Mean Field Games with Poisson Point Processes and Impulse Control

by

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Abstract

This thesis considers mean field games in a continuous time competitive Markov decision process framework. Each player's state has pure jumps modeled by a self-weighted compound Poisson process subject to impulse control. We focus on analyzing the steady-state (or stationary) equation system of the mean field game. The best response is determined as a threshold policy and the stationary distribution of the state is derived in terms of the threshold value. The numerical solution of the equation system is developed. We further generalize the model to an unbounded state space.

Acknowledgements

First I wish to express my sincere gratitude to my supervisor Dr. Minyi Huang, who introduced me to the MDP and MFG and gave me this opportunity to work on the project since 2016 summer. I've been his student for almost two years. During the two years, he not only gave me suggestions on course selection, but also helped me to find my academic interest and learn how to do a research paper. Moreover, his rigorous attitude towards academics profoundly influenced me. Without his guidance and help, this thesis would not have been possible.

When I finished my first Master program in biological mathematics, I realized that I still wanted to follow my heart in studying statistics, especially in stochastic. So I would like to give a big thanks to my parents and my husband for supporting my choice to enter this program.

Last but not the least, I want to thank my colleagues Fangyi and Xiang for helping me solve the problems in using software.

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Chapter 1

Introduction

1.1 Background of Mean Field Games

Mean field game theory studies stochastic dynamic decision problems with a large number of non-cooperative players which are individually insignificant but collectively have a significant impact on a particular player. It starts with a large but finite population and overcomes the curse of dimensionality by considering an infinite population limit model [13, 15, 17]. This area has undergone a surge of research activities and continued to expand. The readers are referred to [3, 5, 6, 11] for an overview on mean field game theory.

Mean field games have many current and potential applications. Some examples include power systems [16], large population electric vehicle recharging control [22, 24], economics and finance [1, 7, 20], stochastic growth theory [12], bio-inspired oscillator games [25].

This thesis studies mean field games in a continuous time Markov decision process (MDP) setup. The dynamics of the individual agents take a specific form of piecewise deterministic Markov processes [9] subject to impulse control. Each player has binary action space. Without active control, its state jumps up as a self-weighted compound Poisson process, and the state can be set to zero by an impulse control. Such modeling features have their motivation from security investment games and vaccination games [18, 23], and maintenance problems [2]. In the mean field game context, Poisson processes as a source of uncertainty have arisen in natural resource exploration problems [21]. Mean field games with pure jump Markov processes are considered in [27] where a nonlinear Markov semigroup approach is developed. We start by considering a finite horizon game and this is further used to derive the steady-state equation system which is the focus of our analysis. We identify the best response as a threshold policy, which in turn determines the stationary state distribution by simple equations.

A key reason for our interest in such a threshold solution structure is that it is rare to have closed form solutions in mean field games except linear quadratic cases [13, 19], which motivates us the quest of problem formulations allowing relatively simple solution structures.

1.2 The Current Research Work

The model considered in this thesis is closely related to a discrete time MDP framework [14]. In [14], Huang and Ma studied a model in which each player has finite state space $S=[0,1]$ and binary action space $A=\{a_0,a_1\}$. According to their transition kernel of player's state, if a player has a current state $x=x_0\in[0,1]$, it can only evolve to a state $x=x_1\in[x_0,1]$ without taking action, or we can take action to bring the state back to 0. The corresponding cost function and value function are given in [14]. They have proved that the value function $V(t,x)$ is continuous and strictly increasing on the state space S. In [14] they also examined a stationary form of the model which is independent of time t. The uniqueness of the solution of the stationary equation is proved in [14].

Although our continuous time modeling has similarity with the discrete binary MDP mean field model in [14], the route of analysis is very different. Due to the use of differential dynamic programming, the value iteration approach cannot be applied. Instead, we need to examine variational inequalities. The solution of our model leads to specific type of first order mean field equation system consisting of a first order variational inequality for solving an optimal impulse control problem, and a first order integral equations for the stationary state distribution, which can be viewed as a Volterra equation. These two equations are linked together by the threshold parameter from the best response.

First order mean field games have been analyzed in other backgrounds; see e.g. [4] for existence analysis. Another related work is [1] which studies stationary mean field equilibria in the discrete time and state MDP setting. In [1], Adlakha, Johari and Weintraub give the definitions of Markov perfect equilibriums (MPE) and stationary equilibriums (SE). One key result of [1] is Adlakha et al. provide the conditions under which one can use stationary equilibrium (SE) instead of Markov perfect equilibrium (MPE). They also use three examples in dynamic industries to illustrate the relation between MPE and SE.

The thesis is organized as follows. Chapter 2 introduces the continuous time Markov decision process model and stationary mean field game, and derives and analyzes the quasi variational inequality. In Chapter 3, we analyze the structure of the solution of the value function and the associated best response strategy. Chapter 3 also determines the stationary state distribution under a threshold policy. We also present several numerical solutions to MFG equation system. Chapter 4 introduces the unbounded state space model and related results. Chapter 5 concludes the thesis.

Chapter 2

The Mean Field Game Model

2.1 Dynamics and Costs

Consider a system of n players denoted by $\mathscr{A}_i, 1 \leq i \leq n$. For simplicity, assume that all the players are homogeneous or symmetric. The state X_t^i of $\mathscr{A}_i$ at time t takes value from the state space $S = [0,1]$. The state can be interpreted as an unfitness or risk level. The action of each player at time t is denoted by $a_t^i \in A = \{a_0, a_1\}$, where A denotes a binary action space. In this setting, a_0 means inaction and a_1 stands for active control.

If player $\mathscr{A}_i$ always takes inaction, its state evolves as a self-weighted compound Poisson process:

$$X_t^i = X_0^i + \sum_{j=1}^{N(t)} (1 - X_{\tau_j^-}^i) Y_j, \quad t \in [0,T], \tag{2.1}$$

where $\{N(t), t \geq 0\}$ is a Poisson process with arrival rate λ, τ_j is the jth jump time of $\{N(t), t \geq 0\}$, and $\{Y_i, i = 1,2,...\}$ is a sequence of i.i.d random variables with p.d.f $f_Y(y)$. We assume that the Poisson process $\{N(t), t \geq 0\}$ is independent of $\{Y_i, i = 1,2,...\}$. From (2.1), we can see that the sample path of X_t^i is right continuous and has left limit. Moreover, the sample path of the state monotonically increases to 1.

Now we consider the controlled model and examine the transition of the state from any initial time state pair (s,x). If $X_s^i = x$ and $a_h^i = a_0$ for all $h \in [s,t]$, then

$$X_t^i = x + \sum_{j, s < \tau_j \leq t} \left(1 - X_{\tau_j^-}\right) Y_j, \tag{2.2}$$

where τ_j is a jump time of $\{N(t), t \geq 0\}$.

If $X_s^i = x$ and $a_s^i = a_1$,

$$X_{s^+}^i = 0. \tag{2.3}$$

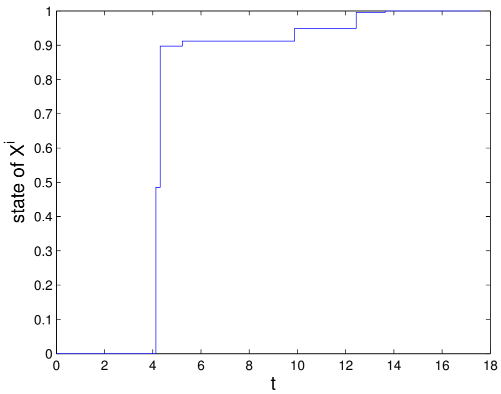

Figure 2.1: Sample path of the player

In other words, the state is reset to zero after an impulse control. Here we can define our impulse control π [10] as

$$\pi := (\tau_1, -x_{\tau_1^-}, \tau_2, -x_{\tau_2^-}, \cdots),$$

where π is a series of stopping times $\tau_j, j = 1, 2, \cdots$ and impulses $-x_{\tau_j^-}$ as the magnitude of the state change. For instance, at $t = \tau_j$, $X_{\tau_j^+} = X_{\tau_j^-} - x_{\tau_j^-} = 0$.

Define the population average state

$$X_t^{(N)} = \frac{1}{N} \sum_{i=1}^{N} X_t^i.$$

The cost function of $\mathscr{A}_i$ is defined as:

$$J^i = E \int_0^T e^{-\rho s} l\big(X_s^i, X_s^{(N)}\big) ds + \gamma E \sum_{j, 0 \le s_j \le T} \mathrm{e}^{-\rho s_j} . 1_{(a_{s_j}^i = a_1)}.$$

To obtain a solution of the mean field game, we approximate $X_t^{(N)}$ by a continuous function $z_t, t \in [0, T]$. Denote the limiting cost function of $\mathscr{A}_i$ as follows:

$$\bar{J}^i = E \int_0^T e^{-\rho s} l\big(X_s^i, z_s\big) ds + \gamma E \sum_{j, 0 \le s_j \le T} \mathrm{e}^{-\rho s_j} 1_{(a_{s_j}^i = a_1)}, \tag{2.4}$$

where $l(\cdot, \cdot)$ is the cost rate function; ρ is the time discount factor; γ is the effort cost; and $s_0 < s_1 < s_2 < \cdots$ is a sequence of time instants to take action a_1. We introduce the assumption:

A1) $l(x, z) > 0$ and for a fixed z (respectively x), $l(\cdot, z)$ (respectively, $l(x, \cdot)$) is strictly increasing. Define

$$J^i\big(t, x, a^i(\cdot)\big) \quad = \quad E \int_t^T e^{-\rho(s-t)} l\big(X_s^i, z_s\big) ds + \gamma E \sum_{j, t \le s_j \le T} e^{-\rho(s_j - t)} 1_{(a_{s_j}^i = a_1)}. \tag{2.5}$$

Define the value function

$$V(t,x) = \inf_{a^i(\cdot)} J^i\big(t,x,a^i(\cdot)\big). \tag{2.6}$$

There is one example in [14] for the motivation of the bounded state space $S = [0,1]$. The example of [14] considers the state x as the unfitness level for a network node. In this case, 0 and 1 stand for the ideal condition and the corrupted condition for the node respectively.

