+2$, theorem II.4.1 reduces to the theorem of Chang, Jennings and Ree. But when $\text{tr } a = \text{tr } b = 2$, then $s_m = m+1, t_n = n+1$ for all integers m, n so that u, v, w of theorem II.4.1 are given by:

$$u = k \ell m n$$

$$v = 4k \ell m n - k\ell - kn - \ell m - mn$$

$$w = 4k \ell m n - 2k\ell - 2\ell m - 2mn - 2nk + 2 - 2 \cos 2 \theta$$

where $\theta = p\pi/q$. Hence, with $\gamma=2\lambda+2, u\gamma^2 - v\gamma + w = 0$ becomes

$$\begin{aligned} & k \ell m n (4\lambda^2 + 8\lambda + 4) - 2(\lambda+1)(4k \ell m n - k\ell - \ell m - mn - nk) \\ & + 4k \ell m n - 2k\ell - 2\ell m - 2nk + 2 - 2 \cos 2 \theta = 0 \end{aligned}$$

that is

$$2k \ell m n \lambda'^2 + 2\lambda'(k+m)(\ell+n) + 1 - \cos 2 \theta = 0$$

where $\lambda' = -\lambda$. (Indeed, F_{λ} is free if and only if $F_{-\lambda}$ is free.)

Thus

$$\lambda' = [-(k+m)(\ell+n) \pm \{(k+m)^2(\ell+n)^2 - 16k\ell m n \sin^2 \theta\}^{1/2}] / 4k\ell m n$$

which is the same as equation (3). Hence, the theorem.

One may expect that for large values of $\alpha, \beta, \gamma, F_{\alpha, \beta, \gamma}$ is free of rank 2. However, we note that the matrices

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \gamma - \beta \\ -1/\gamma - \beta & \beta \end{pmatrix}$$

where β, γ are related by the equations

$$\gamma - \beta = \delta, \quad \beta = \delta + 2\cos 2k\bar{\alpha}/n, \quad (k, n) = 1, \quad n > 2$$

is not free on two generators for any $\delta \in \mathbb{C}$, as shown in example II.2.5.

Hence, the largeness of the trace values does not ensure the freeness of the groups $\langle a, b \rangle$.

CHAPTER III

III.0 INTRODUCTION

Let $\langle a, b \rangle$ be an irreducible subgroup of $SL(2, \mathbb{C})$ and suppose that $\langle a, b \rangle$ is not a free group on two generators. Does there exist an integer n such that $\langle a^n, b^n \rangle$ is free of rank 2? If any one of the a or b is of finite order then the answer to this question is surely in the negative. In Section III.1, we show that if α, β are any two complex numbers, $|\alpha| > 2$, $|\beta| > 2$ and $\alpha = \text{tr } a$, $\beta = \text{tr } b$, then there exists an integer n such that $\langle a^n, b^n \rangle$ is free. Moreover, we prove that if for non-trivial elements $a, b \in SL(2, \mathbb{R})$, the trace of one of a and b is 2 while that of the other is > 2 , then also $\langle a^n, b^n \rangle$ is free for sufficiently large n .

In Section III.2, we give a necessary and sufficient condition for an infinite irreducible subgroup $\langle a, b \rangle$ of $SL(2, \mathbb{C})$ to be solvable-by-finite. Platonov [21] proved a conjecture of Kargaplov [11] that a linear group G over a field of characteristic zero in which every two-generator subgroup is solvable-by-finite, is itself solvable-by-finite. Wehrfritz [28] extended Platonov's results to include some other classes of groups. In Section III.2 of this chapter, we show that a subgroup G of $SL(2, \mathbb{C})$ is reducible if and only if every two-generator subgroup of G is reducible. We use this result to give a necessary and sufficient condition for a subgroup G of $SL(2, \mathbb{C})$ to contain an irreducible solvable-by-finite group.

