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A Bound on the Rank of
Solvable Primitive Permutation Groups

by

Suzanne Seager

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics

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Abstract^c

A Bound on the Rank of
Solvable Primitive Permutation Groups

Suzanne Seager

Theorem Let $\bar{G}$ be a maximal solvable primitive permutation group of degree d and rank r . Then either

$$r > 24^{1/3} d^{0.02747-1.5}$$

or $\bar{G}$ is permutation isomorphic to the wreath product $S(p^m) \text{ wr } K$, where $S(p^m)$ is the group of all semilinear transformations on $\text{GF}(p^m)$, K is a maximal solvable transitive subgroup of S_k , $d = p^{mk}$, and

$$r > 24^{1/3} (1.188)^k - 1.5$$

This theorem is proved by considering separately the cases that the stabilizer of $\bar{G}$ is primitive or imprimitive as a linear group and constructing bounds in each of these two cases. A class of groups acting as a limiting case is found. This extends work of Foulser and Dornhoff.

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Table of Contents

	Page
Abstract	iii
Acknowledgements	iv
List of Tables	vi
List of Symbols	vii
 Introduction	 1
 Chapter I. Primitive Solvable Permutation Groups	 3
 Chapter II. The Primitive Case	
2.1 Maximal Solvable Primitive Linear Groups	 5
2.2 A Bound on the Number of Orbits	7
2.3 The Case $n = 2m$	9
2.4 The Number of Orbits for $n = 2m$.	12
2.5 A General Bound	18
 Chapter III. The Imprimitive Case	
3.1 The Wreath Product	23
3.2 The Number of Orbits of a Wreath Product	 27
3.3 Some Small Cases	29
3.4 The Case K is Primitive	35
3.5 A Bound on $r(k,h)$	38
3.6 A Limiting Case	42
3.7 An Exact Formula for $r(k,h)$	45
 Chapter IV. The General Bound	
4.1 The Product Action of the Wreath Product	 56
4.2 Degrees with Small Rank	58
4.3 The Main Theorem	60
 References	 65

List of Tables

	Page
Table I. Formulas for $r^*(p, n, n/2)$	16
Table II. Values of $r^*(p, n, n/2)$	17
Table III. Values of $r^*(p^n)$ less than 1000	22
Table IV. Formulas for $(h)R_k$	33
Table V. Values of $(h)R_t$ at most 1000	34
Table VI. Values of a_h and a'_h	44
Table VII. Values of $r(k, h)$ at most 1000	55
Table VIII. Semilinear-Component Groups	58
Table IX. Non-Semilinear-Component Groups	59

List of Symbols

$[x, y]$ is the commutator $x^{-1}y^{-1}xy$ of x and y .

(m, n) is the greatest common divisor of m and n .

(a/p) is the Legendre symbol; $(a/p) = 1$ if and only if a is a quadratic residue of p .

$GF(p^n)$ is the Galois field with p^n elements.

$Gal(K/P)$ is the Galois group of field K over field P .

$Sym(X)$ is the symmetric group on the set X .

S_n is the symmetric group on n elements.

$GL(n, p)$ is the general linear group of degree n over $GF(p)$.

$SL(n, p)$ is the special linear group of degree n over $GF(p)$.

$Sp(2l, p)$ is the symplectic group of degree $2l$ over $GF(q)$.

$C_G(H)$ is the centralizer of the group H in the group G .

$|G|$ is the order of the group G .

$G|_U$ is the restriction of the group G to the set U .

$o(g)$ is the order of the group element g .

$Fix(g)$ is the set of all fixed points of the element g .

Symbols Defined in the Text

	Page
$T(p^n)$	3
$S(p^n)$	4
$r^*(p, n, m)$	8
$r^*(p^n)$	18
$(h)K$	29
$r(k, h)$	38
$H \text{ wr } K$	23, 25, 56

Introduction

In 1957 Huppert [5] proved that with six exceptions, any solvable 2-transitive permutation group is semilinear. This was extended in one direction by Passman [11], who found the exceptional cases for solvable $3/2$ -transitive groups. In another direction Foulser [4] and Dornhoff [5] found the exceptions for primitive solvable permutation groups of rank 3, and in addition Foulser considered the possible exceptions for rank 4. Dornhoff also showed that for any primitive non-semilinear solvable permutation group $\bar{G}$ such that the stabilizer of $\bar{G}$ is a primitive linear group, there exists some function of the rank of $\bar{G}$ which bounds the degree. In this thesis we shall extend this to find a relationship between the degree and the rank of a primitive solvable permutation group.

In Chapter I we give the standard analysis of a primitive solvable permutation group $\bar{G}$, by which the stabilizer G of $\bar{G}$ is found to be equivalent to an irreducible linear group. This may be semilinear, in which case $\bar{G}$ will have rank 2 whatever its degree, but otherwise some restriction can be found on the rank of $\bar{G}$, and thus in the next two chapters we consider irreducible linear groups which are not semilinear.

In Chapter II we consider the case that G is a solvable primitive linear group. We use the results of Suprunenko [14]

to analyze the structure of G , and we find a lower bound on the number of orbits of G given the degree, and a class of groups which act as a limiting case.

In Chapter III we consider the case that G is a solvable imprimitive linear group. Here we establish a lower bound on the number of orbits of G based on the degrees and numbers of orbits of its primitive components.

Finally in Chapter IV we combine the results of each of these cases to give a bound on the rank of G .

Chapter I. Primitive Solvable Permutation Groups

The standard analysis of primitive solvable permutation groups follows (see, for example, [12] or [4]):

Theorem 1.1 Let $\bar{G}$ be a primitive solvable subgroup of degree d , and let V be a minimal normal subgroup of $\bar{G}$. Then V is elementary abelian of order p^n , for some prime p and $n \geq 1$, and is a regular normal subgroup of $\bar{G}$ (so $d = p^n$). Let $G = \bar{G}_0$ be the stabilizer of $\bar{G}$. Then G acts as an irreducible subgroup of $GL(V)$.

The converse is also true ([1]):

Theorem 1.2 Let G be an irreducible solvable subgroup of $GL(V)$. Then the split extension $\bar{G} = V \cdot G$ is a primitive solvable permutation group acting on V .

So now let $\bar{G}$ be a primitive solvable permutation group of degree $d = p^n$, as in Theorem 1.1, with stabilizer G an irreducible solvable subgroup of $GL(n, p)$ ($= GL(V)$). Then the rank of $\bar{G}$ is equal to the number of orbits of G . Since we wish to minimize rank for a fixed degree, we may assume that $\bar{G}$ is a maximal solvable subgroup of the symmetric group S_d , and thus that G is a maximal irreducible solvable subgroup of $GL(n, p)$.

Suppose first that V can be identified with the field $K = GF(p^n)$ so that $G = T(p^n)$, the group of all semilinear

transformations of V of the form $x \mapsto ax^\sigma$, with a in K , $a \neq 0$, and σ a field automorphism in $\text{Gal}(K)$. Then $\bar{G} = S(p^n)$, the semilinear group of all transformations of V of the form $x \mapsto ax^\sigma + b$, with a, b in K , $a \neq 0$, and σ in $\text{Gal}(K)$. $S(p^n)$ is solvable of order $np^n(p^n-1)$ and rank 2 (see [12] Chapter 19).

So we will now be concerned with the maximal solvable irreducible subgroups of $GL(n, p)$ which are not semilinear. By Theorem 1.2, finding the possible numbers of orbits of such groups will determine the possible ranks of maximal solvable primitive permutation groups which are not semilinear.

Chapter II. The Primitive Case

1. Maximal Solvable Primitive Linear Groups

The following material is from Suprunenko [14], Chapter 20.

Let G be a maximal primitive solvable subgroup of $GL(n, p)$. Let F be the maximal abelian normal subgroup of G , V the centralizer of F in G , and A/F the maximal abelian normal subgroup of G/F contained in V/F . The invariant series $G \triangleright V \triangleright A \triangleright F \triangleright 1$ is unique, and is called the Suprunenko series of G . We shall now study the factors of this series more closely.

Let $P = GF(p)$. Then $F = K^*$, for some extension field K of P . Let m be the index of P in K ; then $m \mid n$, and $m = n$ if and only if G is semilinear. Further G/V is isomorphic to a subgroup of $Gal(K/P)$, and V is an absolutely irreducible subgroup of $GL(n/m, K)$.

Let $n/m = q_1^{l_1} \dots q_k^{l_k}$ be the canonical decomposition of n/m , and let Q_j/F be a Sylow q_j -subgroup of A/F , $j=1, \dots, k$. Then the Q_j commute among themselves elementwise. For each j , Q_j can be represented in the form

$$Q_j = \langle u_1 \rangle \langle v_1 \rangle \dots \langle u_{l_j} \rangle \langle v_{l_j} \rangle^F$$

where $[u_i, v_i] = \eta$ for some η in F such that $\eta^{q_j} = 1$,

$(u_i)^{q_j}, (v_i)^{q_j} \in F$, and elements in distinct pairs commute.

Now for each x in V suppose

$$xu_i x^{-1} = \lambda_i(u_1)^{\alpha_{1i}} \dots (u_{\ell_j})^{\alpha_{\ell_j i}} (v_1)^{\gamma_{1j}} \dots (v_{\ell_j})^{\gamma_{\ell_j i}}$$

and

$$xv_i x^{-1} = \mu_i(u_1)^{\beta_{1i}} \dots (u_{\ell_j})^{\beta_{\ell_j i}} (v_1)^{\delta_{1i}} \dots (v_{\ell_j})^{\delta_{\ell_j i}}$$

for λ_i, μ_i in K^* , $0 \leq \alpha_{ri}, \beta_{ri}, \gamma_{ri}, \delta_{ri} < q_j$, $i=1, \dots, \ell_j$.

Then we define a homomorphism $\psi_j: V \rightarrow \text{Sp}(2\ell_j, q_j)$ as follows:

$$\psi_j(x) = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1\ell_j} & \beta_{11} & \dots & \beta_{1\ell_j} \\ \vdots & & \vdots & & & \vdots \\ \alpha_{\ell_j 1} & \dots & \alpha_{\ell_j \ell_j} & \beta_{\ell_j 1} & \dots & \beta_{\ell_j \ell_j} \\ \gamma_{11} & \dots & \gamma_{1\ell_j} & \delta_{11} & \dots & \delta_{1\ell_j} \\ \vdots & & \vdots & & & \vdots \\ \gamma_{\ell_j 1} & \dots & \gamma_{\ell_j \ell_j} & \delta_{\ell_j 1} & \dots & \delta_{\ell_j \ell_j} \end{pmatrix}$$

Note that since A is a normal subgroup of G we could as easily have defined $\psi_j: G \rightarrow \text{GL}(2\ell_j, q_j)$. Then the mapping

$\bar{\psi}: V/A \rightarrow \prod_{j=1}^k \text{Sp}(2\ell_j, q_j)$ induced on V/A by the ψ_j is a

monomorphism. Thus V/A is isomorphic to a solvable subgroup of the direct product of the k symplectic groups $\text{Sp}(2\ell_j, q_j)$, $j=1, \dots, k$.

Thus, based on its Suprunenko series, we have the following structure for G :

$$\begin{array}{l}
G \\
| \\
V/A \hookrightarrow \prod_{j=1}^k \text{Sp}(2\ell_j, q_j), \text{ where } \prod_{j=1}^k q_j^{2\ell_j} = |A/F| \\
A \\
| \\
A/F \cong \prod_{j=1}^k \prod_{i=1}^{2\ell_j} \text{GF}(q_j) \\
F \\
| \\
F = K^*, [K:P] \mid h. \\
1
\end{array}$$

2. A Bound on the Number of Orbits

Suppose G has the structure given above. Let $s_1(q_j^{\ell_j})$ be the maximal order of a solvable subgroup of $\text{Sp}(2\ell_j, q_j)$ and let $s_1(n/m) = \prod_{j=1}^k s_1(q_j^{\ell_j})$. Then $|\psi_j(V)| \leq s_1(q_j^{\ell_j})$ and so $|V/A| \leq s_1(n/m)$. Since $|G| = |G/V| |V/A| |A/F| |F|$,

$$|G| \leq m s_1(n/m) (n/m)^2 (p^m - 1)$$

Now consider the action of G on Q_j/F by conjugation. This action is completely reducible, since for any normal abelian subgroup B/F of Q_j/F , $Q_j/F = (B/F) \times (C_{Q_j}(B)/F)$ by Isaacs [7] Theorem 2. Thus if we extend the definition of ψ_j to $\psi_j: G \rightarrow \text{GL}(2\ell_j, q_j)$ we have $\psi_j(G)$ completely reducible for $j=1, \dots, k$. Let $R = \{x \in G \mid \psi_j(x) = 1, j=1, \dots, k\}$. Then $A \leq R$, $V \cap R = A$, and so $|R/A| \leq |G/V| \leq m$, and we have the monomorphism $\bar{\psi}: G/R \rightarrow \prod_{j=1}^k \text{GL}(2\ell_j, q_j)$, with $\bar{\psi}(G/R) = \prod_{j=1}^k \psi_j(G)$.

Let $s_2(q_j^{\ell_j})$ be the maximal order of a solvable completely reducible subgroup of $GL(2\ell_j, q_j)$, and $s_2(n/m) = \prod_{j=1}^k s_2(q_j^{\ell_j})$.

