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PRECISIONSM RESOLUTION TARGETS

Non-Linear Analysis of a Gyroscopic Rotor

by

JEFFREY S. CAIN, B.Eng.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Master of Engineering

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of
Mechanical and Aerospace Engineering
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Abstract

The dry tuned gyro (DTG) is an integral part of inertial measurement and control systems. The gyro rotor provides a frame of reference, fixed in inertial space, which can be used to measure the angular rates of the gyro case. Inertial measurement and control systems are used in a variety of applications including marine, aircraft and spacecraft. This document deals with the development and application of the non-linear, unsymmetric equations of motion of the gyro rotor and its gimbal suspension. The three dimensional kinematics of the gyro and gimbal were developed first. From the kinematic equations, the dynamic equations were derived using both a vectorial mechanics approach and Lagrange's equations. These two methods served to verify one another. The dynamic equations form the basis of a numerical simulation so that specific gyros can be modelled. The results suggest methods which may increase the accuracy of the gyroscope.

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Chapter 1

Introduction

Dry tuned gyroscopes (DTG) are an integral part of modern inertial measurement and control systems. They are used in many different types of vehicles such as satellites, aircraft and ships. The DTG is also used in airborne gravimetric systems and in the control of tracking radars. Common applications of the DTG are in the guidance of satellites, or unmanned space vehicles, and in underwater vehicles. The gyro provides a frame of reference which is fixed in inertial space and is used to measure the angular rates of the vehicle on which it is mounted. Increased vehicle accuracy requirements place high expectations on the DTG. A good example is Sweden's Odin spacecraft for which the required pointing accuracy is ± 7 arcseconds. These high performance applications make it crucial that a detailed understanding of the DTG internal dynamics be known.

The tuned gyro has existed for about 30 years and there are several analyses of it in the literature; however, most make simplifying assumptions for ease of calculation. These simplifications include the use of symmetry and linearization of equations. Linear analyses of the dry tuned gyro have been well documented by Craig[1], Lawrence[2] and Mansour and Lacchini[3] as well as many others; although, a comprehensive non-linear investigation of the dynamics of the system has not been done. When this system is analyzed, the tendency is to use non-spinning coordinates and

this may compound any errors in the mathematical representation. Staley[4] has performed a linear analysis of the DTG where the spinning coordinates were kept until the last step, when the results were transformed to a non-spinning coordinate frame. The current research is directed at a complete mathematical model of the gyroscope with as few simplifying assumptions as possible. With this model, a computer simulation was created for "numerical experimentation".

In general, the mechanical components of a gyroscope consist of a rotor attached to a shaft by a gimbal, or universal joint. When the shaft and rotor are spun, the rotor develops an angular momentum which 'fixes' the rotor in inertial space. The gimbal joint frees the rotor from rotations of the gyro case. From a coordinate system based on the gyro case, a rotation of the case would effectively be seen as a rotation of the rotor. The resulting angular displacement can be measured by using small magnetic sensors or pickoffs. In a 'strapdown' gyro, the rotor would then be nulled, or set to a zero rotation with respect to the case. It should be noted that a 'strapdown' gyro is a gyro which is mounted directly on the vehicle rather than on a gimballed platform and then the vehicle. In this configuration the gyro directly measures the angular rates of the vehicle. In practice there are several other factors which also cause the rotor to drift off its null position. These include mechanical and electrical noise as well as improperly set null positions. Once the dynamics of the gyro are understood, it may be possible to have the natural response of the gyroscopic system partially or completely cancel the drift by artificially offsetting the rotor.

This requires the development of the equations of motion of the gyroscope for which the kinematics of the gimbal joint must be first examined. The gimbal is essentially a universal, or Hooke's, joint which uses flexures rather than bearings or 'pins and jewels'. A specific solution of the gimbal kinematics, where the input and output shafts lie in the same plane, is well documented in the literature (see Martin[6] and Schmelz[5]); however, the case of a gyro requires a more general solution where the input and output shafts are not necessarily in the same plane. An example of the problem presented, due to this joint, is that the angular velocity of the output shaft is a function of the orientation of the output shaft with respect

to the input shaft and the angular velocity of the input shaft.

Once the kinematics of the gimbal have been derived, it is possible to calculate the equations of motion. The gyro can be considered as two coupled rigid bodies; a deceptively simple problem. When the rotational dynamics are considered, the problem becomes difficult, rapidly revealing itself to be non-linear. Also, for the general case, no assumptions of symmetry can be made, either in the rotor or the gimbal. Flexure stiffness must be considered unequal in orthogonal directions and similarly, the viscous damping must be unequal. The nonlinear equations were derived using vectorial mechanics and Lagrange's equations. By using these two methods, one method provided a check on the other. These methods were chosen for their clarity and simplicity of use.

The non-linear equations provide an accurate model of the gyro system. A computer simulation, using these equations, has been constructed, modelling the response of the gyro to angular displacements of the rotor. Since the simulation uses equations with no linearizing assumptions; any practical angular displacement can be modelled. The computer simulation was then verified by comparing the predicted drift rates to that of a real gyro. A Litton CSG-2 gyro at the DREO Laboratories in Ottawa was used as a comparison.

The gyro drift data was measured under a variety of conditions, and these results were compared to the output from a computer simulation of the same gyro. The comparison verified that the computer simulation accurately predicted the trends of the experimental gyro and would be a valid method of predicting the response of a gyro to specific initial conditions.

The output from the numeric simulation includes the rotor angular momentum and shaft reaction forces as well as the rotor angular position and velocity. The angular position data can be used to provide estimates on the torques required to maintain the current position of the rotor. The torques are those which would be required to maintain the null for drift error minimization. This will result in more accurate gyros and hence more accurate inertial sensors and attitude control systems.

Chapter 2

Kinematics of a Hooke's Joint

The gimbal, one of the most important components of the tuned rotor gyroscope, allows the rotational axis of the rotor to be oriented in a different direction than that of the spin motor shaft axis. That is, it allows the rotor axis to tilt about the x and y axes. The gimbal in gyroscopic rotor systems is based on Hooke's joint, Figure 2.1.

This joint, first discussed by Amicus¹, was developed in 1663 by the English physicist Robert Hooke for use in clocks and sundials where the transmission of rotary movement through an angled joint was necessary[5].

Today, this joint is commonly referred to as a universal joint and can be found in many applications, not the least of which are gyroscopes. With the addition of a second Hooke's joint 90 degrees to the first then the constant velocity joint is formed. Without the implementation of the Hooke's joint, the tuned rotor strapdown gyro would not be possible. Typically, the Hooke's joint uses bearings; however, in the gyro application, this was found to lead to instabilities in the rotor dynamics, therefore flexures have replaced the bearings.

The motions of the gimbal affect the motions of the rotor. For the rotor to rotate in place, the gimbal spin axis must oscillate at twice the spin frequency, Figure 2.2. This oscillation causes the flexures (which connect the rotor to the

¹Amicus, from the 17th century, described a type of universal joint in his unpublished work *Chronometria Mechanica Nova*

Figure 2.1: Schematic of a Hooke's Joint[5]

gimbal and the gimbal to the shaft) to exert torsional forces on the rotor, affecting its dynamic response. Thus, the motions of this joint must be fully described before the dynamics of the gyroscope can be understood.

2.1 Classical Analysis of a Hooke's Joint

A characteristic of the Hooke's joint that must be determined is the relationship between the angular rotation of the input axis to that of the output axis. Although he never published the relationship, Hooke knew that the universal joint did not transmit rotary movement steadily, as seen from the fact that he graduated the faces of sundials. In 1824 a French engineer, Jean-Victor Poncelet, proved, with the aid of spherical geometry, that the motion of Hooke's joint is non-uniform[5]. A later proof used the principle of projective geometry[6].

Both solutions consider two shafts lying in a plane at an angle of β to each other, Figure 2.3. Shaft 2 can be taken as the input axis, shaft 3 the output axis and the joint at 4 as the gimbal. Figure 2.4(a) shows that as shaft 2 rotates point B will move in a circular path as shown in Figure 2.4(b). This will cause point D to rotate in a circular path as well and hence result in the rotation of shaft 3. If the rotation of shaft 3 is taken to be θ_3 then point B will move to B'. From Figure 2.4(b)

$$OF = R \cos \theta_3 \quad (2.1)$$

and

$$FB' = R \sin \theta_3 \quad (2.2)$$

Shafts 2 and 3 lie at an angle of β to each other and

$$\begin{aligned} OG &= OF \cos \beta \\ &= R \cos \theta_3 \cos \beta \end{aligned} \quad (2.3)$$

Figure 2.2: Oscillation of a gimbal[7].

Figure 2.3: Two shafts lying at an angle β to each other[6].

Figure 2.4: Projection analysis of two shafts connected by a Hooke's joint[6].

At this point it is essential to note that point B has moved a specific distance perpendicular to the plane of the shafts, FB' in Figure 2.4(b). This distance is also GB' in Figure 2.4(c). Therefore

$$GB' = R \sin \theta_3 \quad (2.4)$$

From Figure 2.4(c) it can be seen that

$$\begin{aligned} \tan \theta_2 &= \frac{GB'}{OG} \\ &= \frac{R \sin \theta_3}{R \cos \theta_3 \cos \beta} \end{aligned} \quad (2.5)$$

This simplifies to

$$\tan \theta_3 = \tan \theta_2 \cos \beta \quad (2.6)$$

Equation 2.6 describes the relationship between the input shaft rotation and the output shaft rotation. This classical result can also be obtained using the spherical techniques used by Poncelet[5]. Derivatives of Equation 2.6 show that if the input shaft rotation is taken to be uniform with time then the output shaft angular rotation will oscillate about the input shaft position at twice the frequency of the input axis angular velocity.

2.2 General Analysis of Hooke's Joint

The classical solution has been used for nearly 150 years and is valid for the special case when the coordinate system used allows the relation between the input shaft and output shaft to be described by only one angle. In this case the angle β fully describes the system. In the general solution, two angles are needed to describe the position of the output shaft with respect to the input shaft, see Figure 2.5. A general method for relating the input shaft to the output shaft is to rotate the coordinate system, with the use of rotation matrices, of the input shaft to that of the output shaft. For this type of analysis three angles must be used; one about the x-axis, one about the y-axis and one about the z-axis. It is the order of these rotations that is important. One possible order is to apply the z-axis rotation about the input shaft

Figure 2.5: Position of output and input shafts in a 3 dimensional sphere.

axis, then the x and y axes of rotation are defined by the x and y axes of the gimbal itself. The result is called the 'gimbal' set of coordinates.

The order of rotation of the two gimbal angles can be arbitrarily set about the x-axis, followed by the y-axis. Thus, with the full rotation matrix from the input shaft coordinates to a rotor, or output shaft, a set of coordinates can be shown to be

$$A = (\theta)_y(\phi)_x(\omega_s t)_z \quad (2.7)$$

where

$(\theta)_y$ = rotation by θ about its y-axis, etc.

$\omega_s t$ = input shaft rotation

ϕ = x-axis gimbal angle

θ = y-axis gimbal angle

This rotation sequence is shown pictorially in Figure 2.6. Expanding Equation 2.7 gives

$$A = \begin{pmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_\phi & S_\phi \\ 0 & -S_\phi & C_\phi \end{pmatrix} \begin{pmatrix} C_{\omega_s t} & S_{\omega_s t} & 0 \\ -S_{\omega_s t} & C_{\omega_s t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 2.6: Rotations about a 'gimbal' set of coordinates.

where, for example,

C_ϕ represents $\cos \phi$

S_ϕ represents $\sin \phi$

Therefore

$$A = \begin{pmatrix} C_\theta C_{\omega_s t} - S_\theta S_\phi S_{\omega_s t} & C_\theta S_{\omega_s t} + S_\theta S_\phi C_{\omega_s t} & -S_\theta C_\phi \\ -C_\phi S_{\omega_s t} & C_\phi C_{\omega_s t} & S_\phi \\ S_\theta C_{\omega_s t} + C_\theta S_\phi S_{\omega_s t} & S_\theta S_{\omega_s t} - C_\theta S_\phi C_{\omega_s t} & C_\theta C_\phi \end{pmatrix} \quad (2.8)$$

A second method of describing the position of the rotor is obtained using a set of 'case' referenced angles. The order of the rotations is about the x and y-axes first and then a rotation about the z-axis. The x and y-axes are essentially fixed to the gyroscope case, hence the name 'case' referenced angles. This can be written as

$$A' = (\gamma)_z (\theta')_y (\phi')_x \quad (2.9)$$

where

γ = the output shaft rotation

ϕ' = the x-axis rotor angle

θ' = the y-axis rotor angle

Figure 2.7: Rotations about a 'case' referenced coordinate set.

This rotation sequence can be seen in Figure 2.7. Expanding Equation 2.9 gives

$$A' = \begin{pmatrix} C_\gamma & S_\gamma & 0 \\ -S_\gamma & C_\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_{\theta'} & 0 & -S_{\theta'} \\ 0 & 1 & 0 \\ S_{\theta'} & 0 & C_{\theta'} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\phi'} & S_{\phi'} \\ 0 & -S_{\phi'} & C_{\phi'} \end{pmatrix}$$

which multiplies out to

$$A' = \begin{pmatrix} C_\gamma C_{\theta'} & C_\gamma S_{\theta'} S_{\phi'} + S_\gamma C_{\phi'} & -C_\gamma S_{\theta'} C_{\phi'} + S_\gamma S_{\phi'} \\ -S_\gamma C_{\theta'} & -S_\gamma S_{\theta'} S_{\phi'} + C_\gamma C_{\phi'} & S_\gamma S_{\theta'} C_{\phi'} + C_\gamma S_{\phi'} \\ S_{\theta'} & -C_{\theta'} S_{\phi'} & C_{\theta'} C_{\phi'} \end{pmatrix} \quad (2.10)$$

Both Equation 2.8 and Equation 2.10 describe the same orientation so they must be equal. Therefore

$$A = A' \quad (2.11)$$

To find the relationship between the input and output angular rotations, take

$$\frac{A_{21}}{A_{22}} = \frac{A'_{21}}{A'_{22}}$$

which, after some simplification becomes

$$\frac{S_{\omega_s t}}{C_{\omega_s t}} = \frac{S_\gamma C_{\theta'}}{C_\gamma C_{\phi'} - S_\gamma S_{\theta'} S_{\phi'}}$$

This can be rearranged to give

$$\frac{S_\gamma}{C_\gamma} = \frac{C_{\phi'} S_{\omega_s t}}{C_{\theta'} C_{\omega_s t} + S_{\phi'} S_{\theta'} S_{\omega_s t}} \quad (2.12)$$

Equation 2.12 is the general solution for the relationship between the input and output angular rotations. Note that γ is in terms of 'case' referenced angles. This was done so the solution would be consistent with the classic solution, which is also in terms of a 'case' referenced angle, β . It is possible to set the initial conditions such that Equation 2.12 is equal to the classic solution, Equation 2.6 from Section 2.1. Setting the y-axis rotation to zero will place the two shafts into the xz plane, which matches the initial conditions used by the classical solution. Equation 2.12 becomes

$$\tan \gamma = \cos \phi' \tan \omega_s t \quad (2.13)$$

where

γ is equivalent to θ_3

$\omega_s t$ is equivalent to θ_2

This shows that the classic solution can be found from the general solution.

2.3 Conversion of Case Referenced and Gimbal Referenced Angles

Section 2.2 shows that the position of the rotor can be described in two different coordinate sets. Equation 2.11 shows these two methods can be related. For a complete analysis of the gimbal joint in the gyro the relationships between the 'case' referenced angles and the 'gimbal' referenced angles must be known.

From Equations 2.8, 2.10 and 2.11 it can be seen that

$$C_{\omega_s t} A_{31} + S_{\omega_s t} A_{32} = C_{\omega_s t} A'_{31} + S_{\omega_s t} A'_{32}$$

This gives

$$\begin{aligned} C_{\omega_s t} (S_\theta C_{\omega_s t} + C_\theta S_\phi S_{\omega_s t}) \\ + S_{\omega_s t} (S_\theta S_{\omega_s t} - C_\theta S_\phi C_{\omega_s t}) = C_{\omega_s t} S_{\theta'} - S_{\omega_s t} C_{\theta'} S_{\phi'} \end{aligned}$$

which can be simplified. S_θ can now be isolated to give

$$S_\theta = S_{\theta'} C_{\omega,t} - C_{\theta'} S_{\phi'} S_{\omega,t} \quad (2.14)$$

The relation $A_{23} = A'_{23}$ can be used to find

$$S_\phi = S_{\theta'} S_\gamma C_{\phi'} + S_{\phi'} C_\gamma \quad (2.15)$$

Equations 2.14 and 2.15 give the 'gimbal' referenced angles in terms of the 'case' referenced angles. Similarly, the 'case' referenced angles can be found in terms of the 'gimbal' referenced angles. The relation $A'_{31} = A_{31}$ can be used to find

$$S_{\theta'} = S_\theta C_{\omega,t} + C_\theta S_\phi S_{\omega,t} \quad (2.16)$$

And the relation $A_{13} S_\gamma + A_{23} C_\gamma$ gives

$$S_{\phi'} = C_\gamma S_\phi - S_\theta C_\phi S_\gamma \quad (2.17)$$

It is now necessary to find γ in terms of the 'gimbal' referenced angles, so that γ may be calculated and used in Equations 2.15 and 2.17. From

$$\frac{A_{21}}{A_{11}} = \frac{A'_{21}}{A'_{11}}$$

and with some algebra

$$\tan \gamma = \frac{C_\phi S_{\omega,t}}{C_\theta C_{\omega,t} - S_\theta S_\phi S_{\omega,t}} \quad (2.18)$$

The relationship between γ and the 'case' referenced angles has already been calculated in Equation 2.12 to be

$$\tan \gamma = \frac{C_{\phi'} S_{\omega,t}}{C_{\theta'} C_{\omega,t} + S_{\phi'} S_{\theta'} S_{\omega,t}}$$

Using the relationships found above, any transformation between the two reference planes is possible.

2.4 Calculation of Gimbal Angular Rates

To fully describe the motion of the gimbal it is now necessary to relate the 'case' referenced gimbal rates to the 'gimbal' referenced rates. There are two methods to do this, the first is to find the derivatives of the relations found in Section 2.3. The second method is to rotate the angular rates using the methods described in Section 2.2; equate the two systems and then solve.

2.4.1 Calculation Using Derivatives of Angular Position

The first method to calculate the gimbal rates is to take the derivative of the relationships in Section 2.3. Assuming that all angular positions are known, the relationships between the angular rates can be found.

Taking the derivative of Equation 2.14 gives

$$\dot{\theta} = \frac{1}{C_{\theta}} \left[(C_{\theta'} C_{\omega_s t} + S_{\theta'} S_{\phi'} S_{\omega_s t}) \dot{\theta}' - C_{\theta'} C_{\phi'} S_{\omega_s t} \dot{\phi}' - (S_{\theta'} S_{\omega_s t} + C_{\theta'} S_{\phi'} C_{\omega_s t}) \omega_s \right] \quad (2.19)$$

Similarly, $\dot{\phi}$ can be found by taking the derivative of Equation 2.15

$$\dot{\phi} = \frac{1}{C_{\phi}} \left[(C_{\gamma} S_{\theta'} C_{\phi'} - S_{\gamma} S_{\phi'}) \dot{\gamma} + S_{\gamma} C_{\theta'} C_{\phi'} \dot{\theta}' - (S_{\gamma} S_{\theta'} S_{\phi'} - C_{\phi'} C_{\gamma}) \dot{\phi}' \right] \quad (2.20)$$

Equations 2.19 and 2.20 give the 'gimbal' referenced angles as a function of the 'case' referenced angles. The inverse relationship can be found using the same method. Taking the derivative of Equation 2.17 gives

$$\dot{\phi}' = \frac{1}{C_{\phi'}} \left[(C_{\phi} C_{\gamma} + S_{\phi} S_{\phi'} S_{\gamma}) \dot{\phi} - C_{\phi} C_{\phi'} S_{\gamma} \dot{\theta} - (S_{\phi} S_{\gamma} + S_{\phi} C_{\phi} C_{\gamma}) \dot{\gamma} \right] \quad (2.21)$$

Finally, taking the derivative of Equation 2.16 gives

$$\dot{\theta}' = \frac{1}{C_{\theta'}} \left[(C_{\theta} C_{\omega_s t} - S_{\theta} S_{\phi} S_{\omega_s t}) \dot{\theta} + C_{\theta} C_{\phi} S_{\omega_s t} \dot{\phi} + (C_{\theta} S_{\phi} C_{\omega_s t} - S_{\theta} S_{\omega_s t}) \omega_s \right] \quad (2.22)$$

For a full description, $\dot{\gamma}$ must be found in terms of the 'case' referenced angles and also in terms of the 'gimbal' referenced angles. This can be accomplished by taking the derivatives of Equations 2.12 and 2.18 and are shown below as Equations 2.23 and 2.24, respectively.

$$\dot{\gamma} = - \frac{C_{\theta'} \left[C_{\phi'} \left[\dot{\theta}' S_{\phi'} S_{\omega_s t}^2 - \omega_s \right] + S_{\phi'} \dot{\phi}' S_{\omega_s t} C_{\omega_s t} \right] + S_{\theta'} S_{\omega_s t} \left[\dot{\phi}' S_{\omega_s t} - \dot{\theta}' C_{\phi'} C_{\omega_s t} \right]}{(1 + \tan^2 \gamma) [C_{\theta'} C_{\omega_s t} + S_{\phi'} S_{\theta'} S_{\omega_s t}]^2} \quad (2.23)$$

$$\dot{\gamma} = \frac{C_{\theta} \left[C_{\phi} \left(\dot{\theta} S_{\phi} S_{\omega_s t}^2 + \omega_s \right) - S_{\phi} \dot{\phi} S_{\omega_s t} C_{\omega_s t} \right] + S_{\theta} S_{\omega_s t} \left[\dot{\theta} C_{\phi} C_{\omega_s t} + \dot{\phi} S_{\omega_s t} \right]}{(\tan^2 \gamma + 1) [C_{\theta} C_{\omega_s t} - S_{\phi} S_{\theta} S_{\omega_s t}]^2} \quad (2.24)$$

The relationships between the four angular velocities have now been fully described. It can be seen that the above equations are large and complex and suggest

that another method of calculating these relationships may be better. This leads to the calculation of the gimbal angular rates using a procedure which rotates the angular velocities into the proper positions.

2.4.2 Calculation Using Rotation of Angular Velocities

There is a second method of calculating the relationships between the two sets of coordinates. By definition, the rotational velocities are set about certain axes. These axes can be transformed into the final orientation of the rotor. If the 'case' referenced coordinate set is used then $\dot{\phi}'$ must be rotated about the y axis and then the z axis. The angular velocity about the y axis, $\dot{\theta}'$, must be rotated about the z axis. These rotations and the addition of the rotational velocity about the output axis fully describe the angular velocities of the output shaft. In other words

$$\omega_r = \begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix} + (\gamma)_z \begin{pmatrix} 0 \\ \dot{\theta}' \\ 0 \end{pmatrix} + (\gamma)_z(\theta')_y \begin{pmatrix} \dot{\phi}' \\ 0 \\ 0 \end{pmatrix}$$

This can be simplified to give

$$\omega_r = \begin{pmatrix} C_\gamma C_{\theta'} \dot{\phi}' + S_\gamma \dot{\theta}' \\ -S_\gamma C_{\theta'} \dot{\phi}' + C_\gamma \dot{\theta}' \\ \dot{\gamma} + S_{\theta'} \dot{\phi}' \end{pmatrix} \quad (2.25)$$

A similar analysis can be made using the 'gimbal' set of coordinates. In this case the angular rate about the input shaft is rotated about the x and y gimbal axes to place it in the orientation of the output shaft. The rate about the x axis must be transformed as well. The relation becomes

$$\omega'_r = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + (\theta)_y \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + (\theta)_y(\phi)_x \begin{pmatrix} 0 \\ 0 \\ \omega_s \end{pmatrix}$$

This can be simplified to

$$\omega'_r = \begin{pmatrix} C_\theta \dot{\phi} - S_\theta C_\phi \omega_s \\ \dot{\theta} + S_\theta \omega_s \\ S_\theta \dot{\phi} + C_\theta C_\phi \omega_s \end{pmatrix} \quad (2.26)$$

Both Equation 2.25 and 2.26 describe the same final orientation hence they must be equal. Therefore

$$\omega_r = \omega'_r \quad (2.27)$$

Now the relationships between the 'gimbal' referenced angular rates and the 'case' referenced rates can be shown to be

$$\dot{\phi} = \frac{1}{C_\theta} (C_\gamma C_{\theta'} \dot{\phi}' + S_\gamma \dot{\theta}' + S_\theta C_\phi \omega_s) \quad (2.28)$$

and

$$\dot{\theta} = C_\gamma \dot{\theta}' - S_\gamma C_{\theta'} \dot{\phi}' - S_\phi \omega_s \quad (2.29)$$

Similarly, the relationships between the 'case' and 'gimbal' referenced rates are found to be

$$\dot{\phi}' = \frac{1}{C_{\theta'}} [C_\gamma C_\theta \dot{\phi} - S_\gamma \dot{\theta} - (C_\gamma S_\theta C_\phi + S_\gamma S_\phi) \omega_s] \quad (2.30)$$

and

$$\dot{\theta}' = S_\gamma C_\theta \dot{\phi} + C_\gamma \dot{\theta} - (S_\gamma S_\theta C_\phi - C_\gamma S_\phi) \omega_s \quad (2.31)$$

The last relationship to be found from Equation 2.27 is the relationship of $\dot{\gamma}$ with the angular rates about the other axes.

$$\dot{\gamma} = S_\theta \dot{\phi} + C_\theta C_\phi \omega_s - S_\phi \dot{\phi}' \quad (2.32)$$

Thus, the angular rates have been fully described in terms of the two coordinate systems.