2.2 Quasi Variational Inequality

This section is to use a quasi variational inequality to characterize the value function. Denote

$$\lambda_1 = \lambda EY.$$

If V is differentiable we obtain

$$0 = \min\Big[l\big(x,z_t\big) + V_t(t,x) + \lambda_1(1-x)V_x(t,x) - \rho V(t,x), \quad V(t,0) - V(t,x) + \gamma\Big], \tag{2.7}$$

where the terminal condition is

$$V(T,x) = 0, \quad x \in S.$$

Next we show the derivation of (2.7). By dynamic programming, the value function of $\mathscr{A}_i$ satisfies:

$$V(t,x) = \min\left[E\Big(\int_t^{t+\Delta t} e^{-\rho(s-t)} l(X_s^i, z_s)ds + e^{-\rho\Delta t} V(t+\Delta t, X_{t+\Delta t}^i)\Big| X_t^i = x\Big), \quad V(t,0) + \gamma\right]. \tag{2.8}$$

The method of using Δt is essentially to introduce a small increment of time and then apply the optimality principle of dynamic programming. If we take a small Δt, then

$$\begin{aligned} E\big(\int_t^{t+\Delta t} e^{-\rho(s-t)} l(X_s^i, z_s)ds | X_t^i = x\big) &= e^{-\rho\Delta t} l(x,z_t)\Delta t, \\ &= (1-\rho\Delta t) l(x,z_t)\Delta t, \\ &= l(x,z_t)\Delta t + o(\Delta t), \end{aligned}$$

which gives

$$V(t,x) = \min\left[l(x,z_t)\Delta t + o(\Delta t) + e^{-\rho\Delta t} E\Big(V(t+\Delta t, X_{t+\Delta t}) | X_t^i = x\Big), \quad V(t,0) + \gamma\right]. \tag{2.9}$$

Moreover,

$$
\begin{aligned}
&E\left(e^{-\rho\Delta t}V(t+\Delta t, X_{t+\Delta t})|X_t^i = x\right) \\
&= V(t,x) + \Delta t \cdot V_t(t,x) + \lambda \cdot EY \cdot (1-x) \cdot \Delta t \cdot V_x(t,x) - \rho \Delta t V(t,x) + o(\Delta t). \qquad (2.10)
\end{aligned}
$$

By (2.9) and (2.10), we obtain

$$
\begin{aligned}
V(t,x) = \min\big[&l(x,z_t)\Delta t + V(t,x) + \Delta t V_t(t,x) + \lambda EY(1-x)\Delta t V_x(t,x) - \rho \Delta t V(t,x), \\
&V(t,0) + \gamma\big]. \qquad (2.11)
\end{aligned}
$$

If we deduct $V(t,x)$ from both sides of (2.11), then

$$
0 = \min\Big[\Delta t\big(l(x,z_t) + V_t(t,x) + \lambda_1(1-x)V_x(t,x) - \rho V(t,x)\big), \quad V(t,0) - V(t,x) + \gamma\Big]. \qquad (2.12)
$$

Without loss of generality, we can let the first term in the bracket on the RHS of (2.12) be divided by Δt and get

$$
0 = \min\Big[l\big(x,z_t\big) + V_t(t,x) + \lambda_1(1-x)V_x(t,x) - \rho V(t,x), \quad V(t,0) - V(t,x) + \gamma\Big].
$$

Following the standard approach in mean field games, we introduce the consistency condition:

$$
z_t = E\left[X_t^i\right], \qquad (2.13)
$$

where X_t^i is under the best response strategy determined by (2.7). By (2.7) we can see that at each fixed t, at least one of the two terms in (2.7) is equal to 0. Moreover, $V(t,0) - V(t,x) + \gamma \geq 0$.

2.3 Stationary Mean Field Game

In this thesis, we will not directly analyze the finite horizon mean field game equation system (2.7)-(2.13). Instead, we study a stationary (or steady-state) version below. From (2.7) we introduce

$$
0 = \min[l\big(x,z\big) + \lambda_1(1-x)V_x(x) - \rho V(x), \quad V(0) - V(x) + \gamma]. \qquad (2.14)
$$

Corresponding to (2.13), we introduce

$$
z = \int_0^1 x\pi\big(\mathrm{d}x\big), \qquad (2.15)
$$

where π is the stationary distribution, if it exists, of the closed loop state process, under a policy determined from (2.14).

Denote

$$f_0(x) = l(x,z) + \lambda_1(1-x)V_x(x) - \rho V(x) \tag{2.16}$$

and

$$f_1(x) = V(0) - V(x) + \gamma. \tag{2.17}$$

Consider the ODE

$$0 = l(x,z) + \lambda_1(1-x)\hat{V}_x(x) - \rho \hat{V}(x).$$

Here $\hat{V}(x)$ can be interpreted as the infinite horizon value function with cost

$$J^i_\infty\big(a^i(\cdot)\big) = E\int_0^\infty e^{-\rho s} l\big(X^i_s, z\big)ds + \gamma E\sum_j e^{-\rho s_j} 1_{(a^i_{s_j}=a_1)}.$$

Depending on the specific model and parameters, the second term on the right hand side of (2.14) may not be in effect. Then (2.14) reduces to

$$0 = l(x,z) + \lambda_1(1-x)\hat{V}_x(x) - \rho \hat{V}(x), \quad 0 < x < 1, \tag{2.18}$$

for which we need to introduce a boundary condition

$$\hat{V}(1) = \int_0^\infty e^{-\rho t} l(1,z)dt = \frac{l(1,z)}{\rho}. \tag{2.19}$$

The reason is that in this case the second term in (2.14) is not used, implying it is optimal to never take action a_1. The consideration of the stationary equation system has the benefit of obtaining structural results without dealing with partial differential equations. Furthermore, the study of this stationary equation system is potentially useful for studying the mean field game with long-run average cost. One may do this by applying a vanishing discount approach to (2.14)-(2.15).

2.4 Solution of the ODE and γ_{max}

Lemma 1 *i) Under the boundary condition* (2.19)*, the ODE* (2.18) *has a unique solution:*

$$\hat{V}(x) = (1-x)^{-\frac{\rho}{\lambda_1}} \int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1} l(y,z)}{\lambda_1} dy. \tag{2.20}$$

ii) $\hat{V}(x)$ *is an increasing function of* x *on* $[0,1]$.

Proof We use the method of variation of constants. Rewrite (2.18) as the following:

$$\hat{V}_x(x) - \frac{\rho}{\lambda_1(1-x)}\hat{V}(x) = -\frac{l(x,z)}{\lambda_1(1-x)}, \tag{2.21}$$

where we denote $p(x) = -\frac{\rho}{\lambda_1(1-x)}$ and $q(x) = -\frac{l(x,z)}{\lambda_1(1-x)}$. According to the variation of constants, we have

$$\hat{V}(x) = e^{-\int p(x)dx}\Big(\int q(x)e^{\int p(x)dx}dx + C_0\Big).$$

If we substitute in

$$\int p(x)dx = \frac{\rho}{\lambda_1}\log(1-x) + A, \quad A : const$$

and combine the constants, we have

$$\hat{V}(x) = (1-x)^{-\frac{\rho}{\lambda_1}}\Big(\int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} l(y,z)dy + C_0\Big). \tag{2.22}$$

Then we substitute in the boundary condition $\lim_{x\to 1}\hat{V}(x) = \frac{l(1,z)}{\rho}$ in order to get C_0:

$$\begin{aligned}
\hat{V}(1) = &\lim_{x\to 1}\frac{(1-x)^{-\frac{\rho}{\lambda_1}}}{\lambda_1}\int_x^1 (1-y)^{\frac{\rho}{\lambda_1}-1}\big(l(1,z) + l(y,z) - l(1,z) + C_0\big)dy,\\
&= \frac{(1-x)^{-\frac{\rho}{\lambda_1}}}{\lambda_1} l(1,z)\int_x^1 (1-y)^{\frac{\rho}{\lambda_1}-1}dy + \frac{(1-x)^{-\frac{\rho}{\lambda_1}}}{\lambda_1}\int_x^1 (1-y)^{\frac{\rho}{\lambda_1}-1}\big(l(y,z) - l(1,z)\big)dy\\
&+ \frac{(1-x)^{-\frac{\rho}{\lambda_1}}}{\lambda_1} C_0\int_x^1 (1-y)^{\frac{\rho}{\lambda_1}-1}dy,\\
&= \frac{l(1,z)}{(1-x)^{\frac{\rho}{\lambda_1}}}\int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1}dy + \varepsilon + (1-x)^{-\frac{\rho}{\lambda_1}}C_0,\\
&= \frac{l(1,z)}{\rho} + \varepsilon + (1-x)^{-\frac{\rho}{\lambda_1}}C_0,
\end{aligned}$$

where

$$\|\varepsilon\| \le \frac{\int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}\left(l(y,z)-l(1,z)\right)}{\lambda_1}}{(1-x)^{\frac{\rho}{\lambda_1}}}dy.$$

For $\forall\varepsilon > 0, \quad \exists\sigma > 0 \quad s.t. \quad \|l(y,z) - l(1,z)\| \le \sigma, \quad y \in [0,1)$. Thus, only when $C_0 = 0$, $\lim_{x\to 1}\hat{V}(x) = \frac{l(1,z)}{\rho}$.