Then $|\psi_j(G)| \leq s_2(q_j^{\ell_j})$ so $|G/R| \leq s_2(n/m)$. Then since $|G| = |G/R| |R/A| |A/F| |F|$, we have

$$|G| \leq s_2(n/m) m(n/m)^2 (p^m - 1).$$

Now let $s(n/m) = \min(s_1(n/m), s_2(n/m))$. Then we have

$$|G| \leq s(n/m) m(n/m)^2 (p^m - 1) \quad (1)$$

But by Wolf [15] Theorem 3.1, $s_2(n/m) \leq (n/m)^{2\alpha} / (24^{1/3})$, for $\alpha = (3\log 48 + \log 24) / 3\log 9 = 2.24399\dots$, and so

$$|G| \leq m(n/m)^{2+2\alpha} (p^m - 1) / (24^{k/3}) \quad (2)$$

Now define $r^*(p, n, m)$ to be the minimal possible number of nontrivial orbits of a maximal solvable subgroup G of $GL(n, p)$ such that its maximal abelian normal subgroup has order $p^m - 1$. Then G acts on a space of $p^n - 1$ elements, and so $r^*(p, n, m) \geq (p^n - 1) / |G|$. Thus using (1) we have

$$r^*(p, n, m) \geq (p^n - 1) / s(n/m) m(n/m)^2 (p^m - 1) \quad (3)$$

or, using (2),

$$r^*(p, n, m) \geq 24^{k/3} (p^n - 1) / m(n/m)^{2+2\alpha} (p^m - 1) \quad (4)$$

and so we have a lower bound on the number of orbits.

3. The Case $n = 2m$

We now construct a specific class of maximal solvable primitive groups. We assume that the structure of G is as above with $n = 2m$. The case $m = 1$ was developed in Suprunenko [14] Chapter 21, and that for $m = 2$ in Zhavrid [16]. The following repeats their constructions for arbitrary m .

Since $n/m = 2$, we have $A = \langle a \rangle \langle b \rangle F$, where $a^2, b^2 \in F$ and $[a, b] = -1$, and $F = K^*$ for $K = GF(p^m)$. By Theorem 21.5 and Lemma 21.4 of Suprunenko, there are two possibilities for a and b as elements of $GL(2, K)$:

(i) K contains an element i of order 4, and

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ or}$$

(ii) K contains no element of order 4, and

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} \theta & \omega \\ \omega & -\theta \end{pmatrix}, \text{ where } \theta, \omega \in P, \\ \text{and } \theta^2 + \omega^2 = -1.$$

But in the first case, we may replace a by $a' = ia$, and b by $b' = ib$ to get a' and b' such that $(a')^2 = (b')^2 = -1$ and $[a', b'] = -1$, and so as in Theorem 21.5 of Suprunenko we may instead have

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} \theta & \omega \\ \omega & -\theta \end{pmatrix}, \theta, \omega \in P, \theta^2 + \omega^2 = -1.$$

Thus in either case we may assume

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} \theta & \omega \\ \omega & -\theta \end{pmatrix}, \text{ where } \theta, \omega \in P, \theta^2 + \omega^2 = -1,$$

and $\omega = 0$ if and only if there is an element in P of order 4.

Further, if we let ρ be an element of order $p^m - 1$ in K^* , then K is generated by the element $c = \rho I$, and thus $A = \langle a, b, c \rangle$.

Now we wish to find V such that $\psi(V) = \text{Sp}(2, 2)$. But

$\text{Sp}(2, 2) = \text{GL}(2, 2)$ and is generated by $\tau_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and

$\tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and thus we want v_1, v_2 such that $\psi(v_i) = \tau_i$,

$i=1, 2$. But $\psi(v) = \tau_1$ if and only if $vav^{-1} = \lambda a$ and

$vbv^{-1} = \mu ab$, for some λ, μ in K^* , and thus $\psi(v) = \tau_1$ if and

only if $v \in v_1 A$ for $v_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Similarly $\psi(v) = \tau_2$ if

and only if $v \in v_2 A$ for $v_2 = \begin{pmatrix} \theta & \omega - 1 \\ \omega + 1 & -\theta \end{pmatrix}$. Thus we have

$V = \langle v_1, v_2, a, b, c \rangle$ and $\psi(V) = \text{GL}(2, 2)$.

Now we want to find G such that $G/V \hookrightarrow \text{Gal}(K/P)$. Let σ

be the mapping $x \rightarrow x^P$ for all x in K . Then σ generates

$\text{Gal}(K/P)$ and so we want to find h in $\text{GL}(n, p)$ such that

$hAh^{-1} = A$ and $huh^{-1} = u^\sigma$ for all u in F ; i.e. such that

$\psi(h) \in \text{GL}(2, 2)$ and $hch^{-1} = c^\sigma$. Now if $\psi(h) \in \text{GL}(2, 2)$ then

since $\psi(V) = \text{GL}(2, 2)$ there exists some v in V with

$\psi(v) = \psi(h)$ and thus replacing h with hv^{-1} we may assume

that $\psi(h) = I$ and thus that $hah^{-1} = \lambda a$ and $hbh^{-1} = \mu b$

for some λ, μ in K^* .

But this is true if and only if $h \in gA$, where $g = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}$

for some Σ in $GL(m, p)$ satisfying $\Sigma v \Sigma^{-1} = v^\sigma$ for all v in K .

Thus we have constructed $G = \langle g, v_1, v_2, a, b, c \rangle$, and we must now show that it is maximal solvable primitive.

Theorem 2.1 Let $G = \langle g, v_1, v_2, a, b, c \rangle$ as above. Then G is a maximal solvable primitive subgroup of $GL(n, p)$, and any maximal solvable primitive subgroup of $GL(n, p)$ satisfying $n \neq 2m$ is conjugate in $GL(n, p)$ to G .

Proof: Let $V = \langle v_1, v_2, a, b, c \rangle$. Then V is normal in G and by Lemma 21.5 of Suprunenko, V is primitive as a subgroup of $GL(2, K)$. But then V is primitive as a subgroup of $GL(n, p)$. For, (following the proof of Lemma 5 of Konyukh [8]), suppose $L_1 + \dots + L_t$ is a system of imprimitivity for V as a subgroup of $GL(n, p)$. Let f be an element of F , and let x_i be in L_i for some i . Suppose $f(x_i) = x_j \in L_j$. Then $(I + f)(x_i) = x_i + x_j$. But $I + f \in F$, and so $(I + f): L_i \rightarrow L_s$ for some s . Thus $x_i + x_j \in L_s$, and so $i = j = s$. Therefore $f(L_i) = L_i$, and so L_i is F -invariant for $i=1, \dots, t$, and so each L_i is a K -space. But then $L_1 + \dots + L_t$ is a system of imprimitivity for V as a subgroup of $GL(2, K)$, a contradiction. Thus G is primitive, and G is solvable since each of G/V , V/A , A/F , and F is solvable. Finally, by the construction of G , G is maximal and any maximal solvable primitive subgroup of $GL(n, p)$ satisfying $n \neq 2m$ is conjugate in $GL(n, p)$ to G .

4. The Number of Orbits for $n = 2m$

Let $G = \langle g, v_1, v_2, a, b, c \rangle$ as above. Then we have

$$g = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}, v_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, v_2 = \begin{pmatrix} \theta & \omega-1 \\ \omega+1 & -\theta \end{pmatrix}, a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} \theta & \omega \\ \omega & -\theta \end{pmatrix}, \text{ and } c = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \text{ where } \langle \rho \rangle = K^* \text{ for } K = GF(p^m),$$

$\theta, \omega \in P$ such that $\theta^2 + \omega^2 = -1$, and $\Sigma \in GL(m, p)$ such that $\Sigma \rho \Sigma^{-1} = \rho^\sigma$, where $\sigma: x \rightarrow x^p$ generates $\text{Gal}(K/P)$ ($P = GF(p)$).

Further $V = \langle v_1, v_2, a, b, c \rangle$, $A = \langle a, b, c \rangle$, and $F = \langle c \rangle$. We consider first the conjugacy classes of G .

Now $gvg^{-1} = v$ for $v = a, b, v_1, v_2$, and thus gF centralizes V/F in G/F . Therefore G/F is equivalent to $\langle gF \rangle \times V/F$. But V/F has 24 elements comprising five conjugacy classes, each class C_i having length ℓ_i and representative $c_i F$ of order o_i as follows:

C_i	ℓ_i	o_i	c_i
1	6	4	$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
2	6	2	$\begin{pmatrix} \theta & \omega-1 \\ \omega+1 & -\theta \end{pmatrix}$
3	3	2	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
4	8	3	$\begin{pmatrix} \theta-\omega-1 & \theta+\omega-1 \\ \theta+\omega+1 & -\theta+\omega-1 \end{pmatrix}$
5	1	1	I

Conjugacy Classes of V/F

Thus G/F has $5m$ conjugacy classes $C_{11}, \dots, C_{5m}$, where the class C_{ij} has length ℓ_i and class representative $g^j c_i F$, for $i=1, \dots, 5$ and $j=1, \dots, m$. We can now count the number of orbits of G .

Theorem 2.2 Let $G = \langle g, v_1, v_2, a, b, c \rangle$ as above, $m = n/2$. Let $\alpha_1 = -1$, $\alpha_2 = -2$, $\alpha_3 = -1$, $\alpha_4 = -3$, and $\alpha_5 = 1$. Define

$$\phi_j^i = \begin{cases} p^{(m,j)+1} & \text{if } \alpha_i \mid m/(m,j) \\ 1 & \text{else if } p = 3, i \neq 4 \\ 2 & \text{else if } 2 \mid (m,j) \text{ or } (\alpha_i/p) = 1 \\ 0 & \text{else} \end{cases}$$

Then the number of nontrivial orbits of G is

$$r^*(p, n, n/2) = \frac{n/2}{12} \sum_{j=1}^5 \sum_{i=1}^m \ell_i \phi_j^i$$

Proof: G acts on the space $P^{(n)}$ which can be identified with the space $K^{(2)}$. Since $F = \langle c \rangle$ is the maximal abelian normal subgroup of G , and F acts on $K^{(2)}$ as multiplication by elements of K , the number of nontrivial orbits of G on $K^{(2)}$ is equal to the number of orbits of G/F on $\bar{K}^{(2)}$, the projective space obtained from $K^{(2)}$. But, by Burnside's Lemma, the number of orbits of G/F on $\bar{K}^{(2)}$ is

$\sum_{x \in G/F} \text{Fix}(x) / |G/F|$, and so, using the conjugacy class

structure of G , we have

$$r^*(p, n, n/2) = \frac{1}{24m} \sum_{j=1}^m \sum_{i=1}^5 \ell_i \text{Fix}(g^j c_i F)$$

Thus to prove the theorem it suffices to show that

$$\text{Fix}(g^j c_i F) = \phi_j^i \text{ for } i=1,2,\dots,5, j=1,2,\dots,m.$$

Let $i \neq 1$ and consider the action of $g^j c_1 F$ on the projective space $\bar{K}^{(2)} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 \\ v \end{pmatrix} \mid v \in K \right\}$ of $p+1$ points. Since $g^j c_1 = \begin{pmatrix} \Sigma^j & -\Sigma^j \\ \Sigma^j & \Sigma^j \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not a fixed point.

Suppose $\begin{pmatrix} 1 \\ v \end{pmatrix}$ is a fixed point. Then $g^j c_1 \begin{pmatrix} 1 \\ v \end{pmatrix} = v' \begin{pmatrix} 1 \\ v \end{pmatrix}$ for some v' in K , and so $(1 - v^{(\sigma^j)})_v = (1 + v^{(\sigma^j)})_{v'}$, or

$$v^{(\sigma^j)} = \frac{v-1}{v+1}, \quad v \neq 1, -1 \quad (5)$$

Let $t = o(\sigma^j) = m/(m, j)$. Then $v^{(\sigma^{tj})} = v$. But (5) implies $v^{(\sigma^{4j})} = v$. Thus we have

$$v^{(\sigma^{(4,t)j})} = v \quad (6)$$

So we consider now three cases:

(i) $2 \nmid t$. Then (6) implies $v^{(\sigma^j)} = v$, and so by (5), $v^2 + 1 = 0$. This has two solutions in K if $(-1/p) = 1$ or $2 \mid (m, j)$ and none otherwise, since v must be in the field $\text{GF}(p^{(m, j)})$, the fixed field of $\sigma^{(m, j)}$ in K . Further any solution of $x^2 + 1 = 0$ in $\text{GF}(p^{(m, j)})$ generates a fixed point and thus the number of fixed points is two if $(-1/p) = 1$ or $2 \mid (m, j)$ and none otherwise.

(ii) $2 \mid t$, $4 \nmid t$. Then (6) implies $v^{(\sigma^{2j})} = v$, and so by (5), $v = -1/v$, and as above the number of fixed points

is two if $(-1/p) = 1$ or $2 \mid (m, j)$ and none otherwise.

(iii) $4 \mid t$. Then $(4j, m) = 4(j, m)$, and so since $v(\sigma^{4j}) = v$ and $v(\sigma^m) = v$, we have $v(\sigma^{4(j, m)}) = v$. But $j/(j, m) \equiv 1$ or $3 \pmod{4}$. Thus $v(\sigma^j) = v(\sigma^{(j, m)})$ or $v(\sigma^j) = v(\sigma^{3(j, m)})$. If $v(\sigma^j) = v(\sigma^{(j, m)})$, then by (5) $v(\sigma^{(j, m)})_v + v(\sigma^{(j, m)}) - v + 1 = 0$, so v is a solution of $x^{p^{(m, j)}+1} + x^{p^{(m, j)}} + x + 1 = 0$ in the field $K = GF(p^m)$.

But any solution to this equation in any extension field of $P = GF(p)$ satisfies $x^{p^{4(m, j)}} = x$, and thus satisfies $x^{p^m} = x$ and is in $K = GF(p^m)$. Thus the number of fixed points is equal to the number of solutions of the equation above, which is $p^{(m, j)} + 1$. Similarly if $v(\sigma^j) = v(\sigma^{3(j, m)})$, then applying $\sigma^{3(j, m)}$ to (5) gives

$$v = \frac{v(\sigma^{(j, m)}) - 1}{v(\sigma^{(j, m)}) + 1}$$

and the same argument again gives $p^{(m, j)} + 1$ fixed points.

Thus we have shown that for $i = 1$, the number of fixed points of $g_{i, F}^j$ is ϕ_j^i . The cases $i = 2, 3, 4, 5$ follow similarly.