III.1 FREENESS OF THE GROUP $\langle a^n, b^n \rangle$ FOR SOME INTEGER n , $a, b \in \text{SL}(2, \mathbb{C})$

Suppose that $\langle a, b \rangle$ is an irreducible subgroup of $\text{SL}(2, \mathbb{R})$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. By theorem II.2.1, $\langle a, b \rangle$ is free provided that $|\alpha| > 2$, $|\beta| > 2$ and $|\gamma| \neq \alpha\beta + 2$ or $\alpha\beta - 2$. For $a, b \in \mathbb{C}$, suppose that $|\alpha| > 2$, $|\beta| > 2$ and $\langle a, b \rangle$ not a free group of rank 2. What can we say about the freeness of the group $\langle a^n, b^n \rangle$ for some integer n ? In the following paragraphs, we shall discuss this question.

It has already been shown that if $\text{tr } a = 2 = \text{tr } b$, then a, b can be reduced simultaneously into the form:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \gamma-2 \\ 0 & 1 \end{pmatrix}, \gamma \neq 2$$

respectively. In this case, a positive integer n can be chosen such that

$$|n(\gamma-2)| \geq 4$$

so that $\langle a^n, b \rangle$ is free, by a result of Chang, Jennings and Ree [2], even though $\langle a, b \rangle$ need not be free. Consequently, $\langle a^n, b^n \rangle$ is free for some integer n sufficiently large. However, if the traces of a and b both are not equal to 2, then it is not so obvious that we can conclude about the freeness of $\langle a^n, b^n \rangle$. We shall show that if $|\alpha| > 2$, $|\beta| > 2$, and $\langle a, b \rangle$ is irreducible then there always exists an integer n such that $\langle a^n, b^n \rangle$ becomes a free group. We shall also prove that if the trace of one of the a and b is 2 while that of the other is > 2 , and a, b non-trivial elements in $\text{SL}(2, \mathbb{R})$ then $\langle a^n, b^n \rangle$ is free for sufficiently large n .

III.1.1 Theorem

Let $\langle a, b \rangle$ be an irreducible subgroup of $SL(2, \mathbb{C})$ with $\alpha = \text{tr } a$, $\beta = \text{tr } b$. Suppose that $|\alpha| > 2$, $|\beta| > 2$. Then, $\langle a^n b^n \rangle$ is free for some sufficiently large n .

Proof

Step 1. By theorem II.1.6, we can take a and b as

$$a = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

and further assume that $|\lambda| > |\lambda^{-1}|$, $|\mu| > |\mu^{-1}|$ (just invert the matrices if necessary). Then

$$\underline{a}(z) = \frac{\lambda z}{\xi z + \lambda^{-1}}, \quad \underline{b}(z) = \frac{\mu z + \eta}{\mu^{-1}}$$

and

$$\underline{a}^n(z) = \frac{\lambda^n z}{\xi' (\lambda^n - \lambda^{-n}) z + \lambda^{-n}}, \quad \underline{b}^n(z) = \frac{\mu^n z + \eta' (\mu^n - \mu^{-n})}{\mu^{-n}}$$

where

$$\xi' = \xi / (\lambda - \lambda^{-1}), \quad \eta' = \eta / (\mu - \mu^{-1})$$

Step 2. If $z \neq 0$, then

$$\underline{a}^n(z)' = \frac{\lambda^n z}{\xi' (\lambda^n - \lambda^{-n}) z + \lambda^{-n}} \rightarrow 1/\xi' \quad \text{as } n \rightarrow \infty$$

and if $z \neq 1/\xi'$ and $z \neq \infty$,

$$\underline{a}^{-n}(z) = \frac{\lambda^{-n} z}{\xi' (\lambda^{-n} - \lambda^n) z + \lambda^n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Similarly, if $z \neq 0$, then

$$\underline{b}^n(z) = \frac{\mu^n z + \eta' (\mu^n - \mu^{-n})}{\mu^{-n}} \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

and if $z \neq -\eta'$ and $z \neq \infty$,

$$\underline{b}^{-n}(z) = \frac{\mu^{-n} z + \eta' (\mu^{-n} - \mu^n)}{\mu^n} \rightarrow -\eta' \quad \text{as } n \rightarrow \infty$$

