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Full Name of Author — Nom complet de l'auteur

COLLEEN DIANE CUTLER

Date of Birth — Date de naissance

JULY 13 / 57

Country of Birth — Lieu de naissance

CANADA

Permanent Address — Résidence fixe

#1207 — 2080 PEMBINA HWY.
WINNIPEG, MANITOBA
R3T 2G9

Title of Thesis — Titre de la thèse

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DR. DONALD A. DAWSON

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SOME MEASURE-THEORETIC AND TOPOLOGICAL RESULTS
FOR MEASURE-VALUED AND SET-VALUED STOCHASTIC PROCESSES

by

© Colleen Cutler, B.Sc., M.Sc.

A thesis submitted to the Faculty of Graduate
Studies for partial fulfilment of the requirements
for the degree of Doctor of Philosophy

Department of Mathematics
and Statistics

Carleton University

Ottawa, Ontario

October, 1984

The undersigned recommend to the Faculty
of Graduate Studies acceptance of the thesis
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FOR MEASURE-VALUED AND SET-VALUED STOCHASTIC PROCESSES"
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fulfilment of the requirements for the degree of
Doctor of Philosophy

Chris G. Courtiza

External Examiner

Donald A. Dawson

Thesis Supervisor

B. 007. Puttaswara

Chairman, Department of Mathematics
and Statistics

Carleton University

October, 1984

To Sherif

ABSTRACT

Given two measure-valued stochastic processes $X(t)$ and $Y(t)$ it is shown that the Lebesgue decomposition of $X(t)$ with respect to $Y(t)$ results in two measure-valued processes, the absolutely continuous component $X_{ac}(Y)(t)$ and the singular component $X_s(Y)(t)$. As a consequence the joint measurability of the Radon-Nikodym derivative $\frac{dX_{ac}(Y)(t)}{dY(t)}$ is established.

When $X(t)$ is a finite random measure process in $\mathbb{R}^N$ a decomposition of $X(t)$ with respect to Hausdorff-Besicovitch dimension can be obtained. The existence of a measure-valued process $\hat{X}(t)$ on $[0, N]$ is established, where the normalized version $\frac{\hat{X}(t, E)}{X(t, \mathbb{R}^N)}$ gives the

proportion of mass of $X(t)$ which can be supported on sets of dimension lying in E . For a finite measure μ in $\mathbb{R}^N$ there exists a representation of μ as a measure-valued integral $\mu(\cdot) = \int \phi(\cdot, a) \mu(da)$ with respect to dimension.

Topologies on the space of closed subsets F are studied and applied to the support process of a random measure process. The measurability of the dimension map on the closed sets is established.

A set-valued Markov process in $[0, 1]$ is constructed and analyzed via semigroup theory.

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INTRODUCTION

The theory of random measures, arising initially from the study of point processes on the line, has found wide application in the physical and biological sciences, dealing as it does with spatially-distributed data. Kallenberg [11] and Jagers [10] provide a thorough treatment of the measure-theoretic background of this theory. The initial portion of this thesis is devoted to establishing further measure-theoretic results in this area. Specifically, given a measure-valued stochastic process in $\mathbb{R}^N$ we can consider decomposing it into two quantities, the absolutely continuous component and the singular component with respect to Lebesgue measure. The absolutely continuous component (assuming it is nonzero) is supported on a (random) set of positive Lebesgue measure (or "large" set) while the singular component is carried by sets of Lebesgue measure zero (or "small" sets). In order to distinguish between these small sets (in terms of size) we use the notion of Hausdorff-Besicovitch dimension. In $\mathbb{R}^N$ a set of positive Lebesgue measure always has dimension N while sets of zero Lebesgue measure have dimension ranging between 0 and N ; a countable set, for example, always has dimension 0. Dawson and Hochberg, in [7] and [8], have successfully exploited this notion of dimension to show that certain measure-valued processes are singular.

Chapters 2 and 3 of this thesis are chiefly devoted to developing the measure-theoretic background of the ideas given above. In Chapter 2 we present the Lebesgue decomposition of a measure-valued stochastic process. Given any σ -finite measure μ we can, via the usual Lebesgue decomposition theorem for measures, split μ into its absolutely continuous and singular components. For measure-valued processes we

show that this decomposition results in two new measure-valued processes and, in its most general form, can be carried out to obtain the decomposition of one random measure process $X(t)$ with respect to another $Y(t)$. These results depend largely on the properties of the weak topology on the space of finite measures and Chapter 1 deals with the basics of this topic.

In Chapter 3 we introduce the notion of decomposing a finite measure in $\mathbb{R}^N$ with respect to Hausdorff-Besicovitch dimension and then extend this decomposition to processes. Given a (nonzero) measure-valued process $X(t)$ in $\mathbb{R}^N$ we have an associated process $\hat{X}(t)$ on $[0, N]$ (which may be normalized to be a probability measure-valued process) where the quantity $\hat{X}(t, E)$ essentially represents the portion of $X(t)$ which can be supported on sets with dimension lying in E . We also give a representation of a finite measure μ as a measure-valued integral $\mu(\cdot) = \int \phi(\cdot, \alpha) \hat{\mu}(d\alpha)$ with respect to dimension. The results here rely heavily on the topological properties set out in Chapter 1 as well as some results developed in Chapter 2.

Another method of studying the behavior of a process $X(t)$ with respect to dimension is to examine the quantity $\dim(\text{Supp } X_s(t))$, the dimension of the support of the singular part of $X(t)$. (The quantity $\dim(\text{Supp } X_{ac}(t))$, where $X_{ac}(t)$ is the absolutely continuous component, is only of use in determining when $X(t)$ is purely singular as it always takes value N when $X_{ac}(t)$ is nonzero.) To make precise the notion of support process we need to specify an appropriate topology on F , the space of closed subsets. The first sections of Chapter 4 are devoted to bringing together and comparing the most likely candidates for a topology on F (the material here is largely taken from Michael [15] and Matheron [14]). Some theory of random sets has

been developed, notably by Choquet [4], Kendall [12], and Matheron [14], but the literature is not wide, probably due to apparent lack of applications. A very recent paper by Norberg [16] indicates an awakening of interest in this area. In this spirit, having introduced the notion of a set-valued process through support processes in Chapter 4 (and proving the measurability of the dimension map of a random closed set) we turn our attention to the construction of a non-trivial example of a set-valued Markov process (the "random Cantor set process") in Chapter 5. This process has the advantage of exhibiting sufficient peculiarities to be interesting while being simple enough to analyze in a fairly complete way. We use the standard techniques of semigroup theory and the topological results of Chapter 4 in describing this model. In the last section of the chapter we examine an associated measure-valued Cantor process which helps illustrate the earlier concepts of Lebesgue and dimension decomposition.

CHAPTER 1

The Weak Topology On $M(S)$

Given any topological space X , the Borel σ -field (or Borel σ -algebra) on X will be denoted by $\mathcal{B}(X)$ and will always mean the smallest σ -field containing the open sets of X . Here we will consider a metric space (S, ρ) and its corresponding Borel σ -field $\mathcal{B}(S)$. A finite Borel measure μ on S is a nonnegative measure defined on $\mathcal{B}(S)$ such that $\mu(S) < \infty$. We will refer to the collection of finite Borel measures on S as $M(S)$. Later, once the topology of weak convergence (or, simply, weak topology) has been defined, we will, unless otherwise stated, assume that $M(S)$ carries this topology.

In this chapter we establish some notation and present certain background material concerning the weak topology on $M(S)$. Most of this material is quite standard and can be found (occasionally in somewhat less general form) in one or more of Billingsley [1], Jagers [10], Kallenberg [11], Parthasarathy [17], and Varadhan [21]. Of particular importance are the results concerning the open, closed, and measurable subsets of $M(S)$ as well as the nature of measurable maps into $M(S)$; these results are used repeatedly in subsequent chapters and thus are presented in detail. We also consider some extensions to $M_R(S)$, the space of locally finite, or Radon, measures on S under the vague topology (a Borel measure μ is locally finite if $\mu(K) < \infty$ for every compact set $K \subset S$).

Let $C_b(S)$ denote the space of all continuous bounded real-valued functions $f: S \rightarrow \mathbb{R}$. A sequence $\{\mu_n\}_{n=1}^{\infty}$ in $M(S)$ converges weakly to

$\mu \in M(S)$ (denoted $\mu_n \xrightarrow{w} \mu$) if $\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu$ for every $f \in C_b(S)$. More generally, if $\{\mu_\alpha\}_{\alpha \in D}$ is a net in $M(S)$ (where D is a directed set) then $\mu_\alpha \xrightarrow{w} \mu$ if $\lim_{\alpha} \int f d\mu_\alpha = \int f d\mu$ for every $f \in C_b(S)$. The advantage in using nets is that convergence of nets completely and uniquely determines a topology. This fact, coupled with the following well-known result, will be used to show the equivalence of different bases for the (as yet undefined) weak topology on $M(S)$.

Theorem 1.1: Let $\mu \in M(S)$ and let $\{\mu_\alpha\}_{\alpha \in D}$ be a net in $M(S)$. Then the following are equivalent:

- (i) $\mu_\alpha \xrightarrow{w} \mu$
- (ii) $\lim_{\alpha} \int f d\mu_\alpha = \int f d\mu$ for every bounded uniformly-continuous function $f: S \rightarrow \mathbb{R}$.
- (iii) $\lim_{\alpha} \mu_\alpha(S) = \mu(S)$ and $\lim_{\alpha} \inf \mu_\alpha(G) \geq \mu(G)$ for every open set $G \subseteq S$.
- (iv) $\lim_{\alpha} \mu_\alpha(S) = \mu(S)$ and $\lim_{\alpha} \sup \mu_\alpha(F) \leq \mu(F)$ for every closed set $F \subseteq S$.
- (v) $\lim_{\alpha} \mu_\alpha(B) = \mu(B)$ for all μ -continuity sets $B \in \mathcal{B}(S)$.
(B is a μ -continuity set if $\mu(\bar{B} \setminus B^\circ) = 0$ where $\bar{B}$ is the closure of B and B° is the interior of B .)

proof: See, for example, Parthasarathy [17], Theorem 6.1, pg. 40.

In dealing with $M(S)$ rather than $M_1(S)$, the space of probability measures (as in [17]), we need only add the condition $\lim_{\alpha} \mu_\alpha(S) = \mu(S)$ in (iii) and (iv). $\square$

We now define the weak topology on $M(S)$ as follows:

The collection S_f of sets $\{\nu \in M(S) \mid |\int f d\nu - \int f d\mu| < \epsilon\}$ for $\mu \in M(S)$, $\epsilon > 0$, $f \in C_b(S)$, is a subbasis for a Hausdorff topology $\mathcal{W}$ on $M(S)$. That this topology is Hausdorff follows from the fact that if $\mu \neq \nu$ then there

exists $f \in C_b(S)$ such that $\int f d\mu \neq \int f d\nu$. (This is easily seen by employing the regularity of the measures μ and ν and also proves the uniqueness of weak limits.) Taking $\epsilon = |\int f d\mu - \int f d\nu|$ then $N_1(\mu) = \{\gamma \in M(S) \mid |\int f d\gamma - \int f d\mu| < \epsilon/2\}$ and $N_2(\nu) = \{\gamma \in M(S) \mid |\int f d\gamma - \int f d\nu| < \epsilon/2\}$ are disjoint neighborhoods (nbds.) of μ and ν respectively.

The basis obtained by taking all finite intersections of members of S_f consists of sets of the form $\{\nu \in M(S) \mid |\int f_i d\nu - \int f_i d\mu_i| < \epsilon_i, i=1, \dots, n\}$

where $f_i \in C_b(S)$, $\epsilon_i > 0$, $n \in \mathbb{Z}^+$, and $\mu_i \in M(S)$, but can be replaced by the smaller and more elegant subcollection B_f of sets of the form

$\{\nu \in M(S) \mid |\int f_i d\nu - \int f_i d\mu| < \epsilon, i=1, \dots, n\}$, $f_i \in C_b(S)$, $\mu \in M(S)$, $\epsilon > 0$, $n \in \mathbb{Z}^+$.

To show that B_f is in fact a basis and generates the topology $\mathcal{W}$ let

$\mu \in M(S)$ and choose arbitrarily a basic nbd. of μ (say) $N(\mu) = \{\nu \in M(S) \mid |\int f_i d\nu - \int f_i d\mu| < \epsilon_i, i=1, \dots, n\}$. Set $\epsilon = \min_{1 \leq i \leq n} (\epsilon_i - |\int f_i d\mu - \int f_i d\mu|)$ and let

$M(\mu) = \{\nu \in M(S) \mid |\int f_i d\nu - \int f_i d\mu| < \epsilon, i=1, \dots, n\}$. Then $\mu \in M(\mu)$, $M(\mu) \in B_f$,

and it is easily seen that $M(\mu) \subseteq N(\mu)$. This shows that B_f is a basis

for $\mathcal{W}$. Note that we have also verified that the members of B_f

"centered at μ " (that is, sets of the form $M(\mu) = \{\nu \in M(S) \mid |\int f_i d\nu - \int f_i d\mu| < \epsilon, i=1, \dots, n\}$ for fixed μ) form a local basis at μ ; if G

is any open set containing μ there exists a set $M(\mu)$ such that $M(\mu) \subseteq G$.

Using the basis B_f it is trivial to verify that convergence of a net $\{\mu_\alpha\}_{\alpha \in D}$ to μ in the topology $\mathcal{W}$ is equivalent to the weak convergence $\mu_\alpha \xrightarrow{\mathcal{W}} \mu$ defined above. Thus $\mathcal{W}$ is called the topology of weak convergence, or weak topology, on $M(S)$.

We will now develop two other topological bases on $M(S)$ and then show that they both generate $\mathcal{W}$.

Let S_F be the collection of sets of the form $\{\nu \in M(S) \mid \nu(F) < \mu(F) + \epsilon\}$

$v(S) > \mu(S) - \delta$, $\mu \in M(S)$, $\epsilon > 0$, $\delta > 0$, F a closed subset of S . Then S_F is a subbasis for a topology on $M(S)$ and the collection of finite intersections of members of S_F is thus a basis for this topology. As in the case of B_F we can consider a smaller equivalent basis B'_F consisting of the sets $\{v \in M(S) \mid v(F_i) < \mu(F_i) + \epsilon, v(S) > \mu(S) - \epsilon, i=1, \dots, n\}$, $\mu \in M(S)$, $\epsilon > 0$, F_i closed, $n \in \mathbb{Z}^+$. To show this we choose $\mu \in M(S)$ and $N(\mu) = \{v \in M(S) \mid v(F_i) < \mu_i(F_i) + \epsilon_i, v(S) > \mu_i(S) - \delta_i, i=1, \dots, n\}$ an arbitrary basic nbd. of μ . Set $\epsilon = \min_{1 \leq i \leq n} (\mu_i(F_i) - \mu(F_i) + \epsilon_i, \mu(S) - \mu_i(S) + \delta_i)$ and let $M(\mu) = \{v \in M(S) \mid v(F_i) < \mu(F_i) + \epsilon, v(S) > \mu(S) - \epsilon, i=1, \dots, n\}$. Then certainly $\mu \in M(\mu)$ and $M(\mu) \in B'_F$. Now suppose $v \in M(\mu)$. Then $v(F_i) < \mu(F_i) + \epsilon \leq \mu_i(F_i) + \mu_i(F_i) - \mu(F_i) + \epsilon_i = \mu_i(F_i) + \epsilon_i$ for each $i=1, \dots, n$, and also $v(S) > \mu(S) - \epsilon \geq \mu_i(S) - (\mu(S) - \mu_i(S) + \delta_i) = \mu_i(S) - \delta_i$ for each $i=1, \dots, n$. Therefore $M(\mu) \subseteq N(\mu)$ which gives the desired result. Note we have also shown that the sets of the form $M(\mu)$, for fixed μ , form a local basis at μ . We will refer to B_F as the closed-set basis.

In an analogous manner we can create the open-set basis B_g whose members are sets of the form $\{v \in M(S) \mid v(G_i) > \mu(G_i) - \epsilon, v(S) < \mu(S) + \epsilon, i=1, \dots, n\}$, $\mu \in M(S)$, $\epsilon > 0$, G_i an open subset of S for $i=1, \dots, n$, $n \in \mathbb{Z}^+$. B_g generates the same topology as that generated by the subbasis S_g of sets $\{v \in M(S) \mid v(G) > \mu(G) - \epsilon, v(S) < \mu(S) + \delta\}$, $\mu \in M(S)$, $\epsilon > 0$, $\delta > 0$, G an open subset of S .

Theorem 1.2: The bases B_F and B_g both generate the weak topology on $M(S)$.

proof: Consider B_F . Let $\mu \in M(S)$ and suppose $\{\mu_\alpha\}_{\alpha \in D}$ is a net such that $\mu_\alpha \xrightarrow{w} \mu$. Let $M(\mu) = \{v \in M(S) \mid v(F_i) < \mu(F_i) + \epsilon, v(S) > \mu(S) - \epsilon, i=1, \dots, n\}$ be a local basic nbd. of μ . Then, by (iv) of Theorem 1.1, we have $\limsup_\alpha \mu_\alpha(F_i) \leq \mu(F_i)$ for each $i=1, \dots, n$. Therefore there exists $\beta_i \in D$

such that $\sup_{\alpha \geq \beta_i} \mu_\alpha(F_i) < \mu(F_i) + \epsilon$ for each $i=1, \dots, n$. Also, since $\lim_\alpha \mu_\alpha(S) = \mu(S)$ (an immediate consequence of weak convergence) there exists $\beta_{n+1} \in D$ such that if $\alpha \geq \beta_{n+1}$ then $\mu_\alpha(S) > \mu(S) - \epsilon$. Now D is a directed set and so there exists $\beta \in D$ such that $\beta \geq \beta_i$, $i=1, \dots, n+1$. Therefore when $\alpha \geq \beta$ we have $\mu_\alpha(F_i) < \mu(F_i) + \epsilon$, $i=1, \dots, n$ and $\mu_\alpha(S) > \mu(S) - \epsilon$. That is, $\alpha \geq \beta$ implies $\mu_\alpha \in M(\mu)$. Hence we can conclude that $\mu_\alpha \rightarrow \mu$ in the topology generated by $\mathcal{B}_F$.

Conversely, suppose $\mu_\alpha \rightarrow \mu$ in the $\mathcal{B}_F$ -topology. Let $\epsilon > 0$ and let F be any closed subset of S . Then there exists $\beta \in D$ such that $\alpha \geq \beta$ implies $\mu_\alpha \in M(\mu) = \{\nu \in M(S) \mid \nu(F) < \mu(F) + \epsilon, \nu(S) > \mu(S) - \epsilon\}$, which gives $\limsup_\alpha \mu_\alpha(F) \leq \mu(F) + \epsilon$ and $\liminf_\alpha \mu_\alpha(S) \geq \mu(S) - \epsilon$. Since $\epsilon > 0$ was arbitrary this gives (a) $\limsup_\alpha \mu_\alpha(F) \leq \mu(F)$ and (b) $\liminf_\alpha \mu_\alpha(S) \geq \mu(S)$. Thus (iv) of Theorem 1.1 is satisfied and so $\mu_\alpha \xrightarrow{w} \mu$.

Since, as noted earlier, the convergence behaviour of nets uniquely determines the topology we can conclude that $\mathcal{B}_F$ generates the weak topology. The proof for $\mathcal{B}_g$ is analogous, using (iii) of Theorem 1.1. $\square$

Note: We have just seen three different bases for the weak topology.

It should be noted, however, that the collection of sets

$\{\nu \in M(S) \mid |\nu(B_i) - \mu(B_i)| < \epsilon, i=1, \dots, n\}$, $\epsilon > 0$, $n \in \mathbb{Z}^+$, B_i a μ -continuity

set for each $i=1, \dots, n$, is not a basis for $\mathcal{W}$, contrary to a statement

made in Billingsley [1], pg. 236, equation (8). An easy counter-

example is the following. Take $S=[0,1]$, λ =Lebesgue measure on $[0,1]$,

and Dirac measures $\delta_n(B) = \begin{cases} 0 & 1/n \notin B \\ 1 & 1/n \in B \end{cases}$ $n=1,2,3,\dots$. Let A be the

half-open interval $(0, 1/4]$. Then $|\delta_n(A) - \lambda(A)| = 1 - 1/4 = 3/4$ for all $n \geq 4$ and

also $\delta_n \xrightarrow{w} \delta_0$ where δ_0 is the Dirac measure at $\{0\}$. However,

$\delta_0(A) = 0$ and so $|\delta_0(A) - \lambda(A)| = 1/4$. This shows that the set

$\{\nu \in M(S) \mid |\nu(A) - \lambda(A)| \geq 1/2\}$ is not closed in the weak topology and hence $\{\nu \in M(S) \mid |\nu(A) - \lambda(A)| < 1/2\}$ is not open (note that A is a λ -continuity set).

We now discuss the general topological nature of $M(S)$ under the weak topology.

Theorem 1.3: The Borel measures with finite support (that is, purely atomic measures having only a finite number of atoms) are dense in $M(S)$.

proof: Let $\mu \in M(S)$, and let $F_1, \dots, F_k$ be any collection of closed subsets of S . Consider the partition of S consisting of the sets $A_1 \cap \dots \cap A_k$ where $A_j = F_j$ or $A_j = F_j^c$, $j=1, \dots, k$. Let $B_1, \dots, B_n$ denote the nonempty members of this partition. Pick $x_i \in B_i$ for each $i=1, \dots, n$ and define the atomic measure ν with atoms $x_1, \dots, x_n$ by $\nu(\{x_i\}) = \mu(B_i)$.

$$\begin{aligned} \text{Then } \nu(S) &= \sum_{i=1}^n \nu(\{x_i\}) = \sum_{i=1}^n \mu(B_i) = \mu(S), \text{ and } \nu(F_j) = \sum_{x_i \in F_j} \nu(\{x_i\}) \\ &= \sum_{B_i \subseteq F_j} \mu(B_i) = \mu(F_j). \end{aligned}$$

Therefore $\forall \epsilon \{ \nu \in M(S) \mid \nu(F_i) < \mu(F_i) + \epsilon, \nu(S) > \mu(S) - \epsilon, i=1, \dots, n \}$ for any $\epsilon > 0$. Thus every basic local nbd of μ contains a Borel measure of finite support. $\square$

Theorem 1.4: If S is a separable metric space then $M(S)$ is separable.

proof: Since S is separable there exists a countable dense subset $S_0 \subseteq S$. Let M_0 be the collection of all atomic measures taking only rational values on their atoms and having finite support with this support lying in S_0 . We claim that M_0 is dense in $M(S)$. (M_0 is clearly countable.) In view of Theorem 1.3 it is sufficient to prove that M_0 is dense in the set of atomic measures with finite

support. Let μ be such a measure with support $\{x_1, \dots, x_k\}$. For each $i=1, \dots, k$ choose a sequence $\{x_{in}\}_{n=1}^{\infty}$ from S_0 such that $\lim_{n \rightarrow \infty} x_{in} = x_i$ and define the atomic measures v_n with atoms $x_{1n}, \dots, x_{kn}$ by $v_n(\{x_{in}\}) = \alpha_{in}$ where $\{\alpha_{in}\}_{n=1}^{\infty}$ is a sequence of rational numbers such that $\lim_{n \rightarrow \infty} \alpha_{in} = \mu(\{x_i\})$. Then $v_n \in M_0$ for each n and it is easily seen that $v_n \xrightarrow{w} \mu$. $\square$

Theorem 1.5: If S is a separable metric space then $M(S)$ is metrizable.

proof: The construction of a metric on $M(S)$ is well-known (see, for example, Varadhan [21], Theorem 2, pg. 21). Since S is a separable metric space it can be embedded in a compact metric space (Y, ρ^*) . The collection $U_b(S, \rho^*)$ of bounded uniformly-continuous functions (w.r.t. ρ^*) on S can be identified with $C_b(\bar{S}, \rho^*)$, the bounded continuous functions on the closure $\bar{S}$ of S in Y . Since $\bar{S}$ is compact $C_b(\bar{S}, \rho^*)$ is separable in the sup norm and so we can choose a countable dense subset $\{f_1, f_2, \dots\}$ of $U_b(S, \rho^*)$. (w.l.o.g. we can take $f_1 \equiv 1$.) We can then define a metric D on $M(S)$ by

$$D(\mu_1, \mu_2) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{| \int f_j d\mu_1 - \int f_j d\mu_2 |}{1 + | \int f_j d\mu_1 - \int f_j d\mu_2 |}.$$

It is easily verified that if $\{\mu_{\alpha}\}_{\alpha \in D}$ is a net then $\mu_{\alpha} \xrightarrow{w} \mu$ iff $\lim_{\alpha} D(\mu_{\alpha}, \mu) = 0$. Thus D generates the weak topology. This also means we can dispense with our consideration of nets since sequences are sufficient in a metric space. $\square$

Theorem 1.6: S is separable iff $M(S)$ is separable and metrizable.

proof: If S is separable Theorems 1.4 and 1.5 give the separability

and metrizable of $M(S)$. The converse is shown by embedding S as a subspace of $M(S)$ via the map $x \rightarrow \delta_x$ where δ_x is the Dirac measure (or unit mass) at $\{x\}$. Since $M(S)$ is second-countable the subspace S must be second-countable and hence separable. $\square$

Our next step is to characterize measurable maps into $M(S)$ when S is separable. To do this we will need a well-known result of Dynkin's (to be applied also in subsequent chapters). Recall that a π -system on a set Ω is any collection of subsets of Ω which is closed under finite intersections. A d -system on Ω is a collection of subsets which contains Ω and is closed under proper differences and increasing limits. (That is, $\mathcal{D}$ is a d -system if (i) $\Omega \in \mathcal{D}$, (ii) $A, B \in \mathcal{D}$, $A \subseteq B$ implies $B \setminus A \in \mathcal{D}$, and (iii) when $A_n \in \mathcal{D}$, $A_n \subseteq A_{n+1} \forall n$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.)

Theorem 1.7 (Dynkin's Lemma): The smallest d -system containing a particular π -system coincides with the σ -algebra generated by the π -system.

proof: See, for example, Billingsley [3], Theorem 3.2, pg. 34. $\square$

Theorem 1.8: Let S be a separable metric space. The Borel σ -field $\mathcal{B}(M(S)) = \sigma(\pi_B | \mathcal{B}\mathcal{B}(S))$, the σ -algebra generated by the "projection maps" π_B , where $\pi_B: M(S) \rightarrow \mathbb{R}$ is defined by $\pi_B(\mu) = \mu(B)$.

proof: We show first that $\mathcal{B}(M(S)) \subseteq \sigma(\pi_B | \mathcal{B}\mathcal{B}(S))$ by using the closed-set basis $\mathcal{B}_F$. Any member of $\mathcal{B}_F$ can be expressed in the form $\{\mu \in M(S) | \mu(F_i) < r_i, \mu(S) > r, i=1, \dots, n\}$ for some numbers $r, r_1, \dots, r_n \in \mathbb{R}$ and $\{\mu \in M(S) | \mu(F_i) < r_i, \mu(S) > r, i=1, \dots, n\} = \pi_S^{-1}((r, \infty)) \cap \bigcap_{i=1}^n \pi_{F_i}^{-1}([0, r_i))$. Since the righthand side (RHS) set clearly belongs to $\sigma(\pi_B | \mathcal{B}\mathcal{B}(S))$ this

shows that $B_F \subseteq \sigma(\pi_B | B \in \mathcal{B}(S))$. Now the separability of S implies the separability and metrizability (and hence second-countability) of $M(S)$. Therefore the basis B_F generates $\mathcal{B}(M(S))$ and so we must have $\mathcal{B}(M(S)) \subseteq \sigma(\pi_B | B \in \mathcal{B}(S))$.

To show the converse, let $C = \{B \in \mathcal{B}(S) | \pi_B \text{ is measurable}\}$. Then C is a d -system (this is easily seen by noting that if $A \subseteq B$ then $\pi_{B \setminus A} = \pi_B - \pi_A$ and if $A_n \subseteq A_{n+1}$ then $\pi_{\bigcup A_n} = \lim_{n \rightarrow \infty} \pi_{A_n}$ pointwise). Now consider the collection F of closed subsets of S . Then $F \subseteq C$ since if F is closed then $\pi_F^{-1}((-\infty, r)) = \{\mu \in M(S) | \mu(F) < r\}$ which is clearly an open set in $M(S)$. Thus π_F is measurable. Now clearly F is a π -system and $\sigma(F) = \mathcal{B}(S)$. Applying Dynkin's Lemma we then obtain $C = \mathcal{B}(S)$, giving the desired result. (Note this part of the proof does not require separability.) $\square$

Corollary 1.8.1: Let (X, X) be any measurable space, S a separable metric space. A map $f: X \rightarrow M(S)$ is measurable iff $\pi_B \circ f$ is measurable for each $B \in \mathcal{B}(S)$.

proof: From Theorem 8 the maps π_B are measurable and thus the measurability of f implies the measurability of $\pi_B \circ f$. Conversely if we consider any subbasic set $\{\mu \in M(S) | \mu(F) < r\}$, F closed, then $f^{-1}(\{\mu \in M(S) | \mu(F) < r\}) = (\pi_F \circ f)^{-1}((-\infty, r))$ and thus the measurability of $\pi_B \circ f$ for all $B \in \mathcal{B}(S)$ implies the measurability of f . $\square$

We now summarize the results concerning the bases and measurable subsets of $M(S)$ in the following theorem:

Theorem 1.9: (i) The collection $\mathcal{B}_f$ of sets of the form $\{\mu \in M(S) | |\int f_i d\mu - \int f_i d\nu| < \epsilon, i=1, \dots, n\}$, $\mu \in M(S)$, $\epsilon > 0$, $n \in \mathbb{Z}^+$, $f_i \in C_b(S)$ $i=1, \dots, n$, is a basis for $M(S)$.

