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Properly stratified endomorphism algebras

by

Xueqing Chen

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics

Carleton University
Ottawa, Ontario, Canada

May, 2002

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Properly stratified endomorphism algebras

Submitted by

Xueqing Chen, B.Sc., M.Sc.

in partial fulfillment of the requirements
for the degree of Doctor of Philosophy.

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May, 2002

Abstract

Properly stratified algebras were introduced by Dlab (see [D4]) by means of properly stratifying idempotents inducing recollements of triangulated categories (in the sense of Beilinson, Bernstein and Deligne). Properly stratified algebras are a natural generalization of quasi-hereditary algebras, and appear in a recent work of Futorny, König and Mazorchuk on a generalization of the category $\mathcal{O}$ of Bernstein, Gelfand and Gelfand.

The objective of this thesis is to construct a class of properly stratified algebras as endomorphism algebras of particular modules over local commutative self-injective algebras.

At the beginning, we recall some definitions, the homological properties and some characterizations of quasi-hereditary algebras.

Then, we introduce a series of stratified algebras that have appeared in the literature, such as standardly stratified algebras, CPS-stratified algebras and properly stratified algebras, which are closely related to each other. A number of results, properties and characterizations of quasi-hereditary algebras can be translated to standardly stratified algebras and properly stratified algebras. As an analogue of BGG-algebras, we define a duality of the standardly stratified algebras and show that every standardly stratified algebra whose module category has a strong duality, is properly stratified. The constructed endomorphism algebras are just the standardly stratified algebras with strong duality.

Later, we study the properties of the constructed algebras and give some examples to illustrate our theory.

Finally, we provide some concrete examples of properly stratified algebras. In particular, we describe a canonical construction of a shallow properly stratified algebra over a given quiver.

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Introduction

Quasi-hereditary algebras were introduced by Cline-Parshall-Scott ([CPS1]) in order to deal with highest weight categories as they appeared in the representation theory of semi-simple complex Lie algebras and algebraic groups. The connection between highest weight categories and quasi-hereditary algebras appears to connect classical Lie theory with the theory of finite dimensional algebras in an entirely new way. Initiated in the language of quasi-hereditary algebras, its applications immediately extended to a number of areas: The theory of perverse sheaves, Iwahori-Hecke algebras, Koszul algebras, Schur algebras, Auslander algebras and cellular algebras, to name a few. The notions of stratified algebras, strictly stratified algebras, CPS-stratified algebras, standardly stratified algebras and properly stratified algebras are examples of these generalization. For details, the reader may consult papers of Cline-Parshall-Scott, Dlab-Ringel, König, Ágoston-Dlab-Lukács, Dlab, Wick and others. The category $\mathcal{O}$ defined by Bernstein-Gelfand-Gelfand motivated the introduction of concepts of quasi-hereditary and BGG-algebras. There is a block decomposition of BGG-category $\mathcal{O}$ and the representation theory of each block corresponds to the representation theory of a finite dimensional quasi-hereditary algebra. The properly stratified algebras appear in the context of representations of Lie algebra in recent works of Futorny-König-Mazorchuk. In [FKM], they construct a generalization of BGG-category $\mathcal{O}$ whose blocks are the categories of modules over properly stratified algebra respectively. As shown by Dlab in [D4], the properly stratified algebras have a natural characterization in terms of the recollement of triangulated categories of Beilinson-Bernstein-Deligne and possess many basic properties (appropriately modi-

fied) of quasi-hereditary algebras. The latter are just the properly standard algebras of finite global dimension.

The main result Theorem 4.1.4 of this thesis, is a characterization of the properly stratified algebras as the endomorphism algebras of particular modules over local commutative self-injective algebras. The theorem provides necessary and sufficient conditions on the direct summands of a module X over R so that the endomorphism algebra $A = \text{End}_R(X)$ is properly stratified. A similar characterization was obtained earlier for the case of quasi-hereditary algebras by Dlab-Heath-Marko in [DHM].

The properly stratified algebras, and in particular quasi-hereditary algebras of Cline-Parshall-Scott, which appear in applications have many additional properties, such as self-duality, homological duality and the fact shown by Ágoston-Dlab-Lukács (to be published in *Algebras and Representation Theory* shortly), that all the standard modules have linear projective resolutions, i.e., are standardly Koszul in their terminology. Here, infinite dimensional algebras of a particular type appear naturally as a part of the theory.

The thesis is organized as follows.

Chapter 1 collects some basic definitions and notations which we need throughout the thesis. For a given algebra, we emphasize that the standard and costandard modules depend on the given partial order set of all non-isomorphic simple modules. In Chapter 2, we recall various definitions of quasi-hereditary algebras which appeared in the literature. We prove the equivalence of those definitions and explain why we can consider the linear order on the set of all non-isomorphic simple modules to present our theory. We also list some basic properties of quasi-hereditary algebras. Chapter 3 summarizes well known generalizations of quasi-hereditary algebras. We

introduce a series of stratified algebras such as CPS-stratified, standardly stratified and properly stratified, which are closely related to each other. As an analogue of BGG-algebras, we define a duality of the standardly stratified algebras and show that every standardly stratified algebra whose module category has a strong duality, is properly stratified. Chapter 4 is devoted to the proof of the main Theorem 4.1.4 described earlier. In addition, we provide a number of examples and a number of additional properties of the properly stratified algebras which relate to our results. The final Chapter 5 illustrates some further results on properly stratified algebras and some canonical constructions of them.

Chapter 1

Preliminaries

The purpose of this chapter is to establish all the facts that we shall use in the subsequent chapters. Throughout the thesis, we fix an algebraically closed field K . All algebras A , unless stated otherwise, will be finite dimensional over the field K , and all modules over A will be finitely generated (or equivalently, finite dimensional over K). We denote by $\text{mod-}A$ the category of all finitely generated right A -modules and by $A\text{-mod}$ the category of all finitely generated left A -modules. The composition of two homomorphisms $\alpha : X_1 \rightarrow X_2$ and $\beta : X_2 \rightarrow X_3$ in $\text{mod-}A$ will be denoted by $\beta\alpha$. The composition of two homomorphisms $\gamma : Y_1 \rightarrow Y_2$ and $\delta : Y_2 \rightarrow Y_3$ in $A\text{-mod}$ will be denoted by $\gamma\delta$.

Recall that the *Jacobson radical* of A , denoted by $\text{rad } A$, is the intersection of all maximal right ideals. Moreover $A/\text{rad } A$ is an artinian semi-simple K -algebra. The right socle of A , denoted by $\text{soc } A$, is the right ideal of A generated by the set of all simple right ideals of A . In general, for an A -module X , the radical $\text{rad } X$ is

the intersection of all maximal submodules of X and $\text{soc } X$ is the submodule of X generated by the set of all simple submodules of X .

An element e of A is an idempotent if $e^2 = e$. Two idempotents e and f are said to be orthogonal if $ef = fe = 0$. An idempotent e is primitive if e cannot be decomposed into a sum of non-zero orthogonal idempotents. We know that the right regular module A_A can be decomposed as $A_A = \bigoplus_{\lambda \in \Lambda} e_\lambda A$, where each e_λ is an idempotent and each $e_\lambda A$ is indecomposable, and $\sum_{\lambda \in \Lambda} e_\lambda = 1$. Thus, these e_λ 's are primitive idempotents. In fact we can write $A_A = \bigoplus_{i=1}^n (\bigoplus_{\lambda_i \in \Lambda_i} e_{\lambda_i} A)$, where $\{\Lambda_i : 1 \leq i \leq n\}$ is a partition of Λ such that $e_\lambda A \cong e_\mu A$ if and only if $\lambda, \mu \in \Lambda_i$ for some $1 \leq i \leq n$. Taking one representative λ_i from each Λ_i , we call the set $\{e_{\lambda_i} : 1 \leq i \leq n\}$ satisfying the preceding conditions a complete set of primitive orthogonal idempotents. Now, A is Morita equivalent to $\hat{A} = \text{End}(\bigoplus_{i=1}^n e_{\lambda_i} A)$. Thus, whenever we consider properties of algebra that are invariant under Morita equivalence, without loss of generality we can deal with the latter simpler algebra. This algebra is basic in the following sense: There is a complete set $\{e_{\lambda_i} : 1 \leq i \leq n\}$ of primitive orthogonal idempotents such that $\sum_{i=1}^n e_{\lambda_i} = 1$. Thus, an algebra A is basic if and only if $A/\text{rad } A$ is isomorphic to a product of division algebras. We denote by D the vector space duality with respect to K . The K -dual of ${}_A A$ is the injective right A -module $D({}_A A) = \text{Hom}_K({}_A A, K)$ which can also be called the coregular module. We shall use the following notations:

$$P(\lambda) = e_\lambda A \quad , \text{ the projective module corresponding to } e_\lambda.$$

$$S(\lambda) = e_\lambda A / e_\lambda \text{rad } A \quad , \text{ the simple module corresponding to } e_\lambda.$$

$$I(\lambda) = D(Ae_\lambda) \quad , \text{ the injective module corresponding to } e_\lambda.$$

$P(\lambda)$ is the projective cover of $S(\lambda)$ and $I(\lambda)$ the injective envelope of $S(\lambda)$. If X is

an A -module, we denote the Jordan-Hölder multiplicity of $S(\lambda)$ in X by $[X : S(\lambda)]$. Given a set $\mathcal{X}$ of A -modules, then for an A -module X , we define the trace of $\mathcal{X}$ in X as the submodule of X generated by all homomorphic images of the A -modules of $\mathcal{X}$.

1.1 Path Algebras

The path algebras over a quiver $\mathcal{Q}$ over a field K will be used to provide examples to illustrate our results. Recall that a (labelled) quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t : \mathcal{Q}_1 \rightarrow \mathcal{Q}_0)$ is given by a set $\mathcal{Q}_0$ of vertices, $\{1, 2, \dots, n\}$, and a finite set $\mathcal{Q}_1$ of arrows labelled by lowercase Greek letters. An arrow ρ starts at the vertex $s(\rho)$ and terminates at $t(\rho)$. Sometimes, we will indicate this by $s(\rho) \xrightarrow{\rho} t(\rho)$. A nontrivial path σ is a sequence $\sigma = \sigma_1 \sigma_2 \dots \sigma_k$ of arrows which satisfy $t(\sigma_i) = s(\sigma_{i+1})$ for all $1 \leq i < k$. Often, we shall denote it by the following diagram

$$\sigma = \bullet \xrightarrow{\sigma_1} \bullet \xrightarrow{\sigma_2} \bullet \dots \xrightarrow{\sigma_k} \bullet.$$

This path starts at $s(\sigma_1)$ and terminates at $t(\sigma_k)$. k is the length of σ . We will denote by $s(\sigma)$ and $t(\sigma)$ the starting and terminating vertex of a path σ . For each vertex i we denote by ϵ_i the trivial path of length zero which starts and terminates at i . The path algebra $K\mathcal{Q}$ is the K -vector space with basis consisting of all paths of $\mathcal{Q}$, and with the product given by the composition of paths: For two paths $\sigma, \tau \in K\mathcal{Q}$

$$\sigma\tau = \begin{cases} \text{composition} & \text{if } t(\sigma) = s(\tau) \\ 0 & \text{otherwise} \end{cases}$$

This product can be extended to the entire space $K\mathcal{Q}$ using distributivity. In this way, we obtain a basic associative algebra $K\mathcal{Q}$.

For example, if $\mathcal{Q}$ is the quiver $1\bullet \xrightarrow{\sigma_1} 2\bullet \xrightarrow{\sigma_2} 3\bullet$, then $\{\epsilon_1, \epsilon_2, \epsilon_3, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$ is a basis of $K\mathcal{Q}$. The product of paths σ_1 and σ_2 is the path $\sigma_1\sigma_2$. On the other hand, the product $\sigma_2\sigma_1$ is zero. Some other products are $\sigma_1\sigma_1 = 0$, $\epsilon_2\sigma_1 = 0$, $\sigma_1\epsilon_1 = 0$, $\epsilon_1\epsilon_1 = \epsilon_1$, $\epsilon_1\epsilon_2 = 0$, etc.

Example 1.1.1. If $\mathcal{Q}$ consists of one vertex and one loop, i.e., $\mathcal{Q} = \bullet \bigcirc \alpha$, then $K\mathcal{Q} \simeq K[T]$. If $\mathcal{Q}$ has one vertex and n loops, then $K\mathcal{Q}$ is the free associative algebra on n letters.

Set $A = K\mathcal{Q}$. We note the following facts:

- (1) The set $\{\epsilon_i \mid i \in \mathcal{Q}\}$ is a complete set of primitive orthogonal idempotents of $K\mathcal{Q}$.
- (2) A has an identity given by $\sum_{i=1}^n \epsilon_i$.
- (3) The spaces $\epsilon_i A$, $A\epsilon_j$ and $\epsilon_i A\epsilon_j$ have as bases the paths starting at i , terminating at j and starting at i and terminating at j , respectively.
- (4) Distinct ϵ_i 's are not equivalent, i.e., $\epsilon_i A \not\cong \epsilon_j A$ for $i \neq j$.
- (5) $A_A = \bigoplus_{i=1}^n \epsilon_i A$, and each $\epsilon_i A$ is an indecomposable projective right A -module.

(6) A is finite dimensional $\Leftrightarrow \mathcal{Q}$ has no oriented cycles .

(7) A is a hereditary algebra.

A module M over $K\mathcal{Q}$ can be identified with a representation of $\mathcal{Q}$ which is given by a set of K -vector spaces indexed by $\mathcal{Q}_0$ along with a set of linear maps indexed by $\mathcal{Q}_1$. Thus $M = (M_i, \phi_\alpha \mid i \in \mathcal{Q}_0, \alpha \in \mathcal{Q}_1)$, where $\phi_\alpha \in \text{Hom}_K(M_i, M_j)$ if α is an arrow from i to j . For instance, let $S(i)$ be the representation with

$$S(i)_j = \begin{cases} K & \text{when } j = i \\ 0 & \text{otherwise} \end{cases} \quad \text{and } S(i)_\alpha = 0 \text{ for all } \alpha \in \mathcal{Q}_1.$$

Then $S(i)$ is a simple module and, in the case that $K\mathcal{Q}$ is finite dimensional, any simple module has such a form. Since there are finite dimensional K -algebras which are not hereditary, not every basic finite dimensional K -algebra is isomorphic to the path algebra of a quiver. But if A is a basic finite dimensional algebra over an algebraically closed field K , one of the fundamental theorems of the theory assures that A is a special quotient algebra of a path algebra $K\mathcal{Q}$ for some finite quiver $\mathcal{Q}$.

These special quotient algebras of a path algebra can be determined by admissible relations. An admissible relation on $\mathcal{Q}$ over a field K is a K -linear combination of paths of length ≥ 2 . Without loss of generality, we can assume that all those paths have the same starting vertex and the same terminating vertex. Given a set $\mathcal{R}$ of admissible relations, consider the $K\mathcal{Q}$ -ideal $\mathcal{I}$ generated by $\mathcal{R}$, and call such an ideal admissible. It is clear that the quotient algebra $K\mathcal{Q}/\mathcal{I}$ can be finite dimensional even if $K\mathcal{Q}$ is infinite dimensional. Again a module over $K\mathcal{Q}/\mathcal{I}$ can be identified with a representation of $\mathcal{Q}$, where the set of maps will satisfy the additional conditions that the linear combinations of homomorphisms corresponding to relations in $\mathcal{R}$ are

zero. Let us reiterate that it is well known that any basic algebra over algebraically closed field K is isomorphic to $KQ/\mathcal{I}$ for some quiver Q and some admissible ideal $\mathcal{I}$.

Example 1.1.2. Let Q be the quiver

$$\begin{array}{ccccc} & \alpha_1 & & \alpha_2 & & \alpha_3 \\ 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 3 & \xrightarrow{\quad} & 4 \\ & \beta_1 & & \beta_2 & & \beta_3 \end{array}$$

and $\mathcal{I} = \langle \beta_3\alpha_3, \beta_1\alpha_1 - \alpha_2\beta_2, \beta_2\alpha_2 - \alpha_3\beta_3 \rangle$.

The right regular module A_A is given as follows:

Remark 1.1.3. Since the regular representations of algebras will often be presented by diagrams of the above type in this thesis, it may be appropriate to describe the structure of the diagram in the above example.

Here, the algebra has 4 indecomposable projective modules $P(i)$, $1 \leq i \leq 4$, and each of them is described by one of the (direct) component of the diagram. Consider the (first) component corresponding to $P(1)$ in more detail. The other components are constructed similarly.

The following set of paths is a K -basis of $P(1)$:

$$\left\{ \begin{aligned} &\epsilon_1, \alpha_1, \alpha_1 \beta_1, \alpha_1 \alpha_2, \alpha_1 \beta_1 \alpha_1 (= \alpha_1 \alpha_2 \beta_2), \alpha_1 \alpha_2 \alpha_3, \alpha_1 \beta_1 \alpha_1 \beta_1 (= \alpha_1 \alpha_2 \beta_2 \beta_1), \\ &\alpha_1 \beta_1 \alpha_1 \alpha_2 (= \alpha_1 \alpha_2 \alpha_3 \beta_3), \alpha_1 \beta_1 \alpha_1 \beta_1 \alpha_1 (= \cdots = \alpha_1 \alpha_2 \alpha_3 \beta_3 \beta_2), \\ &\alpha_1 \beta_1 \alpha_1 \beta_1 \alpha_1 \beta_1 (= \cdots = \alpha_1 \alpha_2 \alpha_3 \beta_3 \beta_2 \beta_1) \end{aligned} \right\}.$$

We can display this basis (together with the A -action on its elements) as follows:

In this way, we have described the composition series of the module $P(1)$. To simplify the diagram, we need only specify the respective composition factors $S(j)$ (of the form $\sigma A / \text{rad } \sigma A$ for the paths σ of the basis) of $P(1)$ and write simply j instead of $S(j)$.

1.2 Standard and Costandard Modules

Let $\{\epsilon_\lambda \mid \lambda \in \Lambda\}$ be a complete set of primitive orthogonal idempotents of a K -algebra A . Thus $S(\lambda)$ for $\lambda \in \Lambda$ are all non-isomorphic simple modules. We now assume that $(\Lambda, \leq)$ is a partially ordered set. In what follows, a K -algebra A , together with a partially ordered index set $(\Lambda, \leq)$ of the simple A -modules, will be denoted by (A, Λ) .

Definition 1.2.1. *Given (A, Λ) , the standard module $\Delta(\lambda)$ is the largest quotient module of $P(\lambda)$ with composition factors of the form $S(\mu)$, $\mu \leq \lambda$, i.e., $\Delta(\lambda) = P(\lambda) / \text{trace}(\bigoplus_{\mu \leq \lambda} P(\mu) \rightarrow P(\lambda))$. Similarly, the costandard module $\nabla(\lambda)$ is the largest submodule of $I(\lambda)$ whose composition factors are of the form $S(\mu)$, $\mu \leq \lambda$, i.e., $\nabla(\lambda) = \bigcap_{\mu \leq \lambda} \text{Ker}(I(\lambda) \rightarrow I(\mu))$.*

We will denote the sets of standard and costandard modules by $\Delta = \{\Delta(\lambda) \mid \lambda \in \Lambda\}$ and $\nabla = \{\nabla(\lambda) \mid \lambda \in \Lambda\}$, respectively. An A -module X is said to have a Δ -filtration (respectively, ∇ -filtration) if there is a filtration

$$0 = X_{m+1} \subset X_m \subset \cdots \subset X_{t+1} \subset X_t \cdots \subset X_1 = X$$

with $X_t/X_{t+1} \simeq \Delta(\lambda_t)$ (respectively, $X_t/X_{t+1} \simeq \nabla(\lambda_t)$) for some $\lambda_t \in \Lambda$ for all $1 \leq t \leq m$. The set of all modules with a Δ -filtration will be denoted by $\mathcal{F}(\Delta)$ and the set of all modules with a ∇ -filtration by $\mathcal{F}(\nabla)$.

The sets Δ and ∇ depend, in an essential way, on the given partial ordering of Λ . We will say that two partially ordered sets Λ and Λ' used as index sets for simple A -modules are equivalent provided the sets $\{\Delta_\Lambda(\lambda) \mid \lambda \in \Lambda\}$ and $\{\Delta_{\Lambda'}(\lambda') \mid \lambda' \in \Lambda'\}$ coincide, and $\{\nabla_\Lambda(\lambda) \mid \lambda \in \Lambda\}$ and $\{\nabla_{\Lambda'}(\lambda') \mid \lambda' \in \Lambda'\}$ coincide.

