ON A SUM OF CUBES OF DEGREES OF
IRREDUCIBLE COMPLEX CHARACTERS

by

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A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of
Master of Science

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submitted by Yaroslav Bezverkhnyev
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Abstract

This thesis deals with the sum of n th powers of degrees of irreducible complex characters. It was inspired by a question from the Kourovka notebook posed by L.S. Kazarin. Kazarin's question was to classify finite groups G such that the sum of cubes of degrees of irreducible complex characters of G is bounded by a certain function that depends on the order of the group G .

Since the Kourovka question proved too difficult to settle, we focused on the properties that the sum of cubes (and n th powers) of degrees of irreducible complex characters possesses and on the relations with other interesting quantities from character theory.

The main result that we prove here is similar to Isaacs and Passman's bound on the index of the abelian subgroup in terms of the largest character degree. We examine the bound on the index of the solvable radical of the group in terms of the "average" character degree.

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My thanks go to I.M. Isaacs whose book has served as a basis for my knowledge of character theory.

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Chapter 1

Basic Results

1.1 The Kourovka question

The main reason for this thesis was an interesting question posed in the Kourovka notebook, which is a book of unsolved problems in group theory (See Problem 12.34 on page 135 in [Kou]). The question posed by L.S. Kazarin is as follows:

“Describe the finite groups G such that the sum of cubes of degrees of all irreducible complex characters is at most $|G| \cdot \log_2 |G|$. The question is interesting for applications in the theory of signal processing.”

First of all, the author would like to point out that we are not going to study applications of this problem to signal processing. Instead we are going to analyze and study its usefulness in a purely theoretic sense.

Numerous attempts to actually answer Kazarin’s question in its entirety were not successful. Thus, the author has focused on studying the properties of the sum of cubes and, more generally, n th powers of degrees of irreducible complex characters. Although we are mainly interested in the case of cubes (because it is related to the Kourovka question), some theorems shall be proven for the general case of n th powers, as the proofs do not differ greatly. All the information that was a result of this process is

compiled in this thesis.

Since the original question from Kourouva notebook mentions finite groups, we shall do the same. Therefore, throughout this thesis we only consider finite groups.

The author would like to point out that the main point of reference about character theory results is the very well written book by I.M. Isaacs (See [Isa]). Therefore, we shall begin by quoting some results from it in order to provide sufficient background for readers not familiar with character theory. However, some knowledge of group theory is required.

1.2 Introductory results from character theory

We begin this section with quoting some basic character theory results from the book by Isaacs.

A *representation* Γ of degree n of a group G is a homomorphism $\Gamma : G \mapsto GL(n, \mathbb{C})$, where $GL(n, \mathbb{C})$ is the multiplicative group of invertible $n \times n$ matrices over $\mathbb{C}$.

A *character* χ of G *afforded* by representation Γ is the function given by $\chi(g) = \text{tr } \Gamma(g)$ for all $g \in G$, where tr is the trace function of a matrix.

A representation Γ of degree n is *irreducible* if there does not exist $T \in GL(n, \mathbb{C})$ and representations F_1, F_2 of smaller degrees such that

$$T^{-1}\Gamma(g)T = \begin{bmatrix} F_1(g) & H(g) \\ 0 & F_2(g) \end{bmatrix}$$

for all $g \in G$.

A character χ of G is *irreducible* if it is afforded by some irreducible representation Γ of G . The set of all irreducible characters of G is denoted by $\text{Irr}(G)$. Every character φ of G is a unique sum of irreducible characters, called the *irreducible constituents* of φ .

Result 1 If a character χ is afforded by a representation of degree n , then $\chi(1) = n$.

Characters of degree 1 are called *linear characters*. The set of all linear characters of G is denoted by $\text{Lin}(G)$.

Result 2 The number of linear characters of a group G is $|\text{Lin}(G)| = [G : G']$, i.e., it is the index of a derived subgroup G' . (Corollary 2.23 in [Isa]).

Result 3 The number of irreducible characters of G is equal to the number of conjugacy classes of G . (Corollary 2.7 in [Isa]).

Result 4 The group G is abelian iff every irreducible character is linear. (Corollary 2.6 in [Isa]).

Result 5 $\sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G|$ and $\chi(1)$ divides $|G|$ for every $\chi \in \text{Irr}(G)$. (Corollary 2.7 and Theorem 3.11 in [Isa]).

Result 6 If A is an abelian subgroup of G such that $[G : A] = n$, then $\chi(1) \leq n$ for all $\chi \in \text{Irr}(G)$. (Exercise 2.9b in [Isa]).

A *class function* on a group G is a function $\varphi : G \rightarrow \mathbb{C}$ which is constant on conjugacy classes. Note that characters are class functions.

Let φ and ϑ be class functions on a group G . Then

$$[\varphi, \vartheta] = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\vartheta(g)} \quad (1.1)$$

is the *inner product* of φ and ϑ .

Let $G = H \times K$ and let φ and ϑ be class functions on H and K , respectively. Define $\chi = \varphi \times \vartheta$ by $\chi(hk) = \varphi(h)\vartheta(k)$ for $h \in H$ and $k \in K$.

Result 7 If $G = H \times K$, then $\text{Irr}(G) = \{\varphi \times \vartheta \mid \varphi \in \text{Irr}(H), \vartheta \in \text{Irr}(K)\}$. (Theorem 4.21 in [Isa]).

Let H be a subgroup of G . The *restriction* χ_H of a character χ of G is defined as $\chi_H(h) = \chi(h)$ for all $h \in H$, i.e., we restrict the values of character χ to the elements of H only (so χ_H is a character of H).

Let H be a subgroup of G and let φ be a class function of H . Then φ^G , the *induced class function* on G , is given by

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^\circ(xgx^{-1}), \quad (1.2)$$

where φ° is defined by $\varphi^\circ(h) = \varphi(h)$ if $h \in H$ and $\varphi^\circ(y) = 0$ if $y \notin H$.

Let $H \triangleleft G$. If ϑ is a class function of H and $g \in G$, define $\vartheta^g : H \mapsto \mathbb{C}$ by $\vartheta^g(h) = \vartheta(ghg^{-1})$. We say that ϑ^g is *conjugate* to ϑ in G . If ϑ is a character of H , then ϑ^g is a character of H and ϑ^G is a character of G .

Result 8 The degree of the restricted character χ_H is $\chi_H(1) = \chi(1)$. The degree of the induced character φ^G is $\varphi^G(1) = \frac{|G|}{|H|} \varphi(1)$.

Frobenius Reciprocity If H is a subgroup of G , φ is a class function on H and ϑ is a class function on G , then

$$[\varphi, \vartheta_H] = [\varphi^G, \vartheta]. \quad (1.3)$$

(Lemma 5.2 in [Isa]).

Clifford's Theorem Let $H \triangleleft G$ and let $\chi \in \text{Irr}(G)$. Let ϑ be an irreducible constituent of χ_H and suppose $\vartheta = \vartheta_1, \vartheta_2, \dots, \vartheta_t$ are the distinct conjugates of ϑ in G . Then

$$\chi_H = e \sum_{i=1}^t \vartheta_i, \quad (1.4)$$

where $e = [\chi_H, \vartheta]$. (Theorem 6.2 in [Isa]).

Gallagher's Theorem Let $N \triangleleft G$ and let $\chi \in \text{Irr}(G)$ be such that $\chi_N = \vartheta \in \text{Irr}(N)$.

Then the characters $\beta\chi$ for $\beta \in \text{Irr}(G/N)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of ϑ^G . (Corollary 6.17 in [Isa]).

Note that the characters $\beta \in \text{Irr}(G/N)$ are actually characters of G that satisfy $N \subseteq \ker \chi$, where $\ker \chi = \{g \in G \mid \chi(g) = \chi(1)\}$.

1.3 Basic properties of $r_n(G)$

A more general approach to the Kurovka problem is to investigate the behavior of the sum of n th powers of complex character degrees.

Let G be a finite group and $Irr(G)$ be the set of complex irreducible characters of G .

We begin by defining

$$r_n(G) := \frac{1}{|G|} \sum_{\chi \in Irr(G)} \chi(1)^n.$$

Since $\sum_{\chi \in Irr(G)} \chi(1)^2 = |G|$ (See Result 5), we get that $r_2(G)$ is always equal to 1. Moreover, $r_3(G)$ is a weighted average of the character degrees, where a character degree $\chi(1)$ has the weight $\frac{\chi(1)^2}{|G|}$. We shall study this quantity $r_n(G)$ for $n \geq 3$ as it is very closely related to the original question from the Kurovka notebook (see chapter 3 for $r_1(G)$). Once again, all the groups mentioned from this point on are assumed to be finite.