Next we prove $\hat{V}(x)$ is an increasing function of x on $[0,1]$. According to (2.20),

$$\begin{aligned}\hat{V}(x) &= (1-x)^{-\frac{\rho}{\lambda_1}} \int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1} l(y,z)}{\lambda_1} dy, \\ &> (1-x)^{-\frac{\rho}{\lambda_1}} \int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1} l(x,z)}{\lambda_1} dy, \\ &= \frac{l(x,z)}{\rho}, \quad x \in [0,1).\end{aligned}$$

Since $l(x,z)$ is an increasing function of x, we have proved that $\hat{V}(x)$ is greater than an increasing function on $[0,1)$. Further we note that $\hat{V}(x) > \frac{l(x,z)}{\rho}$ gives $\hat{V}_x(x) > 0$ because $\hat{V}_x(x) = \frac{\rho}{\lambda_1(1-x)}\hat{V}(x) - \frac{1}{\lambda_1(1-x)} l(x,z)$. Therefore $\hat{V}(x)$ is monotonically increasing on $[0,1]$. □

Our ODE solution (2.20) is obtained when $f_0(x) = 0, \quad x \in [0,1]$. This is an extreme case for our stationary QVI when it's optimal to never take action and the threshold is 1. Define:

$$\gamma_{max} = \hat{V}(1) - \hat{V}(0) = \frac{l(1,z)}{\rho} - \int_0^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1} l(y,z)}{\lambda_1} dy, \tag{2.23}$$

and $\hat{V}(0) + \gamma - \hat{V}(x) \geq 0, \forall x \in (0,1)$ implies

$$\gamma \geq \gamma_{max}. \tag{2.24}$$

Remark 1 *We can rewrite* (2.20) *as*

$$\begin{aligned}\hat{V}(x) = (1-x)^{-\frac{\rho}{\lambda_1}} \big(\hat{V}(0) + \int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1} l(y,z)}{\lambda_1} dy \\ - \int_0^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1} l(y,z)}{\lambda_1} dy\big),\end{aligned} \tag{2.25}$$

where

$$\hat{V}(0) = \int_0^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1} l(y,z)}{\lambda_1} dy.$$

If $\hat{V}(0)$ takes any other value, one can not obtain a bounded solution on $[0,1]$.

Chapter 3

Analysis of the MFG Equation System

3.1 The Best Response

Our first objective is to obtain information on the structure of V and the associated optimal policy under certain smoothness assumptions on V. For the analysis in the chapter, we consider a general $z \in [0,1]$ for (2.14) without being required to satisfy (2.15). In contrast to (2.24), we consider the other region for γ, i.e.,

$$0 \leq \gamma \leq \gamma_{max}. \tag{3.1}$$

We introduce the assumption:
A2) $\frac{dV(x)}{dx}$ exists at each $x \in [0,1]$ and $V(x)$ satisfies (2.14) and (3.1).
Define

$$D = \{x | V(x) = V(0) + \gamma, \quad 0 \leq x \leq 1\}.$$

The first and second terms in the RHS of (2.14) correspond to inaction a_0 and action a_1, respectively.

Theorem 2 *Assume (A1)-(A2). Then there exists $x^* \in [0,1]$ such that*
i) For $x \in [0, x^*)$,

$$\begin{aligned} &l(x,z) + \lambda_1(1-x)V_x(x) - \rho V(x) = 0, \\ &V(0) - V(x) + \gamma > 0. \end{aligned}$$

ii) For $x \in (x^*, 1]$,

$$\begin{aligned} &V(0) - V(x) + \gamma = 0, \\ &l(x,z) + \lambda_1(1-x)V_x(x) - \rho V(x) > 0. \end{aligned}$$

iii) $V_x(x^*) = 0$.

iv) $V \in C^1$.

Proof We first show that D is not empty. When D is empty, $V(x)$ is given by (2.20) and satisfies $l(x,z)+\lambda_1(1-x)V_x(x)-\rho V(x)=0$ and (2.19). According to Lemma 1 and Remark 1, we have $\gamma \geq \gamma_{max}$. Recall (3.1), then we have $\gamma = \gamma_{max}$, which implies $x^* = 1 \in D$. So D cannot be empty.

Now we can assume D is nonempty. Note that D is a closed set. Define

$$x_{min} = \inf D, \quad x_{max} = \sup D.$$

We need to prove that D is an interval of the form $[x^*, 1]$.

Step 1. If $x_{min} = 1$, $f_0(x) = 0$ for all $x \in [0,1]$.

Step 2. Next we consider the case $x_{min} < 1$. It is impossible to have $x_{min} = 0$ since $x_{min} = 0$ would imply $V(0) = V(0) + \gamma$. Since $\gamma > 0$, this is a contradiction. Subsequently, we only consider $x_{min} \in (0,1)$.

Step 3. We show it is impossible to have

$$0 < x_{min} = x_{max} < 1. \tag{3.2}$$

We prove by contradiction. Assume (3.2) holds so that $l(x,z)+\lambda_1(1-x)V_x(x)-\rho V(x)=0$ for all $x \in [0,1]$. Here $x_{min} = x_{max} = x^*$. At $x = x^* < 1$, $V(x^*) = V(0)+\gamma > \frac{l(1,z)}{\rho}$. This single point doesn't affect the solution of the ODE thus our $V(x)$ satisfies (2.20) for all $x \in [0,1]$. According to lemma 1, (2.20) is increasing on $[0,1]$ and $V(1) = \frac{l(1,z)}{\rho}$. We get a contradiction here. Hence, (3.2) does not hold.

Step 4. Now we only need to consider $0 < x_{min} < x_{max} \leq 1$. We first prove the interval $[x_{min}, x_{max}] = D$, and then show $D = [x_{min}, 1]$. We prove by contradiction. If $[x_{min}, x_{max}] \neq D$, then by Rolle's Theorem and (A2), there exists $x_k \notin D$, $x_{min} < x_k < x_{max}$ such that $V_x(x)|_{x=x_k} = 0$. Since x_k is not in D, then it must satisfy:

$$\begin{aligned} l(x_k,z)-\rho V(x_k) &= 0, \\ V(x_k) &< V(0)+\gamma. \end{aligned}$$

Then

$$V(x_k) = \frac{l(x_k,z)}{\rho} < V(0)+\gamma. \tag{3.3}$$

However, at $x = x_{min}$:

$$V(x_{min}) = V(0)+\gamma = \frac{l(x_{min},z)}{\rho}. \tag{3.4}$$

Since $l(x,z)$ is increasing in x, we have

$$x_{min} < x_k, \quad l(x_{min},z) < l(x_k,z).$$

Therefore by (3.3) and (3.4) we have

$$V(x_k) = \frac{l(x_k,z)}{\rho} > \frac{l(x_{min},z)}{\rho} = V(x_{min}).$$

This contradicts with $V(x_{min}) > V(x_k)$. Thus $D = [x_{min}, x_{max}]$.

Next we prove $x_{max} = 1$. We again prove by contradiction here. If $x_{max} < 1$, then by (A2):

$$V_x(x)|_{x=x_{max}} = 0.$$

We also have

$$V_x(x)|_{x=x_{min}} = 0.$$

Thus, at $x = x_{min}$ and $x = x_{max}$,

$$\frac{l(x_{min},z)}{\rho} = V\left(x_{min}\right) = V\left(x_{max}\right) = \frac{l(x_{max},z)}{\rho}.$$

But

$$x_{min} < x_{max}, \quad l(x_{min},z) < l(x_{max},z).$$

We get a contradiction. Thus, $x_{max} = 1$. We conclude $D = [x_{min}, 1]$. Here x_{min} is the threshold x^*. $\square$

Remark 2 *There is a simple interpretation for the nonemptiness property on D. It implies the second term in the variational inequality* (2.14) *has a non-trivial role. In other words, the optimal strategy does not take inaction forever regardless of the state value. Intuitively, this means γ should not be too large. By Theorem 2, a neat result is that V can be solved from an ODE on $[0,x^*)$ and $V(x) = V(0) + \gamma$ for $x \geq x^*$. Theorem 2 - iv) may be called a smooth-fit property, which is well known for optimal switching problems [29, 30, 31].*

We provide some necessary conditions on $V(0)$ and x^*.

Remark 3 *When $\gamma = \gamma_{max} = \frac{l(1,z)}{\rho} - \int_0^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} l(y,z)$, our threshold x^* reaches 1. We further give a necessary condition for γ: if we have an interior threshold $0 < x^* < 1$, then $\gamma < \gamma_{max}$.*

Lemma 3 *Given an interior threshold $0 < x^* < 1$, the solution of the ODE*

$$0 = l(x,z) + \lambda_1(1-x)V_x(x) - \rho V(x) = 0, \quad 0 < x < x^*. \tag{3.5}$$

under the boundary condition $V(x^*) = \frac{l(x^*,z)}{\rho}$ *is*

$$V(x) = (1-x)^{-\frac{\rho}{\lambda_1}} \Big(\int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} l(y,z)dy + \int_{x^*}^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} \big(l(x^*,z) - l(y,z)\big)dy \Big). \tag{3.6}$$

In this case, $\gamma < \gamma_{max}$.

Proof To solve the ODE $f_0(x) = 0$ defined in (2.16) under the boundary condition $V(x^*) = \frac{l(x^*,z)}{\rho}, 0 < x^* < 1$, we use the variation of constants and get

$$V(x) = (1-x)^{-\frac{\rho}{\lambda_1}} \big(\int_x^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} l(y,z)dy + C_0 \big).$$

We use the boundary condition to determine the constant C_0:

$$C_0 = \int_{x^*}^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} \Big(l(x^*,z) - l(y,z) \Big) dy. \tag{3.7}$$

Then we check if $\gamma < \gamma_{max} = \frac{l(1,z)}{\rho} - \int_0^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} l(y,z)dy$ when given that $x^* < 1$.