2.4.3 Comparison of Angular Rate Calculation Methods

In the previous two sections alternative methods of calculating the angular rates of the gimbal have been presented. The results from both methods should be the same; however, at a glance this similarity is not clear. A comparison of the results from the two methods is necessary to verify that they are, in fact, the same. First Equations 2.21 and 2.30 must be compared and proved equal.

Proof 1

Substituting Equation 2.32 into Equation 2.21 gives

$$\dot{\phi}' = \frac{1}{C_{\phi'}} [(C_{\phi}C_{\gamma} + S_{\theta}S_{\phi}S_{\gamma})\dot{\phi} - C_{\theta}C_{\phi}S_{\gamma}\dot{\theta} - (S_{\phi}S_{\gamma} + S_{\theta}C_{\phi}C_{\gamma})(S_{\theta}\dot{\phi} + C_{\theta}C_{\phi}\omega_s - S_{\theta'}\dot{\phi}')]]$$

which can be simplified to

$$\begin{aligned} [C_{\phi'} - S_{\theta'}(S_{\phi}S_{\gamma} + S_{\theta}C_{\phi}C_{\gamma})]\dot{\phi}' \\ = C_{\theta}^2C_{\phi}C_{\gamma}\dot{\phi} - C_{\theta}C_{\phi}S_{\gamma}\dot{\theta} - [S_{\phi}S_{\gamma} + S_{\theta}C_{\phi}C_{\gamma}]C_{\theta}C_{\phi}\omega_s \end{aligned} \quad (2.33)$$

From Equation 2.11

$$A_{13} = A'_{13}$$

Therefore

$$S_{\theta}C_{\phi} = C_{\gamma}S_{\theta'}C_{\phi'} - S_{\gamma}S_{\phi'} \quad (2.34)$$

Substitute Equations 2.34 and 2.15 into Equation 2.34. After simplification this gives

$$C_{\phi'}C_{\theta'}^2\dot{\phi}' = C_{\theta}^2C_{\phi}C_{\gamma}\dot{\phi} - C_{\theta}C_{\phi}S_{\gamma}\dot{\theta} - S_{\theta'}C_{\phi'}C_{\theta}C_{\phi}\omega_s \quad (2.35)$$

But from $A_{33} = A'_{33}$

$$C_{\theta}C_{\phi} = C_{\theta'}C_{\phi'} \quad (2.36)$$

which can be substituted into Equation 2.35. After simplification it becomes

$$C_{\theta'}\dot{\phi}' = C_{\theta}C_{\gamma}\dot{\phi} - S_{\gamma}\dot{\theta} - S_{\theta'}C_{\phi'}\omega_s \quad (2.37)$$

Now calculating $A_{23}S_{\gamma} - A_{13}C_{\gamma}$, which gives

$$C_{\phi'}S_{\theta'} = C_{\gamma}S_{\theta}C_{\phi} + S_{\gamma}S_{\phi} \quad (2.38)$$

and substituting this into Equation 2.37 gives

$$C_{\theta'}\dot{\phi}' = C_{\theta}C_{\gamma}\dot{\phi} - S_{\gamma}\dot{\theta} - (C_{\gamma}S_{\theta}C_{\phi} + S_{\gamma}S_{\phi})\omega_s \quad (2.39)$$

Equation 2.39 and Equation 2.30 are equivalent and therefore Equation 2.21 and 2.30 are the same.

Proof 2

It is now necessary to compare Equations 2.22 and 2.31. This can be done by a similar method as in the case above. From $A_{11} = A'_{11}$ it can be seen that

$$C_\gamma C_{\theta'} = C_\theta C_{\omega_s t} - S_\theta S_\phi S_{\omega_s t} \quad (2.40)$$

Substituting into Equation 2.22 gives

$$\dot{\theta}' = C_\gamma \dot{\theta} + \frac{C_\theta C_\phi}{C_{\theta'}} S_{\omega_s t} \dot{\phi} + \frac{C_\theta S_\phi C_{\omega_s t} - S_\theta S_{\omega_s t}}{C_{\theta'}} \omega_s \quad (2.41)$$

From Equation 2.36

$$C_{\phi'} = \frac{C_\theta C_\phi}{C_{\theta'}}$$

Substitute into Equation 2.41

$$\dot{\theta}' = C_\gamma \dot{\theta} + C_{\phi'} S_{\omega_s t} \dot{\phi} + \frac{C_\theta S_\phi C_{\omega_s t} - S_\theta S_{\omega_s t}}{C_{\theta'}} \omega_s \quad (2.42)$$

From $A_{32} = A'_{32}$

$$S_{\phi'} = \frac{C_\theta S_\phi C_{\omega_s t} - S_\theta S_{\omega_s t}}{C_{\theta'}} \quad (2.43)$$

Substitute this into Equation 2.42, gives

$$\dot{\theta}' = C_\gamma \dot{\theta} + C_{\phi'} S_{\omega_s t} \dot{\phi} + S_{\phi'} \omega_s \quad (2.44)$$

Now combine $A_{21} = A'_{21}$ and Equation 2.36 to get

$$S_{\omega_s t} = \frac{S_\gamma C_\theta}{C_{\phi'}}$$

This can be substituted in Equation 2.44 to achieve

$$\dot{\theta}' = C_\gamma \dot{\theta} + S_\gamma C_\theta \dot{\phi} + S_{\phi'} \omega_s \quad (2.45)$$

Finally replace $S_{\phi'}$ in Equation 2.45 with the relationship described in Equation 2.17 and simplify.

$$\dot{\theta}' = C_\gamma \dot{\theta} + S_\gamma C_\theta \dot{\phi} - (S_\theta C_\phi S_\gamma - S_\phi C_\gamma) \omega_s \quad (2.46)$$

Equation 2.46 and Equation 2.31 are equal, and this means that Equations 2.31 and 2.22 are the same.

Proof 3

Next, Equations 2.20 and 2.28 must be proved to be the same. Starting with Equation 2.20 and replacing $\dot{\gamma}$ with Equation 2.32, gives

$$\begin{aligned} C_\phi \dot{\phi} = & [C_\gamma S_{\theta'} C_{\phi'} - S_\gamma S_{\phi'}] [S_\theta \dot{\phi} + C_\theta C_\phi \omega_s - S_{\theta'} \dot{\phi}'] \\ & + S_\gamma C_{\theta'} C_{\phi'} \dot{\theta}' + [C_{\phi'} C_\gamma - S_\gamma S_{\theta'} S_{\phi'}] \dot{\phi}' \end{aligned}$$

This simplifies to

$$\begin{aligned} [C_\phi - C_\gamma S_{\theta'} C_{\phi'} S_\theta + S_\gamma S_{\phi'} S_\theta] \dot{\phi} = & [C_\gamma S_{\theta'} C_{\phi'} - S_\gamma S_{\phi'}] C_\theta C_\phi \omega_s \\ & + S_\gamma C_{\theta'} C_{\phi'} \dot{\theta}' + [C_{\phi'} C_\gamma - C_\gamma S_{\theta'}^2 C_{\phi'}] \dot{\phi}' \end{aligned} \quad (2.47)$$

Using Equations 2.34 and 2.36 in the above and simplifying gives

$$C_\theta \dot{\phi} = S_\theta C_\phi \omega_s + S_\gamma \dot{\theta}' + C_\gamma C_{\theta'} \dot{\phi}' \quad (2.48)$$

Equation 2.48 is equivalent to Equation 2.28 and thus verifies that Equations 2.28 and 2.20 are the same.

Proof 4

Finally, Equations 2.19 and 2.29 must be shown to be the same. First, substitute Equation 2.15 into Equation 2.29

$$\dot{\theta} = -S_\gamma C_{\theta'} \dot{\phi}' + C_\gamma \dot{\theta}' - \frac{\omega_s}{C_\theta} (S_{\theta'} S_{\omega_s t} + C_{\theta'} S_{\phi'} C_{\omega_s t}) \quad (2.49)$$

From $A_{23} = A'_{23}$

$$S_\gamma = \frac{S_{\omega_s t} C_\phi}{C_{\theta'}}$$

and from $A_{33} = A'_{33}$

$$C_\theta = \frac{C_{\phi'} C_{\theta'}}{C_\phi}$$

Substituting these into Equation 2.49 results in

$$C_\theta \dot{\theta} = \frac{C_\gamma C_{\phi'} C_{\theta'}}{C_\phi} \dot{\theta}' - C_\phi C_\theta S_{\omega_s t} \dot{\phi}' - \omega_s (S_{\theta'} S_{\omega_s t} + C_{\theta'} S_{\phi'} C_{\omega_s t}) \quad (2.50)$$

but from $A_{22} = A'_{22}$

$$C_\gamma C_{\phi'} = C_\phi C_{\omega_s t} + S_\gamma S_{\theta'} S_{\phi'}$$

which can be substituted into Equation 2.50 to give

$$\begin{aligned} C_\theta \dot{\theta} = C_{\theta'} C_{\omega_s t} \dot{\theta}' &+ \frac{S_\gamma C_{\theta'} S_{\theta'} S_{\phi'} \dot{\theta}'}{C_\phi} - C_\phi C_\theta S_{\omega_s t} \dot{\phi}' \\ &- \omega_s (S_{\theta'} S_{\omega_s t} + C_{\theta'} S_{\phi'} C_{\omega_s t}) \end{aligned} \quad (2.51)$$

Finally, from $A_{21} = A'_{21}$

$$S_\gamma C_{\theta'} = S_{\omega_s t} C_\phi$$

Combining this and Equation 2.52 gives

$$\begin{aligned} C_\theta \dot{\theta} = C_{\theta'} C_{\omega_s t} \dot{\theta}' &+ S_{\omega_s t} S_{\theta'} S_{\phi'} \dot{\theta}' - C_\phi C_\theta S_{\omega_s t} \dot{\phi}' \\ &- \omega_s (S_{\theta'} S_{\omega_s t} + C_{\theta'} S_{\phi'} C_{\omega_s t}) \end{aligned} \quad (2.52)$$

Equation 2.52 is equal to Equation 2.19 and this proves that Equation 2.19 and 2.29 describe the same relationship.

The above four proofs show that the two methods of calculating the angular rates of the gimbal provide the same results. However by inspection it can be seen that the relationships found in Section 2.4.2 are simpler making them more practical for use in simulations. It is clear that with relations from either method, the description of the gimbal joint in a gyroscopic rotor system is complete. The angular accelerations can be calculated through use of similar methods; however, the acceleration transformations were not needed in this analysis of the gimbal kinematics.

Chapter 3

Dynamics of a Gyroscopic Rotor

The dynamics of the gyroscope rotor can be found now that the kinematics of the Hooke's joint are known. When the gyroscope was first designed in the 1940's it incorporated a spinning flywheel on an unmodified Hooke's joint. This gyro was found to be very unstable due to the dynamic inertia effect of the gimbal.

Two decades later the "Gyroflex" gyro was developed. Again, this gyro used a rotor on a universal joint; however, flexures were used in place of the normal bearings in the gimbal.

Permanent magnets were attached to the gyro case and provided an attractive force on the rotor. This force effectively cancelled the flexure spring rate and freed the rotor from experiencing any torque. There were several problems with this design and it's performance was inadequate for practical use.

In 1963, a gyro was developed using the dynamic inertia effects of the gimbal (instead of magnets) to create a virtual negative spring rate. This gyro is known as a Dry Tuned Gyroscope (DTG) and is still in use today.[7]

3.1 Description of the Dry Tuned Gyro

The gyro, an important sensor used for many applications, essentially consists of a rotating disk, known as the rotor, mounted on a gimbaled joint. Ideally, the joint isolates the rotor from any external moments, thus the axis of the rotor would remain inertially fixed despite any rotational movement of the object to which it is

Figure 3.1: Schematic of a Dry Tuned Gyro[9]

attached. Figure 3.1 shows a general schematic of a gyroscope. As seen from the figure, the gyroscope is not completely isolated from external angular displacements because of the flexures, or torsional springs. These flexures, used by the gimbal to connect the rotor and the shaft, produce positive spring rates which can affect the rotor. It should be stressed that Figure 3.1 is only a schematic of a gyro and that the gimbal actually resembles Figure 3.2. These flexures are carefully designed so the centre of deflection remains constant, Figure 3.3. This property limits the possibility of translation of the rotor with respect to the case.

Dynamically, the gyroscope is a complicated device. The large angular momentum of the rotor keeps the rotor fixed in space; however, if the rotor experiences a torque about a normal to the spin axis (ie. the x-axis) it will change its position in space. This change in position does not occur as a rotation about the x-axis, as would be expected, but about the y-axis. In other words, the effects of a torque applied to the rotor will be felt 90 degrees from the point of application. A simplified

Figure 3.2: Typical gimbal assembly for a DTG[9]

Figure 3.3: Gimbal assembly before and after a rotation of the rotor about one axis[9]

vector analysis approach is shown in Figure 3.4.

In Figure 3.4 a simple rotor, with moment of inertia, I , and an angular velocity of ω has a moment $\mathbf{M}$ applied to it. Originally, the rotor has an angular momentum of

$$\mathbf{H} = I\omega$$

The moment $\mathbf{M}$ is applied normal to $\mathbf{H}$ and causes a change of $d\mathbf{H}$, or $d(I\omega)$ in time dt . However, it can be shown for any mass system that $\mathbf{M} = \dot{\mathbf{H}}$ and from this it can be seen that the direction of $d\mathbf{H}$ is in the direction of $\mathbf{M}$. Therefore the angular momentum of the rotor rotates through an angle of α about the y-axis[8]. Hence, the application of a moment about the x-axis results in a precession about the y-axis.

The converse of this phenomenon is true as well. A rotation of the rotor about the y-axis causes a moment about the x-axis. It is from this basic premise that the possibility of a negative spring rate exists. For a simplified analysis, the following derivation will only consider the torques caused by the flexures. In Chapter 2 it was shown that if the rotor is offset from null by even the smallest angle, the gimbal must oscillate at twice the spin frequency of the rotor. Thus, at any point in time, there is a positive torsional force on the rotor of $2K_y\theta$, see Figure 3.5, due to the difference in the angular positions of the rotor and the gimbal. However, this motion also creates torques, as described above, and if the rotational velocity of the rotor is properly set these torques will be equal and opposite to those due to the torsional springs. This is known as the tuning condition of the gyro.

From Euler's equations, and assuming that the gimbal is symmetric

$$M_{gy} = I_{gy}\dot{\omega}_{gy} - (I_{gz} - I_{gx})\omega_{gz}\omega_{gx} \quad (3.1)$$

where the subscript 'g' refers to gimbal quantities. For the torsional forces to be equal and opposite then

$$M_{gy} = -2K_y\theta \quad (3.2)$$

Therefore Equation 3.1 becomes

$$-2K_y\theta = I_{gy}\dot{\omega}_{gy} - (I_{gz} - I_{gx})\omega_{gz}\omega_{gx} \quad (3.3)$$

Figure 3.4: Simple rotor with transverse moment applied[8].

It is now necessary to find ω_g as a function of θ and ϕ^1 . Using the method already described in Section 2.4.2 to find the relationship, the angular rate of the input shaft must be rotated about the x-axis and then the rate about this axis must be added.

$$\omega_g = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + (\phi)_x \begin{pmatrix} 0 \\ 0 \\ \omega_s \end{pmatrix} \quad (3.4)$$

This can be simplified to give

$$\omega_g = \begin{pmatrix} \dot{\phi} \\ S_\phi \omega_s \\ C_\phi \omega_s \end{pmatrix} \quad (3.5)$$

Substituting the values for ω_g from Equation 3.5 into Equation 3.3 gives

$$-2K_y \theta = I_{gy} \dot{\phi} C_\phi \omega_s - (I_{gz} - I_{gx}) \dot{\phi} C_\phi \omega_s \quad (3.6)$$

Since the angular offset of the rotor is normally small, this equation can be linearized by stating that

$$S_\phi \approx \phi$$

and

$$C_\phi \approx 1$$

This gives

$$-2K_y \theta = (I_{gy} - I_{gz} + I_{gx}) \dot{\phi} \omega_s \quad (3.7)$$

Define ϕ in terms of an initial offset as

$$\phi = \phi_0 \cos 2\omega_s t$$

and take the derivative of this with respect to time

$$\dot{\phi} = 2\omega_s \phi_0 \sin 2\omega_s t$$

¹Note that θ and ϕ describe the rotor position in terms of the 'gimbal' oriented coordinate system introduced in Chapter 2

The angle θ can also be defined in a similar way, while noting that it is phase shifted from ϕ by 90 degrees. Therefore

$$\theta = \phi_0 \sin 2\omega_s t$$

Now substitute the above relations into Equation 3.7

$$-2K_y \phi_0 \sin 2\omega_s t = -2(I_{gy} + I_{gz} - I_{gx}) \omega_s^2 \phi_0 \sin 2\omega_s t \quad (3.8)$$

Simplify and solve for ω_s

$$\omega_s^2 = \frac{K_y}{I_{gy} + I_{gz} - I_{gx}} \quad (3.9)$$

It should be noted that this equation does not take all the moments experienced by the gimbal into account and so it does not give the full tuning condition, which is[1]

$$\omega_s^2 = \frac{2K_x + 2K_y}{I_{gy} + I_{gz} - I_{gx}} \quad (3.10)$$

however, most of the components are there. If the spin velocity of the input shaft is set by the tuning condition then the rotor is essentially isolated from the flexure connections. The proper tuning condition for a DTG, Equation 3.10, can be found by using a linear analysis of the gyroscopic system and setting the boundary conditions properly.

3.2 Derivation of the Equations of Motion of a Gyroscopic Rotor

The methods for finding the equations of motion for two connected rigid bodies, in this instance, the gyro, are numerous. Two of these methods were used in solving this problem, namely the vectorial mechanics approach and Lagrange's equation, serving as a cross check for each other. A comparison between the ease of use of the two methods is made in Section 3.2.3.

3.2.1 Derivation of the Equations of Motion using Vector Mechanics

The description of a rotating object, which is not undergoing translation can be found using Euler's equations, shown below

$$\mathbf{M} = \begin{bmatrix} \dot{H}_x - H_y\omega_z + H_z\omega_y \\ \dot{H}_y - H_z\omega_x + H_x\omega_z \\ \dot{H}_z - H_x\omega_y + H_y\omega_x \end{bmatrix} \quad (3.11)$$

In this equation $\mathbf{H}$ is the angular momentum and is defined as

$$\mathbf{H} = \begin{bmatrix} I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ I_y\omega_y - I_{xy}\omega_x - I_{yz}\omega_z \\ I_z\omega_z - I_{xz}\omega_x - I_{yz}\omega_y \end{bmatrix} \quad (3.12)$$

Where I_{rx} , I_{ry} , I_{rz} and I_{gx} , I_{gy} , I_{gz} are the principle moments of inertia of the rotor and gimbal, respectively, and are defined, in general, by

$$I_x = \int (z^2 + y^2) dm \quad (3.13)$$

$$I_y = \int (z^2 + x^2) dm \quad (3.14)$$

$$I_z = \int (x^2 + y^2) dm \quad (3.15)$$

Also, I_{rxy} , I_{rxz} , I_{ryz} and I_{gxy} , I_{gxz} , I_{gyz} are the products of inertia of the rotor and gimbal, respectively, and are defined by

$$I_{xy} = \int xy dm \quad (3.16)$$

$$I_{xz} = \int xz dm \quad (3.17)$$

$$I_{yz} = \int yz dm \quad (3.18)$$

The derivative of Equation 3.12 can be found by simply noting that the ω terms are functions of time but that the moments and products of inertia can be considered as constants.

Using Equation 3.11, the general descriptions for the rotor and gimbal can be found. Therefore, the momentum equations for the rotor can be shown to be

$$\begin{aligned}
 M_{r_x} = I_{r_x} \dot{\omega}_{r_x} & - I_{r_{xy}} \dot{\omega}_{r_y} - I_{r_{xz}} \dot{\omega}_{r_z} \\
 & - \omega_{r_x} (I_{r_y} \omega_{r_y} - I_{r_{zy}} \omega_{r_z} - I_{r_{yz}} \omega_{r_z}) \\
 & + \omega_{r_y} (I_{r_x} \omega_{r_x} - I_{r_{zx}} \omega_{r_z} - I_{r_{yz}} \omega_{r_z})
 \end{aligned} \quad (3.19)$$

$$\begin{aligned}
 M_{r_y} = I_{r_y} \dot{\omega}_{r_y} & - I_{r_{xy}} \dot{\omega}_{r_x} - I_{r_{yz}} \dot{\omega}_{r_z} \\
 & - \omega_{r_x} (I_{r_x} \omega_{r_x} - I_{r_{zx}} \omega_{r_z} - I_{r_{yz}} \omega_{r_z}) \\
 & + \omega_{r_z} (I_{r_x} \omega_{r_x} - I_{r_{xy}} \omega_{r_y} - I_{r_{zx}} \omega_{r_z})
 \end{aligned} \quad (3.20)$$

$$\begin{aligned}
 M_{r_z} = I_{r_z} \dot{\omega}_{r_z} & - I_{r_{xz}} \dot{\omega}_{r_x} - I_{r_{yz}} \dot{\omega}_{r_y} \\
 & - \omega_{r_y} (I_{r_x} \omega_{r_x} - I_{r_{xy}} \omega_{r_y} - I_{r_{xz}} \omega_{r_z}) \\
 & + \omega_{r_x} (I_{r_y} \omega_{r_y} - I_{r_{xy}} \omega_{r_x} - I_{r_{yz}} \omega_{r_z})
 \end{aligned} \quad (3.21)$$

which can be simplified to give

$$\begin{aligned}
 M_{r_x} = I_{r_x} \dot{\omega}_{r_x} & + (I_{r_x} - I_{r_y}) \omega_{r_y} \omega_{r_z} - I_{r_{xy}} (\dot{\omega}_{r_y} - \omega_{r_x} \omega_{r_z}) \\
 & - I_{r_{xz}} (\dot{\omega}_{r_z} + \omega_{r_x} \omega_{r_y}) + I_{r_{yz}} (\omega_{r_x}^2 - \omega_{r_y}^2)
 \end{aligned} \quad (3.22)$$

$$\begin{aligned}
 M_{r_y} = I_{r_y} \dot{\omega}_{r_y} & + (I_{r_x} - I_{r_z}) \omega_{r_x} \omega_{r_z} - I_{r_{xy}} (\dot{\omega}_{r_x} + \omega_{r_y} \omega_{r_z}) \\
 & + I_{r_{xz}} (\omega_{r_x}^2 - \omega_{r_z}^2) - I_{r_{yz}} (\dot{\omega}_{r_z} - \omega_{r_x} \omega_{r_y})
 \end{aligned} \quad (3.23)$$

$$\begin{aligned}
 M_{r_z} = I_{r_z} \dot{\omega}_{r_z} & + (I_{r_y} - I_{r_x}) \omega_{r_x} \omega_{r_y} + I_{r_{xy}} (\omega_{r_y}^2 - \omega_{r_x}^2) \\
 & - I_{r_{xz}} (\dot{\omega}_{r_x} - \omega_{r_y} \omega_{r_z}) - I_{r_{yz}} (\dot{\omega}_{r_y} + \omega_{r_x} \omega_{r_z})
 \end{aligned} \quad (3.24)$$

Similar equations can be written that describe the gimbal.

$$\begin{aligned}
 M_{g_x} = I_{g_x} \dot{\omega}_{g_x} & + (I_{g_x} - I_{g_y}) \omega_{g_y} \omega_{g_z} - I_{g_{xy}} (\dot{\omega}_{g_y} - \omega_{g_x} \omega_{g_z}) \\
 & - I_{g_{xz}} (\dot{\omega}_{g_z} + \omega_{g_x} \omega_{g_y}) + I_{g_{yz}} (\omega_{g_x}^2 - \omega_{g_y}^2)
 \end{aligned} \quad (3.25)$$

$$\begin{aligned}
 M_{g_y} = I_{g_y} \dot{\omega}_{g_y} & + (I_{g_x} - I_{g_z}) \omega_{g_x} \omega_{g_z} - I_{g_{xy}} (\dot{\omega}_{g_x} + \omega_{g_y} \omega_{g_z}) \\
 & + I_{g_{xz}} (\omega_{g_x}^2 - \omega_{g_z}^2) - I_{g_{yz}} (\dot{\omega}_{g_z} - \omega_{g_x} \omega_{g_y})
 \end{aligned} \quad (3.26)$$

Figure 3.5: Freebody diagram of the DTG rotor.

$$\begin{aligned}
 M_z = I_{gz} \dot{\omega}_{gz} &+ (I_{gy} - I_{gz}) \omega_{gz} \omega_{gy} + I_{gz} (\omega_{gy}^2 - \omega_{gz}^2) \\
 &- I_{gxx} (\dot{\omega}_{gz} - \omega_{gy} \omega_{gz}) - I_{gyz} (\dot{\omega}_{gy} + \omega_{gz} \omega_{gy})
 \end{aligned} \quad (3.27)$$

From the freebody diagram of the rotor, Figure 3.5, it can be seen that the moments experienced by the rotor also equal

$$\mathbf{M}_r = \begin{bmatrix} 2F_z R \\ -2(D_y \dot{\theta} + K_y \theta) \\ 2F_x R \end{bmatrix} \quad (3.28)$$

where R is the radius of the gimbal, D_y is the viscous damping coefficient for the flexure and K_y represents the stiffness of the flexure.