In general, both the standard and costandard modules of (A, Λ) will change if we refine the partial order of $(\Lambda, \leq)$. We shall restrict our considerations to the so-called admissible partial orders $(\Lambda, \leq)$ (adapted in the terminology of [DR5]), i.e., those satisfying the following condition: For every A -module X with $X/\text{rad } X \simeq S(\lambda_1)$ and $\text{soc } X \simeq S(\lambda_2)$, where λ_1 and λ_2 are incomparable, there is $\mu \in \Lambda$ such that $\mu > \lambda_1$ and $\mu > \lambda_2$ and $S(\mu)$ is a composition factor of X , i.e., $Xe_\mu \neq 0$. By a remark in [DR5], we may weaken the condition as follows: We only have to require the existence of some $\mu \in \Lambda$ with $\mu > \lambda_1$ or $\mu > \lambda_2$ such that $Xe_\mu \neq 0$. Indeed, in case the weaker condition is satisfied, assume that there exists a module X with top $S(\lambda_1)$, socle $S(\lambda_2)$, where λ_1 and λ_2 are incomparable, and such that there is no $\mu \in \Lambda$ with $\mu > \lambda_1$, $\mu > \lambda_2$ and $Xe_\mu \neq 0$. We may assume that X is of smallest possible length, and we know there is at least a μ with $Xe_\mu \neq 0$ and either $\mu > \lambda_1$ or $\mu > \lambda_2$. Assume we have $\mu > \lambda_1$. Now X has a submodule X' with top $S(\mu)$. Note that μ and λ_2 are not comparable. The minimality of X implies that there is a ν such that $\nu > \mu$ and $\nu > \lambda_2$ and $X'\epsilon_\nu \neq 0$, but $\nu > \mu > \lambda_1$, $\nu > \lambda_2$, and $X\epsilon_\nu \neq 0$, contrary to our assumption. As an example of non-admissible partial orderings, the reader should have in mind any non-semisimple algebras with discrete ordering of simple modules. If Λ' is a refinement of Λ , and Λ is admissible, then clearly $\Delta_\Lambda(\lambda) = \Delta_{\Lambda'}(\lambda)$ and $\nabla_\Lambda(\lambda) = \nabla_{\Lambda'}(\lambda)$ for all $\lambda \in \Lambda$, thus Λ and Λ' are equivalent, and Λ' is also admissible. Thus for an admissible Λ , the standard and costandard modules do not change under any refinement of $(\Lambda, \leq)$, thus in such a case, we may replace $(\Lambda, \leq)$ by the equivalent index set $\{1, 2, \dots, n\}$ with its natural order.

This will be *modus operandi* for the rest of the thesis. Thus, in what follows, $\mathbf{e}$ will denote a complete sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ of primitive orthogonal idempotents

and (A.e) the algebra A together with the respective linear order of the set of all isomorphic classes of simple A -modules. For a given (A.e), let us summarize some of the basic properties of standard and costandard modules (including their characterization cf. [DR5] and [D2]). We will not formulate the dual statements.

Lemma 1.2.2. *(Characterization of standard modules) For an A -module X and $i \in \Lambda$, the following assertions are equivalent:*

- (i) $X \simeq \Delta(i)$;
- (ii) $\text{top } X \simeq S(i)$, all composition factors of X are of the form $S(j)$ with $j \leq i$,
and $\text{Ext}_A^1(X, S(j)) \neq 0$ implies $j > i$;
- (iii) $X \simeq P(i)$, $\text{trace}(\bigoplus_{j < i} P(j) \rightarrow P(i))$.

Lemma 1.2.3. *Let X be an A -module, and $i, j \in \Lambda$. Then:*

- (i) $\text{Hom}_A(\Delta(i), X) \neq 0$ implies $[X : S(i)] \neq 0$;
- (ii) $\text{Hom}_A(\Delta(i), \Delta(j)) \neq 0$ implies $i \leq j$;
- (iii) $\text{Hom}_A(\Delta(i), \nabla(j)) \neq 0$ implies $i = j$.

Lemma 1.2.4. *Let X be an A -module, and $i, j \in \Lambda$. Then,*

- (i) $\text{Ext}_A^1(\Delta(i), X) \neq 0$ then $[X : S(j)] \neq 0$ for some $j > i$;
- (ii) $\text{Ext}_A^1(\Delta(i), \Delta(j)) \neq 0$, then $i < j$;
- (iii) $\text{Ext}_A^1(\Delta(i), \nabla(j)) = 0$.

Lemma 1.2.5 (see, e.g., [DR5] or [D2]). *The family $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms and under direct summands.*

Let us note that a module X is called Schurian if the endomorphism algebra of X is a semi-simple algebra.

Lemma 1.2.6. *The following statements are equivalent for all $i \in \Lambda$*

- (i) $\Delta(i)$ is Schurian:
- (i)^{*} $\nabla(i)$ is Schurian:
- (ii) $[\Delta(i) : S(i)] = 1$:
- (ii)^{*} $[\nabla(i) : S(i)] = 1$

Chapter 2

Quasi-Hereditary Algebras

2.1 Equivalence of Several Definitions of Quasi-Hereditary Algebras

Let us first recall that the original recursive definition of quasi-hereditary algebras was given in [CPS1].

Definition 2.1.1. *Let $\mathcal{X}$ be the class of finite dimensional algebras A over K defined recursively as follows: A has a non zero ideal J such that*

- (i) J is projective as a right A -module;*
- (ii) $\text{Hom}_A(J_A, A/J) = 0$ and $J(\text{rad } A)J = 0$;*
- (iii) if $J \neq A$, A/J belongs to $\mathcal{X}$.*

$\mathcal{X}$ is said to be the class of quasi-hereditary algebras.

Trivially, all semi-simple, as well as hereditary, algebras are quasi-hereditary. Cline-Parshall-Scott also provided in [CPS1] and [PS] an equivalent definition of quasi-hereditary algebras in forms of the so-called highest weight categories. They also shown that the concept of quasi-hereditary is Morita equivalent.

A structural definition of quasi-hereditary algebras has been given by Dlab-Ringel in [DR2] as follows.

Definition 2.1.2. *An ideal J of A is called a heredity ideal if $J^2 = J$, $J(\text{rad } A)J = 0$ and J_A as right A -module is projective.*

A finite dimensional K -algebra A is quasi-hereditary if there exists a chain of ideals:

$$0 = J_{m+1} \subset J_m \subset \cdots \subset J_{i+1} \subset J_i \cdots \subset J_1 = A \quad (*)$$

such that J_i/J_{i+1} is a heredity ideal in A/J_{i+1} , $1 \leq i \leq m$.

The above chain is called a heredity chain of A .

Remark 2.1.3. *Each heredity chain $(*)$ of a quasi-hereditary algebra $(A, \mathbf{e})$ gives rise to a partial order on the index set $\lambda \in \Lambda$. As before, denoting by $S(\lambda)$ the respective non-isomorphic simple A -modules, define the partial order in the following way: $\lambda < \mu$ if and only if there is an ideal J_i in the heredity chain such that $S(\lambda)$ appears in the top of module A/J_i , but $S(\mu)$ does not.*

This partial order of the set Λ is admissible.

Remark 2.1.4. *Of course, the partial order of Remark 2.1.3 can be refined to a linear order of the set $\{S(\lambda) \mid \lambda \in \Lambda\}$ as explained in Chapter 1. Let us describe such a refinement in more detail. Observe that any heredity ideal J is generated*

as a two-sided ideal by an idempotent. Moreover, if e is any idempotent in J , then the ideal AeA generated by e is a heredity ideal of A , and J/AeA is a heredity ideal of A/AeA . It follows that we can refine any heredity chain to a heredity chain so that, in addition, J_i/J_{i+1} is generated by a primitive idempotent, and we call such a heredity chain a saturated one. By a proposition in [D1] the saturated heredity chain gives rise to a fixed order $\mathbf{e} = (e_1, e_2, \dots, e_n)$ of the complete set of primitive orthogonal idempotents of A , such that $J_i = A(e_i + e_{i+1} + \dots + e_n)A$ for $1 \leq i \leq n$. As a consequence, $(\Lambda, \leq)$ is equivalent to a totally ordered set and we can replace it by $\{1, 2, \dots, n\}$ with its natural order.

Let us recall the module theoretic definition taken from the paper [DR5] of Dlab-Ringel.

As before, let $(A, \mathbf{e})$ be a finite dimensional K -algebra together with a linear order of the set of all non-isomorphic simple A -modules $S(i)$, or, equivalently, of the set of all projective cover $P(i)$ of $S(i)$, $1 \leq i \leq n$.

Definition 2.1.5. *An algebra $(A, \mathbf{e})$ is said to be quasi-hereditary with respect to $\mathbf{e}$, if*

- (i) $\text{End}_A(\Delta(i))$ is a division algebra (i.e., $\Delta(i)$ is Schurian) for all $1 \leq i \leq n$;
- (ii) $A_A \in \mathcal{F}(\Delta)$.

In fact, it is easy to see that Definition 2.1.5 is equivalent to the following definition.

Definition 2.1.6. *An algebra $(A, \mathbf{e})$ is said to be quasi-hereditary (with respect to $\mathbf{e}$), if for each $1 \leq i \leq n$,*

- (i) $\text{End}_A(\Delta(i))$ is a division algebra (i.e., $\Delta(i)$ is Schurian) and
- (ii) The projective module $P(i)$ has a filtration by standard modules $\Delta(j)$, $j \geq i$ and $[P(i) : \Delta(i)] = 1$.

Proposition 2.1.7. *These above four definitions of a quasi-hereditary algebra are equivalent.*

Proof. The equivalence between Definition 2.1.1 and Definition 2.1.2 is straightforward (see [DR1]). Let us only mention that in the presence of the assumption that J is projective as right A -module, the condition $\text{Hom}_A(J_A, A/J) = 0$ and $J(\text{rad } A)J = 0$ can be replaced by the alternative condition $J = J^2$ and $J(\text{rad } A)J = 0$.

The equivalence between Definition 2.1.2 and Definition 2.1.5 can be seen easily as follows: For a saturated heredity chain,

$$J_i = A(\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n)A = \text{trace}(\bigoplus_{j \geq i} P(j) \rightarrow A), \quad 1 \leq i \leq n.$$

Thus the saturated heredity chain defines a trace filtration of the regular module A_A . The fact that J_i/J_{i+1} is a right projective A/J_{i+1} -module means that J_i/J_{i+1} is isomorphic to a finite direct sum of $\Delta(i)$'s and thus $A_A \in \mathcal{F}(\Delta)$. For an arbitrary idempotent ideal J of A , $J = A\epsilon A$ for some idempotent $\epsilon \in A$ and $J\text{rad } AJ = 0$ means that $\text{End}_A(\epsilon A)$ is a product of division algebras. In particular, for a saturated heredity chain of A , $(J_i/J_{i+1})_A$ is a finite direct sum of $\Delta(i)$. Observe that $\text{End}_A(\Delta(i))$ is a division algebra for all $1 \leq i \leq n$. Hence, $(A, \mathbf{e})$ is quasi-hereditary by Definition 2.1.5. The proof of the converse implication follows from the fact that

the filtration (ii) of Definition 2.1.5 can be assumed to be

$$0 = A_{n+1} \subset A_n \subset \cdots \subset A_{i+1} \subset A_i \cdots \subset A_1 = A$$

where $A_i/A_{i+1} \simeq \hat{\oplus}_{finite} \Delta(i)$ for all $1 \leq i \leq n$. Thus A_i/A_{i+1} is clearly a heredity ideal of A/A_{i+1} and the above chain is a heredity chain.

The equivalence of Definition 2.1.5 and Definition 2.1.6 will be shown latter in Remark 2.2.5 after we state the Bernstein-Gelfand-Gelfand reciprocity law.

□

2.2 Homological Properties

Although the definitions in the previous section are all given in terms of right A -modules, the following Lemma of [DR2] implies that the concept of a quasi-hereditary algebra is “two-sided”.

Lemma 2.2.1. *Let e be an idempotent of a finite dimensional K -algebra A . If the right module $(AcA)_A$ or the left module ${}_A(AcA)$ is projective, then the multiplication map $\eta : Ac \hat{\oplus}_{eAc} eA \rightarrow AcA$ is bijective. Conversely, assume that $e(\text{rad } A)e = 0$. Then, if η is bijective, both modules $(AcA)_A$ and ${}_A(AcA)$ are projective.*

The following are the basic characterizations of quasi-hereditary algebras (see [D1] and [DR5]). In the next chapter we will see that these characterizations can be extended, in fact, to a more general class of standardly stratified algebras.

Theorem 2.2.2. *Assume all standard modules $\Delta(i)$ of (A, e) are Schurian. Then the following conditions are equivalent:*

$$(i) \ A_A \in \mathcal{F}(\Delta);$$

$$(i)^* \ D({}_A A) \in \mathcal{F}(\nabla) ;$$

$$(ii) \ \mathcal{F}(\Delta) = \{X \mid \text{Ext}_A^1(X, \nabla) = 0\};$$

$$(ii)^* \ \mathcal{F}(\nabla) = \{Y \mid \text{Ext}_A^1(\Delta, Y) = 0\};$$

$$(iii) \ \mathcal{F}(\Delta) = \{X \mid \text{Ext}_A^t(X, \nabla) = 0 \text{ for all } t \geq 1\} ;$$

$$(iii)^* \ \mathcal{F}(\nabla) = \{Y \mid \text{Ext}_A^t(\Delta, Y) = 0 \text{ for all } t \geq 1\};$$

$$(iv) \ \text{Ext}_A^2(\Delta, \nabla) = 0.$$

Here, $\text{Ext}_A^1(X, \nabla) = 0$ denotes the fact that $\text{Ext}_A^1(X, \nabla(i)) = 0$ for all $1 \leq i \leq n$; similarly for other notation. Thus, $\text{Ext}_A^2(\Delta, \nabla) = 0$ means that $\text{Ext}_A^2(\Delta(i), \nabla(j)) = 0$ for all $1 \leq i, j \leq n$.

Let $(A, \mathbf{e})$ be a quasi-hereditary algebra. For X in $\mathcal{F}(\Delta)$, the number of copies of $\Delta(i)$ in any Δ -filtration of X is an invariant and is denoted by $[X : \Delta(i)]$. This is a consequence of the following lemma (cf. [DR5] or [D2]):

Lemma 2.2.3. *Let $(A, \mathbf{e})$ be a quasi-hereditary algebra. The restriction of functor $\text{Hom}_A(-, \nabla(i))$ to $\mathcal{F}(\Delta)$ is exact and for any $X \in \mathcal{F}(\Delta)$, we have*

$$\dim_K \text{Hom}_A(X, \nabla(i)) = [X : \Delta(i)].$$

Applying this Lemma to a projective module $P(j)$, we have the following Bernstein-Gelfand-Gelfand reciprocity law:

Proposition 2.2.4. *Let $(A, \mathbf{e})$ be a quasi-hereditary algebra. Then, for all $1 \leq i, j \leq n$,*

$$[P(i) : \Delta(j)] = [\nabla(j) : S(i)].$$

Remark 2.2.5. *It is worthwhile to recall the equivalence of the definition 2.1.5 and 2.1.6 in light of this Lemma: $[P(i) : \Delta(j)] \neq 0$ implies $i \leq j$ and $[P(i) : \Delta(i)] = 1$.*

We like to mention a few additional facts about quasi-hereditary algebras. First of all, every hereditary algebra is trivially a quasi-hereditary algebra: in fact, hereditary algebras are just those algebras which are quasi-hereditary with respect to any linear order of the simple modules (see [D1]).

Example 2.2.6. *The algebra $K\mathcal{Q}/\mathcal{I}$, where $K\mathcal{Q}$ is the path algebra over the quiver*

$$1 \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{matrix} 2$$

and $\mathcal{I} = \langle \beta\alpha \rangle$, is the “smallest” quasi-hereditary algebra (with respect to natural order) which is not hereditary: its right and left regular representations are given as follows:

$$A_A = \begin{matrix} 1 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ \downarrow \\ 1 \end{matrix}, \quad {}_A A = \begin{matrix} 1 \\ \uparrow \\ 2 \\ \uparrow \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ \uparrow \\ 1 \end{matrix}.$$

Dlab-Ringel [DR6] have shown that all K -algebras of global dimension 2 (and thus in particular, Auslander algebras) are quasi-hereditary. The following example of a path algebra is an algebra of global dimension 3 which is not quasi-hereditary.

Example 2.2.7. The algebra $A = KQ/\mathcal{I}$, where Q is the quiver

$$\begin{array}{ccc} & \xrightarrow{\alpha} & \\ 1 & \xrightarrow{\beta} & 2 \\ & \xleftarrow{\gamma} & \end{array}$$

and ideal $\mathcal{I} = \langle \beta\gamma, \gamma\alpha \rangle$. Its right and left regular representation are given as follows:

$$A_A = \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 2 \quad 2 \\ \downarrow \quad \downarrow \\ 1 \quad 1 \\ \downarrow \quad \downarrow \\ 2 \quad 2 \end{array} \cong \begin{array}{c} 2 \\ \downarrow \\ 1 \\ \downarrow \\ 2 \end{array} \quad , \quad {}_A A = \begin{array}{c} 1 \\ \uparrow \\ 2 \\ \uparrow \\ 1 \end{array} \cong \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 1 \\ \uparrow \quad \uparrow \\ 2 \quad 2 \\ \uparrow \quad \uparrow \\ 1 \quad 1 \end{array} .$$

Let us also recall that Dlab-Ringel [DR3] have shown that every finite dimensional K -algebra R is the endomorphism algebra of a projective A -module over a quasi-hereditary K -algebra A . Moreover, a quasi-hereditary algebra can have arbitrary finite global dimension.

Chapter 3

Stratification of Algebras

3.1 Trace Filtrations

Recall that we deal with an algebra A together with a complete sequence $\mathbf{e} = (e_1, e_2, \dots, e_n)$ of primitive orthogonal idempotents. Without loss of generality, we will always assume that $(A, \mathbf{e})$ is basic and connected. Thus, we can write $\sum_{i=1}^n e_i = 1$. In addition to e_i , we shall consider the related idempotents $\varepsilon_i = e_i + e_{i+1} + \dots + e_n$ for $1 \leq i \leq n$ and $\varepsilon_{n+1} = 0$. For $1 \leq i \leq n$, $P(i) = e_i A$ is the right indecomposable projective A -module and $S(i)$ the simple top of $P(i)$. Thus the right regular representation decomposes into a direct sum $A_A = \oplus_{i=1}^n P(i)$ of pairwise non-isomorphic right indecomposable projective A -modules. Since K is algebraically closed, $\dim_K \text{End}_A(S(i)) = 1$ for all $1 \leq i \leq n$.

The standard module $\Delta(i)$ is by definition the maximal factor module of $P(i)$ without composition factors $S(j)$ for $j > i$. We will also need particular factor

modules of the standard modules.

Definition 3.1.1. *Given $(A, \mathbf{e})$, define the proper standard modules $\bar{\Delta}(i)$ as the maximal factor modules of $\Delta(i)$ such that the multiplicity condition*

$$[\bar{\Delta}(i) : S(i)] = \dim_K \operatorname{Hom}_A(P(i), \bar{\Delta}(i)) = 1$$

holds for $1 \leq i \leq n$.

Note that $\operatorname{End}_A \bar{\Delta}(i) \simeq \operatorname{End}_A(S(i))$, so $\bar{\Delta}(i)$ is a Schurian module.

In terms of the idempotents e_i and ε_i , we can write

$$\Delta(i) = e_i A / e_i A \varepsilon_{i+1} A, \quad 1 \leq i \leq n,$$

and

$$\bar{\Delta}(i) = e_i A / e_i \operatorname{rad} A \varepsilon_i A, \quad 1 \leq i \leq n,$$

respectively.

The left standard module $\Delta^o(i)$ and the left proper standard module $\bar{\Delta}^o(i)$ is defined similarly for any $1 \leq i \leq n$. Moreover, we define the right costandard A -module $\nabla(i)$ and the right proper costandard A -modules $\bar{\nabla}(i)$ for $1 \leq i \leq n$ by

$$\nabla(i) = \operatorname{Hom}_K(\Delta^o(i), K),$$

$$\bar{\nabla}(i) = \operatorname{Hom}_K(\bar{\Delta}^o(i), K).$$

Thus $\nabla(i)$ is the maximal submodule of the injective envelope $I(i)$ of $S(i)$ without composition factors $S(j)$ for $i < j$, while $\bar{\nabla}(i)$ is the maximal submodule of $\nabla(i)$ that satisfies the multiplicity condition

$$[\bar{\nabla}(i) : S(i)] = \dim_K \operatorname{Hom}_A(\bar{\nabla}(i), I(i)) = 1.$$

We use the notation $\Delta = \{\Delta(1), \Delta(2), \dots, \Delta(n)\}$, and we define the set $\bar{\Delta}$, ∇ and $\bar{\nabla}$ similarly.

Given a class of modules $\mathcal{C} \subseteq \text{mod-}A$, we denote by $\mathcal{F}(\mathcal{C})$ the full subcategory of $\text{mod-}A$ containing all modules that are filtered by modules in $\mathcal{C}$. In particular, we shall be mostly interested in the categories $\mathcal{F}(\Delta)$, $\mathcal{F}(\bar{\Delta})$, $\mathcal{F}(\nabla)$, $\mathcal{F}(\bar{\nabla})$.

Definition 3.1.2. *Given an A -module, define its trace filtration (with respect to $\mathbf{e}$) by*

$$X = X^{(1)} \supseteq X^{(2)} \supseteq \dots \supseteq X^{(i)} \supseteq X^{(i+1)} \supseteq \dots \supseteq X^{(n)} \supseteq X^{(n+1)} = 0,$$

where $X^{(i)} = \text{trace}(\varepsilon_i A \rightarrow X)$.