Step 3. Choose disjoint open sets A_1, A_2, B_1, B_2 of $C^* = CU(\infty)$ such that $0 \in A_1$, $1/\xi' \in A_2$, $\infty \in B_1$, $-\eta' \in B_2$. Such sets always exist, for example, for suitably small $\epsilon > 0$, one can take:

$$A_1 = \{z \in C^* : |z| < \epsilon\}, \quad A_2 = \{z \in C^* : |z - 1/\xi'| < \epsilon\}$$

$$B_1 = \{z \in C^* : |1/z| < \epsilon\}, \quad B_2 = \{z \in C^* : |z + \eta'| < \epsilon\}$$

where $1/\xi' \neq -\eta'$ for otherwise $\xi\eta = -(\lambda - \lambda^{-1})(\mu - \mu^{-1})$, which is a condition of reducibility of $\langle a, b \rangle$ by II.1.8. Put

$$A = A_1 \cup A_2, \quad B = B_1 \cup B_2$$

Step 4. For each $z \in B$, $z \neq \infty$

$$\underline{a}^n(z) \rightarrow 1/\xi' \in A \quad \text{and} \quad \underline{a}^{-n}(z) \rightarrow 0 \in A \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \underline{a}^n(\infty) = \underline{a}^{-n}(\infty) = 1/\xi'$$

Hence, for sufficiently large n ,

$$\underline{a}^{\pm n}(B) \subset A$$

Likewise, for all $z \in A$, $z \neq 0$

$$\underline{b}^n(z) \rightarrow \infty \in B \quad \text{and} \quad \underline{b}^{-n}(z) \rightarrow -\eta' \in B \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \underline{b}^n(0) = \infty, \quad \underline{b}^{-n}(0) = -\eta'$$

Hence, for sufficiently large n ,

$$b^{-n}(A) \subset B$$

By the Ping Pong lemma 1.2.1 of Macbeath $\langle a^n, b^n \rangle$ is a free group on two generators a^n and b^n for sufficiently large n , as required.

As has already been mentioned, if $\text{tr } a = \alpha = 2$, $\text{tr } b = \beta = 2$ and $\langle a, b \rangle$ is an irreducible subgroup then $\langle a^n, b^n \rangle$ is free for some positive integer n . In the next theorem, we show that a similar statement is valid if the trace of at least one of the matrices a or b is 2 while that of the other > 2 and $a, b \in \text{SL}(2, \mathbb{R})$.

III.1.2 Theorem

Let $\langle a, b \rangle$ be an irreducible subgroup of $\text{SL}(2, \mathbb{R})$ such that the trace of at least one of the matrices a, b , say of a is 2, and $\text{tr } b > 2$. Then, $\langle a^n, b^n \rangle$ is free for some positive integer n .

Proof

If the trace of at least one of the matrices a or b , say of a , is 2, then from the proof of theorem II.1.1, we can conjugate a and b and transform them simultaneously into the form:

$$a_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & \gamma - \beta \\ -1/(\gamma - \beta) & \beta \end{pmatrix}$$

where $\beta = \text{tr } b$, $\gamma = \text{tr } ab$ and $\gamma \neq \beta$ in the case when $\langle a, b \rangle$ is irreducible. Clearly, $\langle a^n, b^n \rangle$ is free if and only if $\langle a_1^n, b_1^n \rangle$ is free. We have the following two cases to discuss.