We can use this theorem to calculate the exact formulas for the number of nontrivial orbits of G as follows:

For any prime $p > 2$ let $\beta_i = \begin{cases} 1 & \text{if } (-i/p) = 1 \\ 0 & \text{if } (-i/p) = -1 \end{cases}$
 for $i = 1, 2, 3$, and $i \neq p$, and let $\beta_3 = \begin{cases} 0 & \text{if } 2 \mid m \\ \frac{1}{2} & \text{if } 2 \nmid m \end{cases}$ if $p = 3$.

Then using elementary results of number theory (see [9] Chapter 5) we can summarize the values of β_i as follows:

$p \pmod{24}$	1	5	7	11	13	17	19	23
$(\beta_1 \beta_2 \beta_3)$	111	100	001	010	101	110	011	000

Using this notation we summarize the first few formulas for the number of orbits of G given $n = 2m$ in Table I.

Table I. Formulas for $r^*(p, n, n/2)$

n	$n/2$	$r^*(p, n, n/2)$
2	1	$(p+1+18\beta_1+12\beta_2+16\beta_3)/24$
4	2	$(p^2+10p+57+12\beta_1+16\beta_3)/48$
6	3	$(p^3+18p+19+54\beta_1+36\beta_2+16\beta_3)/72$
8	4	$(p^4+10p^2+32p+117+32\beta_3)/96$
$2q$	q	$(p^q+(q-1)p+q+18q\beta_1+12q\beta_2+16q\beta_3)/24q$ (q prime, $q > 3$)
12	6	$(p^6+10p^3+18p^2+36p+171+36\beta_1+16\beta_3)/144$
16	8	$(p^8+10p^4+32p^2+64p+213+64\beta_3)/192$
18	9	$(p^9+18p^3+54p+73+162\beta_1+108\beta_2+16\beta_3)/216$
20	10	$(p^{10}+10p^5+4p^2+40p+285+60\beta_1+80\beta_3)/240$

Using this Table, we can calculate $r^*(p, n, n/2)$ for small p , and all values of $r^*(p, n, n/2)$ less than 1000 when $n > 2$ are summarized in Table II.

Table II. Values of $r^*(p, n, n/2)$ [illegible]

5. A General Bound

Let $r^*(p^n) = \min \{r^*(p, n, m) \mid m \mid n, m \neq n\}$, so $r^*(p^n)$ is the least possible number of nontrivial orbits of a maximal solvable primitive non-semilinear subgroup of $GL(p, n)$.

Theorem 2.3 $r^*(p^n) > p^{n/2}/12n$ with the possible exceptions of $p^n = 17^4, 19^4, 7^6, 5^8, 7^8, 11^8, 13^8, 7^9, 3^{16}$ or 5^{16} . Further when n is even and $p \neq 2$, $r^*(p^n) \geq r^*(p, n, n/2)$ with the possible exceptions of $p^n = 3^4, 5^4, 7^4, 11^4, 13^4, 17^4, 19^4, 23^4, 29^4, 31^4, 5^6, 7^6, 3^8, 5^8, 7^8, 11^8, 13^8, 7^9, 3^{12}, 3^{16}$, and 5^{16} , and thus the groups constructed in §2.3 act as a limiting case.

Proof: We wish to show that $r^*(p, n, m) > p^{n/2}/12n$ for $m \mid n$, $m \neq n$, other than for the exceptions above. By (3) it is enough to show that

$$(p^{n-1}/m(n/m))^2 s(n/m) (p^{m-1}) > p^{n/2}/12n$$

or that

$$p^{n/2-m} \geq (n/m) s(n/m)/12 \quad (7)$$

For $n/m = 2$, $s(n/m) = 6$ and this result is trivially true, so we may assume $1 \leq m \leq n/3$. Suppose first that $m \geq 5$ and $n/m \geq 8$ (so $m \leq n/8$ and $n \geq 40$). Then $m \leq n/8$ implies that $p^{n/2-m} \geq p^{3n/8}$, and $m \geq 5$ implies (using (4)) that

$(n/m)s(n/m)/12 < (n/5)^{1+2\alpha}/(24^{1/3}12)$. Thus we want $p^{3n/8} > n^{1+2\alpha}/(24^{1/3}5^{1+2\alpha}12)$. This is true for $n \geq 40$ and $p \geq 2$. Thus we need now only consider $m < 5$ or $n/m < 8$.

Suppose now $n/m < 8$. From Suprunenko Chapter 21 the maximal order of a solvable irreducible subgroup of $Sp(2,q) = SL(2,q)$ is $\max(2(q+1), s_q)$, where s_q is 24 if $q \equiv \pm 3 \pmod{8}$ and 48 otherwise, and the maximal order of a reducible subgroup of $SL(2,q)$ is $q(q-1)$. Further by Dixon [2] the maximal order of a solvable subgroup of $Sp(2,4)$ is 72. Thus we have the following upper bounds for $s(n/m)$:

n/m	2	3	4	5	6	7
$s(n/m)$	6	24	72	24	144	48

So, considering both sides of (7), we can summarize the possible p^n for which (7) may not hold as follows:

m	$p^{n/2-m}$	$(n/m)s(n/m)/12$	Possible Exceptions
$n/3$	$p^{n/6}$	6	$2^3, 3^3, 5^3, \dots, 31^3$ $2^6, 3^6, 5^6, 2^9, 3^9, 2^{12}$
$n/4$	$p^{n/4}$	24	$2^4, 3^4, 5^4, \dots, 23^4$ $2^8, 3^8, 2^{12}, 2^{16}$
$n/5$	$p^{3n/10}$	10	$2^5, 3^5, 2^{10}$
$n/6$	$p^{n/3}$	72	$2^6, 3^6, 5^6, 7^6, 2^{12}, 2^{18}$
$n/7$	$p^{5n/14}$	28	$2^7, 3^7$

But by Suprunenko Corollary 20.3.1 every prime divisor of n/m must divide $p^m - 1$. This eliminates all of the above exceptions except $7^3, 13^3, 19^3, 31^3, 2^6, 5^6, 2^{12}$ ($m = n/3$), $3^4, 5^4, 7^4, 11^4, 13^4, 17^4, 19^4, 23^4, 3^8$ ($m = n/4$) and 7^6 ($m = n/6$). Further by Foulser [4] Theorem 6.34,

$$r^*(p, 3, 1) = (p^2 + 34p + 361 + 144\delta_1 + 108\delta_2) / 216 \quad (8)$$

where $\delta_1 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{9} \\ 0 & \text{otherwise} \end{cases}$ and $\delta_2 = \begin{cases} 1 & \text{if } (3/p) = 1 \\ 0 & \text{otherwise} \end{cases}$

And Theorem 1.1 and Theorem 1.2 of Foulser give the following lower bounds for $r^*(p^n)$:

p^n	2^6	5^6	2^{12}	3^4	5^4	7^4	11^4	13^4	3^8	7^6
$r^*(p^n)$	3	4	4	2	3	2	4	4	4	4

Thus we can eliminate all of the possible exceptions except $7^6, 17^4, 19^4$ and 23^4 . Finally we can eliminate 23^4 using (3), leaving the three possible exceptions $7^6, 17^4$ and 19^4 .

So now we suppose $m = 1, 2, 3$ or 4 and $n \geq 8m$. Using (4) we want

$$p^{n/2-m} > n^{1+2\alpha} / (24^{1/3} 12m^{1+2\alpha}) \quad (9)$$

For fixed m let $f(n, p) = p^{n/2-m} - n^{1+2\alpha} / (24^{1/3} 12m^{1+2\alpha})$, so (9) is true if and only if $f(n, p) > 0$. Then for fixed n $f(n, p)$ increases with p , and for fixed p $f(n, p)$ increases with n provided either $p \geq 5$, or $p \geq 3$ and $n \geq 10$, or $p \geq 2$ and $n \geq 16$. Thus we can summarize the possible

exceptions as follows:

m	p^n such that $f(n,p) < 0$
1	$2^8, 2^9, \dots, 2^{55}, 3^8, 3^9, \dots, 3^{29}, 5^8, 5^9, \dots, 5^{16}$ $7^8, 7^9, \dots, 7^{12}, 11^8, 11^9, 13^8$
2	$2^{16}, 2^{18}, \dots, 2^{40}, 3^{16}, 3^{18}, \dots, 3^{20}$
3	$2^{24}, 2^{27}, 2^{30}, 2^{33}$
4	-

If we test each of these cases as above, the only new exceptions that we find are $5^8, 7^8, 11^8, 13^8, 7^9, 3^{16}, 5^{16}$. Thus we have proved $r^*(p^n) > p^{n/2}/12n$ with ten possible exceptions.

Finally, note that by Theorem 2.2,

$$p^{n/2}/12n < r^*(p, n, n/2) < p^{n/2}/6n$$

for all $n > 9$, $p > 2$. Thus $r^*(p^n) \geq r^*(p, n, n/2)$ can be proved as above.

Now by using this theorem, Table I, formula (8), and bounds (3) and (4) we can calculate all possible values of p and n such that there may exist a maximal solvable primitive subgroup of $GL(p, n)$ with at most 1000 nontrivial orbits, and these results are summarized in Table III.

Table III. Values of $r^*(p^n)$ less than 1000

- (i) $n = 2$.
 $r^*(p^2)$ < 1000 for all odd primes $p \leq 23957$
 $= 1000$ for $p = 23967, 23981$
 > 1000 for $p = 23977$ and $p \geq 23993$.

The exact formula for $r^*(p^2) = r^*(p, 2, 1)$ is in Table I.

- (ii) $n > 2$

p	n	r*	p	n	r*	p	n	r*	p	n	r*
3	4	2	5	10	27	79	4	148	149	4	495
2	6	2	61	3	33	163	3	151	313	3	505
3	6	2	2	20	30†	83	4	162	151	4	508
7	4	2	67	3	33	23	6	175	157	4	548
3	10	3	13	6	35	181	3	183	331	3	561
7	3	3	37	4	38	89	4	185	337	3	581
5	4	3	73	3	38	193	3	205	163	4	589
5	6	4	2	24	39†	199	3	217	167	4	617
3	8	4	79	3	43	97	4	218	349	3	621
7	6	4†	2	21	43†	101	4	235	173	4	661
5	8	4†	41	4	45	211	3	241	3	22	672
7	8	4†	43	4	49	103	4	244	367	3	683
3	16	4†	47	4	57	3	20	258	373	3	705
2	15	4†	97	3	61	107	4	262	179	4	706
2	12	4†	103	3	67	223	3	267	37	6	714
2	18	4†	53	4	71	109	4	272	181	4	722
11	4	4†	17	6	74	13	8	272†	379	3	727
13	4	4†	109	3	75	229	3	281	397	3	795
13	3	5	59	4	86	113	4	291	191	4	801
17	4	5†	11	8	86†	241	3	309	193	4	818
19	4	6†	61	4	92	29	6	347	409	3	841
3	12	6†	3	18	95	127	4	364	7	12	850
19	3	7	127	3	97	271	3	385	197	4	851
31	3	11	19	6	101	131	4	386	199	4	868
23	4	11†	67	4	109	277	3	401	421	3	889
7	9	13†	139	3	113	283	3	417	17	8	907
3	14	14	71	4	121	137	4	421	433	3	939
37	3	15	5	12	123	31	6	422	439	3	963
43	4	17	73	4	128	139	4	433	41	6	969
11	6	22	151	3	131	5	14	466	211	4	973
29	4	22†	7	10	141	2	30	486†			
31	4	26†	157	3	141	307	3	487			

† Lower Estimate

Chapter III. The Imprimitive Case

1. The Wreath Product

The material in this section is primarily from Suprunenko [14] Chapters 3 and 15.

Let U be a linear space over $GF(p)$, H a subgroup of $GL(U)$, and K a subgroup of the symmetric group S_k , $k > 1$. Let V be the cartesian product U^k as a linear space over $GF(p)$, so we can write $V = \{(u_1, \dots, u_k) \mid u_i \in U\}$.

Let $G = \{(g_1, \dots, g_k; s) \mid g_i \in H \text{ and } s \in K\}$, and for

$g = (g_1, \dots, g_k; s)$ in G , define $g: V \rightarrow V$ by

$$g((u_1, \dots, u_k)) = (u'_1, \dots, u'_k), \text{ for } u'_i = g_{s^{-1}(i)}^{-1}(u_{s^{-1}(i)})$$

For $f = (f_1, \dots, f_k; r)$ in G we have

$$gf = (g_{r(1)}f_1, \dots, g_{r(k)}f_k; sr) \text{ in } G$$

and so G is a subgroup of $GL(V)$. G is called the wreath product of the linear group H and the permutation group K and is denoted $G = H \text{ wr } K$.

Theorem 3.1 Let H be an irreducible subgroup of $GL(U)$ and K a transitive subgroup of S_k , $k > 1$. Then the group $G = H \text{ wr } K$ is imprimitive, and is irreducible except when U is a one-dimensional space over $GF(2)$.

Proof: Suprunenko Lemma 15.4.

Now suppose G is an irreducible imprimitive subgroup of $GL(V)$, for V a linear space over $GF(p)$, and let $V = U_1 + \dots + U_k$ be a system of imprimitivity for G . Then we can define a mapping $\psi: G \rightarrow S_k$ by $\psi(g): i \rightarrow j$, where $g(U_i) = U_j$, for any g in G . Let $K = \psi(G)$. Then K is a transitive subgroup of S_k . Now let $G_1 = \{g \text{ in } G \mid g(U_1) = U_1\}$, and let $H = G_1|_{U_1}$. Then we have the following theorem:

Theorem 3.2 G is similar as a linear group to a subgroup of the wreath product $H \text{ wr } K$.

Proof: Suprunenko Lemma 15.5.

This construction leads to the following theorem:

Theorem 3.3 Let G be an imprimitive irreducible maximal solvable subgroup of $GL(V)$, where V is a linear space of finite dimension over $GF(p)$. Then G is similar as a linear group to a wreath product $H \text{ wr } K$, where H is a maximal primitive solvable subgroup of $GL(U)$, K is a maximal transitive solvable subgroup of the symmetric group S_k , and U is a subspace of V such that $[U:GF(p)]k = n$, for $n = [V:GF(p)]$.