(ii) The collection B_F of sets of the form $\{v \in M(S) \mid v(F_i) < u(F_i) + \epsilon, v(S) > u(S) - \epsilon, i=1, \dots, n\}$, $u \in M(S)$, $\epsilon > 0$, $n \in \mathbb{Z}^+$, F_i closed $i=1, \dots, n$, is a basis for $M(S)$. These sets can also be expressed in the form $\{v \in M(S) \mid v(F_i) < r_i, v(S) > r, i=1, \dots, n\}$, where $r, r_1, \dots, r_n \in \mathbb{R}$. In particular the sets $\{v \in M(S) \mid v(F) < r\}$, F closed, $r \in \mathbb{R}$, are open in the weak topology.

(iii) The collection B_G of sets $\{v \in M(S) \mid v(G_i) > u(G_i) - \epsilon, v(S) < u(S) + \epsilon, i=1, \dots, n\}$, $u \in M(S)$, $\epsilon > 0$, $n \in \mathbb{Z}^+$, G_i open $i=1, \dots, n$, is a basis for $M(S)$. We have the alternate representation of these sets as $\{v \in M(S) \mid v(G_i) > r_i, v(S) < r, i=1, \dots, n\}$ where $r, r_1, \dots, r_n \in \mathbb{R}$. In particular the sets $\{v \in M(S) \mid v(G) > r\}$, G open, $r \in \mathbb{R}$, are open in $M(S)$.

(iv) Sets of the form $\{v \in M(S) \mid v(B) < r\}$, $B \in \mathcal{B}(S)$, $r \in \mathbb{R}$ are measurable in $M(S)$ and in fact generate $\mathcal{B}(M(S))$ when S is separable. $\square$

(Many of the results in the following chapters will require that S be a Polish space; that is, in addition to S being separable we have the existence of an equivalent metric in which S is complete (we will sometimes refer to this latter property as topological completeness).

We now want to show that if S is Polish then $M(S)$ is also Polish.

Lemma 1.1: If S is a compact metric space then $M(S)$ is Polish,

proof: Since S is compact $C(S)$, the space of continuous real-valued functions on S , is separable and we can choose a countable dense subset $\{f_1, f_2, \dots\}$ of $C(S)$ (taking $f_1 \equiv 1$). Then the metric D defined

$$\text{by } D(\mu, \nu) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{| \int f_j d\mu - \int f_j d\nu |}{1 + | \int f_j d\mu - \int f_j d\nu |} \text{ generates the weak topology}$$

(see the proof of Theorem 1.5). We will show that $M(S)$ is complete

in the metric D . Clearly a sequence $\{\mu_n\}_{n=1}^\infty$ is Cauchy in D iff $\{\int f_j d\mu_n\}_{n=1}^\infty$ is a real Cauchy sequence for each $j=1,2,\dots$. Since the f_j 's are dense in $C(S)$ under the sup norm it follows that $\{\int f d\mu_n\}_{n=1}^\infty$ is a real Cauchy sequence for each $f \in C(S)$. Let $\Lambda(f) = \lim_{n \rightarrow \infty} \int f d\mu_n$. Then Λ is a bounded linear functional on $C(S)$ (note this uses the fact that $\{\mu_n(S)\}_{n=1}^\infty$ is a bounded sequence, which can be seen by choosing $f \equiv 1$). Since S is compact the Riesz Representation Theorem implies there exists a unique positive measure $\mu \in M(S)$ such that $\Lambda(f) = \int f d\mu$ for every $f \in C(S)$. Therefore $\mu_n \xrightarrow{w} \mu$ and so $(M(S), D)$ is complete.

Theorem 1.10: If S is a Polish space then $M(S)$ is Polish.

proof: The proof of the topological completeness of $M(S)$ follows that of Parthasarathy in [17], Theorem 6.5, pg. 46. The only difference is that we must use directly from Lemma 1.1 the fact that if S^* is compact then $M(S^*)$ is topologically complete. (When considering the space of probability measures $M_1(S^*)$, as in [17], the compactness of S^* actually implies the compactness of $M_1(S^*)$, from which completeness of course follows.) $\square$

As a consequence of Theorem 1.10 the product space $M(S) \times \dots \times M(S)$ (for any finite dimension) is Polish in the product topology.

We now discuss some extensions to $M_R(S)$, the space of Radon measures under the vague topology. As the techniques and results are similar to the weak topology case the proofs are omitted for the sake of brevity. An excellent reference for this material is Kallenberg [11]. S will be assumed to be a locally compact separable metric space.

A net $\{\mu_\alpha\}_{\alpha \in D}$ in $M_R(S)$ converges vaguely to $\mu \in M_R(S)$ (denoted $\mu_\alpha \xrightarrow{v} \mu$) if $\lim_\alpha \int f d\mu_\alpha = \int f d\mu$ for each $f \in C_K(S)$, where $C_K(S)$ denotes the space of continuous functions $f: S \rightarrow \mathbb{R}$ which vanish outside a compact subset of S . As in the case of weak convergence we have various equivalent conditions for vague convergence.

Theorem 1.11: Let $\mu \in M_R(S)$ and let $\{\mu_\alpha\}_{\alpha \in D}$ be a net in $M_R(S)$. Then the following are equivalent:

- (i) $\mu_\alpha \xrightarrow{v} \mu$.
- (ii) There exist compact sets $\{K_n\}_{n=1}^\infty$ satisfying $K_n \subseteq K_{n+1}^o$ and $\bigcup_{n=1}^\infty K_n = S$ such that $\lim_\alpha \mu_\alpha(K_n) = \mu(K_n)$ for every n , and for every compact set K we have $\limsup_\alpha \mu_\alpha(K) \leq \mu(K)$.
- (iii) There exist compact sets $\{\tilde{K}_n\}_{n=1}^\infty$ satisfying $\tilde{K}_n \subseteq K_{n+1}^o$ and $\bigcup_{n=1}^\infty \tilde{K}_n = S$ such that $\lim_\alpha \mu_\alpha(\tilde{K}_n) = \mu(\tilde{K}_n)$ for every n , and for every relatively compact open set G we have $\liminf_\alpha \mu_\alpha(G) \geq \mu(G)$.
- (iv) For every relatively compact μ -continuity set B we have $\lim_\alpha \mu_\alpha(B) = \mu(B)$.

Conditions (ii) and (iii) can be replaced by the equivalent condition

- (ii)' $\limsup_\alpha \mu_\alpha(K) \leq \mu(K)$ for every compact set K and $\liminf_\alpha \mu_\alpha(G) \geq \mu(G)$ for every relatively compact open set G . $\square$

The notions of weak and vague convergence are related as follows: If $\mu \in M(S)$ and $\{\mu_n\}_{n=1}^\infty$ is a sequence in $M(S)$ then $\mu_n \xrightarrow{w} \mu$ iff $\mu_n \xrightarrow{v} \mu$ and $\lim_{n \rightarrow \infty} \mu_n(S) = \mu(S)$. The collection

$\{v \in M_R(S) \mid |f dv - f du| < \epsilon\}$, $u \in M_R(S)$, $f \in C_K(S)$, $\epsilon > 0$, forms a subbasis for a topology (called the vague topology) on $M_R(S)$ in which convergence corresponds to the vague convergence defined above. $M_R(S)$ can be shown to be Polish in the vague topology, and the Borel sets of $M_R(S)$ are generated by the projection maps π_B where B runs over the class of relatively compact Borel sets of S . As a consequence of the local compactness of S the maps π_B are measurable for all Borel sets B (however in some cases these will be extended real-valued functions as $\mu(B) = \infty$ is possible).

As a last remark we note that a locally compact separable metric space is always Polish and some frequently used properties of locally compact spaces can be found in Lemmas 4.2.1, 4.2.2, and 4.2.3 of Chapter 4.

CHAPTER 2

A Lebesgue Decomposition Theorem For Random Measures

Here a random measure X will be defined to be a measurable mapping of a probability space $(\Omega, \mathcal{E}, P)$ into $M(S)$. S will be assumed to be a separable metric space. (When necessary we will additionally require that S be Polish.) Later, in the case that S can be taken to be a locally compact separable metric space, we will consider an extension to measurable mappings into $M_R(S)$.

1. The Main Result

We first need the following three standard lemmas:

Lemma 2.1.1: There exists a countable algebra $\mathcal{A}$ generating the Borel sets of S .

proof: This is a consequence of the second countability of S . Let G_0 be a countable basis for S and G_0^* the set of all complements of members of G_0 . Let $A_0 = G_0 \cup G_0^*$. Then for each nonnegative integer j define A_{j+1} inductively by

$$A_{j+1} = \begin{cases} A_{j\sigma_F} & j \text{ even} \\ A_{j\delta_F} & j \text{ odd} \end{cases}$$

where $A_{j\sigma_F} = \{ \bigcup_{i=1}^n A_i \mid A_i \in A_j, n \text{ a positive integer} \}$ is the set of all

finite unions of members of A_j , and $A_{j\delta_F} = \{ \bigcap_{i=1}^n A_i \mid A_i \in A_j, n \text{ a positive integer} \}$ is the set of all finite intersections of members of A_j .

Clearly $A_j \subseteq A_{j+1}$ for each j . Letting $A = \bigcup_{j=0}^{\infty} A_j$ then it is

easily seen that A is countable and an algebra. Since $G_0 \subseteq A$ it follows that A generates $\mathcal{B}(S)$. $\square$

Lemma 2.1.2: If ν is a σ -finite measure on any measurable space $(X, \mathcal{E})$ and A is an algebra generating $\mathcal{E}$ then for every $E \in \mathcal{E}$ there exists $A \in A_{\sigma\delta}$ such that $E \subseteq A$ and $\nu(A \setminus E) = 0$. Consequently for every $\epsilon > 0$ there exists $A' \in A_{\sigma}$ such that $E \subseteq A'$ and $\nu(A' \setminus E) < \epsilon$.

proof: This follows from Royden [19], Proposition 7, pg. 256. $\square$

Lemma 2.1.3: Let μ and ν be two measures (not necessarily σ -finite) on $(X, \mathcal{E})$. Then $\mu \ll \nu$ iff for every $\epsilon > 0$ there exists $B \in \mathcal{E}$ such that $\nu(B) \leq \epsilon$ and $\mu(X \setminus B) \leq \epsilon$.

proof: By definition $\mu \ll \nu$ if there exists $B \in \mathcal{E}$ such that $\nu(B) = \mu(X \setminus B) = 0$, which certainly implies the ϵ -criterion above.

Conversely, the ϵ -criterion implies that for every n there exists

$B_n \in \mathcal{E}$ such that $\nu(B_n) \leq 1/2^n$ and $\mu(X \setminus B_n) \leq 1/2^n$. Letting $B = \liminf B_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} B_m$ it is then trivial to verify $\nu(B) = \mu(X \setminus B) = 0$. $\square$

We can now obtain the following two results:

Theorem 2.1.1: Let ν be a σ -finite measure on $\mathcal{B}(S)$ and let $C_\nu = \{\mu \in M(S) \mid \mu \ll \nu\}$. Then C_ν is a measurable subset of $M(S)$ and has the representation

$$C_\nu = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\substack{A \in A \\ \nu(A) < 1/n}} \{\mu \in M(S) \mid \mu(A) < 1/m\}, \text{ where}$$

A is any algebra generating $\mathcal{B}(S)$.

proof: Since μ is finite then $\mu \ll \nu$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\nu(B) < \delta$ then $\mu(B) < \epsilon$ (see Rudin [20], pg. 132).

Therefore we can write

$$C_v = \bigcup_{\epsilon > 0} \bigcup_{\delta > 0} \bigcap_{\substack{B \in \mathcal{B}(S) \\ v(B) < \delta}} \{\mu \mid \mu(B) < \epsilon\}$$

$$C_v = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{\substack{B \in \mathcal{B}(S) \\ v(B) < 1/n}} \{\mu \mid \mu(B) < 1/m\} \quad (1)$$

Of course $\{\mu \mid \mu(B) < 1/m\}$ is a measurable set in the weak topology. We will show that we can replace $\mathcal{B}(S)$ in expression (1) by any algebra A which generates $\mathcal{B}(S)$. Let A be such an algebra and

$$\text{let } C_v^* = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{\substack{A \in A \\ v(A) < 1/n}} \{\mu \mid \mu(A) < 1/m\}.$$

Clearly $C_v \subseteq C_v^*$. To show the opposite inclusion let $\mu \in C_v^*$. Then for every $\epsilon > 0$ there exists $\delta_A(\epsilon) > 0$ such that whenever $A \in A$ and $v(A) < \delta_A(\epsilon)$ then $\mu(A) < \epsilon$.

claim: For every $\epsilon > 0$ and for every $B \in \mathcal{B}(S)$, $v(B) < \delta_A(\epsilon/2)$ implies $\mu(B) < \epsilon$.

For suppose $v(B) < \delta_A(\epsilon/2)$. Then there exists $E \in A$ such that $B \subseteq E$ and $v(E \setminus B) < \delta_A(\epsilon/2) - v(B)$ (from Lemma 2.1.2). Now E can be expressed as $E = \bigcup_{i=1}^{\infty} E_i$ where $E_i \in A$ for each $i=1,2,\dots$

Since μ is finite there exists n such that $\mu(\bigcup_{i=1}^n E_i) < \epsilon/2$. Let

$$A = \bigcup_{i=1}^n E_i. \quad \text{Then } A \in A \text{ and we have } A = (A \setminus B) \cup (A \cap B) \subseteq (E \setminus B) \cup B.$$

Therefore $v(A) \leq v(E \setminus B) + v(B) < \delta_A(\epsilon/2) - v(B) + v(B) = \delta_A(\epsilon/2)$ and so $\mu(A) < \epsilon/2$.

Now $\widehat{B} = (B \setminus A) \cup (B \cap A) \subseteq (E \setminus A) \cup A$ and so we have $\mu(B) \leq \mu(E \setminus A) + \mu(A) < \epsilon/2 + \epsilon/2 = \epsilon$ which proves the claim.

Therefore it has been shown that $\mu \ll \nu$ and so $\mu \in C_\nu$. Therefore $C_\nu = C_\nu^*$. That is, we can write

$$C_\nu = \{\mu \mid \mu \ll \nu\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{\substack{A \in \mathcal{A} \\ \nu(A) < 1/n}} \{\mu \mid \mu(A) < 1/m\},$$

and choosing $\mathcal{A}$ to be a countable algebra (as in Lemma 2.1.1) we can conclude the measurability of C_ν . $\square$

Theorem 2.1.2: Let ν be a σ -finite measure on $\mathcal{B}(S)$. Then

$S_\nu = \{\mu \in M(S) \mid \mu \ll \nu\}$ is a measurable subset of $M(S)$

and can be expressed as $S_\nu = \bigcup_{n=1}^{\infty} \bigcup_{\substack{A \in \mathcal{A} \\ \nu(A) < 1/n}} \{\mu \mid \mu(S \setminus A) < 1/n\}$ where $\mathcal{A}$ is an algebra generating $\mathcal{B}(S)$.

proof: Using Lemma 2.1.3 we can write $S_\nu = \bigcup_{n=1}^{\infty} \bigcup_{\substack{B \in \mathcal{B}(S) \\ \nu(B) < 1/n}} \{\mu \in M(S) \mid \mu(S \setminus B) < 1/n\}$. We will show that, as in the case of Theorem 2.1.1, we can replace $\mathcal{B}(S)$ in the above expression by any generating algebra $\mathcal{A}$.

This is equivalent to showing that whenever there exists $B \in \mathcal{B}(S)$ such that $\nu(B) < \epsilon$ and $\mu(S \setminus B) < \epsilon$ then there exists $A \in \mathcal{A}$ such that $\nu(A) < 2\epsilon$ and $\mu(S \setminus A) < 2\epsilon$. Let $B \in \mathcal{B}(S)$ and suppose $\nu(B) < \epsilon$, $\mu(S \setminus B) < \epsilon$. From Lemma 2.1.2 there exists $E \in \mathcal{A}_\sigma$ such that $B \subseteq E$ and $\nu(E \setminus B) < \epsilon$.

Since $E = \bigcup_{i=1}^{\infty} E_i$ where $E_i \in \mathcal{A}$ and μ is finite there exists n such that

$\mu(E \setminus \bigcup_{i=1}^n E_i) < \epsilon$. Let $A = \bigcup_{i=1}^n E_i$. Then $A \in \mathcal{A}$ and $A \subseteq E = (E \setminus B) \cup B$ so

$v(A) \leq v(E \setminus B) + v(B) < \epsilon + \epsilon = 2\epsilon$. Also $S \setminus A = (S \setminus E) \cup (E \setminus A) \subseteq (S \setminus B) \cup (E \setminus A)$, giving $\mu(S \setminus A) \leq \mu(S \setminus B) + \mu(E \setminus A) < \epsilon + \epsilon = 2\epsilon$.

Therefore we have $S_v = \{\mu \mid \mu \perp v\} = \bigcup_{n=1}^{\infty} \bigcup_{A \in A} \{\mu \mid \mu(S \setminus A) < 1/n\}$ where $v(A) < 1/n$

A is any algebra generating $\mathcal{B}(S)$. Choosing A to be countable we have the desired result. $\square$

The representations of C_v and S_v given in Theorems 2.1.1 and 2.1.2 enable us to extend our measurability results to two dimensions.

Theorem 2.1.3: The sets $\{(\mu, v) \in M(S) \times M(S) \mid \mu < v\}$ and $\{(\mu, v) \in M(S) \times M(S) \mid \mu \perp v\}$ are measurable subsets of $M(S) \times M(S)$.

proof: Using the representations of C_v and S_v in Theorems 2.1.1 and 2.1.2 we can write

$$\{(\mu, v) \mid \mu < v\} = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{A \in A} [\{\mu \mid \mu(A) < 1/n\} \times \{v \mid v(A) < 1/m\} \cup M(S)$$

$$\times \{v \mid v(A) \geq 1/m\}]$$

$$\text{and } \{(\mu, v) \mid \mu \perp v\} = \bigcup_{n=1}^{\infty} \bigcup_{A \in A} \{\mu \mid \mu(A) < 1/n\} \times \{v \mid v(S \setminus A) < 1/n\}$$

where A is a countable algebra generating $\mathcal{B}(S)$. $\square$

We now wish to present the decomposition theorem for random measures. We will need the following lemma and Kuratowski mapping theorem.

Lemma 2.1.4: If S is a separable metric space the addition map $+: M(S) \times M(S) \rightarrow M(S)$ is continuous with respect to the product topology.

proof: As noted in Chapter 1, if S is a separable metric space then $M(S)$ is metrizable and therefore also $M(S) \times M(S)$ is metrizable in

the product topology. Hence the continuity of a map from $M(S) \times M(S)$ into $M(S)$ is determined by its behavior w.r.t. convergent sequences.

Let $(\mu_n, \nu_n) \rightarrow (\mu, \nu)$ in the product topology. Then $\mu_n \xrightarrow{w} \mu$ and $\nu_n \xrightarrow{w} \nu$. Let B be a continuity set of $\mu + \nu$. It follows then that B is a continuity set of both μ and ν since $\mu + \nu$ dominates each of them. Therefore $\mu_n(B) \rightarrow \mu(B)$ and $\nu_n(B) \rightarrow \nu(B)$, and so $(\mu_n + \nu_n)(B) = \mu_n(B) + \nu_n(B) \rightarrow \mu(B) + \nu(B) = (\mu + \nu)(B)$. Thus $\mu_n + \nu_n \xrightarrow{w} \mu + \nu$. Equivalently $+(\mu_n, \nu_n) \rightarrow +(\mu, \nu)$. Therefore $+$ is continuous. $\square$

Theorem 2.1.4 (Kuratowski): Let X_1, X_2 be Polish spaces, $E_1 \subseteq X_1$, $E_2 \subseteq X_2$, where E_1 is a Borel set of X_1 . Let ψ be a measurable one-one map of E_1 into X_2 such that $\psi(E_1) = E_2$. Then E_2 is a Borel set.
proof: See Parthasarathy [17], Theorem 3.9, pg. 21. $\square$

- We can now obtain our main result for this chapter:

Theorem 2.1.5 (Lebesgue Decomposition Theorem for Random Measures):

Let X and Y be two random measures mapping the probability space $(\Omega, \mathcal{E}, P)$ into $(M(S), \mathcal{B}(M(S)))$ where S is a Polish space. Denote the Lebesgue decomposition of X with respect to Y by $X = X_{ac}(Y) + X_s(Y)$ where, for each $\omega \in \Omega$, $X_{ac}(Y)(\omega) \ll Y(\omega)$ and $X_s(Y)(\omega) \perp Y(\omega)$. Then $X_{ac}(Y)$ and $X_s(Y)$ are both measurable maps of $(\Omega, \mathcal{E}, P)$ into $(M(S), \mathcal{B}(M(S)))$. That is, they are random measures.

proof: By Theorem 1.10, $M(S)$ is a Polish space. In particular it is second-countable and so $\mathcal{B}(M(S) \times M(S)) = \mathcal{B}(M(S)) \times \mathcal{B}(M(S))$. That is, the Borel σ -field on the product space $M(S) \times M(S)$ coincides with the σ -field generated by the products of one-dimensional Borel sets. As a consequence any function f mapping Ω into $M(S) \times M(S)$ is measurable iff each component map $\pi_i \circ f: \Omega \rightarrow M(S)$ is measurable, $i=1,2$.

(π_1 and π_2 are the projection maps of $M(S) \times M(S)$ onto $M(S)$, $\pi_1(\mu, \nu) = \mu$, $\pi_2(\mu, \nu) = \nu$.) By hypothesis X and Y are each measurable and so, by the above discussion, (X, Y) is a jointly measurable map of Ω into $M(S) \times M(S)$, taking $\omega \rightarrow (X(\omega), Y(\omega))$.

For any two finite measures μ and ν we can uniquely write, using the usual Lebesgue decomposition theorem, $\mu = \mu_{ac}(\nu) + \mu_s(\nu)$ where $\mu_{ac}(\nu) \ll \nu$ and $\mu_s(\nu) \perp \nu$. Define the maps $P: M(S) \times M(S) \rightarrow M(S)$ and $Q: M(S) \times M(S) \rightarrow M(S)$ by $P(\mu, \nu) = \mu_{ac}(\nu)$ and $Q(\mu, \nu) = \mu_s(\nu)$. Noting that we can express $X_{ac}(Y)$ and $X_s(Y)$ as the compositions $X_{ac}(Y) = P \circ (X, Y)$ and $X_s(Y) = Q \circ (X, Y)$ we need only show that P and Q are measurable maps.

Let $N = \{(\mu_1, \mu_2, \nu) \mid \mu_1 \ll \nu, \mu_2 \perp \nu\}$. Then N is a measurable subset of $M(S) \times M(S) \times M(S)$ since we can write $N = N_1 \cap N_2$ where

$$N_1 = \{(\mu_1, \mu_2, \nu) \mid \mu_1 \ll \nu\} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{A \in \mathcal{A}} \{(\mu_1, \mu_2, \nu) \mid \mu_1(A) < 1/n\} \times M(S)$$

$$\times \{\nu \mid \nu(A) < 1/m\} \cup M(S) \times M(S) \times \{\nu \mid \nu(A) \geq 1/m\},$$

$$N_2 = \{(\mu_1, \mu_2, \nu) \mid \mu_2 \perp \nu\} = \bigcap_{n=1}^{\infty} \bigcup_{A \in \mathcal{A}} [M(S) \times \{\mu_2 \mid \mu_2(A) < 1/n\} \\ \times \{\nu \mid \nu(S \setminus A) < 1/n\}], \text{ and}$$

$\mathcal{A}$ is a countable algebra generating $\mathcal{B}(S)$. (This follows from the proof of Theorem 2.1.3.) Clearly each of N_1 and N_2 is measurable; hence $N = N_1 \cap N_2$ is measurable. Define the mapping $A: M(S) \times M(S) \times M(S) \rightarrow M(S) \times M(S)$ by $A(\mu, \gamma, \nu) = (\mu + \gamma, \nu)$. In view of Lemma 2.1.4 A must be continuous (hence measurable). Also, the uniqueness portion of the usual Lebesgue decomposition theorem for measures shows that $A|N$ (the restriction of A to N) is one-one. Noting that $M(S) \times M(S) \times M(S)$ and $M(S) \times M(S)$ are Polish spaces (from Theorem 1.10) we can apply the Kuratowski theorem to conclude that if $M \subseteq N$

is measurable then $A(M)$ is measurable. Let $B \in \mathcal{B}(S)$, $\alpha \in \mathbb{R}$, and let

$$N_{B,\alpha} = \{(\mu_1, \mu_2, \nu) \mid \mu_1 \ll \nu, \mu_1(B) < \alpha, \mu_2 \perp \nu\}. \text{ Then } N_{B,\alpha} = N$$

$\cap [\{\mu_1 \mid \mu_1(B) < \alpha\} \times M(S) \times M(S)]$ and therefore is a measurable subset of N . Therefore $A(N_{B,\alpha})$ is measurable. But also

$$\begin{aligned} A(N_{B,\alpha}) &= \{(\mu_1 + \mu_2, \nu) \mid \mu_1 \ll \nu, \mu_1(B) < \alpha, \mu_2 \perp \nu\} \\ &= \{(\mu, \nu) \mid \mu_{ac}(\nu)(B) < \alpha\} \\ &= P^{-1}(\{\mu \mid \mu(B) < \alpha\}) \\ &= (\pi_B \circ P)^{-1}((-\infty, \alpha)). \end{aligned}$$

From Corollary 1.8.1 it follows that P is measurable.

Similarly, let $M_{B,\alpha} = \{(\mu_1, \mu_2, \nu) \mid \mu_1 \ll \nu, \mu_2 \perp \nu, \mu_2(B) < \alpha\}$.

Then $M_{B,\alpha}$ is measurable, $M_{B,\alpha} \subseteq N$, and so $A(M_{B,\alpha})$ is measurable.

Since $A(M_{B,\alpha}) = \{(\mu, \nu) \mid \mu_s(\nu)(B) < \alpha\} = Q^{-1}(\{\mu \mid \mu(B) < \alpha\})$

$= (\pi_B \circ Q)^{-1}((-\infty, \alpha))$ it follows that Q is also measurable. $\square$

We would like to extend this decomposition theorem to stochastic measure processes. Recall that a measurable stochastic measure process $\{X(t)\}_{t \in T}$ (where T is an interval in $\mathbb{R}$) is a mapping $X: T \times \Omega \rightarrow M(S)$ which is measurable with respect to the product σ -algebra $\mathcal{B}(T) \times \mathcal{E}$ on $T \times \Omega$. If $\{E_t\}_{t \in T}$ is an increasing sequence of sub- σ -fields of $\mathcal{E}$ then $\{X(t)\}_{t \in T}$ is progressively measurable if $X: [0, t] \times \Omega \rightarrow M(S)$ is $\mathcal{B}([0, t]) \times E_t$ -measurable for every $t \in T$.

Theorem 2.1.6 (Lebesgue Decomposition Theorem for Random Measure

Processes): Let S be a Polish space, $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$

measurable stochastic measure processes defined on $(\Omega, \mathcal{E}, P)$. Then

$\{X_{ac}(Y)(t)\}_{t \in T}$ and $\{X_s(Y)(t)\}_{t \in T}$ are measurable stochastic processes.

If $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$ are both progressively measurable then

$\{X_{ac}(Y)(t)\}_{t \in T}$ and $\{X_s(Y)(t)\}_{t \in T}$ are progressively measurable.

proof: We will prove the case where $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$ are measurable; the progressively measurable case is analogous.

Since $M(S)$ is Polish then $B(M(S) \times M(S)) = B(M(S)) \times B(M(S))$.

Therefore $(X, Y): T \times \Omega \rightarrow M(S) \times M(S)$ is $B(T) \times E$ -measurable. Using the measurability of the maps P and Q defined in the proof of

Theorem 2.1.5 we then see that the compositions $X_{ac}(Y) = P \circ (X, Y)$ and

$X_s(Y) = Q \circ (X, Y)$ are measurable functions of $T \times \Omega$ into $M(S)$.

Therefore $\{X_{ac}(Y)(t)\}_{t \in T}$ and $\{X_s(Y)(t)\}_{t \in T}$ are measurable processes. $\square$

2. One-Dimensional Measurability Results; Decomposition Into Atomic And Diffuse Components.

Analogous results to Theorems 2.1.5 and 2.1.6 can be obtained when we consider decomposing a random measure X (or random measure process $X(t)$) with respect to a fixed σ -finite measure ν . (If $S = \mathbb{R}^N$ for some N this allows us to decompose X (or $X(t)$) with respect to n -dimensional Lebesgue measure.) This turns out to be equivalent to showing that direct sums of measurable subsets of $M(S)$ are measurable. $M(S)$ is not a vector space since signed measures are not allowed but some properties of vector spaces carry over. If M and N are subsets of $M(S)$ we can define their sum $M + N = \{\mu + \nu \mid \mu \in M, \nu \in N\}$. M and N will be called orthogonal (denoted $M \perp N$) if each element of $M + N$ has a unique representation as a sum of an element from M and an element from N . In this case we write $M + N$ as the direct sum $M \oplus N$.

We begin by proving some measurability results regarding sums

and direct sums of sets in $M(S)$. These results are applied twice in this section, first to prove the one-dimensional version of the Lebesgue decomposition theorem for random measures and then to obtain a similar decomposition theorem concerning the splitting of a random measure into its atomic and diffuse components. These techniques will be used yet a third time in the dimension decomposition theorem for random measures in Chapter 3.