We begin with some simple properties of $r_n(G)$.

Proposition 1.1. $r_n(G) \geq 1$, equality occurs iff G is abelian.

Proof. Since

$$r_n(G) = \frac{\sum_{\chi \in Irr(G)} \chi(1)^n}{\sum_{\chi \in Irr(G)} \chi(1)^2}$$

and character degrees are positive integers, then it is quite easy to see that $r_n(G) \geq 1$.

Now assume $r_n(G) = 1$. Then $\sum_{\chi \in Irr(G)} \chi(1)^n = \sum_{\chi \in Irr(G)} \chi(1)^2$ and, therefore,

$$\sum_{\chi \in Irr(G)} \chi(1)^2 \cdot (\chi(1)^{n-2} - 1) = 0.$$

Each term in the sum on the left is non-negative as $\chi(1) \geq 1$ for every irreducible character χ . Hence, $\chi(1) = 1$ for every character $\chi \in Irr(G)$, so all of the irreducible characters of G are linear. This implies that G is abelian (by Result 4). The converse is trivial. $\square$

Proposition 1.2. For any groups H and K , $r_n(H \times K) = r_n(H) \cdot r_n(K)$.

Proof. Since $\text{Irr}(H \times K) = \{\varphi \times \vartheta \mid \varphi \in \text{Irr}(H), \vartheta \in \text{Irr}(K)\}$ (see Result 7), we have

$$\begin{aligned}
 r_n(H \times K) &= \frac{1}{|H \times K|} \sum_{\chi \in \text{Irr}(H \times K)} \chi(1)^n \\
 &= \frac{1}{|H| \cdot |K|} \sum_{\substack{\psi \in \text{Irr}(H) \\ \varphi \in \text{Irr}(K)}} \psi(1)^n \cdot \varphi(1)^n \\
 &= \left(\frac{1}{|H|} \sum_{\psi \in \text{Irr}(H)} \psi(1)^n \right) \times \left(\frac{1}{|K|} \sum_{\varphi \in \text{Irr}(K)} \varphi(1)^n \right) \\
 &= r_n(H) \cdot r_n(K),
 \end{aligned}$$

as desired. □

There is an interesting link between $r_n(G)$ and $Pr(G)$, where

$$Pr(G) = \{ (x, y) \in G \times G \mid xy = yx \} / |G|^2$$

is the probability that two group elements chosen at random will commute. It turns out to be a rather interesting and relevant link and, therefore, we shall introduce some results from probabilistic group theory. It should be noted that not all of the results cited below will be used in this thesis, however, they are interesting on their own.

1. $Pr(H \times K) = Pr(H) \cdot Pr(K)$. See [Gus].
2. If G is nonabelian, then $Pr(G) \leq \frac{5}{8}$. Equality is observed for a dihedral group of order 8. The origins of this inequality are unknown. See [Dix] and [Gus].
3. If $Pr(G) = \frac{5}{8}$, then G is nilpotent. See [Gus]
4. If G is simple and nonabelian, then $Pr(G) \leq \frac{1}{12}$. Equality occurs for the alternating group A_5 . See [Gus]
5. If $N \triangleleft G$, then $Pr(G) \leq Pr(G/N) \cdot Pr(N)$. See [Mac].

For more results of similar nature one should look at [Dix], [Gus] and [Mac].

Lemma 1.1. *Let $k(G)$ denote the number of conjugacy classes of the group G . Then*

$$Pr(G) = \frac{k(G)}{|G|}.$$

Proof. We shall give a proof as it appeared in a paper by Gustafson (See [Gus]). Denote $C = \{ (x, y) \in G \times G \mid xy = yx \}$, then $Pr(G) = \frac{|C|}{|G|^2}$. Note that for each $x \in G$, the number of pairs (x, y) with $xy = yx$ is $|C_G(x)|$, where $C_G(x)$ is the centralizer of x in G . Therefore, $|C| = \sum_{x \in G} |C_G(x)|$. However, if x and y are conjugate elements of G , then so are their centralizers. Moreover, the number of elements in the conjugacy class of x is $[G : C_G(x)]$. Let $x_1, x_2, \dots, x_k$ be the representatives of the conjugacy classes in G , here $k = k(G)$. Then we have

$$|C| = \sum_{x \in G} |C_G(x)| = \sum_{i=1}^k [G : C_G(x_i)] \cdot |C_G(x_i)| = k \cdot |G|.$$

Hence, $Pr(G) = \frac{k|G|}{|G|^2} = \frac{k}{|G|}$, as was necessary to prove. $\square$

Thus one can think of $Pr(G)$ as $r_0(G)$. Now let us examine the connection between $r_n(G)$ and $Pr(G)$. We need the following lemma first.

Lemma 1.2. *The inequality*

$$\sqrt[n]{\frac{\sum_{i=1}^k a_i^n}{k}} \leq \sqrt[n+1]{\frac{\sum_{i=1}^k a_i^{n+1}}{k}}$$

holds for all real positive numbers a_i . Moreover, equality is attained iff all a_i 's are equal.

Proof. This lemma follows immediately from a more general result found in [HLP] (See page 26), which states in part:

For all real numbers r and vectors $a = (a_1, \dots, a_k)$ of real positive numbers we denote

$$M_r(a) = \sqrt[r]{\frac{\sum_{i=1}^k a_i^r}{k}}.$$

Then for $0 < r < s$ we have $M_r(a) \leq M_s(a)$ with strict inequality unless all the a_i 's are equal. $\square$

Lemma 1.3. $\frac{r_{n+1}(G)^n}{r_n(G)^{n+1}} \geq \frac{1}{Pr(G)}$, with equality occurring iff G is abelian. In particular, $r_3(G) \geq \frac{1}{\sqrt{Pr(G)}}$.

Proof. From Lemma 1.2 we know that $M_n(C) \leq M_{n+1}(C)$ where C is just a vector containing all character degrees. Moreover, equality occurs iff all the elements of C are equal, i.e. all the characters are linear. Raising both sides of this inequality to $n(n+1)$ th power we obtain

$$\left(\frac{\sum_{\chi \in Irr(G)} \chi(1)^n}{k} \right)^{n+1} \leq \left(\frac{\sum_{\chi \in Irr(G)} \chi(1)^{n+1}}{k} \right)^n$$

where $k = k(G)$ is the number of irreducible characters of G (and also the number of conjugacy classes of G).

Since $\sum_{\chi \in Irr(G)} \chi(1)^n = r_n(G) \cdot |G|$ and $\sum_{\chi \in Irr(G)} \chi(1)^{n+1} = r_{n+1}(G) \cdot |G|$, the inequality simplifies to

$$\left(\frac{r_n(G) \cdot |G|}{k} \right)^{n+1} \leq \left(\frac{r_{n+1}(G) \cdot |G|}{k} \right)^n.$$

Cancelling k and $|G|$ yields $\frac{|G|}{k} \leq \frac{r_{n+1}(G)^n}{r_n(G)^{n+1}}$, where $\frac{|G|}{k} = \frac{1}{Pr(G)}$.

As mentioned above, equality is attained iff all the characters $\chi \in Irr(G)$ are linear, however, this means that the group G is abelian (by Result 4). This completes the proof. $\square$

Going back to the sum of the cubes, we can use the lemma above to deduce a rather interesting corollary.

Corollary 1.1. *If G is a nonabelian group, then $r_3(G) > \sqrt{\frac{8}{5}}$.*

The statement of this corollary follows from Lemma 1.3 and the fact that $Pr(G) \leq \frac{5}{8}$ for any nonabelian group G . Also note that the inequality is strict since $r_n(G)$ is always a rational number. This implies that the bound in Corollary 1.1 is not the best, however, the improvement can be made quite easily.

