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Feature Extraction from Digital Imagery using the Hough Transform

by

Colleen Patterson

**A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of**

Master of Engineering

**Department of Systems and Computer Engineering
Carleton University
Ottawa Ont
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Acceptance

The undersigned recommend to the Faculty of Graduate Studies and Research acceptance of the thesis “Feature Extraction from Digital Imagery using the Hough Transform”, submitted by Colleen Patterson in partial fulfillment of the requirements for the degree of Master of Engineering.

A handwritten signature in cursive script, reading "H. Schwartz", is written over a horizontal line.

Thesis Supervisor

A handwritten signature, appearing to be "C. Patterson", is written over a horizontal line.

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Abstract

This thesis investigated the improved capability to detect a feature in a noisy image using a statistical analysis technique with the Hough Transform. The noise model assumed to describe background noise in an image resulted in the statistical analysis being based upon an underlying binomial distribution. The use of the statistical analysis provided an improvement in the capability to detect small features in the outer regions of the image. It also reduced the number of false features resulting from background noise being detected.

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Chapter 1

INTRODUCTION

1.1 Background

Image processing systems are often used to detect and identify features found within an image. One way to describe an image processing system is to sub-divide it into three subsystems as shown in Figure 1.1.

The first subsystem, Image Preparation, takes the image to be analyzed and converts it to an appropriate digital form. It is responsible for quantizing the image into discrete areas, or pixels, and determining the state of each pixel, on or off. It passes the information on the state of the pixels to the second subsystem for analysis. The second subsystem, Feature Detection, will use the information on the state of the pixels to find patterns that meet criteria associated with the fundamental features used in the images. Common fundamental features are lines, circles and ellipses. The Feature Detection Subsystem will identify locations within the image that contain specified features. The third subsystem, Identification, will compare the location and types of features found by the feature detection subsystem and make a decision on what is contained in the image. For example, if four lines are found that are 90° to each other with common end points, a square may be identified.

Figure 1.1 Image Processing System

The Hough Transform is a feature detection method used in many image processing systems to find features such as lines, circles and ellipses. However, the Hough Transform performs poorly when background noise is present and the feature of interest is small. This thesis will demonstrate an alternative method of implementing the Hough Transform which significantly improves its ability to find small features of interest within background noise. In order to limit the scope of the thesis, analysis will be presented for only one feature of interest, the line.

A substantial amount of research has been published since Peter Hough introduced the Hough Transform in 1962. Predominately, the research has been slanted towards making the computation of the Hough Transform more efficient. However, some papers have explored the use of statistical analysis techniques to quantify and improve the performance of the Hough Transform in both clean and noisy images. The work of Cohen and Toussiant [1] is closely related to the investigation undertaken in this thesis. The main contribution of their work was an alternative method to quantize the parameters associated with the Hough Transform in order to mitigate the effects of background noise in the image. A second paper published by Maitre developed an expression for the performance of the Hough Transform based upon the statistics which he assumed

described the behaviour of features within an image [2]. A third paper that made use of statistics to improve the Hough Transform was published by Pricen, Illingworth and Kittler [3]. Their proposed implementation of the Hough Transform combined the use of multiple hypothesis tests in conjunction with the Hough Transform in order to improve performance. They did not address the issue of background noise in the image. Each of these papers will be discussed in more detail in Chapter 2 following a description of the Hough Transform.

1.2 Objectives and Methodology

The objective of this thesis is to determine if statistical analysis techniques can be used with the standard Hough Transform to improve its ability to find lines in a noisy image with only the image size and noise level given as *a priori* knowledge.

The approach is based on using the standard Hough Transform with no modifications to how the transform is actually taken. The deviation from the standard Hough Transform is in the analysis of the results of the transform by using a statistical test. Cohen and Toussaint [1] make an oblique reference to the technique presented in this thesis as a potential alternative to their proposed approach but they did not provide any indication that they investigated the technique.

1.3 Thesis Outline

Chapter 2 describes the Hough Transform, including a review of published work relevant to this thesis, and provides a description of the technique being proposed as an improved method to analyze the results of the Hough Transform. The effect of noise on the Hough Transform and the statistics of the noise are described in Chapter 3. Chapter 4 is a description of the Test Images that will be used in the thesis. Chapter 5 presents the comparative results of the ability of two different approaches to correctly find the test

lines. Chapter 6 contains concluding remarks.

1.4 Symbol Dictionary

The following symbols are used in the thesis:

m : slope of a line

b : y-intercept

ρ : length of a normal vector

θ : angle of the normal vector measured with respect to the positive x axis

$\Delta\rho$: quantization size of the length of the normal vector

$\Delta\theta$: quantization size of the angle of the normal vector

l : length of a line

M : largest possible value of ρ

μ , $E[x]$: expected value of a random variable

σ^2 , $\text{VAR}[x]$: variance of a random variable

p : probability of a pixel being turned on by noise

N : maximum value of an accumulator array bin

χ^2 : chi-squared test statistic

p_{bin} : probability of an accumulator array bin having a given value or greater

Chapter 2

REVIEW OF THE HOUGH TRANSFORM

2.1 Hough Transform

The Hough Transform is a Feature Detection algorithm that identifies the location of straight lines, circles and ellipses by a set of parameters. One choice of parameters used in the detection of lines makes use of the equation of a line:

$$y = mx + b \quad (1)$$

The parameters are (m, b) , where m is the slope and b is the y-intercept of a standard Cartesian X-Y plane. When using the slope-intercept parameterization, it is possible to have an infinite slope corresponding to a line that is vertical. It is difficult to handle an equation in which one of the parameters can approach values of $\pm \infty$. A second method of parameterization is called the normal parameterization which describes the line by the parameters (ρ, θ) , where θ is the angle and ρ is the radius of the normal to the line [4]. The angle is measured from the positive x axis. This is the parameterization which will be used in this thesis and is shown in Figure 2.1.

There are two domains which must be defined: the image space and the transform space. The image space is defined as an (x, y) coordinate system with the origin at the centre of the image. The maximum value for x and y are determined by the dimensions of the image. The units of the (x, y) system can be chosen for convenience, pixels, inches, cm etc. In this thesis, all units in the image space will be in pixels. The Image

Preparation subsystem will be responsible to quantize the image space into discrete (x, y) areas or pixels.

The transform space is defined by the parameters, which for a normal parameterization results in the transform space being quantized into discrete (ρ, θ) areas of size $\Delta\rho$ and $\Delta\theta$. The relationship between the image space and the transform space is given by:

$$\rho = x \cos \theta + y \sin \theta \quad . \quad (2)$$

The quantized angle can either have values $0 \leq \theta < 360$ or $0 \leq \theta < 180$. When θ is allowed to exceed 180° , all the calculated ρ will be positive. When θ is restricted to have a value less than 180° , the calculated ρ will be both positive and negative. The choice of the between the two ranges of θ has no impact on the performance of the Hough Transform. In this thesis, the quantized angle was allowed to have values $0 \leq \theta < 180$.

The Hough Transform is implemented by first examining each pixel. For each pixel there will be a set of criteria that, when met, will result in the pixel being classified as being in an “on” state. When a pixel is determined to be “on”, all the possible (ρ, θ) combinations that describe the normal to each line that could contain the point are calculated. The (ρ, θ) combinations are calculated by solving for ρ for each quantization level of θ , $\Delta\theta$, using (2). The number of possible (ρ, θ) combinations for each (x, y) point is determined by the angle quantization size, $\Delta\theta$. Each (x,y) point will produce a sinusoid type curve in the transform space.

A simple example of an image space containing a line is shown in Figure 2.1 and the corresponding transform space is shown in Figure 2.2. Figure 2.1 contains a line with end points (x, y) (1,5) and (5,1). The equation of the line is:

$$y = -x + 6 \quad . \quad (3)$$

Using basic geometry shown in Figure 2.1, the radius and angle which describe the

normal to a line with this equation can be found.

$$\begin{aligned}
 180 &= 90 + \theta + \phi & \rho &= x \cos \theta \\
 \theta &= 90 - \arctan(6/6) & \rho &= 6 \cos 45 \\
 \theta &= 45^\circ & \rho &= 4.2426
 \end{aligned}
 \tag{4}$$

Figure 2.1 Image Space with 5 point line showing the geometry associated with the Normal Parameterizaion

As the image is scanned, the points $(x, y)=(1,5) (2,4) (3,3) (4,2) (5,1)$ are found to be “on”. Using equation (2), all the (ρ, θ) combinations possible for each point are found. Consider the point $(1,5)$. For each angle θ , the corresponding ρ is:

$$\rho_0 = 1 \cos 0^\circ + 5 \sin 0^\circ = 1$$

$$\rho_{10} = 1 \cos 10^\circ + 5 \sin 10^\circ = 1.85$$

$$\rho_{20} = 1 \cos 20^\circ + 5 \sin 20^\circ = 2.65$$

:

$$\rho_{170} = 1 \cos 170^\circ + 5 \sin 170^\circ = -0.12$$

The plot of each value of ρ for the corresponding θ is shown in Figure 2.2 as the curve labelled ρ_1 . To accentuate the sinusoid shape of the curve, the discrete (ρ, θ) points have been drawn as a smooth line. The (ρ, θ) curves corresponding to the other points that are “on”, $(2,4)$, $(3,3)$, $(4,2)$ and $(5,1)$, can be found using the same method and are shown in Figure 2.3 as ρ_2 , ρ_3 , ρ_4 and ρ_5 respectively.

The normal associated with the line in the image space that best describes the points that are “on” corresponds to the intersection of the curves in the parameter space. From Figure 2.2, the intersection is found at $(\rho, \theta) (4.2, 45^\circ)$. Comparing this to the normal calculated in (4), it can be seen that they are the same.

Figure 2.2 Plot of ρ vs θ for the line with endpoints (5,1) and (1,5)

The graph in Figure 2.2 describes the transform space and provides a visual tool that allows the normal associated with the line to be easily determined. Since the purpose of the Hough Transform is to automate visual analysis, the transform space is implemented using a two dimensional array with the rows corresponding to different values of the radius and the columns corresponding to the different values of θ . Each cell within the array contains the number of points in the image space associated with the occurrence of each (ρ, θ) combination. This array is called an accumulator array since it accumulates the number of times each (ρ, θ) combination results from equation (2) and the cells are called bins. The array size is determined by the quantization size of both the radius, $\Delta\rho$, and the angle, $\Delta\theta$, as well as the image space dimensions.

For the image space in Figure 2.1 with $\Delta\rho = 1$ and $\Delta\theta = 10^\circ$, the accumulator array will have 13 rows and 18 columns. When the array is used to record information, it is necessary to round the value of ρ obtained from equation (2) to the same level of

resolution as $\Delta\rho$. A value of ρ will be assigned when (5) is satisfied :

$$\left(\rho_{calculated} - \frac{\Delta\rho}{2} \right) \leq \rho < \left(\rho_{calculated} + \frac{\Delta\rho}{2} \right) \quad (5)$$

The populated accumulator array is shown in Figure 2.3.

The information in the accumulator array reveals the number of pixels that would form a line normal to the (ρ, θ) value of the bin. For example, at $(\rho, \theta) = (4, 30^\circ)$ the bin value of 3 indicates that there is a line of length 3 pixels normal to a vector of length 4 pixels at an angle of 30° .

The standard Hough Transform detects lines from the information in the accumulator array using bin value thresholds. The threshold chosen is related to the minimum line length to be detected. The value of each bin of the accumulator array is compared to this threshold and if the bin value is equal to or greater than the threshold, a feature is found. Assume in this example that the threshold is set at a bin value of 5. For the accumulator array in Figure 2.3, two bins have a value equal to the threshold. This means two lines of length 5 would be identified in the image space, one along the line normal to a vector of length 4 at an angle of 40° and the second on a line normal to a vector of length 4 at an angle of 50° .

	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170
6																		
5	1	1	1	1			1	1	1	1	1							
4	1	1	2	3	5	5	3	2	1	1	1	1	1					
3	1	2	2	1			1	2	2	1		1		1				
2	1	1							1	1	1	1	1	1	1	1		
1	1									1	1	1	1		1		1	
0											1		1	1	1	1		1
-1												1		1		1	1	1
-2													1	1	1	1	1	1
-3															1		1	
-4																1	1	1
-5																		1
-6																		

Figure 2.3 Accumulator Array for the line with endpoints (5,1) and (1,5)

This simple example has illustrated the standard Hough Transform method to detect lines in the image space. In finding two lines where only one exists, it has also highlighted a limitation associated with the transform. The characteristics and limitations will be discussed in detail in the remainder of this chapter.

