Sample Size Determination for Markovian Queueing Models

by

Tianyi Dai

A thesis submitted to the Faculty of Graduate and Postdoctoral Affairs

in partial fulfillment of the requirements for the degree of

Master of Science

in

Statistics

Carleton University

Ottawa, Ontario

Abstract

In this thesis, we focus on the sample size of two variants of the standard M/M/1 queueing model. The reason is that variants of the standard M/M/1 queueing model are extensively used in the real world. There are many fields in which queueing models can be utilized. In these applications, parameter plays an extremely important role. For example, the paper (Choudhury and Borthakur, 2008) studied inference for parameters of the M/M/1 queueing model. Therefore, in order to guarantee the precision of parameters estimated in these queueing models, the sample size determination is proposed in this thesis. Firstly, we show how a Bayesian approach could be applied to these models to obtain the minimal sample size required by the given precision. Then, we will illustrate in detail how to use R, a statistical software, to compute the sample size for these models.

Acknowledgements

I would like to express my sincere gratitude to my advisor, Dr. Yiqiang Zhao, for the continuous support and guidance of my master's study.

Also, I would like to thank my family, especially my parents, for supporting me throughout my entire life.

Contents

Chapter 1
Introduction

This introduction chapter highlights the research problem for the thesis, main contributions made, and the organization of the thesis.

1.1. Problem statement

Queueing models are widely applied in the real world. For example, banks and hospitals are using queueing models to guarantee the efficiency of operations. Among all queueing models, the standard M/M/1 queueing model is the most basic one; however, in view of its characteristics, there are many fields in real life that can use this model. Therefore, we will study two variants of the standard M/M/1 queueing model in this thesis. Unlike the usual setting, our queueing models will not necessarily have known parameter systems. Therefore, knowing how to attain a precise estimator of the parameter is extremely important to our target because it would be futile to calculate the limiting probabilities without having the value of the estimator. In order to ensure the accuracy of the parameter estimation, the sample size determination has been proposed. It is well known that sample size plays an important role in statistical tests. In general, the larger the sample size is , the more accurate the estimated parameter will be in representing the true value of the parameter. However, studies have shown that if the sample size is larger than the minimal one required, it may cause the waste of scarce resources and lead to exposing more resources than necessary to any related risks. Therefore, studying an appropriate sample size determination is crucial to the field of statistics.

There are several methods we can use to acquire sample size. In this thesis, we will study how to use Bayesian methods to determine the sample size for estimating parameters for variants of the standard M/M/1 queueing model. With regard to the Bayesian method, the paper (Joseph and Belisle, 1997) showed that there are three commonly used criteria for Bayesian sample size determination: the average coverage criterion (ACC), the average length criterion (ALC), and the worst outcome criterion (WOC). Adcock (1988) also provided details of the derivation process of ACC. We will be using both the average coverage criterion and the worst outcome criterion in this thesis.

Before using these criteria, we should know two basic components. The first one that we should realize is how to obtain the posterior distribution; and the second one is the required sample size for estimation of the standard M/M/1 queueing system. For the first point, Robert (2001, pp. 117) had an explicit demonstration of relations between posterior distributions and the exponential family. For the second point, Quinino and Cruz (2017) accomplished this in their paper.

1.2. Main contributions

Quinino and Cruz (2017) only provided an approach for determining the sample size for the standard M/M/1 queueing system. In this thesis, we study what kind of change will happen for determining the sample size for the M/M/1 queue with balking, and for the M/M/c queueing system, and whether or not we could use the same method to obtain the required sample size for the M/M/1 with multiple vacations, and for the M/M/c queue with balking. Furthermore, we will use R, a different tool than Matlab, to analyze the sample size.

1.3. Organization of the thesis

This thesis is organized as follows:

Chapter 2 introduces some preliminaries, which are expressions for the exponential family of distributions, sample size determination, two ways of parameter estimation, and a review of two papers, which are closely related to this thesis.

Chapter 3 checks two models, the M/M/1 queue with balking and the M/M/c queueing system to determine if they belong to the exponential family.

Chapter 4 uses R to analyze the sample size of these two models that we studied in Chapter 3.

Chapter 5 studies two other models, the M/M/1 queue with multiple vacation and the M/M/c queue with balking, to check whether or not we could use the same process to attain the sample size.

Chapter 6 provides a conclusion.

Chapter 2
Preliminaries

In this chapter, we first introduce the exponential family of distributions, and some important properties of the family required for this thesis, including the conjugate prior. We then outline the three most commonly used methods for determining the minimal required sample size for a given precision level. We continue this chapter by reviewing two concepts in estimation, and we finally provide a literature review on two papers, which are closely related to this thesis.

2.1. Exponential family

In the paper of Quinino and Cruz (2017), they used the property of the exponential family to acquire the natural conjugate prior for the standard M/M/1 queueing model. Thus, for the M/M/1 queue with balking, and the M/M/c queueing system, this property still might be available.

Firstly, we should have some basic knowledge of the exponential family of distributions.

An exponential family is a set of probability distributions of a certain expression, which is selected for mathematical convenience. The exponential family is defined as the set of probability distributions whose probability density function (in the continuous case) or probability mass function (in the discrete case) can be expressed in the following form:

$$f(x|\theta) = C(\theta)h(x)\exp\{R(\theta) \cdot T(x)\},$$

where $C(\theta), h(x), R(\theta), T(x)$ are known functions, and θ is named as the parameter of the exponential family.

There is a special case of the exponential family, which we call the natural exponential family. According to Robert (2001, pp. 117), it can be rewritten as

$$f(x|\theta) = h(x)\exp\{\theta \cdot x - \psi(\theta)\}.$$

In this form, $\psi(\theta)$ is a function of θ.

There are many distributions which belong to the exponential family.

Example 1 Let a random variable X be normally distributed with mean μ and variance σ^2. If σ^2 is known, but μ is unknown, then

$$\begin{aligned} f(x|\mu) &= \frac{1}{\sigma}\frac{1}{(2\pi)^{1/2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sigma}\frac{1}{(2\pi)^{1/2}}\exp\left\{x\left(\frac{\mu}{\sigma^2}\right) - x^2\frac{1}{2\sigma^2} - \mu^2\frac{1}{2\sigma^2}\right\} \\ &= \frac{1}{\sigma}\frac{1}{(2\pi)^{1/2}}\exp\left(-x^2\frac{1}{2\sigma^2}\right)\exp\left\{x\left(\frac{\mu}{\sigma^2}\right) - \mu^2\frac{1}{2\sigma^2}\right\}. \end{aligned}$$

We let $\theta = \frac{\mu}{\sigma^2}$, so $f(x|\mu) = \frac{1}{\sigma}\frac{1}{(2\pi)^{1/2}}\exp\left(-x^2\frac{1}{2\sigma^2}\right)\exp\left\{x\theta - \frac{\sigma^2}{2}\theta^2\right\}$, which also shows that $h(x) = \frac{1}{\sigma}\frac{1}{(2\pi)^{1/2}}\exp\left(-x^2\frac{1}{2\sigma^2}\right), \psi(\theta) = \frac{\sigma^2}{2}\theta^2$.

Example 2 If a random variable $X \sim P(\lambda)$, where $P(\lambda)$ is the Poisson distribution with parameter λ, then

$$f(x|\lambda) = e^{-\lambda}\frac{\lambda^x}{x!} = \frac{1}{x!}e^{-\lambda}e^{ln\lambda^x} = \frac{1}{x!}e^{xln\lambda-\lambda}.$$

We let $\theta = ln\lambda$, so $f(x|\lambda) = \frac{1}{x!}e^{\theta x - e^{\theta}}$, which also shows that $\psi(\theta) = exp\,(\theta)$.

2.2. Conjugate prior

Now that we understand what an exponential family is, we can now learn how to acquire the conjugate prior, given the information we have. In the paper of Quinino and Cruz (2017), they used the exponential family of distributions to infer the conjugate prior of distributions. We can still use this method in this thesis.

Definition of conjugate prior: If the prior and the posterior distribution belong to the same family of distributions, they are conjugate distributions. For this case, the prior distribution is also called the conjugate prior for the likelihood function.

In order to specify the prior and the posterior distributions, we need the following proposition.

Proposition 1 (Robert, 2001, pp. 120) For the natural exponential family

$$f(x|\theta) = h(x)\exp\{\theta \cdot x - \psi(\theta)\},$$

a conjugate prior is given by

$$\pi(\theta) = \varphi(\theta|\mu, \lambda) = K(\mu, \lambda)e^{\theta\mu - \lambda\psi(\theta)},$$

and the corresponding posterior is

$$\pi(\theta|x) = \varphi(\theta|\mu + x, \lambda + 1).$$

In this proposition, $K(\mu, \lambda)$ is the normalizing constant of the density.

Proof: Based on Bayes' theorem, we know that

$$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{\int_\Theta f(x|\theta)\pi(\theta)d\theta}.$$

Then, we have

$$f(x|\theta)\pi(\theta) = h(x)e^{\theta \cdot x - \psi(\theta)}K(\mu, \lambda)e^{\theta\mu - \lambda\psi(\theta)} = h(x)K(\mu, \lambda)e^{\theta(x+\mu)-(1+\lambda)\psi(\theta)},$$

and

$$\int_\Theta f(x|\theta)\pi(\theta)d\theta = h(x)K(\mu, \lambda)\int_\Theta e^{\theta(x+\mu)-(1+\lambda)\psi(\theta)}d\theta$$

$$= h(x)K(\mu,\lambda) \int_{\Theta} \frac{\pi(\theta)}{K(x+\mu,1+\lambda)} d\theta$$

$$= \frac{h(x)K(\mu,\lambda)}{K(x+\mu,1+\lambda)}.$$

Thus,

$$\pi(\theta|x) = \frac{h(x)K(\mu,\lambda)e^{\theta(x+\mu)-(1+\lambda)\psi(\theta)}}{h(x)K(\mu,\lambda)K(x+\mu,1+\lambda)^{-1}} = K(x+\mu,1+\lambda)e^{\theta(x+\mu)-(1+\lambda)\psi(\theta)}.$$

This formula shows that both the posterior distribution and the prior distribution belong to the natural exponential family.

2.3. **Sample size determination (SSD)**

According to Cao (2009), there are three criteria commonly used for determining the sample size.

Average length criterion (ALC)

ALC was introduced by Joseph and Belisle (1997). The first formula is to determine the length of the posterior credible interval for the fixed coverage, or $1-\alpha$ for data x

$$\int_{a(x,n)-\frac{l'(x,n)}{2}}^{a(x,n)+\frac{l'(x,n)}{2}} f(\theta|x,n)\, d\theta = 1-\alpha,$$

where $l'(x,n)$ is the length of the $100(1-\alpha)\%$ posterior credible interval for data x, and $a(x,n)$ is referred to as the function related to x and n.

Then, the second formula is to find the smallest integer n such that

$$\int_{\chi} l'(x,n)f(x)\, dx \le l,$$

where χ is the data space for x, and $f(x)$ is the marginal prior distribution of x.

Average coverage criterion (ACC)

ACC, proposed by Adcock (1988), is to fix the posterior interval length l.

The purpose of ACC is to find the smallest integer n such that

$$\int_{\chi} \left\{ \int_{a(x,n)-\frac{l}{2}}^{a(x,n)+\frac{l}{2}} f(\theta|x,n)\, d\theta \right\} f(x)\, dx \ge 1-\alpha.$$

According to Adcock (1988), we can obtain the expression of n based on the following steps.

First, we need to define the data space χ for x as

$$\chi(x) = [E(\theta|x) - d, E(\theta|x) + d],$$

where d is specified. Sometimes, the data x will rely on the unknown parameter σ^2 or both μ and σ^2, where μ and σ^2 are both unknown parameters. In the rest of thesis, we assume that data x is distributed as the normal distribution $\mathrm{N}(\mu, \sigma^2)$. In other words, $\chi(x) = \chi(x, \sigma^2)$. Thus, the equation, $\chi(x) = [E(\theta|x) \pm d]$, indicates that we have already known σ^2. In the rest of the thesis, we assume that $\sigma^2 = 1$. Second, let data $x_i, i = 1,2, \ldots, n$ be a random sample from the normal distribution $N(\theta, 1)$. Then, the distribution of the sample mean $\bar{X}$ is $N(\theta, \frac{\sigma^2}{n})$. Third, we suppose that θ has the informative prior distribution $N\left(\theta_0, \frac{\sigma^2}{m}\right), m > 0$, where m is a measure of the quantity of prior information.

Using these three steps, we will obtain two distributions:

First, $\theta|\bar{X} \sim N(\lambda, \frac{\sigma^2}{n+m})$, where $\lambda = \frac{n\bar{X}+m\theta_0}{n+m}$. It can be inferred directly from the paper of Gopalan (2015). Second, $\bar{X} \sim N(\theta_0, \frac{\sigma^2}{p})$, where $\frac{1}{p} = \frac{1}{n} + \frac{1}{m}$. $(E(\bar{X}) = E[E(\bar{X}|\theta)] = E(\theta) = \theta_0, Var(\bar{X}) = E[Var(\bar{X}|\theta)] + Var[E(\bar{X}|\theta)] = E\left(\frac{\sigma^2}{n}\right) + Var(\theta) = \frac{\sigma^2}{n} + \frac{\sigma^2}{m})$.

Additionally, we can rewrite the ACC function,

$$\int_{\chi} \left\{ \int_{a(x,n)-\frac{l}{2}}^{a(x,n)+\frac{l}{2}} f(\theta|x,n)\, d\theta \right\} f(x)\, dx \geq 1 - \alpha$$

to

$$E_{\bar{x}} P[\theta \in \chi(x)|\bar{X}] = 1 - \alpha.$$

Also, the expectation of $P[\theta \in \chi(x)|\bar{X}]$ is trivial because it does not rely on $\bar{X}$. Thus,

$$E_{\bar{x}} P[\theta \in \chi(x)|\bar{X}] = P[\theta \in \chi(x)|\bar{X}] = P[|\theta - \lambda| \leq d|\bar{X}] = 1 - \alpha.$$

Thus, the sample size n could be calculated from

$$n + m \geq \frac{Z_{\alpha/2}^2 \sigma^2}{d^2}.$$

Worst outcome criterion (WOC)

WOC, also introduced by Joseph and Belisle (1997), ensures the desired coverage rate and interval length over all possible datasets, and it constitutes a conservative sample size.

WOC sample size is decided by the smallest integer n that satisfies

$$\inf_{x \in \mathcal{J}} \left\{ \int_{a(x,n)-\frac{l}{2}}^{a(x,n)+\frac{l}{2}} f(\theta|x,n)\, d\theta \right\} \geq 1 - \alpha,$$

where $\mathcal{J}$ is an appropriately chosen subset from the data space χ for x.

According to Joseph and Belisle (1997), we could also attain the expression of sample size *n* based on the following steps. The step that we are going to use here is the same as that for ACC's. The only difference is for $\inf_{x \in \mathcal{J}}\{P[\theta \in (\lambda - d, \lambda + d)|\bar{X}]\}$. In the current case, $P[\theta \in (\lambda - d, \lambda + d)|\bar{X}]$ does not depend on data *x*. Thus,

$$\inf_{x \in \mathcal{J}}\{P[\theta \in (\lambda - d, \lambda + d)|\bar{X}]\} = P[\theta \in (\lambda - d, \lambda + d)|\bar{X}] = 1 - \alpha.$$

In the end, we can obtain the expression of sample size *n*, which is exactly the same as ACC's

$$n + m \geq \frac{Z_{\alpha/2}^2 \sigma^2}{d^2}.$$

2.4. **Parameter estimation**

There are two classical methods we can take to estimate the parameter. The first one is point estimation. We all know the definition of point estimation; however, point estimation has a few disadvantages. Firstly, not every parameter has an unbiased estimator (Recall that: $\hat{\theta}$ is an unbiased estimator of the parameter θ if $E(\hat{\theta}) = \theta$). Secondly, it requires a very large sample size. However, sometimes it is impossible to obtain enough data (Recall that: A sequence of estimator $\hat{\theta}_n$ is said to be consistent iff it converges in probability to the true value of the parameter, i.e. $\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| > \varepsilon\right) \to 0$).

The second method is interval estimation. There are two important kinds of interval estimations. The first kind is the confidence interval, which is also a frequentist method. The most important characteristic of this frequentist method is that it does not require the prior or posterior distribution. The second kind of interval estimation is that of a credible interval, which is included in the Bayesian method. There are some merits to using the Bayesian method. Firstly, unlike the frequentist method the Bayesian method offers a natural way to absorb data into the prior distribution and posterior distribution. Secondly, the Bayesian method could provide an explainable answer of any parameters that we estimate. Thirdly, the condition for a large number of models is simple.

In this thesis, we will use the Bayesian method to estimate the sample size.

2.5. Literature review 1

We will discuss the M/M/1 queueing system with multiple vacations in Chapter 5, so a more basic model of the M/M/1 queueing system with multiple vacations will be introduced here. The model, also known as the MWV model from Selvaraju, and Goswami (2013), deals with multiple working vacations.

Notation

λ is the arrival rate based on the Poisson Process. μ_b is the service rate during the normal busy period, which is exponentially distributed. Then we assume that the system is stable, equivalently, $\rho = \frac{\lambda}{\mu_b} < 1$. Moreover, once the system becomes empty, the server will take a working vacation. μ_v is the service rate during the working vacation, which is exponentially distributed, with $\mu_v < \mu_b$. θ is the rate of the duration of a working vacation, which is exponentially distributed as well. On the one hand, the server continues to take vacations until there is at least one customer waiting in the system. Meanwhile, the server changes the rate from μ_v to μ_b, and starts the busy period. On the other hand, if there are no customers waiting in the system when the server completes the vacation, then the server takes another working vacation instantly. The server returns to serving customers by changing the service rate from a lower one to a higher one, and if, upon the completion of a vacation, there is at least one customer waiting in the system for receiving service, the server serves customers at the higher rate until the next vacation.

The customer who is in the system is assumed to be impatient, and this impatience is illustrated as follows. ξ is the exponential rate of an impatient timer *T*, which the customer activates whenever they arrive to the system in the working vacation. Also, *T* is independent of the number of customers in the system at any given moment. The customer continues to stay in the system if the server finishes the working vacation before *T* expires, whereas the customer leaves the system.

From the above information, there are two points that we should know. Firstly, only during the period of working vacation, the customer is impatient. Secondly, the customer begins getting service immediately upon its arrival, and does not become impatient when the customer arrives in the system, and finds the server free during the working vacation.

Thus, Selvaraju and Goswami (2013) established the following model.

Model

Based on the information we described above, the system can be modeled by a continuous time Markov Chain with two-dimensional discrete states, denoted by $\Delta = \{(N_t, J_t), t \geq 0\}$, where N_t represents the total number of customers in the system, and J_t represents the state of the server. That is,

$$J_t = \begin{cases} 1, & \text{the server is in the busy period at time } t \\ 0, & \text{the server is in working vacation at time } t \end{cases}.$$

The state space of the Markov Chain is $E = \{\{(0,0)\} \cup \{(i,j)\}, i = 1,2, \ldots, j = 0,1\}$.