Since $\text{trace}(\varepsilon_i A \rightarrow X) = X\varepsilon_i A$, the above can be rewritten as follows:

$$X = X\varepsilon_1 A \supseteq X\varepsilon_2 A \supseteq \dots \supseteq X\varepsilon_i A \supseteq X\varepsilon_{i+1} A \supseteq \dots \supseteq X\varepsilon_n A \supseteq X\varepsilon_{(n+1)} A = 0.$$

It is easy to see that trace filtrations are compatible with direct sums. Applied to the right regular representation of A , we obtain a filtration of the algebra A by the idempotent ideals $J_i = A\varepsilon_i A$:

$$A = J_1 \supset J_2 \supset \dots \supset J_i \supset J_{i+1} \supset \dots \supset J_n \supset J_{n+1} = 0.$$

Thus, a choice $\mathbf{e}$ of order of idempotents amounts to a choice of a saturated chain of idempotent ideals of A . Observe that the right module

$$J_i/J_{i+1} \simeq \bigoplus_{j=1}^t e_j A\varepsilon_i A / e_j A\varepsilon_{i+1} A$$

with the last summand isomorphic to $\Delta(i)$. In general $X^{(i)}/X^{(i+1)}$ is a module over A/J_{i+1} , whose projective A/J_{i+1} -cover is a finite direct sum of $\Delta(i)$'s. The trace

filtration is related to Δ -filtration as follows: Every $X \in \mathcal{F}(\Delta)$ has a canonical Δ -filtration given by a refinement filtration of the trace filtration

$$X = X^{(1)} \supseteq X^{(2)} \supseteq \cdots X^{(i)} = X\varepsilon_i A \supseteq X^{(i+1)} \supseteq \cdots \supseteq X^{(n)} \supseteq X^{(n+1)} = 0,$$

such that $X^{(i)}/X^{(i+1)}$ is a finite direct sum of $\Delta(i)$'s. (cf [D2]). Similarly any $X \in \mathcal{F}(\bar{\Delta})$ has a canonical $\bar{\Delta}$ -filtration given by a refinement of the proper trace filtration which itself is a refinement of the trace filtration of X :

$$\begin{aligned} \cdots X^{(i)} &= X^{(i,0)} \supseteq X^{(i,1)} \supseteq \cdots \supseteq X^{(i,j)} = \text{trace}(\varepsilon_i A \rightarrow \text{rad } X^{(i,j+1)}) = \\ &= X(\varepsilon_i \text{rad } A \varepsilon_i)^j A \supseteq \cdots \supseteq X^{(i,h_i)} = X^{(i+1)} \supseteq \cdots ; \end{aligned}$$

where for each $1 \leq i \leq n$, the module $X^{(i,j)}/X^{(i,j+1)}$ is a finite direct sum of $\bar{\Delta}(i)$'s for every $0 \leq j \leq h_i - 1$.

Lemma 3.1.3. *The following statements hold for $(A, \mathbf{e})$:*

- (i) $\text{Hom}_A(\bar{\Delta}(i), \bar{\Delta}(j)) = 0$ for $i > j$;
- (ii) $\text{Hom}_A(\bar{\nabla}(i), \bar{\nabla}(j)) = 0$ for $i < j$;
- (iii) $\text{Hom}_A(\Delta(i), \bar{\nabla}(j)) = 0$ for $i \neq j$;
- (iv) $\text{Ext}_A^1(\bar{\Delta}(i), \bar{\Delta}(j)) = 0$ for $i > j$;
- (v) $\text{Ext}_A^1(\bar{\nabla}(i), \bar{\nabla}(j)) = 0$ for $i < j$;
- (vi) $\text{Ext}_A^1(\Delta(i), \bar{\nabla}(j)) = 0$ for every i, j .

Lemma 3.1.4 (see [ADL1]). *The subcategory $\mathcal{F}(\bar{\Delta})$ is closed under kernels of epimorphisms and under direct summands.*

The next Lemma will be useful in identifying the set $\tilde{\Delta} = (\tilde{\Delta}(1), \tilde{\Delta}(2), \dots, \tilde{\Delta}(n))$.

Lemma 3.1.5 (see [AHLU]). *Let $(A, \mathbf{e})$ be given and assume that a set $\{\Phi(1), \Phi(2), \dots, \Phi(n)\}$ of A -modules with $\text{top } \Phi(i) \simeq S(i)$ satisfies the multiplicity conditions $[\Phi(i) : S(i)] = 1$ for every i and $[\Phi(i) : S(j)] = 0$ for $j > i$. If, moreover $P(i) \in \mathcal{F}(\Phi(i), \Phi(i-1), \dots, \Phi(n))$ for all $1 \leq i \leq n$, then $\Phi(i) \simeq \tilde{\Delta}(i)$ for all i .*

3.2 Standardly Stratified Algebras

Definition 3.2.1. *An algebra $(A, \mathbf{e})$ is called standardly stratified if $A_A \in \mathcal{F}(\tilde{\Delta})$.*

Thus, every quasi-hereditary algebra is standardly stratified. As in case of quasi-hereditary algebras, there is a number of equivalent definitions of these algebras. Let us point out that the concept of a standardly stratified algebras was introduced independently in [D3] and [CPS3], where general stratification of an algebra A is studied.

Definition 3.2.2. *A finite dimensional K -algebra A is standardly stratified if there exists a chain of idempotent ideals*

$$0 = J_{n+1} \subset J_n \subset \dots \subset J_{i+1} \subset J_i \dots \subset J_1 = A$$

such that

- (i) *The chain is saturated, i.e., J_i/J_{i+1} contains no proper non-zero idempotent ideals of A/J_{i+1} for each $1 \leq i \leq n$:*

(ii) J_i/J_{i-1} is a projective left A/J_{i-1} -module for each $1 \leq i \leq n$.

It is easy to see that every saturated chain of idempotent ideals gives rise to a fixed order $\mathbf{e} = (e_1, e_2, \dots, e_n)$ of the complete set of primitive orthogonal idempotents of A , such that $J_i = A(e_1 + e_{i-1} + \dots + e_n)A$ as in the case of heredity chains. Note that in Definition 3.2.2 we assume that the chain is saturated. The reason for this assumption is the fact that chains of idempotent ideals satisfying (ii) can not be, in general, refined. The following Example 3.2.3 illustrates this property.

Example 3.2.3. Let $A = KQ/\mathcal{I}$, where Q is

and $\mathcal{I} = \langle \alpha\delta, \delta\beta \rangle$. The left regular representation of A is given as follows:

$${}_A A = \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ \begin{array}{c} 2 \\ \updownarrow \\ 3 \end{array} \quad \begin{array}{c} 3 \\ \updownarrow \\ 2 \end{array} \end{array} \cong \begin{array}{c} 2 \\ \updownarrow \\ 3 \end{array} \oplus \begin{array}{c} 3 \\ \updownarrow \\ 2 \end{array}.$$

$0 \subset J_2 = A(e_2 + e_3)A \subset J_1 = A$ is a chain with projective modules J_2 and J_1/J_2 : no refinement exists preserving (ii) of Definition 3.2.2.

Remark 3.2.4. Let us mention that, similarly to Definition 2.1.6, we can define the standardly stratified algebras as those having filtration of $P(i)$ by proper standard modules $\bar{\Delta}(j)$, $j \geq i$, for all $1 \leq i \leq n$.

In analogy to Theorem 2.2.2, we can formulate the following characterizations of standardly stratified algebras (cf. [D3] and [ADL1]).

Theorem 3.2.5. *Let $(A, \mathbf{e})$ be a K -algebra. Then the following statements are equivalent:*

- (i) $(A, \mathbf{e})$ is standardly stratified :
- (ii) $\mathcal{F}(\tilde{\Delta}) = \{X \mid \text{Ext}_{A_1}^1(X, \nabla) = 0\}$:
- (iii) $\mathcal{F}(\tilde{\Delta}) = \{X \mid \text{Ext}_{A_1}^t(X, \nabla) = 0 \text{ for all } t \geq 1\}$:
- (iv) $\text{Ext}_{A_1}^2(\tilde{\Delta}, \nabla) = 0$:
- (v) $\text{Ext}_{A_1}^t(\tilde{\Delta}, \nabla) = 0 \text{ for all } t \geq 1$:
- (vi) $\dim_K A = \sum_{i=1}^n \dim_K \Delta^o(i) \dim_K \tilde{\Delta}(i)$.

Again $\text{Ext}_{A_1}^1(X, \nabla) = 0$ denotes the fact that $\text{Ext}_{A_1}^1(X, \nabla(j)) = 0$ for all $1 \leq j \leq n$. etc.

Unlike in the case of quasi-hereditary algebras, $A_A \in \mathcal{F}(\tilde{\Delta})$ does not, in general, imply ${}_A A \in \mathcal{F}(\tilde{\Delta}^o)$. However, we have a remarkable relation discovered by Dlab in [D3]; see also [ADL2] and [L]:

Theorem 3.2.6. *Let $(A, \mathbf{e})$ be a K -algebra, then the following two statements are equivalent:*

- (i) A is standardly stratified, i.e., $A_A \in \mathcal{F}(\tilde{\Delta})$:
- (ii) $D({}_A A) \in \mathcal{F}(\nabla)$, i.e., ${}_A A \in \mathcal{F}(\Delta^o)$.

The following example illustrates the behavior described in Theorem 3.2.6.

Example 3.2.7. Let $A = KQ/\mathcal{I}$, where

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2 \circlearrowleft$$

and $\mathcal{I} = \langle \alpha\beta, \beta\alpha, \alpha^2 \rangle$. Thus, the right and left regular representations are

$$A_A = \begin{array}{c} 1 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \end{array} \quad , \quad {}_A A = \begin{array}{c} 1 \\ \downarrow \\ 2 \\ \swarrow \quad \searrow \\ 2 \quad 1 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 2 \quad 1 \end{array} .$$

Theorem 3.2.6 allows us to formulate immediately dual conditions, characterizing standardly stratified algebras, such as

$$(ii)^* \mathcal{F}(\nabla) = \{Y \mid \text{Ext}_A^1(\bar{\Delta}, Y) = 0\}, \text{ etc.}$$

As in the case of quasi-hereditary algebras, we have also a (general) Bernstein-Gelfand-Gelfand law.

Theorem 3.2.8 (cf. [ADL1]). *Let $(A, \mathbf{e})$ be a standardly stratified algebra, then*

$$[P(i) : \bar{\Delta}(j)] = [\nabla(j) : S(i)] \quad \text{for all } 1 \leq i, j \leq n.$$

Finally, let us also mention two characterizations of quasi-hereditary algebras within the class of standardly stratified algebras.

Theorem 3.2.9 (cf. [ADL1]). *Let $(A, \mathbf{e})$ be a standardly stratified algebra. Then the following statements are equivalent:*

$$(i) \text{ gl.dim } A < \infty :$$

- (ii) $\Delta(i) = \bar{\Delta}(i)$ for all $1 \leq i \leq n$;
- (iii) $(A, \mathbf{e})$ is quasi-hereditary.

3.3 CPS-Stratified Algebras and Properly

Stratified Algebras

Definition 3.3.1 ([CPS3]). *An ideal J of A is called a stratifying ideal if the following three conditions are satisfied:*

- (i) J is an idempotent ideal, hence of the form $J = AeA$ for some $e = e^2 \in A$;
- (ii) the multiplication map induces a bijection $Ae \otimes_{eAe} eA \rightarrow AeA$;
- (iii) $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for all $i > 0$.

One consequence is that if $J = AeA$ is a right projective module, then J is a stratifying ideal.

Definition 3.3.2. *Given an algebra A , if there is a non-trivial chain of ideals $0 = J_{m+1} \subset J_m \subset \cdots \subset J_{i+1} \subset J_i \subset \cdots \subset J_1 = A$ such that J_i/J_{i+1} is a stratifying ideal in A/J_{i+1} for all $1 \leq i \leq m$, then A will be called CPS-stratified.*

From the definition of stratifying ideal, we know that if A is CPS-stratified, then A^{opp} is also CPS-stratified. In their paper [ADL2], Ágoston-Dlab-Lukács introduced an extension of the class of standardly stratified algebras to the class of strictly stratified algebras, and shown that all these algebras are CPS-stratified. For

monomial algebras, the latter two concepts coincide. [ADL2] provides an example of a CPS-stratified algebra which is not strictly (and thus also not standardly) stratified. To see that the algebra in Example 3.3.3 is CPS-stratified, one can use the characterization of CPS-stratifying ideals in terms of their projective resolutions (see e.g. [ADL2]).

Example 3.3.3. Let $A = KQ/\mathcal{I}$, where Q is the quiver

and $\mathcal{I} = \langle \alpha o, \alpha \beta \alpha \beta, o^2, \gamma \beta \alpha, \gamma o \beta, o \beta \alpha \beta, o \beta - \beta \alpha \beta \rangle$. Thus, the right regular representation is as follows:

Now, following the idea of [D4], let us introduce the following definition.

Definition 3.3.4. A non-zero idempotent e of an algebra A is said to be properly stratifying if, for $B = A/AeA$, the following two conditions are satisfied:

- (i) $\text{Ext}_A^k(B_A, B_A) = 0$ for all $k > 0$, and
- (ii) B_A and ${}_A B$ have finite projective dimensions.

We shall use conditions (i) and (ii) in the above definition together with the following basic result to describe the structure of the properly stratified algebras defined in Definition 3.3.6 below.

Theorem 3.3.5 (see [D3] or [L]). *Let e be a primitive idempotent of a finite dimensional K -algebra A . Then the left A -module ${}_A(AeA)$ is projective if and only if the right module- A $(AeA)_A$ is filtered by copies of (the right proper standard module) $\bar{\Delta}(e) = eA/e\text{rad } AeA$ and the right module- A $(AeA)_A$ is projective if and only if the left module- A ${}_A(AeA)$ is filtered by copies of (the left proper standard module) $\bar{\Delta}^o(e) = Ae/Ae\text{rad } Ae$. Moreover, the multiplicity $[(AeA)_A : \bar{\Delta}(e)] = \dim_K(Ae)$ and the right projective A -module eA is filtered by $\bar{\Delta}(e)$ with multiplicity $\dim_K(eAe) = \text{composition length of } eAe$.*

Definition 3.3.6. *Given $(A, \mathbf{e})$, call the complete sequence $\mathbf{e} = (e_1, e_2, \dots, e_n)$ of primitive orthogonal idempotents of A a properly stratifying sequence if for each $1 \leq i \leq n$, the idempotent $\bar{e}_i = e_i + J_{i+1}$ of the algebra A/J_{i+1} , where $J_{i+1} = Ae_{i+1}A$, is properly stratifying. Finally, $(A, \mathbf{e})$ is said to be a properly stratified algebra (with respect to the order $\mathbf{e}$) if $\mathbf{e}$ is a properly stratifying sequence.*

Lemma 3.3.7 (see [D4]). *The primitive idempotent e_n of $(A, \mathbf{e})$ is properly stratifying if and only if the right module $(Ae_nA)_A$ belongs to $\mathcal{F}(\Delta(n)) \cap \mathcal{F}(\bar{\Delta}(n))$*

Proof. Since e_n is properly stratifying, both B_A and ${}_AB$, where $B = A/Ae_nA$, have finite projective dimensions. Furthermore, $\text{Ext}_A^k(B_A, B_A) = 0$ for all $k > 0$ implies that both $(Ae_nA)_A$ and ${}_A(Ae_nA)$ have (minimal) projective resolutions whose terms are the direct sums of copies of e_nA and Ae_n , respectively (see [ADL2]). Consequently, $(Ae_nA)_A$ and ${}_A(Ae_nA)$ are either projective or have infinite projective

dimension. Indeed, the last term of a finite minimal projective resolution is a submodule of the radical of the proceeding term. Thus, it has a smaller Loewy length which is under our condition impossible. We conclude that B_A and ${}_A B$ have finite projective dimension if and only if both $(Ae_n A)_A$ and ${}_A(Ae_n A)$ are projective. However, this is equivalent according to Theorem 3.3.5 to the condition that $(Ae_n A)_A$ is filtered both by $\Delta(n)$ and $\bar{\Delta}(n)$. $\square$

Remark 3.3.8. *The condition $(Ae_n A)_A \in \mathcal{F}(\Delta(n)) \cap \mathcal{F}(\bar{\Delta}(n))$ is equivalent to ${}_A(Ae_n A) \in \mathcal{F}(\Delta^o(n)) \cap \mathcal{F}(\bar{\Delta}^o(n))$ or to the condition $(Ae_n A)_A \in \mathcal{F}(\Delta(n))$ and $\Delta(n) \in \mathcal{F}(\bar{\Delta}(n))$.*

Corollary 3.3.9. *A primitive idempotent e of a finite dimensional K -algebra A is properly stratifying if and only if $(AeA)_A$ and ${}_A(AeA)$ are projective.*

Following the above lemma, we can give another definition of properly stratified algebras.

Definition 3.3.10. *A finite dimensional K -algebra A is properly stratified if there exists a chain of idempotent ideals*

$$0 = J_{n+1} \subset J_n \subset \cdots \subset J_{i+1} \subset J_i \cdots \subset J_1 = A$$

such that

- (i) *The chain is saturated which means J_i/J_{i+1} contains no proper non-zero idempotent ideals of A/J_{i+1} for each $1 \leq i \leq n$:*
- (ii) *J_i/J_{i+1} is projective module as right A/J_{i+1} -module and left A/J_{i+1} -module for each $1 \leq i \leq n$.*

We call such a chain of ideals a properly stratifying chain.

All the definitions related to idempotents can be now changed to involve the corresponding idempotent ideals. For each properly stratifying primitive idempotent e , we call the corresponding ideal AeA a properly stratifying ideal. It is trivial to see that every properly stratifying ideal is a CPS-stratifying ideal.

Lemma 3.3.11. *Let A be a basic finite dimensional algebra (in general, we can assume A just be a semiprimary ring), J a properly stratifying ideal of A and $e \in A$ an idempotent with $J \subseteq AeA$. Then eJe is a properly stratifying ideal in eAe and the right eAe -module Je_{eAe} and the left ${}_{eAe}eJ$ both are projective.*

Proof. Since $J^2 = J$ and $J \subseteq AeA$, there exists an idempotent $f \in A$ with $J = AfA$, $f = e f e$. $J = AfA$, as right module, is an epimorphic image of some direct sum $\oplus fA$, and since J_A is projective, J_A is isomorphic to a direct summand of $\oplus fA$ implies that Je_{eAe} is isomorphic to a direct summand of $\oplus fAe$. Since f is an idempotent in AeA , we know that Je_{eAe} is projective. It follows that $eAfAe_{eAe}$ is projective. Similarly, since ${}_AJ$ is projective, we also have ${}_{eAe}eJ$ projective. As a result we know eJe is a properly stratified ideal of eAe .

□

Proposition 3.3.12. *Let A be a finite dimensional K -algebra. Let e be an idempotent of A , let $C = eAe$, and assume that the multiplication map $Ae \otimes_{eAe} eA \rightarrow AeA$ is bijective. Let J be an ideal with $J \subseteq AeA$. The following conditions are equivalent:*

- (i) J is a properly stratifying ideal of A ;
- (ii) eJe is a properly stratifying ideal of C , the C -module $(Je)_C$ and ${}_C(eJ)$ are projective, and the map $Je \otimes_C eJ \rightarrow J$ is bijective;

- (iii) $\epsilon J\epsilon$ is a properly stratifying ideal of C , the C -module $((1-\epsilon)J\epsilon)_C$ and ${}_{C'}(\epsilon J(1-\epsilon))$ are projective, and the map $(1-\epsilon)J\epsilon \otimes_C \epsilon J(1-\epsilon) \rightarrow (1-\epsilon)J(1-\epsilon)$ is bijective.

Proof. The proof is similar to the proof of Proposition 8 in [DR1]. $\square$

By using the above result, we can give an inductive procedure for constructing properly stratified algebras (cf. [D4]).

The following theorem summarizes the homological properties of properly stratified algebras.

Theorem 3.3.13 (see [D4]). *Let $(A, \mathbf{e})$ be a K -algebra. Then the following statements are equivalent:*

- (i) $(A, \mathbf{e})$ is properly stratified :
- (ii) $A_A \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\Delta})$:
- (iii) $A_A \in \mathcal{F}(\Delta)$ and $\Delta(i) \in \mathcal{F}(\bar{\Delta})$ for all $1 \leq i \leq n$:
- (iv) $\text{Ext}_A^2(\Delta, \bar{\nabla}) = \text{Ext}_A^2(\bar{\Delta}, \nabla) = 0$:
- (v) $\mathcal{F}(\bar{\Delta}) = \{X \mid \text{Ext}_A^1(X, \nabla) = 0\}$ and $\mathcal{F}(\Delta) = \{X \mid \text{Ext}_A^1(X, \bar{\nabla}) = 0\}$:
- (vi) $\dim_K A = \sum_{i=1}^n \dim_K \Delta(i) \dim_K \bar{\nabla}(i) = \sum_{i=1}^n \dim_K \bar{\Delta}(i) \dim_K \nabla(i)$.

This theorem indicates the way how the results and canonical constructions concerning standardly stratified algebras can be specialized or translated to properly stratified algebras. In particular, for a properly stratified algebra $(A, \mathbf{e})$, we have the following reciprocity relations.

Lemma 3.3.14. *Let $(A, \mathbf{e})$ be a properly stratified algebra, then*

$$[P(i) : \Delta(j)] = [\bar{\nabla}(j) : S(i)] \text{ for all } 1 \leq i, j \leq n.$$

and

$$[P(i) : \bar{\Delta}(j)] = [\nabla(j) : S(i)] \text{ for all } 1 \leq i, j \leq n.$$

As a consequence of Theorem 3.3.13 and Lemma 2.2 in [D3], we see that “properly stratified” is a two-sided concept. In other words, A is properly stratified if and only if A^{opp} is properly stratified. It is easy to see that if A is quasi-hereditary, then A must be properly stratified. The following example shows that not all properly stratified algebra are quasi-hereditary.