Proposition 1.3. *If G is a nonabelian group, then $r_n(G) \geq \frac{2^{n-2}+1}{2}$. Equality is attained by the dihedral group D_8 . In particular, $r_3(G) \geq \frac{3}{2}$.*

Proof. Assume the contrary, i.e. for some nonabelian group G we have $r_n(G) < \frac{2^{n-2}+1}{2}$. Let m_s denote the number of irreducible characters of G that have degree s . Then, the inequality $r_n(G) < \frac{2^{n-2}+1}{2}$ is equivalent to

$$\sum_s m_s \cdot s^n < \frac{2^{n-2}+1}{2} \cdot \sum_s m_s \cdot s^2.$$

This implies

$$\sum_s m_s \cdot s^2 \cdot (2s^{n-2} - 2^{n-2} - 1) < 0$$

equivalently,

$$(1 - 2^{n-2})m_1 + \sum_{s \geq 2} m_s \cdot s^2 \cdot (2s^{n-2} - 2^{n-2} - 1) < 0.$$

Hence, $(2^{n-2} - 1)m_1 > \sum_{s \geq 2} m_s \cdot s^2 \cdot (2s^{n-2} - 2^{n-2} - 1)$. However,

$$\begin{aligned} \sum_{s \geq 2} m_s \cdot s^2 \cdot (2s^{n-2} - 2^{n-2} - 1) &\geq \sum_{s \geq 2} m_s \cdot s^2 \cdot (2 \cdot 2^{n-2} - 2^{n-2} - 1) \\ &= (2^{n-2} - 1) \cdot \sum_{s \geq 2} m_s \cdot s^2 \\ &= (2^{n-2} - 1) \cdot (|G| - m_1) \end{aligned}$$

yielding $m_1 > |G| - m_1$ and so $|G| < 2m_1$. Since m_1 is equal to the number of linear characters of G , which in turn is equal to $[G : G']$ (see Result 2), we obtain

$$|G| < 2[G : G'] = 2 \cdot \frac{|G|}{|G'|}.$$

Hence, $|G'| < 2$, i.e. $|G'| = 1$ and so G is an abelian group, a contradiction. Therefore, for a nonabelian group G , we have $r_n(G) \geq \frac{2^{n-2}+1}{2}$.

Since the dihedral group D_8 has 4 linear characters and one character of degree 2, we obtain $r_n(D_8) = \frac{4+2^n}{8} = \frac{2^{n-2}+1}{2}$. Thus, D_8 attains a lower bound of this proposition. $\square$

The method used to establish this proposition is quite useful, and will be used to prove Proposition 1.5.

Let us introduce a few more interesting facts about $r_n(G)$.

Proposition 1.4. *If G possesses an abelian subgroup A of index m , then $r_n(G) \leq m^{n-2}$.*

Proof. Result 6 says that if G possesses an abelian subgroup A of index m , then for any $\chi \in \text{Irr}(G)$ we have $\chi(1) \leq m$. Applying this fact we obtain

$$\sum_{\chi \in \text{Irr}(G)} \chi(1)^n = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 \cdot \chi(1)^{n-2} \leq m^{n-2} \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = m^{n-2} |G|$$

and, therefore, $r_n(G) \leq m^{n-2}$. $\square$

So it seems natural to pose a question of describing groups for which $r_n(G)$ is bounded by a specific number. As it turns out, answering this question is not a trivial task at all. Let us consider the following result.

Proposition 1.5. *If G is a nonabelian group with $r_n(G) \leq \frac{3^{n-2}+1}{2}$, then $|G|$ must be even.*

Proof. Suppose that $|G|$ is odd, so, by Result 5, all irreducible characters have odd degree. Rewriting the inequality $r_n(G) \leq \frac{3^{n-2}+1}{2}$ we obtain

$$2 \sum_s m_s \cdot s^n \leq (3^{n-2} + 1) \cdot \sum_s m_s \cdot s^2$$

where, once again, m_s denotes the number of irreducible characters of degree s . Rewriting this as

$$\sum_s m_s \cdot s^2 \cdot (2s^{n-2} - 3^{n-2} - 1) \leq 0$$

yields

$$-(3^{n-2} - 1)m_1 + \sum_{s \geq 3} m_s \cdot s^2 \cdot (2s^{n-2} - 3^{n-2} - 1) \leq 0.$$

Note that the term involving $s = 2$ is not present since $m_2 = 0$ by our assumption. Now let us estimate the remaining sum from below,

$$\begin{aligned} \sum_{s \geq 3} m_s \cdot s^2 \cdot (2s^{n-2} - 3^{n-2} - 1) &\geq \sum_{s \geq 3} m_s \cdot s^2 \cdot (2 \cdot 3^{n-2} - 3^{n-2} - 1) \\ &= (3^{n-2} - 1) \cdot \sum_{s \geq 3} m_s \cdot s^2 = (3^{n-2} - 1) \cdot (|G| - m_1) \end{aligned}$$

Once again m_2 is not present. Thus,

$$(3^{n-2} - 1)m_1 \geq (3^{n-2} - 1) \cdot (|G| - m_1) ,$$

yielding $|G| - m_1 \leq m_1$ and so $|G| \leq 2m_1$. Since m_1 is the number of linear characters, we have $m_1 = [G : G']$ which leads to $|G'| \leq 2$. Since, by our assumption $|G|$ is odd, the possibility of $|G'| = 2$ can be ruled out. This means that $|G'| = 1$ and so G is abelian - a contradiction. Therefore, $|G|$ must be even. $\square$

In particular, for $n = 3$ the proposition above says that if $r_3(G) \leq 2$ then $|G|$ must be even.

The method we applied in Propositions 1.3 and 1.5 did not succeed in producing similar results when considering $r_3(G) \leq 3$, etc.

Chapter 2

Examples

2.1 Dihedral groups

Here we shall calculate $r_3(G)$ for some groups. We begin with the dihedral groups.

Recall that if G is a dihedral group of order $2n$, denoted as D_{2n} , then it can be described as follows:

$$G = \{ a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \}.$$

It is not hard to show that $G' = \begin{cases} \langle a \rangle, & n \text{ is odd;} \\ \langle a^2 \rangle, & n \text{ is even} \end{cases}$ and so the number of linear characters of D_{2n} is $|G : G'| = \begin{cases} 2, & n \text{ is odd;} \\ 4, & n \text{ is even.} \end{cases}$

The information about the number of conjugacy classes of D_{2n} is also easy to obtain (See page 133 in [Dum]): $k(G) = \begin{cases} \frac{n+3}{2}, & n \text{ is odd;} \\ \frac{n}{2} + 3, & n \text{ is even.} \end{cases}$

So now we are ready to calculate $r_3(G)$ for the dihedral group. We just have to keep in mind that the maximum character degree for D_{2n} is 2 as it has an abelian subgroup of index 2, namely $\langle a \rangle$.

First, consider the case when n is odd, that is, $n = 2m + 1$. Here we have $k(G) = m + 2$ and $|G : G'| = 2$. This means that we have 2 linear characters and m characters of degree 2. Thus,

$$r_3(G) = \frac{2 \cdot 1^3 + m \cdot 2^3}{2n} = \frac{8m + 2}{2n} = \frac{4n - 2}{2n} = 2 - \frac{1}{n}.$$

Now consider the case when n is even, i.e. $n = 2m$. Here we have $k(G) = m + 3$ and $|G : G'| = 4$. Therefore, we have 4 linear characters and $m - 1$ characters of degree 2. Hence,

$$r_3(G) = \frac{4 \cdot 1^3 + (m - 1) \cdot 2^3}{2n} = \frac{8m - 4}{2n} = \frac{4n - 4}{2n} = 2 - \frac{2}{n}.$$

Putting the above calculated information together we get the following result.

Lemma 2.1. *For a dihedral group D_{2n} , we have $r_3(D_{2n}) = \begin{cases} 2 - \frac{1}{n}, & n \text{ is odd;} \\ 2 - \frac{2}{n}, & n \text{ is even.} \end{cases}$*

It should be noted that D_{2n} satisfies the hypothesis of Proposition 1.5 with respect to $r_3(G)$. Also it is not hard to verify that D_8 is the only dihedral group which attains the lower bound with respect to $r_3(G)$ in Proposition 1.3.

2.2 Extraspecial p -groups

A p -group G such that $G' = Z(G)$ and $|G'| = p$ is said to be an *extraspecial p -group*. It is known that the order of such a group is p^{2n+1} for some $n \in \mathbb{N}$ and that it has p^{2n} linear characters and $p - 1$ characters of degree p^n (See page 49 in [Ber]).

Using this information we can calculate $r_3(G)$. Namely,

$$r_3(G) = \frac{p^{2n} \cdot 1^3 + (p - 1) \cdot p^{3n}}{p^{2n+1}} = p^n - p^{n-1} + \frac{1}{p}. \quad (2.1)$$

This means that $r_3(G) > p^n - p^{n-1} = \left(1 - \frac{1}{p}\right) \cdot p^n = \left(1 - \frac{1}{p}\right) \cdot \sqrt{\frac{|G|}{p}}$. By setting $p = 2$ we obtain

$$r_3(G) > \frac{1}{2} \cdot \sqrt{\frac{|G|}{2}}. \quad (2.2)$$

This inequality is satisfied by infinitely many extraspecial 2-groups. Therefore, a universal upper bound (i.e. a bound which does not depend on the order of the group) for $r_3(G)$ does not exist.