Stationary Distribution

The stationary probability can be defined as $p_{i,j} = lim_{t\to\infty} P\{N_t = i, J_t = j\}, i = 0,1,2, \ldots, j = 0,1$. Therefore, the balance equations can be established as follows:

$$\lambda p_{0,0} = \mu_v p_{1,0} + \mu_b p_{1,1}, k = 0,$$

$$[\lambda + \mu_v + \theta + (k-1)\xi] p_{k,0} = \lambda p_{k-1,0} + (\mu_v + k\xi) p_{k+1,0}, k \geq 1,$$

$$(\lambda + \mu_b) p_{1,1} = \theta p_{1,0} + \mu_b p_{2,1}, k = 1,$$

$$(\lambda + \mu_b) p_{k.1} = \lambda p_{k-1,1} + \theta p_{k,0} + \mu_b p_{k+1,1}, k \geq 2.$$

Then, the partial probability generating function can be defined as

$$P_0(z) = \sum_{k=0}^{\infty} z^k p_{k,0}, P_1(z) = \sum_{k=1}^{\infty} z^k p_{k,1}, 0 < z < 1$$

with $P_0(1) + P_1(1) = 1, z = 1$.

2.6. Literature review 2

This model was studied in Quinino and Cruz (2017).

For the posterior distribution $\pi(\rho|x)$, and sample size n, if we want to estimate ρ with a posterior $100(1-\alpha)\%$ credible region, we could seek the smallest n such that the width w of the posterior distribution $\pi(\rho|x)$ has a coverage probability of at least $1-\alpha$.

For instance, we could let $d = \frac{w}{2}$, and seek the smallest n such that

$$\int_{\hat{\rho}-d}^{\hat{\rho}+d} \pi(\rho|x)\, d\rho \geq 1-\alpha,$$

where $\hat{\rho}$ represents the mean of the posterior distribution..

Then, according to the ACC and WOC, we obtain

ACC

$$\sum_{\forall x}\left[\int_{\hat{\rho}-d}^{\hat{\rho}+d}\pi(\rho|x)\,d\rho\right]f(x)\geq 1-\alpha,$$

WOC

$$\inf_{x}\left[\int_{\hat{\rho}-d}^{\hat{\rho}+d}\pi(\rho|x)\,d\rho\right]\geq 1-\alpha.$$

Chapter 3
M/M/1 queue with balking and M/M/c queue

In this chapter, we use the method described in the previous chapters to determine the sample size with a $100(1-\alpha)\%$ credible region for two variants of the standard M/M/1 queue.

3.1. The First Model: M/M/1 queue with balking (Ross, 2014, pp. 500):

In the M/M/1 queuing system, upon arrival a customer will join the system with probability $\alpha_k = \alpha\ (k \geq 1), \alpha_0 = 1$. Thus, a customer will leave the system with probability $1-\alpha_k = 1-\alpha$. Now, let $\lambda_0 = \lambda,$ and $\lambda_k = \alpha\lambda\ (k \geq 1)$ denote the arrival rate and let $\mu_k = \mu\ (k \geq 1)$ denote the service rate when there are k customers in the system.

Furthermore, let P_k denote the limiting probability when there are k customers in the system.

Then, we can get the balance equation as follows:

$$\lambda P_0 = \mu P_1, k = 0,$$

$$(\lambda\alpha + \mu)P_1 = \lambda P_0 + \mu P_2, k = 1,$$

$$(\lambda\alpha + \mu)P_k = \lambda\alpha P_{k-1} + \mu P_{k+1}, k \geq 2.$$

Therefore,

$$\mu P_1 = \lambda P_0, k = 0,$$

$$\mu P_{k+1} = \lambda\alpha P_k, k \geq 1.$$

Then, for $\rho = \frac{\alpha\lambda}{\mu}, \rho < 1$,

$$P_k = \frac{\lambda}{\mu}\alpha P_{k-1} = \frac{\lambda\lambda}{\mu\mu}\alpha\alpha P_{k-2} = \left(\frac{\lambda}{\mu}\right)^k \alpha^{k-1}P_0 = \rho^k\alpha^{k-1}P_0, k \geq 1.$$

Using the fact that $\sum_{n=0}^{\infty} P_k = 1$, we obtain

$$P_0 + \sum_{k=1}^{\infty} \rho^k\alpha^{k-1}P_0 = 1.$$

Thus,

$$P_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \rho^k\alpha^{k-1}} = \frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)}$$

and

$$P_k = \rho^k\alpha^{k-1}\frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)}, k \geq 1.$$

Now, we can define the sample as $x = \{x_1, x_2, \cdots, x_n\}$, where x_i represents the number of customers left behind by a departing customer. We assume that $y = \sum_{i=1}^{n} x_i$.

The relation between the exponential family and the M/M/1 queue with balking

We need to determine whether or not P_k can be converted into the form of $h(k)\exp\{\theta k - \psi(\theta)\}$.

First, $h(k)$ is a function of *n*, so $h(k) = \alpha^{k-1}$. Second,

$$\ln\left[\rho^k\frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)}\right] = k\ln\rho - \{\ln[1 + \rho(1 - \alpha)] - \ln(1 - \rho\alpha)\}.$$

If assume that $\theta = \ln\rho$, then $\rho = e^\theta$. Thus,

$$\ln\left[\rho^k\frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)}\right] = \theta k - \ln\frac{1 + e^\theta(1 - \alpha)}{1 - e^\theta\alpha}.$$

Then we obtain

$$\rho^k\frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)} = \exp\left\{\ln\left[\rho^k\frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)}\right]\right\} = \exp\left\{\theta k - \ln\frac{1 + e^\theta(1 - \alpha)}{1 - e^\theta\alpha}\right\},$$

where $\psi(\theta) = \ln\frac{1+e^\theta(1-\alpha)}{1-e^\theta\alpha}$.

Therefore, we know that the original equation,

$$P_k = \rho^k\alpha^{k-1}\frac{1 - \rho\alpha}{1 + \rho(1 - \alpha)},$$

could be converted into

$$P_k = \alpha^{k-1}\exp\left\{\theta k - \ln\frac{1+e^{\theta}(1-\alpha)}{1-e^{\theta}\alpha}\right\},$$

which implies that P_k belongs to the exponential family.

We have defined the sample x, and the size of the sample is n, so the corresponding likelihood function could be generated as follows

$$L(x|\rho) = \rho^{y}\alpha^{y-n}\left[\frac{1-\rho\alpha}{1+\rho(1-\alpha)}\right]^{n},$$

where $y = \sum_{i=1}^{n} x_i$. On the other hand, the natural conjugate prior distribution for ρ is inferred as follows

$$\pi(\rho) = \varphi(\rho|a,b) = K(a,b)e^{a\ln\rho - b\ln\frac{1+\rho(1-\alpha)}{1-\rho\alpha}} = K(a,b)\frac{\rho^{a}(1-\rho\alpha)^{b}}{[1+\rho(1-\alpha)]^{b}},$$

where constants $a > 0, b > 0$ and $K(a,b)$ is the normalizing constant of the density. Thus,

$$K(a,b) = \frac{1}{\int_0^1 \frac{\rho^{a}(1-\rho\alpha)^{b}}{[1+\rho(1-\alpha)]^{b}}d\rho}.$$

Since we have the conjugate prior distribution, the corresponding posterior distribution for ρ is

$$\pi(\rho|x) = K(a+y, b+n)\frac{\rho^{a+y}(1-\rho\alpha)^{b+n}}{[1+\rho(1-\alpha)]^{b+n}}.$$

3.2. The second model: the M/M/c queue (Ross, 2014, pp. 500):

The M/M/c queueing system is a c server system where customers arrive and enter service if any of the c servers are free. Customers will join the queue if all servers are busy. Let $\lambda_k = \lambda\ (k \geq 0)$ denote the arrival rate and $\mu_k = \begin{cases} k\mu, \text{if } k \leq c \\ c\mu, \text{if } k \geq c \end{cases}$ denote the service rate when there are k customers in the system. Now, let P_k denote the limiting probability when there are k customers in the system. Then, we can get the balance equations as follows:

$$\lambda P_0 = \mu P_1, k = 0,$$

$$(\lambda+\mu)P_1 = \lambda P_0 + 2\mu P_2, k = 1,$$

$$(\lambda + k\mu)P_k = \lambda P_{k-1} + (k+1)\mu P_{k+1}, k \leq c-1,$$

$$(\lambda + c\mu)P_k = \lambda P_{k-1} + c\mu P_{k+1}, k \geq c,$$

and obtain

$$\mu P_1 = \lambda P_0, k = 0,$$

$$2\mu P_2 = \lambda P_1, k = 1,$$

$$(k+1)\mu P_{k+1} = \lambda P_k, k \le c-1,$$

$$c\mu P_{k+1} = \lambda P_k, k \ge c.$$

Then, for $\rho = \frac{\lambda}{\mu}, \rho < 1$,

when $k \le c-1$,

$$P_{k+1} = \frac{1}{k+1}\frac{\lambda}{\mu}P_k = \frac{1}{k+1}\frac{1}{k}(\frac{\lambda}{\mu})^2 P_{k-1} = \frac{1}{(k+1)!}(\frac{\lambda}{\mu})^{k+1}P_0 = \frac{1}{(k+1)!}\rho^{k+1}P_0;$$

when $k \ge c$,

$$P_{k+1} = \frac{1}{c}\frac{\lambda}{\mu}P_k = \frac{1}{c^2}(\frac{\lambda}{\mu})^2 P_{k-1} = \frac{1}{c^{k+1-c}}(\frac{\lambda}{\mu})^{k+1-c}P_c = \frac{1}{c^{k+1-c}}(\frac{\lambda}{\mu})^{k+1-c}\frac{1}{c!}(\frac{\lambda}{\mu})^c P_0$$
$$= \frac{1}{c^{k+1-c}}\frac{1}{c!}(\frac{\lambda}{\mu})^{k+1}P_0 = \frac{1}{c^{k+1-c}}\frac{1}{c!}\rho^{k+1}P_0.$$

Using the fact that $\sum_{k=0}^{\infty} P_k = 1$, we obtain

$$\sum_{k=0}^{c}\frac{1}{k!}\rho^k P_0 + \sum_{k=c+1}^{\infty}\frac{1}{c^{k-c}}\frac{1}{c!}\rho^k P_0 = 1.$$

Thus,

$$\left(\sum_{k=0}^{c}\frac{1}{k!}\rho^k + \frac{c^c}{c!}\sum_{k=c+1}^{\infty}\frac{\rho^k}{c^k}\right)P_0 = 1,$$

$$\left(\sum_{k=0}^{c}\frac{1}{k!}\rho^k + \frac{c^c}{c!}\frac{\frac{\rho^{c+1}}{c^{c+1}}}{1-\frac{\rho}{c}}\right)P_0 = 1,$$

$$\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{1}{c!}\rho^c + \frac{1}{c!}\frac{\rho^{c+1}}{c-\rho}\right)P_0 = 1,$$

$$\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right)P_0 = 1,$$

$$P_0 = \frac{1}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}},$$

and

$$P_k = \begin{cases} \dfrac{\dfrac{\rho^k}{k!}}{\sum_{k=0}^{c-1}\dfrac{1}{k!}\rho^k + \dfrac{\rho^c}{c!}\dfrac{c}{c-\rho}}, k \le c, \\ \dfrac{\dfrac{\rho^k}{c^{k-c}}\dfrac{1}{c!}}{\sum_{k=0}^{c-1}\dfrac{1}{k!}\rho^k + \dfrac{\rho^c}{c!}\dfrac{c}{c-\rho}}, k \ge c+1. \end{cases}$$

Now, we can define the sample as $x = \{x_1, x_2, \cdots, x_n\}$, where x_i represents the number of customers left behind by a departing customer. Then we assume that $y = \sum_{i=1}^{n} x_i$.

The relation between the exponential family and the M/M/c queue

We need to detect whether or not P_k can be converted into the form of $h(k)\exp\{\theta k - \psi(\theta)\}$.

Firstly, (a) when $k \le c$,

$$h(k) = \frac{1}{k!};$$

(b) when $k \ge c+1$,

$$h(k) = \frac{1}{c^{k-c}}\frac{1}{c!}.$$

Secondly, (a) when $k \le c$,

$$\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}}\right] = k\ln\rho - \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right).$$

If assume that $\theta = \ln\rho$, then $\rho = e^{\theta}$. Thus,

$$\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}}\right] = \theta k - \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}e^{\theta k} + \frac{e^{\theta c}}{c!}\frac{c}{c-e^{\theta}}\right).$$

Then we obtain

$$\begin{aligned} \frac{\rho^k}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}} &= \exp\left\{\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}}\right]\right\} \\ &= \exp\left\{\theta k - \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}e^{\theta k} + \frac{e^{\theta c}}{c!}\frac{c}{c-e^{\theta}}\right)\right\}, \end{aligned}$$

where $\psi(\theta) = \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}e^{\theta k} + \frac{e^{\theta c}}{c!}\frac{c}{c-e^{\theta}}\right)$.

Therefore, we know that the original equation,

$$P_k = \frac{\frac{\rho^k}{k!}}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}},$$

could be converted into

$$P_k = \frac{1}{k!}\exp\left\{\theta k - \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}e^{\theta k} + \frac{e^{\theta c}}{c!}\frac{c}{c-e^{\theta}}\right)\right\},$$

which implies that $P_k\ (k \le c)$ belongs to the exponential family.

(b) when $k \ge c+1$,

$$\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}}\right] = k\ln\rho - \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right),$$

which is the same as $k \le c$. Thus,

$$\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}} = \exp\left\{\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}}\right]\right\}$$
$$= \exp\left\{\theta k - \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}e^{\theta k} + \frac{e^{\theta c}}{c!}\frac{c}{c-e^{\theta}}\right)\right\},$$

where $\psi(\theta) = \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}e^{\theta k} + \frac{e^{\theta c}}{c!}\frac{c}{c-e^{\theta}}\right)$.

Therefore, we know that the original equation,

$$P_k = \frac{\frac{\rho^k}{c^{k-c}}\frac{1}{c!}}{\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}},$$

could be converted into

$$P_k = \frac{1}{c^{k-c}}\frac{1}{c!}\exp\left\{\theta k - \ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}e^{\theta k} + \frac{e^{\theta c}}{c!}\frac{c}{c-e^{\theta}}\right)\right\},$$

which implies that $P_k\ (k \ge c+1)$ belongs to the exponential family.

We have defined sample x, and the size of the sample is n, and then we need to assume that $x_i \le c\ (i = 1,2,\dots,m)$ and $x_i \ge c+1\ (i = m+1, m+2, \dots, n)$, so the corresponding likelihood function could be generated as follows:

$$L(x|\rho) = \frac{1}{x_1!\,x_2!\cdots x_k!\,(c!)^{n-m}c^{x_{m+1}+\cdots+x_n-(n-m)c}}\frac{\rho^y}{(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho})^n},$$

where $y = \sum_{i=1}^{n} x_i$. On the other hand, the natural conjugate prior distribution for ρ could be inferred as follows

$$\pi(\rho) = \varphi(\rho|a,b) = K(a,b)e^{a\ln\rho - b\ln\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right)} = K(a,b)\frac{\rho^a}{\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right)^b},$$

where constants $a > 0, b > 0$ and $K(a,b)$ is the normalizing constant of the density. Thus,

$$K(a,b) = \frac{1}{\int_0^1 \frac{\rho^a}{\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right)^b} d\rho}.$$

Since we have the conjugate prior distribution, the corresponding posterior distribution for ρ could be obtained

$$\pi(\rho|x) = K(a+y, b+n)\frac{\rho^{a+y}}{\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right)^{b+n}}.$$

Chapter 4
Numerical Results

In order to use ACC and WOC to obtain sample size, we need to use R to acquire it. There are two main functions that we will use in this chapter. The first is **descdist**, which provides a skewness-kurtosis graph to help choose the best candidate to fit a given dataset. The second is **fitdist**, which estimates parameters and provides graphs for a given distribution. All codes are in the Appendix.

4.1. The first model

$$\pi(\rho) = K(a,b)\frac{\rho^a(1-\rho\alpha)^b}{[1+\rho(1-\alpha)]^b},$$

where

$$K(a,b) = \frac{1}{\int_0^1 \frac{\rho^a(1-\rho\alpha)^b}{[1+\rho(1-\alpha)]^b} d\rho}.$$

This model is a variant of standard M/M/1 queue, so we will use the value of a and b that R. C. Quinino, and F. R. B. Cruz (2016) proposed in their paper.

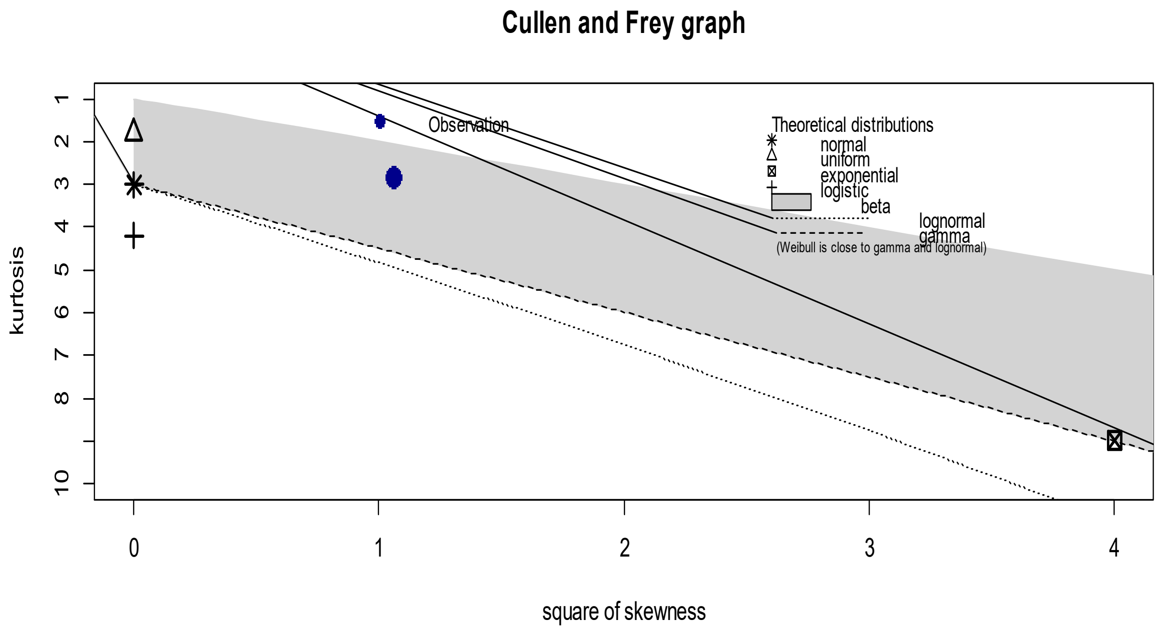
4.1.1. $\boldsymbol{a = 1, b = 1, \alpha = \frac{1}{2}}$

$$K(1,1) = \frac{1}{\int_0^1 \frac{\rho(1-\frac{1}{2}\rho)}{1+\frac{1}{2}\rho} d\rho}$$

Using R, we can obtain the value of $K(1,1) = 3.901996$. Thus,

$$\pi(\rho) = 3.901996 \frac{\rho(1-\frac{1}{2}\rho)}{1+\frac{1}{2}\rho}.$$

In order to use the formula in ACC and WOC, we need to do the distribution fitting for ρ to detect whether or not it fits a normal distribution.