(i) $\gamma - \beta = x > 0$. In this case

$$a_1^n b_1 = \begin{pmatrix} 0 & x \\ -1/x & nx + \beta \end{pmatrix}$$

Hence $\langle a_1^n, b_1 \rangle$ is, by theorem II.2.1, free provided that

$$nx + \beta \geq 2\beta + 2$$

that is, if

$$nx \geq \beta + 2 \quad (1)$$

Since $x = \gamma - \beta > 0$, and β is a fixed real number > 2 , a positive integer n can be chosen such that the inequality (1) is satisfied.

Hence, $\langle a_1^n, b_1 \rangle$ and consequently, $\langle a^n, b^n \rangle$ is free for some positive integer n .

(ii) $\gamma - \beta = -x < 0$ so that $x > 0$. In this case, we consider the group $\langle a_1^n, b_1^{-1} \rangle$

Since

$$b_1^{-1} = \begin{pmatrix} \beta & x \\ -1/x & 0 \end{pmatrix} = b'$$

we have

$$a_1^n \cdot b' = \begin{pmatrix} \beta & x \\ n\beta - 1/x & nx \end{pmatrix}$$

so $\langle a_1^n, b' \rangle$ is, by theorem II.2.1, free if

$$nx + \beta \geq 2\beta + 2$$

that is if,

$$nx \geq \beta + 2 \quad (2)$$

As before, since $x > 0$ and β are fixed real numbers > 2 , an n can be chosen such that the inequality (2) is satisfied. Hence $\langle a_1^n, b \rangle$ and so also $\langle a^n, b^n \rangle$ is free for some positive integer n . This completes the proof of the theorem.

III.2 SOLVABLE-BY-FINITE SUBGROUPS OF $SL(2, \mathbb{C})$

Up to now, we had been mainly concerned with free subgroups of $SL(2, \mathbb{C})$ generated by a pair of elements a, b of $SL(2, \mathbb{C})$. In the present section, we shall examine the nature of irreducible solvable-by-finite subgroups of $SL(2, \mathbb{C})$ which are generated by pairs of matrices in $SL(2, \mathbb{C})$. By a theorem of J. Tits [25], every linear group is either solvable-by-finite or contains a non-cyclic free subgroup. Theorem III.2.2 of this section completely characterizes the 2-generator irreducible infinite solvable-by-finite subgroups of $SL(2, \mathbb{C})$. A necessary and sufficient condition for a subgroup G of $SL(2, \mathbb{C})$ to be reducible will also be given in the present section. This will be used to investigate the group $\langle a, b \rangle$ having an irreducible solvable-by-finite group. First, we prove the following.

III.2.1 Lemma

The following statements are equivalent for an irreducible subgroup $\langle a, b \rangle$ of $SL(2, \mathbb{C})$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$.

- (i) $[a^2, b^2] = I$, (ii) $\alpha\beta = 0$, (iii) $a^2 = -I$ or $b^2 = -I$.

Proof

(i) implies (ii). By Cayley-Hamilton theorem

$$a^2 - \alpha a + I = 0, \quad b^2 - \beta b + I = 0 \quad (1)$$

so that $[a^2, b^2] = I$ implies

$$(\alpha a - I)(\beta b - I) = (\beta b - I)(\alpha a - I)$$

that is, $\alpha\beta ab = \beta\alpha ba$ so that $\alpha\beta = 0$ or $ab = ba$. However, $ab = ba$ is impossible because of irreducibility of $\langle a, b \rangle$ and the fact that abelian subgroups of $SL(2, \mathbb{C})$ are reducible. Hence, $[a^2, b^2] = I$ implies $\alpha\beta = 0$.

(ii) implies (iii). Suppose that $\alpha\beta = 0$. Then $\alpha = 0$ or $\beta = 0$. From equation (1), we have

$$a^2 = -I \text{ or } b^2 = -I$$

(iii) implies (i) is obvious.