Step 1: If $x^* < 1$, then

$$(1-x^*)^{\frac{\rho}{\lambda_1}} < 1. \tag{3.8}$$

Step 2: Since $l(x,z)$ is an increasing function of x on $[0,1]$, $l(1,z) > l(x^*,z)$. (3.8) is multiplied by $\frac{1}{\rho}\big(l(1,z) - l(x^*,z)\big)$ on both sides:

$$\frac{1}{\rho}\big(l(1,z) - l(x^*,z)\big) > \frac{1}{\rho}\big(l(1,z) - l(x^*,z)\big)(1-x^*)^{\frac{\rho}{\lambda_1}}. \tag{3.9}$$

Step 3: Since

$$\begin{aligned} \frac{1}{\rho}\big(l(1,z) - l(x^*,z)\big)(1-x^*)^{\frac{\rho}{\lambda_1}} &= \int_{x^*}^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} \big(l(1,z) - l(x^*,z)\big)dy, \\ &> \int_{x^*}^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} \big(l(y,z) - l(x^*,z)\big)dy. \end{aligned}$$

According to (3.9),

$$\frac{1}{\rho}\big(l(1,z) - l(x^*,z)\big) > \int_{x^*}^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} \big(l(y,z) - l(x^*,z)\big)dy. \tag{3.10}$$

Step 4:

$$
\begin{aligned}
\gamma &= V(x^*) - V(0),\\
&= \frac{l(x^*,z)}{\rho} - \left(\int_0^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1} l(y,z)dy + \int_{x^*}^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1}\Big(l(x^*,z)-l(y,z)\Big)dy\right).
\end{aligned}
$$

$$
\gamma_{max} - \gamma = \frac{1}{\rho}\big(l(1,z)-l(x^*,z)\big) - \Big(\int_{x^*}^1 \frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1}\big(l(y,z)-l(x^*,z)\big)dy\Big). \tag{3.11}
$$

According to (3.10), $\gamma_{max} - \gamma > 0$. □

Lemma 4 *Under the assumption of Theorem 2, further assume $x^* \in (0,1)$. Then*

$$
\begin{cases}
V(0)+\gamma & = (a-ax^*)^{-\frac{1}{a}}\Big(a^{\frac{1}{a}}V(0) \\
& \quad - \int_0^1 \frac{b(a-ay)^{\frac{1}{a}}}{a(1-y)} l(y,z)dy \\
& \quad + \int_{x^*}^1 \frac{b(a-ay)^{\frac{1}{a}}}{a(1-y)} l(y,z)dy\Big),\\
V(0)+\gamma & = b\cdot l(x^*,z),
\end{cases} \tag{3.12}
$$

where $a = \frac{\lambda_1}{\rho}, b = \frac{1}{\rho}$.

Proof When $x \in (0,x^*)$, $V(x)$ satisfies the ODE $l(x,z) + \lambda_1 \cdot (1-x)\cdot V_x - \rho V = 0$. We solve the ODE to give

$$
\begin{aligned}
V(x) = (a-ax)^{-\frac{1}{a}}\Big(a^{\frac{1}{a}}V(0) - \int_0^1 \frac{b(a-ay)^{\frac{1}{a}}}{a(1-y)} l(y,z)dy \\
+ \int_x^1 \frac{b(a-ay)^{\frac{1}{a}}}{a(1-y)} l(y,z)dy\Big), \quad x \in [0,x^*).
\end{aligned} \tag{3.13}
$$

When $x \in (x^*,1)$, $V(x)$ is a constant, i.e.,

$$
V(x) = V(0)+\gamma. \tag{3.14}
$$

When $x = x^*$, by continuity of V,

$$
\begin{aligned}
V(x^*) &= V(0)+\gamma \\
&= (a-ax^*)^{-\frac{1}{a}}\Big(a^{\frac{1}{a}}V(0) - \int_0^1 \frac{b(a-ay)^{\frac{1}{a}}}{a(1-y)} l(y,z)dy \\
&\quad + \int_{x^*}^1 \frac{b(a-ay)^{\frac{1}{a}}}{a(1-y)} l(y,z)dy\Big).
\end{aligned} \tag{3.15}
$$

This condition ensures that the $V(x)$ in (3.13) equals the $V(x)$ in (3.14) at $x = x^*$.

By Theorem 2 - iii),

$$V(x^*) = b \cdot l(x^*, z). \tag{3.16}$$

Finally (3.15) and (3.16) lead to (3.12). □

3.2 The Limiting Distribution

By Theorem 2, the best response is a threshold policy. Below we determine the corresponding stationary state distribution π for X_t^i under the best response strategy. We consider the case of an interior threshold $x^* \in (0,1)$. To derive the stationary distribution π, we consider a modified version $\{\check{X}_t^i, t \geq 0\}$ of the controlled Markov process $\{X_t^i, t \geq 0\}$. If the Markov process $\{X_t^i, t \geq 0\}$ jumps to some value $k > c$ at time t, we set $\check{X}_t^i = 0$. Otherwise $\check{X}_t^i = X_t^i$. The modified process is stochastically equivalent to the original one, i.e., they have the same finite dimensional distributions. Now we may regard the state space of $\{X_t^i, t \geq 0\}$ as $[0, c]$. We consider a general distribution for the i.i.d. sequence Y_i with p.d.f $f_Y(y), \quad y \in [0,1]$. Let Q be the transition kernel of $\{X_t^i, t \geq 0\}$ so that $Q(h, x, B) = P\big(X_h^i \in B | X_0^i = x\big), h \geq 0$.

Lemma 5 *We have*

Case 1:

$$Q(h, 0, \{0\}) \quad = \quad 1 - \Big(1 - \int_c^1 f_Y(y)dy\Big)\lambda h + O(h^2),$$

Case 2:

$$Q(h, x, \{0\}) = \lambda h \int_{\frac{c-x}{1-x}}^1 f_Y(y)dy + O(h^2), \quad 0 < x < c,$$

Case 3:

$$Q(h, 0, B) = h \int_B \lambda f_Y(y)dy + O(h^2),$$

we consider 3 *cases when neither the initial state nor the target state is* 0*:*

Case 4a:

$$Q(h, x, B) \quad = \quad \lambda h \int_{\frac{l-x}{1-x}}^{\frac{m-x}{1-x}} f_Y(y)dy + O(h^2), \quad 0 < x < c,$$

$$B = (l, m) \subset (x, c),$$

Case 4b:

$$Q(h, x, B) = O(h^2), when \quad B \subset (0, x),$$

Case 4c:

$$Q(h, x, \{x\}) = 1 - \lambda h + O(h^2).$$

The term $O(h^2)$ is uniform w.r.t B.

Proof Case 1: $Q\big(h,0,\{0\}\big)$.

$$\begin{aligned}
Q\big(h,0,\{0\}\big) &= P\big(X_h=0|X_0=0\big),\\
&= P\big(X_h=0|X_0=0,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=0\big)\\
&+ P\big(X_h=0|X_0=0,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=0\big)\\
&+ P\big(X_h=0|X_0=0,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=0\big),\\
&= 1\cdot\big(1-\lambda h+o(h)\big)+\int_c^1 f_Y(y)dy\big(\lambda h+o(h)\big)+o(h),\\
&= 1-\Big(1-\int_c^1 f_Y(y)dy\Big)\lambda h+o(h).
\end{aligned}$$

Similarly,
Case 2: $Q\big(h,x,\{0\}\big), x\in(0,c)$.

$$\begin{aligned}
Q\big(h,x,\{0\}\big) &= P\big(X_h=0|X_0=x\big),\\
&= P\big(X_h=0|X_0=x,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h=0|X_0=x,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h=0|X_0=x,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=x\big),\\
&= 0\cdot\big(1-\lambda h+o(h)\big)+\int_{\frac{c-x}{1-x}}^1 f_Y(y)dy\big(\lambda h+o(h)\big)+o(h),\\
&= \lambda h\int_{\frac{c-x}{1-x}}^1 f_Y(y)dy+o(h)\big(\int_{\frac{c-x}{1-x}}^1 f_Y(y)dy+1\big).
\end{aligned}$$

Case 3: $Q\big(h,0,B\big),\quad B=(l,m)\subset(0,c),\quad 0<l<m\leq c.$

$$\begin{aligned}
Q\big(h,0,B\big) &= P\big(X_h\in B|X_0=0\big),\\
&= P\big(X_h\in B|X_0=0,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=0\big)\\
&+ P\big(X_h\in B|X_0=0,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=0\big)\\
&+ P\big(X_h\in B|X_0=0,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=0\big),\\
&= 0+\int_l^m f_Y(y)dy\big(\lambda h+o(h)\big)+o(h),\\
&= \lambda h\int_l^m f_Y(y)dy+o(h)\big(\int_l^m f_Y(y)dy+1\big),\\
&= h\int_B \lambda f_Y(y)dy+o(h)..
\end{aligned}$$

Case 4a: $Q\big(h,x,B\big),\quad B=(l,m),\quad x\in(0,c),\quad B\subset(x,c),\quad x\leq l<m\leq c.$

$$\begin{aligned}
Q\big(h,x,B\big) &= P\big(X_h\in B|X_0=x\big),\\
&= P\big(X_h\in B|X_0=x,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h\in B|X_0=x,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h\in B|X_0=x,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=x\big),\\
&= 0\cdot\big(1-\lambda h+o(h)\big)+\int_{\frac{l-x}{1-x}}^{\frac{m-x}{1-x}} f_Y(y)dy\big(\lambda h+o(h)\big)+o(h),\\
&= \lambda h\int_{\frac{l-x}{1-x}}^{\frac{m-x}{1-x}} f_Y(y)dy+o(h).
\end{aligned}$$

Case 4b: $Q\big(h,x,B\big),B=(l,m),B\subset(0,x),x\in(0,c),0<l<m\leq x.$

$$\begin{aligned}
Q\big(h,x,B\big) &= P\big(X_h\in B|X_0=x\big),\\
&= P\big(X_h\in B|X_0=x,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h\in B|X_0=x,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h\in B|X_0=x,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=x\big),\\
&= 0\cdot\big(1-\lambda h+o(h)\big)+0\cdot\big(\lambda h+o(h)\big)+o(h),\\
&= o(h).
\end{aligned}$$

Case 4c: $Q\big(h,x,\{x\}\big),x\in(0,c).$

$$\begin{aligned}
Q\big(h,x,\{x\}\big) &= P\big(X_h=x|X_0=x\big),\\
&= P\big(X_h=x|X_0=x,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h=x|X_0=x,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h=x|X_0=x,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=x\big),\\
&= 1\cdot\big(1-\lambda h+o(h)\big)+0+0,\\
&= 1-\lambda h+o(h).
\end{aligned}$$

Thus, $Q(h,x,B)$ is uniformly w.r.t B. □

Now we continue to determine the stationary distribution $\pi(dx)$.