Example 3.3.15. *Let $A = KQ/\mathcal{I}$, where Q is the quiver*

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2 \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}.$$

and $\mathcal{I} = \langle \beta\alpha, \tau^2 \rangle$. Thus, the right and left regular representation are

$$A_A = \begin{array}{c} 1 \\ \downarrow \\ 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \quad 1 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \quad 1 \end{array}, \quad {}_A A = \begin{array}{c} 1 \\ \downarrow \\ 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \quad 1 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \quad 1 \end{array}.$$

Until now, we have defined a series of algebras which have various “stratifications”. The relation among those algebras which we have considered is as follows

(cf. [ADL2]):

$$\begin{aligned} \text{hereditary algebras} &\subset \text{quasi-hereditary algebras} \\ &\subset \text{properly stratified algebras} \subset \text{standardly stratified algebras} \\ &\subset \text{strictly stratified algebras} \subset \text{CPS-stratified algebras.} \end{aligned}$$

3.4 Standardly Stratified Algebras with Duality

In their study of representations of complex Lie-algebras and algebraic groups, Cline-Parshall-Scott introduced the quasi-hereditary algebras to describe the so-called highest weight categories. The case when quasi-hereditary algebras have duality on their module categories which fixes simple modules is of particular interest. Following [CPS3], we shall call such duality a strong duality. Quasi-hereditary algebras with strong duality are often called BGG-algebras. As an example of BGG-algebras one may think of Schur algebras or q -Schur algebras. Let us point out that Bernstein-Gelfand-Gelfand in [BGG] showed that their category $\mathcal{O}$ of representations of complex semi-simple Lie algebras has strong duality and proved the so-called *BGG* reciprocity law for $\mathcal{O}$. Motivated by this important development, we study standardly stratified algebras with strong duality, and later, in Chapter 4, construct a class of algebras which have this property.

Definition 3.4.1. *Let $(A, \mathbf{e})$ be as before, a finite dimensional K -algebra over an algebraically closed field K . A duality on the category of the right A -modules is contravariant equivalence $\Phi : \text{mod-}A \rightarrow \text{mod-}A$ such that Φ^2 is naturally equivalent to the identity functor on $\text{mod-}A$.*

Proposition 3.4.2. *Let Φ be a duality on $\text{mod-}A$. Then the following properties are equivalent:*

- (i) $\Phi(S(i)) \simeq S(i)$ for all $1 \leq i \leq n$, i.e., Φ is a strong duality;
- (ii) $\Phi(P(i)) \simeq I(i)$ for all $1 \leq i \leq n$;
- (iii) $\Phi(\Delta(i)) \simeq \nabla(i)$ for all $1 \leq i \leq n$;
- (iv) $\Phi(\bar{\Delta}(i)) \simeq \bar{\nabla}(i)$ for all $1 \leq i \leq n$.

Theorem 3.4.3. *If a standardly stratified K -algebra $(A, \mathbf{e})$ has strong duality, then $(A, \mathbf{e})$ is properly stratified.*

Proof. By Proposition 3.4.2, a strong duality Φ maps indecomposable projective to indecomposable injective module and standard module to costandard module. By Theorem 3.2.6, a standardly stratified algebra $(A, \mathbf{e})$ satisfies $I(i) \in \mathcal{F}(\nabla)$. Clearly, exactness of Φ implies that a module $X \in \mathcal{F}(\nabla)$ if and only if $\Phi(X) \in \mathcal{F}(\Delta)$. Consequently, for each $1 \leq i \leq n$, $\Phi(I(i)) \simeq P(i) \in \mathcal{F}(\Delta)$, and thus $(A, \mathbf{e})$ is properly stratified. $\square$

Corollary 3.4.4. *If a standardly stratified K -algebra $(A, \mathbf{e})$ has strong duality, then $\dim_K A = \sum_{i=1}^n \dim_K \Delta(i) \dim_K \bar{\Delta}(i) = \sum_{i=1}^n \dim_K \nabla(i) \dim_K \bar{\nabla}(i)$.*

Proof. This is a direct consequence of 3.3.13 (vi). $\square$

Corollary 3.4.5. *Let $(A, \mathbf{e})$ be a basic standardly stratified algebra. If there is an anti-automorphism $\theta : A \rightarrow A$ such that $\theta(\epsilon_i)A \simeq \epsilon_i A$, then A is properly stratified.*

Proof. Clearly, such an automorphism induces a strong duality Φ on $\text{mod} - A$ (cf. [CPS2] or [X]): For an A -module X_A , define $\Phi(X) = \text{Hom}_K(X, K)$ with ϕa for $\phi \in \Phi(X)$, $a \in A$ given by

$$(\phi a)(x) = \phi(x\theta(a)), x \in X;$$

moreover, for a homomorphism $\alpha : X \rightarrow Y$, $\Phi(\alpha)$ maps $\gamma \in \Phi(Y)$ to $\gamma\alpha \in \Phi(X)$. Thus in view of Theorem 3.4.3, $(A, \mathbf{e})$ is properly stratified. □

Remark 3.4.6. *Observe that, given an anti-automorphism θ of the algebra A , each left A -module ${}_A X$ can be given a right A -module structure by*

$$x.a = \theta(a).x, \text{ where } x \in X, a \in A.$$

Thus, there is an equivalence

$$\delta : A - \text{mod} \rightarrow \text{mod} - A,$$

given by $\delta({}_A X) = X_A$ as described above with $\alpha : {}_A X \rightarrow {}_A Y$ mapped to $\alpha : X_A \rightarrow Y_A$.

Consequently, if ${}_A X$ is a projective or injective A -module, so is $\delta(X) = X_A$ projective or injective, respectively. The duality in Corollary 3.4.5 is simply a composition of the duality $M_A \rightarrow \text{Hom}_K(M_A, K)$ and δ .

Chapter 4

Endomorphism Algebras of Modules over Local Commutative Self-injective K -algebras

4.1 Preliminaries

Throughout this chapter, R will be a local commutative self-injective K -algebra over an algebraically closed field K , $\dim_K R < \infty$. Furthermore, A will be the endomorphism algebra of a finite direct sum $X = \oplus_{i=1}^n X(i)$ of pairwise non-isomorphic local-colocal left R -modules $X(i)$, i.e. R -modules such that both $X(i)/\text{rad } X(i)$ and $\text{soc } X(i)$ are simple. For each $1 \leq i \leq n$, we define the idempotent $\epsilon_i = p_i m_i$ of the algebra A , where $p_i : X \rightarrow X(i)$ and $m_i : X(i) \rightarrow X$ are the canonical projection and embedding, respectively. The structure of the module X guaranties

that the endomorphism algebra A is basic and connected, and $\sum_{i=1}^n e_i = 1$. In addition, we shall consider the related idempotents $\varepsilon_i = e_i + e_{i+1} + \cdots + e_n$ for $1 \leq i \leq n$; for convenience, put $\varepsilon_{n+1} = 0$. For all $1 \leq i \leq n$, $S(i) = e_i A / \text{rad } e_i A$ is the simple right A -module corresponding to e_i , $P(i) = e_i A$ is its projective cover, and $I(i) = \text{Hom}_K(Ae_i, K)$ its injective hull. Since A is a basic K -algebra, we have $S(i) \simeq S(j)$ if and only if $i = j$. It is easy to show that $\dim_K \text{End}_A(S(i)) = 1$ for all $1 \leq i \leq n$. Since R is self-injective, for any local-colocal module $X(i)$, $1 \leq i \leq n$, there exists an embedding $X(i) \rightarrow_R R$. The image of this embedding is a local ideal of R which will be denoted by $\underline{X}(i)$. As an R -module, $X(i) \simeq \underline{X}(i)$ for all $1 \leq i \leq n$. Note that since every $\underline{X}(i)$ is local, there is some $\underline{x}_i \in R$ such that $\underline{X}(i) = R\underline{x}_i$.

Now, given any R -homomorphism $f : X(i) \rightarrow X(j)$, there exists an extension $\bar{f} : {}_R R \rightarrow {}_R R$. Since every endomorphism of ${}_R R$ is given by multiplication by an element $r \in R$, we can say that f is induced by multiplication by this element. Indeed, f is induced by the R -homomorphism $\underline{f} : \underline{X}(i) \rightarrow \underline{X}(j)$, where $\underline{f} = \bar{f}|_{\underline{X}(i)}$ (see the following diagram).

$$\begin{array}{ccccc}
 0 & \longrightarrow & X(i) & \xrightarrow{\quad} & R \\
 & & \downarrow f & \searrow & \downarrow \bar{f} \\
 & & & & \underline{X}_i \\
 & & & \uparrow \underline{f} & \uparrow \\
 0 & \longrightarrow & X(j) & \xrightarrow{\quad} & R \\
 & & \downarrow & \searrow & \downarrow \\
 & & & & \underline{X}_j
 \end{array}$$

Consequently, given a local-colocal R -module X' , the ideal $\underline{X}'$ of R is determined uniquely, i.e., for two local-colocal R -modules U and V , $U \simeq V$ implies $\underline{U} = \underline{V}$.

Let us fix an isomorphism between $X(i)$ and $\underline{X}(i)$ and denote it by ϕ_i , for all $1 \leq i \leq n$. Moreover, for each ϕ_i , we define $x_i = \phi_i^{-1}(\underline{x}_i)$; thus, $X(i) = Rx_i$. Given

any map $g : X(i) \rightarrow X(j)$, we have $\phi_i^{-1}g\phi_j \in \text{Hom}_R(\underline{X}(i), \underline{X}(j))$. The mapping $g \rightarrow \phi_i^{-1}g\phi_j$ is now seen to be a K -homomorphism, usually denoted by $\text{Hom}(\phi_i^{-1}, \phi_j)$, of $\text{Hom}_R(X(i), X(j))$ into $\text{Hom}_R(\underline{X}(i), \underline{X}(j))$. Thus we have

$$\text{Hom}(\phi_i^{-1}, \phi_j) : \text{Hom}_R(X(i), X(j)) \rightarrow \text{Hom}_R(\underline{X}(i), \underline{X}(j)).$$

Clearly, since R is commutative, $\text{Hom}(\phi_i^{-1}, \phi_j)$ is an R -homomorphism between the R -modules $\text{Hom}_R(X(i), X(j))$ and $\text{Hom}_R(\underline{X}(i), \underline{X}(j))$. In fact, $\text{Hom}(\phi_i^{-1}, \phi_j)$ is an R -isomorphism. A benefit of this interpretation of $\text{Hom}_R(X(i), X(j))$ lies in the fact that whenever we want to study properties of homomorphisms between $X(i)$ and $X(j)$, we can trace them back to homomorphisms between $\underline{X}(i)$ and $\underline{X}(j)$ and these are multiplications by elements in R . In the following, we will write $\underline{f} = \phi_i^{-1}f\phi_j$ for $f \in \text{Hom}_R(X(i), X(j))$. For composition of R -homomorphisms $fg : X(i) \xrightarrow{f} X(j) \xrightarrow{g} X(k)$, we have also the following commutative diagram:

$$\begin{array}{ccc} X(i) & \xrightarrow{\phi_i} & \underline{X}(i) \\ f \downarrow & & \underline{f} \downarrow \\ X(j) & \xrightarrow{\phi_j} & \underline{X}(j) \\ g \downarrow & & \underline{g} \downarrow \\ X(k) & \xrightarrow{\phi_k} & \underline{X}(k) \end{array} .$$

which shows that $\underline{fg} = \underline{f}\underline{g}$.

Now if $f : X(i) \rightarrow X(j)$, then $(\underline{x}_i)\underline{f} = r\underline{x}_i$ for a suitable $r \in R$, and thus $\text{Im } \underline{f} \subseteq \underline{X}(i) \cap \underline{X}(j)$ and $\text{Im } f$ is isomorphic to a submodule of $X(i)$.

For $X(i)$, $1 \leq i \leq n$, let $DX(i)$ denote the usual K -dual of $X(i)$, i.e., $\text{Hom}_K(X(i), K)$. Since each $X(i)$ is a factor module of R_R , it has a natural structure of a local commutative self-injective K -algebra and thus $DX(i) \simeq X(i)$ for all

$1 \leq i \leq n$.

The following simple lemma will be used repeatedly in what follows.

Lemma 4.1.1. *The following three statements are equivalent:*

(1) $\underline{X}(i) \subseteq \underline{X}(j) \subseteq R$:

(2) *there is a monomorphism from $X(i)$ to $X(j)$:*

(3) *there is an epimorphism from $X(j)$ to $X(i)$.*

For any two local ideals $\underline{X}(j)$ and $\underline{X}(k)$ such that $\underline{X}(j) \subseteq \underline{X}(k)$, we define $\underline{m}_{jk} : \underline{X}(j) \hookrightarrow \underline{X}(k)$ to be the natural inclusion map, i.e., $(\underline{x}_j)\underline{m}_{jk} = \underline{x}_j$, and $\underline{p}_{kj} : \underline{X}(k) \rightarrow \underline{X}(j)$ to be the natural projection map, i.e., $(\underline{x}_k)\underline{p}_{kj} = \underline{x}_j$.

The following R -homomorphisms will be used often throughout this section: for any two local-colocal modules $X(j)$ and $X(k)$ such that $\underline{X}(j) \subseteq \underline{X}(k)$, we define

$$m_{jk} : X(j) \rightarrow X(k)$$

to be the monomorphism corresponding to the inclusion $\underline{m}_{jk}$, i.e., $m_{jk} = \phi_j \underline{m}_{jk} \phi_k^{-1}$, and we let

$$p_{kj} : X(k) \rightarrow X(j)$$

to be the epimorphism corresponding to $\underline{p}_{kj}$, i.e., $p_{kj} = \phi_k \underline{p}_{kj} \phi_j^{-1}$.

Lemma 4.1.2. *Let $A = \text{End}_R(\bigoplus_{i=1}^n X(i))$, then there is an anti-automorphism $\theta : A \rightarrow A$ such that $\theta(e_i) = e_i$ for all $1 \leq i \leq n$.*

Proof. We know that any element in A is a sum of maps of the form

$$f : X \xrightarrow{p_i} X(i) \xrightarrow{f_{ij}} X(j) \xrightarrow{m_j} X.$$

For any f , we define

$$\theta(f) : X \xrightarrow{p_i} X(j) \simeq D(X(j)) \xrightarrow{D(f_{ji})} D(X(i)) \simeq X(i) \xrightarrow{m_i} X.$$

It is easy to show (i) $\theta(fg) = \theta(g)\theta(f)$, (ii) $\theta^2(f) = f$ and $\theta(e_i) = e_i$ for all $f, g \in A$ and all $1 \leq i \leq n$. Therefore θ is an anti-automorphism of the algebra A . $\square$

As a consequence, referring to Section 3.4, we have the following result.

Lemma 4.1.3. *The algebra A has strong duality, i.e., there is a duality functor $\Phi : \text{mod-}A \longrightarrow \text{mod-}A$ such that $\Phi(S(i)) \simeq S(i)$, $\Phi(P(i)) \simeq I(i)$, $\Phi(\Delta(i)) \simeq \nabla(i)$ and $\Phi(\tilde{\nabla}(i)) \simeq \tilde{\Delta}(i)$ for all $1 \leq i \leq n$.*

Now, let R be a local commutative self-injective finite dimensional K -algebra. Let $\mathcal{X} = (X(1) = R_R, X(2), \dots, X(n))$ be a sequence of local-colocal R -modules indexed by a linearly ordered set $\Lambda = \{1, 2, \dots, n\}$ reflecting inclusions of the corresponding local ideals, i.e., $\underline{X}(j) \subseteq \underline{X}(i)$ implies $i \leq j$. Let $X = \bigoplus_{i=1}^n X(i)$, $\underline{X} = \sum_{i=1}^n \underline{X}(i)$, $\widetilde{X}(i) = \bigoplus_{t=i+1}^n X(t)$ and $\widetilde{\underline{X}}(i) = \sum_{t=i+1}^n \underline{X}(t)$

For any i , we define the subset Λ_i of Λ by

$$\Lambda_i = \{t \in \Lambda : \underline{X}(t) \subseteq \underline{X}(i)\}.$$

Consider the endomorphism algebra $A = \text{End}_R(\bigoplus_{i=1}^n X(i))$ and the respective order of the primitive idempotents $\mathbf{e} = (e_1, e_2, \dots, e_n)$. Thus, we consider the algebra A together with the order of its simple A -modules: $(A, \mathbf{e})$. Furthermore, consider the ordered sets

$$\Delta = (\Delta(1), \Delta(2), \dots, \Delta(n))$$

and

$$\bar{\Delta} = (\bar{\Delta}(1), \bar{\Delta}(2), \dots, \bar{\Delta}(n))$$

of the right standard and proper standard modules of $(A, \mathbf{e})$.

The main result of this thesis is the following theorem:

Theorem 4.1.4. *Let R be a local commutative self-injective algebra over an algebraically closed field K . Let $\mathcal{X} = (X(1) = R_R, X(2), \dots, X(n))$ be a sequence of local-colocal R -modules reflecting inclusions of the corresponding local ideals, i.e., $\underline{X}(j) \subseteq \underline{X}(i)$ implies $i \leq j$. Let $X = \bigoplus_{i=1}^n X(i)$, $A = \text{End}_R(X)$ and $\mathbf{e} = (e_1, e_2, \dots, e_n)$ be the sequence of the canonical idempotents of A . Then $(A, \mathbf{e})$ is a properly stratified algebra whose standard modules reflect the structure of $\mathcal{X}$ if and only if*

$$(i) \quad \underline{X}(i) \cap \underline{X}(j) = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{X}(t) \text{ for any } 1 \leq i, j \leq n;$$

$$(ii) \quad \underline{X}(j) \cap \widetilde{\underline{X}(j)} = \sum_{t=j+1}^n (\underline{X}(j) \cap \underline{X}(t)) \text{ for any } 1 \leq j \leq n.$$

Remark 4.1.5. *Condition (i) implies that among those local ideals there exists a unique minimal local ideal $\underline{X}(n)$, i.e., $\underline{X}(n) \subseteq \underline{X}(l)$ for any $1 \leq l \leq n$.*

4.2 Proof of Sufficiency

The proof will be achieved through a sequence of lemmas. From now on, we assume (i) and (ii) in Theorem 4.1.4 are satisfied.

Lemma 4.2.1. *$\{\underline{x}_i \mid 1 \leq i \leq n\}$ are K -linearly independent.*

Proof. First, we know $\underline{x}_i \in \text{rad } R$ for $i \neq 1$. If we assume that $\{\underline{x}_i \mid i \in \Lambda\}$ are K -linearly dependent, then there exist some numbers $k_1, k_2, \dots, k_n$ in K such that $k_1 \underline{x}_1 + k_2 \underline{x}_2 + \dots + k_n \underline{x}_n = 0$. Let k_i be the first non zero number. Then we have $\underline{x}_i = -\frac{k_{i+1}}{k_i} \underline{x}_{i+1} - \frac{k_{i+2}}{k_i} \underline{x}_{i+2} - \dots - \frac{k_n}{k_i} \underline{x}_n$, which implies $\underline{X}(i) = \underline{X}(i) \cap (\sum_{t=i+1}^n \underline{X}(t))$. But we know that for all $i+1 \leq t \leq n$, $\underline{X}(t) \cap \underline{X}(i) \subseteq \text{rad}(\underline{X}(i))$. So condition (ii) cannot be satisfied. It is a contradiction. Therefore $\{\underline{x}_i \mid i \in \Lambda\}$ are K -linearly independent. Moreover, we see that the cardinality $|\Lambda|$ of Λ satisfies $|\Lambda| \leq \dim_K R$. $\square$

Lemma 4.2.2. *Every R -homomorphism $h : X(j) \rightarrow X(k)$ with $\underline{X}(j) \subseteq \underline{X}(k)$ can be written as a composition $h'm_{jk}$ with h' an endomorphism of $X(j)$.*

Proof. For the given h , we have $\underline{h} : \underline{X}(j) \rightarrow \underline{X}(k)$. Since $\text{Im } \underline{h}$ is contained in $\underline{X}(j)$, $\underline{h}$ can be written as a composition $\underline{h}'\underline{m}_{jk}$ with $\underline{h}' : \underline{X}(j) \rightarrow \underline{X}(j)$. Then correspondingly, we know h can be written as a composition $h'm_{jk}$ with h' an endomorphism of $X(j)$. $\square$

Lemma 4.2.3. *Every R -homomorphism $fg : X(i) \xrightarrow{f} X(j) \oplus X(j) \xrightarrow{g} X(k)$ with $\underline{X}(j) \subseteq \underline{X}(k)$ can be written as a composition $h'm_{jk}$ with h' an R -homomorphism from $X(i)$ to $X(j)$.*

Proof. We can assume $f = (f_1, f_2)$ and $g = (g_1, g_2)^T$, then $fg = f_1g_1 + f_2g_2$. From the above lemma, we know that there exist R -homomorphisms $g'_1, g'_2 : X(j) \rightarrow X(j)$ such that $g_1 = g'_1m_{jk}$ and $g_2 = g'_2m_{jk}$. So $fg = (f_1g'_1 + f_2g'_2)m_{jk}$. Let $h' = f_1g'_1 + f_2g'_2$. Then h' is an R -homomorphism from $X(i)$ to $X(j)$ and $fg = h'm_{jk}$. $\square$

Lemma 4.2.4. *Every R -homomorphism $f : X(i) \rightarrow X(j)$ factors through $\bigoplus_{t \in \Lambda_i \cap \Lambda_j} X(t)$.*

Proof. We will find the factorization through the map $\underline{f} : \underline{X}(i) \rightarrow \underline{X}(j)$. Since $\underline{f}$ is induced by multiplication, there is $r \in R$ such that $(\underline{x}_i)\underline{f} = r\underline{x}_i$. We know $\text{Im } \underline{f} \subseteq \underline{X}(i) \cap \underline{X}(j) = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{X}(t)$, hence $r\underline{x}_i = \sum_{t \in \Lambda_i \cap \Lambda_j} r_t \underline{x}_i$ with $r_t \underline{x}_i \in \underline{X}(t)$ for $r_t \in R$. For any $t \in \Lambda_i \cap \Lambda_j$, let us define $\underline{f}_t : \underline{X}(i) \rightarrow \underline{X}(t)$ with $(\underline{x}_i)\underline{f}_t = r_t \underline{x}_i$, then it is easy to show that $\underline{f}_t$ is a well-defined R -homomorphism. Hence, $\underline{f} = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{f}_t \underline{m}_{tj}$. By the corresponding map from the homomorphism between two local modules to the homomorphism between two local ideals, we get $f = \phi_i \underline{f} \phi_j^{-1} = \sum_{t \in \Lambda_i \cap \Lambda_j} \phi_i \underline{f}_t \phi_i^{-1} \phi_t \underline{m}_{tj} \phi_j^{-1} = \sum_{t \in \Lambda_i \cap \Lambda_j} f_t m_{tj}$. So we can write $f = gp$ with $g : X(i) \rightarrow \bigoplus_{t \in \Lambda_i \cap \Lambda_j} X(t)$ and $g = (f_t)_{t \in \Lambda_i \cap \Lambda_j}$, and $p = (m_{tj})_{t \in \Lambda_i \cap \Lambda_j}^T$. Indeed, g and p are all well defined R -homomorphism and the factorization is complete. $\square$

Lemma 4.2.5. *If an R -homomorphism $f : X(i) \rightarrow X(j)$ with $\underline{X}(i) \subseteq \underline{X}(j)$ cannot be factored through $\widetilde{X(i)}$, then for any $\underline{X}(j) \subseteq \underline{X}(k)$, $f m_{jk} : X(i) \xrightarrow{f} X(j) \xrightarrow{m_{jk}} X(k)$ cannot be factored through $\widetilde{X(i)}$.*

Proof. If we assume $f m_{jk}$ can be factored through $\widetilde{X(i)}$, then there exist $g : X(i) \rightarrow \widetilde{X(i)}$ and $h : \widetilde{X(i)} \rightarrow X(k)$ such that $f m_{jk} = gh$

$$\begin{array}{ccccc}
 & & X(j) & & \\
 & \nearrow f & & \searrow m_{jk} & \\
 (\dagger) & X(i) & & & X(k) \\
 & \searrow g & & \nearrow h & \\
 & & \widetilde{X(i)} & &
 \end{array}$$

We can write $g = (g_{i+1}, g_{i+2}, \dots, g_n)$ and $h = (h_{i+1}, h_{i+2}, \dots, h_n)^T$ with $f m_{jk} = \sum_{t=i+1}^n g_t h_t$. For any $i+1 \leq t \leq n$, we know g_t can be factored through $\bigoplus_{s \in \Lambda_i \cap \Lambda_t} X(s)$

by Lemma 4.2.4 and so we can write $g_t = g'_t \eta_t$ with $g'_t : X(i) \rightarrow \bigoplus_{s \in \Lambda_i \cap \Lambda_t} X(s)$ and $\eta_t : \bigoplus_{s \in \Lambda_i \cap \Lambda_t} X(s) \rightarrow X(t)$.