Picture 1

We know from Picture 1 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9996505, 0.3842381^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.3842381^2} = 6.773281849.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 6.773281849, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9596.873921, n_{min} = 9597.$$

When $d = 0.05$,

$$n \geq 1529.810271, n_{min} = 1530.$$

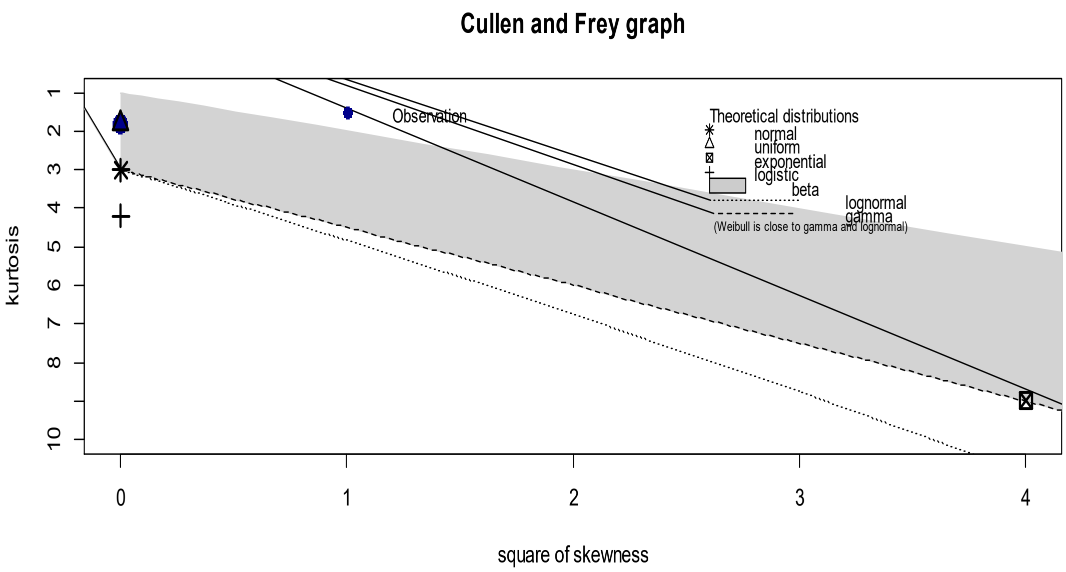
4.1.2. $\boldsymbol{a = 1, b = 0, \alpha = \frac{1}{2}}$

$$K(1,0) = \frac{1}{\int_0^1 \rho \, d\rho} = 2$$

Thus,

$$\pi(\rho) = 2\rho.$$

Using the previous method,



Picture 2

We know from Picture 2 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.0000000, 0.5779273^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.5779273^2} = 2.994012298.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 2.994012298, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9600.653191, n_{min} = 9601.$$

When $d = 0.05$,

$$n \geq 1533.58954, n_{min} = 1534.$$

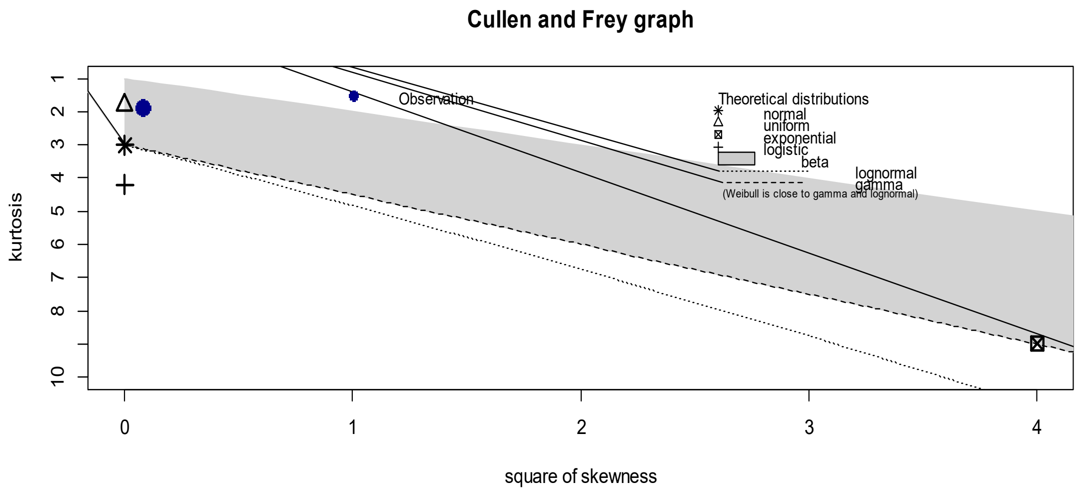
4.1.3. $\boldsymbol{a = 0, b = 1, \alpha = \frac{1}{2}}$

$$K(0,1) = \frac{1}{\int_0^1 \frac{1 - \frac{1}{2}\rho}{1 + \frac{1}{2}\rho} d\rho}$$

Using R, we can obtain the value of $K(0,1) = 1.608078$. Thus,

$$\pi(\rho) = 1.608078 \frac{1 - \frac{1}{2}\rho}{1 + \frac{1}{2}\rho}.$$

Using the previous method,



Picture 3

We know from Picture 3 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.0000721, 0.3064319^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.3064319^2} = 10.6495695.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 10.6495695, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9592.997634, n_{min} = 9593.$$

When $d = 0.05$,

$$n \geq 1525.933983, n_{min} = 1526.$$

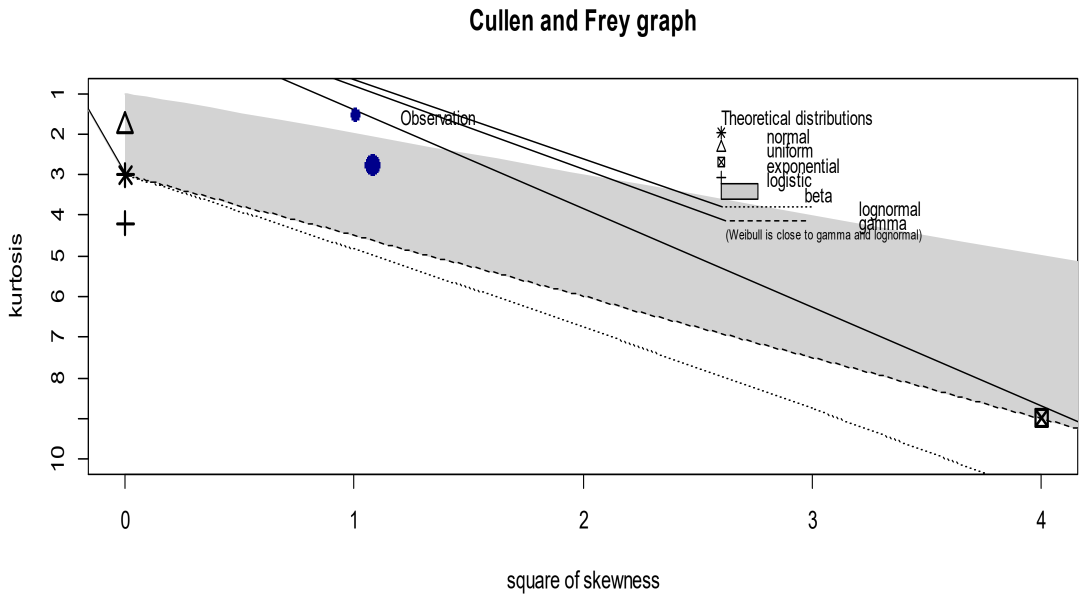
4.1.4. $\boldsymbol{a = 5, b = 2, \alpha = \frac{1}{2}}$

$$K(5,2) = \frac{1}{\int_0^1 \frac{\rho^5 (1 - \frac{1}{2}\rho)^2}{(1 + \frac{1}{2}\rho)^2} d\rho}$$

Using R, we can obtain the value of $K(5,2) = 36.04982$. Thus,

$$\pi(\rho) = 36.04982\frac{\rho^5(1-\frac{1}{2}\rho)^2}{(1+\frac{1}{2}\rho)^2}.$$

Using the previous method,



Picture 4

We know from Picture 4 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.001002, 1.197805^2).$$

Then, we can easily acquire

$$m = \frac{1}{1.197805^2} = 0.6969919413.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 0.6969919413, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9602.950211, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1535.886561, n_{min} = 1536.$$

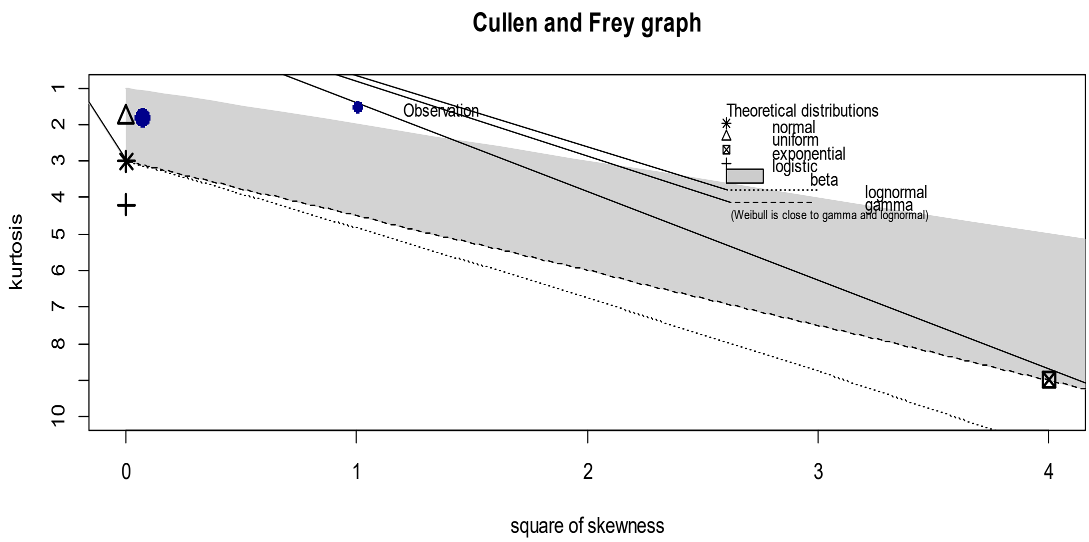
4.1.5. $\boldsymbol{a = 2, b = 5, \alpha = \frac{1}{2}}$

$$K(2,5) = \frac{1}{\int_0^1 \frac{\rho^2(1-\frac{1}{2}\rho)^5}{(1+\frac{1}{2}\rho)^5} d\rho}$$

Using R, we can obtain the value of $K(2,5) = 78.162$. Thus,

$$\pi(\rho) = 78.162 \frac{\rho^2(1-\frac{1}{2}\rho)^5}{(1+\frac{1}{2}\rho)^5}.$$

Using the previous method,



Picture 5

We know from Picture 5 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9991615, 0.4904106^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.4904106^2} = 4.157959958.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 4.157959958, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9599.489243, n_{min} = 9600.$$

When $d = 0.05$,

$$n \geq 1532.425593, n_{min} = 1533.$$

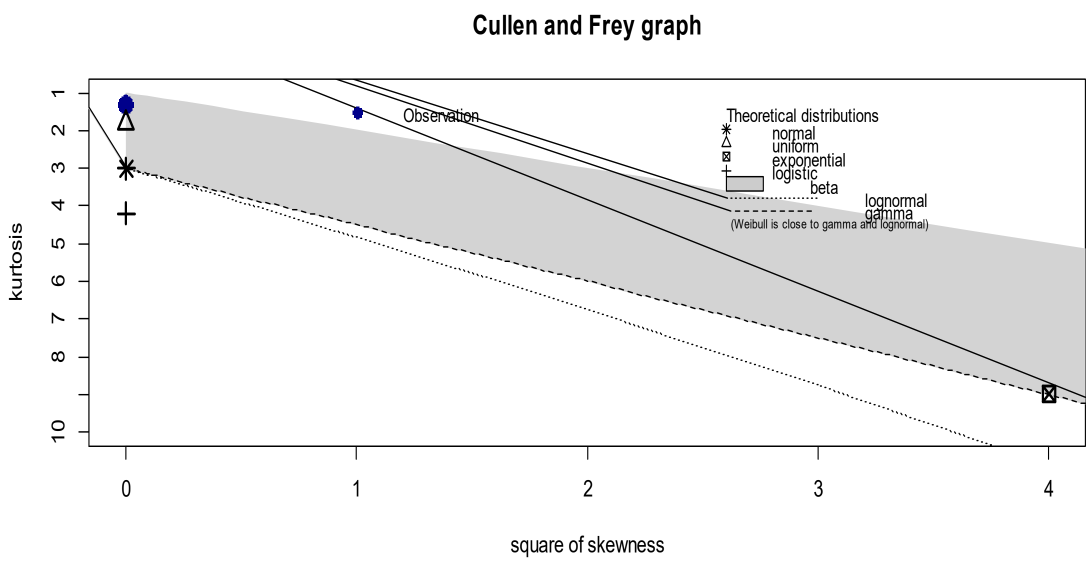
4.1.6. $\boldsymbol{a = 5, b = 5, \alpha = \frac{1}{2}}$

$$K(5,5) = \frac{1}{\int_0^1 \frac{\rho^5(1-\frac{1}{2}\rho)^5}{(1+\frac{1}{2}\rho)^5} d\rho}$$

Using R, we can obtain the value of $K(5,5) = 420.2389$. Thus,

$$\pi(\rho) = 420.2389 \frac{\rho^5(1-\frac{1}{2}\rho)^5}{(1+\frac{1}{2}\rho)^5}.$$

Using the previous method,



Picture 6

We know from Picture 6 that ρ fits the normal distribution. Then, we need to obtain the

parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9998647, 0.7928446^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.7928446^2} = 1.590830302.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 1.590830302, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9602.056373, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1534.992722, n_{min} = 1535.$$

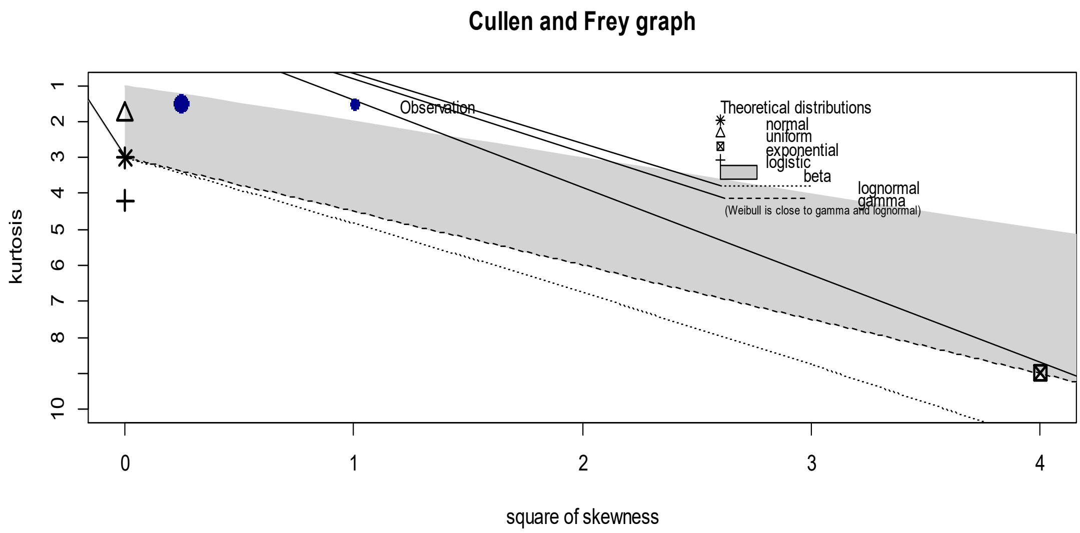
4.1.7. $\boldsymbol{a = 15, b = 15, \alpha = \frac{1}{2}}$

$$K(15,15) = \frac{1}{\int_0^1 \frac{\rho^{15}(1-\frac{1}{2}\rho)^{15}}{(1+\frac{1}{2}\rho)^{15}} d\rho}$$

Using R, we can obtain the value of $K(15,15) = 25865840$. Thus,

$$\pi(\rho) = 25865840 \frac{\rho^{15}(1-\frac{1}{2}\rho)^{15}}{(1+\frac{1}{2}\rho)^{15}}.$$

Using the previous method,



Picture 7

We know from Picture 7 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9999007, 1.0976147^2).$$

Then, we can easily acquire

$$m = \frac{1}{1.0976147^2} = 0.8300421955.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 0.8300421955, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9602.817161, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1535.75351, n_{min} = 1536.$$

In order to make results clear, we put all sample sizes in Table 1.

	Sample size when $d = 0.02$	Sample size when $d = 0.05$
$a = 1, b = 1, \alpha = \frac{1}{2}$	9597	1530
$a = 1, b = 0, \alpha = \frac{1}{2}$	9601	1534
$a = 0, b = 1, \alpha = \frac{1}{2}$	9593	1526
$a = 5, b = 2, \alpha = \frac{1}{2}$	9603	1536
$a = 2, b = 5, \alpha = \frac{1}{2}$	9600	1533
$a = 5, b = 5, \alpha = \frac{1}{2}$	9603	1535
$a = 15, b = 15, \alpha = \frac{1}{2}$	9603	1536

Table 1

From Table 1, the sample size when $d = 0.02$ and the sample size when $d = 0.05$ have the same trend. Moreover, the sample size is not strongly affected by the parameters, and when the value of parameters become larger, the sample size also gradually becomes stable. From these results, we also find that the sample size does not change greatly for different parameters, in accordance with the definition of WOC that the sample size is conservative.