Similarly, by examining a freebody diagram of the gimbal, Figure 3.6, the moments experienced by the gimbal can be obtained.

Figure 3.6: Freebody diagram of the DTG gimbal.

Figure 3.7: DTG rotor and gimbal reaction forces at the flexure.

$$\mathbf{M}_g = \begin{bmatrix} -2F'_z R - 2(K_x \phi + D_x \dot{\phi}) \\ 2G_z R + 2(K_y \theta + D_y \dot{\theta}) \\ 2G_y R - 2F'_x R \end{bmatrix} \quad (3.29)$$

From the setup of the problem, the forces F_x , F_z and F'_x , F'_z act at the same point but at different angles. This can be seen in Figure 3.7 and it follows that the relationship between the two sets of forces is

$$F'_x = F_x \cos \theta - F_z \sin \theta \quad (3.30)$$

$$F'_z = F_z \cos \theta + F_x \sin \theta \quad (3.31)$$

Substituting Equations 3.30 and 3.31 into Equation 3.29 gives

$$\mathbf{M}_g = \begin{bmatrix} -(2F_z R C_\theta + 2F_x R S_\theta) - 2(K_x \phi + D_x \dot{\phi}) \\ 2G_z R + 2(K_y \theta + D_y \dot{\theta}) \\ 2G_y R - (2F_x R C_\theta - 2F_z R S_\theta) \end{bmatrix} \quad (3.32)$$

Equations 3.28 and 3.22 to 3.24 can be combined to give

$$\begin{aligned} 2F_z R = I_{rz} \dot{\omega}_{rz} &+ (I_{rz} - I_{ry}) \omega_{ry} \omega_{rz} - I_{rzy} (\dot{\omega}_{ry} - \omega_{rz} \omega_{ry}) \\ &- I_{rzz} (\dot{\omega}_{rz} + \omega_{rz} \omega_{ry}) + I_{ryz} (\omega_{rz}^2 - \omega_{ry}^2) \end{aligned} \quad (3.33)$$

$$\begin{aligned} -2(D_y \dot{\theta} + K_y \theta) = I_{ry} \dot{\omega}_{ry} &+ (I_{rz} - I_{rz}) \omega_{rz} \omega_{ry} - I_{rzy} (\dot{\omega}_{rz} + \omega_{ry} \omega_{rz}) \\ &+ I_{rzz} (\omega_{rz}^2 - \omega_{ry}^2) - I_{ryz} (\dot{\omega}_{rz} - \omega_{rz} \omega_{ry}) \end{aligned} \quad (3.34)$$

$$\begin{aligned} 2F_z R = I_{rz} \dot{\omega}_{rz} &+ (I_{ry} - I_{rz}) \omega_{ry} \omega_{rz} + I_{rzy} (\omega_{ry}^2 - \omega_{rz}^2) \\ &- I_{rzz} (\dot{\omega}_{rz} - \omega_{ry} \omega_{rz}) - I_{ryz} (\dot{\omega}_{ry} + \omega_{rz} \omega_{ry}) \end{aligned} \quad (3.35)$$

Similarly, Equations 3.32 and 3.25 to 3.27 can be combined to give

$$\begin{aligned} -(2F_z RC_\theta + 2F_z RS_\theta) - 2(K_x \phi + D_x \dot{\phi}) = I_{gz} \dot{\omega}_{gz} &+ (I_{gz} - I_{gy}) \omega_{gy} \omega_{gz} \\ &- I_{gzy} (\dot{\omega}_{gy} - \omega_{gz} \omega_{gy}) - I_{gzz} (\dot{\omega}_{gz} + \omega_{gz} \omega_{gy}) \\ &+ I_{gyz} (\omega_{gz}^2 - \omega_{gy}^2) \end{aligned} \quad (3.36)$$

$$\begin{aligned} 2G_z R + 2(K_y \theta + D_y \dot{\theta}) = I_{gy} \dot{\omega}_{gy} &+ (I_{gz} - I_{gy}) \omega_{gz} \omega_{gy} \\ &- I_{gzy} (\dot{\omega}_{gz} + \omega_{gy} \omega_{gz}) + I_{gzz} (\omega_{gz}^2 - \omega_{gy}^2) \\ &- I_{gyz} (\dot{\omega}_{gz} - \omega_{gz} \omega_{gy}) \end{aligned} \quad (3.37)$$

$$\begin{aligned} 2G_y R - (2F_z RC_\theta - 2F_z RS_\theta) = I_{gz} \dot{\omega}_{gz} &+ (I_{gy} - I_{gz}) \omega_{gz} \omega_{gy} \\ &+ I_{gzy} (\omega_{gy}^2 - \omega_{gz}^2) - I_{gzz} (\dot{\omega}_{gz} - \omega_{gy} \omega_{gz}) \\ &- I_{gyz} (\dot{\omega}_{gy} + \omega_{gz} \omega_{gy}) \end{aligned} \quad (3.38)$$

Now substitute Equations 3.33 and 3.35 into Equation 3.36. This gives

$$\begin{aligned} -2(K_x \phi + D_x \dot{\phi}) - [I_{rz} \dot{\omega}_{rz} C_\theta &+ (I_{rz} - I_{ry}) \omega_{ry} \omega_{rz} C_\theta - I_{rzy} (\dot{\omega}_{ry} - \omega_{rz} \omega_{ry}) C_\theta \\ &- I_{rzz} (\dot{\omega}_{rz} + \omega_{rz} \omega_{ry}) C_\theta + I_{ryz} (\omega_{rz}^2 - \omega_{ry}^2) C_\theta \\ &+ (I_{ry} - I_{rz}) \omega_{rz} \omega_{ry} S_\theta + I_{rz} \dot{\omega}_{rz} S_\theta + I_{rzy} (\omega_{ry}^2 - \omega_{rz}^2) S_\theta \\ &- I_{rzz} (\dot{\omega}_{rz} - \omega_{ry} \omega_{rz}) S_\theta - I_{ryz} (\dot{\omega}_{ry} + \omega_{rz} \omega_{ry}) S_\theta] \\ = I_{gz} \dot{\omega}_{gz} &+ (I_{gz} - I_{gy}) \omega_{gy} \omega_{gz} - I_{gzy} (\dot{\omega}_{gy} - \omega_{gz} \omega_{gy}) \\ &- I_{gzz} (\dot{\omega}_{gz} + \omega_{gz} \omega_{gy}) + I_{gyz} (\omega_{gz}^2 - \omega_{gy}^2) \end{aligned} \quad (3.39)$$

This can be rearranged to give

$$\begin{aligned}
& -2(K_x\phi + D_x\dot{\phi}) - [I_{rx}(\dot{\omega}_{rx}C_\theta - \omega_{rx}\omega_{ry}S_\theta) - I_{ry}(\omega_{ry}\omega_{rx}C_\theta - \omega_{rx}\omega_{ry}S_\theta) \\
& \quad + I_{rx}(\omega_{ry}\omega_{rx}C_\theta + \dot{\omega}_{rx}S_\theta) - I_{ry}(\dot{\omega}_{ry}C_\theta - \omega_{rx}\omega_{rx}C_\theta - \omega_{ry}^2S_\theta + \omega_{rx}^2S_\theta) \\
& \quad - I_{rx}(\dot{\omega}_{rx}C_\theta + \omega_{rx}\omega_{ry}C_\theta + \dot{\omega}_{rx}S_\theta - \omega_{ry}\omega_{rx}S_\theta) \\
& \quad + I_{ry}(\omega_{rx}^2C_\theta - \omega_{ry}^2C_\theta - \dot{\omega}_{ry}S_\theta - \omega_{rx}\omega_{rx}S_\theta)] \\
& = I_{gx}\dot{\omega}_{gx} + (I_{gz} - I_{gy})\omega_{gy}\omega_{gz} - I_{gzy}(\dot{\omega}_{gy} - \omega_{gz}\omega_{gy}) \\
& \quad - I_{gzz}(\dot{\omega}_{gz} + \omega_{gz}\omega_{gy}) + I_{gyz}(\omega_{gz}^2 - \omega_{gy}^2)
\end{aligned} \tag{3.40}$$

It is now necessary to replace ω_r and ω_g with their functions of θ and ϕ . Equation 3.5 describes the relationship for ω_g as shown below

$$\omega_g = \begin{pmatrix} \dot{\phi} \\ S_\phi\omega_s \\ C_\phi\omega_s \end{pmatrix}$$

Equation 2.26 gives the relationship for ω_r

$$\omega_r = \begin{pmatrix} C_\theta\dot{\phi} - S_\theta C_\phi\omega_s \\ \dot{\theta} + S_\phi\omega_s \\ S_\theta\dot{\phi} + C_\theta C_\phi\omega_s \end{pmatrix}$$

The derivatives of ω_r and ω_g are required for a full description of the system and can be found by finding the derivatives of Equations 3.5 and 2.26. These give

$$\dot{\omega}_g = \begin{pmatrix} \ddot{\phi} \\ \dot{\phi}C_\phi\omega_s \\ -\dot{\phi}S_\phi\omega_s \end{pmatrix} \tag{3.41}$$

$$\dot{\omega}_r = \begin{pmatrix} -S_\theta\dot{\theta}\dot{\phi} + C_\theta\ddot{\phi} - C_\theta C_\phi\dot{\theta}\omega_s + S_\theta S_\phi\dot{\theta}\omega_s \\ \ddot{\theta} + C_\phi\dot{\phi}\omega_s \\ C_\theta\dot{\theta}\dot{\phi} + S_\theta\ddot{\phi} - S_\theta C_\phi\dot{\theta}\omega_s - C_\theta S_\phi\dot{\phi}\omega_s \end{pmatrix} \tag{3.42}$$

Substituting Equations 3.5, 2.26, 3.41 and 3.42 into Equation 3.40 and then simplifying gives

$$\begin{aligned}
& I_{rx} \left[-2S_\theta C_\theta \dot{\theta} \dot{\phi} + C_\theta^2 \ddot{\phi} + (S_\theta^2 - C_\theta^2) C_\phi \dot{\theta} \omega_s + S_\theta^2 S_\phi C_\phi \omega_s^2 \right] \\
& - I_{ry} \left[S_\phi C_\phi \omega_s^2 + C_\phi \dot{\theta} \omega_s \right] \\
& + I_{rx} \left[2S_\theta C_\theta \dot{\theta} \dot{\phi} + S_\theta^2 \ddot{\phi} + (C_\theta^2 - S_\theta^2) \dot{\theta} \omega_s C_\phi + C_\theta^2 S_\phi C_\phi \omega_s^2 \right] \\
& + I_{gz} \ddot{\phi} - (I_{gy} - I_{gz}) S_\phi C_\phi \omega_s^2 \\
& - I_{rxy} \left[C_\theta \ddot{\theta} - S_\theta \dot{\theta}^2 - 2S_\theta S_\phi \dot{\theta} \omega_s + (C_\phi^2 - S_\phi^2) S_\theta \omega_s^2 \right] \\
& - I_{rxz} \left[2(C_\theta^2 - S_\theta^2) \dot{\theta} \dot{\phi} + 2C_\theta S_\theta \ddot{\phi} - 4S_\theta C_\theta C_\phi \dot{\theta} \omega_s - 2S_\theta C_\theta S_\phi C_\phi \omega_s^2 \right] \\
& - I_{ryz} \left[S_\theta \ddot{\theta} + C_\theta \dot{\theta}^2 + 2C_\theta S_\phi \omega_s \dot{\theta} + (S_\phi^2 - C_\phi^2) C_\theta \omega_s^2 \right] \\
& + I_{gyz} (C_\phi^2 - S_\phi^2) \omega_s^2 + 2(K_x \phi + D_x \dot{\phi}) = 0
\end{aligned} \tag{3.43}$$

Solving for $\ddot{\phi}$ and rearranging, Equation 3.43 can be rewritten as

$$\begin{aligned}
(I_{rx} C_\theta^2 + I_{ry} S_\theta^2 - 2I_{rxz} S_\theta C_\theta + I_{gz}) \ddot{\phi} = & \\
& (I_{rxy} C_\theta + I_{ryz} S_\theta) \ddot{\theta} \\
& - I_{rx} \left[-2S_\theta C_\theta \dot{\theta} \dot{\phi} + (S_\theta^2 - C_\theta^2) C_\phi \dot{\theta} \omega_s + S_\theta^2 S_\phi C_\phi \omega_s^2 \right] \\
& + I_{ry} \left[S_\phi C_\phi \omega_s^2 + C_\phi \dot{\theta} \omega_s \right] \\
& - I_{rx} \left[2S_\theta C_\theta \dot{\theta} \dot{\phi} + (C_\theta^2 - S_\theta^2) \dot{\theta} \omega_s C_\phi + C_\theta^2 S_\phi C_\phi \omega_s^2 \right] \\
& + (I_{gy} - I_{gz}) S_\phi C_\phi \omega_s^2 \\
& + I_{rxy} \left[-S_\theta \dot{\theta}^2 - 2S_\theta S_\phi \dot{\theta} \omega_s + (C_\phi^2 - S_\phi^2) S_\theta \omega_s^2 \right] \\
& + I_{rxz} \left[2(C_\theta^2 - S_\theta^2) \dot{\theta} \dot{\phi} - 4S_\theta C_\theta C_\phi \dot{\theta} \omega_s - 2S_\theta C_\theta S_\phi C_\phi \omega_s^2 \right] \\
& + I_{ryz} \left[C_\theta \dot{\theta}^2 + 2C_\theta S_\phi \omega_s \dot{\theta} + (S_\phi^2 - C_\phi^2) C_\theta \omega_s^2 \right] \\
& - I_{gyz} (C_\phi^2 - S_\phi^2) \omega_s^2 - 2(K_x \phi + D_x \dot{\phi})
\end{aligned} \tag{3.44}$$

Now let

$$A = I_{rx} C_\theta^2 + I_{ry} S_\theta^2 - 2I_{rxz} S_\theta C_\theta + I_{gz} \tag{3.45}$$

$$B = I_{rxy} C_\theta + I_{ryz} S_\theta \tag{3.46}$$

$$\begin{aligned}
C = & -I_{rx} \left[-2S_\theta C_\theta \dot{\theta} \dot{\phi} + (S_\theta^2 - C_\theta^2) C_\phi \dot{\theta} \omega_s + S_\theta^2 S_\phi C_\phi \omega_s^2 \right] \\
& + I_{ry} \left[S_\phi C_\phi \omega_s^2 + C_\phi \dot{\theta} \omega_s \right] \\
& - I_{rx} \left[2S_\theta C_\theta \dot{\theta} \dot{\phi} + (C_\theta^2 - S_\theta^2) \dot{\theta} \omega_s C_\phi + C_\theta^2 S_\phi C_\phi \omega_s^2 \right] \\
& + (I_{gy} - I_{gz}) S_\phi C_\phi \omega_s^2 \\
& + I_{rx} \left[-S_\theta \dot{\theta}^2 - 2S_\theta S_\phi \dot{\theta} \omega_s + (C_\phi^2 - S_\phi^2) S_\theta \omega_s^2 \right] \\
& + I_{rx} \left[2(C_\theta^2 - S_\theta^2) \dot{\theta} \dot{\phi} - 4S_\theta C_\theta C_\phi \dot{\theta} \omega_s - 2S_\theta C_\theta S_\phi C_\phi \omega_s^2 \right] \\
& + I_{ry} \left[C_\theta \dot{\theta}^2 + 2C_\theta S_\phi \omega_s \dot{\theta} + (S_\phi^2 - C_\phi^2) C_\theta \omega_s^2 \right] \\
& - I_{gy} (C_\phi^2 - S_\phi^2) \omega_s^2 - 2(K_x \phi + D_x \dot{\phi})
\end{aligned} \tag{3.47}$$

Equation 3.44 can now be rewritten as²

$$A\ddot{\phi} = B\ddot{\theta} + C \tag{3.48}$$

It is now necessary to find a relationship for $\ddot{\theta}$. This can be found from Equation 3.23. Substituting Equations 3.5, 2.26, 3.41 and 3.42 into Equation 3.34 and solving for $\ddot{\theta}$ gives

$$\begin{aligned}
I_{ry} \ddot{\theta} = & (C_\theta I_{rx} + S_\theta I_{ry}) \ddot{\phi} \\
& - I_{rx} \left[S_\theta C_\theta \dot{\phi}^2 + (C_\theta^2 - S_\theta^2) C_\phi \omega_s \dot{\phi} - S_\theta C_\theta C_\phi^2 \omega_s^2 \right] \\
& - I_{ry} \left[C_\phi \dot{\phi} \omega_s \right] \\
& + I_{rx} \left[S_\theta C_\theta \dot{\phi}^2 + (C_\theta^2 - S_\theta^2) C_\phi \omega_s \dot{\phi} - S_\theta C_\theta C_\phi^2 \omega_s^2 \right] \\
& + I_{rx} \left[2S_\theta S_\phi \dot{\phi} \omega_s + C_\theta S_\phi C_\phi \omega_s^2 \right] \\
& + I_{rx} \left[(S_\theta^2 - C_\theta^2) (\dot{\phi}^2 - C_\phi^2 \omega_s^2) + 4S_\theta C_\theta C_\phi \dot{\phi} \omega_s \right] \\
& + I_{ry} \left[-2C_\theta S_\phi \dot{\phi} \omega_s + S_\theta S_\phi C_\phi \omega_s^2 \right] \\
& - 2(K_y \theta + D_y \dot{\theta})
\end{aligned} \tag{3.49}$$

By setting

$$D = I_{ry} \tag{3.50}$$

$$E = C_\theta I_{rx} + S_\theta I_{ry} \tag{3.51}$$

²Note that C and C_θ are not the same quantity. C is a constant, C_θ is the short form for $\cos \theta$

$$\begin{aligned}
F = & -I_{r_z} [S_\theta C_\theta \dot{\phi}^2 + (C_\theta^2 - S_\theta^2) C_\phi \omega_s \dot{\phi} - S_\theta C_\theta C_\phi^2 \omega_s^2] \\
& -I_{r_y} [C_\phi \dot{\phi} \omega_s] \\
& +I_{r_x} [S_\theta C_\theta \dot{\phi}^2 + (C_\theta^2 - S_\theta^2) C_\phi \omega_s \dot{\phi} - S_\theta C_\theta C_\phi^2 \omega_s^2] \\
& +I_{r_{zy}} [2S_\theta S_\phi \dot{\phi} \omega_s + C_\theta S_\phi C_\phi \omega_s^2] \\
& +I_{r_{zx}} [(S_\theta^2 - C_\theta^2) (\dot{\phi}^2 - C_\phi^2 \omega_s^2) + 4S_\theta C_\theta C_\phi \dot{\phi} \omega_s] \\
& +I_{r_{yz}} [-2C_\theta S_\phi \dot{\phi} \omega_s + S_\theta S_\phi C_\phi \omega_s^2] \\
& -2(K_y \theta + D_y \dot{\theta})
\end{aligned} \tag{3.52}$$

Therefore, Equation 3.49 can be rewritten as

$$D\ddot{\theta} = E\ddot{\phi} + F \tag{3.53}$$

Using Equations 3.48 and 3.53, $\ddot{\theta}$ and $\ddot{\phi}$ can be isolated. Therefore

$$\ddot{\phi} = \frac{BF + CD}{AD - BE} \tag{3.54}$$

and

$$\ddot{\theta} = \frac{EC + AF}{AD - BE} \tag{3.55}$$

These results hold as long as $AD - BE \neq 0$, which, for a practical gyroscopic rotor and gimbal set, will always be true.

3.2.2 Derivation of the Equations of Motion using the Lagrangian Method

It is possible to find the equations of motion through use of Lagrange's equation. In general Lagrange's equation takes the form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \tag{3.56}$$

Where T is the kinetic energy of the system, $\dot{q}_j$ are the derivatives of a set of coordinates and Q_j are the forces acting on the system[9]. If the forces are derivable from a potential and if the forces are conservative, then Lagrange's equation becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \tag{3.57}$$

Where L is known as the Lagrangian and is defined as

$$L = T - V \quad (3.58)$$

and V is the potential energy in the system.

In general, the kinetic energy of a system is

$$T = \frac{1}{2} \mathbf{v} \cdot \mathbf{G} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{H} \quad (3.59)$$

however, if there is no translation, then

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{H} \quad (3.60)$$

where $\mathbf{H}$ is the angular momentum of an object about a point (see Equation 3.12).

