An initial value problem for vortex Rossby waves on a beta-plane

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Abstract

In this thesis, an initial value problem is presented and we consider the problem analytically. We define a vortex with a circular motion in a configuration for an inviscid fluid flow on a horizontal plane under the influence of the Coriolis force. We consider a wave generated by a specified initial condition which gives an initial value problem. We present a barotropic vorticity equation describing the fluid flow under the influence of the Coriolis force. We also present an initial value problem that comprises the barotropic vorticity equation in an annular region defined in terms of polar coordinates. We consider a background vortex flow, initially as a constant and then with a specified angular velocity that depends on the radial distance from the center of the domain.

We obtain a nonlinear equation which we solve by looking for a perturbation series solution. This equation can be written in terms of several small parameters. We consider the relative magnitude of the parameters and different profiles for the background flow angular velocity, including a case where it is constant and one where it varies with respect to the radial variable. The nonlinear wave interactions give rise to a wave-induced mean flow component, which modifies the background vortex flow, as well as a higher wavenumber component. We also find additional terms with a wavenumber of one which represents the beta-gyre effect.

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Chapter 1

Introduction

Waves occur naturally in the Earth's atmosphere. Atmospheric waves that are generated by the effects of the rotation waves are known as Rossby waves (named after Carl-Gustaf Rossby, the meteorologist who first identified them in 1939). They are generally observed as large-scale (or planetary-scale) oscillations in the jet streams, and they have profound effects on weather and climate. On a smaller scale, there are Rossby waves that occur within cyclonic vortices in the Earth's atmosphere, as a result of the rotation of the vortex and also influenced by the rotation of the earth. They are known as vortex Rossby waves.

A vortex is a region in a fluid in which the flow travels around to form a circle. Vortices can be observed as smoke rings or a whirlpool. There are different types of cyclonic vortices in the Earth's atmosphere. These include tropical cyclones which are typically generated over the tropical oceans. Hurricanes and typhoons are typical tropical cyclones. It is a rapidly rotating storm system, recognized by a donut-like shape with a low-pressure center and extremely strong winds. The earliest recorded tropical cyclones are estimated to have occurred around 6000 BC [1]. The cyclone is usually generated on the ocean, occasionally going to the land and destroys everything in front of its way. A recent example is a tropical

storm named Elsa, located east southeast of Barbados and observed to be moving towards the west-northwest. People keep tracking the movement of the storm to prevent any damage it may cause. Thus, it is very necessary to study cyclones, to learn their behaviour so that we could predict the cyclone before it comes, to prepare everything we could to defend against the catastrophe.

When the winds speed is extremely great, the eyewall replacement cycles may occur. Fortner [2] observed the phenomenon in 1956 and documented it. Some of the outer rainbands may be strengthened and form into a ring of storms and strong winds, looking like an outer eyewall of the cyclone. In the process, the vortex Rossby waves have been observed within tropical cyclones. They take the form of concentric rings of rainbands that propagate outward from the eye at the center of the cyclone. It has been suggested that vortex Rossby waves may play a role in the eyewall replacement cycle firstly by MacDonald [3].

Vortex Rossby waves have been studied mathematically, using two-dimensional horizontal configurations [4] or three-dimensional configurations that take into account vertical variations of the properties of the atmosphere [5]. The Planetary Rossby waves arise from the effects of the Coriolis force, like vorticity of the Earth, occurring in the atmosphere and ocean, but vortex Rossby waves describe the radial gradient of the vorticity of the vortex itself in the cyclone. After decades of research and investigation, for example, [6, 7], [8 - 11] illustrate that the influence of inertia-gravity waves is less significant compared with vortex Rossby waves.

In the past 20 years, the theory of the vortex Rossby waves was being developed and discussed in a significant amount. Montgomery and Schecter [8] continued the exact solution discussed by Montgomery and Kallenbach [12], who firstly integrated the linearized equation on an f -plane, and considered the case of a symmetric vortex decreasing monotonically with the radius. They investigate the condition that the core's potential vorticity of a tropical vortex being

vertically aligned and horizontally axisymmetric, with the case of a critical radius where the mean angular velocity of the vortex equals the phase speed of the vortex waves. They concluded that the amplitude of the core vortex Rossby waves could decay exponentially with a decay rate constant in a small time. The constant can be measured and it depends on the gradient of the potential vorticity at the critical radius. On the other hand, Brunet and Montgomery [13] discussed on linear initial value problem, in the condition of a smooth circular vortex of finite depth.

There is a critical radius within the tropical cyclone during the formation of the secondary eyewall. Hence, a wave mean-flow interaction might happen in such a critical layer. Montgomery [12] deduced that the interaction between the mean-flow and the vortex waves might be the reason to accelerate the speed of the angular wind in the vortex and introduced vortex wave dynamics.

More recently, Campbell and Nikitina [14] developed a two-dimensional barotropic model that represents Rossby waves propagation in a cyclonic vortex starting from a nonlinear equation to a linear equation. Their next step [15] was to include the nonlinearity and the effect of the rotation of the Earth, which is known as the beta-effect, in the model equations.

In these papers, the goal was to study the behaviour of the vortex Rossby waves in the vicinity of the vortex of a critical radius, where the angular velocity of the vortex flow is equal to the phase speed of the wave. In this configuration, the waves are generated at the inner boundary of an annular region on a horizontal plane described in terms of polar coordinates. At the initial time, the wave vorticity is zero and the oscillatory inner boundary condition generates waves with wavenumber k , a positive integer that propagates radially outwards and encounters a critical radius. The effects of the nonlinear wave interactions and the effects of the rotation of the Earth were investigated by Campbell and Nikitina

[15]. The nonlinearity gives rise to a wave mode of wavenumbers $2k$ and a mean flow term with wavenumber zero that grows linearly with the time t . The beta-effect gives rise to wave modes with wavenumber $(k + 1)$ and $(k - 1)$.

The balance between the effects of nonlinearity and rotation gives rise to an additional effect that is commonly called the beta drift or beta-gyre effect which causes an additional spiral motion. The influence of it has been investigated by previous researchers.

Suytrin and Flierl [16] studied a piecewise constant vorticity profile with an initially axisymmetric vortex. They defined a polar coordinate frame translating with the movement of the center of the vortex. This allowed them to investigate the beta-drift analytically but the solution procedure was dependent on the piecewise constant assumption of the vorticity. Their work showed that the beta gyres consist of two parts which are generated by advection of the planetary vorticity and the different displacements of vorticity contours. Reznik and Dewar [17] considered more general initial vortex profiles in order to obtain a more complete solution analytically for large values of the radial variable and the time variable. They looked at the evolution of vortices that are isolated under the perspective of two-dimensional flows. Montgomery et al. [18] used a model based on the shallow water equations where vertical variations are taken into account over a shallow depth of the fluid. They derived linear and nonlinear equations and carried out numerical simulations to show that the movement of the vortex resulting from the beta-gyre effect is always attained in a finite depth fluid. However, the long-time behaviour is not well modelled in their equation. Finally, Llewellyn-Smith [19] carried out an asymptotic analysis to find a solution in powers of a small parameter defined by the beta-effect. He first studied the general angular vorticity profiles and then considered a case with constant angular vorticity. Then he used the non-isolated vortices and obtained the $\mathcal{O}(1)$ solution.

In Campbell and Nikitina's work [15], the formulation does not include the term that generates the beta-gyre. The location of the vortex is fixed and the vortex waves are forced at the fixed inner boundary of the annular domain. No beta-gyre effect could be observed.

In this thesis, we carry out an investigation of this effect by studying a two-dimensional configuration on a horizontal plane. The domain is an annular region described in terms of polar coordinates, whereas the waves are generated by a perturbation imposed at the initial time, rather than by an inner boundary condition. This gives an initial value problem and allows us to investigate the possibility of the beta-gyre effect, as well as the effects of nonlinearity. We choose the background vortex angular velocity and the wave phase speed to ensure not having the critical radius. We consider different profiles for the background flow angular velocity, a case where it is constant and one where it varies with respect to the radial variable. The nonlinear wave interactions give rise to a wave-induced mean flow component. We also find a beta-gyre term with a wavenumber of one. In our formulation, we are able to obtain exact expressions for these terms. However, in such case, the nonlinear wave interactions give rise to a wave-induced mean flow component, which modifies the background vortex flow, as well as a higher wavenumber component.

The following represents an overview of the thesis. In chapter 2, we describe the beta-plane approximation in which the effect of the Coriolis force is approximated by a linear variation with latitude. Under this approximation, we present the barotropic vorticity equation, which is a single partial differential equation describing a barotropic (two-dimensional horizontal) fluid flow under the influence of the Coriolis force. We also present an initial value problem comprising the barotropic vorticity equation in an annular region defined in terms of polar coordinates. There is a background vortex flow with a specified angular velocity

that depends on the radial distance from the center of the domain. A perturbation to the wave vorticity is specified as an initial condition. Such flow generated a wave that propagates forward in time.

In chapters 3 and 4, we examine this initial-boundary value problem by writing it in terms of several small parameters: β the parameter that represents the Coriolis force (the effect of the rotation of the Earth), ε which measures the vortex wave amplitude and the extent of nonlinearity of the equations, γ which is the ratio $\frac{\beta}{\varepsilon}$, α which measures the size of the vortex and μ which measures the size of the initial vortex wave forcing region within the vortex. The ratio $\gamma = \frac{\beta}{\varepsilon}$ multiplies the term known as the beta-gyre or beta-drift term [16 - 21].

In chapter 3, we consider the case where the angular velocity of the background vortex is constant and then in chapter 4, we consider a more realistic background vortex angular velocity profile. In each case, we find a solution for the wave vorticity and streamfunction as a perturbation series in terms of the relevant small parameters.

In chapter 5, some concluding remarks and suggestions for further work are given.

Finally, the basic equations of fluid dynamics are presented. These equations are based on the laws of conservation of mass, momentum and energy and are derived in Appendix A. The concepts of the vorticity and streamfunction of a fluid flow are defined. Mathematical descriptions of the Coriolis force and centripetal force are given. Also, the details of definitions and charts are given in Appendix A.

Chapter 2

Vortex Rossby wave equations and configuration

In this chapter, we introduce an initial value problem to model a wave perturbation on a background vortex fluid flow in the form of a vortex. This problem describes vortex Rossby waves in a cyclonic vortex in the atmosphere. We consider a 2-D fluid flow on a horizontal plane governed by equations (A.18) and (A.24) that are derived in Appendix A. We first derive an expression for the Coriolis force $\vec{F}_c$ in expression (A.15) and then consider a special case that leads to the "beta-plane" approximation. Then we write the equations in terms of the streamfunction and vorticity.

2.1 The Coriolis Force

To explain Coriolis force, we consider the atmosphere or ocean on a rotating sphere (our Earth). The depth scale of a flow H (from the ground to the upper stratosphere is about $50km$) is much less than the longitudinal scale L which is of several thousand kilometers. Hence, we have $\frac{H}{L} \ll 1$. The scale of the horizontal

components of the fluid velocity is U and the scale of the vertical component is W .

A three-dimensional region in atmosphere may be defined in terms of Cartesian coordinates (x, y, z) with x and y being the horizontal coordinates and z the vertical coordinate. The fluid flow in this region may be described by the continuity equation (A.12), with velocity vector $\vec{u} = (u, v, w)$, where u , v and w are the velocity components in the x , y and z directions, respectively. The continuity equation can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where the scale of the variables x and y is L , the scale of z is H , the scale of u and v is U , and the scale of w is W . This means that the first two terms, $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are of scale $\frac{U}{L}$ and the third term is of scale $\frac{W}{H}$. As $\frac{U}{L}$ and $\frac{W}{H}$ must balance, we have that $\frac{W}{U} \sim \frac{H}{L} \ll 1$. Hence, the vertical velocity is much less than the horizontal velocity components.

To study large-scale flows, we need to use spherical coordinates. However, if the horizontal length scale L is much less than the radius of the Earth, that is, 6371Km, we can ignore the curvature of the Earth, and use a local Cartesian system on a tangent plane.

The earth rotates at a rate of $\Omega = \frac{2\pi \text{rad}}{\text{day}} = 0.73 \times 10^{-4} \text{s}^{-1}$ around the polar axis. We can use local Cartesian coordinate system (x, y, z) , where x is to the East describing the longitude, y is to the north describing the latitude and z is going upwards to describe the height altitude. We also use (u, v, w) to describe the velocity in the Cartesian system.

The components of the Earth's angular velocity $\vec{\Omega}$ are $(0, |\vec{\Omega}| \cos \theta, |\vec{\Omega}| \sin \theta) = (0, \Omega \cos \theta, \Omega \sin \theta)$ where $\Omega = |\vec{\Omega}|$. Then we represent the Coriolis force in terms of the angular velocity. Consider a frame of reference (x_1, x_2, x_3) , rotating at a centri-form angular velocity Ω with respect to a fixed frame (X_1, X_2, X_3) . In

Figure 2.1: Schematic diagram of the rotation of the Earth with a Cartesian coordinate system

our case, $(x_1, x_2, x_3) = (x, y, z)$, the local coordinates and (X_1, X_2, X_3) is a fixed extended frame.

Let $\vec{P}$ be a vector in the rotating frame, we have $\vec{P} = P_1\vec{i}_1 + P_2\vec{i}_2 + P_3\vec{i}_3$, where $\vec{i}_i$ is the unit vector in the rotating frame. To a fixed observer in the fixed frame, the unit vector $\vec{i}_i$ changes with time. As to this observer, we have:

$$\begin{aligned} \left(\frac{dP}{dt}\right)_{fixed} &= \frac{d}{dt}(P_1\vec{i}_1 + P_2\vec{i}_2 + P_3\vec{i}_3) \\ &= \vec{i}_1\frac{dP_1}{dt} + \vec{i}_2\frac{dP_2}{dt} + \vec{i}_3\frac{dP_3}{dt} + P_1\frac{d\vec{i}_1}{dt} + P_2\frac{d\vec{i}_2}{dt} + P_3\frac{d\vec{i}_3}{dt} \end{aligned} \quad (2.1)$$

Figure 2.2: Schematic diagram of the rotation of the Earth with angular velocity Ω at latitude θ

To an observer in the rotating frame, the equation becomes:

$$\left(\frac{d\vec{P}}{dt}\right)_{\text{rotating}} = \vec{i}_1 \frac{dP_1}{dt} + \vec{i}_2 \frac{dP_2}{dt} + \vec{i}_3 \frac{dP_3}{dt} \quad (2.2)$$

Hence, we have:

$$\left(\frac{d\vec{P}}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{P}}{dt}\right)_{\text{rotating}} + P_1 \frac{d\vec{i}_1}{dt} + P_2 \frac{d\vec{i}_2}{dt} + P_3 \frac{d\vec{i}_3}{dt} \quad (2.3)$$

Now we need to define the rates of change $\frac{d\vec{i}}{dt}$ in terms of $\vec{\Omega}$.

Figure 2.3: Representation of a vector $\vec{P} = \vec{\Omega}$ in a fixed frame (X_1, X_2, X_3) and a rotating frame (x_1, x_2, x_3)

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{g} - 2\vec{\Omega} \times \vec{u}$$

where $\vec{\Omega}$ is the angular velocity of the Earth and hence, we have the Coriolis acceleration to be $2\vec{\Omega} \times \vec{u}$.

Observe that, in time dt , each unit vector $\vec{i}$ traces a cone with radius $\sin \alpha$ where α is the constant angle between $\vec{i}$ and Ω as the diagram shows.

Suppose in time dt , the tip of the unit vector travels an angle of $d\theta$ along the circumference of the base of the cone. If the unit vector is $\vec{i}$ at the initial time t , and $\vec{i} + d\vec{i}$ at time $t + dt$, then the magnitude of $d\vec{i}$ is the arclength of the cone, $r d\theta$.

Hence, we conclude that

$$d\vec{i} = \sin \alpha d\theta \implies \frac{d\vec{i}}{dt} = \sin \alpha \frac{d\theta}{dt} \quad (2.4)$$

However, notice that $\frac{d\theta}{dt}$ = the angular velocity rate of change of the angle with time = $|\vec{\Omega}|$. Hence, we conclude

$$\frac{d\vec{i}}{dt} = |\vec{\Omega}| \sin \alpha = |\vec{\Omega}| |\vec{i}| \sin \alpha \quad (2.5)$$

where α is the angle between $\vec{\Omega}$ and $\vec{i}$.

So the magnitude of the rate of change of $\vec{i}$ is $|\vec{\Omega}| |\vec{i}| \sin \alpha$ and its direction is perpendicular to the plane containing $\vec{\Omega}$ and $\vec{i}$. Recall the cross product (in the Appendix A), we have

$$\frac{d\vec{i}_j}{dt} = \vec{\Omega} \times \vec{i}_j \text{ for each } j = 1, 2, 3 \quad (2.6)$$

So we can plug (2.6) to our previous equation (2.3) to get

$$\begin{aligned} \left(\frac{d\vec{P}}{dt} \right)_{fixed} &= \left(\frac{d\vec{P}}{dt} \right)_{rotating} + P_1 \frac{d\vec{i}_1}{dt} + P_2 \frac{d\vec{i}_2}{dt} + P_3 \frac{d\vec{i}_3}{dt} \\ &= \left(\frac{d\vec{P}}{dt} \right)_{rotating} + P_1 \vec{\Omega} \times \vec{i}_1 + P_2 \vec{\Omega} \times \vec{i}_2 + P_3 \vec{\Omega} \times \vec{i}_3 \\ &= \left(\frac{d\vec{P}}{dt} \right)_{rotating} + \vec{\Omega} \times \vec{P} \end{aligned} \quad (2.7)$$

We are interested in the velocity vector $\vec{u} = \frac{d\vec{r}}{dt}$. Set $\vec{P} = \vec{r}$, the position vector of a point in the rotating frame. Then we have

$$\vec{u}_{fixed} = \vec{u}_{rotating} + \vec{\Omega} \times \vec{r} \quad (2.8)$$

Now if we differentiate with respect to the time, we attain the acceleration $\left(\frac{d\vec{u}}{dt} \right)_{fixed}$:

$$\left(\frac{d\vec{u}}{dt}\right)_{fixed} = \left(\frac{d\vec{u}}{dt}\right)_{rotating} + \vec{\Omega} \times \vec{u}_{fixed} \quad (2.9)$$

where

$$\begin{aligned} \left(\frac{d\vec{u}}{dt}\right)_{rotating} &= \frac{d}{dt} \left(\vec{i}_1 \frac{dP_1}{dt} + \vec{i}_2 \frac{dP_2}{dt} + \vec{i}_3 \frac{dP_3}{dt} \right) \\ &= \vec{\Omega} \times \vec{u}_{rotating} + \left(\frac{d^2 \vec{P}}{dt^2} \right) \end{aligned} \quad (2.10)$$

Letting $\left(\frac{d\vec{u}}{dt}\right)_{fixed} = \vec{a}_F$, $\vec{u}_{rotating} = \vec{u}_R$ and $\left(\frac{d^2 \vec{P}}{dt^2}\right)_{rotating} = \vec{a}_R$ we substitute (2.8) into (2.9) to obtain:

$$\begin{aligned} \vec{a}_F &= \left(\frac{d\vec{u}}{dt}\right)_r + \vec{\Omega} \times (\vec{u}_{rotating} + \vec{\Omega} \times \vec{r}) \\ &= \vec{a}_F + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \end{aligned} \quad (2.11)$$

The true originated acceleration, i.e., the acceleration in the fixed frame equals to the acceleration in the rotating frame plus the Coriolis acceleration and the centripetal acceleration, where the Coriolis acceleration $2\vec{\Omega} \times \vec{u}_R$ acts perpendicular to the plane containing $\vec{\Omega}$ and $\vec{u}_R$.

Let $\vec{R}$ be the vector perpendicular to the axis of rotation, as the diagram shows, then we have

$$\begin{aligned} \vec{\Omega} \times \vec{r} &= |\vec{\Omega}| |\vec{r}| \sin \gamma \\ &= |\vec{\Omega}| |\vec{R}| \cdot 1 \\ &= |\vec{\Omega}| |\vec{R}| \sin 90 \text{ deg} = \vec{\Omega} \times \vec{R} \end{aligned} \quad (2.12)$$

Hence, we conclude that

Figure 2.4: Centripetal Acceleration

$$\begin{aligned}
 \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) &= \vec{\Omega} \times (\vec{\Omega} \times \vec{R}) \\
 &= (\vec{\Omega} \cdot \vec{R})\vec{\Omega} - (\vec{\Omega} \cdot \vec{\Omega})\vec{R} \\
 &= |\vec{\Omega}|^2 \vec{R} \\
 &\text{as } (\vec{\Omega} \cdot \vec{R}) = |\vec{\Omega}||\vec{R}| \cos 90^\circ = 0
 \end{aligned} \tag{2.13}$$

So the centripetal acceleration acts vertically.