In general, $[a^n, b^n] = I$ if and only if $a^n = \pm I, b^n = \pm I$. Indeed, we can write $a^n = \alpha_{n-1} a - \alpha_{n-2} I, b^n = \beta_{n-1} b - \beta_{n-2} I$ for some $\alpha_{n-1}, \alpha_{n-2}, \beta_{n-1}, \beta_{n-2} \in \mathbb{C}$. Then, $[a^n, b^n] = I$ if and only if

$$(\alpha_{n-1} a - \alpha_{n-2} I)(\beta_{n-1} b - \beta_{n-2} I) = (\beta_{n-1} b - \beta_{n-2} I)(\alpha_{n-1} a - \alpha_{n-2} I)$$

that is, if and only if

$$\alpha_{n-1} \beta_{n-1} ab = \beta_{n-1} \alpha_{n-1} ba$$

which is equivalent to

$$\alpha_{n-1} \beta_{n-1} = 0 \text{ or } ab = ba$$

But the latter of the above equations is impossible because of the irreducibility of $\langle a, b \rangle$. Hence, $[a^n, b^n] = I$ if and only if $\alpha_{n-1} = 0$ or $\beta_{n-1} = 0$, that is, $a^n = -\alpha_{n-2}I$ or $b^n = -\beta_{n-2}I$. Since the only scalars in $SL(2, C)$ are $\pm I$, $[a^n, b^n] = I$ if and only if $a^n = \pm I$ or $b^n = \pm I$, as required.

Let $a, b \in SL(2, C)$ with $\text{tr } a = \alpha$, $\text{tr } b = \beta$, $\text{tr } ab = \gamma$. It is easy to see that if any two of the α, β, γ are equal to zero, that is, if any two of the equations

$$a^2 = -I, \quad b^2 = -I, \quad (ab)^2 = -I$$

hold, then $\langle a, b \rangle$ is isomorphic, modulo $\pm I$, to a dihedral group. We show that in the case when $\langle a, b \rangle$ is an irreducible infinite subgroup of $SL(2, C)$, this becomes a necessary and sufficient condition for $\langle a, b \rangle$ to be solvable-by-finite.

III.2.2 Theorem

An infinite irreducible subgroup $G = \langle a, b \rangle$ of $SL(2, C)$ is solvable-by-finite if and only if G is isomorphic, modulo $\pm I$ to the infinite dihedral group.

Proof

Clearly, if G is isomorphic, modulo $\pm I$, to the infinite dihedral group, then G is solvable-by-finite, in fact, even solvable.

Conversely, suppose that G is an infinite irreducible solvable-by-finite subgroup of $SL(2, C)$. By Malcev's theorem (Theorem 6.4, Dixon [4]), G has an abelian normal subgroup A of finite index in G . Let $C_G(A)$ denote the centralizer of A in G . Then, by 10.11 of [5],

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$(G:C_G(A)) \leq 2$. Since G is infinite and $(G:A)$ is finite, A is infinite and so must contain some non-scalar elements. By problem 10.9 of [5], we may suppose that A consists of diagonal matrices so it contains some elements of the form:

$$x = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \xi \neq \xi^{-1}$$

Now, if

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$$

and

$$gx = \begin{pmatrix} \alpha\xi & \beta\xi^{-1} \\ \gamma\xi & \delta\xi^{-1} \end{pmatrix} = \begin{pmatrix} \alpha\xi & \beta\xi^{-1} \\ \gamma\xi^{-1} & \delta\xi \end{pmatrix} = xg$$

then

$$\beta\xi = \beta\xi^{-1}, \quad \gamma\xi = \gamma\xi^{-1}, \quad \xi \neq \xi^{-1}$$

that is, $\beta=0$ and $\gamma=0$. Consequently, $C_G(A)$ consists of the diagonal matrices only and so $C_G(A)$ is abelian. Since $(G:C_G(A)) \leq 2$, $a^2, b^2 \in C_G(A)$ and hence $[a^2, b^2] = I$ so that $a^2 = -I$ or $b^2 = -I$ by lemma III.2.1. Similarly, either $a^2 = -I$ or $(ab)^2 = -I$ and $b^2 = -I$ or $(ab)^2 = -I$. Hence, at least two of a^2, b^2 and $(ab)^2$ equal $-I$. Since

$$\langle a, b \rangle = \langle a, ab \rangle = \langle b, ab \rangle$$

is irreducible, $G = \langle a, b \rangle$ is isomorphic, modulo $\pm I$, to the infinite dihedral group.