Let us examine the behavior of $r_3(G)$ with regards to largest character degree of the group. We have seen in the previous section that $r_3(D_{2n})$ is relatively close to 2 which is the largest character degree of the dihedral group. The largest character degree of the group G is usually denoted as $b(G)$, i.e. $b(G) = \max \{ \chi(1) \mid \chi \in Irr(G) \}$.

Since $b(G)^2 < \sum_{\chi \in Irr(G)} \chi(1)^2 = |G|$, we can easily obtain the following inequality:

$$b(G) < \sqrt{|G|}. \quad (2.3)$$

Following the same idea as in Proposition 1.4 we can obtain an upper bound on $r_3(G)$ in terms of $b(G)$.

Corollary 2.1. $r_3(G) \leq b(G)$. Equality occurs iff G is abelian.

So in the case of extraspecial p -groups, $r_3(G) > p^n - p^{n-1} = \left(1 - \frac{1}{p}\right) \cdot b(G)$, i.e.

$$\left(1 - \frac{1}{p}\right) \cdot b(G) < r_3(G) \leq b(G). \quad (2.4)$$

A similar bound which is in terms of $|G|$ is the following:

$$\left(1 - \frac{1}{p}\right) \cdot \sqrt{\frac{|G|}{p}} < r_3(G) < \sqrt{|G|} \quad (2.5)$$

2.3 Metacyclic groups

An excellent paper written by B.G. Basmaji (see [Bas]) gives a detailed description of constructing irreducible representations of metacyclic groups. For our purposes we only need the character degrees. We are going to follow Basmaji's notation very closely.

A finite group G is said to be a *metacyclic group* if G has a cyclic normal subgroup A such that G/A is also cyclic. Dihedral groups are an example of metacyclic groups.

Any metacyclic group G can be written as follows:

$$G = \{ a, b \mid a^n = b^m = 1, a^k = b^t, b^{-1}ab = a^r \} \quad (2.6)$$

where $t \mid m$, $r^t \equiv 1 \pmod{n}$ and $kr \equiv k \pmod{n}$, therefore, $\gcd(r, n) = 1$.

We shall also need the following information.

Let U be the set of all complex n th roots of unity and define a map $\sigma : U \mapsto U$ by $\sigma(\xi) = \xi^r$ for all $\xi \in U$. For each $\xi \in U$ define the size of the orbit of ξ under $\langle \sigma \rangle$ by $t(\xi)$. Also let $s(\xi)$ be the order of ξ in U . In his paper, Basmaji constructs irreducible representations $T_{\xi, \theta}$ for some specific θ . Since we are interested in character degrees only, we shall sum up the corresponding results of Basmaji's paper into one lemma.

Lemma 2.2. *For a metacyclic group described above, the following conditions hold:*

1. *The degree of each irreducible representation $T_{\xi, \theta}$ is $t(\xi)$.*
2. *Each representation $T_{\xi, \theta}$ is counted up to equivalence $t(\xi)$ times as ξ runs over U .*
3. *For each ξ there are $t/t(\xi)$ different θ 's producing inequivalent representations $T_{\xi, \theta}$.*

Based on the lemma above we can easily calculate the sum of cubes of irreducible character degrees for a metacyclic group. Namely,

$$\sum_{\chi \in \text{Irr}(G)} \chi(1)^3 = \sum_{\xi \in U} t(\xi)^3 \cdot \frac{t}{t(\xi)} \cdot \frac{1}{t(\xi)} = t \cdot \sum_{\xi \in U} t(\xi).$$

Since the order of our metacyclic group G is equal to $n \cdot t$ (See Proposition 3 in [Bas]), we obtain

$$r_3(G) = \frac{\sum_{\xi \in U} t(\xi)}{n} \quad (2.7)$$

It is easy to see that the action of $\langle \sigma \rangle$ on U is equivalent to the action of r , not to be confused with $r_3(G)$, on $\mathbb{Z}_n$, where $\langle r \rangle$ is a subgroup of $\mathbb{Z}_n^*$ and the action of $\langle r \rangle$ on $\mathbb{Z}_n$ is multiplicative.

Let $x_1, x_2, \dots, x_e$ be representatives of orbits under the action of $\langle r \rangle$. Then $\sum_{\xi \in U} t(\xi) = \sum_{i=1}^e |O(x_i)|^2$, where $|O(x_i)|$ denotes the size of the orbit having a representative x_i .

Before we proceed to further calculate $\sum_{\xi \in U} t(\xi)$, we need some results.

Proposition 2.1. *The action of $\langle r \rangle$ on $\mathbb{Z}_n$ is trivial iff G is abelian.*

Proof. Assume $\langle r \rangle$ acts trivially on $\mathbb{Z}_n$, i.e. for each $x \in \mathbb{Z}_n$ we have $xr \equiv x \pmod{n}$. This implies $(r-1)x \equiv 0 \pmod{n}$. Now take an element $u \in \mathbb{Z}_n^*$, then $(r-1)u \equiv 0 \pmod{n}$ and so $r-1 \equiv 0 \pmod{n}$, in other words, $n \mid r-1$. In this case every orbit is going to have size 1 and, therefore, $r_3(G) = 1$. This implies that G is abelian. The converse is trivial. $\square$

Note that in the proposition above “ G is abelian” can be replaced by “ $n \mid r-1$ ”. The next result gives us a rough bound on the value of $r_3(G)$.

Proposition 2.2. *If there are e orbits under the action of $\langle r \rangle$, then $\frac{n}{e} \leq r_3(G) < n$. We have $r_3(G) = \frac{n}{e}$ iff G is abelian.*

Proof. If we denote the size of the orbit $O(x_i)$ by a_i , then

$$r_3(G) = \frac{\sum a_i^2}{\sum a_i}$$

as $i = 1, 2, \dots, e$. Clearly $r_3(G) < \sum a_i = n$ as the orbits $O(x_i)$ partition $\mathbb{Z}_n$. Applying Cauchy's inequality we obtain:

$$\sqrt{\frac{\sum a_i^2}{e}} \geq \frac{\sum a_i}{e}$$

and so

$$\frac{\sum a_i^2}{\sum a_i} \geq \frac{\sum a_i}{e} = \frac{n}{e}$$

with equality occurring iff all a_i 's are equal, i.e. when G is abelian. $\square$

So now we can assume that G is a nonabelian metacyclic group and proceed to study the action of $\langle r \rangle$ on $\mathbb{Z}_n$ in order to determine $\sum_{\xi \in U} t(\xi)$.

Take an arbitrary $x \neq 0$ from $\mathbb{Z}_n$, say $\gcd(x, n) = d$. Hence, there exists $u \in \mathbb{Z}_n$ such that $ux \equiv d \pmod{n}$. The orbit of x will look like: $O(x) = \{x, xr, xr^2, \dots, xr^s = x\}$. So $x(r^s - 1) \equiv 0 \pmod{n}$, hence $ux(r^s - 1) \equiv 0 \pmod{n}$. Therefore, $r^s \equiv 1 \pmod{\left(\frac{n}{d}\right)}$ and since s is the first such integer, we deduce that $s = \text{order of } r \text{ in } \mathbb{Z}_{n/d}^*$.

Note that every element in $O(x)$ has the same gcd with n , since r and n are coprime. So if we have $y \notin O(x)$ such that $\gcd(y, n) = \gcd(x, n) = d$, then $|O(y)| = |O(x)| = \text{order of } r \text{ in } \mathbb{Z}_{n/d}^*$ which we shall denote by $s(d')$ (here $d' = n/d$). The number of elements of $\mathbb{Z}_n$ which have gcd d with n is $\varphi(d')$, where $\varphi(x)$ is Euler's φ -function. Hence,

$$\sum_{\xi \in U} t(\xi) = \sum_x |O(x)|.$$

Here we are not summing over the orbit representatives so we do not have $|O(x)|^2$ in the sum (instead we are summing over all x). Also,

$$\sum_{\gcd(x,n)=d} |O(x)| = \varphi(d') \cdot s(d')$$

implying

$$\sum_x |O(x)| = \sum_{d|n} \varphi(d') \cdot s(d') = \sum_{d|n} \varphi(d) \cdot s(d).$$

Finally,

$$r_3(G) = \frac{\sum_{d|n} \varphi(d) \cdot s(d)}{n} \quad (2.8)$$

where $s(d)$ is the order of r in $\mathbb{Z}_d^*$.