Ignoring damping, the kinetic energy for the rotor-gimbal system is

$$T = \frac{1}{2} \boldsymbol{\omega}_g \cdot \mathbf{H}_g + \frac{1}{2} \boldsymbol{\omega}_r \cdot \mathbf{H}_r \quad (3.61)$$

which can be expanded to give

$$\begin{aligned} T = & \frac{1}{2} \left(I_{rx} \omega_{rx}^2 + I_{ry} \omega_{ry}^2 + I_{rz} \omega_{rz}^2 + I_{gx} \omega_{gx}^2 + I_{gy} \omega_{gy}^2 + I_{gz} \omega_{gz}^2 \right) \\ & - (I_{rxy} \omega_{rx} \omega_{ry} + I_{rxz} \omega_{rx} \omega_{rz} + I_{ryz} \omega_{ry} \omega_{rz} \\ & + I_{gxy} \omega_{gx} \omega_{gy} + I_{gxz} \omega_{gx} \omega_{gz} + I_{gyz} \omega_{gy} \omega_{gz}) \end{aligned} \quad (3.62)$$

The potential energy for the system can be shown to be

$$V = K_x \phi^2 + K_y \theta^2 \quad (3.63)$$

Substitute Equations 3.62 and 3.63 into the Lagrangian, Equation 3.58, and this gives

$$\begin{aligned} L = & \frac{1}{2} \left(I_{rx} \omega_{rx}^2 + I_{ry} \omega_{ry}^2 + I_{rz} \omega_{rz}^2 + I_{gx} \omega_{gx}^2 + I_{gy} \omega_{gy}^2 + I_{gz} \omega_{gz}^2 \right) \\ & - (I_{rxy} \omega_{rx} \omega_{ry} + I_{rxz} \omega_{rx} \omega_{rz} + I_{ryz} \omega_{ry} \omega_{rz} \\ & + I_{gxy} \omega_{gx} \omega_{gy} + I_{gxz} \omega_{gx} \omega_{gz} + I_{gyz} \omega_{gy} \omega_{gz}) \\ & - (K_x \phi^2 + K_y \theta^2) \end{aligned} \quad (3.64)$$

The equations of motion for this system can be found from Lagrange's equations, specifically

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0 \quad (3.65)$$

and

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (3.66)$$

In Section 3.2.1, it was found that

$$\omega_g = \begin{pmatrix} \dot{\phi} \\ S_\phi \omega_s \\ C_\phi \omega_s \end{pmatrix}$$

and that

$$\omega_r = \begin{pmatrix} C_\theta \dot{\phi} - S_\theta C_\phi \omega_s \\ \dot{\theta} + S_\phi \omega_s \\ S_\theta \dot{\theta} + C_\theta C_\phi \omega_s \end{pmatrix}$$

From these two definitions the required partial derivatives can be found. For the first Lagrange equation, where $q_j = \phi$

$$\begin{aligned} \frac{\partial \omega_{r_z}}{\partial \phi} &= S_\theta S_\phi \omega_s \\ \frac{\partial \omega_{r_y}}{\partial \phi} &= C_\phi \omega_s \\ \frac{\partial \omega_{r_x}}{\partial \phi} &= -C_\theta S_\phi \omega_s \\ \frac{\partial \omega_{g_z}}{\partial \phi} &= 0 \\ \frac{\partial \omega_{g_y}}{\partial \phi} &= C_\phi \omega_s \\ \frac{\partial \omega_{g_x}}{\partial \phi} &= -S_\phi \omega_s \end{aligned} \quad (3.67)$$

$$\begin{aligned} \frac{\partial \omega_{r_z}}{\partial \phi} &= C_\theta \\ \frac{\partial \omega_{r_y}}{\partial \phi} &= 0 \\ \frac{\partial \omega_{r_x}}{\partial \phi} &= S_\theta \end{aligned}$$

$$\begin{aligned}
\frac{\partial \omega_{gz}}{\partial \phi} &= 1 \\
\frac{\partial \omega_{gy}}{\partial \phi} &= 0 \\
\frac{\partial \omega_{gx}}{\partial \phi} &= 0
\end{aligned} \tag{3.68}$$

Now substitute the Lagrangian into the first Lagrange's equation, Equation 3.65. This gives

$$\begin{aligned}
& \frac{d}{dt} \left[I_{rz} \omega_{rz} \frac{\partial \omega_{rz}}{\partial \dot{\phi}} + I_{ry} \omega_{ry} \frac{\partial \omega_{ry}}{\partial \dot{\phi}} + I_{rx} \omega_{rx} \frac{\partial \omega_{rx}}{\partial \dot{\phi}} + \right. \\
& I_{gz} \omega_{gz} \frac{\partial \omega_{gz}}{\partial \dot{\phi}} + I_{gy} \omega_{gy} \frac{\partial \omega_{gy}}{\partial \dot{\phi}} + I_{gx} \omega_{gx} \frac{\partial \omega_{gx}}{\partial \dot{\phi}} + \\
& - \left(I_{ry} \left(\frac{\partial \omega_{rz}}{\partial \dot{\phi}} \omega_{ry} + \omega_{ry} \frac{\partial \omega_{ry}}{\partial \dot{\phi}} \right) + I_{rz} \left(\frac{\partial \omega_{ry}}{\partial \dot{\phi}} \omega_{rz} + \omega_{rz} \frac{\partial \omega_{rz}}{\partial \dot{\phi}} \right) \right. \\
& + I_{ry} \left(\frac{\partial \omega_{gz}}{\partial \dot{\phi}} \omega_{ry} + \omega_{ry} \frac{\partial \omega_{gz}}{\partial \dot{\phi}} \right) + I_{gz} \left(\frac{\partial \omega_{ry}}{\partial \dot{\phi}} \omega_{gz} + \omega_{gz} \frac{\partial \omega_{ry}}{\partial \dot{\phi}} \right) \\
& + I_{gz} \left(\frac{\partial \omega_{gz}}{\partial \dot{\phi}} \omega_{gz} + \omega_{gz} \frac{\partial \omega_{gz}}{\partial \dot{\phi}} \right) + I_{gy} \left(\frac{\partial \omega_{gz}}{\partial \dot{\phi}} \omega_{gy} + \omega_{gy} \frac{\partial \omega_{gz}}{\partial \dot{\phi}} \right) \left. \right) \Bigg] \\
& - \left[I_{rz} \omega_{rz} \frac{\partial \omega_{rz}}{\partial \dot{\phi}} + I_{ry} \omega_{ry} \frac{\partial \omega_{ry}}{\partial \dot{\phi}} + I_{rx} \omega_{rx} \frac{\partial \omega_{rx}}{\partial \dot{\phi}} + \right. \\
& I_{gz} \omega_{gz} \frac{\partial \omega_{gz}}{\partial \dot{\phi}} + I_{gy} \omega_{gy} \frac{\partial \omega_{gy}}{\partial \dot{\phi}} + I_{gx} \omega_{gx} \frac{\partial \omega_{gx}}{\partial \dot{\phi}} + \\
& - \left(I_{ry} \left(\frac{\partial \omega_{rz}}{\partial \dot{\phi}} \omega_{ry} + \omega_{ry} \frac{\partial \omega_{ry}}{\partial \dot{\phi}} \right) + I_{rz} \left(\frac{\partial \omega_{ry}}{\partial \dot{\phi}} \omega_{rz} + \omega_{rz} \frac{\partial \omega_{rz}}{\partial \dot{\phi}} \right) \right. \\
& + I_{ry} \left(\frac{\partial \omega_{gz}}{\partial \dot{\phi}} \omega_{ry} + \omega_{ry} \frac{\partial \omega_{gz}}{\partial \dot{\phi}} \right) + I_{gz} \left(\frac{\partial \omega_{ry}}{\partial \dot{\phi}} \omega_{gz} + \omega_{gz} \frac{\partial \omega_{ry}}{\partial \dot{\phi}} \right) \\
& + I_{gz} \left(\frac{\partial \omega_{gz}}{\partial \dot{\phi}} \omega_{gz} + \omega_{gz} \frac{\partial \omega_{gz}}{\partial \dot{\phi}} \right) + I_{gy} \left(\frac{\partial \omega_{gz}}{\partial \dot{\phi}} \omega_{gy} + \omega_{gy} \frac{\partial \omega_{gz}}{\partial \dot{\phi}} \right) \left. \right) \Bigg] + 2K_x \phi = 0
\end{aligned} \tag{3.69}$$

Substitute Equations 3.67 and 3.68 into Equation 3.69 and simplify

$$\begin{aligned}
& I_{rz} [\dot{\omega}_{rz} C_\theta - \omega_{rz} \dot{\theta} S_\theta - \omega_{rz} S_\theta S_\phi \omega_s] \\
& - I_{ry} \omega_{ry} C_\phi \omega_s \\
& + I_{rz} [\dot{\omega}_{rz} S_\theta + \omega_{rz} \dot{\theta} C_\theta + \omega_{rz} C_\theta S_\phi \omega_s] \\
& + I_{gz} \dot{\omega}_{gz} - I_{gy} \omega_{gy} C_\phi \omega_s + I_{gz} \omega_{gz} S_\phi \omega_s
\end{aligned}$$

$$\begin{aligned}
& - I_{r_{yy}} [\dot{\omega}_{r_y} C_\theta - \omega_{r_y} \dot{\theta} S_\theta - \omega_{r_y} S_\theta S_\phi \omega_s - \omega_{r_z} C_\phi \omega_s] \\
& - I_{r_{zz}} [\dot{\omega}_{r_z} C_\theta - \omega_{r_z} \dot{\theta} S_\theta + \dot{\omega}_{r_z} S_\theta + \omega_{r_z} \dot{\theta} C_\theta \\
& \quad - \omega_{r_z} S_\theta S_\phi \omega_s + \omega_{r_z} C_\theta S_\phi \omega_s] \\
& - I_{r_{yz}} [\dot{\omega}_{r_y} S_\theta + \omega_{r_y} \dot{\theta} C_\theta - \omega_{r_z} C_\phi \omega_s + \omega_{r_y} C_\theta S_\phi \omega_s] \\
& - I_{g_{yy}} [\dot{\omega}_{g_y} - \omega_{g_z} C_\phi \omega_s] - I_{g_{zz}} [\dot{\omega}_{g_z} + \omega_{g_z} S_\phi \omega_s] \\
& + I_{g_{yz}} [\omega_{g_z} C_\phi \omega_s - \omega_{g_y} S_\phi \omega_s] + 2K_x \phi = 0
\end{aligned} \tag{3.70}$$

Equation 2.26 and 3.5 can now be substituted into Equation 3.70. This can then be simplified to get

$$\begin{aligned}
& I_{r_z} [-2S_\theta C_\theta \dot{\theta} \dot{\phi} + C_\theta^2 \ddot{\phi} + (S_\theta^2 - C_\theta^2) C_\phi \dot{\theta} \omega_s + S_\theta^2 S_\phi C_\phi \omega_s^2] \\
& - I_{r_y} [S_\phi C_\phi \omega_s^2 + C_\phi \dot{\theta} \omega_s] \\
& + I_{r_z} [2S_\theta C_\theta \dot{\theta} \dot{\phi} + S_\theta^2 \ddot{\phi} + (C_\theta^2 - S_\theta^2) \dot{\theta} \omega_s C_\phi + C_\theta^2 S_\phi C_\phi \omega_s^2] \\
& + I_{g_z} \ddot{\phi} - (I_{g_y} - I_{g_z}) S_\phi C_\phi \omega_s^2 \\
& - I_{r_{yy}} [C_\theta \ddot{\theta} - S_\theta \dot{\theta}^2 - 2S_\theta S_\phi \dot{\theta} \omega_s + (C_\phi^2 - S_\phi^2) S_\theta \omega_s^2] \\
& - I_{r_{zz}} [2(C_\theta^2 - S_\theta^2) \dot{\theta} \dot{\phi} + 2C_\theta S_\theta \ddot{\phi} - 4S_\theta C_\theta C_\phi \dot{\theta} \omega_s - 2S_\theta C_\theta S_\phi C_\phi \omega_s^2] \\
& - I_{r_{yz}} [S_\theta \ddot{\theta} + C_\theta \dot{\theta}^2 + 2C_\theta S_\phi \omega_s \dot{\theta} + (S_\phi^2 - C_\phi^2) C_\theta \omega_s^2] \\
& + I_{g_{yz}} (C_\phi^2 - S_\phi^2) \omega_s^2 + 2K_x \phi = 0
\end{aligned} \tag{3.71}$$

Equation 3.71 above is equivalent to Equation 3.43, found using the vectorial approach. The second Lagrange Equation, where $q_j = \theta$, must now be found.

Again, from the definitions of ω_g and ω_r , the partial derivatives can be found.

$$\begin{aligned}
\frac{\partial \omega_{r_z}}{\partial \theta} &= -S_\theta \dot{\phi} - C_\theta C_\phi \omega_s \\
\frac{\partial \omega_{r_y}}{\partial \theta} &= 0 \\
\frac{\partial \omega_{r_z}}{\partial \theta} &= C_\theta \dot{\phi} - S_\theta C_\phi \omega_s \\
\frac{\partial \omega_{g_z}}{\partial \theta} &= 0 \\
\frac{\partial \omega_{g_y}}{\partial \theta} &= 0
\end{aligned}$$

$$\frac{\partial \omega_{\theta z}}{\partial \theta} = 0 \quad (3.72)$$

$$\begin{aligned} \frac{\partial \omega_{r z}}{\partial \theta} &= 0 \\ \frac{\partial \omega_{r y}}{\partial \theta} &= 1 \\ \frac{\partial \omega_{r x}}{\partial \theta} &= 0 \\ \frac{\partial \omega_{\theta z}}{\partial \theta} &= 0 \\ \frac{\partial \omega_{\theta y}}{\partial \theta} &= 0 \\ \frac{\partial \omega_{\theta x}}{\partial \theta} &= 0 \end{aligned} \quad (3.73)$$

The Langrangian and Equation 3.66 can be combined and then simplified with the use of Equations 3.72 and 3.73 to get

$$\begin{aligned} & I_{r z} [\omega_{r z} (S_{\theta} \dot{\phi} + C_{\theta} C_{\phi} \omega_s)] + I_{r y} \dot{\omega}_{r y} \\ & - I_{r z} [\omega_{r z} (C_{\theta} \dot{\phi} - S_{\theta} C_{\phi} \omega_s)] \\ & - I_{r z y} [\dot{\omega}_{r z} + \omega_{r y} (S_{\theta} \dot{\phi} + C_{\theta} C_{\phi} \omega_s)] \\ & - I_{r z z} [\omega_{r z} (S_{\theta} \dot{\phi} + C_{\theta} C_{\phi} \omega_s) - \omega_{r z} (C_{\theta} \dot{\phi} + S_{\theta} C_{\phi} \omega_s)] \\ & - I_{r y z} [\dot{\omega}_{r z} - \omega_{r y} (C_{\theta} \dot{\phi} + S_{\theta} C_{\phi} \omega_s)] \\ & + 2K_y \theta = 0 \end{aligned} \quad (3.74)$$

Now Equations 2.26 and 3.5 and their derivatives can be substituted into Equation 3.75, which can then be simplified to give

$$\begin{aligned} & I_{r z} [S_{\theta} C_{\theta} \dot{\phi}^2 + (C_{\theta}^2 - S_{\theta}^2) C_{\phi} \omega_s \dot{\phi} - S_{\theta} C_{\theta} C_{\phi}^2 \omega_s^2] \\ & + I_{r y} [\ddot{\theta} + C_{\phi} \dot{\phi} \omega_s] \\ & - I_{r z} [S_{\theta} C_{\theta} \dot{\phi}^2 + (C_{\theta}^2 - S_{\theta}^2) C_{\phi} \omega_s \dot{\phi} - S_{\theta} C_{\theta} C_{\phi}^2 \omega_s^2] \\ & - I_{r z y} [C_{\theta} \ddot{\phi} + 2S_{\theta} S_{\phi} \dot{\phi} \omega_s + C_{\theta} S_{\phi} C_{\phi} \omega_s^2] \\ & - I_{r z z} [(S_{\theta}^2 - C_{\theta}^2) (\dot{\phi}^2 - C_{\phi}^2 \omega_s^2) + 4S_{\theta} C_{\theta} C_{\phi} \dot{\phi} \omega_s] \\ & - I_{r y z} [S_{\theta} \ddot{\phi} - 2C_{\theta} S_{\phi} \dot{\phi} \omega_s + S_{\theta} S_{\phi} C_{\phi} \omega_s^2] \\ & + 2K_y \theta = 0 \end{aligned} \quad (3.75)$$

With some rewriting it is possible to show that Equation 3.75 is equivalent to Equation 3.49. Thus, the Lagrangian method has verified the solution arrived at by the vectorial solution. Note that damping has been ignored in the Lagrangian solution. It is possible to include it into the Lagrange equations but to do so would add to the complexity of the problem, and it was not necessary for this calculation.

Now that Equation 3.71 and 3.75 have been obtained, it is possible for $\ddot{\theta}$ and $\ddot{\phi}$ to be isolated and solved, as was done in the vectorial solution.

3.2.3 Comparison of Vectorial Approach and Lagrangian Method

The vectorial and Lagrangian approaches were used to provide a cross-check and allow convergence to identical results. Without a cross-check such as this, algebraic errors can easily surface simply due to the tedious nature of the manipulation. An additional benefit in using the two methods has been the opportunity for comparison on a non-trivial problem.

The vectorial approach requires a good understanding of the problem and the forces involved. Without a proper understanding, the initial equations would not be set up correctly. The system had to be broken into two separate freebody diagrams and then analyzed. The vectorial method required more algebraic manipulation than the Lagrangian method³. One advantage of the vectorial approach is that the system is fully described (ie. the shaft forces can be easily found.)

The Lagrangian method does not require as complete an understanding of the problem. Nor does it require that the analyst deal with internal forces. However, the algebra, while simpler than in the vectorial approach, is still complex and the possibility for error remains a problem. The Lagrangian method also requires the proper use of partial derivatives.

Overall, the Lagrangian method required less time and had less chance of an algebraic error. This method arrived at the dynamic equations quickly without the

³It should be noted that many of the intermediate steps for the vectorial approach were not shown in Section 3.2.1

Figure 3.8: Freebody diagram of the DTG shaft.

precise understanding of the problem as required by the vectorial approach.

3.3 Calculation of the Forces on the Input Shaft

Now that the dynamics of the system have been found it is a straight forward procedure to find the forces on the input shaft. It is necessary to know the forces because they must be reacted to by the rotor case and eventually the 'bus' which contains the instrument. If the rotor is offset by an angle, any vibrational effects caused by the shaft forces must be known and countered. This is especially important for use on spacecraft.

From Figure 3.8 it can be seen that all forces have been balanced (ie. no force reactions are required by the shaft), but the act of balancing the forces has created couples. These couples act on the shaft and, hence, the gyro case. First, resolve the

forces G_y and G_z into the rotating shaft coordinates. This gives

$$\mathbf{G} = (-\phi)_x \begin{pmatrix} 0 \\ G_y \\ G_z \end{pmatrix} \quad (3.76)$$

$$\mathbf{G} = \begin{pmatrix} 0 \\ C_\phi G_y - S_\phi G_z \\ S_\phi G_y + C_\phi G_z \end{pmatrix} \quad (3.77)$$

Now it should be noted that the forces in the y-direction cause a moment about the z-axis and the forces in the z-direction cause a moment about the y-axis. Thus, including the torsional forces caused by the flexure, the moments experienced by the shaft are

$$\mathbf{M}_{\text{shaft}} = \begin{pmatrix} 2(K_x \phi + D_x \dot{\phi}) \\ 2R(S_\phi G_y + C_\phi G_z) \\ 2R(C_\phi G_y - S_\phi G_z) \end{pmatrix} \quad (3.78)$$

Forces G_y and G_z can be found from Equations 3.38 and 3.37. Once calculated they can be substituted into the above equation to find the moments acting on the shaft.

Chapter 4

Simulation of a Gyro

In Chapter 3, a set of differential equations were found that describe the motion of the gyro rotor, see Equations 3.54 and 3.55. These equations can be solved to find the response of the rotor to a set of conditions which would define the initial angular position and rate of the rotor.

It may be possible to solve the set of differential equations analytically; however, this set of equations is very complex and would be extraordinarily difficult to solve. For all intents and purposes the equations can only be solved analytically if the product of inertia terms were set to zero and then the equations linearized. The equations can then be solved using state space or complex variable techniques, see Staley[4].

It is the point of this research to fully describe the gyro, so the above simplifications cannot be made. The solution, then, is to use a numerical technique to solve the problem instead of an analytical one. Since a numerical technique must be used to solve the set of equations the further extension of this idea to a computer simulation is practical.

4.1 The Runge-Kutta-Fehlberg Numeric Technique

There are many different techniques to numerically integrate a set of differential equations. They range from Taylor series methods to multi-step methods. The

Runge-Kutta technique is a popular method and is commonly used in situations involving rotating rigid bodies[10][11].

The Runge-Kutta technique is similar to the Taylor series method except explicit definitions of the derivatives are not needed beyond the first derivative. A fourth order Runge-Kutta is given by[12]

$$y_{n+1} = y_n + \frac{1}{6} (k_0 + 2k_1 + 2k_2 + k_3)$$

where

$$\begin{aligned} k_0 &= hf(y_n, t_n) \\ k_1 &= hf\left(y_n + \frac{1}{2}k_0, t_n + \frac{h}{2}\right) \\ k_2 &= hf\left(y_n + \frac{1}{2}k_1, t_n + \frac{h}{2}\right) \\ k_3 &= hf(y_n + k_2, t_n + h) \end{aligned}$$

and h is the step size.

The Runge-Kutta method has several advantages; it is easy to code, is numerically stable for most problems and the step size can be changed at any time during the calculation. A difficulty with the Runge-Kutta method is that there is no way to estimate the error. This implies that picking a step size which results in small errors is difficult.

One way to avoid this downfall and still keep the advantages is to combine a fourth order and a fifth order Runge-Kutta. A comparison of the results from the two calculations gives an error estimate which is then used for step size control. This method is known as the Runge-Kutta-Fehlberg method[13].

The Runge-Kutta-Fehlberg method has become very popular in numeric integration[14] and was the method used to solve Equations 3.54 and 3.55. A coded version of the fourth-fifth order Runge-Kutta-Fehlberg (RKF) technique, found in Forsythe[12], called RKF45 was used in the computer simulation of the gyro¹.

¹It should be noted that the subroutine was coded in Fortran 77

4.2 Numeric Simulation of a Gyro

The purpose of the simulation is to take known gyro specifications and initial conditions and find the response of the system. When the simulation was constructed, care was taken to ensure it was as user friendly as possible in both its input and output. The code for the computer simulation can be found in Appendix C.

Throughout this work two coordinate systems have been used which result in 'case' referenced angles and 'gimbal' referenced angles. The 'gimbal' referenced angles are used in the equations that mathematically describe the response of the gyro, however it is difficult to measure these angles directly and so using them for the initial conditions would be impractical. It is easier to measure the 'case' referenced angles. Thus the initial condition data is entered into the simulation using these angles. Similarly, data is output in 'case' referenced angles.

The frequency of output was also a consideration when this simulation was constructed. Linear analyses of the system, see Staley[4], have shown that there are four natural frequencies. Two of these are near the spin frequency of the input shaft and two are roughly twice the spin frequency. This implies that the calculation step size (ie. the difference between the time entering the RKF subroutine and the time exiting the routine) for the non-linear equations should be at least an order of magnitude smaller than the input shaft rotational period. A step size of one hundredth of the input shaft spin speed was arbitrarily chosen so as to not miss any potential high frequency response and to avoid the possibility of aliasing.

To show the high frequency response of the system to the initial conditions it is necessary to use the calculation frequency as the output rate. However, for long simulations, eg. 10 seconds, using this output rate would result in data files that are over 20 megabytes in size. This size of file is unwieldy when analyzing the information. In runs this large, the high frequency data may not be needed and reducing the frequency of output is the obvious solution for lowering the number of data points and hence the output file size. Thus output step size, as a multiple of the calculation step size, is able to be chosen before a simulation is begun.

Whenever discrete sampling of an analog frequency is performed, aliasing could

Figure 4.1: Example of Aliasing[19].

occur. Aliasing occurs when, because of discrete sampling, high frequencies appear as low frequencies. An example can be seen in Figure 4.1. Passing the data through a low pass filter before analysis removes any high frequencies so that aliasing does not occur. A low pass filter was not used in this case because of the difference in relative magnitudes of the nutation (the high frequency oscillation of the rotor) and the precession.

Choice of output is an important consideration when creating a simulation. This simulation calculates the angular position (ie. tilt of the rotor), it's angular momentum and the forces experienced by the shaft. This information gives a thorough description of the response of the gyro to the initial conditions.

4.2.1 Overview of the Simulation

Now that the input/output interfaces of the simulation have been introduced, a brief discussion of the internal operation of the program should be made. The flow chart, Figure 4.2, gives a good overall view of the internal procedure of the program.

The user interface includes the main menu and sub-menus accessed by the user to enter the particular gyro specifications (ie. the moments of inertia), initial conditions, and start and stop times.

When the user begins the simulation the initial angular positions and rates of the rotor are converted from 'case' to 'gimbal' referenced angles using the equations

Figure 4.2: Flow chart showing the basic procedure used for the gyro simulation.

derived in Chapter 2. This conversion prepares the initial conditions for the numerical integration. The program then calculates the required step size, arbitrarily chosen as one hundredth of ω_s , using the following relation

$$t_{\text{step}} = \frac{\pi}{50\omega_s} \quad (4.1)$$

where ω_s has units of radians/second. Once the step size is known, the next iteration time step can be calculated using

$$t_{c+1} = t_c + t_{\text{step}} \quad (4.2)$$

where t_c is the current time, and t_{c+1} is the next time step.

With the gyro defined and the initial conditions in the proper coordinate system, the RKF method can now be used to solve the set of differential equations. Once the equations have been solved for that time step, the current time and the 'stop' time, t_s , must be compared. If $t_c \geq t_s$, then the simulation is finished and the user menu is updated and activated. If $t_c < t_s$, then t_c has to be compared to the next output time, t_o . If $t_c > t_o$, t_c is incremented by t_{step} and the integration cycle is repeated, if $t_c \leq t_o$ then the data is output to a file and the next output time is calculated. The rotor position and rate are known and so the angular momentum can be calculated using Euler's Equation. The shaft forces can be found using Equation 3.78.

The angular position and the rates must be converted to the 'case' coordinate frame using the equations found in Sections 2.3 and 2.4. The angular momentum

can be rotated to a non-spinning, case fixed coordinate system using the rotation matrix

$$A = (-\omega_s)_z (-\phi)_x (-\theta)_y \quad (4.3)$$

The shaft forces are already in the shaft coordinate frame and need only be moved to the non-rotating coordinate system. The rotor torques are in the rotor coordinate frame and demonstrate the magnitude of the control torques needed to maintain the current position of the rotor.

This data is then output to a file and using the RKF method again, the cycle is repeated. It should be noted that the program was coded using the Fortran 77 standard. This was done because Fortran 77 is well tested and often used for numerical analysis, as well as the fact that RKF45 was already coded in Fortran 77. A third reason for this language being chosen was that Fortran 77 compilers were readily available on several platforms, specifically PC, SGI and SUN Sparc2 based platforms.

4.3 Evaluation of the Simulation

In order to evaluate the simulation it is necessary to compare the results of the simulations to that of an actual gyro. This can be done by measuring drift rates of a real gyro and comparing them with the calculated drift rates of the simulated one. A Litton CSG-2 gyro was tested at the DREO Laboratories on December 18, 1995. The specifications for this gyro are given in Table A.2²

The setup at DREO consisted of a CSG-2 gyro mounted on a Contraves gyro spin table. The gyro was positioned so that the spin axis pointed up, the x-axis pickoff pointed north and the y-axis pickoff pointed west. The rotor pickoffs were connected to a rebalance loop, internally known as R1-3, which actively maintains the rotor position at null. It does this through use of the pickoffs for position measurements of the rotor and then uses magnetic torquers to rebalance the rotor.

²See Appendix A.