4.2. The second model

$$\pi(\rho) = K(a,b)\frac{\rho^a}{\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right)^b}$$

where

$$K(a,b) = \frac{1}{\int_0^1 \frac{\rho^a}{\left(\sum_{k=0}^{c-1}\frac{1}{k!}\rho^k + \frac{\rho^c}{c!}\frac{c}{c-\rho}\right)^b} d\rho}$$

This model is also a variant of the standard M/M/1 queueing system, so we will still use the value of a and b that R. C. Quinino, and F. R. B. Cruz (2016) proposed in their paper. For the M/M/c queueing model, we know that when $c = 1$, it represents the standard M/M/1 queueing model, which is the one that R. C. Quinino, and F. R. B. Cruz (2016) studied. Thus, we mainly study the situation when $c = 2, c = 3$, and $c = 7$.

4.2.1. $\boldsymbol{c = 2}$

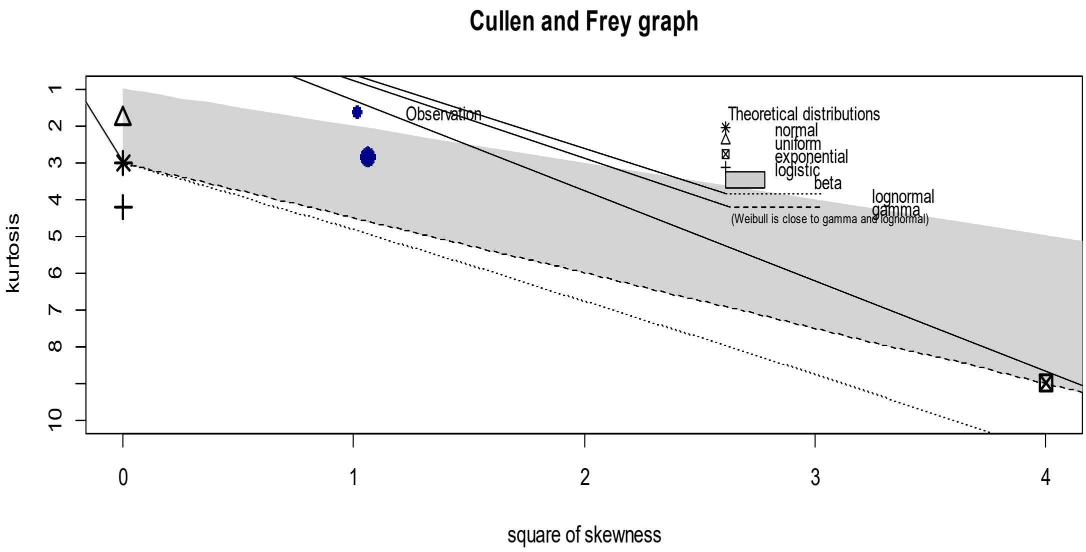
4.2.1.1. $\boldsymbol{a = 1, b = 1}$

$$K(1,1) = \frac{1}{\int_0^1 \frac{\rho}{\sum_{k=0}^{1} \frac{1}{k!}\rho^k + \frac{\rho^2}{2-\rho}} d\rho}$$

Using R, we obtain the value of $K(1,1) = 3.901996$. Thus,

$$\pi(\rho) = 3.901996 \frac{\rho}{\sum_{k=0}^{1} \frac{1}{k!}\rho^k + \frac{\rho^2}{2-\rho}}.$$

Next, we are going to do the distribution fitting.



Picture 8

We know from Picture 8 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9996505, 0.3842381^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.3842381^2} = 6.773281849.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 6.773281849, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9596.873921, n_{min} = 9597.$$

When $d = 0.05$,

$$n \geq 1529.810271, n_{min} = 1530.$$

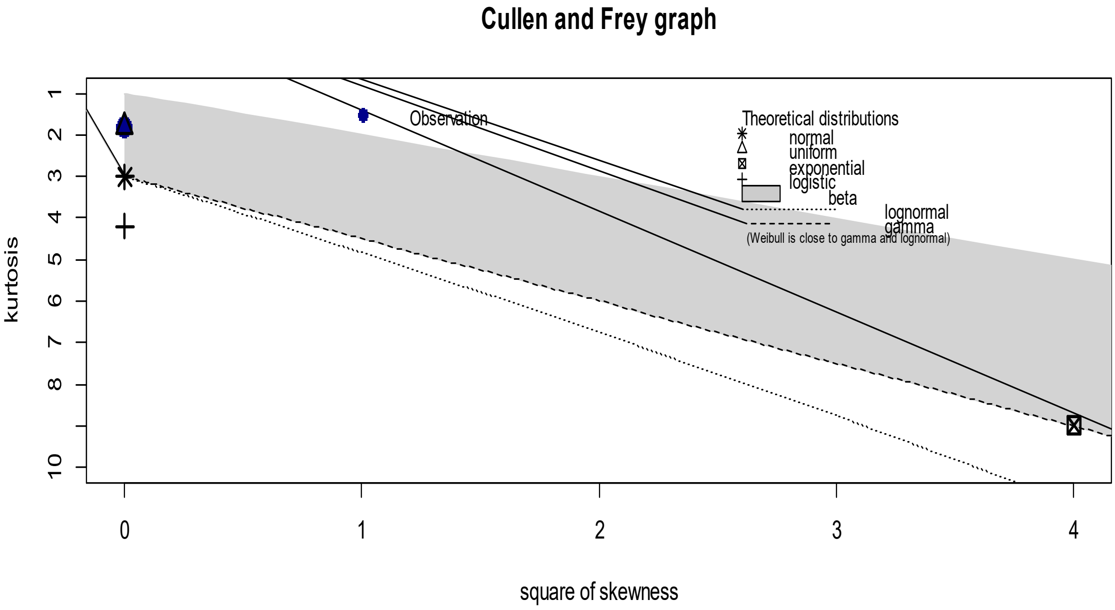
4.2.1.2. $a = 1, b = 0$

$$K(1,0) = \frac{1}{\int_0^1 \rho \, d\rho} = 2$$

Thus,

$$\pi(\rho) = 2\rho.$$

Next, we are going to do the distribution fitting.



Picture 9

We know from Picture 9 that ρ fits the normal distribution. Then, we need to obtain the

parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.0000000, 0.5779273^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.5779273^2} = 2.994012298.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 2.994012298, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9600.653191, n_{min} = 9601.$$

When $d = 0.05$,

$$n \geq 1533.58954, n_{min} = 1534.$$

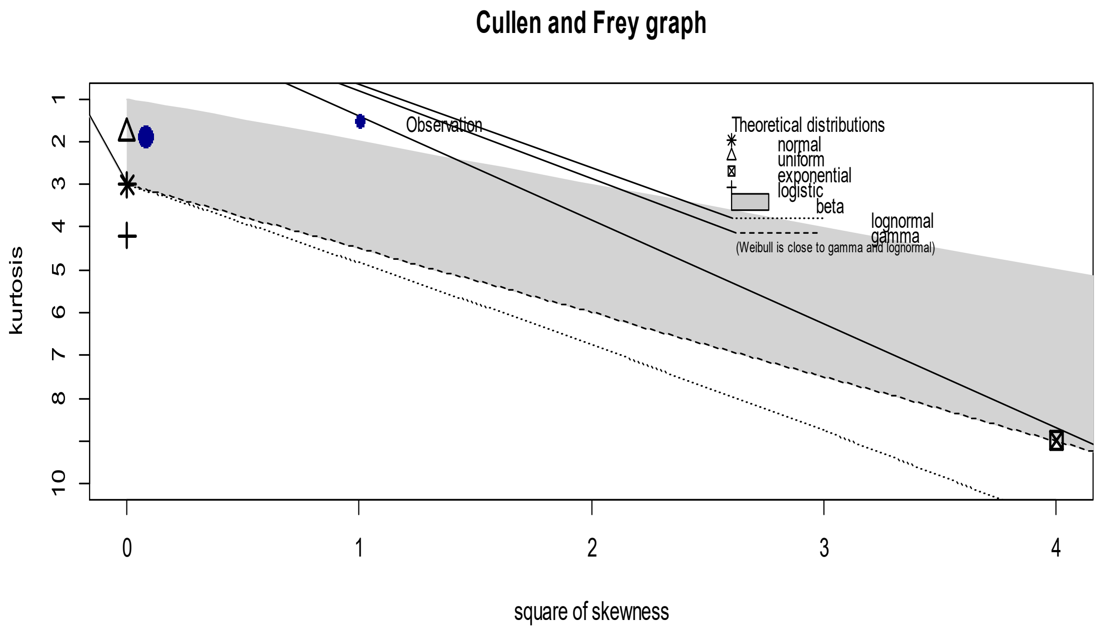
4.2.1.3. $\boldsymbol{a = 0, b = 1}$

$$K(0,1) = \frac{1}{\int_0^1 \frac{1}{\sum_{k=0}^{1} \frac{1}{k!}\rho^k + \frac{\rho^2}{2-\rho}} d\rho}$$

Using R, we obtain the value of $K(0,1) = 1.608078$. Thus,

$$\pi(\rho) = 1.608078 \frac{1}{\sum_{k=0}^{1} \frac{1}{k!}\rho^k + \frac{\rho^2}{2-\rho}}.$$

Next, we are going to do the distribution fitting.



Picture 10

We know from Picture 10 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.0000721, 0.3064319^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.3064319^2} = 10.6495695.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 10.6495695, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9592.997634, n_{min} = 9593.$$

When $d = 0.05$,

$$n \geq 1525.933983, n_{min} = 1526.$$

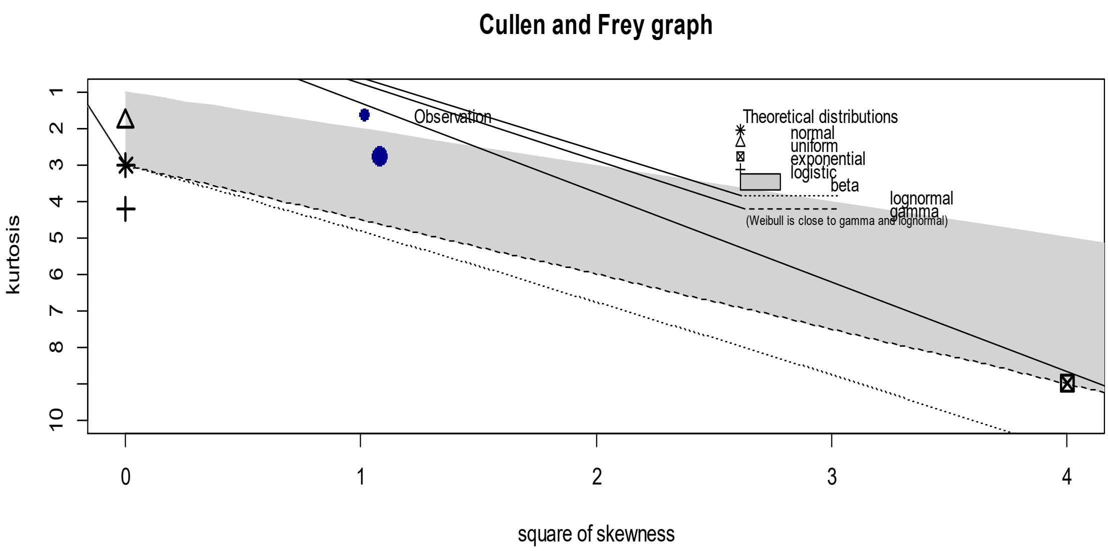
4.2.1.4. $\boldsymbol{a = 5, b = 2}$

$$K(5,2)=\frac{1}{\int_0^1 \frac{\rho^5}{\left(\sum_{k=0}^{1}\frac{1}{k!}\rho^k+\frac{\rho^2}{2-\rho}\right)^2}d\rho}$$

Using R, we obtain the value of $K(5,2)=36.04982$. Thus,

$$\pi(\rho)=36.04982\frac{\rho^5}{\left(\sum_{k=0}^{1}\frac{1}{k!}\rho^k+\frac{\rho^2}{2-\rho}\right)^2}.$$

Next, we are going to do the distribution fitting.



Picture 11

We know from Picture 11 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.001002, 1.197805^2).$$

Then, we can easily acquire

$$m=\frac{1}{1.197805^2}=0.6969919413.$$

Using the formula that we have in ACC and WOC, we obtain

$$n\geq\frac{Z_{0.025}^2}{d^2}-0.6969919413, Z_{0.025}=-1.959964.$$

When $d=0.02$,

$$n \geq 9602.950211, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1535.886561, n_{min} = 1536.$$

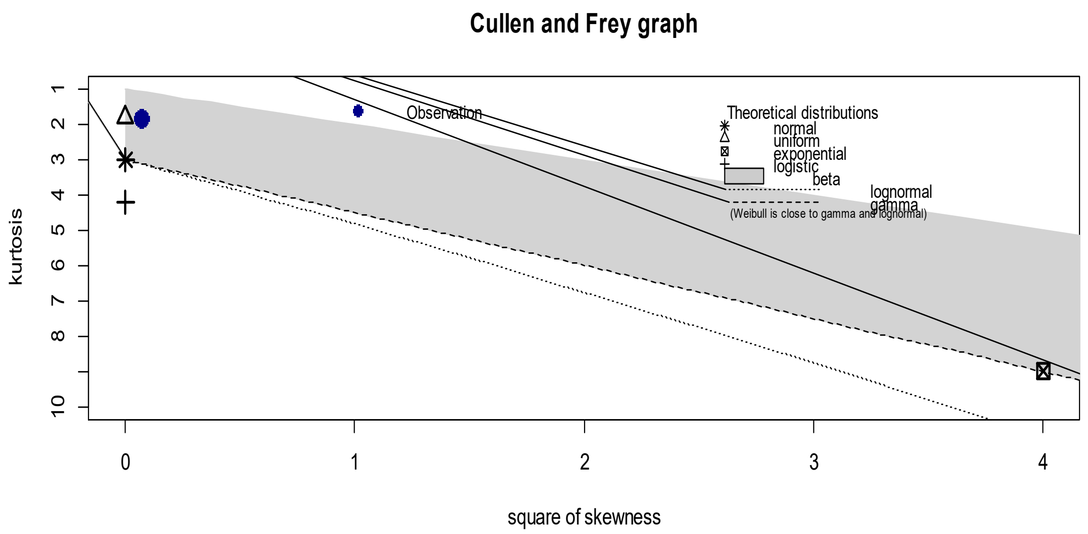
4.2.1.5. $\boldsymbol{a = 2, b = 5}$

$$K(2,5) = \frac{1}{\int_0^1 \frac{\rho^2}{\left(\sum_{k=0}^{1} \frac{1}{k!}\rho^k + \frac{\rho^2}{2-\rho}\right)^5} d\rho}$$

Using R, we obtain the value of $K(2,5) = 78.162$. Thus,

$$\pi(\rho) = 78.162 \frac{\rho^2}{\left(\sum_{k=0}^{1} \frac{1}{k!}\rho^k + \frac{\rho^2}{2-\rho}\right)^5}.$$

Next, we are going to do the distribution fitting.



Picture 12

We know from Picture 12 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9991615, 0.4904106^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.4904106^2} = 4.157959958.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 4.157959958, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9599.489243, n_{min} = 9600.$$

When $d = 0.05$,

$$n \geq 1532.425593, n_{min} = 1533.$$

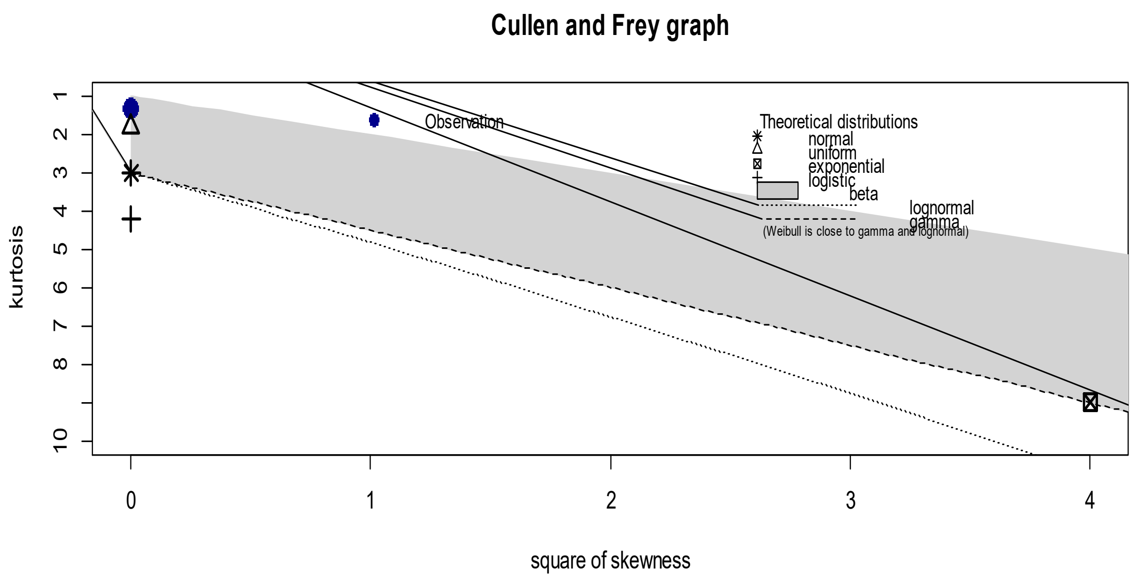
4.2.1.6. $\boldsymbol{a = 5, b = 5}$

$$K(5,5) = \frac{1}{\int_0^1 \frac{\rho^5}{\left(\sum_{k=0}^{1} \frac{1}{k!}\rho^k + \frac{\rho^2}{2-\rho}\right)^5} d\rho}$$

Using R, we obtain the value of $K(5,5) = 420.2389$. Thus,

$$\pi(\rho) = 420.2389 \frac{\rho^5}{\left(\sum_{k=0}^{1} \frac{1}{k!}\rho^k + \frac{\rho^2}{2-\rho}\right)^5}.$$

Next, we are going to do the distribution fitting.