Chapter 3

$f(G)$ and $r_3(G)$

In their recently published book Berkovich and Zhmud' mention a quantity similar to $r_n(G)$ (See Chapter 11 in [Ber]). It is denoted by $f(G)$ and is equal to $\frac{1}{|G|} \cdot \sum_{\chi \in \text{Irr}(G)} \chi(1)$. So, in fact, $f(G) = r_1(G)$. It seems natural to study the relation between $f(G)$ and $r_n(G)$, and we are going to do it for the case $n = 3$. That is the purpose of this chapter.

First we shall state the main theorem from Chapter 11 in [Ber].

A prime divisor p of the order of a nonabelian group G is said to be *essential* if a Sylow p -subgroup of G does not belong to $Z(G)$.

Theorem 3.1. *Let p be the smallest essential prime divisor of the order of a nonabelian group G . If $f(G) > 1/p$, then one of the following statements holds.*

1. G contains an abelian subgroup of index p ,
2. $|G'| = p$,
3. $p = 2$, $G = G_1 * G_2$ (central product), $|G_1'| = |G_2'| = 2$, $f(G_1) = 3/4 = f(G_2)$, $|G'| = 4$,
4. $\text{cd}(G) = \{1, p\}$ where $\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$. $G/Z(G)$ is of order p^3 and exponent p .

Conversely, if G is a group with properties (1) - (4), then $f(G) > 1/p$.

Here are some more interesting results from [Ber] (See Lemmas 6, 7 in Chapter 11).

Lemma 3.1. *Let H be a subgroup of a group G . Then the following are true.*

1. $Pr(G) \leq Pr(H)$.
2. If there are α characters of degree $n \notin cd(H)$ in $Irr(G)$, then $f(H) \geq f(G) + \frac{n\alpha}{|G|}$.
In particular, $f(H) \geq f(G)$.
3. $f(G) = f(H)$ iff $Pr(G) = Pr(H)$. Then $G' = H'$, and all normal subgroups of H are normal in G .
4. If $H \triangleleft G$, then $f(G) \leq f(H) - \frac{1}{|H|} \cdot (1 - f(G/H))$.

Lemma 3.2. $f(G)^2 \leq Pr(G)$, with equality iff G is abelian.

It should be noted that in [Ber], $Pr(G)$ is denoted as $mc(G)$.

We begin by proving:

Lemma 3.3. $f(G) \cdot r_3(G) \geq 1$, with equality occurring iff G is abelian.

Proof. Here we are going to use Cauchy-Schwarz inequality.

$$|G|^2 = \left(\sum_{\chi \in Irr(G)} \chi(1)^2 \right)^2 = \left(\sum_{\chi \in Irr(G)} \chi(1)^{1/2} \cdot \chi(1)^{3/2} \right)^2 \leq \sum_{\chi \in Irr(G)} \chi(1) \cdot \sum_{\chi \in Irr(G)} \chi(1)^3$$

and, therefore,

$$1 \leq \left(\frac{1}{|G|} \sum_{\chi \in Irr(G)} \chi(1) \right) \cdot \left(\frac{1}{|G|} \sum_{\chi \in Irr(G)} \chi(1)^3 \right) = f(G) \cdot r_3(G).$$

Note that equality occurs if and only if the ratio $\frac{\chi(1)^{1/2}}{\chi(1)^{3/2}}$ is constant for each character $\chi \in Irr(G)$. The latter implies that all $\chi(1)$'s are equal and, therefore, G is abelian. $\square$

Let us illustrate another proof of Lemma 1.3 for $r_3(G)$, using the results above. By Lemma 3.2 we have, $f(G)^2 \leq Pr(G)$. Multiplying both sides by $r_3(G)^2$ we obtain

$$(f(G)r_3(G))^2 \leq Pr(G) \cdot r_3(G)^2.$$

However, Lemma 3.3 tells us that $(f(G)r_3(G))^2 \geq 1$ and so

$$1 \leq (f(G)r_3(G))^2 \leq Pr(G) \cdot r_3(G)^2$$

yielding $r_3(G)^2 \geq \frac{1}{Pr(G)}$. Note that by Lemmas 3.2 and 3.3 the equality is attained iff G is abelian.

Referring to Section 2.1 about dihedral groups we can deduce the following result.

Corollary 3.1. *If G is a dihedral group of order $2n$ where n is odd, then*

$$\lim_{n \rightarrow \infty} f(G) \cdot r_3(G) = 1.$$

Proof. From Lemma 2.1 we know that for n odd, $r_3(G) = 2 - \frac{1}{n}$. Say $n = 2m + 1$ for some $m \in \mathbb{Z}$. Recalling the results from Section 2.1 about character degrees, we can establish that for odd n we shall have 2 linear and m nonlinear characters of degree 2. Thus $f(G) = \frac{2 \cdot 1 + m \cdot 2}{2n} = \frac{n+1}{2n}$. Hence, $\lim_{n \rightarrow \infty} f(G) \cdot r_3(G) = \frac{1}{2} \cdot 2 = 1$, as desired. $\square$

Calculations with small groups and Corollary 3.1 suggest that there may be a universal upper bound for the quantity $f(G)r_3(G)$, i.e. a bound which does not depend on the order of the group G . As we shall see, however, such a universal bound does not exist. It is sufficient to look at the extraspecial p -groups for a counterexample.

Lemma 3.4.

1. *For infinitely many extraspecial 2-groups G , we have $f(G)r_3(G) \approx \frac{1}{4}\sqrt{\frac{|G|}{2}}$.*
2. *For all groups G , the following holds, $f(G)r_3(G) \leq |G|^{1/2}$.*

Proof. Since

$$f(G) = \frac{1}{|G|} \sum_{\chi \in Irr(G)} \chi(1) = \frac{\sum_{\chi \in Irr(G)} \chi(1)}{\sum_{\chi \in Irr(G)} \chi(1)^2}$$

we obtain $f(G) \leq 1$. By Equation 2.3 and Corollary 2.1 and the fact that $f(G) \leq 1$ we can easily establish the second part of this lemma.

Now suppose that G is an extraspecial p -group of order p^{2n+1} . From Section 2.2 we know that $r_3(G) \approx p^n - p^{n-1}$. Since G has p^{2n} linear characters and $p - 1$ characters of degree p^n , we obtain

$$f(G) = \frac{p^{2n} \cdot 1 + (p - 1) \cdot p^n}{p^{2n+1}} \approx \frac{1}{p},$$

hence, $f(G)r_3(G) \approx p^{n-1} - p^{n-2}$. If $p = 2$, we get $f(G)r_3(G) \approx 2^{n-2}$. However, $|G| = 2^{2n+1}$ so $2^{n-2} = \frac{1}{4}\sqrt{\frac{|G|}{2}}$ implying that $f(G)r_3(G) \approx \frac{1}{4}\sqrt{\frac{|G|}{2}}$, as was necessary to prove. $\square$

So even though a universal upper bound for $f(G)r_3(G)$ does not exist, a bound in terms of the group order can be established.

Chapter 4

Normal Subgroups

Generally we do not have a lot of information about how the characters of G behave when restricted to a subgroup H . However, the situation is different when $H \triangleleft G$, since then we have a wealth of information about the restriction of characters to H .

Here we shall explore how the quantities $r_n(G)$ and $r_n(H)$ are related if H is a normal subgroup of G , and prove a theorem which is an analogue of the following theorem of Isaacs and Passman (See Proposition 1.6 in [Isa2]).

Theorem 4.1. *Let $N \triangleleft G$ with G/N nonabelian. If all the irreducible characters of G have degree $\leq n$, then all the irreducible characters of N have degree $\leq \frac{n}{2}$.*

We can establish the same kind of “inductive” structure for $r_n(G)$.

Theorem 4.2. *Let G be a finite group and H a normal subgroup of G . Then the following statements hold:*

1. *If G/H is nonabelian, then $r_n(G) \geq \frac{2^{n-2}+1}{2} \cdot r_n(H)$.*
2. *If G/H is abelian, then $r_n(G) \geq r_n(H)$.*

These inequalities are the best possible. They can be attained in the first case when $G = D_8 \times H$, and in the second case when $G = A \times H$, where D_8 is the dihedral group of order 8 and A is any abelian group.