In the equation of motion, we can replace $\rho \left(\frac{D\vec{u}}{Dt} \right)_{fixed}$ by $\rho(\vec{a}_F + 2\vec{\Omega} \times \vec{u}_R - |\vec{\Omega}|^2 \vec{r})$. The multiplication of ρ brings us the force per unit volume.

In general, the vertical component of the Coriolis force is negligible compared with the dominant terms in equation (A.15) the vertical equation of motion. Hence, we can approximate the Coriolis force by

$$(-2v\Omega \sin \theta, 2u\Omega \sin \theta, 0) \tag{2.14}$$

The quantity $f = 2\Omega \sin \theta$ is called the planetary vorticity or the Coriolis parameter.

Hence, the Coriolis force can be written as $(-fv, fu, 0)$ and equation (A.15) becomes

$$\begin{aligned} \frac{D\vec{u}}{Dt} &= -\frac{1}{\rho}\nabla p + v\nabla^2\vec{u} - 2\vec{\Omega} \times \vec{u} + \vec{g} \\ \Rightarrow \left\{ \begin{array}{l} \frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv \\ \frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu \\ \frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho} \end{array} \right. \end{aligned} \quad (2.15)$$

2.2 Barotropic fluid flow

The beta-plane approximation

Comparing with the horizontal scale of flow of about thousands of kilometers, the atmosphere and the ocean are very thin layers of a few kilometers as noted in section 2.1. The trajectories of fluid elements are nearly horizontal when their vertical velocities are much smaller.

In this thesis, we further consider the situation where the density of the fluid is a function of pressure only. Such a fluid flow is said to be barotropic. For a barotropic fluid, surfaces of constant density are the same as surfaces of constant pressure. If we consider a background density that varies in the vertical direction $\bar{\rho}$, then the surfaces of constant density are horizontal planes. As a first approximation, we can consider a 2-D problem involving a horizontal plane defined in terms of Cartesian coordinates x and y with constant ρ . In that case, the momentum equation is approximated by

$$\left\{ \begin{array}{l} \frac{Du}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial x} + (2\Omega \sin \theta)v \\ \frac{Dv}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial y} - (2\Omega \sin \theta)u \end{array} \right.$$

For a barotropic fluid flow, we have $\Omega = |\vec{\Omega}|$ the angular velocity of the Earth.

In the f-plane approximation, we approximate the Coriolis parameter f by a constant f_0 . We can do this because the variation of θ is important only for phenomena having long time scales or long length scales. For many purposes, we simply consider $f_0 = 2\Omega \sin \theta_0$, where θ_0 is the central latitude of the region that we are interested in.

In the β -plane approximation, we approximate the Coriolis parameter f by a linear function of latitude and approximate the angle θ by the arc-length y with the help of Taylor series. We use $y = R\theta$ to be the arc-length approximation where R indicates the radius of the Earth, approximately $6371km$. We let $f = f_0 + \beta y$, where

$$\beta = \left(\frac{df}{dy} \right)_{\theta_0} = \left(\frac{df}{d\theta} \frac{d\theta}{dy} \right)_{\theta_0} = \frac{2\Omega \cos \theta_0}{R}$$

The vorticity equation

This section presents the derivation of the barotropic vorticity equation. By our previous discussion, we have:

$$\begin{cases} \frac{Du}{Dt} = \frac{1}{\rho} \frac{\partial p}{\partial x} + fv \\ \frac{Dv}{Dt} = \frac{1}{\rho} \frac{\partial p}{\partial y} - fu \end{cases} \quad (2.16)$$

where $f = 2\Omega \sin \theta$.

We rewrite these two equations in terms of the streamfunction which was defined in (A.24) and then combine them as a single equation. In terms of Ψ , we have

$$\begin{aligned} \frac{D}{Dt}(-\Psi_y) &= \left(\frac{\partial}{\partial t} + (-\Psi_y) \frac{\partial}{\partial x} + \Psi_x \frac{\partial}{\partial y} \right) (-\Psi_y) \\ &= (-\Psi_{yt} + \Psi_y \Psi_{xy} - \Psi_x \Psi_{yy}) \\ &= \frac{1}{\rho} \frac{\partial p}{\partial x} + f \Psi_x; \end{aligned} \quad (2.17)$$

$$\begin{aligned}
\frac{D}{Dt}(\Psi_x) &= \left(\frac{\partial}{\partial t} + (-\Psi_y) \frac{\partial}{\partial x} + \Psi_x \frac{\partial}{\partial y} \right) (\Psi_x) \\
&= (\Psi_{xt} - \Psi_y \Psi_{xx} + \Psi_x \Psi_{xy}) \\
&= \frac{1}{\rho} \frac{\partial p}{\partial y} + f \Psi_y
\end{aligned} \tag{2.18}$$

where the subscripts in t , x and y denote the partial differentiation with respect to these variables. We differentiate (2.17) with respect to y and (2.18) with respect to x and obtain

$$-\Psi_{yyt} + \Psi_y \Psi_{xyy} + \Psi_x \Psi_{yyy} = \frac{1}{\rho} \frac{\partial p}{\partial x \partial y} + \beta \Psi_x + f \Psi_{xy} \tag{2.19}$$

and

$$\Psi_{xxt} - \Psi_y \Psi_{xxx} - \Psi_x \Psi_{xxy} = \frac{1}{\rho} \frac{\partial p}{\partial x \partial y} + f \Psi_{xy} \tag{2.20}$$

We then subtract (2.20) from (2.19) to obtain

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi_t + \Psi_y \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi_x - \Psi_x \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi_y = \beta \Psi_x \tag{2.21}$$

This can be written as

$$\nabla^2 \Psi_t + \Psi_x \nabla^2 \Psi_y - \Psi_y \nabla^2 \Psi_x + \beta \Psi_x = 0 \tag{2.22}$$

where $\nabla^2 \Psi = \xi$ is the vorticity, which is defined in Appendix A.3. Equation (2.22) is called the barotropic vorticity equation.

We now describe the non-dimensionalization of the equation. The non-dimensional variables and parameters are described below.

Up to this point, all the functions, variables and parameters in our equations are considered to be dimensional. The SI units for these quantities are given in Appendix A.4. We consider reference values for each of the variables, based on characteristic values measured in the atmosphere.

We let L represent the characteristic horizontal length scale of the vortex with the length unit m , and U represents the characteristic speed of the tangential flow with the speed unit ms^{-1} . Based on these, the characteristic time scale is $\frac{L}{U}$ with the unit s and the characteristic value of the streamfunction is UL with the unit m^2s^{-1} . We can then denote the dimensional quantities by $x^*, y^*, t^*, \psi^*, \beta^*$ and define corresponding non-dimensional quantities by

$$\left\{ \begin{array}{l} x = \frac{x^*}{L} \\ y = \frac{y^*}{L} \\ t = \frac{Ut^*}{L} \\ \Psi = \frac{\Psi^*}{UL} \\ \beta = \frac{L^2\beta^*}{U} \end{array} \right. \quad (2.23)$$

As β is the Coriolis force effect, we can denote the corresponding dimensional quantity β^* by

$$\beta^* = \frac{2\Omega_E}{R_E} \cos \theta_0 \quad (2.24)$$

where θ_0 is the latitude of the center of the vortex, Ω_E is the angular velocity of the Earth's rotation and R_E is the radius of the Earth.

In terms of the non-dimensional quantities, the barotropic vorticity equation takes the same form as (2.22). From the research of previous scholars [22], we notice that the value of Earth's angular velocity is about $\Omega_E \approx 7 \times 10^{-5}s^{-1}$ and $\cos \theta_0 = \cos 30 \approx 0.8$. Hence we have

$$\beta^* \approx 2 \times 10^{-11} m^{-1} s^{-1}$$

For a hurricane, the wind speed should be about $30 - 100 ms^{-1}$, and the diameter is up to about $500 km = 5 \times 10^5 m$. In this thesis, the diameter is about

10 nondimensional units. From (2.23) we have

$$L = \frac{x^*}{x} = \frac{y^*}{y} = 5 \times 10^4 m$$

The vortex wind speed $\bar{v}$ is close to 1 when $r = 1$, so we can have $U = 100 m s^{-1}$.

Then we obtain

$$\beta = \frac{L^2 \beta^*}{U} = \frac{25 \times 10^8 m^2 \times 2 \times 10^{-11} m^{-1} s^{-1}}{100 m s^{-1}} = 5 \times 10^{-4}$$

Based on this, we choose β to be a small parameter in the analysis.

2.3 Vortex Rossby waves on a β -plane

In this section, we describe a model for vortex Rossby waves in a cyclonic vortex on a β -plane described in terms of polar coordinates r and λ . The wave is generated by an initial perturbation to a background steady vortex flow.

In the most fundamental scenario, we have the flow in the cyclonic vortex with the steady uniform rotation with a zero radial component and an azimuthal component, denoted as $\bar{v}(r)$. Hence, we have $\bar{\psi}_\lambda = 0$ and $\bar{\psi}_r = \bar{v}(r)$. Then we can rewrite the background vortex streamfunction $\bar{\psi}(r)$ which is the steady basic flow component, and the background angular velocity $\bar{\Omega}(r)$ to be

$$\bar{\psi}'(r) = \bar{v}(r) , \bar{\Omega}(r) = \frac{\bar{v}(r)}{r} \quad (2.25)$$

In this case, the prime represents the derivative with respect to r .

We also introduce the representation of the components of the velocity in the directions by v_{radial} and $v_{azimuthal}$ from [15], and the continuity equation is

$$\frac{1}{r} \frac{\partial}{\partial r} r v_{radial} + \frac{1}{r} \frac{\partial}{\partial \lambda} v_{azimuthal} = 0 \quad (2.26)$$

The velocity components can be expressed in terms of the streamfunction as

$$v_{radial} = -\frac{1}{r} \frac{\partial \Psi}{\partial \lambda}, \quad v_{azimuthal} = \frac{\partial \Psi}{\partial r} \quad (2.27)$$

Each component of the total vorticity and the total streamfunction can each be decomposed into apart from the steady basic vortex and a relatively small time-dependent perturbation. We can write the streamfunction as

$$\begin{aligned} \Psi(r, \lambda, t) &= \bar{\psi}(r) + \varepsilon \psi(r, \lambda, t) \\ v_{radial}(r, \lambda, t) &= \varepsilon u(r, \lambda, t) \\ v_{azimuthal}(r, \lambda, t) &= \bar{v}(r) + \varepsilon v(r, \lambda, t) \end{aligned} \quad (2.28)$$

where all the functions ψ, u and v are $O(1)$ with $\varepsilon \ll 1$.

We write the derivatives of the total streamfunction and the perturbation streamfunction as:

$$\left\{ \begin{array}{ll} \frac{\partial \Psi}{\partial r} = v_{azimuthal} = \bar{v} + \varepsilon \psi_r \\ \frac{1}{r} \frac{\partial \Psi}{\partial \lambda} = & -v_{radial} = -\varepsilon u \\ \psi_r = & v \\ \psi_\lambda = & -ru \end{array} \right. \quad (2.29)$$

In our case of a two-dimensional vortex, rewriting the barotropic vorticity equation (2.22) in polar coordinates r and λ is more straightforward, where r represents the radial distance from the center of the vortex to the point and λ represents the directional angle measured. We have $x = r \cos \lambda$ and $y = r \sin \lambda$. Thus, we have the following expression:

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial r} = \cos \lambda, \quad \frac{\partial x}{\partial \lambda} = -r \sin \lambda \\ \frac{\partial y}{\partial r} = \sin \lambda, \quad \frac{\partial y}{\partial \lambda} = r \cos \lambda \end{array} \right. \quad (2.30)$$

Using the Chain Rule, we can find Ψ_r and Ψ_λ with the expression of Ψ_x and Ψ_y as:

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial \Psi}{\partial y} \frac{\partial y}{\partial r} = \Psi_x \cos \lambda + \Psi_y \sin \lambda \quad (2.31)$$

$$\frac{\partial \Psi}{\partial \lambda} = \frac{\partial \Psi}{\partial x} \frac{\partial x}{\partial \lambda} + \frac{\partial \Psi}{\partial y} \frac{\partial y}{\partial \lambda} = -\Psi_x r \sin \lambda + \Psi_y r \cos \lambda \quad (2.32)$$

By the equations above we can express Ψ_x and Ψ_y in terms of Ψ_r and Ψ_λ .

$$\begin{cases} \Psi_x = \cos \lambda \Psi_r - \frac{1}{r} \sin \lambda \Psi_\lambda \\ \Psi_y = \sin \lambda \Psi_r + \frac{1}{r} \cos \lambda \Psi_\lambda \end{cases} \quad (2.33)$$

Then we can substitute (2.33) into (2.22) to obtain:

$$\begin{aligned} \Psi_x \nabla^2 \Psi_y - \Psi_y \nabla^2 \Psi_x &= (\cos \lambda \Psi_r - \frac{1}{r} \sin \lambda \Psi_\lambda) \nabla^2 (\sin \lambda \Psi_r + \frac{1}{r} \cos \lambda \Psi_\lambda) \\ &\quad - (\sin \lambda \Psi_r + \frac{1}{r} \cos \lambda \Psi_\lambda) \nabla^2 (\cos \lambda \Psi_r - \frac{1}{r} \sin \lambda \Psi_\lambda) \\ &= \Psi_r \nabla^2 (\sin \lambda \cos \lambda \Psi_r + \frac{1}{r} \cos^2 \lambda \Psi_\lambda) \\ &\quad - \frac{1}{r} \Psi_\lambda \nabla^2 (\sin^2 \lambda + \frac{1}{r} \cos \lambda \sin \lambda \Psi_\lambda) \\ &\quad - \Psi_r \nabla^2 (\cos \lambda \sin \lambda \Psi_r - \frac{1}{r} \sin^2 \lambda \Psi_\lambda) \\ &\quad - \frac{1}{r} \Psi_\lambda \nabla^2 (\cos^2 \lambda \Psi_r - \frac{1}{r} \sin \lambda \cos \lambda \Psi_\lambda) \\ &= \frac{1}{r} \Psi_r \nabla^2 \Psi_\lambda - \frac{1}{r} \Psi_\lambda \nabla^2 \Psi_r \end{aligned} \quad (2.34)$$

The Laplacian operator appearing in equation (2.34) is in terms of derivatives with respect to x and y . Making the change of variables to polar coordinates r and λ , we have

$$\nabla^2 \psi = \psi_{rr} + \frac{1}{r} \psi_r + \frac{1}{r^2} \psi_{\lambda\lambda}$$

Hence, we can write equation (2.22) as:

$$\nabla^2 \Psi_t - \frac{1}{r} \Psi_\lambda \nabla^2 \Psi_r + \frac{1}{r} \Psi_r \nabla^2 \Psi_\lambda - \frac{\beta}{r} \Psi_\lambda \sin \lambda + \beta \Psi_r \cos \lambda = 0 \quad (2.35)$$

which is the equation of the model which is studied throughout the entire thesis.

We substitute $\Psi = \bar{\psi}(r) + \varepsilon\psi(r, \lambda, t)$ from (2.28) into (2.35) and obtain the following:

$$\begin{aligned}
& \nabla^2 \Psi_t - \frac{1}{r} \Psi_\lambda \nabla^2 \Psi_r + \frac{1}{r} \Psi_r \nabla^2 \Psi_\lambda - \frac{\beta}{r} \Psi_\lambda \sin \lambda + \beta \Psi_r \cos \lambda \\
&= \nabla^2 \varepsilon \psi_t - \frac{1}{r} (\varepsilon \psi_\lambda) \nabla^2 (\bar{v} + \varepsilon \psi_r) \\
&+ \frac{1}{r} (\bar{v} + \varepsilon \psi_r) \nabla^2 (\varepsilon \psi_\lambda) \\
&- \frac{\beta}{r} \varepsilon \psi_\lambda \sin \lambda + \beta (\bar{v} + \varepsilon \psi_r) \cos \lambda \\
&= \nabla^2 \varepsilon \psi_t - \frac{1}{r} (\varepsilon \psi_\lambda \nabla^2 \varepsilon \psi_r - \varepsilon \psi_r \nabla^2 \varepsilon \psi_\lambda) \\
&- \frac{1}{r} (\varepsilon \psi_\lambda \nabla^2 \bar{v} + \bar{v} \nabla^2 \varepsilon \psi_\lambda) \\
&- \frac{\beta}{r} \varepsilon \psi_\lambda \sin \lambda + \beta \bar{v} \cos \lambda + \beta \varepsilon \psi_r \cos \lambda = 0
\end{aligned}$$

We divide this by ε and obtain

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + \frac{\bar{v}}{r} \frac{\partial}{\partial \lambda} \right) \nabla^2 \psi - \frac{\varepsilon}{r} (\psi_\lambda \nabla^2 \psi_r - \psi_r \nabla^2 \psi_\lambda) \\
&- \frac{1}{r} \psi_\lambda \nabla^2 \bar{v} - \frac{\beta}{r} \psi_\lambda \sin \lambda \\
&+ \frac{\beta}{\varepsilon} \bar{v} \cos \lambda + \beta \psi_r \cos \lambda = 0
\end{aligned} \tag{2.36}$$

where we denote the ratio $\frac{\beta}{\varepsilon}$ by $\gamma = \frac{\beta}{\varepsilon}$.

We note that $\nabla^2 \bar{v} = \bar{v}_{rr} + \frac{1}{r} \bar{v}_r + \frac{1}{r^2} \bar{v}_{\lambda\lambda} = \frac{d}{dr} (\bar{v}' + \frac{\bar{v}}{r})$. By rearranging terms, we can obtain the following equation:

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + \frac{\bar{v}}{r} \frac{\partial}{\partial \lambda} \right) \nabla^2 \psi - \frac{1}{r} \psi_\lambda \frac{d}{dr} (\bar{v}' + \frac{\bar{v}}{r}) - \frac{\beta}{r} \psi_\lambda \sin \lambda + \gamma \bar{v} \cos \lambda \\
&+ \beta \psi_r \cos \lambda = \frac{\varepsilon}{r} (\psi_\lambda \nabla^2 \psi_r - \psi_r \nabla^2 \psi_\lambda)
\end{aligned} \tag{2.37}$$

We note that we can separate the equation into four groups of terms.

There is a linear $\mathcal{O}(1)$ part of $(\frac{\partial}{\partial t} + \frac{\bar{v}}{r} \frac{\partial}{\partial \lambda}) \nabla^2 \psi - \frac{1}{r} \psi_\lambda \frac{d}{dr} (\bar{v}' + \frac{\bar{v}}{r})$, an $\mathcal{O}(\beta)$ part of $-\frac{\beta}{r} \psi_\lambda \sin \lambda + \beta \psi_r \cos \lambda$, an $\mathcal{O}(\gamma)$ term of $(\gamma \bar{v} \cos \lambda)$ and an $\mathcal{O}(\varepsilon)$ nonlinear part of $-\frac{\varepsilon}{r} (\psi_\lambda \nabla^2 \psi_r - \psi_r \nabla^2 \psi_\lambda)$.

In chapter 3 and chapter 4, we consider different balances of the parameters β and ε and the ratio γ . Thus we have 3 parameters in (2.37): β which measures the Coriolis effect, ε which measures the amplitude of the wave relative to the background vortex flow, and γ which gives the ratio of the relative importance of the Coriolis effect to the nonlinear effect.