<i>Offset Dial Position</i>	<i>Measured DC Offset</i>		<i>PO Offset</i>	
	<i>X-Axis (Volt)</i>	<i>Y-Axis (Volt)</i>	<i>X-Axis (mrad)</i>	<i>Y-Axis (mrad)</i>
-2	-0.483	-0.482	-1.858	-1.854
-1	-0.137	-0.137	-0.527	-0.527
0	0.000	0.000	0.000	0.000
1	0.137	0.137	0.527	0.527
2	0.481	0.480	1.850	1.846

Table 4.1: Rotor Offset from Null Position.

4.3.1 Calculation of Rotor Offset Using Pickoff Data

The R1-3 rebalance circuit allows the rotor to be offset to precise rotor angles by adding or removing electronic resistances to the current used to rebalance the rotor. Through use of this circuit the rotor could be set to five distinct positions about both of the transverse axes. The precise pickoff (PO) offset for each of the settings had to be determined by measuring the PO DC offset when the rebalance circuit was fed 5 Vrms. See Table 4.1 for the readings taken. These readings were then used in Equation 4.4 to determine the PO offset in Vrms[15].

$$\text{PO Offset (Volts RMS)} = \left(\frac{\text{DC Offset} - \text{DC Offset at Zero}}{\text{DG}} \right) \quad (4.4)$$

where the DG is the demodulator gain in the circuit, measured as 0.2 Vdc/Vrms for both axes. These results can then be converted to radians using Equation 4.5.

$$\text{PO Offset (mrad)} = \text{PO Offset (Vrms)} \left(\frac{1 \text{ mrad}}{1.30 \text{ Vrms}} \right) \quad (4.5)$$

See Table 4.1 for the results. The CSG-2 gyro rotor can now be adjusted to the five positions listed for each axis.

It is important to note that these angles represent the rotor position in the yz plane (x-axis PO) and the xz plane (y-axis PO) based on a coordinate system fixed to the case. Also, a x-axis pickoff offset is actually a rotation about the y-axis and vice versa. Except in a few special cases, these angles do not correspond to the 'case'

Figure 4.3: Rotor position in 'case' and 'pickoff' referenced angles.

referenced angles used in the analysis of the gyro. Thus, it is necessary to develop a method of converting between pickoff angles and 'case' referenced angles.

In Figure 4.3 it can be seen that the rotation about the x-axis is the same whether measured by 'pickoff' referenced or by 'case' referenced angles. The rotation about the y-axis is different. The 'pickoff' referenced angle would be ζ , while the 'case' referenced angle is θ' . So it is only necessary to find a conversion between these two angles.

The first step is to redefine the rotation matrix for the 'case' referenced angles. From Equation 2.10 the rotation matrix is defined thus

$$A' = \begin{pmatrix} C_\gamma C_{\theta'} & C_\gamma S_{\theta'} S_{\phi'} + S_\gamma C_{\phi'} & -C_\gamma S_{\theta'} C_{\phi'} + S_\gamma S_{\phi'} \\ -S_\gamma C_{\theta'} & -S_\gamma S_{\theta'} S_{\phi'} + C_\gamma C_{\phi'} & S_\gamma S_{\theta'} C_{\phi'} + C_\gamma S_{\phi'} \\ S_{\theta'} & -C_{\theta'} & C_{\theta'} C_{\phi'} \end{pmatrix} \quad (4.6)$$

Now it is necessary to define a rotation matrix in which ζ is defined. It is possible to do this by interchanging the order of rotation so that

$$A'' = (\gamma')_z (\alpha)_x (\zeta)_y \quad (4.7)$$

Simplifying this gives

$$A'' = \begin{pmatrix} C_{\gamma'}C_{\zeta} + S_{\gamma'}S_{\alpha}S_{\zeta} & S_{\gamma'}C_{\alpha} & -C_{\gamma'}S_{\zeta} + S_{\gamma'}S_{\alpha}C_{\zeta} \\ -S_{\gamma'}C_{\zeta} + C_{\gamma'}S_{\alpha}S_{\zeta} & C_{\gamma'}C_{\alpha} & S_{\gamma'}S_{\zeta} + C_{\gamma'}S_{\alpha}C_{\zeta} \\ C_{\alpha}S_{\zeta} & -S_{\alpha} & C_{\alpha}C_{\zeta} \end{pmatrix} \quad (4.8)$$

Both Equation 2.10 and 4.8 describe the same orientation and so

$$A' = A''$$

It is necessary to solve for ζ and this can be done by using the following relationship

$$\begin{aligned} \frac{A''_{31}}{A''_{33}} &= \frac{A'_{31}}{A'_{33}} \\ \frac{C_{\alpha}S_{\zeta}}{C_{\alpha}C_{\zeta}} &= \frac{S_{\theta'}}{C_{\theta'}C_{\phi'}} \\ \tan \zeta &= \frac{\tan \theta'}{C_{\phi'}} \end{aligned} \quad (4.9)$$

Therefore a relationship between ζ and θ' . Note that if ϕ' is zero, meaning there is no rotation of the rotor about the x-axis, then $\theta' = \zeta$ as would be expected. It should also be noted that for this relation to work $\phi' \neq 90$ deg. If this were to happen then the x-axis pickoff would not be able to sense any displacement of the rotor. This case is impractical and need not be considered.

Later it will also be necessary to find a relationship between $\dot{\zeta}$ and $\dot{\theta}'$. This can be done by calculating the derivative of Equation 4.9 and solving for $\dot{\theta}'$. This gives

$$\dot{\theta}' = \cos^2 \theta' \left(\frac{\cos \phi'}{\cos^2 \zeta} \dot{\zeta} - \sin \phi' \tan \zeta \dot{\phi}' \right) \quad (4.10)$$

It should be noted that for very small angles $\dot{\theta}' \approx \dot{\zeta}$

4.3.2 Experimental Procedure and Results

To obtain the experimental results, the Litton CSG-2 gyro was placed on the spin table, connected to the rebalance loop circuitry and then allowed to spin up to the operating speed. When the gyro was at operating temperature and the rotor was at the proper spin speed, drift readings were taken.

Three distinct sets of data were obtained during this experiment. These sets included pure rotor offsets about the x-axis, pure rotor offsets about the y-axis and offsets about both the x and y-axes. In each case, the five possible rotor offsets were used³. Drift data was collected over a period of two minutes for each case, at an approximate frequency of 2 Hz. The data collected over this two minute period was averaged and then the mean of the five values, one for each rotor position, was calculated and subtracted. This was done to remove any gravity drift terms. The results are shown in Table 4.2. These drift terms have units of volts DC and can be converted to degrees per hour using Equation 4.11.

$$\text{Drift Rate} = SF \times \frac{V}{514.993} \quad (4.11)$$

where V is the voltage output of the torquers and SF is a scaling factor ($SF_{x\text{-axis}} = 1663.245 \text{ deg/hr/mA}$ and $SF_{y\text{-axis}} = 1656.858 \text{ deg/hr/mA}$). The value of 514.993 is the scaling resistance, in ohms, used by the R1-3 circuit.

The data still cannot be compared to the simulation drift rates as it is not in the 'case' set of coordinates. It is necessary to convert these values using Equation 4.10, because what has been measured is the 'pickoff' referenced angles and not the 'case' referenced ones. Plots of the rotor offset versus drift rate can be seen in Figures 4.4 and 4.5.

4.3.3 Simulation Procedure and Results

Now that the experimental results have been obtained it is possible to compare the results, or at the very least, the trends, of the computer simulation of the gyro and the actual gyro. Using the gyro specifications given in Table A.2 and the same initial conditions as the experimental gyro, drift rates can be obtained. See Appendix B for the calculation of these rates from the simulation data. Plots of the rotor offset versus drift rate can be seen in Figures 4.6 and 4.7.

³As discussed in Section 4.3.

Figure 4.4: Experimental drift rate of a Litton CSG-2 DTG due to a rotor offset about the y-axis.

Figure 4.5: Experimental drift rate of a Litton CSG 2 DTG due to a rotor offset about the x-axis.

Figure 4.6: CSG-2 simulation drift rate due to a rotor offset about the y-axis.

Figure 4.7: CSG-2 simulation drift rate due to a rotor offset about the x-axis.

<i>Rotor Offset</i>				<i>Mean Drift Rate</i>			
<i>About X-Axis</i> (mrad)	<i>About Y-Axis</i> (mrad)	ϕ' (mrad)	θ' (mrad)	<i>X-Axis</i> (m Volts)	<i>Y-Axis</i> (m Volts)	$\dot{\phi}'$ (Arcsec/hr)	$\dot{\theta}'$ (Arcsec/hr)
0.000	-1.858	0.000	-1.858	-0.7966	-0.0119	-9261.43	-137.62
0.000	-0.527	0.000	-0.527	-0.2252	-0.0076	-2618.50	-87.58
0.000	0.000	0.000	0.000	-0.0002	0.0023	-2.41	26.27
0.000	0.527	0.000	0.527	0.2267	0.0094	2636.20	109.31
0.000	1.850	0.000	1.850	0.7952	0.0077	9245.87	89.62
-1.854	0.000	-1.854	0.000	-0.0229	0.8957	-260.11	10373.61
-0.527	0.000	-0.527	0.000	0.0003	0.2607	3.63	3019.01
0.000	0.000	0.000	0.000	0.0162	-0.0009	188.61	-10.17
0.527	0.000	0.527	0.000	0.0214	-0.2621	248.37	-3035.52
1.846	0.000	0.000	0.000	-0.0150	-0.8934	-174.49	-10346.92

Table 4.2: Experimental Drift Rate versus Rotor Angular Position

4.3.4 Discussion of the Simulation and Experimentation Results

The drift rates of the Litton CSG-2 gyro for several different rotor offsets have now been found using both experimental measurement and computer simulation based on the derived non-linear differential equations. The results can now be compared.

When the drift rates about the x and y axes for the case with an initial rotor offset about the y-axis are compared, Figures 4.4 and 4.6, it can be seen that the trends of the data and their magnitudes are very similar. Both figures show that for a pure rotor offset about the y-axis there is a large drift about the x-axis and only a small one about the y-axis. This result is consistent with intuition. This implies that the rotor is moving from a pure rotation about the y-axis to a position that involves rotations about both axes.

It is also evident from Figures 4.4 and 4.6 that as the initial rotor offset increases so does the drift rate. The simulation drift data shows that the relationship between the drift rate and the angular offset is not linear. It is actually in the form of a cubic.

The cubic has a point of inversion at the origin and has a convex form. This means that as the rotor offset increases, the rate of change of the drift rate decreases. This trend is found in the drifts about both axes; however, it is more evident in the x-axis drift. This result is not as clear in the experimental data but taking the standard deviation of the data, which was up to 100 degrees/hr, into account the trend can be seen.

Figures 4.5 and 4.7 show the drift rates for the experimental CSG-2 gyro and the simulation, respectively, when the rotor was offset about the x-axis. Again, the trend of the data is similar and the cubic relationship is apparent. This implies that the results are consistent and repeatable. The results are also what would be expected from intuition. The cubic relationship is not clear in the experimental data but when the standard deviation of the data is taken into account the cubic relationship can be obtained. The standard deviation for the x-axis drift is large when compared to the data itself and this makes any trends difficult to see. It should be noted that the drift about this axis is only in the range of 200 arcseconds/hr or 0.0556 degrees/hr and so it is very small and difficult to measure accurately.

There are many reasons for the differences between the experimental and simulation values. The first involves the fact that many of the specifications were not exact. A few examples of this include: the orthogonal flexure stiffness were assumed to be the same, however this is most likely not the case; the gimbal inertia could be more than a factor of ten less than the rotor inertia; and damping was assumed to be zero. Damping would not be zero; even if the mechanics were assumed to be ideal then air resistance would still cause slight damping. This source of error is most likely the prominent reason why there are errors in the magnitudes.

A second known source of error is the fact that the spin axis of the gyro was oriented pointing up. This causes the gyro to measure the Earth rate, that is, the rotation of the Earth about its own axis. This effect is most easily demonstrated by considering an ideal rotor with two degrees of freedom positioned on the Earth so its spin axis is perpendicular to the surface, and x-axis pointing north and its y-axis west, see Figure 4.8. Due to its angular momentum the rotor will remain fixed in inertial space; however, the Earth rotates about the axis once every 23 hours and

Figure 4.8: Gyroscope rotor fixed on the surface of the Earth.

56 minutes, thus the spin rate of the Earth is

$$\begin{aligned}\omega_E &= \frac{360 \text{ degrees}}{23.933 \text{ hours}} \\ &= 15.04 \frac{\text{degrees}}{\text{hour}}\end{aligned}$$

If the gyro was located at the equator (a latitude of 0°) then this spin rate would be seen as a rotation of the rotor about the x-axis (the axis pointing north). At the Earth's north or south pole the rate would be about the spin axis and thus not able to be measured. At any latitude between these extremes the rate sensed is some component of the 15.04 degrees/hr. The latitude of the test site at DREO is 45.349° north. The rate measured about the x-axis can be calculated (see Figure 4.9)

$$\begin{aligned}d_E &= \omega_E \cos \theta \\ &= 15.04 \times \cos 45.349 \\ &= 10.57 \text{ deg/hr}\end{aligned}$$

where d_E is the drift measured by the gyro about the x-axis, ω_E is the Earth spin rate and θ is the latitude of the gyro.

When the gyro is maintained at the null position, ie zero rotor offset, then the Earth rate can be eliminated using one of two ways. The first is to orient the spin axis so it is parallel with the Earth's spin axis, the second is to take the mean of the

Figure 4.9: Calculation of Earth rate at a latitude of θ .

x-axis drift data and subtract it from each point. The second method was used in this case. This eliminates the constant rate; however, since this experiment involved shifting the spin axis for each case, all of the Earth rate cannot be eliminated. To calculate the Earth rate measured by the gyro, the offset about the y-axis must be used. Note that offsets about the x-axis would not cause the Earth rate to come into effect. The effect of the Earth rate, after subtracting the mean, can be found by

$$d_{E_m} = \omega_E \cos(\theta + \Delta\theta) - \omega_E \cos \theta \quad (4.12)$$

A table containing the modified Earth rate for each of the 5 rotor offsets about the y-axis can be seen in Table 4.3. These rates effect the x-axis drift ($\dot{\phi}'$) for the case where there is a rotor offset about the y-axis.

The above two reasons explain the presence of the errors in the magnitudes. The important aspect of the comparison of the experimental and simulation drifts is that the trends are the same. This implies that the simulation and specifically, the set of differential equations, Equations 3.54 and 3.55, are valid methods of predicting the response of a gyroscope rotor to an initial offset.

θ'_0 (mrad)	dE_m (arcsec/hr)
-1.858	71.50
-0.527	20.29
0.000	0.00
0.527	20.30
1.850	71.63

Table 4.3: Modified Earth Rate for Different Rotor Offsets about Y-Axis

<i>Specification</i>	<i>Value</i>
J	$9.677 \times 10^{-5} \text{ kg m}^2$
I_{r_i}	$5.419 \times 10^{-5} \text{ kg m}^2$
I_{g_i}	$1.209 \times 10^{-6} \text{ kg m}^2$
I_{g_t}	$6.165 \times 10^{-7} \text{ kg m}^2$
K_x	$7.633 \times 10^{-4} \text{ Nm}$
K_y	$3.816 \times 10^{-4} \text{ Nm}$
D_x	$0.000 \times 10^0 \text{ Nms}$
D_y	$0.000 \times 10^0 \text{ Nms}$
ω_s	5000.0 rpm

Table 4.4: Test Gyro Specifications.

4.4 General Results from the Simulation

With the validity of the computer simulation established, it is possible to explore some general cases. The effects of changing the rotor offset by small angles (> 2 mrad) has already been discussed in the previous section. Large rotor offsets can now be simulated using a fictitious gyro with the specifications given in Table 4.4.

The effect of rotor tilt on precession will be discussed first. Figures 4.10, 4.11 and 4.12 show the precession of the rotor when the initial conditions are $\phi'_0 = 10$ deg, $\phi'_0 = 5$ deg and $\phi'_0 = 1$ deg, respectively. The period of precession decreases as the rotor offset increases. The above figures also show the cyclic motion of the rotor as

it precesses.

The rotor moves from a position defined by a pure rotation about the x-axis to one defined by a pure rotation about the y-axis, and then back again. The motion repeats itself and thus the path of the motion forms a circle, as shown in Figure 4.13. The line plotted in Figure 4.13 can be taken as a tracing of a normal to the rotor on the inside of a sphere. Finally, note that if the rotor is offset about both axes, see Figures 4.14 and 4.15, the same characteristics as an offset about only one axis can be seen with the exception of a phaseshift.

Now that the long term characteristics have been discussed, the short term effects can be explored. Figures 4.16 and 4.17 show the nutation of the rotor for $\phi'_0 = 10\text{ deg}$ and $\phi'_0 = 5\text{ deg}$. The period of nutation remains the same as the offset angle decreases. It is also apparent there is a beat frequency, the period of which is approximately eight times that of the nutation and remains constant as the rotor offset decreases. This beat frequency means that the nutation of the rotor goes to zero periodically. It appears that, while the magnitude is affected, the period of the nutation and 'beat' remain unaffected by the rotor offset. The period of these two frequencies is dependent on the input shaft spin speed. This is demonstrated by comparing Figures 4.16 and 4.18.

Finally, the effect of damping on the response of the system has to be explored. The damping of the test gyro was arbitrarily set to 8.0×10^{-6} Nms about both axes. The effect on the rotor, with an initial rotor offset of 10 deg about the x-axis, is shown in Figure 4.19. The damping causes the angular offset of the rotor to decrease until it is zero, demonstrated by the spiral pattern in Figure 4.19. The larger the magnitude of the damping, the greater the rate of decrease of the rotor offset. The inverse is true as well.

Thus in a real gyro, where damping is most definitely present, there will be a tendency for the rotor offset to drift to zero. The rate of this drift is dependent on the magnitude of the damping. The larger the damping, the faster the rotor will move to a zero offset position. Gyros, in practice, have had the damping reduced as much as possible, so the damping should be small, implying that a control system (re-balance loop) may not need to take account of it during the design process.

Figure 4.10: Rotor angular position versus time for an arbitrarily chosen DTG, assumed to have no damping and an initial rotor offset of $\phi'_0 = 10$ deg.

Figure 4.11: Rotor angular position versus time for an arbitrarily chosen DTG, assumed to have no damping and an initial rotor offset of $\phi'_0 = 5$ deg.

Figure 4.12: Rotor angular position versus time for an arbitrarily chosen DTG, assumed to have no damping and an initial offset of $\phi'_0 = 1$ deg.

Figure 4.13: Rotor angular position about the y-axis versus rotor angular position about the x-axis for an arbitrarily chosen DTG, assumed to have no damping and an initial rotor offset of $\phi'_0 = 10$ deg, for a simulation with a duration of 2 seconds.

Figure 4.14: Rotor angular position versus time for an arbitrarily chosen DTG, assumed to have no damping and initial rotor offsets of $\phi'_0 = 10$ deg and $\theta'_0 = 10$ deg.

Figure 4.15: Rotor angular position about the y-axis versus rotor angular position about the x-axis for an arbitrarily chosen DTG, assumed to have no damping and initial rotor offsets of $\phi'_0 = 10$ deg and $\theta'_0 = 10$ deg, for a simulation with a duration of one second.

Figure 4.16: Rotor angular position versus time for an arbitrarily chosen DTG, assumed to have no damping and an initial rotor offset of $\phi'_0 = 10$ deg.

Figure 4.17: Rotor angular position versus time for an arbitrarily chosen DTG, assumed to have no damping and an initial rotor offset of $\phi'_0 = 5$ deg.

Figure 4.18: Rotor angular position versus time for an arbitrarily chosen DTG, assumed to have no damping and an initial rotor offset of $\phi'_0 = 10$ deg and $\omega_s = 5000$ rpm.

Figure 4.19: Angular displacement about the y-axis versus angular displacement about the x-axis for an arbitrarily chosen DTG, assumed to have damping of $D_x = D_y = 8.0 \times 10^{-6}$ Nms and $\phi'_0 = 10$ deg.

4.5 Calculation of Rotor Torques

Now that the angular position of the rotor is known, an estimate of the torques required to maintain a specific rotor position can be made. By ignoring nutation, the precession torques can be estimated by considering the precession term alone.[8]

$$\mathbf{M} = \mathbf{p} \times \mathbf{I}\boldsymbol{\Omega} \quad (4.13)$$

where $\mathbf{M}$ represents the precession torques, $\mathbf{I}$ is the inertia matrix, $\boldsymbol{\Omega}$ is the spin velocity and $\mathbf{p}$ is the precession velocity. In other words

$$\mathbf{I} = \begin{bmatrix} I_{rx} & -I_{rxy} & -I_{rxz} \\ -I_{rxy} & I_{ry} & -I_{ryz} \\ -I_{rxz} & -I_{ryz} & I_{rz} \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} \dot{\phi}' \\ \dot{\theta}' \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 \\ 0 \\ \omega_s \end{bmatrix}$$

Expanding Equation 4.13 and solving gives

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} \dot{\phi}' \\ \dot{\theta}' \\ 0 \end{bmatrix} \times \begin{bmatrix} I_{rx} & -I_{rxy} & -I_{rxz} \\ -I_{rxy} & I_{ry} & -I_{ryz} \\ -I_{rxz} & -I_{ryz} & I_{rz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_s \end{bmatrix} \\ &= \begin{bmatrix} \dot{\theta}' I_{rx} \omega_s \\ -\dot{\phi}' I_{rx} \omega_s \\ \dot{\theta}' I_{rxz} \omega_s - \dot{\phi}' I_{ryz} \omega_s \end{bmatrix} \end{aligned} \quad (4.14)$$

From Figures 4.10 to 4.12 it can be seen that, with no damping, the rotor position varies with time in a steady sinusoidal pattern. It is also apparent that the average of this oscillation over one period is zero and the absolute maximum, D_m , corresponds

to a combination of the initial offsets. Assuming the period of the oscillation to be τ and noticing that the precession of the rotor about the x-axis is shifted by 90 degrees from the precession about the y-axis, the following approximate relationship for the angular positions can be found

$$\phi' \approx D_m \cos \left(\frac{2\pi}{\tau} t + \alpha \right) \quad (4.15)$$

$$\theta' \approx D_m \sin \left(\frac{2\pi}{\tau} t + \alpha \right) \quad (4.16)$$

where α represents a phase shift.

By taking the derivative of Equations 4.15 and 4.16, the precession velocity, $\mathbf{p}$, can be found for any time, t .

$$\dot{\phi}' \approx -D_m \frac{2\pi}{\tau} \sin \left(\frac{2\pi}{\tau} t + \alpha \right) \quad (4.17)$$

$$\dot{\theta}' \approx D_m \frac{2\pi}{\tau} \cos \left(\frac{2\pi}{\tau} t + \alpha \right) \quad (4.18)$$

Therefore, after simplification Equation 4.13 becomes

$$\mathbf{M} = \begin{bmatrix} I_{rx} \frac{2\pi}{\tau} D_m \omega_s \cos \left(\frac{2\pi}{\tau} t + \alpha \right) \\ I_{ry} \frac{2\pi}{\tau} D_m \omega_s \sin \left(\frac{2\pi}{\tau} t + \alpha \right) \\ \omega_s D_m \frac{2\pi}{\tau} \left(I_{rxz} \cos \left(\frac{2\pi}{\tau} t + \alpha \right) + I_{ryz} \sin \left(\frac{2\pi}{\tau} t + \alpha \right) \right) \end{bmatrix} \quad (4.19)$$

Equation 4.19 can now be used to solve for the precession torques for specific cases. An example can be performed using the data from Figure 4.10, the case of the test gyro with a initial offset 10 degrees about the x-axis. From Figure 4.10 it can be seen that the period of precession, τ , is 1.73 seconds and that $D_m = 0.175$ radians. This can be substituted into Equation 4.19, along with the gyro specifications found in Table 4.4 and the observation that $\alpha = 0$, to give

$$\mathbf{M} = \begin{bmatrix} 0.032 \cos 3.63t \\ 0.032 \sin 3.63t \\ 0 \end{bmatrix} \quad (4.20)$$

Therefore, for the above case, the maximum torque due to the precession is approximately 32 milli-Nm. Even though this is only an estimation of the torques required

to maintain the position of the rotor, the above calculation shows that the torques will be very small.

In Section 4.3.4 it was shown that as the rotor offset decreases the period of precession increases, in other words the magnitude of τ increases. Also, as the rotor offset decreases, the absolute magnitude of the precession decreases, ie. D_m decreases. This means that for smaller rotor offsets the torques caused by the precession of the rotor will be smaller in magnitude as well. See Table 4.5 for the precession torques at several different rotor offsets.

4.5.1 Estimation of Current Required to Rebalance an Offset Rotor

The torques needed to rebalance the rotor are generated electro-magnetically. A permanent magnet is fixed to the rotor and a coil of wire, known as the torquer coil, is fixed to the case near the permanent magnet. Placing a current, i , in the coil results in a magnetic field which interacts with the field from the permanent magnet on the rotor creating a force couple which rebalances the rotor. By varying the torquer coil current, the torques can be varied. The torque generated can be calculated using the following equation[17]

$$T = 2BNiaR_r \quad (4.21)$$

where T is the torque, B is the flux density of the magnetic field caused by the permanent magnet, N is the number of turns of wire in the coil, i is the current supplied, a is the length of the coil in the field and R_r is the rotor radius.