Let $h'_t = \eta_t h_t$. We have the commutative diagram

$$\begin{array}{ccc}
 & X(t) & \\
 g_t \nearrow & & \searrow h_t \\
 X(i) & & X(k) \\
 g'_t \searrow & & \nearrow h'_t \\
 & \bigoplus_{s \in \Lambda_i \cap \Lambda_t} X(s) &
 \end{array}$$

Now we replace $(\dagger)$ by

$$\begin{array}{ccc}
 & X(j) & \\
 f \nearrow & & \searrow m_{jk} \\
 (\dagger\dagger) \quad X(i) & & X(k) \\
 q' \searrow & & \nearrow h' \\
 & \bigoplus_{t=i+1}^n (\bigoplus_{s \in \Lambda_i \cap \Lambda_t} X(s)) &
 \end{array}$$

where $g' = (g'_{i+1}, g'_{i+2}, \dots, g'_n)$ and $h' = (h'_{i+1}, h'_{i+2}, \dots, h'_n)^T$. Here for some indexes s and s' , $X(s) = X(s')$, and we can use Lemma 4.2.3 to make the direct summands distinct by reducing the extra direct summands since we know $\underline{X}(s) \subseteq \underline{X}(i) \subseteq \underline{X}(k)$ for any $s \in \Lambda_i \cap \Lambda_t$ and any t such that $i+1 \leq t \leq n$. Let $\Lambda_0 = \{l \mid l \in \Lambda_i \cap \Lambda_t, i+1 \leq$

$t \leq n\}$, we can replace $(\dagger\dagger)$ by the following commutative diagram:

$$\begin{array}{ccc}
 & X(j) & \\
 f \nearrow & & \searrow m_{jk} \\
 X(i) & & X(k) \\
 g \searrow & & \nearrow h \\
 & \varinjlim_{l \in \Lambda_0} X(l) &
 \end{array}$$

where $\dot{g} = (\dot{g}_l)_{l \in \Lambda_0}$ and $\dot{h} = (\dot{h}_l)_{l \in \Lambda_0}^T$. Since $\underline{X}(l) \subseteq \underline{X}(i) \subseteq \underline{X}(j) \subseteq \underline{X}(k)$ for any $l \in \Lambda_0$, we know $\dot{h}_l = \dot{h}'_l m_{lk}$ for some $\dot{h}'_l : X(l) \rightarrow X(l)$ by Lemma 4.2.2. But $m_{lk} = m_{lj} m_{jk}$, we have $f m_{jk} = \sum_{l \in \Lambda_0} \dot{g}_l \dot{h}'_l m_{lj} m_{jk}$. So $f = \sum_{l \in \Lambda_0} \dot{g}_l \dot{h}'_l m_{lj}$, which means that f can be factored through $\varinjlim_{l \in \Lambda_0} X(l)$. But $\Lambda_0 \subseteq \{i+1, i+2, \dots, n\}$, and we know f can be factored through $\widetilde{X(i)}$. This is a contradiction. $\square$

The following lemma is a dual result of the above lemma.

Lemma 4.2.6. *If an R -homomorphism $h : X(j) \rightarrow X(k)$ with $\underline{X}(j) \subseteq \underline{X}(k)$ cannot be factored through $\widetilde{X(j)}$, then for any $\underline{X}(j) \subseteq \underline{X}(i)$, $p_{ij} h : X(i) \xrightarrow{p_{ij}} X(j) \xrightarrow{h} X(k)$ also cannot be factored through $\widetilde{X(j)}$.*

Proof. For any local module $X(k)$, we have $\underline{X}(k) \subseteq \underline{X}(1) = R$. So if we can prove that $p_{ij} h m_{k1} : X(i) \xrightarrow{p_{ij}} X(j) \xrightarrow{h} X(k) \xrightarrow{m_{k1}} X(1)$ cannot be factored through $\widetilde{X(j)}$, then automatically, $p_{ij} h$ cannot be factored through $\widetilde{X(j)}$. By the above lemma, we know that if $h : X(j) \rightarrow X(k)$ with $\underline{X}(j) \subseteq \underline{X}(k)$ cannot be factored through $\widetilde{X(j)}$, then for $\underline{X}(k) \subseteq \underline{X}(1)$, $h m_{k1}$ also cannot be factored through $\widetilde{X(j)}$. So to prove the lemma, we only need to prove the case when $k = 1$.

If we assume $p_{ij}h$ can be factored through $\widetilde{X(j)}$, then there exist g and q such that $gq = p_{ij}h$, where $g = (g_{j+1}, g_{j+2}, \dots, g_n)$ and $q = (q_{j+1}, q_{j+2}, \dots, q_n)^T$

$$\begin{array}{ccc}
 & X(j) & \\
 p_{ij} \nearrow & & \searrow h \\
 X(i) & & X(1) \\
 \searrow g & & \nearrow q \\
 & \widetilde{X(j)} &
 \end{array}$$

At first, we analyze the factorization of homomorphisms between local ideals, and then trace them back to homomorphisms between local modules. Since every R -homomorphism from one local ideal to another is induced by multiplication by elements in R , we can assume that $(\underline{x}_j)h = r'\underline{x}_j$, $(\underline{x}_t)g_t = r_t\underline{x}_t$ and $(\underline{x}_t)q_t = r'_t\underline{x}_t$, for $j+1 \leq t \leq n$ with $r', r'_t \in R$. Since $p_{ij}h = gq = \sum_{t=j+1}^n g_t q_t$, we know $r'\underline{x}_j = \sum_{t=j+1}^n r'_t r_t \underline{x}_t$. By using condition (ii) $\underline{X}(j) \cap (\widetilde{\underline{X}(j)}) = \sum_{t=j+1}^n (\underline{X}(j) \cap \underline{X}(t))$, we have $r'\underline{x}_j = \sum_{t=j+1}^n \dot{r}_t \underline{x}_j$, with $\dot{r}_t \underline{x}_j \in \underline{X}(j) \cap \underline{X}(t)$ for $j+1 \leq t \leq n$ and $\dot{r}_t \in R$. After defining $g'_t : \underline{X}(j) \rightarrow \underline{X}(t)$ with $(r\underline{x}_j)g'_t = r\dot{r}_t \underline{x}_j$, we know $h = \sum_{t=j+1}^n g'_t m_{t1}$. Then we have $h = \sum_{t=j+1}^n g'_t m_{t1}$. So we know that h can be factored through $\widetilde{X(j)}$. This is a contradiction. The lemma follows. $\square$

Now we turn to considering the stratification of the endomorphism algebra A .

Lemma 4.2.7. *For every $i \in \Lambda$,*

$$\tilde{\mathcal{B}}(i) = \{p_i m_{ij} m_j \mid \underline{X}(i) \subseteq \underline{X}(j)\}$$

is a K -basis for the right proper standard module $\tilde{\Delta}(i)$, where m_{ij} denotes the monomorphism from $X(i)$ to $X(j)$ corresponding to $\underline{m}_{ij}$ and $p_i m_{ij} m_j$ denotes the

corresponding elements in $\tilde{\Delta}(i)$. Moreover, if we define

$$\mathcal{B}(i) = \{p_i \alpha_t m_{ij} m_j : \underline{X}(i) \subseteq \underline{X}(j)\}$$

where $\{\alpha_t : 1 \leq t \leq d_i\}$ is a K -basis of

$$\{\alpha \in \text{End}_R(X(i)) : \alpha \text{ cannot be factored through } \widetilde{X(i)}\},$$

then $\mathcal{B}(i)$ is a K -basis of $\Delta(i)$.

In fact,

$$\Delta(i) = \epsilon_i A / \epsilon_i A (\sum_{t \in \Lambda_i \setminus \{i\}} \epsilon_t) A \quad (*)$$

$$\tilde{\Delta}(i) = \epsilon_i A / \epsilon_i \text{rad } A (\sum_{t \in \Lambda_i} \epsilon_t) A \quad (**)$$

Proof. By definition, $\Delta(i) = \epsilon_i A / \epsilon_i A \varepsilon_{i+1} A$ and $\tilde{\Delta}(i) = \epsilon_i A / \epsilon_i \text{rad } A \varepsilon_i A$. To prove (*) and (**), it is sufficient to show that every R -homomorphism $f : X(i) \rightarrow X(k)$ with $k > i$ but $k \not\leq \Lambda_i$ (which also means i and k are incomparable in Λ) can be factored through $\oplus_{t \in \Lambda_i \setminus \{i\}} X(t)$. First, f cannot be monomorphism since if f is monic, then $\underline{X}(i) \subseteq \underline{X}(k)$ will imply $i \geq k$ which contradicts our assumption $k > i$. Then by Lemma 4.2.4, we get that f factors through $\oplus_{t \in \Lambda_i \cap \Lambda_k} X(t)$. Since i and k are incomparable, we know $\Lambda_i \cap \Lambda_k \subseteq \Lambda_i \setminus \{i\}$ and f can factor through $\oplus_{t \in \Lambda_i \setminus \{i\}} X(t)$.

Regarding $\tilde{\mathcal{B}}(i)$, it is clear that this is a linearly independent set and every monomorphism $X(i) \rightarrow X$ is a linear combination of maps from $\tilde{\mathcal{B}}(i)$. Assuming $\underline{X}(i) \subseteq \underline{X}(j)$, we know that there is no monomorphism $f : X(i) \rightarrow X(j)$ such that f can be factored through $\oplus_{t \in \Lambda_i \setminus \{i\}} X(t)$. Moreover, since any monomorphism

$f : X(i) \rightarrow X(j)$ cannot be factored through $X(i)$ by a noninvertible endomorphism of $X(i)$, i.e., $p_i f m_j \notin e_i \operatorname{rad} A (\sum_{t \in \Lambda_i} e_t) A$, the K -span of $\tilde{\mathcal{B}}(i)$ must sit inside $\tilde{\Delta}(i)$.

Finally, if $f : X(i) \rightarrow X(j)$ is not a monomorphism for any two given local-colocal modules, there exist two possibilities for the relation between $\underline{X}(i)$ and $\underline{X}(j)$:

(1) When $\underline{X}(i) \subseteq \underline{X}(j)$, we have seen that $\underline{f}$ is induced by multiplication by an element r in R , furthermore, this r must be in $\operatorname{rad} R$. We can define a new R -homomorphism $\underline{g} : \underline{X}(i) \rightarrow \underline{X}(i)$ such that $(x_i)\underline{g} = (x_i)\underline{f}$, so that we have $\underline{f} = \underline{g}m_{ij}$. Since $\underline{g}$ is not an invertible R -homomorphism, it must be in fact a nilpotent endomorphism of $\underline{X}(i)$. Similarly we know that f can be factored through a non-invertible endomorphism of $X(i)$, which implies $p_i f m_j \in e_i (\operatorname{rad} A) e_i A \subseteq e_i (\operatorname{rad} A) (\sum_{t \in \Lambda_i} e_t) A$.

(2) When $\underline{X}(i) \not\subseteq \underline{X}(j)$, f can be factored through $\oplus_{t \in \Lambda_i \cap \Lambda_j} X(t)$ by Lemma 4.2.4. Since $\Lambda_i \cap \Lambda_j \subseteq \Lambda_i \setminus \{i\}$, $p_i f m_j \in e_i \operatorname{rad} A (\sum_{t \in \Lambda_i \setminus \{i\}} e_t) A \subseteq e_i \operatorname{rad} A (\sum_{t \in \Lambda_i} e_t) A$.

Therefore, $\tilde{\mathcal{B}}(i)$ is a K -basis for $\tilde{\Delta}(i)$. The proof for $\mathcal{B}(i)$ is similar. $\square$

Remark 4.2.8. Let us point out that the above lemma describes the structure of the proper standard modules: the factorizations $m_{ik} = m_{ij}m_{jk}$ correspond to the embeddings $\underline{X}(i) \subseteq \underline{X}(j) \subseteq \underline{X}(k)$, so the lattice of the submodules of $\tilde{\Delta}(i)$ is anti-isomorphic to the lattice of ideals of R which contains $\underline{X}(i)$. In particular, every ideal $\underline{X}(i)$ is a subset of $\underline{X}(1) = R$. Thus the proper standard module $\tilde{\Delta}(i)$ has a simple socle generated by m_{i1} , i.e., $\operatorname{soc} \tilde{\Delta}(i) \simeq S(1)$ for all $1 \leq i \leq n$.

The direct consequence of the above lemma is the fact that the proper standard module $\tilde{\Delta}(i)$ remains unchanged under any linear order of Λ which reflects the inclusions. Another immediate consequence of the above lemma is the multiplicity formula for the composition factors of the proper standard module:

$$[\bar{\Delta}(i) : S(k)] = 1 \text{ if } \underline{X}(i) \subseteq \underline{X}(k):$$

$$[\bar{\Delta}(i) : S(k)] = 0 \text{ otherwise.}$$

Lemma 4.2.9. *If $\underline{X}(j) \subseteq \underline{X}(i)$, then $e_i A \varepsilon_j A / e_i A \varepsilon_{j+1} A \simeq \Delta(j)$. If $\underline{X}(j) \not\subseteq \underline{X}(i)$, then $e_i A \varepsilon_j A = e_i A \varepsilon_{j+1} A$.*

Proof. If $\underline{X}(j) \not\subseteq \underline{X}(i)$, then any R -homomorphism $f : X(i) \rightarrow X(j)$ cannot be surjective. In this case, there are two possibilities for the relation between $\underline{X}(i)$ and $\underline{X}(j)$:

(1) If $\underline{X}(i) \subset \underline{X}(j)$, then $i > j$ and $e_i A \varepsilon_j A = e_i A \varepsilon_{j+1} A = e_i A$.

(2) If $\underline{X}(i)$ and $\underline{X}(j)$ are incomparable then by Lemma 4.2.4, any R -homomorphism from $X(i)$ to $X(j)$ can be factored through $\bigoplus_{s \in \Lambda_i \cap \Lambda_j} X(s)$. But $\Lambda_i \cap \Lambda_j \subseteq \Lambda_j \setminus \{j\} \subseteq \{j+1, j+2, \dots, n\}$, therefore $e_i A \varepsilon_j A \subseteq e_i A \varepsilon_{j+1} A$ which also implies $e_i A \varepsilon_j A = e_i A \varepsilon_{j+1} A$.

If $\underline{X}(j) \subseteq \underline{X}(i)$, we know that p_{ij} is the surjective R -homomorphism from $\underline{X}(i)$ to $\underline{X}(j)$ which maps $\underline{x}_i$ to $\underline{x}_j = r \underline{x}_i$ for some $r \in R$. Given any R -homomorphism $\underline{h} : \underline{X}(i) \rightarrow \underline{X}(j)$, suppose $(\underline{x}_i) \underline{h} = r' \underline{x}_i$ for some $r' \in R$. Since $\text{Im } \underline{h} \subseteq \underline{X}(j)$, there exists some $r_j \in R$ such that $(\underline{x}_i) \underline{h} = r' \underline{x}_i = r_j \underline{x}_j$. Hence we can write $\underline{h} = p_{ij} \underline{q}$ with $\underline{q} : \underline{X}(j) \rightarrow \underline{X}(j)$, $(\underline{x}_j) \underline{q} = r_j \underline{x}_j$. As a consequence, we have the corresponding R -homomorphism $h : X(i) \rightarrow X(j)$, $h = p_{ij} q$ with q an endomorphism of $X(j)$. Suppose $f' : X(i) \rightarrow X(j)$ is another surjective R -homomorphism then there is an automorphism g of $X(j)$ such that $f' = p_{ij} g$. Let $P_{ij} = p_i p_{ij} m_j$ be an element in A . It is easy to show that $e_j A \simeq e_i P_{ij} e_j A$ and $e_j A \varepsilon_{j+1} A \simeq e_i P_{ij} e_j A \varepsilon_{j+1} A$, and we have $\Delta(j) \simeq e_i P_{ij} e_j A / e_i P_{ij} e_j A \varepsilon_{j+1} A$. Since any R -homomorphism from $X(i)$ to $X(j)$ is the composition of p_{ij} and one endomorphism of $X(j)$, we have $e_i A e_j A = e_i P_{ij} e_j A$.

Thus, $\epsilon_i A \epsilon_j A \simeq \epsilon_j A$. We also have $\epsilon_i A \varepsilon_j A / \epsilon_i A \varepsilon_{j+1} A \simeq \epsilon_i A \epsilon_j A / \epsilon_i A \varepsilon_{j+1} A \cap \epsilon_i A \epsilon_j A$ hence $\epsilon_i A \varepsilon_j A / \epsilon_i A \varepsilon_{j+1} A$ is isomorphic to a quotient of $P(j) = \epsilon_j A$. So to finish our proof, we only need to show $\epsilon_i P_{ij} \epsilon_j A \varepsilon_{j+1} A = \epsilon_i A \varepsilon_{j+1} A \cap \epsilon_i A \epsilon_j A$. To prove this equality is equivalent to showing that for any $1 \leq l \leq n$, $\epsilon_i P_{ij} \epsilon_j A \varepsilon_{j+1} A \epsilon_l = \epsilon_i A \varepsilon_{j+1} A \epsilon_l \cap \epsilon_i A \epsilon_j A \epsilon_l = \epsilon_i A \varepsilon_{j+1} A \epsilon_l \cap \epsilon_i P_{ij} \epsilon_j A \epsilon_l$.

For any $1 \leq l \leq n$, there are three possibilities for the relation between $\underline{X}(l)$ and $\underline{X}(j)$:

(1) If $\underline{X}(l) \subset \underline{X}(j)$, then $l > j$ and $X(l)$ is a direct summand of $\widetilde{X(j)}$. Moreover, $\epsilon_i P_{ij} \epsilon_j A \epsilon_l \subseteq \epsilon_i P_{ij} \epsilon_j A \varepsilon_{j+1} A \epsilon_l$.

(2) If $\underline{X}(l)$ and $\underline{X}(j)$ are incomparable, then any R -homomorphism from $X(j)$ to $X(l)$ can be factored through $\hat{\oplus}_{s \in \Lambda_j \cap \Lambda_l} X(s)$. But $\Lambda_j \cap \Lambda_l \subseteq \Lambda_j \setminus \{j\} \subseteq \{j+1, j+2, \dots, n\}$. So $\epsilon_i P_{ij} \epsilon_j A \epsilon_l \subseteq \epsilon_i P_{ij} \epsilon_j A \varepsilon_{j+1} A \epsilon_l$.

