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A STUDY OF A CLASS OF SELF-BIT SYNCHRONIZERS

by

Peter William Herbert Willcock, B.Sc., B.E., M.E.

A Thesis submitted to the Faculty
of Graduate Studies in partial fulfillment
of the requirements for the degree
of Doctor of Philosophy

Faculty of Engineering
Carleton University

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.....

Thesis Supervisor

.....

Chairman, Division of
Systems Engineering

Carleton University
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ABSTRACT

This thesis is concerned with self-bit synchronization, i.e. with the derivation of bit timing at a communications receiver from a data signal rather than from a transmitted reference signal. The data signal is assumed to be received in additive white Gaussian noise at a known fixed bit rate and is not specially coded for timing purposes.

A survey of some sources of timing errors is given including sources in the channel and in the receiver. It is shown that in a number of cases Gaussian statistics or sinusoidal variations may be used to represent sources of timing errors.

The performance of a matched filter receiver with timing errors is examined in detail. Two different timing error probability density functions are used to examine the receiver probability of error both when the timing error is independent of and dependent on the data bit energy-to-noise ratio.

The maximum a posteriori probability criterion is used to derive a class of self-bit synchronizers, and a practical device is proposed. The steady state performance of this device is derived and compared with simulation results. An implementation of the device on an analog computer is described and its performance is reported. The probability of error of a matched filter receiver whose timing is derived from this device is presented. Some discussion is also given of the transient performance of the device when the data signal is a square wave and of the practical implications of various possible steady state timing errors.

It is concluded from this study that self-bit synchronizers, and in particular the device studied in detail in this thesis, can provide good timing estimates from the signals assumed. They are not difficult or complex to build, and matched filters give close to theoretical performance when these devices are used to provide timing.

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LIST OF PRINCIPAL SYMBOLS

A	data pulse amplitude
d_k	value of k^{th} data pulse = $\pm A$
E	bit energy
e_k	loop error signal formed at $\hat{T}_k + \Delta T/2$
$g(\phi)$	equivalent phase-locked loop nonlinearity
$g_0(\phi)$	normalized equivalent phase-locked loop nonlinearity
G_1	integrator gain
G_2	loop filter gain
k_{VCO}	VCO gain
$n(t)$	received noise
N_0	noise power spectral density of $n(t)$ in watts/Hz
$P[E]$	probability of error for a matched filter receiver
$p_s(\phi)$	stationary timing error density
$r(t)$	received signal
$r_s(t), s(t)$	received data signal
$S(\omega, \phi)$	equivalent phase-locked loop additive noise source
$S_0(\omega, \phi)$	normalized equivalent phase-locked loop additive noise source
T	bit length
$\hat{T}_k$	timing estimate corresponding to k^{th} bit transition time
U	E/N_0
δ_s	ratio of bit rate to linear loop bandwidth, absolute value device
δ_{ss}	ratio of bit rate to linear loop bandwidth, square law device

(List of Principal Symbols continued)

ΔT	time shift between integrator intervals
θ	unknown constant time shift
ϕ_k, ϵ_k	timing error at time KT
ϕ, ϵ	constant timing error
ω_k	VCO frequency in k^{th} interval

CHAPTER 1

THE SYNCHRONIZATION PROBLEM

1.0 INTRODUCTION

It seems fair to say that statistical communication theory began to grow in the late 1930's and early 1940's with the appearance of a number of famous papers, reports, and books. One thinks of Wiener's work on time series, Van Vleck's work on clipped noise, and in particular North's report on the detection of pulsed signals in noise because in this latter report the matched filter for digital signals appeared. The matched filter is significant because it is the receiver which detects an equally likely binary signal in additive white Gaussian noise with minimum probability of error. However, it is assumed in the derivation of this result that the receiver can generate a perfect replica of the signal and that the receiver knows the time of arrival of the signal. In the radar case the matched filter output is used to estimate the signal arrival time. The arrival time is determined by detecting the filter peak output and this detection operation requires additional circuitry [1].

The two assumptions that the receiver can generate a perfect replica of the signal and that the receiver knows the time of arrival of the signal are a fundamental part of the problem of reception of signals with low probability of error. If the time of arrival is not known exactly, then for a matched filter receiver, the signal to noise ratio at the sampling time will be degraded, and the receiver probability of error will be higher than it would be if the time of arrival were known. If a perfect replica of the signal cannot be generated then the

matched filter cannot be built exactly because its impulse response is a time reversed signal element.

In spite of the great amount of work done in communication theory since 1940, little effort had been devoted to the problem of synchronization, even by 1963 when the chairman of a panel discussion had to say that "the most significant thing which this panel can accomplish is to cause a general awakening among engineers to the fact that there is a synchronization problem, and that it exists both as a theoretical and as a practical problem, and to stimulate thinking in both circles..." [2]. It appears from the literature that the thinking has begun in the last few years, i.e., since about 1966.

1.1 OUTLINE OF THE SYNCHRONIZATION PROBLEM

In digital communications systems there are basically 3 types of synchronization: a) phase sync, b) bit sync, c) block sync. These are briefly outlined below, and a much fuller discussion may be found in [2], and particularly in Sviridenko and Dmitriyev [3] which is a good survey of the Russian and English literature up to 1968.

Phase sync refers to the establishment of a locally generated carrier wave at the receiver which has the same phase and frequency as the received carrier. This may be accomplished by extracting the received carrier frequency with a narrow band filter and using the filter output to control a stable local oscillator within certain narrow limits. The requirement for phase sync is avoided if the receiver is an incoherent receiver, i.e., it does not require an accurate local carrier. An example is an AM receiver with envelope detection, but incoherent reception generally gives poorer performance than coherent

reception.

Bit sync refers to the detection of the time of occurrence of individual data bits which comprise a code word. A bit may be a square pulse as in telegraph, a fixed number of cycles of a sine wave of given frequency as in FSK, etc. Bit sync may be acquired in many ways, for example it may be derived from frame or word sync by frequency division, or it may be harmonically related to the carrier and derived from the local carrier, or it may be derived from some feature of the bits themselves. However, it should be noted that none of the systems that the author has encountered actually derive bit sync from word sync. Bit sync is usually required to provide a receiver time base for all other receiver operations, such as binary word decoding, as shown in fig. 1.1.

Block sync here encompasses several things such as frame sync, and word sync, and applies to the demodulated data signal (see fig. 1.2). In some systems such as time division multiplex systems, it is necessary to group the data into large blocks called frames. For example a frame may be one sampling sequence of all the subchannels. The system must then be able to detect the start and end of frames in order to decommutate the received signal correctly. Also the data may be encoded into code words before transmission and then the receiver must be able to detect the start and finish of code words. In most cases the code words are chosen so that this can be done easily.

In its most general form, then, the synchronization problem is the problem of determining the frequency, phase, and time of arrival of a signal for which these quantities are unknown. This problem has been studied by Gumacos [4] and Selin and Tuteur [4] by using search

FIGURE 1.1 A TYPICAL DATA RECEIVER

FIGURE 1.2 A TYPICAL TIME DIVISION MULTIPLEX FRAME

procedures for scanning cells in the time frequency plane. However, much work has treated phase sync and bit sync as separate problems and assumed that some other part of the receiver obtains phase sync (for example). The extensive literature on phase locked loops contains many examples of the former [5], and [6] is an example of the latter. In this thesis only bit sync is examined.

The relationship between the sync problem and the arrival time problem in radar also should be mentioned. In radar, the arrival time of a pulse is important information in itself and cannot be predicted a priori. In digital communication systems, however, there is some a priori information on the arrival time and length of pulses. This can be derived from the stream of approximately periodic pulses which have already arrived and from the fact that the bit rate is usually bounded by close upper and lower limits. The a priori information must be used in the case of zero signals which are frequently used in digital systems but which do not have any use in radar. In addition, the time between transmission and reception is not important in a digital communication system (within reasonable limits) except as an aid in obtaining bit sync.

There are a number of reasons why a receiver cannot determine exactly the start and finish of data bits. These will be discussed in more detail in chapter 2, but briefly there are 2 main causes. The first cause is random variations in the channel. For example, if a filtered sine wave carrier is used by a regenerative repeater to regenerate the data pulses, additive noise introduces random variations in the sine wave zero crossings and hence in the lengths of the regenerated pulses. Multipath propagation is another example. The second cause is random variations in receiver circuitry. For example, an

oscillator always has small random phase deviations and so it never gives uniformly spaced zero crossings which could be used to establish timing marks.

1.2 SYNCHRONIZATION PROBLEM TREATED IN THIS THESIS

This thesis is concerned with the problem of self-bit synchronization which is the acquisition of bit sync directly from the received data stream, rather than from any additional transmitted reference carriers, or from the use of any special data coding designed to ease synchronization. However, comparison is made with transmitted reference systems at appropriate points.

Most bit sync schemes proposed, or in use today, use one of the following methods:

- a) a separate synchronization reference carrier or other synchronization signal is transmitted and used to establish bit sync at the receiver. In one such system [7] the synchronization signal is a wideband PN sequence and the low rate data signal is placed in one of the PN sequence spectral nulls. In another example, [8], a PN sequence which occupies the same frequency band as the information signal is used for synchronization. These systems are shown in block diagram form in fig. 1.3;
- b) the data rate is a subharmonic of the data subcarrier and so bit sync can occur at only a finite number of points in a bit interval [9]. This simplifies the estimation of bit sync somewhat but requires knowledge of carrier phase;

FIGURE 1.3a A SYSTEM WITH THE DATA SPECTRUM IN A NULL OF THE SYNCHRONIZATION SIGNAL SPECTRUM [7]

FIGURE 1.3b A SYSTEM WITH THE DATA AND SYNCHRONIZATION
SIGNAL SPECTRA OVERLAPPING [8]

- c) the data and a PN synchronization signal are added mod 2 to form a single signal [10]. The receiver can separate the data and synchronization signal (see fig. 1.4);
- d) the data is encoded so that the data stream spectrum has a discrete component which can be filtered out [11]. The code used is a pair-selected ternary code which tends to reduce the occurrence of long strings of 1's or 0's.

In some cases the timing information is used to operate a receiver which is designed on the assumption of perfect timing, e.g., (a) above.

Each of these schemes places some limitation on the designer, e.g., increased bandwidth, increased transmitter power, code complexity, loss of channel capacity because of sync information, restriction of data rate. The limitations may be small, as for example if one reference carrier is used for many data subchannels, but they can be eliminated if reliable self-bit sync can be achieved. Furthermore, it is desirable to know how well a self-bit sync system can operate and how complex it may be. Of course, it is possible that such a system will introduce its own constraints, e.g., the system performance may be strongly dependent on signal design which in turn may imply large bandwidth or high peak-to-average power ratio. However, the self-bit synchronization problem has not been examined in much detail in the literature.

In this thesis binary data signals using rectangular signal pulses are considered. The data signal is assumed to be received in additive white Gaussian noise. In chapter 2 some sources and statistics of timing jitter are discussed. Chapter 3 contains a discussion of the

FIGURE 1.4 A SYSTEM HAVING THE DATA AND SYNCHRONIZATION SIGNALS ADDED MOD 2 BEFORE TRANSMISSION [10]

matched filter receiver with the assumption that the timing information is derived from a separate transmitted reference sine wave. Previous work on bit synchronizers is surveyed in chapter 4; and in chapter 5 a class of self-bit synchronizers is proposed based on the maximum a posteriori probability criterion. The steady state performance of certain of these devices is analyzed in chapter 6, and chapter 7 presents the results of experimental measurements of the performance of one of these devices which was simulated on an analog computer. Chapter 8 gives the results of digital computer simulation of the devices previously analyzed and built. The starting transient performance of one of these devices is also discussed in chapter 6 for the case where the data signal is a periodic rectangular wave. Conclusions and suggested areas of further study are presented in chapter 9.

In summary, the thesis presents:

- a) a discussion of sources of timing jitter and their statistics;
- b) a survey of previous work on bit synchronizers;
- c) a study of the effects of timing error on the performance of a matched filter receiver for binary data signals;
- d) a derivation of a class of optimum bit synchronizers;
- e) steady state and transient analyses of some suboptimum realizations of these synchronizers;
- f) measured and simulated steady state performance of the synchronizers analyzed in (e).

A number of similar devices have recently been proposed and the relation between them and the work done in this thesis is discussed in detail in

section 5.4.

References cited in the thesis are listed at the ends of the chapters in which they are cited. Appendices are also included at the end of each chapter.

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CHAPTER 2

SOURCES OF BIT TIMING ERRORS

2.0 INTRODUCTION

Most communication channels have time-varying parameters, as do the active devices used in communications equipment. Thus a perfectly stable, periodic, bit rate and perfectly stable receiver timing are mathematical idealizations, although they are often used to approximate slowly varying physical situations. In most communication problems it is necessary to describe the time-varying parameters statistically by probability density functions, etc. Hence, in this chapter some of the physical mechanisms which give rise to bit rate jitter and timing errors are examined and some elementary statistical descriptions are given. These statistics are often difficult to obtain analytically because they involve the solution of level-crossing problems. An example is the problem of the time of initiation of switching of a trigger circuit from one state to another when the input signal to the trigger circuit contains noise [1].

In this chapter it is assumed that a digital signal is received in additive noise over a channel which may be dispersive and time varying. The receiver contains a synchronizer which attempts to estimate bit sampling times for the receiver, and may also be used to provide timing pulses to operate other circuitry in the receiver such as decoders. In digital communication systems the transmitted signals are usually sequences of contiguous equal length pulses. Thus, in considering bit rate jitter we are concerned with those mechanisms in the transmitter and channel which cause the received pulses to have

unequal length. In considering timing jitter we are concerned with those mechanisms which cause the synchronizer in the receiver to construct a sequence of timing pulses which deviate in time from the theoretically optimum sampling times.

2.1 SOURCES OF JITTER

2.1.1 Jitter of Transmitter Bit Rate

A possible source of bit rate jitter is the transmitter itself. It may derive its bit rate from a local oscillator which is affected by noise and drift. Usually the oscillator is a crystal oscillator and this source of bit rate jitter may be considered negligible. Crystal oscillators under 20 g.vibrations at 2000 Hz have been noted to have a jitter of less than .04 radians peak-to-peak [2], and under static conditions one might expect stabilities of 1 part in 10^6 or better.

However, in some cases the transmitter might be deriving its clock from some received signal, as for example in a repeater or small remote measuring station. In biomedical or industrial telemetry there may be situations when the simplest possible oscillators are used. Then it may be necessary to investigate the effect of bit rate jitter by modelling the oscillator by an equation of the form $s(t) = A \cos(\omega t + n(t))$ where $n(t)$ is a narrow band Gaussian process. In order to find the zero crossing jitter statistics the level crossings of $\omega t + n(t)$ would have to be examined.

2.1.2 Time-varying Channel Phase

There are several channel mechanisms which give rise to a varying phase.

1. MODULATION SYSTEMS. If the data channel passes through an FDM multiplex system, for example, then it undergoes a number of frequency translations. Even though the system oscillators are locked to an extremely stable crystal source they still have phase variations caused by power supply ripple, ringing supply pickup, inherent phase variations, etc. In addition there are phase variations from carrier transmission systems, such as microwave links or wire line carrier systems used to transmit the FDM signal. These phase variations can severely affect the waveform of the data pulse if they are large enough. Values of 2° to 15° peak-to-peak phase jitter have been measured on circuits in the U.S., and the jitter generally seems to have components at 60 Hz, 120 Hz, the ringing frequency 20 Hz, and a random wideband component [3]. The components at harmonics of the power line frequency are often the most serious cause of jitter [4]. Further, it has been observed that the jitter on a sine wave at the center of a voice channel may be uncorrelated with that on a sine wave at the edge of the channel [3]. Hence a sine wave pilot may not give a very good timing estimate, although if the jitter is small the estimate may be good enough.

Recently phase jitter on voice band data channels over the Bell System switched network was measured in a 1000 sample survey [5] of toll connections from end office to end office. A 1700 Hz tone was transmitted and the peak-to-peak phase jitter was measured in the band 12-768 Hz at the receive end. The phase jitter distribution was not normal, and a sample of the results for the 10, 50, and 90 percent points is given in table 2.1.

The same study measured the absolute value of the frequency offset at the receiver. A precise 1200 Hz tone was transmitted and a

TABLE 2.1

Percentage Points from Cumulative Distributions of
Peak-to-Peak Phase Jitter in Degrees on Toll
Connections in a 12-768 Hz band [5].

	All	0-180 mi.	180-725 mi.	725-2900 mi.
10%	0	0	2	2
50%	3	2	4	5
90%	7	6	8	9

frequency counter measured the frequency of the received tone. The amount of frequency offset was very small being less than 0.2 Hz on 90% of all circuits measured.

In addition, A.T.&T. has set an objective for the Bell System network of 7.2° peak-to-peak phase jitter, and attempts to limit rapid phase changes to $\pm 30^\circ$ [6].

Further, step phase changes occur due to switching of communication systems caused by failure, maintenance, etc.

2. TAPE RECORDERS. A tape recorder is a fairly common channel element. For example, some satellites record data for playback on a more favourable pass, data is often recorded for later analysis, and small tape recorders are being used for auxiliary computer memory. There are several mechanisms which cause random phase variations in a tape recorder channel [7]-[10]. Flutter, which is a variation in tape speed past the head, may have sinusoidal components due to mechanical linkages and a random Gaussian component. The power spectrum of the random component may be rectangular between 10 Hz and 10 kHz in recorders with servomechanism speed controls, but extends down to very low frequencies in those without servos. In addition skew, which is a variation in the angle between the tape axis and the recorder head, contributes to phase variations and the amount depends on the position of the track on the tape.

3. PCM CHANNELS. An extensive body of literature exists on the analysis of PCM regenerative repeater timing circuitry such as shown in fig. 2.1. A few examples are [11]-[13]. The most important effect is a buildup of timing jitter in a chain of regenerative repeaters which

FIGURE 2.1 A PCM REPEATER TIMING CIRCUIT

is caused by pulse pattern effects in the timing circuitry. A particular pulse sequence may give rise to a certain timing offset due to frequency offset of the tuned circuit or AM-PM conversion in the limiter in the timing circuitry for example. As the sequence travels through the repeaters the offset tends to occur at each repeater and build up.

Thus the timing error can give rise to phase modulation of the message signal so that a data signal which may be part of the message encounters a channel with varying phase. The effect is illustrated in fig. 2.2. The PCM pulse jitter process has been shown by measurement [12] to have a Gaussian amplitude probability density function to a very good approximation. The measured power spectrum is essentially low-pass as shown in fig. 2.3a. The mean square phase deviation was 10° after 84 repeaters of a T1 system, although this quantity can be controlled by system design. Thus the samples of a message channel being carried by the system are jittered by the system and the message output samples can be written

$$s_0(t) = \sum_{n=-\infty}^{\infty} s(nT) \delta(t-nT-\theta_n)$$

if the message samples are $s(nT)$, and $\theta_n = \theta(nT)$ are the jitter process samples. Then the phase modulation property for small slowly varying jitters can be shown as follows.

$$\begin{aligned} \text{Let } r(t) &= \sum_{n=-\infty}^{\infty} s_n \cdot \delta(t-nT-J_n) \quad \text{when } J_n \text{ is the jitter of the impulse} \\ &\quad \text{from time } nT. \\ &= \sum_{n=-\infty}^{\infty} s(nT) \cdot \delta(t-nT-J_n) \\ &= \sum_{n=-\infty}^{\infty} s(t-\theta(t)) \cdot \delta(t-nT-\theta(t)) \end{aligned}$$

FIGURE 2.2 MODULATION OF A MESSAGE SIGNAL BY SAMPLE JITTER

FIGURE 2.3a POWER SPECTRUM OF TIMING JITTER ON A T1 SYSTEM [12]

FIGURE 2.3b SAMPLE JITTER PROCESS DEFINITIONS

$$= s(t-\theta(t)) \cdot \sum_{-\infty}^{\infty} \delta(t-nT-\theta(t))$$

where $\theta(t)$ is a jitter process such that

$$\left[t-nT-\theta(t) \right] \Big|_{t=nT+J_n} = 0 \quad \text{i.e., } \theta(nT+J_n) = J_n,$$

and then

$$s(t-\theta(t)) \Big|_{t=nT+J_n} = s(nT)$$

Now let $\tau = t-\theta(t)$.

$$\text{Thus} \quad r(t) = s(t-\theta(t)) \cdot \sum_{-\infty}^{\infty} \delta(\tau-nT)$$

and using the Poisson Sum Formula,

$$\sum_{-\infty}^{\infty} \delta(\tau-nT) = \frac{1}{T} \sum_{-\infty}^{\infty} e^{jn\omega_0\tau} \quad \text{where } \omega_0 = 2\pi/T$$

$$\begin{aligned} r(t) &= s(\tau) \cdot \left[\frac{1}{T} + \frac{2}{T} \sum_{n=1}^{\infty} \cos(n\omega_0\tau) \right] \\ &= s(t-\theta(t)) \cdot \frac{1}{T} \left[1 + 2 \sum_{n=1}^{\infty} \cos[n\omega_0(t-\theta(t))] \right]. \end{aligned}$$

Thus if $\theta(t)$ is slowly varying

$$r(t) \approx \frac{1}{T} \cdot s(t-\theta(t))$$

and also in practical systems the message is usually oversampled which tends to increase the values of $n\omega_0$ and improve the approximation. A similar result has been obtained by Glave [18].

4. DOPPLER SHIFT AND OSCILLATOR FREQUENCY OFFSET. If the transmitter and receiver are moving relative to one another, then the received bit rate depends on their relative velocity and the velocity

C = CONSTANT SIGNAL VELOCITY

$r(t)$ = RADIAL DISTANCE TO
RECEIVER AT TIME t

$\dot{r}(t)$ = RADIAL VELOCITY

$\dot{r}(t) < 0$ FOR TRANSMITTER
MOVING TOWARD
RECEIVER

FIGURE 2.4 DOPPLER EFFECT DEFINITIONS

of propagation of the signal. With reference to figure 2.4 it can be shown that [14] if the transmitter velocity is u then the received bit rate f_r is given by

$$f_r = f_s \frac{\sqrt{1-u^2/c^2}}{1+\dot{r}(t)/c} \quad (2.2.1)$$

where f_s is the transmitted bit rate.

Two of the important effects of Doppler shift are those of bit slippage and variable received data rate. In order to examine a simple bit slippage situation, assume $f_r > f_s$ and that the receiver clock is perfectly stable and running at exactly f_s . Then the bit transition points for the received signal at rate f_r and for the local clock at rate f_s are shown in figure 2.5. Bit slippage is said to occur when the local timing marks differ from the received timing marks by 1 bit. In the case shown, the receiver will then lose a bit, i.e., a bit slip is said to occur. By rearranging equation (2.2.1)

$$f_s - f_r = f_s \frac{1+a-\sqrt{1-u^2/c^2}}{1+a} \triangleq f_s \frac{d}{1+a}$$

where $a = \dot{r}(t)/c$, and if we assume $\dot{r}(t)$ is constant then

$$\text{number of bits difference in } t \text{ secs} = \frac{d}{1+a} f_s t = 1 \text{ for a bit slip}$$

$$\therefore \text{time to bit slip} = t = \frac{1}{f_s} \frac{1+a}{d} \quad (2.2.2)$$

if the receiver clock starts in exact synchronism with a received bit transition point. Equation (2.2.2) is plotted in figure 2.6.

The problem of variable received data rate may be described as follows. Even if the receiver tracks the variations in received bit

FIGURE 2.5 BIT SLIPPAGE CAUSED BY LOCAL CLOCK AND RECEIVED SIGNAL BIT RATE DIFFERENCES

FIGURE 2.6 GRAPH OF EQUATION (2.2.2)

rate accurately, the data is probably processed at a constant rate, say R. Thus if the received bit rate increases above R, more bits are received than are processed and so a store has to be provided to hold the excess received bits during periods of high received bit rate.

Oscillator frequency offset is the situation in which the transmitter bit rate and the receiver clock frequency are not exactly the same. Then the same effect of bit slippage occurs. Frequency offset also occurs on carrier systems and some measured results in the Bell System have been noted under point 1 above.

5. MULTIPATH. A multipath channel causes distortion of the data signal zero crossings and so affects the synchronizer performance. Brandinger and Goldman [15] made some measurements of this for an FSK signal transmitted over an HF channel with a bit length of 10 msec. The output of the receiver discriminator was hard limited and differentiated to give impulses at the received signal zero crossings. The probability distribution and autocorrelation function of the jitter of the impulses about the nominal transition times were computed using signal to noise ratios of more than 20 dB. Some results are shown in figures 2.7 and 2.8, and in general it was found that the jitter density could be represented by the Laplace density function

$$p(x) = Ne^{-2N|x|} \quad \text{where } 1 < N < 5.$$

The average value of the correlation coefficient $\rho(\tau)$ was $1/e$ at $\tau = .67$ seconds, i.e., the jitter was highly correlated.

FIGURE 2.7 DISTRIBUTION OF JITTER ON AN FSK SYSTEM [15]

FIGURE 2.8 AUTOCORRELATION FUNCTION OF JITTER ON AN FSK SYSTEM [15]

2.1.3 Fixed Channel Phase Errors

There are situations in which signals transmitted over separate parallel channels undergo different absolute delays. This gives rise to a difference in arrival time. In other cases component tolerances in parts of a receiver may cause time delays which are unknown. Some examples of these which are discussed further below are the simultaneous transmission of a signal over coaxial and microwave transmission systems, component tolerances in a synchronizer, and tape recorder head misalignment.

1. FILTER PHASE ERRORS AND TRANSMISSION DELAYS. Normal tolerances in circuit design produce unknown fixed delays in a transmission path. If paths are switched during transmission then a change in the channel fixed delay occurs. If the data receiver derives its timing from a separate sync signal, then the sync signal and data signal channels may have different delays. Furthermore, component tolerances within the receiver and its synchronizer may give rise to unknown fixed timing errors.

For example, for a 3000 mile microwave system the transmission delay is about .016 seconds while it is about .02 seconds for a 3000 mile coaxial system [16]. Thus $f_s \times .004$ pulses will be lost in switching from coaxial to microwave where f_s is the data bit rate in pulses per second.

2. TAPE RECORDER HEAD MISALIGNMENT. Separate record and playback heads are not usually parallel and hence a different delay exists in different tracks. This would cause a transmitted reference to have a different delay from the data if the two signals were recorded on

different tracks.

2.1.4 Intersymbol Interference and Noise

Neither of these affect the number of pulses per second arriving at the receiver. However, intersymbol interference alters the transmitted pulse shape and changes the time positions of the data signal zero crossings. Hence it gives rise to timing errors at the output of a synchronizer which, for example, hard limits the received signal to estimate data transitions.

It is easy to see that limited intersymbol interference gives rise to a time shift of zero crossings which depends on the data signal. As the interference overlaps more pulses more possible shifts arise and they occur at random depending on the pulse pattern. An example of this type is shown in figure 2.9 where a 1,1,-1 or -1,-1,1 sequence produces one shift and a 1,-1,1 or -1,1,-1 sequence produces another shift. It is clear that intersymbol interference on a periodic transmitted reference signal would give rise to a constant time shift similar to those discussed above in section 2.1.3.

Additive noise received with the data signal will cause additional jitter in the received signal zero crossings and hence will add to the synchronizer timing jitter.

An extensive measurement program [17] conducted in the Bell System produced the results shown in figure 2.10 for peak distortion which is a measure of the jitter on the demodulated bit transitions. The data set used was an FM digital subset, and peak distortions of 20% or less were considered acceptable. This study did not investigate the effects of phase jitter in the channel which are included in the results.

FIGURE 2.9 EXAMPLE OF THE EFFECT OF INTERSYMBOL INTERFERENCE ON ZERO CROSSINGS

FIGURE 2.10 PEAK DISTORTION FOR 600 B.P.S. CALLS ON THE DDD NETWORK WITH NO EQUALIZATION [17]

2.1.5 Data Pattern

Synchronizer performance is usually sensitive to the data pattern, i.e., the number and position of 1's and 0's in a binary wave (for example). This has already been discussed with respect to inter-symbol interference, but there are other factors to be considered. For example, if the signal element is a non-return-to-zero pulse, then if a long sequence of 1's or 0's occurs in the data signal there are no zero crossings nor is there any signal element shape to give synchronizing information. Then the synchronizer timing estimates may drift because of internal noise or oscillator drift.

2.2 CONCLUSIONS

In many common situations the local timing estimates are jittered about the optimum sampling times. The jitter may be caused by synchronizer components or by channel variations and noise. A number of practical jitter models were described.

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CHAPTER 3

TIMING ERRORS IN MATCHED FILTERS

3.0 INTRODUCTION

As has already been noted, most theoretical analyses of data communications assume perfect bit synchronization. There have not been many attempts to show the effects of bit synchronization error on common systems. In this chapter the effect of timing errors in matched filters is considered.

Communication theory indicates that the matched filter is the optimum receiver for signals received in additive white Gaussian noise, and it often appears as a component of more complicated receivers. However, the output of a matched filter must be sampled at the correct time, and the assumption is made in most work that this time is known exactly to the receiver, i.e., the receiver is synchronized with the transmitter. The presence of timing error means that this assumption does not hold and so it is of interest to examine the effects of timing errors on matched filter receiver performance.

This problem has received some attention in the past. Kaneko [1] calculated the probability of error for binary rectangular pulses when the timing is derived from a filtered sine wave in Gaussian noise. In the same article he considered the problem of optimum power division between the synchronizing signal and the information signal. Wintz and Hancock [2] derived an expression for the probability of error for an FSK system when the timing error is a Gaussian random variable and the pulses have a Gaussian envelope. However, they did not consider the

effect of adjacent pulses, stating that "this point is of course of minor consequence when the system is operating in the near optimum region (reasonably good delay measurements) and the signals contain little energy near their end points".

A related study is that of Matskov and Danielyan [3] in which the effect variation of pulse length on the output signal of a filter matched to a PN sequence is examined.

Another approach which can be taken to reduce the effect of timing errors is to use the optimum linear smoothing filter as a receiver rather than the matched filter. The receiver is then optimum for all time rather than at one point in time as is the case for the matched filter. This has been done by Nichols [4] as an example of the filtering and smoothing theory which he developed.

In this chapter we assume a receiver as shown in figure 3.1. The incoming pulses are binary, equally likely + or -, independent, and observed in additive, white, Gaussian noise. The bit interval is assumed to be known exactly but a constant unknown timing error exists. The synchronizer may derive its timing from either the data or a separate synchronization signal. The average probability of error is computed for some common signal shapes and timing error probability density functions. This model can represent several physical situations:

- a) random component errors in the synchronizer;
- b) slowly varying phase changes in the synchronization signal or pattern dependencies in synchronizer performance;
- c) different transmission delays in data and sync signal paths.

FIGURE 3.1 MATCHED FILTER RECEIVER

3.1 DERIVATION OF P[E] FOR CONSTANT UNKNOWN TIMING ERROR

Assume $s_1(t) = -s_2(t) = \begin{cases} s(t) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$

and let $\int_0^T s^2(t) dt = E.$

Also assume that $s_1(t)$ and $s_2(t)$ are equally likely to occur in any bit interval T independent of any other bit interval, that T and $s(t)$ are known exactly to the receiver, and that $n(t)$ is additive white Gaussian noise of two-sided spectral density $N_0/2$. As shown in figure 3.1 the receiver samples with a constant unknown timing error ϵT where $-1/2 \leq \epsilon \leq 1/2$.

Consider the case $\epsilon > 0$. Then for a given value of ϵ ,

$$P[E|\epsilon] = P[r_1 > r_2 | s_2, s_2] P[s_2, s_2] + P[r_1 > r_2 | s_2, s_1] P[s_2, s_1] \\ + P[r_2 > r_1 | s_1, s_1] P[s_1, s_1] + P[r_2 > r_1 | s_1, s_2] P[s_1, s_2]$$

where $[s_i, s_j]$ denotes the event:

$$[s_i \text{ occurred in } 0 \leq t \leq T, s_j \text{ occurred in } T \leq t \leq 2T].$$

Thus

$$P[E|\epsilon] = \frac{1}{4} \left\{ P[r_1 - r_2 > 0 | s_2, s_2] + P[r_1 - r_2 > 0 | s_2, s_1] \right. \\ \left. + P[r_2 - r_1 > 0 | s_1, s_1] + P[r_2 - r_1 > 0 | s_1, s_2] \right\} \quad (3.1.1)$$

for independent equally likely pulses. But,

$$\begin{aligned}
r_2|_{s_2, s_2} &= \int_{\epsilon T}^T r(t) s_2(t-\epsilon T) dt + \int_T^{(1+\epsilon)T} r(t) s_2(t-\epsilon T) dt \\
&\quad + \int_{\epsilon T}^{(1+\epsilon)T} n(t) s_2(t-\epsilon T) dt \\
&= \int_{\epsilon T}^T s(t) s(t-\epsilon T) dt + \int_T^{(1+\epsilon)T} s(t-T) s(t-\epsilon T) dt \\
&\quad - \int_{\epsilon T}^{(1+\epsilon)T} n(t) s(t-\epsilon T) dt \\
&\triangleq R(-\epsilon T) + R\{(1-\epsilon)T\} + n_1
\end{aligned}$$

where n_1 is a Gaussian random variable and $R(\tau)$ is the autocorrelation function of $s(t)$. (A short discussion of the use of the convolution integral for computing the outputs of filter-and-dump or filter-and-sample realizations of matched filters is given in Appendix 3A.) Similarly,

$$\begin{aligned}
r_1|_{s_2, s_2} &= - \int_{\epsilon T}^T s(t) s(t-\epsilon T) dt - \int_T^{(1+\epsilon)T} s(t) s(t-\epsilon T) dt \\
&\quad + \int_{\epsilon T}^{(1+\epsilon)T} n(t) s_1(t-\epsilon T) dt \\
&\triangleq -R(\epsilon T) - R\{(1-\epsilon)T\} + n_2
\end{aligned}$$

so that

$$(r_1 - r_2) | s_2, s_2 = -2R(\epsilon T) - 2R((1-\epsilon)T) + 2 \int_{\epsilon T}^{(1+\epsilon)T} n(t) s(t-\epsilon)T dt$$

which is a Gaussian random variable with mean $-2[R(\epsilon T) + R((1-\epsilon)T)]$ and variance $2E N_0$. Thus

$$\begin{aligned} P[r_1 - r_2 > 0 | s_2, s_2] &= \frac{1}{\sqrt{4\pi E N_0}} \int_0^\infty \exp \left[-\frac{(x - m_1)^2}{4E/N_0} \right] dx \\ &= \frac{1}{2} \left[1 + \operatorname{erf} (m_1 / 2\sqrt{E N_0}) \right] \end{aligned} \quad (3.1.2)$$

where $m_1 = -2[R(\epsilon T) + R((1-\epsilon)T)]$.