Proof: Suprunenko Theorem 15.5..

Paralleling the construction of the wreath product of a linear group and a permutation group is the construction

of the wreath product of two permutation groups. Let M, N be subgroups of $\text{Sym}(X)$, $\text{Sym}(Y)$ respectively, where $|X| = m$ and $|Y| = n$. Let $Z = X \times Y$ and let $K = \{(r_1, \dots, r_m; s) \mid r_j \in M, s \in N\}$. For $r = (r_1, \dots, r_m; s)$ define $r: Z \rightarrow Z$ by

$$r(x, y) = (r_y(x), s(y))$$

For $t = (t_1, \dots, t_m; s')$ in K we have

$$rt = (r_{s'(1)} t_1, \dots, r_{s'(m)} t_m; ss') \text{ in } K$$

and so K is a subgroup of $\text{Sym}(Z)$. K is called the wreath product of M and N in its imprimitive action and is again denoted $K = M \text{ wr } N$. And as above we have the following theorem:

Theorem 3.4 Let K be a maximal solvable imprimitive subgroup of S_k . Then K is similar as a permutation group to a wreath product $M \text{ wr } N$, where M is a maximal solvable primitive subgroup of S_m , N is a maximal solvable transitive subgroup of S_n , and $k = mn$.

These wreath products can be combined, and in fact the combinations are all associative.

Theorem 3.5 Let K_i be a subgroup of $\text{Sym}(X_i)$, $i = 1, 2, 3$. Then $(K_1 \text{ wr } K_2) \text{ wr } K_3$ and $K_1 \text{ wr } (K_2 \text{ wr } K_3)$ are similar.

Proof: Huppert [6] Lemma 15.4.

Theorem 3.6 Let H be a subgroup of $GL(U)$ and let M, N be subgroups of $Sym(X)$ and $Sym(Y)$ respectively, where $|X| = m$ and $|Y| = n$. Then $H \wr (M \wr N)$ and $(H \wr M) \wr N$ are similar.

Proof: Let $G_1 = (H \wr M) \wr N$ and $G_2 = H \wr (M \wr N)$. Then G_1 acts on $(U^m)^n$ and G_2 acts on $U^{(mn)}$ so we identify $(U^m)^n$ and $U^{(mn)}$ via $v = ((v_{11}, \dots, v_{1m}), \dots, (v_{n1}, \dots, v_{nm})) = (v_{11}, \dots, v_{1m}, \dots, v_{n1}, \dots, v_{nm})$. Let $g \in G_1$, say $g = (f_1, \dots, f_n; t)$ for t in N and $f_i = (h_{i1}, \dots, h_{im}, s_i)$ in $H \wr M$, h_{ij} in H and s_i in M . Define $\psi: G_1 \rightarrow G_2$ by

$$\psi(g) = (h_{11}, \dots, h_{1m}, \dots, h_{n1}, \dots, h_{nm}; r)$$

for $r = (s_1, \dots, s_n; t)$ in $M \wr N$. Then for any v in $(U^m)^n$, $v = (w_1, \dots, w_n)$ with $w_i = (v_{i1}, \dots, v_{im})$, we have

$$g(v) = (w'_1, \dots, w'_n) \text{ for } w'_i = f_{t^{-1}(i)}^{-1}(w_{t^{-1}(i)}).$$

But then $w'_i = (v'_{i1}, \dots, v'_{im})$ for $v'_{ij} = h_{i'j}(v_{i'j})$, where $i' = t^{-1}(i)$ and $j' = (s_{t^{-1}(i)})^{-1}(j)$. Also we have

$$\psi(g)(v) = (h_{r^{-1}(11)}^{-1}(v_{r^{-1}(11)}), \dots, h_{r^{-1}(nm)}^{-1}(v_{r^{-1}(nm)})).$$

$$\text{But } r(i'j') = (s_1, \dots, s_n, t)(i'j')$$

$$= t(i')s_{i'}(j') = t(t^{-1}(i))s_i(s_{t^{-1}(i)}^{-1}(j))$$

$$= ij.$$

Thus $r^{-1}(ij) = i'j'$, and so $h^{-1}(ij) (v^{-1}(ij)) = h_{i'j'}(v_{i'j'})$
 $= v_{ij}$. Therefore $\psi(g)(v) = g(v)$, and so G_1 and G_2 are
 similar as linear groups.

Thus any maximal solvable imprimitive subgroup G of $GL(n, p)$ can be written $G = H \text{ wr } K_1 \text{ wr } K_2 \text{ wr } \dots \text{ wr } K_s$,
 where H is a maximal solvable primitive subgroup of $GL(n_1, p)$,
 K_i is a maximal solvable primitive subgroup of S_{k_i} , $i=1, \dots, s$,
 and $n = n_1 k_1 k_2 \dots k_s$. Further, by associativity,
 $G = (H \text{ wr } K_1 \text{ wr } K_2 \text{ wr } \dots \text{ wr } K_{s-1}) \text{ wr } K_s$, and so any
 maximal solvable imprimitive subgroup G of $GL(n, p)$ can be
 written in the form $G = H' \text{ wr } K$, where H' is a maximal
 solvable irreducible subgroup of $GL(n_2, p)$, K is a
 maximal solvable primitive subgroup of S_k , and $n = n_2 k$.

2. The Number of Orbits of a Wreath Product

Theorem 3.7 Let G be a maximal solvable imprimitive
 subgroup of $GL(n, p)$, with $G = H \text{ wr } K$, where H is an
 irreducible solvable subgroup of $GL(m, p)$ with h orbits
 and K is a maximal solvable primitive subgroup of S_k ,
 $n = km$. Let r' be the number of orbits of G . Then

$$r' = \frac{1}{|K|} \sum_{x \in K} h^{c(x)}$$

where $c(x)$ is the number of cycles in x for each permutation
 x in K .

Proof: Let $P = GF(p)$ and let $V = P^{(n)}$, so that $GL(n, p) = GL(V)$. Let $V = V_1 + \dots + V_k$ be the imprimitivity decomposition corresponding to the wreath product $G = H \wr K$ so each V_i is isomorphic to $U = P^{(m)}$, and $GL(m, p) = GL(U)$. Let $R_1, \dots, R_h$ be the orbits of H on U , and let $R_{i1}, \dots, R_{ih}$ be the corresponding orbits on V_i , $i=1, \dots, k$. Let $R(j_1, \dots, j_k) = R_{1j_1} + \dots + R_{kj_k}$ for each $(j_1, \dots, j_k)$ in X^k , $X = \{1, \dots, h\}$. Then each $R(j_1, \dots, j_k)$ is wholly contained in an orbit of $H \wr K$, any two such are disjoint, and the union of all such sets is V . Define an action of K on X^k by $s: (j_1, \dots, j_k) \rightarrow (i_1, \dots, i_k)$ if and only if $(1, \dots, 1; s)(R(j_1, \dots, j_k)) = R(i_1, \dots, i_k)$, where $(1, \dots, 1; s) \in H \wr K$. Then K is equivalent to a subgroup of $\text{Sym}(X^k)$ and $(j_1, \dots, j_k), (i_1, \dots, i_k)$ are in the same orbit of K on X^k if and only if $R(j_1, \dots, j_k), R(i_1, \dots, i_k)$ are in the same orbit of $H \wr K$. Thus the number of orbits of G on V is equal to the number of orbits of K on X^k , and so by Burnside's Lemma we have

$$r' = \frac{1}{|K|} \sum_{x \in K} |\text{Fix}(x)|$$

But $\text{Fix}(x) = \{(j_1, \dots, j_k) \text{ in } X^k \mid \text{all } j_i \text{ with } i \text{ in the same cycle of } x \text{ are equal}\}$, for any x in K . Thus we have

$|\text{Fix}(x)| = |X|^{c(x)} = h^{c(x)}$, and thus substituting this in the above equation proves the theorem.

As a result of this theorem note that for any permutation group K , if H and H' have the same number of orbits then H wr K and H' wr K will have the same number of orbits. Thus we can define $(h)K$ to be the number of orbits of H wr K for any linear group H having h orbits. We shall now compute $(h)K$ for certain K .

3. Some Small Cases

Following the notation of Pálffy [10], let R_q denote the group of linear transformations $x \rightarrow ax + b$, for a, b in $GF(q)$, $a \neq 0$, and q prime, and let $R_4 = S_4$. Then by Suprunenko [14] Theorem 4.11, R_q is the unique maximal solvable transitive subgroup of S_q and R_q is primitive. Further $|R_q| = q(q-1)$ for q prime, $|R_4| = 24$, and $R_2 = S_2$, $R_3 = S_3$. Then considering the cycle structure of R_q for $q \geq 5$ we find one element with q cycles, $(q-1)$ elements with one cycle, and for each $k \mid (q-1)$, $\phi(k)q$ elements with $1 + (q-1)/k$ cycles (where $\phi(k)$ here is the Euler function). Thus by Theorem 3.7 we have

$$(h)R_q = (h^q + (q-1)h + \sum_{\substack{k \mid (q-1) \\ k \neq 1}} q\phi(k)h^{(1+(q-1)/k)})/q(q-1) \quad (1)$$

Further by Foulser [4] Lemma 2.6 we have

$$(h)R_k = \binom{k+h-1}{k} \quad \text{for } k = 2, 3, 4 \quad (2)$$

Now suppose K is a maximal solvable primitive subgroup of S_8 . Since the stabilizer of K must be an irreducible subgroup of $GL(3,2)$, the only possibility ([14] Chapter 21) is the semilinear group $S(2^3)$. So let $R_8 = S(2^3)$. Then R_8 can be generated by the elements $(2\ 3\ 4\ 5\ 6\ 7\ 8)$, $(3\ 4\ 6)(5\ 8\ 7)$ and $(1\ 2)(3\ 5)(4\ 8)(6\ 7)$, and so the cycle structure of the elements of R_8 is as follows:

Structure	Number of Elements	Number of Cycles
1-7	48	2
2-6	56	2
1-1-3-3	56	4
2-2-2-2	7	4
identity	1	8

Thus $(h)R_8 = (h^8 + 63h^4 + 104h^2)/P68$.

Now suppose K is a maximal solvable primitive subgroup of S_9 . Here ([14] Chapter 21) there are three possibilities for the stabilizer of K as a maximal solvable irreducible subgroup of $GL(2,3)$, and so we consider each of these individually:

(i) K_1 , the semilinear group $S(3^2)$. Then K_1 has 144 elements and is generated by the elements $(4\ 7)(5\ 8)(6\ 9)$, $(2\ 5\ 7\ 8\ 3\ 9\ 4\ 6)$, and $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$, and so the cycle structure of the elements of K_1 is as follows:

Structure	Number of Elements	Number of Cycles
1-8	36	2
6-3	24	2
1-4-4	54	3
3-3-3	8	3
1-2-2-2-2	9	5
1-1-1-2-2-2	12	6
identity	1	9

Thus $(h)K_1 = (h^9 + 12h^6 + 9h^5 + 62h^3 + 60h^2)/144$.

(ii) K_2 with imprimitive stabilizer. Then K_2 has 72 elements and is generated by the elements $(2\ 3)(5\ 6)(8\ 9)$, $(2\ 4)(3\ 7)(6\ 8)$ and $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$, and so the cycle structure of the elements of K_2 is as follows:

Structure	Number of Elements	Number of Cycles
6-3	24	2
4-4-1	18	3
3-3-3	8	3
2-2-2-2-1	9	5
1-1-1-2-2-2	12	6
identity	1	9

Thus $(h)K_2 = (h^9 + 12h^6 + 9h^5 + 26h^3 + 24h^2)/72$.

(iii) K_3 with stabilizer having the Suprunenko series $V_6 A_4 F_2 1$. Then K_3 has 432 elements and is generated by the elements $(2\ 3)(4\ 6)(7\ 8)$, $(2\ 8\ 4\ 5\ 3\ 6\ 7\ 9)$ and $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$, and so the cycle structure of the elements of K_3 is as follows:

Structure	Number of Elements	Number of Cycles
8-1	108	2
6-3	72	2
6-3-1	72	3
4-4-1	54	3
3-3-3	56	3
3-3-1-1-1	24	5
2-2-2-2-1	9	5
2-2-2-1-1-1	36	6
identity	1	9

Thus $(h)K_3 = (h^9 + 36h^6 + 33h^5 + 182h^3 + 180h^2)/432$.

Since $(h)K_3$ is less than either $(h)K_1$ or $(h)K_2$ for any h , we let $R_9 = K_3$. Then for $k = 4, 8, 9$, or q , for any prime q , we have an exact formula for the number of orbits of H wr R_k whenever H is an irreducible subgroup of $GL(m, p)$ with h orbits. These formulas are summarized for values of k up to 23 in Table IV.

Table IV. Formulas for $(h)R_k$

k	$(h)R_k$
2	$(h^2+h)/2$
3	$(h^3+3h^2+2h)/6$
4	$(h^4+6h^3+11h^2+6h)/24$
5	$(h^5+5h^3+10h^2+2h)/20$
7	$(h^7+7h^4+14h^3+14h^2+6h)/42$
8	$(h^8+63h^4+104h^2)/168$
9	$(h^9+36h^6+33h^5+182h^3+180h^2)/432$
11	$(h^{11}+11h^6+44h^3+44h^2+10h)/110$
13	$(h^{13}+13h^7+26h^5+26h^4+26h^3+52h^2+12h)/156$
17	$(h^{17}+17h^9+34h^5+68h^3+136h^2+16h)/272$
19	$(h^{19}+19h^{10}+38h^7+38h^4+114h^3+114h^2+18h)/342$
23	$(h^{23}+23h^{12}+230h^3+230h^2+22h)/506$

Using the above formulas, we can calculate all values of $(h)R_k$ which are less than or equal to 1000, and these are summarized in Table V.