2.2 Characteristics of the Hough Transform

The Hough Transform has certain inherent characteristics that need to be highlighted in order to understand some of the limitations associated with the transform. There are also external factors that will affect the ability to find lines using the Hough

Transform. In this section, both the inherent and external factors that impact the performance of the transform relevant to the thesis will be presented.

2.2.1 Finite Image Space

For practical reasons, the Hough Transform is taken of an image space with finite dimensions. Finite dimensions result in decreasing possible line lengths for increasing ρ when the origin of the coordinate system is at the centre of the finite image. Consider the two lines in Figure 2.4 described by the normals (ρ_1, θ_1) (1.4, 45°) and (ρ_2, θ_2) (4.2, 45°). The first line has length of 8.5 and has seven (x,y) points on it. The second line has length of 2.8 and has three (x, y) points on it. When noise is added to the image space, depending on the amount of noise and the image size, it is conceivable that more pixels on the longer line will be turned “on” due to noise than the length of a feature at large ρ .

Another characteristic of a finite image space that has square or rectangular dimensions instead of circular dimensions is that the maximum ρ is different for each angle. If the difference in maximum ρ is not taken into account when populating the accumulator array, the end point of the normal vector can be found outside the image space. The image space in Figure 2.4 is a square which is 10 by 10 with the origin of an x-y Cartesian coordinate system in the centre. The quadrants are numbered Q1 to Q4 as shown in the diagram. The maximum radius and corresponding angle with the positive x axis for each quadrant was found using basic geometry and are listed in Table 2.1.

Figure 2.4 Image Space of size 10 pixels by 10 pixels

quadrant	$\rho_{\max}$	θ
Q1	5.7	45°
Q2	6.4	141.3°
Q3	-7.1	45°
Q4	-6.4	128.7°

Table 2.1 $\rho_{\max}$, θ for image space shown in Figure 2.5

2.2.2 Inherent Noise

Noise in the accumulator array can be defined as bins that have a non-zero value

that do not describe the normal to a feature in the image space. The Hough Transform has inherent noise since it allows each (x, y) point to augment multiple bins, not all of which correspond to a feature in the image space [5]. The inherent noise associated with the five pixel line in Figure 2.1 can be seen in the accumulator array in Figure 2.3 as the bins that have a value greater than zero but less than the threshold value of 5. For a single feature, the contribution of this inherent noise is limited. If multiple features exist, this noise source becomes significant.

2.2.3 Threshold determination

The determination of the bin value threshold is critical to the performance of the standard Hough Transform and usually requires *a priori* knowledge of the number and size of features to be found. When the threshold is set too low, too many false features will be found. Consider the example of the image space in Figure 2.1 and the corresponding accumulator array in Figure 2.3. Even with the threshold set at the exact length of the line, two lines were found where only one existed. However, if the threshold is set too high, features that have a maximum size less than the threshold will be missed. Consider the image space in Figure 2.4. If a threshold of four or larger is used to find lines in a accumulator array based on this image, a line at (ρ_2, θ_2) would never be found. Using a simple threshold, it is difficult to determine the probability of error associated with finding a line of a given length.

2.2.4 Exact location of the line

The Hough Transform will find the normal that best describes the line across the image space as opposed to finding the normal to the line bound by the end points of the feature. Using the 10 by 10 image in Figure 2.4, say there are two lines with end points (x,y) $(-2, 4)$ $(0, 2)$ and $(2, 0)$ $(4,-2)$. The accumulator array will indicate a single line with a normal $(1, 45^\circ)$ with six points on it rather than two separate lines which lie on the line.

2.2.5 Parameter Quantization

The choice of quantization levels in both the image space and the transform space will impact the accuracy of the Hough Transform. Van Veen and Groen [7] investigated the effect of quantization in both the image space ($\Delta x, \Delta y$) and the transform space ($\Delta \rho, \Delta \theta$). They found that peaks in the accumulator array will be spread across multiple bins in both the radius direction and the theta direction, that is spreading across both columns and rows of the accumulator array. The number of bins over which the peak will be spread is related to the length of the line in the image space, the width of the line in the image space and the quantization levels used in the transform space. Van Veen and Groen suggested that for a line of length l in the image space, the ideal relationship between $\Delta \theta$ and $\Delta \rho$ is:

$$\Delta \rho \approx l \sin\left(\frac{\Delta \theta}{2}\right) \quad (6)$$

Yuen and Ma also looked at the effect of quantization levels and found that under sampling can be caused by inappropriate quantization of $\Delta \theta$ and $\Delta \rho$ [8]. They suggested that to avoid discontinuities in the transform plane the following relationship should be respected:

$$M \leq \frac{\Delta \rho}{\Delta \theta} \quad (7)$$

where M is the maximum distance from the origin (the largest ρ possible) and $\Delta \theta$ is in radians. Unlike the relationship between $\Delta \rho$ and $\Delta \theta$ suggested by Van Veen and Groen [7], this one does not require any knowledge about the feature.

2.2.6 Noise in the Image Space

The presence of speckle noise in the digitized image that is passed from the Image

Preparation subsystem to the Feature Detection subsystem is an external factor that affects the Hough Transform. Each pixel that is erroneously determined to be “on” because of noise will augment $n = \lfloor 180/\Delta\theta \rfloor$ bins which are not associated with the line of interest.

The following example will show the effect of speckle noise. Figure 2.5(a) is an image space of dimensions 500 by 400 pixels. It contains a line that is 12 pixels in length. A Cartesian x-y coordinate system with the origin at the centre of the image was used to reference all pixel locations in the image. The line has end points at (-202, 156) and (-204, 144). This image contains no background noise.

The Hough Transform of the image was taken using the normal parameterization with $\Delta\rho = 2$ $\Delta\theta = 5^\circ$. The resulting accumulator array for the image was sorted by bin values and the bins with the 9 largest values for the image is shown in Figure 2.5(b).

Assume that there was enough knowledge about the feature that a threshold of 12 was set. This means that any bins with a value of 12 or greater would be identified as having a feature. For the noiseless image in Figure 2.5, the location of the line would be correctly identified.

Figure 2.6 shows an enlarged view of the area of the image where the 12 pixels are “on” forming the line. The choice of the quantization of x and y have resulted in the line being represented by the pixels shown. It is interesting to note that the single line is broken down into three smaller lines of length 4 pixels which could be described by the normals (ρ, θ) of (-204, 0), (-203, 0) and (-202, 0). Using the quantization of $\Delta\rho = 2$, the normals would become (-204, 0) and (-202, 0).

Figure 2.7 (a) is the same image space as in Figure 2.5(a) corrupted with 1% background noise. The noise corruption was introduced on a per pixel basis: that is, each

pixel is independent of all other pixels with respect to noise corruption. The Hough Transform of the image was taken using the normal parameterization with $\Delta\rho = 2$ and $\Delta\theta = 5^\circ$. The resulting accumulator array for the image was sorted by bin values and the bins with the 9 largest values for the image is shown in Figure 2.7(b). When the noise is added to the image space, the line is still visible to the human eye but is no longer clearly identifiable to the computer using the Hough Transform.

Looking at the bin with the largest value, it indicates that the line described by a normal of (154, 90) was augmented 23 times. This means the Hough Transform has found a line parallel to the x axis at height $y = 154$ that is 23 pixels in length. A visual inspection of the Image Space near the top of the image does not reveal any line patterns that would dominate the image. Using a threshold of 12 pixels results in the Hough Transform finding numerous features that exceed the threshold when only one feature actually exists. The bin of interest has the 9th largest bin value.

(a)

radius (ρ)	angle (θ)	bin value
226	170	12
242	160	11
-202	0	8
216	175	8
248	155	8
-190	5	7
234	165	6
236	165	6
250	150	6

(b)

Figure 2.5 Clean Image Space and sorted Partial Accumulator Array

Figure 2.6 Enlarged view of 12 pixel line

(a)

radius (ρ)	angle (θ)	bin value
154	90	23
-202	0	22
-86	70	22
32	55	22
-56	105	21
-44	95	21
8	125	21
128	25	21
226	170	20

(b)

Figure 2.7 Noisy Image Space and sorted Partial Accumulator Array

2.3 Previous Performance Measures

Several papers have been published with respect to the accuracy of the Hough Transform proposing methods to quantify and, in some papers, improve its performance. A complete review of all the methods will not be presented here but a very good survey of the work published up to 1988 can be found in [9].

Cohen and Toussaint investigated the ability of the Hough Transform to detect features in noisy images through the use of maximum entropy quantization [1]. They used a circular image space corrupted with independent, additive, uniformly distributed noise. They made use of the knowledge that the values of bins associated with large ρ are smaller than those associated with small ρ to develop a probability density function for ρ in order to quantize $\Delta\rho$ into equal probability subintervals instead of equal length subintervals. The aim was to have the value associated with each bin in the accumulator array to be approximately the same if only noise were present in the image space. This would allow the peak values in the accumulator array associated with a feature to be more reliably identified. Cohen and Toussaint's probability density function was disputed in subsequent work completed by Alagar and Thiel as representing "only the conditional distribution of ρ " [10]. Work completed by Maitre [2] shows agreement with Alagar and Thiel [10]. Maitre attempted to derive probability density functions for use with a rectangular image but was unable to "due to the complexity of the problem" [2].

Alagar and Thiel compared the results of using uniform quantization, maximum entropy quantization and a hybrid of the two in three aspects: precision, recall and goodness. Precision was defined as the ratio of the number of times a correct line was found to the number of lines detected. Recall was defined as the ratio of the number of correct lines found to the number of actual lines in the Image space. Goodness was defined as the ratio of the difference between the number of correct lines found and the number of false lines found to the total number of actual lines in the Image space. The

difference between the three quantization methods was not significant in the three areas of comparison [10].

Cohen and Toussaint [1] also suggested that when the noise distribution was unknown, the probability, p' , that a feature point will augment a bin the accumulator array can be estimated. The estimate could be “obtained from pictures containing only background or noise and no lines nor circles” [1]. The accumulator array bin values, h , can then be replaced with the standardized bin value $(h-\mu)/\sigma$, where $\mu=Np'$ and $\sigma^2 = Np'(1-p')$ given N as the number of points in the picture [1]. Any bins that show a large standardized bin value would be identified as a feature. They made the argument that in the limit, each standardized bin value would behave as a Gaussian distribution providing the same confidence level by applying the same threshold to all bins. In their paper, no results were provided for this method.

In addition to trying to develop a probability density function for p in a rectangular retina, Maitre also investigated the statistics associated with features, specifically associated with lines, to develop a prediction tool for the performance of the Hough Transform for a given image space [2]. His development was based upon reducing the image space to a “band” around the feature of interest. The coordinate system is rotated so that the x axis is parallel to the feature. The y -axis dimension is reduced to 2δ , ranging from $\pm\delta$. Within this band, the feature points are uniformly distributed about the rotated x -axis. Using the band as his reduced image space, Maitre was able to develop an expression to determine a signal to noise ratio (SNR) where the SNR is defined as “the ratio of the maximal signal count to the maximal noise count”[2] :

$$SNR = \frac{n_s L^2}{n_n \delta^2} \quad (8)$$

where:

n_s is related to the number of feature points in the band

n_n is related to the number of noise points in the band
 L is the length of the line in the band
 δ is $\frac{1}{2}$ of the width of the band in the rotated coordinates y -axis.

Princen, Illingworth and Kittler provided a formal mathematical definition of the Hough Transform [11] and developed test statistics based on the definition of the transform for the parameters ρ and θ associated with the normal as well as a test statistic for the width of the line, w [3]. In developing test statistics, the central limit theorem was employed to approximate the accumulator array bin behaviour as Gaussian. This assumption is only valid when the number of pixels associated with a bin describing a feature would be sufficiently large and indicated that the approximation would not be “uniformly good” for all bins [3]. They used this approximation in order to make their test statistics applicable to various implementations of the Hough Transform including the standard implementation as previously described.