Similarly $r_2 - r_1 | s_1, s_2 = -2R(\epsilon T) + 2R((1-\epsilon)T) + n$

where n is a zero-mean Gaussian random variable with variance $2E N_0$, so

$$P[r_2 - r_1 > 0 | s_1, s_2] = \frac{1}{2} \left[1 + \operatorname{erf} (m_2 / 2\sqrt{E N_0}) \right] \quad (3.1.3)$$

where $m_2 = -2[R(\epsilon T) - R((1-\epsilon)T)]$. Then, since

$$P[r_2 - r_1 > 0 | s_1, s_2] = P[r_1 - r_2 > 0 | s_2, s_1] \text{ and } P[r_2 - r_1 > 0 | s_1, s_1] = P[r_1 - r_2 > 0 | s_2, s_2],$$

we get from (3.1.1) - (3.1.3)

$$\begin{aligned} P[E | \epsilon] &= \frac{1}{4} \left[2 + \operatorname{erf} (m_1 / 2\sqrt{E N_0}) + \operatorname{erf} (m_2 / 2\sqrt{E N_0}) \right] \\ &= \frac{1}{4} \left\{ 2 - \operatorname{erfc} \left[\sqrt{\frac{E}{N_0}} \frac{R(\epsilon T) + R((1-\epsilon)T)}{R(0)} \right] - \operatorname{erfc} \left[\sqrt{\frac{E}{N_0}} \frac{R(\epsilon T) - R((1-\epsilon)T)}{R(0)} \right] \right\} \end{aligned}$$

and

$$P[E] = \frac{1}{2} \int_0^{1/2} p(\epsilon) \left\{ 1 - \frac{1}{2} \operatorname{erf} \left[\sqrt{\frac{E}{N_0}} \frac{R(\epsilon T) + R((1-\epsilon)T)}{R(0)} \right] \right. \\ \left. - \frac{1}{2} \operatorname{erf} \left[\sqrt{\frac{E}{N_0}} \frac{R(\epsilon T) - R((1-\epsilon)T)}{R(0)} \right] \right\} d\epsilon. \quad (3.1.4)$$

Similar results are obtained for $\epsilon < 0$ with $R((1-\epsilon)T)$ replaced by $R((1+\epsilon)T)$, so that since $R(\cdot)$ is an even function we can write

$$P[E] = \frac{1}{2} \int_{-1/2}^{1/2} p(\epsilon) \left\{ 1 - \frac{1}{2} \operatorname{erf} \left[\sqrt{\frac{E}{N_0}} \frac{R(|\epsilon|T) + R((1-|\epsilon|)T)}{R(0)} \right] \right. \\ \left. - \frac{1}{2} \operatorname{erf} \left[\sqrt{\frac{E}{N_0}} \frac{R(|\epsilon|T) - R((1-|\epsilon|)T)}{R(0)} \right] \right\} d\epsilon \quad (3.1.5)$$

$$= \int_0^{1/2} p(\epsilon) \left\{ 1 - \frac{1}{2} \operatorname{erf} \left[\sqrt{\frac{E}{N_0}} \frac{R(\epsilon T) + R((1-\epsilon)T)}{R(0)} \right] \right. \\ \left. - \frac{1}{2} \operatorname{erf} \left[\sqrt{\frac{E}{N_0}} \frac{R(\epsilon T) - R((1-\epsilon)T)}{R(0)} \right] \right\} d\epsilon \quad (3.1.6)$$

if $p(\epsilon)$ is symmetrical about $\epsilon = 0$, which is also given in [5].

Note that if $p(\epsilon) = \delta(\epsilon)$, i.e., perfect timing, then since $R(0) = E$ and $R(T) = 0$, (3.1.5) gives

$$P[E] = \frac{1}{2} [1 - \operatorname{erf}(\sqrt{E/N_0})]$$

which is the standard result for a matched filter with perfect timing.

3.2 P[E] FOR JITTERED TIMING

In this section it is shown that $P[E]$ for the case shown in figure 3.2 is given by an equation of the same form as (3.1.4). Here it is assumed that at each nominal sampling instant KT the timing is in error by $\epsilon_K T$ where the ϵ_K are random variables.

Now assume that a filter-and-sample receiver, i.e., the matched filter, with impulse response $s(T-t)$ which may be non-zero over $0 \leq t \leq T$ only, is sampled at $KT + \epsilon_K T$ for the K^{th} pulse and at $(K+1)T + \epsilon_{K+1} T$ for the $(K+1)^{\text{st}}$ pulse. Sampling does not affect the filter and hence the sample at $(\epsilon_K + K)T$ does not affect the output at $(K+1 + \epsilon_{K+1})T$. Therefore, to determine the filter output at time $(K+1 + \epsilon_{K+1})T$ we convolve the input up to this time with the impulse response $s(T-t)$. This is exactly the same procedure as was used in section 3.1, but there the ϵ_K were constant having the value ϵ . Thus it can be seen that equation (3.1.4) with ϵ replaced by ϵ_K gives $P[E]$ for the present case.

Thus the analysis of section 3.1 holds under the following conditions:

- a) a filter-and-dump implementation where the timing error is a fixed unknown quantity and the samples are spaced by T secs. which is the length of the impulse response (a practical assumption since the same clock drives the sampler and the dump circuit);
- b) a filter-and-sample implementation where the samples are jittered about the nominal time with first order density $p(\epsilon)$. The samples are assumed to draw negligible energy from the filter and the filter impulse response is assumed to be independent of the sample timing.

FIGURE 3.2 TIMING ERROR DEFINITIONS FOR JITTERED TIMING

3.3 NUMERICAL RESULTS

3.3.1 Introduction

Equation (3.1.4) was evaluated numerically for several cases to determine the effect of timing error for some common signals. It is assumed in this section that $T=1$, $R(0)=E=1$, and let $\eta = \sqrt{E/N_0}$.

The probability density function of the timing error is assumed in section 3.3.3 to be

$$p(\epsilon) = \frac{1}{I_0(\alpha)} e^{\alpha \cos(2\pi\epsilon)} \quad (3.3.1)$$

for

$$-\frac{1}{2} \leq \epsilon \leq \frac{1}{2} \quad \text{and} \quad 0 \leq \alpha \leq \infty$$

which is plotted in figure 3.3 for several values of α . This function was chosen for the following reasons:

- a) it represents a broad class of probability density functions indexed by one parameter α and so may represent the timing error of many synchronizers;
- b) (3.3.1) appears to be easier to deal with analytically than the probability density function of the phase of a sine wave in Gaussian noise which is another reasonable choice for $p(\epsilon)$. The relation between this latter function and (3.3.1) is discussed in Appendix 3B;
- c) (3.3.1) describes the stationary phase error of a phase-locked loop, (PLL), and one method of synchronization is to use a phase-locked loop to receive a transmitted reference sine wave.

FIGURE 3.3 PHASE-LOCKED LOOP PHASE ERROR DENSITY FUNCTION

In section (3.3.4) the probability density function used is that of the phase of a sine wave in Gaussian noise. This is used because Wintz and Luecke [5] measured the timing error probability density function of their suboptimum synchronizer and found it to be close to this probability density function.

The signals used are a rectangular pulse and a raised cosine pulse. It is clear from (3.1.4) that $P[E]$ is minimized by choosing

$$R(t) = \begin{cases} R(0) & 0 \leq t \leq 1/2 \\ 0 & \text{elsewhere} \end{cases} \quad (3.3.2)$$

for symmetrical $p(\epsilon)$. In this case (3.1.6) becomes

$$P[E] = \frac{1}{2} [1 - \text{erf}(\sqrt{E/N_0})]$$

which is the result for a binary matched filter receiver with perfect timing. But (3.3.2) is not an autocorrelation function because its Fourier Transform has negative values and an autocorrelation function must have a non-negative Fourier Transform [6]. There is, then, a signal design problem of finding the pulse shape which minimizes (3.1.4) and one approach would be to find an autocorrelation function which is close to (3.3.2) in some sense. In this connection it may be noted that Boas and Kac [7] prove the following theorem:

Let $f(x)$ be the Fourier Transform of a non-negative function $Q(t)$ where $-\infty \leq t \leq \infty$, so that

$$f(x) = \int_{-\infty}^{\infty} Q(t) e^{jxt} dt,$$

and let $f(x)$ vanish for $|x| \geq 1$. Then

$$|f(x)| \leq f(0) \cdot \cos\left(\frac{\pi}{1+[1/x]}\right)$$

where $[t]$ means the least integer not less than t .

Thus if we identify $|S(\omega)|^2$ with $Q(t)$ and consider pulse shapes such that $R(t)$ vanishes for $|t| \geq T/2$, then

$$|R(x)| \leq E \cdot \cos\left(\frac{\pi}{1+[1/x]}\right) \quad (3.3.3)$$

where $x=2t/T$. Equation (3.3.3) and the autocorrelation functions used in this section are shown in figure 3.4. The parameters of the rectangular and raised cosine pulses have been chosen so that they are both T seconds long and both have the same energy. Equation (3.3.3) tells us that we cannot find a pulse shape which vanishes for $|t| \geq T/2$ and has an energy E and comes any closer to (3.3.2). Figure 3.4(b) illustrates this point by showing some autocorrelation functions for pulse shapes which vanish for $|t| > T/2$.

3.3.2 Simplification of $P[E]$ Expression

For a rectangular pulse let

$$R(t) = \begin{cases} E(T-|t|)/T & 0 \leq |t| \leq T \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$R(T-|t|) = \begin{cases} E|t| & 0 \leq |t| \leq T \\ 0 & \text{elsewhere} \end{cases}$$

and from (3.1.5)

$$P[E] = \frac{1}{2} - \frac{1}{4} \operatorname{erf}(\eta) - \frac{1}{2} \int_0^{1/2} \operatorname{erf}(\eta(1-2x)) p(x) dx \quad (3.3.3)$$

FIGURE 3.4a PULSE AUTOCORRELATION FUNCTIONS FOR PULSES OF LENGTH T

FIGURE 3.4b PULSE AUTOCORRELATION FUNCTIONS FOR PULSES OF LENGTH $T/2$

where $\eta = \sqrt{E/N_0}$ and $p(x)$ is symmetrical about $x=0$.

For a raised cosine pulse $S(t) = A[1 - \cos(\omega_0 t)]/2$ where $\omega_0 = 2\pi/T$ and $0 \leq t \leq T$,

$$R(t) = \begin{cases} 2E \left[(T - |t|) (1 + 1.5 \cos(\omega_0 t)) + (3/2\omega_0) \sin(\omega_0 |t|) \right] / 3T & 0 \leq |t| \leq T \\ 0 & \text{elsewhere.} \end{cases}$$

But since $R(0) = E$, then

$$R(t) + R(T-t) = 2E(1 + 1.5 \cos(\omega_0 t)) / 3$$

$$R(t) - R(T-t) = 2E(1 - 2t/T) (1 + 1.5 \cos(\omega_0 t)) / 3 + 2E \sin(\omega_0 t) / \omega_0 T$$

and from (3.1.5)

$$\begin{aligned} P[E] &= \frac{1}{2} - \frac{1}{2} \int_0^{1/2} \text{erf}[2\eta(1 + 1.5 \cos(2\pi x)) / 3] p(x) dx \\ &\quad - \frac{1}{2} \int_0^{1/2} \text{erf}[2\eta(1 - 2x) (1 + 1.5 \cos(2\pi x)) / 3 + \eta \sin(2\pi x) / \eta] p(x) dx \end{aligned} \quad (3.3.4)$$

3.3.3 Results for PLL Probability Density Function

Curves of $P[E]$ vs η are shown in figure 3.5 for

$$p(\epsilon) = e^{\alpha \cos(2\pi\epsilon)} / I_0(\alpha) \quad -\frac{1}{2} \leq \epsilon \leq \frac{1}{2}$$

using several values of α .

Results where $P[E] : 10^{-7}$ may not be significant as the approximation used for the erf function had a maximum error of 10^{-7} , although the error was much better than that at several argument values which were tried.

FIGURE 3.5 PROBABILITY OF ERROR FOR MATCHED FILTER SYSTEM AND TIMING ERROR INDEPENDENT OF DATA SIGNAL TO NOISE RATIO - TIMING ERROR DESCRIBED BY PHASE-LOCKED LOOP PHASE ERROR DENSITY

Several points can be made from the results.

1. In most cases the curves follow the ideal matched filter curve at low values of E/N_0 and then break away as E/N_0 increases. In those cases where this is not observed, it is thought that this is only because of the limited range of E/N_0 used. This means that timing errors can place a severe limitation on communication system performance.
2. For any reasonable synchronizer the value of α which describes its timing error pdf should be proportional to E/N_0 in some way. That is, as E/N_0 increases, the timing error should become more peaked about $\epsilon=0$ becoming a delta function for infinite E/N_0 . This means that a system error rate will not be given by curves with fixed α .
3. For $E/N_0=15$ the following values are obtained

Pulse	$\alpha=0$	$\alpha=1.5$	$\alpha=5$	$\alpha=10$
Raised cos.	$.306 \times 10^{-1}$	$.464 \times 10^{-2}$	$.139 \times 10^{-4}$	$.777 \times 10^{-7}$
Rectangular	$.364 \times 10^{-1}$	$.596 \times 10^{-2}$	$.353 \times 10^{-4}$	$.112 \times 10^{-5}$

The conclusion is that full length raised cosine pulses are better than full length rectangular pulses and synchronizer design pays off well ($\alpha=5$, $\alpha=10$) when the former are used.

4. For $E/N_0=1$ the following values are obtained

Pulse	$\alpha=0$	$\alpha=1.5$	$\alpha=5.0$	$\alpha=10$
Raised cos.	.202	.120	$.869 \times 10^{-1}$	$.824 \times 10^{-1}$
Rectangular	.168	.118	$.936 \times 10^{-1}$	$.883 \times 10^{-1}$

From this it can be said that

- a) since at $\alpha=10$, $P[E]=.883 \times 10^{-1}$ for rectangular pulses, and from figure 3.5 $P[E]=.83 \times 10^{-1}$ at $E/N_0=4$ and $\alpha=0$ for rectangular pulses, then synchronizer design can gain about 6 dB in SNR. That is, by improving α from 0 to 10 one can achieve the same $P[E]$ at $E/N_0=1$;
- b) there is negligible difference in performance between the rectangular and raised cosine pulses at low E/N_0 .

5. For a fixed E/N_0 synchronizer design improves the probability of error, but after a certain point the rate of improvement decreases as shown by the table below for rectangular pulses.

E/N_0	$\alpha=0$	$\alpha=1.5$	$\alpha=5.0$	$\alpha=10.0$	$\alpha=\infty$
.1	.370	.349	.337	.334	.327
1.0	.168	.118	$.936 \times 10^{-1}$	$.883 \times 10^{-1}$	$.186 \times 10^{-1}$
5.0	$.634 \times 10^{-1}$	$.163 \times 10^{-1}$	$.248 \times 10^{-2}$	$.153 \times 10^{-2}$	$.182 \times 10^{-3}$
10.0	$.446 \times 10^{-1}$	$.814 \times 10^{-2}$	$.149 \times 10^{-3}$	$.271 \times 10^{-4}$	$.388 \times 10^{-5}$

The most improvement is gained from $\alpha=0$ to $\alpha=5$ for E/N_0 of the order of 1 or less, and little is gained by making $\alpha>5$. For higher E/N_0 a much better synchronizer is required as shown by the fact that for $E/N_0=10$ and $\alpha=10$ the probability of error can still be improved by an order of magnitude.

The reason for these results can be seen by looking at the output of a matched filter. Figure 3.6 shows the waveforms at the output of a matched filter for rectangular pulses. From figure 3.3 it

FIGURE 3.6 MATCHED FILTER OUTPUT FOR RECTANGULAR DATA PULSES

is seen that changing α from 5 to 10 makes the timing error density function more peaked, i.e., the timing error is more likely to be near zero. The additive white Gaussian noise has a variance $EN_0/2$ at the matched filter output, and its standard deviation is shown on figure 3.6. Let us consider only positive timing errors and examine the probability of error on the first pulse shown in figure 3.6. If the second pulse is the same then

$$P[E|\epsilon] = \frac{1}{2}[1 - \text{erf}(\sqrt{E/N_0})]$$

and if the second pulse has the opposite sign then

$$P[E|\epsilon] = \frac{1}{2} - \frac{1}{2} \text{erf}[(1-2\epsilon)\sqrt{E/N_0}].$$

These equations state that when the following pulse has the same sign the probability of error is independent of ϵ , i.e., improving the timing error has no effect. However, when the following pulse has the opposite sign the probability of error does depend on ϵ as shown in figure 3.7. These statements are also evident from figure 3.6 itself.

As α is increased from its lowest value of 0, i.e., as the timing error probability density function becomes more peaked, the sampling pulse is more likely to be near the peak of the output waveform at $t=T/2$. This means that the noise effects will be reduced and the probability of error will be improved. It could be expected from figure 3.6 that as α is increased the probability of error would improve rapidly at first, and then the rate of improvement would diminish as the timing error became more concentrated near the peak of the output waveform. This is indeed what is found in the table, i.e., at $E/N_0=10$

FIGURE 3.7 PROBABILITY OF ERROR AS A FUNCTION OF UNKNOWN TIME SHIFT FOR MATCHED FILTER SYSTEM WITH RECTANGULAR PULSES

there is more than an order of magnitude improvement in $P[E]$ as α goes from 1.5 to 5, but less than one order of magnitude improvement as α goes from 5 to 10.

The preceding discussion has been for the case where the timing error probability density function does not change when the data signal bit energy to noise density ratio E/N_0 varies. This might be true when synchronization information is obtained from a separate channel or where the data signal is affected by independent Gaussian intermodulation noise. The discussion also applied to cases where the circuitry of a synchronizer could be changed to give improved synchronizer performance at a constant E/N_0 .

We now consider the case where the timing error probability density function does depend on E/N_0 . It is assumed that the noise power density is the same at the inputs to the matched filter detector and to the synchronizer.

Assume, for example, that bit synchronization is provided by a phase-locked loop with $\alpha = \frac{A_C^2}{N_0 W_L}$ [9]

where A_C is the amplitude of a synchronizing sine wave

W_L is the loop bandwidth (2 sided)

α is the sine wave carrier to noise ratio in the loop bandwidth.

The data signal power is E/T and the sine wave power is $A_C^2/2$. Let $A_C^2/2 = \beta(E/T)$ so that $\alpha = \beta(E/N_0) \cdot 2/W_L T$.

A typical value of $2/W_L T$ is 25 [9], so the probability of error has been plotted in figure 3.8 for $\alpha=25 \beta(E/N_0)$ with $\beta=.1, .01$, and $.001$. These curves show, for example, the decrease in $P[E]$ as E/N_0 is increased provided a fixed ratio is maintained between data signal energy

FIGURE 3.8 PROBABILITY OF ERROR FOR MATCHED FILTER SYSTEM WITH SYNCHRONIZATION SIGNAL POWER A CONSTANT FRACTION OF DATA SIGNAL POWER

and synchronization signal energy. Under these conditions the rectangular pulse shape is slightly better than the raised cosine at low E/N_0 and vice versa at high E/N_0 . The curves also show the effect of increasing the synchronization signal energy while keeping the data signal energy fixed. For example if the data signal energy is such that $E/N_0=4$, the $P[E]$ goes from .077 at $\beta=.001$ to .00278 at $\beta=.1$, i.e., approximately a 10% increase in transmitter energy devoted entirely to synchronization signal.

Figure 3.9 shows $P[E]$ as a function of β for various E/N_0 when the total transmitter energy is kept fixed. This shows the optimum power division for this system. It should be noted that in these curves E/N_0 is the total transmitter energy to noise density ratio, and that the data signal energy to noise density ratio is given by $(1+\beta)^{-1}E/N_0$. The E/N_0 shown on the curves is also the data signal bit energy to noise ratio when $\beta=0$. It is evident that $\beta=.1$ is the optimum power division for these cases. If α were larger, e.g., $\alpha=50\beta(1+\beta)^{-1}E/N_0$, the curves would be lower and the optimum power division points would occur at a lower value of β because the higher value of α makes the timing error density more peaked.

3.3.4 Results for Sine Wave in Gaussian Noise

In this section it is assumed that the timing error probability density function is given by the phase density of a sine wave in narrow-band Gaussian noise, i.e.,

$$p(\theta) = \frac{e^{-\eta^2}}{2\pi} \left\{ 1 + \sqrt{\pi} \eta \cos(\theta) [1 + \operatorname{erf}(\eta \cos(\theta))] e^{\eta^2 \cos^2(\theta)} \right\}$$

FIGURE 3.9 PROBABILITY OF ERROR FOR MATCHED FILTER SYSTEM WITH SYNCHRONIZATION REFERENCE SIGNAL AND CONSTANT TRANSMITTER POWER

for $-\pi \leq \theta \leq \pi$ with $\eta^2 = A_C^2 / 2\sigma^2$

A_C = sine wave peak amplitude

σ^2 = noise variance.

If this density is used for the timing error pdf in the matched filter $P[E]$ equation (3.1.6), then for constant η^2 curves of $P[E]$ vs. E/N_0 are obtained as in figure 3.10. E/N_0 is the data signal bit energy to noise power density ratio, and the noise in the data channel is assumed to be independent of the noise in the synchronization channel.

Since η^2 is the SNR at the band pass filter output and α was defined as the SNR in the PLL linearized loop equivalent noise bandwidth, the band pass filter ideal phase detector can be compared with the PLL by setting $\eta^2 = \alpha$ and computing $P[E]$. This has been done in table 3.1 where results are also presented for $\eta^2 = \alpha/2$ which has been used in discussion of experimental PLL phase densities [10]. This latter case corresponds to equal modulation bandwidth because the PLL bandwidth is W_L centered about $\omega=0$ while the band pass filter bandwidth is W_L centered about some carrier frequency (and thus $2W_L$ total 2 sided bandwidth). Then the probability density functions are very similar with the PLL density having a lower peak as shown in figure 3.11.

It can be seen from table 3.1a that the band pass filter ideal phase detector gives only slightly better performance than the PLL when $\eta^2 = \alpha$, whereas it gives worse performance than the PLL when $\eta^2 = \alpha/2$. This behaviour is expected from the shapes of the timing error density functions shown in figure 3.11. However, the equal SNR case, $\eta^2 = \alpha$, means that the one-sided bandwidth of the band pass filter must be the same as the one-sided PLL equivalent noise bandwidth. The case $\eta^2 = \alpha/2$

TABLE 3.1a
Probability of Error for $\eta^2=\alpha$ and Rectangular Pulses

E/N_0	$\alpha=1$	$\eta^2=1$	$\alpha=4$	$\eta^2=4$
.1	.354	.344	.339	.335
.2	.299	.286	.278	.274
.4	.231	.213	.203	.198
.8	.154	.133	.121	.115
1	.130	.109	.961 x 10 ⁻¹	.902 x 10 ⁻¹
2	.676 x 10 ⁻¹	.462 x 10 ⁻¹	.344 x 10 ⁻¹	.300 x 10 ⁻¹
4	.327 x 10 ⁻¹	.153 x 10 ⁻¹	.640 x 10 ⁻²	.452 x 10 ⁻²
8	.182 x 10 ⁻¹	.628 x 10 ⁻²	.683 x 10 ⁻³	.315 x 10 ⁻³
10	.155 x 10 ⁻¹	.513 x 10 ⁻²	.349 x 10 ⁻³	.162 x 10 ⁻³
20	.999 x 10 ⁻²	.313 x 10 ⁻²	.868 x 10 ⁻⁴	.652 x 10 ⁻⁴

TABLE 3.1b
Probability of Error for $\eta^2=\alpha/2$ and Rectangular Pulses

E/N_0	$\alpha=1.5$	$\eta^2=\alpha/2$	$\alpha=5$	$\eta^2=\alpha/2$
.1	.349	.347	.337	.338
.2	.292	.289	.276	.277
.4	.221	.218	.201	.202
.8	.142	.138	.119	.120
1	.118	.114	.936 x 10 ⁻¹	.947 x 10 ⁻¹
2	.548 x 10 ⁻¹	.513 x 10 ⁻¹	.324 x 10 ⁻¹	.336 x 10 ⁻¹
4	.216 x 10 ⁻¹	.193 x 10 ⁻¹	.538 x 10 ⁻²	.647 x 10 ⁻²
8	.990 x 10 ⁻²	.893 x 10 ⁻²	.376 x 10 ⁻³	.111 x 10 ⁻²
10	.814 x 10 ⁻²	.743 x 10 ⁻²	.149 x 10 ⁻³	.774 x 10 ⁻³
20	.490 x 10 ⁻²	.464 x 10 ⁻²	.176 x 10 ⁻⁴	.405 x 10 ⁻³

FIGURE 3.10 PROBABILITY OF ERROR FOR MATCHED FILTER SYSTEM AND TIMING ERROR INDEPENDENT OF DATA SIGNAL TO NOISE RATIO - TIMING ERROR DESCRIBED BY PHASE DENSITY OF SINE WAVE IN GAUSSIAN NOISE

FIGURE 3.11 COMPARISON OF PHASE-LOCKED LOOP PHASE ERROR
 DENSITY AND PHASE DENSITY OF SINE WAVE IN
 GAUSSIAN NOISE

means that the one-sided bandwidth of the bandpass filter is twice that of the PLL equivalent noise bandwidth.

3.4 BOUNDS FOR $P[E]$

In this section some bounds for $P[E]$ are derived for rectangular data pulses with the timing error probability density function given by the PLL phase error density. These bounds serve two purposes. The first is as a check on the numerical calculations; the theoretical $P[E]$ must fall between the upper and lower bounds on $P[E]$. The second is that the bounds show that $P[E] \rightarrow 0$ as $E/N_0 \rightarrow \infty$, i.e., there is not a residual error probability at high SNR. Unfortunately it does not seem possible to find asymptotic expressions for $P[E]$ at high SNR from the bounds as will be seen below. A number of useful bounds for the error function are derived in Appendix 3C.

3.4.1 Rectangular Data Pulses - Uniform Timing Error

Good bounds can be derived for the case of $p(\epsilon)=1$. For a lower bound, we have from (3.3.3)

$$\begin{aligned} P[E] &= \int_0^{1/2} \left(1 - \frac{1}{2} \operatorname{erf}(\eta) - \frac{1}{2} \operatorname{erf}[\eta(1-2u)]\right) p(u) du \\ &= \frac{1}{2} \left[1 - \frac{1}{2} \operatorname{erf}(\eta)\right] - \frac{1}{2} \int_0^{1/2} \operatorname{erf}[\eta(1-2u)] du \end{aligned}$$

$$\text{for } p(u)=1, \eta=\sqrt{E/N_0}. \quad (3.4.1)$$

As noted in Appendix 3C, equation (3C.2), a good upper bound for the erf function is

$$\text{erf}(x) \leq \begin{cases} 2/\sqrt{\pi} \arctan(x) & 0 \leq x \leq 1.226 \\ 1 & x \geq 1.226 \end{cases}$$

Therefore we use

$$\text{erf}[\eta - 2\eta u] \leq \begin{cases} 1 & 0 \leq u \leq u_L \\ 2/\sqrt{\pi} \arctan[\eta - 2\eta u] & u_L \leq u \leq \frac{1}{2} \end{cases} \quad (3.4.2)$$

where u_L is such that $\eta - 2\eta u_L = 1.226$

$$\text{or } u_L = \frac{\eta - 1.226}{2\eta}.$$

Clearly $\eta \geq 1.226$ is required for use of this bound, i.e., "high" SNR.

$$\int_0^{1/2} \text{erf}[\eta - 2\eta u] du \leq \int_0^{u_L} du + \frac{2}{\sqrt{\pi}} \int_{u_L}^{1/2} \arctan[\eta - 2\eta u] du = .5 - \frac{.259}{\eta}$$

$$P[E] \geq \frac{1}{4} - \frac{1}{4} \text{erf}(\eta) + \frac{.1293}{\eta} \quad \text{for } \eta \geq 1.226 \quad (3.4.3)$$

where $1.226 = \tan(\sqrt{\pi}/2)$. A lower bound for $0 \leq \eta \leq 1.226$ can be derived as follows.

From Appendix 3C, equation (3C.1), we have an upper bound on the error function integral of (3.4.1) by letting $p=2/\sqrt{\pi}$, $s=1$, so that for $\eta \leq 1.226$

$$\begin{aligned} \int_0^{1/2} \text{erf}[\eta - 2\eta u] du &\leq \frac{2}{\sqrt{\pi}} \int_0^{1/2} \arctan(\eta - 2\eta u) du \\ &= \frac{1}{\sqrt{\pi}} \arctan(\eta) - \frac{1}{2\eta\sqrt{\pi}} \log_e(1+\eta^2). \end{aligned}$$

Substituting in (3.4.1)

$$P[E] > \frac{1}{2} - \frac{1}{4} \operatorname{erf}(\eta) - \frac{1}{2\sqrt{\pi}} \arctan(\eta) + \frac{1}{4\eta\sqrt{\pi}} \log_e(\eta^2+1). \quad (3.4.4)$$

Note that $\frac{1}{4\sqrt{\pi}\eta} \log_e(\eta^2+1) \rightarrow 0$ as $\eta \rightarrow 0$ as do $\operatorname{erf}(\eta)$ and $\arctan(\eta)$ so that

$P[E] \rightarrow \frac{1}{2}$ as $\eta \rightarrow 0$. The bound is also valid for all SNR and as $\eta \rightarrow \infty$ we get $\frac{1}{2} - \frac{\sqrt{\pi}}{4}$ which is poor because for large η the argument of $\operatorname{erf}[\eta-2\eta u]$ is large for most u and the arctangent is a poor approximation for arguments >1 . However, the bound is very tight for the region $0 \leq \eta \leq 1.226$. Hence for a lower bound on $P[E]$ we have

$$P[E] > \frac{1}{2} - \frac{1}{4} \operatorname{erf}(\eta) - \frac{1}{2\sqrt{\pi}} [\arctan(\eta) - \frac{1}{2\eta} \log_e(\eta^2+1)] \quad 0 \leq \eta \leq 1.226 \quad (3.4.5)$$

$$\text{and} \quad P[E] > \frac{1}{4} - \frac{1}{4} \operatorname{erf}(\eta) + \frac{.1293}{\eta} \quad \eta \geq 1.226 \quad (3.4.6)$$

The bounds are compared with the true computed values in table 3.2.

For an upper bound we use equation (3C.9) of Appendix 3C so that

$$\begin{aligned} P[E] &\leq \frac{1}{2} - \frac{1}{4} \operatorname{erf}(\eta) - \frac{1}{2} \left[\frac{1}{2} - \frac{1}{2} \int_0^{1/2} e^{-x^2} + e^{-2x^2} du \right] \quad \text{where } x = \eta(1-2u) \\ &= \frac{1}{4} + \left(\frac{\sqrt{\pi}}{16\eta} - \frac{1}{4} \right) \operatorname{erf}(\eta) + \frac{\sqrt{\pi}}{16\sqrt{2}\eta} \operatorname{erf}(\sqrt{2}\eta) \end{aligned} \quad (3.4.7)$$

which is also given in table 3.2. It is not as tight as the lower bounds. In both cases, however, $P[E] \propto \frac{1}{\eta}$ (approximately) as $\eta \rightarrow \infty$ thus showing that there is not any residual $P[E]$ at high E/N_0 .

TABLE 3.2

Bounds on the Probability of Error for Rectangular
Pulses and Uniform Timing Error

E/N ₀	Exact P[E]	Eq. 3.4.5	Eq. 3.4.6	Eq. 3.4.7
.1	.370	.370		.402
.2	.321	.321		.359
.4	.259	.259		.301
.8	.1898	.188		.231
1	.168	.166		.207
2	.109		.103	.141
4	$.716 \times 10^{-1}$		$.658 \times 10^{-1}$	$.955 \times 10^{-1}$
8	$.499 \times 10^{-1}$		$.457 \times 10^{-1}$	$.668 \times 10^{-1}$
10	$.446 \times 10^{-1}$		$.409 \times 10^{-1}$	$.598 \times 10^{-1}$
20	$.315 \times 10^{-1}$		$.289 \times 10^{-1}$	$.423 \times 10^{-1}$
40	$.223 \times 10^{-1}$		$.204 \times 10^{-1}$	$.299 \times 10^{-1}$
80	$.158 \times 10^{-1}$		$.144 \times 10^{-1}$	$.211 \times 10^{-1}$
100	$.141 \times 10^{-1}$		$.129 \times 10^{-1}$	$.189 \times 10^{-1}$

3.4.2 Rectangular Timing Pulses - General Timing Error

In this section bounds are presented for $p(\epsilon) = \frac{e^{\alpha \cos(2\pi\epsilon)}}{I_0(\alpha)}$.

The algebra is very lengthy and so only the simplest bounds have been attempted.

For a lower bound we have from equation (3.4.1)

$$\begin{aligned}
 P[E] &= \frac{1}{2} - \frac{1}{4} \operatorname{erf}(\eta) - \frac{1}{2} \int_0^{1/2} P(u) \operatorname{erf}[\eta - 2\eta u] du \\
 &> \frac{1}{2} - \frac{1}{4} \operatorname{erf}(\eta) - \frac{1}{2} \int_0^{1/2} P_{u_B}(u) \operatorname{erf}_{u_B}[\eta - 2\eta u] du \\
 &\quad \text{where } P_{u_B}(u) \text{ denotes upper bound etc.} \\
 &= \frac{1}{2} - \frac{1}{4} \operatorname{erf}(\eta) - \frac{1}{4\eta I_0(\alpha)} \int_0^\eta e^{-\alpha \cos(\pi x/\eta)} \operatorname{erf}_{u_B}(x) dx \quad (3.4.8)
 \end{aligned}$$

Now bound $\cos(\pi x/\eta)$ below by the straight line approximation shown in figure 3.12 and use the simple error function approximation

$$\operatorname{erf}(x) \leq \sqrt{1 - e^{-2x^2}} \leq 1 - \frac{1}{2} e^{-2x^2} \quad 0 \leq x \leq \infty$$

from equation (3C.4).

$$\begin{aligned}
 P[E] &> \frac{1}{4} - \frac{1}{4} \operatorname{erf}(\eta) + \frac{\sqrt{\pi/2}}{16\eta \cdot I_0(\alpha)} \left\{ e^{-\alpha + \alpha^2} [\operatorname{erf}(\eta/2\sqrt{2} - \alpha) + \operatorname{erf}(\alpha)] \right. \\
 &\quad + e^{(\alpha/\eta)^2 - \sqrt{2}\alpha} [\operatorname{erf}(\eta/\sqrt{2} - \alpha/\eta) - \operatorname{erf}(\eta/2\sqrt{2} - \alpha/\eta)] \\
 &\quad \left. + e^{\alpha^2} [\operatorname{erf}(\eta - \alpha) - \operatorname{erf}(\eta/2 - \alpha)] + e^{\alpha^2} [\operatorname{erf}(\eta\sqrt{2}) - \operatorname{erf}(\eta)] \right\} \quad (3.4.9)
 \end{aligned}$$

FIGURE 3.12 APPROXIMATION TO COSINE FUNCTION

where $a = (\sqrt{2}-1)\alpha/\eta$

$$b = \sqrt{2}\eta(1/2 + 1/\pi)$$

$$c = \alpha\pi/\eta 2\sqrt{2}$$

$$d = (\alpha\pi/\eta)^2/8 - \pi\alpha/2,$$

and numerical values of (3.4.9) are given in table 3.3. As can be seen, the bound is not very tight but does follow the general shape of the theoretical curves.

A similar upper bound can be derived, but both it and (3.4.9) are difficult to compute because some of the exponentials become extremely large and are multiplied by the differences of two erf functions which are almost equal. This occurs at low E/N_0 and accounts for the absence of some entries in table 3.3.