If all β , ε and γ are sufficiently small, that is to say, β, γ and $\varepsilon \ll 1$, we can neglect every term except the leading order linear part, and solve the linear equation, as [14] did. The vortex wave fluctuations have an amplitude of about $5 - 10ms^{-1}$ superimposed on a typical background speed of wind about $30 - 100ms^{-1}$. In this case, a more realistic consideration takes the two parameters ε and β both small but non-negligible. As noted in section 2.2, the nondimensional parameter β is around 5×10^{-4} . The ratio γ may be $\mathcal{O}(1)$ or smaller. There are 5 more possibilities retaining the additional terms. They are listed as the following:

- (i) $\beta \ll \varepsilon^2 \ll \varepsilon \ll 1$. In this case, $\gamma = \frac{\beta}{\varepsilon}$ is also small and the influence of the term of $\mathcal{O}(\varepsilon)$ dominates.
- (ii) $\beta \sim \varepsilon^2 \ll \varepsilon \ll 1$. In this case, $\gamma = \frac{\beta}{\varepsilon} \sim \varepsilon$ and the influence of the term of $\mathcal{O}(\gamma)$ is the same as the term of $\mathcal{O}(\varepsilon)$.
- (iii) $\varepsilon^2 \ll \beta \ll \varepsilon \ll 1$. In this case, $\beta \ll \varepsilon \ll \gamma = \frac{\beta}{\varepsilon} \ll 1$ and the influence of the term of $\mathcal{O}(\gamma)$ dominates.
- (iv) $\beta \sim \varepsilon \ll 1$. In this case, $\gamma = \frac{\beta}{\varepsilon} \sim 1$ and the other terms affect less.
- (v) $\varepsilon \ll \beta \ll 1$. In this case, $\gamma = \frac{\beta}{\varepsilon} \gg 1$ so the single term $\beta\psi_r \cos \lambda$ is dominant which gives an inconsistent situation.

We shall examine cases (i) - (iv).

2.4 Initial-boundary value problem

In cases (iii) and (iv), the dominant terms in equation (2.37) are the $\mathcal{O}(1)$ linear terms and the $\mathcal{O}(\gamma)$ non-homogeneous term at leading orders. Hence, the equation becomes:

$$\left(\frac{\partial}{\partial t} + \frac{\bar{v}}{r} \frac{\partial}{\partial \lambda}\right) \nabla^2 \psi - \frac{1}{r} \psi_\lambda \frac{d}{dr} \left(\bar{v}' + \frac{\bar{v}}{r}\right) + \gamma \bar{v} \cos \lambda = 0 \quad (2.38)$$

We discuss this equation first and then consider the other cases where the $\mathcal{O}(\varepsilon)$ nonlinear terms are included. In this section, we will describe the complete initial value and the boundary value, and define the necessary terms and conditions. In the next few sections, all things will be discussed with details to analyze this equation.

Equation (2.38) is defined on the annular domain

$$r_1 \leq r < \infty, 0 \leq \lambda < 2\pi, 0 < t < \infty \quad (2.39)$$

The domain is shown in Figure 2.5. It is in the form of an annulus, representing a cyclone with its "eye" at its center, in which there is no wind speed.

Hence, we specify the initial condition as:

$$\psi(r, \lambda, 0) = A(r)e^{ik\lambda} + \text{c.c.} \quad (2.40)$$

where $A(r)$ is a specified function satisfying $A(r) \rightarrow 0$ as $r \rightarrow \infty$ and c.c. denotes the complex conjugate. Especially, we shall consider $A(r) = A_0 e^{-\mu r^2}$ where $\mu \lesssim 1$. The constant k is a specified positive integer, $k \geq 1$. This initial condition generates a Rossby wave perturbation with wavenumber k . At $r = r_1$, we have

$$\begin{aligned} \psi(r_1, \lambda, t) &= A(r_1)e^{ik\lambda} + \text{c.c.} \\ \psi(r, \lambda, t)_{r \rightarrow \infty} &= \text{finite} \end{aligned} \quad (2.41)$$

Figure 2.5: The spatial domain for equation (2.38)

We will also define the wave vorticity as

$$\xi = \nabla^2 \psi = \psi_{rr} + \frac{1}{r} \psi_r + \frac{1}{r^2} \psi_{\lambda\lambda} \quad (2.42)$$

The wave vorticity itself has a initial condition as

$$\begin{aligned} \xi(r, \lambda, 0) &= B(r)e^{ik\lambda} + \text{c.c.} \\ &= [A''(r) + \frac{1}{r}A'(r) - \frac{k^2}{r^2}A(r)]e^{ik\lambda} + \text{c.c.} \end{aligned} \quad (2.43)$$

The behaviour of $A(r)$ is straight-forward as $A(r) = A_0 e^{-\mu r^2}$. The value of μ determines how fast the function approaches zero. However, the behaviour of $B(r)$ is not obvious to readers, hence, we need to represent a diagram. The behaviour of $A(r)$ and $B(r)$ are represented below. We will give each function two plots: one is a contour plot of $A(r)e^{ik\lambda} + \text{c.c.}$ and $B(r)e^{ik\lambda} + \text{c.c.}$ in terms of the polar coordinates r and λ and the other sets the r as the horizontal axis. In the following diagram, we choose $\mu = 0.1$ and r ranges from 0.1 to 8 and the

constant $A_0 = 1$.

(a) The initial wave streamfunction amplitude $A(r)$

(b) The initial wave streamfunction (2.40) with $k = 2$ and $r_1 = 0.1$

Figure 2.6: Linear and contour plots of $A(r)$

(a) The initial wave vorticity amplitude $B(r)$

(b) The initial wave vorticity (2.43) with $k = 2$ and $r_1 = 0.1$

Figure 2.7: Linear and contour plots of $B(r)$

We need to define the function $\bar{\Omega}$, the angular velocity. Recall from (2.25), we have $\bar{\Omega}(r) = \frac{\bar{v}(r)}{r}$. According to previous researchers like [13], we shall consider $\bar{\Omega}(r) = \Omega_0 e^{-\alpha r^2}$, where $\alpha \ll 1$ and Ω_0 is a specified constant. In this case, we have

4 different parameters: $\varepsilon, \beta, \alpha$ and μ . We can look for a solution as a perturbation series in terms of these parameters. The relative magnitudes of these parameters determine the form of the solution.

We will first consider the simpler case where the angular velocity is constant: $\bar{\Omega} = \Omega_0$. In this situation, we have three parameters ε, β and μ .

We should omit the $r_1 = 0$ case since the problem is not well defined in that case. Next, in section 3.3 we will use the nonlinear analysis to find the asymptotic expression of the equation.

In chapter 4, we will repeat the process with the case that $\bar{\Omega}$ is a function of r . In section 4.1, the size of each parameter is discussed. Section 4.2 represents a basic linear initial boundary value problem and section 4.3 enlarges it to be a nonlinear analysis.

Chapter 3

The case with the constant background angular velocity

3.1 Solution of the linearized equation

In this chapter, we consider the special case where the mean vortex angular velocity $\bar{\Omega}$ is constant and the mean velocity $\bar{v} = \bar{\Omega}r$ a linear function of r . We obtain solutions for the PDE in equation (2.37) in the domain (2.39) with initial conditions (2.40), (2.43) and boundary conditions (2.41). We first consider the cases (iii) $\beta \ll \varepsilon \ll \gamma = \frac{\beta}{\varepsilon} \ll 1$ and (iv) $\gamma \sim 1$ where the $\mathcal{O}(1)$ linear terms and the $\mathcal{O}(\gamma)$ term are dominant in equation (2.37). Then we have:

$$\left(\frac{\partial}{\partial t} + \frac{\bar{v}}{r} \frac{\partial}{\partial \lambda}\right) \nabla^2 \psi - \frac{1}{r} \psi_\lambda \frac{d}{dr} \left(\bar{v}' + \frac{\bar{v}}{r}\right) + \gamma \bar{v} \cos \lambda = 0 \quad (3.1)$$

With $\bar{\Omega}$ being constant and $\bar{v} = \bar{\Omega}r$, the factor of $\frac{d}{dr} \left(\bar{v}' + \frac{\bar{v}}{r}\right)$ is zero. So (3.1) can be written as:

$$\left(\frac{\partial}{\partial t} + \frac{\bar{v}}{r} \frac{\partial}{\partial \lambda}\right) \xi + \frac{\beta}{2\varepsilon} \bar{v} (e^{i\lambda} + e^{-i\lambda}) = 0 \quad (3.2)$$

where $\xi = \nabla^2 \psi$.

We solve equation (3.2) on the domain (2.39). As defined in section 2.4, we impose the initial conditions

$$\begin{aligned}\psi(r, \lambda, 0) &= A(r)e^{ik\lambda} + \text{c.c.} \\ \xi(r, \lambda, 0) &= B(r)e^{ik\lambda} + \text{c.c.}\end{aligned}\tag{3.3}$$

where $B(r) = A'' + \frac{1}{r}A' - \frac{k^2}{r^2}A$. We choose the function A so that $A \rightarrow 0$ as $r \rightarrow \infty$. An example is that $A(r) = e^{-\mu r^2}$. Hence, we have the boundary condition that the solution of the homogeneous equation corresponding to (3.2) remains finite as $r \rightarrow \infty$. This may not be possible with the non-homogeneous form.

We also specify the boundary condition

$$\begin{aligned}\psi(r_1, \lambda, t) &= A(r_1)e^{ik\lambda} + \text{c.c.} \\ \xi(r_1, \lambda, t) &= B(r_1)e^{ik\lambda} + \text{c.c.}\end{aligned}\tag{3.4}$$

The solution of (3.2) satisfying the initial condition (3.3) can be written as the sum of the solution of the corresponding homogeneous equation $(\frac{\partial}{\partial t} + \bar{\Omega}\frac{\partial}{\partial \lambda})\xi = 0$ satisfying the initial condition, and a particular solution of the non-homogeneous equation (3.2) satisfying a zero initial condition.

Thus, we can write the solution of (3.2) as

$$\begin{aligned}\psi(r, \lambda, t) &= \hat{\psi}_k(r, t)e^{ik\lambda} + \gamma\hat{\psi}_1(r, t)e^{i\lambda} + \text{c.c.} \\ \xi(r, \lambda, t) &= \hat{\xi}_k(r, t)e^{ik\lambda} + \gamma\hat{\xi}_1(r, t)e^{i\lambda} + \text{c.c.}\end{aligned}\tag{3.5}$$

where $\hat{\psi}_k(r, t)e^{ik\lambda} + \text{c.c.}$ is the solution of the homogeneous partial differential equation

$$\left(\frac{\partial}{\partial t} + \bar{\Omega}\frac{\partial}{\partial \lambda}\right)\nabla^2\psi = 0\tag{3.6}$$

satisfying $\psi(r, \lambda, 0) = A(r)e^{ik\lambda} + \text{c.c.}$

and $\gamma\hat{\psi}_1(r, t)e^{i\lambda} + \text{c.c.}$ is a particular solution of the non-homogeneous partial differential equation (3.2).

$$\left(\frac{\partial}{\partial t} + \bar{\Omega}\frac{\partial}{\partial \lambda}\right) \nabla^2 \psi = -\gamma \bar{v} \cos \lambda \quad (3.7)$$

satisfying $\psi(r, \lambda, 0) = 0$

and

$$\begin{aligned} \hat{\psi}_k(r, 0) &= A(r), \hat{\xi}_k(r, 0) = B(r); \\ \hat{\psi}_k(r_1, t) &= A(r_1), \hat{\xi}_k(r_1, t) = B(r_1); \\ \hat{\psi}_k &\longrightarrow 0, \hat{\xi}_k \longrightarrow 0 \text{ as } r \longrightarrow \infty \end{aligned} \quad (3.8)$$

$$\begin{aligned} \hat{\psi}_1(r, 0) &= \hat{\xi}_1(r, 0) = 0 \\ \hat{\psi}_1(r_1, t) &= \hat{\xi}_1(r_1, t) = 0 \end{aligned} \quad (3.9)$$

The following subsections solve these two equations separately. By superposition, the solution of (3.6) and (3.7) gives the solution to the initial boundary value problem. We solve for $\hat{\xi}_k$ and $\hat{\xi}_1$ and then solve the following equations to obtain $\hat{\psi}_k$ and $\hat{\psi}_1$:

Solution $\hat{\psi}_k$

To solve equation (3.6), notice that in our assumption ψ has only an exponential parameter involving λ . Hence, $\frac{\partial}{\partial \lambda} \longrightarrow ik$ and $\frac{\partial}{\partial \bar{\lambda}} \longrightarrow -ik$ for the complex conjugate part, and we obtain (3.6) as the following equation:

$$\left(\frac{\partial}{\partial t} + ik\bar{\Omega}\right) \hat{\xi}_k = 0 \quad (3.10)$$

$r_1 \leq r < \infty; 0 < t < \infty;$

with initial condition $\hat{\xi}_k(r, 0) = B(r)$

The solution of the initial value problem (3.10) is:

$$\hat{\xi}_k(r, t) = B(r)e^{-ik\bar{\Omega}t} \quad (3.11)$$

We then solve for $\hat{\psi}_k$:

$$\begin{aligned} \left(\hat{\psi}_k\right)_{rr} + \frac{1}{r}\hat{\psi}_{kr} - \frac{k^2}{r^2}\hat{\psi}_k &= B(r)e^{-ik\bar{\Omega}t} \\ &= \left(A'' + \frac{1}{r}A' - \frac{k^2}{r^2}A\right)e^{-ik\bar{\Omega}t} \end{aligned} \quad (3.12)$$

We note that this is a Cauchy-Euler equation, which has solutions for the homogeneous equation of the form r^p for some number p . Then we have $p(p-1)r^{p-2} + pr^{p-2} - k^2r^{p-2} = 0$. Hence, we have $p^2 - k^2 = 0$ to get $p = \pm k$. Then we know that the two linearly independent solutions are $y_1 = r^k, y_2 = r^{-k}$. The Wronskian of the two solutions is $W = y_1y_2' - y_1'y_2 = -2kr^{-1}$.

A particular solution of the non-homogeneous equation (3.12) is $\phi(r) = u_1y_1 + u_2y_2$ where

$$u_1(r) = - \int \frac{y_2 B(r)}{W(y_1, y_2)} dr = \frac{1}{2k} \int \frac{B(r)}{r^{k-1}} dr$$

Similarly,

$$u_2(r) = \int \frac{y_1 B(r)}{W(y_1, y_2)} dr = -\frac{1}{2k} \int r^{k+1} B(r) dr$$

Putting them together, we can derive

$$\begin{aligned} \phi(r) &= u_1 r^k + u_2 r^{-k} \\ &= \frac{1}{2k} r^k \int \frac{B(\tilde{r})}{\tilde{r}^{k-1}} d\tilde{r} - \frac{1}{2k} \frac{1}{r^k} \int \tilde{r}^{k+1} B(\tilde{r}) d\tilde{r} \\ &= \frac{1}{2k} \int \left(\frac{r^k}{\tilde{r}^{k-1}} - \frac{\tilde{r}^{k+1}}{r^k} \right) B(\tilde{r}) d\tilde{r} \end{aligned}$$

where $\tilde{r}$ is the variable of integration. Thus, we have

$$\hat{\psi}_k(r, t) = \phi(r)e^{-ik\bar{\Omega}t} + c_1(t)r^k + c_2(t)r^{-k} \quad (3.13)$$

where $c_1(t)$ and $c_2(t)$ are to be determined. In order for the solution of the homogeneous equation to be finite as $r \rightarrow \infty$, we set $c_1(t) = 0$. This gives

$$\hat{\psi}_k(r, t) = \phi(r)e^{-ik\bar{\Omega}t} + c_2(t)r^{-k} \quad (3.14)$$

At $t = 0$, we have

$$\hat{\psi}_k(r, 0) = \phi(r) + c_2(0)r^{-k} = A(r) \quad (3.15)$$

We also consider the boundary condition at $r = r_1$:

$$\hat{\psi}_k(r_1, t) = \phi(r_1)e^{-ik\bar{\Omega}t} + c_2(t)r_1^{-k} = A(r_1) \quad (3.16)$$

Substituting $\phi(r) = A(r) - c_2(0)r^{-k}$ into (3.14) gives

$$\hat{\psi}_k(r, t) = A(r)e^{-ik\bar{\Omega}t} + (c_2(t) - c_2(0)e^{-ik\bar{\Omega}t})r^{-k} \quad (3.17)$$

From (3.16), we also have $c_2(t)r_1^{-k} = -\phi(r_1)e^{-ik\bar{\Omega}t} + A(r_1)$ and $c_2(0)r_1^{-k} = -\phi(r_1) + A(r_1)$. Hence, we obtain

$$(c_2(t) - c_2(0)e^{-ik\bar{\Omega}t})r_1^{-k} = A(r_1)(1 - e^{-ik\bar{\Omega}t}) \quad (3.18)$$

Substituting the expression (3.18) into (3.17) gives

$$\begin{aligned} \hat{\psi}_k(r, t) &= A(r)e^{-ik\bar{\Omega}t} + \frac{A(r_1)r^{-k}}{r_1^{-k}}(1 - e^{-ik\bar{\Omega}t}) \\ &= (A(r) - A(r_1)r_1^k r^{-k})e^{-ik\bar{\Omega}t} + A(r_1)r_1^k r^{-k} \end{aligned} \quad (3.19)$$

which satisfies initial and boundary conditions (3.8).

Solution of $\hat{\psi}_1$

We substitute $\psi = \gamma\hat{\psi}_1(r, t)e^{i\lambda} + c.c.$ into (3.7) to find a particular solution $\hat{\xi}_1(r, t)$ and $\hat{\psi}_1(r, t)$. In this case, we will have $\frac{\partial}{\partial\lambda} = i$ and $\frac{\partial}{\partial\bar{\lambda}} = -i$ for the complex conjugate part. We then solve

$$\begin{aligned} \left(\frac{\partial}{\partial t} + i\bar{\Omega}\right)\hat{\xi}_1 &= \frac{-\bar{\Omega}r}{2} \\ \text{with } \hat{\xi}_1(r, 0) &= 0 \end{aligned} \quad (3.20)$$

We can solve this using the integrating factor $e^{\int i\bar{\Omega}dt} = e^{i\bar{\Omega}t}$ to make the ODE become

$$e^{i\bar{\Omega}t}(\hat{\xi}_1)_t + e^{i\bar{\Omega}t}i\bar{\Omega}\hat{\xi}_1 = -\frac{1}{2}e^{i\bar{\Omega}t}\bar{\Omega}r$$

Hence, we have

$$\begin{aligned} e^{i\bar{\Omega}t}\hat{\xi}_1 &= \int -e^{i\bar{\Omega}t}\frac{\bar{\Omega}r}{2}dt \\ &= -\frac{\bar{\Omega}r}{2}\frac{e^{i\bar{\Omega}t}}{i\bar{\Omega}} + C(r) \\ \implies \hat{\xi}_1 &= \frac{ir}{2} + C(r)e^{-i\bar{\Omega}t} \end{aligned} \quad (3.21)$$

where $C(r)$ is the integration constant with respect to t . Applying the zero initial condition from (3.9), we find $C(r) = -\frac{ir}{2}$ and so we have

$$\hat{\xi}_1(r, t) = \frac{ir}{2}(1 - e^{-i\bar{\Omega}t}) \quad (3.22)$$

Hence, we have the following ODE to solve for $\hat{\psi}_1(r, t)$:

$$\begin{aligned} \hat{\psi}_{rr} + \frac{1}{r}\hat{\psi}_r - \frac{1}{r^2}\hat{\psi} &= \frac{ir}{2}(1 - e^{-i\bar{\Omega}t}) \\ \text{satisfying } \hat{\psi}_1(0, r) &= 0 \end{aligned} \quad (3.23)$$

Again, this is a non-homogeneous Cauchy-Euler equation. So the solution of the homogeneous equation are of the form r^n for some number n and we have

that

$$n(n-1)r^{n-2} + nr^{n-2} - r^{n-2} = 0$$

which implies $n^2 - 1 = 0$ to make $n = \pm 1$. Hence, we have the homogeneous solutions:

$$y_1 = r, \quad y_2 = r^{-1} \quad (3.24)$$

and the Wronskian of the two solutions is $W(y_1, y_2) = -\frac{2}{r}$.