Proof. Consider the following families of sets: $\{Y = Irr(\chi_H) \mid \chi \in Irr(G)\}$ and $\{X = Irr(\varphi^G) \mid \varphi \in Irr(H)\}$. Thus each Y is the set of all irreducible characters of H which are constituents of χ_H ; similarly, each X is the set of all irreducible characters of G which are constituents of φ^G . Hence, $X \subset Irr(G)$ and $Y \subset Irr(H)$. We shall first establish that these sets of the form X and Y partition $Irr(G)$ and $Irr(H)$ respectively.

Note that if $\chi \in Irr(G)$ and $\varphi \in Irr(H)$, then

$$\chi \in Irr(\varphi^G) \iff \varphi \in Irr(\chi_H). \quad (4.1)$$

Indeed, by Frobenius Reciprocity:

$$\chi \in Irr(\varphi^G) \iff [\chi, \varphi^G] \geq 1, \text{ but } 1 \leq [\chi, \varphi^G] = [\chi_H, \varphi] \iff \varphi \in Irr(\chi_H).$$

Take two sets $X_1 = Irr(\vartheta_1^G), X_2 = Irr(\vartheta_2^G)$ and suppose there exists a character χ such that $\chi \in X_1$ and $\chi \in X_2$. Frobenius reciprocity tells us that $\vartheta_1 \in Irr(\chi_H)$ and $\vartheta_2 \in Irr(\chi_H)$. However, by Clifford's theorem (See Equation 1.4), $\chi_H = a_\chi \cdot \sum_{i=1}^t \varphi_i$ for some integer a_χ and where all of φ_i 's are conjugate in G . This implies that (without loss of generality) $\vartheta_1 = \varphi_1$ and $\vartheta_2 = \varphi_2$, so ϑ_1 and ϑ_2 are conjugate in G .

Now take an arbitrary character ξ from X_1 . Then, since $\varphi_1 \in Irr(\xi_H)$, we get $\xi_H = b_\xi \cdot \sum_{i=1}^t \varphi_i$ meaning that $\varphi_2 = \vartheta_2 \in Irr(\xi_H)$ and so $\xi \in Irr(\vartheta_2^G) = X_2$. Thus, $X_1 \subseteq X_2$, similarly we can establish $X_2 \subseteq X_1$. Therefore, $X_1 = X_2$. Hence, sets X_1 and X_2 are either disjoint or coincide.

Similarly we can show that the sets Y_1 and Y_2 are either disjoint or coincide. Moreover, because of Equation 4.1, there is a bijection between families of sets $\{X_i\}_i$ and $\{Y_i\}_i$. Thus, these sets partition $Irr(G)$ and $Irr(H)$ respectively.

It is sufficient to prove the following inequality for each pair X, Y of sets under this correspondence:

$$\sum_{\chi \in X} \frac{\chi(1)^n}{|G|} \geq \alpha \cdot \sum_{\varphi \in Y} \frac{\varphi(1)^n}{|H|} \quad (4.2)$$

where α is either 1 or $\frac{2^{n-2}+1}{2}$ depending on whether G/H is abelian or not.

Denote the left and right sums in Equation 4.2 by S_G and S_H respectively. To make our life easier we shall estimate their ratio. Before we proceed with proving Equation 4.2, we need some identities.

Say $|Y| = t$, i.e. χ_H has t irreducible constituents in H and, therefore, by Clifford's Theorem $\chi_H = a \sum_{j=1}^t \varphi_j$ where $\varphi_j \in Y$, a is a positive integer and Y is, once again, a set of G -conjugate characters in $\text{Irr}(H)$. All of φ_j 's have equal degrees since they are conjugate in G . Therefore, $\chi(1) = \chi_H(1) = at\varphi(1)$ for any $\varphi \in Y$. More generally,

$$\forall \chi_i \in X : \quad \chi_i(1) = a_i t \varphi(1). \quad (4.3)$$

Note that the a_i 's will not necessarily be all equal since they depend on the restriction of character χ_i to H , more generally, $a_i = [(\chi_i)_H, \varphi]$. Denote $g = |G|$, $h = |H|$. Recall that the degree of the induced character can be found as follows $\varphi^G(1) = \frac{g}{h} \varphi(1)$. However, $\varphi^G(1) = \sum_{\chi_i \in X} a_i \chi_i(1)$ due to Frobenius reciprocity. Thus,

$$\frac{g}{h} \varphi(1) = \sum_{\chi_i \in X} a_i \chi_i(1) = \sum_{\chi_i \in X} a_i^2 t \varphi(1) = t \varphi(1) \cdot \sum_{\chi_i \in X} a_i^2$$

because of Equation 4.3. Simplifying a bit we obtain:

$$\frac{g}{h} = t \sum_{\chi_i \in X} a_i^2. \quad (4.4)$$

So now let us take a look at the ratio S_G/S_H .

$$S_G = \frac{1}{g} \cdot \sum_{\chi_i \in X} \chi_i(1)^n = \frac{1}{g} \cdot \sum_{\chi_i \in X} a_i^n t^n \varphi(1)^n = \frac{t^n \varphi(1)^n}{g} \cdot \sum_{\chi_i \in X} a_i^n$$

as a result of Equation 4.3. Also,

$$S_H = \frac{1}{h} \cdot \sum_{\varphi_j \in Y} \varphi_j(1)^n = \frac{t \varphi(1)^n}{h}$$

since all $\varphi_j(1)$ are equal and the size of Y is t . Therefore,

$$\frac{S_G}{S_H} = \frac{\frac{t^n \varphi(1)^n}{g} \cdot \sum_{\chi_i \in X} a_i^n}{\frac{t \varphi(1)^n}{h}} = \frac{t^{n-1} \cdot \sum_{\chi_i \in X} a_i^n}{\frac{g}{h}} = \frac{t^{n-1} \cdot \sum_{\chi_i \in X} a_i^n}{t \cdot \sum_{\chi_i \in X} a_i^2}$$

as a consequence of Equation 4.4. Thus,

$$\frac{S_G}{S_H} = t^{n-2} \cdot \frac{\sum_{\chi_i \in X} a_i^n}{\sum_{\chi_i \in X} a_i^2}. \quad (4.5)$$

It is important to note that all a_i 's are integers since they are the multiplicities of characters from Y that appear in $(\chi_i)_H$.

Now we have to consider some special cases.

Case 1 Assume $t \geq 2$. Then $S_G/S_H \geq 2^{n-2} > \frac{2^{n-2}+1}{2}$ since $\frac{\sum_{\chi_i \in X} a_i^n}{\sum_{\chi_i \in X} a_i^2} \geq 1$. And so in this case the inequality 4.2 holds.

Case 2 Assume $t = 1$ and all a_i 's are at least 2. This means that

$$\frac{\sum_{\chi_i \in X} a_i^n}{\sum_{\chi_i \in X} a_i^2} \geq 2^{n-2} \cdot \frac{\sum_{\chi_i \in X} a_i^2}{\sum_{\chi_i \in X} a_i^2} = 2^{n-2} > \frac{2^{n-2}+1}{2}.$$

Hence, in this case inequality 4.2 is also valid.

Case 3 Assume $t = 1$ and there exists i for which $a_i = 1$. This means that $1 = a_i = [(\chi_i)_H, \varphi]$ and so $(\chi_i)_H = \varphi$, i.e. $(\chi_i)_H$ is irreducible.

Using a result due to Gallagher (See Section 1.2), we conclude that the characters $\beta\chi$ for $\beta \in \text{Irr}(G/H)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of φ^G . Thus, $X = \text{Irr}(\varphi^G) = \{\beta\chi \mid \beta \in \text{Irr}(G/H)\}$. Now let us reexamine the ratio S_G/S_H .

Clearly, $S_H = \frac{\varphi(1)^n}{h}$ since $Y = \{\varphi\}$ in this case. Also, $S_G = \frac{1}{g} \cdot \sum_{\chi_i \in X} \chi_i(1)^n$ where $\chi_i = \beta_i\chi$ for some $\beta_i \in \text{Irr}(G/H)$. Then

$$S_G = \frac{1}{g} \sum_{\beta_i \in \text{Irr}(G/H)} \beta_i(1)^n \cdot \chi(1)^n = \frac{\chi(1)^n}{g} \sum_{\beta_i \in \text{Irr}(G/H)} \beta_i(1)^n.$$

Hence,

$$\frac{S_G}{S_H} = \frac{\frac{\chi(1)^n}{g} \cdot \sum_{\beta_i \in \text{Irr}(G/H)} \beta_i(1)^n}{\frac{\varphi(1)^n}{h}} = \frac{\chi(1)^n}{\varphi(1)^n} \cdot \frac{1}{g/h} \sum_{\beta_i \in \text{Irr}(G/H)} \beta_i(1)^n.$$

However, since $\chi_H = \varphi$ we get $\chi(1) = \varphi(1)$. This leads to:

$$\frac{S_G}{S_H} = \frac{1}{g/h} \sum_{\beta_i \in \text{Irr}(G/H)} \beta_i(1)^n.$$

The quantity on the right side is exactly $r_n(G/H)$. Therefore, we obtain a nice relation:

$$\frac{S_G}{S_H} = r_n(G/H). \quad (4.6)$$

Applying the results of Propositions 1.3 and 1.1, we know that if G/H is non-abelian then $r_n(G/H) \geq \frac{2^{n-2}+1}{2}$ (equality is attained by D_8) and if G/H is abelian then $r_n(G/H) = 1$. Therefore, inequality 4.2 is satisfied in this case as well.