Theorem 2.2.1: Let S be a separable metric space, $F, K, M, N, G_1, G_2 \subseteq M(S)$. Then we have:

- (i) If F and K are both compact then $F+K$ is compact.
- (ii) If F is closed and K compact then $F+K$ is closed.
- (iii) If M is measurable and $\nu \in M(S)$ then $M \oplus \nu$ is measurable.
- (iv) If S is compact and M, N are F_σ sets then $M+N$ is F_σ . In particular, if G_1, G_2 are open then G_1+G_2 is F_σ .
- (v) If S is Polish and M, N are measurable, $M \perp N$, then $M \oplus N$ is measurable.

proof: (i) Suppose $\mu_n = \nu_n + \gamma_n$ where $\nu_n \in F$ and $\gamma_n \in K$. Then F compact implies there exists a convergent subsequence $\nu_{n_k} \xrightarrow{w} \nu$ for some $\nu \in F$. K compact implies $\{\gamma_{n_k}\}_{k=1}^\infty$ has a convergent subsequence $\gamma_{n_{k_j}} \xrightarrow{w} \gamma$ for

some $\gamma \in K$. Therefore $\nu_{n_{k_j}} + \gamma_{n_{k_j}} \xrightarrow{w} \nu + \gamma \in F+K$, and so $F+K$ is compact.

(ii) Let $\gamma_n \in F+K$ for every n and suppose $\gamma_n \xrightarrow{w} \gamma \in M(S)$. Then $\gamma_n = \mu_n + \nu_n$ where $\mu_n \in F, \nu_n \in K$. Since K is compact there exists $\nu \in K$ and a subsequence $\{\nu_{n_k}\}$ such that $\nu_{n_k} \xrightarrow{w} \nu$. If B is a continuity set

of both γ and v then

$$\begin{aligned} v(B) &= \lim_{k \rightarrow \infty} v_{n_k}(B) \leq \lim_{k \rightarrow \infty} (\mu_{n_k}(B) + v_{n_k}(B)) = \lim_{k \rightarrow \infty} (\mu_{n_k} + v_{n_k})(B) \\ &= \gamma(B). \end{aligned}$$

Therefore $v(B) \leq \gamma(B)$ whenever B is a continuity set of both γ and v . It can be shown that the collection of such sets is an algebra which generates $\mathcal{B}(S)$ and if the relation $v(B) \leq \gamma(B)$ holds for all members of a generating algebra then in fact it holds for all Borel sets. Thus we can write $v \leq \gamma$. It then follows that the difference $\gamma - v$ is a nonnegative finite measure which is dominated by γ . Now $\mu_{n_k} + v \xrightarrow{w} \gamma$ since $\mu_{n_k}(B) = \gamma_{n_k}(B) - v_{n_k}(B) + \gamma(B) - v(B)$ whenever B is a continuity set of γ . Therefore, if C is a closed subset of S we have

$$\limsup_k \mu_{n_k}(C) + v(C) = \limsup_k (\mu_{n_k}(C) + v(C)) \leq \gamma(C)$$

$$\text{so } \limsup_k \mu_{n_k}(C) \leq \gamma(C) - v(C) = (\gamma - v)(C).$$

Noting that also $\mu_{n_k}(S) + (\gamma - v)(S)$ we conclude $\mu_{n_k} \xrightarrow{w} \gamma - v$. Since F is closed $\gamma - v \in F$. Therefore $\gamma = (\gamma - v) + v \in F + K$ and so $F + K$ is a closed subset of $M(S)$.

(iii) Since a singleton $\{v\}$ is compact, $F \oplus v$ is closed whenever F is closed (apply (i)). Now let $E = \{M \in \mathcal{B}(M(S)) \mid M \oplus v \text{ is measurable}\}$.

Then the collection of all closed sets belongs to E . If $\{M_n\}_{n=1}^{\infty}$ is a sequence from E then $(\bigcup_n M_n) \oplus v = \bigcup_n (M_n \oplus v)$ countable union of measurable sets. Therefore $\bigcup_n M_n \in E$. If $M \in E$ then $M^C \oplus v =$

$$\{\mu \mid \mu \geq v\} \setminus (M \oplus v). \text{ Since } \{\mu \mid \mu \geq v\} = \bigcap_{A \in \mathcal{A}} \{\mu \mid \mu(A) \geq v(A)\} \text{ where } \mathcal{A}$$

is a countable algebra generating $\mathcal{B}(S)$ then $\{\mu \mid \mu \geq v\}$ is measurable

and so $M^C \otimes \nu$ is measurable. Therefore $M^C \in \mathcal{E}$. This shows that $\mathcal{E}$ is a σ -algebra and since it contains the closed sets it must coincide with $\mathcal{B}(S)$.

(iv) If S is compact then the sets $\{u \mid \mu(S) \leq n\}$ are compact in $M(S)$ (see Varadhan [21], Theorem 3, pg. 27). Therefore $M(S)$ is σ -compact since we can write $M(S) = \bigcup_{n=1}^{\infty} \{u \mid \mu(S) \leq n\}$. Therefore

every closed set is a K_G set (a countable union of compact sets).

Let F and C be closed sets. Then we can write $C = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact and so $F+C = F + \bigcup_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} (F+K_n)$ which is an F_G set (using (i)). Therefore the sum of two closed sets is an F_G set. If M and N are two F_G sets then we have:

$$\begin{aligned} M + N &= \bigcup_n F_n + \bigcup_m C_m && F_n, C_m \text{ closed sets} \\ &= \bigcup_n \bigcup_m (F_n + C_m) \end{aligned}$$

and since $F_n + C_m$ is F_G it follows that $M+N$ is F_G . As every open set in a metric space is F_G it follows that the sum of two open sets is F_G .

(v) Suppose S is Polish, M and N two orthogonal measurable subsets of $M(S)$. Then $M \times N$ is a measurable "rectangle" in the product space $M(S) \times M(S)$ and the addition map $+: M(S) \times M(S) \rightarrow M(S)$ is one-one when its domain is restricted to $M \times N$. Noting that $M \otimes N = +(M \times N)$ an application of Lemma 2.1.4 and Theorem 2.1.4 shows that $M \otimes N$ is measurable. $\square$

Theorem 2.2.2 (One-Dimensional Lebesgue Decomposition Theorem for Random Measures): Let S be a Polish space, and ν any σ -finite measure defined on the Borel sets of S . If X is a random measure

mapping into $M(S)$ then $X_{ac}(v)$ and $X_S(v)$ are both random measures.

If $\{X(t)\}_{t \in T}$ is a (progressively) measurable random measure process then both $\{X_{ac}(v)(t)\}_{t \in T}$ and $\{X_S(v)(t)\}_{t \in T}$ are (progressively) measurable.

proof: Define the maps $P_v : M(S) \rightarrow M(S)$ and $Q_v : M(S) \rightarrow M(S)$ by $P_v(\mu) = \mu_{ac}(v)$, $Q_v(\mu) = \mu_S(v)$ (notation the same as in Theorem 2.1.5).

Since $X_{ac}(v) = P_v \circ X$ and $X_S(v) = Q_v \circ X$, it is sufficient to prove the maps P_v and Q_v are measurable.

The collection of sets $\{\mu | \mu(B) < \alpha\}$, $B \in \mathcal{B}(S)$, $\alpha \in \mathbb{R}$, generates the Borel sets of $M(S)$, and we have:

$$\begin{aligned} P_v^{-1}(\{\mu | \mu(B) < \alpha\}) &= \{\mu | P_v(\mu)(B) < \alpha\} = \{\mu | \mu_{ac}^v(B) < \alpha\} \\ &= A_v \oplus S_v \end{aligned}$$

$$\begin{aligned} \text{where } A_v &= \{\gamma | \gamma \ll v \text{ and } \gamma(B) < \alpha\} = \{\gamma | \gamma \ll v\} \cap \{\gamma | \gamma(B) < \alpha\} \\ &= C_v \cap \{\gamma | \gamma(B) < \alpha\} \end{aligned}$$

and $S_v = \{\gamma | \gamma \perp v\}$. From Theorems 2.1.1 and 2.1.2 it follows that A_v and S_v are measurable, and from Theorem 2.2.1(v) we can conclude $A_v \oplus S_v$ is measurable. Therefore P_v is a measurable function.

Similarly

$$\begin{aligned} Q_v^{-1}(\{\mu | \mu(B) < \alpha\}) &= \{\mu | \mu_S(v)(B) < \alpha\} \\ &= C_v \oplus D_v \end{aligned}$$

$$\begin{aligned} \text{where } C_v &= \{\gamma | \gamma \ll v\}, \quad D_v = \{\gamma | \gamma \perp v \text{ and } \gamma(B) < \alpha\} \\ &= S_v \cap \{\gamma | \gamma(B) < \alpha\}. \end{aligned}$$

Therefore Q_v is also a measurable function.

Now suppose $\{X(t)\}_{t \in T}$ is (progressively) measurable. It is immediate that the compositions $X_{ac}(v)(t) = P_v \circ X(t)$ and $X_S(v)(t) = Q_v \circ X(t)$ are (progressively) measurable. $\square$

We now wish to consider the decomposition of a random measure X into its atomic and diffuse components $X = X_a + X_d$. ($X_a(\omega)$ is the purely atomic measure whose atoms $A(\omega)$ are the atoms of $X(\omega)$ and satisfies $X_a(\omega, \{x\}) = X(\omega, \{x\})$ for each $x \in A(\omega)$.) Some work along this line is done by Kallenberg in [11], page 10 using techniques very different from those to be used here (namely Theorem 2.2.1). Our approach affords very simple, elegant proofs of the results to be presented.

We will assume that S is a locally compact separable metric space. As a consequence there exist compact sets $\{S_n\}_{n=1}^{\infty}$ such that $S = \bigcup_n S_n$. Define the set $M = \{\delta_{x,\alpha} \mid \alpha \geq 0, x \in S\}$ where

$$\delta_{x,\alpha}(B) = \begin{cases} \alpha & x \in B \\ 0 & x \notin B \end{cases} \quad \text{That is, } M \text{ consists of all point masses}$$

plus the zero measure 0 . We can write $M = \bigcup_{n=1}^{\infty} M_n$, where

$$M_n = \{\delta_{x,\alpha} \mid x \in S_n, \alpha \geq 0\}, \text{ and then write } M_n = \bigcup_{m=1}^{\infty} M_{n,m} \text{ with}$$

$$M_{n,m} = \{\delta_{x,\alpha} \mid x \in S_n, 0 \leq \alpha \leq m\}. \text{ Thus } M = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} M_{n,m}.$$

Lemma 2.2.1: For each n and m , $M_{n,m}$ is compact in $M(S)$.

proof: If $\{\delta_{x_k, \alpha_k}\}_{k=1}^{\infty}$ is a sequence in $M_{n,m}$ then $\{x_k\}_{k=1}^{\infty}$ and

$\{\alpha_k\}_{k=1}^{\infty}$ are sequences in S_n and $[0, m]$ respectively. Using the

compactness of S_n and $[0, m]$ we obtain a subsequence $\{\delta_{x_{k_j}, \alpha_{k_j}}\}_{j=1}^{\infty}$ and

points $x \in S_n, \alpha \in [0, m]$ such that $x_{k_j} \rightarrow x$ and $\alpha_{k_j} \rightarrow \alpha$. It then

follows that $\delta_{x_{k_j}, \alpha_{k_j}} \xrightarrow{w} \delta_{x, \alpha}$. Thus $M_{n,m}$ is compact. $\square$

Corollary 2.2.1.1: M is σ -compact in $M(S)$.

proof: This follows from Lemma 2.2.1 and the fact that

$$M = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} M_{n,m}.$$

Lemma 2.2.2: For every positive integer k , the sum $\sum_{i=1}^k M =$

$M + \dots + M$ (k terms) is σ -compact.

proof: We use induction on k . The case $k=1$ is true by Corollary

2.2.1.1. For $k > 1$ we have $\sum_{i=1}^k M = \left(\sum_{i=1}^{k-1} M \right) + M$

$$= \left(\bigcup_{j=1}^{\infty} K_j \right) + M \text{ where each } K_j \text{ is compact (using induction hypothesis)}$$

$$= \left(\bigcup_{j=1}^{\infty} K_j \right) + \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} M_{n,m}$$

$$= \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (K_j + M_{n,m})$$

From (i) of Theorem 2.2.1 the sum $K_j + M_{n,m}$ is compact. Hence

$\sum_{i=1}^k M$ is σ -compact. $\square$

Let P_F be the set of all purely atomic measures in $M(S)$ having only a finite number of atoms. (For convenience we include $0 \in P_F$.)

Note that $\mu \in \sum_{i=1}^k M$ iff μ is purely atomic and has no more than k

atoms. Thus $P_F = \bigcup_{k=1}^{\infty} \left(\sum_{i=1}^k M \right)$. This leads to the following result:

Theorem 2.2.3: P_F is a measurable (in fact σ -compact) subset of $M(S)$.

proof: This follows immediately from Lemma 2.2.2 and the relation

$$P_F = \bigcup_{k=1}^{\infty} \left(\sum_{i=1}^k M \right). \quad \square$$

Now let P be the set of all purely atomic measures in $M(S)$.

Lemma 2.2.3: $P = \bigcap_{n=1}^{\infty} (P_F + H_n)$ where $H_n = \{\mu \mid \mu(S) \leq 1/n\}$.

proof: Suppose $\nu \in P$. Since ν is purely atomic and $\nu(S) < \infty$ then for every n there exists a finite number of atoms (say) $x_1, \dots, x_m$ of ν such that $\nu(S \setminus \{x_1, \dots, x_m\}) \leq 1/n$. Define $\gamma = \sum_{j=1}^m \nu(\{x_j\}) \delta_{x_j} =$

$\sum_{j=1}^m \delta_{x_j, \alpha_j}$ where $\alpha_j = \nu(\{x_j\})$, and let $\mu = \nu - \gamma$. Then $\nu = \gamma + \mu$,

$\gamma \in P_F$, and $\mu(S) = \nu(S) - \gamma(S) = \nu(S) - \nu(\{x_1, \dots, x_m\}) = \nu(S \setminus \{x_1, \dots, x_m\})$

$\leq 1/n$. Thus $\mu \in H_n$, and so $\nu \in (P_F + H_n)$. Since n was arbitrary

we have $\nu \in \bigcap_{n=1}^{\infty} (P_F + H_n)$. That is, $P \subseteq \bigcap_{n=1}^{\infty} (P_F + H_n)$.

Conversely, suppose $\nu \in \bigcap_{n=1}^{\infty} (P_F + H_n)$. Then for each n we can find

$\gamma \in P_F$ and $\mu \in H_n$ such that $\nu = \gamma + \mu$. It follows that the atoms of γ must also be atoms of ν . Letting A_ν denote the atoms of ν we have $\gamma(S) = \gamma(A_\nu) \leq \nu(A_\nu)$ and since $\mu \in H_n$ then $\mu(S) \leq 1/n$. Thus $\nu(S) = \gamma(S) + \mu(S) \leq \nu(A_\nu) + 1/n$. Since n was arbitrary it follows that $\nu(S) = \nu(A_\nu)$ and hence ν is purely atomic. Therefore

$\bigcap_{n=1}^{\infty} (P_F + H_n) \subseteq P$. This completes the proof. $\square$

Theorem 2.2.4: The collection P of purely atomic measures in $M(S)$ is a measurable subset of $M(S)$.

proof: From Theorem 2.2.3 P_F is σ -compact so we can write

$P_F = \bigcup_{j=1}^{\infty} K_j$ where each K_j is compact in $M(S)$. Since

$H_n = \{\mu \mid \mu(S) \leq 1/n\}$ is closed in the weak topology the sum $K_j + H_n$ is also closed (see (ii) of Theorem 2.2.1). Therefore

$$P = \bigcap_{n=1}^{\infty} (P_F + H_n) = \bigcap_{n=1}^{\infty} \left(\bigcup_{j=1}^{\infty} K_j + H_n \right) = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} (K_j + H_n) \text{ which is an}$$

$F_{\sigma\delta}$ (hence measurable) set. $\square$

Theorem 2.2.5: The collection $\mathcal{D}$ of diffuse measures in $M(S)$ (that is, measures having no atoms) is a measurable subset of $M(S)$.

proof: We will first show that if K is a compact subset of S then

$\mathcal{D}_K = \{\mu \mid \mu \text{ is diffuse over } K\}$ is measurable.

claim: $\mathcal{D}_K = \mathcal{D}_K^*$ where $\mathcal{D}_K^* = \bigcup_{n=1}^{\infty} \bigcup_{\substack{G' \subseteq G_0 \\ G' \text{ finite} \\ \cup G' \supseteq K}} \{ \mu \mid \mu(G) \leq 1/n \}$,

G_0 is a countable basis for S and G' refers to a finite sub-collection of sets from G_0 . (The collection of all such possible G' is therefore countable and hence $\mathcal{D}_K^*$ is a measurable set.)

proof of claim: Suppose $\mu \in \mathcal{D}_K$. Then, given $n \in \mathbb{Z}^+$, for every $x \in K$ there exists a basic open set $G(x) \in G_0$ such that $x \in G(x)$ and $\mu(G(x)) \leq 1/n$. (This follows since μ is finite and no point of K is an atom of μ .) The collection $\{G(x)\}_{x \in K}$ covers K . Since K is compact there exists a finite subcover $G(x_1), \dots, G(x_m)$ so take $G' = \{G(x_1), \dots, G(x_m)\}$. This shows $\mathcal{D}_K \subseteq \mathcal{D}_K^*$. To show the opposite inclusion let $\mu \in \mathcal{D}_K^*$. Then for every $n \in \mathbb{Z}^+$ there exists a finite covering G' of K by basic open sets with $\mu(G) \leq 1/n$ for every $G \in G'$. Now $x \in K$ implies $x \in G$ for some $G \in G'$ which implies $\mu(\{x\}) \leq \mu(G) \leq 1/n$. Since this can be done for every n it follows that $\mu(\{x\}) = 0$;

that is, μ has no atoms in K . This completes the proof of the claim.

Therefore D_K is a measurable (in fact $F_{\sigma\delta}$) set. Now let

$\{K_n\}_{n=1}^{\infty}$ be a sequence of compact sets such that $\bigcup_{n=1}^{\infty} K_n = S$. Then

$$D = \bigcup_{n=1}^{\infty} D_{K_n} \text{ so } D \text{ is also an } F_{\sigma\delta} \text{ set. } \square$$

It is obvious that each $\mu \in M(S)$ has a unique representation $\mu = \mu_a + \mu_d$ where μ_a is purely atomic and μ_d is diffuse. (Equivalently we can express $M(S)$ as $M(S) = P \oplus D$.) Define the maps $A : M(S) \rightarrow M(S)$ and $D : M(S) \rightarrow M(S)$ by $A(\mu) = \mu_a$ and $D(\mu) = \mu_d$.

Theorem 2.2.6 (Decomposition of Random Measures Into Atomic And Diffuse Components): Let X be a random measure. Then the atomic and diffuse components $X_a = A \circ X$ and $X_d = D \circ X$ of X are both random measures.

proof: We need only show that A and D are measurable maps. Let $B \in \mathcal{B}(S)$ and $\alpha \in \mathbb{R}$. Then $A^{-1}(\{\mu | \mu(B) < \alpha\}) = \{\mu | \mu_a(B) < \alpha\} = (P \cap \{\nu | \nu(B) < \alpha\}) \oplus D$.

Similarly $D^{-1}(\{\mu | \mu(B) < \alpha\}) = \{\mu | \mu_d(B) < \alpha\} = P \oplus (D \cap \{\nu | \nu(B) < \alpha\})$.

Since direct sums of measurable sets are measurable it follows that A and D are measurable. $\square$

This result extends in the obvious manner when $\{X(t)\}_{t \in T}$ is a measurable (or progressively measurable) random measure process.

3. Joint Measurability Of The Radon-Nikodym Derivative

As a consequence of Theorem 2.1.5 we can obtain the measurability of the Radon-Nikodym derivative $\frac{dX_{ac}(Y)}{dY} : \Omega \times S \rightarrow \mathbb{R}$ with respect to the product σ -algebra $\mathcal{E} \times \mathcal{B}(S)$ where X and Y are two random

measures mapping the probability space $(\Omega, \mathcal{E}, P)$ into $M(S)$.

We will first need the following result:

Theorem 2.3.1: Let $(\Omega, \mathcal{E})$ be any measurable space, P a probability measure and μ any finite measure on $\mathcal{E}$, where $\mu \ll P$. Let $\{\mathcal{E}_n\}_{n=1}^{\infty}$ be an increasing sequence of sub- σ -fields of $\mathcal{E}$ such that $\bigcup_{n=1}^{\infty} \mathcal{E}_n$ generates $\mathcal{E}$.

If f_n is the Radon-Nikodym derivative of μ w.r.t P when restricted to $\mathcal{E}_n$ then $f_n \rightarrow f$ P -a.s. where $f = \frac{d\mu}{dP}$ on $\mathcal{E}$.

proof: See Theorem 35.8, pg. 419, in [3]. It is a consequence of the martingale convergence theorem. $\square$

We can now obtain the joint measurability of the Radon-Nikodym derivatives of random measures.

Theorem 2.3.2: Let X and Y be two random measures mapping $(\Omega, \mathcal{E}, P)$ into $(M(S), \mathcal{B}(M(S)))$ where S is a Polish space. Then there exists a

version of the Radon-Nikodym derivative $\frac{dX_{ac}(Y)}{dY} : \Omega \times S \rightarrow \mathbb{R}$ which is measurable with respect to the product σ -algebra $\mathcal{E} \times \mathcal{B}(S)$ on $\Omega \times S$.

proof: Since $X_{ac}(Y)$ is a random measure (by Theorem 2.1.5) it is sufficient to prove that whenever X and Y are two random measures such that $X(\omega) \ll Y(\omega), \forall \omega \in \Omega$ then there exists a jointly measurable version of the Radon-Nikodym derivative $\frac{dX}{dY}(\omega, s)$.

Since S is separable there exists a countable algebra $\mathcal{A}$ generating the Borel sets $\mathcal{B}(S)$ (see Lemma 2.1.1) and so there exists an increasing sequence of finite Borel fields $\{\mathcal{A}_n\}_{n=1}^{\infty}$ such that $\bigcup_n \mathcal{A}_n = \mathcal{A}$ generates $\mathcal{B}(S)$.

Let $\{A_{1(n)}, \dots, A_{m_n(n)}\}$ be the atoms of $\mathcal{A}_n$. For each n define

$$f_n(\omega, s) = \sum_{j=1}^{m_n} g_{j(n)}(\omega) I_{A_{j(n)}}(s)$$

$$\text{where } g_{j(n)}(\omega) = \begin{cases} \frac{X(\omega, A_{j(n)})}{Y(\omega, A_{j(n)})} & \text{if } Y(\omega, A_{j(n)}) \neq 0 \\ 0 & \text{if } Y(\omega, A_{j(n)}) = 0. \end{cases}$$

Then each product function $g_{j(n)}(\omega) I_{A_{j(n)}}(s)$ is jointly measurable with respect to $E \times A_n$, and hence the finite sum $f_n(\omega, s)$ is jointly measurable with respect to $E \times A_n$. (Note that $X(\omega, A_{j(n)}) = \pi_{A_{j(n)}} \circ X(\omega, \cdot)$)

claim: $f_n(\omega, \cdot)$ is a Radon-Nikodym derivative of $X(\omega)$ with respect to $Y(\omega)$ restricted to A_n .

proof: $f_n(\omega, \cdot) : S \rightarrow \mathbb{R}$ is clearly measurable with respect to A_n .

If $A \in A_n$, then A is a finite disjoint union of atoms of A_n .

Therefore it is sufficient to check that $\int_{A_{k(n)}} f_n(\omega, s) Y(\omega, ds)$

$= X(\omega, A_{k(n)})$ for each atom $A_{k(n)}$ of A_n .

$$\int_{A_{k(n)}} f_n(\omega, s) Y(\omega, ds) = \int_{A_{k(n)}} \sum_{j=1}^{m_n} g_{j(n)}(\omega) I_{A_{j(n)}}(s) Y(\omega, ds)$$

$$= \sum_{j=1}^{m_n} g_{j(n)}(\omega) \int_{A_{k(n)}} I_{A_{j(n)}}(s) Y(\omega, ds)$$

$$= g_{k(n)}(\omega) Y(\omega, A_{k(n)})$$

$$= \begin{cases} \frac{X(\omega, A_{k(n)})}{Y(\omega, A_{k(n)})} \cdot Y(\omega, A_{k(n)}) = X(\omega, A_{k(n)}) & \text{when } Y(\omega, A_{k(n)}) \neq 0 \\ 0 & \text{when } Y(\omega, A_{k(n)}) = 0 \end{cases}$$

since $X(\omega) < Y(\omega)$.

Therefore $\int_{A_{k(n)}} f_n(\omega, s) Y(\omega, ds) = X(\omega, A_{k(n)})$ as required,

proving the claim.

Note that when $Y(\omega, \Omega) \neq 0$ it is equivalent to say that $f_n(\omega, \cdot)$ is a Radon-Nikodym derivative of $X^*(\omega)$ w.r.t. $Y^*(\omega)$ restricted to A_n , where $X^*(\omega) = \frac{X(\omega)}{Y(\omega, \Omega)}$ and $Y^*(\omega) = \frac{Y(\omega)}{Y(\omega, \Omega)}$ normalized so that $Y^*(\omega)$ is a probability measure.

Let $f(\omega, s) = \liminf_{n \rightarrow \infty} f_n(\omega, s)$. Then $f(\cdot, \cdot)$ is measurable with respect to $E \times \mathcal{B}(S)$.

Since $Y^*(\omega)$ is a probability measure when $Y(\omega, \Omega) \neq 0$ we can apply Theorem 2.3.1 to conclude that for all such ω $f_n(\omega, \cdot) \rightarrow g_\omega(\cdot)$ $Y^*(\omega)$ -a.s. where g_ω is a Radon-Nikodym derivative of $X^*(\omega)$ with respect to $Y^*(\omega)$ on $\mathcal{B}(S)$. It follows that g_ω is a Radon-Nikodym derivative of $X(\omega)$ w.r.t. $Y(\omega)$ on $\mathcal{B}(S)$ and $f_n(\omega, \cdot) \rightarrow g_\omega(\cdot)$ $Y(\omega)$ -a.s. on S .

Since $f(\omega, \cdot) = g_\omega(\cdot)$ $Y(\omega)$ -a.s. and $f(\omega, \cdot)$ is $\mathcal{B}(S)$ -measurable it follows that $f(\omega, \cdot)$ is a Radon-Nikodym derivative of $X(\omega)$ with respect to $Y(\omega)$ on $\mathcal{B}(S)$. (Note that if $Y(\omega, \Omega) = 0$ then $X(\omega, \Omega) = 0$ and $f_n(\omega, \cdot) = f(\omega, \cdot) \equiv 0$ so this case is trivial.)

Therefore the proof is complete, taking $\frac{dX}{dY}(\omega, s) = f(\omega, s)$. $\square$

Theorem 2.3.3: Let $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$ be two measurable random measure processes mapping into $M(S)$ where S is a Polish space. Then

there exists a version of the Radon-Nikodym derivative $\frac{dX_{ac}(Y)}{dY}$:

$T \times \Omega \times S \rightarrow \mathbb{R}$ which is jointly measurable with respect to the product σ -algebra $\mathcal{B}(T) \times \mathcal{E} \times \mathcal{B}(S)$. If $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$ are progressively measurable then for each $t \in T$ we can find a version

$\frac{dX_{ac}(Y)}{dY} : [0, t] \times \Omega \times S \rightarrow \mathbb{R}$ which is jointly measurable w.r.t.

$\mathcal{B}([0, t]) \times \mathcal{E}_t \times \mathcal{B}(S)$.

proof: The proof follows that of Theorem 2.3.2 except here we define

$$g_{j(n)}(t, \omega) = \begin{cases} \frac{X(t, \omega, A_{j(n)})}{Y(t, \omega, A_{j(n)})} & \text{if } Y(t, \omega, A_{j(n)}) \neq 0 \\ 0 & \text{if } Y(t, \omega, A_{j(n)}) = 0 \end{cases}$$

Each $g_{j(n)}(t, \omega)$ is jointly measurable in t and ω because X and Y are jointly measurable in t and ω by assumption. $\square$

Remark: It should be noted that the results in this chapter have made no use of the underlying probability measure P on the space $(\Omega, \mathcal{E}, P)$.

This is easily amended by noting that if $X, Y: (\Omega, \mathcal{E}, P) \rightarrow M(S)$ are two random measures then

$[X \ll Y] = \{\omega \mid X(\omega) \ll Y(\omega)\} = (X, Y)^{-1}(\{(\mu, \nu) \mid \mu \ll \nu\})$ and hence is a measurable subset of Ω .

Similarly $[X \perp Y] = \{\omega \mid X(\omega) \perp Y(\omega)\} = (X, Y)^{-1}(\{(\mu, \nu) \mid \mu \perp \nu\})$ is a measurable subset of Ω .

Thus we can say $X \ll Y$ P-a.s. if $P([X \ll Y]) = 1$ while $X \perp Y$ P-a.s. if $P([X \perp Y]) = 1$.

If $X \ll Y$ P-a.s. then there exists a jointly measurable function $f: \Omega \times S \rightarrow \mathbb{R}$ such that $f = \frac{dX}{dY}$ P-a.s. (We can define f as in Theorem 2.3.2 and note that $f(\omega, \cdot)$ is a version of the Radon-Nikodym derivative for almost all ω .)