Note that if H is an irreducible linear group with h orbits, then $H \text{ wr } R_{k_1}$ is an irreducible linear group with $(h)R_{k_1}$ orbits, and thus since wreath products are associative, $(h)(R_{k_1} \text{ wr } R_{k_2}) = ((h)R_{k_1})R_{k_2}$. Thus we can calculate the number of orbits for strings of wreath products; e.g. $(2)(R_8 \text{ wr } R_3) = ((2)R_8)R_3 = (10)R_3 = 220$.

Table V. Values of $(h)R_t$ at most 1000

[illegible]

4. The Case K is Primitive

Theorem 3.8 Let K be a maximal solvable primitive subgroup of S_k , where $k = q^s$, q prime, $s > 1$. Then there exist transitive solvable subgroups K_i of S_{k_i} , $i = 1, 2$ such that $k = k_1 k_2$ and $(h)(K_1 \text{ wr } K_2) < (h)K$, except when $k = 4$, or $k = 8$ and $h = 2, 3, 4$, or $k = 9$ and $h = 2, 3, 4, 5, 6$.

Proof: Let $K_i = R_q^{(i)} = \underbrace{R_q \text{ wr } R_q \text{ wr } \dots \text{ wr } R_q}_i$

for $i = 0, 1, \dots, s$, and let $h_i = (h)K_i$, so $h_0 = h$ and $h_{i+1} = (h_i)R_q$ for $i = 0, 1, \dots, s-1$. We will show that $(h)(K_{s-1} \text{ wr } R_q) < (h)K$.

By Palfy [10] Theorem 1 we have

$$|K| \leq 24^{1/3} (k^{c_q})$$

where

$$c_2 = 1 + \log_4 6 \cdot 24^{1/3} = 3.05664\dots,$$

$$c_3 = 1 + \log_9 48 \cdot 24^{1/3} = 3.24399\dots,$$

$$c_5 = 1 + \log_{25} 96 \cdot 24^{1/3} = 2.74710\dots,$$

$$c_7 = 1 + \log_{49} 144 \cdot 24^{1/3} = 2.54918\dots,$$

$$c_q = 1 + \log_q (q-1) \cdot 24^{1/3} < 2.5 \text{ for } q > 7.$$

Thus by Theorem 3.7 we have

$$(h)K > \frac{1}{|K|} h^k \geq 24^{1/3} h^k / (k^{c_q}) \quad (3)$$

Suppose first that $q \geq 11$. Then by (1) we have

$$h_{i+1} < 2h_i^q / (q(q-1)) \quad i=0,1,\dots,s-1, \text{ and so}$$

$$h_s < (2/q(q-1))^{(q^s-1)/(q-1)} h(q^s) \quad (4)$$

But $h_s = (h)(K_{s-1} \text{ wr } R_q)$, and so by combining (3) and (4) we have $(h)K > (h)(K_{s-1} \text{ wr } R_q)$ if

$$(q(q-1))^{(q^s-1)/(q-1)} > 2^{(q^s-1)/(q-1)} (q^s)^{Cq/24^{1/3}}$$

or if $q^{(q^{s-1})} > q^{2.5s}$, which is true for all $s > 1$, $q > 7$.

Now suppose $q = 7$. Then by (1), $h_{i+1} < h_i^7/12$, and using this as above gives the same result. Similarly for $q = 5$ we can use $h_{i+1} < h_i^5/5$.

Now let $q = 3$. Then (1) gives $h_{i+1} < h_i^3/2$, and so

$$(h)K > (h)(K_{s-1} \text{ wr } R_3) \text{ if}$$

$$(1/2)^{(3^s-1)/2} < 24^{1/3} / (3^{3.244s})$$

which is true for all $s > 3$. Further for $h > 10$, (1) gives

$$h_{i+1} < 286h_i^3/1331, \text{ and as above this gives}$$

$(h)K > (h)(K_{s-1} \text{ wr } R_3)$ for $s > 1$. Thus we need only consider $s = 2, 3$ and $h = 2, 3, \dots, 10$. But then using (3) and the

exact formula for $(h)(R_3 \text{ wr } R_3 \text{ wr } R_3)$ from (2) we can

eliminate the case $s = 3$. For $s = 2$, the three possibilities

for K were discussed in §3.3; using the exact formulas for

these K and that for $(h)(R_3 \text{ wr } R_3)$ eliminates the cases

$s = 2$, $h = 7, 8, 9, 10$, and in fact we have

$(h)R_9 < (h)(R_3 \text{ wr } R_3)$ for $h = 2, 3, 4, 5, 6$. This completes the case $q = 3$.

Finally, suppose $q = 2$. Then we consider separately five possibilities for s :

(i) $s = 2$. Then using the exact formulas from (2) we have $(h)R_4 < (h)(R_2 \text{ wr } R_2)$ for all $h \geq 2$.

(ii) $s = 3$. Then as in §3.3, the only possibility for K is R_8 . Again using the exact formulas, we have $(h)(R_2 \text{ wr } R_4) < (h)R_8$ for $h > 4$, and in fact $(h)R_8 < (h)(R_2 \text{ wr } R_4) \leq (h)(R_4 \text{ wr } R_2)$ for $h = 2, 3, 4$.

(iii) $s = 4$. Then by Suprunenko [14] Chapter 21 there are only two possibilities for K : the semilinear group $S(2^4)$ of order 960 and the group with imprimitive stabilizer of order 1152. Thus $(h)K > h^{16}/1152$. But $(h)(R_4 \text{ wr } R_4) < (5/27)^5 h^{16}$, and so $(h)K > (h)(R_4 \text{ wr } R_4)$ for all $h > 2$. For $h = 2$, $(h)(R_8 \text{ wr } R_2) = 55$, and $(h)K > 2^{16}/1152 > 56$, and thus the result is true.

(iv) s odd, $s > 3$. Then by (2) $(h)R_2 < h^2$ and $(h)R_4 < 5h^4/16$, and so we have

$$(h)(R_4^{((s-1)/2} \text{ wr } R_2) < [(5/16)^{(s-1)/2-1}/3] 2^s h^{2^s}$$

and $(h)K > 24^{1/3} h^{2^s} / (2^{3.06s})$. Thus we want

$$(16/5)^2 (4^{(s-1)/2} - 1)/3 > 2^{3.06s/24} 1/3$$

which is true for $s \geq 5$.

(v) s even, $s > 4$. Then as above we have

$$(h)(R_4^{(s/2)}) < (5/16) (4^{s/2} - 1)/3 h^{2^s}$$

and so as above we get $(h)K > (h)(R_4^{((s/2)-1)} \text{ wr } R_4)$.

This completes the proof of the theorem.

Now define $r(k, h) = \min\{(h)K \mid K \text{ is a transitive solvable subgroup of } S_k\}$. Then by the above theorem and the associativity of wreath products, $r(k, h) = (h)K$ for some K of the form $K = R_{k_1} \text{ wr } \dots \text{ wr } R_{k_s}$, where $s \geq 1$, each k_i is either prime or one of 4, 8, or 9, and $k = k_1 k_2 \dots k_s$.

5. A Bound on $r(k, h)$

Let $d_2 = d_3 = 8$, and $d_h = 4$ for all $h > 3$.

Let $b_h = \begin{cases} 1.45 & h = 2 \\ 1.49 & h = 3 \\ 1.48 & h > 3 \end{cases}$ and $a_h = [(r(d_h, h) + b_h)/24^{1/3}]^{1/d_h}$.

Let $\bar{r}(k, h) = 24^{1/3} (a_h)^k - b_h$, for all $k, h \geq 2$. Then we have a lower bound on the number of orbits of imprimitive solvable groups as follows:

Theorem 3.6 $\bar{r}(k, h) \leq r(k, h)$ for all $k, h \geq 2$.

Proof: By definition $r(k,h) = (h)K$ for some K of the form $K = R_{k_1} \text{ wr } \dots \text{ wr } R_{k_s}$ where $k = k_1 \dots k_s$. We use induction on s .

Suppose first that $s = 1$, so $K = R_k$. We consider the possibilities for h and K :

(i) $h = 2$ or 3 . Then $b_2 = 1.45$, $b_3 = 1.49$, $a_2 = 1.188$ and $a_3 = 1.5064$, and so by direct calculation we find that $\bar{r}(k,h) \leq r(k,h)$ for $h = 2, 3$ and $k = 2, 3, 4, 5, 7, 8, 9$ and 11 . Thus we need only consider $K = R_q$, q prime, $q > 11$. But $\bar{r}(q,h) = 24^{1/3} a_h^q - b_h < h^q / (q(q-1)) < r(q,h)$ for $q > 11$.

(ii) $h > 3$, $k = 4$. Then we have

$$\bar{r}(4,h) = 24^{1/3} [((r(4,h) + 1.48) / 24^{1/3})^{1/4}]^4 - 1.48 = r(4,h).$$

(iii) $h > 3$, $k = q$, q prime. Then using the exact formula for $r(4,h)$ from (2) we find that for $h > 3$, $r(4,h) + 1.48 < (h+1.5)^4 / 24$. Thus we have

$$\bar{r}(q,h) < 24^{1/3} [(h+1.5)^4 / 24^{4/3}]^{q/4}. \text{ But } r(q,h) > h^q / (q(q-1)).$$
Thus it is enough to show that

$$24^{1/3} q(q-1) < (24^{1/3} h / (h+1.5))^q$$

But this is true for $h > 3$ and $q \geq 7$, or $h > 5$ and $q = 5$, or $h > 13$ and $q = 3$, or $h > 7$ and $q = 2$. Then testing individually with the exact formulas the fifteen cases that remain, we find that the result is true for all $q \geq 2$, $h > 3$.

(iv) $h > 3$, $k = 8$ or 9 . Then by Theorem 3.8 we need only consider $h = 4$, $k = 8$, or $h = 4, 5, 6$, $k = 9$. Testing these four cases individually gives us the result.

Thus we have $\bar{r}(k, h) \leq r(k, h)$ for $s = 1$. Now suppose $s > 1$. Let $K' = R_{k_1} \text{ wr } \dots \text{ wr } R_{k_{s-1}}$, $k' = k_1 k_2 \dots k_{s-1}$, and $h' = (h)K'$, so $K = K' \text{ wr } R_{k_s}$, $(h)K = (h')R_{k_s}$, and we assume by induction that $\bar{r}(k', h) \leq r(k', h)$. Then we have $r(k, h) = (h)K = (h')R_{k_s} = r(k_s, h')$, and so

$$\begin{aligned} \bar{r}(k, h) &= 24^{1/3} a_h^{k_s} - b_h \\ &= 24^{1/3} [(\bar{r}(k', h) + b_h)/24^{1/3}]^{k_s} - b_h \\ &\leq 24^{1/3} [(r(k', h) + b_h)/24^{1/3}]^{k_s} - b_h \\ &< 24^{1/3} [(h' + 1.49)/24^{1/3}]^{k_s} - 1.45 \end{aligned}$$

Let $f(k_s, h') = 24^{1/3} [(h' + 1.49)/24^{1/3}]^{k_s} - 1.45$. Then we want $f(k_s, h') \leq r(k_s, h')$ so again we consider the possibilities for k_s :

(i) $k_s = 4$. Then $f(4, h') \leq r(4, h')$ if and only if $(h' + 1.49)^4/24 - 1.45 \leq (h'^4 + 6h'^3 + 11h'^2 + 6h')/24$, and this is true for $h' > 60$. Thus we need only consider $h' \leq 60$. But there are only thirty-six combinations of K' and h that give $(h)K' \leq 60$ (listed on the next page) and considering the exact formulas in each of these cases we find that they all give $\bar{r}(k, h) \leq (h)(K' \text{ wr } R_4)$.

h K' with $(h)K' \leq 60$ (from Table V)	
2	$R_2, R_3, R_4, R_5, R_7, R_8, R_9, R_{11}, R_2 \text{ wr } R_3, R_2 \text{ wr } R_5,$ $R_4 \text{ wr } R_3, R_3 \text{ wr } R_2, R_3 \text{ wr } R_3, R_3 \text{ wr } R_4, R_5 \text{ wr } R_2,$ $R_5 \text{ wr } R_3, R_7 \text{ wr } R_2, R_8 \text{ wr } R_2, R_2 \text{ wr } R_3 \text{ wr } R_2$
3	$R_3, R_4, R_5, R_2, R_2 \text{ wr } R_3, R_3 \text{ wr } R_2$
4	R_2, R_3, R_4
5, 6	R_2, R_3
7, 8, 9, 10	R_2

(ii) $k_s = q$, q prime. Then to show $f(q, h') \leq r(q, h')$ it is enough to show $f(q, h') \leq (h')^q / (q(q-1))$, or to show $24^{1/3} q(q-1) \leq (24^{1/3} h' / (h' + 1.49))^q$. But this is true for $h' \geq 3$ and $q \geq 11$, or $h' \geq 4$ and $q = 7$, or $h' \geq 6$ and $q = 5$, or $h' \geq 13$ and $q = 3$, or $h' \geq 8$ and $q = 2$. Thus (from Table V) we need only consider the following possibilities for K' , h and q :

h	K'	q
2	R_2	2, 3, 5, 7
2	R_3	2, 3, 5
2	R_4	2, 3, 5
2	R_5	2, 3
2	R_7	3
2	$R_2 \text{ wr } R_3$	3
2	R_8	3
3	R_2	2, 3
3	R_3	3
4	R_2	3

Considering the exact formulas in each of these cases, we

find that they all give $\bar{r}(k,h) < (h)(K' \text{ wr } R_q)$.

(iii) $k_s = 8$ or 9 . Then again we need only consider individually the cases where $k_s = 8$ and $h' = 3, 4$, or $k_s = 9$ and $h' = 3, 4, 5, 6$. As above, $\bar{r}(k,h) < (h)(K' \text{ wr } R_{k_s})$ in each of these cases.

This completes the proof of the theorem.

6. A Limiting Case

Let $d_2 = d_3 = 8$ and $d_h = 4$ for $h > 3$ as before.