Picture 13

We know from Picture 13 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9998647, 0.7928446^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.7928446^2} = 1.590830302.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 1.590830302, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9602.056373, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1534.992722, n_{min} = 1535.$$

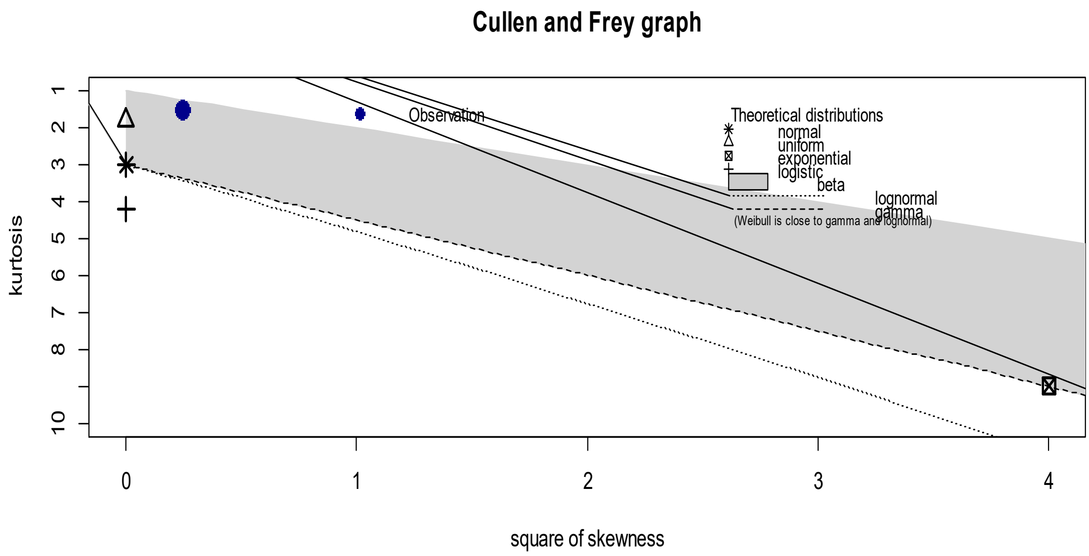
4.2.1.7. $\boldsymbol{a = 15, b = 15}$

$$K(15,15) = \frac{1}{\int_0^1 \frac{\rho^{15}}{\left(\sum_{k=0}^{1} \frac{1}{k!} \rho^k + \frac{\rho^2}{2-\rho}\right)^{15}} d\rho}$$

Using R, we obtain the value of $K(15,15) = 25865840$. Thus,

$$\pi(\rho) = 25865840 \frac{\rho^{15}}{\left(\sum_{k=0}^{1} \frac{1}{k!} \rho^k + \frac{\rho^2}{2-\rho}\right)^{15}}.$$

Next, we are going to do the distribution fitting.



Picture 14

We know from Picture 14 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9999007, 1.0976147^2).$$

Then, we can easily acquire

$$m = \frac{1}{1.0976147^2} = 0.8300421955.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 0.8300421955, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9602.817161, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1535.75351, n_{min} = 1536.$$

4.2.2. $c = 3$

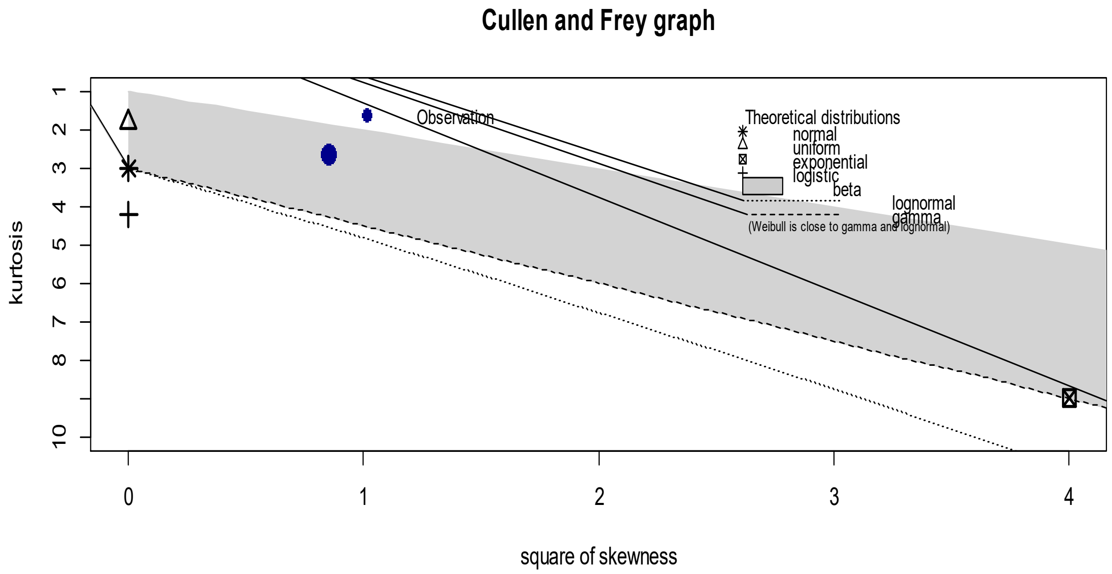
4.2.2.1. $a = 1, b = 1$

$$K(1,1) = \frac{1}{\int_0^1 \frac{\rho}{\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}} d\rho}$$

Using R, we obtain the value of $K(1,1) = 3.796481$. Thus,

$$\pi(\rho) = 3.796481 \frac{\rho}{\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}}.$$

Next, we are going to do the distribution fitting.



Picture 15

We know from Picture 15 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9996904, 0.3960265^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.3960265^2} = 6.376046928.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 6.376046928, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9597.271156, n_{min} = 9598.$$

When $d = 0.05$,

$$n \geq 1530.207506, n_{min} = 1531.$$

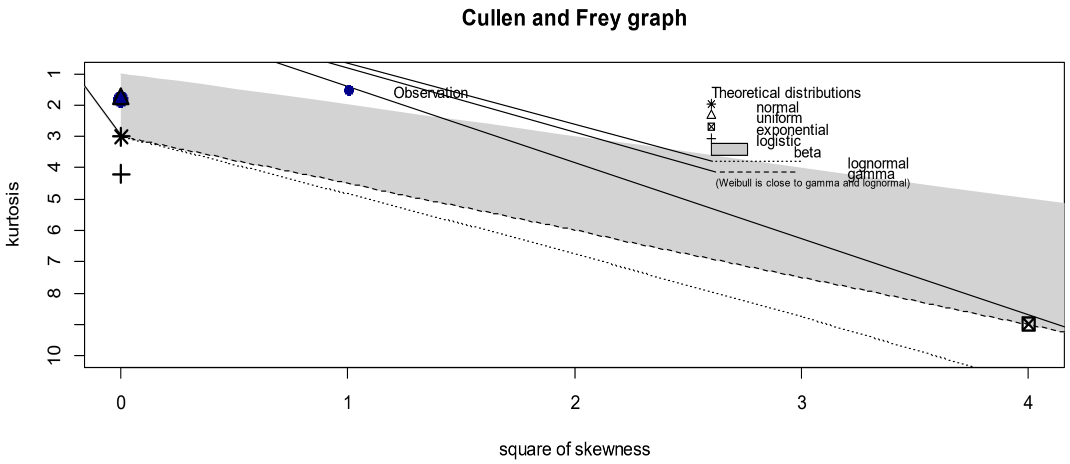
4.2.2.2. $a = 1, b = 0$

$$K(1,0) = \frac{1}{\int_0^1 \rho \, d\rho} = 2$$

Thus,

$$\pi(\rho) = 2\rho.$$

Next, we are going to do the distribution fitting.



Picture 16

We know from Picture 16 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.0000000, 0.5779273^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.5779273^2} = 2.994012298.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 2.994012298, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9600.653191, n_{min} = 9601.$$

When $d = 0.05$,

$$n \geq 1533.58954, n_{min} = 1534.$$

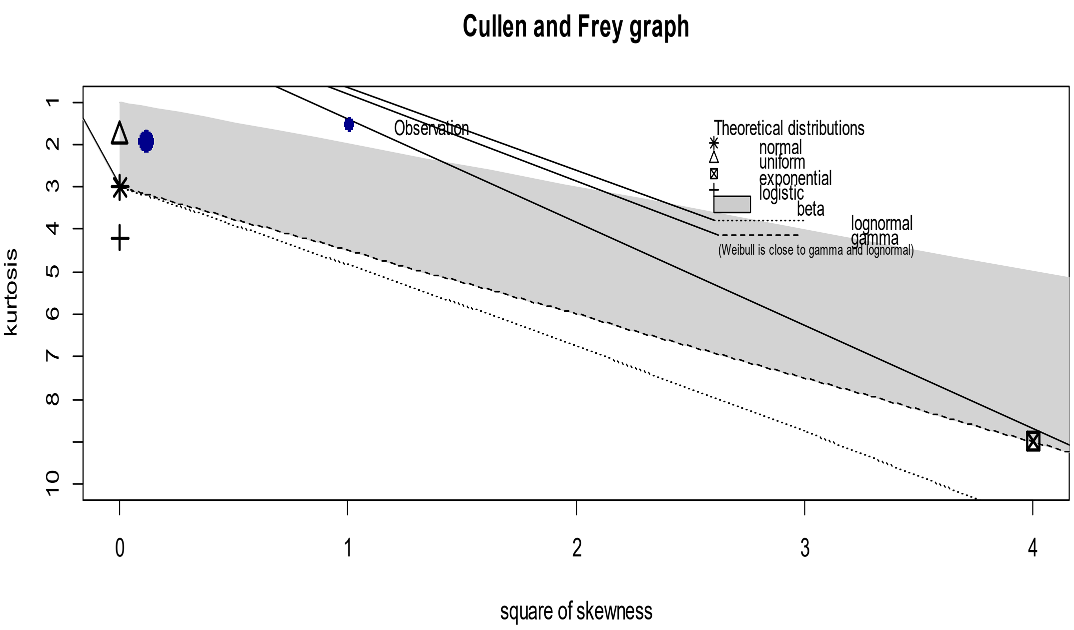
4.2.2.3. $\boldsymbol{a = 0, b = 1}$

$$K(0,1) = \frac{1}{\int_0^1 \frac{1}{\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}} d\rho}$$

Using R, we obtain the value of $K(0,1) = 1.58458$. Thus,

$$\pi(\rho) = 1.58458 \frac{1}{\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}}.$$

Next, we are going to do the distribution fitting.



Picture 17

We know from Picture 17 that ρ fits the normal distribution. Then, we need to obtain the

parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.000081, 0.288740^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.288740^2} = 11.99460901.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 11.99460901, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9591.652594, n_{min} = 9592.$$

When $d = 0.05$,

$$n \geq 1524.588944, n_{min} = 1525.$$

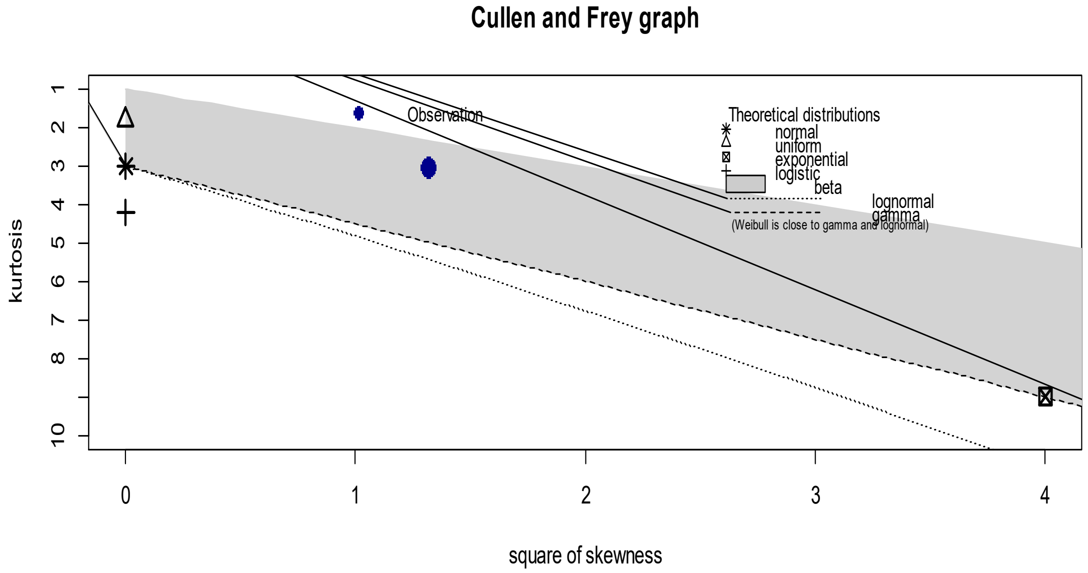
4.2.2.4. $\boldsymbol{a = 5, b = 2}$

$$K(5,2) = \frac{1}{\int_0^1 \frac{\rho^5}{\left(\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}\right)^2} d\rho}$$

Using R, we obtain the value of $K(5,2) = 32.60169$. Thus,

$$\pi(\rho) = 32.60169 \frac{\rho^5}{\left(\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}\right)^2}.$$

Next, we are going to do the distribution fitting.



Picture 18

We know from Picture 18 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.001155, 1.239653^2).$$

Then, we can easily acquire

$$m = \frac{1}{1.239653^2} = 0.650728351.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 0.650728351, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9602.996475, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1535.932824, n_{min} = 1536.$$

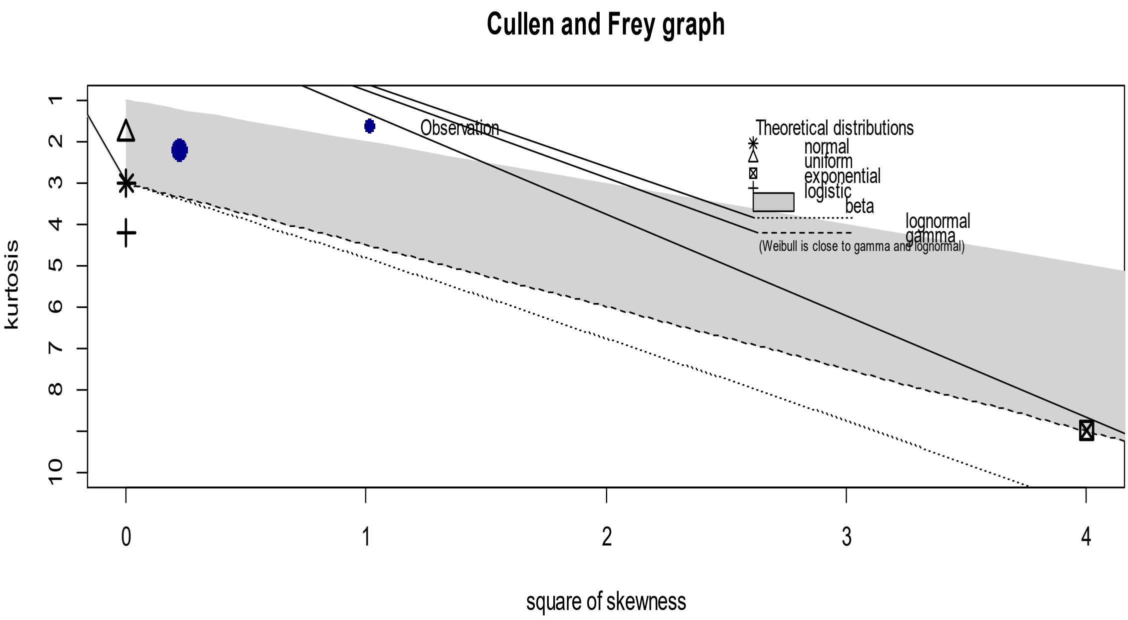
4.2.2.5. $\boldsymbol{a = 2, b = 5}$

$$K(2,5) = \frac{1}{\int_0^1 \frac{\rho^2}{\left(\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}\right)^5} d\rho}$$

Using R, we obtain the value of $K(2,5) = 72.06215$. Thus,

$$\pi(\rho) = 72.06215 \frac{\rho^2}{\left(\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}\right)^5}.$$

Next, we are going to do the distribution fitting.



Picture 19

We know from Picture 19 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9992299, 0.4403521^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.4403521^2} = 5.157032358.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 5.157032358, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9598.490171, n_{min} = 9599.$$

When $d = 0.05$,

$$n \geq 1531.42652, n_{min} = 1532.$$

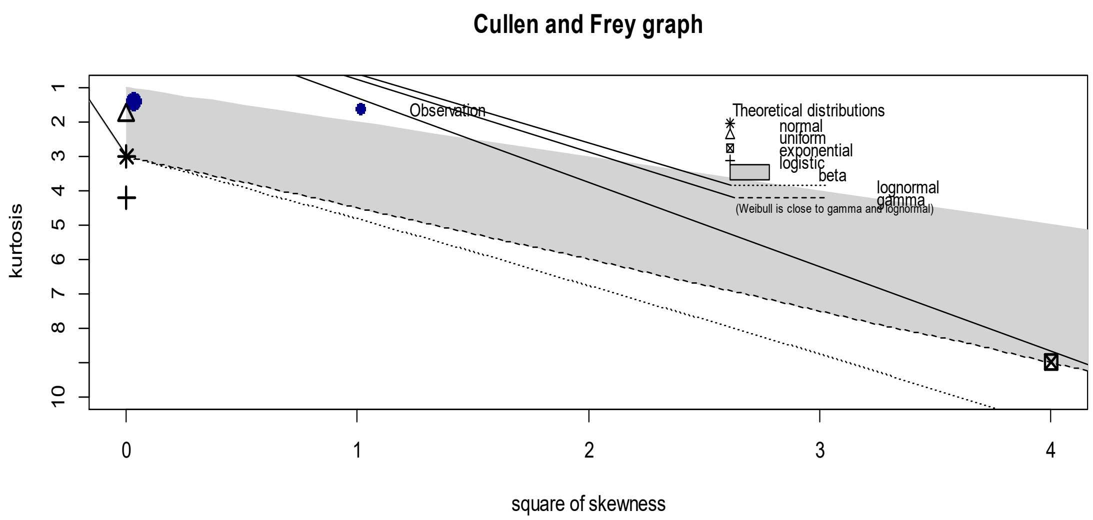
4.2.2.6. $\boldsymbol{a = 5, b = 5}$

$$K(5,5) = \frac{1}{\int_0^1 \frac{\rho^5}{\left(\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}\right)^5} d\rho}$$

Using R, we obtain the value of $K(5,5) = 347.1034$. Thus,

$$\pi(\rho) = 347.1034 \frac{\rho^5}{\left(\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}\right)^5}.$$

Next, we are going to do the distribution fitting.