4. Extensions To The Vague Topology On $M_R(S)$.

The major results of the preceding three sections extend easily to $M_R(S)$.

Theorem 2.4.1: $\{(\mu, \nu) \in M_R(S) \times M_R(S) \mid \mu \ll \nu\}$ and $\{(\mu, \nu) \in M_R(S) \times M_R(S) \mid \mu \perp \nu\}$ are measurable subsets of $M_R(S) \times M_R(S)$.

proof: Since S is locally compact we can choose a sequence $\{K_j\}_{j=1}^{\infty}$ of compact subsets of S such that $K_j \subseteq K_{j+1}$ for every j and $\bigcup_{j=1}^{\infty} K_j = S$.

For each $\mu \in M(S)$ define μ_j by $\mu_j(B) = \mu(B \cap K_j)$. Hence each μ_j is a finite measure and it is easily seen, given two measures

$\mu, \nu \in M_R(S)$, that $\mu \ll \nu$ iff $\mu_j \ll \nu_j$ for every j and $\mu \perp \nu$ iff

$\mu_j \perp \nu_j$ for every j . Therefore, applying Theorem 2.1.3, we obtain

$$\{(\mu, \nu) \mid \mu \ll \nu\} = \bigcap_{j=1}^{\infty} \{(\mu, \nu) \mid \mu_j \ll \nu_j\} = \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{A \in \mathcal{A}} \{(\mu, \nu) \mid \mu(A \cap K_j) < 1/n\} \times \{\nu(A \cap K_j) \geq 1/m\} \cup M_R(S) \times \{\nu(A \cap K_j) \geq 1/m\}$$

and

$$\{(\mu, \nu) \mid \mu \perp \nu\} = \bigcap_{j=1}^{\infty} \{(\mu, \nu) \mid \mu_j \perp \nu_j\} = \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{A \in \mathcal{A}} \{(\mu, \nu) \mid \mu(A \cap K_j) < 1/n\} \times \{\nu((S \setminus A) \cap K_j) < 1/n\}$$

$\times \{\nu((S \setminus A) \cap K_j) < 1/n\}$

where $\mathcal{A}$ is a countable algebra generating $\mathcal{B}(S)$. $\square$

Theorem 2.4.2 (Lebesgue Decomposition Theorem for Random Locally

Finite Measures): Let X and Y be two measurable maps of $(\Omega, \mathcal{E}, P)$ into $(M_R(S), \mathcal{B}(M_R(S)))$ where S is a locally compact separable metric space.

Denote the Lebesgue decomposition of X with respect to Y by

$X = X_{ac}(Y) + X_s(Y)$. Then $X_{ac}(Y)$ and $X_s(Y)$ are measurable maps of $(\Omega, \mathcal{E}, P)$ into $(M_R(S), \mathcal{B}(M_R(S)))$.

proof: Since the usual Lebesgue decomposition theorem for measures applies to σ -finite (and hence locally finite) measures the proof here is completely analogous to that of Theorem 2.1.5. $\square$

Note: The extension of this result to measurable and progressively measurable stochastic random measure processes in $M_R(S)$ is immediate.

Theorem 2.4.3 (Joint Measurability of Radon-Nikodym Derivative): Let

X and Y be two random measures mapping $(\Omega, \mathcal{E}, P)$ into $(M_R(S), \mathcal{B}(M_R(S)))$ where S is a locally compact separable metric space. Then there

exists a version of the Radon-Nikodym derivative $\frac{dX_{ac}(Y)}{dY} : \Omega \times S \rightarrow \mathbb{R}$

which is measurable with respect to the product σ -algebra $\mathcal{E} \times \mathcal{B}(S)$ on $\Omega \times S$.

proof: In view of Theorem 2.4.2 it is sufficient to consider random measures X and Y such that $X(\omega) \ll Y(\omega)$ for every $\omega \in \Omega$. Let

$\{K_j\}_{j=1}^{\infty}$ be a sequence of compact sets such that $K_j \subseteq K_{j+1}$ and

$S = \bigcup_{j=1}^{\infty} K_j$. Define X_j and Y_j by $X_j(\omega, B) = X(\omega, B \cap K_j)$ and

$Y_j(\omega, B) = Y(\omega, B \cap K_j)$. Then X_j and Y_j are finite random measures

and the proof of Theorem 2.3.2 is valid, showing that a jointly

measurable version f_j of $\frac{dx_j}{dy_j}$ exists for each j . Letting $f(\omega, s) =$

$\liminf_j f_j(\omega, s)$ it follows that f is jointly measurable, $f(\omega, \cdot) =$

$\lim_{j \rightarrow \infty} f_j(\omega, \cdot) Y(\omega) - \text{a.s. for each } \omega$, and f is a version of $\frac{dx}{dy}$. $\square$

Note: It has been brought to my attention that in a recent paper by Dai Yonglong [6] (in Chinese) the joint measurability of the Radon-Nikodym derivative has been established by a different technique for random locally finite measures on a complete probability space. The Lebesgue decomposition theorem for random measures can be deduced from this.

Theorem 2.4.4 (Decomposition of Random Locally Finite Measures into Atomic and Diffuse Components): Let X be a measurable mapping of $(\Omega, \mathcal{E}, P)$ into $(M_R(S), \mathcal{B}(M_R(S)))$. Then the atomic and diffuse components X_a and X_d are measurable.

proof: Here we need to do some modifications of the proof for finite measures. We will use the same notation as in Section 2. We note that the measurability of the set $\mathcal{D}$ of diffuse measures in $M_R(S)$ can be established by a proof analogous to that of Theorem 2.2.5. The only necessary change is to take G_0 to be a countable basis of relatively compact sets. We therefore need only to establish the measurability of $\mathcal{P}$, the collection of atomic measures. It is easily seen that $M_{n,m} = \{\delta_{x,\alpha} | x \in S_n, 0 \leq \alpha \leq m\}$ is compact in the vague topology, and that the sum of two compact sets in the vague topology is also compact. Hence $M = \{\delta_{x,\alpha} | x \in S, \alpha \geq 0\}$ is σ -compact and $\mathcal{P}_F$, the set of purely atomic measures in $M_R(S)$ having only a finite number of atoms, is

also σ -compact (the proofs here are analogous to those of Lemma

2.2.1, Lemma 2.2.2, and Theorem 2.2.3). We now claim that we can

express P as $P = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (P_F + C_{n,m})$ where $C_{n,m} = \{\mu \in M_R(S) \mid \mu(S_m^O) \leq 1/n\}$

and $\{S_m\}_{m=1}^{\infty}$ is a sequence of compact subsets of S such that $S = \bigcup_{m=1}^{\infty} S_m$

and $S_m \subseteq S_{m+1}^O$ for each m . The proof of this is similar to that

of Lemma 2.2.3. Now since P_F is σ -compact and thus can be expressed

as $P_F = \bigcup_{j=1}^{\infty} K_j$, where each K_j is compact in $M_R(S)$, we can write

$P = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} (K_j + C_{n,m})$. Each $C_{n,m}$ is closed in the vague

topology and it is easily seen that the sum $K_j + C_{n,m}$ is also closed.

Hence P is measurable and we may apply the proof of Theorem 2.2.6

to conclude the measurability of x_a and x_d . $\square$

CHAPTER 3

Dimension Decomposition Of Finite Measures On $\mathbb{R}^N$ 1. Hausdorff-Besicovitch Dimension

Let $E \subseteq \mathbb{R}^N$ and $\alpha \in \mathbb{R}$. If $E = \emptyset$ define $H^\alpha(E) = 0$ and for $E \neq \emptyset$ let

$$H^\alpha(E) = \lim_{\delta \rightarrow 0^+} \inf_{\substack{\bigcup_{i=1}^{\infty} B_i \supseteq E \\ d(B_i) < \delta}} \sum_{i=1}^{\infty} (d(B_i))^\alpha$$

where B_i is an open ball, and
 $d(B_i) = \sup_{x, y \in B_i} \rho(x, y)$ is the
 diameter of B_i .

Then H^α is an outer measure on the subsets of $\mathbb{R}^N$ and is a measure when its domain is restricted to the Borel sets $\mathcal{B}(\mathbb{R}^N)$. The quantity $H^\alpha(E)$ is called the α -outer measure of E . It is easily seen that we obtain the same value $H^\alpha(E)$ when using countable coverings of closed balls rather than open balls. H^α is in fact a special case of a Hausdorff outer measure; a more general definition allows the replacement of the function t^α by any nonnegative increasing right-continuous function $h(t)$. (See Rogers [18] and Federer [9] for thorough discussions of this topic.)

It is a well-known result that if $E \neq \emptyset$ then there exists a unique value β , $0 \leq \beta \leq N$, such that $H^\alpha(E) = \infty$ for $\alpha < \beta$ and $H^\alpha(E) = 0$ for $\alpha > \beta$. This quantity β is defined to be the Hausdorff-Besicovitch dimension of E , and we will denote this as $\beta = \dim(E)$. We now list the basic properties of $\dim(E)$. (These properties will be used repeatedly without comment in the next section.)

Theorem 3.1.1: Let $E, E_1, E_2, \dots$ be subsets of $\mathbb{R}^N$. Then:

- (i) $0 \leq \dim(E) \leq N$.
- (ii) if E is countable then $\dim(E) = 0$.
- (iii) if $\lambda(E) > 0$ then $\dim(E) = N$, where λ denotes N -dimensional Lebesgue measure.
- (iv) $E_1 \subseteq E_2$ implies $\dim(E_1) \leq \dim(E_2)$.
- (v) $\dim(\bigcup_{n=1}^{\infty} E_n) = \sup_n \dim(E_n)$.

proof: See, for example, Billingsley [2], pg. 137. These are all elementary properties, following immediately from the definition of $\dim(E)$ and the fact that H^α is an outer measure for each α . $\square$

This notion of the dimension of a set enables us to distinguish (in terms of "size") between sets of Lebesgue measure zero. If we consider a measure $\mu \in M(\mathbb{R}^N)$ it may be of interest to know the proportion of mass of μ which is concentrated over sets of certain dimension. In the following section we take up the problem of decomposing μ with respect to dimension.

2. The Dimension Decomposition

Theorem 3.2.1: Let $\mu \in M(\mathbb{R}^N)$. Then, for every $0 \leq \alpha \leq N$, μ has a unique decomposition $\mu = \mu_\alpha + \mu'_\alpha$ where $\mu_\alpha, \mu'_\alpha \in M(\mathbb{R}^N)$, μ_α can be supported on a set D satisfying $\dim(D) \leq \alpha$, and μ'_α has the property that $\mu'_\alpha(B) = 0$ whenever $\dim(B) \leq \alpha$. (As a consequence we have $\mu_\alpha \perp \mu'_\alpha$.)

proof: We prove uniqueness first. Let $\alpha \in [0, N]$. Suppose $\mu = \nu + \nu' = \gamma + \gamma'$, where $\nu, \nu', \gamma, \gamma'$ have the properties stated in the theorem. Then $\dim(B) \leq \alpha$ implies $\nu'(B) = \gamma'(B) = 0$ which implies

$\nu(B) = \gamma(B) = \mu(B)$. By hypothesis there exist $D_1, D_2 \in \mathcal{B}(\mathbb{R}^N)$ such that $\dim(D_1) \leq \alpha$, $\dim(D_2) \leq \alpha$, and $\nu(\mathbb{R}^N \setminus D_1) = \gamma(\mathbb{R}^N \setminus D_2) = 0$.

Let $D = D_1 \cup D_2$. Then $\dim(D) \leq \alpha$ and D supports both ν and γ .

$$\begin{aligned} \text{Then, for every } B \in \mathcal{B}(\mathbb{R}^N), \quad \nu(B) &= \nu(B \cap D) && \text{since } D \text{ supports } \nu \\ &= \mu(B \cap D) && \text{since } \dim(B \cap D) \leq \alpha \\ &= \gamma(B \cap D) && \text{also since } \dim(B \cap D) \leq \alpha \\ &= \gamma(B) && \text{since } D \text{ supports } \gamma \end{aligned}$$

Therefore $\nu = \gamma$ so we must also have $\nu' = \gamma'$, proving uniqueness.

To show existence, define $\mu_\alpha(B) = \sup_{\dim(D) \leq \alpha} \mu(B \cap D)$.

μ_α is clearly nonnegative and easily seen to be countably subadditive, as

$$\begin{aligned} \mu_\alpha\left(\bigcup_{n=1}^{\infty} B_n\right) &= \sup_{\dim(D) \leq \alpha} \mu\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap D\right) \\ &\leq \sup_{\dim(D) \leq \alpha} \sum_n \mu(B_n \cap D) \leq \sum_n \sup_{\dim(D) \leq \alpha} \mu(B_n \cap D) = \sum_n \mu_\alpha(B_n). \end{aligned}$$

To prove μ_α is a measure we need only verify superadditivity.

Let $B_1, B_2 \in \mathcal{B}(\mathbb{R}^N)$ such that $B_1 \cap B_2 = \emptyset$. Let $\epsilon > 0$:

There exists D_1 and D_2 such that $\dim(D_1) \leq \alpha$, $\dim(D_2) \leq \alpha$, and

$$\mu(B_1 \cap D_1) > \mu_\alpha(B_1) - \epsilon/2, \quad \mu(B_2 \cap D_2) > \mu_\alpha(B_2) - \epsilon/2.$$

Let $D = D_1 \cup D_2$. Then $\dim(D) \leq \alpha$ and $\mu(B_1 \cap D) \geq \mu(B_1 \cap D_1) > \mu_\alpha(B_1)$

$$- \epsilon/2, \quad \mu(B_2 \cap D) \geq \mu(B_2 \cap D_2) > \mu_\alpha(B_2) - \epsilon/2.$$

Therefore

$$\mu_\alpha(B_1 \cup B_2) = \sup_{\dim(E) \leq \alpha} \mu((B_1 \cup B_2) \cap E) \geq \mu((B_1 \cup B_2) \cap D)$$

$$= \mu(B_1 \cap D) + \mu(B_2 \cap D) \geq \mu_\alpha(B_1) + \mu_\alpha(B_2) - \epsilon.$$

Since $\epsilon > 0$ was arbitrary we have $\mu_\alpha(B_1 \cup B_2) \geq \mu_\alpha(B_1) + \mu_\alpha(B_2)$ and countable additivity now follows.

Therefore μ_α is a measure.

Furthermore $\mu_\alpha \leq \mu$ since $\mu_\alpha(B) = \sup_{\dim(D) \leq \alpha} \mu(B \cap D) \leq \mu(B)$. (1)

However μ_α and μ agree on sets having dimension not exceeding α since $\dim(B) \leq \alpha$ implies $\mu_\alpha(B) = \sup_{\dim(D) \leq \alpha} \mu(B \cap D) = \mu(B \cap B) = \mu(B)$. (2)

Defining $\mu'_\alpha = \mu - \mu_\alpha$ then, in view of (1) and (2), μ'_α is a nonnegative measure satisfying $\mu'_\alpha(B) = 0$ whenever $\dim(B) \leq \alpha$.

To show that μ_α can be supported on a set D having $\dim(D) \leq \alpha$ we note that there exists a sequence of sets D_n , $\dim(D_n) \leq \alpha$, such that

$$\mu_\alpha(\mathbb{R}^N) = \lim_{n \rightarrow \infty} \mu(\mathbb{R}^N \cap D_n) = \lim_{n \rightarrow \infty} \mu(D_n).$$

Set $D = \bigcup_{n=1}^{\infty} D_n$. Then $\dim(D) \leq \alpha$ and so $\mu(D) = \mu(\mathbb{R}^N \cap D)$

$$\leq \mu_\alpha(\mathbb{R}^N).$$

But also $\mu(D) \geq \lim_{n \rightarrow \infty} \mu(D_n) = \mu_\alpha(\mathbb{R}^N)$ so $\mu(D) = \mu_\alpha(\mathbb{R}^N)$.

Since μ_α is finite we have $\mu_\alpha(\mathbb{R}^N \setminus D) = 0$ and therefore D supports μ_α . This completes the proof. $\square$

Remark 1: We cannot conclude that the dimension of the support of μ_α (where the support of μ_α , denoted by $\text{Supp } \mu_\alpha$, is the smallest closed set on which μ_α takes all its mass - see Chapter 4) does not exceed α . For example, if μ is a discrete measure assigning positive mass to all points with rational coordinates then $\mu = \mu_0$ since $\dim(\mathbb{Q}^N) = 0$, but $\text{Supp } \mu_0 = \text{Supp } \mu = \mathbb{R}^N$.

Remark 2: If $\mu(B) > 0$ the quantity $\frac{\mu_\alpha(B)}{\mu(B)}$ can be interpreted as the

proportion of mass on B which is supported on a set of dimension not exceeding α .

Lemma 3.2.1: For all $\mu \in M(\mathbb{R}^N)$ we have $\mu_N = \mu$ and if $\alpha_1 \leq \alpha_2$ then $\mu_{\alpha_1} \leq \mu_{\alpha_2}$. That is, μ_α is an increasing function of α on $[0, N]$.

Furthermore, if $\mu \ll \lambda$ then $\mu_\alpha = 0$ for $\alpha < N$ and if μ is discrete then $\mu = \mu_0$.

proof: Clearly, $\mu_N = \mu$ since $B \in \mathcal{B}(\mathbb{R}^N)$ implies $\dim(B) \leq N$. μ_α is increasing in α since for $\alpha_1 \leq \alpha_2$ we have

$$\mu_{\alpha_1}(B) = \sup_{\dim(D) \leq \alpha_1} \mu(B \cap D) \leq \sup_{\dim(D) \leq \alpha_2} \mu(B \cap D) = \mu_{\alpha_2}(B).$$

If $\mu \ll \lambda$ and $\dim(D) < N$ then $\mu(D) = 0$ since $\lambda(D) = 0$.

Therefore $\mu_\alpha = 0$ when $\alpha < N$. If μ is discrete then there exists a countable set A such that $\mu(\mathbb{R}^N \setminus A) = 0$. Therefore $\mu(B) = \mu(B \cap A)$, $\dim(A) = 0$, and so $\mu = \mu_0$. $\square$

Lemma 3.2.2: μ_α is a right-continuous function of α on $[0, N]$. That is, for every $B \in \mathcal{B}(\mathbb{R}^N)$ we have $\mu_\alpha(B) = \lim_{\beta \rightarrow \alpha^+} \mu_\beta(B)$.

proof: Clearly $\lim_{\beta \rightarrow \alpha^+} \mu_\beta(B)$ exists with $\mu_\alpha(B) \leq \lim_{\beta \rightarrow \alpha^+} \mu_\beta(B)$ since $\mu_\alpha(B)$ is increasing in α .

It is enough to show that if $\beta_n \rightarrow \alpha$ then $\lim_{n \rightarrow \infty} \mu_{\beta_n}(B) \leq \mu_\alpha(B)$.

For every n there exists D_n such that $\dim(D_n) \leq \beta_n$ and $\mu_{\beta_n}(B) = \mu(B \cap D_n)$.

Let $D'_n = \bigcup_{k=n}^{\infty} D_k$. Then $\dim(D'_n) = \sup_{k \geq n} \dim(D_k) \leq \sup_{k \geq n} \beta_k = \beta_n$.

$\mu(B \cap D'_n) = \mu_{\beta_n}(B)$, and $D'_n \supseteq D'_{n+1}$ for every n . Let $D = \bigcap_n D'_n$.

Then $\dim(D) \leq \dim(D_n') \leq \beta_n$ for every n implies $\dim(D) \leq \alpha$.

Also $\mu(B \cap D) = \lim_{n \rightarrow \infty} \mu(B \cap D_n') = \lim_{n \rightarrow \infty} \mu_{\beta_n}(B)$ which shows that

$\mu_\alpha(B) \geq \lim_{n \rightarrow \infty} \mu_{\beta_n}(B)$ and completes the proof. $\square$

Theorem 3.2.2: Let $\mu \in M(\mathbb{R}^N)$. Then for each $B \in \mathcal{B}(\mathbb{R}^N)$ there exists a dimension measure $\hat{\mu}_B \in M([0, N])$ such that $\hat{\mu}_B([0, \alpha]) = \mu_\alpha(B)$ for each $0 \leq \alpha \leq N$. If $\mu(B) > 0$ then the normalized version $\hat{\mu}_B = \frac{\mu_B}{\mu(B)}$ is a probability measure on $[0, N]$.

proof: Define the function $F_B: \mathbb{R} \rightarrow \mathbb{R}$ by $F_B(\alpha) = \begin{cases} 0 & \alpha < 0 \\ \mu_\alpha(B) & 0 \leq \alpha \leq N \\ \mu(B) & \alpha \geq N \end{cases}$

Since $\mu_N(B) = \mu(B)$ and μ_α is increasing on $[0, N]$ it follows that F_B is an increasing function of α , and from Lemma 3.2.2, F_B is also right-continuous in α . Therefore F_B is the distribution function of a finite measure $\hat{\mu}_B$ on $[0, N]$ with $\hat{\mu}_B([0, \alpha]) = F_B(\alpha) = \mu_\alpha(B)$ and taking total mass $\hat{\mu}_B([0, N]) = \mu(B)$. $\square$

We now list some properties of $\hat{\mu}_B$.

Theorem 3.2.3: Let $\mu, \mu_1, \mu_2, \dots$ belong to $M(\mathbb{R}^N)$. Then:

(i) for every $c \geq 0$ $(c\mu)_B = c\hat{\mu}_B$ for each $B \in \mathcal{B}(\mathbb{R}^N)$.

(ii) if $\nu = \sum_n \mu_n$ is a finite measure on $\mathbb{R}^N$ then

$$\hat{\nu}_B = \sum_n (\hat{\mu}_n)_B \text{ for each } B \in \mathcal{B}(\mathbb{R}^N).$$

(iii) if $\{B_n\}_{n=1}^\infty$ is a disjoint sequence of Borel sets then

$$\hat{\mu}_{\cup B_n} = \sum_n \hat{\mu}_{B_n}.$$

proof: (i) It is sufficient to show $(c\mu)_\alpha = c\mu_\alpha$ for each $0 \leq \alpha \leq N$.

$$\text{Clearly } (c\mu)_\alpha(B) = \sup_{\dim(D) \leq \alpha} c\mu(B \cap D) = c \sup_{\dim(D) \leq \alpha} \mu(B \cap D) = c\mu_\alpha(B)$$

and so the result follows.

(ii) We show $v_\alpha = \sum_n (\mu_n)_\alpha$ for each $0 \leq \alpha \leq N$. There exists a

Borel set D with $\dim(D) \leq \alpha$ such that D supports v_α . Therefore

$$v_\alpha(B) = v_\alpha(B \cap D) = v(B \cap D) = \sum_n \mu_n(B \cap D) \leq \sum_n (\mu_n)_\alpha(B).$$

Conversely, for each n there exists a set D_n with $\dim(D_n) \leq \alpha$ such

that D_n supports $(\mu_n)_\alpha$. Let $D = \bigcup_{n=1}^{\infty} D_n$. Then $\dim(D) \leq \alpha$ and D

supports each $(\mu_n)_\alpha$. Therefore we obtain

$$\sum_n (\mu_n)_\alpha(B) = \sum_n (\mu_n)_\alpha(B \cap D) = \sum_n \mu_n(B \cap D) = v(B \cap D) \leq v_\alpha(B).$$

Thus $v_\alpha = \sum_n (\mu_n)_\alpha$ and so it follows that $\hat{v}_B = \sum_n (\hat{\mu}_n)_B$.

(iii) It is sufficient to prove that $\hat{\mu}_{\bigcup_n B_n}([0, \alpha]) = \sum_n \hat{\mu}_{B_n}([0, \alpha])$.

for each $0 \leq \alpha \leq N$. Now $\hat{\mu}_{\bigcup_n B_n}([0, \alpha]) = \mu_\alpha(\bigcup_n B_n) = \sum_n \mu_\alpha(B_n) = \sum_n \hat{\mu}_{B_n}([0, \alpha])$

as required. $\square$

Note: For notational convenience we will denote $\hat{\mu}_{\mathbb{R}^N}$ and $\hat{\mu}_{\mathbb{R}^N}$ by $\hat{\mu}$ and $\hat{\mu}$.

Theorem 3.2.4: Let $\mu \in M(\mathbb{R}^N)$ and suppose μ has no continuous singular component. That is, given the Lebesgue decomposition $\mu = \mu^1 + \mu^2$ of μ with respect to λ then μ^1 is discrete and $\mu^2 \ll \lambda$. It follows

then that $\hat{\mu} = \mu^1(\mathbb{R}^N)\delta_0 + \mu^2(\mathbb{R}^N)\delta_N$ where δ_0 and δ_N are the Dirac measures at 0 and N respectively.

proof: Clearly μ^1 discrete implies $\hat{\mu}^1 = \mu^1(\mathbb{R}^N)\delta_0$ since $\mu^1 = \mu_0^1$.

Similarly $\mu^2 \ll \lambda$ implies $\hat{\mu}^2 = \mu^2(\mathbb{R}^N)\delta_N$ as $\mu_\alpha^2 = 0$ for $\alpha < N$.

Then from (ii) of Theorem 3.2.3 we obtain $\hat{\mu} = \hat{\mu}^1 + \hat{\mu}^2 = \mu^1(\mathbb{R}^N)\delta_0 + \mu^2(\mathbb{R}^N)\delta_N$. $\square$

Thus an absolutely continuous or discrete measure μ produces an atomic dimension measure $\hat{\mu}$. A continuous singular measure might also produce an atomic dimension measure. For example, the Cantor measure

ν (representing the uniform distribution over the Cantor set in $[0,1]$) has dimension measure $\hat{\nu} = \delta_{\frac{\log 2}{\log 3}}$, the unit mass at $\frac{\log 2}{\log 3}$

(see Section 5 of Chapter 5). A future research problem would be to characterize the types of measures which can be the dimension measure of some measure on $\mathbb{R}^N$. In particular it would be interesting to know whether or not we can find $\mu \in M(\mathbb{R}^N)$ such that $\hat{\mu}$ is a diffuse measure.

We now consider the measurability of $\hat{\mu}_B$ as a map of $M(\mathbb{R}^N)$ into $M([0,N])$.

Theorem 3.2.5: For every $0 \leq \alpha \leq N$, $D_\alpha = \{\nu \in M(\mathbb{R}^N) \mid \nu = \nu_\alpha\}$
 $= \{\nu \in M(\mathbb{R}^N) \mid \nu \text{ can be supported on a set of dimension not exceeding } \alpha\}$
 is a G_δ (hence measurable) subset of $M(\mathbb{R}^N)$.

proof: We will show that $D_\alpha = E_\alpha$, where

$$E_\alpha = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{\{B_i\}_{i=1}^{\infty}} \{v | v(\mathbb{R}^N \setminus \bigcup_{i=1}^{\infty} B_i) < 1/j\}.$$

$d(B_i) < 1/n$, B_i an open ball

$$\sum_i (d(B_i))^\alpha + 1/k < 1/n$$

E_α is a G_δ set since $\bigcup_i B_i$ open implies $\mathbb{R}^N \setminus \bigcup_i B_i$ closed which implies

$\{v | v(\mathbb{R}^N \setminus \bigcup_i B_i) < 1/j\}$ is an open set in the weak topology.

We first show that $D_\alpha \subseteq E_\alpha$. If $v \in D_\alpha$ then there exists a set D with $\dim(D) \leq \alpha$ such that D supports v . Since $\dim(D) \leq \alpha$ implies $H^{\alpha + 1/k}(D) = 0$ for every $k \in \mathbb{Z}^+$ (where $H^{\alpha + 1/k}$ is the Hausdorff $\alpha + 1/k$ -outer measure), then for every $k, n \in \mathbb{Z}^+$ there exists a $1/n$ -covering (that is, $d(B_i) < 1/n$, $\bigcup_i B_i \supseteq D$) such that $\sum_i (d(B_i))^\alpha + 1/k < 1/n$, and since $\bigcup_i B_i \supseteq D$ we have $v(\mathbb{R}^N \setminus \bigcup_i B_i) \leq v(\mathbb{R}^N \setminus D) = 0$. Therefore $v \in E_\alpha$.

To show the opposite inclusion, for each $v \in E_\alpha$ and each $k \in \mathbb{Z}^+$ we will construct a set D^k with $\dim(D^k) \leq \alpha + 1/k$ such that D^k supports v .

Let $v \in E_\alpha$.

claim: Let $k \in \mathbb{Z}^+$ be fixed. For every $\delta > 0$ there exists a δ -collection $\{A_i\}_{i=1}^{\infty}$ (that is, $d(A_i) < \delta$) of open balls such that $\sum_i (d(A_i))^\alpha + 1/k < \delta$ and $v(\mathbb{R}^N \setminus \bigcup_i A_i) = 0$.

proof of claim: Since $v \in E_\alpha$ then for every $\delta > 0$ and $j \in \mathbb{Z}^+$ there exists a δ_j -collection of open balls $\{B_{ji}\}_{i=1}^{\infty}$ such that

$$\sum_i (d(B_{ji}))^{\alpha + 1/k} < \delta/2^j \text{ and } v(\mathbb{R}^N \setminus \cup_{i,j} B_{ji}) < 1/j.$$

Then $\{B_{ji}\}_{i=1}^{\infty} \prod_{j=1}^{\infty}$ is also a δ -collection with $\sum_j \sum_i (d(B_{ji}))^{\alpha + 1/k}$

$$< \sum_j \delta/2^j = \delta \text{ and } v(\mathbb{R}^N \setminus \cup_{j,i} B_{ji}) \leq v(\mathbb{R}^N \setminus \cup_{i,j} B_{ji}) < 1/j \text{ for every}$$

$j \in \mathbb{Z}^+$ and so $v(\mathbb{R}^N \setminus \cup_{j,i} B_{ji}) = 0$, which proves the claim.