A particular solution of the non-homogeneous equation (3.23) is $u_{11}y_1 + u_{12}y_2$, where

$$\begin{aligned} u_{11} &= \int \frac{y_2 \hat{\xi}_1(r, t)}{W(y_1, y_2)} dr \\ &= - \int \frac{ir}{4} (1 - e^{-i\bar{\Omega}t}) \\ &= -\frac{ir^2}{8} (1 - e^{-i\bar{\Omega}t}) \\ u_{12} &= \int \frac{y_1 \hat{\xi}_1(r, t)}{W(y_1, y_2)} dr \\ &= - \int \frac{ir^3}{4} (1 - e^{-i\bar{\Omega}t}) \\ &= -\frac{ir^4}{16} (1 - e^{-i\bar{\Omega}t}) \end{aligned} \quad (3.25)$$

Thus, the particular solution $u_{11}r + u_{12}r^{-1}$ is

$$\begin{aligned} \hat{\psi}_{1p} &= \left(-\frac{ir^3}{16} - \frac{ir^3}{8}\right)(1 - e^{-i\bar{\Omega}t}) \\ &= -\frac{3ir^3}{16}(1 - e^{-i\bar{\Omega}t}) \end{aligned} \quad (3.26)$$

Hence, we have

$$\hat{\psi}_1(r, t) = -\left(\frac{3ir^3}{16}\right)(1 - e^{-i\bar{\Omega}t}) + c_{11}(t)r + c_{12}(t)r^{-1} \quad (3.27)$$

We want the solution of the homogeneous equation to remain finite as $r \rightarrow \infty$, thus we must have $c_{11}(t) = 0$. Applying the zero initial condition on $\hat{\psi}_1(r, 0) = 0$ tells us that $c_{12}(0) = 0$.

If we apply the boundary condition $\hat{\psi}_1(r_1, t) = 0$, we have

$$\hat{\psi}_1(r_1, t) = \frac{3ir_1^3}{16}(1 - e^{i\bar{\Omega}t}) + c_{12}(t)r_1^{-1} = 0 \quad (3.28)$$

which gives us

$$c_{12}(t) = -\frac{3ir_1^4}{16}(1 - e^{i\bar{\Omega}t})$$

Thus, the solution is

$$\hat{\psi}_1(r, t) = -\frac{3i}{16}\left(r^3 - \frac{r_1^4}{r}\right)(1 - e^{i\bar{\Omega}t}) \quad (3.29)$$

where the r^3 term in the $\mathcal{O}(\gamma)$ terms in the solution approaches infinity as r goes to infinity, which is a secular term. This secular behaviour results from the choice of constant background angular velocity. In chapter 4, we choose a more physically realistic background angular velocity profile that goes to zero as $r \rightarrow \infty$ which gives a solution to the non-homogeneous equation that does not exhibit this secular behaviour.

Thus, the solution of the initial boundary value problem is here presented.

$$\begin{aligned} \psi(r, \lambda, t) &= \hat{\psi}_k(r, t)e^{ik\lambda} + \gamma\hat{\psi}_1(r, t)e^{i\lambda} + c.c. \\ &= ((A(r) - A(r_1)r_1^k r^{-k})e^{-ik\bar{\Omega}t} + A(r_1)r_1^k r^{-k})e^{ik\lambda} \\ &\quad - \gamma\frac{3i}{16}\left(r^3 - \frac{r_1^4}{r}\right)(1 - e^{-i\bar{\Omega}t})e^{i\lambda} + c.c. \end{aligned} \quad (3.30)$$

$$\xi(r, \lambda, t) = B(r)e^{ik(\lambda - \bar{\Omega}t)} + \gamma\frac{ir}{2}e^{i\lambda}[1 - e^{-i\bar{\Omega}t}] + c.c. \quad (3.31)$$

The r^3 term in the $\mathcal{O}(\gamma)$ streamfunction solution and the r term in the $\mathcal{O}(\gamma)$ vorticity solution are secular terms, which mean that the solution approaches

infinity as $r \rightarrow \infty$. The secular term in the ψ solution becomes $\mathcal{O}(1)$ at the radius where $\gamma r^3 \sim 1$, i.e., at $r \sim \left(\frac{\varepsilon}{\beta}\right)^{\frac{1}{3}}$.

A contour plot of this solution is shown in the following figure. To plot the solution, we set the following parameters.

$$\left\{ \begin{array}{l} \mu = 0.01, \\ A_0 = 1, \\ 0.1 \leq r \leq 5, \\ \Omega_0 = 1, \\ k = 2, \\ \gamma = 0.001, \\ r \sim 5, \\ t = 1 \end{array} \right. . \quad (3.32)$$

The plots below, and all plots in this thesis are at fixed value of t . The first diagram illustrates the picture in a whole, while the second diagram takes a closer look at the central part of the solution. The last plot represents the diagram with λ on the horizontal axis and r on the vertical axis. As r becomes larger, the secular term becomes larger and t is $\mathcal{O}(1)$ for $r \sim \left(\frac{\varepsilon}{\beta}\right)^{\frac{1}{3}} = 10$.

Figure 3.1: Contour plots of the solution $\psi(r, \lambda, t)$ given in (3.30)

Figure 3.2: Cartesian plots of the solution $\psi(r, \lambda, t)$ given in (3.30)

3.2 Nonlinear Problem

In this section, we examine the nonlinear equation comprising the $\mathcal{O}(1)$, $\mathcal{O}(\gamma)$ and $\mathcal{O}(\varepsilon)$ terms that appear at the dominant order in equation (2.37) of case (ii). We have

$$\beta \ll \gamma \sim \varepsilon, \beta \sim \varepsilon^2$$

We set $\beta = \varepsilon^2$ and as before, $\bar{\Omega}$ is a constant. Then, equation (2.37) becomes:

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \bar{\Omega} \frac{\partial}{\partial \lambda} \right) \xi + \frac{\varepsilon \bar{v}}{2} (e^{i\lambda} + e^{-i\lambda}) - \frac{\varepsilon}{r} (\psi_\lambda \xi_r - \psi_r \xi_\lambda) + \mathcal{O}(\varepsilon^2) = 0 \\ & \text{with initial condition } \psi(r, \lambda, 0) = A(r)e^{ik\lambda} + \text{c.c.} \\ & \text{and } \xi(r, \lambda, 0) = B(r)e^{ik\lambda} + \text{c.c.} \end{aligned} \tag{3.33}$$

As before we have the boundary conditions (3.4).

The equation (3.33) is the equation of the form:

$$\begin{aligned} & \text{Linear differential Equation of } \mathcal{O}(1) + \varepsilon(\text{non-homogeneous term}) \\ & + \varepsilon(\text{nonlinear Jacobian}) + \mathcal{O}(\varepsilon^2) = 0 \end{aligned} \tag{3.34}$$

We look for a solution of the same asymptotic form:

$$\begin{aligned}\psi &= \psi^{(0)} + \varepsilon \psi^{(1)} + \mathcal{O}(\varepsilon^2) + \dots \\ \xi &= \xi^{(0)} + \varepsilon \xi^{(1)} + \mathcal{O}(\varepsilon^2) + \dots\end{aligned}\tag{3.35}$$

Substituting (3.35) into (3.33), we have the leading order equation

$$\left(\frac{\partial}{\partial t} + \bar{\Omega} \frac{\partial}{\partial \lambda} \right) \xi^{(0)} = 0$$

with the same initial condition of function (2.37) and the solution of $\xi^{(0)} = \hat{\xi}_k e^{ik\lambda} + \text{c.c.}$ with $\hat{\xi}_k$ given in (3.11). Applying the initial condition gives $\xi^{(0)}(r, \lambda, t) = B(r) e^{ik(\lambda - \bar{\Omega}t)}$ as before. Following the method above, we have the ψ_k solution found earlier. We then solve for $\psi^{(0)}$ and apply the initial and boundary conditions to obtain the same linear solution as in section 3.1. Solving the same equation (3.12) for $\psi^{(0)}$ gives that $\psi^{(0)} = \hat{\psi}_k e^{ik\lambda} + \text{c.c.}$ with $\hat{\psi}_k$ given in equation (3.19).

$$\begin{aligned}\psi^{(0)} &= (A(r) - A(r_1) r_1^k r^{-k}) e^{-ik\bar{\Omega}t} e^{ik\lambda} + A(r_1) r_1^k r^{-k} e^{ik\lambda} + \text{c.c.} \\ \xi^{(0)} &= \nabla^2 \psi^{(0)} = B(r) e^{-ik\bar{\Omega}t} e^{ik\lambda} + \text{c.c.}\end{aligned}\tag{3.36}$$

At the next order of $\mathcal{O}(\varepsilon)$, we have that

$$\left(\frac{\partial}{\partial t} + \bar{\Omega} \frac{\partial}{\partial \lambda} \right) \xi^{(1)} = -\bar{\Omega} r \cos \lambda - (\psi_\lambda^{(0)} \xi_r^{(0)} - \psi_r^{(0)} \xi_\lambda^{(0)})\tag{3.37}$$

which satisfies zero boundary and initial conditions as the following:

$$\begin{aligned}\hat{\psi}^{(1)}(0, t) &= 0 \\ \hat{\psi}^{(1)}(r_1, 0) &= 0 \\ \hat{\xi}^{(1)}(0, t) &= 0 \\ \hat{\xi}^{(1)}(r_1, 0) &= 0\end{aligned}\tag{3.38}$$

Linear equation (3.37) is a non-homogeneous linear equation for $\hat{\xi}^{(1)}$. The solution $\hat{\xi}^{(1)}$ is the sum of a function of the form $\hat{\xi}_1(r, t) e^{i\lambda} + \text{c.c.}$ arising from

the $\mathcal{O}(\gamma)$ term and $\mathcal{O}(\varepsilon)$ on the right hand side and a function arising from the nonlinear term on the right hand side. The function $\hat{\xi}_1(r, t)$ has the same the solution (3.22) we found in section 3.1. We now solve for the $\mathcal{O}(\varepsilon)$ part of the solution. First we evaluate the derivatives of $\psi^{(0)}$ and $\xi^{(0)}$ that appear in the nonlinear terms in (3.37). Differentiating the expressions (3.36) gives

$$\begin{aligned}
\psi_\lambda^{(0)} &= ik(A(r) - A(r_1)r_1^k r^{-k})e^{ik\lambda}e^{-ik\bar{\Omega}t} + ikA(r_1)r_1^k r^{-k}e^{ik\lambda} \\
&\quad - ik(A(r) - A(r_1)r_1^k r^{-k})e^{-ik\lambda}e^{ik\bar{\Omega}t} - ikA(r_1)r_1^k r^{-k}e^{-ik\lambda} \\
\psi_r^{(0)} &= (A'(r) + kA(r_1)r_1^k r^{-k-1})e^{ik\lambda}e^{-ik\bar{\Omega}t} - kA(r_1)r_1^k r^{-k-1}e^{ik\lambda} \\
&\quad + (A'(r) + kA(r_1)r_1^k r^{-k-1})e^{-ik\lambda}e^{ik\bar{\Omega}t} - kA(r_1)r_1^k r^{-k-1}e^{-ik\lambda} \\
\xi_\lambda^{(0)} = \nabla^2 \psi_\lambda^{(0)} &= ik(A''(r) + \frac{1}{r}A'(r) - \frac{k^2}{r^2}A(r))e^{-ik\bar{\Omega}t}e^{ik\lambda} \\
&\quad - ik(A''(r) + \frac{1}{r}A'(r) - \frac{k^2}{r^2}A(r))e^{ik\bar{\Omega}t}e^{-ik\lambda} \\
&= ikB(r)e^{-ik\bar{\Omega}t}e^{ik\lambda} - ikB(r)e^{ik\bar{\Omega}t}e^{-ik\lambda} \\
\xi_r^{(0)} = \nabla^2 \psi_r^{(0)} &= (A'''(r) + \frac{1}{r}A''(r) - \frac{k^2+1}{r^2}A'(r) + \frac{2k^2}{r^3}A(r))e^{ik\lambda}e^{-ik\bar{\Omega}t} \\
&\quad + (A'''(r) + \frac{1}{r}A''(r) - \frac{k^2+1}{r^2}A'(r) + \frac{2k^2}{r^3}A(r))e^{-ik\lambda}e^{ik\bar{\Omega}t} \\
&= B'(r)e^{-ik\bar{\Omega}t}e^{ik\lambda} + B'(r)e^{ik\bar{\Omega}t}e^{-ik\lambda}
\end{aligned} \tag{3.39}$$

For simplicity, we define $a = A(r_1)r_1^k$. Hence, the nonlinear Jacobian on the right hand side of (3.37) is

$$\begin{aligned}
& (\psi_\lambda^{(0)} \nabla^2 \psi_r^{(0)} - \psi_r^{(0)} \nabla^2 \psi_\lambda^{(0)}) \\
&= -ikar^{-k} e^{ik\bar{\Omega}t} (B'(r) + \frac{k}{r} B(r)) - ikar^{-k} e^{-ik\bar{\Omega}t} (B'(r) + \frac{k}{r} B(r)) \\
&+ ikar^{-k} e^{2ik\lambda} e^{-ik\bar{\Omega}t} (B'(r) - \frac{k}{r} B(r)) - ikar^{-k} e^{-2ik\lambda} e^{ik\bar{\Omega}t} (B'(r) - \frac{k}{r} B(r)) \\
&+ ike^{2ik\lambda} e^{-2ik\bar{\Omega}t} ((A(r) - ar^{-k})B'(r) + (A'(r) + kar^{-k-1}B(r))) \\
&- ike^{-2ik\lambda} e^{2ik\bar{\Omega}t} ((A(r) - ar^{-k})B'(r) + (A'(r) + kar^{-k-1}B(r))) \\
&+ ik(A(r) - ar^{-k})B'(r) - ik(A(r) - ar^{-k})B'(r) \\
&+ ik(A'(r) - ar^{-k-1})B(r) - ik(A'(r) - ar^{-k-1})B(r)
\end{aligned} \tag{3.40}$$

where $A(r)$ and $B(r)$ are two real functions. We note that the last two lines are equal to zero since each of these expressions is a pure imaginary functions plus its conjugate.

Hence (3.40) can be simplified as

$$\begin{aligned}
& (\psi_\lambda^{(0)} \nabla^2 \psi_r^{(0)} - \psi_r^{(0)} \nabla^2 \psi_\lambda^{(0)}) \\
&= -ikar^{-k} e^{-ik\bar{\Omega}t} (B'(r) + \frac{k}{r} B(r)) + \text{c.c.} \\
&+ ikar^{-k} e^{2ik\lambda} e^{-ik\bar{\Omega}t} ((B'(r) - \frac{k}{r} B(r)) + \text{c.c.}) \\
&+ ike^{2ik\lambda} e^{-2ik\bar{\Omega}t} ((A(r) - ar^{-k})B'(r) + (A'(r) + kar^{-k-1}B(r))) + \text{c.c.}
\end{aligned} \tag{3.41}$$

The terms that are independent of λ , i.e., the zero wave-number terms give the wave-induced mean flow. If we denote the right hand side of (3.41) as a function $P(r, \lambda, t) + \text{c.c.}$, we can then rewrite equation (3.37) of the order ε as:

$$\left(\frac{\partial}{\partial t} + \bar{\Omega} \frac{\partial}{\partial \lambda} \right) \xi^{(1)} = -\frac{r\bar{\Omega}}{2} e^{i\lambda} - P(r, \lambda, t) + \text{c.c.} \tag{3.42}$$

The solution of equation (3.42) is the sum of the linear combination of the solutions of the homogeneous equation, and of the sum of the particular solutions

corresponding to the terms on the right hand side. In this case, the solution will be determined separately. For simplicity, we define three expressions as the following:

$$\begin{aligned} G_1 &= ar^{-k}(B'(r) + \frac{k}{r}B(r)) \\ G_2 &= ar^{-k}(B'(r) - \frac{k}{r}B(r)) \\ G_3 &= [(A(r) - ar^{-k})B'(r) + (A'(r) + kar^{-k})B(r)] \end{aligned}$$

The particular solutions are functions proportional to $e^{-ik\bar{\Omega}t}$ (the zero wavenumber component, independent of lambda), $e^{-ik(2\lambda - \bar{\Omega}t)}$ and $e^{-2ik(\lambda - \bar{\Omega}t)}$ (both of them are corresponding to the wavenumber $2k$).

The zero wavenumber component represents a mean flow that is generated as a result of the nonlinear wave interaction. The $2k$ wavenumber term represents smaller scale waves that are generated as a result of the nonlinear wave interaction. Hence, we can write the particular solutions as $\hat{\xi}_2^{(1)} = \phi_0(r, t) + \phi_2(r, t)e^{2ik\lambda}$.

To solve equation (3.42), we write $\xi^{(1)}$ as the sum of 3 functions $\xi_1^{(1)}$, $\xi_2^{(1)}$ and $\xi_3^{(1)}$. Then we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \bar{\Omega} \frac{\partial}{\partial \lambda} \right) \xi_1^{(1)} &= -\frac{r\bar{\Omega}}{2}e^{i\lambda} + c.c. \\ \left(\frac{\partial}{\partial t} + \bar{\Omega} \frac{\partial}{\partial \lambda} \right) \xi_2^{(1)} &= -P(r, \lambda, t) + c.c. \\ \left(\frac{\partial}{\partial t} + \bar{\Omega} \frac{\partial}{\partial \lambda} \right) \xi_3^{(1)} &= 0 \end{aligned} \tag{3.43}$$

We will solve $\xi_1^{(1)}$ and $\xi_3^{(1)}$ first, then $\xi_2^{(1)}$. The equation $\xi_1^{(1)}$ is the same as (3.20). We have

$$\xi_1^{(1)} = -e^{ik\bar{\Omega}t} \int e^{ik\bar{\Omega}t} \frac{-1}{2} r \bar{\Omega} e^{i\lambda} dt + c.c.$$

Hence, the solution is the same as (3.21):

$$\xi_1^{(1)} = \frac{ir}{2}e^{i\lambda} + f_1(r)e^{-ik\bar{\Omega}t} + \text{c.c.} \quad (3.44)$$

where $f_1(r)e^{-ik\bar{\Omega}t}$ is the homogeneous solution for $\xi_3^{(1)}$ and $f_1(r)$ is an undetermined function only dependent on r . We then solve equation (3.10) to obtain $\psi^{(1)}$ and we obtain the second line of (3.12).