3.5 CONCLUSIONS

In this chapter a simple binary matched filter system with unknown timing error has been examined. The probability of error expression was derived and computed for a representative timing error density. As a result the following conclusions were reached for a system with timing jitter density independent of data channel SNR:

1. $P[E]$ breaks away from the perfect timing curve as E/N_0 increases, but does go to zero as E/N_0 becomes large. This is because the timing error ϵT is restricted to $[-T/2, T/2]$ and hence in the noiseless case perfect detection is still possible.
2. Raised cosine pulses are better than rectangular pulses for high α . Intuitively one would expect this to be so

TABLE 3.3

Bounds on the Probability of Error for Rectangular
Pulses and PLL-type Timing Error Density

E/N_0	$\alpha=1.5$		$\alpha=5$	
	Exact $P[E]$	Eq. 3.4.9	Exact $P[E]$	Eq. 3.4.9
.1	.349			
.2	.292			
.4	.221	.178	.201	.157
.8	.142	.108	.119	.102
1	.118	$.864 \times 10^{-1}$	$.936 \times 10^{-1}$	$.774 \times 10^{-1}$
2	$.548 \times 10^{-1}$	$.336 \times 10^{-1}$	$.325 \times 10^{-1}$	$.209 \times 10^{-1}$
4	$.216 \times 10^{-1}$	$.109 \times 10^{-1}$	$.538 \times 10^{-2}$	$.204 \times 10^{-2}$
8	$.990 \times 10^{-2}$	$.521 \times 10^{-2}$	$.376 \times 10^{-3}$	$.720 \times 10^{-3}$
10	$.814 \times 10^{-2}$	$.440 \times 10^{-2}$	$.149 \times 10^{-3}$	$.640 \times 10^{-4}$
20	$.490 \times 10^{-2}$	$.281 \times 10^{-2}$	$.176 \times 10^{-4}$	$.835 \times 10^{-5}$
40	$.322 \times 10^{-2}$	$.188 \times 10^{-2}$	$.727 \times 10^{-5}$	$.463 \times 10^{-5}$
80	$.221 \times 10^{-2}$	$.129 \times 10^{-2}$	$.439 \times 10^{-5}$	$.287 \times 10^{-5}$
100	$.196 \times 10^{-2}$	$.114 \times 10^{-2}$	$.382 \times 10^{-5}$	$.249 \times 10^{-5}$

since the raised cosine has a flatter autocorrelation function near $\tau=0$.

3. A synchronizer which gives $5 < \alpha < 10$ will achieve essentially the best possible system performance for $E/N_0 \leq 1$. $\alpha > 10$ is required for $E/N_0 \geq 10$.

For a system in which a fraction $\beta/(1+\beta)$ of the transmitter power is devoted to the synchronization signal where $0 < \beta < 1$, it was found that for $\beta = .1$ and raised cosine pulses the system performance was essentially perfect for $E/N_0 > 1$.

If the synchronization channel noise power density is the same as the data signal channel noise power density, and if the transmitter power is kept fixed, then $\beta = .1$ gives the optimum power division for $E/N_0 \leq 15$. This is for a sine wave reference signal received by a PLL with $W_L T = .08$.

APPENDIX 3A

In this appendix the calculation of the output of a matched filter when the sampling time is jittered is discussed. Let the matched filter have an impulse response $s(T-t) = h(t)$ of finite duration T , and consider first the case where the filter sampling time is in error.

For purposes of illustration a message $r(t)$ as shown in figure 3.13a is assumed, and consider the sampling operation at $t=3T+\epsilon$ where $\epsilon>0$ is the sampling time error. Then from figure 3.13b, since

$$y(t) = \int_0^t r(t-\tau)h(\tau) d\tau$$

then

$$\begin{aligned} y(3T+\epsilon) &= \int_0^{\epsilon} s(\epsilon-\tau)s(T-\tau)d\tau + \int_{\epsilon}^T s(T+\epsilon-\tau)s(T-\tau) d\tau \\ &= \int_T^{T+\epsilon} s(p-T)s(p-\epsilon)dp + \int_{\epsilon}^T s(p)s(p-\epsilon) dp. \end{aligned}$$

Thus for $\epsilon>0$ there is a contribution from the k^{th} and $(k+1)^{\text{th}}$ pulses. The case $\epsilon<0$ follows in a similar way with the $(k-1)^{\text{th}}$ and k^{th} pulses contributing.

The above analysis holds for several practical cases -

1. a filter where a finite duration impulse response is realized by a delay and subtraction circuit and the sampling timing is not synchronized properly with the incoming signal;
2. a filter where the finite duration impulse response is realized by an integrate-and-dump circuit or by shift

FIGURE 3.13 ILLUSTRATION OF WAVEFORMS IN CONVOLUTION

registers and the dumping and sampling are simultaneous and not synchronized properly with the received signal;

3. a correlation receiver in which the local signal is also synchronized with the sampling and dumping operations.

APPENDIX 3B

In this appendix the probability density function for the phase of a sine wave in narrow band Gaussian noise and the probability density function for the phase error of a PLL are examined. For the former we have

$$p(\theta) = \frac{e^{-\eta^2}}{2\pi} \left\{ 1 + \eta\sqrt{\pi} \cos(\theta) \cdot [1 + \operatorname{erf}(\eta \cos(\theta))] \right\} e^{\eta^2 \cos^2(\theta)} \quad (3B.1)$$

and for the latter when the input to the PLL is a sine wave in band-pass Gaussian noise [10]

$$p(\theta) = \frac{\exp[\alpha \cos(\theta)]}{2\pi I_0(\alpha)} \quad (3B.2)$$

where for (3B.1) $\eta^2 = \frac{A_c^2}{2\sigma^2} = \text{SNR}$

A_c = sine wave amplitude

σ^2 = noise variance

and for (3B.2) $\alpha = \frac{A_c^2}{N_0 W_L}$

A_c = sine wave amplitude

$N_0/2$ = 2 sided noise spectral density amplitude

W_L = 2 sided linearized loop equivalent noise bandwidth.

The two densities can be compared in several ways. They can be compared on the basis of equal SNR's or equal modulation bandwidths as already discussed in section 3.3.4. They can also be compared on the basis of their moments as is done below.

Since the distributions are symmetrical the odd moments are zero, and a formula for the moments of (3B.2) can be derived as follows:

from eq. (9.6.34) of the NBS Handbook [11]

$$\exp[\alpha \cos(\theta)] = I_0(\alpha) + 2 \sum_{k=1}^{\infty} I_k(\alpha) \cos(k\theta)$$

so that

$$\begin{aligned} m^{\text{th}} \text{ moment} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{k=1}^{\infty} \frac{I_k(\alpha)}{I_0(\alpha)} \cos(k\theta) \right] \theta^m d\theta \\ &= \frac{\pi^m}{m+1} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{I_k(\alpha)}{I_0(\alpha)} \int_{-\pi}^{\pi} \theta^m \cos(k\theta) d\theta, \quad m \text{ even.} \end{aligned}$$

Integrating by parts

$$\begin{aligned} \int_{-\pi}^{\pi} \theta^m \cos(k\theta) d\theta &= \frac{2m\pi^{m-1}}{k^2} (-1)^k \left[1 - \frac{(m-1)(m-2)}{(k\pi)^2} + \frac{(m-1)(m-2)(m-3)(m-4)}{(k\pi)^4} \dots \right. \\ &\quad \left. \dots \pm \frac{(m-1)\dots(2)}{(k\pi)^{m-2}} \right] \end{aligned} \quad (3B.3)$$

so that

$$\begin{aligned} m^{\text{th}} \text{ moment} &= \frac{\pi^m}{m+1} + 2m\pi^{m-2} \left[\sum_{k=1}^{\infty} F_k(\alpha) \frac{(-1)^k}{k^2} - \frac{(m-1)(m-2)}{\pi^2} \sum_{k=1}^{\infty} F_k(\alpha) \frac{(-1)^k}{k^4} \dots \right. \\ &\quad \left. \pm \frac{(m-1)\dots(2)}{\pi^{m-2}} \sum_{k=1}^{\infty} F_k(\alpha) \frac{(-1)^k}{k^m} \right] \end{aligned} \quad (3B.4)$$

where

$$\begin{aligned}
F_k(\alpha) &= \frac{I_k(\alpha)}{I_0(\alpha)} = \frac{1}{I_0(\alpha)} \left[\frac{-2(k-1)}{\alpha} I_{k-1}(\alpha) + I_{k-2}(\alpha) \right] \quad k \geq 2 \\
&= \frac{-2(k-1)}{\alpha} F_{k-1} + F_{k-2}(\alpha)
\end{aligned}$$

and $I_0(\alpha)$, $I_1(\alpha)$ are tabulated. The terms of the series converge quite rapidly, and for $m=2$ we have

$$\sigma_\theta^2 = \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k I_k(\alpha)}{k^2 I_0(\alpha)}$$

as given also by Viterbi [12]. The first 6 central moments of (3B.1) and (3B.2) are given in table 3.4 which was computed using a 25 point Simpson's Rule.

TABLE 3.4

Moments of Timing Error Densities

Moment	$\alpha=1.5$		$\alpha=5.0$		$\alpha=10.0$	
	Eq. 3B.1	Eq. 3B.2	Eq. 3B.1	Eq. 3B.2	Eq. 3B.1	Eq. 3B.2
1	0	0	0	0	0	0
2	.958	1.091	.279	.227	.117	.106
3	0	0	0	0	0	0
4	3.661	4.10	.474	.177	.0620	.0344
5	0	0	0	0	0	0
6	21.99	23.85	2.14	.285	.137	.0204

APPENDIX 3C

In this appendix some bounds for error functions and integrals of error functions are derived.

Erf upper bound #1:

Let

$$\text{erf}(x) = p \int_0^x e^{-st^2} dt$$

But

$$e^{-st^2} \leq \frac{1}{1+st^2}$$

so that

$$\text{erf}(x) \leq \frac{p}{\sqrt{s}} \cdot \arctan(x\sqrt{s}) \quad (3C.1)$$

where in the common definitions of $\text{erf}(x)$ either $p=2/\sqrt{\pi}$ and $s=1$, or $p=1/\sqrt{2\pi}$ and $s=1/2$.

Figure 3.14 shows that a better bound is obtained by using

$$\text{erf}(x) \leq \begin{cases} \frac{2}{\sqrt{\pi}} \arctan(x) & 0 \leq x \leq y \\ 1 & x \geq y \end{cases} \quad (3C.2)$$

where

$$y = \tan(\sqrt{\pi}/2) \approx 1.226$$

Erf upper bound #2:

Let

$$y(x) = p \int_0^x e^{-st^2} dt$$

so that

FIGURE 3.14 COMPARISON OF $\text{Erf}(x)$ AND $\text{Arctan}(x)$

$$\begin{aligned}
y^2(x) &= p^2 \int_0^x \int_0^x e^{-(t^2+w^2)s} dt dw \\
&< p^2 \int_{r=0}^{\sqrt{2}x} \int_{\theta=0}^{\pi/2} e^{-sr^2} r dr d\theta \\
&= \frac{\pi p^2}{4s} (1 - e^{-2sx^2})
\end{aligned}$$

$$\therefore \quad \text{erf}(x) < \sqrt{\frac{\pi}{4} \frac{p^2}{s} (1 - e^{-2sx^2})} \quad (3C.3)$$

By a binomial expansion

$$\sqrt{1 - e^{-2sx^2}} \leq 1 - \frac{1}{2} e^{-2sx^2} \quad \text{for } x \geq 0$$

so that for $p=2/\sqrt{\pi}$ and $s=1$

$$\text{erf}(x) < 1 - \frac{1}{2} e^{-2x^2} \quad \text{for } x \geq 0 \quad (3C.4)$$

By direct calculation it can be seen that

$$\sqrt{1 - e^{-2x^2}} \leq 1 - \frac{1}{2} e^{-\sqrt{2}x} - \frac{1}{2} e^{-2\sqrt{2}x} \quad 0 \leq x \leq \frac{1}{2}$$

so that for $p=2/\sqrt{\pi}$ and $s=1$

$$\text{erf}(x) < 1 - \frac{1}{2} e^{-\sqrt{2}x} - \frac{1}{2} e^{-2\sqrt{2}x} \quad 0 \leq x \leq \frac{1}{2} \quad (3C.5)$$

Erf upper bound #3:

Integrating by parts

$$\int_a^\infty e^{-x^2} dx = \frac{e^{-a^2}}{2a} - \int_a^\infty \frac{e^{-x^2}}{2x^2} dx$$

But

$$\int_a^{\infty} \frac{e^{-x^2}}{x^2} dx < \frac{1}{a^3} \int_a^{\infty} x e^{-x^2} dx = \frac{e^{-a^2}}{2a^3} \quad [13]$$

so that

$$\int_a^{\infty} e^{-x^2} dx > \frac{e^{-a^2}}{2a} \left(1 - \frac{1}{2a^2}\right).$$

Thus

$$\begin{aligned} \operatorname{erf}(x) &= 1 - \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt \\ &\leq 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 - \frac{1}{2x^2}\right] \quad x > 0 \end{aligned} \quad (3C.6)$$

Erf lower bound:

Let

$$y(x) = p \int_0^x e^{-st^2} dt$$

so that

$$\begin{aligned} y^2(x) &= p^2 \int_0^x \int_0^x e^{-s(t^2+w^2)} dt dw \\ &> p^2 \int_{r=0}^x \int_{\theta=0}^{\pi/2} e^{-sr^2} r dr d\theta = \frac{\pi p^2}{4s} (1 - e^{-sx^2}) \end{aligned}$$

Thus

$$y(x) > \sqrt{\frac{\pi p^2}{4s}} \sqrt{1 - e^{-sx^2}}$$

and for $p = \frac{2}{\sqrt{\pi}}$, $s=1$

$$\operatorname{erf}(x) > \sqrt{1 - e^{-x^2}} \quad (3C.7)$$

Since $0 \leq 1 - e^{-x^2} \leq 1$ implies $\sqrt{1 - e^{-x^2}} > 1 - e^{-x^2}$ then

$$\operatorname{erf}(x) > 1 - e^{-x^2}, \quad (3C.8)$$

and also by direct calculation

$$\sqrt{1 - e^{-x^2}} \geq 1 - \frac{1}{2} e^{-x^2} - \frac{1}{2} e^{-2x^2} \quad \text{for } 0 \leq x < \infty$$

so that

$$\operatorname{erf}(x) > 1 - \frac{1}{2} e^{-x^2} - \frac{1}{2} e^{-2x^2} \quad (3C.9)$$

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CHAPTER 4

PREVIOUS WORK ON SELF-BIT SYNCHRONIZERS

4.0 INTRODUCTION

In addition to the bit synchronization schemes mentioned in Chapter 2, several different types of self-bit synchronizers have been built and used. However, attempts at designing self-bit synchronizers which are optimum in some sense have been relatively recent. In 1968 and 1969 several articles appeared [1]-[3] describing simultaneous work on maximum likelihood (ML) or maximum a posteriori probability (MAP) synchronizers. Several other approaches have been described [4], [5], and the discrete maximum principle has been applied to the design of a suboptimum synchronizer [6]. A Bayesian approach has been outlined briefly in [7] and in much more detail by Nitadori [8] who gives some references to the Japanese literature. Some of these synchronizers are discussed in the following sections.

Most of the work on self-bit synchronizers seems to have been spurred by the need for PCM telemetry systems in the U.S. space program and by industrial development of PCM systems. Usually only NRZ (non-return-to-zero) binary signals have been considered.

4.1 LIMITER-DIFFERENTIATOR TYPE

The simplest way to derive bit timing for the case of binary rectangular pulses received in additive noise would be to hard-limit, differentiate, and full-wave rectify the received signal, and to use the resulting sequence of impulses to control the phase of a local oscillator. One such synchronizer is shown in figure 4.1 and it is the

FIGURE 4.1 LIMITER-DIFFERENTIATOR TYPE SYNCHRONIZER [9]

synchronizer used in the Apollo program. Timing errors in this device include static timing error due to tolerances in filter delays and slight frequency offset in the VCO, and random errors due to noise-induced jitter in the VCO and variations in the data transition density. Two points worth noting from the study of this synchronizer presented in [9] are: a) that the timing error is about 9-10% of a bit period and, b) that the overall system has a performance degradation of about 2.5 dB. (Performance degradation is the difference in SNR's required to achieve a given bit error rate with the MSFTP-2 and an ideal matched filter receiver with perfect timing.) This is for an SNR of 13.5 dB and equally likely binary data with rectangular signal pulses.

4.2 DIGITAL TRANSITION TRACKER

The digital transition tracker was invented by Anderson, Hurd, and Lindsey [10], and has been recently described by Hurd and Anderson [5]. Block diagrams of this device are shown in figure 4.2. The device is entirely digital except for the integrators and two digital-to-analog (D/A) converters, and has automatic loop filter bandwidth control-wide for acquisition and narrow for tracking after acquisition. It operates over a range of data rates from 5.6 bps to 250 Kbps and achieves a performance degradation of .03 dB at an SNR of -3 dB. This is important because coded data signals can be recovered at this SNR and lower, and most synchronizers have a performance degradation of 1 to 5 dB in this region. Furthermore, bit slippage usually occurs in the range 0-5 dB SNR [12].

Briefly, the device works as follows. The timing logic samples and dumps the in-phase integrator (and clocks other parts of the device)

FIGURE 4.2 DIGITAL TRANSITION TRACKER [5]

at the assumed transition times. The mid-phase integrator is sampled and dumped at $1/2$ a window length later, i.e., it integrates only over a W second interval which is less than the length of a bit. The output of the in-phase channel indicates the presence of a transition while the output of the mid-phase channel is an indication of the timing error when a transition occurs. The two outputs are used to provide an estimate of the timing error which is independent of the direction of a transition. The loop filter is a digital filter whose output is converted to an analog signal which is used to vary the VCO frequency. A complete description is given in [10].

A related device has also been described by Simon [11] and is shown in figure 4.3.

4.3 MAXIMUM LIKELIHOOD SYNCHRONIZERS

Several workers have studied this type of synchronizer [1]-[3], [8]. It is assumed that a binary signal with unknown static timing error θ , as shown in figure 4.4, is observed in additive white Gaussian noise. The problem then is to choose $\hat{\theta}$ so that $p(\hat{\theta}|r(t)) = \max_{\theta} p(\theta|r(t))$ where $r(t)$ is the received waveform. The expression derived is of the form

$$p(\theta|r(t)) \propto \prod_{j=0}^L \cosh \left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt \right]$$

where LT is the length of the observation interval

$$I_0 = [0, \theta]$$

$$I_j = [\theta + (j-1)T, \theta + jT] \quad j=1, 2, \dots, L-1$$

$$I_L = [\theta + (L-1)T, LT]$$

$$s_j(t) = s(t - \theta - (j-1)T)$$

FIGURE 4.3 DIGITAL DATA TRANSITION TRACKING LOOP [11]

FIGURE 4.4 ILLUSTRATION OF NOTATION IN EQUATION (4.3.1)

as shown in figure 4.4. An equivalent expression is

$$A(\theta) = \sum_{j=0}^L \text{lncosh} \left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt \right] \quad (4.3.1)$$

where lncosh denotes the natural logarithm of the hyperbolic cosine, and an approximate realization of this expression is shown in figure 4.5 as obtained by Sage and McBride [3]. At low values of v , $\text{lncosh}(v) \approx v^2/2$, and this approximation was used in a Monte Carlo simulation [3] giving the performance curves of figure 4.6.

Wintz and Luecke [2] proposed, built, and tested a suboptimum realization of equation (4.3.1) shown in figure 4.7. The reasoning that leads to this realization is that (4.3.1) involves a correlation followed by a nonlinear operation, and then by the summation of several such operations. For low-pass waveforms a low-pass filter can be a reasonable suboptimum correlator, the lncosh can be approximated by a square law device, and the bandpass filter acts as a memory or summing device. They examined the performance of this device extensively and figure 4.8 shows some representative results. These particular curves illustrate the fact that rounded signal pulses give better synchronizer performance than rectangular pulses.

In 1964 Van Horn [4] proposed the synchronizer of figure 4.9 which is very similar to figure 4.5. The idea is simply that correlations at various time shifts will give maximum output when in phase with the incoming signal. The measure of performance used was probability of synchronization error.

REALIZATION OF (4.3.1) FOR NRZ PULSES

FIGURE 4.5 REALIZATION OF EQUATION (4.3.1) FOR NRZ PULSES [8]

FIGURE 4.6 PERFORMANCE OF DEVICE OF FIGURE 4.5 [3]

FIGURE 4.7 REALIZATION OF EQUATION (4.3.1) [2]

FIGURE 4.8 PERFORMANCE OF DEVICE OF FIGURE 4.7 [2]

FIGURE 4.9 CORRELATION TYPE SYNCHRONIZER [4]

4.4 BAYES SYNCHRONIZERS

Pitcher and Rumsey [7] presented a brief outline of the equations for a Bayesian synchronizer. A sequence of independent 0's and 1's is assumed to be received in additive white Gaussian noise. The receiver integrates over each bit interval with a static timing misalignment θ . Then the integrator output samples form a vector $\bar{r} = (\theta s_0 + (1-\theta)s_1 + n_1, \dots, \theta s_{N-1} + (1-\theta)s_N + n_N)$ where the n_i are independent zero mean unit variance Gaussian random variables, and the $s_0, \dots, s_N$ are the data signal. σ is known and θ is assumed uniformly distributed over $[0,1]$.

If the average risk is written as

$$R = \int_0^1 \int_{\bar{r}} c(\theta, \hat{\theta}(\bar{r})) p(\theta, \bar{r}) d\bar{r} d\theta$$

where $c(\theta, \hat{\theta}(\bar{r}))$ is a cost function, then it can be shown that we must choose $\hat{\theta}$ to minimize

$$\int_0^1 k(\bar{r}, \theta) \cdot c(\theta, \hat{\theta}(\bar{r})) d\theta$$

where
$$k(\bar{r}, \theta) = \sum_{s_0=1}^2 \dots \sum_{s_N=1}^2 \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^N (r_i - m_i)^2 \right]$$

$$m_i = \theta s_{i-1} + (1-\theta)s_i.$$

It is also shown that a cost function which will minimize the average number of errors is

$$c(\theta, \hat{\theta}(\bar{r})) = P\left[|n| < \frac{1}{2} - |\theta - \hat{\theta}| \text{ or } |n| > \frac{1}{2} - |\theta - \hat{\theta}|\right]$$

where n is a zero mean unit variance Gaussian random variable and σ is

known. This does not appear to be a practical scheme as it would be very difficult to realize such a $\hat{\theta}$.

4.5 SEQUENTIAL SYNCHRONIZER

Sage and McBride [6] derived a synchronizer by formulating the synchronization problem as an optimal control problem with the $A(\theta)$ of (4.3.1) as part of the cost function.

It can be seen from figure 4.10 that we can write a recursive relation for the timing misalignment if there is a small frequency difference between the transmitter and receiver clocks as

$$\theta_{k+1} = \theta_k + bT.$$

By assuming a constraint

$$\theta_{k+1} = \theta_k + d_k$$

where d_k is a zero mean white Gaussian process, and minimizing the cost function

$$J = \frac{1}{2} P^{-1}(0) [\hat{\theta}_0 - \bar{\theta}_0]^2 + \sum_{k=0}^{L-1} \left(\frac{1}{2} Q^{-1} \hat{d}_k^2 - F(\hat{\theta}_k) \right)$$

where

$\hat{\theta}_k, \hat{d}_k$ are estimates

Q is the variance of the d_k process

P_0 is the initial error variance

$$F(\hat{\theta}_k) = \ln \cosh \left[\frac{2}{N_0} \int_{I_k} r(t) s_k(t) dt \right] = \ln \cosh(F_k)$$

I_k is the k^{th} observation interval

$s_k(t)$ is the locally generated signal which depends on $\hat{\theta}_{k-1}$ and

$$\theta_1 = bT = T - T_T \quad b = \frac{\Delta f}{f}$$

$$\theta_2 = \theta_1 + bT$$

$$\theta_3 = \theta_2 + bT$$

$$\vdots$$

FIGURE 4.10 DEFINITIONS OF TIMING MISALIGNMENT

$\hat{\theta}_k$, the sequential estimation equations

$$\hat{\theta}_{k+1} = \hat{\theta}_k + P_{k+1} F_k \frac{\partial F_k}{\partial \hat{\theta}}$$

$$P_{k+1} = \frac{P_k + Q}{1 - P_k \left[\left(\frac{\partial F_k}{\partial \hat{\theta}_k} \right)^2 + F_k \frac{\partial^2 F_k}{\partial \hat{\theta}_k^2} \right]}$$

can be obtained by using invariant imbedding. The assumption $\ln \cosh(v) = v^2/2$ has been made.

For a half cycle sine wave pulse the equations can be implemented by figure 4.11 with performance shown in figure 4.12, SD=1 curve.

However, this synchronizer has the possibility of false locks, i.e., the cost function is maximized instead of minimized. In addition it was found that the performance improved when the sample data term was omitted from the equations (SD=0 curve figure (4.12) and there were fewer false locks. It is thought that this is due to approximations made in the derivation.

This synchronizer seems to perform quite well particularly at high SNR, although the assumption $\ln \cosh(v) = v^2/2$ is a low SNR assumption. It can be seen from figure 4.12 that the early-late gate synchronizer, which is another term for the digital transition tracker described in section 4.2 above, excels at low SNR but is poorer at high SNR. This is probably due to the coarse one bit quantization used in its error estimate.

FIGURE 4.11 SEQUENTIAL SYNCHRONIZER [6]

FIGURE 4.12 PERFORMANCE OF SEQUENTIAL SYNCHRONIZER [6]

FIGURE 4.13 TAPPED DELAY LINE SYNCHRONIZER [13]

4.6 TAPPED DELAY LINE TYPE

Amoroso [13] has proposed a technique for using tapped delay lines to eliminate intersymbol interference in a timing circuit. If a symmetrical data pulse has zero first derivative at $t=0$ as shown in figure 4.13, then the derivative pulse can give a voltage proportional to timing error for small errors. The tapped delay line cancels out the intersymbol interference at $t = \pm 10$ usec.

4.7 FILTERING TECHNIQUES

Both linear and nonlinear filtering theory using state variables have been applied to the bit synchronization problem.

Farrell and Murtha [14] have described a synchronization procedure using a Kalman filter. An unknown fixed timing error and constant known bit length are assumed and are taken as the two state variables X_1 and X_2 respectively. The received noisy data signal is passed through a filter which is assumed to cause negligible intersymbol interference, and this is followed by a transition detector. The transition detector provides a sequence of observed values for each time instant ($Z_1, Z_2, \dots$) when a data signal transition occurs. The observations can then be written

$$Z_m = X_1 + N_m X_2 + \epsilon_m \quad m=1,2,\dots$$

where N_m is the total number of bit intervals occurring before the m^{th} observed transition

ϵ_m is the timing error in the measurement.

ϵ_m is assumed to be Gaussian with variance σ^2 and it is assumed that the

SNR is high. The correct sequence would be

$$Y_m = X_1 + (m-1)X_2 \quad m=1,2,\dots$$

which is the message model. The Kalman filtering algorithm is then

$$\hat{X}_{M+1} = \hat{X}_M + W_{M+1}[Z_{M+1} - H_{M+1}\hat{X}_M]$$

where Z_{M+1} is the $M+1^{\text{st}}$ observation

$$H_{M+1} = [1 \ N_{M+1}]$$

$$W_{M+1} = P_M H_{M+1}^T [H_{M+1} P_M H_{M+1}^T + \sigma^2]^{-1}$$

$$P_{M+1} = [I - W_{M+1} H_{M+1}] P_M.$$

Lee and Komo have discussed the application of nonlinear filtering theory to the detection of jittered digital signals when the jitter is a Markov process [15], [16], [17]. The idea is to produce an optimum estimate of the data wave by using a slight modification of Kushner's [18] equations for the a posteriori density to take into account the jitter process. Since the partial differential equations for continuous cases are usually solved by a finite difference method, the jitter process is assumed to be discrete with m possible values. A set of ordinary differential equations for a posteriori probability densities of the possible message sequences conditioned on the observations up to the present time are obtained. However, this is an extremely large number of equations and so a sequential algorithm is derived which reduces the number of equations. A further reduction is obtained by performing bit synchronization separately.

4.8 CONCLUSION

From the foregoing survey of current synchronizers it seems that the two best synchronizers are the digital transition tracking loop [10] and Wintz and Luecke's [2] suboptimum synchronizer. The former is particularly well suited to digital implementation, and the latter is the simplest of all the synchronizers and it works well at high SNR for raised cosine pulses. Sage and McBride's [6] sequential synchronizer performs well except for the false lock problem which might be eliminated with more study.

However, it is difficult to evaluate and compare the performance of the various synchronizers because the results are often stated in terms of different parameters, such as timing error variance, or probability of error of a matched filter whose timing is derived from the synchronizer being considered. In addition the performance must be known for a wider range of input signals than is usually considered. For example, it is necessary to know the sensitivity of performance to data transition density, noise density, and bit interval jitter. This latter could be sinusoidal (e.g., tape recorded data), or random with some standard statistics. Peavey [12] reports some initial work on this in an interesting paper in which he shows, for example, that some synchronizers are very sensitive to small sinusoidal variations in bit rate.

It is clear from Peavey's study that while current devices perform well in additive Gaussian noise with a stable bit rate, they tend to perform very poorly in terms of acquisition time and tracking in the presence of bit rate jitter. Bit slippage also occurs at SNR's in the 2-5 dB range.

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CHAPTER 5

MAP SYNCHRONIZERS

5.0 INTRODUCTION

In this chapter the derivation of a synchronizer based on maximum a posteriori, (MAP), estimation theory is presented and an approximate physical realization of the device is described. Some similar devices which have subsequently been described in the literature are also mentioned and related to the device presented here.

Throughout this chapter it is assumed that the transmitter and receiver clocks are perfectly stable with identical known frequencies, but that there is an unknown constant timing misalignment.

5.1 DERIVATION

5.1.1 Synchronization Derived from the Data Signal

The receiver is assumed to observe a sequence of non-overlapping binary pulses in additive white Gaussian noise for NT seconds where T is the bit length. The pulses are independent, equally likely + or -, and of known shape. As shown in figure 5.1, the unknown parameter θ , which is to be estimated, is defined as the time from the start of observation to the first bit transition point (or equivalently to the start of the first complete observed pulse). Thus N-1 complete pulses and two incomplete pulses are observed.

Let the received signal be

$$r(t) = q(t, \bar{v}) + n(t) \quad 0 \leq t \leq NT$$

where $n(t)$ is additive white Gaussian noise with two-sided power

$a_i \neq 1$ FOR CONVENIENCE ONLY

FIGURE 5.1 ILLUSTRATION OF NOTATION FOR SECTION 5.1

spectral density $N_0/2$,

$\bar{v} = (\theta, a_0, a_1, \dots, a_N)$ is an unknown parameter vector with the a_i being ± 1 ,

$q(t, \bar{v})$ is the signal given by

$$q(t, \bar{v}) = \sum_{i=0}^N a_i s[t - \theta - (i-1)T] \quad (5.1.1)$$

with $s(t)$ being the pulse shape with energy E as shown in figure 5.1. We wish to find $p(\theta|r(t))$ and $\hat{\theta}$ a number such that $p(\hat{\theta}|r(t)) = \max_{\theta} p(\theta|r(t))$.