Therefore for every $k \in \mathbb{Z}^+$ there exists a $1/n$ -collection

$\{A_{ni}^k\}_{i=1}^{\infty}$ such that

$$\sum_i (d(A_{ni}^k))^{\alpha + 1/k} < 1/n \text{ and } v(\mathbb{R}^N \setminus \cup_i A_{ni}^k) = 0.$$

Let $A_n^k = \cup_i A_{ni}^k$ and let $D^k = \cap_n A_n^k$.

$$\text{Then } v(\mathbb{R}^N \setminus D^k) = v(\mathbb{R}^N \setminus \cap_n A_n^k) = v(\cup_n (\mathbb{R}^N \setminus A_n^k)).$$

$$\leq \sum_n v(\mathbb{R}^N \setminus A_n^k) = 0$$

and so D^k supports v .

Furthermore $v_{\alpha + 1/k}(D^k) = 0$ since for each $n \in \mathbb{Z}^+$ the collection $\{A_{ni}^k\}_{i=1}^{\infty}$ is a $1/n$ -covering of D^k with $\sum_i (d(A_{ni}^k))^{\alpha + 1/k} < 1/n$.

This implies $\dim(D^k) \leq \alpha + 1/k$. Therefore for each $k \in \mathbb{Z}^+$ we have a set D^k with $\dim(D^k) \leq \alpha + 1/k$ such that D^k supports v . This implies that v can be supported on a set D with $\dim(D) \leq \alpha$. One method is to note that $v_{\alpha + 1/k} = v$ for every $k \in \mathbb{Z}^+$ and so from

right-continuity (see Lemma 3.2.2) we have $v_\alpha = \lim_{k \rightarrow \infty} v_{\alpha + 1/k} = v$.

A more direct method is to note that the set $D = \limsup_k D^k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} D^k$

is a support of v satisfying $\dim(D) \leq \alpha$ (this is in fact the proof of Lemma 3.2.2). $\square$

Theorem 3.2.6: For every $0 \leq \alpha \leq N$, $D'_\alpha = \{v \in M(\mathbb{R}^N) \mid v_\alpha = 0\}$

$= \{v \in M(\mathbb{R}^N) \mid v \text{ has no mass on any set of dimension not exceeding } \alpha\}$ is an $F_{\sigma\delta}$ (hence measurable) subset of $M(\mathbb{R}^N)$.

proof: We will actually show that the complement $(D'_\alpha)^c = \{v \mid v_\alpha \neq 0\}$ is a $G_{\delta\sigma}$ set by proving $(D'_\alpha)^c = E'_\alpha$ where

$$E'_\alpha = \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{\substack{\{B_i\}_{i=1}^{\infty} \\ d(B_i) < 1/n, B_i \text{ an open ball} \\ \sum_i (d(B_i))^\alpha + 1/k < 1/2^n}} \{v \mid v(\bigcup_i B_i) > 1/j\}.$$

E'_α is a $G_{\delta\sigma}$ set since $\bigcup_i B_i$ open implies $\{v \mid v(\bigcup_i B_i) > 1/j\}$ is an open set in the weak topology.

We first show $(D'_\alpha)^c \subseteq E'_\alpha$. $v \in (D'_\alpha)^c$ implies v has some mass on a set of dimension not exceeding α which implies there exists $j \in \mathbb{Z}^+$ and a set D with $\dim(D) \leq \alpha$ such that $v(D) > 1/j$. Now $\dim(D) \leq \alpha$ implies $H^{\alpha + 1/k}(D) = 0$ for each $k \in \mathbb{Z}^+$, so for every $k, n \in \mathbb{Z}^+$ there exists a $1/n$ -covering $\{B_i\}_{i=1}^{\infty}$ of D such that $\sum_i (d(B_i))^\alpha + 1/k < 1/2^n$. Since $\bigcup_i B_i \supseteq D$ we have $v(\bigcup_i B_i) \geq v(D) > 1/j$.

This shows $v \in E'_\alpha$.

Conversely, let $v \in E'_\alpha$. We will show that there exists $j \in \mathbb{Z}^+$ such that for every $k \in \mathbb{Z}^+$ there exists a set D^k with $\dim(D^k) \leq \alpha + 1/k$ and $v(D^k) \geq 1/j$.

Since $v \in E'_\alpha$ there exists $j \in \mathbb{Z}^+$ such that for every $k, n \in \mathbb{Z}^+$ there exists a $1/n$ -collection $\{B_{ni}^k\}_{i=1}^\infty$ satisfying $\sum_1 (d(B_{ni}^k))^\alpha + 1/k < 1/2^n$

and $v(\cup_i B_{ni}^k) > 1/j$.

Let $B_n^k = \cup_i B_{ni}^k$ and define $D^k = \limsup_n B_n^k = \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty B_m^k$.

Since $v(\bigcup_{m=n}^\infty B_m^k) \geq v(B_n^k) > 1/j$ for each n it follows that $v(D^k) \geq 1/j$.

Also, for each n , $\{B_{mi}^k\}_{i=1}^\infty_{m=n}^\infty$ is a covering of D^k satisfying

$d(B_{mi}^k) < 1/n$ for each $m \geq n$ and all i , and

$$\sum_{m=n}^\infty \sum_i (d(B_{mi}^k))^\alpha + 1/k < \sum_{m=n}^\infty 1/2^m = 2^{-(n-1)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This shows $H^{\alpha + 1/k}(D^k) = 0$ so $\dim(D^k) \leq \alpha + 1/k$.

Therefore we can conclude that the set $D = \limsup_k D^k$ has $\dim(D) \leq \alpha$ and $v(D) \geq 1/j$. This shows $v \in (D'_\alpha)^c$. $\square$

Theorem 3.2.7: For each $0 \leq \alpha \leq N$ define the maps $P_\alpha : M(\mathbb{R}^N) \rightarrow \mathcal{D}_\alpha$ and $P'_\alpha : M(\mathbb{R}^N) \rightarrow \mathcal{D}'_\alpha$ by $P_\alpha(\mu) = \mu_\alpha$ and $P'_\alpha(\mu) = \mu'_\alpha$. Then P_α and P'_α are both measurable functions.

proof: Here we employ the techniques of Section 2 in Chapter 2.

To show the measurability of P we need to show that for every

$B \in \mathcal{B}(\mathbb{R}^N)$ and $r \in \mathbb{R}$ the set $P_\alpha^{-1}(\{\mu | \mu(B) < r\})$ is measurable in $M(\mathbb{R}^N)$.

$$\begin{aligned} \text{Now } P_\alpha^{-1}(\{\mu | \mu(B) < r\}) &= \{\mu | \mu_\alpha(B) < r\} \\ &= (D_\alpha \cap \{\mu | \mu(B) < r\}) \oplus D'_\alpha. \end{aligned}$$

That this last quantity is a direct sum follows from the uniqueness portion of Theorem 3.1.1. Since direct sums of measurable sets are measurable (see Theorem 2.2.1(v)) this shows that P_α is measurable.

Similarly $P'_\alpha^{-1}(\{\mu | \mu(B) < r\}) = D'_\alpha \oplus (D_\alpha \cap \{\mu | \mu(B) < r\})$ and so P'_α is measurable. $\square$

Theorem 3.2.8: For each $B \in \mathcal{B}(\mathbb{R}^N)$ the map $D_B: M(\mathbb{R}^N) \rightarrow M([0, N])$ defined by $D_B(\mu) = \hat{\mu}_B$ is measurable.

proof: We need only show that for each Borel set $A \subseteq [0, N]$ the map $\pi_A \circ D_B: M(\mathbb{R}^N) \rightarrow \mathbb{R}$ is measurable. Let $A = \{[0, \alpha] | 0 \leq \alpha \leq N\}$. Note that A is a π -system which generates $\mathcal{B}([0, N])$. We will show that $\pi_A \circ D_B$ is measurable for every $A \in \mathcal{A}$.

$$\begin{aligned} \text{Let } A = [0, \alpha]. \text{ Then } (\pi_{[0, \alpha]} \circ D_B)(\mu) &= \pi_{[0, \alpha]}(\hat{\mu}_B) \\ &= \hat{\mu}_B([0, \alpha]) \\ &= \mu_\alpha(B) = P_\alpha(\mu)(B) = \pi_B \circ P_\alpha(\mu). \end{aligned}$$

Therefore we have shown $\pi_{[0, \alpha]} \circ D_B = \pi_B \circ P_\alpha$. By Theorem 3.2.7 it

then follows that $\pi_{[0, \alpha]} \circ D_B$ is measurable. It is then easily verified

that $A^* = \{A \in \mathcal{B}([0, N]) | \pi_A \circ D_B \text{ is measurable}\}$ is a d -system and

since $A \subseteq A^*$ it follows from Dynkin's lemma that $A^* = \mathcal{B}([0, N])$. $\square$

Corollary 3.2.8.1: Let $X(t): (\Omega, \mathcal{E}, P) \rightarrow M(\mathbb{R}^N)$ be a measure-valued stochastic process in $M(\mathbb{R}^N)$, $t \in T$ where T is an interval in $\mathbb{R}$.

Let $X_{ac}(t)$ and $X_s(t)$ denote the absolutely continuous and singular portions (with respect to Lebesgue measure) of $X(t)$ respectively.

Then $\hat{X}(t)$, $\hat{X}_{ac}(t)$, and $\hat{X}_s(t)$ are random measure processes in $M([0, N])$.

proof: This follows from composition of measurable maps. For

example, $\hat{X}_{ac}(t) = D_{\mathbb{R}^N} \circ X_{ac}(t)$. $\square$

Remark 3: The above corollary also holds for the quantities $\hat{X}(t)_B$, $\hat{X}_{ac}(t)_B$ and $\hat{X}_s(t)_B$ for each $B \in \mathcal{B}(\mathbb{R}^N)$. Furthermore, if $X(t)$ is measurable (or progressively measurable) then so are $\hat{X}(t)_B$, $\hat{X}_{ac}(t)_B$, and $\hat{X}_s(t)_B$.

We can now obtain the following representation theorem for a measure $\mu \in M(\mathbb{R}^N)$.

Theorem 3.2.9: Let $\mu \in M(\mathbb{R}^N)$. Then there exists a mapping

$\Phi: \mathcal{B}(\mathbb{R}^N) \times [0, N] \rightarrow \mathbb{R}$ such that:

- (i) for every $0 \leq \alpha \leq N$, $\Phi(\cdot, \alpha)$ is a probability measure on $\mathcal{B}(\mathbb{R}^N)$.
- (ii) for every $B \in \mathcal{B}(\mathbb{R}^N)$, $\Phi(B, \cdot)$ is a measurable function of α
- (iii) for every $B \in \mathcal{B}(\mathbb{R}^N)$ and $E \in \mathcal{B}([0, N])$ we have

$$\hat{\mu}_B(E) = \int_E \Phi(B, \alpha) \hat{\mu}(d\alpha).$$

In particular $\mu(B) = \int_{[0, N]} \Phi(B, \alpha) \hat{\mu}(d\alpha)$ so we can write

$$\mu(\cdot) = \int_{[0, N]} \Phi(\cdot, \alpha) \hat{\mu}(d\alpha).$$

- (iv) for every $B \in \mathcal{B}(\mathbb{R}^N)$, $\Phi(B, \cdot)|_{(\dim(B), N]} = 0$ $\hat{\mu}$ -a.s., where

$\phi(B, \cdot)|_{(\dim(B), N]}$ is the restriction of $\phi(B, \cdot)$ to $\alpha \in (\dim(B), N]$.

proof: Note that from (iii) of Theorem 3.2.3 we have

$$\hat{\mu} = \hat{\mu}_B + \hat{\mu}|_{\mathbb{R}^N \setminus B} \quad \text{which gives } \hat{\mu}_B \leq \hat{\mu} \text{ and so } \hat{\mu}_B \ll \hat{\mu} \text{ for each } B \in \mathcal{B}(\mathbb{R}^N).$$

Now $\hat{\mu}_B \ll \hat{\mu}$ implies $\hat{\mu}_B(E) = \int_E \phi_B(\alpha) \hat{\mu}(d\alpha)$ where $\phi_B(\cdot)$ is the Radon-

Nikodym derivative $\frac{d\hat{\mu}_B}{d\hat{\mu}}$. ϕ_B is nonnegative, measurable, and unique

up to Borel sets of $\hat{\mu}$ -measure zero. Note we have the following:

$$(a) \quad \phi|_{\mathbb{R}^N} = 1 \quad \hat{\mu}\text{-a.s. and } \phi_\emptyset = 0 \quad \hat{\mu}\text{-a.s.}$$

This follows since $\hat{\mu}(E) = \int_E \hat{\mu}(d\alpha)$ and $\hat{\mu}_\emptyset = \hat{\mu}$.

(b) If $\{B_n\}_{n=1}^\infty$ is a sequence of disjoint sets from $\mathcal{B}(\mathbb{R}^N)$ then

$$\phi_{\cup B_n} = \sum_n \phi_{B_n} \quad \hat{\mu}\text{-a.s.} \quad \text{We prove this as follows.}$$

$$\hat{\mu}_{\cup B_n}(E) = \int_E \phi_{\cup B_n}(\alpha) \hat{\mu}(d\alpha) \text{ but also}$$

$$\hat{\mu}_{\cup B_n}(E) = \sum_n \hat{\mu}_{B_n}(E) = \sum_n \int_E \phi_{B_n}(\alpha) \hat{\mu}(d\alpha) \quad \text{from Theorem 3.2.3(iii)}$$

$$= \int_E \sum_n \phi_{B_n}(\alpha) \hat{\mu}(d\alpha) \quad \text{from monotone convergence.}$$

Therefore, uniqueness of the Radon-Nikodym derivative implies

$$\phi_{\cup B_n} = \sum_n \phi_{B_n} \quad \hat{\mu}\text{-a.s.}$$

(c) For each $B \in \mathcal{B}(\mathbb{R}^N)$, $\phi_B(\cdot)|_{(\dim(B), N]} = 0 \quad \hat{\mu}\text{-a.s.}$

We obtain this by noting that $\hat{\mu}_B([0, N]) = \int_{[0, N]} \phi_B(\alpha) \hat{\mu}(d\alpha)$ but also

$$\hat{\mu}_B([0, N]) = \mu(B) = \mu_{\dim(B)}(B) = \hat{\mu}_B([0, \dim(B)]) = \int_{[0, \dim(B)]} \phi_B(\alpha) \hat{\mu}(d\alpha).$$

Therefore $\int_{(\dim(B), N]} \phi_B(\alpha) \hat{\mu}(d\alpha) = 0$ and the nonnegativity of $\phi_B(\cdot)$

then implies $\phi_B(\cdot) = 0$ μ -a.s.

when restricted to $(\dim(B), N]$.

We would like to choose a version of $\phi_B(\alpha)$ which is a probability measure on $B(\mathbb{R}^N)$ for each $\alpha \in [0, N]$. The remainder of the proof is devoted to showing that this can be done.

(1) The real-valued case ($\mathbb{R}^N = \mathbb{R}$)

Let $\mathbb{Q}$ denote the rational numbers. For each $q \in \mathbb{Q}$, let

$A_q = (-\infty, q]$ and choose a version of ϕ_{A_q} .

Let $q_n \uparrow -\infty$ be some sequence in $\mathbb{Q}$ and let $E^{(1)} = \{\alpha \in [0, 1] \mid \lim_{n \rightarrow \infty} \phi_{A_{q_n}}(\alpha) = 0\}$.

Since $A_{q_n} \uparrow \emptyset$ we have $\phi_{A_{q_n}} \rightarrow \phi_{\emptyset} = 0$ μ -a.s., so

$$\mu([0, 1] \setminus E^{(1)}) = 0.$$

Similarly let $q'_n \uparrow \infty$ be some sequence in $\mathbb{Q}$ and let

$$E^{(2)} = \{\alpha \in [0, 1] \mid \lim_{n \rightarrow \infty} \phi_{A_{q'_n}}(\alpha) = 1\}. \text{ Since } \phi_{A_{q'_n}} \rightarrow \phi_{\mathbb{R}} = 1 \text{ } \mu\text{-a.s.}$$

it follows that $\mu([0, 1] \setminus E^{(2)}) = 0$.

Now, for each $q \in \mathbb{Q}$, let $E_q = \bigcap_{q' < q} \{\alpha \in [0, 1] \mid \phi_{A_{q'}}(\alpha) \leq \phi_{A_q}(\alpha)\}$.

Since $q' < q$ implies $A_{q'} \subset A_q$ which implies $\phi_{A_{q'}} \leq \phi_{A_q}$ μ -a.s. we

must have $\mu(\{0,1\} \setminus E_q) = 0$. Therefore let

$E = E^{(1)} \cap E^{(2)} \cap \dots \cap E^{(q)} \cap \dots$. Then $\mu(\{0,1\} \setminus E) = 0$ and for each

$\alpha \in E$, $\phi_{A_q}(\alpha)$ is an increasing function of q with $\lim_{q \rightarrow -\infty} \phi_{A_q}(\alpha) = 0$,

$\lim_{q \rightarrow \infty} \phi_{A_q}(\alpha) = 1$.

For each $\alpha \in E$ and $x \in \mathbb{R}$ define $F(x, \alpha) = \inf_{\substack{q > x \\ q \in \mathbb{Q}}} \phi_{A_q}(\alpha)$.

It is easily verified that for each $\alpha \in E$ $F(\cdot, \alpha)$ is an increasing right-continuous function with $\lim_{x \rightarrow -\infty} F(x, \alpha) = 0$ and $\lim_{x \rightarrow \infty} F(x, \alpha) = 1$

so $F(\cdot, \alpha)$ is a distribution function (d.f.) on $\mathbb{R}$.

Let $G(x)$ be any d.f. on $\mathbb{R}$ and for $\alpha \in ([0,1] \setminus E)$ set $F(x, \alpha) = G(x)$ for every $x \in \mathbb{R}$. Therefore, for every $\alpha \in [0,1]$, $F(\cdot, \alpha)$ is a d.f. and uniquely defines a probability measure $\phi(\cdot, \alpha)$ on $\mathcal{B}(\mathbb{R})$.

Furthermore note that we can write

$$F(x, \alpha) = \begin{cases} \inf_{\substack{q > x \\ q \in \mathbb{Q}}} \phi_{A_q}(\alpha) & \alpha \in E \\ G(x) & \alpha \in [0,1] \setminus E \end{cases}$$

which shows that $F(x, \cdot)$ is a measurable function of α for each $x \in \mathbb{R}$.

It is then easily shown that $\phi(\cdot, \alpha)$ is a measurable function of α by using monotone class arguments.

Therefore the function $\phi: \mathcal{B}(\mathbb{R}) \times [0,1] \rightarrow \mathbb{R}$ satisfies properties (i) and (ii) of the theorem. Properties (iii) and (iv) will be verified if we show that $\phi(B, \cdot) = \phi_B(\cdot)$ μ -a.s. for each $B \in \mathcal{B}(\mathbb{R})$.

Note that $\alpha \in E$ implies $\phi_{A_q}(\alpha) \leq F(q, \alpha)$ but by choosing a sequence

$q_n \uparrow q$ we obtain $\phi_{A_{q_n}} \uparrow \phi_{A_q}$ μ -a.s. so $\phi_{A_q}(\cdot) \leq F(q, \cdot)$ μ -a.s.

Then $\phi((q_1, q_2], \cdot) = F(q_2, \cdot) - F(q_1, \cdot) = \phi_{A_{q_2}}(\cdot) - \phi_{A_{q_1}}(\cdot)$ μ -a.s.

$$= \phi_{(q_1, q_2]}(\cdot) \quad \mu\text{-a.s.}$$

It then follows that if A belongs to the algebra $\mathcal{A}$ of finite unions of right-closed intervals with rational endpoints then

$\phi(A, \cdot) = \phi_A(\cdot)$ μ -a.s., and since $\mathcal{A}$ generates the Borel sets and it is easily verified that the collection

$E = \{B \in \mathcal{B}(\mathbb{R}) \mid \phi(B, \cdot) = \phi_B(\cdot) \text{ } \mu\text{-a.s.}\}$ is a monotone class

we have

$$\phi(B, \cdot) = \phi_B(\cdot) \quad \mu\text{-a.s. for each } B \in \mathcal{B}(\mathbb{R}).$$

(2) The N-dimensional case ($N \geq 2$)

The method here is similar to the real-valued case.

Let $\bar{x} = (x_1, \dots, x_N)$ denote an arbitrary point in $\mathbb{R}^N$. We will say that $\bar{x} \geq \bar{y}$ if $x_i \geq y_i$ for each $i=1, \dots, N$, and say $\bar{x} > \bar{y}$ if $x_i > y_i$ for each $i=1, \dots, N$.

Note that the collection E of right-closed rectangles $R = \prod_{i=1}^N (q_i', q_i]$,

$q_i, q_i' \in \mathbb{Q}$, is countable. In the special case that

$R = \prod_{i=1}^N (-\infty, q_i]$; $q_i \in \mathbb{Q}$ for every i , we will denote R by $A_{\bar{q}}$ where

$$\bar{q} = (q_1, \dots, q_N).$$

For each $R \in E$ choose a version of ϕ_R . Let $\mathbb{Q}^N$ denote the points of $\mathbb{R}^N$ having rational coordinates.

Similar to the real-valued case we can find a set $E_1 \subseteq [0, N]$ having $\mu([0, N] \setminus E_1) = 0$ and possessing the following properties for each $\alpha \in E_1$:

$$(a') \quad \bar{q} \leq \bar{q}' \text{ implies } \phi_{A_{\bar{q}}}(\alpha) \leq \phi_{A_{\bar{q}'}}(\alpha) \text{ where } \bar{q}, \bar{q}' \in \mathbb{Q}^N$$

$$(b') \quad \lim_{\bar{q} \rightarrow \infty} \phi_{A_{\bar{q}}}(\alpha) = 1 \quad \text{where } \bar{q} \rightarrow \infty \text{ means } q_i \rightarrow \infty \text{ for each } i=1, \dots, N$$

$$\lim_{q_i \rightarrow -\infty} \phi_{A_{\bar{q}}}(\alpha) = 0 \quad \text{for each } i=1, \dots, N, \quad \bar{q} \in \mathbb{Q}^N$$

$$(c') \quad \phi_R(\alpha) \geq 0 \quad \forall R \in E \quad (\text{note each } \phi_B \text{ is nonnegative so (c') is automatically satisfied.})$$

Define, for each $\alpha \in E_1$ and $\bar{x} \in \mathbb{R}^N$, $F(\bar{x}, \alpha) = \inf_{\substack{\bar{q} > \bar{x} \\ \bar{q} \in \mathbb{Q}^N}} \phi_{A_{\bar{q}}}(\alpha)$.

It follows that F is "increasing" in the sense that $\bar{x}_1 \leq \bar{x}_2$ implies $F(\bar{x}_1, \alpha) \leq F(\bar{x}_2, \alpha)$ and $\lim_{\bar{x} \rightarrow \infty} F(\bar{x}, \alpha) = 1$, $\lim_{x_i \rightarrow -\infty} F(\bar{x}, \alpha) = 0$ for

each $i=1, \dots, N$.

It can be shown that the right-hand limit $F(\bar{x}^+, \alpha)$ exists and it is immediate that $F(\bar{x}, \alpha) \leq F(\bar{x}^+, \alpha)$ from the increasing property of $F(\cdot, \alpha)$. However, for every $\bar{q} > \bar{x}$ we also have $F(\bar{x}^+, \alpha) \leq \phi_{A_{\bar{q}}}(\alpha)$

which gives $F(\bar{x}^+, \alpha) \leq F(\bar{x}, \alpha)$ and so $F(\bar{x}, \alpha) = F(\bar{x}^+, \alpha)$. Therefore

$F(\cdot, \alpha)$ is continuous from above.

Now for $\alpha \in E$, we clearly have $\phi_{A_{\bar{q}}}(\alpha) \leq F(\bar{q}, \alpha)$ for each $\bar{q} \in \mathbb{Q}^N$.

Let $\bar{q}_n \rightarrow \bar{q}$. Then $A_{\bar{q}_n} \rightarrow A_{\bar{q}}$ and so $\phi_{A_{\bar{q}_n}}(\cdot) \rightarrow \phi_{A_{\bar{q}}}(\cdot)$ μ -a.s.

which gives $F(\bar{q}, \cdot) \leq \lim_{n \rightarrow \infty} \phi_{A_{\bar{q}_n}}(\cdot) = \phi_{A_{\bar{q}}}(\cdot)$ μ -a.s.

Therefore $F(\bar{q}, \cdot) = \phi_{A_{\bar{q}}}(\cdot)$ μ -a.s.

Therefore we can find a set $E_2 \subseteq E_1$ with $\mu([0, N] \setminus E_2) = 0$ such that for all $\alpha \in E_2$ and all $\bar{q} \in \mathbb{Q}^N$ we have $F(\bar{q}, \alpha) = \phi_{A_{\bar{q}}}(\alpha)$. (1)

Now each rectangle $R = \prod_{i=1}^N (q'_i, q_i]$ has 2^N vertices $\bar{v} = (v_1, \dots, v_N)$

where for each i $v_i = q'_i$ or $v_i = q_i$. Let $\Delta_k(R)$ = the collection of vertices of R where exactly k of the components $v_i = q'_i$.

From the a.s. countable additivity of the ϕ_B and the usual inclusion-exclusion principle we obtain

$$\phi_R(\cdot) = \sum_{k=0}^N (-1)^k \sum_{\bar{v} \in \Delta_k(R)} \phi_{A_{\bar{v}}}(\cdot) \quad \mu\text{-a.s.}, \quad \text{and in view of (1)}$$

$$\text{we have } \phi_R(\cdot) = \sum_{k=0}^N (-1)^k \sum_{\bar{v} \in \Delta_k(R)} F(\bar{v}, \alpha) \quad \mu\text{-a.s.}$$

Since E is countable there exists $E_3 \subseteq E_2$ with $\mu([0, N] \setminus E_3) = 0$

such that for every $\alpha \in E_3$ and $R \in E$ we have

$$\phi_R(\alpha) = \sum_{k=0}^N (-1)^k \sum_{\bar{v} \in \Delta_k(R)} F(\bar{v}, \alpha). \quad (2)$$

Therefore, for every $\alpha \in E_3$, $F(x, \alpha)$ has the following properties:

(a") $F(\cdot, \alpha)$ is continuous from above

(b") $\lim_{x \rightarrow \infty} F(\bar{x}, \alpha) = 1$, $\lim_{x_i \rightarrow -\infty} F(\bar{x}, \alpha) = 0$ for each $i=1, \dots, N$

(c") For every $R \in E$, $\sum_{k=0}^N (-1)^k \sum_{\bar{v} \in \Delta_k(R)} F(\bar{v}, \alpha) \geq 0$ (this follows since

the expression on

the left is

equivalent to

$$\phi_R(\alpha) \geq 0$$

We would like to extend (c") to (c'''): For every rectangle

$R = \prod_{i=1}^N (a'_i, a_i]$, $a'_i, a_i \in \mathbb{R}$, we have $\sum_{k=0}^N (-1)^k \sum_{\bar{v} \in \Delta_k(R)} F(\bar{v}, \alpha) \geq 0$.

We do this by using the right-continuity of F and approximating by

rectangles from E . Given $\epsilon > 0$ there exists $R_\epsilon = \prod_{i=1}^N (q'_i, q_i]$, $q_i \in \mathbb{Q}$,

such that $a'_i \leq q'_i$, $a_i \leq q_i$, and

$$\sum_{k=0}^N (-1)^k \sum_{\bar{v} \in \Delta_k(R)} F(\bar{v}, \alpha) \geq \sum_{k=0}^N (-1)^k \sum_{\bar{v} \in \Delta_k(R_\epsilon)} F(\bar{v}, \alpha) - \epsilon \geq -\epsilon \text{ from (c").}$$

Since $\epsilon > 0$ was arbitrary it follows that $\sum_{k=0}^N (-1)^k \sum_{\bar{v} \in \Delta_k(R)} F(\bar{v}, \alpha) \geq 0$

and so (c''') holds for all $\alpha \in E_3$.

Therefore $F(\bar{x}, \alpha)$ satisfies the requirements of a multivariate d.f.

for each $\alpha \in E_3$ and defines a unique probability measure $\phi(\cdot, \alpha)$ on $B(\mathbb{R}^N)$.

For $\alpha \notin E_3$ we can take $F(\bar{x}, \alpha) = G(\bar{x})$ where $G(\bar{x})$ is some d.f.

As in the real-valued case $F(\bar{x}, \cdot)$ is easily seen to be measurable and

so $\phi(B, \cdot)$ is also measurable.

From (2) we obtain $\phi_R(\alpha) = \sum_{k=0}^N (-1)^k \sum_{\vec{v} \in \Delta_k(R)} F(\vec{v}, \alpha) = \phi(R, \alpha)$ for every

$\alpha \in E_3$, $R \in E$, and so by the same argument as in the real-valued case we can conclude $\phi_B(\cdot) = \phi(B, \cdot)$ μ -a.s. for every $B \in \mathcal{B}(\mathbb{R}^N)$. This completes the proof. $\square$

Corollary 3.2.9.1: If $\mu \in M(\mathbb{R}^N)$ then we have the representations

$$\mu_\alpha(\cdot) = \int_{[0, \alpha]} \phi(\cdot, \beta) \hat{\mu}(d\beta) \text{ and } \mu'_\alpha(\cdot) = \int_{(\alpha, N]} \phi(\cdot, \beta) \hat{\mu}(d\beta)$$

for each $\alpha \in [0, N]$.

proof: This follows immediately from (iii) of Theorem 3.2.9 by

noting $\mu'_\alpha(B) = \hat{\mu}_B([0, \alpha]) = \int_{[0, \alpha]} \phi(B, \beta) \hat{\mu}(d\beta)$. $\square$

In this chapter we have seen that we can study the change in dimension of a random measure process $X(t)$ by observing the quantities $\hat{X}(t)$, $\hat{X}_{ac}(t)$, and $\hat{X}_s(t)$. Another possibility is to study the behavior of $\dim(\text{Supp } \hat{X}_s(t))$. In the next chapter we develop the topological and measure-theoretic background involved in defining this quantity.