Finally, by summing over all sets X and Y we obtain the statements (1) and (2) of this theorem.

Now let us prove that the inequalities obtained are the best possible. By Proposition 1.2 we have $r_n(H \times K) = r_n(H)r_n(K)$. First, let $G = D_8 \times H$, then $r_n(G) = r_n(D_8)r_n(H) = \frac{2^{n-2}+1}{2} \cdot r_n(H)$ as $r_n(D_8) = \frac{2^{n-2}+1}{2}$. Lastly, if A is any abelian group and $G = A \times H$, then $r_n(G) = r_n(A)r_n(H) = r_n(H)$ as $r_n(A) = 1$ by Proposition 1.1. $\square$

Recall that a chain of subgroups $1 = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$ is called a *composition series* if each composition factor G_{i-1}/G_i is a simple group (that is G_i is a maximal proper normal subgroup of G_{i-1}).

The previous theorem leads to a rather interesting corollary.

Corollary 4.1. *If a group G has s nonabelian composition factors, then*

$$r_n(G) \geq \left(\frac{2^{n-2}+1}{2} \right)^s.$$

In particular, $r_3(G) \geq \left(\frac{3}{2}\right)^s$.

Proof. We know that s of these composition factors are nonabelian. Applying the result of Theorem 4.2 we see that if G_{i-1}/G_i is nonabelian, then $r_n(G_{i-1}) \geq \frac{2^{n-2}+1}{2} \cdot r_n(G_i)$; and if G_{i-1}/G_i is abelian we get $r_n(G_{i-1}) \geq r_n(G_i)$.

Starting with $G = G_0$ and proceeding down the chain all the way to $G_n = 1$ we will obtain

$$r_n(G) \geq \left(\frac{2^{n-2}+1}{2} \right)^s \cdot 1^{n-s}$$

as there are s nonabelian factors G_{i-1}/G_i and $n - s$ abelian ones. Hence,

$r_n(G) \geq (\frac{2^{n-2}+1}{2})^s$, as required.

□

Chapter 5

Main Results

We begin by stating a theorem due to Jordan (See Satz 200 in [Spe]).

Theorem 5.1. *(Jordan) There exists a function $J(n)$ such that, if a finite group G has a faithful representation of degree n over $\mathbb{C}$, then G has an abelian normal subgroup of index $\leq J(n)$.*

Over the years various functions satisfying Jordan's theorem have been found. Up until 1971 the best function $J(n)$ was the one derived by Blichfeldt in 1917, namely, $J(n) = n!(6^{n-1})^{\pi(n+1)+1}$. Weisfeiler has made a substantial improvement in 1984 (See [Wei]), where his function was $J(n) = (n+1)!n^{a \log n + b}$ for some constants a and b . It should be noted that Weisfeiler's proof depended on the Classification of Finite Simple Groups. For more information about the function $J(n)$ please look at [Kuz].

The function $J(n)$ from Jordan's theorem has some nice properties (See [Isa2]), namely:

1. $J(s)J(t) \leq J(s+t)$;
2. $J(n) \geq (n+1)!$, because the symmetric group S_{n+1} has a faithful representation of degree n .

Here we shall establish a result similar to Isaacs and Passman's theorem (See Theorem 1 in [Isa2]) which gives a bound on the index of an abelian subgroup based on the largest character degree. We quote their result below.

Theorem 5.2. (*Isaacs, Passman*) *There exists a function f defined on the positive integers with the following property. If G is a finite group all of whose irreducible representations have degree $\leq n$, then G has an abelian subgroup of index $\leq f(n)$. We may take*

$$f(n) = \prod_{i=0}^r J_1\left(\left\lceil \frac{n}{2^i} \right\rceil\right),$$

where $r = \lfloor \log_2 n \rfloor$ and $J_1(m) = \max\{J(m), m^{m+1}\}$.

The result that we are trying to establish is about the index of a solvable radical of a finite nonsolvable group G , and the analogue of the function $f(n)$ for $r_3(G)$. We note that $r_3(G)$ is a weighted average of character degrees and often much smaller than the parameter n in Theorem 5.2, which is the largest character degree.

Before we proceed to the main theorem we need some lemmas.

Lemma 5.1. *Let G be a finite group and X be the following set,*

$$X = \{\chi \in \text{Irr}(G) \mid \chi(1) < \frac{60}{59}r_3(G)\}.$$

If $H = \bigcap_{\chi \in X} \ker \chi$, then H has order < 60 and so is solvable.

Proof. First of all, note that the set X is nonempty as all the linear characters of G belong to it. Let d_i denote the degrees of the irreducible characters χ_i of G . Then,

$$r_3(G) \cdot |G| = \sum_{\chi_i \in X} d_i^3 + \sum_{\chi_i \notin X} d_i^3 > \sum_{\chi_i \notin X} d_i^3 \geq \frac{60}{59}r_3(G) \cdot \sum_{\chi_i \notin X} d_i^2 \quad (5.1)$$

as all the characters that do not belong to X are going to have degrees at least $\frac{60}{59}r_3(G)$. Note that $H \triangleleft G$ because it is an intersection of normal subgroups of G . Thus, we can

view the characters from X as characters of G/H since their kernels contain H (See Lemma 2.22 in [Isa]). Therefore, $\sum_{\chi_i \in X} d_i^2 \leq |G : H|$ which means that

$$\sum_{\chi_i \notin X} d_i^2 = |G| - \sum_{\chi_i \in X} d_i^2 \geq |G| \left(1 - \frac{1}{|H|}\right).$$

Applying Equation 5.1 we obtain, $r_3(G) \cdot |G| > \frac{60}{59} r_3(G) \cdot |G| \left(1 - \frac{1}{|H|}\right)$ and thus $\frac{59}{60} > 1 - \frac{1}{|H|}$ yielding $|H| < 60$. Since all the groups of order < 60 are solvable H must be solvable as well. $\square$

The lemma above can be modified to show the following.

Lemma 5.2. *If G is a finite group and $X = \{\chi \in \text{Irr}(G) \mid \chi(1) \leq 2r_3(G)\}$, then*

$H = \bigcap_{\chi \in X} \ker \chi$ is equal to 1.

Lemma 5.1 yields a rather important corollary.

Corollary 5.1. *If G is a nonsolvable group, then there exists $\chi \in \text{Irr}(G)$ such that $1 < \chi(1) < \frac{60}{59} r_3(G)$.*

Proof. Let X be the same set as defined in Lemma 5.1. Essentially we need to show that one of the characters in X is nonlinear. Assume the contrary, i.e. all the characters in X are linear. Thus, $\text{Lin}(G) = X$. However, $H = \bigcap_{\chi \in X} \ker \chi = \bigcap_{\chi \in \text{Lin}(G)} \ker \chi = G'$ (See Corollary 2.23 in [Isa]), meaning that $H = G'$. Thus, by Lemma 5.1, we have that G' is solvable implying that G is solvable - a contradiction. $\square$

We need the following definitions.

A subgroup H of the group G is said to be *characteristic in G* if H is invariant under any automorphism of the group G .

The maximal normal solvable subgroup of a finite group G is called the *solvable radical of G* , denoted by $\text{rad}(G)$.

Note that the existence of a solvable radical of any finite group is assured by Corollary 7.48 in [Ros]. Moreover, this radical is unique and is a characteristic subgroup of G .

Lemma 3.15 in [Ros] states the following.

Lemma 5.3. *If $H \triangleleft G$ and K is a characteristic subgroup of H then $K \triangleleft G$.*

Based on the lemma above, we can prove the following.