Picture 20

We know from Picture 20 that ρ the fits normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.0001035, 0.8509565^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.8509565^2} = 1.380973294.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 1.380973294, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9602.26623, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1535.202579, n_{min} = 1536.$$

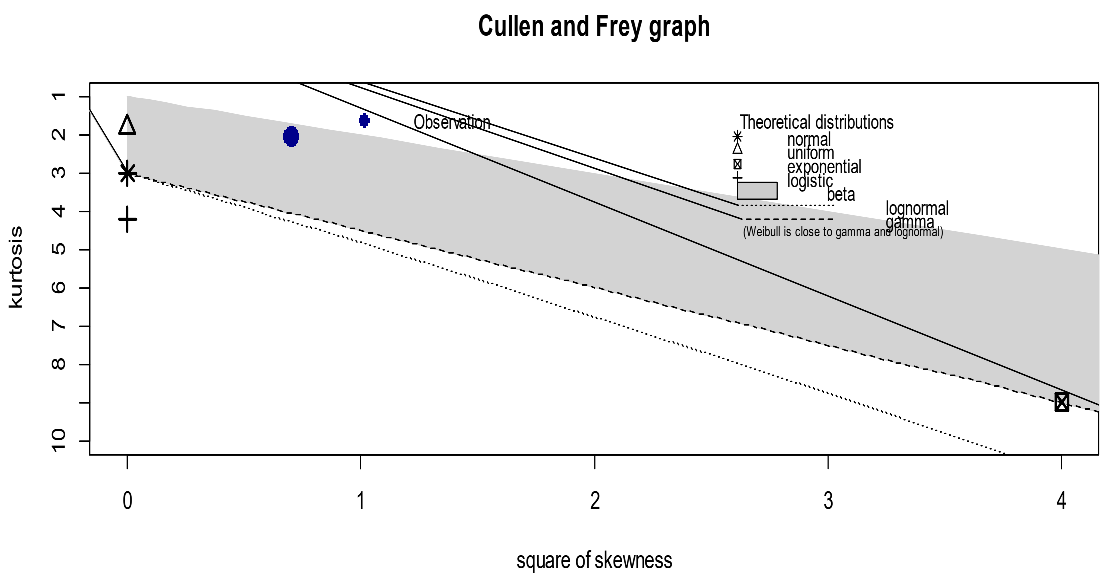
4.2.2.7. $\boldsymbol{a = 15, b = 15}$

$$K(15,15) = \frac{1}{\int_0^1 \frac{\rho^{15}}{\left(\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}\right)^{15}} d\rho}$$

Using R, we obtain the value of $K(15,15) = 12756446$. Thus,

$$\pi(\rho) = 12756446 \frac{\rho^{15}}{\left(\sum_{k=0}^{2} \frac{1}{k!}\rho^k + \frac{\rho^3}{2}\frac{1}{3-\rho}\right)^{15}}.$$

Next, we are going to do the distribution fitting.



Picture 21

We know from Picture 21 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.000639, 1.245293^2).$$

Then, we can easily acquire

$$m = \frac{1}{1.245293^2} = 0.6448473304.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 0.6448473304, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9603.002356, n_{min} = 9604.$$

When $d = 0.05$,

$$n \geq 1535.938705, n_{min} = 1536.$$

4.2.3. $\boldsymbol{c = 7}$

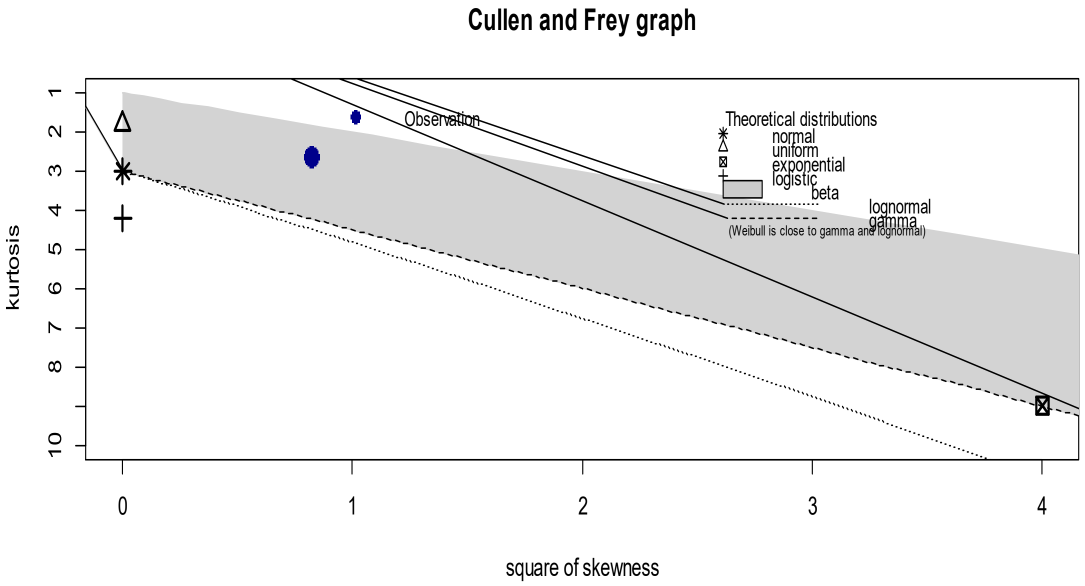
4.2.3.1. $\boldsymbol{a = 1, b = 1}$

$$K(1,1) = \frac{1}{\int_0^1 \frac{\rho}{\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}} d\rho}$$

Using R, we obtain the value of $K(1,1) = 3.784424$. Thus,

$$\pi(\rho) = 3.784424 \frac{\rho}{\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}}.$$

Next, we are going to do the distribution fitting.



Picture 22

We know from Picture 22 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9996962, 0.3975770^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.3975770^2} = 6.326412349.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 6.326412349, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9597.320791, n_{min} = 9598.$$

When $d = 0.05$,

$$n \geq 1530.25714, n_{min} = 1531.$$

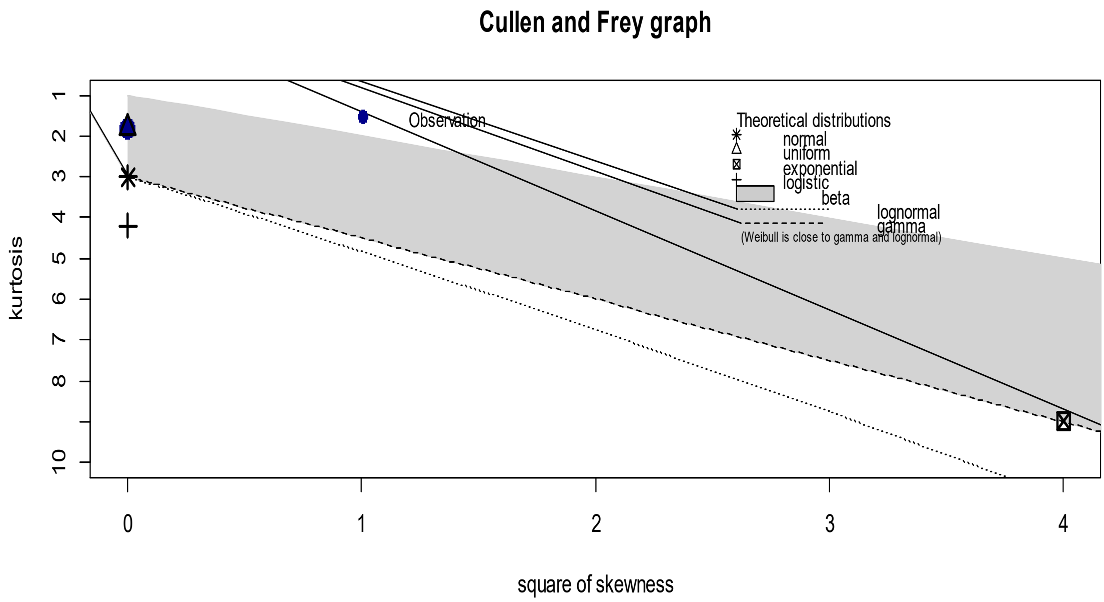
4.2.3.2. $\boldsymbol{a = 1, b = 0}$

$$K(1,0) = \frac{1}{\int_0^1 \rho \, d\rho} = 2$$

Thus,

$$\pi(\rho) = 2\rho.$$

Next, we are going to do the distribution fitting.



Picture 23

We know from Picture 23 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.0000000, 0.5779273^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.5779273^2} = 2.994012298.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 2.994012298, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9600.653191, n_{min} = 9601.$$

When $d = 0.05$,

$$n \geq 1533.58954, n_{min} = 1534.$$

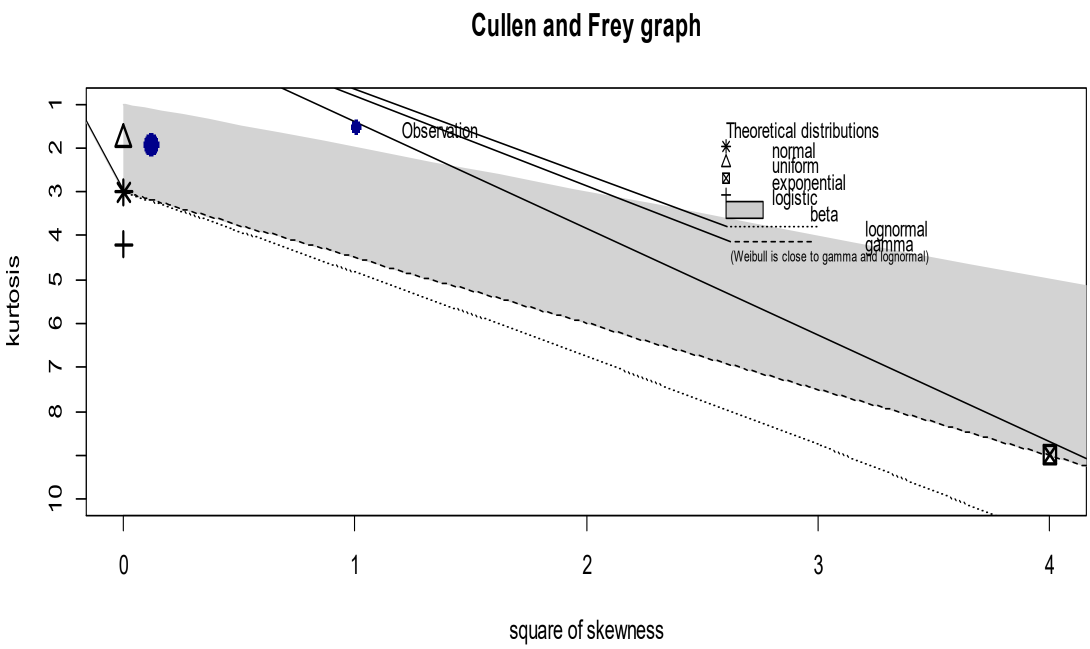
4.2.3.3. $\boldsymbol{a = 0, b = 1}$

$$K(0,1) = \frac{1}{\int_0^1 \frac{1}{\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}} d\rho}$$

Using R, we obtain the value of $K(0,1) = 1.581977$. Thus,

$$\pi(\rho) = 1.581977 \frac{1}{\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}}.$$

Next, we are going to do the distribution fitting.



Picture 24

We know from Picture 24 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.000082, 0.286621^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.286621^2} = 12.1726178.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 12.1726178, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9591.474585, n_{min} = 9592.$$

When $d = 0.05$,

$$n \geq 1524.410935, n_{min} = 1525.$$

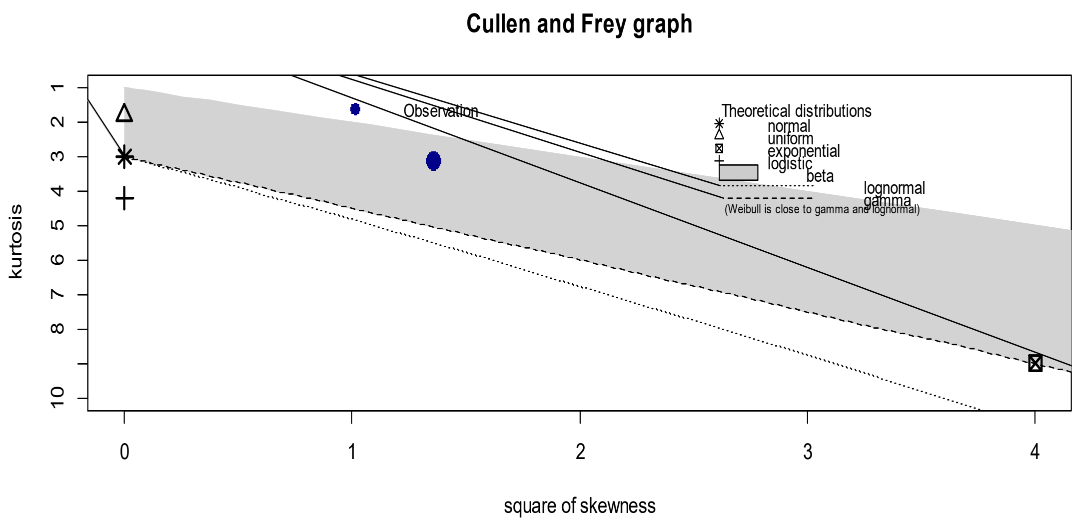
4.2.3.4. $\boldsymbol{a = 5, b = 2}$

$$K(5,2) = \frac{1}{\int_0^1 \frac{\rho^5}{\left(\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}\right)^2} d\rho}$$

Using R, we obtain the value of $K(5,2) = 32.19915$. Thus,

$$\pi(\rho) = 32.19915 \frac{\rho^5}{\left(\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}\right)^2}.$$

Next, we are going to do the distribution fitting.



Picture 25

We know from Picture 25 that ρ fits the normal distribution. Then, we need to obtain the

parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.001179, 1.245732^2).$$

Then, we can easily acquire

$$m = \frac{1}{1.245732^2} = 0.6443929179.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 0.6443929179, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9603.00281, n_{min} = 9604.$$

When $d = 0.05$,

$$n \geq 1535.93916, n_{min} = 1536.$$

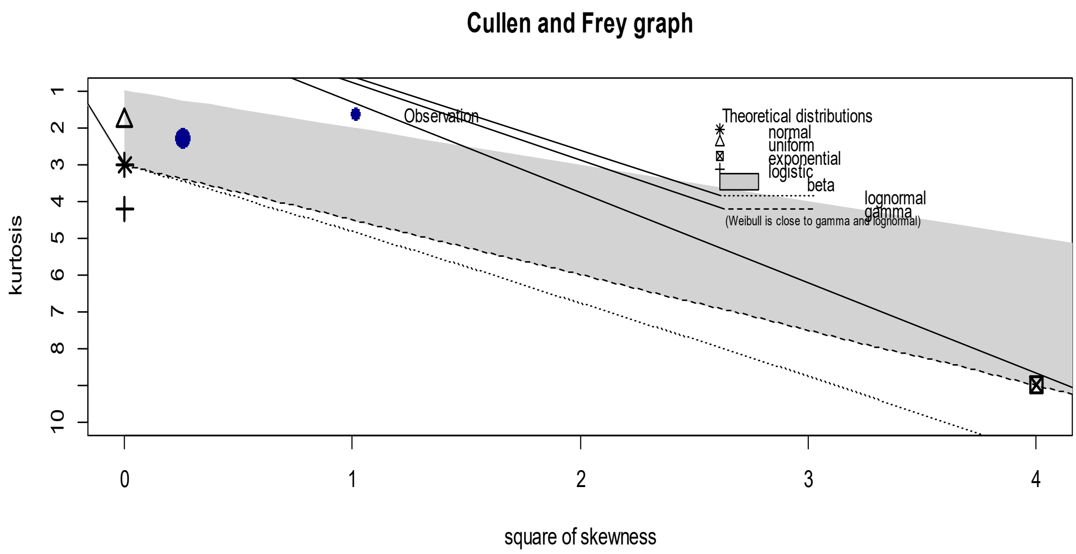
4.2.3.5. $\boldsymbol{a = 2, b = 5}$

$$K(2,5) = \frac{1}{\int_0^1 \frac{\rho^2}{\left(\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}\right)^5} d\rho}$$

Using R, we obtain the value of $K(2,5) = 71.40022$. Thus,

$$\pi(\rho) = 71.40022 \frac{\rho^2}{\left(\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}\right)^5}.$$

Next, we are going to do the distribution fitting.



Picture 26

We know from Picture 26 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(0.9992410, 0.4337879^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.4337879^2} = 5.314288562.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 5.314288562, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9598.332915, n_{min} = 9599.$$

When $d = 0.05$,

$$n \geq 1531.269264, n_{min} = 1532.$$

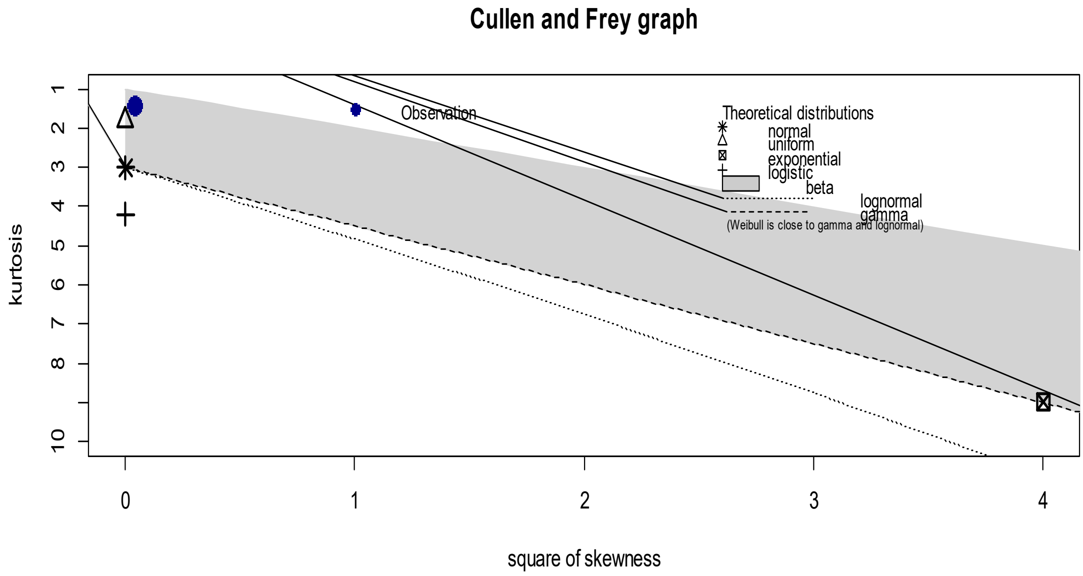
4.2.3.6. $\boldsymbol{a = 5, b = 5}$

$$K(5,5) = \frac{1}{\int_0^1 \frac{\rho^5}{\left(\sum_{k=0}^{6} \frac{1}{k!} \rho^k + \frac{\rho^7}{720} \frac{1}{7-\rho}\right)^5} d\rho}$$

Using R, we obtain the value of $K(5,5) = 339.0503$. Thus,

$$\pi(\rho) = 339.0503 \frac{\rho^5}{\left(\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720} \frac{1}{7-\rho}\right)^5}.$$

Next, we are going to do the distribution fitting.