CHAPTER 4

Topologies On The Space Of Closed Subsets F

In this chapter we wish to make precise the notion of the support process $\text{Supp } X(t)$ of a random measure process in $M(S)$. This necessitates the defining of an appropriate topology on the space of closed subsets of S (often referred to as the hyperspace of S) which we will denote by F . As noted in the introduction some theory of random sets has been developed by Choquet [4], Kendall [12], and Matheron [14]. As the topological background is not widely known the first two sections of this chapter are devoted to summarizing the relevant facts about the three topologies we will consider: the finite topology, the compact topology, and the perhaps better known Hausdorff metric topology (this last actually being a collection of topologies for various choices of metric ρ on S). The main sources of information in these two sections are Christenson and Voxman [5], Matheron [14], and Michael [15]. It should be noted that the Hausdorff metric topology is usually defined on F^- , the space of nonempty closed subsets of S , but we can easily modify this to include $\emptyset$ (see the end of Section 1). Since a common technique in studying a stochastic process is to compactify the underlying space S we derive convergence results for the one-point compactification S^* when S is a locally compact metric space (this is Section 3). In Section 4 we discuss the measurability and general behavior of the support map while the last section deals with the Hausdorff dimension of the support map in Euclidean space.

1. Hausdorff Metric Topologies On F

Let (S, ρ) be a metric space and assume ρ is a bounded metric. This is not a restriction since if ρ is not bounded we can always consider the corresponding standard bounded metric $\bar{\rho}$, where $\bar{\rho}(x, y) = \min(\rho(x, y), 1)$. ρ and $\bar{\rho}$ are uniformly equivalent on S ; in particular (S, ρ) is complete iff $(S, \bar{\rho})$ is complete.

The induced Hausdorff metric d_ρ on the hyperspace F^- is defined by:

$$d_\rho(A, B) = \max \left(\sup_{x \in A} \rho(x, B), \sup_{x \in B} \rho(x, A) \right)$$

where $\rho(x, B) = \inf_{z \in B} \rho(x, z)$ is the usual minimum distance between a point x and a closed set B .

Theorem 4.1.1: (S, ρ) is isometrically embedded in (F^-, d_ρ) .

proof: Consider the mapping $x \rightarrow \{x\}$. It is immediate that

$$d_\rho(\{x\}, \{y\}) = \rho(x, y). \quad \square$$

Notation: Let A be a closed subset of S , $r \geq 0$. By $A + r$ we mean the closed set $A + r = \{x \mid \rho(x, A) \leq r\}$.

Theorem 4.1.2: $d_\rho(A, B) \leq r$ iff $A \subseteq B + r$ and $B \subseteq A + r$.

proof: Clearly $\sup_{x \in A} \rho(x, B) \leq r$ is equivalent to $A \subseteq B + r$

and $\sup_{x \in B} \rho(x, A) \leq r$ is equivalent to $B \subseteq A + r$

and so the result follows. $\square$

Theorem 4.1.3: $d_\rho(A, B) \leq \max(\text{diam } A, \text{diam } B) + \rho(A, B)$.

proof: Here $\text{diam } A = \sup_{x, y \in A} \rho(x, y)$ and $\rho(A, B) = \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$.

We will show $\sup_{x \in B} \rho(x, A) \leq \text{diam } B + \rho(A, B)$.

Let $b_0 \in B$. Let $\epsilon > 0$. Then there exists $b \in B$, $a \in A$ such that

$\rho(a, b) \leq \rho(A, B) + \epsilon$. Therefore we have:

$$\begin{aligned} \rho(b_0, A) &\leq \rho(b_0, a) \leq \rho(b_0, b) + \rho(b, a) \\ &\leq \text{diam } B + \rho(A, B) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary we have $\rho(b_0, A) \leq \text{diam } B + \rho(A, B)$. Since

$b_0 \in B$ was arbitrary we can conclude $\sup_{x \in B} \rho(x, A) \leq \text{diam } B + \rho(A, B)$.

By symmetry we obtain $\sup_{x \in A} \rho(x, B) \leq \text{diam } A + \rho(A, B)$, and so the

desired result follows. $\square$

Note: The upper bound in Theorem 4.1.3 is the best that can be obtained in the sense that it is sometimes achieved by $d_\rho(A, B)$. Take the real line $\mathbb{R}$ with bounded metric $\rho(x, y) = \min(|x - y|, 1)$. Let $a, b \in \mathbb{R}$, $r > 0$, such that $a < b$, $\rho(a, b) + r \leq 1$. Take $A = \{a\}$, $B = [b, b+r]$. Then $d_\rho(A, B) = b - a + r = \rho(A, B) + \text{diam } B$.

Theorem 4.1.4: If two bounded metrics ρ and ρ' on S are uniformly equivalent then the induced Hausdorff metrics d_ρ and $d_{\rho'}$ are uniformly equivalent (in particular, they induce the same topology on F).

proof: ρ and ρ' uniformly equivalent means that for every $r > 0$ there exists $s > 0$ such that for all $x, y \in S$, $\rho(x, y) < s$ implies $\rho'(x, y) < r$ and $\rho'(x, y) < s$ implies $\rho(x, y) < r$. We will show that for every $r > 0$ there exists $r' > 0$ such that for all $A, B \in F$, $d_{\rho'}(A, B) < r'$ implies $d_\rho(A, B) < r$. (The complete result then follows from symmetry, interchanging the roles of ρ and ρ' .) Let $r > 0$ and let $0 < r_0 < r$. Then there exists $r' > 0$ such that $\rho'(x, y) < r'$ implies $\rho(x, y) < r_0$. Now suppose $d_{\rho'}(A, B) < r'$. Then

$\sup_{x \in A} \rho'(x, B) < r'$, and so for each $x \in A$ there exists $y \in B$ such that $\rho'(x, y) < r'$. Then $\rho(x, y) < r_0$ and so $\rho(x, B) < r_0$. Therefore

$\sup_{x \in A} \rho(x, B) \leq r_0 < r$. Similarly $\sup_{x \in B} \rho(x, A) < r$, and therefore

$d_\rho(A, B) < r$ as required. $\square$

Note: Theorem 4.1.4 is important in that two metrics which are simply equivalent (rather than uniformly equivalent) need not induce equivalent Hausdorff metrics. As an example consider $[0, \infty)$ under the equivalent metrics $\rho(x, y) = \min(|x - y|, 1)$ and

$\rho'(x, y) = \left| \frac{x}{1+x} - \frac{y}{1+y} \right|$. The sequence of intervals $\{[0, n]\}_{n=1}^\infty$ is

not Cauchy in d_ρ since $d_\rho([0, n], [0, n+1]) = 1$ for every n but

$\lim_{n \rightarrow \infty} [0, n] = [0, \infty)$ (in d_ρ , as

$$\begin{aligned} d_\rho([0, n], [0, \infty)) &= \sup_{x > 0} \rho'(x, [0, n]) = \sup_{x > n} \left| \frac{x}{1+x} - \frac{n}{1+n} \right| \\ &= 1 - \frac{n}{1+n} = \frac{1}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore d_ρ and $d_{\rho'}$ do not generate the same topologies on F^- .

Theorem 4.1.5: If S is compact the induced Hausdorff metric topology is unique.

proof: It is a well-known result that any two equivalent metrics on a compact space are in fact uniformly equivalent. Thus uniqueness follows from Theorem 4.1.4. $\square$

Note: It will be convenient to be able to consider the Hausdorff metric topology on F (rather than F^-) and we do this by adding $\emptyset$ as an isolated point. Since ρ is a bounded metric it follows that d_ρ is a bounded metric on F^- , so let M be some upper bound such that

$d_p(A, B) \leq M$ for every $A, B \in F^-$. Then, for all $A \in F^-$, define
 $d_p(A, \emptyset) = d_p(\emptyset, A) = M + 1$, and $d_p(\emptyset, \emptyset) = 0$. It is easily verified
 that this extended version of d_p is a metric on F . In subsequent
 discussion we will assume this metric and refer to the Hausdorff
 metric topology on F .

2. The Finite, Compact, And Myope Topologies On F And K

Following the notation of Matheron in [14] let

$$F_A = \{F \in F \mid F \cap A \neq \emptyset\}, \quad A \text{ a subset of } S$$

$$F^A = \{F \in F \mid F \cap A = \emptyset\}, \quad A \text{ a subset of } S.$$

The Vietoris or finite topology on F can be defined for any
 topological space S by the subbasis consisting of sets of the form
 F_G , G open, and F^C , C closed. Noting that $F^{C_1} \cap F^{C_2} = F^{C_1 \cup C_2}$ then
 a typical basis element is of the form:

$$F_{G_1, \dots, G_n}^C = \{F \in F \mid F \cap C = \emptyset, F \cap G_i \neq \emptyset, i=1, \dots, n\}$$

$$n \in \mathbb{Z}^+, \quad C \text{ closed}, \quad G_1, \dots, G_n \text{ open}.$$

The basic nbds. of $\emptyset$ are the sets F^C , C closed, and taking $C = S$
 we note that in fact $\emptyset$ is an isolated point of F in this topology.

In [15] Michael actually only defines the finite topology on F^- via
 basis elements of the form:

$$\langle G_1, \dots, G_n \rangle = \{F \in F^- \mid F \subseteq \bigcup_{i=1}^n G_i, F \cap G_i \neq \emptyset, i=1, \dots, n\}$$

$$G_1, \dots, G_n \text{ open sets of } S$$

but since $\emptyset$ is isolated when extending this topology to F the
 results of Michael (obtained for F^-) which we will use are applicable

to F .

Theorem 4.2.1 (Properties of the Finite Topology): Let S be a separable metric space.

(i) The finite topology is the coarsest topology satisfying the requirement that whenever C is closed in S and G is open in S then $\{F \mid F \subseteq C\}$ is closed in F and $\{F \mid F \subseteq G\}$ is open in F .

(ii) S is topologically embedded in (F, finite) via the mapping $x \mapsto \{x\}$.

(iii) F is Hausdorff and separable in the finite topology.

(iv) S is compact iff F is compact in the finite topology.

(v) If F is either metrizable or second-countable in the finite topology then S is compact.

proof: (i) This property has been referred to as acceptability by Michael. If $G \subseteq S$ is open then $\{F \mid F \subseteq G\} = \{F \mid F \cap (S \setminus G) = \emptyset\} = F^{S \setminus G}$, a subbasis set, and therefore open. If $C \subseteq S$ is closed then $F \setminus \{F \mid F \subseteq C\} = \{F \mid F \cap (S \setminus C) \neq \emptyset\} = F_{S \setminus C}$, a subbasis set. Therefore $\{F \mid F \subseteq C\}$ is closed. The finite topology is the coarsest topology having this property since the sets $F^{S \setminus G}$ and $F_{S \setminus C}$ generate the finite topology.

(ii) Michael refers to this property as admissibility. Let $\psi(x) = \{x\}$. If $G \subseteq S$ is open then $\psi^{-1}(F_G) = \{x \mid \{x\} \in F_G\} = \{x \mid \{x\} \cap G \neq \emptyset\} = \{x \mid x \in G\} = G$. If $C \subseteq S$ is closed then $\psi^{-1}(F_C) = \{x \mid \{x\} \cap C = \emptyset\} = \{x \mid x \notin C\} = S \setminus C$ which is open. This shows ψ is continuous. ψ is also a (relatively) open map of S onto $\psi(S)$ since if $G \subseteq S$ is open then we have $\psi(G) = \{\{x\} \mid x \in G\} = \{\{x\} \mid \{x\} \subseteq G\} = \{F \mid F \subseteq G\} \cap \psi(S)$ which is open in $\psi(S)$. Therefore ψ is an embedding.

(iii) Let F and C be two closed sets such that $F \neq C$. Then w.l.o.g. there exists $x \in F \setminus C$. Since $S \setminus C$ is open and S is a metric space there exists a closed ball $\bar{B}(x, r) \subseteq S \setminus C$. Then $F \cap \bar{B}(x, r)$ is a nbd. of C , $F \cap \bar{B}(x, r)$ is a nbd. of F and $F \cap \bar{B}(x, r) \cap F \cap \bar{B}(x, r) = \emptyset$. This shows the space is Hausdorff.

Since S is second-countable there exists a countable basis of open balls, and since each open ball is a countable union of closed balls then S has a countable "basis" of closed balls. That is, there exists a countable collection C of closed balls such that every open set of S is the union of some subcollection of these balls. If $\mathcal{D}$ is the class consisting of $\emptyset$ and all finite unions of members from C then $\mathcal{D}$ is countable and dense in F . Given an arbitrary basic open set $F \cap G_1 \dots G_n$ (with $(S \setminus C) \cap G_i \neq \emptyset$ for any $i = 1, \dots, n$ since otherwise $F \cap G_1 \dots G_n = \emptyset$) then there exists $\bar{B}_i \in C$ such that $\bar{B}_i \subseteq (S \setminus C) \cap G_i$, for each $i = 1, \dots, n$. Let $D = \bar{B}_1 \cup \dots \cup \bar{B}_n$. Then $D \cap C = \emptyset$ and $D \cap G_i \neq \emptyset$, $i = 1, \dots, n$, and so $D \in F \cap G_1 \dots G_n$. This shows $\mathcal{D}$ is dense in F .

(iv) See Theorem 4.2, page 161, of Michael [15]. The proof is very similar to the proof of the analogous result for F in the compact topology (Theorem 4.2.3 (iii) of this section).

(v) See Theorem 4.6, page 162, of [15]. Michael gives further separation properties of F under the finite topology in Theorem 4.9 of [15]. $\square$

We can now obtain the following result:

Theorem 4.2.2: Let (S, ρ) be a bounded connected metric space which is not compact. Then the finite topology on F is not coarser than the Hausdorff metric topology on F .

proof: We will show the existence of an open set G such that $\{H \in F | H \subseteq G\}$ is not open in the Hausdorff topology. (From Theorem 4.2.1 (i) this set is open in the finite topology.) Since S is not compact we can find an infinite sequence $F = \{x_1, x_2, \dots\}$ which has no convergent subsequence; therefore F is a closed subset of S .

Note we can also assume $F \neq S$ so let $x \in S \setminus F$ and let $r = \rho(x, F)$.

Let $\delta_n = \inf_{m \neq n} \rho(x_n, x_m)$. Since F has no convergent subsequences it

follows that $\delta_n > 0$ for each n . Let $r_n = \min(r, \frac{\delta_n}{3}, \frac{1}{n})$ and

consider the open balls $B_n = B(x_n, r_n)$. Let $\epsilon_n = \inf_{m \neq n} \rho(B_n, B_m)$. Then

$\epsilon_n > 0$ since

$$\begin{aligned} \rho(B_n, B_m) &\geq \rho(x_n, x_m) - (r_n + r_m) \geq \rho(x_n, x_m) - \frac{1}{3}(\delta_n + \delta_m) \\ &\geq \rho(x_n, x_m) - \frac{1}{3}(\rho(x_n, x_m) + \rho(x_m, x_n)) \\ &= \rho(x_n, x_m) - \frac{2}{3}\rho(x_n, x_m) \\ &= \frac{1}{3}\rho(x_n, x_m) \geq \frac{\delta_n}{3}. \end{aligned}$$

Let $G = \bigcup_{n=1}^{\infty} B_n$. Then $F \subseteq G$, G is open in S , and $x \notin G$. In

particular $S \setminus G \neq \emptyset$. We wish to show that $\rho(F, S \setminus G) = 0$. Let

$n \in \mathbb{Z}^+$, and suppose there exists $0 < \epsilon < \epsilon_n$ for which

$B_{n+\epsilon}(S \setminus G) = \emptyset$ (where $B_{n+\epsilon} = B(x_n, r_n + \epsilon)$). Then this implies B_n

must be closed, since if z is a limit point of B_n then $z \notin S \setminus G$

(as $\rho(B_n, S \setminus G) \geq \epsilon$) and $z \notin \bigcup_{m \neq n} B_m$ (since $\rho(B_n, B_m) \geq \frac{\delta_n}{3} > 0$), so

$z \in S \setminus B_n$, giving $z \notin B_n$. But B_n is also an open ball; hence B_n is both open and closed, a contradiction since by hypothesis S is connected. Therefore, for every $0 < \varepsilon < \varepsilon_n$ we must have $B_{n+\varepsilon} \cap (S \setminus G) \neq \emptyset$, and so $\rho(x_n, S \setminus G) \leq r_n \leq 1/n \rightarrow 0$. Since $\rho(F, S \setminus G) = \inf_n \rho(x_n, S \setminus G)$ we can conclude $\rho(F, S \setminus G) = 0$. Then, for every $h > 0$, we must have $(F + h) \cap (S \setminus G) \neq \emptyset$, and since $d_p(F, F+h) \leq h$ this shows that any open ball (in the Hausdorff metric) centered at F meets the set $\{H \mid H \cap (S \setminus G) \neq \emptyset\}$ and hence is not contained in $\{H \mid H \subseteq G\}$. Therefore $\{H \mid H \subseteq G\}$ is not open in the Hausdorff metric topology. $\square$

The compact topology on F is defined when the underlying space S is a locally compact separable metric space (LCSM), the local compactness being essential to the properties of the compact topology. We first list some elementary properties of a LCSM for easy reference:

Lemma 4.2.1: For every open set G in a LCSM there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of compact sets such that $K_n \subseteq K_{n+1}^{\circ} \forall n$, and $\bigcup_{n=1}^{\infty} K_n = G$. In particular this is true for $G = S$. $\square$

Lemma 4.2.2: In a LCSM there exists a standard basis. That is, there exists a countable basis A of relatively compact open sets such that for every open set U and $x \in U$ there exists $B \in A$ such that $x \in B \subseteq \bar{B} \subseteq U$. Note that a standard basis consisting of open balls can be chosen. $\square$

Lemma 4.2.3: For every nonempty compact set K in a LCSM there exists a sequence $\{G_n\}_{n=1}^{\infty}$ of relatively compact open sets such that $\bar{G}_{n+1} \subseteq G_n$, $\bigcap_{n=1}^{\infty} G_n = K$, and for every open set $U \supseteq K$ there exists n such

that $G_n \subseteq U$. (The collection $\{G_n\}_{n=1}^{\infty}$ forms a "local basis at K ".) $\square$

The compact topology on F is defined by the subbasis consisting of the sets F_G and F^K , where G runs over all open sets of S and K runs over all compact sets of S . Since $F^{K_1} \cap F^{K_2} = F^{K_1 \cup K_2}$, a basis element in the compact topology is of the form:

$$F_{G_1 \dots G_n}^K = \{F \in F \mid F \cap K = \emptyset, F \cap G_i \neq \emptyset, i=1, \dots, n\}$$

$n \geq 0$, K compact, G_i open.

The basic nbds. of $\emptyset$ are the sets F^K , $K \in K$, and $\emptyset$ is an isolated point of F iff S is compact, since S compact is equivalent to saying $F^S = \{F \mid F \cap S = \emptyset\} = \{\emptyset\}$ is open. This contrasts with the finite topology in which $\emptyset$ is always isolated. Of course if S is compact, then the finite and compact topologies coincide, while if S is not compact the compact topology is strictly coarser.

Theorem 4.2.3: (Properties of the Compact Topology) Let S be a LCSM.

- (i) If C is closed in S then $F^{S \setminus C} = \{F \mid F \subseteq C\}$ is closed in the compact topology on F . If G is open in S then $F^{S \setminus G} = \{F \mid F \subseteq G\}$ is open in the compact topology iff $S \setminus G$ is compact.
- (ii) S is topologically embedded in $(F, \text{compact})$ via the map $x \mapsto \{x\}$.
- (iii) F is second-countable, compact, and Hausdorff in the compact topology.
- (iv) S is compact iff F is compact in the compact topology.

proof: If C is closed the set $\{F \mid F \cap (S \setminus C) \neq \emptyset\}$ is open and hence its complement $F^{S \setminus C} = \{F \mid F \subseteq C\}$ is closed. If G is open and $S \setminus G$ is compact then $F^{S \setminus G} = \{F \mid F \subseteq G\}$ is open. Now suppose G is open and

$S \setminus G$ is not compact. The collection K of compact subsets of S is a directed set under inclusion (that is, we say $K_1 \leq K_2$ iff $K_1 \subseteq K_2$). We create a net indexed by K as follows: Let $C = S \setminus G$. Since C is closed but not compact the set $C \setminus K$ is not empty for any compact K . Therefore choose $x_K \in C \setminus K$ and define the net $\{F_K\}_{K \in K}$ in F by $F_K = \{x_K\}$. If K_0 is an arbitrary compact set then $K \geq K_0$ implies $C \setminus K \subseteq C \setminus K_0$ which implies $x_K \in (C \setminus K_0)$ so $F_K \cap K_0 = \emptyset$ and $F_K \in F_{K_0}^c$. This shows that $\{F_K\}_{K \in K}$ ultimately belongs to every nbd. of $\emptyset$, and so $F_K \rightarrow \emptyset$. But $F_K \in F_C$ for each K while $\emptyset \notin F_C$. Therefore F_C is not closed and so its complement $F^{S \setminus G} = \{F \mid F \subseteq G\}$ is not open.

(ii) Let $\psi(x) = \{x\}$. Then $\psi^{-1}(F_G) = G$ is open when G is open and $\psi^{-1}(F_K) = S \setminus K$ is open when K is compact so ψ is continuous. If C is closed in S then $\psi(C) = \{F \mid F \subseteq C\} \cap \psi(S)$ which is closed in the subspace topology on $\psi(S)$. Therefore ψ is a closed mapping and so is a homeomorphism of S onto $\psi(S)$.

(iii) The details of the following proofs are on page 4 of Matheron [14].

Second-Countability: Let A be a standard basis for S . Then the

countable collection $\{F_{B_1 \cup \dots \cup B_n}^{\bar{B}_1 \cup \dots \cup \bar{B}_m} \mid B_i, B_j \in A, m, n = 0, 1, 2, \dots\}$ is

a basis for the compact topology on F .

Hausdorff: Let F_1 and F_2 be two closed sets such that $F_1 \neq F_2$.

Then w.l.o.g. there exists $x \in (F_1 \setminus F_2)$ and we can find a basic open set B such that $x \in B \subseteq \bar{B} \subseteq (S \setminus F_2)$ and $\bar{B}$ is compact. Then F_B and $F_{\bar{B}}$ are nbds. of F_1 and F_2 respectively and $F_B \cap F_{\bar{B}} = \emptyset$.

Compactness: The proof uses the following form of Alexander's lemma:

Let S be a subbasis for a topological space X . Then X is compact iff every collection $\{C_\alpha \mid \alpha \in A\}$ of closed sets satisfying $(X \setminus C_\alpha) \in S$ and having the finite intersection property (f.i.p.) has nonempty intersection.

We then take any collection $\{K_\alpha \mid \alpha \in A\}$ of compact sets and $\{G_\beta \mid \beta \in B\}$ of open sets and show that $(\bigcap_{\alpha \in A} F_{K_\alpha}) \cap (\bigcap_{\beta \in B} F_{G_\beta}^G) = \emptyset$ implies the collection $\{F_{K_\alpha}, F_{G_\beta}^G \mid \alpha \in A, \beta \in B\}$ cannot have f.i.p. Thus F is compact.

(iv) Suppose S is compact. As noted earlier then $\{\emptyset\}$ is open in F . Let $\{G_\alpha \mid \alpha \in A\}$ be any open covering of F^- ; then $\{\{\emptyset\}, G_\alpha \mid \alpha \in A\}$ is an open covering of F , and F compact implies there exists a finite subcover $\{\{\emptyset\}, G_{\alpha_i} \mid i=1, \dots, n\}$. It follows that $\{G_{\alpha_i} \mid i=1, \dots, n\}$ covers F^- . Hence F^- is compact. Conversely, suppose F^- is compact. Let

$\{B_\alpha \mid \alpha \in A\}$ be any covering of S by open sets. Then $\{F_\alpha^B \mid \alpha \in A\}$ is a collection of closed sets of F satisfying $\bigcap_{\alpha \in A} F_\alpha^B = F^{\bigcup_{\alpha \in A} B_\alpha} = F^S = \{\emptyset\}$.

Therefore the collection $\{F_\alpha^B \setminus \{\emptyset\} \mid \alpha \in A\}$ consists of closed sets of F^- , and we have $\bigcap_{\alpha \in A} (F_\alpha^B \setminus \{\emptyset\}) = \emptyset$. Therefore $\{F_\alpha^B \setminus \{\emptyset\} \mid \alpha \in A\}$ cannot have f.i.p. as F^- is compact, so there exist $\alpha_1, \dots, \alpha_n$ such that

$\bigcap_{i=1}^n (F_{\alpha_i}^B \setminus \{\emptyset\}) = \emptyset$. Therefore $\bigcap_{i=1}^n F_{\alpha_i}^B = F^{\bigcup_{i=1}^n B_{\alpha_i}} = \{\emptyset\}$ which implies

$\bigcup_{i=1}^n B_{\alpha_i} = S$, so a finite subcover exists and hence S is compact. (This

is an expanded version of the proof given on page 5 of [14].) $\square$

Note it follows then from the Urysohn metrization theorem that

S is metrizable.

From this point on we will assume F to have the compact topology unless otherwise stated. From the definition of the basis elements $F_{G_1 \dots G_n}^K$ it is clear that a sequence $F_n \rightarrow F$ in F iff the following two

conditions hold:

1. Whenever an open set G meets F (that is, $F \cap G \neq \emptyset$) then G meets all but finitely-many F_n .
2. Whenever a compact set K is disjoint from F then K is disjoint from all but finitely-many F_n .

Matheron has shown that we can characterize convergence in F by convergence of certain sequences of points of S . In particular, condition 1, above is equivalent to:

- 1.' If $x \in F$ then x is the limit of a sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \in F_n$ for all but finitely-many n .

Condition 2. above is equivalent to:

- 2.' If $\{F_{n_k}\}_{k=1}^{\infty}$ is any subsequence of $\{F_n\}_{n=1}^{\infty}$, $x_k \in F_{n_k}$ for each k , and $\lim_{k \rightarrow \infty} x_k = x \in S$ exists, then $x \in F$.

We will show that 2. and 2.' are equivalent. Suppose that 2. does not hold so that K is a compact set such that $K \cap F = \emptyset$ but K meets infinitely many F_n . Then there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ such that $F_{n_k} \cap K \neq \emptyset$ for each k . Let $x_k \in (F_{n_k} \cap K)$;

then $\{x_k\}_{k=1}^{\infty}$ has a convergent subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ since K is compact. Let $x = \lim_{j \rightarrow \infty} x_{k_j}$. Then $x \in K$ so $x \notin F$. This shows that

2.' does not hold. Therefore 2.' implies 2. Now suppose 2.' does not hold. Then there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ and a sequence $x_k \in F_{n_k}$ such that $\lim_{k \rightarrow \infty} x_k = x \in S$ but $x \notin F$. Since $S \setminus F$ is open there exists a relatively compact nbd. B such that $x \in B \subseteq \bar{B} \subseteq S \setminus F$, and for all sufficiently large k we have $F_{n_k} \cap \bar{B} \neq \emptyset$ while $F \cap \bar{B} = \emptyset$. This shows that 2. does not hold. Therefore 2. implies 2.'.

The proof of the equivalence of 1. and 1.' is equally straightforward and can be found on page 6 of [14].

We can summarize this as follows:

Theorem 4.2.4: A sequence $F_n \rightarrow F$ in the compact topology iff every point of F is the limit point of a sequence $x_n \in F_n$ (for sufficiently large n) and every convergent sequence $x_k \in F_{n_k}$ (for any subsequence $\{F_{n_k}\}$) has its limit point in F .

Note: A sequence $F_n \rightarrow F$ in the finite topology iff both of the following conditions hold:

1." (Same as 1.)

2." Whenever a closed set C is disjoint from F then C is disjoint from all but finitely many F_n .

It can be shown that 1." is equivalent to 1.' and 2." implies 2.' when S is a metric space. It is easy to construct counterexamples to show that 2." does not imply 2.'.

For the sake of brevity we will call a point $x \in S$ a limit point of the sequence $\{F_n\}_{n=1}^{\infty}$ iff there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ and a sequence $x_k \in F_{n_k}$ such that $x = \lim_{k \rightarrow \infty} x_k$. We will call $x \in S$ an

absolute limit point if there exists a sequence $x_n \in F_n$ (except for perhaps finitely many n) such that $x = \lim_{n \rightarrow \infty} x_n$. We can then define the

quantities $\overline{\lim} F_n =$ set of all limit points of $\{F_n\}_{n=1}^{\infty}$ and $\underline{\lim} F_n =$

set of all absolute limit points of F_n . Theorem 4.2.4 can then be

rephrased as $F_n \rightarrow F$ iff $\overline{\lim} F_n = \underline{\lim} F_n = F$.