Lemma 5.4. *If $H \triangleleft G$ then $\text{rad}(H) \leq \text{rad}(G)$.*

Proof. Since $\text{rad}(H)$ is characteristic in H and H is a normal subgroup of G , we obtain $\text{rad}(H) \triangleleft G$ (by Lemma 5.3). So $\text{rad}(H)$ is a solvable normal subgroup of G and, therefore, must be contained in $\text{rad}(G)$ because $\text{rad}(G)$ is the maximal normal solvable subgroup of G . $\square$

The next lemma establishes a very important “inductive step” necessary for proving the main theorem.

Lemma 5.5. *If G is a nonsolvable group then there exists a normal subgroup H of G such that:*

1. $\ker \chi \leq H$ for some character $\chi \in \text{Irr}(G)$ with $1 < \chi(1) < \frac{60}{59}r_3(G)$;
2. G/H is nonabelian;
3. $[G : H] \leq J(\lfloor \frac{60}{59}r_3(G) \rfloor)$.

Proof. Let X be the set defined in Lemma 5.1, so $X = \{\chi \in \text{Irr}(G) \mid \chi(1) < \frac{60}{59}r_3(G)\}$. Define a set $Y = \{\chi \in X \mid \chi(1) > 1\}$ so Y is the set of nonlinear characters from the set X . Then, the set Y is nonempty because of Corollary 5.1 and so

$$T = \bigcap_{\chi \in X} \ker \chi = \left(\bigcap_{\chi \in Y} \ker \chi \right) \cap \left(\bigcap_{\lambda \in \text{Lin}(G)} \ker \lambda \right) = \left(\bigcap_{\chi \in Y} \ker \chi \right) \cap G'$$

since the intersection of kernels of linear characters of G is equal to G' (See Corollary 2.23 in [Isa]). Note that by Lemma 5.1, T is a solvable subgroup of G .

Now take an arbitrary character χ from the set Y , then $1 < \chi(1) < \frac{60}{59}r_3(G)$. We can view χ as a character of $G/\ker \chi$, moreover, χ would be a faithful character of

$G/\ker \chi$. Then by Jordan's theorem (Theorem 5.1), there exists a normal abelian subgroup $A(\chi)/\ker \chi$ of $G/\ker \chi$ such that

$$[G/\ker \chi : A(\chi)/\ker \chi] \leq J(\chi(1)) \leq J\left(\left\lfloor \frac{60}{59} r_3(G) \right\rfloor\right).$$

However, $[G/\ker \chi : A(\chi)/\ker \chi] = [G : A(\chi)]$ and so $[G : A(\chi)] \leq J\left(\left\lfloor \frac{60}{59} r_3(G) \right\rfloor\right)$. Also, since $A(\chi)/\ker \chi \triangleleft G/\ker \chi$ we obtain $A(\chi) \triangleleft G$. Thus, $A(\chi)$ is a perfect candidate for the role of the subgroup H in this lemma. The only thing that remains to be verified is that $G/A(\chi)$ is nonabelian.

Suppose that for all $\chi \in Y$ we have $G/A(\chi)$ is abelian. Then, for all $\chi \in Y$ we have $G' \leq A(\chi)$. Since $A(\chi)/\ker \chi$ is an abelian group, we get $A(\chi)' \leq \ker \chi$. Then for all $\chi \in Y$, $G'' \leq A(\chi)' \leq \ker \chi$ implying that $G'' \leq \bigcap_{\chi \in Y} \ker \chi$. Clearly $G'' \leq G'$ and thus

$$G'' \leq \left(\bigcap_{\chi \in Y} \ker \chi \right) \cap G' = T.$$

As we have mentioned before, T is a solvable group and, therefore, G'' is also solvable. However, the last statement implies that G is solvable - a contradiction.

Thus, there exists a character $\chi_0 \in Y$ such that $G/A(\chi_0)$ is nonabelian. We may take $H = A(\chi_0)$ to be the group that satisfies all the properties of this lemma. $\square$

And now onto the Main theorem.

Main Theorem. *There exists a realvalued function g defined on the positive integers such that if G is a nonsolvable group and $\text{rad}(G)$ is its solvable radical, then*

$[G : \text{rad}(G)] \leq g(r_3(G))$. We may take

$$g(r) = \prod_{i=1}^t J\left(\left[\left(\frac{2}{3}\right)^{i-1} \cdot \frac{60}{59} r\right]\right),$$

where $t = \left\lceil \frac{\log r}{\log 1.5} \right\rceil$.

Proof. For each group G denote $t = t(G)$ as the least integer such that $\left(\frac{3}{2}\right)^{t-1} < r_3(G) \leq \left(\frac{3}{2}\right)^t$ so $t(G) = \left\lceil \frac{\log r_3(G)}{\log 1.5} \right\rceil$. We proceed by induction on $t(G)$. The result is trivial (with $g(1) = 1$) when $t = 0$.

Suppose that $t(G) > 1$. If G is solvable then the result is trivial ($\text{rad}(G) = G$), so suppose that G is not solvable.

Let H be a subgroup satisfying all of the conditions of Lemma 5.5, then by Theorem 4.2 we have $r_3(H) \leq \frac{2}{3}r_3(G)$ and so $t(H) \leq t(G) - 1$. In the proof of Lemma 5.4 we saw that $\text{rad}(H) \triangleleft G$ and $\text{rad}(H) \leq \text{rad}(G)$, and so

$$[G : \text{rad}(G)] \leq [G : \text{rad}(H)] = [G : H] \cdot [H : \text{rad}(H)] ,$$

where $[G : H] \leq J(\lceil \frac{60}{59}r_3(G) \rceil)$. Since $t(H) \leq t(G) - 1$, by induction we obtain that $[G : \text{rad}(G)] \leq J(\lceil \frac{60}{59}r_3(G) \rceil) \cdot g(r_3(H))$. So we may take $g(r_3(G)) = J(\lceil \frac{60}{59}r_3(G) \rceil) \cdot g(r_3(H))$.

Repeating this step at most $t(G)$ times we shall end up with a solvable (actually abelian) subgroup H_t so $g(r_3(H_t)) = 1$. Thus,

$$g(r_3(G)) = J\left(\left\lceil \frac{60}{59}r_3(G) \right\rceil\right) J\left(\left\lceil \frac{60}{59}r_3(H_1) \right\rceil\right) J\left(\left\lceil \frac{60}{59}r_3(H_2) \right\rceil\right) \dots J\left(\left\lceil \frac{60}{59}r_3(H_{t-1}) \right\rceil\right) .$$

This leads to the function given in the formulation of the theorem. $\square$

Corollary 5.2. *If G is a nonsolvable group then $[G : \text{rad}(G)] \leq J\left(\lceil \frac{180}{59}r_3(G) \rceil\right)$.*

Proof. We are going to use the fact that $J(s)J(t) \leq J(s+t)$. Then

$$g(r) = \prod_{i=1}^t J\left(\left\lceil \left(\frac{2}{3}\right)^{i-1} \cdot \frac{60}{59}r \right\rceil\right) \leq J\left(\sum_{i=1}^t \left\lceil \left(\frac{2}{3}\right)^{i-1} \cdot \frac{60}{59}r \right\rceil\right) \leq J\left(\frac{60}{59}r \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1}\right) ,$$

however $\sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} = \frac{1}{1-\frac{2}{3}} = 3$, yielding $g(r) \leq J\left(\lceil \frac{180}{59}r \rceil\right)$. $\square$

Appendix A

Notations

G	finite group
$\text{tr} A$	trace of a matrix
$\text{Irr}(G)$	set of irreducible characters of group G
$\text{Lin}(G)$	set of linear characters of group G
$ G $	order of the group G
$H \leq G$	H is a subgroup of G
$H \triangleleft G$	H is a normal subgroup of G
$[G : H]$	index of a subgroup H in G
G'	derived subgroup
$Z(G)$	center of the group
G/H	factorgroup
$\text{rad}(G)$	solvable radical of group G
$k(G)$	number of conjugacy classes of G
$\langle x \rangle$	cyclic group generated by element x
$H \times K$	direct product of two groups
$C_G(x)$	centralizer of x in G
$\text{cd}(G)$	set of distinct irreducible character degrees of G

$b(G)$	largest character degree
$f(G)$	sum of character degrees divided by the order of the group
χ	character
χ_H	restriction of a character to a subgroup
φ^G	induced character
$\chi(1)$	character degree
$\ker \chi$	kernel of a character
$[\chi, \varphi]$	inner product of two characters
m_s	number of irreducible characters of degree s
$Pr(G)$	probability that two elements chosen at random will commute
$J(n)$	Jordan's function
$g(n)$	bound on the index of a solvable radical of G
$[\cdot], [\cdot]$	integer part of a real number
$r_n(G)$	sum of n th powers of irreducible character degrees divided by the order of the group

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