Picture 27

We know from Picture 27 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.0001421, 0.8601861^2).$$

Then, we can easily acquire

$$m = \frac{1}{0.8601861^2} = 1.351497228.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 1.351497228, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9602.295706, n_{min} = 9603.$$

When $d = 0.05$,

$$n \geq 1535.232055, n_{min} = 1536.$$

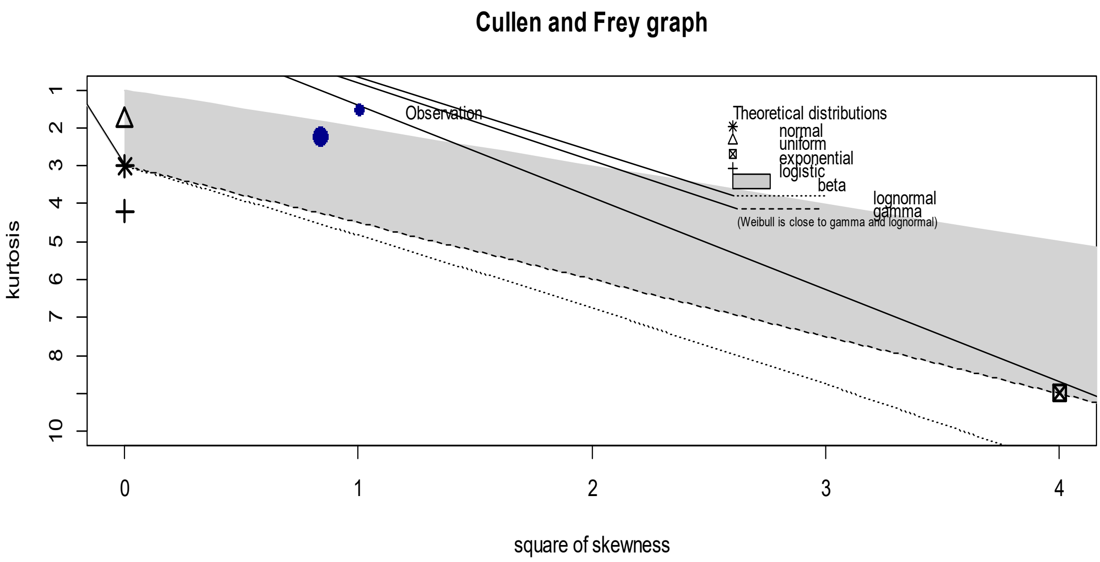
4.2.3.7. $\boldsymbol{a = 15, b = 15}$

$$K(15,15) = \frac{1}{\int_0^1 \frac{\rho^{15}}{\left(\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}\right)^{15}} d\rho}$$

Using R, we obtain the value of $K(15,15) = 11629777$. Thus,

$$\pi(\rho) = 11629777 \frac{\rho^{15}}{\left(\sum_{k=0}^{6} \frac{1}{k!}\rho^k + \frac{\rho^7}{720}\frac{1}{7-\rho}\right)^{15}}.$$

Next, we are going to do the distribution fitting.



Picture 28

We know from Picture 28 that ρ fits the normal distribution. Then, we need to obtain the parameters of this normal distribution. After using R, we know that

$$\rho \sim N(1.000778, 1.274894^2).$$

Then, we can easily acquire

$$m = \frac{1}{1.274894^2} = 0.6152503162.$$

Using the formula that we have in ACC and WOC, we obtain

$$n \geq \frac{Z_{0.025}^2}{d^2} - 0.6152503162, Z_{0.025} = -1.959964.$$

When $d = 0.02$,

$$n \geq 9603.031953, n_{min} = 9604.$$

When $d = 0.05$,

$$n \geq 1535.968302, n_{min} = 1536.$$

In order to make all results clear, we put all sample sizes in Table 2.

	Sample size when $d = 0.02$			Sample size when $d = 0.05$		
	$c = 2$	$c = 3$	$c = 7$	$c = 2$	$c = 3$	$c = 7$
$a = 1, b = 1$	9597	9598	9598	1530	1531	1531
$a = 1, b = 0$	9601	9601	9601	1534	1534	1534
$a = 0, b = 1$	9593	9592	9592	1526	1525	1525
$a = 5, b = 2$	9603	9603	9604	1536	1536	1536
$a = 2, b = 5$	9600	9599	9599	1533	1532	1532
$a = 5, b = 5$	9603	9603	9603	1535	1536	1536
$a = 15, b = 15$	9603	9604	9604	1536	1536	1536

From Table 2, we can obtain the different sample sizes for different parameters. In the first place, it is more likely that it is better to use $a = 1, b = 0$ and $a = 0, b = 1$ when we need $\rho \approx 1$ and $\rho \approx 0$ respectively. For example, for $\rho \approx 1$, we know that $\lambda \approx \mu$, which means that the model under this circumstance is equivalent to standard M/M/1 model. That is the reason that the sample sizes are all the same for $a = 1, b = 0$. For $\rho \approx 0$, it means that $\lambda \ll \mu$. In the other way, it indicates that there is basically no customer in the system. That is why the sample size of this situation is the smallest one among all different parameters.

Chapter 5
Two other models

After discussing two variants of the M/M/1 queueing models, we are going to have a quick look for the other variants of the M/M/1 model to check whether or not we can obtain the sample size from them using the same method.

5.1. M/M/1 queue with multiple vacations

The definition is similar to what we discussed in Review 1.

λ is the arrival rate based on the Poisson Process. μ is the service rate during the normal busy period, which is exponentially distributed. θ is the rate of the duration of a working vacation, which is exponentially distributed. The difference between this model and the model in Review 1 is that the server in the system does not give the service during the vacation. Then, we could define the state

$$J_t = \begin{cases} 1, & \text{the server is in the busy period at time } t, \\ 0, & \text{the server is in working vacation at time } t. \end{cases}$$

The state space of the Markov Chain is $E = \{\{(0)\} \cup \{(i,j)\}, i = 1,2,\dots, j = 0,1\}$. Thus, the balance equation is obtained as follows:

For $j = 0$,

$$\lambda P_0 = \mu P_{11},$$

$$(\lambda + \theta) P_{10} = \lambda P_0,$$

$$(\lambda + \theta) P_{20} = \lambda P_{10},$$

$$(\lambda + \theta) P_{k0} = \lambda P_{k-1,0}.$$

For $j=1$,

$$(\lambda+\mu)P_{11}=\theta P_{10}+\mu P_{21},$$

$$(\lambda+\mu)P_{21}=\lambda P_{11}+\mu P_{31}+\theta P_{20},$$

$$(\lambda+\mu)P_{k1}=\lambda P_{k-1,1}+\mu P_{k+1,1}+\theta P_{k0}.$$

When $j=0$, we can easily acquire

$$P_{k0}=\frac{P_{k0}}{P_{k-1,0}}\frac{P_{k-1,0}}{P_{k-2,0}}\cdots\frac{P_{20}}{P_{10}}\frac{P_{10}}{P_0}P_0=\left(\frac{\lambda}{\lambda+\theta}\right)^k P_0.$$

When $j=1$,

$$P_{11}=\frac{\lambda}{\mu}P_0,$$

$$P_{21}=\left[\left(\frac{\lambda}{\mu}\right)^2+\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\lambda+\theta}\right)\right]P_0,$$

$$P_{31}=\left[\left(\frac{\lambda}{\mu}\right)^3+\left(\frac{\lambda}{\mu}\right)^2\left(\frac{\lambda}{\lambda+\theta}\right)+\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\lambda+\theta}\right)^2\right]P_0,$$

$$\begin{aligned}P_{k1}&=\left[\left(\frac{\lambda}{\mu}\right)^k+\left(\frac{\lambda}{\mu}\right)^{k-1}\left(\frac{\lambda}{\lambda+\theta}\right)+\left(\frac{\lambda}{\mu}\right)^{k-2}\left(\frac{\lambda}{\lambda+\theta}\right)^2+\cdots+\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\lambda+\theta}\right)^{k-1}\right]P_0\\&=\left[\rho^k+\rho^{k-1}\left(\frac{\lambda}{\lambda+\theta}\right)+\rho^{k-2}\left(\frac{\lambda}{\lambda+\theta}\right)^2+\cdots+\rho\left(\frac{\lambda}{\lambda+\theta}\right)^{k-1}\right]P_0\\&=\frac{\rho\left[\rho^k-\left(\frac{\lambda}{\lambda+\theta}\right)^k\right]}{\rho-\frac{\lambda}{\lambda+\theta}}P_0.\end{aligned}$$

Using the fact that $P_0+\sum_{i=1}^{\infty}P_{i0}+\sum_{j=1}^{\infty}P_{j1}=1$, we obtain

$$P_0+\sum_{i=1}^{\infty}\left(\frac{\lambda}{\lambda+\theta}\right)^i P_0+\sum_{j=1}^{\infty}\frac{\rho\left[\rho^j-\left(\frac{\lambda}{\lambda+\theta}\right)^j\right]}{\rho-\frac{\lambda}{\lambda+\theta}}P_0=1,$$

$$P_0+\frac{\frac{\lambda}{\lambda+\theta}}{1-\frac{\lambda}{\lambda+\theta}}P_0+\frac{\rho}{\rho-\frac{\lambda}{\lambda+\theta}}\left(\frac{\rho}{1-\rho}-\frac{\frac{\lambda}{\lambda+\theta}}{1-\frac{\lambda}{\lambda+\theta}}\right)P_0=1,$$

$$P_0+\frac{\lambda}{\theta}P_0+\frac{\rho}{\rho-\frac{\lambda}{\lambda+\theta}}\left(\frac{\rho}{1-\rho}-\frac{\lambda}{\theta}\right)P_0=1,$$

$$P_0=\frac{\theta(1-\rho)}{\lambda+\theta}.$$

Thus,

$$P_{k1} = \frac{\rho\left[\rho^k - \left(\frac{\lambda}{\lambda+\theta}\right)^k\right]}{\rho - \frac{\lambda}{\lambda+\theta}} \frac{\theta(1-\rho)}{\lambda+\theta}.$$

This formula could not be rewritten into the form of $f(k|\rho) = h(k)\exp\{\rho \cdot k - \psi(\rho)\}$, which means that P_{k1} does not belong to the exponential family.

5.2. M/M/c queue with balking

A customer will join the standard M/M/c queue with probability $\alpha_k = \alpha\ (k \geq 1), \alpha_0 = 1$, which means that a customer will leave the system with probability $1 - \alpha_k = 1 - \alpha$. Then, let $\lambda_0 = \lambda,$ and $\lambda_k = \alpha\lambda\ (k \geq 1)$ denote the arrival rate and $\mu_k = \mu\ (k \geq 1)$ denote the service rate when there are k customers in the system. Thus, we can obtain the balance equations as follows:

$$\lambda P_0 = \mu P_1, k = 0,$$

$$(\lambda\alpha + \mu)P_1 = \lambda P_0 + 2\mu P_2, k = 1,$$

$$(\lambda\alpha + 2\mu)P_2 = \lambda\alpha P_1 + 3\mu P_3, k = 2,$$

$$(\lambda\alpha + k\mu)P_k = \lambda\alpha P_{k-1} + (k+1)\mu P_{k+1}, k \leq c - 1,$$

$$(\lambda\alpha + c\mu)P_k = \lambda\alpha P_{k-1} + c\mu P_{k+1}, k \geq c,$$

and obtain

$$\mu P_1 = \lambda P_0, k = 0,$$

$$2\mu P_2 = \lambda\alpha P_1, k = 1,$$

$$3\mu P_3 = \lambda\alpha P_2, k = 2,$$

$$(k+1)\mu P_{k+1} = \lambda\alpha P_k, k \leq c - 1,$$

$$c\mu P_{k+1} = \lambda\alpha P_k, k \geq c.$$

Then, for $\rho = \frac{\lambda}{\mu}, \rho < 1,$

when $k \leq c - 1,$

$$P_{k+1} = \frac{\alpha}{k+1}\frac{\lambda}{\mu}P_k = \frac{\alpha}{k+1}\frac{\alpha}{k}(\frac{\lambda}{\mu})^2 P_{k-1} = \frac{\alpha^{k+1}}{(k+1)!}(\frac{\lambda}{\mu})^{k+1}P_0 = \frac{\alpha^{k+1}}{(k+1)!}\rho^{k+1}P_0;$$

when $k \geq c$,

$$P_{k+1} = \frac{\alpha}{c}\frac{\lambda}{\mu}P_k = \frac{\alpha^2}{c^2}(\frac{\lambda}{\mu})^2 P_{k-1} = \frac{\alpha^{k+1-c}}{c^{k+1-c}}(\frac{\lambda}{\mu})^{k+1-c}P_c = \frac{\alpha^{k+1-c}}{c^{k+1-c}}(\frac{\lambda}{\mu})^{k+1-c}\frac{\alpha^c}{c!}(\frac{\lambda}{\mu})^c P_0$$
$$= \frac{\alpha^{k+1}}{c^{k+1-c}}\frac{1}{c!}(\frac{\lambda}{\mu})^{k+1}P_0 = \frac{\alpha^{k+1}}{c^{k+1-c}}\frac{1}{c!}\rho^{k+1}P_0.$$

Using the fact that $\sum_{k=0}^{\infty} P_k = 1$, we obtain

$$\sum_{k=0}^{c}\frac{\alpha^k}{k!}\rho^k P_0 + \sum_{k=c+1}^{\infty}\frac{\alpha^k}{c^{k-c}}\frac{1}{c!}\rho^k P_0 = 1.$$

Thus,

$$\left(\sum_{k=0}^{c}\frac{\alpha^k}{k!}\rho^k + \frac{c^c}{c!}\sum_{k=c+1}^{\infty}\frac{\alpha^k\rho^k}{c^k}\right)P_0 = 1,$$

$$\left(\sum_{k=0}^{c}\frac{\alpha^k}{k!}\rho^k + \frac{c^c}{c!}\frac{\frac{\alpha^{c+1}\rho^{c+1}}{c^{c+1}}}{1-\frac{\alpha\rho}{c}}\right)P_0 = 1,$$

$$\left(\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c}{c!}\rho^c + \frac{1}{c!}\frac{\alpha^{c+1}\rho^{c+1}}{c-\alpha\rho}\right)P_0 = 1,$$

$$\left(\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}\right)P_0 = 1,$$

$$P_0 = \frac{1}{\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}},$$

and

$$P_k = \begin{cases} \dfrac{\frac{\alpha^k\rho^k}{k!}}{\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}}, k \leq c, \\ \dfrac{\frac{\alpha^k\rho^k}{c^{k-c}}\frac{1}{c!}}{\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}}, k \geq c+1. \end{cases}$$

Then we need to detect whether or not P_k can be converted into the form of

$$h(k)\exp\{\theta k - \psi(\theta)\}.$$

Firstly, (a) when $k \leq c$,

$$h(k) = \frac{\alpha^k}{k!};$$

(b) when $k \geq c+1$,

$$h(k) = \frac{\alpha^k}{c^{k-c}} \frac{1}{c!}.$$

Secondly, (a) when $k \leq c$,

$$\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c \rho^c}{c!}\frac{c}{c-\alpha\rho}}\right] = k\ln\rho - \ln\left(\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c \rho^c}{c!}\frac{c}{c-\alpha\rho}\right).$$

If assume that $\theta = \ln\rho$, then $\rho = e^\theta$. Thus,

$$\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c \rho^c}{c!}\frac{c}{c-\alpha\rho}}\right] = \theta k - \ln\left(\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}e^{\theta k} + \frac{\alpha^c e^{\theta c}}{c!}\frac{c}{c-\alpha e^\theta}\right).$$

Then we obtain

$$\begin{aligned} \frac{\rho^k}{\sum_{k=0}^{c-1} \frac{\alpha^n}{k!}\rho^k + \frac{\alpha^c \rho^c}{c!}\frac{c}{c-\alpha\rho}} &= \exp\left\{\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c \rho^c}{c!}\frac{c}{c-\alpha\rho}}\right]\right\} \\ &= \exp\left\{\theta k - \ln\left(\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}e^{\theta k} + \frac{\alpha^c e^{\theta c}}{c!}\frac{c}{c-\alpha e^\theta}\right)\right\}, \end{aligned}$$

where $\psi(\theta) = \ln\left(\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}e^{\theta k} + \frac{\alpha^c e^{\theta c}}{c!}\frac{c}{c-\alpha e^\theta}\right)$.

Therefore, we know that the original equation,

$$P_k = \frac{\frac{\alpha^k \rho^k}{k!}}{\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}\rho^k + \frac{\alpha^c \rho^c}{c!}\frac{c}{c-\alpha\rho}},$$

could be converted into

$$P_k = \frac{\alpha^k}{k!}\exp\left\{\theta k - \ln\left(\sum_{k=0}^{c-1} \frac{\alpha^k}{k!}e^{\theta k} + \frac{\alpha^c e^{\theta c}}{c!}\frac{c}{c-\alpha e^\theta}\right)\right\},$$

which implies that P_k $(k \leq c)$ belongs to the exponential family.

(b) when $k \geq c+1$,

$$\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k+\frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}}\right]=k\ln\rho-\ln\left(\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k+\frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}\right),$$

which is the same as $k \leq c$. Thus,

$$\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k+\frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}}=\exp\left\{\ln\left[\frac{\rho^k}{\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k+\frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}}\right]\right\}$$

$$=\exp\left\{\theta k-\ln\left(\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}e^{\theta k}+\frac{\alpha^c e^{\theta c}}{c!}\frac{c}{c-\alpha e^{\theta}}\right)\right\},$$

where $\psi(\theta)=\ln\left(\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}e^{\theta k}+\frac{\alpha^c e^{\theta c}}{c!}\frac{c}{c-\alpha e^{\theta}}\right)$.

Therefore, we know that the original equation,

$$P_k=\frac{\frac{\alpha^k\rho^k}{c^{k-c}}\frac{1}{c!}}{\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}\rho^k+\frac{\alpha^c\rho^c}{c!}\frac{c}{c-\alpha\rho}},$$

could be converted into

$$P_k=\frac{\alpha^k}{c^{k-c}}\frac{1}{c!}\exp\left\{\theta k-\ln\left(\sum_{k=0}^{c-1}\frac{\alpha^k}{k!}e^{\theta k}+\frac{\alpha^c e^{\theta c}}{c!}\frac{c}{c-\alpha e^{\theta}}\right)\right\},$$

which implies that P_k $(k \geq c+1)$ belongs to the exponential family. Therefore, we could obtain the sample size of this model using the same way that we used in the previous models.

Chapter 6
Conclusion

In the previous chapters, we studied how to obtain the minimal sample size for the exponential family of distributions based on the condition that the we know the parameter σ^2 for data x. Under this circumstance, we can obtain the exact number of the sample size for the M/M/1 queue with balking, and the M/M/c queueing system. For the M/M/c queue with balking, we could use the same method to achieve the minimal sample size. In the future, if we could guarantee that a queueing system belongs to the exponential family of distributions with known parameter σ^2 for data x, then we could use the same method to acquire the minimal sample size. However, if we can not calculate the value of the parameter σ^2 for data x, then the problem will be more complex. According to Adcock (1988), and Joseph and Belisle (1997), we have an exact method to obtain the minimal sample size under some specific assumptions. Especially in the paper proposed by Joseph and Belisle (1997), they introduced a way to obtain the expressions for the sample size in ACC, ALC, and WOC.

Reference

Choudhury, & Borthakur, Bayesian inference and prediction in the single server Markovian queue, Metrika, Vol.67, pp371-383, 2008.

Cruz, Quinino, & Ho, Bayesian estimation of traffic intensity based on queue length in a multi-server M/M/s queue, Communications in Statistics - Simulation and Computation, Vol.46, pp7319-7331, 2017.