Note: The above definitions of $\overline{\lim} F_n$ and $\underline{\lim} F_n$ are not equivalent

to the usual definitions $\liminf F_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} F_k$ and $\limsup F_n =$

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} F_k.$$

Noting that $\underline{\lim} F_n$ and $\overline{\lim} F_n$ are both closed sets we clearly have $\liminf F_n \subseteq \underline{\lim} F_n$ and $\limsup F_n \subseteq \overline{\lim} F_n$. In fact it can

also be shown that $\underline{\lim} F_n =$ intersection of the limits of all

convergent subsequences of $\{F_n\}_{n=1}^{\infty}$, and $\overline{\lim} F_n =$ union of the limits of all convergent subsequences of $\{F_n\}_{n=1}^{\infty}$ (see page 8 of [14]).

By applying Theorem 4.2.4 it is easy to prove the following

(Corollary 3 on page 7 of [14]):

Theorem 4.2.5: Let $\{F_n\}_{n=1}^{\infty}$ and $\{F'_n\}_{n=1}^{\infty}$ be two sequences in F .

(i) If $F_n \rightarrow F$ (that is, $F_n \supseteq F_{n+1}$ and $\bigcap_{n=1}^{\infty} F_n = F$) then $F_n \rightarrow F$ in the compact topology.

(ii) If $F_n \rightarrow A$ (that is, $F_n \subseteq F_{n+1}$ and $\bigcup_{n=1}^{\infty} F_n = A$) then $F_n \rightarrow A$ in the compact topology. Here A is any subset of S .

(iii). If $F_n \rightarrow F$ and $F'_n \rightarrow F'$ then $F_n \cap F'_n \rightarrow F \cap F'$ in the compact topology and $F_n \cup F'_n \rightarrow F \cup F'$ in the compact topology.

(iv) If $F_n \uparrow A$ and $F'_n \uparrow A$ then $\overline{F_n \cup F'_n} = \overline{A \cup A'} = \overline{A} \cup \overline{A'}$ in the compact topology and $F_n \cap F'_n \rightarrow \overline{A \cap A'} \subseteq \overline{A} \cap \overline{A'}$ in the compact topology.

(v) If $F_n \subseteq F'_n$ for each n and $\lim_n F_n$ and $\lim_n F'_n$ exist then we

have $\lim_n F_n \subseteq \lim_n F'_n$.

We complete this section with a note on the myope topology.

The myope topology is defined on K , the class of compact subsets of S , and is the restriction of the finite topology to K . (In particular, if S is not compact then the myope topology on K is strictly finer than the restriction of the compact topology to K .) K is a well-behaved subspace of F under the finite topology; it is second-countable (while F is not) and in fact coincides with the Hausdorff metric topology restricted to K . (See pages 12-15 of [14]).

3. A Comparison Of The Hausdorff And Compact Topologies

In this section we assume S to be a LCSM. We make some general comparisons of the Hausdorff and compact topologies and then study the relationships between their convergence behavior on F and F^* , where F^* is the collection of closed subsets of the one-point compactification S^* .

Theorem 4.3.1: The compact topology on F is coarser than the Hausdorff topology.

proof: It is sufficient to show that the subbasis sets F^K , $F_{G,K}$ compact, G open, of the compact topology are open in the Hausdorff topology.

First consider F^K . If $K = \emptyset$ then $F^K = F$ so there is nothing to

prove here. Let $K \neq \emptyset$, and suppose $F \in F^K$. If $F = \emptyset$ we are done since $\{\emptyset\}$ is open in the Hausdorff metric topology. Suppose $F \neq \emptyset$.

Then $F \cap K = \emptyset$, and since K is compact and F closed we have

$\rho(F, K) = r > 0$. Then the open ball $B(F, r) \subseteq F^K$ since $A \in B(F, r)$

implies $d_\rho(A, F) < r$ which implies $\sup_{x \in A} \rho(x, F) < r$ so $A \cap K = \emptyset$ and

$A \in F^K$. Therefore F^K is open in the Hausdorff topology. Now suppose

$F \in F_G$. Then $F \cap G \neq \emptyset$ and so there exists $x \in F \cap G$ and $r > 0$ such

that $B(x, r) \subseteq G$. Then $B(F, r) \subseteq F_G$ since $A \in B(F, r)$ implies $\sup_{y \in F} \rho(y, A)$

$< r$ so $\rho(x, A) < r$ which implies there exists $z \in A$ such that

$\rho(x, z) < r$ implies $z \in B(x, r) \subseteq G$ so $A \cap G \neq \emptyset$. Therefore $A \in F_G$

and F_G is open. $\square$

Theorem 4.3.2: Let $\{F_n\}_{n=1}^\infty$ be a sequence in F , K a compact set in S .

Then $F_n \xrightarrow{d_\rho} K$ if both of the following hold:

(i) There exists a compact K_0 and integer N such that $n \geq N$ implies

$F_n \subseteq K_0$.

(ii) $F_n \rightarrow K$ in the compact topology.

proof: Suppose $F_n \xrightarrow{d_\rho} K$. From Theorem 4.3.1 this clearly implies

(ii) holds. If $K = \emptyset$ then $F_n = \emptyset$ for sufficiently large n which

gives (i) in this case. So let $K \neq \emptyset$ and suppose (i) does not hold.

Let G be a relatively compact open set such that $G \supseteq K$. Then there

exists a subsequence $\{F_{n_k}\}_{k=1}^\infty$ such that, for each k , $F_{n_k} \not\subseteq G$, so

there exists some $x_k \in (F_{n_k} \cap (S \setminus G))$ for each k . Therefore

$d_\rho(F_{n_k}, K) \geq \sup_{x \in F_{n_k}} \rho(x, K) \geq \rho(x_k, K) > \rho(S \setminus G, K) > 0$ for every k .

This contradicts the assumption that $F_n \xrightarrow{d_\rho} K$. Therefore (i) must hold.

Conversely, if $K = \emptyset$ and both (i) and (ii) hold it follows that there exists N such that $n \geq N$ implies $F_n = \emptyset$ and so $d_\rho(F_n, \emptyset) = 0$.

Therefore let $K \neq \emptyset$ and suppose $\sup_{x \in F_n} \rho(x, K) \not\rightarrow 0$. Then there exists $\epsilon > 0$ and $x_n \in F_n$ such that $\rho(x_n, K) > \epsilon$ for every n . If (i) holds then the sequence $\{x_n\}$ ultimately belongs to the compact set K_0 so it has a convergent subsequence $x_{n_k} \rightarrow z$ for some $z \in S$. It follows that $z \in K$ since $\rho(z, K) = \lim_{k \rightarrow \infty} \rho(x_{n_k}, K) \geq \epsilon$, but this implies

(ii) cannot hold (apply Theorem 4.2.4). Therefore (i) and (ii)

together imply $\lim_{n \rightarrow \infty} \sup_{x \in F_n} \rho(x, K) = 0$. Now suppose $\sup_{x \in K} \rho(x, F_n) \not\rightarrow 0$.

Then there exists $\epsilon > 0$ and $x_k \in K$ such that $\rho(x_k, F_{n_k}) > \epsilon$ (for some

subsequence $\{F_{n_k}\}$). But K compact implies there exists a subsequence

$\{x_{k_j}\}$ such that $x_{k_j} \rightarrow x$ (where x is some point in K). Therefore, for

k sufficiently large, $\rho(x, F_{n_k}) > \epsilon/2$, and so x cannot be an absolute

limit point of the sequence $\{F_n\}_{n=1}^\infty$. This shows (ii) cannot hold

(again apply Theorem 4.2.4). Therefore (ii) is enough to conclude

$\lim_{n \rightarrow \infty} \sup_{x \in K} \rho(x, F_n) = 0$.

Therefore we obtain $\lim_{n \rightarrow \infty} d_\rho(F_n, K) = \lim_{n \rightarrow \infty} \max(\sup_{x \in F_n} \rho(x, K), \sup_{x \in K} \rho(x, F_n))$

$= 0$ as required. $\square$

Let (S, ρ) be a noncompact LCSM and d the induced Hausdorff metric

on F . Let (S^*, ρ^*) be the one-point compactification of S where $S^* = S \cup \{\infty\}$. Let d^* be the induced Hausdorff metric on F^* where F^* is the hyperspace of closed subsets of S^* . Noting that $\rho^*|_S$ is an equivalent metric on S let d_S^* denote the Hausdorff metric on F induced by $\rho^*|_S$.

Given a sequence $\{F_n\}_{n=1}^\infty$ of closed sets of S such that $F_n \rightarrow F$ in the compact topology we wish to investigate the convergence behavior of F_n w.r.t. d_S^* and of F_n^* w.r.t. d^* where

$$F_n^* = \begin{cases} F_n \cup \{\infty\} & \text{if } F_n \text{ is not compact in } S \\ F_n & \text{otherwise.} \end{cases}$$

(We need to consider F_n^* here because if F_n is not compact in S then it is not closed in S^* until the point ∞ is added.) We will show that the convergence of F_n to F in the compact topology on F always implies convergence of these sets (perhaps modified by adding ∞) w.r.t. d^* except in one particular situation (and even in this case we can find a subsequence which converges as desired). We obtain somewhat similar results when we consider the convergence of F_n to F in the metric d_S^* on F and use these results to conclude that d_S^* is the coarsest Hausdorff metric on F .

We classify a sequence $\{F_n\}_{n=1}^\infty$ in F into one of three groups in the following way:

$\{F_n\}_{n=1}^\infty$ will be called totally-bounded if there exists a compact set K_0 and an integer N such that $n \geq N$ implies $F_n \subseteq K_0$.

$\{F_n\}_{n=1}^\infty$ will be called totally-unbounded if for every compact set K there exists N such that $n \geq N$ implies $F_n \cap (S \setminus K) \neq \emptyset$. A sequence which is neither totally-bounded nor totally-unbounded will be called

a mixture.

In the following F, F_n will always refer to closed subsets of S , and K, K_n will refer to compact subsets of S . We also use the notation

$$K^* = K \cup \{\infty\}, F^* = F \cup \{\infty\}, \text{ and } F_n^* = \begin{cases} F_n \cup \{\infty\} & \text{if } F_n \text{ is not compact in } S \\ F_n & \text{if } F_n \text{ is compact in } S. \end{cases}$$

Lemma 4.3.1: Let S be a LGSM which is not compact and let $F_n \rightarrow F$ in the compact topology on F . Then in S^* we have $\lim F_n^* = F$ or F^* and $\overline{\lim F_n^*} = F$ or F^* .

proof: Since $F_n \subseteq F_n^*$ for each n it follows that $F = \lim F_n \subseteq \lim F_n^*$ in S^* . Conversely, if $x \in \lim F_n^*$ then there exists a sequence $x_n \in F_n^*$ such that $x_n \rightarrow x$. If $x \in S$ (that is, $x \neq \infty$) then we must have $x_n \in S$ for all sufficiently large n , and so $x_n \in F_n$ for all sufficiently large n . Therefore $x \in \lim F_n = F$. This gives $F \subseteq \lim F_n^* \subseteq F \cup \{\infty\} = F^*$ which gives the desired result that either $\lim F_n^* = F$ or $\lim F_n^* = F^*$. The proof for $\overline{\lim F_n^*}$ is analogous, only using subsequences in this case.

Lemma 4.3.2: Let S be a LCSM which is not compact and let $F_n \rightarrow F$ in the compact topology on F .

- (i) If the sequence $\{F_n\}_{n=1}^{\infty}$ is totally-bounded then $F_n \rightarrow F$ (as subsets of S^*) in the compact topology on F^* .
- (ii) If the sequence $\{F_n\}_{n=1}^{\infty}$ is totally-unbounded then $F_n^* \rightarrow F^*$ (as subsets of S^*) in the compact topology on F^* .
- (iii) If the sequence $\{F_n\}_{n=1}^{\infty}$ is a mixture then $\lim F_n^* = F$ and $\overline{\lim F_n^*} = F^*$ in S^* (no convergence in F^*).

proof: (i) In the totally-bounded case it is clear that $\lim F_n = F = \overline{\lim F_n}$ in S^* .

(ii) We first show that $\lim F_n^* = F \cup \{\infty\}$. In view of Lemma 4.3.1 it is sufficient to prove $\infty \in \lim F_n^*$. Let $\{K_m\}_{m=1}^{\infty}$ be an increasing sequence of compact subsets of S such that $K_m \uparrow S$. Since $\{F_n\}_{n=1}^{\infty}$ is totally-unbounded there exists a sequence of integers $N_m > N_{m-1}$ such that $n \geq N_m$ implies $F_n \cap (S \setminus K_m) \neq \emptyset$. For $N_m \leq n < N_{m+1}$ choose $x_n \in F_n \cap (S \setminus K_m)$. Then $n \geq N_m$ implies $x_n \in (S \setminus K_m)$. Therefore, for $n \geq N_1$ we have a sequence $x_n \in F_n$ such that $x_n \rightarrow \infty$ in S^* , which gives $\infty \in \lim F_n^*$ as required. Therefore, using the fact that $\lim F_n^* \subseteq \overline{\lim F_n^*}$ and applying Lemma 4.3.1 we obtain $\lim F_n^* = \overline{\lim F_n^*} = F^*$ in S^* . Therefore $F_n^* \rightarrow F^*$ in F^* .

(iii) In the case of a mixture we clearly have a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ which is totally-bounded and a subsequence $\{F_{j_k}\}_{k=1}^{\infty}$ which is totally-unbounded. Then from (i) we obtain $F_{n_k} \rightarrow F$ in F^* and therefore it follows from Theorem 4.2.4 (in particular see the second note following it) and Lemma 4.3.1 that $\lim F_{n_k}^* = F$. (Note this implies F is compact in S .) Also (ii) implies $F_{j_k}^* \rightarrow F^*$ in F^* and we can conclude $\lim F_n^* = F^*$. Therefore convergence in the compact topology does not occur in this case. $\square$

Remark: Since S^* is compact all Hausdorff metric topologies and the compact topology on F^* are equivalent (this can be deduced from Theorem 4.1.5, Theorem 4.3.1, and Theorem 4.3.2). In particular if a sequence converges in the compact topology on F^* then it converges w.r.t. the metric d^* .

Theorem 4.3.3: Let S be a LCSM which is not compact and let

$\{F_n\}_{n=1}^{\infty}$ be a sequence of closed subsets of S such that $F_n \rightarrow F$ in the compact topology on F .

(i) Suppose the limit set $F = K$, a compact subset of S .

(a) if $\{F_n\}_{n=1}^{\infty}$ is totally-bounded then $d^*(F_n, K) =$

$$d_S^*(F_n, K) \rightarrow 0.$$

(b) if $\{F_n\}_{n=1}^{\infty}$ is totally-unbounded then $d^*(F_n^*, K^*) \rightarrow 0$

but we do not have convergence of F_n to K with respect to d_S^* on F .

(c) if $\{F_n\}_{n=1}^{\infty}$ is a mixture then we do not have convergence in either d^* or d_S^* .

(ii) If the limit set F is not compact in S then we have

$$d^*(F_n^*, F^*) \rightarrow 0 \text{ in } F^* \text{ and } d_S^*(F_n, F) \rightarrow 0 \text{ in } F.$$

proof: (i) (a) Lemma 4.3.2(i) implies that $F_n \rightarrow K$ in the compact topology on F^* so applying the Remark we can conclude $d^*(F_n, K) \rightarrow 0$. Since we are dealing with totally-bounded sets we also have $d_S^*(F_n, K) = d^*(F_n, K)$. Thus convergence occurs w.r.t. both d^* on F^* and d_S^* on F .

(b) From Lemma 4.3.2(ii) $F_n^* \rightarrow K^*$ in the compact topology on F^* and so using the Remark we obtain $d^*(F_n^*, K^*) \rightarrow 0$ as $n \rightarrow \infty$. Let $K \neq \emptyset$. We can show that $d_S^*(F_n, K) \not\rightarrow 0$. This follows since there exists a sequence $x_n \in F_n$ such that $x_n \rightarrow \infty$, and so $d_S^*(F_n, K) \geq$

$$\sup_{x \in F_n} \rho^*(x, K) \geq \rho^*(x_n, K) \text{ and } \lim_{n \rightarrow \infty} \rho^*(x_n, K) = \rho^*(\infty, K) = c > 0 \text{ as } K \text{ is}$$

compact. Therefore $\liminf_{n \rightarrow \infty} d_S^*(F_n, K) \geq c > 0$. If $K = \emptyset$ we clearly

do not have convergence in d_S^* as totally-unbounded implies $F_n \neq \emptyset$

for all sufficiently large n .

(c) From Lemma 4.3.2(iii) we do not have convergence in the compact topology on F^* so we cannot have convergence in the Hausdorff metric d^* . Since there exists a totally-unbounded subsequence we can apply (b) to conclude we cannot have convergence in d_s^* .

(ii) We prove $d^*(F_n^*, F^*) \rightarrow 0$ by showing that $\{F_n\}_{n=1}^\infty$ is totally-unbounded and then applying Lemma 4.3.2(ii) and the Remark. For suppose $\{F_n\}_{n=1}^\infty$ is not totally-unbounded. Then there exists a subsequence $\{F_{n_k}\}_{k=1}^\infty$ and a compact set K such that $F_{n_k} \subseteq K$ for every k .

Therefore $F = \varliminf F_n \subseteq K$ and so F is compact, a contradiction.

Therefore $\{F_n\}_{n=1}^\infty$ is totally-unbounded as claimed.

We now need to show $d_s^*(F_n, F) \rightarrow 0$ which depends on the following:

claim: If A and B are closed sets of S , $A \neq \emptyset$, and B not compact in S , then

$$\sup_{x \in A^*} \rho^*(x, B^*) = \sup_{x \in A} \rho^*(x, B) \text{ where } B^* = B \cup \{\infty\}$$

$$A^* = \begin{cases} A & \text{if } A \text{ compact} \\ A \cup \{\infty\} & \text{otherwise.} \end{cases}$$

proof of claim: $\sup_{x \in A^*} \rho^*(x, B^*) = \sup_{x \in A} \rho^*(x, B^*)$ since $\rho^*(\infty, B^*) = 0$.

Now $\rho^*(x, B^*) = \min(\inf_{z \in B} \rho^*(x, z), \rho^*(x, \infty))$, and B not compact implies

there exists a sequence $x_n \in B$ such that $x_n \rightarrow \infty$ in S^* . Therefore

$$\rho^*(x, \infty) = \lim_{n \rightarrow \infty} \rho^*(x, x_n) \geq \rho^*(x, B) \text{ and so } \rho^*(x, B^*) = \rho^*(x, B). \text{ This}$$

gives $\sup_{x \in A^*} \rho^*(x, B^*) = \sup_{x \in A} \rho^*(x, B^*) = \sup_{x \in A} \rho^*(x, B)$ as claimed.

Now F is not compact and so by the above claim

$$\sup_{x \in F_n} \rho^*(x, F) = \sup_{x \in F_n^*} \rho^*(x, F^*) \leq d^*(F_n^*, F^*) \rightarrow 0. \quad (1)$$

Now if F_n is not compact for some n we have (again using the claim)

$$\sup_{x \in F} \rho^*(x, F_n) = \sup_{x \in F^*} \rho^*(x, F_n^*) \leq d^*(F_n^*, F^*)$$

while if F_n is compact then $F_n^* = F_n$ (by definition) giving

$$\sup_{x \in F} \rho^*(x, F_n) \leq \sup_{x \in F^*} \rho^*(x, F_n) = \sup_{x \in F^*} \rho^*(x, F_n^*) \leq d^*(F_n^*, F^*).$$

Therefore, whether F_n is compact or not we have

$$\sup_{x \in F} \rho^*(x, F_n) \leq d^*(F_n^*, F^*) \rightarrow 0. \quad (2)$$

(1) and (2) together imply $d_s^*(F_n, F) \rightarrow 0$ as required. $\square$

Corollary 4.3.3.1: If $F_n \rightarrow K$ in the compact topology on F and the sequence $\{F_n\}_{n=1}^{\infty}$ is a mixture then there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ such that $d^*(F_{n_k}, K) = d_s^*(F_{n_k}, K) \rightarrow 0$ as $k \rightarrow \infty$.

proof: Since the sequence is a mixture there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ which is totally-bounded. The desired result then follows

from Theorem 4.3.3(i) (a). $\square$

The results can be summarized in the following table:

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Assume $F_n \rightarrow F$ in the compact topology on F .

Limit Set	Type of Sequence	Convergence in F^*	Convergence in F w.r.t. d_s^*
$F = K$ Compact	(a) Totally-Bounded	YES $d^*(F_n, K) \rightarrow 0$	YES $d_s^*(F_n, K) \rightarrow 0$
	(b) Totally-Unbounded	YES $d^*(F_n^*, K^*) \rightarrow 0$	NO
	(c) Mixture	NO, but a sub-subsequence $d^*(F_{n_k}, K) \rightarrow 0$	NO, but for the same subsequence $d_s^*(F_{n_k}, K) \rightarrow 0$
F Noncompact	The sequence will be totally-unbounded	YES $d^*(F_n^*, F^*) \rightarrow 0$	YES $d_s^*(F_n, F) \rightarrow 0$

As in the preceding discussion, $K^* = K \cup \{\infty\}$, $F^* = F \cup \{\infty\}$, and

$$F_n^* = \begin{cases} F_n & \text{if } F_n \text{ compact in } S \\ F_n \cup \{\infty\} & \text{otherwise.} \end{cases}$$

We can force better behavior of the sequence $\{F_n\}_{n=1}^{\infty}$ by imposing the restriction that the sets be connected.

Theorem 4.3.4: Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of connected closed sets in S (where S is a noncompact LCSM) such that $F_n \rightarrow F$ (where $F \neq \emptyset$) in the compact topology on S . Then $d_s^*(F_n, F) \rightarrow 0$ as $n \rightarrow \infty$. If F is compact then $d^*(F_n, F) \rightarrow 0$, and if F is not compact then $d^*(F_n^*, F^*) \rightarrow 0$.

proof: If F is not compact then the result is immediate from Theorem 4.3.3(ii). Therefore suppose F is compact. We will show that

$\{F_n\}_{n=1}^\infty$ is totally-bounded (and hence the result follows from Theorem 4.3.3(i)(a)).

Suppose $\{F_n\}_{n=1}^\infty$ is not totally-bounded. W.l.o.g. we can then assume that $\{F_n\}_{n=1}^\infty$ is totally-unbounded (otherwise apply the following argument to a totally unbounded subsequence). Fix $x \in F$. Then there exists a sequence $x_n \in F_n$ such that $x_n \rightarrow x$, and since we are assuming the F_n 's to be totally-unbounded there exists a sequence $y_n \in F_n$ such that $\rho^*(y_n, \infty) \rightarrow 0$ as $n \rightarrow \infty$. Since S is a noncompact LCSM there exist compact K_1 and K_2 and positive numbers $a_1 > 0$, $a_2 > 0$, such that $F \subseteq K_1^0 \subseteq K_1 \subseteq K_2^0 \subseteq K_2 \subseteq S$, $\rho^*(F, S \setminus K_1^0) = a_1$, $\rho^*(K_2, \infty) = a_2$.
claim: There exists N_1 such that $n \geq N_1$ implies $x_n \in K_1$ and $y_n \in S \setminus K_2^0$.

This is immediate since $x_n \rightarrow x \in F \subseteq K_1^0 \subseteq K_1$ and $y_n \rightarrow \infty$. Therefore $n \geq N_1$ implies $F_n \cap K_1 \neq \emptyset$ and $F_n \cap (S \setminus K_2^0) \neq \emptyset$.

claim: There exists N_2 such that $n \geq N_2 \Rightarrow F_n \cap (K_2^0 \setminus K_1) = \emptyset$.

We prove this as follows. Let $\varepsilon = \min(a_1, a_2)$. Since $F_n^* \rightarrow F^*$ in the metric d^* on F^* (apply Theorem 4.3.3(i)(b)) there exists N_2 such that $n \geq N_2$ implies $\sup_{z \in F_n^*} \rho^*(z, F^*) < \varepsilon$. Now if $z \in (K_2^0 \setminus K_1)$ then $\rho^*(z, \infty) \geq \rho^*(K_2, \infty) = a_2$ while $\rho^*(z, F) \geq \rho^*(F, S \setminus K_1^0) = a_1$. Therefore $\rho^*(z, F^*) \geq \min(a_1, a_2) = \varepsilon$. This shows that $z \notin F_n^*$ when $n \geq N_2$, which proves the claim.

Now let $N = \max(N_1, N_2)$. Then $n \geq N$ implies F_n is not connected, as the disjoint, nonempty, closed sets $F_n \cap K_1$ and $F_n \cap (S \setminus K_2^0)$ form a separation of F_n .

Therefore F_n connected for every n and F compact implies $\{F_n\}_{n=1}^\infty$ is a totally-bounded sequence, which completes the proof. $\square$

Theorem 4.3.5: If S is a LCSM then d_S^* induces the coarsest Hausdorff metric topology on F .

proof: Let ρ be an equivalent metric on S and suppose $d_\rho(F_n, F) \rightarrow 0$.

Then $F_n \rightarrow F$ in the compact topology and if F is not compact then we

can conclude $d_S^*(F_n, F) \rightarrow 0$ by Theorem 4.3.3(ii). If F is compact then

Theorem 4.3.2(i) implies the sequence $\{F_n\}_{n=1}^\infty$ is totally-bounded,

and hence $d_S^*(F_n, F) \rightarrow 0$ by Theorem 4.3.3(i)(a). Therefore convergence in

d_ρ implies convergence in d_S^* . $\square$

4. Applications To The Support Map Of A Random Measure

Let S be a separable metric space. If $\mu \in M(S)$ the support of μ (denoted by $\text{Supp } \mu$) is defined to be the smallest closed set $C \subseteq S$ satisfying $\mu(S \setminus C) = 0$. It is a well-known result that $\text{Supp } \mu$ exists as defined above and also has the alternate characterization:

$$\text{Supp } \mu = \{x \in S \mid \mu(B(x)) > 0 \text{ for every open ball } B(x) \text{ centered at } x\}. \quad (1)$$

Theorem 4.4.1: Let (S, ρ) be a separable metric space. Then

$\text{Supp}: M(S) \rightarrow (F, d_\rho)$ is a measurable mapping.

proof: First recall that $\emptyset$ is an isolated point of F in the

Hausdorff metric topology, and $\text{Supp}^{-1}(\{\emptyset\}) = \{\underline{0}\}$ (where $\underline{0}$ denotes

the zero measure), a singleton and hence measurable. Now let $F \neq \emptyset$

and $r > 0$ such that $\emptyset \notin \bar{B}(F, r)$ where $\bar{B}(F, r)$ denotes the closed ball centered at F with radius r .

Then $\text{Supp}^{-1}(\bar{B}(F, r)) = \{\mu \mid \text{Supp } \mu \in \bar{B}(F, r)\}$

$= \{\mu \mid F \subseteq \text{Supp } \mu + r, \text{ and } \text{Supp } \mu \subseteq F + r\}$ using the

notation of Theorem 4.1.2

$$= \{\mu|_F \subseteq \text{Supp } \mu + r\} \cap \{\mu|_{\text{Supp } \mu} \subseteq F+r\}.$$

Now $\{\mu|_{\text{Supp } \mu} \subseteq F+r\} = \{\mu|_{\mu(S \setminus (F+r))} = 0\}$ which is a closed set in the weak topology ($G = S \setminus (F+r)$ is open in S , and if $\mu_n \xrightarrow{w} \mu$ with $\mu_n(G) = 0$ for every n then it follows that $\mu(G) = 0$).

Now let D_F be a countable dense subset of F . D_F exists since F is a second-countable metric space as a subspace of S (and hence separable).

$$\text{claim: } \{\mu|_{\text{Supp } \mu + r} \supseteq F\} = \bigcap_{z \in D_F} \bigcap_{n=1}^{\infty} \{\mu|_{\mu(B(z, r + 1/n))} > 0\}$$

where $B(z, r + 1/n)$ is the open ball centered at z of radius $r + 1/n$ in S . Note that $\{\mu|_{\mu(B(z, r + 1/n))} > 0\}$ is open in the weak topology.

proof of claim: Suppose $\text{Supp } \mu + r \supseteq F$. Then $\rho(x, \text{Supp } \mu) \leq r$ for every $x \in F$. Therefore, for any $x \in F$ and $n \in \mathbb{Z}^+$ the open ball $B(x, r + 1/n)$ meets $\text{Supp } \mu$ and hence it is a nbd. of some point in $\text{Supp } \mu$. From the characterization in (1) it follows that $\mu(B(x, r + 1/n)) > 0$. This shows the LHS $\subseteq$ RHS. Conversely, let μ belong to the RHS set. Let $z \in D_F$. Since $\mu(B(z, r + 1/n)) > 0$ then $B(z, r + 1/n)$ meets $\text{Supp } \mu$, and so there exists $y_n \in \text{Supp } \mu$ such that $\rho(z, y_n) < r + 1/n$. Therefore

$$\rho(z, \text{Supp } \mu) = \inf_{y \in \text{Supp } \mu} \rho(z, y) \leq \inf_n \rho(z, y_n) \leq r.$$

Now let $x \in F$ be arbitrary and let $\epsilon > 0$. Then there exists $z \in D_F$ such that $\rho(x, z) < \epsilon$ and so $\rho(x, \text{Supp } \mu) \leq \rho(x, z) + \rho(z, \text{Supp } \mu) < \epsilon + r$.

Since $\epsilon > 0$ was arbitrary we have $\rho(x, \text{Supp } \mu) \leq r$.