Let $\bar{a}_j$ be a particular signal vector with $N+1$ coordinates $(a_0, a_1, \dots, a_N)$, and there are $J=2^{N+1}$ possible vectors $\bar{a}_0, \dots, \bar{a}_{J-1}$. Then

$$p(\theta, \bar{a}_j | r(t)) = \frac{p(r(t) | \theta, \bar{a}_j)}{p(r(t))} p(\theta) p(\bar{a}_j). \quad (5.1.2)$$

Choose a complete orthonormal set of functions $\{Q_i(t)\}$ such that

$$n(t) = \text{l.i.m.}_{M \rightarrow \infty} \sum_{i=1}^M n_i Q_i(t)$$

where

$$n_i = \int_0^{NT} n(s) Q_i(s) ds$$

so that the n_i are independent Gaussian random variables with $E[n_i n_j] = \delta_{ij} N_0/2$ and $E[n_i] = 0$. Then approximate $r(t)$ by

$$r_M(t) = \sum_{i=1}^M r_i Q_i(t)$$

where

$$r_i = \int_0^{NT} r(t) Q_i(t) dt \triangleq q_{ij} + n_i.$$

Then we can write

$$p(r_1, \dots, r_M | \theta, \bar{a}_j) = \frac{1}{(\pi N_0)^M} \exp \left[- \sum_{i=1}^M \frac{(r_i - q_{ij})^2}{N_0} \right].$$

Similarly

$$\begin{aligned} p(r_1, \dots, r_M) &= \int_{\theta} \sum_{j=0}^{J-1} p(r_1, \dots, r_M | \theta, \bar{a}_j) p(\theta) p(\bar{a}_j) d\theta \\ &= \int_{\theta} \sum_{j=0}^{J-1} \frac{1}{(\pi N_0)^M} \exp \left[- \sum_{i=1}^M \frac{(r_i - q_{ij})^2}{N_0} \right] \frac{p(\theta)}{2^{N+1}} d\theta \end{aligned}$$

where

$$q_{ij} = \int_0^{NT} q_j(t) Q_i(t) dt$$

with

$$q_j(t) = q(t, \bar{a}_j, \theta)$$

is the signal coordinate when the message vector is the particular vector $\bar{a}_j$, and all message vectors are equiprobable. Thus

$$\frac{p(r_1, \dots, r_M | \theta, \bar{a}_j)}{p(r_1, \dots, r_M)} = \frac{\exp \left[\frac{2}{N_0} \sum_{i=1}^M r_i q_{ij} - \frac{1}{N_0} \sum_{i=1}^M q_{ij}^2 \right]}{\frac{1}{2^{N+1}} \int_{\theta} \sum_{j=0}^{J-1} \exp \left[\frac{2}{N_0} \sum_{i=1}^M r_i q_{ij} - \frac{1}{N_0} \sum_{i=1}^M q_{ij}^2 \right] p(\theta) d\theta}$$

But

$$\int_0^{NT} r_M(t) q_{Mj}(t) dt = \int_0^{NT} \sum_{i=1}^M \sum_{k=1}^M r_i Q_i(t) q_{kj} Q_k(t) dt = \sum_{i=1}^M r_i q_{ij}$$

and

$$\int_0^{NT} q_{Mj}^2(t) dt = \int_0^{NT} \sum_{i=1}^M \sum_{k=1}^M q_{ij} Q_i(t) q_{kj} Q_k(t) dt = \sum_{i=1}^M q_{ij}^2 = NE$$

so that

$$\frac{p(r_1, \dots, r_M | \theta, \bar{a}_j)}{p(r_1, \dots, r_M)} = \frac{\exp\left[\frac{2}{N_0} \int_0^{NT} r_M(t) q_{Mj}(t) dt\right]}{\frac{1}{2^{N+1}} \int_{\theta} \sum_{k=0}^{J-1} \exp\left[\frac{2}{N_0} \int_0^{NT} r_M(t) q_{Mk}(t) dt\right] p(\theta) d\theta}$$

Now define

$$\begin{aligned} \frac{p(r(t) | \theta, \bar{a}_j)}{p(r(t))} &= \lim_{M \rightarrow \infty} \frac{p(r_1, \dots, r_M | \theta, \bar{a}_j)}{p(r_1, \dots, r_M)} \\ &= \frac{\exp\left[\frac{2}{N_0} \int_0^{NT} r(t) q_j(t) dt\right]}{\frac{1}{2^{N+1}} \int_{\theta} \sum_{k=0}^{J-1} \exp\left[\frac{2}{N_0} \int_0^{NT} r(t) q_k(t) dt\right] p(\theta) d\theta} \quad (5.1.3) \end{aligned}$$

The summation in the denominator can be taken further by considering the case $N=1$. Then the message vectors are $\bar{a}_0 = (-1, -1)$, $\bar{a}_1 = (-1, 1)$, $\bar{a}_2 = (1, -1)$, $\bar{a}_3 = (1, 1)$, so that for example

$$q_1(t) = -s(t-\theta+T) + s(t-\theta) = -s_0(t, \theta) + s_1(t, \theta).$$

Thus

$$\begin{aligned} \sum_{k=0}^3 \exp\left[\frac{2}{N_0} \int_0^T r(t) q_k(t) dt\right] &= \exp\left[\frac{-2}{N_0} \int_0^\theta r(t) s_0(t, \theta) dt - \frac{2}{N_0} \int_\theta^T r(t) s_1(t, \theta) dt\right] \\ &+ \exp\left[\frac{-2}{N_0} \int_0^\theta r(t) s_0(t, \theta) dt + \frac{2}{N_0} \int_\theta^T r(t) s_1(t, \theta) dt\right] \\ &+ \exp\left[\frac{2}{N_0} \int_0^\theta r(t) s_0(t, \theta) dt - \frac{2}{N_0} \int_\theta^T r(t) s_1(t, \theta) dt\right] \\ &+ \exp\left[\frac{2}{N_0} \int_0^\theta r(t) s_0(t, \theta) dt + \frac{2}{N_0} \int_\theta^T r(t) s_1(t, \theta) dt\right] \\ &= 4 \cosh\left[\frac{2}{N_0} \int_0^\theta r(t) s_0(t, \theta) dt\right] \cosh\left[\frac{2}{N_0} \int_\theta^T r(t) s_1(t, \theta) dt\right]. \end{aligned}$$

In general for $J=2^{N+1}$ we find that

$$\sum_{j=0}^{J-1} \exp\left[\frac{2}{N_0} \int_0^T r(t) q_j(t) dt\right] = 2^{N+1} \prod_{j=0}^N \cosh\left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt\right]$$

where $I_0 = [0, \theta)$

$$I_k = [\theta + (k-1)T, \theta + kT] \quad k=1, \dots, N-1$$

$$I_N = [\theta + (N-1)T, NT]$$

$$S_k(t, \theta) = s[t - (\theta + (k-1)T)] \quad k=0, \dots, N.$$

Then note that $p(\theta|r(t)) = \sum_{j=0}^{J-1} p(\theta, \bar{a}_j|r(t)) p(\bar{a}_j)$ so that using equations (5.1.2) and (5.1.3)

$$p(\theta|r(t)) = \frac{\prod_{j=0}^N \cosh\left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt\right] p(\theta)}{\int_{\theta} (\text{numerator}) d\theta} \quad (5.1.4)$$

Since the denominator is independent of θ we have for $p(\theta)$ uniform

$$\begin{aligned} \hat{\theta} &= \max_{\theta} \prod_{j=0}^N \cosh\left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt\right] \\ &= \max_{\theta} \sum_{j=0}^N \ln \cosh\left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt\right] \\ &\triangleq \max_{\theta} F(\theta). \end{aligned} \quad (5.1.5)$$

5.1.2 Periodic Reference Signals

The results of section 5.1.1 can easily be modified to cover the case of periodic reference signals from which synchronization is obtained. Now the parameter vector $\bar{v}$ can only take on two values $\bar{a}_0$ and $\bar{a}_1$ each with probability $\frac{1}{2}$ no matter what value N takes. Furthermore, $\bar{a}_0 = -\bar{a}_1$, and so

$$p(r_1, \dots, r_M) = \int_{\theta} \sum_{j=0}^1 \frac{1}{(\pi N_0)^M} \exp \left[- \sum_{i=1}^M \frac{(r_i - q_{ij})^2}{N_0} \right] \frac{1}{2} p(\theta) d\theta$$

and

$$\frac{p(r(t) | \theta, \bar{a}_j)}{p(r(t))} = \frac{\exp \left[\frac{2}{N_0} \int_0^{NT} r(t) q_j(t) dt \right]}{\frac{1}{2} \int_{\theta} \sum_{k=0}^1 \exp \left[\frac{2}{N_0} \int_0^{NT} r(t) q_k(t) dt \right] p(\theta) d\theta}.$$

Now since $\bar{a}_0 = -\bar{a}_1$ then $q_0(t) = -q_1(t)$ and so

$$\frac{1}{2} \sum_{k=0}^1 \exp \left[\frac{2}{N_0} \int_0^{NT} r(t) q_k(t) dt \right] = \cosh \left[\frac{2}{N_0} \int_0^{NT} r(t) q_0(t) dt \right]$$

giving

$$\begin{aligned} p(\theta | r(t)) &= \frac{\cosh \left[\frac{2}{N_0} \int_0^{NT} r(t) q_0(t) dt \right] p(\theta)}{\int_{\theta} (\text{numerator}) d\theta} \\ &= \frac{\cosh \left[\frac{2}{N_0} \sum_{j=0}^N (-1)^j \int_{I_j} r(t) s_j(t, \theta) dt \right] p(\theta)}{\int_{\theta} (\text{numerator}) d\theta}. \end{aligned}$$

Therefore, for $p(\theta)$ uniform

$$\hat{\theta} = \max_{\theta} \cosh \left[\frac{2}{N_0} \int_0^{NT} r(t) q_0(t) dt \right]: \quad (5.1.6)$$

The principal difference between this expression and (5.1.5) is the order of summation and nonlinearity. Here the integrals over each subinterval are summed before performing the nonlinear operation, whereas in (5.1.5) a nonlinear operation is performed on each integral over a subinterval and the results are summed.

5.2 APPROXIMATE DERIVATIVE IMPLEMENTATION

The MAP estimate of θ , $\hat{\theta}$, can be written as a solution of

$$\frac{d}{d\theta} F(\theta) = 0$$

which will be unique if $F(\theta)$ is unimodal and does not have any minima or points of inflexion. Now make an approximation to the derivative of the form

$$f = \frac{1}{\Delta} \left[F(\theta + \Delta/2) - F(\theta - \Delta/2) \right]$$

and use a hill-climbing algorithm to approach $\hat{\theta}$ as follows:

1. choose $\theta = \theta_0$ and calculate f for a fixed Δ and N ,
2. increment θ by Δf and repeat.

The device of figure 5.2a is proposed as an implementation of this algorithm, and figure 5.2b shows the waveforms at various points in the device for six bit intervals when $N=3$.

The definition (5.1.5) of $F(\theta)$ requires a summation of correlations over NT seconds. The device proposed here does this, shifts the local signal phase to the new estimated time shift, and then sums the correlations for another NT seconds.

FIGURE 5.2a APPROXIMATE MAP SYNCHRONIZER

FIGURE 5.2b WAVEFORMS IN THE MAP SYNCHRONIZER

FIGURE 5.2c PRACTICAL REALIZATION OF MAP SYNCHRONIZER

The upper path computes $\text{Incosh} \left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta - \Delta/2) dt \right]$ and the lower path computes $\text{Incosh} \left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta + \Delta/2) dt \right]$. The difference of these is summed over the N intervals $I_0, \dots, I_{N-1}$, and the stable clock provides the simultaneous dump pulses required by interval I_N at NT . The stable clock also times the sample-and-hold circuit to sample the summer output after the $N+1$ st interval following which the summer is reset to zero. The sample-and-hold circuit output remains constant for the next NT seconds and is used to adjust the signal generator phase.

Further approximations may be necessary to make the implementation easier or more practical.

1. Since the waveform into the summer is impulsive, the summer could be an integrator. It also could be built digitally having an output which could only take certain discrete values.
2. A perfectly stable clock is not physically possible, and the bit rate may not be known exactly. Thus it could be useful to abandon the clock, ignore the I_0 interval, and extend the I_N interval to the next local signal crossing. This then is a phase-locked loop, and the summer and sample-and-hold circuit could be replaced by a filter which could give a weighted sum of all past integrations.

3. The Incosh nonlinearity could be replaced by some other approximation such as a square law or absolute value nonlinearity.

In order to have a device which is reasonably practical and which has possibilities of operating under a wider variety of conditions than that of figure 5.2a, the phase-locked loop type of device of figure 5.2c is investigated. The Incosh is approximated by a square law and by an absolute value.

5.3 AN ALTERNATE APPROACH

Rather than the finite difference approximation of the previous section, the maximum of equation (5.1.5) can be found by direct differentiation, i.e., $\hat{\theta}$ is given by

$$\left. \frac{d}{d\theta} \sum_{j=0}^N \text{Incosh} \left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt \right] \right|_{\theta=\hat{\theta}} = 0,$$

$$\text{or} \quad \sum_{j=0}^N \tanh \left[\frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt \right] \frac{2}{N_0} \int_{I_j} r(t) s_j(t, \theta) dt = 0 \text{ at } \theta = \hat{\theta}$$

if the appropriate conditions are satisfied, i.e.,

1. $r(t)s_j(t, \theta)$ is continuous and has continuous derivative with respect to θ in the domain of the (θ, t) plane which includes the limits
2. the integral limits have continuous derivatives in θ . It has also been assumed that $s_j(t, \theta) = 0$ at $t = \theta + (j-1)T$ and $t = \theta + jT$.

Now since this is a derivative it will be >0 if $\theta < \hat{\theta}$ and <0 if $\theta > \hat{\theta}$ so that the same approach to an approximate algorithm or device can be used as in section 5.1. An approximate implementation is given in figure 5.3, where the nonlinearity is theoretically a tanh nonlinearity. However, figure 5.4 shows some approximations which might be used such as the hard limiter or a straight line approximation.

Comparing the devices of figures 5.2 and 5.3 we see

- a) in figure 5.3 a derivative signal is required which is not so in figure 5.2,
- b) figure 5.3 requires fewer sampling and delay operations,
- c) the multiplier before the sample and hold in figure 5.3 can be replaced by a switch if the hard limiter nonlinearity is used.

Thus figure 5.3 is likely to be easier to implement.

5.4 RELATION TO OTHER PUBLISHED WORK

Mengali [1] has also recently proposed the device of figure 5.3 and given a linear analysis leading to an expression for the timing jitter variance at high SNR. His proposal is based on the same MAP estimation approach as that given in this thesis, but he does not point out clearly that he has made the approximations of neglecting the end subintervals and the perfectly stable local clock in his analysis.

Figure 4.3 shows the digital data transition tracking loop analyzed by Simon [2], and it resembles a combination of the two devices derived in this chapter with the addition of the transition detector in the in-phase branch. This device is for rectangular pulse shapes only

FIGURE 5.3 ALTERNATE APPROXIMATE MAP SYNCHRONIZER [1]

FIGURE 5.4 COMPARISON OF HARD LIMITER, ABSOLUTE VALUE, AND Tanh NONLINEARITIES

After the analysis of chapter 6 was substantially completed and the experimental work of chapter 7 was underway, it was discovered that Simon [3] had also presented a nonlinear analysis of the device shown in figure 5.2c with the Incosh replaced by either a square law or an absolute value nonlinearity. However, there are several differences between his work and the work presented in chapters 5 to 8 of this thesis. They are

1. Simon does not give any theoretical analysis which leads to the proposal of his device as has been done here; he simply presents a device and analyzes its performance.
2. The equations derived in Simon [3] are completely different in form from those derived here in chapter 6 although the resulting numerical values for jitter variance are the same.
3. Simon [3] does not give probability of error curves which describe the performance of a matched filter receiver using timing pulses derived from the synchronizer, nor does he compare the synchronizer performance with transmitted reference schemes. Both of these topics are discussed in chapter 6 here
4. The analysis of this thesis is more general with Simon's work [3] being a special case in which the time shift between the two integrators is half a bit interval.
5. In this thesis some consideration is given to the transient response of the synchronizer.
6. This thesis presents the results of an analogue computer realization of the synchronizer as well as a digital computer simulation. The results of these indicate the range of validity of the assumptions underlying the analysis and confirm the analysis within that range.
7. Simon's analysis of the square law device uses high SNR approximations whereas the work in this thesis is valid for all SNR's.

It is also appropriate at this point to refer to the recent book by Stiffler [4], which summarizes maximum likelihood estimation of bit synchronization in chapter 7, and which was published when this thesis was substantially complete. An equation is derived which is a

computer simulation. The results of these indicate the range of validity of the assumptions underlying the analysis and confirm the analysis within that range.

5.5 CONCLUSION

In this chapter several approximations to a MAP synchronizer have been presented which lend themselves to practical realization. In the following chapter the steady state and transient behaviour of the device of figure 5.2c will be analyzed.

References

1. Mengali, U., *A Self-bit Synchronizer Matched to Signal Shape*, I.E.E.E. Trans. Aerospace & Elect. Sys., AES-7, #4, July 1971, pp. 686-693.
2. Simon, M.K., Chapter 4, reference 11.
3. Simon, M.K., *Nonlinear Analysis of an Absolute Value Type of an Early-Late Gate Bit Synchronizer*, I.E.E.E. Trans. on Comm. Tech., COM-18, #5, Oct. 1970, pp. 589-596.

CHAPTER 6

MAP DERIVATIVE DEVICE ANALYSIS

6.0 INTRODUCTION

In section 5.1 a synchronizer that used an approximation to the derivative of the a posteriori density function was proposed, and was illustrated in figure 5.2c. In this chapter this realization is analyzed for steady state performance when the input consists of a train of random rectangular pulses in additive white Gaussian noise. Some consideration is also given to the transient performance when the input is a periodic rectangular wave.

A comparison of Incosh, absolute value, and square law nonlinearities is given in figure 6.1. It is evident that the Incosh is like a square law nonlinearity for low argument values and like an absolute value nonlinearity for high argument values. The analyses will consider the square law and absolute value nonlinearities instead of the Incosh, and the device to be considered is shown in figure 6.2 which is the device of figure 5.2c assuming rectangular pulses.

6.1 STEADY STATE ANALYSIS - ABSOLUTE VALUE NONLINEARITY

6.1.1 Introduction

In the device shown in figure 6.2 the nonlinearity is taken to be an absolute value nonlinearity in this section, and the filter transfer function is assumed to be a constant G_2 . G_1 is an arbitrary gain in the integrators. The integrator dumping is controlled by the VCO and occurs approximately every T seconds where T is the bit length.

It is as assumed that the device is in steady state operation tracking a data signal in additive white Gaussian noise. The data signal

FIGURE 6.1 COMPARISON OF ABSOLUTE VALUE, SQUARE LAW, AND Lncosh NONLINEARITIES

$$r(t) = \sum_{K=-\infty}^{\infty} d_K s(t - KT - \theta T) + n(t), \quad d_K = \pm A$$

FIGURE 6.2 APPROXIMATE MAP DERIVATIVE DEVICE FOR 1 BIT
OBSERVATION INTERVAL AND RECTANGULAR NRZ PULSES

is assumed to be composed of equally likely independent binary rectangular pulses of amplitudes $\pm A$ having an unknown time shift which is taken to be a constant θT . The noise is assumed to have a two-sided spectral density $N_0/2$ watts/Hz. The timing generator forms an estimate of the unknown time shift θT , and it is assumed that the timing estimate remains constant at $\hat{\theta}T$ over a large number of bit intervals, i.e., the loop bandwidth is low. The timing error is then defined as

$$\phi = \theta - \hat{\theta}.$$

The rectangular non-return-to-zero data pulses in noise make it difficult for a synchronizer to operate, because the synchronization information is carried in data signal zero crossings which will not occur at every bit transition time, and the noise causes additional extraneous zero crossings.

An expression for the probability density function of the timing error ϕ is derived in terms of the bit energy E , the noise power density $N_0/2$, the time shift ΔT between the two integrators, and δ_s which is the ratio of the bit rate to the linear loop bandwidth. The variance of ϕ and the probability of error of a matched filter receiver whose sampling times are given by this device are calculated numerically. The timing error probability density function is obtained by representing the device as the phase locked loop of figure 6.4 and using the Fokker-Planck equation. The function $g(\phi)$ is an average loop phase detector characteristic and $n_\phi(t)$ is an equivalent additive noise source with spectrum $S(\omega, \phi)$.

The waveform $x(t)$ at the output of the sample-and-hold circuit is a staircase function which takes the value e_k at time $\hat{T}_k + \Delta T/2$ as

FIGURE 6.3 WAVEFORMS IN THE DEVICE OF FIGURE 6.2

FIGURE 6.4 EQUIVALENT PHASE-LOCKED LOOP

shown in figure 6.3. For a fixed ϕ we consider an ensemble of such waves and call the average value $g(\phi)$. The waves can be considered to consist of a signal term plus a noise term, where for a fixed ϕ the signal is given by

$$\bar{e}_k = E_{ns}\{e_k|\phi\}$$

and the noise is $n_\phi(t) = x(t) - \bar{e}_k$ with spectrum $S(\omega, \phi)$. E_{ns} denotes expectation over the noise and data signal sequence. A similar method of analysis has been used by Simon [1].

6.1.2 Derivation of Phase Detector Characteristic and Noise Source Spectrum

The analysis outlined above is now done illustrated by figures 6.3 to 6.5. Figure 6.5 shows various diagrams which are of use in formulating the equations used. Diagram (a) shows the two integration intervals used in the upper and lower branches. $e_{k,1}$ is the absolute value of the upper integrator output when it is sampled and $e_{k,2}$ is for the lower one. The horizontal axis is a time axis and the timing estimates are shown as being at $\hat{T}_{k-1}$ and $\hat{T}_k$. The important thing is the time between the timing estimate and the bit transition time rather than the absolute time from device turn-on. It is considered that at the end of the $e_{k,2}$ interval the error signal $e_k = e_{k,2} - e_{k,1}$ is formed and held by the sample-and-hold circuit. Diagrams (b) and (c) show two ranges of ϕ which are of interest, namely, $[\Delta/2, 1-\Delta/2]$ where two simultaneous upper and lower integrator intervals contain a bit transition time, and $[-\Delta/2, \Delta/2]$ where the bit transition time falls in adjacent upper and lower integration intervals. The data pulses are labelled d_{k-1} , d_k , d_{k+1} where $d_{(\cdot)} = \pm A$.

Let

$$e_k = G_2 \left| G_1 \int_{\hat{T}_{k-1} + \Delta T/2}^{\hat{T}_k + \Delta T/2} r(t) dt \right| - G_2 \left| G_1 \int_{\hat{T}_{k-1} - \Delta T/2}^{\hat{T}_k - \Delta T/2} r(t) dt \right| \stackrel{\Delta}{=} e_{k,2} - e_{k,1}$$

$$\stackrel{\Delta}{=} \left| s_{k,2} + n_{k,2} \right| - \left| s_{k,1} + n_{k,1} \right|$$

Thus from figure 6.5 for $-\Delta/2 \leq \phi \leq \Delta/2$

$$s_{k,1} = G_1 G_2 \int_{\hat{T}_{k-1} - \Delta T/2}^{\hat{T}_k - \Delta T/2} r_s(t) dt = AG_1 G_2 T [(\phi + \Delta/2) d_{k-1} + (1 - \phi - \Delta/2) d_k]$$

$$s_{k,2} = G_1 G_2 \int_{\hat{T}_{k-1} + \Delta T/2}^{\hat{T}_k + \Delta T/2} r_s(t) dt = AG_1 G_2 T [(1 + \phi - \Delta/2) d_k + (\Delta/2 - \phi) d_{k+1}]$$

(6.1.1)

and for $\Delta/2 \leq \phi \leq 1 - \Delta/2$

$$s_{k,1} = G_1 G_2 T [(\phi + \Delta/2) d_{k-1} + (1 - \phi - \Delta/2) d_k]$$

$$s_{k,2} = G_1 G_2 T [(\phi - \Delta/2) d_{k-1} + (1 - \phi + \Delta/2) d_k]$$

where $d_{(\cdot)} = \pm A$

$$r(t) = r_s(t) + n(t)$$

$r_s(t)$ is the data signal

$n(t)$ is white Gaussian noise with 2-sided spectral density $N_0/2$.

$$\text{Thus, } n_{k,1} = G_1 G_2 \int_{\hat{T}_{k-1}-\Delta T/2}^{\hat{T}_k-\Delta T/2} n(t) dt \text{ and } n_{k,2} = G_1 G_2 \int_{\hat{T}_{k-1}+\Delta T/2}^{\hat{T}_k+\Delta T/2} n(t) dt$$

are zero mean Gaussian random variables with variance $G_1^2 G_2^2 N_0 T/2$.

The values of $s_{k,1}$ and $s_{k,2}$ are given in table 6.1 for the range of ϕ which is of interest, and where

$$\alpha = \begin{cases} 1+2(\phi-\Delta/2) & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ 1-2(\phi-\Delta/2) & \text{for } \Delta/2 \leq \phi \leq 1-\Delta/2 \end{cases}$$

$$\beta = 1-2(\phi+\Delta/2) \quad \text{for } -\Delta/2 \leq \phi \leq 1-\Delta/2. \quad (6.1.2)$$

To obtain $g(\phi)$ we average e_k over the noise first and then over the signal, i.e.,

$$g(\phi) = E_{n,s}\{e_k|\phi\} = E_s\{E_n\{e_k|\phi,s\}\} = E_{n,s}\{e_{k,2}|\phi\} - E_{n,s}\{e_{k,1}|\phi\}.$$

$$\text{and } E_n\{e_{k,j}|\phi,s\} = E_n\{|s_{k,j}+n_{k,j}|\}|\phi,s\}$$

where $s_{k,j}+n_{k,j}$ is a Gaussian random variable with mean $s_{k,j}$ and variance $G_1^2 G_2^2 N_0 T/2$. For a Gaussian random variable y with variance σ^2 and mean m

$$E\{|y|\} = \sigma\sqrt{2/\pi} e^{-m^2/2\sigma^2} + m \cdot \text{erf}(m/\sigma\sqrt{2})$$

where

$$\text{erf}(x) = 2/\sqrt{\pi} \int_0^x e^{-t^2} dt$$

so that

$$E_n\{e_{k,j}|\phi,s_{k,j}\} = G_1 G_2 \sqrt{N_0 T/\pi} e^{-s_{k,j}^2/G_1^2 G_2^2 N_0 T} + s_{k,j} \text{erf}(s_{k,j}/G_1 G_2 \sqrt{N_0 T}). \quad (6.1.3)$$

TABLE 6.1
Values of $s_{k,1}$ and $s_{k,2}$

Range of ϕ	d_{k-1}	d_k	$s_{k,1}$
$[-\Delta/2, 1-\Delta/2]$	-A	-A	$-G_1 G_2 AT$
	-A	A	$G_1 G_2 AT\beta$
	A	-A	$-G_1 G_2 AT\beta$
	A	A	$G_1 G_2 AT$
Range of ϕ	d_k	d_{k+1}	$s_{k,2}$
$[-\Delta/2, \Delta/2]$	-A	-A	$-G_1 G_2 AT$
	-A	A	$-G_1 G_2 AT\alpha$
	A	-A	$G_1 G_2 AT\alpha$
	A	A	$G_1 G_2 AT$
$[\Delta/2, 1-\Delta/2]$	d_{k-1}	d_k	
	-A	-A	$-G_1 G_2 AT$
	-A	A	$G_1 G_2 AT\alpha$
	A	-A	$-G_1 G_2 AT\alpha$
	A	A	$G_1 G_2 AT$

Thus, using table 6.1 and (6.1.2)-(6.1.3) and averaging over all signal sequences

$$E_{n,s}\{e_{k,1}|\phi\} = \frac{1}{2} G_1 G_2 \sqrt{N_0 T / \pi} (e^{-U} + e^{-\beta^2 U}) + \frac{1}{2} G_1 G_2 AT [\text{erf}(\sqrt{U}) + \beta \cdot \text{erf}(\beta \sqrt{U})]$$

$$E_{n,s}\{e_{k,2}|\phi\} = \frac{1}{2} G_1 G_2 \sqrt{N_0 T / \pi} (e^{-U} + e^{-U\alpha^2}) + \frac{1}{2} G_1 G_2 AT [\text{erf}(\sqrt{U}) + \alpha \cdot \text{erf}(\alpha \sqrt{U})]$$

so that

$$g(\phi) = \frac{1}{2} G_1 G_2 \sqrt{N_0 T / \pi} (e^{-U\alpha^2} - e^{-U\beta^2}) + \frac{1}{2} G_1 G_2 AT [\alpha \cdot \text{erf}(\alpha \sqrt{U}) - \beta \cdot \text{erf}(\beta \sqrt{U})] \quad (6.1.4)$$

where $U = E/N_0$.

A normalized phase-detector characteristic, $g_0(\phi)$, can be defined as

$$g_0(\phi) = \frac{g(\phi)}{2ATG_1G_2} = (e^{-\alpha^2 U} - e^{-\beta^2 U}) / 4\sqrt{\pi U} + [\alpha \cdot \text{erf}(\alpha \sqrt{U}) - \beta \cdot \text{erf}(\beta \sqrt{U})] / 4 \quad (6.1.5)$$

For large U

$$g(\phi) \approx \frac{1}{2} G_1 G_2 AT [\alpha \cdot \text{sgn}(\alpha) - \beta \cdot \text{sgn}(\beta)]$$

which is plotted in figure 6.6. It is seen that as $\Delta \rightarrow 0$ the curve becomes rectangular. Figure 6.7 shows $g(\phi)$ for various E/N_0 and Δ . The curves for $\Delta = \frac{1}{16}$, $\frac{1}{8}$ and $\frac{1}{4}$ show the tendency towards a rectangular wave with amplitude 2Δ at $E/N_0 = 10$. The $\Delta = \frac{1}{2}$ curve remains almost triangular up to $E/N_0 = 20$.

Now to find the spectrum $S(\omega, \phi)$ we need to calculate

$$D(m, \phi) = E_{n,s} \left\{ (e_k - \bar{e}_k) (e_{k+m} - \bar{e}_{k+m}) | \phi \right\}$$

where $\bar{e}_j \triangleq E_{n,s} \{e_j | \phi\}$.

FIGURE 6.6 $g_0(\phi)$ FOR HIGH E/N_0 , ABSOLUTE VALUE DEVICE

FIGURE 6.7 $g_0(\phi)$ FOR VARIOUS E/N_0 , ABSOLUTE VALUE DEVICE

Since

$$\bar{e}_k = \bar{e}_{k+m} = g(\phi),$$

then

$$D(m, \phi) = E_{n,s} \{e_k e_{k+m} | \phi\} - g^2(\phi). \quad (6.1.6)$$

First, consider $m=0$. Then

$$E_{n,s} \{e_k^2 | \phi\} = E_{n,s} \left\{ (s_{k,1}^2 + n_{k,1}^2 - 2 |s_{k,1} + n_{k,1}| |s_{k,2} + n_{k,2}| + s_{k,2}^2 + n_{k,2}^2) | \phi \right\} \quad (6.1.7)$$

since

$$E_{n,s} \{s_{k,j} n_{k,j}\} = 0.$$

But (1) $E_{n,s} \{n_{k,j}^2 | \phi\} = G_1^2 G_2^2 N_0 T / 2$ since $n_{k,j}$ is zero mean Gaussian and independent of signal sequence,

(2) by considering table 6.1 and averaging over all signal sequences

$$E_s \{s_{k,1}^2 | \phi\} = \frac{1}{2} (AG_1 G_2 T)^2 (1 + \beta^2)$$

$$E_s \{s_{k,2}^2 | \phi\} = \frac{1}{2} (AG_1 G_2 T)^2 (1 + \alpha^2)$$

(3) from Appendix 6A, for two jointly Gaussian random variables x and y with means m_x and m_y variances σ^2 and correlation coefficient $\rho = (E\{xy\} - m_x m_y) / \sigma^2$

$$\begin{aligned} E\{|x||y|\} &= 2\sigma^2 \left\{ \frac{\sqrt{1-\rho^2}}{\pi} e^{-m_x^2 U - B^2} - e^{-m_x^2 U} \sqrt{U/\pi} m_y \cdot \text{erf}(B) \right. \\ &\quad + e^{-m_y^2 U} \sqrt{U/\pi} m_x \cdot \text{erf}[\sqrt{U/1-\rho^2} (m_x - \rho m_y)] \\ &\quad \left. + (Q_1 - Q_2) (\rho/2 + m_x m_y U) / \sqrt{\pi} \right\} \\ &\triangleq 2\sigma^2 \cdot f(m_x, m_y, \rho) \end{aligned} \quad (6.1.8)$$

where

$$Q_1 = \int_{-m_y \sqrt{U}}^{\infty} e^{-y^2} \operatorname{erf}[(m_x \sqrt{2U} + \sqrt{2\rho} y) / \sqrt{2(1-\rho^2)}] dy$$

$$Q_2 = \int_{m_y \sqrt{U}}^{\infty} e^{-y^2} \operatorname{erf}[(m_x \sqrt{2U} - \sqrt{2\rho} y) / \sqrt{2(1-\rho^2)}] dy$$

$$\sigma^2 = G_1^2 G_2^2 N_0 T / 2$$

$$B = (\rho m_x - m_y) \sqrt{U / (1-\rho^2)}$$

$$M = A G_1 G_2 T.$$

Then, letting $x = (s_{k,1} + n_{k,1})$, $y = (s_{k,2} + n_{k,2})$, so that $Mm_x = s_{k,1}$, $Mm_y = s_{k,2}$, and $\rho = 1-\Delta$, we have after averaging over all signal sequences and using table 6.2 that

$$E_{n,s} \left\{ (|s_{k,1} + n_{k,1}| |s_{k,2} + n_{k,2}|) | \phi \right\} = \begin{cases} \frac{1}{2} \sigma^2 [f(1,1,\rho) + f(1,\alpha,\rho) + f(-\beta,-\alpha,\rho) \\ + f(-\beta,-1,\rho)] & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ \sigma^2 [f(1,1,\rho) + f(-\beta,-\alpha,\rho)] & \text{for } \Delta/2 \leq \phi \leq 1-\Delta/2 \end{cases}$$

$$\triangleq \begin{cases} \frac{1}{2} \sigma^2 \cdot h_0(\beta, \alpha, \rho) & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ \sigma^2 \cdot h_1(\beta, \alpha, \rho) & \text{for } \Delta/2 \leq \phi \leq 1-\Delta/2. \end{cases} \quad (6.1.9)$$

The fact that $f(a,b,\rho) = f(-a,b,\rho)$ has also been used. Thus, from (1)

- (3) above,

TABLE 6.2
Values of $s_{k,1}$ and $s_{k,2}$

$-\Delta/2 \leq \phi \leq \Delta/2$					$\Delta/2 \leq \phi \leq 1-\Delta/2$			
d_{k-1}	d_k	d_{k+1}	$Mm_x = s_{k,1}$	$Mm_y = s_{k,2}$	d_{k-1}	d_k	$Mm_x = s_{k,1}$	$Mm_y = s_{k,2}$
A	A	A	M	M	A	A	M	M
A	A	-A	M	$M\alpha$	A	-A	$-M\beta$	$-M\alpha$
A	-A	A	$-M\beta$	$-M\alpha$	-A	A	$M\beta$	$M\alpha$
A	-A	-A	$-M\beta$	-M	-A	-A	-M	-M
-A	A	A	$M\beta$	M				
-A	A	-A	$M\beta$	$M\alpha$				
-A	-A	A	-M	$-M\alpha$				
-A	-A	-A	-M	-M				

TABLE 6.3
Values of $s_{k,1}$ and $s_{k+1,1}$

d_{k-1}	d_k	d_{k+1}	$Mm_x = s_{k,1}$	$Mm_y = s_{k+1,1}$
A	A	A	M	M
A	A	-A	M	$-M\beta$
A	-A	A	$-M\beta$	$M\beta$
A	-A	-A	$-M\beta$	-M
-A	A	A	$M\beta$	M
-A	A	-A	$M\beta$	$-M\beta$
-A	-A	A	-M	$M\beta$
-A	-A	-A	-M	-M

$$D(0, \phi) = E_{ns} \{e_k^2 | \phi\} - g^2(\phi)$$

$$= \frac{M^2}{2} [2 + \beta^2 + \alpha^2] + G_1^2 G_2^2 N_0 T - g^2(\phi) - 2 \begin{cases} h_0(\beta, \alpha, \rho) \sigma^2 / 2 & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ h_1(\beta, \alpha, \rho) \sigma^2 & \text{for } \Delta/2 \leq \phi \leq 1 - \Delta/2 \end{cases}$$

and where $g(\phi)$ is given by (6.1.5) and $M = AG_1 G_2 T$.

Now consider $m=1$. Then

$$E_{n,s} \{e_k e_{k+1} | \phi\} = E_{n,s} \left\{ \left(|s_{k,1} + n_{k,1}| |s_{k+1,1} + n_{k+1,1}| - |s_{k,1} + n_{k,1}| |s_{k+1,2} + n_{k+1,2}| \right. \right. \\ \left. \left. - |s_{k,2} + n_{k,2}| |s_{k+1,1} + n_{k+1,1}| + |s_{k,2} + n_{k,2}| |s_{k+1,2} + n_{k+1,2}| \right) | \phi \right\}$$

But

- (1) for uncorrelated Gaussian random variables x and y with means m_x and m_y and variances σ^2 , from equation (6.1.8) with $\rho=0$

$$E\{|x| |y|\} = 2\sigma^2 \left\{ \frac{1}{\pi} e^{-(m_x^2 + m_y^2)U} + \sqrt{U/\pi} m_y \cdot \text{erf}(m_y \sqrt{U}) e^{-m_x^2 U} \right. \\ \left. + \sqrt{U/\pi} m_x \cdot \text{erf}(m_x \sqrt{U}) e^{-m_y^2 U} \right. \\ \left. + U m_x m_y \cdot \text{erf}(m_x \sqrt{U}) \cdot \text{erf}(m_y \sqrt{U}) \right\} \quad (6.1.10) \\ \triangleq 2\sigma^2 \cdot f_U(m_x, m_y)$$

where $f_U(a, b) = f_U(a, -b) = f_U(-a, -b)$, and $U = E/N_0$.