III.2.3 Corollary

Suppose that $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0$. Then, $F_{\alpha, \beta, \gamma}$ is solvable-by-finite if and only if two of α, β, γ equal zero.

III.2.4 Corollary

For $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 \neq 0$, $F_{\alpha, \beta, \gamma}$ contains a non-cyclic free subgroup if and only if at most one of the α, β and γ equal zero. (This is equivalent to saying that at most one of the squares a^2 , b^2 and $(ab)^2$ is -1 , by lemma III.2.1.)

Proof

By corollary III.2.3, $F_{\alpha, \beta, \gamma}$ is solvable-by-finite if and only if at least two of α, β and γ equal zero. Thus, $F_{\alpha, \beta, \gamma}$ is not solvable-by-finite if and only if at most one of the α, β and γ equals zero. By Tits's theorem [25], a group which is not solvable-by-finite always contains a non-cyclic free subgroup. Hence, the corollary.

III.2.5 Theorem

A subgroup G of $SL(2, \mathbb{C})$ is reducible if and only if every two generator subgroup of G is reducible.

Proof

G is reducible over $\mathbb{C}$ implies every two generator subgroup of G is reducible.

Conversely, suppose that every two generator subgroup of G is reducible and let G be irreducible. Then, by Burnside's theorem (Theorem 2.7B, Dixon [4]), the dimension of the matrix algebra $\mathbb{C}G$ is 4.

Hence, there exist matrices a, b and c in G such that

$$CG = \{\xi_0 I + \xi_1 a + \xi_2 b + \xi_3 c : \xi_i \in C, i=0,1,2,3\}$$

so that I, a, b, c span CG . Suppose that $\langle a, b \rangle$, $\langle b, c \rangle$ and $\langle c, a \rangle$ are reducible. Then, there exist vectors v_{ab}, v_{bc}, v_{ca} such that v_{ab} is a common eigenvector for a, b ; v_{bc} is a common eigenvector for b, c ; v_{ca} is a common eigenvector for c, a . If v_{ab}, v_{ca} are linearly dependent, then $C v_{ab} = C v_{ca}$ is an invariant subspace of a, b, c and CG is reducible. Consequently, G is reducible. But this contradicts our assumption about the irreducibility of G . Hence, v_{ab} and v_{ca} are linearly independent over C and so form a basis for the 2-dimensional space $v_2(C)$. Transform the matrices a, b, c relative to this basis. If $u = \begin{pmatrix} v_{ab} \\ v_{ca} \end{pmatrix}$, then we can write:

$$u a u^{-1} = \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} = a_1, \delta \neq \delta^{-1}$$

$$u b u^{-1} = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix} = b_1$$

$$u c u^{-1} = \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix} = c_1$$

when $\langle a_1, b_1 \rangle$, $\langle b_1, c_1 \rangle$ and $\langle c_1, a_1 \rangle$ are reducible by our hypothesis about the reducibility of every 2-generator subgroup. Now, $\langle b_1, c_1 \rangle$ is, by theorem II.1.8 reducible if and only if $\xi\eta=0$ or $\xi\eta=-(\lambda-\lambda^{-1})(\mu-\mu^{-1})$.