(3) If $\underline{X}(j) \subseteq \underline{X}(l)$, we know that elements of $\epsilon_i A \varepsilon_{j+1} A \epsilon_l$ are sums of maps of the form $p_i \phi m' m_l$ with $\phi : X(i) \rightarrow \widetilde{X(j)}$, $m' : \widetilde{X(j)} \rightarrow X(l)$ and $m_l : X(l) \rightarrow X$. But any R -homomorphism from $X(t)$ to $X(l)$, where $j+1 \leq t \leq n$, can be factored through $\hat{\oplus}_{s \in \Lambda_t \cap \Lambda_l} X(s)$. Let $\Lambda' = \{k : k \in \Lambda_t \cap \Lambda_l \text{ where } j+1 \leq t \leq n\}$, and by using Lemma 4.2.2, we know that every element in $\epsilon_i A \varepsilon_{j+1} A \epsilon_l$ can be simplified to the form $p_i \delta m m_l$ with $\delta : X(i) \rightarrow \hat{\oplus}_{k \in \Lambda'} X(k)$ and $m = (m_{kl})_{k \in \Lambda'}$. Let $\psi \in \epsilon_i A \varepsilon_{j+1} A \epsilon_l \cap \epsilon_i A \epsilon_j A \epsilon_l$. Then we know that $\psi = p_i \delta m m_l = p_i p_{ij} g m_l$ with $g : X(j) \rightarrow X(l)$. We have the

following commutative diagram:

$$\begin{array}{ccc}
 & \varinjlim_{k \in \Lambda'} X(k) & \\
 \delta \nearrow & & \searrow m \\
 X(i) & & X(l) \\
 p_{ij} \searrow & & \nearrow g \\
 & X(j) &
 \end{array}$$

Since $p_{ij}g = \delta m$, we get that $p_{ij}g$ can be factored through $\varinjlim_{k \in \Lambda'} X(k)$. Moreover, $\Lambda' \subseteq \{j+1, j+2, \dots, n\}$, thus we know that $p_{ij}g$ can be factored through $\widetilde{X(j)}$. Then by Lemma 4.2.6, g can be factored through $\widetilde{X(j)}$. Then $v \in e_i P_{ij} e_j A \varepsilon_{j+1} A e_l$.

For all the three cases, we have shown $e_i A \varepsilon_{j+1} A e_l \cap e_i A e_j A e_l \subseteq e_i P_{ij} e_j A \varepsilon_{j+1} A e_l$. It is trivial to show that $e_i A \varepsilon_{j+1} A \cap e_i A e_j A = e_i P_{ij} e_j A \varepsilon_{j+1} A$. $\square$

Definition 4.2.10. We say that the standard modules of the algebra $(A = \text{End}_R(X), \mathbf{e})$, where $X = \varinjlim_{i=1}^n X(i)$, reflect the structure of $\mathcal{X}$, provided that the standard modules $\Delta(i)$, $1 \leq i \leq n$, have the following property: If $S(k)$ is a composition factor of $\Delta(i)$, then $\underline{X}(k) \supseteq \underline{X}(i)$, $1 \leq k \leq i$.

Proof of Sufficiency

Lemma 4.2.9 yields readily that $A_A \in \mathcal{F}(\Delta)$. Consequently, by Dlab's theorem [D3] (or [L]), ${}_A A \in \mathcal{F}(\bar{\Delta}^o)$. By Lemma 4.1.2, there is an anti-automorphism θ which, in view of Corollary 3.4.5, implies that $(A, \mathbf{e})$ is properly stratified. Moreover, since $\Delta(i)$ can be filtered by $\bar{\Delta}(i)$ for all $1 \leq i \leq n$, by Remark 4.2.8, we know that standard modules reflect the structure of $\mathcal{X}$.

4.3 Proof of Necessity

We assume that the endomorphism algebra $(A = \text{End}_R(X), \mathbf{e})$ is a properly stratified algebra whose standard modules reflect the structure of $\mathcal{X}$, i.e., $\Delta(i)$ only have the composition factors of the form $S(k)$, where $i \in \Lambda_k$, i.e., $\underline{X}(i) \subseteq \underline{X}(k) \subseteq \underline{X}(1) = R$. Thus, we have

$$\begin{aligned} \text{trace}(\varinjlim_{t=i-1}^n P(t) \rightarrow P(i)) &= \text{trace}(\varinjlim_{X(i) \subseteq X(t)} P(t) \rightarrow P(i)) \\ &= \text{trace}(\varinjlim_{i \in \Lambda_k} P(k) \rightarrow P(i)). \end{aligned}$$

We will prove the implication of conditions (i) and (ii) in Lemma 4.3.2 and Lemma 4.3.3. First, we have the following auxiliary result.

Lemma 4.3.1. *Let $f : X(i) \rightarrow X(k)$ be an R -homomorphism with $\underline{X}(i) \subseteq \underline{X}(k)$. If f is monic, then $p_i f m_k \notin e_i(\text{rad } A) \varepsilon_i A$. If f is not monic, and A is properly stratified, then $p_i f m_k \in e_i(\text{rad } A) \varepsilon_i A$. Thus if A is properly stratified, then the multiplicity $[\bar{\Delta}(i) : S(k)] = 1$ for $i \in \Lambda_k$ and $[\bar{\Delta}(i) : S(k)] = 0$ otherwise.*

Proof. The image $\text{Im } f$ of a homomorphism $f : X(i) \rightarrow X(k)$ which factors through $\widetilde{X(i)}$ is isomorphic to a submodule of $\text{rad } X(i)$, since f cannot be a monomorphism. However, if f is a monomorphism then $\text{Im } f \simeq X(i)$ and thus $p_i f m_k \notin e_i(\text{rad } A) \varepsilon_i A$. Now if f is not a monomorphism, then f induces a non-invertible endomorphism of $X(i)$, and therefore, in the case that A is properly stratified, $p_i f m_k$ must belong to $e_i(\text{rad } A) \varepsilon_i A$ since $[\bar{\Delta}(i) : S(i)] = 1$ and $p_i \text{id}_{X(i)} m_i \notin e_i(\text{rad } A) \varepsilon_i A$.

Consequently, since $\bar{\Delta}(i) = e_i A / e_i(\text{rad } A) \varepsilon_i A$, it follows that $[\bar{\Delta}(i) : S(k)] \neq 0$ if and only if $\underline{X}(i) \subseteq \underline{X}(k)$. In fact, in this case $[\bar{\Delta}(i) : S(k)] = 1$. Indeed, if $f_1, f_2 : X(i) \rightarrow X(k)$ are both monomorphisms, then both maps are induced

by multiplication by invertible elements of R and thus $f_2 = \alpha f_1$, with $\alpha \in R$ invertible. Since $p_i(\beta f_1)m_k \in e_i(\text{rad } A)\varepsilon_i A$ for every non-invertible $\beta \in R$, we can write $p_i f_2 m_k - p_i(\bar{\alpha} f_1)m_k \in e_i(\text{rad } A)\varepsilon_i A$ with $\bar{\alpha} \in R/\text{rad } R \simeq K$, and the lemma follows. $\square$

Before we prove the following lemmas, let us recall the Bernstein-Gelfand-Gelfand reciprocity law for properly stratified algebra A :

$$[P(i) : \Delta(j)] = [\bar{\nabla}(j) : S(i)].$$

Since A is properly stratified, $\Delta(i)$ is filtered by $\bar{\Delta}(i)$. So we also know $[\Delta(i) : S(k)] = a \geq 1$ for any k such that $\underline{X}(i) \subseteq \underline{X}(k)$ and $[\Delta(i) : S(k)] = 0$ otherwise. In view of the duality $\Phi : \text{mod-}A \rightarrow \text{mod-}A$ which we have mentioned in Lemma 4.1.3 and Remark 3.4.2, we know that $\Phi(\bar{\nabla}(i)) \simeq \bar{\Delta}(i)$ for all $1 \leq i \leq n$. Then we have $[\bar{\Delta}(i) : S(k)] = [\bar{\nabla}(i) : S(k)]$. So the Bernstein-Gelfand-Gelfand reciprocity law can be written as

$$[P(i) : \Delta(j)] = [\bar{\Delta}(j) : S(i)].$$

From this and Lemma 4.3.1 we can know the filtration of any $P(i)$, and consequently

$$[P(i) : \Delta(j)] = 1 \text{ for } \underline{X}(j) \subseteq \underline{X}(i):$$

$$[P(i) : \Delta(j)] = 0 \text{ otherwise.}$$

By definition, we know that $\Delta(i) = e_i A / e_i A \varepsilon_{i+1} A$. Moreover, since A is properly stratified and $[P(i) : \Delta(j)] \neq 0$ only for $j \in \Lambda_i$, then $e_i A \varepsilon_{i+1} A = e_i A (\sum_{t \in \Lambda_i \setminus \{i\}} e_t) A$ and $\Delta(i) = e_i A / e_i A (\sum_{t \in \Lambda_i \setminus \{i\}} e_t) A$.

Since there exist a nonzero R -homomorphism from $X(n)$ to any $X(k)$ which maps x_n to some element in $\text{soc } X(k)$, we have $[P(n) : S(k)] \neq 0$ for any $1 \leq k \leq n$. Since $\Delta(n) = P(n)$ and the standard modules of A reflect the structure of $\mathcal{X}$, we know $\underline{X}(n) \subseteq \underline{X}(k)$ for any $1 \leq k \leq n$.

Lemma 4.3.2. *If A is properly stratified, then $\underline{X}(i) \cap \underline{X}(j) = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{X}(t)$ for any $1 \leq i, j \leq n$.*

Proof. For any two local ideals $\underline{X}(i)$ and $\underline{X}(j)$, if they are comparable, then $\underline{X}(i) \cap \underline{X}(j) = \underline{X}(i)$ or $\underline{X}(j)$. Then $\Lambda_i \cap \Lambda_j = \Lambda_i$ or Λ_j and the equality follows trivially.

Now we assume that they are not comparable. We know $[P(i) : \Delta(j)] = [\bar{\Delta}(j) : S(i)] = 0$ if $\underline{X}(i)$ and $\underline{X}(j)$ are not comparable. Hence for any R -homomorphism $f : X(i) \rightarrow X(j)$, we know $p_i f m_j \in e_i A \varepsilon_{i-1} A = e_i A (\sum_{t \in \Lambda_i \setminus \{i\}} e_t) A$. Moreover, f is a sum of maps of the form $\rho \omega : X(i) \xrightarrow{\rho} \oplus_{t \in \Lambda_i \setminus \{i\}} X(t) \xrightarrow{\omega} X(j)$. For any $\underline{X}(t)$ such that $t \in \Lambda_i \setminus \{i\}$, if $\underline{X}(t)$ and $\underline{X}(j)$ are comparable, we know that $\underline{X}(t) \subseteq \underline{X}(j)$ must hold since if $\underline{X}(j) \subseteq \underline{X}(t)$, then $\underline{X}(j)$ will be contained in $\underline{X}(i)$ which contradicts to our assumption. If $\underline{X}(t)$ and $\underline{X}(j)$ are not comparable, then any R -homomorphism from $X(t)$ to $X(j)$ is a sum of maps which can be factored through $X(s)$ with $s \in \Lambda_t \setminus \{t\}$. Since Λ is finite and $\underline{X}(n) \subseteq \underline{X}(k)$ for any $1 \leq k \leq n$, we can get after applying the same procedure that f is a sum of maps which can be factored through $\oplus_{t \in \Lambda_i \cap \Lambda_j} X(t)$. So we can write $f = \sum_{t \in \Lambda_i \cap \Lambda_j} f_t m_{tj}$ with $f_t : X(i) \rightarrow X(t)$ for all $t \in \Lambda_i \cap \Lambda_j$. For any $\underline{x} \in \underline{X}(i) \cap \underline{X}(j)$, we can define one R -homomorphism $\underline{f} : \underline{X}(i) \rightarrow \underline{X}(j)$ with $(\underline{x}_i) \underline{f} = \underline{x}$. Since we know that $f = o_i \underline{f} o_j^{-1}$ has the form $f = \sum_{t \in \Lambda_i \cap \Lambda_j} f_t m_{tj}$, we get $\underline{f} = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{f}_t \underline{m}_{tj}$. Therefore

$$\underline{x} \in \sum_{t \in \Lambda_i \cap \Lambda_j} (\underline{X}(i)) \underline{f}_t \underline{m}_{tj} \subseteq \sum_{t \in \Lambda_i \cap \Lambda_j} (\underline{X}(t)) \underline{m}_{tj} \subseteq \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{X}(t).$$

that is $\underline{X}(i) \cap \underline{X}(j) \subseteq \sum_{t \in \Lambda_{i,j}} \underline{X}(t)$. Since the inverse inclusion is trivial, the lemma follows. $\square$

Lemma 4.3.3. *If A is properly stratified, then $\underline{X}(j) \cap \widetilde{\underline{X}(j)} = \sum_{t=j+1}^n (\underline{X}(j) \cap \underline{X}(t))$ for any $1 \leq j \leq n$.*

Proof. From the above lemma, we know that $\underline{X}(j) \cap \underline{X}(t) = \sum_{k \in \Lambda_j \cap \Lambda_t} \underline{X}(k)$, and so $\sum_{t=j+1}^n (\underline{X}(j) \cap \underline{X}(t)) = \sum_{k \in \Lambda_j \setminus \{j\}} \underline{X}(k)$. Now we only need to show $\underline{X}(j) \cap (\sum_{t=j+1}^n \underline{X}(t)) \subseteq \sum_{k \in \Lambda_j \setminus \{j\}} \underline{X}(k)$. Since $\underline{X}(j) \subseteq \underline{X}(1) = R$, it follows that $[P(1) : \Delta(j)] = [\Delta(j) : S(1)] = 1$ for any $1 \leq j \leq n$. In the proof of Lemma 4.2.9, we have defined the element $P_{1j} \in A$, i.e., $P_{1j} = p_1 p_{1j} m_j$. We know that every R -homomorphism $h : X(1) \rightarrow X(j)$ can be written as a composition $p_{1j} h'$ with $h' : X(j) \rightarrow X(j)$, therefore $e_1 A e_j A = e_1 P_{1j} e_j A = e_1 P_{1j} e_j A \varepsilon_j A \simeq e_j A$ and $e_1 A e_j A \varepsilon_{j+1} A = e_1 P_{1j} e_j A \varepsilon_{j+1} A \simeq e_j A \varepsilon_{j+1} A$.

We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \rightarrow & e_j A \varepsilon_{j+1} A & \rightarrow & e_j A & \rightarrow & e_j A / e_j A \varepsilon_{j+1} A & \rightarrow 0 \\ & \downarrow & & \downarrow & & \vdots & \\ 0 \rightarrow & e_1 P_{1j} e_j A \varepsilon_{j+1} A & \rightarrow & e_1 P_{1j} e_j A \varepsilon_j A & \rightarrow & e_1 P_{1j} e_j A \varepsilon_j A / e_1 P_{1j} e_j A \varepsilon_{j+1} A & \rightarrow 0 \end{array}$$

$$\begin{aligned} \text{Then, } \Delta(j) &= e_j A / e_j A \varepsilon_{j+1} A \simeq e_1 P_{1j} e_j A \varepsilon_j A / e_1 P_{1j} e_j A \varepsilon_{j+1} A = \\ &= e_1 A e_j A / e_1 A e_j A \varepsilon_{j+1} A. \end{aligned}$$

Since $[P(1) : \Delta(j)] = 1$, it follows that

$$\Delta(j) \simeq e_1 A \varepsilon_j A / e_1 A \varepsilon_{j+1} A \simeq e_1 A e_j A / e_1 A e_j A \cap e_1 A \varepsilon_{j+1} A.$$

Since $e_1 A e_j A \varepsilon_{j+1} A \subseteq e_1 A e_j A \cap e_1 A \varepsilon_{j+1} A$, we have as a consequence that

$$e_1 A e_j A \varepsilon_{j+1} A = e_1 P_{1j} e_j A \varepsilon_{j+1} A = e_1 A e_j A \cap e_1 A \varepsilon_{j+1} A.$$

We know that $e_j.A\varepsilon_{j-1}.A = e_j.A(\sum_{t \in \Lambda_j \setminus \{j\}} e_t).A$ and $\Delta(j) = e_j.A/e_j.A(\sum_{t \in \Lambda_j \setminus \{j\}} e_t).A$. Then it follows that $e_1.P_{1j}e_j.A\varepsilon_{j-1}.A = e_1.P_{1j}e_j.A(\sum_{t \in \Lambda_j \setminus \{j\}} e_t).A = e_1.Ae_j.A \cap e_1.A\varepsilon_{j-1}.A$.

Let $\underline{x} \in \underline{X}(j) \cap \widetilde{\underline{X}(j)}$. Then there exist $r_l \in R$ for $j \leq l \leq n$ such that $\underline{x} = r_j \underline{x}_j = \sum_{t=j+1}^n r_t \underline{x}_t$. We can define two maps

$$\underline{X}(1) \xrightarrow{p_{1j}} \underline{X}(j) \xrightarrow{f} \underline{X}(j) \xrightarrow{m_{1j}} \underline{X}(1)$$

with $(\underline{x}_j)f = r_j \underline{x}_j$ and

$$\underline{X}(1) \xrightarrow{g} \widetilde{\underline{X}(j)} \xrightarrow{m} \underline{X}(1)$$

with $\underline{g} = (\underline{g}_{j-1}, \underline{g}_{j-2}, \dots, \underline{g}_n)$ and $\underline{m} = (\underline{m}_{(j+1)1}, \underline{m}_{(j+2)1}, \dots, \underline{m}_{n1})^T$ such that $(\underline{x}_t)\underline{g}_t = r_t \underline{x}_t$ for all $j+1 \leq t \leq n$. It follows that $\underline{gm} = p_{1j}f\underline{m}_{j1}$. For the corresponding R -homomorphism among local modules, we have $p_1 p_{1j} f m_{j1} m_1 \in e_1.Ae_j.A \cap e_1.A\varepsilon_{j-1}.A = e_1.P_{1j}e_j.A(\sum_{t \in \Lambda_j \setminus \{j\}} e_t).A$ which implies that $f m_{j1}$ is the sum of maps which can be factored through $\sum_{t \in \Lambda_j \setminus \{j\}} X(t)$. After tracing back to the homomorphisms between local ideals, we have $\underline{x} \in \sum_{t \in \Lambda_j \setminus \{j\}} \underline{X}(t)$. Therefore

$$\underline{X}(j) \cap \widetilde{\underline{X}(j)} = \sum_{t \in \Lambda_j \setminus \{j\}} \underline{X}(t) = \sum_{t=j+1}^n (\underline{X}(j) \cap \underline{X}(t)).$$

□

Corollary 4.3.4. *If $|\Lambda| = \dim_k R$ and the two conditions (i) and (ii) in Theorem 4.1.4 hold, then A is quasi-hereditary.*

Proof. By using the two conditions (i) and (ii), we know $(A, \mathbf{e})$ is properly stratified algebra. We also know

$$\dim_K R = [P(1) : S(1)] = \sum_{i=1}^n [P(1) : \Delta(i)][\Delta(i) : S(1)]$$

and $[P(1) : \Delta(i)] = [\bar{\Delta}(i) : S(1)] = 1$ for $1 \leq i \leq n$ by Bernstein-Gelfand-Gelfand reciprocity law. If $[\Lambda] = \dim_K R$, we get $[\Delta(i) : S(1)] = 1$ and this implies $\Delta(i) = \bar{\Delta}(i)$. Then $(A, \mathbf{e})$ is quasi-hereditary by Theorem 3.2.9. $\square$

Remark 4.3.5. In their paper [DHM], Dlab-Heath-Marko constructed a class of quasi-hereditary algebras (more precisely, lean quasi-hereditary algebras) using a commutative local self-injective algebra R . The construction is as follows: Let R be a commutative local self-injective K -algebra over a splitting field K , $\dim_K R < \infty$. Let $\mathcal{X} = \{X(\lambda) : \lambda \in \Lambda\}$ be a set of local ideals of R indexed by a finite partially ordered set Λ reflecting inclusions: $X(\lambda) \subset X(\mu)$ if and only if $\lambda > \mu$. Let $R = X(\lambda_1)$ belong to $\mathcal{X}$. Then $A = \text{End}_R(\oplus_{\lambda \in \Lambda} X(\lambda))$ is a quasi-hereditary algebra with respect to Λ if and only if (i) $[\Lambda] = \dim_K R$ and (ii) $\text{rad } X(\lambda) = \sum_{\lambda < \kappa} X(\kappa)$. From here we can see our result as a non-trivial generalization of their result. Under the condition $[\Lambda] = \dim_K R$, and after refining the partial order to a linear order, we can see that $\text{rad } X(\lambda) = \sum_{\lambda < \kappa} X(\kappa)$ is equivalent to conditions (i) and (ii). In our construction of properly stratified algebras, we use linearly ordered set of all non-isomorphic simple modules. The stratification of algebra $(A, \mathbf{e})$ corresponding to every linear ordering of Λ which is compatible with the original partial ordered set structure (the inclusion relation among those local ideals) on Λ only depends on the inclusion order but not on the choice of a refining linear order.

4.4 Examples

In this section, we provide some illustrations of the main Theorem 4.1.4. On

right regular representations:

$$A_A = \begin{array}{c} \begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & \swarrow \\ & & & & & 2 & \\ & & & & 1 & \swarrow & \\ & & & 2 & \swarrow & \searrow & \\ & & 1 & \swarrow & 3 & \swarrow & \\ & 1 & \swarrow & 2 & \swarrow & 4 & \swarrow \\ & 1 & \swarrow & 2 & \swarrow & 3 & \swarrow \\ & 1 & \swarrow & 2 & \swarrow & 4 & \swarrow \\ & 1 & \swarrow & 2 & \swarrow & 5 & \swarrow \\ & 1 & \swarrow & 2 & \swarrow & 6 & \swarrow \\ & 1 & \swarrow & 2 & \swarrow & 7 & \\ & 1 & \swarrow & 2 & \swarrow & 6 & \\ & 1 & \swarrow & 2 & \swarrow & 5 & \\ & 1 & \swarrow & 2 & \swarrow & 4 & \\ & 1 & \swarrow & 2 & \swarrow & 3 & \\ & 1 & \swarrow & 2 & \swarrow & 2 & \\ & 1 & \swarrow & 2 & \swarrow & 1 & \end{array} \end{array} \cong P(2) \oplus \cdots \oplus P(7).$$

where $P(i)$ for $2 \leq i \leq 7$ is a submodule of $P(1)$. Thus for $1 \leq i \leq m-1=6$, the standard modules and proper standard modules coincide: $\Delta(i) = \tilde{\Delta}(i)$; however $\Delta(7)$ is filtered by $t-m+1=4$ uniserial modules $\tilde{\Delta}(7)$.