2.4 Proposed Method to Improve the Hough Transform Performance

The method being proposed and investigated in this thesis is similar in approach to the second method suggested by Cohen and Toussaint [1]. It will be shown that if random noise in the image space corrupts each pixel independently, the resulting accumulator array bin values will behave as binominal distributions. This knowledge allows the probability of a particular bin value occurring as a result of only noise in the image space to be calculated from the binominal distribution describing that bin. As the probability of the value occurring if only noise is present in the image space decreases, it becomes more likely that the bin value is actually the result of a feature associated with the bin. This approach requires no modification to how the Hough Transform is taken but does require knowledge of the background noise level in the image space.

Chapter 3

Noise in the Accumulator Array

3.1 Introduction

Performing a statistical analysis of the accumulator array in order to find features is analogous to finding a signal in the presence of noise. The bin(s) in the accumulator array corresponding to a valid feature(s) in the image space can be defined as the signal. The remaining bins which do not correspond to valid features in the image space can be defined as the noise.

The signal can be treated as a deterministic signal since the bin value will be related to the length of the line in the image space. The noise will have both a deterministic component and a random variable component. The deterministic component is shown in Figure 2.4 as the value in the bins that do not completely describe the feature but which are augmented by points within the feature. This noise mechanism will not be formally addressed in this thesis. The Random Variable component will be caused by noise points contained within the image space which result from imperfections in the Image Preparation subsystem.

To conduct a statistical analysis, a test of significance is used which is based upon the initial assumption that there is no significant difference between the observed data and the assumed distribution. Statistical tests are usually presented in the form of a Null Hypothesis statement:

h_0 : the event of interest has not happened.

Two types of errors can be made in statistical tests. The first is a Type I error in which the null hypothesis is rejected when it is correct. This means the test erroneously reveals a significant difference between the data being tested and the anticipated results. The second is a Type II error in which a significant difference is not found when there is one and the null hypothesis is accepted in error [12].

In order to use a statistical test to analysis the accumulator array to find features, the statistical behaviour of the bins must be determined. Correspondingly, the behaviour of the pixels in the image space must also be examined. The notation used to describe statistical behaviour in this thesis is consistent with [13].

3.2 Pixels in the Image Space

The Image Preparation subsystem will be presented with pixels of various shades of grey. In order to determine which pixels are “on” or “off” a threshold must be applied to each pixel. If the grey level is above the threshold, the pixel will be “on”. If the grey level is below the threshold, the pixel will be “off”. For the purpose of this thesis, the noise acting upon individual pixels within the image space has been modelled as independent Gaussian noise.

The Gaussian distribution, often called the Normal distribution, describes a continuous Random Variable. The probability density function of a Gaussian distribution is described by :

$$f_x(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} \quad (9)$$

where:

μ = expected value;

σ^2 = variance.

The expected value is determined from:

$$E[x] = \int_{-\infty}^{\infty} t f_x(t) dt \quad (10)$$

and the variance is determined from:

$$VAR[X] = E[(X - E[X])^2] \quad (11)$$

The probability of a pixel being turned on because of noise can be found by determining the area under the probability distribution function from the threshold level, t , to $+\infty$. For a Gaussian noise distribution, this area can be found by evaluating the integral:

$$\begin{aligned} p = \text{area} &= \int_{x=t}^{\infty} f_x(x) dx \\ &= \int_{x=t}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \end{aligned} \quad (12)$$

By introducing a variable z , where $z = (x - \mu) / \sigma$, and substituting into (12), the integral becomes:

$$\begin{aligned}
& \int_{z=\frac{t-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2/2} \sigma dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{z=\frac{t-\mu}{\sigma}}^{\infty} e^{-z^2/2} dz
\end{aligned} \tag{13}$$

Once the equation is in this form, we recognize that the $Q(\cdot)$ function can be used to evaluate the area described by (12). The area becomes:

$$p = \text{area} = Q\left(\frac{t-\mu}{\sigma}\right) \tag{14}.$$

Using published lookup tables, the probability of a pixel being turned on by Gaussian noise with a mean μ and variance σ^2 can be easily determined. In a similar manner, the threshold t for a given error rate, μ and σ^2 can be determined by solving for t in (14).

3.3 Accumulator Bin Behaviour

A Binomial Distribution is a discrete Random Variable and is the sum of a series of independent Bernoulli trials. A Bernoulli trial is a trial which can have only two outcomes. Outcome 1 occurs with a probability p and outcome 2 occurs with probability $q = 1-p$. With respect to the image space, the two outcomes can be described as:

- outcome 1: pixel “on” because of noise with probability p
- outcome 2: pixel not “on” because of noise with probability q .

The probabilities p and q do not need to be equal but the sum of p and q must be unity.

A binomial distribution can be described by its mean and variance:

$$E[X] = Np \quad (15)$$

$$\sigma^2 = Npq \quad (16)$$

for a set of possible outcomes $S_x = \{0, 1, \dots, N\}$. The probability of an exact outcome within the set of possible outcomes is determined by:

$$p_k = \binom{N}{k} p^k q^{N-k} . \quad (17)$$

The shape of the binomial probability density function is dependent upon the value of the parameter p . When p is small and q is large, the density function will be skewed to the left as shown in Figure 3.1 (a). When $p=q$, the density function will be approximately Gaussian as shown in Figure 3.1 (b). When p is large and q is small, the density function will be skewed to the right as shown in Figure 3.1(c).

Considering an image with no features present, each pixel is a Bernoulli trial with a probability p of being “on” where p is determined using (12). The observed value of each bin within in the accumulator array is the result of the summation of the number of pixels that have been corrupted by noise on a given normal in the image space. If the noise acting on each pixel is independent, each bin is accumulating the results of a series of independent Bernoulli trials and will therefore have the characteristics of a binomial distribution. The maximum value of a bin in the accumulator array is N . The value of N for each bin will be the total number of pixels in the image space that produce the (ρ, θ) combination associated with the bin.

(a) $p=0.04$, $q=0.96$, $N=25$

(b) $p=0.5$, $q=0.5$, $N=25$

(c) $p=0.8$, $q=0.02$, $N=25$

Figure 3.1 Binomial probability density function for different p

The binomial distribution can be shown to move asymptotically towards one of two continuous distributions [12]. If the value of N , p and q are such that $Npq \gg 1$, then a Gaussian distribution with mean $= Np$ and variance $= Npq$ can be used to approximate the binomial distribution. When Np is of order 1, then a Gaussian can not be used to approximate a binomial. The Poisson theorem is then more applicable. The Poisson theorem states:

If $N \rightarrow \infty$, $p \rightarrow 0$ and Np is finite, then

$$\frac{N!}{k!(N-k)!} p^k q^{N-k} \xrightarrow{N \rightarrow \infty} e^{-Np} \frac{(Np)^k}{k!} \quad (18)$$

In this thesis, the noise levels which will be investigated will be 1%, 2%, 3% and 4% which will result in p values of 0.01, 0.02, 0.03 and 0.04 respectively. No bin in the accumulator array has a value of N greater than 800. Most bins have a N value between 300 and 600.

Table 3.1 shows the values of Npq for three values of N at each noise level. When the noise level is at 4%, $p=0.04$, the value of Npq is such that a the binomial distribution can be approximated by a Gaussian distribution. At each of the other noise levels, both the Gaussian and Poisson distribution could be used as approximations to the binomial.

p	$(Npq)_{800}$	$(Npq)_{300}$	$(Npq)_{600}$
0.01	7.92	2.97	5.94
0.02	15.68	5.88	11.76
0.03	23.28	8.73	17.46
0.04	30.72	11.52	23.04

Table 3.1 Values of Npq at different N and p

In the statistical tests that will be conducted in Chapter 5, different confidence levels between 99.9 % and 99.9998% will be used in the data analysis. The confidence level corresponds to the area under the probability distribution curve to the left of the value of interest. In order to see how the three distributions compare at these confidence levels, the tail area of a bin with maximum value bin value, N , of 100 was calculated for different observed values, k , for each distribution. The area to the left of the observed value, k , is presented in Figure 3.2. Only a portion of the values are presented since above an observed value of 14, the area to the right of the curve is negligible and the probability associated with the observed value is ≈ 1 .

Comparing the three distributions for each of the noise levels, the observed value k corresponding to the significance levels between 99.9 % and 99.9998% will be the same for each of the distributions. This means that using the approximations will not increase the chance of a Type 1 or Type 2 error being made.

The Gaussian approximation will be used in the text of this thesis to provide mathematical explanations for behaviour that will be observed. However, neither the Gaussian nor the Poisson distribution was implemented in the algorithms since the computer used to generate the results was able to handle the Binomial distribution at the values of N and p under consideration.

	1% noise		
k	Binomial	Gaussian	Poisson
0	0.366032	0.157439	0.367879
1	0.735762	0.500000	0.735759
2	0.920627	0.842561	0.919699
3	0.981626	0.977788	0.981012
4	0.996568	0.998716	0.996340
5	0.999465	0.999971	0.999406
6	0.999929	1.000000	0.999917
7	0.999992	1.000000	0.999990
8	0.999999	1.000000	0.999999
9	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000
11	1.000000	1.000000	1.000000
12	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000
15	1.000000	1.000000	1.000000

	2% noise		
k	Binomial	Gaussian	Poisson
0	0.132620	0.076564	0.135335
1	0.403272	0.237525	0.406006
2	0.676686	0.500000	0.676676
3	0.858962	0.762475	0.857123
4	0.949170	0.923436	0.947347
5	0.984516	0.983938	0.983436
6	0.995938	0.997863	0.995466
7	0.999068	0.999822	0.998903
8	0.999811	0.999991	0.999763
9	0.999966	1.000000	0.999954
10	0.999994	1.000000	0.999992
11	0.999999	1.000000	0.999999
12	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000
15	1.000000	1.000000	1.000000

	3% noise		
k	Binomial	Gaussian	Poisson
0	0.047553	0.039320	0.049787
1	0.194622	0.120514	0.199148
2	0.419775	0.278867	0.423190
3	0.647249	0.500000	0.647232
4	0.817855	0.721133	0.815263
5	0.919163	0.879486	0.916082
6	0.968772	0.960680	0.966491
7	0.989376	0.990482	0.988095
8	0.996784	0.998311	0.996197
9	0.999126	0.999782	0.998898
10	0.999785	0.999980	0.999708
11	0.999952	0.999999	0.999929
12	0.999990	1.000000	0.999984
13	0.999998	1.000000	0.999997
14	1.000000	1.000000	0.999999
15	1.000000	1.000000	1.000000

	4% noise		
k	Binomial	Gaussian	Poisson
0	0.016870	0.020613	0.018316
1	0.087163	0.062893	0.091578
2	0.232143	0.153717	0.238103
3	0.429476	0.304917	0.433470
4	0.629864	0.500000	0.628837
5	0.788375	0.695083	0.785130
6	0.893608	0.846283	0.889326
7	0.952488	0.937107	0.948866
8	0.981008	0.979387	0.978637
9	0.993156	0.994638	0.991868
10	0.997761	0.998900	0.997160
11	0.999332	0.999823	0.999085
12	0.999817	0.999978	0.999726
13	0.999954	0.999998	0.999924
14	0.999989	1.000000	0.999980
15	0.999998	1.000000	0.999995

Figure 3.2 Cumulative distribution at 4 noise levels for the Binomial, Gaussian and Poisson distribution for a bin with $N = 100$

3.4 Goodness of Fit Test

In order to test the hypothesis that each bin behaves as a binomial distribution, a statistical test was used. Figure 3.3 shows the process that was used to test the null hypothesis. The hypothesis under test can be stated using the conventional method of writing a Null Hypothesis as follows:

h_0 : each bin behaves as a binomial distribution with parameters N and p .

The maximum value for each bin, N , corresponds to the number of pixels that exist in the image space that are on the line normal to the (ρ, θ) value associated with the bin. The value of N for each bin was easily found by taking the Hough Transform of the image space with all the pixels on.

For statistical tests, the Null Hypothesis is rejected at the 5% or 1% significance level [12]. A related concept is the confidence level. "The confidence level is the area of the acceptance region, whereas the significance level would be the area of the rejection region." [12] The 5% significance level, or confidence level of 95%, was used in the background distribution test to accept or reject the Null Hypothesis.

Using the random number generator that is standard in the Borland C++ 4.5 compiler to generate a random seed and a Gaussian random number generator from Numerical Recipes [14], pixels were corrupted in the image space. The corruption of each pixel was independent. The Hough Transform was taken of the corrupted image and the accumulator array was saved to a file. This was repeated until 100 accumulator arrays were generated.