Lemma 6 *$\pi(dx)$ takes the form*

$$\begin{aligned}
\pi(\{0\}) &= \theta_0,\\
\pi(B) &= \int_B p(x)\mathrm{d}x,\quad B\subset(0,c),
\end{aligned}\tag{3.17}$$

where θ_0 and $p(x):[0,c]\to\mathbb{R}$ are given by the equation system

$$\begin{cases} \theta_0 f_Y(x) - p(x) + \int_0^x p(y)\frac{f_Y\left(\frac{x-y}{1-y}\right)}{1-y}dy = 0, \\ \theta_0 + \int_0^c p(x)dx = 1. \end{cases} \tag{3.18}$$

Proof The existence and uniqueness of a solution (θ, p) to (3.18) can be proved by using the method given in [26] for Volterra type equations. If we rewrite the first equation in (3.18) as

$$p(x) - \int_0^x p(y)\frac{f_Y\left(\frac{x-y}{1-y}\right)}{1-y}dy = \theta_0 f_Y(x), \tag{3.19}$$

then according to the theorem of [26, p.16], for a given θ_0, there exists a unique solution

$$\begin{aligned} &p(x) = \theta_0 f_Y(x) + \int_0^x \theta_0 f_Y(s)\Gamma(x,s)ds, \qquad (3.20)\\ &\Gamma(x,s) = \sum_{m=1}^{\infty} K_m(x,s), \\ &K_1(x,s) = \frac{f_Y\left(\frac{x-s}{1-s}\right)}{1-s}, \quad K_m(x,s) = \int_s^x K_1(x,t)K_{m-1}(t,s)dt, \quad m = 2,3,\cdots. \end{aligned}$$

Combining (3.20) with the second equation in (3.18), we can uniquely determine θ_0. The representation of (θ, p) for π is given as follows.

Since we have the relation

$$\int_{x\in[0,c)} \pi(dx)Q(h,x,B) = \pi(B), \tag{3.21}$$

where B is any Borel subset of $[0,c)$. Define $\mathscr{G}$ as the set of functions: $\phi \in \mathscr{G}$ if $\phi : [0,c) \to \mathbb{R}$, and ϕ is uniformly continuous on $(0,c)$. We introduce a test function $\phi \in \mathscr{G}$. Then by (3.21) we have

$$\int_{y\in[0,c)} \phi(y) \int_{x\in[0,c)} \pi(dx)Q(h,x,dy) = \int_{z\in[0,c)} \phi(z)\pi(dz).$$

By exchanging the order of the integration we have

$$\int_{x\in[0,c)} \pi(dx) \int_{y\in[0,c)} \phi(y)Q(h,x,dy) = \int_{z\in[0,c)} \phi(z)\pi(dz). \tag{3.22}$$

The RHS of (3.22) is

$$\int_{z\in[0,c)} \phi(z)\pi(dz) = \theta_0\phi(0) + \int_{z\in(0,c)} \phi(z)\pi(dz). \tag{3.23}$$

For small h, the LHS of (3.22) is given by

$$\begin{aligned}
LHS &= \theta_0 \int_{y\in[0,c)} \phi(y)Q\big(h,0,dy\big) + \int_{x\in(0,c)} p(x)dx \int_{y\in[0,c)} \phi(y)Q\big(h,x,dy\big), \\
&= \theta_0 \left[\phi(0)Q\big(h,0,\{0\}\big) + \int_{y\in(0,c)} \phi(y)Q\big(h,0,dy\big)\right] \\
&+ \int_{x\in(0,c)} p(x)dx \left[\phi(0)Q\big(h,x,\{0\}\big) + \int_{y\in(0,x)} \phi(y)Q\big(h,x,dy\big)\right. \\
&+ \left.\phi(x)Q\big(h,x,\{x\}\big) + \int_{y\in(x,c)} \phi(y)Q\big(h,x,dy\big)\right], \\
&= \theta_0 \left(\phi(0)\Big(1-\big(1-\int_c^1 f_Y(y)dy\big)\lambda h\Big) + h\int_{y\in(0,c)} \phi(y)\lambda f_Y(y)dy + O(h^2)\right) \\
&+ \int_{x\in(0,c)} p(x)dx \left(\lambda h\phi(0)\int_{\frac{c-x}{1-x}}^1 f_Y(y)dy + \phi(x)\big(1-\lambda h\big)\right. \\
&+ \left.\lambda h \int_{y\in(x,c)} \phi(y)\cdot\frac{1}{1-x}\cdot f_Y\big(\frac{y-x}{1-x}\big)dy + O(h^2)\right). \qquad (3.24)
\end{aligned}$$

Since the right hand sides of both (3.23) and (3.24) are equal,

$$\theta_0 \left(-\phi(0)\Big(1-\int_c^1 f_Y(y)dy\Big) + \int_{y\in(0,c)} \phi(y)f_Y(y)dy\right) + o(h)$$

$$= \int_{x\in(0,c)} p(x)\left(\phi(x) - \phi(0)\int_{\frac{c-x}{1-x}}^1 f_Y(y)dy - \int_{y\in(x,c)} \phi(y)\frac{f_Y\big(\frac{y-x}{1-x}\big)}{1-x}dy\right)dx + o(h)$$

If we change the order of integration of the third term within the large bracket in the RHS, we can get,

$$\theta_0 \int_{y\in(0,c)} \phi(y)f_Y(y)dy - \theta_0\phi(0)\int_0^c f_Y(y)dy =$$

$$\int_{y\in(0,c)} \phi(y)\Big(p(y) - \int_{x\in(0,y)} p(x)\frac{f_Y\big(\frac{y-x}{1-x}\big)}{1-x}dx\Big)dy - \phi(0)\int_{x\in(0,c)} p(x)\Big(\int_{\frac{c-x}{1-x}}^1 f_Y(y)dy\Big)dx.$$

In the above equation, if we set $\phi(0) = 0$,

$$\theta_0 \int_{y\in(0,c)} \phi(y)f_Y(y)dy = \int_{y\in(0,c)} \phi(y)\Big(p(y) - \int_{x\in(0,y)} p(x)\frac{f_Y\big(\frac{y-x}{1-x}\big)}{1-x}dx\Big)dy$$

which gives

$$\int_{y\in(0,c)} \phi(y)\left(\theta_0 f_Y(y) - p(y) + \int_{x\in(0,y)} p(x)\frac{f_Y\left(\frac{y-x}{1-x}\right)}{1-x}dx\right)dy = 0.$$

Since $\phi \in \mathscr{G}$ is arbitrary, then,

$$\theta_0 f_Y(y) - p(y) + \int_{x\in(0,y)} p(x)\frac{f_Y\left(\frac{y-x}{1-x}\right)}{1-x}dx = 0.$$

If we interchange the variables x and y, we have,

$$\theta_0 f_Y(x) - p(x) + \int_0^x p(y)\frac{f_Y\left(\frac{x-y}{1-y}\right)}{1-y}dy = 0.$$

□

According to Lemma 6, given the threshold x^*, the stationary distribution $\pi(dx)$ has a point mass at $x = 0$. θ_0 and $p(x)$ have the relation (3.18).

3.3 Numerical Solution

Example 1. (Threshold in the Best Response) We take a particular cost rate function $l(x,z) = x(1+z)$. When fixed $a = 1, b = 1.1$, the equation system (3.12) becomes

$$\begin{cases} \left(1-(1-x^*)^{-1}\right)V(0)+\gamma = -\frac{0.55x^{*2}(z+1)}{1-x^*}, \\ V(0)+\gamma = 1.1\cdot x^*(1+z). \end{cases}$$

which has the solution

$$\begin{cases} x^* = \left(\frac{\gamma}{0.55(1+z)}\right)^{\frac{1}{2}}, \\ V(0) = 1.1(1+z)\left(\frac{\gamma}{0.55(1+z)}\right)^{\frac{1}{2}} - \gamma. \end{cases}$$

If we take $\gamma = 0.5, z = 0.5$, we can get

$$\begin{cases} x^* \approx 0.7785, \\ V(0) \approx 0.7845, \end{cases}$$

which determines an interior threshold x^*. In this case, $V(x)$ is given as

i) $V(x) = \frac{0.825x^2-0.7845}{x-1}$, for $x \in [0,x^*)$,

ii) $V(x) = V(0)+\gamma = 1.2845$, for $x \in [x^*,1]$.

An extreme case is that $x^* = 0$ and $V(0) = 0$ if we take $\gamma = 0$. When $\gamma = 0$, the cost of taking action is 0 so it's always optimal to take action and bring the state x back to 0. Another extreme case is when we slowly increase γ to $\hat{\gamma} = 0.825$ until the threshold x^* hits 1; this gives us an upper bound for γ and we find that at this time

$$V(0) = 1.1 \cdot 1.5 - 0.55 \cdot 1.5 = 0.825,$$
$$V(1) = V(0) + \hat{\gamma} = 1.65. \tag{3.25}$$

The boundary condition (2.19) gives

$$V(1) = \int_0^\infty \mathrm{e}^{-\rho s} l(1,z) ds = 1.65,$$

which is consistent with (3.25). Figure 3.1 shows the value function in this example.