- (2) $(s_{k,1} + n_{k,1})$ and $(s_{k+1,1} + n_{k+1,1})$ are uncorrelated Gaussian random variables with means $s_{k,1}$ and $s_{k+1,1}$ so that after averaging over all signal sequences using table 6.3 and

equation (6.1.10)

$$E_{n,s} \left\{ \left(|s_{k,1} + n_{k,1}| |s_{k+1,1} + n_{k+1,1}| \right) | \phi \right\} = \frac{1}{2} \sigma^2 [f_U(1,1) + f_U(1,-\beta) + f_U(-\beta,\beta) + f_U(-\beta,-1)]$$

$$\triangleq h_2(\beta) \cdot \sigma^2 / 2$$

- (3) $(s_{k,2} + n_{k,2})$ and $(s_{k+1,2} + n_{k+1,2})$ are uncorrelated Gaussian random variables with means $s_{k,2}$ and $s_{k+1,2}$, so that from table 6.4

$$E_{n,2} \left\{ \left(|s_{k,2} + n_{k,2}| |s_{k+1,2} + n_{k+1,2}| \right) | \phi \right\} = \frac{1}{2} \sigma^2 [f_U(1,1) + f_U(1,-\alpha) + f_U(-\alpha,\alpha) + f_U(\alpha,-1)]$$

$$= h_2(\alpha) \cdot \sigma^2 / 2$$

- (4) $(s_{k,1} + n_{k,1})$ and $(s_{k+1,2} + n_{k+1,2})$ are uncorrelated Gaussian random variables with means $s_{k,1}$ and $s_{k+1,2}$, so that from table 6.5

$$E_{n,s} \left\{ \left(|s_{k,1} + n_{k,1}| |s_{k+1,2} + n_{k+1,2}| \right) | \phi \right\} = \frac{1}{2} \sigma^2 [f_U(1,1) + f_U(1,-\alpha) + f_U(-\beta,\alpha) + f_U(-\beta,1)]$$

$$\triangleq h_3(\beta,\alpha) \cdot \sigma^2 / 2 \quad (6.1.11)$$

- (5) $(s_{k,2} + n_{k,2})$ and $(s_{k+1,1} + n_{k+1,1})$ are correlated Gaussian random variables with means $s_{k,2}$ and $s_{k+1,1}$, and $\rho = \Delta$, so that from table 6.6

TABLE 6.4
Values of $s_{k,2}$ and $s_{k+1,2}$

$-\Delta/2 \leq \phi \leq \Delta/2$					$\Delta/2 \leq \phi \leq 1-\Delta/2$				
d_k	d_{k+1}	d_{k+2}	$Mm_x = s_{k,2}$	$Mm_y = s_{k+1,2}$	d_{k-1}	d_k	d_{k+1}	$Mm_x = s_{k,2}$	$Mm_y = s_{k+1,2}$
A	A	A	M	M	A	A	A	M	M
A	A	-A	M	$M\alpha$	A	A	-A	M	$-M\alpha$
A	-A	A	$M\alpha$	$-M\alpha$	A	-A	A	$-M\alpha$	$M\alpha$
A	-A	-A	$M\alpha$	-M	A	-A	-A	$-M\alpha$	-M
-A	A	A	$-M\alpha$	M	-A	A	A	$M\alpha$	M
-A	A	-A	$-M\alpha$	$M\alpha$	-A	A	-A	$M\alpha$	$-M\alpha$
-A	-A	A	-M	$-M\alpha$	-A	-A	A	-M	$-M\alpha$
-A	-A	-A	-M	-M	-A	-A	-A	-M	-M

TABLE 6.5
Values of $s_{k,1}$ and $s_{k+1,2}$

$-\Delta/2 \leq \phi \leq \Delta/2$						$\Delta/2 \leq \phi \leq 1-\Delta/2$				
d_{k-1}	d_k	d_{k+1}	d_{k+2}	$Mm_x = s_{k,1}$	$Mm_y = s_{k+1,2}$	d_{k-1}	d_k	d_{k+1}	$Mm_x = s_{k,1}$	$Mm_y = s_{k+1,2}$
A	A	A	A	M	M	A	A	A	M	M
A	A	A	-A	M	$M\alpha$	A	A	-A	M	$-M\alpha$
A	A	-A	A	M	$-M\alpha$	A	-A	A	$-M\beta$	$M\alpha$
A	A	-A	-A	M	$-M$	A	-A	-A	$-M\beta$	$-M$
A	-A	A	A	$-M\beta$	M	-A	A	A	$M\beta$	M
A	-A	A	-A	$-M\beta$	$M\alpha$	-A	A	-A	$M\beta$	$-M\alpha$
A	-A	-A	A	$-M\beta$	$-M\alpha$	-A	-A	A	$-M$	$M\alpha$
A	-A	-A	-A	$-M\beta$	$-M$	-A	-A	-A	$-M$	$-M$
-A	A	A	A	$M\beta$	M					
-A	A	A	-A	$M\beta$	$M\alpha$					
-A	A	-A	A	$M\beta$	$-M\alpha$					
-A	A	-A	-A	$M\beta$	$-M$					
-A	-A	A	A	$-M$	M					
-A	-A	A	-A	$-M$	$M\alpha$					
-A	-A	-A	A	$-M$	$-M\alpha$					
-A	-A	-A	-A	$-M$	$-M$					

TABLE 6.6
Values of $s_{k,2}$ and $s_{k+1,1}$

$-\Delta/2 \leq \phi \leq \Delta/2$				$\Delta/2 \leq \phi \leq 1-\Delta/2$				
d_k	d_{k+1}	$Mm_x = s_{k,2}$	$Mm_y = s_{k+1,1}$	d_{k-1}	d_k	d_{k+1}	$Mm_x = s_{k,2}$	$Mm_y = s_{k+1,1}$
A	A	M	M	A	A	A	M	M
A	-A	$M\alpha$	$-M\beta$	A	A	-A	M	$-M\beta$
-A	A	$-M\alpha$	$M\beta$	A	-A	A	$-M\alpha$	$M\beta$
-A	-A	$-M$	$-M$	A	-A	-A	$-M\alpha$	$-M$
				-A	A	A	$M\alpha$	M
				-A	A	-A	$M\alpha$	$-M\beta$
				-A	-A	A	$-M$	$M\beta$
				-A	-A	-A	$-M$	$-M$

$$E_{n,s} \left\{ \left(|s_{k,2} + n_{k,2}| |s_{k+1,1} + n_{k+1,1}| \right) | \phi \right\} = \begin{cases} h_1(\alpha, -\beta, \Delta) \cdot \sigma^2 & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ h_0(\alpha, -\beta, \Delta) \cdot \sigma^2/2 & \text{for } \Delta/2 \leq \phi \leq 1-\Delta/2 \end{cases} \quad (6.1.12)$$

where $h_0(\cdot, \cdot, \rho)$ and $h_1(\cdot, \cdot, \rho)$ are defined in (6.1.8) and (6.1.9).

Using (1) - (5) above

$$\begin{aligned} D(1, \phi) &= E_{n,s} \{ e_k e_{k+1} | \phi \} - g^2(\phi) \\ &= \frac{1}{2} \sigma^2 [h_2(\beta) + h_2(\alpha) - h_3(\beta, \alpha)] - g^2(\phi) - \begin{cases} h_1(\alpha, -\beta, \Delta) \cdot \sigma^2 & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ h_0(\alpha, -\beta, \Delta) \cdot \sigma^2/2 & \text{for } \Delta/2 \leq \phi \leq 1-\Delta/2 \end{cases} \end{aligned} \quad (6.1.13)$$

For $m=-1$ the same results apply as for $m=1$. For $|m|>1$ e_k and e_{k+m} are independent, so that $D(m, \phi) = 0$ for $|m|>1$.

It can easily be shown (Appendix 6C) that the autocorrelation function of the noise output of the sample-and-hold circuit is

$$R(\tau) = \begin{cases} (1-|\tau|/T) \bar{e}_k^2 + |\tau|/T \cdot \overline{e_k e_{k+1}} - \bar{e}_k^2 & 0 \leq |\tau| \leq T \\ (\overline{e_k e_{k+1}} - \bar{e}_k^2) (2-|\tau|/T) & T \leq |\tau| \leq 2T \\ 0 & |\tau| \geq 2T \end{cases} \quad (6.1.14)$$

and that

$$S(0, \phi) = T[D(0, \phi) + 2D(1, \phi)] \quad (6.1.15)$$

where $S(0, \phi)$ is the only value of the noise spectrum $S(\omega, \phi)$ that we need to consider if the loop bandwidth is low enough.

Thus, it is assumed that the loop filter is driven by a signal term $g(\phi) = \bar{e}_k$ and an additive zero mean noise source having a flat spectrum $S(0, \phi)$.

6.1.3 Derivation of Timing Error Density Function

Having obtained the equivalent noise spectrum, we can now use the Fokker-Planck equation approach [2] to find the timing error density and variance. The stochastic differential equation which describes the loop of figure 6.4 is shown in Appendix 6F to be

$$\dot{\phi} = -k_{VCO} F(p) [g(\phi) + n_{\phi}(t)] \quad (6.1.16)$$

where p is the differential operator d/dt

$F(p)$ is the loop filter transfer function

$n_{\phi}(t)$ is the equivalent noise source with spectrum $S(\omega, \phi)$.

The loop filter is assumed to have a transfer function

$$F(p) = G_2.$$

Then, if the correlation time of $n_{\phi}(t)$ is much less than the correlation time of the system, the ϕ process is approximately Markov, and so the probability density function of ϕ , $p(\phi, t)$, satisfies approximately a Fokker-Planck equation [2]

$$\frac{\partial}{\partial t} p(\phi, t) = - \frac{\partial}{\partial \phi} [k_1(\phi) p(\phi, t)] + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} [k(\phi) p(\phi, t)] \quad (6.1.17)$$

where

$$k_1(\phi) = \langle \dot{\phi} \rangle$$

$$k(\phi) = 2 \int_{-\infty}^0 \bar{k} [J(\phi_t), J(\phi_{t+\tau})] d\tau$$

$$J(\phi_t) = \text{R.H.S. of (6.1.16) at time } t$$

$$J(\phi_{t+\tau}) = \text{R.H.S. of (6.1.16) at time } t+\tau$$

$$\bar{k}[\phi_t, \phi_{t+\tau}] = \langle \phi_t \phi_{t+\tau} \rangle - \langle \phi_t \rangle \langle \phi_{t+\tau} \rangle$$

$\langle (\cdot) \rangle$ denotes ensemble average of $(\cdot)$ with ϕ held fixed. Then,

if $\lim_{t \rightarrow \infty} p(\phi, t) = p_s(\phi)$,

$$\frac{d}{d\phi} [k_1(\phi) p_s(\phi)] = \frac{1}{2} \frac{d^2}{d\phi^2} [k(\phi) p_s(\phi)]$$

whose solution is

$$p_s(\phi) = C \exp \left[\int_0^\phi \frac{2k_1(y) - \frac{dk(y)}{dy}}{k(y)} dy \right] \quad (6.1.18)$$

which becomes

$$p_s(\phi) = \frac{C}{S(0, \phi)} \exp \left[- \frac{2}{k_{VCO}} \int_0^\phi \frac{g(y)}{S(0, y)} dy \right] \quad (6.1.19)$$

after using (6.1.16) and the definitions following (6.1.17).

Now we can write a normalized power spectrum from (6.1.15) as

$$\begin{aligned}
S_0(0, \phi) &= \frac{S(0, \phi)}{2\sigma^2 T} \\
&= \frac{1}{2} U(2 + \alpha^2 + \beta^2) + 1 - 3g_1^2(\phi) + \frac{1}{2} [h_2(\beta) + h_2(\alpha) - h_3(\beta, \alpha)] \\
&\quad - \begin{cases} \frac{1}{2} h_0(\beta, \alpha, 1 - \Delta) + h_1(\alpha, -\beta, \Delta) & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ \frac{1}{2} h_0(\alpha, -\beta, \Delta) + h_1(\beta, \alpha, 1 - \Delta) & \text{for } \Delta/2 \leq \phi \leq 1 - \Delta/2 \end{cases} \quad (6.1.20)
\end{aligned}$$

where $g_1^2(\phi) = 4Ug_0^2(\phi)$
 $g_0(\phi)$ is defined in equation (6.1.5).

With these definitions $p_s(\phi)$ can be written as

$$p_s(\phi) = \frac{C}{S_0(0, \phi)} \exp \left[-4U\delta_s \int_0^\phi \frac{g_0(x) dx}{S_0(0, x)} \right] \quad (6.1.21)$$

$$\text{where } \delta_s = \frac{1}{k_{VCO} G_1 G_2 A T^2} \triangleq \frac{1}{W_{LS} T}. \quad (6.1.22)$$

W_{LS} is the 2-sided loop bandwidth at high SNR of the equivalent first order loop shown in figure 6.4. This can be shown as follows. For a first order loop with phase detector gain k_d and unity loop filter gain, the 2-sided loop bandwidth at high SNR is [3]

$$W_{LS} = k_d k_{VCO} / 2.$$

From equation (6.1.4) for $g(\phi)$ if we consider only the portion near $\phi=0$, i.e., assume small jitter at high SNR, then

$$k_d = \frac{dg}{d\phi} = 2G_1G_2AT$$

so that

$$W_{LS} = k_{VCO}G_1G_2AT. \quad (6.1.23)$$

It can now be seen that δ_s is the ratio of the bit rate $1/T$ to the loop bandwidth W_{LS} .

Some curves of the variance of ϕ are shown in figure 6.8 and figure 6.9 gives some plots of $S_0(0, \phi)$ and $p_s(\phi)$. It can be seen that the jitter variance is inversely proportional to E/N_0 for $E/N_0 \geq 20$, and that $S_0(0, \phi)$ reaches an asymptotic value at $\phi=0$ and $\phi = \frac{1}{2}$ for $E/N_0 \geq 10$. The power spectrum $S_0(0, \phi)$ also becomes increasingly peaked at $\phi = \frac{1}{4}$ as E/N_0 increases.

6.2 ASYMPTOTIC VALUES - ABSOLUTE VALUE DEVICE

It is shown in Appendix 6B that for high $U = E/N_0$

$$S_0(0, \phi) \rightarrow \frac{1}{2} - \left(\frac{1}{2} - \Delta\right) \cdot \text{sgn}(\beta) + \frac{1}{4}(\alpha - |\beta|)^2 U - \frac{1}{2}(\alpha + |\beta|) e^{-\beta^2 U} \sqrt{U/\pi} e^{-2\beta^2 U/4\pi} \quad (6.2.1)$$

giving

$$S_0(0, 0) \rightarrow \Delta$$

$$S_0(0, \frac{1}{2}) \rightarrow 1 - \Delta$$

$$S_0(0, \Delta/2) \rightarrow \Delta(1 + \Delta U) - (1 - \Delta) e^{-\beta^2 U} \sqrt{U/\pi} e^{-2\beta^2 U/4\pi}$$

where $\beta = 1 - 2\Delta$ at $\phi = \Delta/2$ makes the spectrum become more peaked at $\phi = \Delta/2$ as the SNR becomes larger.

Table 6.7 gives a comparison of the variance using (6.2.1) with that using the exact values of $S_0(0, \phi)$ from (6.1.20). It is evident that the asymptotic expression gives results to within 10% for $E/N_0 \geq 20$.

TABLE 6.7
Asymptotic Variance, Absolute Value Device

δ_s	E/N_0	Normalized Variance		
		Asymp. Val. Eq. 6.2.1	Exact Val. Eq. 6.1.24	$1/8\delta_s (E/N_0)$
5	8	$.249 \times 10^{-2}$	$.388 \times 10^{-2}$	$.313 \times 10^{-2}$
	10	$.224 \times 10^{-2}$	$.312 \times 10^{-2}$	$.250 \times 10^{-2}$
	20	$.157 \times 10^{-2}$	$.146 \times 10^{-2}$	$.125 \times 10^{-2}$
	40	$.784 \times 10^{-3}$	$.780 \times 10^{-3}$	$.625 \times 10^{-3}$
	80	$.391 \times 10^{-3}$	$.391 \times 10^{-3}$	$.312 \times 10^{-3}$
	100	$.313 \times 10^{-3}$	$.313 \times 10^{-3}$	$.250 \times 10^{-3}$
25	8	$.485 \times 10^{-3}$	$.633 \times 10^{-3}$	$.625 \times 10^{-3}$
	10	$.425 \times 10^{-3}$	$.512 \times 10^{-3}$	$.500 \times 10^{-3}$
	20	$.260 \times 10^{-3}$	$.253 \times 10^{-3}$	$.250 \times 10^{-3}$
	40	$.130 \times 10^{-3}$	$.130 \times 10^{-3}$	$.125 \times 10^{-3}$
	80	$.651 \times 10^{-4}$	$.651 \times 10^{-4}$	$.625 \times 10^{-4}$
	100	$.521 \times 10^{-4}$	$.521 \times 10^{-4}$	$.500 \times 10^{-4}$
50	8	$.311 \times 10^{-3}$	$.245 \times 10^{-3}$	$.313 \times 10^{-3}$
	10	$.251 \times 10^{-3}$	$.210 \times 10^{-3}$	$.250 \times 10^{-3}$
	20	$.127 \times 10^{-3}$	$.125 \times 10^{-3}$	$.125 \times 10^{-3}$
	40	$.638 \times 10^{-4}$	$.638 \times 10^{-4}$	$.625 \times 10^{-4}$
	80	$.319 \times 10^{-4}$	$.319 \times 10^{-4}$	$.312 \times 10^{-4}$
	100	$.255 \times 10^{-4}$	$.255 \times 10^{-4}$	$.250 \times 10^{-4}$

FIGURE 6.8 JITTER VARIANCE FOR $\Delta=1/2$, ABSOLUTE VALUE DEVICE

FIGURE 6.9 TIMING ERROR DENSITY AND EQUIVALENT NOISE SPECTRUM FOR $\delta_S=5$, $\Delta=1/2$, ABSOLUTE VALUE DEVICE

If we consider that at high SNR $\phi \ll \frac{1}{2}$ so that little error is caused by letting $S_0(0, \phi) = \Delta$ and $g_0(\phi) = \phi$, then $p_s(\phi)$ is zero mean Gaussian with variance $\Delta/4\delta_s U$. Simon [4] gives the special case $\Delta = \frac{1}{2}$ of this result. Table 6.7 compares this expression for jitter variance with the other values obtained. The Gaussian assumption gives good results for $\delta_s \geq 25$ and $E/N_0 \geq 20$.

6.3 STEADY STATE ANALYSIS - SQUARE LAW NONLINEARITY

As shown in figure 6.1 the Incosh function can also be approximated by a square law over a certain range. In this section the analysis of section 6.1 is repeated using a square law approximation to the Incosh nonlinearities. The analysis is much simpler and only the main points are given, with the device shown in figure 6.2 with the nonlinearity being a square law.

We have

$$e_k = G_2 \left[G_1 \int_{\hat{T}_{k-1} + \Delta T/2}^{\hat{T}_k + \Delta T/2} r(t) dt \right]^2 - G_2 \left[G_1 \int_{\hat{T}_{k-1} - \Delta T/2}^{\hat{T}_k - \Delta T/2} r(t) dt \right]^2 \triangleq e_{k,2} - e_{k,1}$$

$$\triangleq (s_{k,2} + n_{k,2})^2 - (s_{k,1} + n_{k,1})^2 \quad (6.3.1)$$

so that $s_{k,j} + n_{k,j}$ is conditionally Gaussian with mean $s_{k,j}$ and variance $G_1^2 G_2^2 N_0 T/2$. For a Gaussian random variable y with mean m and variance σ^2

$$E\{y^2\} = \sigma^2 + m^2$$

so that it can be shown that

$$g(\phi) \triangleq E_{n,s}\{e_{k,2}|\phi\} - E_{n,s}\{e_{k,1}|\phi\} = \frac{1}{2}(G_1\sqrt{G_2}AT)^2(\alpha^2 - \beta^2) \quad (6.3.2)$$

$$\begin{aligned} \text{or } g_0(\phi)^\Delta &= \frac{g(\phi)}{\frac{1}{2}(G_1\sqrt{G_2}AT)^2} = \alpha^2 - \beta^2 \\ &= \begin{cases} 8\phi(1-\Delta) & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ 4\Delta(1-2\phi) & \text{for } \Delta/2 \leq \phi \leq 1-\Delta/2. \end{cases} \end{aligned} \quad (6.3.3)$$

$g_0(\phi)$ is shown in figure 6.10.

Now to find the power spectrum $S(0, \phi)$ we need to evaluate

$$E_{n,s}\left\{(e_k - \bar{e}_k)(e_{k+m} - \bar{e}_{k+m})|\phi\right\}.$$

Proceeding exactly as in section 6.1 with much simpler algebra, and using the facts that (1) for a Gaussian random variable y with mean m and variance σ^2

$$E\{y^4\} = 3\sigma^4 + 6\sigma^2 m^2 + m^4$$

and (2) for correlated Gaussian random variables x and y with means m_x and m_y , variances σ^2 , and correlation coefficient

$$\rho = \frac{\overline{xy} - m_x m_y}{\sigma^2}$$

$$E\{x^2 y^2\} = 2(\rho\sigma^2)^2 + 4\rho\sigma^2 m_x m_y + (\sigma^2 + m_x^2)(\sigma^2 + m_y^2)$$

we find

FIGURE 6.10 $g_0(\phi)$ FOR SQUARE LAW DEVICE

$$\frac{1}{T} S(0, \phi) = \begin{cases} 8\sigma^4 \Delta (1-\Delta) + 2\sigma^2 M^2 [1-\Delta + \alpha^2 + \beta^2 - (1-\Delta)(\alpha + \beta + \alpha\beta) + 2\Delta\alpha\beta] + \frac{M^4}{4}(\alpha^2 - \beta^2)^2 & \text{for } -\Delta/2 \leq \phi \leq \Delta/2 \\ 8\sigma^4 \Delta (1-\Delta) + 2\sigma^2 M^2 [(\alpha - \beta)^2 - \Delta(-1 + \alpha - \beta - 3\alpha\beta)] + \frac{M^4}{4}(\alpha^2 - \beta^2)^2 & \text{for } \Delta/2 \leq \phi \leq 1 - \Delta/2 \end{cases} \quad (6.3.4)$$

where

$$M = AG_1 \sqrt{G_2 T}$$

$$\sigma^2 = G_1^2 G_2 N_0 T / 2.$$

Now letting $S(0, \phi) = T[2\sigma^4 t_1 + \sigma^2 M^2 t_2 + M^4 t_3]$ with t_1, t_2, t_3 defined in an obvious way from (6.3.4), we find that with $U = E/N_0$

$$\begin{aligned} \frac{g(\phi)}{S(0, \phi)} &= \frac{U^2}{2G_1^2 G_2 A^2 T^3} \frac{\alpha^2 - \beta^2}{\frac{1}{2} t_1 + \frac{1}{2} U t_2 + U^2 t_3} \\ &\triangleq \frac{U^2}{2G_1^2 G_2 A^2 T^3} Q(\phi) \end{aligned} \quad (6.3.5)$$

and

$$p_s(\phi) = \frac{C_1}{S(0, \phi)} \exp \left[-2U^2 (1-\Delta) \delta_{ss} \int_0^\phi Q(x) dx \right] \quad (6.3.6)$$

where

$$\delta_{ss} = \frac{1}{W_{LS} T}$$

$$W_{LS} = 2G_1^2 G_2 k_{VCO} A^2 T^2 (1-\Delta).$$

If the definitions of α and β are used, then $Q(x)$ becomes

$$Q(x) = \begin{cases} \frac{8x}{2\Delta + U[1 + 2\Delta(1-\Delta) - (1-\Delta)^2] + 4U[3 + 4U(1-\Delta)]x^2} & \text{for } 0 \leq x \leq \Delta/2 \\ \frac{1-2x}{\frac{1}{4}[2(1-\Delta) + U(4 + 2\Delta - 3\Delta^2) + 4U^2\Delta] - U(3 + 4U\Delta)x + U(3 + 4U\Delta)x^2} & \text{for } \Delta/2 \leq x \leq 1/2 \end{cases} \quad (6.3.7)$$

Figure 6.11 shows the jitter variance computed from (6.3.6) and figure 6.12 shows $p_S(\phi)$ and $S_0(0, \phi)$ where $S_0(0, \phi)$ is given by

$$S_0(0, \phi) = \frac{S(0, \phi)}{4\sigma^4 T}.$$

6.4 ASYMPTOTIC VALUES - SQUARE LAW DEVICE

An approximate analytical expression for the jitter density and jitter variance can be obtained at low SNR. If U is sufficiently small, from (6.3.7)

$$Q(x) \approx \begin{cases} 4x/\Delta & 0 \leq x \leq \Delta/2 \\ 2(1-2x)/(1-\Delta) & \Delta/2 \leq x \leq 1/2 \end{cases} \quad (6.4.1)$$

and

$$S_0(0, \phi) \approx 2\Delta(1-\Delta)$$

$$p_S(\phi) \approx \begin{cases} C \exp(-4U^2 \delta_{SS} \phi^2 / \Delta) & \text{for } 0 \leq \phi \leq \Delta/2 \\ C \exp[U^2 \delta_{SS} \Delta / (1-\Delta)] \cdot \exp[-4U^2 \delta_{SS} \phi(1-\phi) / (1-\Delta)] & \text{for } \Delta/2 \leq \phi \leq 1/2 \end{cases}$$

Thus $p_S(0) \approx C$, $p_S(\Delta/2) \approx C \exp(-U^2 \delta_{SS} \Delta)$, $p_S(1/2) \approx C \exp(-U^2 \delta_{SS})$ and $p_S(\phi)$ is essentially a uniform distribution since for example

FIGURE 6.11 JITTER VARIANCE FOR $\Delta=1/2$, SQUARE LAW DEVICE

FIGURE 6.12 TIMING ERROR DENSITY AND EQUIVALENT NOISE SPECTRUM FOR $\delta_{SS}=5, \Delta=1/2$, SQUARE LAW DEVICE

at $U=.1$, $\delta_{ss}=5$, $\Delta=1/2$, the exponentials are close to unity.

The constant C can be evaluated approximately as follows.

Since
$$\frac{1}{2} = \int_0^{1/2} p_s(\phi) d\phi$$

then

$$\frac{1}{C} = \frac{\sqrt{\pi\Delta/\delta_{ss}}}{2U} \cdot \text{erf}(U\sqrt{\delta_{ss}\Delta}) + \frac{\sqrt{(1-\Delta)/\delta_{ss}}}{2U} e^{-U^2\delta_{ss}} \int_{-\Delta\sqrt{\delta_{ss}(1-\Delta)}}^0 e^{-x^2} dx.$$

If $U\sqrt{\delta_{ss}(1-\Delta)} \ll 1$ then $e^{x^2} \approx 1+x^2$, and if $U\sqrt{\delta_{ss}\Delta} \leq .5$ then

$\text{erf}(U\sqrt{\delta_{ss}\Delta}) \approx 2U\sqrt{\delta_{ss}\Delta}/\pi$ so that

$$C = (\Delta + (1-\Delta)e^{-U^2\delta_{ss}})^{-1}$$

The assumptions used in deriving this value for C are

$$U \leq .05 \quad \text{in (6.4.1)}$$

$$U\sqrt{\delta_{ss}(1-\Delta)} \ll 1$$

$$U\sqrt{\delta_{ss}\Delta} \leq .5,$$

of which the second is the most stringent because $\Delta \leq 1/2$ and $\delta_{ss} \gg 1$.

The variance may now be calculated from

$$\begin{aligned}
\sigma^2 &= 2 \int_0^{1/2} x^2 p(x) dx \\
&= 2C \int_0^{\Delta/2} x^2 e^{-ax^2} dx + 2bc \int_{\Delta/2}^{1/2} x^2 e^{a(x^2-x)} dx
\end{aligned}$$

where

$$\begin{aligned}
a &= 4U^2 \delta_{ss} / (1-\Delta) \\
b &= \exp[U^2 \delta_{ss} / (1-\Delta)].
\end{aligned}$$

Then if $e^{-ax^2} \approx 1-ax^2$ and $e^{a(x^2-x)} \approx 1+a(x^2-x)$ over the range of integration it can be shown that

$$\begin{aligned}
\sigma^2 &\approx \frac{1}{12} [\Delta + (1-\Delta)e^{-U^2 \delta_{ss}}]^{-1} \exp[U^2 \delta_{ss} \Delta / (1-\Delta)] \\
&\rightarrow \frac{1}{12} \text{ as } U \rightarrow 0
\end{aligned}$$

as expected.

At high SNR it can be assumed as before that $\phi \ll 1/2$ so that

$$S_0(0, \phi) \approx (1-\Delta) [1 + 2\Delta(1-\Delta) - (1-\Delta)^2] U$$

and

$$g_0(\phi) \approx 8\phi(1-\Delta)$$

which makes $p_s(\phi)$ zero mean Gaussian with variance $\Delta(4-3\Delta)/8\delta_{ss} U$. In the special case $\Delta = 1/2$ this becomes $5/32\delta_{ss} U$ which is also given by Simon [4].

6.5 COMPARISON WITH SINE WAVE REFERENCE

The performance of a system using the bit timing estimation devices of this section can be judged by comparing the probability of error when the timing estimates are used in a matched filter with the probability of error obtainable with a transmitted reference system.

For the transmitted reference system, assume that the reference is a sine wave and that the data pulses are rectangular. If timing is derived from the sine wave by a PLL, and if the data signal power and sine wave power are related by $\beta E/T = A_C^2/2$ as in chapter 3, then the probability of error is given in figure 3.9 as a function of β with $2/W_L T = 25$ and 50 , and for various E/N_0 where E is the total transmitter energy. The minima of these curves represent the optimum power division between reference and data signals and are plotted in figure 6.13. In the other curves shown, E/N_0 is the data bit energy to noise ratio which is again the total transmitter energy to noise ratio since these are for self bit synchronizers.

Three features of these curves may be noted. The first is that the absolute value and square law devices give performance which is close to the matched filter if δ_s or $\delta_{ss} \geq 25$ for $E/N_0 < 10$ at least. Errors due to the finite number of integration intervals prevented extending the calculations to higher values of E/N_0 .

The second feature is that for a comparable linear loop bandwidth the self bit synchronizers give almost the same performance as the transmitted reference system. The systems being compared have the same linear loop bandwidth and the same transmitter energy to noise ratio. However, it has not been shown that this similar performance

FIGURE 6.13 PROBABILITY OF ERROR FOR MATCHED FILTER SYSTEM WHOSE
 TIMING IS DERIVED FROM ABSOLUTE VALUE DEVICE, SQUARE
 LAW DEVICE, AND SINE WAVE REFERENCE

is true for all signal to noise ratios. It is expected that at high SNR the self bit synchronizer curves will change to a much lower slope and will not follow the matched filter curve as noted in the third point below.

The third feature already mentioned is that the absolute value and square law device curves for δ_s and $\delta_{ss} = 5$ break away from the matched filter curve at $E/N_0 \approx 10$. This might be expected from the jitter variance curves of figures 6.8 and 6.11 where the corresponding curves do not attain their high SNR asymptotic slope until $E/N_0 \approx 10$. That is, the latter curves fall off rapidly as E/N_0 increases from 1 to about 10, and then fall off much more slowly as E/N_0 increases above 10. This behaviour is then reflected in the probability of error curves.

The variances at high SNR of the three devices considered here may also be compared. The absolute value device and the square law device have variances $1/8\delta_s(E/N_0)$ and $5/32\delta_s(E/N_0)$ respectively for $\Delta = 1/2$, while the first order PLL has a variance $1/\alpha$ [3] in the linear region. In the latter case α is proportional to E/N_0 . It cannot be concluded, however, that since all three devices have variances which are inversely proportional to E/N_0 at high SNR then the asymptotic probability of error should be the same. This is because in the transmitted reference system $\alpha = \frac{25\beta}{1+\beta} E/N_0$ and for each E/N_0 the optimum value of β is used, which value is not necessarily independent of E/N_0 .

Also shown in figure 6.13 are measured error rates for a binary PSK system [9] and for the Bell System type 303 data set [10]. Thus δ_s should be 25 because of other degradations such as matched filter realization.

6.6 TRANSIENT ANALYSIS, ABSOLUTE VALUE DEVICE

6.6.1 Introduction

This section contains an approximate transient analysis for the absolute value device assuming that a data signal is already present without noise when the synchronizer begins operation. The data signal used in this analysis is a periodic rectangular wave because any data transmitted during this transient period would probably not be received correctly, and a rectangular wave may give a shorter transient than a random data wave. The VCO frequency is assumed to be exactly the same as the data wave fundamental frequency just before operation begins. Furthermore, the analysis is for a first order loop, i.e., the loop filter transfer function is a constant.

The transient operation is illustrated in figure 6.14. Some further assumptions about the device operation are necessary in order to proceed with an analysis. The VCO waveform is represented as a sine wave, and it is assumed the timing pulses are produced by a timing circuit whose operation is triggered as the sine wave makes an upward crossing of some negative trigger level. Such a point is labelled L in figure 6.14, and at this point the timing circuit dumps the upper integrator. At $\Delta T/2$ seconds later the timing estimate $\hat{T}_k$ occurs and another $\Delta T/2$ seconds later the lower integrator is dumped and the VCO frequency is changed by the new error signal. This is exactly the manner in which the device was built as described in chapter 7.

The time origin is labelled $\hat{T}_0$ and the integrators start operating at some time γT after $\hat{T}_0$ where $-\Delta/2 \leq \gamma \leq 1 - \Delta/2$. In order to reduce the complexity of the analysis it was assumed that $\gamma = \Delta/2$, i.e.,

FIGURE 6.14 WAVEFORMS FOR TRANSIENT ANALYSIS

the first integrator operation starts at $\Delta T/2$ which is the same time as the error pulses are formed. ϕT is then the time from $\hat{T}_0$ to the first data wave transition time and $\Delta/2 - 1 \leq \phi \leq 1 - \Delta/2$ is assumed. It is further assumed that the VCO transients have died out before $\hat{T}_0$.

In subsequent sections equations for the ϕ_k will be derived, together with appropriate initial conditions under the further restriction that $\Delta = 1/2$ in order to simplify the analysis. Three regions have to be considered, namely $-1 + \Delta/2 \leq \phi_k \leq -\Delta/2$, $-\Delta/2 \leq \phi_k \leq \Delta/2$, and $\Delta/2 \leq \phi_k \leq 1 - \Delta/2$, and then in addition the cases where ϕ_k , ϕ_{k-1} , and ϕ_{k-2} belong to different regions.

In summary, the assumptions used in this chapter are:

1. periodic rectangular data signal of known period T
2. no noise
3. loop filter transfer function is a constant
4. $\Delta = 1/2$
5. data signal already present at the start of synchronizer operation
6. VCO transients have died out and its rest frequency is exactly $1/T$ Hz at the start of integrator operation
7. integrator operation begins $\Delta T/2$ seconds after the first upward VCO output wave zero crossing.

6.6.2 Initial Conditions

In this section the behaviour of the device over the first interval is analyzed. The value of e_0 is always zero if the integrators are turned on at $\hat{T}_0 + \Delta T/2$.

Case A, $\Delta/2 \leq \phi \leq 1-\Delta/2$:

From figure 6.14 we can write

$$\begin{aligned} G_1^{-1} S_{1,1} &= d_0(\phi - \Delta/2)T + d_1(\hat{T}_1 - \hat{T}_0 - \Delta T/2 - \phi_0 T) \\ &= AT[2\phi - 1] \end{aligned}$$

if we let $d_0 = A$ and $\hat{T}_1 - \hat{T}_0 = T$.

Also

$$\begin{aligned} G_1^{-1} S_{1,2} &= d_0(\phi_0 - \Delta/2)T + d_1(T + \Delta T/2 - \phi T) \\ &= AT(2\phi - 1 - \Delta). \end{aligned}$$

(The G_1^{-1} arises because from figure 6.2 the integrators have a gain G_1).