Figure 3.3 Process to Verify Background Distribution

When testing to determine how well a theoretical probability distribution describes gathered data, the test is commonly referred as a *goodness of fit* test [15]. A standard goodness of fit test is the chi-squared (χ^2) test. Based on measuring the mean square error between the observed value and the expected value, the χ^2 statistic is calculated using:

$$\chi^2 = \sum \frac{(O - E)^2}{E} \quad (19)$$

where: O is the observed value

E is the expected value.

In this thesis, the chi-squared test will be used to determine if the accumulator array bins behave as a binomial distribution when only random noise is present in the image space. The binomial parameters, N and p , must be identified for each bin to allow the probability of each outcome, p_k , to be calculated for the set of all possible outcomes $S_x = \{0, 1, \dots, N\}$. Once p_k is calculated and the number of simulations, Y, determined, the expected number of observations for each possible outcome can be found by multiplying the two terms, Yp_k . The number of times each possible outcome occurs is the observed value, O_k . The χ^2 value is then calculated:

$$\chi^2 = \sum_{k=0}^N \frac{(O_k - Yp_k)^2}{Yp_k} \quad (20)$$

Harald Cramer provides a proof showing that as $Y \rightarrow \infty$, the random variable χ^2 calculated by (19) and (20) tends toward a χ^2 distribution with N degrees of freedom [15].

A large value of χ^2 will result when the number of times each member of S_x is observed in Y simulations deviates significantly from the expected number of times associated with the Binomial distribution. The level of significance is related to the area under the χ^2 distribution curve from the value calculated by (20) and $+\infty$. The smaller

the area, the more significant the difference. A χ^2 value that results in the area being equal to 0.05 means that there is a 5% probability that the observed data is from the same distribution as the expected data. This is the 5% significance level.

The test was conducted using an image space of 400 by 500 pixels and a parameterization of $\Delta p = 1$ and $\Delta \theta = 1$. The resulting accumulator array had 115380 bins. Of these bins, 80858 bins had a non-zero maximum value N . The largest maximum bin value was $N = 800$. Any size of image space and parameterization could be used to conduct the chi-squared test. The choice to use the 400 by 500 pixel image space was made to be consistent with the test images that will be presented in Chapter 4. The number of bins that were rejected at the 5% significance level for different values of the parameter p are presented in Table 3.2.

Noise Level (p)	number of bins at 5%
0.01	3
0.02	3
0.03	6
0.04	28

Table 3.2 Number of bins Significant at different Noise Levels

Since a 5% significance level was used in conjunction with the chi-squared test, it was anticipated that the null hypothesis would be rejected for 4043 bins, 5% of 80858. However, as shown in Table 3.2, significantly fewer bins were rejected at each of the different values of p considered. The reduced number of bins being rejected is a result of a limitation of the chi-squared test. In order for the test to be valid, the minimum number of expected observations must be greater than or equal to five[12]. In the remainder of this chapter, the technique implemented to ensure that this condition was respected and the consequence of the technique will be described.

Consider a binomial distribution with $N = 6$ and $p = 0.01$. If one hundred simulations are conducted, $Y=100$, the expected number of observations, Yp_k , of each member of S_x is shown in Table 3.3. As the 100 simulations are conducted, the number of times each member of S_x actually occurs is counted to determine O_k . The χ^2 test statistic is then calculated using (20):

$$\chi^2 = \frac{(O_0 - Yp_0)}{Yp_0} + \frac{(O_1 - Yp_1)}{Yp_1} + \dots + \frac{(O_6 - Yp_6)}{Yp_6}$$

where O_0 is the number of times a 0 is observed, O_1 is the number of 1's observed, O_2 is the number of 2's observed, O_3 is the number of 3's observed, O_4 is the number of 4's observed, O_5 is the number of 5's observed and O_6 is the number of 6's observed.

k	p_k	Yp_k
0	0.94148	94.148
1	0.05706	5.706
2	0.00144	0.144
3	1.94E-05	1.94E-03
4	1.47E-07	1.47E-05
5	5.94E-10	5.94E-8
6	1E-12	1E-10

Table 3.3 Expected number of observations for $Y=100$, $N=6$, $p=0.01$

As previously stated, the χ^2 test is valid when the term Yp_k is ≥ 5 [12]. For the distribution presented in Table 3.3, 5×10^{12} simulations would need to be run in order to satisfy this condition. An alternative to conducting a large number of simulations is to group terms until the combined Yp_k is ≥ 5 [12]. For the distribution presented in Table 3.2 the terms associated with the values of $k=1, 2, 3, 4, 5$ and 6 can be combined into one

term k' and the χ^2 test statistic is found using (20):

$$\chi^2 = \frac{(O_0 - Yp_0)}{Yp_0} + \frac{(O_{k'} - Yp_{k'})}{Yp_{k'}}$$

where:

$$p_{k'} = \sum_{k=1}^6 p_k = 0.0585 .$$

This will allow the condition of $Yp_k \geq 5$ to be satisfied by conducting only 100 simulations. However, it will weaken the ability to identify a significant observation increasing the possibility of a Type II error.. Assume that in 100 simulations, $O_0 = 93$, $O_1 = 5$ and $O_2 = 2$. Using (20) without grouping of terms to respect the $Yp_k \geq 5$ condition, the value of $\chi^2 = 24.0039$. This is significant at the 5% level for a χ^2 distribution with 6 degrees of freedom. Using (20) and combining observations in order to respect the $Yp_k \geq 5$ condition, the value of $\chi^2 = 0.2401$. This is not significant at the 5% level for a χ^2 distribution with 6 degrees of freedom.

The algorithm used to obtain the results presented in Table 3.2 combined observations to ensure the minimum value of $Yp_k \geq 5$. This is the most likely reason that so few bins were rejected, or found to be significant, at the 5% level.

While investigating the cause of fewer bins being rejected at the given significance level, the effect of the parameters associated with the binomial distribution were also examined. As shown in Figure 3.1, the shape of the binomial distribution will change for different values of the parameter p . The shape of the binomial distribution associated with the data presented in Table 3.3 will be skewed to the left. Table 3.4 contains the values of p_k and Yp_k for a binomial distribution with $N=6$, $p = 0.5$ and $Y=100$. In order to satisfy the condition of $Yp_k \geq 5$ for each member of S_x , $Y=320$

simulations would need to be conducted. This is significantly less than required for the binomial with parameters $N = 6$ and $p = 0.01$.

k	p_k	Yp_k
0	0.015625	1.56
1	0.09375	9.37
2	0.234375	23.43
3	0.3125	31.25
4	0.234375	23.43
5	0.09375	9.37
6	0.015625	1.56

Table 3.4 Expected number of observations for $Y=100$, $N=6$, $p=0.5$

The value of N will also affect the number of simulations required to ensure the minimum expected observations criteria will be satisfied. Table 3.5 contains the values of p_k for a binomial with $N = 12$ and $p = 0.01$. The number of simulations required to satisfy the minimum expected observations without combination of observations is 5×10^{24} . This suggests that as N increases by a factor of x , the number of simulations will increase by an exponential power of x .

As identified earlier, the bins in the accumulator used in the chi-squared test could have a value between $N = 0$ and $N = 800$. As well, the background noise level, or the parameter p , was small which resulted in the binomial distribution for each bin being skewed to the left. As discussed in this section, the combination of the large value of N and small value of p would require a tremendous number of simulations to be conducted in order for the minimum expected number of observations to be greater than or equal to 5 without combination of observations.

k	p_k
0	0.8864
1	0.1074
2	0.00597
3	0.000201
4	4.6E-06
5	7.4E-08
6	8.7E-10
7	7.5E-12
8	4.8E-14
9	2.1E-16
10	6.5E-19
11	1.2E-21
12	1E-24

Table 3.5 p_k for a binomial with $N = 12$, $p = 0.01$.

The hypothesis that random noise in the image space will result in the accumulator array bins behaving as a binomial distribution has not been satisfactorily proved to be true by using the chi-squared test. However, the hypothesis will be accepted for the purpose of this thesis.

Chapter 4

Test Image Description

4.1 Introduction

Three bitmaps files were generated each containing a single line. The first image space, shown in Figure 4.1 (a), has a line 12 pixels long with end points at (-202, 156) and (-204, 144). The second image space, shown in Figure 4.2 (a), has a line 42 pixels long with end points at (62, 148) and (77, 107). The third image space, shown in Figure 4.3 (a), has a line 18 pixels long with end points at (-23, 60) and (-40, 43). The origin of the co-ordinate system for all three images is the centre of the image space.

The bitmaps were corrupted by Gaussian noise on a pixel by pixel basis at a given background noise level: 0%, 1%, 2%, 3% and 4%. The Hough Transform was taken and the accumulator array was populated. For each image, 25 accumulator arrays were populated at each noise level. Figure 4.1 (b), 4.2 (b) and 4.3 (b) shows each line at 4% noise corruption.

Each bitmap was created using Microsoft Paint. Using Paint allowed the exact placement of the line to be determined. As well, it ensured that the image being corrupted was not already affected by external noise sources. Each pixel within the bitmap corresponds to an x-y point.

(a)

(b)

Figure 4.1 Test Image 1, 12 pixel line at 0% noise and 4% noise

(a)

(b)

Figure 4.2 Test Image 1, 42 pixel line at 0% noise and 4% noise

(a)

(b)

Figure 4.3 Test Image 1, 18 pixel line at 0% noise and 4% noise

4.2 Quantization of $\Delta\rho$ and $\Delta\theta$

Prior to taking the Hough Transform, the quantization level of the parameters $\Delta\rho$ and $\Delta\theta$ must be determined. The image space that was used is 400 by 500 pixels. The relationship between $\Delta\rho$ and $\Delta\theta$ developed by Van Veen and Groen [7] was presented in Chapter 2 and is repeated here and rearranged to solve for $\Delta\theta$:

$$\Delta\rho \geq l \sin\left(\frac{\Delta\theta}{2}\right) \rightarrow \Delta\theta \leq 2 \arcsin\left(\frac{l}{\Delta\rho}\right) \quad (21)$$

One of the two parameters must be set. The choice of fixing $\Delta\rho$ and solving equation (6) for each value of $\Delta\theta$ is the most straight forward computation. Setting $\Delta\rho = 1$, the ideal $\Delta\theta$ for each line would be:

line 1 $l=12, \Delta\theta \approx 9.56^\circ - 9^\circ$

line 2 $l=42, \Delta\theta \approx 2.73^\circ - 2^\circ$

line 3 $l=18, \Delta\theta \approx 6.37^\circ - 6^\circ$.

An alternative quantization was calculated using the work of Yuen and Ma [8]. For an image space of 400 by 500 and setting $\Delta\rho = 1$, $\Delta\theta$ is found from:

$$M \leq \frac{\Delta\rho}{\Delta\theta} \rightarrow \Delta\theta \leq \frac{\Delta\rho}{M}$$

where

$$M = \sqrt{\left(\frac{400}{2}\right)^2 + \left(\frac{500}{2}\right)^2} = 320.1562$$

$$\Delta\theta \leq \frac{1}{320.1562} = 0.0031 \text{ rads} = 0.179^\circ$$

Van Veen and Groen's relationship (21) was developed to minimize the spreading of the peak cell value across cells in the ρ direction [7]. The amount of peak spread, n_p , is related to $\Delta\rho$ and $\Delta\theta$ by :

$$n_p = \left\lfloor \frac{l \sin\left(\frac{1}{2}\Delta\theta\right)}{\Delta\rho} \right\rfloor + 2 \quad (22)$$

where $\lfloor x \rfloor$ is the largest integer less than x . If the peak in the accumulator array is spread across more than two rows of bins, that is $n_p > 2$, the parameterization is such that over sampling is occurring [7]. Using the quantization of $\Delta\rho = 1$ and $\Delta\theta = 0.16^\circ$, the amount of peak spread, n_p , is calculated for each image space using (22):

line 1 $l=12, n_p = 2$

line 2 $l=42, n_p = 2$

line 3 $l=18, n_p = 2$.

Since $n_p \leq 2$, additional peak spread would not be induced using the quantization $\Delta\rho = 1$ and $\Delta\theta = 0.16^\circ$, and this parameterization was used in preliminary simulations. Rearranging (21) to solve for l , the longest line which can be found at this parameterization is:

$$\frac{\Delta\rho}{\sin\left(\frac{\Delta\theta}{2}\right)} \geq l \rightarrow l \leq \frac{1}{\sin\left(\frac{0.16}{2}\right)} \quad (23)$$

$l = 716$

For $\Delta\rho = 1$ and $\Delta\theta = 0.16^\circ$, the accumulator array has dimensions 641 by 1125, a total array size of 721,125 bins. In doing initial simulations, both to test the noise distribution hypothesis and to test the ability to find features, this level of quantization resulted in an

excessive number of bins being significant at the 1% and 5% level.