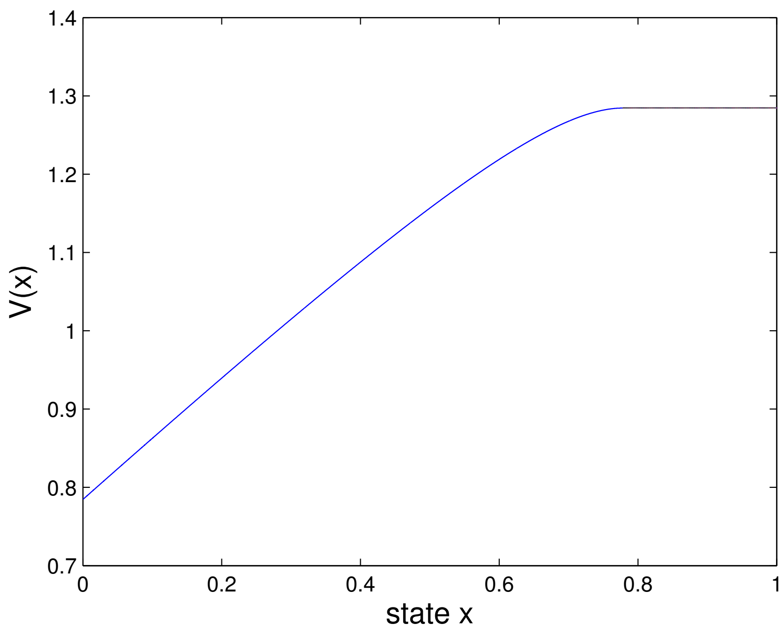

Figure 3.1: x - $V(x)$

Example 2. (Limiting State Distribution under Threshold Policy) Y $\sim$ Unif $[0,1]$ and $f_Y(x) = 1$ and denote the threshold $c \in [0,1)$. The first equation in (3.18) becomes

$$\theta_0 = p(x) + \int_0^x \frac{p(y)}{1-y} dy, \tag{3.26}$$

where $0 \le x \le c$. Then we take the derivative of x on both sides of (3.26):

$$0 = p'(x) + \frac{p(x)}{1-x}. \tag{3.27}$$

Solving the ODE (3.27), we get

$$p(x) = c_1(x-1), \tag{3.28}$$

where the constant c_1 is to be determined. Substituting (3.28) into the second equation of (3.18) gives:

$$\theta_0 = 1 + c_1 \cdot c - \frac{c_1}{2} \cdot c^2. \tag{3.29}$$

To solve for c_1, we substitute (3.29) into (3.18):

$$1 + c_1 \cdot c - \frac{c_1}{2} \cdot c^2 = -c_1, \quad c_1 = \frac{1}{\frac{c^2}{2} - 1 - c}.$$

Therefore, θ_0 and $p(x)$ are

$$\theta_0 = \frac{1}{1 + c - \frac{c^2}{2}}, \quad p(x) = \frac{1}{1 + c - \frac{c^2}{2}}(1 - x).$$

Taking $c = 0.5$, the figure of $p(x)$ is as follow:

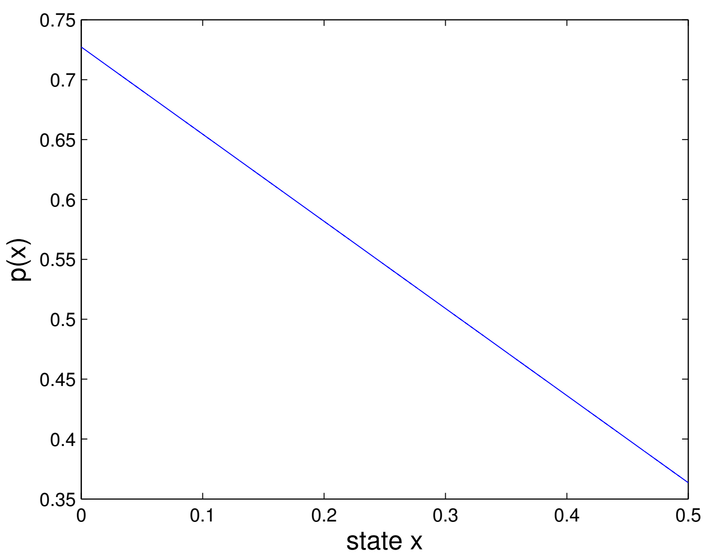

Figure 3.2: x - $p(x)$

Example 3. (Solving the Mean Field Game Equations) We propose an iterative procedure to solve the equation system.

$$\begin{cases} 0 = \min\Big[l(x,z) + \lambda_1(1-x)V_x(x) - \rho V(x), V(0) - V(x) + \gamma\Big], \\ z = \int_0^1 x\pi(\mathrm{d}x). \end{cases}$$

Step 1. Given a proper initial z_0 and an initial $V(0)$, use Lemma 4 to get the threshold $c = x^*$ and update $V(0)$ by

$$V(0)_{\text{new}} = 0.98 \cdot V(0)_{\text{old}} + 0.02 \cdot V(0)_{\text{new}}$$

Step 2. If the threshold $c=0$, this means we take actions all the time, thus the state x is always 0 implying $z_1=0$. If the threshold $c>1$, this means we do not intervene the system, thus the state will eventually go to 1 implying $z_1=1$. If the threshold is between 0 and 1, we can get z_1 by using

$$z_1=\int_0^c x\pi(\mathrm{d}x)=\int_0^c x\cdot p(x)\mathrm{d}x.$$

Step 3. Replace z_0 by z_{new}:

$$z_{new}=0.98\cdot z_0+0.02\cdot z_1.$$

Then set $z_0=z_{new}$ and back to step 1.

Now let's consider a specific example with $\gamma=0.3,\quad l(x,z)=x(1+z),\quad \lambda=\frac{1}{1.1},\quad \rho=\frac{1}{1.1},\quad Y\sim\textbf{Unif}[0,1]$.

$$\begin{aligned}\gamma_{max}&=\frac{l(1,z)}{\rho}-\int_0^1\frac{(1-y)^{\frac{\rho}{\lambda_1}-1}}{\lambda_1}l(y,z)dy,\\&=0.55(1+z).\end{aligned}$$

When $z=0,\quad \gamma_{max}=0.55$; when $z>0,\quad \gamma_{max}>0.55$. Since $\gamma=0.3<\gamma_{max}$ for all $z\in[0,1]$, this gives us an interior threshold. In this case, we can actually solve for the theoretical z_{new} for each z_{old}. Given z, by Example 1, we can evaluate x^*:

$$x^*=\left(\frac{\gamma}{0.55(1+z)}\right)^{\frac{1}{2}}.$$

We substitute x^* in z_{new}:

$$z_{new}=\int_0^{x^*}\frac{1}{1+x^*-\frac{x^{*2}}{2}}(1-x)dx=\frac{x^*-\frac{1}{2}x^{*2}}{1+x^*-\frac{x^{*2}}{2}}.$$

Then we update $z=0.98z_{old}+0.02z_{new}$.

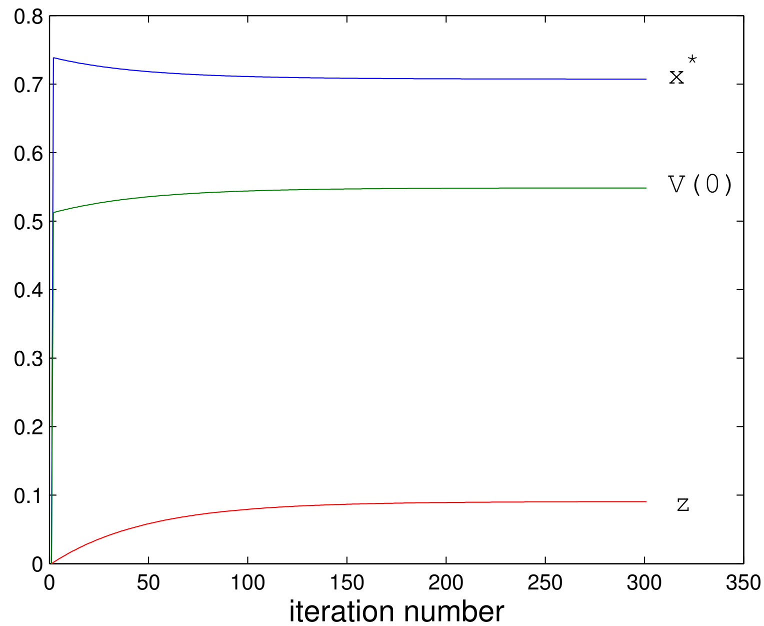

Figure 3.3: $x^{*}, V(0) \quad and \quad z$

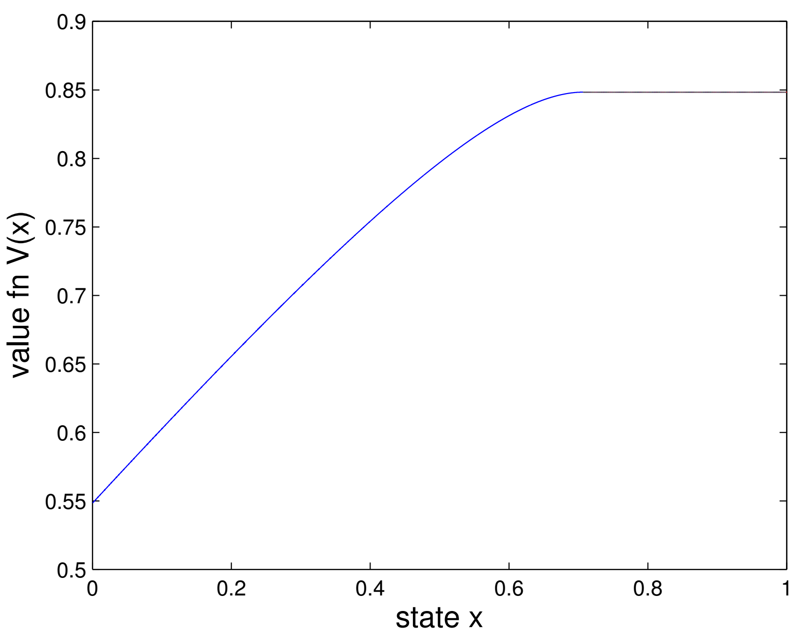

Figure 3.4: x - $V(x)$

Chapter 4

Unbounded State Space Model

In this chapter we consider a model with unbounded state space $S = [0, \infty)$, where the state $x \in S$ can be interpreted as the hazard rate or failure rate. The motivation of the unbounded state space can be found in [28]. In that paper, the authors consider a model of maintaining a machine of which the failure rate is increasing with its age. In our model, player $\mathscr{A}_i$'s state will evolve as a compound Poisson process:

$$X_t^i = X_0^i + \sum_{j=1}^{N(t)} Y_j, \quad t \in [0, T], \tag{4.1}$$

if the player always takes inaction, where $Y \sim f_Y(y), y \in [0, \infty)$. In this model we eliminate the self-weighted term $(1 - X^i_{\tau_j^-})$ in (2.1) and this leads to the new quasi variational inequality:

$$\begin{aligned} 0 = \min\big[& l(x, z_t) + V_t(t, x) + \lambda_1 V_x(t, x) - \rho V(t, x), \\ & V(t, 0) - V(t, x) + \gamma\big], \end{aligned} \tag{4.2}$$

and the stationary QVI will become

$$0 = \min\left[l(x, z) + \lambda_1 V_x(x) - \rho V(x), V(0) - V(x) + \gamma\right]. \tag{4.3}$$

The next lemma is similar to Lemma 4.

Lemma 7 *We assume threshold* $x^* \in (0, \infty)$. *Then*

$$\begin{cases} V(0) + \gamma & = e^{\frac{x^*}{a}} \big(V(0) + b \int_0^{x^*} (-\frac{1}{a}) e^{-\frac{y}{a}} l(y, z) dy\big), \\ V(0) + \gamma & = b \cdot l(x^*, z), \end{cases} \tag{4.4}$$

where $a = \frac{\lambda_1}{\rho}, b = \frac{1}{\rho}$.

Proof When $x \in (0, x^*)$, $V(x)$ satisfies the ODE $l(x,z) + \lambda_1 \cdot V_x - \rho V = 0$. We solve the ODE to give

$$V(x) = e^{\frac{x}{a}}\Big(V(0) + b\int_0^x (-\frac{1}{a})e^{-\frac{y}{a}} l(y,z)dy\Big), \quad x \in [0, x^*). \tag{4.5}$$

When $x \in (x^*, 1)$, $V(x)$ is a constant, i.e.,

$$V(x) = V(0) + \gamma. \tag{4.6}$$

When $x = x^*$, by continuity of V,

$$\begin{aligned} V(x^*) &= V(0) + \gamma \\ &= e^{\frac{x^*}{a}}\Big(V(0) + b\int_0^{x^*} (-\frac{1}{a})e^{-\frac{y}{a}} l(y,z)dy\Big). \end{aligned} \tag{4.7}$$

This condition ensures that the $V(x)$ in (4.5) equals the $V(x)$ in (4.6) at $x = x^*$.

For continuity at $x = x^*$,

$$V(x^*) = b \cdot l(x^*, z). \tag{4.8}$$

Finally (4.7) and (4.8) lead to (4.4). □

The transition kernel is given by:

Lemma 8 *We denote $c = x^* \in (0,1)$, and have*

Case 1:

$$Q(h, 0, \{0\}) = 1 - \lambda h + \lambda h \int_c^{\infty} f_Y(y)dy + O(h^2),$$

Case 2:

$$Q(h, x, \{0\}) = \lambda h \int_{c-x}^{\infty} f_Y(y)dy + O(h^2), x \in (0, c),$$

Case 3:

$$Q(h, 0, B) = \lambda h \int_B f_Y(y)dy + O(h^2), B = (l, m) \subset (0, c), \quad 0 < l < m \le c,$$

Case 4a:

$$Q_a(h, x, B) = \lambda h \int_{l-x}^{m-x} f_Y(y)dy + O(h^2),$$

$$B = (l, m) \subset (x, c),$$

Case 4b:

$$Q(h, x, B)_b = O(h^2), \text{ when } \quad B \subset (0, x),$$

Case 4c:

$$Q(h,x,\{x\})_c = 1-\lambda h + O(h^2), \quad x \in (0,c).$$

The term $O(h^2)$ is uniform w.r.t B.

Proof Case 1: $Q\big(h,0,\{0\}\big)$.

$$\begin{aligned}
Q\big(h,0,\{0\}\big) &= P\big(X_h=0|X_0=0\big),\\
&= P\big(X_h=0|X_0=0,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=0\big)\\
&+ P\big(X_h=0|X_0=0,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=0\big)\\
&+ P\big(X_h=0|X_0=0,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=0\big),\\
&= 1\cdot\big(1-\lambda h+o(h)\big)+\int_c^\infty f_Y(y)dy\big(\lambda h+o(h)\big)+o(h),\\
&= 1-\Big(1-\int_c^\infty f_Y(y)dy\Big)\lambda h+o(h).
\end{aligned}$$

Case 2: $Q\big(h,x,\{0\}\big), x\in(0,c)$.

$$\begin{aligned}
Q\big(h,x,\{0\}\big) &= P\big(X_h=0|X_0=x\big),\\
&= P\big(X_h=0|X_0=x,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h=0|X_0=x,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h=0|X_0=x,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=x\big),\\
&= 0\cdot\big(1-\lambda h+o(h)\big)+\int_{c-x}^\infty f_Y(y)dy\big(\lambda h+o(h)\big)+o(h),\\
&= \lambda h\int_{c-x}^\infty f_Y(y)dy+o(h).
\end{aligned}$$

Case 3: $Q\big(h,0,B\big), \quad B=(l,m)\subset(0,c), \quad 0<l<m\le c.$

$$\begin{aligned}
Q\big(h,0,B\big) &= P\big(X_h\in B|X_0=0\big),\\
&= P\big(X_h\in B|X_0=0,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=0\big)\\
&+ P\big(X_h\in B|X_0=0,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=0\big)\\
&+ P\big(X_h\in B|X_0=0,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=0\big),\\
&= 0+\int_l^m f_Y(y)dy\big(\lambda h+o(h)\big)+o(h),\\
&= \lambda h\int_l^m f_Y(y)dy+o(h)\big(\int_l^m f_Y(y)dy+1\big),\\
&= h\int_B \lambda f_Y(y)dy+o(h).
\end{aligned}$$

Case 4a: $Q\big(h,x,B\big), \quad B=(l,m), \quad x\in(0,c), \quad B\subset(x,c), \quad x\le l<m\le c.$

$$
\begin{aligned}
Q\big(h,x,B\big) &= P\big(X_h\in B|X_0=x\big),\\
&= P\big(X_h\in B|X_0=x,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h\in B|X_0=x,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h\in B|X_0=x,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=x\big),\\
&= 0\cdot\big(1-\lambda h+o(h)\big)+\int_{l-x}^{m-x} f_Y(y)dy\big(\lambda h+o(h)\big)+o(h),\\
&= \lambda h\int_{l-x}^{m-x} f_Y(y)dy+o(h).
\end{aligned}
$$

Case 4b: $Q\big(h,x,B\big), B=(l,m), B\subset(0,x), x\in(0,c), 0<l<m\le x.$

$$
\begin{aligned}
Q\big(h,x,B\big) &= P\big(X_h\in B|X_0=x\big),\\
&= P\big(X_h\in B|X_0=x,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h\in B|X_0=x,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h\in B|X_0=x,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=x\big),\\
&= 0\cdot\big(1-\lambda h+o(h)\big)+0\cdot\big(\lambda h+o(h)\big)+o(h),\\
&= o(h).
\end{aligned}
$$

Case 4c: $Q\big(h,x,\{x\}\big), x\in(0,c).$

$$
\begin{aligned}
Q\big(h,x,\{x\}\big) &= P\big(X_h=x|X_0=x\big),\\
&= P\big(X_h=x|X_0=x,0\text{ jump in}[0,h]\big)\cdot P\big(0\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h=x|X_0=x,1\text{ jump in}[0,h]\big)\cdot P\big(1\text{ jump in}[0,h]|X_0=x\big)\\
&+ P\big(X_h=x|X_0=x,2\text{ or more jumps in}[0,h]\big)\cdot P\big(2\text{ or more jumps in}[0,h]|X_0=x\big),\\
&= 1\cdot\big(1-\lambda h+o(h)\big)+0+0,\\
&= 1-\lambda h+o(h).
\end{aligned}
$$

Thus, $Q(h,x,B)$ is uniformly w.r.t B. □

Consequently, we can determine the stationary distribution $\pi(dx)$:

Lemma 9 *$\pi(dx)$ takes the form*

$$
\begin{aligned}
\pi(\{0\}) &= \theta_0,\\
\pi(B) &= \int_B p(x)\mathrm{d}x, \quad B\subset(0,c),
\end{aligned}
\tag{4.9}
$$

where θ_0 and $p(x)$ are given by the equation system

$$\begin{cases} \theta_0 f_Y(y) - p(y) + \int_{x\in(0,y)} p(x) f_Y(y-x)\mathrm{d}x = 0, \\ \theta_0 + \int_0^c p(x)dx = 1. \end{cases} \tag{4.10}$$

Proof For the unself-weighted model, $\pi(x)$ takes the form

$$\begin{aligned} \pi(\{0\}) &= \theta_0, \\ \pi(B) &= \int_B p(x)\mathrm{d}x, B \subset (0,c), \end{aligned} \tag{4.11}$$