(b) To indicate a general case, consider $R = K[x]/\langle x^{10} \rangle$, $X_R = R_R \oplus \langle \bar{x}^5 \rangle \oplus \langle \bar{x}^8 \rangle$; then

$$\Delta = \left(\begin{array}{c} 1 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \end{array} , \begin{array}{cc} & 2 & & 3 & \\ & \swarrow \searrow & & \swarrow \searrow & \\ 1 & & 2 & & 2 & & 3 \\ & \swarrow \searrow & & \swarrow \searrow & & \swarrow \searrow & \\ & 1 & & 2 & & 1 & & 2 \\ & & \swarrow \searrow & & \swarrow \searrow & & \swarrow \searrow & \\ & & 1 & & 1 & & 1 & \end{array} \right)$$

generated by the relations $\rho = \sum_{i=1}^m c_i \alpha_i \beta_i$ where $c_i \in K^*$, α_i and β_i are arrows and the paths $\alpha_i \beta_i$ have the same origin and terminus vertices for $1 \leq i \leq m$.

The next result classifies quadratic algebras among properly stratified algebras which we constructed in the above example.

Lemma 4.4.2. *$A = \text{End}(\bigoplus_{i=1}^m X(t_i))$ is quadratic if and only if either $(t_{j+1} - t_j) = 1$ for all $1 \leq j \leq m-1$, or $(t_{j+1} - t_j) = 2$ for all $1 \leq j \leq m-1$ and $(t - t_m) \leq 1$.*

Proof. The proof is straightforward. When $(t_{j+1} - t_j) = 1$ the quiver $\mathcal{Q}_A$ has the form

$$1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} 2 \begin{array}{c} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{array} \cdots m-1 \begin{array}{c} \xrightarrow{\alpha_{m-1}} \\ \xleftarrow{\beta_{m-1}} \end{array} m$$

and

$$\mathcal{I} = \langle \beta_1 \alpha_1 - \alpha_2 \beta_2, \beta_2 \alpha_2 - \alpha_3 \beta_3, \dots, \\ \beta_{m-2} \alpha_{m-2} - \alpha_{m-1} \beta_{m-1}, \beta_{m-1} \alpha_{m-1} \rangle.$$

When $(t_{j+1} - t_j) = 2$, we have to distinguish two cases. If $t_m = t$ the quiver $\mathcal{Q}_A$ has the form

$$\begin{array}{c} \sigma_1 \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} \begin{array}{c} \sigma_2 \\ \circlearrowleft \\ 2 \end{array} \begin{array}{c} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{array} \cdots m-1 \begin{array}{c} \xrightarrow{\alpha_{m-1}} \\ \xleftarrow{\beta_{m-1}} \end{array} m$$

and

$$\mathcal{I} = \langle \sigma_1^2 - \alpha_1 \beta_1, \beta_1 \alpha_1 - \sigma_2^2, \sigma_2^2 - \alpha_2 \beta_2, \beta_2 \alpha_2 - \sigma_3^2, \dots, \\ \sigma_{m-2}^2 - \alpha_{m-2} \beta_{m-2}, \beta_{m-2} \alpha_{m-2} - \sigma_{m-1}^2, \sigma_{m-1}^2 - \alpha_{m-1} \beta_{m-1}, \beta_{m-1} \alpha_{m-1} \rangle.$$

If $t_m = t - 1$, the quiver $\mathcal{Q}_A$ has the form

$$\begin{array}{c} \sigma_1 \\ \circlearrowleft \\ 1 \end{array} \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} \begin{array}{c} \sigma_2 \\ \circlearrowleft \\ 2 \end{array} \begin{array}{c} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{array} \cdots m-1 \begin{array}{c} \xrightarrow{\alpha_{m-1}} \\ \xleftarrow{\beta_{m-1}} \end{array} m \begin{array}{c} \sigma_m \\ \circlearrowleft \end{array}$$

and

$$\begin{aligned} \mathcal{I} = \langle & \sigma_1^2 - \alpha_1 \beta_1, \beta_1 \alpha_1 - \sigma_2^2, \sigma_2^2 - \alpha_2 \beta_2, \beta_2 \alpha_2 - \sigma_3^2, \dots, \sigma_{m-2}^2 - \alpha_{m-2} \beta_{m-2}, \\ & \beta_{m-2} \alpha_{m-2} - \sigma_{m-1}^2, \sigma_{m-1}^2 - \alpha_{m-1} \beta_{m-1}, \sigma_{m-1}^2, \beta_{m-1} \alpha_{m-1} \rangle. \end{aligned}$$

□

Example 4.4.3. Let $R = K[x, y]/\langle xy, x^3 - y^3 \rangle$.

- (a) Let $X = \bigoplus_{i=1}^5 X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x}^2 + \bar{y}^2 \rangle$, $\underline{X}(3) = \langle \bar{y}^2 \rangle$, $\underline{X}(4) = \langle \bar{x}^2 \rangle$ and $\underline{X}(5) = \langle \bar{x}^3 \rangle$. In this case only condition (i) is satisfied. Then $A = \text{End}_R(X)$ is not properly stratified since $[P(1) : S(1)] = 6$ while a Δ -filtration of $P(1)$ requires $[P(1) : S(1)] = 7$.
- (b) Let $X = \bigoplus_{i=1}^5 X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y} \rangle$, $\underline{X}(3) = \langle \bar{y} \rangle$, $\underline{X}(4) = \langle \bar{y}^2 \rangle$ and $\underline{X}(5) = \langle \bar{x}^2 \rangle$. In this case only condition (ii) is satisfied. A_A cannot be filtered by Δ . Hence, A is not properly stratified.
- (c) Let $X = \bigoplus_{i=1}^5 X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y} \rangle$, $\underline{X}(3) = \langle \bar{y} \rangle$, $\underline{X}(4) = \langle \bar{y}^2 \rangle$ and $\underline{X}(5) = \langle \bar{x}^3 \rangle$. In this case both conditions (i) and (ii) are satisfied. Under the usual ordering, $(A, \mathbf{e})$ is properly stratified and the proper standard modules look as follows:

and the projective module $P(1)$ has the following representation:

In fact, $A = K\mathcal{Q}/\mathcal{I}$, where the quiver $\mathcal{Q}$ has the form

and $\mathcal{I}$ is defined by appropriate relations.

- (d) Let $X = \oplus_{i=1}^6 X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y} \rangle$, $\underline{X}(3) = \langle \bar{y} \rangle$, $\underline{X}(4) = \langle \bar{x}^2 + \bar{y}^2 \rangle$, $\underline{X}(5) = \langle \bar{y}^2 \rangle$ and $\underline{X}(6) = \langle \bar{x}^3 \rangle$. Since $\dim_K R = 6$ and conditions (i)

and (ii) are satisfied, we know that A is quasi-hereditary with

$$\Delta = \bar{\Delta} = (\begin{array}{ccccccc} & & & & & & 6 \\ & & & & & & \swarrow \quad \searrow \\ & & & & & & 5 \qquad 4 \\ & & 2 \quad 3 & & 4 & & \swarrow \quad \searrow \\ & \downarrow \quad \downarrow & & \downarrow & & & 5 \qquad 2 \\ 1 & \quad 1 & & 1 & & 3 & \swarrow \quad \searrow \\ & & & & & & 3 \qquad 2 \\ & & & & & & \swarrow \quad \searrow \\ & & & & & & 1 \end{array})$$

and the projective module $P(1)$ has the following representation:

$$P(1) = \begin{array}{ccccccc} & & & & 1 & & \\ & & & & \downarrow & & \\ & & & & 3 & & 2 \\ & & & & \downarrow & & \downarrow \\ 1 & \cdots \bullet & \cdots & 1 & & 5 & 4 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 2 & \cdots \bullet & \cdots & 2 & & 3 & 6 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \cdots \bullet & \cdots & 1 & & 5 & 4 \\ & & & & \downarrow & & \downarrow \\ & & & & 3 & & 2 \\ & & & & \downarrow & & \\ & & & & 1 & & \end{array} .$$

In fact, $A = KQ/\mathcal{I}$, where the quiver Q has the form

$$\begin{array}{ccccc} 1 & \xrightarrow{\beta} & 2 & \xrightarrow{\sigma} & 4 \\ \uparrow \delta & \alpha & \uparrow \nu & \rho & \uparrow \eta \\ 3 & \xrightarrow{\lambda} & 5 & \xrightarrow{\tau} & 6 \\ \downarrow \gamma & \kappa & \downarrow \mu & \pi & \downarrow \zeta \end{array} .$$

and $\mathcal{I}$ is defined by appropriate relations.

(e) To get a properly stratified algebra, it is not necessary to choose $X' \simeq \text{soc } R$ as one term of X . Let $X = \bigoplus_{i=1}^n X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y} \rangle$, $\underline{X}(3) = \langle \bar{y} \rangle$, $\underline{X}(4) = \langle \bar{y}^2 \rangle$. A is properly stratified with

$$\tilde{\Delta} = \left(\begin{array}{ccccccc} & & & & 4 & & \\ & & 2 & 3 & \swarrow & \searrow & \\ & 1 & \cdot & \cdot & \cdot & 2 & \cdot & 3 \\ & & \downarrow & \downarrow & & & & \\ & & 1 & 1 & & & & \\ & & & & & & 1 & \end{array} \right)$$

and the projective module $P(1)$ has the following representation:

$$P(1) = \begin{array}{ccccccc} & & & & 1 & & \\ & & & & \downarrow & \searrow & \\ & & & & 2 & & 3 \\ & \swarrow & & & \downarrow & \swarrow & \downarrow \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & \cdot & \cdot & \cdot & 2 & \cdot & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 4 \\ & \searrow & & & \downarrow & \swarrow & \downarrow \\ & & & & 2 & & 3 \\ & & & & \downarrow & \swarrow & \\ & & & & 1 & & \end{array}.$$

Here, $A = KQ/\mathcal{I}$, where the quiver Q has the form

$$\begin{array}{ccc} 1 & \xrightleftharpoons[\alpha]{\beta} & 2 \\ \uparrow \delta & \gamma & \downarrow \mu \\ 3 & \xrightleftharpoons[\kappa]{\lambda} & 4 \end{array}$$

and $\mathcal{I}$ is defined by appropriate relations.

(f) We have seen earlier that the quiver of the endomorphism algebra contain loops. The following example shows that even if $\Delta(i) \neq \bar{\Delta}(i)$, there may be no loops at the vertex i of the respective quiver: let $X = \oplus_{i=1}^3 X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} - \bar{y} \rangle$ and $\underline{X}(3) = \langle \bar{y}^2 \rangle$. The algebra A is properly stratified with

and the projective module $P(1)$ has the following representation:

Here, $A = K\mathcal{Q}/\mathcal{I}$, where the quiver $\mathcal{Q}$ has the following form

and $\mathcal{I}$ is defined by appropriate relations.

Example 4.4.4. Our method of constructing properly stratified algebras using R is not unique.

(a) Given any R , let ${}_R X = \bigoplus_{i=1}^n R/(\text{rad } R)^i$ where n is the nilpotency index of $\text{rad } A$. Then by Dlab-Ringel's result, we know that the endomorphism algebra of ${}_R X$ is quasi-hereditary (and thus properly stratified).

(b) Consider the four-dimensional algebra $R = K[x, y]/\langle xy, x^2 - y^2 \rangle$. Let

$$X = R \oplus (R\bar{x} \oplus R/R\bar{x}^2)/\langle \bar{x}^2 - (\bar{y} + R\bar{x}^2) \rangle \oplus R\bar{x}^2.$$

In this case, not all direct summands are local-colocal. The standard modules are as follows:

It is easy to see that $\tilde{\Delta} = \left(\begin{array}{c} 2 \\ \vdots \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ \vdots \\ 2 \\ \vdots \\ 1 \end{array} \oplus \begin{array}{c} 3 \\ \vdots \\ 2 \\ \vdots \\ 1 \end{array} \right)$. The right regular representation of $A = \text{End}_R(X)$ is given as follows:

Here, $A = K\mathcal{Q}/\mathcal{I}$, where the quiver $\mathcal{Q}$ has the following form

and

$$\mathcal{I} = \langle \alpha\gamma, \varepsilon^2, \delta\beta, \delta\gamma, \delta\varepsilon\gamma, \gamma\delta - \beta\alpha - \varepsilon\beta\alpha\varepsilon \rangle.$$

This algebra is properly stratified and its module category possesses strong duality. The last statement follows from the fact that the set of the relations defining $\mathcal{I}$ is symmetric.

4.5 Further Properties of the Constructed Algebras

Properly stratified algebras constructed in the previous section enjoy additional properties. Unless otherwise stated, we will assume in this section that $(A = \text{End}_R(X), \mathbf{e})$ is properly stratified, where $X = \bigoplus_{i=1}^n X(i)$ is a direct sum of non-isomorphic left local and colocal modules over a commutative local self-injective finite dimensional K -algebra R and $\mathbf{e} = \{e_1, e_2, \dots, e_n\}$ is a complete sequence of primitive orthogonal idempotents of A . Correspondingly, for each local and colocal module $X(i)$, there exists one local ideal $\underline{X}(i)$ such that $X(i) \simeq \underline{X}(i)$. The linear order $1 < 2 < \dots < n$ is a refinement of the original partial order on the index set Λ , i.e., $\underline{X}(j) \subseteq \underline{X}(i) \subseteq R$ implies $j \geq i$. Recall that A is basic and $\sum_{i=1}^n e_i = 1$.

Definition 4.5.1. Let R and A be finite dimensional algebras (or just say rings). The bimodule ${}_R M_A$ is called *balanced* if R is isomorphic to the algebra $\text{End}_A(M_A, M_A)$ and A is isomorphic to the algebra $\text{End}_R({}_R M, {}_R M)$.

Proposition 4.5.2. Let R , X and A be as above, so we can consider X as a right A -module. Then $R \simeq \text{Hom}_A(X_A, X_A)^{opp}$. Therefore ${}_R X_A$ is balanced.

Proof. We can define a map $\phi : R \rightarrow \text{Hom}_A(X_A, X_A)^{opp}$ by mapping r to $\phi_r : x \mapsto rx$. First we shall prove that ϕ_r is an A -homomorphism for all $r \in R$. Let $a \in A$, $r \in R$ and $x \in X$. Then we have

$$\phi_r(xa) = r(xa) = (rx)a = \phi_r(x)a.$$

Suppose $\phi_r = \phi_s$, then $\phi_r|_{X(1)} = \phi_s|_{X(1)}$. After tracing back to the morphism between local ideals, we get $r\underline{x} = s\underline{x}$ for all $\underline{x} \in \underline{X}(1) = R$. Take $1 \in R$, then $r1 = s1$ and so $r = s$. We see that ϕ is a monomorphism. Finally, we show that ϕ is an epimorphism. Let $\rho \in \text{End}_A(X_A, X_A)^{opp}$, $e_i \in A$ (the idempotent A -endomorphism for ${}_R X$ along $X(i)$, i.e., $e_i = p_i m_i$), then $\rho(xe_i) = \rho(x)e_i$ and so $\rho_i = \rho|_{X(i)} : X(i) \rightarrow X(i)$. Hence $\rho_1 = \rho|_{X(1)}$ is an $e_1 A e_1$ -endomorphism of $X(1)$ and $R \simeq \text{End}_R(X(1), X(1)) \simeq e_1 A e_1$. Consequently, ρ_1 is of the form $r \mapsto sr$ for some $s \in R$. Now let $p_{1j} : X(1) \simeq R \rightarrow X(j)$ be the epimorphism corresponding to $\underline{p}_{1j}$ and let the element $P_{1j} \in A$ be defined as in the Lemma 4.2.9. Then

$$\begin{aligned} \rho(x1) &= \sum_{i=1}^n \rho(xe_i) = \sum_{i=1}^n \rho((x_i')P_{1i}) \\ &= \sum_{i=1}^n (\rho(x_i'))P_{1i} \\ &= \sum_{i=1}^n (sx_i')P_{1i} \\ &= sx. \end{aligned}$$

Since R is commutative, we also have $R \simeq \text{Hom}_A(X_A, X_A)$. □

Remark 4.5.3. *If the bimodule ${}_R X_A$ is balanced, then the number of non-isomorphic indecomposable summands of X_A is equal to the number of non-associate primitive idempotents of R .*

Since R is a local algebra, we have the following:

Proposition 4.5.4. *The module X_A is indecomposable. In fact, $X_A \simeq e_1 A$ and $X_A \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\Delta})$.*

Proof. Since $A = \text{End}_R({}_R X, {}_R X)$, we have $e_1 A_A \simeq \text{Hom}_R({}_R R, {}_R X)_A \simeq X_A$. $\square$

Proposition 4.5.5. *$P(j)$ is isomorphic to a submodule of $P(1)$ for all $1 \leq j \leq n$.*

Proof. In fact, we have shown this statement in the proof of the Lemma 4.2.9 since we know $e_1 P_{1j} e_j A \simeq e_j A$. $\square$

Proposition 4.5.6. *$P(1) \simeq I(1)$, i.e., $P(1)$ is projective-injective. Moreover, $P(1)$ is the unique indecomposable projective-injective module.*

Proof. Let $\xi \in P(1) = e_1 A$. $\xi : X \xrightarrow{p_1} X(1) \xrightarrow{\xi_{1j}} X(j) \xrightarrow{m_j} X$. Since $\underline{X}(j) \subseteq \underline{X}(1) = R$ for all $1 \leq j \leq n$, we can write

$$\xi : X \xrightarrow{p_1} X(1) \xrightarrow{\xi_{1j}} X(j) \xrightarrow{m_j} X(1) \xrightarrow{m_j} X.$$

Now consider the map $\zeta \in Ae_1$ given by

$$\zeta : X \xrightarrow{p_1} X(1) \xrightarrow{p_{1j}} X(j) \xrightarrow{D\xi_{1j}} X(1) \xrightarrow{m_j} X.$$

It is easy to see that $\xi \in \text{Hom}_K(Ae_1, K) \simeq I(1)$ and since $D\zeta = \xi$, it follows that $P(1) \simeq I(1)$. In fact, from the above we can see that $e_1 A = e_1 Ae_1 = Ae_1$. For any other indecomposable projective module $P(j)$ ($j \neq 1$), we have $\text{top } P(j) \simeq S(j)$.

Since $P(j)$ is isomorphic to a submodule of $P(1)$ and $\text{soc } P(1) \simeq \text{soc } I(1)$, we have $\text{soc } P(j) \simeq S(1)$. Since A is basic, $P(j)$ is not injective. $\square$

M. Auslander introduced a concept of representation dimension of Artin algebras in [A]. “It is hoped that this notion gives a reasonable way of measuring how far an Artin algebra is from being of representation finite type.” The methods given there have been effectively applied not only for the representation theory of Artin algebras, but also for the theory of quasi-hereditary algebras.

The representation dimension of an algebra A is defined by

$$\text{rep. dim}(A) = \inf\{\text{gl. dim}(\text{End}_A(X)) \mid X \text{ is a generator and cogenerator}\}.$$

We know that if every indecomposable projective has a simple socle, then the representation dimension of A is finite. This yields the following consequence.

Proposition 4.5.7. *Every indecomposable projective A -module has a simple socle, and $\text{rep. dim}(A)$ is finite.*

We have seen that each proper standard module has a simple socle isomorphic to $S(1)$. Moreover, the socle of each standard module has all composition factors isomorphic to $S(1)$; this follows from the following lemma.

Lemma 4.5.8. *For all $1 \leq i \leq n$ and all $k \neq 1$, $[\text{soc } \Delta(i) : S(1)] \neq 0$ and $[\text{soc } \Delta(i) : S(k)] = 0$.*

Proof. We know that $[\Delta(i) : S(k)] \neq 0$ for $X(i) \subseteq X(k)$ and $[\Delta(i) : S(k)] = 0$ otherwise. If $[\text{soc } \Delta(i) : S(k)] \neq 0$ for some $k \neq 1$, then there exists an element $\bar{f} \in \text{soc } \Delta(i) \subset e_i A / e_i A \varepsilon_{i-1} A = \Delta(i)$ such that $\bar{f} e_k \neq 0$ and $\bar{f} \text{rad } A = 0$. But by

Lemma 4.2.5. we have $\bar{f}e_k p_k m_{k1} m_1 e_1 \neq 0$, where $m_{k1} : X(k) \rightarrow X(1)$ is related to $\underline{m}_{k1} : \underline{X}(k) \hookrightarrow \underline{X}(1)$. This is a contradiction since $e_k p_k m_{k1} e_1 \in \text{rad } A$. $\square$

4.6 CPS's Stratification Hypothesis

The Dlab-Heath-Marko construction of a class of quasi-hereditary algebras has been instrumental in a major study of Cline-Parshall-Scott on the stratification of endomorphism algebras (see [CPS3]). Our construction of properly stratified algebras is different from their construction in [CPS3] where they use “Specht modules” as a bridge to analyze the stratification structure of endomorphism algebras. For comparison, let us first state the terminology used in their construction. Let R and A be algebras and let ${}_R T_A$ be a bimodule. Throughout their paper, the authors assume the following setup:

- (1) $T_A \simeq \oplus_{i=1}^t T_i^{\dot{\cdot} d_i}$, $T_i \in \text{mod } -A$ are distinct and indecomposable;
- (2) ${}_R T \simeq \oplus_{\lambda \in \Lambda} Y_\lambda^{\dot{\cdot} m_\lambda}$, $Y_\lambda \in R - \text{mod}$ are distinct and indecomposable;
- (3) $A = \text{End}_R({}_R T, {}_R T)$.