For a specific gyro B , N , a and R_r can be considered constant. Therefore Equation 4.21 becomes

$$T = Ci \quad (4.22)$$

or

$$i = \frac{T}{C} \quad (4.23)$$

where

$$C = 2BNaR_r$$

<i>Rotor Offset (Degrees)</i>	<i>Maximum Precession Torque (milli-Nm)</i>	<i>Maximum Torquer Coil Current (mA)</i>
10	32	1274.9
5	4.42	176.1
1	0.072	2.9

Table 4.5: Estimate of Torquer Coil Current Required to Re-balance the Rotor for Different Rotor Offsets

To give an idea of the current required to maintain the rotor at a 10 degree offset it is necessary to make a few assumptions before using Equation 4.23. First, assume that the torquer coil consists of 200 turns, ie $N = 100$. Next, assume a conservative value for the flux density and take $B = 0.2$ Tesla. The rotor can be assumed to be four centimetres in diameter so that $a \approx 15.7$ mm. Therefore

$$C = 0.0251 \text{ milli-Nm/mA}$$

Finally, using Equation 4.23 and using $T = 32$ milli-Nm

$$i = 1.27 \text{ Amps}$$

It should be noted that an offset of 10 degrees is extreme and would cause very high precession rates. The current required decreases as the rotor offset decreases, see Table 4.5 for the current required at several rotor offsets. This ‘back of the envelope’ calculation shows the current required to maintain the rotor position can be quite small. It is also apparent that the relationship between the current required and the rotor offset is not linear.

Chapter 5

Applications

A nonlinear dynamic model of the tuned rotor gyro has now been found. This model was then used to develop a computer simulation, and with this simulation the dynamics of the rotor were demonstrated. It was shown in Chapter 4 that, if initially there, the ideal rotor will remain fixed at the null position. However, it was also demonstrated that if the rotor position is adjusted by a small offset about either, or both, axes then a stable and predictable precession occurs.

The gyro is required to be as accurate as possible for as long as possible. As described earlier, the DTG is used as an angular position and rate sensing device. Ideally the gyro remains fixed in space while the vehicle moves about it. The angle between the rotor and the gyro pickoffs can be measured and since the frequency of data acquisition is known, the rate of spin of the vehicle can also be calculated. Although the machining processes used to manufacture the gyro are very precise, there are still small variations and imperfections in the gyro properties. This includes friction in the bearings and slightly unsymmetric rotors and gimbals. These imperfections in the final product vary and imply that each gyro will display a unique personality.

It is difficult to find the correct null position for these gyros so there is always a small drift in the position of the rotor with respect to inertial space. This implies that the inertial measurement unit must be recalibrated occasionally to determine the actual position of the vehicle versus the calculated position. In spacecraft this process is currently performed using an additional sensor (ie. a star tracker or

Earth sensor) which determines the exact orientation of the craft. The system is then recalibrated to match. The frequency of the recalibration process depends on the mission accuracy requirements and the drift rate of the gyro. In current remote sensing missions, like that of the Swedish satellite Odin, the accuracy can be as high as ± 7 arcseconds/hr[16]¹. Cain et al[16] found that the Litton CSG-2 gyro would require a recalibration time of approximately 38 minutes to maintain this accuracy.

If the gyro was not recalibrated at regular intervals then the actual attitude would be quite different from the calculated one. In the Odin spacecraft application, for example, this could mean that the instrumentation is pointed toward deep space rather than the Earth's limb². In severe cases the solar arrays could end up facing away from the Sun and thus the satellite would have no power.

It should be noted that very high accuracies may not be possible with some gyros since the Earth or Sun sensor cannot provide recalibration data fast enough for the gyro to maintain the required accuracy. It can be seen that the accuracy of the gyro is an important component of any mission. Methods to increase this accuracy would be a great plus to the industry and would expand the number of possible missions.

One way to improve the accuracy of the gyro is to improve the manufacturing process and this is constantly being done as better and more efficient processes are developed. A second method can be developed from the unsymmetric, and non-linear set of differential equations found in Chapter 3 and the simulation shown in Chapter 4.

The drift rates of the physical gyro can be found either by long term tests before launch or by determining the drift once in operation. If the specifications of the gyro are known then the gyro can be modelled using a computer simulation. In Chapter 4 it was shown that if the rotor is rotated about either of the gimbal axes then a predictable precession drift rate would be seen.

Using the simulation, the ideal rotor tilt position can be found. This ideal position is one where the predicted drift is opposite in direction but equal in magnitude to the actual drift rate. The null of the actual gyro can then be adjusted to this

¹Note that 1 arcsecond is 1/3600 of a degree.

²The Earth's limb is the edge of the disk formed by the Earth, as seen from space.

<i>Gyro Drift Rate</i>	<i>X-Axis</i>	<i>Y-Axis</i>
<i>Actual¹ (arcsec/hr)</i>	11.0	-8.2
<i>Simulated (arcsec/hr)</i>	-11.07	8.25
<i>Combined² (arcsec/hr)</i>	-0.07	0.05
<i>% Difference (2 vs. 1)</i>	99.3%	99.4%

Table 5.1: Comparison of Drift Rates for a CSG-2 Gyro Before and After Offsetting the Rotor.

optimal position. This causes the natural drift rate of the gyro due to an offset of the rotor and the drift rate due to noise to cancel each other. The result is a lower drift rate for the gyro.

In some cases it may be only possible to cause a minor reduction in the drift of the gyro, however, there may also be cases where the drift can be almost completely eliminated. The advantage of using this method to increase the accuracy of the gyro is that it is fairly quick and easy to implement.

The potential for this method of reducing the drift rate can be theoretically demonstrated by use of the simulation package from Chapter 4 and data from the experiment performed in the paper by Cain et al[16]. From long term tests³ it was determined that the drift of a CSG-2 gyro was 11 arcseconds/hour about the x-axis and 8.2 arcseconds/hour about the y-axis. It is now necessary to assume that the gyro used in the above test is similar to the model of a CSG-2 developed in Appendix A. After several tests of different rotor tilt positions, it was found that the rotor orientation of $\phi'_0 = -0.31$ arcseconds and $\theta'_0 = -0.43$ arcseconds achieves a drift rate of -11.07 arcseconds/hour about the x-axis and 8.25 arcseconds/hour about the y-axis. If the rotor is offset to the positions defined above, the x-axis drift will become 0.07 arcseconds/hour and the y-axis drift 0.05 arcseconds/hour. This is a 99.3% improvement about the x-axis and a 99.4% improvement about the y-axis. See Table 5.1 for a summary of these results.

³The gyro was tested for a continuous 4 day period

Thus, it is possible for significant reductions in the drift rates of a gyro. Re-stated, this results in very accurate position and rate measurements and enables the spacecraft bus to be more versatile and practical for demanding scientific research.

A second aspect of the DTG, as yet not discussed, is the 'turn on to turn on' repeatability. Each time a gyro is started the drift characteristics are different. This aspect of the gyro means that recalibration is needed after each turn-on.

It has always been thought that the 'turn on to turn on' inconsistency has been due to the mechanical components of the gyro. However, it has been shown in this research that a slight offset of the null of the rotor will cause a constant drift about each of the axes. It is possible that the change in the drift characteristics is not just because of the mechanical components but because there is a small shift in the electronic null of the gyro re-balance loop between operation times. It has also been shown that even a small shift could cause large drifts. Investigation of this relationship is beyond the scope of this research, however it is proposed that in the future this explanation for the differences in drift between 'turn on to turn on' should be investigated.

Chapter 6

Summary and Conclusions

The gyro is an important instrument used in many applications to measure angular rates. The dry tuned gyro (DTG) is used for position and angular rate determination in many situations, these include; spacecraft, aircraft and marine applications. The DTG is an extremely complicated device to describe mathematically, and thus only descriptions with simplifying assumptions have been previously developed, despite the use of the DTG for over three decades.

It is important that a good understanding of the dynamics of the gyro is known due to the increasing demand on accuracy in its applications. An example of the increased demand for accuracy on the inertial control package, of which the DTG is an integral part, is the Swedish spacecraft Odin, which requires a minimum pointing accuracy of ± 7 arcseconds. This type of accuracy demand is only a start as scientific missions to explore the furthest reaches of space by remote sensing are developed.

Ignoring the magnetic sensors, torquers and motors, the gyro consists of three main components; the shaft, the gimbal and the rotor. Of these mechanical components the dynamics of the rotor are critical since the position of the rotor is used to measure the angular position and rate of the vehicle. However, before the rotor could be examined, the kinematics of the gimbal had to be explored. Ideally, the gimbal, a version of the Hooke's joint, is the mechanism by which the rotor is freed from external torques and it is also the device through which the motor power is transferred to the rotor. As only two dimensional analyses of the gimbal are found in literature, a general three dimensional description of the kinematics of the gimbal

2 of/de 2

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

was developed, see Chapter 2. These kinematic equations were found using a series of rotation matrices and by defining two sets of coordinate reference frames; the 'case' and 'gimbal'. A full description of the kinematics was found including angular position and rates. The analysis can be extended to find an analytic description for the angular accelerations, however this was not done, as it was outside the scope of the research.

Once the kinematics of the gimbal were determined the dynamics of the rotor could be found. Two methods were used to find the equations that describe the rotor dynamics. The vectorial analysis approach and Lagrange's equations provided a check on each other. As few assumptions as possible were made during the analysis which took asymmetry, damping and varying stiffness of the flexures into account. The equations were developed in the spinning coordinate system to maintain consistency with the physical situation. The results were later converted to the non-spinning coordinate system using the relationships developed for the gimbal. Lastly, it is important to note that no linearizing assumptions were made. This made the dynamic analysis much more difficult but was necessary to obtain a complete and general description. A bonus, because of the use of the two different methods of analysis of the rotor, was that the vector mechanics approach and Lagrange's equation method could be compared. It was found that the Lagrange method was faster to implement and use than the vectorial approach and was easier to do algebraically. Implementing damping was complicated, and hence not included in the Lagrangian analysis, and this method failed to provide any insight into the problem and the dynamics involved which the vectorial mechanics approach did. In both cases, the algebra was difficult and cumbersome leading to the fact that it was easy to make a small algebraic mistake which would be very difficult to find. Determining the dynamics by two methods is easily justified as every time they were used they would check each other. The differential equations that were arrived at can be seen in Chapter 3.

These equations were then used in a computer simulation developed to accept specific gyro specifications including mass moments and products of inertia, stiffness of the gimbal flexures, and damping of the same. The output of the program was

angular position and rate of the rotor, rotor momentum and shaft reaction forces. Although the rotor momentum and shaft reactions were never explicitly used in this research they did supply information to describe the motion of the rotor and the forces associated. The data reduction focused on the use of the angular position data. This data, with the time index, provided the average drift rate of the rotor.

From the angular position data supplied by the simulation and the time required to obtain them, drift rates were calculated. This enabled the simulation to be compared to an actual DTG. A Litton CSG-2 gyro was used for this comparison and the drift data obtained from a series of rotor initial offset tests can be seen in Chapter 4. Accurate CSG-2 gyro specifications could not be found therefore it was necessary to calculate and assume the numeric values of certain gyro specifications. This made it impossible to compare the magnitude of the drift rates found from the experimentation and the simulation, however the trends in the data could be compared.

The trends in the data, in all cases, were very similar and, in fact, even though the assumptions about the simulation gyro had to be made, the magnitudes of the data were similar as well. These differences in magnitude can be explained at least two ways. The first, as already stated, is that assumptions had to be made and the probability of getting them exactly correct is low. The second was the Earth rate; this effect was calculated to be up to approximately 75 arcseconds/hour and would have been compensated for if the error caused by the assumptions had not been so large. A table of the effective Earth rates experienced by the rotor at the individual tilt orientations was presented in Chapter 4.

The comparison of the experimental and the simulation data validated the simulation proving that the simulation, and thus the set of differential equations, gives an accurate prediction of the dynamics of the rotor. With the validity of the simulation proven, it was then possible to explore the general trends of the dynamics of a rotor when tilted about an axis relative to the shaft. The precession was found to be a stable, periodic movement when no damping was present. When damping was present, the rotor slowly moved toward a zero null position. In other words, the magnitude of the precession slowly decreases until it is zero. The rate of change of

the magnitude of the precession was directly proportional to the magnitude of the damping.

Also present in the motion of the rotor is a nutation, or higher frequency oscillation, that had a period approximately one-eighth that of the precession for the cases studied. This nutation was also stable and repeatable. However, there was a beat frequency in the nutation which caused the magnitude on the nutation to vary from zero to its full magnitude. This beat frequency made the total number of natural frequencies of the system to be four in agreement with those discussed by Staley[4] in his linear analysis.

Aside from allowing the dynamics of a DTG to be explored, without the cost of purchasing or manufacturing one, the existence of these dynamic equations provide a method to increase the accuracy of the DTG. Inertial control packages are relied on heavily for spacecraft applications and the accuracy of these packages goes a long way toward determining the limitation of the spacecraft's mission.

All gyros drift from their null position. This drift is caused by many factors including: mechanical imperfections, electronic noise due to the support electronics and improperly set mechanical or electrical null used for rebalance. Once the drift for the particular gyro being used is known then the simulation can be used to find a rotor offset which eliminates, in the worst case some, in the best case all, of this drift. An example of this potential use is seen in Chapter 5 where improvements of up to 99.4% were calculated for the Litton CSG-2 gyro, already a very accurate gyro.

In the last decade or so new types of rate sensing instruments have been developed (ie. the laser gyro), however these instruments are large, require high voltages and have not been accepted for spacecraft applications. Thus, mechanical gyroscopes are still a vital component in an inertial package. It is essential that these mechanical gyros be completely understood so they may be used to their full advantage. Manufacturing techniques can only go so far in the effort to make these devices more accurate. The procedure described in Chapter 5 allows the accuracy of the instruments to be increased even further.

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Appendix A

Calculation of Litton CSG-2 Gyro Specs

Before the Litton CSG-2 gyro can be simulated its specifications must be known. The gyro specification sheet supplied by Litton lists several crucial factors which can be used to calculate some of the exact specifications of the CSG-2 gyro. This implies that not all of the required information can be found. It will then be necessary to estimate some of the values. This will only be done as it becomes necessary. A list of the provided data is shown in Table A.1[18].

In Table A.1 one item listed is F_m , this is known as the figure of merit of a gyroscope and it relates the rotor spin inertia and the inertias of the gimbal[1], see Equation A.1

$$F_m = \frac{I_{r_z}}{I_{g_x} + I_{g_y} - I_{g_z}} \quad (\text{A.1})$$

What should be noted about the CSG-2 gyro is that both the rotor and the gimbal can be assumed symmetric about both of the transverse axis, ie.

$$I_{r_x} = I_{r_z} = I_{r_y}$$

Specification	Value
F_m	240.0
Spin Speed, ω_s	105.5 Hz
Transverse Rotor Inertia	40.5 g-cm ²
Rotor Momentum (at 103 Hz)	45 300 g-cm ² /s

Table A.1: CSG-2 Gyro Specifications as Provided By Litton.

and

$$I_{g_i} = I_{g_z} = I_{g_y}$$

and for simplicity, let

$$I_{r_x} = I_{r_z}$$

$$I_{g_x} = I_{g_z}$$

This means that Equation A.1 becomes

$$F_m = \frac{I_{r_x}}{2I_{g_i} - I_{g_x}} \quad (\text{A.2})$$

If the rotor is assumed to be in a null position when the Litton specifications were taken then the rotor momentum can be shown to be

$$H = I_{r_x} \omega_s$$

where ω_s is defined in Table A.1 and so

$$\begin{aligned} I_{r_x} &= \frac{H}{\omega_s} \\ &= \frac{45300 \frac{\text{g cm}^2}{\text{s}}}{103 \text{ Hz}} \\ &= 7.00 \times 10^{-6} \text{ kg m}^2 \end{aligned}$$

Now Equation A.2 can be used to relate the transverse and spin inertias of the gimbal.

$$2I_{g_i} - I_{g_x} = \frac{F_m}{I_{r_x}} \quad (\text{A.3})$$

Assume that the gimbal inertia is at least ten times smaller than the rotor inertia. If this is true, then

$$I_{g_x} = 7.00 \times 10^{-7} \text{ kg m}^2$$

This value can then be used in Equation A.3 to find

$$I_{g_i} = 3.646 \times 10^{-7} \text{ kg m}^2$$

The tuning condition, introduced in Chapter 3, can now be used to find the stiffness of the flexures. From Equation 3.10

$$\omega_s^2 = \frac{2K_x + 2K_y}{2I_{g_i} - I_{g_x}}$$

Specification	Value
$I_{r,}$	$7.000 \times 10^{-6} \text{ kg m}^2$
$I_{r,}$	$4.050 \times 10^{-6} \text{ kg m}^2$
$I_{g,}$	$7.000 \times 10^{-7} \text{ kg m}^2$
$I_{g,}$	$3.646 \times 10^{-7} \text{ kg m}^2$
K_x	$3.204 \times 10^{-3} \text{ Nm}$
K_y	$3.204 \times 10^{-3} \text{ Nm}$
D_x	$0.000 \times 10^0 \text{ Nms}$
D_y	$0.000 \times 10^0 \text{ Nms}$
ω_s	6330.0 rpm

Table A.2: Litton CSG-2 Gyro Specifications

where ω_s is the spin speed of the rotor.

$$2K_x + 2K_y = 1.28174 \times 10^{-2}$$

Assume that $K_x = K_y$. Therefore

$$4K_x = 1.28174 \times 10^{-2}$$

$$K_x = K_y = 3.204 \times 10^{-3} \text{ Nm}$$

For simplicity, a case where no damping exists will be assumed. The specifications for the Litton CSG-2 gyro used in the experiment are summarized in Table A.2.

Appendix B

Calculation of Simulation Drift Rates

The output from the computer simulation is in the form of angular displacement and velocity versus time. To compare this output with the measured drift rates from the test of the CSG-2 gyro it is necessary to calculate the drift from the angular displacement. The angular velocity cannot be used because it is the instantaneous velocity and is mainly comprised of the nutation of the gyro with a small effect by the precession.

The measured drift from the CSG-2 gyro is based on the positional change of the gyro over the time period. Thus the average velocity, based on the precession rather than the nutation of the rotor, is needed.

Figures B.1 through B.5 show the calculated angular position for a time, t , for the case where the initial offset of the rotor is about the y-axis. The nutation is still apparent in these plots and so it is necessary to find a best fit line and then continue the calculation from there. The mathematical package, MATLAB, was used to fit a third order approximation to the data. The results are shown in Figures B.6 and B.7. The first of the two has the x-axis drift versus time, the second, the y-axis drift. From these plots it is possible to find the angular position at 4.77 seconds, the approximate time step size for the experimental drift calculations. This can be used to calculate the difference in angular position between $t = 0s$ and $t = 4.77s$. The result is the drift rate of the gyro. Before the results are able to be compared, the above must be converted to the same units as the experimental results. The results can be seen in Figure 4.6.

A similar analysis of the output for the case with an initial displacement of the rotor about the x-axis can be performed. The output from the numeric simulation can be seen in Figures B.8 through B.12. A third order approximation of the data can be seen in Figures B.13 and B.14. Finally, the drift rates are shown in Figure 4.7.

Figure B.1: Rotor angular position versus time for the simulated CSG-2 DTG and $\theta'_0 = -1.858$ rad.

Figure B.2: Rotor angular position versus time for the simulated CSG-2 DTG and $\theta'_0 = -0.527$ rad.

Figure B.3: Rotor angular position versus time for the simulated CSG-2 DTG and $\theta'_0 = 0.000$ rad.

Figure B.4: Rotor angular position versus time for the simulated CSG-2 DTG and $\theta'_0 = 0.527$ rad.

Figure B.5: Rotor angular position versus time for the simulated CSG-2 DTG and $\theta'_0 = 1.850$ rad.

Figure B.6: Third order approximation of rotor angular position versus time for an initial offset about the y-axis.

Figure B.7: Third order approximation of rotor angular position versus time for an initial offset about the y-axis.

Figure B.8: Rotor angular position versus time for the simulated CSG-2 DTG and $\phi'_0 = -1.854$ rad.

Figure B.9: Rotor angular position versus time for the simulated CSG-2 DTG and $\phi'_0 = -0.527$ rad.

Figure B.10: Rotor angular position versus time for the simulated CSG-2 DTG and $\phi'_0 = 0.000$ rad.

Figure B.11: Rotor angular position versus time for the simulated CSG-2 DTG and $\phi'_0 = 0.527$ rad.

Figure B.12: Rotor angular position versus time for the simulated CSG-2 DTG and $\phi'_0 = 1.846$ rad.

Figure B.13: Third order approximation of rotor angular position versus time for an initial offset about the x-axis.

Figure B.14: Third order approximation of rotor angular position versus time for an initial offset about the y-axis.

Appendix C

Fortran Source Code for RSIM

The following is a copy of the Fortran 77 source code for the computer program RSIM.

```

C   ROTR2.FOR   -   LAST MODIFIED   March 20, 1996
C   By:         Jeffrey S. Cain

C   THIS PROGRAM RECEIVES THE SPECIFICATIONS FOR A SPECIFIC GYRO
C   AND, USING THE USER DEFINED INITIAL CONDITIONS, CALCULATES THE
C   TIME RESPONSE OF THE GYRO.  THE INFORMATION IS OUTPUT INTO FOUR
C   FILES:
C   <filename>.ang - CASE BASED ANGULAR POSITIONS AND RATES
C   OUTPUT FORM: T G-S(X) ANGLE G-R(Y) ANGLE G-S RATE G-R RATE
C   <filename>.mom - CASE BASED ROTOR ANGULAR MOMENTUM VECTOR
C   OUTPUT FORM: T HX HY HZ
C   <filename>.rea - SHAFT BASED REACTION FORCES
C   OUTPUT FORM: T REACTX REACTY REACTZ

C   REQUIRED FILES:  rotr2.for, intr2.for, rkf45m.for, datan4.for,
C                   scrptfile.for

C   THIS SUBROUTINE IMPLEMENTS THE NON-LINEAR EQUATIONS OF A
C   GYROSCOPIC ROTOR AND CALCULATES THE DERIVATIVES OF THE
C   ANGLES AND ANGULAR RATES.
C   CALLED FROM: RKF45 AND CALREA

      SUBROUTINE ROTOR(T,Y,YP)
      DOUBLE PRECISION T, Y(4), YP(4)
      DOUBLE PRECISION A, B, C, D, E, F, DENOM
      DOUBLE PRECISION S1, C1, S2, C2
      DOUBLE PRECISION DSIN, DCOS
      DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      DOUBLE PRECISION KX, KY, DX, DY, RADIUS
      DOUBLE PRECISION WS, PERIOD
      DOUBLE PRECISION PIE
      INTEGER I/O/

      COMMON /RSPC/ IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      COMMON /GSPC/ IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      COMMON /SANDD/ KX, KY, DX, DY, RADIUS
      COMMON /ROTAR/ WS, PERIOD
      COMMON /PIE/ PIE

C   FOR THE FOLLOWING CALCULATIONS Y(I) MUST BE IN RADIANS
C   THEREFORE THE VALUES MUST BE CONVERTED FROM ARCSECONDS
C   TO RADIANS
      DO 199 I = 1,4
        Y(I) = Y(I) * ( PIE / 648000.0D0 )
199  CONTINUE

C   BY DEFINITION THE FIRST DERIVATIVES OF THE ANGLES ARE EQUIVALENT
C   TO THE ANGULAR RATES
      YP(1) = Y(3)
      YP(2) = Y(4)

C   PRE CALCULATE THE SINES AND COSINES OF THE ANGLES TO FACILITATE
C   COMPUTATION
      S1 = DSIN(Y(1))
      S2 = DSIN(Y(2))
      C1 = DCOS(Y(1))
      C2 = DCOS(Y(2))

C   THE EQUATIONS HAVE BEEN BROKEN DOWN INTO