A second quantization level was chosen of $\Delta\rho=1$ and $\Delta\theta=1$ resulting in an accumulator array with 115,380 bins. At this second value of parameterization, $n_p = 2$ for each image space so the peak will not be unduly spread in the ρ direction. The maximum line length which can be found without peak spread is found from (23) to have a value of $l = 114$ pixels. According to Yuen and Ma, this will have the same affect as under sampling the image space [8].

4.3 Theoretical Test Line Location

The purpose of analysing the accumulator array is to determine the length and angle of the normal that will describe the location of a line in the image space. To be able to quantify how well a specific analysis technique works, the true or theoretical normal must be known. For each image space, the true position of the line was found using two techniques. First, the theoretical normal was calculated using basic geometry. Second, the Hough Transform was taken of each image space without corrupting the background with noise. In the algorithm, this is the equivalent to taking the transform at 0% noise.

4.3.1 Line 1

Line 1, the 12 pixel line, is described by the equation:

$$y = 6x + 1368 \quad (24)$$

and using geometry, the theoretical normal is $(\rho, \theta) = (224.9, 170.54^\circ)$. Allowing for rounding, and using $\Delta\rho = 1$, $\Delta\theta = 1$, this corresponds to the bin $(\rho, \theta) = (225, 171^\circ)$. When the Hough Transform was taken of a clean image, the largest bin value was 12, which is exactly the number of pixels in the line. There were two bins with value 11 and two with value 10. The five largest bin values with the radius and angle information are shown in Table 4.1.

radius (ρ)	angle (θ)	bin value
235	165	12
228	169	11
230	168	11
224	171	10
226	170	10

Table 4.1 Top 5 bin values, Line 1 at 0% noise, $\Delta\rho=1$ $\Delta\theta=1^\circ$

The theoretical normal most closely corresponds to bin (224, 171°) which has value of 10. The discrepancy between the theoretical normal and the observed maximum normal appears quite large, $\Delta\rho$ of 11 pixels and $\Delta\theta$ of 6° . The difference is caused by two of the characteristics associated with the Hough Transform identified in Chapter 2: quantization of the image space and the inability to find the exact line location bound by the end points of the line

The quantization of the image space has caused the pixels to be turned “on” in three groups of four pixels as shown in Figure 4.4. The geometrical calculation is based on an assumption of a straight line existing between the two end points for which a slope can be calculated. This condition does not exist and results in the slope of the quantized line not actually being equal to six, $m \neq 6$, as indicated in (24). As a result, the angle θ will be calculated incorrectly since the angle is calculated from:

$$\theta = 90 - \arctan(m) \quad (25)$$

and the radius ρ will also be incorrectly calculated since the radius is calculated from:

$$\rho = x \cos \theta . \quad (26)$$

As described in Chapter 2, the Hough Transform finds the normal to a line across

the entire image space and not the line bound by the end points of the actual line segment. Figure 4.5 shows two lines, one associated with the normal found by the maximum bin value, $(235, 161^\circ)$ and one associated with the theoretical normal, $(224, 171^\circ)$. The two lines intersect in the region of the 12 pixel line. This region has been redrawn in Figure 4.6 and Figure 4.7 to show which pixels actually are on the two lines.

Figure 4.6 shows the pixels that lay on the line associated with the normal $(235, 165^\circ)$ as triangles. Comparing this with Figure 4.4, it can be seen that the line associated with this normal covers all 12 pixels of the line. Figure 4.7 shows the pixels that lay on the line associated with the theoretical normal $(224, 171^\circ)$ as triangles. Comparing this with Figure 4.4, it can be seen that the line misses two of the pixels at location $(-204, 148)$ and $(-202, 153)$ which are shown as circles in Figure 4.7. Because of the quantization in ρ and θ , these pixels are described by the lines normal to $(225, 171^\circ)$ and $(223, 171^\circ)$ respectively. This can be seen graphically and can be verified using (2):

$$\begin{aligned}\rho &= x \cos \theta + y \sin \theta \\ \text{for } (-204, 148) \\ \rho &= -204 \cos(171) + 148 \sin(171) \\ \rho &= 224.64 \rightarrow 225 \\ \text{for } (-202, 153) \\ \rho &= -202 \cos(171) + 153 \sin(171) \\ \rho &= 223.4475 \rightarrow 223\end{aligned}$$

Figure 4.4 Enlarged view of 12 pixel line

Figure 4.5 Image Space with lines associated with the theoretical and primary bins

Figure 4.6 Enlarged view of region of Line 1 and line associated with primary bin

Figure 4.7 Enlarged view of region of Line 1 and line associated with theoretical bin

When the Hough Transform was taken of the image space containing line 1 with the parameterization that results from using (21), $\Delta\rho = 1$ and $\Delta\theta = 9^\circ$, the top five bins were found to be as shown in Table 4.2.

radius (ρ)	angle (θ)	bin value
224	171	10
239	162	6
240	162	6
249	153	6
-204	0	4

Figure 4.2 Top 5 bin values, Line 1 at 0% noise, $\Delta\rho=1$ $\Delta\theta=9$

Comparing Table 4.2 to Table 4.1 shows that determining the parameters in the transform space based on the line length will yield results close to the expected theoretical values. However, one of the objectives of the thesis is to investigate the use of statistical tests without any *a priori* knowledge of the features in an image. Accordingly, when noise was introduced in the image space, Line 1 was declared to be found if any of the top five values shown in Table 4.1 were found as a result of analysis of the accumulator array.

4.3.2 Line 2

Line 2, the 42 pixel line, is described by the equation:

$$y = -2.73x + 317.47 \quad (27)$$

and using geometry, the theoretical normal is $(\rho, \theta) = (109.1, 20.1^\circ)$. Allowing for rounding and using $\Delta\rho = 1$ and $\Delta\theta = 1^\circ$, this corresponds to bin $(109, 20^\circ)$. The Hough

Transform was taken at 0% noise and the five largest bin values, bin radius and angle information are shown in Table 4.3.

radius (ρ)	angle (θ)	bin value
109	20	39
111	21	30
112	22	26
107	19	24
114	23	21

Table 4.3 Top 5 bin values, Line 2 at 0% noise, $\Delta\rho=1$ $\Delta\theta=1$

The bin with the maximum value corresponds directly to the theoretical normal. The bin value of 39 is less than the expected value of 42. Three points on the line, (63, 144), (74, 117) and (77, 109) do not contribute to (109, 20°) but do contribute to (108, 20°), (110, 20°) and (110, 20°) respectively.

In order for line 2 to be found, bin (109, 20°) had to be found.

4.3.3 Line 3

Line 3, the 18 pixel line, is described by the equation:

$$y = x + 83 \quad (28)$$

and using geometry, the theoretical normal is $(\rho, \theta) = (58.69, 135^\circ)$. Allowing for rounding, and using $\Delta\rho = 1$, $\Delta\theta = 1$, this corresponds to bin (59, 135°). The Hough Transform was taken at 0% noise and the five largest bin values are shown in Table 4.4.

radius (ρ)	angle (θ)	bin value
59	135	18
59	134	18
58	137	16
59	133	16
58	138	14

Table 4.4 Top 5 bin values, Line 3 at 0% noise, $\Delta\rho=1$ $\Delta\theta=1$

Two bins shared the same maximum bin value of 18 at the same radius but at different angles. Line 3 was considered to be found if either bin (59, 135°) and (59, 134°) was found.

4.4 Bin Value behaviour in the presence of noise

As noise is introduced into the image space, the bin values in the accumulator array will start to increase. With $\Delta\theta = 1^\circ$, each pixel that is “on”, either due to noise or as a result of a feature, will augment 180 bins in the accumulator array. The effect on the bins that correspond to Line 1, Line 2 and Line 3 will be examined in this section. The values presented are average values from 25 accumulator arrays generated for each image space at each noise level.

Since it was shown in Chapter 3 that each bin behaves as a Binomial Distribution in the presence of noise, it is possible to predict the bin values at each noise level by making use of (15). The predicted bin value associated with each line was determined by:

$$l + (N - l)p \quad (29)$$

where: $l=12$ for Line 1;

$l=42$ for Line 2;

$l=18$ for Line 3.

In the algorithm used, the pixels that were on the line were not affected by noise. This reduces the maximum number of pixels associated with the bins of interest from N to $(N-l)$. This is reflected in (29).

4.4.1 Line 1

Five accumulator array bins are associated with the 12 pixel line. In Table 4.5, the observed average value and the predicted value, using (29), for each bin at each noise level are presented. The average of the observed values is very close to the expected value at all noise levels for each of the five bins. At all noise levels, the difference between the five bins is not very large. Figure 4.8 plots the average of the five bins against the noise level and shows the trend for the bin value to increase as the noise level increases.

ρ	θ	N	1%		2%		3%		4%	
			expect	observe	expect	observe	expect	observe	expect	observe
235	165	234	14.34	14.08	16.68	16.8	19.02	18.64	21.36	21.36
228	169	298	13.98	13.69	16.96	16.2	19.94	19.56	22.92	23.48
230	168	278	13.78	14.04	16.56	16.28	19.34	19.72	22.12	22.72
224	171	353	13.53	13.32	17.06	16.44	20.59	20.6	24.12	23.88
226	170	325	13.25	13.16	16.5	15.64	19.75	19.96	23	22.28

Table 4.5 Line 1 average bin values at different noise levels

The expected variance and the observed variance are shown in Table 4.6. The observed

variance deviates from the expected for every bin at all noise levels.

ρ	θ	N	1%		2%		3%		4%	
			expect	observe	expect	observe	expect	observe	expect	observe
235	165	234	2.32	2.41	4.59	6.67	6.81	8.07	8.99	9.91
228	169	298	2.95	3.56	5.84	4.75	8.67	10.26	11.44	6.43
230	168	278	2.75	1.71	5.45	3.54	8.09	7.88	10.68	9.96
224	171	353	3.49	2.64	6.92	3.84	10.27	16.00	13.56	16.44
226	170	325	3.22	4.64	6.37	7.66	9.46	7.62	12.48	14.88

Table 4.6 Line 1 variance at different noise levels

4.4.2 Line 2

Only one bin is associated with the 42 pixel line. Table 4.7 presents the observed average value across the 25 simulations and the expected value at each noise level. The observed average value at each noise level is plotted in Figure 4.9. A gradual increase from a value of 39 at 0% noise to an average value of 54.2 at 4% noise is observed.

ρ	θ	N	1%		2%		3%		4%	
			expect	observe	expect	observe	expect	observe	expect	observe
235	165	428	43.28	43.32	47.56	46.6	51.84	51.56	56.12	54.2

Table 4.7 Line 2 average bin values at different noise levels

The expected variance and the observed variance are presented in Table 4.8. At 2% and 4% noise, the observed variance is very different from the expected variance. By inspection, this was determined to be a result of an inadequate sample size. The

maximum bin parameter, N , is relatively large while the noise level, p , is relatively small. The nature of the binomial distribution with large N and small p results in the distribution being skewed as shown in Figure 3.1 (a). The skew is such that as the probability of a value occurring decreases dramatically the further the value is from the mean. If a small sample size is used, most of the observed values will be close to the mean. This will result in the sample variance being smaller than the variance associated with underlying binomial distribution. As the sample size is increased, the probability of observing values further away from the mean increases and the sample variance will move asymptotically towards the variance associated with the underlying binomial distribution.

The sample size was increased by increasing the number of simulations to 100 at both 2% and 4% noise. At 2% noise, the mean was found to be 47 and the variance was 8. At 4% noise, the mean was found to be 54 and the variance was 16.67. These results agree closely with the expected mean and variance at the two noise levels.

p	θ	N	1%		2%		3%		4%	
			expect	observe	expect	observe	expect	observe	expect	observe
235	165	428	4.24	4.31	8.39	5.67	12.45	10.26	16.44	13.75

Table 4.8 Line 2 variance at different noise levels

4.4.3 Line 3

Two bins are associated with the 18 pixel line. The expected value and the average observed value of the two bins are presented in Table 4.9 The average value of both bins across the 25 simulations at each noise level is plotted in Figure 4.10. The bin that corresponds to the theoretical normal to the line, (59, 135), increases from a value of 18 to an average value of 47.2 while the bin (59, 134) increases from a value of 18 to an

average value of 39.04.