The function $\xi_3^{(1)}$ is the solution of the corresponding homogeneous equation in (3.43) and takes the form

$$\xi_3^{(1)} = f_3(r, \lambda - \bar{\Omega}t) \quad (3.45)$$

The function $\xi_2^{(1)}$ arises from the nonhomogeneous terms of (3.41). We have all nonlinear terms consisting of factors of $e^{-ik\bar{\Omega}t}$ and $e^{ik\lambda}$. Noticing that the ODE only has t and λ derivatives, we can write each term in the Jacobian (3.41) as $F(t, \lambda)G(r)$. Then we only need to focus on the function F . We can write the function $P(r, \lambda, t)$ as the sum of 3 terms, plus their complex conjugates.

$$\left(\frac{\partial}{\partial t} + \bar{\Omega}\frac{\partial}{\partial \lambda}\right)\xi_2^{(1)} = ike^{-ik\bar{\Omega}t}G_1(r) - ike^{ik(2\lambda - \bar{\Omega}t)}G_2(r) - ike^{2ik(\lambda - \bar{\Omega}t)}G_3(r) + \text{c.c.} \quad (3.46)$$

To solve (3.46), we consider the three terms separately to find a particular solution corresponding to each of the non-homogeneous terms. Then, the solution will be the sum of the three particular solutions.

$$\left(\frac{\partial}{\partial t} + \bar{\Omega}\frac{\partial}{\partial \lambda}\right)\xi_{2.1}^{(1)} = ike^{-ik\bar{\Omega}t}G_1(r) \quad (3.47)$$

$$\left(\frac{\partial}{\partial t} + \bar{\Omega}\frac{\partial}{\partial \lambda}\right)\xi_{2.2}^{(1)} = ike^{-ik\bar{\Omega}t}G_2(r)e^{2ik\lambda} \quad (3.48)$$

$$\left(\frac{\partial}{\partial t} + \bar{\Omega}\frac{\partial}{\partial \lambda}\right)\xi_{2.3}^{(1)} = ike^{-2ik\bar{\Omega}t}G_3(r)e^{2ik\lambda} \quad (3.49)$$

Note that the right hand side of (3.47) is independent of λ . We seek a particular solution of the form $\xi_{2.1}^{(1)}(r, t)$ independent of λ . This function is the zero wavenumber component of the solution at this order. It presents the change in the mean flow velocity that results from the nonlinear wave integration. We can integrate directly with respect to t and obtain $-ik\bar{\Omega}t$. Hence, $\xi_{2.1}^{(1)} = -\frac{1}{\bar{\Omega}}e^{-ik\bar{\Omega}t}G_1(r)$. In equation (3.48) and (3.49), the right hand side is a function of r multiplied by the complex exponential of t and λ . As the derivative of exponential functions are themselves exponential, we seek a solution of $\xi_{2.2}$, $\xi_{2.2} = f_2(r)e^{-ik\bar{\Omega}t}e^{2ik\lambda}$. In this case, we obtain the equation

$$(-ik\bar{\Omega} + 2ik)f_2(r) = ikG_2(r)$$

hence, we have

$$f_2(r) = \frac{1}{2 - \bar{\Omega}}G_2(r)$$

In equation (3.49), the non-homogeneous term is proportional to $e^{2ik(\lambda - \bar{\Omega}t)}$ and so it is a solution of the homogeneous equation. Hence, we look for a particular solution of the form $\xi_{2.3}^{(1)} = f_3(r)te^{2ik(\lambda - \bar{\Omega}t)}$. Substituting it into the equation (3.49) gives

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \bar{\Omega}\frac{\partial}{\partial \lambda}\right)\xi_{2.3}^{(1)} &= f_3(r)e^{2ik(\lambda - \bar{\Omega}t)} - 2ik\bar{\Omega}f_3(r)te^{2ik(\lambda - \bar{\Omega}t)} \\ &\quad + 2ik\bar{\Omega}f_3(r)te^{2ik(\lambda - \bar{\Omega}t)} \\ &= ikG_3(r)e^{2ik(\lambda - \bar{\Omega}t)} \\ &\implies f_3(r) = ikG_3(r) \end{aligned} \tag{3.50}$$

Hence, we have the solution as the following:

$$\xi_2^{(1)} = -\frac{1}{\bar{\Omega}}e^{-ik\bar{\Omega}t}G_1(r) + \frac{1}{2 - \bar{\Omega}}G_2(r)e^{-ik\bar{\Omega}t}e^{2ik\lambda} + ikG_3(r)te^{2ik(\lambda - \bar{\Omega}t)} + \text{c.c.} \tag{3.51}$$

Then we have the corresponding ODE:

$$\begin{aligned}\nabla^2 \psi_1^{(1)} = \xi^{(1)} &= \frac{ir}{2} e^{i\lambda} + f_1(r) e^{-i\bar{\Omega}t} e^{i\lambda} + f_3(r, \lambda - \bar{\Omega}t) \\ &+ -\frac{1}{\bar{\Omega}} e^{-ik\bar{\Omega}t} G_1(r) + \frac{1}{2 - \bar{\Omega}} G_2(r) e^{2ik\lambda} e^{-ik\bar{\Omega}t} + ik G_3(r) t e^{2ik(\lambda - \bar{\Omega}t)}\end{aligned}\quad (3.52)$$

Then we apply the zero initial condition on (3.52) to obtain

$$\begin{aligned}\xi^{(1)}(r, \lambda, 0) &= f_3(r, \lambda) + \frac{ir}{2} e^{i\lambda} + f_1(r) e^{i\lambda} - \frac{1}{\bar{\Omega}} G_1(r) + \frac{1}{2 - \bar{\Omega}} G_2(r) e^{2ik\lambda} = 0 \\ \implies f_3(r, \lambda) &= \frac{1}{\bar{\Omega}} G_1(r) - \frac{1}{2 - \bar{\Omega}} G_2(r) e^{2ik\lambda} \\ f_1(r) &= -\frac{ir}{2}\end{aligned}\quad (3.53)$$

Hence, we have that

$$\begin{aligned}\xi^{(1)}(r, \lambda, t) &= \frac{1}{\bar{\Omega}} G_1(r) (1 - e^{-ik\bar{\Omega}t}) + \frac{ir}{2} e^{i\lambda} (1 - e^{-i\bar{\Omega}t}) \\ &+ \frac{1}{2 - \bar{\Omega}} G_2(r) e^{2ik\lambda} (e^{-ik\bar{\Omega}t} - 1) + ik G_3(r) t e^{2ik(\lambda - \bar{\Omega}t)}\end{aligned}\quad (3.54)$$

In equation (3.47) the function $\xi_{2.1}^{(1)}$ on the right hand side depends on r and t only. So we have

$$\nabla^2 \psi_{2.1}^{(1)} = \frac{\partial}{\partial r^2} \psi + \frac{1}{r} \frac{\partial}{\partial r} \psi$$

In equation (3.48) and (3.49) that the function $\xi_{2.2}^{(1)}$ and $\xi_{2.3}^{(1)}$ on the right hand side is proportional to $e^{2ik\lambda}$. Thus, ψ for $\psi_{2.2}^{(1)}$ and $\psi_{2.3}^{(1)}$ takes the form $\psi = \hat{\psi}(r, t) e^{2ik\lambda} + c.c.$ and hence,

$$\nabla^2 \psi = \left(\frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{4k^2}{r^2} \right) \hat{\psi} e^{2ik\lambda}$$

The solution of $\nabla^2 \psi^{(1)} = \xi^{(1)}$ with $\xi^{(1)}$ given in (3.54) will be solved at the next subsection.

Solving the DE

In this section, we will solve those equations (3.47) term by term. On the right hand side of equation (3.54), the term $\frac{1}{\Omega}G_1(r)(1 - e^{-ik\bar{\Omega}t})$ represents the zero wavenumber, that is, the wave induced mean flow. The term $\frac{ir}{2}e^{i\lambda}(1 - e^{-i\bar{\Omega}t})$ has a wavenumber 1 and it arises from the γ term. The other terms have a wave number of $2k$. By adding up the particular solution and the homogeneous solution, we can obtain the final solution of $\hat{\psi}^{(1)}$. The process is represented below.

We consider each of these equations separately.

$$(\hat{\psi}_{2.1}^{(1)})_{rr} + \frac{1}{r}(\hat{\psi}_{2.1}^{(1)})_r = -\frac{1}{\Omega}e^{-ik\bar{\Omega}t}G_1(r) = -\frac{a}{\Omega}r^{-k}(B'(r) + \frac{k}{r}B(r))e^{-ik\bar{\Omega}t} \quad (3.55)$$

$$(\hat{\psi}_{2.2}^{(1)})_{rr} + \frac{1}{r}(\hat{\psi}_{2.2}^{(1)})_r - \frac{4k^2}{r^2}(\hat{\psi}_{2.2}^{(1)}) = \frac{1}{\Omega}e^{ik(2\lambda - \bar{\Omega}t)}G_2(r) \quad (3.56)$$

$$(\hat{\psi}_{2.3}^{(1)})_{rr} + \frac{1}{r}(\hat{\psi}_{2.3}^{(1)})_r - \frac{4k^2}{r^2}(\hat{\psi}_{2.3}^{(1)}) = \frac{1}{\Omega}te^{2ik(\lambda - \bar{\Omega}t)}G_3(r) \quad (3.57)$$

We can use the method of variation of parameters to find particular solutions of the equation (3.55). However, for (3.56) and (3.57), this method gives the solution in terms of an integral, not in closed form.

We can write equation (3.55) as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{\psi}_r) = -\frac{a}{\Omega}r^{-k}(B'(r) + \frac{k}{r}B(r))e^{-ik\bar{\Omega}t} \quad (3.58)$$

or

$$\frac{\partial}{\partial r}(r\hat{\psi}_r) = -\frac{a}{\Omega}(r^{-k+1}B'(r) + kr^{-k}B(r))e^{-ik\bar{\Omega}t} \quad (3.59)$$

We use integration by parts to obtain

$$\begin{aligned}
& \int [r^{-k+1}B'(r) + kr^{-k}B(r)]dr \\
&= r^{-k+1}B(r) + (k-1) \int r^{-k}B(r)dr + k \int r^{-k}B(r)dr \\
&= r^{-k+1}B(r) + (2k-1) \int r^{-k}B(r)dr \\
&= r^{-k+1}B(r) + (2k-1) \int r^{-k}(A'' + \frac{1}{r}A' - \frac{k^2}{r^2}A)dr \\
&= r^{-k+1}B(r) + (2k-1) \int (r^{-k}A'' + r^{-k-1}A' - k^2r^{-k-2}A)dr
\end{aligned} \tag{3.60}$$

Using integration by parts on $\int r^{-k}A''dr$, we obtain

$$\int r^{-k}A''dr = r^{-k}A' + k \int r^{-k-1}A'dr = r^{-k}A' + k(r^{-k-1}A + (k+1) \int r^{-k-2}Adr)$$

Using integration by parts on $\int r^{-k-1}A'dr$, we obtain

$$\int r^{-k-1}A'dr = r^{-k-1}A + (k+1) \int r^{-k-2}Adr$$

Hence (3.60) becomes

$$\begin{aligned}
& \int [r^{-k+1}B'(r) + kr^{-k}B(r)]dr \\
&= r^{-k+1}B(r) + (2k-1)[r^{-k}A' + kr^{-k-1}A \\
&\quad + (k^2+k) \int r^{-k-2}Adr + r^{-k-1}A + (k+1) \int r^{-k-2}Adr] \\
&= r^{-k+1}B(r) + (2k-1)[r^{-k}A' + (k+1)r^{-k-1}A + (1-k^2) \int r^{-k-2}Adr]
\end{aligned} \tag{3.61}$$

Hence we have

$$r\hat{\psi}_r = -\frac{a}{\Omega(r)}[r^{-k+1}B(r) + (2k-1)(r^{-k}A' + (k+1)r^{-k-1}A + (1-k^2) \int r^{-k-2}Adr)] \tag{3.62}$$

Dividing by r , we have the dominant terms of the solution

$$(\hat{\psi}_{2.1}^{(1)})_r \sim -\frac{a}{\bar{\Omega}}(r^{-k}B(r) + (2k-1)r^{-k-1}A'(r)) \quad (3.63)$$

Since $v_{azimuthal} = \psi_r$, this gives us an asymptotic expression for the wave-induced mean azimuthal velocity

$$\bar{v}^{(1)} \sim -\frac{1}{\bar{\Omega}}(r^{-k}A''(r) + \mathcal{O}(r^{-k-1}))e^{-ik\bar{\Omega}(r)t} \quad (3.64)$$

which indicates that the mean flow changes with t .

The other two solutions of (3.56) and (3.57) are still given by the method of variation of parameters. We have that the solutions of the homogeneous equation

$$(\hat{\psi}_{2.2}^{(1)})_{rr} + \frac{1}{r}(\hat{\psi}_{2.2}^{(1)})_r - \frac{4k^2}{r^2} = 0$$

are $y_1 = r^{2k}$, $y_2 = r^{-2k}$ with the Wronskian being $W = -\frac{4k}{r}$. In this case, we can derive the particular solution as we know that from the previous part, in equation (3.19). Hence, we have $\hat{\psi}_{2.2}^{(1)} = u_1y_1 + u_2y_2$ and

$$\begin{aligned} \hat{\psi}_{p2}(r, t) &= \frac{1}{4k}y_1(r) \int_{r_1}^r y_2(\tilde{r})G_2(\tilde{r})d\tilde{r} - \frac{1}{4k}y_2(r) \int_{r_1}^r y_1(\tilde{r})G_2(\tilde{r})d\tilde{r} \\ &= \frac{1}{4k} \int_{r_1}^r (r^{2k}\tilde{r}^{-2k} - \tilde{r}^{2k}r^{-2k})G_2(\tilde{r})d\tilde{r} \end{aligned} \quad (3.65)$$

where $\tilde{r}$ is the variable of integral. We have set the lower limit as r_1 and the upper limit as r , so that $\psi_p = 0$ at $r = r_1$. Each term in the integrand is of terms multiplying $e^{-\mu^2\tilde{r}}$, so we cannot obtain a closed of expression and we have to write them as integrals.

Similarly, we have the homogeneous solution of $\hat{\psi}_{2.3}^{(1)}$ is

$$\hat{\psi}_{p3}(r, t) = \frac{1}{4k} \int_{r_1}^r (r^{2k}\tilde{r}^{-2k} - \tilde{r}^{2k}r^{-2k})G_3(\tilde{r})d\tilde{r} \quad (3.66)$$

Summarizing all above, we obtain the solution of the DE (3.54) being the summation of the particular solutions and the homogeneous solutions.

$$\begin{aligned}
\psi^{(1)} &= \hat{\psi}_{2,1}^{(1)}(1 - e^{-ik\bar{\Omega}t}) \\
&+ \frac{1}{2 - \bar{\Omega}} \hat{\psi}_{p2} e^{2ik\lambda} (e^{-ik\bar{\Omega}t} - 1) \\
&+ ikt \hat{\psi}_{p3} e^{2ik(\lambda - \bar{\Omega}t)} \\
&+ \gamma \frac{3i}{16} (r^3 - \frac{r_1^4}{r}) (1 - e^{-i\bar{\Omega}t}) e^{i\lambda} \\
&+ c_1(t)r^{-1} + c_2(t)r^{-2k} + c.c.
\end{aligned} \tag{3.67}$$

This solution can only be written in an integral form as we have terms $\hat{\psi}_{p2}$ and $\hat{\psi}_{p3}$. It can not be written in a closed form.

In this chapter, we considered the equation (2.37) with the case of the constant mean vortex angular velocity $\bar{\Omega}$ in the annulus domain (2.39) write the barotropic vorticity equation as a perturbation series. We discussed the case (iii) $\beta \ll \varepsilon \ll \gamma = \frac{\beta}{\varepsilon} \ll 1$ and (iv) $\gamma \sim 1$ to get the equation (3.1). We then solve this linear problem to obtain the dominant order solution of the equation (3.30) and the vorticity (3.31), with the diagram presented in Figure 3.1 and 3.2. Then we considered the nonlinear problem with case (ii) where $\beta \ll \gamma = \frac{\beta}{\varepsilon} \sim \varepsilon$ and seek for a solution of the asymptotic form. The solution of vorticity is given in (3.54). However the solution of $\mathcal{O}(\varepsilon)$ is given in equation (3.67) and unable to be written in a closed form. In the next chapter, we will discuss about the case where the background angular velocity $\bar{\Omega}$ is a function of r .

Chapter 4

The case with radially varying background angular velocity

4.1 Relative magnitude of parameters

In chapter 2, we introduced an initial boundary value problem describing vortex Rossby wave evolution in a cyclonic vortex with the β effect, given by equation (2.37) with the initial condition (2.40) and the boundary conditions (2.41).

In chapter 3, we considered a case where the background angular velocity $\bar{\Omega}$ is constant. In section 3.1, we linearized the equation and found the expressions for the wave vorticity (3.31) and the wave stream function solution (3.30). In section 3.2, we considered the nonlinear equation and found the asymptotic form of the solution. However, the solution for the streamfunction cannot be written in a closed form, so we have the expression of (3.67). The case with constant $\bar{\Omega}$ gives a good starting point to derive solutions and analyze the behaviour of the solutions.

However, we saw in chapter 3 that this choice of $\bar{\Omega}$ gives solutions with secular terms that go to infinity as $r \rightarrow \infty$. In this chapter, we consider a more realistic

configuration where $\bar{\Omega}$ depends on the radial variable r , which allows us to avoid this situation. Martinez [27,28] used an exponential profile for the basic flow angular velocity, which they noted gives a reasonable representation of typical flows observed in tropical cyclones. Based on these, we choose $\bar{\Omega}(r) = \Omega_0 e^{-\alpha r^2}$ where Ω_0 and α are specified constant. We can choose the function $A(r) = e^{-\mu r^2}$, so that the initial condition becomes

$$\psi(r, \lambda, 0) = A(r)e^{ik\lambda} = e^{-\mu r^2}e^{ik\lambda} \quad (4.1)$$

In this case, there are four non-dimensional parameters: ε which measures the amplitude of the wave relative to the background vortex flow and determines the magnitude of the nonlinear terms in equation (2.37), β which measures the effect of the gradient of the Coriolis parameter, α which measures the scale of the vortex, and μ which measures the scale of the initial wave profile. Since the wave is a small perturbation to the background vortex, we assume the background flow amplitude is $\mathcal{O}(1)$ as in [23] and [24].

We seek solutions that take the asymptotic form:

$$\begin{aligned} \psi &\sim \psi^{(0)} + \alpha\psi^{(01)} + \mathcal{O}(\alpha^2) + \mathcal{O}(\varepsilon) + \psi^{(\gamma)} \\ \xi &\sim \xi^{(0)} + \alpha\xi^{(01)} + \mathcal{O}(\alpha^2) + \mathcal{O}(\varepsilon) + \xi^{(\gamma)} \end{aligned} \quad (4.2)$$

The relative magnitude of the parameters is discussed below.

The function $A(r)$ defines the wave and $\bar{\Omega}(r)$ defines the background vortex. The vortex should be larger than the wave and the wave should be inside the vortex, so the function $A(r)$ has to go to zero faster than $\bar{\Omega}(r)$ as $r \rightarrow \infty$. Hence, we have $\mu > \alpha$.

In section 4.2, the fundamental model of $\bar{\Omega}$ without talking about the relative magnitude of two values is firstly considered.

We need to decide the relative magnitude of the two parameters: α and ε , in

this chapter. There are three possibilities: either $\varepsilon \ll \alpha$, $\alpha \sim \varepsilon$ or $\alpha \ll \varepsilon$.

The case where $\varepsilon \ll \alpha$ will be discussed in section 4.2, where the equation is linear up to $\mathcal{O}(\alpha)$ and the nonlinear $\mathcal{O}(\varepsilon)$ terms appear at higher order. α dominates the equation hence, we could work out the first few terms of the linear solution in terms of α , and then we use these terms as the linear solution, as we did in equation (3.36).

In the case where $\varepsilon \sim \alpha$, they are at a similar magnitude, the $\mathcal{O}(\alpha)$ terms of the linear solution will appear at the same order as the $\mathcal{O}(\varepsilon)$ term of the nonlinear solution. In the case where $\alpha \ll \varepsilon$, the $\mathcal{O}(\varepsilon)$ term of the nonlinear solution will appear before the $\mathcal{O}(\alpha)$ term of the linear solution. We will do another nonlinear analysis as in chapter 3 and considering the first term of the linear solution, which produces a mixed solution of both. The relative magnitudes of parameters and how they affect the solution will be discussed in section 4.3.

4.2 Linear solution: ($\varepsilon \ll \alpha$)

In this section, we consider the case where $\bar{\Omega}(r) = \Omega_0 e^{-\alpha r^2}$ with $\varepsilon \ll \alpha$ and with $\gamma \ll 1$. Ω_0 is a specified constant. In that case, we can examine the leading-order behaviour of the solution of the initial value problem by neglecting the nonlinear terms and the terms of order β in equation (2.37), as we did in section 4.1. We consider the case where we also have $\gamma \ll \alpha$ and neglect the non-homogeneous term of order γ .