Joseph, & Bélisle, Bayesian Sample Size Determination for Normal Means and Differences Between Normal Means, Journal of the Royal Statistical Society. Series D (The Statistician), Vol.46, pp209-226, 1997.

Adcock, A Bayesian Approach to Calculating Sample Sizes, Journal of the Royal Statistical

Society. Series D (The Statistician), Vol.37, pp433-439, 1988.

Robert, The Bayesian Choice, Springer, pp117, 2001.

Quinino, & Cruz, Bayesian sample sizes in an M/M/1 queueing systems, The International Journal of Advanced Manufacturing Technology, Vol.88, pp995-1002, 2017.

Cao, Lee, & Alber, Comparison of Bayesian sample size criteria: ACC, ALC, and WOC, Journal of Statistical Planning and Inference, Vol.139, pp4111-4122, 2009.

Selvaraju, & Goswami, Impatient customers in an M/M/1 queue with single and multiple working vacations, Computers & Industrial Engineering, Vol.65, pp207-215, 2013.

Ross, Introduction to Probability Models, Elsevier, pp50, 2014.

Gopalan, Quantification of Observed prior and likelihood information in parametric Bayesian modeling, 2015.

Appendix

4.1.1. $\mathbf{a = 1, b = 1, \alpha = \frac{1}{2}}$

```
> f<-function(ρ)(ρ*(1-ρ/2))/(1+ρ/2)
> integrate(f,0,1)
0.2562791 with absolute error < 2.8e-15
> 1/0.2562791
```

```
[1] 3.901996

> f<-function(ρ) (3.901996*ρ*(1-ρ/2))/(1+ρ/2)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  1.338953
median:  1.170599
mean:  0.9996505
estimated sd:  0.3844301
estimated skewness:  -1.029572
estimated kurtosis:  2.824877

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
        estimate Std. Error
mean 0.9996505 0.01214460
sd   0.3842381 0.00858727
```

4.1.2. $\mathbf{a = 1, b = 0, \alpha = \frac{1}{2}}$

```
> f<-function(ρ) 2*ρ
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  2
median:  1
mean:  1
estimated sd:  0.5782162
estimated skewness:  1.008477e-16
estimated kurtosis:  1.8

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
        estimate Std. Error
mean 1.0000000 0.01826654
sd   0.5779273 0.01291622
```

4.1.3. $\mathbf{a = 0, b = 1, \alpha = \frac{1}{2}}$

```
> f<-function(ρ) (1-ρ/2)/(1+ρ/2)
> integrate(f,0,1)
0.6218604 with absolute error < 6.9e-15
> 1/0.6218604
[1] 1.608078

> f<-function(ρ) (1.608078*(1-ρ/2))/(1+ρ/2)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0.536026   max:  1.608078
median:  0.9648468
mean:  1.000072
estimated sd:  0.3065851
estimated skewness:  0.2828056
estimated kurtosis:  1.899248

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
       estimate   Std. Error
mean 1.0000721 0.009685385
sd   0.3064319 0.006848274
```

4.1.4. $\mathbf{a = 5, b = 2, \alpha = \frac{1}{2}}$

```
> f<-function(ρ)((ρ^5)*((1-ρ/2)^2))/((1+ρ/2)^2)
> integrate(f,0,1)
0.02773939 with absolute error < 3.1e-16
> 1/0.02773939
[1] 36.04982

> f<-function(ρ) (36.04982*(ρ^5)*((1-ρ/2)^2))/((1+ρ/2)^2)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  4.005536
median:  0.4055605
mean:  1.001002
estimated sd:  1.198403
estimated skewness:  1.039412
```

estimated kurtosis: 2.753299

```
> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
         estimate Std. Error
mean 1.001002 0.03785898
sd   1.197805 0.02677026
```

4.1.5. $\mathbf{a = 2, b = 5, \alpha = \frac{1}{2}}$

```
> f<-function(ρ)((ρ^2)*((1-ρ/2)^5))/((1+ρ/2)^5)
> integrate(f,0,1)
0.01279394 with absolute error < 1.4e-16
> 1/0.01279394
[1] 78.162

> f<-function(ρ) (78.162*(ρ^2)*((1-ρ/2)^5))/((1+ρ/2)^5)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:   0     max:   1.649445
median:   1.039523
mean:   0.9991615
estimated sd:   0.4906557
estimated skewness:   -0.2659688
estimated kurtosis:   1.825664

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
          estimate Std. Error
mean 0.9991615 0.01550040
sd   0.4904106 0.01096023
```

4.1.6. $\mathbf{a = 5, b = 5, \alpha = \frac{1}{2}}$

```
> f<-function(ρ)((ρ^5)*((1-ρ/2)^5))/((1+ρ/2)^5)
> integrate(f,0,1)
0.002379599 with absolute error < 5.8e-17
> 1/0.002379599
[1] 420.2389

> f<-function(ρ) (420.2389*(ρ^5)*((1-ρ/2)^5))/((1+ρ/2)^5)
```

```
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0    max:  1.999351
median:  1.021181
mean:  0.9998647
estimated sd:  0.7932409
estimated skewness:  -0.03259927
estimated kurtosis:  1.307932

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 0.9998647 0.02505942
sd   0.7928446 0.01771956
```

4.1.7. $\mathbf{a = 15, b = 15, \alpha = \frac{1}{2}}$

```
> f<-function(ρ)((ρ^15)*((1-ρ/2)^15))/((1+ρ/2)^15)
> integrate(f,0,1)
3.866103e-08 with absolute error < 1.3e-14
> 1/3.866103e-08
[1] 25865840

> f<-function(ρ) (25865840*(ρ^15)*((1-ρ/2)^15))/((1+ρ/2)^15)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0    max:  2.78551
median:  0.3711465
mean:  0.9999007
estimated sd:  1.098163
estimated skewness:  0.4972349
estimated kurtosis:  1.510811

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 0.9999007 0.03469228
```

sd   1.0976147 0.02453106

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
           estimate Std. Error
mean 0.9999007 0.03469228
sd   1.0976147 0.02453106

4.2.1.1. $\mathbf{a = 1, b = 1}$

> f<-function(ρ) ρ/(1+ρ+((ρ^2)/(2-ρ)))
> integrate(f,0,1)
0.2562791 with absolute error < 2.8e-15
> 1/0.2562791
[1] 3.901996

> f<-function(ρ) 3.901996*ρ/(1+ρ+((ρ^2)/(2-ρ)))
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min: 0   max: 1.338953
median: 1.170599
mean: 0.9996505
estimated sd: 0.3844301
estimated skewness: -1.029572
estimated kurtosis: 2.824877

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
           estimate Std. Error
mean 0.9996505 0.01214460
sd   0.3842381 0.00858727

4.2.1.2. $\mathbf{a = 1, b = 0}$

> f<-function(ρ) 2*ρ
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min: 0   max: 2
median: 1
mean: 1

estimated sd: 0.5782162
estimated skewness: 1.008477e-16
estimated kurtosis: 1.8

```
> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 1.0000000 0.01826654
sd   0.5779273 0.01291622
```

4.2.1.3. $\mathbf{a = 0, b = 1}$

```
> f<-function(ρ) 1/(1+ρ+((ρ^2)/(2-ρ)))
> integrate(f,0,1)
0.6218604 with absolute error < 6.9e-15
> 1/0.6218604
[1] 1.608078

> f<-function(ρ) 1.608078/(1+ρ+((ρ^2)/(2-ρ)))
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0.536026   max:  1.608078
median:  0.9648468
mean:  1.000072
estimated sd:  0.3065851
estimated skewness:  0.2828056
estimated kurtosis:  1.899248

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate  Std. Error
mean 1.0000721 0.009685385
sd   0.3064319 0.006848274
```

4.2.1.4. $\mathbf{a = 5, b = 2}$

```
> f<-function(ρ) (ρ^5)/((1+ρ+((ρ^2)/(2-ρ)))^2)
> integrate(f,0,1)
0.02773939 with absolute error < 3.1e-16
> 1/0.02773939
[1] 36.04982

> f<-function(ρ) 36.04982*(ρ^5)/((1+ρ+((ρ^2)/(2-ρ)))^2)
```

```
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0    max:  4.005536
median:  0.4055605
mean:  1.001002
estimated sd:  1.198403
estimated skewness:  1.039412
estimated kurtosis:  2.753299

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
     estimate Std. Error
mean 1.001002 0.03785898
sd   1.197805 0.02677026
```

4.2.1.5. $\mathbf{a = 2, b = 5}$

```
> f<-function(ρ) (ρ^2)/((1+ρ+((ρ^2)/(2-ρ)))^5)
> integrate(f,0,1)
0.01279394 with absolute error < 1.4e-16
> 1/0.01279394
[1] 78.162

> f<-function(ρ) 78.162*(ρ^2)/((1+ρ+((ρ^2)/(2-ρ)))^5)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0    max:  1.649445
median:  1.039523
mean:  0.9991615
estimated sd:  0.4906557
estimated skewness:  -0.2659688
estimated kurtosis:  1.825664

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 0.9991615 0.01550040
sd   0.4904106 0.01096023
```

4.2.1.6. $\mathbf{a = 5, b = 5}$

```
> f<-function(ρ) (ρ^5)/((1+ρ+((ρ^2)/(2-ρ)))^5)
> integrate(f,0,1)
0.002379599 with absolute error < 5.8e-17
> 1/0.002379599
[1] 420.2389

> f<-function(ρ) 420.2389*(ρ^5)/((1+ρ+((ρ^2)/(2-ρ)))^5)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  1.999351
median:  1.021181
mean:  0.9998647
estimated sd:  0.7932409
estimated skewness:  -0.03259927
estimated kurtosis:  1.307932

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 0.9998647 0.02505942
sd   0.7928446 0.01771956
```

4.2.1.7. $\mathbf{a = 15, b = 15}$

```
> f<-function(ρ) (ρ^15)/((1+ρ+((ρ^2)/(2-ρ)))^15)
> integrate(f,0,1)
3.866103e-08 with absolute error < 1.3e-14
> 1/3.866103e-08
[1] 25865840

> f<-function(ρ) 25865840*(ρ^15)/((1+ρ+((ρ^2)/(2-ρ)))^15)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  2.78551
median:  0.3711465
mean:  0.9999007
estimated sd:  1.098163
estimated skewness:  0.4972349
```

```
estimated kurtosis:  1.510811

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 0.9999007 0.03469228
sd   1.0976147 0.02453106
```

4.2.2.1. $\mathbf{a = 1, b = 1}$

```
> f<-function(ρ) ρ/(1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))
> integrate(f,0,1)
0.2634018 with absolute error < 2.9e-15
> 1/0.2634018
[1] 3.796481

> f<-function(ρ) 3.796481*ρ/(1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  1.381771
median:  1.150449
mean:  0.9996904
estimated sd:  0.3962245
estimated skewness:  -0.9218964
estimated kurtosis:  2.643166

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate   Std. Error
mean 0.9996904 0.012517200
sd   0.3960265 0.008850743
```

4.2.2.2. $\mathbf{a = 1, b = 0}$

```
> f<-function(ρ) 2*ρ
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  2
median:  1
mean:  1
```

estimated sd: 0.5782162
estimated skewness: 1.008477e-16
estimated kurtosis: 1.8

```
> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 1.0000000 0.01826654
sd   0.5779273 0.01291622
```

4.2.2.3. $\mathbf{a = 0, b = 1}$

```
> f<-function(ρ) 1/(1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))
> integrate(f,0,1)
0.6310822 with absolute error < 7e-15
> 1/0.6310822
[1] 1.58458

> f<-function(ρ) 1.58458/(1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0.5762109   max:  1.58458
median:  0.9603515
mean:  1.000081
estimated sd:  0.2888843
estimated skewness:  0.3360583
estimated kurtosis:  1.926928

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
     estimate  Std. Error
mean 1.000081 0.009126197
sd   0.288740 0.006452848
```

4.2.2.4. $\mathbf{a = 5, b = 2}$

```
> f<-function(ρ) (ρ^5)/((1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))^2)
> integrate(f,0,1)
0.03067326 with absolute error < 3.4e-16
> 1/0.03067326
[1] 32.60169

> f<-function(ρ) 32.60169*(ρ^5)/((1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))^2)
```

```
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:   0      max:   4.310967
median:   0.3742159
mean:   1.001155
estimated sd:   1.240273
estimated skewness:   1.14789
estimated kurtosis:   3.056884

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
        estimate Std. Error
mean 1.001155 0.03918169
sd      1.239653 0.02770555
```

4.2.2.5. $\mathbf{a = 2, b = 5}$

```
> f<-function(ρ) (ρ^2)/((1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))^5)
> integrate(f,0,1)
0.01387691 with absolute error < 1.5e-16
> 1/0.01387691
[1] 72.06215

> f<-function(ρ) 72.06215*(ρ^2)/((1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))^5)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:   0      max:   1.5579
median:   1.053364
mean:   0.9992299
estimated sd:   0.4405722
estimated skewness:   -0.469405
estimated kurtosis:   2.198182

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
          estimate    Std. Error
mean 0.9992299 0.013918198
sd      0.4403521 0.009841424
```

4.2.2.6. $\mathbf{a = 5, b = 5}$

```
> f<-function(ρ) (ρ^5)/((1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))^5)
> integrate(f,0,1)
0.002880986 with absolute error < 3.2e-17
> 1/0.002880986
[1] 347.1034

> f<-function(ρ) 347.1034*(ρ^5)/((1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))^5)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  2.216835
median:  0.8869296
mean:  1.000104
estimated sd:  0.8513819
estimated skewness:  0.1746592
estimated kurtosis:  1.410011

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 1.0001035 0.02689616
sd   0.8509565 0.01901834
```

4.2.2.7. $\mathbf{a = 15, b = 15}$

```
> f<-function(ρ) (ρ^15)/((1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))^15)
> integrate(f,0,1)
7.839174e-08 with absolute error < 4.9e-15
> 1/7.839174e-08
[1] 12756446

> f<-function(ρ) 12756446*(ρ^15)/((1+ρ+(ρ^2)/2+((ρ^3)/(6-(2*ρ))))^15)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  3.323173
median:  0.212824
mean:  1.000639
estimated sd:  1.245916
estimated skewness:  0.8370415
```

```
estimated kurtosis:    2.044505

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
         estimate Std. Error
mean 1.000639 0.03935995
sd      1.245293 0.02783161
```

4.2.3.1. $\mathbf{a = 1, b = 1}$

```
> f<-function(ρ) ρ/(1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))
> integrate(f,0,1)
0.264241 with absolute error < 2.9e-15
> 1/0.264241
[1] 3.784424

> f<-function(ρ)
3.784424*ρ/(1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:    0      max:    1.392209
median:    1.147685
mean:    0.9996962
estimated sd:    0.3977757
estimated skewness:    -0.9070495
estimated kurtosis:    2.621885

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
           estimate    Std. Error
mean 0.9996962 0.012566207
sd      0.3975770 0.008885397
```

4.2.3.2. $\mathbf{a = 1, b = 0}$

```
> f<-function(ρ) 2*ρ
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:    0      max:    2
median:    1
```

```
mean:    1
estimated sd:    0.5782162
estimated skewness:    1.008477e-16
estimated kurtosis:    1.8

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
          estimate Std. Error
mean 1.0000000 0.01826654
sd      0.5779273 0.01291622
```

4.2.3.3. $\mathbf{a = 0, b = 1}$

```
> f<-function(ρ) 1/(1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))
> integrate(f,0,1)
0.6321205 with absolute error < 7e-15
> 1/0.6321205
[1] 1.581977

> f<-function(ρ)
1.581977/(1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:    0.5819757      max:    1.581977
median:    0.9595175
mean:    1.000082
estimated sd:    0.2867643
estimated skewness:    0.3449694
estimated kurtosis:    1.930656

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
         estimate    Std. Error
mean 1.000082 0.009059223
sd      0.286621 0.006405487
```

4.2.3.4. $\mathbf{a = 5, b = 2}$

```
> f<-function(ρ)
(ρ^5)/((1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))^2)
> integrate(f,0,1)
0.03105672 with absolute error < 3.4e-16
> 1/0.03105672
```

```
[1] 32.19915

> f<-function(ρ) 32.19915*(ρ^5)/((1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))^2)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0    max:  4.357664
median:  0.3701689
mean:  1.001179
estimated sd:  1.246355
estimated skewness:  1.164007
estimated kurtosis:  3.104752

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
      estimate Std. Error
mean 1.001179 0.03937382
sd   1.245732 0.02784142
```

4.2.3.5. $\mathbf{a = 2, b = 5}$

```
> f<-function(ρ) (ρ^2)/((1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))^5)
> integrate(f,0,1)
0.01400556 with absolute error < 1.6e-16
> 1/0.01400556
[1] 71.40022

> f<-function(ρ) 71.40022*(ρ^2)/((1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))^5)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0    max:  1.546075
median:  1.054225
mean:  0.999241
estimated sd:  0.4340048
estimated skewness:  -0.5042894
```

```
estimated kurtosis:    2.272521

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
          estimate    Std. Error
mean 0.9992410 0.013710725
sd      0.4337879 0.009694715
```

4.2.3.6. $\mathbf{a = 5, b = 5}$

```
> f<-function(ρ)
(ρ^5)/((1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))^5)
> integrate(f,0,1)
0.002949415 with absolute error < 3.3e-17
> 1/0.002949415
[1] 339.0503

> f<-function(ρ)
339.0503*(ρ^5)/((1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))^
5)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:    0      max:    2.284481
median:    0.8697169
mean:    1.000142
estimated sd:    0.8606161
estimated skewness:    0.2100095
estimated kurtosis:    1.442499

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
          estimate Std. Error
mean 1.0001421 0.02718788
sd      0.8601861 0.01922462
```

4.2.3.7. $\mathbf{a = 15, b = 15}$

```
> f<-function(ρ)
(ρ^15)/((1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ))))^15)
> integrate(f,0,1)
8.598617e-08 with absolute error < 4.5e-15
> 1/8.598617e-08
[1] 11629777
```

```
> f<-function(ρ)
11629777*(ρ^15)/((1+ρ+(ρ^2)/2+(ρ^3)/6+(ρ^4)/24+(ρ^5)/120+(ρ^6)/720+((ρ^7)/(5040-(720*ρ)))
)^15)
> ρ<-seq(from=0,to=1,by=0.001)
> y<-f(ρ)
> descdist(y)
summary statistics
------
min:  0   max:  3.557474
median:  0.1962966
mean:  1.000778
estimated sd:  1.275532
estimated skewness:  0.9152648
estimated kurtosis:  2.227377

> fitdist(y,"norm")
Fitting of the distribution ' norm ' by maximum likelihood
Parameters:
     estimate Std. Error
mean 1.000778 0.04029556
sd   1.274894 0.02849318
```