Let $a'_h = [(r(d_h, h) + 1.5)/24^{1/3}]^{(1/d_h)}$. For $k > 1$, $h \geq 2$ let $\tilde{r}(k, h) = 24^{1/3} a_h^k - 1.5$. Then $\tilde{r}(k, h)$ acts as an upper bound for a limiting case as follows:

Theorem 3.9 For any $N > 0$ and $h \geq 2$ there exists some $k > N$ such that $r(k, h) \leq \tilde{r}(k, h)$.

Proof: Fix $h \geq 2$. Let $K_t = R_{d_h} \text{ wr } R_4^{(t-1)}$. Let

k_t be the degree of K_t , so $k_t = 4^{t-1} d_h$. Then

$r(k_t, h) \leq (h)K_t$, and we will prove by induction on t that

$$(h)K_t \leq \tilde{r}(k_t, h).$$

If $t = 1$, then $(h)K_t = (h)R_{d_h} = r(d_h, h)$, and

$\tilde{r}(k_t, h) = \tilde{r}(d_h, h) = 24^{1/3} (a'_h)^{d_h} - 1.5 = r(d_h, h)$ by the definition of a'_h . Thus $(h)K_1 \leq \tilde{r}(k_1, h)$.

Now suppose $t > 1$. Let $h' = (h)K_{t-1}$ and $\tilde{h}' = \tilde{r}(k_{t-1}, h)$, and we assume $h' \leq \tilde{h}'$. Note that since $t > 1$, $(h)K_{t-1} \geq (2)R_4 = 5$, and so $\tilde{h}' > 4$. Thus

$$\begin{aligned}
 (h)K_t &= (h)K_{t-1} \text{ wr } R_4 \\
 &= r(4, h') \\
 &\leq r(4, \tilde{h}') \\
 &= (\tilde{h}'^4 + 6\tilde{h}'^3 + 11\tilde{h}'^2 + 6\tilde{h}')/24 \\
 &< (\tilde{h}' + 1.5)^4/24 - 1.5 \\
 &= [(24^{1/3} a_h^{(k_{t-1})} - 1.5) + 1.5]^4/24 - 1.5 \\
 &= 24^{1/3} a_h^{(4k_{t-1})} - 1.5 \\
 &= \tilde{r}(k_t, h)
 \end{aligned}$$

Thus the result is true for all $t \geq 1$ and $h \geq 2$, and so the theorem is proved.

Thus by combining Theorem 3.8 and Theorem 3.9 we have that for any $h \geq 2$ and $N > 0$ there exists a transitive solvable subgroup K of S_k , where $k > N$, such that

$$24^{1/3} a_h^k - 1.5 \leq (h)K \leq 24^{1/3} a_h^k - 1.5$$

Table VI gives a comparison of the values of a_h and a'_h for small h .

Table VI. Values of a_h and a'_h .

h	a_h	a'_h
2	1.188069	1.188716
3	1.506405	1.506429
4	1.885414	1.886060
5	2.231148	2.231304
6	2.578356	2.578457
7	2.926171	2.926240
8	3.274136	3.274185
9	3.622078	3.622114
10	3.969935	3.969962

Note: The values of a_h and a'_h can be improved by taking higher values of d_h (which means the values of b_h will also increase, approaching 1.5). For example, taking $d_2 = 32$ gives $a_2 = [(r(32,2)+1.499)/24^{1/3}]^{1/32} = 1.188087117$ and $a'_2 = [(r(32,2)+1.5)/24^{1/3}]^{1/32} = 1.188087169$.

Theorem 3.8 is still true when this a_2 is used, but the number of exceptional cases that must be checked in the proof is much larger. The values of d_h used must be of the form $8 \cdot 4^t$ when $h = 2, 3$ and $4 \cdot 4^t$ when $h > 3$, for some $t \geq 0$, and it is possible that if large d_h are used in Theorem 3.8 there may be some exceptions involving $k < d_h$.

7. An Exact Formula for $r(k,h)$

Define $\lambda(k,h) = t$, where $r(k,h) = (h)(R_t \text{ wr } K')$, K' a transitive solvable subgroup of S_k , with $k = k't$, and t

maximal such that $r(k,h)$ can be written in this form.

So for example $\lambda(6,3) = 3$, since $(3)(R_3 \text{ wr } R_2) = 55$ and $(3)(R_2 \text{ wr } R_3) = 56$, and these are the only two maximal transitive solvable subgroups of S_6 .

Now given $h, k \geq 2$, let $h_0 = h$, $k_0 = k$, and $t_0 = \lambda(k_0, h_0)$. Let $k_{i+1} = k_i/t_i$, $h_{i+1} = r(t_i, h_i)$, and $t_{i+1} = \lambda(k_{i+1}, h_{i+1})$, $i=0,1,\dots,s-1$ where s is such that $k_s = 1$. Then by the definition of $\lambda(k,h)$, $r(k,h) = h_s$. Thus $\lambda(k,h)$ can be used to calculate $r(k,h)$ exactly for any k,h .

So let q_i be the (i) th prime, for all $i \geq 1$, and for any $k > 1$ let $s(k,i) = q_1^{a_1} \dots q_{i-1}^{a_{i-1}}$ where $k = q_1^{a_1} \dots q_m^{a_m}$ is the prime decomposition of k . Then we can calculate $\lambda(k,h)$ as follows:

Theorem 3.8 For any $k \geq 2$, $h \geq 2$ we have

$$\lambda(k,h) = \begin{cases} 4 & \text{if } 4 \mid k \\ 3 & \text{else if } 3 \mid k \\ 5 & \text{else if } 5 \mid k \\ 2 & \text{else if } 2 \mid k \\ q & \text{else if } q \text{ is the least prime with } q \mid k \end{cases}$$

with the following exceptions:

$$\lambda(k,2) = \begin{cases} 11 & \text{if } 11 \mid k, s(k,11) \mid 9,10,12,14, \text{ or } 16 \\ 7 & \text{else if } 7 \mid k, s(k,7) \mid 16 \text{ or } 90, \\ & \text{or } 4 \mid k, 8 \nmid k \\ 5 & \text{else if } 5 \mid k, s(k,5) \mid 12 \\ & \text{or } 16 \mid k \text{ and } 9,32 \nmid k \\ 9 & \text{else if } 9 \mid k, s(k,3) = 1,2,4, \text{ or } 16 \\ 8 & \text{else if } 8 \mid k \\ 13 & \text{else if } 13 \mid k, s(k,13) \mid 12 \\ 17 & \text{else if } 17 \mid k, s(k,17) \mid 6 \\ 19 & \text{else if } 19 \mid k, s(k,19) \mid 2 \end{cases}$$

$$\lambda(k,3) = \begin{cases} 7 & \text{if } 7 \mid k, s(k,7) = 4 \cdot 3 \cdot 5^\alpha \text{ or } 4 \cdot 5^\alpha \text{ } (\alpha > 0) \\ & \text{or } s(k,7) \mid 6 \\ 5 & \text{else if } 5 \mid k, s(k,5) = 16 \text{ or } s(k,5) \mid 12 \\ 8 & \text{else if } 8 \mid k \\ 9 & \text{else if } 9 \mid k \\ 11 & \text{else if } 11 \mid k, s(k,11) \mid 2 \text{ or } 3 \\ 13 & \text{else if } 13 \mid k, s(k,13) \mid 6 \end{cases}$$

$$\lambda(k,4) = \begin{cases} 9 & \text{if } 9 \mid k, 32 \nmid k \\ 5 & \text{else if } 5 \mid k, s(k,5) \mid 24 \\ 8 & \text{else if } 8 \mid k, k \equiv 2 \pmod{4}, \text{ and } 128,96 \nmid k \\ 7 & \text{else if } 7 \mid k, s(k,7) \mid 6 \\ 11 & \text{else if } 11 \mid k, s(k,11) \mid 2 \end{cases}$$

$$\lambda(k,5) = \begin{cases} 9 & \text{if } 9 \mid k, 4 \nmid k \\ 5 & \text{else if } 5 \mid k, s(k,5) \mid 6,8 \\ 7 & \text{else if } 7 \mid k, s(k,7) \mid 6 \end{cases}$$

$$\lambda(k,6) = \begin{cases} 5 & \text{if } 5 \mid k, s(k,5) \mid 18 \\ 9 & \text{else if } 9 \mid k, 4 \nmid k \\ 7 & \text{else if } 7 \mid k, s(k,7) \mid 2 \end{cases}$$

$$\lambda(k,7) = \begin{cases} 5 & \text{if } 5 \mid k, s(k,5) \mid 18 \\ 7 & \text{else if } 7 \mid k \end{cases}$$

$$\lambda(k,h) = \begin{cases} 5 & \text{if } 5 \mid k, s(k,5) \mid 6 \\ 7 & \text{else if } 7 \mid k, s(k,7) \mid 2 \end{cases}$$

for $h = 8, 9, 10$

$$\lambda(k,h) = 7 \quad \text{if } 7 \mid k, s(k,7) \mid 2$$

for $h = 11, 12, 13, 14, 15$

Note: Apart from the exceptions, using this theorem to produce K such that $r(k,h) = (h)K$ results in K having maximal order as a transitive solvable subgroup of S_k .

([10] Theorem 2).

Proof: If k is prime, or $k = 4$, or $k = 8$ and $h = 2, 3, 4$ or $k = 9$ and $h = 2, 3, 4, 5, 6$ then $r(k,h) = (h)R_k$ and so $\lambda(k,h) = k$. So suppose now that $k = t_1 t_2$, where t_i is either prime, or 4, or 8 with $h = 2, 3, 4$ or 9 with $h = 2, 3, 4, 5, 6$ and $t_1 \neq t_2$. Then to find $\lambda(k,h)$ we must compare $(h)(R_{t_1} \text{ wr } R_{t_2})$ and $(h)(R_{t_2} \text{ wr } R_{t_1})$.

Suppose first that $t_1 = 3$ and $t_2 = q$, q prime, $q > 3$. Then $r(3,h) = (h^3 + 3h^2 + 2h)/6$ and so

$$(h + 0.8)^3/6 < (h)R_3 < (h + 1)^3/6 \quad \text{for all } h \leq 2.$$

Similarly,

$$h^q/(q(q-1)) < (h)R_q < h^q/(q(q-1)) + h^{(q+1)}/2$$

and thus we have

$$(h)(R_3 \text{ wr } R_q) < [(h+1)^3/6]^q/q(q-1) + [(h+1)^3/6]^{(q-1)/2}$$

and

$$(h)(R_q \text{ wr } R_3) > (h^q/q(q-1) + 0.8)^3/6$$

and so we want

$$\begin{aligned} & [(h+1)^3/6]^q/(q(q-1)) [1 + q(q-1)/[(h+1)^3/6]^{(q-1)/2}] \\ & < [h^q/(q(q-1)) + 0.8]^3/6 \end{aligned}$$

Taking logarithms, it is enough if

$$\begin{aligned} & q \ln[(h+1)^3/6] - \ln[q(q-1)] + q(q-1)/[(h+1)^3/6]^{(q-1)/2} \\ & < 3 \ln[h^q/q(q-1) + 0.8] - \ln 6 \end{aligned}$$

or

$$\begin{aligned} & 3q \ln(h+1) - (q-1)\ln 6 - \ln[q(q-1)] + q(q-1)/[4.5^{(q-1)/2}] \\ & < 3q \ln(h) - 3 \ln[q(q-1)] \end{aligned}$$

or

$$\begin{aligned} & 3q[\ln(h+1) - \ln(h) - (\ln 6)/3] + \ln 6 + 2\ln[q(q-1)] \\ & + q(q-1)/[4.5^{(q-1)/2}] < 0 \end{aligned}$$

Now $\ln(h+1) - \ln(h) - (\ln 6)/3 < -0.19$ for $h \geq 2$. Thus we want

$$-0.57q + \ln 6 + 2\ln[q(q-1)] + q(q-1)/[4.5^{(q-1)/2}] < 0$$

and this is true for $q \geq 29$. Similarly, by considering $h > 2$ we find that the result is true for $h \geq 3$ and $q \geq 17$, for $h \geq 4$ and $q \geq 11$, for $h \geq 6$ and $q \geq 7$, and for $h \geq 16$ and $q \geq 5$. But this leaves twenty-five possibilities for q and h which we can check individually using the exact formulas, and we find that $(h)(R_3 \text{ wr } R_q) < (h)(R_q \text{ wr } R_3)$ except for the following cases:

$$(h)(R_5 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_5), h = 2, 3, \dots, 10$$

$$(h)(R_7 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_7), h = 2, 3, 4, 5$$

$$(h)(R_{11} \text{ wr } R_3) < (h)(R_3 \text{ wr } R_{11}), h = 2, 3$$

$$(2)(R_{13} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{13})$$

$$(2)(R_{17} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{17})$$

$$(2)(R_{19} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{19})$$

By similarly considering every other combination of t_1 and t_2 we find that

$$(h)(R_4 \text{ wr } R_3) \leq (h)(R_3 \text{ wr } R_4)$$

$$(h)(R_4 \text{ wr } R_2) \leq (h)(R_2 \text{ wr } R_4)$$

$$(h)(R_3 \text{ wr } R_2) \leq (h)(R_2 \text{ wr } R_3) \quad \text{for all } h \geq 2 \quad (5)$$

$$(h)(R_5 \text{ wr } R_2) \leq (h)(R_2 \text{ wr } R_5)$$

and otherwise

$$(h)(R_{t_1} \text{ wr } R_{t_2}) < (h)(R_{t_2} \text{ wr } R_{t_1}) \text{ whenever } t_1 < t_2$$

with the following exceptions:

- $$\begin{aligned}
& (h)(R_8 \text{ wr } R_t) < (h)(R_t \text{ wr } R_8) \quad h = 2, 3, 4, \text{ for all } t \\
& (h)(R_9 \text{ wr } R_t) < (h)(R_t \text{ wr } R_9) \quad h = 2, 3, 4, 5, 6, \quad t \neq 8 \\
& (2)(R_{11} \text{ wr } R_7) < (2)(R_7 \text{ wr } R_{11}) \\
& (2)(R_7 \text{ wr } R_5) < (2)(R_5 \text{ wr } R_7) \\
& (2)(R_{11} \text{ wr } R_5) < (2)(R_{11} \text{ wr } R_5) \\
& (h)(R_5 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_5) \quad h = 2, 3, \dots, 10 \\
& (h)(R_7 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_7) \quad h = 2, 3, 4, 5 \\
& (h)(R_{11} \text{ wr } R_3) < (h)(R_3 \text{ wr } R_{11}) \quad h = 2, 3 \\
& (2)(R_{13} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{13}) \\
& (h)(R_5 \text{ wr } R_4) < (h)(R_4 \text{ wr } R_5) \quad h = 2, 3, 4, 5 \\
& (h)(R_7 \text{ wr } R_4) < (h)(R_4 \text{ wr } R_7) \quad h = 2, 3 \\
& (2)(R_{11} \text{ wr } R_4) < (2)(R_4 \text{ wr } R_{11}) \\
& (2)(R_{13} \text{ wr } R_4) < (2)(R_4 \text{ wr } R_{13}) \\
& (h)(R_7 \text{ wr } R_2) < (h)(R_2 \text{ wr } R_7) \quad h = 2, 3, \dots, 16 \\
& (h)(R_{11} \text{ wr } R_2) < (h)(R_2 \text{ wr } R_{11}) \quad h = 2, 3, 4 \\
& (h)(R_{13} \text{ wr } R_2) < (h)(R_2 \text{ wr } R_{13}) \quad h = 2, 3 \\
& (2)(R_{17} \text{ wr } R_2) < (2)(R_2 \text{ wr } R_{17}) \\
& (2)(R_{19} \text{ wr } R_2) < (2)(R_2 \text{ wr } R_{19}) \\
& (2)(R_{23} \text{ wr } R_2) < (2)(R_{23} \text{ wr } R_2) \\
& (2)(R_{17} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{17}) \\
& (2)(R_{19} \text{ wr } R_3) < (2)(R_3 \text{ wr } R_{19})
\end{aligned}$$

Since none of these exceptions include primes greater than 23, it follows that for any prime $q > 23$, $\lambda(k, h) = q$ if and only if $q \mid k$ and $t \nmid k$ for any $t < q$. So we consider now under which circumstances $\lambda(k, h) = t$ although $q \mid k$, $q < t$.

Suppose first that $t = 5$. Then we want to know when $\lambda(k, h) = 5$ although k is divisible by some prime less than 5, and so we may assume $k = 2^a 3^b 5$, for some a, b such that $a + b > 0$. Let $d = a/2$ if a is even, and $(a-1)/2$ if a is odd, and let $c = a - 2d$. Then by (5) we know that

$r(2^a 3^b, h) = (h) (R_4^{(d)} \text{ wr } R_3^{(b)} \text{ wr } R_2^{(c)})$, except possibly when $h = 2, 3, 4$ and we may have $(h) (R_8 \text{ wr } K')$ for some K' , or $h = 2, 3, 4, 5, 6$ and we may have $(h) (R_9 \text{ wr } K')$ for some K' .

• Thus we wish to know when

$$r(k, h) = (h) (R_5 \text{ wr } R_4^{(d)} \text{ wr } R_3^{(b)} \text{ wr } R_2^{(c)}) \quad (6)$$

or possibly if $h = 2$,

$$r(k, 2) = (2) (R_5 \text{ wr } R_9 \text{ wr } R_4^{(d)} \text{ wr } R_3^{(b-2)} \text{ wr } R_2^{(c)}) \quad (7)$$

We consider the possibilities for d and b :

(i) $d > 1$. Then we know that the only possibilities for $(h) (R_5 \text{ wr } R_4) < (h) (R_4 \text{ wr } R_5)$ are $h = 2, 3, 4, 5$, and so

$$\begin{aligned} & (h) (R_4 \text{ wr } R_5 \text{ wr } R_4^{(d-1)} \text{ wr } R_3^{(b)} \text{ wr } R_2^{(c)}) \\ & < (h) (R_5 \text{ wr } R_4^{(d)} \text{ wr } R_3^{(b)} \text{ wr } R_2^{(c)}) \end{aligned}$$

for all $h > 5$, and thus (6) can only be true for $h = 2, 3, 4, 5$.

But $(h) (R_4^{(2)} \text{ wr } R_5) < (h) (R_5 \text{ wr } R_4^{(2)})$ for $h = 4, 5$, and so as above we need only consider $h = 2, 3$. But for $h = 2, 3$,

$$(h) (R_5 \text{ wr } R_4^{(2)}) < (h) (R_8 \text{ wr } R_5 \text{ wr } R_2) < (h) (R_4^{(2)} \text{ wr } R_5),$$

and so (6) is true for $k = 2^4 5$. But

$$(h)(R_8 \text{ wr } R_4 \text{ wr } R_5) < (h)(R_5 \text{ wr } R_4^{(2)} \text{ wr } R_2), h = 2, 3$$

and so (6) is not true for $k = 2^5 5$. Further

$$(h)(R_8 \text{ wr } R_4 \text{ wr } R_5 \text{ wr } R_2) < (h)(R_5 \text{ wr } R_4^{(3)}), h = 2, 3$$

and so as above (6) is not true for $d > 2$. Also

$$(3)(R_8 \text{ wr } R_3 \text{ wr } R_5 \text{ wr } R_2) < (3)(R_5 \text{ wr } R_4^{(2)} \text{ wr } R_3)$$

and so (6) is not true for $h = 3$, $d = 2$, and $b > 0$. Thus the only remaining possibility is $h = 2$, $d = 2$. But

$$(2)(R_8 \text{ wr } R_4 \text{ wr } R_3 \text{ wr } R_5) < (2)(R_5 \text{ wr } R_4^{(2)} \text{ wr } R_3 \text{ wr } R_2)$$

and

$$(2)(R_8 \text{ wr } R_3^{(2)} \text{ wr } R_5 \text{ wr } R_2) < (2)(R_5 \text{ wr } R_4^{(2)} \text{ wr } R_3^{(2)})$$

and so (6) is not true for $h = 2$, $d = 2$ unless $c + b = 0$; i.e. $k = 2^4 5$.

(ii) $d = 1$. As above we may assume $h < 6$. Then since $(h)(R_5 \text{ wr } R_4) < (h)(R_5 \text{ wr } R_5)$ for $h = 2, 3, 4, 5$, (6) is true for $k = 2^2 5$. Also, $(h)(R_5 \text{ wr } R_4 \text{ wr } R_3) < (h)(R_4 \text{ wr } R_3 \text{ wr } R_5)$ for $h = 2, 3, 4$, and so (6) is true for $k = 2^2 3^1 5$, $h = 2, 3, 4$, but not for $h = 5$. $(5)(R_5 \text{ wr } R_4 \text{ wr } R_2) < (5)(R_4 \text{ wr } R_5 \text{ wr } R_2)$ and $(4)(R_5 \text{ wr } R_4 \text{ wr } R_2) < (4)(R_8 \text{ wr } R_5)$, but for $h = 2, 3$, $(h)(R_8 \text{ wr } R_5) < (h)(R_5 \text{ wr } R_4 \text{ wr } R_2)$ and so (6) is true for

$k = 2^3 5$ when $h = 4, 5$ but not when $h = 2, 3$. Thus the only possibility left to consider is $b > 1$. But since
 $(5)(R_4 \text{ wr } R_3 \text{ wr } R_5) < (5)(R_5 \text{ wr } R_4 \text{ wr } R_3)$, (6) is not true for $h = 5$, $b > 0$. Also,

$$(h)(R_9 \text{ wr } R_4 \text{ wr } R_5) < (h)(R_5 \text{ wr } R_4 \text{ wr } R_3^{(2)})$$

for $h = 2, 3, 4$ and so (6) is not true for $b > 1$.

(iii) $d = 0$, $b > 0$. Then we know that

$(h)(R_5 \text{ wr } R_3) < (h)(R_3 \text{ wr } R_5)$ if and only if $h < 11$, and so (6) is true for $b = 1$ and $h < 11$, and we may assume now $h < 11$. Now, $(6)(R_5 \text{ wr } R_3^{(2)}) < (6)(R_9 \text{ wr } R_5)$ but
 $(h)(R_9 \text{ wr } R_5) < (h)(R_5 \text{ wr } R_3^{(2)})$ for $h = 2, 3, 4, 5$, and thus

$$\begin{aligned} (h)(R_9 \text{ wr } R_5 \text{ wr } R_3^{(b-2)} \text{ wr } R_2^{(c)}) \\ < (h)(R_5 \text{ wr } R_3^{(b)} \text{ wr } R_2^{(c)}) \end{aligned}$$

for all $b > 1$, $h = 2, 3, 4, 5$, and (6) is not true in these cases. So we need now only consider $h = 6, 7, 8, 9, 10$. But
 $(h)(R_3^{(2)} \text{ wr } R_5) < (h)(R_5 \text{ wr } R_3^{(2)})$ for $h = 8, 9, 10$, and so we need only consider $h = 6, 7$. We have

$(7)(R_5 \text{ wr } R_3^{(2)}) < (7)(R_3^{(2)} \text{ wr } R_5)$, and so (6) is true for $k = 3^2 5$ and $h = 6, 7$. Also for $k = 2^1 3^2 5$ we have

$(6)(R_5 \text{ wr } R_3^{(2)} \text{ wr } R_2) < (6)(R_9 \text{ wr } R_5 \text{ wr } R_2)$ and
 $(7)(R_5 \text{ wr } R_3^{(2)} \text{ wr } R_2) < (7)(R_3^{(2)} \text{ wr } R_5 \text{ wr } R_2)$, and so (6) is true. But for $b > 2$ we have

(6) $(R_9 \text{ wr } R_3 \text{ wr } R_5) < (6) (R_5 \text{ wr } R_3^{(3)})$ and
 (7) $(R_3^{(3)} \text{ wr } R_5) < (7) (R_5 \text{ wr } R_3^{(3)})$, and so (6) is not true
 for $b > 2$.

(iv) $d = 0, b = 0$. But then the only possibility is
 $c = 1$, and we know $(h) (R_5 \text{ wr } R_2) < (h) (R_2 \text{ wr } R_5)$ for all h .
 Thus (6) is always true in this case.

By a similar investigation we find that no new
 possibilities for λ are added by (7). Thus by collecting
 all the cases above when (6) is true, we find that
 $\lambda(k, h) = 5$ for $k = 2^a 3^b 5$ only as follows:

k	h
$2^1 5$	any h
$2^2 5$	2, 3, 4, 5
$2^4 5$	2, 3
$2^4 3^1 5$	2
$2^2 3^1 5$	2, 3, 4
$2^3 5$	4, 5
$2^3 3^1 5$	4
$2^1 3^1 5$	2, 3, ..., 10
$3^2 5$	6, 7
$2^1 3^2 5$	6, 7

Similar analyses for $t = 2, 3, 4, 7, 8, 9, 11, 13, 17, 19, 23$
 complete the proof of the theorem.

Table VII. Values of $r(k,h)$ less than 1000

h	k	$r(k,h)$	h	k	$r(k,h)$	h	k	$r(k,h)$
2	2	3	3	7	78	29	2	435
2	3	4	12	2	78	13	3	455
2	4	5	7	3	84	4	7	460
2	5	6	13	2	91	6	5	462
3	2	6	14	2	105	2	25	462
2	7	10	2	18	105	30	2	465
2	8	10	8	3	120	2	22	465
3	3	10	15	2	120	9	4	495
2	6	10	6	4	126	31	2	496
4	2	10	2	20	126	4	8	496
2	9	14	16	2	136	2	17	522
3	4	15	3	9	140	32	2	528
5	2	15	17	2	153	14	3	560
4	3	20	9	3	165	2	27	560
6	2	21	18	2	171	33	2	561
2	10	21	19	2	190	34	2	595
3	5	24	5	5	201	35	2	630
7	2	28	20	2	210	5	6	630
2	11	30	4	6	210	36	2	666
4	4	35	7	4	210	15	3	680
2	12	35	10	3	220	3	12	680
5	3	35	2	21	220	37	2	703
8	2	36	2	24	220	10	4	715
9	2	45	21	2	231	2	28	715
10	2	55	22	2	253	2	32	715
2	14	55	23	2	276	38	2	741
2	16	55	11	3	286	39	2	780
3	6	55	24	2	300	16	3	816
6	3	56	3	10	300	40	2	820
2	15	56	25	2	325	41	2	861
11	2	66	8	4	330	42	2	903
5	4	70	26	2	351	43	2	946
2	13	74	12	3	364	7	5	952
3	8	75	27	2	378	17	3	969
4	5	76	28	2	406	44	2	990

Chapter IV. The General Bound

1. The Product Action of the Wreath Product

We consider now another form of the wreath product of permutation groups, which is analogous to the action of the wreath product on a linear space. Let K, N be subgroups of $\text{Sym}(X)$, $\text{Sym}(Y)$ respectively, where $|X| = k$, $|Y| = n$. Let $G = \{(r_1, \dots, r_n; s) \mid r_i \in K, s \in N\}$. For the element $r = (r_1, \dots, r_n; s)$ of G define an action of r on X^n by

$$r(x_1, \dots, x_n) = (r_{s^{-1}(1)}^{-1}(x_{s^{-1}(1)}), \dots, r_{s^{-1}(n)}^{-1}(x_{s^{-1}(n)})).$$

Then for any $t = (t_1, \dots, t_n; s') \in G$ we have

$$rt = (r_{s'(1)}^{-1}t_1, \dots, r_{s'(n)}^{-1}t_n; ss') \in G,$$

and so G is a subgroup of $\text{Sym}(X^n)$, and is isomorphic as an abstract group to the wreath product $K \wr N$ in its imprimitive action on $X \times Y$. G is called the wreath product $K \wr N$ in its product action on X^n .

Theorem 4.1 Let N be a transitive subgroup of S_n , and K a primitive subgroup of $\text{Sym}(X)$, K not regular of prime order. Then $K \wr N$ is primitive in its product action on X^n .