```

```

C      COMPONENTS - SEE CHAPTER 3 OF THESIS BY JEFFREY CAIN FOR
C      DESCRIPTION AND DERIVATION OF EQUATIONS
      A = IRX * C2**2 + IRZ * S2**2 - 2 * IRXZ * S2 * C2 + IGX
      B = IRXY * C2 + IRYZ * S2
      C = - WS**2 * C1 * ( IRYZ * C2 - IRXY * S2 + IGYZ )
1      + WS * C1 * ( WS * S1 * ( (IRX - IRZ) * C2**2 -
2      2 * IRXZ * S2 * C2 +
3      IGY - IGZ - IRX + IRY )
4      + Y(4) * ( 2 * (IRX - IRZ) * C2**2 -
5      4 * IRXZ * S2 * C2 - IRX +
6      IRY + IRZ ) )
7      + WS**2 * S1**2 * (IRYZ * C2 - IRXY * S2 + IGYZ)
8      - 2 * Y(4) * WS * S1 * (IRXY * S2 - IRYZ * C2)
9      + 4 * Y(3) * Y(4) * C2**2 * IRXZ - Y(4)**2 * IRXY * S2
8      + C2 * ( 2 * Y(3) * Y(4) * (IRX - IRZ) * S2 +
7      Y(4)**2 * IRYZ )
6      - 2 * ( Y(3) * Y(4) * IRXZ + KX * Y(1) + DX * Y(3) )

      D = IRY
      E = C2 * IRXY + S2 * IRYZ
      F = C1**2 * ( 2 * IRXZ * WS**2 * C2**2 +
1      WS**2 * (IRX - IRZ) * S2 * C2 - IRXZ * WS**2 )
2      + C1 * ( S1 * ( IRXY * WS**2 * C2 + IRYZ * WS**2 * S2 ) +
3      Y(3) * ( 2 * WS * (IRZ - IRX) * C2**2 +
4      4 * IRXZ * WS * S2 * C2 +
5      WS * (IRX - IRY - IRZ) ) )
6      + S1 * ( 2 * Y(3) * WS * ( IRXY * S2 - IRYZ * C2 ) )
7      - 2 * Y(3)**2 * IRXZ * C2**2 + Y(3)**2 * IRXZ
8      - C2 * Y(3)**2 * (IRX - IRZ) * S2
9      - 2 * ( KY * Y(2) + DY * Y(4) )

      DENOM = A * D - B * E

      YP(3) = ( B * F + C * D ) / DENOM
      YP(4) = ( E * C + A * F ) / DENOM
C      CONVERT THE VALUES OF YP AND Y TO ARCSECONDS AGAIN
      DO 200 I = 1,4
          Y(I) = Y(I) * ( 648000 / PIE )
          YP(I) = YP(I) * (648000 / PIE )
200      CONTINUE

      RETURN
      END

C      THIS IS THE MAIN PROGRAM

*$noextensions
      EXTERNAL ROTOR
      DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      DOUBLE PRECISION KX, KY, DX, DY, RADIUS
      DOUBLE PRECISION WS, PERIOD
      DOUBLE PRECISION Y(4), Z(4), G
      DOUBLE PRECISION PIE
      DOUBLE PRECISION H(3)
      DOUBLE PRECISION T, TOUT, RELERR/1.0D-10/
      DOUBLE PRECISION ABSERR/0.000000000000D0/
      DOUBLE PRECISION TFINAL, TPRINT, WORK(27)

```

```

DOUBLE PRECISION TSTART
DOUBLE PRECISION REAC(3)
INTEGER IWORK(5), IFLAG, NEQN/4/, REPE, FLAG
INTEGER I/O/, J, NUMSTEPS
CHARACTER*8 FNAME
COMMON /RSPC/   IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
COMMON /GSPC/   IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
COMMON /SANDD/  KX, KY, DX, DY, RADIUS
COMMON /ROTAR/  WS, PERIOD
COMMON /ANGLE/  Y, Z, G
COMMON /PIE/    PIE
COMMON /ANGMOM/ H
COMMON /FILEN/  FNAME
COMMON /FLG/    FLAG
COMMON /REACT/  REAC
COMMON /COUNT/ NUMSTEPS

C      FIRST INITIALIZE ALL THE VARIABLES TO ZERO
      CALL INITVAL(T,TFINAL,REPE)

C      INTRO CONTAINS THE USER INTERFACE SUBROUTINES THEREFORE
C      THIS ROUTINE MUST BE CALLED FIRST AND THEN REFERENCED
C      AT THE END OF EVERY SIMULATION
      NUMSTEPS = 1
      CALL INTRO(T,TFINAL,REPE)
      J = NUMSTEPS - 1

C      CONVERT THE SHAFT SPIN VELOCITY FROM RPM TO RAD/SEC
      WS = WS * 2.0D0 * PIE / 60.0D0

C      SET A FLAG FOR THE RUNGE-KUTTA ROUTINE
      IFLAG = 1

C      DEFINE THE CURRENT AND STARTING TIMES FOR THE INITIAL CALL
C      OF THE RUNGE-KUTTA ROUTINE
      TSTART = T
      TOUT = T

C      CONVERT THE INITIAL CONDITIONS FROM A 'CASE' ORIENTED
C      SYSTEM TO A 'GIMBAL' ORIENTED ONE
      CALL CONGRA(T)

C      OPEN SEVERAL FILES TO SAVE THE SIMULATION DATA IN
      CALL OPENFILE(FNAME)

C      CONVERT THE INITIAL CONDITION ANGLES TO ARCSECONDS FROM
C      RADIANS. THIS INCREASES THE MAGNITUDE OF THE NUMBERS
C      AND LIMITS THE NUMBER OF ERRORS CAUSED BY THE USE OF
C      SMALL NUMBERS.
      DO 500 I = 1,4
         Y(I) = Y(I) * ( 648000 / PIE )
500    CONTINUE

C      CALL THE RUNGE-KUTTA ROUTINE AND SOLVE THE SET OF DIFERENTIAL
C      EQUATIONS NUMERICALLY
      CALL RKF45(ROTOR,NEQN,Y,T,TOUT,RELERR,ABSERR,IFLAG,WORK,IWORK)

C      CONVERT THE RESULTANT ANGLES FROM ARCSECONDS TO RADIANS SO THAT
C      THEY CAN BE USED IN THE FOLLOWING COVERSIONS AND CALCULATIONS.
      DO 501 I = 1,4
         Y(I) = Y(I) * ( PIE / 648000 )
501    CONTINUE

```

```

501  CONTINUE
C    CALCULATE THE PERIOD OF THE SHAFT ROTATION
      PERIOD = 2.0D0 * PIE / WS
C    BASE THE STEP SIZE FOR THE NUMERICAL INTEGRATION ON THE
C    PERIOD OF SHAFT ROTATION
      TPRINT = PERIOD / 100
      J = J + 1
      IF ( J .LT. NUMSTEPS ) GO TO 509
      J = 0
C    CONVERT THE OUTPUT RESULTS FROM 'GIMBAL' TO 'CASE'
      CALL CONCRA(T)
C    CALCULATE THE ANGULAR MOMENTUM OF THE ROTOR AT TIME T
      CALL CALMOM(T)
C    CALCULATE THE MOMENTS/TORQUES AND FORCES EXPERIENCED
C    BY THE SHAFT DUE TO THE GIMBAL OSCILLATION AND OTHER
C    DYNAMIC EFFECTS
      CALL CALREA(T)
C    WRITE(*,*) REAL(T), REAL(Y(3)), REAL(Y(4))
C    OUTPUT THE 'CASE' ANGLES TO FILE '.ANG'
      WRITE(9,11) REAL(T), REAL(Z(1)), REAL(Z(2)),
1      REAL(Z(3)), REAL(Z(4))
C    OUTPUT THE ANGULAR MOMENTUM OF THE ROTOR
C    TO FILE '.MOM'
      WRITE(10,12) REAL(T), REAL(H(1)), REAL(H(2)), REAL(H(3))
C    OUTPUT THE SHAFT FORCES TO FILE '.REA'
      WRITE(11,12) REAL(T), REAL(REAC(1)), REAL(REAC(2)),
1      REAL(REAC(3))
C    SINCE IFLAG IS A WARNING VARIABLE IN RKF45M, USE THE OUTPUT
C    TO DETECT ANY PROBLEMS. SEE RKF45M.FOR FOR A DETAILED
C    EXPLANATION OF THE FLAG INTERPRETATION.
509  GO TO (80,20,30,40,50,60,70,80), IFLAG
C    NO PROBLEMS - ADD ONE TIME STEP TO OUTPUT TIME AND RECALCULATE
20   TOUT = T + TPRINT
      IF (T + 0.00001D0 .LT. TFINAL) GO TO 10
C    CLOSE ANY OPEN FILES
      CLOSE(9)
      CLOSE(10)
      CLOSE(11)
      GO TO 5
30   WRITE(6,31) RELERR,ABSERR
      GO TO 10
C    THE NUMBER OF CALCULATIONS
C    USED TO GET THE REQUIRED ACCURACY IS NOT THAT IMPORTANT.
C    USER IS NOT REQUIRED TO KNOW THAT THIS ERROR HAS OCCURED.
C    THE PROGRAM WILL ACKNOWLEDGE THE PROBLEM AND THEN CONTINUE.
C40  WRITE(6,41)
40   J = J - 1
      GO TO 10
50   ABSERR = 1.0D-9
      WRITE(6,31) RELERR,ABSERR

```



```

        GO TO 10
60      RELERR = 10.0DO * RELERR
        WRITE(6,31) RELERR,ABSERR
        IFLAG = 2
        GO TO 10
C      AGAIN, TOO MUCH OUTPUT IS NOT THAT CRITICAL EITHER
C70     WRITE(6,71)
70      J = J - 1
        IFLAG = 2
        GO TO 10
80      WRITE(6,81)
        STOP
11      FORMAT(F10.5,4E16.8)
12      FORMAT(F10.5,3E16.8)
31      FORMAT(17H TOLERANCES RESET, 2E12.3)
41      FORMAT(11H MANY STEPS)
71      FORMAT(12H MUCH OUTPUT)
81      FORMAT(14H IMPROPER CALL)
        END

C      THIS SUBROUTINE CONVERTS ANGLES FROM 'CASE' COORDINATE
C      SYSTEMS TO 'GIMBAL' COORDINATE SYSTEMS
        SUBROUTINE CONGRA(T)
        DOUBLE PRECISION T
        DOUBLE PRECISION WS, PERIOD
        DOUBLE PRECISION Y(4), Z(4), G
        DOUBLE PRECISION NUM, DENOM
        DOUBLE PRECISION DCOS, DSIN, DASIN, DATAN4
        COMMON /ROTAR/ WS, PERIOD
        COMMON /ANGLE/ Y, Z, G
C      CALCULATE GAMMA - TO BE USED LATER
C      DATAN4 IS USED SO THAT THE RESULT WILL BE IN THE RANGE
C      OF 0 TO 2*PI
        NUM = DSIN(WS*T) * DCOS(Z(1))
        DENOM = DCOS(WS*T) * DCOS(Z(2)) + DSIN(Z(1)) * DSIN(Z(2)) *
1          DSIN(WS*T)
        G = DATAN4(NUM,DENOM)
C      CONVERT THE ANGLES AND ANGULAR RATES - THESE EQUATIONS ARE BASED
C      ON THOSE DERIVED IN CHAPTER 2.
        Y(1) = DASIN( DSIN(G) * DCOS(Z(1)) * DSIN(Z(2)) +
1          DCOS(G) * DSIN(Z(1)) )
        Y(2) = DASIN( DSIN(Z(2)) * DCOS(WS*T) -
1          DCOS(Z(2)) * DSIN(Z(1)) * DSIN(WS*T) )
        Y(3) = ( DCOS(G) * DCOS(Z(2)) * Z(3) + DSIN(G) * Z(4) +
1          DSIN(Y(2)) * DCOS(Y(1)) * WS ) / DCOS(Y(2))
        Y(4) = - DSIN(G) * DCOS(Z(2)) * Z(3) + DCOS(G) * Z(4) -
2          DSIN(Y(1)) * WS
        RETURN
        END

C      THIS SUBROUTINE CONVERTS ANGLES FROM 'GIMBAL' COORDINATE
C      SYSTEMS TO 'CASE' COORDINATE SYSTEMS
        SUBROUTINE CONCRA(T)
        DOUBLE PRECISION T

```

```

      DOUBLE PRECISION T
      DOUBLE PRECISION WS, PERIOD
      DOUBLE PRECISION Y(4), Z(4), G
      DOUBLE PRECISION NUM, DENOM
      DOUBLE PRECISION DCOS, DSIN, DASIN, DATAN4
      COMMON /ROTAR/ WS, PERIOD
      COMMON /ANGLE/ Y, Z, G
C     CONVERT THE ANGLES USING EQUATIONS DERIVED IN CHAPTER 2.
      Z(2) = DASIN( DCOS(WS*T) * DSIN(Y(2)) +
1              DSIN(WS*T) * DSIN(Y(1)) * DCOS(Y(2)) )
      Z(1) = DASIN( ( DCOS(WS*T) * DSIN(Y(1)) * DCOS(Y(2)) -
1              DSIN(WS*T) * DSIN(Y(2)) ) /DCOS(Z(2)) )
C     FIND GAMMA FOR USE IN THE CONVERSION OF THE ANGULAR RATES
C     AGAIN DATAN4 IS USED
      NUM = DSIN(WS*T) * DCOS(Z(1))
      DENOM = DCOS(WS*T) * DCOS(Z(2)) + DSIN(Z(1)) * DSIN(Z(2)) *
1      DSIN(WS*T)
      G = DATAN4(NUM,DENOM)
C     CONVERT THE ANGULAR RATES
      Z(3) = ( DCOS(G) * ( DCOS(Y(2)) * Y(3) -
1              DSIN(Y(2)) * DCOS(Y(1)) * WS ) -
2      DSIN(G) * ( Y(4) + DSIN(Y(1)) * WS ) ) / DCOS(Z(2))
      Z(4) = DSIN(G) * DCOS(Y(2)) * Y(3) + DCOS(G) * Y(4) -
1      ( DSIN(G) * DSIN(Y(2)) * DCOS(Y(1)) -
2      DCOS(G) * DSIN(Y(1)) ) * WS
      RETURN
      END
C     CALCULATE THE ANGULAR MOMENTUM OF THE ROTOR.
C     THIS MOMENTUM IS THEN OUTPUT IN THE 'CASE' COORDINATE SYSTEM.
      SUBROUTINE CALMOM(T)
      DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      DOUBLE PRECISION KX, KY, DX, DY, RADIUS
      DOUBLE PRECISION WS, PERIOD
      DOUBLE PRECISION Y(4), Z(4), G
      DOUBLE PRECISION H(3)
      DOUBLE PRECISION HTEMP(3),HT(3),T
      DOUBLE PRECISION DCOS,DSIN

      COMMON /RSPC/ IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      COMMON /GSPC/ IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      COMMON /SANDD/ KX, KY, DX, DY, RADIUS
      COMMON /ROTAR/ WS, PERIOD
      COMMON /ANGLE/ Y, Z, G
      COMMON /ANGMOM/ H
C     CALCULATE THE ANGULAR VELOCITIES OF THE ROTOR.
C     SEE CHAPTER 3.
      WRX = DCOS(Y(2)) * Y(3) - DSIN(Y(2)) * DCOS(Y(1)) * WS
      WRY = Y(4) + DSIN(Y(1)) * WS
      WRZ = DSIN(Y(2)) * Y(3) + DCOS(Y(2)) * DCOS(Y(1)) * WS
C     CALCULATE THE ANGULAR MOMENTUM OF THE ROTOR. THIS IS
C     BASED ON EULER'S EQUATIONS
      HTEMP(1) = IRX*WRX - IRXY*WRY - IRXZ*WRZ
      HTEMP(2) = IRY*WRY - IRXY*WRX - IRYZ*WRZ

```

```

C      CONVERT THE ANGULAR MOMENTUM CALCULATED ABOVE FROM THE
C      'GIMBAL' ORIENTED SYSTEM TO THE 'CASE' ORIENTED ONE.
      HT(1) = DCOS(Y(2)) * HTEMP(1) + DSIN(Y(2)) * HTEMP(3)
      HT(2) = DSIN(Y(1)) * DSIN(Y(2)) * HTEMP(1) +
1         DCOS(Y(1)) * HTEMP(2) -
2         DSIN(Y(1)) * DCOS(Y(2)) * HTEMP(3)
      HT(3) = - DCOS(Y(1)) * DSIN(Y(2)) * HTEMP(1) +
1         DSIN(Y(1)) * HTEMP(2) +
2         DCOS(Y(1)) * DCOS(Y(2)) * HTEMP(3)
      H(1) = DCOS(WS*T) * HT(1) - DSIN(WS*T) * HT(2)
      H(2) = DSIN(WS*T) * HT(1) + DCOS(WS*T) * HT(2)
      H(3) = HT(3)
      RETURN
      END

C      THIS SUBROUTINE CALCULATES THE THE REACTION FORCES ON THE
C      SHAFT DUE TO THE DYNAMICS OF THE ROTOR AND GIMBAL
      SUBROUTINE CALREA(T)

      DOUBLE PRECISION WRX, WRY, WRZ, DWRX, DWRY, DWRZ
      DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      DOUBLE PRECISION KX, KY, DX, DY, RADIUS
      DOUBLE PRECISION WS, PERIOD
      DOUBLE PRECISION Y(4), Z(4), G, DERY(4)
      DOUBLE PRECISION REAC(3)
      DOUBLE PRECISION HG(3), HR(3), DHG(3), DHR(3)
      DOUBLE PRECISION MG(3), MR(3), GF(3)
      DOUBLE PRECISION DSIN, DCOS

      COMMON /RSPC/ IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      COMMON /GSPC/ IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      COMMON /SANDD/ KX, KY, DX, DY, RADIUS
      COMMON /ROTAR/ WS, PERIOD
      COMMON /ANGLE/ Y, Z, G
      COMMON /REACT/ REAC

C      CALCULATE THE ANGULAR ACCELERATIONS OF THE SYSTEM IN GIMBAL FIXED
C      COORDINATES.
      CALL ROTOR(T,Y,DERY)

C      CALCULATE THE ANGULAR VELOCITIES AND ACCELERATIONS EXPERIENCED BY
C      THE GIMBAL AND THE ROTOR RESPECTIVELY
      WGX = Y(3)
      WGY = DSIN(Y(1)) * WS
      WGZ = DCOS(Y(1)) * WS
      DWGX = DERY(3)
      DWGY = Y(3) * DCOS(Y(1)) * WS
      DWGZ = - Y(3) * DSIN(Y(1)) * WS
      WRX = DCOS(Y(2)) * Y(3) - DCOS(Y(2)) * DCOS(Y(1)) * WS
      WRY = Y(4) + DSIN(Y(1)) * WS
      WRZ = DSIN(Y(2)) * Y(3) + DCOS(Y(2)) * DCOS(Y(1)) * WS
      DWRX = - DSIN(Y(2)) * Y(4) * Y(3) + DCOS(Y(2)) * DERY(3)
1         - DCOS(Y(2)) * DCOS(Y(1)) * Y(4) * WS
2         + DSIN(Y(1)) * DSIN(Y(2)) * Y(3) * WS
      DWRY = DCOS(Y(1)) * Y(3) * WS + DERY(4)

```

```

      2      + DSIN(Y(1)) * DSIN(Y(2)) * Y(3) * WS
      DWRV = DCOS(Y(1)) * Y(3) * WS + DERY(4)
      DWRZ = DCOS(Y(2)) * Y(4) * Y(3) + DSIN(Y(2)) * DERY(3)
      1      - DSIN(Y(2)) * DCOS(Y(1)) * Y(4) * WS
      2      - DCOS(Y(2)) * DSIN(Y(1)) * Y(3) * WS
C      CALCULATE THE MOMENT VECTORS FOR THE ROTOR AND THE GIMBAL
C      TO DO THIS - FIRST CALCULATE THE ANGULAR MOMENTUM VECTOR FOR
C      THE GIMBAL
      HG(1) = IGX * WGX - IGXY * WGY - IGXZ * WGZ
      HG(2) = IGY * WGY - IGXY * WGX - IGYZ * WGZ
      HG(3) = IGZ * WGZ - IGXZ * WGX - IGYZ * WGY
C      NOW CALCULATE THE ANGULAR MOMENTUM VECTOR FOR THE ROTOR
      HR(1) = IRX * WRX - IRXY * WRY - IRXZ * WRZ
      HR(2) = IRY * WRY - IRXY * WRX - IRYZ * WRZ
      HR(3) = IRZ * WRZ - IRXZ * WRX - IRYZ * WRY
C      NOW THE DERIVATIVES OF THE ANGULAR MOMENTUM VECTORS CAN BE
C      CALCULATED
      DHG(1) = IGX * DWGX - IGXY * DWGY - IGXZ * DWGZ
      DHG(2) = IGY * DWGY - IGXY * DWGX - IGYZ * DWGZ
      DHG(3) = IGZ * DWGZ - IGXZ * DWGX - IGYZ * DWGY
      DHR(1) = IRX * DWRX - IRXY * DWRV - IRXZ * DWRZ
C      DHR(2) = IRY * DWRV - IRXY * DWRX - IRYZ * DWRZ
      DHR(3) = IRZ * DWRZ - IRXZ * DWRX - IRYZ * DWRV
C      FINALLY THE MOMENTUM VECTOR CAN BE CALCULATED
      MG(1) = DHG(1) - HG(2) * WGZ + HG(3) * WGY
      MG(2) = DHG(2) - HG(3) * WGX + HG(1) * WGZ
      MG(3) = DHG(3) - HG(1) * WGY + HG(2) * WGX
      MR(1) = DHR(1) - HR(2) * WRZ + HR(3) * WRY
C      MR(2) = DHR(2) - HR(3) * WRX + HR(1) * WRZ
      MR(3) = DHR(3) - HR(1) * WRY + HR(2) * WRX
C      THE MOMENTS CAUSED BY THE FLEXURE FORCES ARE
      GF(1) = 0.0D0
      GF(2) = MG(3) + MR(3) * DCOS(Y(2)) - MR(1) * DSIN(Y(2))
      GF(3) = MG(2) - 2 * ( KY * Y(2) + DY * Y(4) )
C      ROTATING THESE MOMENTS INTO THE SHAFT COORDINATE FRAME AND
C      ADD IN THE TORSION FORCES DUE TO THE SPRINGS AND DAMPING
      REAC(1) = 2 * ( KX * Y(1) + DX * Y(3) )
      REAC(2) = GF(3) * DCOS(Y(1)) - GF(2) * DSIN(Y(1))
      REAC(3) = GF(3) * DSIN(Y(1)) + GF(2) * DCOS(Y(1))
      RETURN
      END

C$include intr3.for
C$include rkf45m.for
C$include datan4.for
C$include scrptfile.for
      INCLUDE 'intr3.for'
      INCLUDE 'rkf45m.for'
      INCLUDE 'datan4.for'
      INCLUDE 'scrptfile.for'

C      THIS SUBROUTINE INITIALIZES MOST OF THE VARIABLES
C      TO ZERO.