Since $\bar{\Omega}(r) = \Omega_0 e^{-\alpha r^2}$, we obtain

$$\begin{aligned}\bar{\Omega}'(r) &= -2\alpha r \Omega_0 e^{-\alpha r^2} = -2\alpha r \bar{\Omega}(r) \\ \bar{\Omega}''(r) &= (-2\alpha + 4\alpha^2 r^2) \Omega_0 e^{-\alpha r^2} = (-2\alpha + 4\alpha^2 r^2) \bar{\Omega}(r)\end{aligned}\tag{4.3}$$

We can substitute (4.3) into the second term of equation (2.37), the coefficients of the Ψ_λ term is:

$$\begin{aligned}
\frac{1}{r} \frac{d}{dr} (\bar{v}' + \frac{\bar{v}}{r}) &= \frac{1}{r} \frac{d}{dr} (r\bar{\Omega}' + \bar{\Omega} + \bar{\Omega}) \\
&= \frac{1}{r} (r\bar{\Omega}'' + \bar{\Omega}' + 2\bar{\Omega}') = \bar{\Omega}'' + \frac{3}{r} \bar{\Omega}' \\
&= \Omega_0 (4\alpha^2 r^2 - 2\alpha - 6\alpha) e^{-\alpha r^2}
\end{aligned} \tag{4.4}$$

The initial condition is (4.1) and we assume $\alpha \ll \mu \sim 1$. As before, we define the wave vorticity by

$$\xi = \nabla^2 \psi = \psi_{rr} + \frac{1}{r} \psi_r + \frac{1}{r^2} \psi_{\lambda\lambda}$$

Then, the linear equation is

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \bar{\Omega} \frac{\partial}{\partial \lambda} \right) \xi - (\bar{\Omega}'' + \frac{3}{r} \bar{\Omega}') \psi_\lambda = 0 \\
\Rightarrow &\left(\frac{\partial}{\partial t} + \Omega_0 e^{-\alpha r^2} \frac{\partial}{\partial \lambda} \right) \xi + \Omega_0 (8\alpha - 4\alpha^2 r^2) e^{-\alpha r^2} \psi_\lambda = 0
\end{aligned} \tag{4.5}$$

with initial conditions $\psi(r, \lambda, 0) = A(r) e^{ik\lambda} + \text{c.c.}$

and $\xi(r, \lambda, 0) = B(r) e^{ik\lambda} + \text{c.c.}$

where

$$\psi(r, \lambda, t) = \hat{\psi}(r, t) e^{ik\lambda} + \text{c.c.}$$

$$\xi(r, \lambda, t) = \hat{\xi}(r, t) e^{ik\lambda} + \text{c.c.}$$

We seek a solution in the form of a perturbation series in powers of the parameter α :

$$\begin{aligned}
\hat{\psi}(r, t) &\sim \psi^{(0)}(r, t) + \alpha \psi^{(01)}(r, t) + \alpha^2 \psi^{(02)}(r, t) \\
\hat{\xi}(r, t) &\sim \xi^{(0)}(r, t) + \alpha \xi^{(01)}(r, t) + \alpha^2 \xi^{(02)}(r, t)
\end{aligned} \tag{4.6}$$

We substitute (4.6) into equation (2.37) to obtain

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + ik\bar{\Omega}(r) \right) (\hat{\psi}_{rr}^{(0)} + \frac{1}{r}\hat{\psi}_r^{(0)} - \frac{k^2}{r^2}\hat{\psi}^{(0)}) \\
& + \alpha \left(\frac{\partial}{\partial t} + ik\bar{\Omega}(r) \right) (\hat{\psi}_{rr}^{(01)} + \frac{1}{r}\hat{\psi}_r^{(01)} - \frac{k^2}{r^2}\hat{\psi}^{(01)}) \\
& + \alpha^2 \left(\frac{\partial}{\partial t} + ik\bar{\Omega}(r) \right) (\hat{\psi}_{rr}^{(02)} + \frac{1}{r}\hat{\psi}_r^{(02)} - \frac{k^2}{r^2}\hat{\psi}^{(02)}) \\
& = -\alpha 8ik\bar{\Omega}(r)\hat{\psi}^{(0)} - \alpha^2 8ik\bar{\Omega}(r)\hat{\psi}^{(01)} + 4\alpha^2 r^2 k\bar{\Omega}(r)\hat{\psi}^{(0)} \\
& + \mathcal{O}(\alpha^3) + \mathcal{O}(\varepsilon) + \mathcal{O}(\gamma) + \mathcal{O}(\beta)
\end{aligned} \tag{4.7}$$

We can solve recursively for the functions $\psi^{(0)}$, $\psi^{(01)}$ and $\psi^{(02)}$ and apply the initial conditions. The initial conditions and boundary conditions are assigned to each functions separately.

$$\begin{aligned}
\psi^{(0)}(r, \lambda, 0) &= A(r)e^{ik\lambda} + \text{c.c.}, \quad \psi^{(0)}(r_1, \lambda, t) = A(r_1)e^{ik\lambda} + \text{c.c.}; \\
\psi^{(01)}(r, \lambda, 0) &= 0, \quad \psi^{(01)}(r_1, \lambda, t) = 0; \\
\psi^{(02)}(r, \lambda, 0) &= 0, \quad \psi^{(02)}(r_1, \lambda, t) = 0
\end{aligned} \tag{4.8}$$

Solution of $\hat{\psi}^{(0)}$

At the leading order, we have

$$\left(\frac{\partial}{\partial t} + ik\bar{\Omega}(r) \right) (\hat{\psi}_{rr}^{(0)} + \frac{1}{r}\hat{\psi}_r^{(0)} - \frac{k^2}{r^2}\hat{\psi}^{(0)}) = 0 \tag{4.9}$$

and we denote

$$\hat{\xi}^{(0)} = (\hat{\psi}_{rr}^{(0)} + \frac{1}{r}\hat{\psi}_r^{(0)} - \frac{k^2}{r^2}\hat{\psi}^{(0)}) \tag{4.10}$$

which is the leading order component of the wave vorticity, with the initial condition

$$\hat{\xi}^{(0)}(r, \lambda, 0) = B(r)e^{ik\lambda} \tag{4.11}$$

The solution satisfying this initial condition is

$$\hat{\xi}^{(0)} = B(r)e^{-ik\bar{\Omega}(r)t} \quad (4.12)$$

Then we substitute (4.12) into (4.10) to obtain:

$$(\hat{\psi}_{rr}^{(0)} + \frac{1}{r}\hat{\psi}_r^{(0)} - \frac{k^2}{r^2}\hat{\psi}^{(0)}) = B(r)e^{-ik\bar{\Omega}(r)t} \quad (4.13)$$

To find a solution $\hat{\psi}^{(0)}$ we consider the function $\tilde{\psi}(r, t) = A(r)e^{-ik\bar{\Omega}(r)t}$ to satisfy:

$$\begin{aligned} \tilde{\psi}_{rr} + \frac{1}{r}\tilde{\psi}_r - \frac{k^2}{r^2}\tilde{\psi} &= [A'' + \frac{1}{r}A' - \frac{k^2}{r^2}A - 2ik\bar{\Omega}'(r)tA' \\ &\quad - k^2(\bar{\Omega}'(r))^2t^2A - ik^3\frac{\bar{\Omega}'(r)t}{r}A - ik\bar{\Omega}''(r)tA]e^{-ik\bar{\Omega}(r)t} \end{aligned} \quad (4.14)$$

The derivatives $\bar{\Omega}'(r)$ and $\bar{\Omega}''(r)$ are given by (4.3).

We substitute (4.3) into (4.14) to obtain:

$$\begin{aligned} \tilde{\psi}_{rr} + \frac{1}{r}\tilde{\psi}_r - \frac{k^2}{r^2}\tilde{\psi} &= \left(A'' + \frac{1}{r}A' - \frac{k^2}{r^2}A\right)e^{-ik\bar{\Omega}(r)t} \\ &\quad - \alpha[4ikr\bar{\Omega}(r)tA' + 4ik\bar{\Omega}(r)tA]e^{-ik\bar{\Omega}(r)t} + \mathcal{O}(\alpha^2) \\ &= B(r)e^{-ik\bar{\Omega}(r)t} - \alpha[4ikr\bar{\Omega}(r)tA' + 4ik\bar{\Omega}(r)tA]e^{-ik\bar{\Omega}(r)t} + \mathcal{O}(\alpha^2) \end{aligned} \quad (4.15)$$

Thus, the function $\tilde{\psi} = A(r)e^{-ik\bar{\Omega}(r)t}$ satisfies equation (4.13) at leading order. This tells us that a particular solution of (4.13) can be written as $\tilde{\psi}(r, t) + \mathcal{O}(\alpha)$, or

$$A(r)e^{-ik\bar{\Omega}(r)t} + \mathcal{O}(\alpha)$$

Hence at leading order, we have

$$\hat{\psi}^{(0)}(r, t) = c_1(t)r^k + c_2(t)r^{-k} + A(r)e^{-ik\bar{\Omega}(r)t} \quad (4.16)$$

Apply the initial condition and the boundary condition (4.8), we set $c_1(t) = 0$ to satisfy the assumption that the homogeneous solution of $\hat{\psi}(r, t)$ is finite as $r \rightarrow \infty$. Hence, we have

$$\begin{aligned}
\hat{\psi}^{(0)}(r_1, 0) &= c_2(0)r_1^{-k} + A(r_1) = A(r_1) \\
\implies c_2(0) &= 0 \\
\hat{\psi}^{(0)}(r_1, t) &= (A(r_1) - c_2(0)r_1^{-k})e^{-ik\bar{\Omega}(r_1)t} + c_2(t)r_1^{-k} = A(r_1) \quad (4.17) \\
\implies c_2(t)r_1^{-k} &= A(r_1)(1 - e^{-ik\bar{\Omega}(r_1)t}) \\
\implies c_2(t) &= A(r_1)(1 - e^{-ik\bar{\Omega}(r_1)t})r_1^k
\end{aligned}$$

We obtain, at the leading order,

$$\begin{aligned}
\hat{\psi}^{(0)} &= A(r)e^{-ik\bar{\Omega}(r)t} + A(r_1)(1 - e^{-ik\bar{\Omega}(r_1)t})r_1^k r^{-k} \\
&= A(r)e^{-ik\bar{\Omega}(r)t} + ar^{-k}(1 - e^{-ik\bar{\Omega}(r_1)t})
\end{aligned} \quad (4.18)$$

where $a = A(r_1)r_1^{-k}$.

Figure 4.1 represents the behaviour of the functions $\hat{\xi}^{(0)}$ and $\hat{\psi}^{(0)}$ given by (4.12) and (4.18). We will use parameters similar to (3.32). However, we changed the domain of r from $0.1 \leq r \leq 5$ to $0.1 \leq r \leq 8$ in order to show a larger interval of r values. The choice of parameters for the flows is listed below.

$$\left\{ \begin{array}{l} \mu = 0.01, \\ A_0 = 1, \\ 0.1 \leq r \leq 8, \\ \Omega_0 = 1, \\ k = 2, \\ \gamma = 0.001, \\ r \sim 10 \end{array} \right. . \quad (4.19)$$

(a) Plot $\hat{\xi}^{(0)}(r, t)$ as a function of r according to (4.12)

(b) Contour plot $\xi^{(0)}(r, \lambda, t)$ as a function of r according to (4.12)

Figure 4.1: Plots of $\hat{\xi}^{(0)}$

(a) Plot $\hat{\psi}^{(0)}(r, t)$ as a function of r according to (4.18)

(b) Contour plot $\psi^{(0)}(r, \lambda, t)$ as a function of r according to (4.18)

Figure 4.2: Plots of $\hat{\psi}^{(0)}$

Solution of $\hat{\psi}^{(01)}$

Having solved $\hat{\psi}^{(0)}$ at leading order, we return to equation (4.7) and solve for $\hat{\xi}^{(01)}$ and $\hat{\psi}^{(01)}$ which are the $\mathcal{O}(\alpha)$ terms in the series (4.6). At $\mathcal{O}(\alpha)$ from equation (4.15), we have

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + ik\bar{\Omega}(r) \right) \left(\hat{\psi}_{rr}^{(01)} + \frac{1}{r} \hat{\psi}_r^{(01)} - \frac{k^2}{r^2} \hat{\psi}^{(01)} \right) \\
& = -8ik\bar{\Omega}(r) \hat{\psi}^{(0)} - \left(\frac{\partial}{\partial t} + ik\bar{\Omega}(r) \right) 4ik\bar{\Omega}(r)t(rA_r + A)e^{-ik\bar{\Omega}(r)t}
\end{aligned} \tag{4.20}$$

with the zero initial condition.

Defining $\hat{\xi}^{(01)} = \hat{\psi}_{rr}^{(01)} + \frac{1}{r} \hat{\psi}_r^{(01)} - \frac{k^2}{r^2} \hat{\psi}^{(01)}$, we can rewrite the solution as

$$\left(\frac{\partial}{\partial t} + ik\bar{\Omega}(r) \right) \hat{\xi}^{(01)} = -8ik\bar{\Omega}(r) \hat{\psi}^{(0)} - \left(\frac{\partial}{\partial t} + ik\bar{\Omega}(r) \right) 4ik\bar{\Omega}(r)t(rA_r + A)e^{-ik\bar{\Omega}(r)t} \tag{4.21}$$

This equation can be solved by multiplying with the integrating factor $u(t) = e^{\int ik\bar{\Omega}(r)dt} = e^{ik\bar{\Omega}(r)t}$, which gives

$$\begin{aligned}
\frac{\partial}{\partial t} [e^{ik\bar{\Omega}(r)t} \hat{\xi}^{(01)}] &= -8ik\bar{\Omega}(r)A(r) - 8ika\bar{\Omega}(r)r^{-k}(1 - e^{-ik\bar{\Omega}(r_1)t})e^{ik\bar{\Omega}(r)t} \\
&\quad - \frac{\partial}{\partial t} [4ik\bar{\Omega}(r)t(rA_r + A)]
\end{aligned} \tag{4.22}$$

In order to satisfy the zero initial condition that $\hat{\xi}^{(1)}(r, 0) = 0$, we integrate this equation from 0 to t , setting the lower limit of the integration at 0 so that the solution will satisfy the zero initial condition, we obtain

$$\begin{aligned}
e^{ik\bar{\Omega}(r)t} \hat{\xi}^{(01)} &= -8ik\bar{\Omega}(r) \int_0^t \hat{\psi}^{(0)} e^{ik\bar{\Omega}(r)t} dt - 4ik\bar{\Omega}(r)t(rA_r + A) \\
&= \int_0^t (-8ik\bar{\Omega}(r)A(r) - 8ika\bar{\Omega}(r)r^{-k}e^{ik\bar{\Omega}(r)t} \\
&\quad + 8ika\bar{\Omega}(r)r^{-k}e^{-ik\bar{\Omega}(r_1)t}e^{ik\bar{\Omega}(r)t}) dt - 4ik\bar{\Omega}(r)t(rA_r + A) \\
\hat{\xi}^{(01)} &= -8ikt\bar{\Omega}(r)A(r)e^{-ik\bar{\Omega}(r)t} + 8ar^{-k}(e^{-ik\bar{\Omega}(r)t} - 1) \\
&\quad - \frac{8\bar{\Omega}(r)ar^{-k}}{\bar{\Omega}(r) - \bar{\Omega}(r_1)}(e^{-ik\bar{\Omega}(r)t} - e^{-ik\bar{\Omega}(r_1)t}) \\
&\quad - 4ik\bar{\Omega}(r)t(rA_r + A)e^{-ik\bar{\Omega}(r)t}
\end{aligned} \tag{4.23}$$

The equation for the $\mathcal{O}(\alpha)$ stream function perturbation is

$$\hat{\psi}_{rr}^{(01)} + \frac{1}{r}\hat{\psi}_r^{(01)} - \frac{k^2}{r^2}\hat{\psi}^{(01)} = \hat{\xi}^{(01)} \quad (4.24)$$

Hence, we have the equation

$$\begin{aligned} (\hat{\psi}_{rr}^{(01)} + \frac{1}{r}\hat{\psi}_r^{(01)} - \frac{k^2}{r^2}\hat{\psi}^{(01)}) &= -8ikt\bar{\Omega}(r)A(r)te^{-ik\bar{\Omega}(r)t} + 8ar^{-k}(e^{-ik\bar{\Omega}(r)t} - 1) \\ &\quad - \frac{8\bar{\Omega}(r)ar^{-k}}{\bar{\Omega}(r) - \bar{\Omega}(r_1)}(e^{-ik\bar{\Omega}(r)t} - e^{-ik\bar{\Omega}(r_1)t}) \\ &\quad - 4ik\bar{\Omega}(r)t(rA_r + A)e^{-ik\bar{\Omega}(r)t} \end{aligned} \quad (4.25)$$

with the zero initial condition. As before the solutions of the corresponding homogeneous equation are $y_1 = r^k$ and $y_2 = r^{-k}$ and their Wronskian is $\frac{-2k}{r}$. Hence, the solution is

$$\hat{\psi}^{(01)}(r, t) = \hat{\psi}_p^{(01)}(r, t) + c_1(t)r^{-k} + c_2(t)r^k \quad (4.26)$$

where $c_2(t) = 0$. The particular solution, similar to the previous calculation, is determined by the integral formula.

$$\begin{aligned} \hat{\psi}_p^{(01)} &= r^k \int \frac{\tilde{r}r^{-k}}{2k}\hat{\xi}^{(01)}d\tilde{r} - r^{-k} \int \frac{\tilde{r}r^k}{2k}\hat{\xi}^{(01)}d\tilde{r} \\ &= \frac{r^k}{2k} \int \tilde{r}^{1-k}\hat{\xi}^{(01)}d\tilde{r} - \frac{r^{-k}}{2k} \int \tilde{r}^{k+1}\hat{\xi}^{(01)}d\tilde{r} \end{aligned} \quad (4.27)$$

However, it cannot be evaluated in a closed form.

If $\frac{\beta}{\varepsilon} = \gamma \ll \alpha$, then the $\mathcal{O}(\gamma)$ term appears at the higher order, so we cannot evaluate it until we have found the $\mathcal{O}(\alpha)$ streamfunction solution (4.27) in a closed form.

If $1 \gtrsim \gamma \gtrsim \alpha$ then we can evaluate the $\mathcal{O}(\gamma)$ term independently at $\hat{\psi}_k$ the same way we did for the constant $\bar{\Omega}$ case in chapter 3. In such way, the $\mathcal{O}(\gamma)$ term $(\hat{\psi}^{(\gamma)}(r, t)e^{i\lambda} + c.c.)$ is proportional to $e^{i\lambda}$ with the corresponding vorticity

$(\hat{\xi}^{(\gamma)}(r, t)e^{i\lambda} + c.c.)$ as we did for the function (3.30) and (3.31). In this case, we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} + i\bar{\Omega}(r)\right)\hat{\xi}^{(\gamma)} &= -\frac{r\bar{\Omega}(r)}{2} \\ \text{and } \hat{\psi}_{rr}^{(\gamma)} + \frac{1}{r}\hat{\psi}_r^{(\gamma)} - \frac{1}{r^2}\hat{\psi}^{(\gamma)} &= \hat{\xi}^{(\gamma)} \end{aligned} \quad (4.28)$$

that is proportional to $e^{i\lambda}$. The equation is the same as (3.20) with the function $\bar{\Omega}(r)$. Hence, we will use the same procedure to solve it. We obtain the same $\hat{\xi}$ as before in (3.21):

$$\hat{\xi}^{(\gamma)} = \frac{ir}{2} + C(r)e^{-i\bar{\Omega}t} \quad (4.29)$$

Applying the zero initial condition on (4.29), we have

$$C(r) = \frac{-ir}{2} \quad (4.30)$$

which gives

$$\hat{\xi}^{(\gamma)}(r, t) = \frac{ir}{2}(1 - e^{-i\bar{\Omega}t}) \quad (4.31)$$

Thus, the part of the vorticity equation that results from the $\mathcal{O}(\gamma)$ nonhomogeneous term is

$$\begin{aligned} \xi^{(\gamma)}(r, \lambda, t) &= \hat{\xi}^{(\gamma)}(r, t)e^{i\lambda} + c.c. \\ &= \frac{ir}{2}e^{i\lambda} \left(1 - e^{-i\bar{\Omega}(r)t}\right) + c.c. \end{aligned} \quad (4.32)$$

This function represents the "beta-gyre" part of the vorticity solution which appears at $\mathcal{O}(\gamma) = \mathcal{O}(\frac{\beta}{\varepsilon})$. In chapter 3, where we considered a constant $\bar{\Omega}$ profile, this part of the solution was a secular term which would go to infinity as $r \rightarrow \infty$. Here, $\xi^{(\gamma)} \rightarrow 0$ as $r \rightarrow \infty$, which is the expected physically realistic behaviour. To find information about the corresponding term in the streamfunction solution, we solve

$$\hat{\psi}_{rr}^{(\gamma)} + \frac{1}{r}\hat{\psi}_r^{(\gamma)} - \frac{1}{r^2}\hat{\psi}^{(\gamma)} = \frac{ir}{2}(1 - e^{-i\bar{\Omega}t}) \quad (4.33)$$

from (4.28).