Thus

$$\frac{|S_{1,2}| - |S_{1,1}|}{G_1 AT} = \begin{cases} \Delta & \text{for } \Delta/2 \leq \phi \leq 1/2 \\ 2 + \Delta - 4\phi & \text{" } 1/2 \leq \phi \leq (1+\Delta)/2 \\ -\Delta & \text{" } (1+\Delta)/2 \leq \phi \leq 1-\Delta/2 \end{cases} \quad (6.6.1)$$

Case B, $-\Delta/2 \leq \phi \leq \Delta/2$:

From figure 6.14

$$\begin{aligned} G_1^{-1} S_{1,1} &= d_1(\hat{T}_1 - \hat{T}_0 - \Delta T) \\ &= -AT(1 - \Delta) \end{aligned}$$

$$\begin{aligned} G_1^{-1} S_{1,2} &= d_1(\hat{T}_1 - \hat{T}_0 - \Delta T/2 + \phi T) + d_2(\Delta/2 - \phi)T \\ &= -AT(1 - \Delta + 2\phi). \end{aligned}$$

Thus

$$\frac{|S_{1,2}| - |S_{1,1}|}{G_1AT} = \begin{cases} 2(\Delta-1-\phi) & \text{for } -\Delta/2 \leq \phi \leq (\Delta-1)/2 \\ 2\phi & \text{" } (\Delta-1)/2 \leq \phi \leq \Delta/2 \end{cases} \quad (6.6.2)$$

Case C, $-1+\Delta/2 \leq \phi \leq -\Delta/2$:

From figure 6.14

$$\begin{aligned} G_1^{-1}S_{1,1} &= d_1(1+\phi-\Delta/2)T + d_2(-\phi-\Delta/2)T \\ &= -AT(1+2\phi) \end{aligned}$$

$$\begin{aligned} G_1^{-1}S_{1,2} &= d_1(1+\phi-\Delta/2)T + d_2(\Delta/2-\phi)T \\ &= -AT(1+2\phi-\Delta)T. \end{aligned}$$

Thus

$$\frac{|S_{1,2}| - |S_{1,1}|}{G_1AT} = \begin{cases} \Delta & -1+\Delta/2 \leq \phi \leq -1/2 \\ -2-4\phi+\Delta & -1/2 \leq \phi \leq -1/2+\Delta/2 \\ -\Delta & -1/2+\Delta/2 \leq \phi \leq -\Delta/2 \end{cases} \quad (6.6.3)$$

Calculation of ϕ_2 :

The following points can be noted:

1. $\hat{T}_1 = T, \phi_0 = \phi_1 = \phi$
2. $e_0 = 0$ and e_1 can be calculated from (6.6.1) - (6.6.3) above
3. $\Delta T \omega_0 + \omega_2[(\hat{T}_2 - \Delta T/2) - (\hat{T}_1 + \Delta T/2)] = 2\pi$ from the VCO sine wave zero crossings.

Thus from 1-3, since

$$\phi_2 - \phi_1 = 1 - (\hat{T}_2 - \hat{T}_1) / T$$

then

$$\begin{aligned} \phi_2 - \phi_1 &= \frac{1}{2} \frac{\omega_2 - \omega_0}{\omega_2} \quad \text{if } \Delta = 1/2 \\ &= - \frac{1}{2\delta_s} \frac{f_1}{1 - \delta_s^{-1} f_1} \end{aligned} \quad (6.6.4)$$

where $e_1 \triangleq AG_1 G_2 T f_1$

$$f_1 = [|S_{1,2}| - |S_{1,1}|] / ATG_1$$

G_2 is the loop filter transfer function.

Hence ϕ_2 can be calculated from (6.6.1) - (6.6.4).

Calculation of ϕ_3 :

Since ϕ_3 is still affected by the first interval through e_1 , it cannot be calculated by using the single region equations which are developed in subsequent sections. It can be calculated by using equation (8.0.5) written as

$$\phi_3 = \phi_2 - \frac{1}{2\delta_s} \frac{f_2(\text{data}) + f_1(\text{data})}{1 - f_2(\text{data})/\delta_s}$$

and figure 6.14 to determine $f_2(\text{data})$; $f_1(\text{data})$ has been given above. The results are shown in table 6.8.

TABLE 6.8

Equations for ϕ_2 and ϕ_3

Range of ϕ_0, ϕ_1	Range of ϕ_2	Equation for ϕ_2	Equation for ϕ_3
0, 1/4	0, 1/4	$\phi_1(1 - \frac{1}{\delta_s - 2\phi_1})$	$\phi_2 - \frac{\phi_2 + 2\phi_1}{\delta_s - 2(\phi_2 + \phi_1)}$
"	1/4, 1/2	"	$\phi_2 - \frac{.25 + 2\phi_1}{\delta_s - .5 - 2\phi_1}$ if $\phi_2 + \phi_1 \leq 1/2$
			$\phi_2 - \frac{.75}{\delta_s - 1.5 + 2\phi_1}$ if $\phi_2 + \phi_1 \geq 1/2$
1/4, 1/2	1/4, 1/2	$\phi_1 - \frac{.25}{\delta_s - .5}$	$\phi_2 - \frac{1.25 - \phi_2 - \phi_1}{\delta_s - 2(1 - \phi_2 - \phi_1)}$
"	1/2, 3/4	"	"
1/2, 3/4	"	$\phi_1 - \frac{1.25 - 2\phi_1}{\delta_s - 2.5 + 4\phi_1}$	$\phi_2 - \frac{2.25 - \phi_2 - 3\phi_1}{\delta_s - 2(1 - \phi_2 - \phi_1)}$

d

6.6.3 Single Region Equations

In this section it is assumed that all ϕ_k of interest lie in one of the regions $[-1+\Delta/2, -\Delta/2]$, $[-\Delta/2, \Delta/2]$, or $[\Delta/2, 1-\Delta/2]$, and that $k \geq 4$. The latter is required because for $k < 4$ the initial e_1 is still affecting the device operation and the equations presented in this section do not hold. As in (8.0.5) we can write

$$\phi_k \approx \phi_{k-1} - \frac{1}{2\delta_s} [f_{k-1}(\text{data}) + f_{k-2}(\text{data})] \quad (6.6.5)$$

if $|f_{k-1}(\text{data})/\delta_s| \ll 1$,

and it is shown in Appendix 6D that if $\Delta=1/2$ we can write (6.6.5) as

$$\phi_k \approx \phi_{k-1}(1+1/\delta_s) + (2\phi_{k-2} + \phi_{k-3})/\delta_s - 2/\delta_s \quad \text{for } 1/4 \leq \phi_{(.)} \leq 3/4 \quad (6.6.6)$$

$$\phi_k \approx \phi_{k-1}(1-1/\delta_s) - (2\phi_{k-2} + \phi_{k-3})/\delta_s \quad \text{for } -1/4 \leq \phi_{(.)} \leq 1/4 \quad (6.6.7)$$

$$\phi_k \approx \phi_{k-1}(1+1/\delta_s) + (2\phi_{k-2} + \phi_{k-3})/\delta_s + 2/\delta_s \quad \text{for } -3/4 \leq \phi_{(.)} \leq -1/4. \quad (6.6.8)$$

Equation (6.6.8) has the same transient solution as equation (6.6.6); the steady state term has the opposite sign. All three equations can be written in the general form

$$\phi_k + s\phi_{k-1} + t\phi_{k-2} + t/2 \phi_{k-3} = u$$

and if the solution is of the form $\phi_k = \phi_{ss} + u\beta^k$ then

$$\beta^3 + s\beta^2 + t\beta + t/2 = 0. \quad (6.6.9)$$

The roots of this equation are given by [8]

$$\beta_1 = (\rho_1 + \rho_2) - s/3$$

$$\beta_2 = -(\rho_1 + \rho_2)/2 - s/3 + j\sqrt{3}(\rho_1 - \rho_2)/2$$

$$\beta_3 = -(\rho_1 + \rho_2)/2 - s/3 - j\sqrt{3}(\rho_1 - \rho_2)/2$$

$$\begin{aligned} \text{where } \rho_1 &= [r + (q^3 + r^2)^{1/2}]^{1/3} & \rho_2 &= [r - (q^3 + r^2)^{1/2}]^{1/3} \\ q &= t/3 - s^2/9 & r &= (st - 3t/2)/6 - s^3/27. \end{aligned}$$

The roots are plotted in figures 6.15 and 6.16 for various values of δ_s . If the roots are β_1 , β_2 , and β_3 , then the solutions of (6.6.6) - (6.6.8) are

$$\phi_k = \phi_{ss} + c_1 \beta_1^k + c_2 \beta_2^k + c_3 \beta_3^k$$

where the c_j are constants to be determined from initial conditions.

The steady state solutions ϕ_{ss} are $\phi_{ss} = 1/2$ for (6.6.6), $\phi_{ss} = 0$ for (6.6.7), and $\phi_{ss} = -1/2$ for (6.6.8).

It is of interest to compute the points of emergence, the value of δ_s for which the root loci cross the unit circle, and the asymptotic behaviour of the root locus as $\delta_s \rightarrow 0$ or ∞ . These points are shown in figures 6.15 and 6.16 and the derivations are given in Appendix 6E. However, it must be remembered that $|\delta_s f_{k-1}(\text{data})| \ll 1$ is required, and so the root loci may not describe the device behaviour very well for small δ_s .

It can be seen that for $\delta_s > 2.39$ the transient solution will die out in equation (6.6.7), whereas in equations (6.6.6) and (6.6.8) the solution always has an exponentially increasing term. Hence the synchronizer will settle at $\delta_{ss} = 0$.

FIGURE 6.16 ROOTS OF EQUATIONS (6.6.6) AND (6.6.8)

6.6.4 Dual Region Equations

In this section the equations are presented for the cases where the ϕ_k of interest may be in $[-\Delta/2, \Delta/2]$ or in $[\Delta/2, 1-\Delta/2]$. We can use the approximate equation (6.6.5) and use the values for $f_{k-1}(\text{data})$ and $f_{k-2}(\text{data})$ given in table 6.9 which were derived directly from figure 6.14.

6.6.5 Discussion of Transient Behaviour

It is evident from the preceding discussion that the transients are of two types

- a) a damped sinusoid if the initial timing error is near zero
- b) a rising exponential followed by a damped sinusoid if the initial timing error is near $T/2$.

Figure 6.17 illustrates these transients. If the initial timing error ϕ_0 is exactly $1/2$, then the device will not lock at this value in the noiseless case considered here because the first loop error signal is not zero, but will move slowly towards its steady state operating point.

Let us define settling time as the time required for the timing error to be less than .01. Then as ϕ_0 increases from zero to $1/2$ the settling time increases, and it decreases again as ϕ_0 increases from $1/2$ to 1. An indication of the range of settling times encountered is given in table 6.10 which is the result of a number of simulated transients at various values of ϕ_0 . The range of δ_s shown in table 6.10 is the range which is suitable for design purposes because from figure 6.15 the loop is unstable for $\delta_s \leq 2.4$ and critical damping occurs at $\delta_s = 18$.

TABLE 6.9a

Dual Region Equations

Range of Variables			f_{k-1}	f_{k-2}
θ_{k-1}	θ_{k-2}	θ_{k-3}		
$[-1/4, 1/4]$	$[-1/4, 1/4]$	$[+1/4, 3/4]$	$2(\theta_{k-1} + \theta_{k-2})$	$1/2 + 2\theta_{k-2}$ if $\theta_{k-2} + \theta_{k-3} \leq 1/2$ $3/2 - 2\theta_{k-2}$ if $\theta_{k-2} + \theta_{k-3} \geq 1/2$
"	$[+1/4, 3/4]$	"	$1/2 + 2\theta_{k-1}$ if $\theta_{k-1} + \theta_{k-2} \leq 1/2$ $3/2 - 2\theta_{k-1}$ if $\theta_{k-1} + \theta_{k-2} \geq 1/2$	$2(1 - \theta_{k-2} - \theta_{k-3})$
"	"	$[-1/4, 1/4]$	$1/2 + 2\theta_{k-1}$ if $\theta_{k-1} + \theta_{k-2} \leq 1/2$ $3/2 - 2\theta_{k-1}$ if $\theta_{k-1} + \theta_{k-2} \geq 1/2$	$1/2 + 2\theta_{k-3}$ if $\theta_{k-2} + \theta_{k-3} \leq 1/2$ $3/2 - 2\theta_{k-2}$ if $\theta_{k-2} + \theta_{k-3} \geq 1/2$
$[+1/4, 3/4]$	"	"	$2(1 - \theta_{k-1} - \theta_{k-2})$	$1/2 + 2\theta_{k-3}$ if $\theta_{k-2} + \theta_{k-3} \leq 1/2$ $3/2 - 2\theta_{k-2}$ if $\theta_{k-2} + \theta_{k-3} \geq 1/2$
"	$[-1/4, 1/4]$	"	$1/2 + 2\theta_{k-2}$ if $\theta_{k-1} + \theta_{k-2} \leq 1/2$ $3/2 - 2\theta_{k-2}$ if $\theta_{k-1} + \theta_{k-2} \geq 1/2$	$2(\theta_{k-2} + \theta_{k-3})$
"	"	$[1/4, 3/4]$	$1/2 + 2\theta_{k-2}$ if $\theta_{k-1} + \theta_{k-2} \leq 1/2$ $3/2 - 2\theta_{k-2}$ if $\theta_{k-1} + \theta_{k-2} \geq 1/2$	$1/2 + 2\theta_{k-3}$ if $\theta_{k-2} + \theta_{k-3} \leq 1/2$ $3/2 - 2\theta_{k-3}$ if $\theta_{k-2} + \theta_{k-3} \geq 1/2$

TABLE 6.9b

Dual Region Equations

Range of Variables			f_{k-1}	f_{k-2}
θ_{k-1}	θ_{k-2}	θ_{k-3}		
$[-1/4, 1/4]$	$[-1/4, 1/4]$	$[-3/4, -1/4]$	$2(\theta_{k-1} + \theta_{k-2})$	$-1/2 + 2\theta_{k-2}$ if $\theta_{k-2} + \theta_{k-3} \geq -1/2$ $-3/2 - 2\theta_{k-3}$ if $\theta_{k-2} + \theta_{k-3} \leq -1/2$
"	$[-3/4, -1/4]$	$[-1/4, 1/4]$	$-1/2 + 2\theta_{k-1}$ if $\theta_{k-1} + \theta_{k-2} \geq -1/2$ $-3/2 - 2\theta_{k-2}$ if $\theta_{k-1} + \theta_{k-2} \leq -1/2$	$-1/2 + 2\theta_{k-3}$ if $\theta_{k-2} + \theta_{k-3} \geq -1/2$ $-3/2 - 2\theta_{k-2}$ if $\theta_{k-2} + \theta_{k-3} \leq -1/2$
"	"	$[-3/4, -1/4]$	$-1/2 + 2\theta_{k-1}$ if $\theta_{k-1} + \theta_{k-2} \geq -1/2$ $-3/2 - 2\theta_{k-2}$ if $\theta_{k-1} + \theta_{k-2} \leq -1/2$	$-2(1 + \theta_{k-2} + \theta_{k-3})$
$[-3/4, -1/4]$	$[-1/4, 1/4]$	$[-1/4, 1/4]$	$-1/2 + 2\theta_{k-2}$ if $\theta_{k-1} + \theta_{k-2} \geq -1/2$ $-3/2 - 2\theta_{k-1}$ if $\theta_{k-1} + \theta_{k-2} \leq -1/2$	$2(\theta_{k-2} + \theta_{k-3})$
"	"	$[-3/4, -1/4]$	$-1/2 + 2\theta_{k-2}$ if $\theta_{k-1} + \theta_{k-2} \geq -1/2$ $-3/2 - 2\theta_{k-1}$ if $\theta_{k-1} + \theta_{k-2} \leq -1/2$	$-1/2 + 2\theta_{k-2}$ if $\theta_{k-2} + \theta_{k-3} \geq -1/2$ $-3/2 - 2\theta_{k-3}$ if $\theta_{k-2} + \theta_{k-3} \leq -1/2$
"	$[-3/4, -1/4]$	$[-1/4, 1/4]$	$-2(1 + \theta_{k-1} + \theta_{k-2})$	$-1/2 + 2\theta_{k-3}$ if $\theta_{k-2} + \theta_{k-3} \geq -1/2$ $-3/2 - 2\theta_{k-2}$ if $\theta_{k-2} + \theta_{k-3} \leq -1/2$

TABLE 6.10
Settling Time in Bit Intervals

δ_s	Initial Timing Error ϕ_0	
	.5	.1
5	24.5	14.1
8	18.5	8.1
10	16.5	8.1
12	18.5	7.1
14	20.5	7.1
16	20.5	8.1

FIGURE 6.17 TRANSIENT RESPONSE OF ABSOLUTE VALUE DEVICE

The values of δ_s used in this section are much less than the value of 25 which was found to be suitable for steady state operation with a random data signal. Thus in a practical system a method of changing loop bandwidth at the end of the acquisition period would be necessary.

6.7 ABSOLUTE VALUE DEVICE OPERATING POINTS

6.7.1 Introduction

The differential equation describing the operation of the equivalent PLL was given in section 6.1.3 as

$$\dot{\phi} = -k_{VCO}F(\rho)[g(\phi)+n_{\phi}(t)] \quad (6.7.1)$$

where $g(\phi)$, which is called the loop S-curve [1], was given in (6.1.4) and was plotted in figure 6.7. If we just consider the part

$$\dot{\phi} = -k_{VCO}G_2g(\phi)$$

where $F(\rho) = G_2$ has been used, stable operating points occur whenever $g(\phi)$ has a positive slope as it crosses the zero ordinate axis, i.e., at $\phi = 0, \pm 1, \pm 2, \dots$. However, this assumed that the PLL gain was $-k_{VCO}$. If the PLL gain is k_{VCO} , then the differential equation of operation is

$$\dot{\phi} = k_{VCO}F(\rho)[g(\phi) + n_{\phi}(t)] \quad (6.7.2)$$

and the stable operating points are at $\phi = \pm 1/2, \pm 3/2, \dots$.

There are a number of implications of this which are discussed in this section, namely, the steady state and transient performance differences at the two sets of operating points, and some practical circuit design differences.

6.7.2 Circuit Design Differences

Since it was shown in section 6.1 that $g(\phi)$ is an increasing function at $\phi = 0$, then (6.7.1) requires either a VCO with a negative gain constant or a sign changing operation in the loop. The latter could be achieved by forming $(e_{k,1} - e_{k,2})$ at the loop summer instead of $(e_{k,2} - e_{k,1})$ as assumed in the theory here. On the other hand (6.7.2) requires a VCO with a positive gain constant.

The value of ϕ desired as a stable operating point depends on the implementation of the matched filter receiver. Theoretically for a rectangular data wave the matched filter outputs should be sampled at the bit transition times because then the SNR is maximum at the filter outputs. Then a synchronizer with a stable lock point at $\phi = 1/2$ would require a $T/2$ second delay of its timing estimates. But this is not necessarily a disadvantage; it may be wise to have an adjustable delay. This is because the matched filter realization may be approximate and have optimum sampling times other than the bit transition times, or the circuitry of the synchronizer may give rise to a slight fixed shift in its operating point due to component tolerances etc.

The above discussion shows that there are no significant practical advantages or disadvantages to either choice of operating point.

6.7.3 Steady State Performance Differences

It is shown in this section that for the case $\Delta = 1/2$ there is no difference in steady state performance at either operating point $\phi = 0$ or $\phi = 1/2$. For all other Δ the performance at $\phi = 0$ is better than that at $\phi = 1/2$ at high SNR.

The analysis for an operating point $\phi = 0$ has been given in section 6.1 resulting in the equation

$$p_s(\phi) = \frac{C}{S_0(0, \phi)} \exp \left[-4U\delta_s \int_0^{\phi} \frac{g_0(x) dx}{S_0(0, x) dx} \right] \quad 0 \leq \phi \leq 1/2 \quad (6.7.3)$$

with $g_0(x)$ given by (6.1.5) and $S_0(0, x)$ by (6.1.20), and $p_s(-\phi) = p_s(\phi)$ also.

For an operating point $\phi = 1/2$, equation (6.7.2) leads to

$$p_s(\phi) = \frac{C}{S_0(0, \phi)} \exp \left[4U\delta_s \int_{1/2}^{\phi} \frac{g_0(x) dx}{S_0(0, x) dx} \right] \quad 1/2 \leq \phi \leq 1 \quad (6.7.4)$$

and $p_s(1-\phi) = p_s(\phi)$. It is shown in Appendix 6G that $g_0(\phi) = -g_0(1-\phi)$ so that (6.7.4) can be written

$$p_s(\phi) = \frac{C}{S_0(0, \phi)} \exp \left[-4U\delta_s \int_0^{\phi-1/2} \frac{g_0(1/2-v)}{S_0(0, v+1/2)} dv \right] \quad 1/2 \leq \phi \leq 1 \quad (6.7.5)$$

It is also shown in Appendix 6G that if $\Delta = 1/2$ then

$g_0(1/2-v) = g_0(v)$ for $0 \leq v \leq 1/2$, and that $S_0(0, v+1/2) = S_0(0, v)$, so that

$$p_s(\phi) = \frac{C}{S_0(0, \phi)} \exp \left[-4U\delta_s \int_0^{\phi-1/2} \frac{g_0(v)}{S_0(0, v)} dv \right] \quad 1/2 \leq \phi \leq 1 \quad (6.7.6)$$

whose right hand side is the same as that of (6.7.3). Thus the steady state performance is the same for $\Delta = 1/2$ at both operating points.

For other values of Δ conclusions can most easily be drawn from the high SNR asymptotic results derived earlier and (6.7.5). It was shown in section (6.2) that at high E/N_0 , $S_0(0,0) \rightarrow \Delta$ and $S_0(0,1/2) \rightarrow 1-\Delta$, and for $0 < \Delta < 1/2$ we have $\Delta < 1-\Delta$. This means that the noise in the loop is greater at $\phi = 1/2$ than at $\phi = 0$. From (6.1.5) it can be seen that at high $U = E/N_0$

$$g_0(\phi) = 1/4 [\alpha \cdot \text{sgn}(\alpha) - \beta \cdot \text{sgn}(\beta)]$$

Using these results in (6.7.5) and assuming that the jitter is small enough then

$$p_s(\phi) \rightarrow C_1 \exp \left[-\frac{2U\delta_s}{1-\Delta} (\phi-1/2)^2 \right]$$

which is a Gaussian density with mean $1/2$ and variance $(1-\Delta)/4U\delta_s$. This may be compared with the Gaussian density derived in section 6.2 for the $\phi = 0$ operating point which had zero mean and variance $\Delta/4U\delta_s$. Thus we see that the variance is larger at the $\phi = 1/2$ point than at the $\phi = 0$ point.

6.7.4 Transient Performance Differences

The transient performance was discussed in section 6.6 for the device which settles at $\phi = 0, \pm 1, \dots$. The equations which describe the transient performance of the device which settles at $\phi = \pm 1/2, \pm 3/2 \dots$ are very similar. For example, corresponding to (6.6.6) - (6.6.8) we have

$$\phi_k \approx \phi_{k-1}(1-1/\delta_s) - (2\phi_{k-2} + \phi_{k-3})/\delta_s + 2/\delta_s \quad \text{for } 1/4 \leq \phi_{(.)} \leq 3/4 \quad (6.7.7)$$

$$\phi_k \approx \phi_{k-1}(1+1/\delta_s) + (2\phi_{k-2} + \phi_{k-3})/\delta_s \quad \text{for } -1/4 \leq \phi_{(.)} \leq 1/4 \quad (6.7.8)$$

$$\phi_k \approx \phi_{k-1}(1-1/\delta_s) - (2\phi_{k-2} + \phi_{k-3})/\delta_s - 2/\delta_s \quad \text{for } -3/4 \leq \phi_{(.)} \leq -1/4. \quad (6.7.9)$$

The difference is that the exponential region now falls about $\phi = 0$ and the damped sinusoid about $\phi = 1/2$. Furthermore, the error signal from the first interval is 2ϕ when $(\Delta-1)/2 \leq \phi \leq \Delta/2$ and Δ for $\Delta/2 \leq \phi \leq 1/2$.

Thus there is now a longer transient time when $\phi_0 = 0$. In the case analyzed in section 6.6, the corresponding exponential region fell about $\phi = 1/2$, and for $\phi_0 = 1/2$ the large error signal from the first interval helped to speed up the transient response.

6.7.5 Summary of Differences

The following points have been made about device performance at operating points $\phi = 0, \pm 1, \pm 2, \dots$ as opposed to those at $\phi = \pm 1/2, \pm 3/2, \dots$:

1. there are not any significant differences in circuit design
2. the steady state performance is the same for $\Delta = 1/2$ but worse at $\phi = \pm 1/2, \dots$ for other Δ

3. the transient performance is better at $\phi = 0, \pm 1, \dots$ operating points.

Therefore operating points $\phi = 0, \pm 1, \dots$ should be used because of their better steady state and transient performance.

6.8 CONCLUSION

The steady state performances of the absolute value and square law devices have been analyzed, and some asymptotic jitter variance results have been obtained. These devices have been compared with a sine wave reference system, and for the range of E/N_0 examined the self bit synchronization systems are better because the matched filter receiver has a lower probability of error when they are used to provide its timing.

Some discussion of the transient behaviour of the absolute value device was given. It was shown that a lower value of δ_s is required for fast transient response with a periodic data signal, than is required for good steady state performance with a random data signal.

It is also of interest to note that in the work in Appendix 6D it is shown that the error signal $|f_k(\text{data})|$ is proportional to the timing error and is independent of data pulse polarity (assuming that a transition has occurred). This provides a practical reason for considering this device as opposed to the theoretical approach from the MAP criterion used in chapter 5.

Further discussion of the possible steady state timing errors for the absolute value device showed that a steady state timing error

of 0 is most desirable. This particular choice gives better transient and steady state performance than the other choice of $T/2$.

APPENDIX 6A

EVALUATION OF $E\{|xy|\}$ FOR NON-ZERO MEAN CORRELATED GAUSSIAN RANDOM VARIABLES

$$\begin{aligned}
 E\{|xy|\} &= \iint_0^{\infty} xy \cdot p(x,y) dx dy + \iint_{-\infty}^0 xy \cdot p(x,y) dx dy \\
 &= \int_{x=-\infty}^0 \int_{y=0}^{\infty} xy \cdot p(x,y) dx dy + \int_{x=0}^{\infty} \int_{y=-\infty}^0 xy \cdot p(x,y) dx dy \\
 &\triangleq I_1 + I_2 - I_3 - I_4
 \end{aligned} \tag{6A.1}$$

where

$$p(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\sigma^2(1-\rho^2)} \left[(x-m_x)^2 - 2\rho(x-m_x)(y-m_y) + (y-m_y)^2\right]\right\}.$$

To evaluate I_1 :

$$\text{Let } U_1 = \frac{x-m_x}{\sigma\sqrt{2(1-\rho^2)}} \quad \text{and} \quad U_2 = \frac{y-m_y}{\sigma\sqrt{2(1-\rho^2)}} \quad \text{so that}$$

$$I_1 = \frac{\sqrt{1-\rho^2}}{\pi} \int_{U_1=-m_x/a}^{\infty} \int_{U_2=-m_y/a}^{\infty} (aU_1+m_x)(aU_2+m_y) \cdot \exp[-(U_1^2 - 2\rho U_1 U_2 + U_2^2)] dU_1 dU_2$$

$$\text{where} \quad a = \sigma\sqrt{2(1-\rho^2)}$$

Now use the following transformation due to Rice [5]. Let

$$\bar{U}^T D \bar{U} = (U_1 U_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = U_1^2 a_{11} + 2a_{12} U_1 U_2 + a_{22} U_2^2$$

where we choose $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = -\rho$, and let

$$U_1 = h_1 y_1 + h_2 d_{21} y_2$$

$$U_2 = h_2 d_{22} y_2$$

$$\text{where } h_1 = 1/\sqrt{a_{11}} \quad h_2 = \frac{1}{\sqrt{a_{11}(a_{11}a_{22}-a_{21}^2)}} = \frac{1}{\sqrt{1-\rho^2}}$$

$$d_{21} = -a_{21} \quad d_{22} = a_{11}$$

$$\text{i.e., } U_1 = y_1 + \rho y_2 / \sqrt{1-\rho^2} \quad U_2 = y_2 / \sqrt{1-\rho^2}.$$

The Jacobian is $1/\sqrt{1-\rho^2}$ and the limits of integration become $y_{22} = -m_y/\sigma\sqrt{2} \leq y_2 \leq \infty$ and $y_{12} = -m_x/a - \rho y_2/\sqrt{1-\rho^2} \leq y_1 \leq \infty$.
Thus

$$\begin{aligned} I_1 &= \frac{1}{\pi} \int_{y_{22}}^{\infty} \int_{y_{12}}^{\infty} (a y_1 + \sigma \rho \sqrt{2} y_2 + m_x) (\sigma \sqrt{2} y_2 + m_y) e^{-y_1^2 - y_2^2} dy_1 dy_2 \\ &= \sqrt{1-\rho^2} e^{-m_x^2/2\sigma^2} \left\{ \frac{\sigma^2(1-\rho^2)}{2\pi} \cdot \exp[-(\rho m_x - m_y)^2/a^2] \right. \\ &\quad \left. - \sigma \sqrt{(1-\rho^2)/8\pi} (\rho m_x - m_y) [1 - \operatorname{erf}((\rho m_x - m_y)/a)] \right\} \\ &\quad + \frac{1}{2\sqrt{\pi}} \int_{y_{22}}^{\infty} (\sigma \rho \sqrt{2} y_2 + m_x) (\sigma \sqrt{2} y_2 + m_y) [1 + \operatorname{erf}(-y_{12})] e^{-y_2^2} dy_2. \end{aligned}$$

I_2 , I_3 , and I_4 can be treated in the same manner and we find that

$$\begin{aligned}
I_2 = & \sqrt{1-\rho^2} e^{-m_x^2/2\sigma^2} \left\{ \frac{\sigma^2(1-\rho^2)}{2\pi} \cdot \exp \left[-(\rho m_x - m_y)^2/a^2 \right] \right. \\
& + \sigma(1-\rho^2)/8\pi(\rho m_x - m_y) \left[1 + \operatorname{erf}((\rho m_x - m_y)/a) \right] \left. \right\} \\
& + \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{y_{22}} (\sigma\rho\sqrt{2}y_2 + m_x) (\sigma\sqrt{2}y_2 + m_y) [1 - \operatorname{erf}(-y_{12})] e^{-y_2^2} dy_2
\end{aligned}$$

$$I_3 = -(\text{1st term of } I_1)$$

$$+ \frac{1}{2\sqrt{\pi}} \int_{y_{22}}^{\infty} (\sigma\rho\sqrt{2}y_2 + m_x) (\sigma\sqrt{2}y_2 + m_y) [1 - \operatorname{erf}(-y_{12})] e^{-y_2^2} dy_2$$

$$I_4 = -(\text{1st term of } I_2)$$

$$+ \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{-y_{22}} (\sigma\rho\sqrt{2}y_2 + m_x) (\sigma\sqrt{2}y_2 + m_y) [1 + \operatorname{erf}(-y_{12})] e^{-y_2^2} dy_2.$$

After a great deal more algebra we find that from (6A.1)

$$\begin{aligned}
E\{|xy|\} = & \sigma\sqrt{2/\pi} \exp(-m_x^2/2\sigma^2) \left[\sqrt{2(1-\rho^2)}/\pi \sigma e^{-b^2} - m_y \operatorname{erf}(b) \right] \\
& + \sigma m_x \sqrt{2/\pi} \exp(-m_y^2/2\sigma^2) \operatorname{erf}((m_x - \rho m_y)/a) \\
& + (\sigma^2\rho + m_x m_y) (Q_1 - Q_2) / \sqrt{\pi}
\end{aligned} \tag{6A.2}$$

where $a = \sigma\sqrt{2(1-\rho^2)}$

$$b = (\rho m_x - m_y)/a$$

$$Q_1 \triangleq Q_1(m_x, m_y, \rho, \sigma) = \int_{-m_y/\sigma\sqrt{2}}^{\infty} e^{-y_2^2} \operatorname{erf}((m_x + \rho\sigma\sqrt{2}y_2)/a) dy_2$$

$$Q_2 \triangleq Q_2(m_x, m_y, \rho, \sigma) = \int_{m_y/\sigma\sqrt{2}}^{\infty} e^{-y_2^2} \operatorname{erf}((m_x - \rho\sigma\sqrt{2}y_2)/a) dy_2$$

It may be noted that $Q_j(-m_x, -m_y, \rho, \sigma) = -Q_j(m_x, m_y, \rho, \sigma)$.

This expression has been checked by simulation at various values of ρ , m_x , m_y , and σ . The random variables x and y were generated from independent zero mean unit variance Gaussian random variables x_1 and x_2 by the transformation

$$x = \sigma x_1 + m_x$$

$$y = \sigma(\rho x_1 + x_2\sqrt{1-\rho^2}) + m_y,$$

and $E\{|xy|\}$ was obtained by averaging over 10,000 trials. The results are given in table 6.11 which shows good agreement between the simulation results and equation (6A.2).

A further partial check is made by considering the case $m_x = m_y = 0$. Then using the fact that

$$\int_0^{\infty} e^{-y^2} \operatorname{erf}(ay) dy = (1/\sqrt{\pi}) \arctan(a)$$

we can reduce (6A.2) to

TABLE 6.11

Simulated and Theoretical Results for $E\{|xy|\}$ where
 x and y are Correlated Gaussian Random Variables

σ	ρ	m_x	m_y	Simulated $E\{ xy \}$	Theoretical $E\{ xy \}$
.5	.3	.5	1	.642	.638
.5	-.6	.5	2	1.066	1.064
1	.3	.5	1	1.148	1.139
1	-.6	.5	2	1.594	1.597
5	.3	.5	1	17.02	17.11
5	-.6	.5	2	19.28	19.56
.5	0	.5	1	.592	.588
.5	0	.5	2	1.17	1.17
1	0	.5	1	1.05	1.04
1	0	.5	2	1.81	1.81
5	0	.5	1	16.26	16.31
5	0	.5	2	17.24	17.26

$$E\{|xy|\} = 2\sigma^2(\sqrt{1-\rho^2} + \rho \arcsin(\rho))/\pi$$

which agrees with the result given for this case in Papoulis [6] equation (7.109).

In addition, if $\rho=0$ then

$$\begin{aligned} E\{|xy|\} &= \sigma\sqrt{2/\pi} \left[m_y \operatorname{erf}(m_y/\sigma\sqrt{2}) e^{-m_x^2/2\sigma^2} + m_x \operatorname{erf}(m_x/\sigma\sqrt{2}) e^{-m_y^2/2\sigma^2} \right] \\ &+ (2\sigma^2/\pi) \cdot \exp[-(m_x^2+m_y^2)/2\sigma^2] \\ &+ m_x m_y \operatorname{erf}(m_x/\sigma\sqrt{2}) \operatorname{erf}(m_y/\sigma\sqrt{2}) \end{aligned}$$

APPENDIX 6B

ASYMPTOTIC VALUES OF $S(0, \phi)$, ABSOLUTE VALUE DEVICE

For large $U = E/N_0$, equation (6.1.8) tends to

$$2\sigma^2 \frac{\rho/2 + Um_x m_y}{\sqrt{\pi}} (Q_1 - Q_2). \quad (6B.1)$$

For $m_y > 0$, $Q_2 \rightarrow 0$ as U becomes large, and

$$\begin{aligned} Q_1 &\rightarrow \int_{-\infty}^{\infty} e^{-y^2} \operatorname{erf}[(m_x \sqrt{2U} + \sqrt{2\rho}y) / \sqrt{2(1-\rho^2)}] dy \\ &\approx \sqrt{\pi} \cdot \operatorname{sgn}(m_x) \end{aligned}$$

because for large U the term $m_x \sqrt{2U}$ is much larger than $\sqrt{2\rho}y$ for any small value of y , and for other values of y the e^{-y^2} term causes any variation of the erf term to have a negligible effect on the integral.