The set Λ indexes the distinct isomorphism classes of simple objects in $\text{mod } -A$.

An ideal I in an algebra A is called a stratifying ideal provided that

- (1) $J = AeA$ for some idempotent $e \in A$;
- (2) Multiplication induces an isomorphism

$$Ae \otimes_{eAe} eA \simeq J$$

and $\text{Tor}_n^{A^e}(Ae, eA) = 0$ for all $n > 0$.

By a stratification (of length n) of an algebra A , we mean a chain

$$0 = J_0 \subsetneq J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_n = A$$

of ideals such that J_i/J_{i-1} is a stratifying ideal in A/J_{i-1} (for $i = 1, 2, \dots, n$). V. Dlab calls such algebras “CPS-stratified”. If each J_i/J_{i-1} is projective (as a right A/J_{i-1} -module), then we call the stratification standard. If an algebra A has a standard stratification, we say that the algebra A^{opp} is standardly stratified.

CPS’s Stratification Hypothesis. *Let R , A and T be as above and assume that Λ is a partially ordered set. For each $\lambda \in \Lambda$, there is given a fixed R -submodule $S_\lambda \xrightarrow{t_\lambda} Y_\lambda$ and a fixed (increasing) filtration $F_\lambda = F_\lambda^{[0, t(\lambda)]}$ of Y_λ .*

These data are subject to the following conditions (A)-(D):

(A) *For $\lambda \in \Lambda$, there is specified a fixed sequence $\nu_{\lambda,0}, \nu_{\lambda,1}, \dots, \nu_{\lambda,t(\lambda)-1}$ in Λ such that*

$\nu_{\lambda,0} = \lambda$ and for $i > 0$, $\nu_{\lambda,i} > \lambda$. For $0 \leq i < t(\lambda)$, there is a fixed isomorphism $S_{\nu_{\lambda,i}} \simeq Gr^i F_\lambda$.

(B) *for $\lambda, \mu \in \Lambda$, $0 \leq i < t(\lambda)$, let $Y_\lambda/F_\lambda^i \xrightarrow{f} Y_\mu$ be a morphism. Then there exists a morphism $Y_{\nu_{\lambda,i}} \xrightarrow{t} Y_\mu$ making the following diagram commutative:*

$$\begin{array}{ccc} Y_\lambda/F_\lambda^i & \xrightarrow{f} & Y_\mu \\ \epsilon_{\nu_{\lambda,i}} \uparrow & & \uparrow t \\ S_{\nu_{\lambda,i}} & \xrightarrow{t_{\nu_{\lambda,i}}} & Y_{\nu_{\lambda,i}}. \end{array}$$

In this diagram, the morphism $\epsilon_{\nu_{\lambda,i}}$ is defined by composing the fixed isomorphism $S_{\nu_{\lambda,i}} \simeq Gr^i F_\lambda$ with the natural inclusion $Gr^i F_\lambda \hookrightarrow Y_\lambda/F_\lambda^i$.

(C) For $\lambda, \mu \in \Lambda$, let

$$\phi_{\lambda, \mu} = \text{Hom}_R(\iota_\lambda, Y_\mu) : \text{Hom}_R(Y_\lambda, Y_\mu) \rightarrow \text{Hom}_R(Y_\lambda, Y_\mu).$$

If $\phi_{\lambda, \mu} \neq 0$, then $\mu \leq \lambda$.

(D) For $\lambda, \mu \in \Lambda$, let $\alpha_{\lambda, \mu} = \#\{i \mid \nu_{\lambda, i} = \mu\}$, and m_λ is the multiplicity of Y_λ as a summand of ${}_R T$. Then

$$\sum_{\lambda, \mu} m_\lambda \alpha_{\lambda, \mu} \nu_\mu = \dim A.$$

Theorem 3.1.3 in [CPS3]. Assume that the stratification hypothesis (A) – (D) holds for the triple R, T and A . Then A has a standard stratification.

The stratification hypothesis presents conditions which guarantee that an endomorphism algebra $A = \text{End}_R(T)$ has a standard stratification. There exists a converse to this result.

Let $T \in \text{mod} - A$ and consider the following data:

- (1) $R \stackrel{\text{def}}{=} \text{End}_A(T)^{\text{opp}}$, thus $T \in R - \text{mod}$;
- (2) For $\lambda \in \Lambda$, $S_\lambda \stackrel{\text{def}}{=} \text{Hom}_A(\Delta(\lambda), T) \in R - \text{mod}$;
- (3) For $Y_\lambda \stackrel{\text{def}}{=} \text{Hom}_A(P_\lambda, T) \in R - \text{mod}$;
- (4) For $\lambda \in \Lambda$, $\iota_\lambda \stackrel{\text{def}}{=} (S_\lambda \hookrightarrow Y_\lambda)$ corresponding to canonical projection $P(i) \rightarrow \Delta(i)$.

Theorem 3.3.2 in [CPS3]. Assume that A has standard stratification with respect to Λ and let $T \in \text{mod} - A$. Using the above notation, assume that:

(1) $A \simeq \text{End}_R(T)$:

(2) $\text{Ext}_A^1(\Delta(\lambda), T) = 0$ for all $\lambda \in \Lambda$:

(3) For all $\lambda \in \Lambda$, $\Delta(\lambda) \in \text{mod} - A$ is a submodule of a module in $\text{add}_A T$.

Then the stratification hypothesis holds for R, T, A .

In our case, letting $T = X$, from the above we know: (1) $A \simeq \text{End}_R({}_R X, {}_R X)$ and $R \simeq \text{End}_A(X_A, X_A)$. (2) $\text{Ext}_A^1(\Delta(i), X_A) = 0$ for all $1 \leq i \leq n$, since $X_A \simeq P(1) \simeq I(1)$. (3) for all $1 \leq i \leq n$, $\Delta(i)$ is a submodule of a module in $\text{add}_A X$ (where $\text{add}_A X$ denotes the full subcategory of $\text{mod} - A$ having objects which are isomorphic to a direct summands of finite direct sum of copies of X_A). since $\text{soc } \Delta(i)$ only has $S(1)$ as composition factor and $X_A \simeq I(1) \simeq P(1)$. After we define $S_i \stackrel{\text{def}}{=} \text{Hom}_A(\Delta(i), X_A)$, we can see that all conditions in Theorem 3.3.2 of [CPS3] are satisfied. Therefore Cline-Parshall-Scott's stratification hypothesis (3.1.1 of [CPS3]) holds for $R, {}_R X_A, A$. Consequently, for a commutative local self-injective algebra R , the assumptions (i) and (ii) in Theorem 4.1.4 imply CPS's hypothesis 3.1.1 (A)-(D) of [CPS3].

Chapter 5

A Particular Class of Properly Stratified Algebras

This final chapter is appended to provide some concrete illustrations of properly stratified algebras. In particular, we are going to describe a canonical construction of “a smallest properly stratified algebra over a given quiver” in the sense of Theorem 5.5. Such algebras are analogs of the shallow algebras of Dlab-Ringel in [DR3] (cf. also [D2]) in the following sense.

Definition 5.1. *A properly stratified algebra $(A, \mathbf{e})$ is said to be shallow if $\text{rad } \bar{\Delta}(i)$ and $\text{rad } \bar{\Delta}^o(i)$ are semi-simple, and the multiplicities $[\text{rad}^2 \Delta(i) : S(i)] = 0$ for all $1 \leq i \leq n$.*

Here, as in the other parts of this thesis, we consider a basic K -algebra A over an algebraically closed field K , with n isoclasses of simple modules which are linearly ordered: hence the notation $(A, \mathbf{e})$. The symbols $\Delta(i)$, $\Delta^o(i)$, $\bar{\Delta}(i)$ and $\bar{\Delta}^o(i)$ denote

the respective right and left standard modules, and the right and left proper standard modules (see Chapter 3).

Recall that a finite labelled quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is given by a finite sequence of vertices $\mathcal{Q}_0 = (1, 2, \dots, n)$, and a finite set $\mathcal{Q}_1$ of arrows. We will use lowercase Greek alphabet to denote arrows. Each arrow $\alpha \in \mathcal{Q}$ has its start vertex $s(\alpha)$ and its terminal vertex $t(\alpha)$:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Let us point out that there may be more than one arrow with the same start and terminal vertices.

In what follows, the symbol $i \rightarrow j$ will denote any of the arrows α with $s(\alpha) = i$ and $t(\alpha) = j$. Similarly, $i \rightarrow j \rightarrow k$ will denote all paths of length 2 from vertex i to vertex k via vertex j .

Now, we will define the basic canonical construction of a shallow properly stratified K -algebra over a given labelled quiver.

Definition 5.2. *Let $\mathcal{Q}$ be a labelled quiver. Define $S(\mathcal{Q}) = K\mathcal{Q}/\mathcal{I}$, where*

$$\mathcal{I} = \langle i \rightarrow j \rightarrow k \mid i = j = k \text{ or } j < \text{Max}\{i, k\} \rangle.$$

Remark 5.3. *Clearly, the K -algebra $S(\mathcal{Q})$ satisfies $\text{rad}^1(S(\mathcal{Q})) = 0$. Indeed, $S(\mathcal{Q})$ has a K -basis consisting of all vertices (representing the trivial paths of length zero), arrows and paths*

$$v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_m \quad (m \geq 2),$$

where $v_i > \text{Max}\{v_{i-1}, v_{i+1}\}$, or $v_{i-1} = v_i > v_{i+1}$, or $v_{i-1} < v_i = v_{i+1}$ for all $1 \leq i \leq m-1$. It turns out that all such paths satisfy $v_0 < v_1$, $v_1 > v_2$ or

$v_0 = v_1 > v_2$ or $v_0 < v_1 = v_2$, $v_2 > v_3$, and thus $m \leq 3$, i.e., $\text{rad}^1(S(\mathcal{Q})) = 0$. Of course, if the quiver $\mathcal{Q}$ has no loops (i.e., no arrows α with $s(\alpha) = t(\alpha)$), then $S(\mathcal{Q})$ is the canonical shallow quasi-hereditary algebra of Dlab-Ringel ([DR4] and [D2]), and $\text{rad}^3(S(\mathcal{Q})) = 0$.

Proposition 5.4. *$S(\mathcal{Q})$ is a quadratic shallow properly stratified algebra.*

Proof. Trivially, $S(\mathcal{Q})$ is a quadratic algebra. Now $\text{rad } \bar{\Delta}(i)$ has a K -basis consisting of all $i \rightarrow j$, $1 \leq j \leq i-1$, and is therefore semi-simple. Similarly, $\text{rad } \bar{\Delta}^o(i)$ has a K -basis consisting of all $i \leftarrow j$, $1 \leq j \leq i-1$, and is semi-simple, as well. Finally, $\text{rad}^2 \Delta(i)$ has a K -basis consisting of all paths $i \rightarrow j \rightarrow k$, where $k \leq j = i$. Hence, $\text{rad}^2 \Delta(i)$ is semi-simple and all simple direct summands $S(k)$ satisfy $k < i$. Consequently $[\text{rad}^2 \Delta(i) : S(i)] = 0$, as required.

Making a reference to Remark 5.3, one can see immediately that $S(\mathcal{Q})$ (with respect to the natural order of the simple modules) is properly stratified. $\square$

Generalizing a result of [DR4] on shallow quasi-hereditary algebras, we can formulate the following statement.

Theorem 5.5. *Let $(A, \mathbf{e})$ be a properly stratified algebra. Let $\mathcal{Q}_A$ be its labelled quiver. Then*

$$\dim_K S(\mathcal{Q}_A) \leq \dim_K A.$$

Moreover, $\dim_K S(\mathcal{Q}_A) = \dim_K A$ if and only if A is a shallow properly stratified algebra.

Proof. We are going to give a proof based on the number theoretic characterization of the properly stratified algebras of Dlab (Theorem 5(vi) of [D4]): $(A, \mathbf{e})$ is properly

stratified if and only if

$$\dim_K A = \sum_{i=1}^n \dim_K \Delta_A(i) \dim_K \bar{\Delta}_A^o(i) = \sum_{i=1}^n \dim_K \bar{\Delta}_A(i) \dim_K \Delta_A^o(i).$$

Of course, besides the algebra A , also $S = S(\mathcal{Q})$ satisfies a similar condition

$$\dim_K S = \sum_{i=1}^n \dim_K \Delta_S(i) \dim_K \bar{\Delta}_S^o(i) = \sum_{i=1}^n \dim_K \bar{\Delta}_S(i) \dim_K \Delta_S^o(i).$$

Now, Theorem 5.5 follows immediately from the following lemma.

□

Lemma 5.6. *Let $(A, \mathbf{e})$ be a properly stratified algebra and $\mathcal{Q}_A$ be its labelled quiver. Let $S = S(\mathcal{Q}_A)$ be the canonical shallow properly stratified algebra defined above. Then*

$$\dim_K \bar{\Delta}_S(i) \leq \dim_K \bar{\Delta}_A(i) \text{ for all } 1 \leq i \leq n \quad (*)$$

and thus

$$\dim_K \Delta_S(i) \leq \dim_K \Delta_A(i) \text{ for all } 1 \leq i \leq n \quad (**)$$

Moreover, an equality in $(*)$ means that both $\text{rad } \bar{\Delta}_S(i)$ and $\text{rad } \bar{\Delta}_A(i)$ are semi-simple. In this case, the addition equality in $(**)$ means that

$$[\Delta_S(i) : \bar{\Delta}_S(i)] = [\Delta_A(i) : \bar{\Delta}_A(i)].$$

Both inequalities $(*)$ and $(**)$, as well as the corresponding statements that follow them, hold for the left standard modules $\Delta^o(i)$ and the left proper standard modules $\bar{\Delta}^o(i)$.

Proof. Consider the representation of the algebra A in the form

$$A \simeq K\mathcal{Q}_A/\mathcal{I}_A \text{ and } A_A = P(1) \oplus \cdots \oplus P(i) \cdots \oplus P(n).$$

Evidently, the set of all arrows $i \rightarrow j$ for $j < i$ in $\mathcal{Q}_A$ which forms a K -basis of $\text{rad } \tilde{\Delta}_S(i)$, can be extended to a K -basis of $\tilde{\Delta}_A(i) \simeq e_i A / e_i (\text{rad } A) e_i A$, this yields the inequality $(\star)$. In a shallow fashion, using Remark 5.3 and the fact that standard modules are filtered by proper standard modules, we can derive the inequality $(\star\star)$. Then rest follows easily. $\square$

Now in order to complete the proof of Theorem 5.5, observe that, in view of Lemma 5.6, always $\dim_K S(\mathcal{Q}_A) \leq \dim_K A$ and that the equality may occur if and only if all relations in $(\star)$ and $(\star\star)$ are equalities, i.e., when $(A, \mathbf{e})$ is shallow.

Given a labelled quiver $\mathcal{Q}$, we can easily express the K -dimension of the shallow properly stratified algebra $S(\mathcal{Q})$. To this end, define the following 3 numbers:

$$\begin{aligned} d_{\mathcal{Q}}(i) &= \text{card}\{ \alpha \in \mathcal{Q} \mid s(\alpha) = i \text{ and } t(\alpha) < i \}, \\ l_{\mathcal{Q}}(i) &= \text{card}\{ \alpha \in \mathcal{Q} \mid s(\alpha) = i \text{ and } t(\alpha) = i \}, \\ u_{\mathcal{Q}}(i) &= \text{card}\{ \alpha \in \mathcal{Q} \mid s(\alpha) < i \text{ and } t(\alpha) = i \}. \end{aligned}$$

Observe that $a_{\mathcal{Q}} = \sum_{i=1}^n (d_{\mathcal{Q}}(i) + l_{\mathcal{Q}}(i) + u_{\mathcal{Q}}(i))$ is the total number of arrows in $\mathcal{Q}$ and that

$$a_{\mathcal{Q}} = \dim_K \text{rad } S(\mathcal{Q}) / \text{rad}^2 S(\mathcal{Q}).$$

Proposition 5.7. *Given a labelled quiver $\mathcal{Q}$,*

$$\dim_K S(\mathcal{Q}) = \sum_{i=1}^n (d_{\mathcal{Q}}(i) + 1)(l_{\mathcal{Q}}(i) + 1)(u_{\mathcal{Q}}(i) + 1).$$

Proof. Taking into account the already mentioned number theoretical characterization of properly stratified algebras of [D4] and the structure of the K -basis of $S(\mathcal{Q})$ described in Remark 5.3, the formula follows readily. $\square$

Remark 5.8. *Let us point out that, in the case that the quiver $\mathcal{Q}$ has no loops, the K -dimension of the (canonical) shallow quasi-hereditary algebra $S(\mathcal{Q})$ is $\sum_{i=1}^n (d_{\mathcal{Q}}(i) + 1)(u_{\mathcal{Q}}(i) + 1)$.*

Let us specialize these results to the shallow algebras with strong duality.

Definition 5.9. *A quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is said to be symmetric if there is a one-to-one correspondence Φ on $\mathcal{Q}_1$ such that*

$$\Phi(s(\alpha) \xrightarrow{\alpha} t(\alpha)) = t(\beta) \xleftarrow{\beta} s(\beta)$$

where $s(\alpha) = t(\beta)$, $s(\beta) = t(\alpha)$ and $\Phi(\rho) = \rho$ for all loops ρ .

Proposition 5.10. *The shallow algebra $S(\mathcal{Q})$ has strong duality if and only if $\mathcal{Q}$ is a symmetric (labelled) quiver.*

Proof. Let $\mathcal{Q}$ be a symmetric quiver and Ψ the correspondence of Definition 5.9. Since the defining relations of the shallow algebra $S(\mathcal{Q})$ are symmetric, Ψ can be extended to an anti-automorphism of $S(\mathcal{Q})$ by $\Psi(e_i) = e_i$ and $\Psi(\alpha\beta) = \Psi(\beta)\Psi(\alpha)$. Hence, by Remark 3.4.6, $S(\mathcal{Q})$ has strong duality. The converse statement is obvious. $\square$

Remark 5.11. *If the labelled quiver in Proposition 5.10 is symmetric, we have*

$$\dim_K S(\mathcal{Q}) = \sum_{i=1}^n (d_{\mathcal{Q}}(i) + 1)^2 (l_{\mathcal{Q}}(i) + 1).$$

Thus, if $\mathcal{Q}_n$ is “the complete quiver” on n vertices (i.e., the quiver having one arrow $i \rightarrow j$ for every choice of $1 \leq i, j \leq n$), then

$$\dim_K S(\mathcal{Q}_n) = \sum_{i=1}^n 2 \cdot i^2 = \frac{n(n+1)(2n+1)}{3}.$$

For $n = 4$, the right regular representation of $S = S(\mathcal{Q}_4)$ is as follows:

Let us conclude this final chapter with two illustrations.

Example 5.12. Let $\mathcal{Q}$ be the labelled quiver

and $\mathcal{I} = \langle \alpha\varepsilon, \beta\alpha, \beta\gamma, \alpha^2, \alpha\varepsilon, \delta\alpha, \delta\gamma, \eta^2 \rangle$. Then the right and the left representations of the canonical shallow properly stratified algebra $S = S(\mathcal{Q}) = K\mathcal{Q}/\mathcal{I}$ are given as follows:

$${}_SS = \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 2 \quad 3 \\ \swarrow \searrow \quad \swarrow \searrow \\ 1 \quad 2 \quad 1 \quad 2 \quad 3 \\ \swarrow \searrow \quad \swarrow \searrow \\ 1 \quad 2 \quad 1 \quad 2 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \\ 1 \end{array} \oplus \begin{array}{c} 3 \\ \swarrow \quad \searrow \\ 1 \quad 2 \quad 3 \\ \swarrow \searrow \quad \swarrow \searrow \\ 1 \quad 2 \quad 1 \quad 2 \end{array}.$$

The values of $d_Q(i)$, $l_Q(i)$ and $u_Q(i)$ are given in the following table

i	1	2	3
d_Q	0	1	1
l_Q	0	1	1
u_Q	0	1	2

Thus, $\dim_K S(Q) = 1 + 8 + 12 = 21$.

Example 5.13. Let Q be the symmetric labelled quiver

(a) For $\mathcal{I} = \langle \varepsilon^2, \varepsilon\alpha, \beta\alpha, \beta\varepsilon, \alpha^2 \rangle$, the right and the left regular representations of the shallow properly stratified algebra $S = S(Q) = KQ/\mathcal{I}$ are given as follows:

$$S_S = \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \searrow \quad \swarrow \searrow \\ 1 \quad 2 \quad 1 \quad 2 \\ \swarrow \searrow \\ 1 \quad 2 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \\ 1 \end{array}, \quad {}_SS = \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \searrow \quad \swarrow \searrow \\ 1 \quad 2 \quad 1 \quad 2 \\ \swarrow \searrow \\ 1 \quad 2 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \\ 1 \end{array}.$$

(b) For $\mathcal{T}' = \langle \varepsilon^2 - \alpha\beta, \varepsilon\alpha - \alpha\phi, \beta\alpha, \beta\varepsilon - \phi\beta, \phi^2 \rangle$, the algebra $S' = K\mathcal{Q}/\mathcal{T}'$ is also a shallow properly stratified K -algebra (with strong duality). Its right and the left regular representations are as follows:

$$S'_{S'} = \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \end{array}, \quad S'_S = \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \end{array}.$$

Clearly, $S \not\cong S'$. However, in accordance with Theorem 5.5, $\dim_K S = \dim_K S'$ ($=2+8=10$). This illustrates the fact that a quiver does not determine the shallow algebra over it. Only the dimension of a shallow algebra is uniquely determined by the quiver.

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