```

```

SUBROUTINE INITVAL(T,TFINAL,REPE)
  DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
  DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
  DOUBLE PRECISION KX, KY, DX, DY, RADIUS
  DOUBLE PRECISION WS, PERIOD
  DOUBLE PRECISION Y(4), Z(4), G
  DOUBLE PRECISION H(3)
  DOUBLE PRECISION REAC(3)
  DOUBLE PRECISION T, TFINAL
  INTEGER I, REPE

  COMMON /RSPC/   IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
  COMMON /GSPC/   IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
  COMMON /SANDD/  KX, KY, DX, DY, RADIUS
  COMMON /ROTAR/  WS, PERIOD
  COMMON /ANGLE/  Y, Z, G
  COMMON /ANGMOM/ H
  COMMON /REACT/  REAC

  IRX   = 0.0D0
  IRY   = 0.0D0
  IRZ   = 0.0D0
  IRXY  = 0.0D0
  IRXZ  = 0.0D0
  IRYZ  = 0.0D0
  IGX   = 0.0D0
  IGY   = 0.0D0
  IGZ   = 0.0D0
  IGXY  = 0.0D0
  IGXZ  = 0.0D0
  IGYZ  = 0.0D0
  KX    = 0.0D0
  KY    = 0.0D0
  WS    = 0.0D0
  PERIOD = 0.0D0
  RADIUS = 0.0D0
  DO 2 I = 1,4
    Y(I) = 0.0D0
    Z(I) = 0.0D0
    IF (I .LE. 3) THEN
      H(I) = 0.0D0
      REAC(I) = 0.0D0
    END IF
  CONTINUE
  DX = 0.0D0
  DY = 0.0D0
  G = 0.0D0
  T = 0.0D0
  TFINAL = 0.0D0
  REPE = 0
  RETURN
END

C      OPEN THE THREE REQUIRED OUTPUT FILES USING EITHER THE
C      DEFAULT FILE NAME OR THE USER INPUT ONE.
SUBROUTINE OPENFILE(FNAME)
  CHARACTER*8 FNAME
  CHARACTER*12 FILENAME
  INTEGER R/9/, I

  DO 500 I = 8,1,-1
    IF (FNAME(I:I) .EQ. ' ') R = I
500  CONTINUE

```

```
FILENAME(1:12) = '          '  
FILENAME(1:R-1) = FNAME  
FILENAME(R:R+3) = '.ang'  
OPEN(9,FILE=FILENAME,STATUS='UNKNOWN',ACCESS='SEQUENTIAL')  
FILENAME(R:R+3) = '.mom'  
OPEN(10,FILE=FILENAME,STATUS='UNKNOWN',ACCESS='SEQUENTIAL')  
FILENAME(R:R+3) = '.rea'  
OPEN(11,FILE=FILENAME,STATUS='UNKNOWN',ACCESS='SEQUENTIAL')  
RETURN  
END
```

```
C THIS SUBROUTINE CAN BE CONSIDERED AS A SECONDARY
C MAIN PROGRAM. ROTR2 CALLS THIS ROUTINE FOR ALL USER
C INTERFACES. THIS SUBROUTINE THEN OUTPUTS AN INTRODUCTION
SUBROUTINE INTRO(T,TFINAL,REPE)
DOUBLE PRECISION T, TFINAL
DOUBLE PRECISION WS, PERIOD
DOUBLE PRECISION PIE, DATAN
INTEGER CHOICE, REPE, FLAG
CHARACTER*8 FNAME
CHARACTER*12 SFILE
LOGICAL SFLAG

COMMON /PIE/      PIE
COMMON /ROTAR/    WS, PERIOD
COMMON /FILEN/    FNAME
COMMON /FLG/      FLAG
COMMON /SCRIPT/   SFLAG

CALCULATE PI AS ACCURATELY AS POSSIBLE WITH THE
CURRENT MACHINE
PIE = 4.0D0 * DATAN(1.0D0)

CONVERT THE SHAFT SPIN VELOCITY FROM RADIAN/SEC
TO RPM
WS = WS * 60.0D0 / ( 2.0D0 * PIE )

IF THE USER HAS ALREADY SEEN THE INTRODUCTION
THEN SKIP IT
IF (REPE.GT.0) GO TO 103
WRITE THE INTRODUCTION TO THE SCREEN
WRITE (*,10)
FORMAT(/,/,,/,/,/,/,/,/,/,/,/,/)
1  /,' *****'
2  /,' '*
2  /,' '* "RSIM" '*'
3  /,' '* A Non-Linear Rotor Simulation '*'
4  /,' '* '*'
5  /,' '* by '*'
6  /,' '* Jeffrey S. Cain '*'
7  /,' '* March 1996 '*'
8  /,' '* '*'
9  /,' '* This program simulates a two axis non-linear '*'
8  /,' '* gyroscope or momentum wheel. The inputs '*'
7  /,' '* include the start and end times of the '*'
6  /,' '* simulation. It also requires rotor and '*'
5  /,' '* gimbal specifications, and initial conditions '*'
4  /,' '* Symmetric and unsymmetric cases can be '*'
3  /,' '* considered with this program. '*'
2  /,' '* '*'
1  /,' *****')

SET THE FLAG TO INDICATE THAT THE PROGRAM IS OPERATING UNDER
A SCRIPT FILE COMMAND SYSTEM TO FALSE
SFLAG = .FALSE.

OUTPUT INTRODUCTION
WRITE(*,20)
FORMAT('Press enter to continue, or input the script file name.')
20
```

```

      READ(*,'(A)') SFILE
C     FLAG TRACKS WHETHER OR NOT THE USER IS USING SYMMETRIC
C     OR UNSYMMETRIC MODE
      FLAG = 1
C     REPE IS USED AS A COUNTER FOR THE DEFAULT OUTPUT FILE NAMES
C     AS WELL AS A CHECK TO SEE IF THE USER HAS ALREADY SEEN THE
C     INTRO SCREEN
103    REPE = REPE + 1
C     AUTOMATICALLY INCREMENT THE FILENAME SO THAT THE NEXT RUN
C     WILL NOT OVERWRITE THE PREVIOUS RUN'S DATA
      WRITE(FNAME,500) REPE
500    FORMAT('rotor',I3.3)
C     IF THE USER HAS ENTERED A SCRIPT FILE NAME THEN PUT THE
C     PROGRAM UNDER ITS CONTROL
      IF (SFILE(1:1) .NE. ' ') THEN
          CALL SCRPTFLE(SFILE,T,TFINAL)
          RETURN
      END IF

C     THE PROGRAM WILL BE USER CONTROLLED SO CONTINUE
C     GO TO THE APPROPRIATE MENU DEPENDING ON THE TYPE OF
C     MODEL
105    IF (FLAG.EQ.1) CALL MENU(T,TFINAL,CHOICE)
      IF (FLAG.EQ.2) CALL MENU2(T,TFINAL,CHOICE)
C     THE USER HAS CHOSEN AN OPTION - GO TO THE
C     APPROPRIATE SUBROUTINE
      IF ((CHOICE.LT.1).OR.(CHOICE.GT.10)) GO TO 105
      GO TO (200,210,220,230,240,250,260,270,280,290) CHOICE
C     MISCELLANEOUS PROGRAM FEATURES NOT COVERED BY ANY OF THE
C     OTHER SUBROUTINES
200    CALL OPTIONS(FLAG)
      GO TO 105
210    RETURN
C     LOAD A PREVIOUSLY SAVED RUN - GYRO SPECS AND INITIAL COND.
220    CALL LORUN(T,TFINAL)
      GO TO 105
C     SAVE THE CUURENT GYRO SPECS AND INITIAL CONDITIONS
230    CALL SAVRUN(T,TFINAL)
      GO TO 105
C     LOAD A PREVIOUSLY SAVED GYRO - SPECIFICATIONS ONLY
240    CALL LOSPC
      GO TO 105
C     SAVE THE CURRENT GYRO SPECIFICATIONS ONLY
250    CALL SAVSPC
      GO TO 105
C     INPUT NEW GYRO SPECIFICATIONS - INSPC - SYMMETRIC
C                                     INSPC2 - UNSYMMETRIC
260    IF (FLAG.EQ.1) THEN
          CALL INSPC
      ELSE
          CALL INSPC2
      END IF
      GO TO 105
C     INPUT NEW INITIAL CONDITIONS

```



```

270 CALL INPINIT  
GO TO 105  
C INPUT NEW START AND END TIMES FOR THE SIMULATION  
280 CALL INPTIME(T,TFINAL)  
GO TO 105  
290 STOP  
END  
  
C THIS IS A USER MENU FOR USE WITH SYMMETRIC SIMULATIONS  
C NOTE THAT THE DATA IS RETAINED EVEN IF THE USER  
C SWITCHES BETWEEN MENU TYPES  
SUBROUTINE MENU(T,TFINAL,CHOICE)  
DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ  
DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ  
DOUBLE PRECISION KX, KY, DX, DY, RADIUS  
DOUBLE PRECISION WS, PERIOD  
DOUBLE PRECISION Y(4), Z(4), G  
DOUBLE PRECISION T,TFINAL  
INTEGER CHOICE  
  
COMMON /RSPC/ IRX, IRY, IRZ, IRXY, IRXZ, IRYZ  
COMMON /GSPC/ IGX, IGY, IGZ, IGXY, IGXZ, IGYZ  
COMMON /SANDD/ KX, KY, DX, DY, RADIUS  
COMMON /ROTAR/ WS, PERIOD  
COMMON /ANGLE/ Y, Z, G  
  
C OUTPUT MENU TO THE SCREEN  
WRITE(*,110) IRX, IGX, IRZ, IGZ, KX, KY, DX, DY, RADIUS  
110 FORMAT(/,/ ,/,/,/,/,/,/,/,/,/,/,/,/  
1 //,'ROTOR-GIMBAL SPECIFICATIONS'  
2 //,'Irt (kg m-2): ',D15.9,' Igt (kg m-2): ',D15.9,  
3 //,'Irs (kg m-2): ',D15.9,' Igs (kg m-2): ',D15.9,  
4 //,'Kx (G-S)(Nm): ',D15.9,' Ky (R-G)(Nm): ',D15.9,  
5 //,'Dx (Nms): ',D15.9,' Dy (Nms): ',D15.9,  
6 //,'Radius (mm): ',D15.9)  
WRITE(*,120) Z(1), Z(2), Z(3), Z(4), WS, T, TFINAL  
120 FORMAT(/,'INITIAL CONDITIONS'  
1 //,'X-Axis (G-S)' Y-Axis (R-G)  
2 //,'Angle (Rad): ',D16.9,' Angle (Rad): ',D16.9,  
3 //,'Ang Rate : ',D16.9,' Ang Rate : ',D16.9,  
4 //,'Shaft Spin Velocity (Ws) (RPM): ',D15.9,  
5 //,'Start Time (s): ',D15.9,' End Time (s): ',D15.9)  
WRITE(*,130)  
130 FORMAT(/,  
1 '*****')  
C PROMPT USER FOR CHOICE OF OPTIONS AND RETRIEVE THE  
C RESULT  
CALL MCHOICE(CHOICE)  
RETURN  
END  
  
C THIS SUBROUTINE IS SIMILIAR TO THE ABOVE HOWEVER  
C IT IS FOR THE UNSYMMETRIC CASE  
SUBROUTINE MENU2(T,TFINAL,CHOICE)
```

```

DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
DOUBLE PRECISION KX, KY, DX, DY, RADIUS
DOUBLE PRECISION WS, PERIOD
DOUBLE PRECISION Y(4), Z(4), G
DOUBLE PRECISION T,TFINAL
INTEGER CHOICE

COMMON /RSPC/  IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
COMMON /GSPC/  IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
COMMON /SANDD/ KX, KY, DX, DY, RADIUS
COMMON /ROTAR/ WS, PERIOD
COMMON /ANGLE/ Y, Z, G

C      OUTPUT MENU TO SCREEN
      WRITE(*,110) IRX, IGX, IRY, IGY, IRZ, IGZ
110    FORMAT(/,/,/,/,/,/, 'ROTOR-GIMBAL SPECIFICATIONS'
1      /, 'Irx (kg m^-2): ', D15.9, '      Igx (kg m^-2): ', D15.9,
2      /, 'Iry (kg m^-2): ', D15.9, '      Igy (kg m^-2): ', D15.9,
3      /, 'Irz (kg m^-2): ', D15.9, '      Igz (kg m^-2): ', D15.9)
      WRITE(*,115) IRXY, IGXY, IRXZ, IGXZ, IRYZ, IGYZ, KX, KY, DX, DY, RADIUS
115    FORMAT('Irx (kg m^-2): ', D15.9, '      Igxy (kg m^-2): ', D15.9,
4      /, 'Irxz (kg m^-2): ', D15.9, '      Igxz (kg m^-2): ', D15.9,
5      /, 'Iryz (kg m^-2): ', D15.9, '      Igyz (kg m^-2): ', D15.9,
6      /, 'Kx (G-S)(Nm): ', D15.9, '      Ky (R-G)(Nm): ', D15.9,
7      /, 'Dx (Nms): ', D15.9, '      Dy (Nms): ', D15.9,
8      /, 'Radius (mm): ', D15.9)
      WRITE(*,120) Z(1), Z(2), Z(3), Z(4), WS, T, TFINAL
120    FORMAT(/, 'INITIAL CONDITIONS'
1      /, 'X-Axis (G-S)                                Y-Axis (R-G)'
2      /, 'Angle (Rad): ', D16.9, '      Angle (Rad): ', D16.9,
3      /, 'Ang Rate : ', D16.9, '      Ang Rate : ', D16.9,
4      /, 'Shaft Spin Velocity (Ws) (RPM): ', D15.9,
5      /, 'Start Time (s): ', D15.9, '      End Time (s): ', D15.9)
      WRITE(*,130)
130    FORMAT(
1      '*****')

C      PROMPT USER FOR CHOICE OF OPTION AND RETRIEVE RESULT
      CALL MCHOICE(CHOICE)
      RETURN
      END

C      OFFER THE USER A RANGE OF OPTIONS OF WHAT TO DO NEXT. THIS
C      SUBROUTINE DISPLAYS THE OPTIONS, ACCEPTS THE USER INPUT AND
C      RETURNS THE INPUT TO INTRO WHERE THE PROPER SUBROUTINE IS
C      CALLED
      SUBROUTINE MCHOICE(CHOICE)
      INTEGER CHOICE
      WRITE(*,100)
100    FORMAT('Please choose one of the following'
1      /, '(1) Change Options          (2) Begin Simulation'
2      /, '(3) Load Simulation Data    (4) Save Simulation Data'
3      /, '(5) Load R-G Specifications (6) Save R-G Specifications'
4      /, '(7) Enter R-G Specifications (8) Enter Initial Conditions'
5      /, '(9) Enter Start and End Times (10) Exit')

```

```

      READ(*,*) CHOICE
      RETURN
      END

C      THIS SUBROUTINE CONTAINS OPTIONS FOR SWITCHING BETWEEN
C      THE SYMMETRIC AND UNSYMMETRIC MODES AND CHANGING THE
C      FILENAME FOR THE OUTPUT DATA AND THE FREQUENCY OF OUTPUT STEPS
      SUBROUTINE OPTIONS(FLAG)
      REAL TEMP
      INTEGER FLAG, CHOICE, NUMSTEPS
      LOGICAL SFLAG
      CHARACTER*3 SYM
      CHARACTER*8 FNAME

      COMMON /FILE/  FNAME
      COMMON /COUNT/ NUMSTEPS
      COMMON /SCRIPT/ SFLAG

C      SYMMETRIC/UNSYMMETRIC MODE
      IF (FLAG.EQ.1) THEN
        WRITE(SYM(1:3),'(A)') 'OFF'
      ELSE
        WRITE(SYM(1:3),'(A)') 'ON '
      END IF

      IF (SFLAG) THEN
        READ(1,*) CHOICE
        IF ((CHOICE.LT.1).OR.(CHOICE.GT.4)) STOP
      ELSE
        WRITE(*,11) SYM, FNAME, NUMSTEPS
        READ(*,*) CHOICE
      END IF

      IF (CHOICE.EQ.1) THEN
        IF (FLAG.EQ.1) THEN
          FLAG = 2
        ELSE
          FLAG = 1
        END IF
      END IF

C      CHANGE THE OUTPUT FILENAME
      IF ( CHOICE .EQ. 2 ) THEN
        IF (SFLAG) THEN
          READ(1,'(A)') FNAME
        ELSE
          WRITE(*,15)
          READ(*,'(A)') FNAME
        END IF
      END IF

      IF ( CHOICE .EQ. 3 ) THEN
        IF (SFLAG) THEN
          READ(1,*) TEMP
        ELSE
          WRITE(*,16)
          READ(*,*) TEMP
        END IF
        IF ( TEMP .GT. 100.00000 ) GOTO 20
        IF ( TEMP .LT. 0.0000001 ) GOTO 20
        NUMSTEPS = INT( 100 / TEMP )
      END IF
20

```

```
C      OPEN THE GYRO SPEC FILE
      CALL OPFILE(FILENAME)
      READ(8,15) IRX, IRY
      READ(8,15) IRXY, IRXZ
```

```

      READ(8,15) IRYZ, IRZ
      READ(8,15) IGX, IGY
      READ(8,15) IGXY, IGXZ
      READ(8,15) IGYZ, IGZ
      READ(8,15) KX, KY
      READ(8,15) DX, DY
      READ(8,16) RADIUS
      CLOSE(8)
C      OPEN THE INITIAL COND FILE
      WRITE(FILENAME(R:R+3), '(A)') '.run'
      CALL OPFILE(FILENAME)
      READ(8,15) Z(1), Z(2)
      READ(8,15) Z(3), Z(4)
      READ(8,17) WS, T, TFINAL
      CLOSE(8)
15      FORMAT(2D15.9)
16      FORMAT(D15.9)
17      FORMAT(3D15.9)
      RETURN
      END

C      SIMILIAR TO THE ABOVE EXCEPT THAT ONLY THE GYRO
C      SPECS ARE LOADED
      SUBROUTINE LOSPC
      DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      DOUBLE PRECISION KX, KY, DX, DY, RADIUS
      DOUBLE PRECISION WS, PERIOD
      DOUBLE PRECISION Y(4), Z(4), G
      INTEGER I, R
      CHARACTER*8 FLNAME
      CHARACTER*12 FILENAME

      COMMON /RSPC/   IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      COMMON /GSPC/   IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      COMMON /SANDD/  KX, KY, DX, DY, RADIUS
      COMMON /ROTAR/  WS, PERIOD
      COMMON /ANGLE/  Y, Z, G

      R = 9
      WRITE (*,10)
10      FORMAT(/,/,/, 'Please enter the filename...')
      READ(*, '(A)') FLNAME
      DO 500 I=8,1,-1
         IF (FLNAME(I:I).EQ.' ') R = I
500      CONTINUE
      FILENAME(1:12) = '          '
      FILENAME(1:R-1) = FLNAME
      FILENAME(R:R+3) = '.spc'
      CALL OPFILE(FILENAME)
      READ(8,15) IRX, IRY
      READ(8,15) IRXY, IRXZ
      READ(8,15) IRYZ, IRZ
      READ(8,15) IGX, IGY

```

```

      READ(8,15) IGXY, IGXZ
      READ(8,15) IGYZ, IGZ
      READ(8,15) KX, KY
      READ(8,15) DX, DY
      READ(8,16) RADIUS
      CLOSE(8)
15  FORMAT(2D15.9)
16  FORMAT(D15.9)
      RETURN
      END

C      SAVE THE CURRENT GYRO SPECS FOR LATER USE
      SUBROUTINE SAVSPC
      DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      DOUBLE PRECISION KX, KY, DX, DY, RADIUS
      DOUBLE PRECISION WS, PERIOD
      DOUBLE PRECISION Y(4), Z(4), G
      INTEGER I, R
      CHARACTER*8 FLNAME
      CHARACTER*12 FILENAME

      COMMON /RSPC/   IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      COMMON /GSPC/   IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      COMMON /SANDD/  KX, KY, DX, DY, RADIUS
      COMMON /ROTAR/  WS, PERIOD
      COMMON /ANGLE/  Y, Z, G

      R = 9
      WRITE (*,10)
10  FORMAT(/,/,'Please enter the filename...')
      READ(*,'(A)') FLNAME
      DO 500 I = 8,1,-1
         IF (FLNAME(I:I) .EQ. ' ') R = I
500  CONTINUE
      FILENAME(1:12) = ' '
      FILENAME(1:R-1) = FLNAME
      FILENAME(R:R+3) = '.spc'
      CALL OPFILE(FILENAME)
      WRITE(8,15) IRX, IRY
      WRITE(8,15) IRXY, IRXZ
      WRITE(8,15) IRYZ, IRZ
      WRITE(8,15) IGX, IGY
      WRITE(8,15) IGXY, IGXZ
      WRITE(8,15) IGYZ, IGZ
      WRITE(8,15) KX, KY
      WRITE(8,15) DX, DY
      WRITE(8,16) RADIUS
      CLOSE(8)
15  FORMAT(2D15.9)
16  FORMAT(D15.9)
      RETURN
      END

```

```

C      SAVE THE CURRENT SETUP FOR LATER USE
      SUBROUTINE SAVRUN(T,TFINAL)
      DOUBLE PRECISION IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      DOUBLE PRECISION IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      DOUBLE PRECISION KX, KY, DX, DY, RADIUS
      DOUBLE PRECISION WS, PERIOD
      DOUBLE PRECISION Y(4), Z(4), G
      DOUBLE PRECISION T,TFINAL
      INTEGER I, R
      CHARACTER*8 FLNAME
      CHARACTER*12 FILENAME

      COMMON /RSPC/   IRX, IRY, IRZ, IRXY, IRXZ, IRYZ
      COMMON /GSPC/   IGX, IGY, IGZ, IGXY, IGXZ, IGYZ
      COMMON /SANDD/  KX, KY, DX, DY, RADIUS
      COMMON /ROTAR/  WS, PERIOD
      COMMON /ANGLE/  Y, Z, G

      R = 9
      WRITE (*,10)
10     FORMAT(/,/,/, 'Please enter the filename...')
      READ(*, '(A)') FLNAME
      DO 500 I=8,1,-1
          IF (FLNAME(I:I) .EQ. ' ') R = I
500    CONTINUE
      FILENAME(1:12) = ' '
      FILENAME(1:R-1) = FLNAME
      FILENAME(R:R+3) = '.spc'
      CALL OPFILE(FILENAME)
      WRITE(8,15) IRX, IRY
      WRITE(8,15) IRXY, IRXZ
      WRITE(8,15) IRYZ, IRZ
      WRITE(8,15) IGX, IGY
      WRITE(8,15) IGXY, IGXZ
      WRITE(8,15) IGYZ, IGZ
      WRITE(8,15) KX, KY
      WRITE(8,15) DX, DY
      WRITE(8,16) RADIUS
      CLOSE(8)
      FILENAME(R:R+3) = '.run'
      CALL OPFILE(FILENAME)
      WRITE(8,15) Z(1), Z(2)
      WRITE(8,15) Z(3), Z(4)
      WRITE(8,17) WS, T, TFINAL
      CLOSE(8)
15     FORMAT(2D15.9)
16     FORMAT(D15.9)
17     FORMAT(3D15.9)
      RETURN
      END

C      THIS SUBROUTINE SIMPLY OPENS FILES
      SUBROUTINE OPFILE(FILENAME)
      CHARACTER*12 FILENAME

```



```
IF (CHOICE .EQ. 1) THEN
  WRITE(*,22)
  READ(*,*) IRX
END IF
IF (CHOICE .EQ. 3) THEN
  WRITE(*,23)
  READ(*,*) IRY
END IF
IF (CHOICE .EQ. 5) THEN
  WRITE(*,26)
  READ(*,*) IRZ
END IF
IF (CHOICE .EQ. 7) THEN
  WRITE(*,27)
  READ(*,*) IRXY
END IF
IF (CHOICE .EQ. 9) THEN
  WRITE(*,28)
  READ(*,*) IRXZ
END IF
IF (CHOICE .EQ. 11) THEN
  WRITE(*,29)
  READ(*,*) IRYZ
END IF
IF (CHOICE .EQ. 2) THEN
  WRITE(*,30)
  READ(*,*) IGX
END IF
IF (CHOICE .EQ. 4) THEN
  WRITE(*,31)
  READ(*,*) IGY
END IF
IF (CHOICE .EQ. 6) THEN
  WRITE(*,34)
  READ(*,*) IGZ
END IF
IF (CHOICE .EQ. 8) THEN
  WRITE(*,35)
  READ(*,*) IGXY
END IF
IF (CHOICE .EQ. 10) THEN
  WRITE(*,36)
  READ(*,*) IGXZ
END IF
IF (CHOICE .EQ. 12) THEN
  WRITE(*,37)
  READ(*,*) IG'Z
END IF
IF (CHOICE .EQ. 13) THEN
  WRITE(*,38)
  READ(*,*) KX
END IF
IF (CHOICE .EQ. 14) THEN
  WRITE(*,42)
  READ(*,*) KY
END IF
IF (CHOICE .EQ. 15) THEN
  WRITE(*,46)
  READ(*,*) DX
END IF
```


[illegible]

```
        GO TO 215
      END IF
      GO TO 105
15      FORMAT(D15.9)
20      FORMAT(/,/, 'Please choose which parameter to change:')
22      FORMAT('Start Time (s): ')
26      FORMAT('End Time (s): ')
215     CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SCRPTFLE(SFILE,T,TFINAL)
      INTEGER CHOICE
      LOGICAL SFLAG
      CHARACTER*12 SFILE
      COMMON /SCRIPT/ SFLAG
      OPEN(1,FILE=SFILE,ACCESS='SEQUENTIAL',STATUS='UNKNOWN')
      SFLAG = .TRUE.
300    READ(1,*) CHOICE
      IF ((CHOICE.LT.1).OR.(CHOICE.GT.10)) STOP
      GO TO (100,110,120,130,130,130,130,130,130,130) CHOICE
100    CALL OPTIONS(1)
      GO TO 300
110    RETURN
120    CALL LORUN(T,TFINAL)
      GO TO 300
130    CLOSE(1)
      STOP
      END
```

```

C      This function calculates the arctan of an angle and the result
C      is between 0 and 2*pi. The inputs to the function are the
C      numerator and the denominator. The function does not perform a
C      divide by zero check, so the main program must perform the check.
C      Inputs: NUM - the numerator in the datan function
C              DENOM - the denominator in the datan function
C      Internal Variables: PIE - value of pi
C      Note that all the variables are double precision.
C
C      Last Modified: Mar. 20, 1996

DOUBLE PRECISION FUNCTION DATAN4(NUM,DENOM)
DOUBLE PRECISION NUM, DENOM, PIE
DOUBLE PRECISION DATAN

C      Calculate the value of pi to 15 decimal places within the
C      accuracy of the CPU being used.
PIE = 4 * DATAN(1.0D0)

C      Determine the quadrant that the angle is in and calculate the
C      arctan.
      IF ((NUM.GE.0.0D0).AND.(DENOM.GE.0.0D0)) DATAN4 =
1      DATAN(NUM/DENOM)
      IF ((NUM.GE.0.0D0).AND.(DENOM.LT.0.0D0)) DATAN4 =
1      DATAN(NUM/DENOM) + PIE
      IF ((NUM.LT.0.0D0).AND.(DENOM.LT.0.0D0)) DATAN4 =
1      DATAN(NUM/DENOM) + PIE
      IF ((NUM.LT.0.0D0).AND.(DENOM.GE.0.0D0)) DATAN4 =
1      DATAN(NUM/DENOM)+2.0D0*PIE

      RETURN
      END

```

END

2 1 1 1 9 6

FIN