For $m_y < 0$, $Q_1 \rightarrow 0$ as U becomes large, and

$$Q_2 \rightarrow \sqrt{\pi} \cdot \operatorname{sgn}(m_x)$$

so that

$$Q_1 - Q_2 \rightarrow \sqrt{\pi} \operatorname{sgn}(m_x) \cdot \operatorname{sgn}(m_y)$$

and (6B.1) becomes

$$2\sigma^2 \left[\rho/2 \cdot \operatorname{sgn}(m_x) \cdot \operatorname{sgn}(m_y) + U |m_x| |m_y| \right] \quad (6B.2)$$

It must be noted too that $\beta = 1 - \Delta - 2\phi$ is 0 at $\phi = (1 - \Delta)/2$ so that terms involving $e^{-\beta^2 U}$ have to be kept in any asymptotic expressions. With

this and (6B.2) in mind, and using the definitions of $f(m_x, m_y, \rho)$ and $f_U(m_x, m_y)$ given in (6.1.8) and (6.1.10), we find that for large U

$$f(1, 1, \rho) \rightarrow \rho/2 + U$$

$$f(1, \alpha, \rho) \text{ and } f(-\alpha, -1, \rho) \rightarrow \rho/2 + U\alpha$$

$$f(-\alpha, \beta, \rho) \text{ and } f(\beta, -\alpha, \rho) \rightarrow \alpha\sqrt{U/\pi} e^{-\beta^2 U} e^{-\rho/2 \cdot \text{sgn}(\beta) + U\alpha|\beta|}$$

$$f(-\alpha, -\beta, \rho) \text{ and } f(-\beta, -\alpha, \rho) \rightarrow \alpha\sqrt{U/\pi} e^{-\beta^2 U} e^{+\rho/2 \cdot \text{sgn}(\beta) + U\alpha|\beta|}$$

$$f(-\beta, -1, \rho) \rightarrow \sqrt{U/\pi} e^{-\beta^2 U} e^{+\rho/2 \cdot \text{sgn}(\beta) + U|\beta|}$$

$$f(1, -\beta, \rho) \rightarrow \sqrt{U/\pi} e^{-\beta^2 U} e^{-\rho/2 \cdot \text{sgn}(\beta) + U|\beta|}$$

$$f_U(1, 1) \rightarrow U$$

$$f_U(1, -\beta) \rightarrow \sqrt{U/\pi} e^{-\beta^2 U} e^{+U|\beta|}$$

$$f_U(-\beta, -\beta) \rightarrow 1/\pi e^{-2\beta^2 U} e^{+2\sqrt{U/\pi}|\beta|} e^{-\beta^2 U} e^{+U|\beta|^2}$$

$$f_U(1, -\alpha) \rightarrow U\alpha$$

$$f_U(-\beta, \alpha) \rightarrow \sqrt{U/\pi} \alpha e^{-\beta^2 U} e^{+U\alpha|\beta|}$$

$$f_U(-\alpha, \alpha) \rightarrow U\alpha^2$$

where $\alpha > 0$ so that $\alpha = |\alpha|$. Then substituting in (6.1.20) we find after some algebra that

$$\begin{aligned} S_0(0, \phi) \rightarrow & \frac{1}{2} - \left(\frac{1}{2} - \Delta\right) \cdot \text{sgn}(\beta) + \frac{1}{4} U(\alpha - |\beta|)^2 - \frac{1}{2} \sqrt{U/\pi} e^{-\beta^2 U} (\alpha + |\beta|) \\ & - \frac{1}{4} \pi e^{-2\beta^2 U} \quad 0 \leq \phi \leq 1/2 \end{aligned} \quad (6B.3)$$

Some values of interest are

$$S_0(0,0) \approx \Delta$$

$$S_0(0,\Delta/2) \approx \Delta(1+\Delta U) - (1-\Delta)\sqrt{U/\pi} e^{-\beta^2 U} - 1/4\pi \cdot e^{-2\beta^2 U}$$

$$S_0(0,1/2) \approx 1-\Delta$$

with $\beta = 1-2\Delta$. Table 6.12 compares $S_0(0,\phi)$ as given by the exact equation (6.1.20) and the high SNR approximate equation (6B.3).

TABLE 6.12
Exact and Asymptotic Values of Normalized Noise
Power Spectrum

Δ	U	$S_0(0,0)$		$S_0(0,\Delta/2)$		$S_0(0,1/2)$	
		Exact	Asymp.	Exact	Asymp.	Exact	Asymp.
1/2	10	.480	.427	2.028	2.03	.480	.427
1/2	20	.499	.492	4.159	4.161	.499	.492
1/4	20	.25	.25	1.499	1.498	.616	.564
1/4	40	.25	.25		2.75		.677

APPENDIX 6C

AUTOCORRELATION FUNCTION AND POWER SPECTRUM FOR THE SAMPLE-AND-HOLD CIRCUIT OUTPUT

Let the output of the sample and hold circuit be $x(t)$ as shown in figure 6.18 for a particular case where the error ϕ and delay θ are 0. The autocorrelation function and power spectrum to be derived here are functions of ϕ , but the dependence on ϕ appears in the pulse amplitudes e_k so that in the derivation the waveform $x(t)$ can be taken as shown. The wave $x(t)$ is split into 2 parts as shown so that

$$x(t) = y(t) + z(t)$$

and
$$R_{xx}(\tau) = R_{yy}(\tau) + R_{yz}(\tau) + R_{zy}(\tau) + R_{zz}(\tau)$$

where
$$R_{xx}(\tau) = E\{x(t+\tau)x(t)\} \text{ etc.}$$

The function $R_{yy}(\tau)$ is computed by replacing each pulse in $y(t)$ by pulses of height $\overline{e_k e_{k+m}}$ and width $(T-\tau)$ for $0 \leq \tau \leq T$ as shown for $m=0$ in figure 6.19, and then letting

$$R_{yy}(\tau) = \frac{T-\tau}{2T} \cdot \overline{e_k e_{k+m}}.$$

This procedure is equivalent to assuming a uniformly distributed random time shift in the members of the ensemble. It should be noted that this is in addition to the timing error ϕ which is assumed fixed for all members of the ensemble presently under consideration. This method is explained in more detail in Lee [7].

FIGURE 6.18 ILLUSTRATION PERTAINING TO SAMPLE-AND-HOLD
CIRCUIT OUTPUT AUTOCORRELATION FUNCTION
ANALYSIS

For $R_{yy}(\tau)$ and $R_{zz}(\tau)$ we get

$$\begin{aligned} \frac{1}{2T} [T-\tau] \overline{e_k^2} &= \frac{1}{2} [1-\tau/T] \overline{e_k^2} & 0 \leq \tau \leq T \\ \frac{1}{2T} [\tau-T] \overline{e_k e_{k+2}} &= \begin{cases} \frac{1}{2} [\tau/T-1] \overline{e_k^2} & T \leq \tau \leq 2T \\ \frac{1}{2} [3-\tau/T] \overline{e_k^2} & 2T \leq \tau \leq 3T \\ \frac{1}{2} [\tau/T-3] \overline{e_k^2} & 3T \leq \tau \leq 4T \\ \vdots & \end{cases} \end{aligned}$$

and for $R_{yz}(\tau)$ and $R_{zy}(\tau)$

$$\begin{aligned} \frac{1}{2T} [\tau] \overline{e_k e_{k+1}} &= \frac{1}{2} [\tau/T] \overline{e_k e_{k+1}} & 0 \leq \tau \leq T \\ \frac{1}{2T} [2T-\tau] \overline{e_k e_{k+1}} &= \frac{1}{2} [2-\tau/T] \overline{e_k e_{k+1}} & T \leq \tau \leq 2T \\ \frac{1}{2T} [\tau-2T] \overline{e_k e_{k+2}} &= \frac{1}{2} [\tau/T-2] \overline{e_k^2} & 2T \leq \tau \leq 3T \\ \frac{1}{2T} [4T-\tau] \overline{e_k e_{k+2}} &= \frac{1}{2} [4-\tau/T] \overline{e_k^2} & 3T \leq \tau \leq 4T \end{aligned}$$

so that

$$R_{xx}(\tau) = \begin{cases} \overline{e_k^2} - (\overline{e_k^2} - \overline{e_k e_{k+1}}) |\tau|/T & 0 \leq |\tau| \leq T \\ \overline{e_k e_{k+1}} + (\overline{e_k e_{k+1}} - \overline{e_k^2}) (1 - |\tau|/T) & T \leq |\tau| \leq 2T \\ \overline{e_k^2} & |\tau| \geq 2T. \end{cases}$$

But the noise is $x(t) - E[x(t)] = x_T(t)$

so that

$$\begin{aligned}
 R_{x_R x_R}(\tau) &= R_{xx}(\tau) - \bar{e}_k^2 \\
 &= \begin{cases} (1-|\tau|/T) \bar{e}_k^2 + \bar{e}_k \bar{e}_{k+1} \cdot |\tau|/T - \bar{e}_k^2 & 0 \leq |\tau| \leq T \\ (\bar{e}_k \bar{e}_{k+1} - \bar{e}_k^2) (2-|\tau|/T) & T \leq |\tau| \leq 2T \\ 0 & |\tau| \geq 2T \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 S(0, \phi) &= \int_{-\infty}^{\infty} R_{x_R x_R}(\tau) d\tau \\
 &= T[(\bar{e}_k^2 - \bar{e}_k^2) + 2(\bar{e}_k \bar{e}_{k+1} - \bar{e}_k^2)] \\
 &= T[D(0, \phi) + 2 \cdot D(1, \phi)]
 \end{aligned}$$

FIGURE 6.19 SAMPLE-AND-HOLD CIRCUIT OUTPUT NOISE AUTOCORRELATION FUNCTION

APPENDIX 6D

Derivation of (6.6.6):

For $\Delta/2 \leq \phi_k \leq 1-\Delta/2$ from equation (8B.1) we have

$$\begin{aligned}
 G_1 A T \cdot f_{k-1}(\text{data}) &= |s_{k-1,2}| - |s_{k-1,1}| \\
 f_{k-1}(\text{data}) &= |(\phi_{k-2} - \Delta/2) d_{k-2} + d_{k-1}(1 + \Delta/2 - \phi_{k-1})| \\
 &\quad - |d_{k-2}(\phi_{k-2} + \Delta/2) + d_{k-1}(1 - \Delta/2 - \phi_{k-1})| \\
 &= \begin{cases} 2\Delta & \text{if } \phi_{k-2} + \phi_{k-1} \leq 1 - \Delta \\ 2(1 - \phi_{k-1} - \phi_{k-2}) & \text{if } 1 - \Delta \leq \phi_{k-2} + \phi_{k-1} \leq 1 + \Delta \\ -2\Delta & \text{if } 1 + \Delta \leq \phi_{k-2} + \phi_{k-1} \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 A T \cdot f_{k-2}(\text{data}) &= |s_{k-2,2}| - |s_{k-2,1}| \\
 f_{k-2}(\text{data}) &= |d_{k-3}(\phi_{k-3} - \Delta/2) + d_{k-2}(1 + \Delta/2 - \phi_{k-2})| \\
 &\quad - |d_{k-3}(\phi_{k-3} + \Delta/2) + d_{k-2}(1 - \Delta/2 - \phi_{k-2})| \\
 &= \begin{cases} 2\Delta & \text{if } \phi_{k-3} + \phi_{k-2} \leq 1 - \Delta \\ 2(1 - \phi_{k-2} - \phi_{k-3}) & \text{if } 1 - \Delta \leq \phi_{k-3} + \phi_{k-2} \leq 1 + \Delta \\ -2\Delta & \text{if } 1 + \Delta \leq \phi_{k-3} + \phi_{k-2} \end{cases}
 \end{aligned}$$

where $d_{k-3} = 1$ has been assumed with a square wave data signal so that

$d_j = -d_{j+1}$ for all j . Now if $\Delta = 1/2$ then $1 - \Delta = 1/2$, $1 + \Delta = 3/2$,

$1/4 \leq \phi_k \leq 3/4$, so that $1 - \Delta \leq \phi_j + \phi_{j-1} \leq 1 + \Delta$ always. So

$$[f_{k-1}(\text{data}) + f_{k-2}(\text{data})] = 2(2 - \phi_{k-1} - 2\phi_{k-2} - \phi_{k-3})$$

and substituting this in (6.6.5) we get

$$\phi_k \approx \phi_{k-1}(1+1/\delta_s) + (2\phi_{k-2} + \phi_{k-3})/\delta_s - 2/\delta_s$$

Derivation of (6.6.7):

For $-\Delta/2 \leq \phi_k \leq \Delta/2$ from equation (8B.1) we proceed as above and find that

$$\begin{aligned} f_{k-1}(\text{data}) &= |d_{k-1}(1-\Delta/2+\phi_{k-2}) + d_k(\Delta/2-\phi_{k-1})| \\ &\quad - |(\Delta/2+\phi_{k-2})d_{k-2} + d_{k-1}(1-\Delta/2-\phi_{k-1})| \\ &= 2(\phi_{k-1} + \phi_{k-2}) \end{aligned}$$

$$\begin{aligned} f_{k-2}(\text{data}) &= |d_{k-2}(1-\Delta/2+\phi_{k-3}) + d_{k-1}(\Delta/2-\phi_{k-2})| \\ &\quad - |(\Delta/2+\phi_{k-3})d_{k-3} + d_{k-2}(1-\Delta/2-\phi_{k-2})| \\ &= 2(\phi_{k-2} + \phi_{k-3}) \end{aligned}$$

so that
$$\phi_k \approx \phi_{k-1}(1+1/\delta_s) - (2\phi_{k-2} + \phi_{k-3})/\delta_s$$

Derivation of (6.6.8):

For $-1+\Delta/2 \leq \phi_k \leq -\Delta/2$ we find from (8B.1) that

$$\begin{aligned} f_{k-1}(\text{data}) &= |d_{k-1}(1-\Delta/2+\phi_{k-2}) + d_k(\Delta/2-\phi_{k-1})| \\ &\quad - |d_{k-1}(1+\Delta/2+\phi_{k-2}) - d_k(\phi_{k-1}+\Delta/2)| \\ &= -2(1+\phi_{k-1} + \phi_{k-2}) \end{aligned}$$

$$\begin{aligned} f_{k-2}(\text{data}) &= |d_{k-2}(1-\Delta/2+\phi_{k-3}) + d_{k-1}(\Delta/2-\phi_{k-2})| \\ &\quad - |d_{k-2}(1+\Delta/2+\phi_{k-3}) - d_{k-1}(\phi_{k-2}+\Delta/2)| \\ &= -2(1+\phi_{k-2} + \phi_{k-3}) \end{aligned}$$

so that
$$\phi_k \approx \phi_{k-1}(1-1/\delta_s) + (2\phi_{k-2} + \phi_{k-3})/\delta_s + 2/\delta_s$$

APPENDIX 6E POINTS OF EMERGENCE

Equation (6.6.7):

In this case (6.6.9) becomes

$$\beta^3 - (1 - 1/\delta_s)\beta^2 + (2/\delta_s)\beta + 1/\delta_s = 0 \quad (6E.1)$$

or
$$\delta_s(\beta - 1)\beta^2 + (\beta + 1)^2 = 0. \quad (6E.2)$$

From these 2 equations we can say that

- a) for any $\delta_s > 0$ there is a root in the range $[-1, 0]$ from (6E.2) and figure 6.20
- b) for $\delta_s = \infty$ from (6E.1) we have a double root at $\beta = 0$ and a single root at $\beta = 1$
- c) for $\delta_s = 0$ from (6E.2) we have a double root at $\beta = -1$.

Thus we can always factor a real root d , say, out of (6E.1) and write

$$(\beta + d)[\beta^2 - (1 - 1/\delta_s + d)\beta + [2/\delta_s + d(1 - 1/\delta_s + d)]] = 0. \quad (6E.3)$$

The root locus can now be plotted manually as follows:

1. choose a value of $d \in [0, 1]$ which gives 1 root at $\beta = -d$
2. calculate the corresponding δ_s from (6E.2) with $\beta = -d$
3. calculate the other 2 roots as the roots of the quadratic term in (6E.3).

The points of emergence can be found by noting that at these points we have 2 equal real roots so that from the quadratic term of (6E.3)

FIGURE 6.20 GRAPH OF TERMS IN EQUATION (6E.2)

$$(1-1/\delta_s+d)^2-4[2/\delta_s+d(1-1/\delta_s+d)] = 0$$

or

$$\delta_s = \frac{(5-d) \pm 2(6-2d+d^2)^{1/2}}{1-2d-3d^2}.$$

At the same time (6E.2) must be true with $\beta=-d$, so that d is given by the solutions of

$$\frac{(1-d)^2}{d^2(1+d)} = \frac{(5-d) \pm 2(6-2d+d^2)^{1/2}}{1-2d-3d^2}$$

in the range $[0,1]$. Let these values be d_1 and d_2 . The points of emergence then are

$$(1-1/\delta_s+d_1) \quad \text{where} \quad \delta_s = \frac{(1-d_1)^2}{d_1^2(1+d_1)}$$

$$(1-1/\delta_s+d_2) \quad \text{where} \quad \delta_s = \frac{(1-d_2)^2}{d_2^2(1+d_2)}.$$

The points of crossing the unit circle centered at the origin can be found from the fact that then the square magnitudes of the quadratic roots equal 1. Then in a similar manner we find that we must solve

$$\frac{(1-d)^2}{d^2(1+d)} - \frac{2-d}{1-d-d^2} = 0$$

in the range $[0,1]$.

Equation (6.6.7):

In this case (6.6.9) becomes

$$\beta^3 - (1+1/\delta_s)\beta^2 - (2/\delta_s)\beta - 1/\delta_s = 0 \quad (6E.4)$$

or

$$\delta_s(\beta-1)\beta^2-(\beta+1)^2 = 0 \quad (6E.5)$$

For these 2 equations we can say that

- a) from Descartes Rule of Signs and (6E.4) there is 1 real positive root d , say, for all positive δ_s . In addition, from (6E.5) it can be seen that $d>1$ if $\delta_s>0$ because only then is there a negative term and a positive term which can cancel,
- b) for $\delta_s=\infty$ from (6E.4) we have a double root at $\beta=0$ and a single root at $\beta=1$,
- c) for $\delta_s=0$ from (6E.5) we have a double root at $\beta=-1$.

Thus we can always factor a real root out of (6E.4) and write

$$(\beta-d) [\beta^2-(1+1/\delta_s-d)\beta-[2/\delta_s-d(1+1/\delta_s-1)]] = 0 \quad (6E.6)$$

The root locus can then be plotted as follows:

1. choose a value of $d>1$ which gives 1 root at $\beta=d$
2. calculate the corresponding δ_s from (6E.5) with $\beta=d$
3. calculate the other 2 roots as the roots of the quadratic term in (6E.6).

APPENDIX 6F

DIFFERENTIAL EQUATION OF OPERATION OF THE LOOP

In this appendix the differential equation which describes the operation of the loop of figure 6.2 is derived. From figures 6.2 and 6.4 we can write

$$\hat{T}_{k+1} = \hat{T}_k + \phi_k T - \phi_{k+1} T$$

or
$$T(\phi_{k+1} - \phi_k) = T - (\hat{T}_{k+1} - \hat{T}_k). \quad (6F.1)$$

But the timing error ϕ_k can be regarded as a sample value at time $\hat{T}_k$ of a continuous process $\phi(t)$, i.e., $\phi_k = \phi(\hat{T}_k)$, and if $\phi(t)$ is slowly varying then to a good approximation

$$\frac{\phi_{k+1} - \phi_k}{\hat{T}_{k+1} - \hat{T}_k} = \left. \frac{d\phi}{dt} \right|_{t=\hat{T}_k}. \quad (6F.2)$$

Now let $W(t)$ be the number of timing estimates per second,

i.e.
$$W(t) = \frac{1}{T} - k_{VCO} e(t)$$

where $e(t)$ is the input signal to the VCO. In addition,

$$W(\hat{T}_k) = \frac{1}{\hat{T}_{k+1} - \hat{T}_k}.$$

Thus
$$\frac{T}{\hat{T}_{k+1} - \hat{T}_k} - 1 = T \cdot W(\hat{T}_k) - 1 = -T k_{VCO} e(\hat{T}_k) \quad (6F.3)$$

and from (6F.1)-(6F.3)

$$\frac{d\phi}{dt} = -k_{VCO}e(t) = -k_{VCO}F(s) \cdot g(\phi)$$

since the equations are true for all $\hat{T}_k$ and all processes are slowly varying.

APPENDIX 6G

Proof that $g_0(\phi) = -g_0(1-\phi)$:

For $\Delta/2 \leq \phi \leq 1-\Delta/2$, $g_0(\phi)$ as given by (6.1.5) can be written

$$g_0(\phi) = - \frac{e^{-\Delta^2 U}}{2\sqrt{\pi U}} \sinh[2\Delta U(1-2\phi)] e^{-U(1-2\phi)^2} \\ + \frac{1}{4} (1+\Delta-2\phi) \cdot \text{erf}[(1+\Delta-2\phi)\sqrt{U}] - \frac{1}{4} (1-\Delta-2\phi) \cdot \text{erf}[(1-\Delta-2\phi)\sqrt{U}] \quad (6G.1)$$

If we replace ϕ by $1-\phi$ on the right hand side then $1-2\phi$ becomes $2\phi-1$ and because $\text{erf}(x) = -\text{erf}(-x)$ the right hand side becomes

$$\frac{e^{-\Delta^2 U}}{2\sqrt{\pi U}} \sinh[2\Delta U(1-2\phi)] e^{-U(1-2\phi)^2} \\ + \frac{1}{4} (1-\Delta-2\phi) \cdot \text{erf}[(1-\Delta-2\phi)\sqrt{U}] - \frac{1}{4} (1+\Delta-2\phi) \cdot \text{erf}[(1+\Delta-2\phi)\sqrt{U}]$$

which is the negative of the right hand side of (6G.1).

For $-\Delta/2 \leq \phi \leq \Delta/2$, $g_0(\phi)$ can be written

$$g_0(\phi) = - \frac{e^{-(1-\Delta)^2 U}}{2\sqrt{\pi U}} \sinh[4\phi(1-\Delta)U] e^{-4\phi^2 U} \\ + \frac{1}{4} (1-\Delta+2\phi) \cdot \text{erf}[(1-\Delta+2\phi)\sqrt{U}] - \frac{1}{4} (1-\Delta-2\phi) \cdot \text{erf}[(1-\Delta-2\phi)\sqrt{U}] \quad (6G.2)$$

If we replace ϕ by $-\phi$ on the right hand side, then the right hand side becomes

$$\frac{e^{-(1-\Delta)^2 U}}{2\sqrt{\pi U}} \sinh[4\phi(1-\Delta)U] e^{-4\phi^2 U}$$

$$- \frac{1}{4} (1-\Delta+2\phi) \cdot \operatorname{erf}[(1-\Delta+2\phi)\sqrt{U}] + \frac{1}{4} (1-\Delta-2\phi) \cdot \operatorname{erf}[(1-\Delta-2\phi)\sqrt{U}]$$

which is the negative of the right hand side of (6G.2). Since $g_0(\phi)$ for $-\Delta/2 \leq \phi \leq 0$ is the same as $g_0(\phi)$ for $1-\Delta/2 \leq \phi \leq 1$, then it has been shown that

$$g_0(\phi) = -g_0(1-\phi) \quad 0 \leq \phi \leq 1.$$

Proof that $g_0(1/2-\phi) = g_0(\phi)$, $0 \leq \phi \leq 1/2$, $\Delta=1/2$:

It must be shown that $g_0(\phi)$ for $0 \leq \phi \leq 1/4$ is the same as $g_0(1/2-\phi)$ for $0 \leq \phi \leq 1/4$. Over the former range $g_0(\phi)$ is given by (6G.2) and let this be $g_{01}(\phi)$; over the latter range $g_0(\phi)$ is given by (6G.1) and let this be $g_{02}(\phi)$. Now if $\Delta=1/2$

$$g_{01}(\phi) = - \frac{e^{-U/4}}{2\sqrt{\pi U}} e^{-4\phi^2 U} \sinh[2\phi U] + \frac{1}{4} (1/2+2\phi) \cdot \operatorname{erf}[(1/2+2\phi)\sqrt{U}]$$

$$- \frac{1}{4} (1/2-2\phi) \cdot \operatorname{erf}[(1/2-2\phi)\sqrt{U}];$$

(6G.3)

and

$$g_{02}(\phi) = - \frac{e^{-U/4}}{2\sqrt{\pi U}} e^{-U(1-2\phi)^2} \sinh[U(1-2\phi)] + \frac{1}{4} (3/2-2\phi) \cdot \operatorname{erf}[(3/2-2\phi)\sqrt{U}]$$

$$- \frac{1}{4} (1/2-2\phi) \cdot \operatorname{erf}[(1/2-2\phi)\sqrt{U}].$$

(6G.4)

Replace ϕ by $1/2-\phi$ in the right hand side of (6G.4) so that $1-2\phi$ becomes 2ϕ , $3/2-2\phi$ becomes $1/2+2\phi$, and $1/2-2\phi$ becomes $-1/2+2\phi$, and it is identical to (6G.3) because $\operatorname{erf}(-x) = -\operatorname{erf}(x)$.

Proof that $S_0(0, \phi+1/2) = S_0(0, \phi)$, $0 \leq \phi \leq 1/2$, $\Delta=1/2$:

We take (6.1.20) and prove the necessary symmetry term by term.

$\alpha^2 + \beta^2$:

For $-\Delta/2 \leq \phi \leq \Delta/2$, $\alpha^2 + \beta^2 = 2[(1-\Delta)^2 + 4\phi^2]$ which has even symmetry about $\phi=0$.

For $\Delta/2 \leq \phi \leq 1-\Delta/2$, $\alpha^2 + \beta^2 = 2[1 + \Delta^2 - 4\phi + 4\phi^2]$ which has even symmetry about $\phi=1/2$. With $\Delta=1/2$ the appropriate equality follows.

$h_2(\alpha) + h_2(\beta)$:

For $-\Delta/2 \leq \phi \leq \Delta/2$, since $\alpha = 1/2 + 2\phi$ and $\beta = 1/2 - 2\phi$, it follows from the definition of $h_2(\beta)$ following (6.1.10) that $h_2(\beta)$ and $h_2(\alpha)$ are mirror images of each other about the $\phi=0$ axis and hence $h_2(\alpha) + h_2(\beta)$ has even symmetry about $\phi=0$.

For $\Delta/2 \leq \phi \leq 1-\Delta/2$ similar considerations lead to even symmetry of $h_2(\alpha) + h_2(\beta)$ about $\phi=1/2$.

$h_2(\beta, \alpha)$:

Proof of even symmetries about $\phi=0$ and $\phi=1/2$ follows from the facts that $f_U(1, -\alpha)$ and $f_U(1, -\beta)$ are mirror images about $\phi=0$ or $\phi=1/2$ and that $f_U(-\beta, \alpha)$ also has even symmetry about $\phi=0$ and $\phi=1/2$.

$[h_0(\beta, \alpha, 1/2)/2 + h_1(\alpha, -\beta, 1/2)]$ and $[h_0(\alpha, -\beta, 1/2)/2 + h_1(\beta, \alpha, 1/2)]$:

After a great deal of algebra it may be shown that these terms possess the necessary symmetry.

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CHAPTER 7

EXPERIMENTAL RESULTS

7.0 EXPERIMENTAL CIRCUIT

The absolute value device of section 6.1 was set up on the EAI 580 hybrid computer using a Wavetek 115 oscillator as the voltage controlled oscillator. The timing pulses were obtained from a Hewlett Packard 8005A Pulse Generator which was driven by the Wavetek 115. A block diagram of the experimental synchronizer is shown in figure 7.1. Some features which may be of interest are:

1. The integrate-and-dump circuits were built using the fact that the integrators can be controlled by external logic signals at the IC and OP control terminals. Table 7.1 gives the integrator state for various logic signal combinations, and integration occurs if $OP=5v$ and $IC=0v$, while the integrator is reset to the initial condition if $OP=5v$ and $IC=5v$. An initial condition of $0v$ causes the dump operation, and it was found that the integrators could be reset in 0.2 msec.
2. The integrator outputs had to be sampled just before the integrators were reset to zero and $400\text{ }\mu\text{sec.}$ delay lines were used to delay the reset logic pulses.
3. The absolute value nonlinearities were built by using an integration network and an operational amplifier wired as an electronic switch to choose the integrator output if it was positive and minus the integrator output if it was negative. The complement output of a comparator which compared an integrator output to ground was used to determine

TABLE 7.1

Integrator Mode as a Function of IC and
OP Logic Voltages

IC in Volts	OP in Volts	Mode
0	5	OP
5	0	IC
0	0	HOLD
5	5	IC

whether the integrator output was positive or negative.

4. The Wavetek 115 was used as a voltage controlled oscillator because it had a good linear operating curve as shown in figure 7.2, and because the frequency could be continuously varied by a dial. This latter feature was convenient because the data generator only operated at certain fixed bit rates.
5. The data signal was a random wave of rectangular pulses generated by sampling the output of a noise generator at periodic intervals and making the pulse sign + or - according to the sample sign. This was done in the Hewlett Packard 3722A Signal Generator.

Table 7.2 gives some data on the equipment used in the experiment. The bit rate of 100 b.p.s. was selected as a compromise between several factors such as

- a) low bit rate so that the noise bandwidth would be much greater than the bit rate and the noise would approximate the white noise assumption used in the analysis of chapter 6,
- b) low bit rate so that the integrator dump time would be a negligible fraction of the bit length,
- c) low bit rate to avoid problems with stray capacitance etc,
- d) high bit rate for easier implementation of low pass filters in the measurement circuits,
- e) high bit rate for ease of viewing on an oscilloscope.

The jitter variance was measured by the circuitry of figure 7.3. Briefly, the idea is to generate a sawtooth wave in synchronism with

TABLE 7.2
Experimental Equipment Data

Bit Rate	- 100 b.p.s.
Bit Amplitude	- .1v
1390B Gaussian Noise Generator B/W	- 3 dB down at approx. 5 Hz and 40 kHz
	- 5 mV/Hz r.m.s. voltage spectral density
Timing Pulse Width	- .2 msec.
EAI Unit Gain Invertor B/W	> 125 kHz
Integrator Gain	- 100
Sample-and-hold gain	- 10
Square-law Gain	- .108

FIGURE 7.2 WAVETEK 115 VCG FREQUENCY VS VOLTAGE CHARACTERISTIC

the data oscillator clock, and then to sample this with the jittered timing estimate pulses. The sampler output is then a stream of jittered pulses which are also amplitude modulated in proportion to the jitter. A measurement of the jitter variance can then be made, (as shown in Appendix 7A), by passing the sampler output pulse stream through a square law device followed by a low pass filter.

This method of measuring jitter variance has two advantages over a method which forms a jitter histogram and computes the variance, namely

1. simplicity of circuitry,
2. any equipment problems which occur during an experimental run show up as a variation on the meter at the output of the jitter variance measuring circuit. A histogram, on the other hand, might not indicate any problem, and even if it did indicate a problem this would not be known until the histogram was examined. Thus the circuit used here saved time in the experimental work from this point of view.

Some disadvantages were

1. length of time to take a measurement. The long time constant of the low pass filters made it necessary to wait 5 minutes or more at each variance measurement for the reading to settle,
2. some delay in operation of relays and a clock signal with less than 100% duty cycle limited the range over which jitter could be measured,

3. dynamic range of the samples could also have limited the larger jitter measurements.

7.1 EXPERIMENTAL PROCEDURE

In this section the calculation and setting of various parameters are discussed. The synchronizer parameters such as loop gain, VCO frequency, signal amplitude, were set before the synchronizer was actually turned on. When it was turned on it achieved lock.

The signal amplitude was measured on an oscilloscope at the output of potentiometer P₂₀, while the bit rate was determined by the data signal generator panel switch setting and confirmed by the oscilloscope. The VCO frequency was adjusted so that there was no detectable drift between it and the data generator clock when both were viewed on a dual beam oscilloscope.

From the theory of chapter 6, equation (6.1.26),

$$\delta_s = 1/(G_1 G_2 A T^2 k_{VCO})$$

and if we let

G_1 = (integrator gain) x (sample-and-hold gain)

G_2 = potentiometer P₀₀ value,

then the required setting of G_2 is known for any desired δ_s value since A, T, and k_{VCO} are known and G_1 is known from handbook data. The required value of P₀₀ was set with the potentiometer in the circuit using the digital voltmeter to measure the setting.

The value of E/N₀ was varied by changing the value of potentiometer P₀₂ while keeping the data signal and the noise generator output

voltage constant. The value of E/N_0 for any setting of P_{02} can be calculated as follows. Let R be the input resistance of the synchronizer, so that the data pulse energy at the input is A^2T/R . Let the mean square noise voltage across R in a one-sided bandwidth B be v_n^2 , and let the one-sided noise power spectral density be N_0 watts/Hz. Then $N_0B = v_n^2/R$ and $E/N_0 = A^2TB/v_n^2$. But $v_n^2 = P_{02}^2 V_n^2$ if V_n^2 is the noise generator mean square voltage output and P_{02} is the setting of potentiometer number 02. Thus

$$E/N_0 = A^2TB/(P_{02}^2 V_n^2).$$

From the handbook for the GR1390B noise generator $V_n^2/B = 25 \times 10^{-6}$ so that for $A = .1$, $T = 10^{-2}$, we have

$$E/N_0 = 4/P_{02}^2.$$

The gain of the square law device in the variance measuring circuit was measured by passing a triangle wave through the device and measuring the peak input and output values on an oscilloscope.

The ramp generator pot settings were determined by trial and error so that a large ramp slope was obtained without overloading amplifiers. The final settings were such that the ramp crossed zero volts at the theoretical timing point, i.e., $T/4$ from a data bit transition time. The sample pulse shape was then measured on an oscilloscope.

7.2 EXPERIMENTAL RESULTS

Figure 7.4 shows the measured and theoretical variances for the absolute value device with $\Delta = 1/2$ and $\delta_s = 5, 25, 50$. Each experimental

FIGURE 7.4 EXPERIMENTAL JITTER VARIANCE,
ABSOLUTE VALUE DEVICE WITH $\Delta=1/2$

point is the average of 8 readings taken over a period of time as discussed above. The theoretical and experimental results are in good agreement for $\delta_s = 25$ and 50 , but for $\delta_s = 5$ they are not. The same type of agreement will be shown in chapter 8 for digital computer simulation of this device.

Some typical timing error sample functions are shown in figure 7.5, and it is evident that those for $\delta_s = 25$ and 50 are much smoother than those for $\delta_s = 5$. It is thought that the theoretical assumption of constant timing error used in the analysis is not sufficiently good at $\delta_s = 5$. At $\delta_s = 25$ or 50 the assumption is better because of the much smoother timing error sample functions.

The points labelled " $E/N_0 = \infty$ " are the readings when the input noise was zero. This jitter is due to internal noise in the synchronizer.

7.3 CONCLUSION

It has been shown that the absolute value device can be built, and the theoretical results of chapter 6 have been verified for loop parameters of interest. The experimental system also showed that the internal noise of a synchronizer limits its performance as the input signal to noise ratio becomes large.