As in the case of constant $\bar{\Omega}$, the solutions of the corresponding homogeneous equation for $\hat{\psi}^{(\gamma)}$ are r and r^{-1} , with the corresponding Wronskian $W = -\frac{2}{r}$. Hence, we obtain the particular solution:

$$\begin{aligned} u_{11} &= \int \frac{r}{2} r^{-1} \frac{ir}{2} (1 - e^{-i\bar{\Omega}(r)t}) dr \\ &= \int \frac{ir}{4} (1 - e^{-i\bar{\Omega}t}) dr \\ u_{12} &= - \int \frac{r}{2} r \frac{ir}{2} (1 - e^{-i\bar{\Omega}t}) dr \\ &= \int \frac{ir^3}{4} (e^{-i\bar{\Omega}(r)t} - 1) dr \end{aligned} \quad (4.34)$$

In the case of chapter 3, with the constant $\bar{\Omega}$, we were able to integrate function (4.34) exactly and get the solution (3.28). However, with the varying function $\bar{\Omega}(r)$, we cannot obtain any closed form as we are integrating a function $e^{-i\Omega_0 e^{-\alpha r^2} t}$.

However, we can still deduce that as $r \rightarrow \infty$, $\bar{\Omega}(r) \rightarrow 0$. Hence, $e^{-ik\bar{\Omega}(r)t} \rightarrow e^0 = 1$ and the integrand will approach 0. Hence, the solution is bounded as $r \rightarrow \infty$ and the boundary condition is satisfied. The diagram of $\hat{\xi}_1$ is plotted below. Notice that in the situation we have $k = 1$, hence, it requires us to use a wider range of radius. In this diagram we set everything the same as chapter 2, with the only difference of r being $0.1 \leq r \leq 15$.

We observe that both values of $\hat{\xi}^{(\gamma)}$ and $\xi^{(\gamma)}$ is symmetric in a sense of absolute value, which is corresponding with the rotation of the vortex in real life.

The angular velocity defines a starting speed of rotation at the vortex and changes with respect to time and radial distance. However, with this more detailed assumption of a changing angular velocity, we can have a more close-to-

(a) Plot $\hat{\xi}^{(\gamma)}(r, t)$ as a function of r according to (4.33)

(b) Contour plot $\xi^{(\gamma)}(r, \lambda, t)$ as a function of r according to (4.33)

Figure 4.3: Plots of $\hat{\xi}^{(\gamma)}$ under beta-gyre effect

natural solution, with the "wind eye" at the center of the hurricane. However, the solutions are only valid for "early time", that is to say, we have an extra t term when differentiating $\bar{\Omega}(r)$. In this case, we have αt and the solution is valid for $\alpha t < \mathcal{O}(1)$, or for $t < \mathcal{O}(\alpha^{-1})$.

4.3 Nonlinear solutions for different parameter values

The case where $\alpha \sim \varepsilon \sim \gamma$

In this subsection, we consider the case where all three parameters are of the same order of magnitude, $\alpha \sim \varepsilon \sim \gamma = \frac{\beta}{\varepsilon} \ll 1$. None of them take the dominant position. Hence, all the terms corresponding to the three parameter affect the behaviour of the solution and the order of terms do not affect the solution of each other term. As before, we have $\beta \ll \varepsilon$ so the $\mathcal{O}(\beta)$ term in the solution appears at higher order. Since $\bar{\Omega}(r) = \Omega_0 e^{-\alpha r^2}$, the linear solution is of the form

$\psi^{(0)} + \alpha\psi^{(01)}$. Following the same idea of the asymptotic series by (3.35), we now have

$$\begin{aligned}\psi &= \psi^{(0)} + \alpha\psi^{(01)} + \gamma\psi^{(\gamma)} + \varepsilon\psi^{(1)} + \mathcal{O}(\alpha^2) + \mathcal{O}(\gamma^2) \\ \xi &= \xi^{(0)} + \alpha\xi^{(01)} + \gamma\xi^{(\gamma)} + \varepsilon\xi^{(1)} + \mathcal{O}(\alpha^2) + \mathcal{O}(\gamma^2)\end{aligned}\tag{4.35}$$

Each term was calculated and discussed in chapters 3 and 4. $\psi^{(0)} = \hat{\psi}^{(0)}e^{ik\lambda}$ where $\hat{\psi}^{(0)} = \hat{\psi}_k$ is given by (3.19), $\psi^{(01)} = \hat{\psi}^{(01)}e^{i\lambda}$ and $\hat{\psi}^{(01)}$ is given by (4.27), $\psi^{(\gamma)} = \hat{\psi}^{(\gamma)}e^{i\lambda}$ with $\hat{\psi}^{(\gamma)}$ can only be written in the integral form, $\psi^{(1)} = \hat{\psi}^{(1)}e^{i\lambda}$ with $\hat{\psi}^{(1)}$ given by (3.67). Similarly, $\xi^{(0)} = \hat{\xi}_ke^{ik\lambda}$ and $\hat{\xi}_k$ is given by (3.11), $\xi^{(01)} = \hat{\xi}^{(01)}e^{i\lambda}$ and $\hat{\xi}^{(01)}$ is given by (4.23), $\xi^{(\gamma)}$ is given by (4.32) and $\xi^{(1)}$ is given by (3.54).

As before we have (3.36) and (3.37), so the λ derivatives $\psi_\lambda^{(0)}, \xi_\lambda^{(0)}$ are the same, as before given by (3.10). The r derivatives $\hat{\psi}_r^{(0)}$ and $\hat{\xi}_r^{(0)}$ are

$$\begin{aligned}\psi_r^{(0)} &= (A'(r) + kar^{-k-1})e^{ik\lambda}e^{-ik\bar{\Omega}(r)t} \\ &\quad - ik(A(r) - ar^{-k})\bar{\Omega}'(r)te^{ik\lambda}e^{-ik\bar{\Omega}t} \\ &\quad - kar^{-k-1}e^{ik\lambda} + \text{c.c.}\end{aligned}\tag{4.36}$$

where $a = A(r_1)r_1^{-k}$ and

$$\xi_r^{(0)} = (B'(r) - ik\bar{\Omega}'(r)t)e^{ik\lambda}e^{-ik\bar{\Omega}(r)t} + \text{c.c.}\tag{4.37}$$

The same additional terms arise from differentiating $\bar{\Omega}(r)$ in the exponent of $e^{-ik\bar{\Omega}t}$, including the factor $-ik\bar{\Omega}'(r)te^{-ik\bar{\Omega}t}$. With $\bar{\Omega}(r) = \Omega_0e^{-\alpha r^2}$, we have $\bar{\Omega}'(r) = -2\alpha\Omega_0re^{-\alpha r^2}$ and

$$\frac{\partial}{\partial r}e^{ik\bar{\Omega}(r)t} = -2ik\alpha r\bar{\Omega}(r)te^{ik\bar{\Omega}(r)t}\tag{4.38}$$

so the additional terms arising from differentiating $\bar{\Omega}(r)$ are $\mathcal{O}(\alpha)$ and $\mathcal{O}(t)$. This means that these terms appear at $\mathcal{O}(t\alpha\varepsilon)$ in the nonlinear equation (4.35), which

is negligible but only for a certain time. Thus, the $\mathcal{O}(\varepsilon)$ terms $\hat{\psi}^{(1)}$ and $\hat{\xi}^{(1)}$ appearing the expansions (4.33) take the same form as the expressions for the $\mathcal{O}(\varepsilon)$ terms (3.39) and (3.40) in section 3.2.

The case where $\alpha \ll \varepsilon \sim \gamma$

In this subsection, we consider the case where $\alpha \ll \varepsilon \sim \frac{\beta}{\varepsilon} = \gamma$ in the nonlinear problem given by equation (3.33) with the initial conditions (2.40) and the boundary conditions (2.41). Again we look for a solution in the form of the perturbation series (4.35). In this case, the $\mathcal{O}(\varepsilon)$ terms appear before the higher order $\mathcal{O}(\alpha)$ terms.

We look for series of the form

$$\begin{aligned}\psi &= \psi^{(0)} + \varepsilon\psi^{(01)} + \gamma\psi^{(\gamma)} + \mathcal{O}(\alpha) \\ \xi &= \xi^{(0)} + \varepsilon\xi^{(01)} + \gamma\xi^{(\gamma)} + \mathcal{O}(\alpha)\end{aligned}\tag{4.39}$$

As before, the leading order linear solution $\psi^{(0)}$ is (4.18)

The $\mathcal{O}(\gamma)$ terms in the perturbation series (4.39) are the same as those evaluated in section 3.2 given by (3.67) and (3.54). As in section 4.3, the $\mathcal{O}(\varepsilon)$ terms $\hat{\psi}^{(1)}$ and $\hat{\xi}^{(1)}$ in the expansions (4.39) take the same form as (3.22) and (3.67) in section 3.1 with r derivatives given by (4.36) and (4.37) in section 4.3 and λ derivatives still given by (3.36) and (3.37) in section 3.2. The additional $\mathcal{O}(\alpha)$ terms that arise differentiating $\bar{\Omega}(r)$ appear at $\mathcal{O}(\alpha\varepsilon)$ in the nonlinear equation (4.20).

The case where $\alpha \gg \varepsilon$

In this subsection, we consider the case where $\varepsilon \ll \alpha \sim \frac{\beta}{\varepsilon} = \gamma$ with the same nonlinear equation, initial conditions (2.40) and boundary conditions (2.41). In this case, the $\mathcal{O}(\alpha)$ terms appear before the nonlinear $\mathcal{O}(\varepsilon)$ terms. Hence the

$\mathcal{O}(\alpha)$ terms would only affect the nonlinear part of the solution at higher order $\mathcal{O}(\alpha\varepsilon)$ and $\mathcal{O}(\alpha^2\varepsilon)$ cases in 4.3.2. Hence, by contrast to the section 4.4, the nonlinear equation can be written as

We look for series of the form

$$\begin{aligned}\psi &= \psi^{(0)} + \alpha\psi^{(01)} + \mathcal{O}(\alpha^2) + \gamma\psi^{(\gamma)} + \mathcal{O}(\varepsilon) \\ \xi &= \xi^{(0)} + \alpha\xi^{(01)} + \mathcal{O}(\alpha^2) + \gamma\xi^{(\gamma)} + \mathcal{O}(\varepsilon)\end{aligned}\tag{4.40}$$

The function $\psi^{(0)}, \psi^{(01)}, \xi^{(0)}, \xi^{(01)}$ are those obtained in section 4.2 and are given by (4.18), (4.23) (4.12) and (4.23).

The $\mathcal{O}(\gamma)$ term of ψ does not have a closed form, but the term of ξ is the same as those evaluated in section 4.2, 4.32).

The $\mathcal{O}(\alpha)$ terms appear in the nonlinear products at $\mathcal{O}(\alpha\varepsilon)$ but since $\alpha \ll 1$, these terms do not affect the $\mathcal{O}(\varepsilon)$ terms in the solution which still take the form (3.22).

In this chapter, we discussed the configuration where the background angular velocity $\bar{\Omega}$ depends on the radial variable r . In this case, we obtain another non-dimensional parameter μ that measures the scale of the initial wave profile. Then we built the fundamental model of $\bar{\Omega}(r)$, and discussed the nonlinear terms relative to the magnitudes of parameters afterwards.

Chapter 5

Conclusions

5.1 Summary and discussion

In this thesis, we described and examined a barotropic model to represent vortex waves forced by an initial perturbation in a vortex on a beta-plane. We formulated the model in terms of the streamfunction and vorticity of the wave. In chapter 3 we considered the case where the vortex has a constant background angular vorticity in the annulus domain (2.39). We solved such a linear problem to obtain the analytic solution of the dominant order and the vorticity. However in the next order, the $\mathcal{O}(\frac{\beta}{\varepsilon})$ beta-gyre term are secular in the case with constant $\bar{\Omega}$. In the more realistic choice of $\bar{\Omega}(r)$ used in chapter 4, these terms go to zero as $r \rightarrow \infty$, as required in the physical problem. We also considered a nonlinear approach to the solution and obtained a solution that cannot be written as a closed form in the next order of $\mathcal{O}(\varepsilon)$.

In chapter 4 we considered a more realistic case where the background angular velocity of the vortex varies with radial distance from the center. In this case, we need to discuss the relative magnitude of the scale μ , which measures the initial wave profile, with other parameters we already have. Then we build the

fundamental model and obtained the solution that is the sum of the linear solution with wavenumber k satisfying the initial condition, terms with wavenumber zero and wavenumber $2k$ arising from the nonlinear products, and a "beta-gyre" term arising from the nonhomogeneous term at $\mathcal{O}(\frac{\beta}{\varepsilon})$. The linear solution includes additional terms of $\mathcal{O}(\alpha)$, $\mathcal{O}(\alpha^2)$ and so on. These terms include factors of t . Since we have terms of $\mathcal{O}(\alpha t)$, the solution is valid for $t < \mathcal{O}(\alpha^{-1})$. If t exceeds the limitation, we may need to introduce the large t scale to obtain the solution at a more precise level. However, it will not be discussed in this thesis and may be considered as future work. The beta-gyre effect includes a term proportional to $e^{i(\lambda - \bar{\Omega}(r)t)}$. Since $\bar{\Omega}(r) > 0$, it represents an oscillation with wavenumber 1 involving in the positive λ direction, i.e., in the counter-clockwise direction. This is consistent with the conclusions of Montgomery et al. [18].

5.2 Suggestions for further work

The barotropic initial-boundary-value problem studied in this thesis is a useful starting point for further investigations. This section illustrates some proposals for possible future related work.

The $\mathcal{O}(t)$ terms in $\psi^{(01)}$ in the linear solution in chapter 4 indicate that the solution is valid only for $t < \mathcal{O}(\alpha^{-1})$. We can do a late t linear solution with a slow t variable such as $\tau = \alpha t$ to investigate this further. In this case, we will have

$$\begin{aligned}\psi &= \psi^{(0)} + \alpha\psi^{(01)} + \gamma\psi^{(\gamma)} + \varepsilon\psi^{(1)} + \mathcal{O}(\alpha^2) + \mathcal{O}(\gamma^2) + \mathcal{O}(\tau) \\ \xi &= \xi^{(0)} + \alpha\xi^{(01)} + \gamma\xi^{(\gamma)} + \varepsilon\xi^{(1)} + \mathcal{O}(\alpha^2) + \mathcal{O}(\gamma^2) + \mathcal{O}(\tau)\end{aligned}\tag{5.1}$$

The future work can also take into account the case where $\beta \ll \varepsilon$, so that the $\mathcal{O}(\beta)$ terms in the solution appear at higher order. Or, those can consider the case where $\beta \sim \varepsilon$. That means $\gamma \sim \mathcal{O}(1)$. They could evaluate the $\mathcal{O}(\beta)$ terms at the same order as the nonlinear terms. In evaluating the nonlinear $\mathcal{O}(\varepsilon)$ terms,

the products of $\mathcal{O}(\gamma\varepsilon)$ and $\mathcal{O}(\gamma^2\varepsilon)$ must be included as well as the $\mathcal{O}(\varepsilon)$ products.

It is also suggested to consider the case where there is a critical radius in the domain, like in Campbell and Nikitina, [14] and [15] considered, but with the wave forced by an initial condition, as we have been doing here.

Those who are interested can set up a 3-D configuration taking into account the vertical variations. The barotropic solutions could be used for comparison. The analysis would be more complicated but the numerical solutions would be more feasible. The cases mentioned in this thesis can be solved numerically if needed. One can use a Fourier spectral method in the λ direction and work with wavenumber 1 plus wavenumbers in multiples of k . The products of these modes will generate modes $(k \pm 1)$ and so on.

Appendix A

Equations of Fluid Dynamics

A.1 Basic definitions

In this Appendix, we give an outline of the derivation of the partial differential equations that describe fluid motion.

First, the basic definitions of fluid motion, mass and momentum are introduced. Then, the equations of fluid motion are derived and discussed. They are based on physical laws of conservation of mass, momentum and the energy of a fluid. This chapter closely follows the fluid mechanics textbook by Kundu and Cohen [25].

Definition A.1.1 (Categorization of matter). All forms of matter could be roughly distinguished as being *fluid* or *solid*. A solid is a kind of substance that does not deform continuously under an applied shear stress. It also has a preferred shape that it relaxes to, when the external forces on it are withdrawn. A fluid deforms continuously under an applied shear stress. In other words, it does not have a preferred shape.

Definition A.1.2 (Categorization of fluid). Generally speaking, fluids are classified into two kinds, *liquids* and *gas*. A gas always expands to fill the volume

of room that stores it, while the volume of a liquid does not completely fill a container that is larger than the volume of the liquid.

A fluid is composed of a large number of molecules in constant motion, experiencing continuous collisions with each other. At the microscopic scale, a fluid is discontinuous or discrete. Hence, we can study the motions of molecules in terms of the kinetic theory or statistical mechanics to help us understand the mechanics of fluid. However, we are often interested in the *average manifestation* of the molecular motion which transforms the result we get from the microscopic scale to give us information about macroscopic properties.

When we have sufficiently large enough molecular density of the fluid and size of the region of interest, we often consider average properties to explain macroscopic phenomena and ignore discrete microscopic molecules. We use the term *continuum* to discuss the continuous distribution of the fluid. This is the situation when we study the dynamics of the atmosphere.

Definition A.1.3 (Continuum). A fluid is said to exist in continuum if the matter in its body (i.e. molecules) is continuously distributed and fills the entire space it occupies. In a continuum, the properties of the fluid are defined at every point in the space it occupies, and they are the averages of the molecular characteristics in a small volume surrounding the point of interest.

Remark 1. *This continuum approximation is valid in the situation where the Knudsen number is much less than unity. The Knudsen number is defined as $K_n = \frac{l}{L}$, where l is the mean free path of the molecules, and L is the length scale of interest.*

After understanding the matter we are going to investigate, we now need to define some basic concepts that can be applied to relate to our fluid problem.

Definition A.1.4 (Pressure). In a static fluid in a container, forces are exerted on the boundaries of the container due to the free motion of molecules. The *pressure*, p , is defined as the statistical average of the magnitude of the force per unit area. It is sometimes named *absolute pressure* to distinguish it from the *gauge pressure*, which is defined as the difference between the absolute pressure and atmospheric pressure. $p_{gauge} = p - p_{atm}$. In this definition, the atmospheric pressure is considered to be a constant.

Definition A.1.5 (Density). The density of a fluid is defined as the mass per unit volume, $\rho = \frac{m}{V}$.

Definition A.1.6 (Compressibility of a fluid). All fluids are compressible to some extent. A compressible fluid is a fluid in which significant density variations occur during its changing of pressure or temperature. When the change of density is sufficiently small or negligible, we call such a fluid incompressible.

Remark 2. *In the language of mathematics, the incompressibility can be expressed by saying that the partial derivative of the density with respect to the temperature is zero. i.e.*

$$\frac{\partial \rho}{\partial T} = 0 \tag{A.1}$$

Definition A.1.7 (Steadiness of a fluid flow). A fluid flow is said to be steady if none of its properties change with time. A fluid flow can be described by functions representing properties such as density pressure or temperature. Thus, a flow is steady if none of these functions depend on time. Hence, and naturally, a time dependent flow is defined as unsteady flow or transient flow.

Definition A.1.8 (Surface tension). Microscopically, whenever two immiscible fluids are in contact, there is a discontinuity of the density. The unbalanced attractive intermolecular forces make the interface behaviour of the two fluids

behave as if it were a stretched membrane under tension. Molecules try to pull the interface molecules inward toward the center. The net effect of the attractive forces is for the interface to contract until an equilibrium state is attained. Hence, the magnitude of the tensile force acting on an open line segment lying on the surface is defined as the *surface tension*, and it is calculated as the force per unit length, Ω , with the unit Nm^{-1} .

Definition A.1.9 (Kinematics). Kinematics is the branch of mechanics that describes the motion of the fluid particles and the quantities that describe the fluid in space and time. Kinematics deals with the variables describing the fluid such as displacement, velocity, acceleration, and rotation *without referring* to the forces that cause those motions.

Remark 3. *In this appendix, we shall define the equations for a three-dimensional flow that can be described and investigated properly by using three independent spatial coordinates. Cartesian coordinates may locate a single point P via coordinates triplets (x, y, z) or (x_1, x_2, x_3) with the corresponding velocity components (u, v, w) or (u_1, u_2, u_3) . Other coordinate systems such as the Cylindrical polar coordinates system and Spherical polar coordinates system may be used. The choice of the coordinate system is made depending on the geometry of the region that the fluid occupies.*

With the help of the coordinate system, we have a proper way to describe the fluid motion now. There are two ways of describing the fluid motion: *Euclidean description* and *Lagrangian description*.