But $\xi\eta=0$ implies either $\xi=0$ or $\eta=0$ and in both cases a_1, b_1, c_1 generate a reducible group so that CG is reducible, a contradiction. Hence,

$\xi\eta = -(\lambda-\lambda^{-1})(\mu-\mu^{-1})$. and $\xi\eta \neq 0$ implies $\lambda-\lambda^{-1} \neq 0$, $\mu-\mu^{-1} \neq 0$. Now,

$$a_1 b_1 = \begin{pmatrix} \delta\lambda & 0 \\ \delta^{-1}\xi & \delta^{-1}\lambda^{-1} \end{pmatrix}$$

so the reducibility of $\langle a_1 b_1, c_1 \rangle$ implies

$$\delta^{-1}\xi\eta = 0 \text{ or } \delta^{-1}\xi\eta = -(\delta\lambda-\delta^{-1}\lambda^{-1})(\mu-\mu^{-1})$$

Using $\xi\eta = -(\lambda-\lambda^{-1})(\mu-\mu^{-1})$, the second of the above equations gives

$$\delta^{-1}(\lambda-\lambda^{-1})(\mu-\mu^{-1}) = (\delta\lambda-\delta^{-1}\lambda^{-1})(\mu-\mu^{-1})$$

As $\mu-\mu^{-1} \neq 0$, we obtain

$$\delta^{-1}(\lambda-\lambda^{-1}) = \delta\lambda-\delta^{-1}\lambda^{-1}$$

that $\delta^{-1}\lambda = \delta\lambda$ which yields $\delta = \delta^{-1}$ so that $a_1 = I$, a contradiction. Hence,

$\xi\eta = -(\lambda-\lambda^{-1})(\mu-\mu^{-1}) \neq 0$ also leads to a contradiction. Therefore, $\langle a_1, b_1, c_1 \rangle$ and hence also $\tilde{G}$ is reducible.

III.2.6 Theorem

Let G be an irreducible subgroup of $SL(2, C)$. Then, G contains an irreducible solvable-by-finite subgroup if and only if G has an element x such that $x^2 = -I$.

Proof

Suppose that G contains an irreducible solvable-by-finite subgroup H . Since H is irreducible over C , there exist, by theorem III.2.5, elements a, b in H such that $\langle a, b \rangle$ is an irreducible solvable-by-finite

subgroup of H . If $\langle a, b \rangle$ is infinite, then by theorem III.2.2, $\langle a, b \rangle$ is isomorphic, modulo $\pm I$, to the infinite dihedral group. Hence, $\langle a, b \rangle$ and therefore also G contains an element x such that $x^2 = -I$. If, on the other hand, $\langle a, b \rangle$ is a finite irreducible subgroup of G then the finite irreducible subgroups of $GL(2, C)$ are listed in Miller, Blickfeldt and Dickson [16b], pp. 215-226 (see also Dornhoff [6a] pp. 144-148), and all of them have even order modulo the scalars. Hence, $\langle a, b \rangle$ must contain a non-scalar element x such that x^2 is scalar, since $x \in SL(2, C)$, $x^2 - (\text{tr } x)x + I = 0$. Since x^2, I are scalar and x is not, $\text{tr } x = 0$ and hence $x^2 = -I$.

Conversely, suppose that G has an element x such that $x^2 = -I$. If G is finite, then there is nothing to prove, so G is infinite. Now, by Burnside's theorem (theorem 2.8A Dixon [4]), an irreducible linear group is finite if and only if the set of trace values of elements of G is finite. Since G is infinite, there exists an element $a \in G$, $a \neq x$ such that

$$\text{tr } a = \alpha \neq 0, \pm 1, \pm \sqrt{2}, \pm 1/\sqrt{2} \quad (1)$$

Now consider the group $\langle x, y \rangle$ where $y = axa^{-1}$. Then $x^2 = y^2 = -I$ so $\langle x, y \rangle$ is solvable-by-finite and

$$\begin{aligned} \text{tr } xy &= \text{tr } xa x a^{-1} \\ &= \text{tr } xax(\alpha I - a), \text{ since } a^2 - \alpha a + I = 0 \\ &= \alpha \text{tr } x^2 a - \text{tr}(xa)^2 \\ &= -\alpha^2 - (\gamma^2 - 2) \text{ if } \gamma = \text{tr } xa \\ &= 2 - \alpha^2 - \gamma^2 \end{aligned}$$