ρ	θ	N	1%		2%		3%		4%	
			expect	observe	expect	observe	expect	observe	expect	observe
59	135	733	25.33	25.2	32.66	32.96	39.99	38.68	47.32	47.2
59	134	517	23.17	23.16	28.34	27.24	33.51	33.56	38.68	39.04

Table 4.9 Line 3 average bin values at different noise levels

The expected and observed variance are presented in Table 4.10. At all noise levels, the difference between the expected and observed variance is quite significant. As for line 2, it was determined that the difference was the result of an inadequate sample size. Additional simulations were run at each noise level. The number of simulations, the observed mean and observed variance for each bin are shown in Table 4.11. Even at 400 simulations, the bin with the larger value of N still has a significant deviation between the observed and expected variance. This means that more than 400 simulations would be required for bins with a maximum value $N \geq 733$.

ρ	θ	N	1%		2%		3%		4%	
			expect	observe	expect	observe	expect	observe	expect	observe
59	135	733	7.26	4.75	14.37	18.71	21.33	12.31	28.15	18.08
59	134	517	5.12	6.97	10.13	8.19	15.04	12.01	19.85	19.37

Table 4.10 Line 3 variance at different noise levels

noise level	# of simulations		bin (59, 135°)	bin (59, 134°)
1%	100	mean	25	22
		variance	5.80	5.90
2%	200	mean	32	27
		variance	11.43	11.37
3%	400	mean	39	32
		variance	18.45	13.14
4%	400	mean	46	38
		variance	22.58	19.09

Table 4.11 Line 3 variance at different noise levels with increased sample size

Figure 4.8 Line 1 bin values

Figure 4.9 Line 2 bin values

Figure 4.10 Line 3 bin values

Chapter 5

Finding Features

5.1 Introduction

Using the images described in the previous chapter, a Monte Carlo experiment based on 25 simulations was conducted. Each image space was corrupted with background noise using a different noise generator for each simulation. The Hough Transform was taken of each of the corrupted images and the accumulator array was saved. The image was corrupted at four different noise levels, 1%, 2%, 3% and 4%. This resulted in 25 accumulator arrays being populated for each image space at each noise level. The results of five different techniques used to analyse the accumulator arrays are presented in this chapter. The five different techniques performed on each of the 25 accumulator arrays generated for each test image at each noise level is summarized in Figure 5.1.

Two distinct approaches were taken in the analysis. The first involved using an approach consistent with the traditional method of analysing the Hough Transform accumulator array. The occurrence of a feature was determined based on the accumulator array bin value alone.

The second approach used the alternative method of analysing the accumulator array being proposed by this thesis. For each bin, the probability of the observed bin value, k , was calculated making use of the maximum value of the bin, N , and the

background noise level, p and equation (17). The occurrence of a feature was based on the probability of the bin value occurring, p_k , if only noise was present in the image space.

The first technique as shown in Figure 5.1, used a bin threshold value to determine the presence of a feature in the image space. The second technique as shown in Figure 5.1 sorted the accumulator array bins by bin value and captured the bins with the largest 20 values. Both of these techniques are consistent with the traditional method of analysing the Hough Transform accumulator array.

The third technique as shown in Figure 5.1 used a significance level threshold to identify the existence of a feature in the image space. The fourth technique as shown in Figure 5.1 sorted the accumulator array bins capturing the 20 bins with the smallest probability of occurrence associated with the observed bin value. The fifth technique as shown in Figure 5.1 used a bin value threshold adjusted for the level of background noise to identify the existence of a feature in the image space. The final three techniques are based upon the proposed method to analysis the Hough Transform accumulator array.

Two measures of performance were of interest in analysing the results associated with the two approaches. The first performance measure determined the ability of the two approaches to correctly find a known feature in the image space at different background noise levels. Of the five techniques described above, the second and fourth techniques were used to assess this performance criteria. The second performance measure was related to the number of false features found in the image space by the traditional method to analyse the accumulator array and the proposed method to analyse the accumulator array. Of the five techniques described above, the first, third and fifth techniques were used to assess this performance criteria.

Figure 5.1 Algorithm to Generate and Analyse the Accumulator Array

5.2 Global Maximum

One technique used in the analysis of the accumulator array involves searching for the global maximum value of a specific accumulator array bin attribute. The bins in the accumulator array are sorted by the desired attribute. Once sorted, the bin which maximizes the attribute is identified as being associated with a feature in the image space. This technique will be used to measure the relative performance to correctly identify a known feature in the image space.

For the traditional analysis of the accumulator array, the attribute that the bins are sorted by is the bin value. The bins are sorted by descending bin value and the bin with the largest value is identified as being associated with a feature in the image space.

The method being proposed in this thesis uses the probability of the bin value or greater occurring as the attribute. The purpose of the bin analysis is to identify the observed bin value that has the least likely chance of occurring if only noise is present in the image space. This will maximize the probability that the bin is associated with a feature. The accumulator array bins are sorted by ascending order of the probability of the bin value or greater occurring. The bin with the value that has the least probability of occurring due to noise is identified as the one most likely to correspond to a feature.

In the data that will be presented in the following two sections, the top twenty bins were recorded after the bins were sorted instead of a single bin. The use of the global maximum in this analysis will demonstrate the improvement in finding small lines using the analysis technique being proposed in this thesis over the analysis technique used in the standard Hough Transform.

5.2.1 Bin Value

Following the approach used to analyze the accumulator array that is consistent with the standard Hough transform, the bins in each accumulator array were sorted based on descending bin value. For each accumulator array, the bins with the largest 20 values were recorded. The (ρ, θ) combinations describing the 20 bins were compared to the (ρ, θ) combinations associated with the test lines as identified in section 4.3. For each accumulator array, the instance of the test line was noted under one of four headings: primary line, second line, 3rd to 20th line and line not captured. The test line was found as the primary line if a bin corresponding to one of the bins associated with the test line had the largest value. Similarly, if a bin corresponding to the test line had the second highest value, it was identified as the second line found. Bins associated with the test line that corresponded to having the third largest to twentieth largest value were captured as between 3 and 20. Finally, the number of times that the bins corresponding to the test line were not one of the 20 bins with the largest value, it was identified as the line not being captured.

The number of times each line was found and the where it ranked in the top 20 for each noise level is presented in Tables 5.1, 5.2, 5.3 and 5.4.

The 42 pixel line was found at every noise level as one of the top 20 lines and was the primary line 95 out of 100 times. The 18 pixel line was found in 99 of the 100 simulations and at 1% and 2% noise as the primary line in each accumulator array. At 4% noise, the 18 pixel line was not captured in one simulation as one of the top twenty lines. The 12 pixel line was not found in 78 of the 100 simulations. It was found as either the primary or secondary line in 14 of the arrays at 1% noise.

	primary line	second line	between 3-20	not captured
Line 1 (12 pixel)	3	11	7	4
Line 2 (42 pixel)	25			
Line 3 (18 pixel)	25			

Table 5.1 Number of times lines found in 1% noise using bin value

	primary line	second line	between 3-20	not captured
Line 1 (12 pixel)			1	24
Line 2 (42 pixel)	25			
Line 3 (18 pixel)	25			

Table 5.2 Number of times lines found in 2% noise using bin value

	primary line	second line	between 3-20	not captured
Line 1 (12 pixel)				25
Line 2 (42 pixel)	24	1		
Line 3 (18 pixel)	17	4	4	

Table 5.3 Number of times lines found in 3% noise using bin value

	primary line	second line	between 3-20	not captured
Line 1 (12 pixel)				25
Line 2 (42 pixel)	21	3	1	
Line 3 (18 pixel)	13	7	4	1

Table 5.4 Number of times lines found in 4% noise using bin value

5.2.2 Probability of Bin Value

The proposed alternative for analysing the accumulator array was based upon the hypothesis that only noise was present in the image space. As a result of this hypothesis, a bin value that had a low probability of occurrence would indicate a feature was present in the image space.

For each accumulator array bin, the probability of the observed bin value or greater occurring was calculated using a binomial distribution. The probability of occurrence, p_{bin} , was calculated as the sum of the binomial probabilities of the observed bin value up to and including the maximum value for the bin, N . The individual probabilities were calculated using equation (17). The equation describing the calculation is:

$$p_{bin} = \sum_{k=binvalue}^N \left(\frac{N}{k} \right) p^k q^{N-k} \quad (30)$$

where p_{bin} = the probability of the bin value or greater occurring

N = maximum bin value

p = noise level in the image space.

The bins were sorted based on ascending probability value and the bins with the lowest twenty values of p_{bin} were recorded. The 20 bins were then examined to determine if any of them corresponded to a bin associated with a test line as identified in section 4.3. The instance of each test line was recorded using the same four categories as described in the previous section.

The number of times each line was found and the where it ranked in the top 20 for each noise level is presented in Tables 5.5, 5.6, 5.7 and 5.8.

	primary line	second line	between 3-20	not captured
Line 1 (12 pixel)	24	1		
Line 2 (42 pixel)	25			
Line 3 (18 pixel)	24	1		

Table 5.5 Number of times lines found in 1% noise using p_{bin}

	primary line	second line	between 3-20	not captured
Line 1 (12 pixel)	13	3	9	
Line 2 (42 pixel)	25			
Line 3 (18 pixel)	22	1	2	

Table 5.6 Number of times lines found in 2% noise using p_{bin}

	primary line	second line	between 3-20	not captured
Line 1 (12 pixel)	13	3	4	5
Line 2 (42 pixel)	24	1		
Line 3 (18 pixel)	23	2		

Table 5.7 Number of times lines found in 3% noise using p_{bin}

	primary line	second line	between 3-20	not captured
Line 1 (12 pixel)	6	3	7	9
Line 2 (42 pixel)	23	2		
Line 3 (18 pixel)	8	3	12	2

Table 5.8 Number of times lines found in 4% noise using p_{bin}

5.2.3 Comparison of Techniques

Comparing the results shown in Tables 5.1 through 5.8 for line 1, the 12 pixel line, the improved capability of the proposed method to find small lines at each noise level is immediately obvious. At 1% noise, the 12 pixel line can only be found as the primary line 3 times using the traditional method of analysing the accumulator array bins. However, when the same accumulator arrays are analysed using the value of p_{bin} , the 12 pixel line, is found as the primary line 24 times. At 2% and 3% noise, the line is never found as the primary line using the traditional method of analysis. Using the proposed method, however, it was found as the primary line 13 times at 2% noise and 13 times at 3% noise. Figure 4.1 (b) shows the image space with the 12 pixel line with 4% background noise. It is difficult to identify the location of the line visually. The proposed method, however, was able to find it as the primary line 6 times out of 25. In fact the proposed method did not identify the line in only 9 out of the 25 tests. This is a significant improvement over the traditional method of analysis which never found the line at 4% noise.

There was no significant advantage using the proposed probability approach over the traditional analysis based on bin value for the relatively long line of 42 pixels.

At 1% and 2% noise there was no difference between the two methods in the ability to correctly find the 18 pixel line. At 3% noise, the proposed probability approach found the line more often as the primary line than the traditional analysis using the observed bin values. At 4% noise, the use of bin values performed better than the use of the probability of the bin value.

5.2.4 Minimum Line Length

The decrease in the ability to find the 18 pixel line at 4% noise using the proposed

analysis approach suggests that there will be a minimum line length l that can be found for each bin related to the maximum bin value N and the image space noise level p . In order to develop this relationship, the Gaussian approximation to the Binomial distribution will be used. The relationship developed will only be valid for bins that can be approximated by a Gaussian distribution. For each of the bins that correspond to the test lines used in this thesis, $Npq \gg 1$ and the Gaussian approximation is valid.

The mean bin value is given by $\mu = Np$ and the standard deviation is given by $\sigma = (Npq)^{1/2}$. The probability of the bin value or greater occurring corresponds to the area under the probability distribution curve that describes the bin behaviour in the presence of image space noise which falls to the right of the observed bin value. For a Gaussian distribution, this area is found from equation (14) using the mean and the standard deviation as stated above.

$$area = Q\left(\frac{binvalue - Np}{\sqrt{Npq}}\right) \quad (31)$$

When the accumulator array bins were sorted using p_{bin} as the bin attribute, the value of p_{bin} was recorded for each of the 20 bins with the lowest p_{bin} value. When a test line was found as the primary or second line as shown in Tables 5.5, 5.6, 5.7 and 5.8, p_{bin} had a value equal to or less than 2×10^{-6} . As was explained in the previous section, p_{bin} corresponds to the area under the binomial distribution describing the bin behaviour to the right of the observed bin value. This is the same area indicated in (31).