Proof: L. Scott [13] p.329.

The relationship between the product action of the wreath product and the wreath product of a linear group and a permutation group can be seen in the following theorem:

Theorem 4.2 Let $\bar{G}$ be a primitive solvable permutation group acting on the set V , with stabilizer G a subgroup of the linear group $GL(V)$, so $\bar{G} = V \cdot G$. Suppose G has the form $G = H \text{ wr } K$, where K is a transitive solvable subgroup of S_k , and H is a primitive subgroup of $GL(W)$ for some space W such that $V = W^k$. Let $\bar{H}$ be the primitive permutation group $W \cdot H$ (as in Theorem 1.2). Then $\bar{G}$ is permutation isomorphic to the wreath product $\bar{H} \text{ wr } K$ in its product action on W^k .

Proof: Let $\bar{g} \in \bar{G}$. Then $\bar{g} = vg$ for some $v \in V$, $g \in G$, and for any $x \in V$, $\bar{g}(x) = vg(x) = g(x) + v$. Since $G = H \text{ wr } K$, $g = (h_1, \dots, h_k; r)$ for some $h_i \in H$, $r \in K$. Since $V = W^k$, $v = (w_1, \dots, w_k)$ for some $w_i \in W$. So define $\psi: \bar{G} \rightarrow \bar{H} \text{ wr } K$ by

$$\psi(\bar{g}) = (w_{r(1)}h_1, \dots, w_{r(k)}h_k; r),$$

where $w_{r(i)}h_i$ is in $\bar{H}$ since $\bar{H} = W \cdot H$. Then for any $x \in V$,

$x = (x_1, \dots, x_k) \in W^k$, we have (for $x'_i = x_{r^{-1}(i)}$, $i=1, \dots, k$)

$$\begin{aligned} \psi(\bar{g})(x) &= (w_{r(1)}h_1, \dots, w_{r(k)}h_k; r)(x_1, \dots, x_k) \\ &= (w_1h_{r^{-1}(1)}(x'_1), \dots, w_kh_{r^{-1}(k)}(x'_k)) \\ &= (h_{r^{-1}(1)}(x'_1) + w_1, \dots, h_{r^{-1}(k)}(x'_k) + w_k) \\ &= (h_1, \dots, h_k; r)(x_1, \dots, x_k) + (w_1, \dots, w_k) \\ &= g(x) + v \\ &= \bar{g}(x). \end{aligned}$$

Thus $\bar{G}$ and $\bar{H} \text{ wr } K$ are permutation isomorphic in $\text{Sym}(W^k)$.

2. Degrees with Small Rank

Now let $\bar{G}$ be a maximal solvable primitive permutation group of degree $d = p^n$ and rank r . We look at the cases in which r can be at most 10.

Let G be the stabilizer of $\bar{G}$, so G is a maximal solvable irreducible subgroup of $GL(n, p)$. Suppose first that G has a semilinear component, so $G = (T(p^m) \text{ wr } K)$ where K is a maximal solvable transitive subgroup of S_k , $k \geq 1$, and $n = km$. Then by Theorem 4.2 $\bar{G} = (S(p^m) \text{ wr } K)$. Further the rank of $\bar{G}$ is the number of orbits of G , so $r = 2$ if $k = 1$ and $r \geq r(k, 2)$ if $k > 1$. Thus by using Table VII we can list all such groups with rank at most 10:

Table VIII. Semilinear-Component Groups

Group	Rank
$S(p^m)$	2
$S(p^m) \text{ wr } R_2$	3
$S(p^m) \text{ wr } R_3$	4
$S(p^m) \text{ wr } R_4$	5
$S(p^m) \text{ wr } R_5$	6
$S(p^m) \text{ wr } R_7$	10
$S(p^m) \text{ wr } R_8$	10
$S(p^m) \text{ wr } R_2 \text{ wr } R_3$	10
$S(p^m) \text{ wr } R_3 \text{ wr } R_2$	10

Now suppose G has no semilinear component, and $\bar{G}$ has rank r . Then $G = H \text{ wr } K$, where H is a maximal solvable primitive non-semilinear subgroup of $GL(p^m)$, K is a maximal solvable transitive subgroup of S_k , $k \geq 1$ and $n = km$. Let h be the number of orbits of H , so $h \geq (r^*(p^m)+1)$. Then $r = h$ if $k = 1$ and $r = (h)K \geq r(k, h)$ if $k > 1$. Thus from Table III and Table VII we can compute the least possible rank of $\bar{G}$ given its degree p^n as follows:

Table IX. Non-Semilinear-Component Groups

Rank	Degree
2	$3^2, 5^2, 7^2, 11^2, 23^2$
3	$13^2, 17^2, 19^2, 29^2, 31^2, 47^2, 3^4, 5^4, 7^4, 11^4, 23^4, 2^6, 3^6$
4	$37^2, 41^2, 43^2, 53^2, 59^2, 71^2, 7^3, 5^6, 7^6, 11^6, 23^6, 3^{10}$
5	$61^2, 67^2, 79^2, 83^2, 3^8, 5^8, 7^8, 11^8, 23^8$ $(13^4, 2^{12}, 2^{15}, 3^{16}, 2^{18})^\dagger$
6	$97^2, 109^2, 113^2, 127^2, 131^2, 13^3, 17^4, 19^4, 29^4, 31^4, 47^4$ $7^8, 5^{10}, 7^{10}, 11^{10}, 23^{10}, 3^{12}$
7	$137^2, 139^2, 141^2, 151^2, 167^2, 19^3$
8	$157^2, 163^2, 173^2, 179^2, 191^2$
9	$181^2, 197^2, 199^2$
10	$193^2, 211^2, 223^2, 227^2, 239^2, 37^4, 41^4, 43^4, 53^4, 59^4$ $71^4, 13^6, 17^6, 19^6, 29^6, 31^6, 47^6, 5^{12}, 7^{12}, 11^{12}, 23^{12}$ $3^{14}, 5^{14}, 7^{14}, 11^{14}, 23^{14}, 5^{16}, 7^{16}, 11^{16}, 23^{16}, 3^{18}, 3^{20}$

† Lower estimates

3. The Main Theorem

Theorem 4.3 Let $\bar{G}$ be a maximal solvable primitive permutation group of rank r and degree d . Then either

$$r \leq 24^{1/3} d^{0.02747-1.5} \quad (1)$$

or $\bar{G}$ is permutation isomorphic to $S(p^m)$ wr K , where K is a maximal transitive solvable subgroup of S_k , $d = p^{mk}$, and

$$r > 24^{1/3} (1.188)^k - 1.5 \quad (2)$$

Proof: Let G be the stabilizer of $\bar{G}$ and suppose $G = H$ wr K , where H is a maximal solvable primitive subgroup of $GL(m, p)$, K is a maximal transitive solvable subgroup of S_k , $k \geq 1$, and $d = p^n$, $n = mk$ (by Theorem 1.1 and Theorem 3.3).

Suppose first that $H = T(p^m)$. Then by Theorem 4.2 $\bar{G}$ is permutation isomorphic to $S(p^m)$ wr K , and by Theorem 3.6 we have (2) (for $h = 2$, $a_h = 1.188$).

So now suppose $H \neq T(p^m)$. Let h be the number of orbits of H , and suppose $k > 1$. Then by Theorem 3.6,

$$r = (h)K \geq r(k, h) \geq 24^{1/3} a_h^k - 1.5$$

Thus it is enough to show that $d^{0.02747} < a_h^k$. But

$k = \log_{(p^m)}(d)$, and thus $a_h^k = d^{\ln(a_h)/\ln(p^m)}$. Thus it is enough to show that $0.02747 < \ln(a_h)/\ln(p^m)$, or

$$p < a_h^{(1/0.02747m)} \quad (3)$$

Now $r(4, h) > h^4/24$, and $r(8, h) > h^8/168 > h^8/(24^{(8-1)/3})$,
 and so $a_h = [(r(d_h, h) + b_h)/24^{1/3}]^{1/d_h} > h/24^{1/3}$ for all h .
 By Theorem 2.3 we know that $h > p^{m/2}/12m$ with the possible
 exceptions of $p^m = 17^4, 19^4, 7^6, 5^8, 7^8, 11^8, 13^8, 7^9, 3^{16}, 5^{16}$.
 Thus for (3) to hold it is enough if

$$p < [p^{m/2}/(24^{1/3}12m)]^{1/(0.02747m)}$$

or

$$p^{0.48253} > (24^{1/3}12m)^{1/m} \quad (4)$$

Now $(24^{1/3}12m)^{1/m}$ decreases with m for $m > 1$, and thus if
 (4) is true for $m = m_0$, it is true for all $m > m_0$. So
 consider first $m = 2$. Then (4) is true for $m = 2$ if and
 only if $p > 88.56$, or $p \geq p_2$ for $p_2 = 89$. Similarly for
 each $m \leq 20$ there is some p_m with (4) true for all $p \geq p_m$:

m	2	3	4	5	6	8	12	20
p_m	89	29	13	11	7	5	3	2

This gives sixty-four more possible exceptions.

So let us now consider the seventy-four possible
 exceptions. Since a_h increases as h increases we need only
 show that for each possible exception p^m there is some $h' > h$
 and some $m' > m$ such that p, m', h' satisfy (3). Thus,
 using Table 1, we can eliminate exceptions as follows:

h'	$a_{h'}$	m'	$a_{h'} (1/0.02747m')$	Exceptions Eliminated
2	1.188	2	23.003	$2^2, 3^2, 5^2, \dots, 23^2$
		2	1733.05	$29^2, 31^2, \dots, 83^2$
3	1.506	4	41.63	$2^3, 3^3, \dots, 23^3, 2^4, 3^4, \dots, 41^4$
		6	12.01	$2^5, 3^5, 5^5, 7^5, 2^6, 3^6, 5^6, 7^6$
4	1.885	10	10.07	$2^7, 3^7, 5^7, 2^8, 3^8, 5^8$ $2^9, 3^9, 5^9, 7^9, 2^{10}, 3^{10}$
		8	38.54	$11^8, 13^8$
5	2.231	16	6.21	5^{16}
		19	4.66	$3^{16}, 2^{11}, 2^{12}, \dots, 2^{19}$

Thus all of the possible exceptions have been eliminated, and the case $k > 1$ is complete.

So now suppose $k = 1$. Then $G = H$ wr $K = H$ since $K = S_1$, and so by Theorem 2.3 we have

$$r \geq r^*(p, n) + 1 > p^{n/2}/12n$$

with ten possible exceptions. Thus we want

$$p^{n/2}/12n > 24^{1/3} p^{0.02747n} - 1.5 \quad (5)$$

If $n \geq 20$, then (5) is true for all $p \geq 2$. For each $n < 20$ there exists some p_n such that for all $p \geq p_n$ (5) is true, and these p_n can be summarized as follows:

n	2	3	4	6	7	11	19
p_n	53	19	11	7	5	3	2

so we have forty-nine further possible exceptions.

So let us consider the fifty-nine possible exceptions. Since $24^{1/3}d^{0.02747}-1.5$ increases with d , let d_r be the greatest d having $r \geq 24^{1/3}d^{0.02747}-1.5$; i.e.

$d_r = [(r+1.5)/24^{1/3}]^{1/0.02747}$. Then by computing d_r and using Table IX we can eliminate possible exceptions as follows:

r	d_r	Exceptions Eliminated
2	1142.34	$2^2, 3^2, \dots, 31^2, 2^3, 3^3, 5^3, 7^3, 2^4, 3^4, 5^4$
3	10740371.	$37^2, \dots, 53^2, 7^4, 11^4, \dots, 31^4, 11^3, 13^3, 17^3, 19^3$ $2^5, 2^6, \dots, 2^{12}, 3^5, 3^6, \dots, 3^{12}, 5^8, 7^5, 7^6, 7^8$
4	1.59×10^{10}	$11^8, 13^8, 7^9, 2^{13}, 2^{14}, \dots, 2^{19}, 3^{16}$

Thus all possible exceptions have been eliminated and the case $k = 1$ is complete.

This completes the proof of the theorem.

Theorem 4.4 For any $N > 0$ there exists a maximal solvable primitive permutation group $\bar{G}$ of rank r and degree $d > N$ such that $\bar{G}$ has no initial semilinear component and

$$24^{1/3}d^{0.02747}-1.5 < r < 24^{1/3}d^{0.02757}-1.5$$

Proof: Let H be the maximal solvable primitive non-semilinear subgroup of $GL(2,23)$ as constructed in §2.3. Then the number of nontrivial orbits of H is $r^*(23,2,1) = 1$, and so H has 2 orbits. Let K_t be the group of degree $8 \cdot 4^{(t-1)}$ constructed in §3.6. Let $G = H \text{ wr } K_t$. Then G is a maximal irreducible solvable subgroup of $GL(4^{(t+1)}, 23)$. Let $\bar{G}$ be the maximal solvable primitive permutation group with stabilizer G as in Theorem 1.2. Then $\bar{G}$ has degree

$$= 23^{(4^{(t+1)})}$$

and the rank r of $\bar{G}$ is the number of orbits of G . Thus $r = (2)K_t$, and so for $m = \log_{(23^2)}(d) = 8 \cdot 4^{(t-1)}$,

$$r < 24^{1/3} a_h^m - 1.5 < 24^{1/3} (1.18872)^m - 1.5$$

and so

$$r < 24^{1/3} d^{\ln(1.18872)/\ln(23^2)} - 1.5$$

$$< 24^{1/3} d^{0.02757} - 1.5$$

and this completes the proof of the theorem.

Finally note that any subgroup of a maximal primitive solvable permutation group $\bar{G}$ has rank greater than or equal to the rank of $\bar{G}$. Thus by reversing (2) in Theorem 4.3 we have the following:

Theorem 4.5 Let $\bar{G}$ be a primitive solvable subgroup of S_d with no semilinear component. Then if r is the rank of $\bar{G}$, then we have $d < [(r+1.5)/24^{1/3}]^{1/0.02747}$.

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