Hence $\langle x, y \rangle$ is irreducible if and only if

$$(\text{tr } x)^2 + (\text{tr } y)^2 + (\text{tr } xy)^2 - \text{tr } x \cdot \text{tr } y \cdot \text{tr } xy = 4 \neq 0$$

by theorem II.1.1. But $\text{tr } x = \text{tr } y = 0$ (see lemma II.2.1) so the latter condition is equivalent to

$$2 - \alpha^2 - \gamma^2 \neq \pm 2,$$

that is

$$(\alpha^2 + \gamma^2)(\alpha^2 + \gamma^2 + 4) \neq 0$$

Suppose that

$$(\alpha^2 + \gamma^2)(\alpha^2 + \gamma^2 + 4) = 0 \quad (2)$$

then we consider the group $\langle x, y' \rangle$ where $y' = a^2 x a^{-2}$. Once again, $\langle x, y' \rangle$ is solvable-by-finite because $x^2 = y'^2 = -I$. Since $\text{tr } a = \alpha \neq 0$, $a^2 \neq -I$. Also, $\text{tr } a^2 = \alpha^2 - 2$ and

$$\begin{aligned} \text{tr } a^2 x &= \text{tr}(\alpha a - I)x = \alpha \text{tr } ax - \text{tr } x \\ &= \alpha \gamma, \quad \gamma = \text{tr } ax \end{aligned}$$

Moreover,

$$\begin{aligned} \text{tr } xy' &= \text{tr } x a^2 x a^{-2} \\ &= \text{tr}(\alpha x a - x)((\alpha^2 - 1)x - \alpha x a) \\ &= \text{tr}[\alpha(\alpha^2 - 1)x a x - \alpha^2 (x a)^2 - (\alpha^2 - 1)x^2 + \alpha x^2 a] \\ &= \text{tr}[\alpha(\alpha^2 - 1)(-a) - \alpha^2 (x a)^2 - (\alpha^2 - 1)(-I) - \alpha a] \\ &= -\alpha^2(\alpha^2 - 1) - \alpha^2(\gamma^2 - 2) + 2(\alpha^2 - 1) - \alpha^2 \\ &= 2 - \alpha^2 \gamma^2 - (\alpha^2 - 2)^2 \end{aligned}$$

If $\text{tr } xy \neq 2$, then $\langle x, y \rangle$ is the required irreducible solvable-by-finite group. Suppose, on the contrary, that $\alpha^2 \gamma^2 + (\alpha^2 - 2)^2 - 2 = 2$, then

$$(\alpha^2 \gamma^2 + (\alpha^2 - 2)^2)(\alpha^2 \gamma^2 + (\alpha^2 - 2)^2 + 4) = 0 \quad (3)$$

If $\alpha^2 + \gamma^2 = 0$, then (3) becomes

$$(4 - 4\alpha^2)(8 - 4\alpha^2) = 16(1 - \alpha^2)(2 - \alpha^2) = 0$$

giving $\alpha = 1, \pm\sqrt{2}$. If, on the other hand, $\alpha^2 + \gamma^2 + 4 = 0$, then (3) becomes

$$32(1 - 2\alpha^2)(1 - \alpha^2) = 0$$

which yields

$$\alpha = 1, \pm 1/\sqrt{2}$$

However, our choice of α , by (1) is such that $\alpha \neq 0, 1, \pm\sqrt{2}, \pm 1/\sqrt{2}$.

Hence, in case (2) holds, (3) fails to hold and conversely. Therefore, either $\langle x, y \rangle$ or $\langle x, y^* \rangle$ is irreducible, and of course, solvable-by-finite as required.

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