From look up tables, the value of the $Q(.)$ function argument that corresponds to an area of 2×10^{-6} is approximately 4.61. This value will be the lowest value of the $Q(.)$ function argument since the area in (31) will decrease as the argument increases. The smallest line that can be found from each noise level can then be found by solving the following equation:

$$4.61 \leq \frac{binvalue - Np}{\sqrt{Npq}}$$

Substituting

$$binvalue = l + (N - l)p$$

and solving for the line length, l , :

$$4.61 \leq \frac{l + (N - l)p - Np}{\sqrt{Npq}}$$

$$l \geq \frac{4.61(\sqrt{Npq})}{1 - p} \quad (32)$$

Using equation (32), the minimum line length which can be found for each bin by using the proposed accumulator array analysis technique can be calculated. For the bins that were identified as corresponding to a line in the test images, the minimum line length at each noise level were calculated are presented in Tables 5.9, 5.10 and 5.11.

Rho	theta	N	1% noise	2% noise	3% noise	4% noise
235	165	234	7	10	12	14
228	169	298	8	11	14	16
230	168	278	8	11	13	15
224	171	353	9	12	15	17
226	170	325	8	12	14	16

Table 5.9 Minimum line length for bins associated with Line 1

Rho	theta	N	1% noise	2% noise	3% noise	4% noise
109	20	428	9	13	16	19

Table 5.10 Minimum line length for bins associated with Line 2

Rho	theta	N	1% noise	2% noise	3% noise	4% noise
59	135	733	12	17	21	24
59	134	517	10	15	18	21

Table 5.11 Minimum line length for bins associated with Line 3

From the results shown in Table 5.9, the 12 pixel line should be able to be found up to and including 3% noise. This is consistent with the results found in section 5.2.2. The 42 pixel line should be able to be found at all noise levels according to the information in Table 5.10. This is also consistent with the results found in section 5.2.2. Finally, Table 5.11 indicates that the 18 pixel line should be found up to and including 3% noise. Again, this is consistent with the results found in the previous section.

5.3 Threshold Techniques

A second technique that was used to analyse the 25 accumulator arrays employed threshold values. For each bin, a single attribute was compared to an appropriate threshold. Depending on the relationship found between the attribute and the threshold, the bin was identified as being associated with a line in the image space. This technique was used to assess the relative ability of the three different analysis performed to minimize the number of false features identified.

The attribute used for comparison in the standard Hough Transform is the bin

value. The value at which the threshold is set will determine how well the standard Hough Transform will find a feature. If the threshold is set too low, “false features” will be found. This occurs when a bin value is greater than or equal to the threshold but there is no feature in the image space associated with the bin. If the threshold is set too high, true features may be missed. This occurs when a bin that is associated with a feature in the image space has a value less than the threshold.

The approach being proposed in this thesis uses the probability of the bin value occurring as the attribute and a significance level as the threshold. For each bin, the probability of the bin value or greater occurring is calculated assuming the image space contains only noise. If the probability is less than the significance level, a feature is identified. If the probability is greater than the significance level, it is determined that no feature is associated with the bin. False features will be found if the significance level is too high and features will be missed if it is too low.

5.3.1 Traditional Threshold Analysis

Each bin value was compared to nine different thresholds: 10, 15, 20, 25, 30, 35, 40, 45 and 50. As will be seen in the following graphs, the number of bins that exceed the lower thresholds was quite large, even at 1% noise. In order to present the data in a more readable format, bins with a value equal to or larger than a threshold but less than the next threshold were recorded for each threshold level. That is, the bins that were recorded as exceeding the threshold of 10 are the bins which had a value of 10, 11, 12, 13, and 14. The total number of bins with a value greater than a specific threshold can be found by adding the number of bins at thresholds greater than and equal to the threshold of interest.

The average number of bins in the 25 simulations that exceed each of the thresholds at 1% noise for each image space is shown in Figure 5.2. From this diagram it

can be seen the number of bins which exceed a threshold of 10, i.e. having a value of 10, 11, 12, 13 or 14, is approximately 1500 bins for each image space containing the three different lines. The image space that contains the line 42 pixels has an average number of bins that is slightly higher than the other two image spaces. The graph in Figure 5.2 (a) is redrawn in Figure 5.2 (b) to show only the portion of the curves which correspond to a threshold of 15 or higher. It can be seen that the image space with the 42 pixel line has more bins found at each threshold than the other images. The image space containing the 18 pixel line has no bins found at thresholds greater than 35. The image space containing the 12 pixel line has no bins found at thresholds greater than 20.

Figure 5.3 (a) is a plot of the average number of bins in the 25 simulations that exceed each threshold at 2% noise. Comparing the number of bins that exceed a threshold of 10, a threshold of 15 and a threshold of 20 at 2% noise to the number that exceed the same thresholds at 1% noise, there are more bins found at 2% noise. Table 5.12 shows the number of bins found the three thresholds at 1% and 2% noise.

threshold	1% noise	2% noise
10	1459	22092
15	22	3502
20	1	196

Table 5.12 Average number of bins exceeding specific thresholds at 1% and 2% noise

As expected, the image containing the 42 pixel line has slightly higher values at each of the three thresholds, 10, 15 and 20, than the other two images. The graph in Figure 5.3(a) is redrawn in Figure 5.3(b) to clearly show the difference between the number of bins that exceed the higher thresholds.

Figure 5.4 (a) is a plot of the average number of bins in the 25 simulations that exceed each threshold at 3% noise. At 3% noise, more bins were found to exceed each

threshold than in 2% noise. The greatest increase was observed at higher thresholds.

Table 5.13 shows the number of bins found at the two noise levels for thresholds of 15, 20, 25 and 30.

threshold	2% noise	3% noise
15	3502	18443
20	196	4778
25	8	568
30	1	40

Table 5.13 Average number of bins exceeding specific thresholds at 2% and 3% noise

The average number of bins which exceed each threshold is not significantly different for the three image spaces at any threshold value. The trend of the image space with the 42 pixel line to have slightly higher bin values continues. The graph in Figure 5.4 (a) is redrawn in Figure 5.4 (b) so that the number of bins exceeding the higher thresholds for each line can be clearly seen.

Figure 5.5(a) is a plot of the average number of bins in the 25 simulations that exceed each threshold at 4% noise. The shape of the curve has changed from the shape observed at 1%, 2% and 3% noise shown in Figures 5.2(a), 5.3(a) and 5.4(a) respectively. At the lower noise levels, the number of bins that were captured by the threshold of 10 is larger than the number captured at any of the higher thresholds. At 4% noise, the number of bins captured by the threshold of 15, that is the number of bins having a value of 15, 16, 17, 18 and 19, is greater than the number of bins captured by the threshold of 10. As expected, the total number of bins exceeding 10 has actually increased as the noise is increased from 52,332 bins at 3% noise to 64,038 at 4% noise. However, more of the bins have value between 15 and 19 than between 10 and 14. The graph in Figure 5.5(a) has been redrawn in Figure 5.5(b) so that the number of bins exceeding the higher thresholds for each line can be clearly seen.

(a)

(b)

Figure 5.2 Average number of bins exceeding thresholds at 1% noise

(a)

(b)

Figure 5.3 Average number of bins exceeding thresholds at 2% noise

(a)

(b)

Figure 5.4 Number of bins exceeding thresholds at 3% noise

(a)

(b)

Figure 5.5 Number of bins exceeding thresholds at 4% noise

5.3.2 Threshold adjusted for Noise

In chapter 4, the average bin value was shown to increase as background noise in the image space increased. In this section, the results of a second threshold method that adjusts the value of the threshold for each bin to mitigate the effect of rising bin values due to noise are presented. By adjusting for the noise, the number of features found at each threshold should decrease when compared to the number of features found in the previous section at the same thresholds.

The threshold is adjusted by adding the expected bin value caused by noise in the image space to the desired threshold value:

$$t' = Np + t \quad (33)$$

where:

t' is the adjusted threshold

Np is the expected bin value due to noise in the image space

t is the desired threshold.

The bins that have a value greater or equal to t' are captured as bins describing the location of a line. Using the same thresholds as in 5.3.1, t had values of 10, 15, 20, 25, 30, 35, 40, 45 and 50. The results that will be presented in this section allows double counting of bins. That is, a bin that is greater than t' for $t = 20$ will also be included in the number of bins greater than t' at $t = 10$ and $t = 15$.

Figure 5.6 is a plot of the average number of bins that exceed t' for a given threshold t at 1% noise. Figure 5.7 is a plot of the average number of bins that exceed t' for a given threshold t at 2% noise. Comparing the curves in the two figures, it can be seen that the average number of bins exceeding a threshold of 10 increased for each image space as the noise level increased from 1% to 2%. The average number of bins

exceeding thresholds of 15 or higher did not have the same increase in value as the noise level increased. This means that more small lines of size 10, 11, 12, 13 or 14 pixels will be found at 2% noise.

Figure 5.8 (a) is a plot of the average number of bins that exceed t' for a given threshold t at 3% noise. There is a large increase in the average number of bins exceeding a threshold of 10 at 3% noise, approximately 375 bins, compared to the number exceeding the same threshold at 2% noise, approximately 115 bins. The curve in Figure 5.8 (a) has been redrawn in Figure 5.8 (b) to allow the average number of bins exceeding the higher thresholds to be more clearly seen. Comparing Figure 5.8 (b) to Figure 5.6 and Figure 5.7, it can be seen the average number of bins that exceed a threshold of 15 has increased at 3% noise over the number found at 1% and 2% noise. The average number of bins that exceed the thresholds of 20 or greater has not significantly increased from 1% and 2% noise.

Figure 5.9 (a) is a plot of the average number of bins that exceed t' for a given threshold t at 4% noise. The average number of bins found to exceed the thresholds of 10 and 15 have essentially doubled from 3% noise. Figure 5.9 (a) has been redrawn in Figure 5.9 (b) to allow the average number of bins exceeding the higher thresholds to be more clearly seen. Comparing Figure 5.9 (b) to Figure 5.6, Figure 5.7 and Figure 5.8(b), it can be seen the average number of bins that exceed thresholds of 20 or greater have slightly increased as the noise has increased from 1% to 4%.

The average number of bins that exceed each threshold is slightly higher for the image space containing the 42 pixel line at all noise levels.

Figure 5.6 Number of bins exceeding threshold adjusted for 1% noise

Figure 5.7 Number of bins exceeding threshold adjusted for 2% noise

(a)

(b)

Figure 5.8 Number of bins exceeding threshold adjusted for 3% noise

(a)

(b)

Figure 5.9 Number of bins exceeding threshold adjusted for 4% noise

5.3.3 Proposed Threshold Analysis

As stated in the introduction to this chapter, the proposed method to analyze the accumulator array is based on the probability of the observed bin value occurring if no feature is associated with the bin. More precisely, the probability of the observed bin value or greater occurring, p_{bin} , is calculated. In Chapter 3, it was shown that the bins will behave as binomial random variable if only noise is present in the image space. A small value of p_{bin} will indicate that the observed bin value is significantly different than that associated with a binomial random variable. If the bin value deviates from the binomial random variable, then there must be a feature associated with the bin.

In essence, the proposed approach is a Null Hypothesis test. The Null hypothesis that was tested can be stated as:

h_0 : No feature is associated with the bin.

If a Type I error was made, that is if the null hypothesis was rejected when no feature existed in the image space, a “false feature” was found. If a Type II error was made, that is if the null hypothesis was accepted when a feature actually did exist in the image space, a feature was missed.

Four significance levels were used; 1×10^{-3} , 1×10^{-4} , 1×10^{-5} and 2×10^{-6} . The significance levels were chosen based on the results of section 5.2.2 and 5.2.4. The null hypothesis was rejected for bins for which p_{bin} was greater than the significance level under consideration. The significance levels can also be stated as percentage confidence levels. The four confidence levels are 99.9%, 99.99%, 99.999% and 99.9998%.