Since we have the relation

$$\int_{x\in[0,c)} \pi(\mathrm{d}x) Q(h,x,B) = \pi(B), \tag{4.12}$$

where B is any Borel subset of $[0,c)$. We introduce a test function $\phi(x)$ which is bounded and uniformly continuous on $(0,c)$. Then we have

$$\int_{y\in[0,c)} \phi(y) \int_{x\in[0,c)} \pi(\mathrm{d}x) Q(h,x,\mathrm{d}y) = \int_{z\in[0,c)} \phi(z)\pi(\mathrm{d}z).$$

If we exchange the order of the integration and get

$$\int_{x\in[0,c)} \pi(\mathrm{d}x) \int_{y\in[0,c)} \phi(y) Q(h,x,\mathrm{d}y) = \int_{z\in[0,c)} \phi(z)\pi(\mathrm{d}z). \tag{4.13}$$

The RHS of (4.13) is

$$\int_{z\in[0,c)} \phi(z)\pi(\mathrm{d}z) = \theta_0\phi(0) + \int_{z\in(0,c)} \phi(z)\pi(\mathrm{d}z). \tag{4.14}$$

When h is small, LHS of (4.13) is given by

$$\begin{aligned}
LHS &= \theta_0 \int_{y\in[0,c)} \phi(y) Q\big(h,0,\mathrm{d}y\big) + \int_{x\in(0,c)} p(x)\mathrm{d}x \int_{y\in[0,c)} \phi(y) Q\big(h,x,\mathrm{d}y\big) \\
&= \theta_0 \Bigg(\phi(0) Q\big(h,0,\{0\}\big) + \int_{y\in(0,c)} \phi(y) Q\big(h,0,\mathrm{d}y\big) \Bigg) \\
&+ \int_{x\in(0,c)} p(x)\mathrm{d}x \Bigg(\phi(0) Q\big(h,x,\{0\}\big) + \int_{y\in(0,x)} \phi(y) Q\big(h,x,\mathrm{d}y\big) + \phi(x) Q\big(h,x,\{x\}\big) \\
&+ \int_{y\in(x,c)} \phi(y) Q\big(h,x,\mathrm{d}y\big) \Bigg) \\
&= \theta_0 \Bigg(\phi(0)\Big(1 - \big(1 - \int_c^\infty f_Y(y)\mathrm{d}y\big)\lambda h + o(h)\Big) + h \int_{y\in(0,c)} \phi(y)\lambda f_Y(y)\mathrm{d}y + o(h) \Bigg) \\
&+ \int_{x\in(0,c)} p(x)\mathrm{d}x \Bigg(\lambda h \phi(0) \int_{c-x}^\infty f_Y(y)\mathrm{d}y + \phi(0)\cdot o(h) + \phi(x)\big(1 - \lambda h + o(h)\big) \\
&+ \lambda h \int_{y\in(x,c)} \phi(y)\cdot f_Y(y-x)\mathrm{d}y + O(h^2) \Bigg). \qquad (4.15)
\end{aligned}$$

If we deduct the RHS from both LHS and RHS of (4.13) and then divide λh from both sides, we can get:

$$\theta_0 \Bigg(-\phi(0)\Big(1 - \int_c^\infty f_Y(y)\mathrm{d}y\Big) + \int_{y\in(0,c)} \phi(y) f_Y(y)\mathrm{d}y \Bigg) + o(h)$$

$$= \int_{x\in(0,c)} p(x) \Bigg(\phi(x) - \phi(0) \int_{c-x}^\infty f_Y(y)\mathrm{d}y - \int_{y\in(x,c)} \phi(y) f_Y(y-x)\mathrm{d}y \Bigg) \mathrm{d}x + o(h).$$

If we change the order of integration of the third term within the large bracket in the RHS of the above equation, we can get:

$$\theta_0 \int_{y\in(0,c)} \phi(y) f_Y(y)\mathrm{d}y - \theta_0 \phi(0)\Big(1 - \int_c^\infty f_Y(y)\mathrm{d}y\Big) =$$

$$\int_{y\in(0,c)} \phi(y)\Big(p(y) - \int_{x\in(0,y)} p(x) f_Y(y-x)\mathrm{d}x\Big)\mathrm{d}y - \phi(0) \int_{x\in(0,c)} p(x)\Big(\int_{c-x}^\infty f_Y(y)\mathrm{d}y\Big)\mathrm{d}x.$$

In the above equation, if we set $\phi(0) = 0$, then

$$\theta_0 \int_{y\in(0,c)} \phi(y) f_Y(y)\mathrm{d}y = \int_{y\in(0,c)} \phi(y)\Big(p(y) - \int_{x\in(0,y)} p(x) f_Y(y-x)\mathrm{d}x\Big)\mathrm{d}y$$

which gives

$$\int_{y\in(0,c)}\phi(y)\left(\theta_0 f_Y(y)-p(y)+\int_{x\in(0,y)}p(x)f_Y(y-x)\mathrm{d}x\right)\mathrm{d}y=0.$$

Since $\phi(x), x\in(0,c)$ can be any functions, the function in the biggest bracket of above equation must be 0, i.e.,

$$\theta_0 f_Y(y)-p(y)+\int_{x\in(0,y)}p(x)f_Y(y-x)\mathrm{d}x=0.$$

□

In order to be compare our two models, we provide the following table:

Table 4.1: Comparation between the Self-weighted Model and Unbounded Model

	self-weighted model	unself-weighted model
player's state evolution	$X_t^i=X_0^i+\sum_{j=1}^{N(t)}\left(1-X_{\tau_j^-}^i\right)Y_j$	$X_t^i=X_0^i+\sum_{j=1}^{N(t)}Y_j$
state space of X	$X\in S=[0,1]$	$X\in S=[0,\infty)$
QVI	$0=\min\Big[l(x,z_t)+V_t(t,x)+\lambda_1(1-x)V_x(t,x)-\rho V(t,x),V(t,0)-V(t,x)+\gamma\Big]$	$0=\min\Big[l(x,z_t)+V_t(t,x)+\lambda_1 V_x(t,x)-\rho V(t,x),V(t,0)-V(t,x)+\gamma\Big]$
Stationary QVI	$0=\min\Big[l(x,z)+\lambda_1(1-x)V_x(x)-\rho V(x),V(0)-V(x)+\gamma\Big]$	$0=\min\Big[l(x,z)+\lambda_1 V_x(x)-\rho V(x),V(0)-V(x)+\gamma\Big]$
Transition kernel Q	1) $Q(h,0,\{0\})=1-\lambda h+\lambda h\int_c^1 f_Y(y)\mathrm{d}y+o(h)$. 2) $Q(h,x,\{0\})=\lambda h\int_{\frac{c-x}{1-x}}^1 f_Y(y)\mathrm{d}y+o(h), x\in(0,c)$. 3) $Q(h,0,B)=\lambda h\int_B f_Y(y)\mathrm{d}y+o(h), B=(l,m)\subset(0,c), 0<l<m\leq c$. 4a) $Q_a(h,x,B)=\lambda h\int_{\frac{l-x}{1-x}}^{\frac{m-x}{1-x}} f_Y(y)\mathrm{d}y+o(h), B=(l,m)\subset(x,c)$. 4b) $Q_b(h,x,B)=o(h), B=(l,m)\subset(0,x)$. 4c) $Q_c(h,x,\{x\})=1-\lambda h+o(h), x\in(0,c)$.	1) $Q(h,0,\{0\})=1-\lambda h+\lambda h\int_c^\infty f_Y(y)\mathrm{d}y+o(h)$. 2) $Q(h,x,\{0\})=\lambda h\int_{c-x}^\infty f_Y(y)\mathrm{d}y+o(h), x\in(0,c)$. 3) $Q(h,0,B)=\lambda h\int_B f_Y(y)\mathrm{d}y+o(h), B=(l,m)\subset(0,c), 0<l<m\leq c$. 4a) $Q_a(h,x,B)=\lambda h\int_{l-x}^{m-x} f_Y(y)\mathrm{d}y+o(h), B=(l,m), B\subset(x,c)$. 4b) $Q_b(h,x,B)=o(h), B=(l,m)\subset(0,x)$. 4c) $Q_c(h,x,\{x\})=1-\lambda h+o(h), x\in(0,c)$.

Example 4 Consider Y $\sim f_Y(y)=\lambda e^{-\lambda y}$. The first equation in (4.10) becomes

$$\theta_0\cdot\lambda e^{-\lambda x}-p(x)e^{\lambda x}e^{-\lambda x}+\int_0^x p(y)\lambda e^{-\lambda(x-y)}dy=0. \tag{4.16}$$

We multiply $e^{\lambda x}$ to both sides of (4.16) and write $L(x) = p(x)e^{\lambda x}$:

$$\lambda\theta_0 - L(x) + \int_0^x \lambda L(y)dy = 0. \tag{4.17}$$

First we fix θ_0 and then differentiate with respect to x of both sides of (4.17) we obtain

$$L'(x) = \lambda L(x). \tag{4.18}$$

The solution of (4.18) is

$$L(x) = c_1 e^{\lambda x}, \tag{4.19}$$

where c_1 is a constant to be determined. By comparing (4.19) and $L(x) = p(x)e^{\lambda x}$, we know

$$p(x) = c_1.$$

We substitute (4.19) into (4.17) and solve for θ_0:

$$\theta_0 = \frac{c_1}{\lambda}.$$

Finally we substitute $p(x)$ and θ_0 into the second equation of (4.10) and get

$$c_1 = \frac{1}{\frac{1}{\lambda}+1} = \frac{\lambda}{1+\lambda c}.$$

Thus $p(x) = \frac{\lambda}{1+\lambda c}, \quad x \in (0,c]$ and $\theta_0 = \frac{1}{1+\lambda c}$.

Chapter 5

Conclusion

In this thesis we analyze mean field games in a continuous time Markov Decision Processes framework, and we investigate the properties of the stationary equation system. A very simple threshold structure is identified for the stationary value function and the associated best response strategy. For further work, we plan to analyze the property of the value function with a finite time horizon. A further extension is to study the mean field game with a long-run average cost, based on the stationary equation system in this thesis.

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