FIGURE 7.5 EXPERIMENTAL TIMING ERROR SAMPLE FUNCTIONS

APPENDIX 7A MEASUREMENT CIRCUIT OUTPUTS

The variance measuring circuit can be represented as in figure 7.6. Let $s(t)$ be a pulse of width Δ and height 1 centered at the origin as shown in figure 7.6, so that if e_j is the jitter at time jT , then

$$x(t) = \sum_j m(t-jT) s(t-jT-e_j).$$

Let $h(t) = Ae^{-at} U(t)$, where $U(t)$ is the unit step function, and consider an ensemble of waves similar to $x(t)$ but having uniformly distributed random time shift so that the ensemble is stationary. Then

$$E\{z(t)\} = b E\{x^2(t)\} \int_0^{\infty} Ae^{-at} dt = \frac{bA}{a} E\{x^2(t)\}.$$

Referring to figure 7.7, $E\{x^2(t)\}$ can be found as follows:

$$\text{Pulse area} = \int_{e_j - \Delta/2}^{e_j + \Delta/2} m^2 t^2 dt = m^2 \Delta (e_j^2 + \Delta^2/12)$$

so that the average area is $m^2 \Delta (\overline{e_j^2} + \Delta^2/12)$

$$\therefore \overline{x^2(t)} = \frac{\Delta m^2}{T} (\overline{e_j^2} + \Delta^2/12)$$

and

$$\overline{z(t)} = \frac{\Lambda \Delta b m^2}{aT} [\overline{e_j^2} + \Delta^2/12].$$

In addition we have from work below that $\overline{e_j}$ is given by

FIGURE 7.6 ILLUSTRATIONS FOR THE ANALYSIS OF JITTER VARIANCE

FIGURE 7.7 ILLUSTRATION FOR THE CALCULATION OF $E\{x^2(t)\}$

$$\overline{e_j} = \frac{aT}{Am\Delta} \overline{y(t)}$$

where $y(t)$ is the output of the low pass filter when it is driven by $x(t)$. Thus

$$\sigma^2 = E\{(e_j - \overline{e_j})^2\} = \frac{aT}{A\Delta b m^2} \overline{z(t)} - \frac{\Delta^2}{12} - \left(\frac{aT}{Am\Delta}\right)^2 \overline{y(t)^2}$$

The values of a and A are given by

$$a = \frac{1}{R_f C_f} \quad A = \frac{1}{R_i C_f}$$

from the low pass filter realization shown in figure 7.3.

Now for the average value circuit as shown in figure 7.3, we proceed as above so that

$$E\{y(t)\} = \frac{A}{a} E\{x(t)\}$$

where

$$\begin{aligned} E\{x(t)\} &= \frac{1}{T} E\{\text{pulse area}\} \\ &= \frac{1}{T} [m(\overline{e_j} - \Delta/2)\Delta + m(\Delta)\Delta/2] \\ &= \frac{m\Delta}{T} \overline{e_j}. \end{aligned}$$

$$\therefore \quad \overline{e_j} = \frac{aT}{Am\Delta} \overline{y(t)}$$

CHAPTER 8

DIGITAL COMPUTER SIMULATION

8.0 PRELIMINARY ANALYSIS

It was shown in section 6.1 that the two integrators in the synchronizer phase detector operate with a fixed time shift ΔT seconds where $1/T$ is the bit rate. In operation, the VCO frequency is varied from the bit rate by the error signal, but the delay ΔT remains constant. For the first order device with sample-and-hold before the VCO, the VCO frequency changes at the instant an error impulse is formed and remains constant until the next error impulse. Figure 8.1 illustrates the timing operation where the $\hat{T}_j$ are the timing estimates and are assumed to fall in the center of the Δ intervals as shown.

It is assumed that the device is turned on at $t = -\Delta T/2$ and so $\hat{T}_0 = 0$. $\phi T = \phi_0 T$ is the time from $\hat{T}_0$ to the first data transition point. Then it is clear from figure 8.1 that

$$\phi_k T = kT + \phi T - \hat{T}_k. \quad (8.0.1)$$

In figure 8.1, the sine wave represents the VCO operation, which changes frequency in a step manner from ω_{k-1} to ω_k at the instant when e_{k-1} occurs. Then

$$\Delta T \omega_{k-1} + \omega_k [(\hat{T}_k - \Delta T/2) - (\hat{T}_{k-1} + \Delta T/2)] = 2\pi \quad (8.0.2)$$

where

$$\omega_j = \omega_0 \pm 2\pi k_{VCO} e_{j-1}$$

Using (8.0.1) in (8.0.2) it is shown in Appendix 8A that if $\Delta = 1/2$ then

FIGURE 8.1 WAVEFORMS IN THE ABSOLUTE VALUE DEVICE

$$\phi_k - \phi_{k-1} = \pm \pi k_{VCO} \frac{e_{k-1} + e_{k-2}}{\omega_0 \pm 2\pi k_{VCO} e_{k-1}} \quad (8.0.3)$$

where if the loop filter has a transfer function G_2 we can write

$$e_j = G_2 [F(s_{k,2}) - F(s_{k,1})]$$

where F represents the nonlinearity following the integrators. It is shown in Appendix 8B that in the case of $F = 1$ then

$$|s_{k,2}| - |s_{k,1}| = AG_1 T \cdot f_k(\text{data}) \quad (8.0.4)$$

where $f_k(\text{data})$ denotes "a function of data integrated over interval k ", and then (8.0.3) becomes

$$\phi_k = \phi_{k-1} \pm \frac{1}{2} \delta_s^{-1} \frac{f_{k-1}(\text{data}) + f_{k-2}(\text{data})}{1 \pm \delta_s^{-1} f_{k-1}(\text{data})} \quad (8.0.5)$$

This equation can then be used to calculate device performance since it allows fast calculation of the timing errors ϕ_k . It can also be used as a starting point for transient analysis and some other topics which are pursued later.

Similarly, for the square law device where $F = (\cdot)^2$, then (8.0.3) becomes

$$\phi_k = \phi_{k-1} \pm \frac{1}{2} \delta_{ss}^{-1} \frac{z_{k-1}(\text{data}) + z_{k-2}(\text{data})}{1 \pm \delta_{ss}^{-1} z_{k-1}(\text{data})} \quad (8.0.6)$$

where $z_k(\text{data})$ is defined by

$$(s_{k,2})^2 - (s_{k,1})^2 = A^2 G_1^2 T^2 \cdot Z_k(\text{data}) \quad (8.0.7)$$

If the received signal $r(t)$ is a data wave plus additive white Gaussian noise, i.e.

$$r(t) = s(t) + n(t),$$

then we can write

$$s_{k,1} = AG_1 \int_{\hat{T}_{k-1}-\Delta T/2}^{\hat{T}_k-\Delta T/2} \text{sgn}\{s(t)\} dt + G_1 \int_{\hat{T}_{k-1}-\Delta T/2}^{\hat{T}_k-\Delta T/2} n(t) dt$$

$$s_{k,2} = AG_1 \int_{\hat{T}_{k-1}+\Delta T/2}^{\hat{T}_k+\Delta T/2} \text{sgn}\{s(t)\} dt + G_1 \int_{\hat{T}_{k-1}+\Delta T/2}^{\hat{T}_k+\Delta T/2} n(t) dt.$$

It is convenient to define

$$n_{k-1,1} = \int_{\hat{T}_{k-1}+\Delta T/2}^{\hat{T}_k+\Delta T/2} n(t) dt \quad n_{k-1,2} = \int_{\hat{T}_{k-1}+\Delta T/2}^{\hat{T}_k-\Delta T/2} n(t) dt$$

$$n_{k-1,3} = \int_{\hat{T}_k-\Delta T/2}^{\hat{T}_k+\Delta T/2} n(t) dt$$

which are independent zero mean Gaussian random variables. $n_{k-1,1}$ and $n_{k-1,3}$ have variances $\sigma_1^2 = N_0 \Delta T/2$ and $n_{k-1,2}$ has a variance

$$\sigma_2^2 = E\{n_{k-1,2} n_{k-1,2}\} = \frac{N_0}{2} E\{\hat{T}_k - \hat{T}_{k-1} - \Delta T\} = (1-\Delta) T N_0/2.$$

Then $s_{k,1}$ and $s_{k,2}$ are easily formed as the sum of a signal term and 2

independent Gaussian random variables. The signal term can be computed by equation (8B.1).

8.1 SIMULATION RESULTS

Both the absolute value and square law devices were simulated on a Sigma-7 digital computer using equations (8.0.5) and (8.0.6) above. Additive noise was simulated by generating the independent Gaussian random variables $n_{k-1,j}$ described above, and the data signal was generated by a pseudorandom number generator. Figure 8.2 shows a flow chart of the simulation program.

Since the devices are known to lock at a timing error of $T/2$ this was used as the unknown starting time error. The device was run for 500 data bits before computation of jitter mean and variance were begun to allow transients to die out. A further 10,000 bits were processed to compute jitter mean and variance. The variance results are shown in table 8.1 for the absolute value device and for the square law device with $\Delta = 1/2$. Agreement is good between theoretical and experimental results except for δ_s or $\delta_{ss} = 5$ as in chapter 7. Figure 8.3 compares the results of the experimental device, the simulation, and the theory for the absolute value device with $\delta_s = 5$ and $\Delta = 1/2$. The experimental and simulation results agree well, and the effect of internal noise in the experimental device can be seen at $E/N_0 \approx 100$ as the experimental points begin to curve away from the simulation curve. The simulation curve has the same slope as the theoretical curve, but is displaced by about 3 dB. That is, the theoretical curve states that a given jitter variance is achieved at a value of E/N_0 which is 3 dB lower than the value at which the simulation curve passes through that

FIGURE 8.2 SIMULATION PROGRAM FLOW CHART

TABLE 8.1

Simulation Results for Absolute Value and Square Law Devices

E/N ₀	Absolute Value Device			Square Law Device		
	δ_s	Theoretical Variance	Simulated Variance	δ_{ss}	Theoretical Variance	Simulated Variance
100	5	$.313 \times 10^{-3}$	$.610 \times 10^{-3}$	5	$.400 \times 10^{-3}$	$.860 \times 10^{-3}$
80		$.391 \times 10^{-3}$	$.762 \times 10^{-3}$		$.499 \times 10^{-3}$	$.108 \times 10^{-2}$
40		$.780 \times 10^{-3}$	$.153 \times 10^{-2}$		$.101 \times 10^{-2}$	$.222 \times 10^{-2}$
20		$.146 \times 10^{-2}$	$.306 \times 10^{-2}$		$.212 \times 10^{-2}$	$.450 \times 10^{-2}$
10		$.312 \times 10^{-2}$	$.623 \times 10^{-2}$			
100	25	$.521 \times 10^{-4}$	$.550 \times 10^{-4}$	25	$.657 \times 10^{-4}$	$.725 \times 10^{-4}$
80		$.651 \times 10^{-4}$	$.687 \times 10^{-4}$		$.823 \times 10^{-4}$	$.908 \times 10^{-4}$
40		$.130 \times 10^{-3}$	$.137 \times 10^{-3}$		$.166 \times 10^{-3}$	$.183 \times 10^{-3}$
20		$.253 \times 10^{-3}$	$.275 \times 10^{-3}$		$.340 \times 10^{-3}$	$.374 \times 10^{-3}$
10		$.512 \times 10^{-3}$	$.540 \times 10^{-3}$		$.708 \times 10^{-3}$	$.778 \times 10^{-3}$
8		$.633 \times 10^{-3}$	$.662 \times 10^{-3}$		$.902 \times 10^{-3}$	$.990 \times 10^{-3}$
4		$.122 \times 10^{-2}$	$.123 \times 10^{-2}$		$.198 \times 10^{-2}$	$.218 \times 10^{-2}$
2		$.239 \times 10^{-2}$	$.233 \times 10^{-2}$		$.476 \times 10^{-2}$	$.522 \times 10^{-2}$
1		$.521 \times 10^{-2}$	$.507 \times 10^{-2}$			
100	50	$.255 \times 10^{-4}$	$.257 \times 10^{-4}$	50	$.328 \times 10^{-4}$	$.328 \times 10^{-4}$
80		$.319 \times 10^{-4}$	$.322 \times 10^{-4}$		$.406 \times 10^{-4}$	$.411 \times 10^{-4}$
40		$.638 \times 10^{-4}$	$.644 \times 10^{-4}$		$.814 \times 10^{-4}$	$.829 \times 10^{-4}$
20		$.125 \times 10^{-3}$	$.129 \times 10^{-3}$		$.166 \times 10^{-3}$	$.169 \times 10^{-3}$
10		$.251 \times 10^{-3}$	$.254 \times 10^{-3}$		$.346 \times 10^{-3}$	$.352 \times 10^{-3}$
8		$.311 \times 10^{-3}$	$.313 \times 10^{-3}$		$.440 \times 10^{-3}$	$.441 \times 10^{-3}$
4		$.594 \times 10^{-3}$	$.586 \times 10^{-3}$		$.964 \times 10^{-3}$	$.955 \times 10^{-3}$
2		$.117 \times 10^{-2}$	$.112 \times 10^{-2}$		$.227 \times 10^{-2}$	$.223 \times 10^{-3}$
1		$.252 \times 10^{-2}$	$.239 \times 10^{-2}$		$.603 \times 10^{-2}$	$.594 \times 10^{-3}$
.8		$.332 \times 10^{-2}$	$.315 \times 10^{-2}$		$.886 \times 10^{-2}$	$.850 \times 10^{-3}$
.4		$.865 \times 10^{-2}$	$.849 \times 10^{-2}$

FIGURE 8.3 SIMULATED AND EXPERIMENTAL RESULTS FOR ABSOLUTE VALUE DEVICE, $\delta_s=5$, $\Delta=1/2$

jitter variance. As stated before in chapter 7, it is thought that the theoretical assumption of constant timing error does not hold at $\delta_s = 5$ and that this accounts for the 3 dB discrepancy.

8.2 CONCLUSION

The results of digital computer simulation of the absolute value and square law devices have been presented. The agreement between the theoretical results of chapter 6 and the simulation results is good for $\delta_s \geq 25$. For $\delta_s = 5$ the agreement is not good, but the simulation agrees well with the experimental results of chapter 7. This latter fact shows the failure of the theoretical assumption of constant timing error at $\delta_s = 5$.

APPENDIX 8A

From (8.0.2)

$$\Delta T \omega_{k-1} + \omega_k [\hat{T}_k - \hat{T}_{k-1} - \Delta T] = 2\pi \quad (8A.1)$$

and from (8.0.1)

$$\hat{T}_k - \hat{T}_{k-1} = T - \phi_k T + \phi_{k-1} T \quad (8A.2)$$

Substituting (8A.2) in (8A.1) we get

$$\begin{aligned} \phi_k - \phi_{k-1} &= \pm 2\pi k_{VCO} \frac{(1-\Delta) e_{k-1} + \Delta e_{k-2}}{\omega_k} \\ &= \pm \pi k_{VCO} \frac{e_{k-1} + e_{k-2}}{\omega_k} \end{aligned}$$

if $\Delta = 1/2$.

APPENDIX 8B
PROOF OF EQUATION (8.0.4)

By considering figure 8.1 it is clear that

$$G_1^{-1}s_{k,1} = \begin{cases} (\frac{\Delta}{2} + \phi_{k-1})Td_{k-1} + d_k [\hat{T}_k - \hat{T}_{k-1} - T(\frac{\Delta}{2} + \phi_{k-1})] & \text{for } \phi_{k-1} \geq \frac{\Delta}{2}, \phi_k \geq -\frac{\Delta}{2} \\ (\frac{\Delta}{2} + \phi_{k-1})Td_{k-1} + d_k T - d_{k+1}(\frac{\Delta}{2} + \phi_k)T & \text{for } \phi_{k-1} \geq -\frac{\Delta}{2}, \phi_k \leq -\frac{\Delta}{2} \\ d_k(\hat{T}_k - \hat{T}_{k-1}) & \text{for } \phi_{k-1} \leq -\frac{\Delta}{2}, \phi_k \geq -\frac{\Delta}{2} \\ d_k[T + (\phi_{k-1} + \frac{\Delta}{2})T] - d_{k+1}(\frac{\Delta}{2} + \phi_k)T & \text{for } \phi_{k-1} \leq -\frac{\Delta}{2}, \phi_k \leq -\frac{\Delta}{2} \end{cases}$$

$$G_1^{-1}s_{k,2} = \begin{cases} (\phi_{k-1} - \frac{\Delta}{2})Td_{k-1} + d_k[\hat{T}_k - \hat{T}_{k-1} - (\phi_{k-1} - \frac{\Delta}{2})T] & \text{for } \phi_{k-1} \geq \frac{\Delta}{2}, \phi_k \geq \frac{\Delta}{2} \\ (\phi_{k-1} - \frac{\Delta}{2})Td_{k-1} + d_k T + d_{k+1}(\frac{\Delta}{2} - \phi_k)T & \text{for } \phi_{k-1} \geq \frac{\Delta}{2}, \phi_k \leq \frac{\Delta}{2} \\ d_k(\hat{T}_k - \hat{T}_{k-1}) & \text{for } \phi_{k-1} \leq \frac{\Delta}{2}, \phi_k \geq \frac{\Delta}{2} \\ d_k[T - (\frac{\Delta}{2} - \phi_{k-1})T] + d_{k+1}(\frac{\Delta}{2} - \phi_k)T & \text{for } \phi_{k-1} \leq \frac{\Delta}{2}, \phi_k \leq \frac{\Delta}{2} \end{cases}$$

$$\text{for } -\frac{\Delta}{2} \leq \phi_k, \phi_{k-1} \leq 1 - \frac{\Delta}{2} \quad (8B.1)$$

and where

$$s_{k,1} = G_1 \int_{\hat{T}_{k-1} - \Delta T/2}^{\hat{T}_k - \Delta T/2} r(t) dt, \quad s_{k,2} = G_1 \int_{\hat{T}_{k-1} + \Delta T/2}^{\hat{T}_k + \Delta T/2} r(t) dt$$

$$d_j = \pm A.$$

Then, since $\hat{T}_k - \hat{T}_{k-1} = T - \phi_k T + \phi_{k-1} T$, it can be seen from the above that

AG_1T can be factored out of all expressions for $s_{k,1}$ and $s_{k,2}$. Thus we can write in general

$$|s_{k,2}| - |s_{k,1}| = AG_1T \cdot f_k(\text{data})$$

CHAPTER 9

EXTENSIONS AND COMPARISONS

9.0 INTRODUCTION

In Chapter 5 a synchronizer was derived for binary data signals with arbitrary pulse shapes received in additive white Gaussian noise. This device is shown in figure 5.2c. Chapters 6 to 8 have discussed this device when the pulse is rectangular, and figure 6.2 shows the device for this case. In this chapter the steady state performance of the general pulse shape device of figure 5.2c is derived, and this performance is compared with that of other devices.

Transients during initial acquisition were also analyzed in chapter 6. Transients during operation, such as changes in bit rate, are discussed in this chapter.

9.1 STEADY STATE ANALYSIS

The analysis follows that of section 6 and so only the modifications that extend the results of chapter 6 to the general pulse shape case will be given here.

Let the received signal be

$$r_s(t) = Af_s(t) + n(t)$$

where A is the data pulse peak amplitude,

$f_s(t)$ is the data pulse shape having unit peak amplitude and length T seconds,

$n(t)$ is additive white Gaussian noise with two-sided spectral density $N_0/2$ watts/Hz.

Let the unmodulated pulse shape generated by the synchronizer VCO be $f_L(t)$ having unit peak amplitude and length T seconds.

Now define the correlation functions

$$R_{ff_L}(u) \triangleq \int_0^{1-u} f(pT) \cdot f_L[(p+u)T] dp \quad (9.1.1)$$

$$R_{f_L f}(u) \triangleq \int_0^{1-u} f_L(pT) \cdot f[(p+u)T] dp \quad (9.1.2)$$

Then it can be shown that the phase detector characteristic $g(\phi)$ is given by

$$g(\phi) = \frac{1}{2} G_1 G_2 \sqrt{N_0' T / \pi} [e^{-\delta^2 U'} - e^{-\gamma^2 U'} + e^{-\alpha^2 U'} - e^{-\beta^2 U'}] \\ + \frac{1}{2} G_1 G_2 A T [\delta \operatorname{erf}(\delta \sqrt{U'}) - \gamma \operatorname{erf}(\gamma \sqrt{U'}) + \alpha \operatorname{erf}(\alpha \sqrt{U'}) - \beta \operatorname{erf}(\beta \sqrt{U'})] \quad (9.1.3)$$

$$\text{where } \gamma = R_{f_L f}(1-\phi-\Delta/2) + R_{ff_L}(\phi+\Delta/2) \quad -\Delta/2 \leq \phi \leq 1-\Delta/2 \\ \delta = \begin{cases} R_{f_L f}(\Delta/2-\phi) + R_{ff_L}(1+\phi-\Delta/2) & -\Delta/2 \leq \phi \leq \Delta/2 \\ R_{f_L f}(1-\phi+\Delta/2) + R_{ff_L}(\phi-\Delta/2) & \Delta/2 \leq \phi \leq 1-\Delta/2 \end{cases} \quad (9.1.4) \\ \alpha = \begin{cases} R_{f_L f}(\Delta/2-\phi) - R_{ff_L}(1+\phi-\Delta/2) & -\Delta/2 \leq \phi \leq \Delta/2 \\ -R_{f_L f}(1-\phi+\Delta/2) + R_{ff_L}(\phi-\Delta/2) & \Delta/2 \leq \phi \leq 1-\Delta/2 \end{cases} \\ \beta = -R_{f_L f}(1-\phi-\Delta/2) + R_{ff_L}(\phi+\Delta/2) \quad -\Delta/2 \leq \phi \leq 1-\Delta/2 \\ N_0' = R_{f_L f_L}(0) \cdot N_0 \\ U' = U / [R_{ff}(0) \cdot R_{f_L f_L}(0)].$$

Then at high SNR $g_0(\phi)$ is given by

$$g_0(\phi) = \frac{1}{4} \left[|\delta| - |\gamma| + |\alpha| - |\beta| \right]. \quad (9.1.5)$$

It can also be shown that $S_0(0, \phi)$ is given by

$$\begin{aligned} S_0(0, \phi) = & \frac{1}{2} U'[\alpha^2 + \beta^2 + \gamma^2 + \delta^2] + 1 - 3g_1^2(\phi) \\ & + \frac{1}{2} [h_2(\beta, \gamma) + h_2(\alpha, \delta) - h_3(\beta, \alpha, \gamma, \delta)] \\ & - \begin{cases} \frac{1}{2} h_0(\beta, \alpha, \gamma, \delta, \rho_1) + h_1(\alpha, -\beta, \gamma, \delta, \rho_2) & \text{if } -\Delta/2 \leq \phi \leq \Delta/2 \\ \frac{1}{2} h_0(-\alpha, \beta, \gamma, \delta, \rho_2) + h_1(\beta, \alpha, \gamma, \delta, \rho_1) & \text{if } \Delta/2 \leq \phi \leq 1 - \Delta/2 \end{cases} \end{aligned} \quad (9.1.6)$$

where

$$\begin{aligned} h_0(\alpha, \beta, \gamma, \delta, \rho) & \triangleq f(\gamma, \delta, \rho) + f(\gamma, \alpha, \rho) + f(-\beta, -\alpha, \rho) + f(-\beta, -\delta, \rho) \\ h_1(\alpha, \beta, \gamma, \delta, \rho) & \triangleq f(\gamma, \delta, \rho) + f(-\beta, -\alpha, \rho) \\ h_2(\alpha, \delta) & \triangleq f_U(\delta, \delta) + f_U(\delta, -\alpha) + f_U(-\alpha, -\alpha) + f_U(\alpha, -\gamma) \\ h_3(\alpha, \beta, \gamma, \delta) & \triangleq f_U(\gamma, \delta) + f_U(\gamma, -\alpha) + f_U(-\beta, \alpha) + f_U(-\beta, \delta) \\ \rho_1 & = R_{f_L f_L}(1 - \Delta) / R_{f_L f_L}(0) \\ \rho_2 & = R_{f_L f_L}(\Delta) / R_{f_L f_L}(0). \end{aligned} \quad (9.1.7)$$

The functions $f(\cdot, \cdot, \rho)$ and $f_U(\cdot, \cdot)$ were defined in equations (6.1.8) and (6.1.10).

Equations (9.1.1) to (9.1.7) enable one to analyze the steady state performance of the device of figure 5.2c for arbitrary pulse shapes when the local replica is either the same as or different from the received pulse shape. The analysis is valid for all SNR's.

At high SNR equation (9.1.6) becomes

$$S_0(0, \phi) = 1 - \rho_1 + \frac{U'}{4} (\alpha - \beta + \gamma - \delta)^2 + 3U'(\alpha - \beta)(\delta - \gamma). \quad (9.1.8)$$

9.2 High SNR Performance, Raised Cosine Pulses

Let

$$f_S(t) = f_L(t) = \begin{cases} \frac{1}{2} [1 - \cos(\omega_0 t)] & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

where $\omega_0 = 2\pi/T$.

Then it can be shown from (9.1.5) that

$$\begin{aligned} g_0(\phi) &= \frac{1}{16} \sin(2\pi\phi) \left[1 + (1-\Delta) \sin(\pi\Delta) - (3/\pi) \cos(\pi\Delta) \right] \\ &\quad + \frac{1}{2} \phi \left[1 + \frac{1}{2} \cos(\pi\Delta) \cos(2\pi\Delta) \right] \\ &\approx \phi(\pi/8) \left[1 + (4/\pi) + (1-\Delta) \sin(\pi\Delta) - (3/\pi) \cos(\pi\Delta) \right] \\ &\quad \text{for small } \phi, \end{aligned}$$

and from (9.1.8) that

$$S_0(0, \phi) = 1 - \frac{2}{3} (1-\Delta) \left[1 + \frac{1}{2} \cos(2\pi\Delta) \right] - \frac{1}{2\pi} \sin(2\pi\Delta)$$

for small ϕ .

Substituting these values in equation (6.1.19) for the timing error probability density function we find that it is Gaussian with zero mean and variance

$$\sigma^2 = \frac{1}{U\delta_S} \cdot \frac{3G}{4\pi^2 H^2} \quad (9.1.9)$$

Δ	Eq. (9.1.9)	Eq. (9.1.10) (Nengali)	Eq. (9.1.11) (Stiffler)
1/2	$\sigma^2 = \frac{1}{121} \cdot \frac{1}{U\delta_s}$	$\sigma^2 = \frac{1}{26.2} \cdot \frac{1}{U\delta_s}$	$\sigma^2 = \frac{1}{0.8} \cdot \frac{1}{U\delta_s}$
1/4	$\sigma^2 = \frac{1}{170} \cdot \frac{1}{U\delta_s}$	$\sigma^2 = \frac{1}{26.2} \cdot \frac{1}{U\delta_s}$	$\sigma^2 = \frac{1}{1.3} \cdot \frac{1}{U\delta_s}$

TABLE 9.1

Jitter Variance for Raised Cosine Pulses

where $G = 1 + (4/\pi) + (1-\Delta) \sin(\pi\Delta) - (3/\pi) \cos(\pi\Delta)$

$$H = 1 + 2\Delta - (1-\Delta) \cos(2\pi\Delta) - \frac{3}{2\pi} \sin(2\pi\Delta).$$

This result may be compared with the corresponding result for the device of figure 5.3 proposed by Mengali [1]

$$\sigma^2 = \frac{1}{U\delta_s} \cdot \frac{3}{8\pi^2} \quad (9.1.10)$$

and with the result for the device proposed by Stiffler [2]

$$\sigma^2 = \frac{1}{U\delta_s} \cdot \frac{C}{2} \quad (9.1.11)$$

where $C = 1.5$ for $\Delta \leq 1/4$ and increases to 2.5 as Δ increases to $1/2$.

These results are compared in table 9.1, and it is clear that the device proposed in this thesis has a better performance than the other two devices. This implies a lower probability of error when this device is used to derive timing for a matched filter receiver.

9.3 Operating Transients

It would be expected from the work of chapter 6 that the synchronizer of figure 6.2 would behave in the same way as a first order phase-locked loop provided that the timing error during the transient period does not exceed $T/2$ in magnitude. It is shown in this section that this is true. The assumptions are that the received signal is a periodic rectangular wave without any noise and that the synchronizer has locked.

Consider first a step change in signal phase as illustrated in figure 9.1. Then we can write

$$\hat{T}_k - \hat{T}_{k-1} = (1 + \rho - \phi_k + \phi_{k-1})T \quad (9.2.1)$$

and

$$\begin{aligned} \phi_k &= \phi_{k-1}(1 - 1/\delta_s) - (2\phi_{k-2} + \phi_{k-3})/\delta_s + \rho \\ \phi_j &= \phi_{j-1}(1 - 1/\delta_s) - (2\phi_{j-2} + \phi_{j-3})/\delta_s \quad j \neq k \end{aligned} \quad (9.2.2)$$

Thus there will be a damped sinusoidal transient and the synchronizer will return to lock with a steady state error of zero.

For a step change in bit rate, df , equation (9.2.1) becomes

$$\hat{T}_k - \hat{T}_{k-1} = (\phi_{k-1} - \phi_k + 1)(T = \delta T)$$

and (9.2.2) becomes

$$\phi_k = \phi_{k-1}(1 - 1/\delta_s) - (2\phi_{k-2} + \phi_{k-3})/\delta_s - (1 - \Delta)\delta/(1 + \delta)$$

which has a steady state solution

$$\begin{aligned} \phi_{ss} &= - \frac{\delta_s(1 - \Delta)}{4} \frac{\delta}{1 + \delta} \\ &= - \frac{df}{4W_{Ls}} (1 - \Delta). \end{aligned}$$

These results are very similar to the behaviour of a phase-locked loop when step changes of carrier phase or frequency occur [2]. The transients in both cases will be damped sinusoids decaying to the steady state tracking error.

(a) STEP PHASE CHANGE IN K^{th} INTERVAL

(b) STEP CHANGE IN BIT RATE IN K^{th} INTERVAL

FIGURE 9.1 TRANSIENT ANALYSIS NOTATION

The device will lose lock if the step change in frequency, df , is such that

$$df > \Delta/\pi\delta_s.$$

This is because the VCO frequency can be written as

$$\omega_k = \omega_0 \left[1 \pm \frac{1}{\delta_s} f_{k-1}(\text{data}) \right]$$

for the k^{th} interval, and $|f_{k-1}(\text{data})| \leq 2\Delta$. Thus the step frequency change must be within the limit $\Delta/\pi\delta_s$ for the loop to maintain lock.

In view of the similarity of the transient behaviour to that of a first order phase-locked loop, it may be expected that this device will lock if the data signal bit rate is not the same as the synchronizer VCO frequency at initial acquisition.

9.4 Conclusion

In this chapter the phase detector characteristic and the loop noise power spectrum have been presented for the case of arbitrary signal and synchronizer pulse shapes. This allows computation of the jitter variance under a wide variety of situations, and results at high SNR for raised cosine pulses were presented. It was shown that the device of this thesis has a lower jitter variance than some other devices which were derived from the same MAP approach.

It was also shown that the device will maintain lock under step changes in data signal phase and bit rate.

References

1. Mengali, U., *A Self Bit Synchronizer Matched to Signal Shape*, I.E.E.E. Trans. on Aerospace and Elect. Sys., AES-7, No. 4, July 1971, pp. 686-693.
2. Gardner, F.M., *Phaselock Techniques*, Wiley, New York, 1966.

CHAPTER 10

CONCLUSION AND AREAS OF FURTHER STUDY

9.0 CONCLUSION

This thesis is a study of several aspects of the bit synchronization problem in digital communications systems. The aspects studied are:

1. sources of timing errors and their statistics,
2. the effect of timing errors on the performance of matched filter receivers,
3. the derivation and implementation of a class of self-bit synchronizers which are close to optimum,
4. an analysis of the steady state and transient performance of these synchronizers and a comparison with transmitted reference schemes. The steady state performance was also measured by simulation and by building a device.

Items 2-4 above are the original contributions of this thesis, while item 1 is mostly a compilation of existing information.

A number of physical mechanisms in the communication channel and in the bit synchronizer were discussed. It was shown that bit rate jitter may have Gaussian and sinusoidal components, and that step phase changes may occur in a channel resulting in loss of data bits. Component tolerances within the synchronizer and channel give rise to unknown fixed timing errors and noise within the synchronizer causes random timing errors.

The performance of a matched filter receiver for binary signals

was examined, using the assumption that a synchronizer was providing timing estimates whose timing error probability density function was either that of the phase of a sine wave in Gaussian noise or that of the phase error of a first order phase locked loop. It was found that the receiver probability of error departed from the perfectly timed matched filter curve if the SNR at the synchronizer input was independent of the data channel SNR. A probability of error very close to that of the perfectly timed matched filter can be obtained if a transmitted reference is used for synchronization and if its power is about 10% of the data signal power. If in addition the transmitter power is limited, then this is also the optimum power division for transmitter energy to noise ratios of 15 or less in the system considered.

The analysis of the matched filter showed that nearly perfect performance can be achieved with a simple timing system. It also provided a basis for comparison of the self bit synchronizers.

A class of self bit synchronizers was then derived. The first step was to derive the a posteriori density of the unknown time shift in a data signal which is received in additive Gaussian noise with an unknown fixed time shift. Various practical assumptions were made to arrive at reasonable devices which would approximate the MAP estimate of the unknown time shift. These devices were phase locked loops with a nonlinearity in the loop, and the cases where the nonlinearity was an absolute value or a square law were examined in detail.

The Fokker-Planck technique was used to obtain an analytical expression for the timing jitter density, and numerical integration

was used to calculate the jitter variance and probability of error of a matched filter receiver whose timing was derived from one of these devices. It was found that for $\delta_s = 25$ the probability of error was very close to that of a perfectly timed matched filter receiver. A transmitted reference system having the same PLL linear loop bandwidth gave rise to almost the same probability of error as the self bit synchronizers. Thus their relative merits would have to be judged using some other criteria such as loss of lock, acquisition time, cost, etc.

The absolute value device was built on an analog computer and its timing jitter variance was measured. The results were in good agreement with the theoretical results except for the case $\delta_s = 5$. In this latter case it was concluded that the assumptions underlying the theoretical analysis were not valid. The same results were obtained from a digital computer simulation of both the absolute value and square law devices.

In conclusion, it can be said that the devices proposed in this thesis can provide nearly ideal theoretical error probability when used to provide timing estimates for matched filters. Their performance is close to that of sine wave transmitted reference systems with the same linear loop bandwidth. The synchronizers are easy to implement since they are essentially a phase locked loop with nonlinearity in the loop, and the data signal consists of standard rectangular non-return-to-zero pulses. Thus the questions raised in section 1.2 concerning the performance and complexity of self bit synchronizers have been answered for the data signal considered.

9.1 AREAS OF FURTHER STUDY

A number of topics can be studied further. Some are:

1. synchronizers for waveshapes other than rectangular. It is expected that the use of return-to-zero pulse shapes would improve performance, because every T seconds a zero value would occur regardless of the data sequence. Implementation of these would be somewhat more complicated.
2. derivation of a class of synchronizers for situations other than the constant unknown time shift considered here. This could be approached by considering the data pulses to be phase modulated by the channel and deriving a device to estimate the phase modulation.
3. study of the transient response of the synchronizers presented in this thesis under other conditions such as change of bit rate, step changes in bit phase, noisy signals, etc.
4. signal design to minimize probability of error for a matched filter with timing error. A class of synchronizers such as the low-pass filter followed by a nonlinearity and bandpass filter could be studied.