Euclidean and Lagrangian description

Definition A.1.10 (Euclidean description). Assume there is a spatial point $\vec{x}$ in the three-dimensional space with the independent variables of $\vec{x} = (x_1, x_2, x_3)$

(e.g, in the rectangular coordinates it is (x, y, z)), and the flow quantities (dependent variables) are functions of $\vec{x}$ and t : $F(\vec{x}, t)$. For example, we say that the velocity at the point $\vec{x}$ and time t is $V(\vec{x}, t)$. In this case, different fluid particles pass by that point at different times.

Definition A.1.11 (Lagrangian description). In this description, we follow the motion of each individual fluid particle. For each particle, we assign a label $\vec{l}$ to it, which is equal to the position vector of the particle location at the initial time $t = 0$.

The two independent variables then are $\vec{l}$ and t , and the flow variables (i.e. dependent variables) are functions of $\vec{l}$ and t . For example, $\vec{V}(\vec{l}, t)$.

At a given point, we note the position vector $\vec{r}$. This is a dependent variable that illustrates movement with respect of time. So we have $\vec{r} = \vec{r}(\vec{l}, t)$, the location at time t of a particle whose position was $\vec{l}$ at $t = 0$.

In this thesis, the Lagrangian description is useful for describing the paths that the fluid particles are travelling. The Euclidean description will be normally used to describe the kinematics.

Derivatives with respect to time

In this section, we define derivatives based on the two different descriptions that were introduced above.

In the Lagrangian description, we use $\frac{\partial F}{\partial t}$ to define the total rate of change of a quantity $\vec{F}(\vec{l}, t)$ measured as a moving fluid particle with the label $\vec{l}$.

By contrast, in the Euclidean description, the expression $\frac{\partial F}{\partial t}$ indicates the local rate of change of a quantity $\vec{F}(\vec{x}, t)$ at the given point $\vec{x}$, but not the rate that is observed by the fluid particles.

If we need to measure the total rate of change following a moving fluid particle, consider a fluid property $\vec{F}$ for which

$$\vec{F}(\vec{x}, t) = \vec{F}(\vec{l}, t)$$

at the same position and time in those two descriptions. Assuming that the time and length scales are the same, i.e, they share units of measurements, we note that the position vector of this given particle at time t is $\vec{x} = r(\vec{l}, t)$, which implies that

$$\vec{F}(\vec{x}, t) = \vec{F}(\vec{x}(\vec{l}, t), t)$$

Thus, we can conclude that the rate of change following the particle equals to

$$\frac{\partial \vec{F}}{\partial t} + \frac{\partial \vec{F}}{\partial \vec{x}} \left(\frac{\partial \vec{x}}{\partial t} \right) = \frac{\partial \vec{F}}{\partial t} + \vec{u} \cdot \nabla \vec{F} \quad (\text{A.2})$$

where $\frac{\partial \vec{F}}{\partial t}$ is the local rate of change of the velocity at a given point. Notice that the value is zero for a steady flow.

On the other hand, $\frac{\partial \vec{F}}{\partial x}$ stands for the change in $\vec{F}$ as a result of the movement of the particle moving from one location to another, where the value of $\vec{F}$ is different between the two locations.

Remark 4. Notice here x might be (and always is when we discuss real life 3-D regions) a position with dimension 3. In the 3-D case in terms of rectangular coordinates, we have

$$\frac{D\vec{F}}{Dt} = \frac{\partial \vec{F}}{\partial t} + u \frac{\partial \vec{F}}{\partial x} + v \frac{\partial \vec{F}}{\partial y} + w \frac{\partial \vec{F}}{\partial z} \quad (\text{A.3})$$

In general, we have

$$\frac{D\vec{F}}{Dt} = \frac{\partial \vec{F}}{\partial t} + \frac{\partial \vec{F}}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial \vec{F}}{\partial t} + u_i \frac{\partial \vec{F}}{\partial x_i} = \frac{\partial \vec{F}}{\partial t} + \vec{u} \cdot \nabla \vec{F} \quad (\text{A.4})$$

where the index i denotes summation over $i = 1, 2, 3$. This is called the material derivative or particle derivative and it is denoted as $\frac{D}{Dt}$ and $u_i \frac{\partial \vec{F}}{\partial x_i}$ is called the advective derivative.

A.2 Conservation laws

Conservation of mass

We all know that mass can neither be created nor destroyed, despite nuclear reactions and relativism. If we focus on the subject of interest, that is, individual mass elements including molecules, grains, fluid particles, etc., we may track them within a flow field since they will not disappear, and new elements will not appear from nowhere. The following equation represents the conservation of mass in a fluid, assuming that the mass of neighbouring particles is constant. We use the notation $V(t)$ to define the volume occupied by some specific collection of fluid particles. We denote the density of the fluid by the function $\rho(x, t)$. This volume moves and deforms within the flow, so it always has the same mass elements and none leaves from it. Hence, we have:

$$\frac{d}{dt} \int_{V(t)} \rho(x, t) dV = 0 \quad (\text{A.5})$$

To continue, we need to use the Reynolds transport theorem. It is a generalization of Leibniz's rule to a 3-D vector system and it is stated as:

$$\frac{d}{dt} \int_{V(t)} \rho(x, t) dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{A(t)} (\vec{u}, \vec{r}) \rho \cdot \vec{n} dA \quad (\text{A.6})$$

In the equation, we denote the surface area of the fluid volume $A(t)$. The mass of fluid in the control volume is conserved if the left hand side of (A.6) is zero. By the Reynolds transport theorem, letting $\vec{F} = \rho$, we can expand the previous equation to be:

$$\int_{V(t)} \frac{\partial \rho(x, t)}{\partial t} dV + \int_{A(t)} \rho(x, t) \vec{u}(x, t) \cdot \vec{n} dA = 0 \quad (\text{A.7})$$

which is a mass-balance statement between integrated density changes within $V(t)$ and integrated motion of its surface $A(t)$.

Based on (A.5), we know that the surface must move at the local fluid velocity u so that we can keep the particles that we are interested in inside of the fluid. However, (A.7) may be difficult to apply in practice as the equation of $V(t)$, $A(t)$ are determined by the flow itself and normally we cannot specify them. A more practically useful version is obtained by rewriting (A.7) as a partial differential equation.

Remark 5. *For conservation of mass, we have that given a rate of change of time, the total mass of fluid in a control volume V equals to the mass flow going in or out of the control volume through the surface. This can be written as*

$$\frac{d}{dt} \int_{V(t)} \rho(x, t) dV = - \int_{A(t)} \rho \vec{u} \cdot \vec{n} dA = - \int \nabla \cdot (\rho \vec{u}) dV$$

by the divergence theorem to express the surface integral as a volume integral.

In conclusion, we get the following expression:

$$\int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) \right) dV = 0 \quad (\text{A.8})$$

This equality is true if the integrand vanished at every point in the chosen space. If not, then in a small volume around a point will be nonzero to yield a nonzero integral. Hence, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad (\text{A.9})$$

We note that

$$\nabla \cdot (\rho \vec{u}) = \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} \quad (\text{A.10})$$

This is the equation for conservation of mass in the differential form and is also called the continuity equation. Hence, (A.9) can be written as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0 \quad (\text{A.11})$$

where $\frac{D}{Dt}$ is the Lagrangian derivative defined in equation (A.4)

In 3-D flow with $\vec{u} = (u, v, w)$. In a incompressible fluid flow, we have $\frac{D\rho}{Dt} = 0$. This is the incompressible continuity equation. From (A.8) we will also have the zero divergence.

$$\nabla \cdot \vec{u} = 0 \quad (\text{A.12})$$

Conservation of momentum

In this section we will derive the equation for the conservation of momentum law through Newton's second law, which is the fundamental principle.

Definition A.2.1 (Momentum). Momentum is a vector quantity that is defined as the product of mass and velocity.

Theorem 1. Newton's Second Law can: The force applied on a object can be calculated as the product of mass and acceleration.

For a fluid with constant density $\rho = \frac{m}{V}$, where m is the mass of the fluid in a volume V , Newton's Second Law states that

$$\text{Force} = m \frac{D}{Dt} \vec{u}$$

Hence,

$$\text{Force per unit volume} = \frac{m}{v} \frac{D\vec{u}}{Dt} = \rho \frac{D\vec{u}}{Dt}$$

Then have the integral form in the general case with density $\rho(x, t)$

$$\int_{V(t)} \vec{F} dV = \int_{V(t)} \rho \frac{D\vec{u}}{Dt} dV \quad (\text{A.13})$$

representing the Newton's Second Law applying to the total force over volume $V(t)$.

Or we have $\int_{V(r)} (\rho \frac{D\vec{u}}{Dt} - \vec{F}) = 0$. Thus, for any fluid flow for which the integrand is a continuous function of space, we conclude that the integrand must be zero. This must be true for any arbitrary volume $V(t)$.

Hence, we have $\rho \frac{D\vec{u}}{Dt} = \vec{F}$, where the force is a vector, indicating the sum of all forces on the fluid. It includes gravitational force, Coriolis force, pressure gradient force and viscous forces. These specific forces will be defined below.

Definition A.2.2 (Pressure gradient). Pressure gradient is a vector that describes the direction and the rate of pressure which increases most rapidly around the location that we are interested in. It is define as ∇p where p is the pressure. The unit of it is pascals per meter (Pa/m).

Definition A.2.3 (Gravitational force). Also called *gravity*, it is the force that attracts any two objects of mass. It is calculated as

$$F = G \frac{Mm}{r^2} \quad (\text{A.14})$$

with direction towards the counterpart. In this equation, G is the gravitational constant of the quantity $6.674 \cdot 10^{-11} N(m/kg)^2$, which measured by [29]. M and m represent the mass of two objects. For a fluid in the Earth's atmosphere, the gravitational force acts vertically downward towards the surface of the earth.

Definition A.2.4 (Gravitational acceleration). In terms of Cartesian Coordinates where $x_1 = x$ and $x_2 = y$ are the horizontal coordinates and $x_3 = z$ is the vertical coordinate altitude, the gravitational acceleration can be written as the vector $\vec{g} = (0, 0, -g)$ where $g \approx 9.8ms^{-2}$. Thus, the gravitational force is $\rho\vec{g} = (0, 0, -\rho g)$.

Definition A.2.5 (Coriolis force). The Coriolis force is an inertial force that acts on a moving object within a reference that rotates with respect to an inertial frame. A closer discussion will be represented in section 2.1.

Another important force is Viscous force which results from the effects of friction between fluid particles. In this thesis, will be considering a fluid such as in the atmosphere where the viscous force is weak enough relative to the other forces that it can be neglected. The inviscid equation for conservation of momentum is called the Euler equation with the pressure gradient gravitation and Coriolis forces. We have

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{g} + \vec{F}_c \quad (\text{A.15})$$

where p is the pressure and $\vec{F}_c$ is the Coriolis force.

Conservation of energy

The internal energy of a body consists of the kinetic energy and the potential energy. In this subsection, the definitions of energies are stated and the conservation of energy law is expressed.

Definition A.2.6 (Kinetic energy). Kinetic energy is a form of energy that an object or a particle has by reason of its motion. The kinetic energy of a fluid is stored in static pressure p , and dynamic pressure defined as $\frac{1}{2}\rho V^2$, where ρ is the fluid density and V is the fluid velocity.

Definition A.2.7 (Potential energy). The potential energy of a fluid of a unit mass at a height h above the ground level is defined by $P = mgh$. However, in this thesis we are using a 2-D assumption to describe the vortex. Hence, the potential energy of the vortex is constant.

Theorem 2 (Conservation of Energy of a Fluid). The conservation of energy of a fluid can be stated as that the total energy e of the fluid (the sum of the potential energy and the kinetic energy) equals to the work done on the fluid by

body forces, the work done on the fluid by surface forces, and the heat transferred out of the fluid. This can be written as:

$$\rho \frac{De}{Dt} = -p(\nabla \cdot \vec{u}) - \nabla \cdot \vec{q} + \phi \quad (\text{A.16})$$

where ρ , p and $\vec{u}$ are the density pressure and velocity, q is the heat flux vector and ϕ is the rate of viscous dissipation of energy. In this thesis, we will be studying a fluid that is approximated as inviscid ($\phi = 0$) and incompressible ($\nabla \cdot \vec{u} = 0$) on a 2-D horizontal plane (barotropic).

A.3 Vorticity and streamfunction

So far, in Appendix A.1 and A.2, we have presented equations describing the dynamic of an inviscid fluid flow subject to some forces that are present in the atmosphere. In chapter 2, we formulate an initial value problem for vortex Rossby waves written in terms of the vorticity and streamfunction. The model equations are derived from (A.12) and (A.14). In this section, we introduce the concepts of the vorticity and streamfunction.

Definition A.3.1 (Vorticity). The vorticity of a function flow is a vector defined as $\vec{\omega} = \nabla \times \vec{u}$ where $\vec{u}$ is the velocity vector. It gives a measure of the extent of rotation of the fluid particles. Fluid are oriented along different directions, and rotated by different amounts [26].

To define rotation rate, we consider two perpendicular lines of fluid particles. For example, it can be two perpendicular sides of a rectangular fluid element in a 2-D fluid with velocity $\vec{u} = (u_1, u_2)$ where the velocity components u_1 and u_2 are functions of the spacial variables x_1 and x_2 but not constants. The following diagram represents a rectangular fluid element in a fluid with velocity (u_1, u_2) .

Figure A.1: Rectangular Fluid

We let Δx_1 and Δx_2 be the lengths of the two perpendicular sides as shown in Figure A.1. Because of the nonzero velocity gradients in the fluid, the element rotates around the x_3 axis.

We have that the angular velocity of the line AB is $\frac{d\beta}{dt}$ and the angular velocity of the line AC is $\frac{d\alpha}{dt}$, as shown in the Figure A.2.

From the time t , the point A' travels a distance of $u_2 dt$ in the direction of y , and B' travels a distance of $(u_2 + \frac{\partial u_2 \Delta x_1}{\partial x_1}) dt$ in the direction of y . Hence, the distance between A' and B' is $\frac{\partial u_2}{\partial x_1} \Delta x_1 dt$. The angular velocity is $\frac{\frac{1}{\Delta x_1} \frac{\partial u_2}{\partial x_1} \Delta x_1 dt}{dt}$.

Similarly, A' moves a distance of $u_1 dt$ in the x direction and C' moves a distance of $(u_1 - \frac{\partial u_1}{\partial x_2} \Delta x_2) dt$ in the x direction. Hence, the distance between A' and C' is $-\frac{\partial u_1}{\partial x_2} \Delta x_2 dt$ with the corresponding angular velocity of $\frac{1}{dt} \frac{1}{\Delta x_2} \frac{\partial u_1}{\partial x_2} \Delta x_2 dt$.

Hence, we have the average angular velocity to be

Figure A.2: Effect of a nonzero velocity gradient on a rectangular fluid element

$$\frac{1}{2dt} \left(\frac{\partial u_2}{\partial x_1} dt - \frac{\partial u_1}{\partial x_2} dt \right) = \frac{1}{2} \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \quad (\text{A.17})$$

To avoid the factor of $\frac{1}{2}$ in (A.17), we can define the vorticity to double the angular vorticity of the rotation about the x_3 axis.

$$\omega_3 = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \text{ for } \vec{u} = (u_1, u_2, u_3) = (u, v, w) \quad (\text{A.18})$$

Note that this is the z-component of the curl of $\vec{u}$.

In fact, a similar argument in the xz and yz planes gives rotation about the x axis and the y axis respectively, in a 3-D fluid.

$$\vec{\omega} = \nabla \times \vec{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \quad (\text{A.19})$$

can be written as

$$\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}, \text{ where } \varepsilon_{ijk} \text{ is the alternating tensor} \quad (\text{A.20})$$

Definition A.3.2 (Alternating Tensor).

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } ijk \text{ is of cyclic order,} \\ 0, & \text{if any two indices are equal,} \\ -1, & \text{if } ijk \text{ is of anti-cyclic order} \end{cases} \quad (\text{A.21})$$

Definition A.3.3 (Irrotational flow). A fluid motion is said to be **irrotational** if its vorticity vector is zero, i.e.,

$$\vec{\omega} = 0 \implies \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \forall i \neq j$$

In an irrotational flow, the velocity can be written as the gradient of a scalar function $\phi(\vec{x}, t)$ where

$$\vec{u} = \nabla \phi \text{ or } u_i = \frac{\partial \phi}{\partial x_i} \quad (\text{A.22})$$

Remark 6. If we have $\vec{u} = \nabla \phi$, then we have

$$\nabla \times \vec{u} = \nabla \times \nabla \phi = (\phi_{x_2 x_3} - \phi_{x_3 x_2}, \phi_{x_1 x_3} - \phi_{x_3 x_1}, \phi_{x_2 x_1} - \phi_{x_1 x_2}) = 0 \quad (\text{A.23})$$

Definition A.3.4 (Streamfunction). In the special case of a 2-D fluid flow with velocity $\vec{u} = (u, v)$, the streamfunction $\Psi(r, y)$ is defined by

$$\frac{\partial \Psi}{\partial x} = v, \frac{\partial \Psi}{\partial y} = -u \quad (\text{A.24})$$

where $\vec{u} = (u, v)$ in 2-D.

The vorticity in 2-D is defined as $\xi = \nabla^2 \Psi$.

The streamfunction $\Psi(x, y)$ is related to the concept of streamlines in a 2-D fluid flow.

To define streamlines, we consider that at any particular time and at every point in a fluid flow, there is a velocity vector with a specific direction and magnitude.

Definition A.3.5 (Streamlines). Streamlines are *instantaneous* curves that at any given time, are everywhere tangent to the velocity vectors, i.e., tangent to the direction field.

For unsteady flows, the velocity changes with time so the streamline pattern changes with time as well.

Let $d\vec{s} = (dx, dy, dz)$ be an element of arc-length along a streamline in a 3-D fluid flow and $\vec{u} = (u, v, w)$ be the local velocity vector. They are in the same direction so we have

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad (\text{A.25})$$

along a streamline.

Remark 7. As the streamlines are defined by $\frac{dx}{u} = \frac{dy}{v}$, we can also write $udy - vdx = 0$. In terms of the stream function, we have

$$-\frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = 0$$

which implies that

$$d\Psi = 0 \text{ along a streamline} \quad (\text{A.26})$$

we can also write

$$\frac{d\Psi}{dx} = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{dy}{dx} d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} \frac{dy}{dx} dx = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = 0$$

By the similar argument we will also have that $\frac{d\Psi}{dy} = 0$. Hence, we can conclude that Ψ is a constant along a streamline. So the instantaneous streamlines in a flow are given by the curves of constant Ψ .

A.4 Units of measurement

In this section, some general units of measurements and definitions related to this thesis will be represented to help readers understanding the basic terms and units.

To measure the properties of a fluid, we shall use SI units. We use meter for length, kilogram for mass, second for time and Kelvin for temperature. A table of SI units is given. To represent the fluid flow mathematically, we shall make all the fluid quantities in the equations non-dimensionally (i.e. without units) by dividing each fluid quantity by a reference value of the quantity.

Unit Name	Quantity	Symbol	Transition
Length	Meter	m	
Mass	Kilogram	kg	
Time	Second	s	
Temperature	Kelven	K	
Frequency	Hertz	Hz	s^{-1}
Force	Newton	N	$kg\ m\ s^{-2}$
Energy	Joule	J	$N\ m$
Pressure	Pascal	Pa	$N\ m^{-2}$
Power	Watt	W	Js^{-1} or $N\ m\ s^{-1}$

Definition A.4.1 (Vector and Scalar). *Vectors* are quantities that are fully described by both a magnitude and a direction.

Scalars are quantities that are fully described by a magnitude (or numerical value) alone.

Definition A.4.2 (Units of measurement). The density ρ is defined as the mass per unit volume.

The unit is kilograms per meters cubed.

The speed u is defined as the rate of change of the distance travelled per unit of time, which is a scalar.

The velocity $\vec{u}$ is defined as the displacement per unit of time, which is a vector.

The acceleration is defined as the rate of change of velocity per unit of time, which is a vector as well.

Remark 8. *Each component of the vector $\vec{u}$ is the speed in a given direction.*

For example, in our regular 3-dimension coordinates, we have $\vec{u} = (u, v, w) = u\hat{i} + v\hat{j} + w\hat{k}$, where (u, v, w) are the scalar of speeds and $(\hat{i}, \hat{j}, \hat{k})$ indicates the direction of the speeds.

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