By using significance levels instead of straight thresholds, an increasing number of bins should not be found to exceed a given significance level as the background noise

increases. Rather, the number of bins found to exceed a given significance level should be reasonably constant at all noise levels. The number of bins that exceed different significance levels should increase and decrease as the significance level increases and decreases. In other words, the number of bins found to be significant at a level of 1×10^{-3} should be greater than the number of bins found to be significant at a level of 1×10^{-5} .

In collecting the data that is presented in Figure 5.10, there is double counting of bins across significance levels. That is a bin found to be significant at the 1×10^{-3} significance level will also be significant at the other three levels of significance and will be included in the bin count at the other significance levels. This double counting was not removed since the total number of bins found at each significance level was not excessive.

Figure 5.10 (a) is a plot of the average number of bins for which the null hypothesis was rejected at each significance level for the image space containing the 12 pixel line. The number of bins for which the null hypothesis was rejected at the 2×10^{-6} significance level is much lower than the number rejected at the 1×10^{-3} significance level, 3 bins compared to approximately 67 bins. This means the number of Type I errors will increase as the significance level increases resulting in an increasing number of false features being detected.

Figure 5.10 (b) is a plot of the average number of bins for which the null hypothesis was rejected at each significance level for the image space containing the 42 pixel line at each of the four noise levels considered. Figure 5.10 (c) is a plot of the average number of bins for which the null hypothesis was rejected at each significance level for the image space containing the 18 pixel line at each of the four noise levels considered. As with the image space containing the 12 pixel line, the number of bins for which the null hypothesis will be rejected increased as the significance level increased.

(a) average number of significant bins for the image with the 12 pixel line

(b) average number of significant bins for the image with the 42 pixel line

(c) average number of significant bins for the image with the 18 pixel line

Figure 5.10 Number of bins exceeding different significance levels

The average number of bins for which the null hypothesis was rejected was higher at 1% noise for each image space at every significance level except for one case, the image space containing line 3 at a significance level of 1×10^{-3} . As stated earlier, the expectation was that the number of bins for which the null hypothesis would be rejected should have been relatively constant at a given significance level. The observed behaviour is the effect of inherent noise associated with the Hough Transform.

The Hough Transform inherent noise results in bins having a non-zero value even though they are not associated with a feature. If the inherent noise value of these bins is of sufficient size, the significance level at which the test is being conducted may be exceeded prior to corruption of the image space by noise. Consider the image space containing line 2, the 42 pixel line. In section 4.3.2, the bins with the top five bin values at 0% noise were presented in Table 4.6. The largest bin value was 39 and this was defined as the bin associated with the test line. The next largest four bins values were 30, 26, 24 and 21. These bin values resulted from the Hough Transform inherent noise.

When the image space is corrupted by noise, the value of a bin with a non-zero value, due to the inherent noise, will not be the expected value of the binomial as calculated by (15). Instead, the value of the bin will be the expected value found by (15) added to the inherent noise value of the bin. As the noise level increases, the shape of the binomial curve describing the bin will change slightly and the bin value associated with the same significance level will increase slightly. As the bin value associated with the significance level increases, the effect of the inherent noise will be decreased. Therefore, bins that have a non-zero value due to inherent noise may be significant at 1% noise but not at 4% noise.

5.3.4 Comparison of Threshold Techniques

The results of three threshold analysis techniques were presented in the previous

three sections. The first two techniques used bin value thresholds. Both of these techniques resulted in an increasing number of bins exceeding a given threshold as the image space noise level increased. The impact of the results will be considered for each image space.

The image space that contained the 12 pixel line has five bins that have a value equal to or greater than 10, as was shown in Table 4.1. If more than five bins are found that exceed a threshold of 10, the excess bins will be associated with false features being found. From the data presented in Figures 5.2 to 5.9, the total number of bins that on average exceed a threshold of 10 are tabulated in Table 5.14. When a straight threshold is used which is not adjust for noise in the image space, as presented in section 5.3.1, a large number of bins are found using a threshold of 10. Each of the excess bins will result in a false feature being identified in the image space. When the threshold was adjusted for each bin according to the noise level in the image space, as presented in section 5.3.2, a minimum of 3 excess bins were found to exceed the threshold of 10 at 1% noise. However, at the higher noise levels considered, the number of bins increased. At 2% noise, for example, a minimum of 90 excess bins were found to exceed the threshold of 10. This will result in 90 potential false features being identified in the image space. More false features will be identified if the five bins associated with the actual feature are not included in the 95 bins.

Noise Level	non-adjusted threshold	noise adjusted threshold
1%	1,440	9
2%	25,509	95
3%	52,220	374
4%	64,124	762

Table 5.14 Line1: Average number of bins exceeding a threshold of 10

The analysis of the accumulator array associated with the image space containing the 42 pixel line resulted in a similar average number of bins exceeding a threshold of 10 as the image space containing the 12 pixel line. However, the threshold of interest for the 42 pixel line is 35. When no noise was present in the image space, the bin associated with the line in the image space had a value of 39. This would be captured by the threshold of 35. The average number of bins that exceeded a threshold of 35 and a noise adjusted threshold at each noise level is tabulated in Table 5.15. When the threshold was adjusted for the noise level, on average only one bin exceeded a threshold of 35. When a straight threshold was used, the number of bins that exceeded the threshold of 35 increased from 3% noise to 4% noise. At 4% noise, 153 potential false features will be identified at a threshold of 35.

noise level	non-adjusted threshold	noise adjusted threshold
1%	1	1
2%	3	1
3%	7	1
4%	154	1

Table 5.15 Line2: Average number of bins exceeding a threshold of 35

The accumulator array associated with the image space containing the 18 pixel line had four bins that exceeded a value of 15 when no noise was present in the image space as shown in Table 4.4. The average number of bins that exceeded a threshold of 15 at each noise level is tabulated in Table 5.16. With the straight threshold of 15, the number of false features that would be identified varies from 19 at 1% noise to approximately 45,956 at 4% noise. Using a threshold that is adjusted to account of different noise levels in the image space, the number of false features was significantly reduced.

noise level	non-adjusted threshold	noise adjusted threshold
1%	23	4
2%	3686	9
3%	23,771	15
4%	45,956	46

Table 5.16 Line3: Average number of bins exceeding a threshold of 15

In addition to the observation concerning the increase in false features being detected as the noise level in the image increases, a second observation can be made from the recorded data. The second observation reveals when the background noise in the image becomes the dominant factor on the observed bin value instead of the feature(s) present in the image. The noise can be declared as the dominant factor when the number of bins exceeding a given threshold results in a large number of false features being detected. This is the situation for a threshold of 10 for each of the images at 1% noise as shown in Figure 5.3.

The noise can also be declared as the dominant factor when the number of bins exceeding a given threshold increases significantly as the noise level increases. This is the situation as the noise increases from 1% to 2% for the thresholds 10, 15 and 20 and at all thresholds, except 45 and 50, as the noise increases from 2% to 3% and from 3% to 4%.

The results suggest that at 1% noise, the noise will be the dominant influence on the bin with lower bin values which are captured by the threshold value of 10. At 2% noise, the noise is the dominant influence on bins captured by the thresholds 10, 15 and 20. At 3% and 4% noise, the noise appears to be the dominant influence on bins captured by all the thresholds except 45 and 50. Compensating for the noise by adjusting the

threshold on a bin by bin basis by the expected value of the noise value reduced the number of bins found at each threshold. However, there are still more bins found at each threshold than lines in the test images.

When a statistical threshold is used instead of a fixed bin value threshold, the number of bins that would indicate the presence of a line has been significantly reduced at each noise level compared to the number of bins found using the straight bin value threshold as presented in section 5.3.1. Table 5.17 shows the average number of bins exceeding a significance level of 1×10^{-5} and the average number of bins exceeding a straight threshold of 10 for the image space containing Line 1. The average number of bins found to exceed the significance level is much smaller than the average number bins that exceed a straight threshold.

Noise Level	straight threshold	significance level
1%	1,440	5
2%	25,509	2
3%	52,220	1
4%	64,124	1

**Table 5.17 Line1: Average number of bins
exceeding a threshold of 10 and significance level of 1×10^{-5}**

Table 5.18 shows the average number of bins exceeding a significance level of 1×10^{-5} and the average number of bins exceeding a straight threshold of 35 for the image space containing Line 2. At 4% noise, the average number of bins found to exceed the significance level is much smaller than the average number bins that exceed a straight threshold. The decreasing average number of bins exceeding the significance level as the noise increases was explained in section 5.3.3.

Noise Level	straight threshold	significance level
1%	1	21
2%	3	11
3%	7	7
4%	154	6

**Table 5.18 Line 2: Average number of bins
exceeding a threshold of 35 and significance level of 1×10^{-5}**

Table 5.19 shows the average number of bins exceeding a significance level of 1×10^{-5} and the average number of bins exceeding a straight threshold of 15 for the image space containing Line 3. The average number of bins found to exceed the significance level is much smaller than the average number bins that exceed a straight threshold. The decreasing average number of bins exceeding the significance level as the noise increases was explained in section 5.3.3.

Noise Level	straight threshold	significance level
1%	23	6
2%	3686	3
3%	23,771	1
4%	45,956	1

**Table 5.19 Line3: Average number of bins
exceeding a threshold of 15 and significance level of 1×10^{-5}**

In addition to reducing the average number of bins found, using a significance level instead of a bin value threshold has a second benefit. The average number of bins found to exceed a given significance level does not increase with increasing background

noise as it does using a bin value threshold. This will result in a relatively constant amount of data being passed to the third subsystem shown in Figure 1, identification, which in turn will make the implementation of the subsystem more straight forward. That is, the information flowing from the feature detection subsystem will be more synchronous in nature using a threshold based on significance level and more asynchronous in nature using a bin value threshold.

Chapter 6

CONCLUSIONS

The analysis technique proposed by this thesis was shown to provide an improved capability to find short lines in the outer regions of an image space compared to the standard Hough Transform implementation.

This thesis explored the use of statistical analysis of the accumulator array generated by taking the standard Hough Transform of a noisy image space. The technique investigated is supported by Cohen and Toussaint [1] as an alternative to their developed solution and by Princen, Illingworth and Kittler [3]. The technique was based on having no *a priori* knowledge of the length of the feature. A probability of the occurrence of a line can be calculated for each bin using the knowledge that the bin behaviour in the presence of only noise is distributed as a binomial distribution.

The noise model assumed in this thesis was different from the one used by Cohen and Toussaint [1] and Maitre[2]. The work in this thesis was based on the assumption that each pixel in the image space was corrupted by independent noise. As a result, each pixel had a probability p of being corrupted by noise. This assumption allowed a binomial random variable to be used to describe the accumulator array bin value resulting from image space noise. If the noise acting on each pixel is not independent, the analysis would be much more difficult since each pixel would have a unique probability p_i of being corrupted by noise. This would mean the bin behaviour would no longer be described by a simple binomial with parameters (N, p) .

The chi-squared test was used to verify the hypothesis that the bin behaviour was truly a binomial distribution. The actual test proved to be inconclusive as a result of the algorithm used to respect the conditions of validity for the chi-squared test and the number of simulations conducted. The excellent results obtained in identifying features in the test images used in this thesis support the hypothesis that the bins do behave as a binomial distribution in the presence of random image space noise.

A significance level of 2×10^{-6} , or a confidence level of 99.9998%, was found to be appropriate for statistical tests applied to the accumulator array bin values. This level, however, was derived empirically from the data gathered and was not the result of a mathematical derivation. Further investigation would be required to determine if the significance level used in the statistical analysis of the accumulator array bin is dependent upon the parameterization chosen.

Based on the significance level used in the statistical analysis, the minimum line length that can be found that is associated with a given accumulator array bin can be determined. The minimum line length is related to the maximum number of pixels associated with a bin, N . As N increases, the size of the minimum line length that can be found at the same significance level increases.

The statistical analysis performed extremely well in two measures of performance. The first measure, the ability to correctly identify features, was investigated in section 5.2. A significant improvement in the ability to find small lines in a noisy image was observed using the proposed statistical analysis compared to the use of a straight bin threshold analysis. The second measure, the ability to minimize the detection of false features, was investigated in section 5.3. Again, a significant improvement was observed using the proposed statistical analysis compared to the use of a straight bin threshold analysis.

The proposed analysis technique proved an improvement in the ability to detect small features located in the outer region of the image space. Dramatic improvements were not observed for large features nor features located closer to the centre of the image space using the proposed analysis technique. However, the proposed technique did not decrease the ability to detect large features and features located closer to the centre of image.

References

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