Therefore $F \subseteq \text{Supp } \mu + r$ as required, so RHS $\subseteq$ LHS, which completes

the proof of the claim. Note this shows that $\{\mu | \text{Supp } \mu + f \supseteq F\}$ is measurable and completes the proof of the theorem. $\square$

Corollary 4.4.1.1: Suppose S is a LCSM and consider the compact topology on F . Then $\text{Supp} : M(S) \rightarrow (F, \text{compact})$ is a measurable mapping.

proof: This follows from Theorem 4.3.1 and Theorem 4.4.1 but a direct proof is elementary. Let G be an open subset of S . Then we have $\text{Supp}^{-1}(F_G) = \{\mu | \text{Supp } \mu \cap G \neq \emptyset\} = \{\mu | \mu(G) > 0\}$ which is open in the weak topology. This is enough to prove the measurability of $\text{Supp } \mu$ since the collection $F_G, G \text{ open}$, generates the Borel sets of the compact topology. If $K \neq \emptyset$ is compact then there exists a sequence $\{G_n\}_{n=1}^{\infty}$ of relatively compact open sets with $\overline{G_{n+1}} \subseteq G_n$ for every n such that $\bigcap_{n=1}^{\infty} G_n = K$ (see Lemma 4.2.3), and hence $F^K = \bigcup_{n=1}^{\infty} F^{G_n} = \bigcup_{n=1}^{\infty} (F \setminus F_{G_n})$. Therefore the basis for the compact topology belongs to $\sigma(F_G, G \text{ open})$ and from the second-countability of the compact topology it follows that the $F_G, G \text{ open}$, generate the Borel sets. $\square$

Note: Matheron refers to a function ψ from a topological space into $(F, \text{compact})$ as lower semi-continuous (l.s.c.) if it satisfies $\psi^{-1}(F_G)$ is open for every open set G . In view of this and the proof of Corollary 4.4.1.1 we obtain:

Corollary 4.4.1.2: If S is a LCSM then $\text{Supp} : M(S) \rightarrow (F, \text{compact})$ is lower semi-continuous.

Matheron in fact states this result on page 112 of [14] for the case $S = \mathbb{R}^n$ and the space of Radon measure $M_R(\mathbb{R}^n)$ with the vague

topology. It is clear from the proof of Corollary 4.1.1.1 that we could replace $M(S)$ by $M_R(S)$ under the vague topology in Corollary 4.1.1.2.

Theorem 4.4.2: Let S be a separable metric space, $\mu, \mu_n \in M(S)$, and suppose $\mu_n \xrightarrow{w} \mu$. Then $\text{Supp } \mu \subseteq \lim \text{Supp } \mu_n$.

proof: We first note that if $\{F_n\}_{n=1}^\infty$ is a sequence of closed subsets of S then the quantities $\lim F_n$ and $\overline{\lim F_n}$ are well-defined without considering a topology on F or requiring that S be locally compact.

Let $x \in \text{Supp } \mu$. Then there exists a sequence of open balls

$\{B(x, k_m)\}_{m=1}^\infty$ where $k_m \downarrow 0$ and each $B(x, k_m)$ is a μ -continuity set.

Since $x \in \text{Supp } \mu$ we must have $\mu(B(x, k_m)) > 0$ for every m . Therefore

$\lim_{n \rightarrow \infty} \mu_n(B(x, k_m)) = \mu(B(x, k_m)) > 0$ for each m . Therefore, for a fixed

m there exists N_m such that $n \geq N_m$ implies $\mu_n(B(x, k_m)) > 0$. We can

take $N_{m+1} > N_m$. Now $n \geq N_m$ implies $\mu_n(B(x, k_m)) > 0$ so $B(x, k_m) \cap \text{Supp } \mu_n$

$\neq \emptyset$. Therefore there exists $y_n \in \text{Supp } \mu_n$ such that $\rho(y_n, x) < k_m$.

Choose y_n in this way for all n satisfying $N_m \leq n < N_{m+1}$ and do this

for each m . Then for $n \geq N_1$ we have defined a sequence $y_n \in \text{Supp } \mu_n$

such that $y_n \rightarrow x$. Therefore $x \in \lim \text{Supp } \mu_n$. $\square$

Note: The same result holds when S is a LCSM, $\mu, \mu_n \in M_R(S)$, and μ_n

tends vaguely to μ . (The proof of Theorem 4.4.2 applies, the only

necessary modification being that the open balls $B(x, k_m)$ should be

chosen to be relatively compact.) This result also follows from

Matheron's alternate characterization of l.s.c. maps from a

topological space E into F under the compact topology, namely:

A map $\psi : E \rightarrow F$ is l.s.c. iff $\lim \psi(x_n) \supseteq \psi(x)$ for every

sequence $x_n \rightarrow x$. (See page 9, Proposition 1-2-4, of [14]).

Note: It should be realized that Theorem 4.2.2 is the best we can do in terms of relating the behavior of the sets $\text{Supp } \mu_n$ and $\text{Supp } \mu$. For example, consider μ_n to be the probability distribution on $\mathbb{R}$ corresponding to a normal random variable with mean 0 and variance $1/n$. Then μ_n tends weakly to δ_0 , the Dirac measure at 0

(i.e. $\delta_0(B) = \begin{cases} 0 & 0 \notin B \\ 1 & 0 \in B \end{cases}$), while $\text{Supp } \delta_0 = \{0\}$ and $\text{Supp } \mu_n = \mathbb{R}$.

We can differentiate between the behavior of supports with respect to weak convergence and with respect to vague convergence by considering ϵ -supports, where an ϵ -support of a measure μ is a set A such that $\mu(S \setminus A) < \epsilon$.

Theorem 4.4.3: Let S be a separable metric space, $F = \text{Supp } \mu$, and let $F_n \uparrow F$ where each F_n is a closed μ -continuity set. (Such F_n exist by the usual arguments, and if F is a μ -continuity set we can take $F_n = F$ for every n .) Then, given a sequence $\mu_n \xrightarrow{w} \mu$ and a sequence $\epsilon_k \rightarrow 0$, there exists N_k such that $n \geq N_k$ implies F_k is an ϵ_k -support of μ_n .

proof: $\lim_{n \rightarrow \infty} \mu_n(F_k) = \mu(F_k)$ since F_k is a μ -continuity set
 $\quad \quad \quad = \mu(S)$ since $F_k \supseteq F$.

Also $\mu_n(S) \rightarrow \mu(S)$, and so $\lim_{n \rightarrow \infty} \mu_n(S \setminus F_k) = 0$.

Therefore there exists N_k such that $\sup_{n \geq N_k} \mu_n(S \setminus F_k) < \epsilon_k$. $\square$

Note that if we take $F_n = F + r_n = \{x \mid \rho(x, F) \leq r_n\}$ (where the r_n are chosen so that $r_n \downarrow 0$ and each F_n is a μ -continuity set) then $\lim_{n \rightarrow \infty} d_\rho(F_n, F) = 0$ for any equivalent metric ρ on S . If S is a LCSM

then we also have convergence in the compact topology. Thus these ϵ -supports converge to $\text{Supp } \mu$. An analogous result does not exist for the case of vague convergence.

We can now extend the measurability results for the support map to random measures and random measure processes.

Theorem 4.4.4: (a) Let (S, ρ) be a Polish space. If X is a random measure in $\mathcal{M}(S)$ then $\text{Supp } X$ is a random closed set (that is, measurable mapping) in (F, d_0) . If $\{X(t)\}_{t \in T}$ is a measurable or progressively measurable stochastic measure process, then $\{\text{Supp } X(t)\}_{t \in T}$ is respectively measurable or progressively measurable in (F, d_0) . If $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$ are two measurable or progressively measurable stochastic measure processes then the support processes $\{\text{Supp}(X_{ac}(Y)(t))\}_{t \in T}$, $\{\text{Supp}(X_s(Y))(t)\}_{t \in T}$ are both respectively measurable or progressively measurable in (F, d_0) . (Here the notation $X_{ac}(Y)$ and $X_s(Y)$ refers to the Lebesgue decomposition $X = X_{ac}(Y) + X_s(Y)$ as used in Theorem 2.1.5.)

(b) If (S, ρ) is a LCSM the results of (a) also apply with the Hausdorff metric topology replaced by the compact topology on F , and we can consider $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$ as random measure processes in $\mathcal{M}_R(S)$ under the vague topology.

proof: Let $(\Omega, \mathcal{E}, P)$ be the underlying probability space. Then $\text{Supp } X = \text{Supp} \circ X : \Omega \rightarrow F$, $\text{Supp } X(t) = \text{Supp} \circ X : T \times \Omega \rightarrow F$, composition of measurable (or progressively measurable) maps, and hence the results follow. When considering two processes $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$ we apply Theorem 2.1.5 and Theorem 2.4.2 to obtain the measurability (or progressive measurability) of $\{X_{ac}(Y)(t)\}_{t \in T}$.

$\{X_s(Y)(t)\}_{t \in T}$. The results then follow by the composition of the maps $\text{Supp} \circ X_{ac}(Y)$ and $\text{Supp} \circ X_s(Y)$. $\square$

5. The Dimension Of A Random Closed Set In $\mathbb{R}^N$

As in Chapter 3 we will denote the Hausdorff-Besicovitch dimension of a set $E \subseteq \mathbb{R}^N$ by $\dim(E)$. For convenience we will use coverings by open balls so that the α -outer measure of a set $E \subseteq \mathbb{R}^N$ is given by:

$$H^\alpha(E) = \lim_{\delta \rightarrow 0^+} \inf_{\substack{\bigcup_{i=1}^{\infty} B_i \supseteq E \\ \text{diam}(B_i) < \delta \\ B_i \text{ an open ball}}} \sum_{i=1}^{\infty} (\text{diam}(B_i))^\alpha$$

Theorem 4.5.1: Let F be the closed subsets of $\mathbb{R}^N$ under the compact topology.

The dimension map $\dim : F \rightarrow \mathbb{R}$ is measurable.

proof: Let $\{K_\ell\}_{\ell=1}^{\infty}$ be a sequence of compact subsets of $\mathbb{R}^N$ such that $K_\ell \subseteq K_{\ell+1}$, $\bigcup_{\ell=1}^{\infty} K_\ell = \mathbb{R}^N$. Define $\dim_\ell : F \rightarrow \mathbb{R}$ by $\dim_\ell(F) = \dim(F \cap K_\ell)$.

Then, for any $F \in F$, $\dim(F) = \dim(\bigcup_{\ell=1}^{\infty} (F \cap K_\ell)) = \sup_{\ell} \dim(F \cap K_\ell) =$

$\sup_{\ell} \dim_\ell(F)$. Therefore it is sufficient to prove that $\dim_\ell$ is measurable for each ℓ .

$$\{F \mid \dim_\ell(F) \leq \alpha\} = \{F \mid \dim(F \cap K_\ell) \leq \alpha\}$$

$$= \bigcap_{m=1}^{\infty} \{F \mid H^{\alpha+1/m}(F \cap K_\ell) = 0\}$$

$$= \bigcup_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \{F \mid F \cap K_k \subseteq \bigcup_{i=1}^{\infty} B_i\} \quad (1)$$

$$d(B_i) < 1/j$$

$$\sum_{i=1}^{\infty} (d(B_i))^{\alpha+1/m} < 1/k$$

B_i open ball

Let G be any open subset of $\mathbb{R}^N$, K any compact subset of $\mathbb{R}^N$. Then $\{F \mid F \cap K \subseteq G\} = \{F \mid F \cap (K \cap (\mathbb{R}^N \setminus G)) = \emptyset\} = F^{K \cap (\mathbb{R}^N \setminus G)}$ is open in the compact topology since $K \cap (\mathbb{R}^N \setminus G)$ is compact. Therefore $\{F \mid F \cap K_k \subseteq \bigcup_{i=1}^{\infty} B_i\}$ is open, and so expression (1) above shows that $\{F \mid \dim_k(F) \leq \alpha\}$ is in fact a G_δ set in the compact topology (and hence measurable). This completes the proof. $\square$

Corollary 4.5.1.1: Let $\{C(t)\}_{t \in T}$ be a set-valued process in $\mathbb{R}^N$, mapping a probability space $(\Omega, \mathcal{E}, P)$ into $(F, \text{compact})$. Then $\{\dim(C(t))\}_{t \in T}$ is a real stochastic process. If $\{C(t)\}_{t \in T}$ is measurable or progressively measurable then $\{\dim(C(t))\}_{t \in T}$ is respectively measurable or progressively measurable.

proof: This follows by composition of measurable mappings. $\square$

The above corollary can in particular be applied to the quantities $\{\dim(\text{Supp } X(t))\}_{t \in T}$ and $\{\dim(\text{Supp } X_s(t))\}_{t \in T}$, where $X_s(t)$ is the singular portion (with respect to Lebesgue measure on $\mathbb{R}^N$) of a random measure process $\{X(t)\}_{t \in T}$. As noted in the introduction it is the quantity $\dim(\text{Supp } X_s(t))$ which is of particular interest.

CHAPTER 5

The Random Cantor Set Process

For the most part this chapter deals with the construction of a set-valued Markov process in the unit interval $[0,1]$. The construction of this process has as its basis the construction of the usual Cantor set C which is obtained by removing the middle third interval $(1/3, 2/3)$ from $[0,1]$, then removing the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining intervals and so on (see Royden [19], problem 36, pg. 44). We will denote the intervals to be removed as $\{I_{ij}\}_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}}$ since 2^{i-1} intervals (each of length 3^{-i}) are removed at the i^{th} stage of the construction. We can then express the Cantor set as

$$C = \left(\bigcap_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} I_{ij} \right)^c = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} I_{ij}^c$$

where the complements are assumed to be taken in $[0,1]$. The Cantor set process $X(t)$ is constructed by removing the intervals I_{ij} in random order and at random times determined by a collection of "alarm clock" random variables $\{\xi_{ij}\}_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}}$.

In the first two sections we consider the measurability of certain maps into F , the space of closed subsets, and examine the topological nature of the state space of $X(t)$.

1. Certain Measurable Maps Into $F = F(S)$

Let S be a LCSM and let $F = F(S)$ denote the space of closed subsets of S under the compact topology. From Matheron [14] we obtain the following:

Theorem 5.5.1: (i) The union map $U: F \times F \rightarrow F$ defined by

$U(F_1, F_2) = F_1 \cup F_2$ is continuous.

(ii) The intersection map $\cap: F \times F \rightarrow F$ defined by

$\cap(F_1, F_2) = F_1 \cap F_2$ is measurable.

proof: (i) Let G be an open subset of S .

Then $\{(F_1, F_2) \mid (F_1 \cup F_2) \cap G \neq \emptyset\} = \{(F_1, F_2) \mid F_1 \cap G \neq \emptyset\} \cup \{(F_1, F_2) \mid$

$F_2 \cap G \neq \emptyset\}$.

$$= (F_G \times F) \cup (F \times F_G)$$

which is the union of two open sets in $F \times F$.

If K is compact then

$\{(F_1, F_2) \mid (F_1 \cup F_2) \cap K = \emptyset\} = \{(F_1, F_2) \mid F_1 \cap K = \emptyset\} \cap \{(F_1, F_2) \mid F_2 \cap K = \emptyset\}$

$$= (F^K \times F) \cap (F \times F^K)$$

the intersection of two open sets in $F \times F$.

Hence the union map is continuous.

(ii) We will show that $\{(F_1, F_2) \mid (F_1 \cap F_2) \cap K \neq \emptyset\}$ is a closed set. Let $\{(F_{n1}, F_{n2})\}_{n=1}^{\infty}$ be a sequence from this set such

that $(F_{n1}, F_{n2}) \rightarrow (F_1, F_2)$ where $(F_1, F_2) \in F \times F$. Then $F_{n1} \rightarrow F_1$ and $F_{n2} \rightarrow F_2$ in F , and since $F_{n1} \cap F_{n2} \cap K \neq \emptyset$ for any n there exists a point $x_n \in F_{n1} \cap F_{n2} \cap K$ for each n . As K is compact this implies there exists a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ with limit $x \in K$.

It follows that $x \in \overline{\lim} F_{n1} = F_1$ and $x \in \overline{\lim} F_{n2} = F_2$ so

$x \in F_1 \cap F_2 \cap K$ and hence $(F_1 \cap F_2) \cap K \neq \emptyset$. Thus

$\{(F_1, F_2) \mid (F_1 \cap F_2) \cap K \neq \emptyset\}$ is closed, and this shows the intersection map is measurable. $\square$

Corollary 5.1.1.1: (i) If $X_1, \dots, X_n$ are random closed sets from

a probability space $(\Omega, \mathcal{E}, P)$ into F then $X = \bigcup_{i=1}^n X_i$ is a random closed set.

(ii) If $\{X_i\}_{i=1}^{\infty}$ is a sequence of random closed sets from a probability space $(\Omega, \mathcal{E}, P)$ into F then $X = \bigcap_{i=1}^{\infty} X_i$ is also a random closed set.

proof: The proof of Theorem 5.1.1 can be extended easily to finite unions and countable intersections. $\square$

2. The State Space C

The Cantor set process $X(t)$ will take as its value the complement of the union of the intervals I_{ij} which have been removed at time t . While we can regard $X(t)$ as a map into $F([0,1])$ it will be more convenient to consider the minimal state space C consisting of the complements of all possible unions of the I_{ij} 's. That is, $F \in C$ iff $F = \left(\bigcup_{(i,j) \in W'} I_{ij} \right)^C$ where W' is a subset of $W = \{(i,j) \mid i=1,2,\dots, j=1,\dots,2^{i-1}\}$.

For notational convenience we remark that $F \in C$ can be identified uniquely by the representation $\langle F \rangle = \{(i,j) \mid (i,j) \in W'\} = W'$ where $F = \left(\bigcup_{(i,j) \in W'} I_{ij} \right)^C$. Note that $[0,1]$ has the representation

$\langle [0,1] \rangle = \emptyset$. The level of F (denoted $L(F)$) is defined by

$L(F) = \sup\{i \mid (i,j) \in \langle F \rangle\}$ if $F \neq [0,1]$. We set $L([0,1]) = 0$.

If $L(F) < \infty$ then F is called terminating. (Equivalently, a terminating point F is the complement of the union of a finite number of the I_{ij} 's.)

We let $L_N = \{F \mid L(F) \leq N\}$, the collection of all members of C which

terminate by level N . Then $L_T = \bigcup_{N=0}^{\infty} L_N$ is the set of all terminating

points of C .

We also define the truncation maps $T_N: C \rightarrow L_N$ by

$$\langle T_N(F) \rangle = \{(i,j) \in \langle F \rangle \mid i \leq N\} \text{ for } N=1,2,\dots \text{ and } T_N(F) = [0,1].$$

Note that T_N takes on only a finite number of values since L_N has only finitely-many members (in fact L_N has precisely 2^{2N-1} elements).

We may express L_N as $L_N = \{F \mid T_N(F) = F\}$.

Since $[0,1]$ is compact then the compact topology on the closed subsets $F([0,1])$ agrees with the (unique) Hausdorff metric topology on $F([0,1])$ and it is easily seen that if $F_1, F_2 \in C$ then

$$d(F_1, F_2) = 1/2 \max \left(\sup_{(i,j) \in \langle F_1 \rangle \setminus \langle F_2 \rangle} \lambda(I_{ij}), \sup_{(i,j) \in \langle F_2 \rangle \setminus \langle F_1 \rangle} \lambda(I_{ij}) \right)$$

where d is the Hausdorff metric on $F([0,1])$ induced by the usual metric $\rho(x,y) = |x-y|$ on $[0,1]$ (see Section 1 of Chapter 4). Since $\lambda(I_{ij}) = 3^{-i}$ it follows that $d(F_1, F_2) = \frac{3^{-N}}{2}$ where

$N = \inf\{M \mid T_M(F_1) \neq T_M(F_2)\}$ is the first level on which F_1 and F_2 disagree. Basically, two elements of C are close if they agree up to level N (that is, $T_N(F_1) = T_N(F_2)$) for large N .

Lemma 5.2.1: The collection L_T of terminating points is a countable dense subset of C .

proof: L_T is clearly countable and if $F \in C$ then $T_N(F) \in L_T$ and

$$d(T_N(F), F) \leq \frac{3^{-(N+1)}}{2} \rightarrow 0 \text{ as } N \rightarrow \infty. \text{ Hence } L_T \text{ is dense. } \square$$

Now for each N and $A \in L_N$ define $C_{N,A} = \{F \mid T_N(F) = A\}$. Then the collection $\{C_{N,A} \mid A \in L_N\}$ is a finite partition of C for each N .

Lemma 5.2.2: (i) For each N and $A \in L_N$, $C_{N,A}$ is an open ball in

C . Conversely, if B is an open ball in C then $B = C_{N,A}$ for some

N and some $A \in L_N$.

(ii) For each N and $A \in L_N$, $\bar{C}_{N,A}$ is a closed ball in

C . Conversely, if $\bar{B}$ is a closed ball in C then $\bar{B} = \bar{C}_{N,A}$ for some

N and some $A \in L_N$.

proof: We will use the notation $B(A,r)$ to denote the open ball

$\{F | d(F,A) < r\}$ and $\bar{B}(A,r)$ to denote the closed ball $\{F | d(F,A) \leq r\}$.

(i) It is easily seen that $C_{N,A} = B(A, \frac{3^{-N}}{2})$ while if $0 < r \leq 1/6$

we have $B(A,r) = B(T_N(A), \frac{3^{-N}}{2}) = C_{N,T_N(A)}$ where $N = \inf\{M \geq 1 | \frac{3^{-M}}{2} \geq r\}$.

For $r > 1/6$ we have $B(A,r) = C = C_{0,[0,1]}$.

(ii) We can also write $C_{N,A} = \bar{B}(A, \frac{3^{-(N+1)}}{2})$. If $r \geq 1/6$ then

$\bar{B}(A,r) = C$ while for $0 < r < 1/6$ we have $\bar{B}(A,r) = \bar{B}(T_N(A), \frac{3^{-N}}{2}) =$

$C_{N,T_N(A)}$ where $N = \inf\{M \geq 1 | \frac{3^{-M}}{2} > r\}$. $\square$

We can now present an interesting result about C . As a subspace of $F([0,1])$ it exhibits the same topological behavior as the Cantor set C does as a subspace of $[0,1]$.

Theorem 5.2.1: C is a compact separable metric space and is a perfect, nowhere dense, totally disconnected subspace of $F([0,1])$.

proof: We begin by showing that C is complete in the metric d .

Let $\{F_n\}_{n=1}^{\infty}$ be a Cauchy sequence in C . Then $\lim_{m,n \rightarrow \infty} d(F_n, F_m) = 0$

so for each $N \in \mathbb{Z}^+$ there exists $M_N \in \mathbb{Z}^+$ such that $m, n \geq M_N$ implies

$T_N(F_m) = T_N(F_n)$. We can choose the sequence $\{M_N\}_{N=1}^{\infty}$ to be increasing.

Note that for each N we have $T_N(T_{N+1}(F_{M_{N+1}})) = T_N(F_{M_{N+1}}) = T_N(F_{M_N})$

so $T_N(F_{M_N}) \supseteq T_{N+1}(F_{M_{N+1}})$. Set $F = \bigcap_{N=1}^{\infty} T_N(F_{M_N})$. It follows that

$F \in C$ and $T_N(F) = T_N(F_{M_N})$ for each N . Then for $n \geq M_N$ we have

$$T_N(F_n) = T_N(F) \text{ so } d(F_n, F) \leq \frac{3}{2}^{-(N+1)} \text{ and therefore } \lim_{n \rightarrow \infty} d(F_n, F) = 0.$$

Thus C is complete. Since $F([0,1])$ is compact and separable it follows that C is compact and separable. Since C clearly has no isolated points it is perfect (every point of C is an accumulation point of C).

It is easily seen that each $F \in L_T$ is the limit point of a sequence from $F([0,1]) \setminus C$ and since L_T is dense in C it follows that each $F \in C$ is the limit point of a sequence from $F([0,1]) \setminus C$. Thus C is nowhere dense in $F([0,1])$. The fact that C is totally disconnected (that is, the only connected subsets of C are the singletons) follows from

Lemma 5.2.2. If $F_1 \neq F_2$ then there exists N such that $T_N(F_1) \neq T_N(F_2)$ and hence F_1 and F_2 belong to different partition elements $C_{N, T_N(F_1)}$

and $C_{N, T_N(F_2)}$ respectively. Since $C_{N, T_N(F_1)}$ and $C_{N, T_N(F_2)}$ are both

open and closed F_1 and F_2 are separated. $\square$

We now note that the truncation maps T_N are continuous functions of C into itself since T_N takes on only finitely-many values with

$$T_N^{-1}(\{A\}) = C_{N,A} \text{ for each } A \in L_N. \text{ In fact } T_N \text{ is } \underline{\text{supercontinuous}} \\ \text{(that is, } T_N^{-1}(H) \text{ is open for } \underline{\text{all}} \ H \subseteq C) \text{ as } T_N^{-1}(H) = \bigcup_{A \in (H \cap L_N)} T_N^{-1}(\{A\}).$$

We can use the truncation maps to characterize the continuous functions on C .

Theorem 5.2.2: Let (S, d') be a metric space. Then:

(i) a function $f: C \rightarrow S$ is continuous iff $f_N \rightarrow f$ uniformly, where

$$f_N = f \circ T_N.$$

(ii) a function $f: S \rightarrow C$ is continuous iff $T_N \circ f$ is continuous for every N .

proof: (i) From the supercontinuity of T_N it follows immediately

that $f \circ T_N$ is supercontinuous for any function $f: C \rightarrow S$. Thus

$f_N \rightarrow f$ uniformly implies f is continuous, being the limit of a uniformly convergent sequence of continuous functions. Conversely;

suppose f is continuous. The compactness of C implies f is uniformly

continuous and since $\sup_{F \in C} d(T_N(F), F) < \frac{3^{-N}}{2} \rightarrow 0$ as $N \rightarrow \infty$ then

$\sup_{F \in C} d'(f(T_N(F)), f(F)) \rightarrow 0$ as $N \rightarrow \infty$. Noting that $f(T_N(F)) = f \circ T_N(F) =$

$f_N(F)$ we have thus shown that $f_N \rightarrow f$ uniformly.

(ii) If f is continuous then the composition $T_N \circ f$ must be continuous. Conversely, suppose each $T_N \circ f$ is continuous. If B is an open ball of C then $B = C_{N,A}$ for some N and some $A \in L_N$ (see Lemma 5.2.2(i)). Therefore

$$f^{-1}(B) = f^{-1}(C_{N,A}) = f^{-1}(T_N^{-1}(C_{N,A})), \text{ since } C_{N,A} = T_N^{-1}(C_{N,A})$$

$$= (T_N \circ f)^{-1}(C_{N,A})$$

$$= (T_N \circ f)^{-1}(B) \quad \text{which is open.}$$

Therefore f is continuous. $\square$

A real-valued function $f: C \rightarrow \mathbb{R}$ will be called truncated if there exists a nonnegative integer N such that f is constant over

$C_{N,A}$ for each $A \in L_N$. Equivalently, f is truncated if there exists N such that $f = f \circ T_N$. The collection of truncated functions will be denoted by $T(C)$. We will let $C(C)$ denote the continuous functions $f: C \rightarrow \mathbb{R}$ of which $T(C)$ is clearly a subset.

Corollary 5.2.2.1: The truncated functions $T(C)$ are dense in $C(C)$, and the collection $T_0(C)$ of truncated functions taking on only rational values is a countable dense subset of $C(C)$.

proof: That $T(C)$ is dense in $C(C)$ follows immediately from Theorem 5.2.2(i). It is clear that $T_0(C)$ is countable and dense in $T(C)$ and hence $T_0(C)$ is dense in $C(C)$. $\square$

The truncated functions $T(C)$ and the preceding corollary will become important when we discuss the infinitesimal generator of the Cantor set process in Section 4 of this chapter.

We now give an example of a continuous function $f: C \rightarrow \mathbb{R}$ which is not truncated. Define $f(F) = \lambda(F^C)$. Then

$$\begin{aligned} f(F) &= \sum_{(i,j) \in \langle F \rangle} \lambda(I_{ij}) \\ &= \sum_{\substack{(i,j) \in \langle F \rangle \\ i \leq N}} \lambda(I_{ij}) + \sum_{\substack{(i,j) \in \langle F \rangle \\ i > N}} \lambda(I_{ij}) \\ &= f(T_N(F)) + \sum_{\substack{(i,j) \in \langle F \rangle \\ i > N}} \lambda(I_{ij}) \\ &= f_N(F) + \sum_{\substack{(i,j) \in \langle F \rangle \\ i > N}} \lambda(I_{ij}) \end{aligned}$$

$$\begin{aligned} \text{Therefore } \sup_{F \in C} |f_N(F) - f(F)| &= \sup_{F \in C} \sum_{\substack{(i,j) \in \langle F \rangle \\ i > N}} \lambda(I_{ij}) \leq \sum_{i > N} \lambda(I_{ij}) \\ &= \sum_{i=N+1}^{\infty} \frac{2^{i-1}}{3^i} = (2/3)^N + 0 \end{aligned}$$

Let $C_N = \sigma(C_{N,A}, A \in L_N)$. Note C_N is actually a finite field with atoms $C_{N,A}, A \in L_N$. We will refer to C_N as the algebra of N-cylinders since membership of F to $A \in C_N$ is determined only by the behavior of F with respect to the intervals I_{ij} for $i \leq N$.

(Equivalently, membership of F to A depends only on $T_N(F)$.) Note that $C_N \subseteq C_{N+1}$ for each N . Thus $C_F = \bigcup_{N=0}^{\infty} C_N$ is an algebra, which we will call the algebra of finite-dimensional cylinders. From Lemma 5.2.2(i) it is immediate that $\sigma(C_F) = B(C)$. That is, the Borel sets of C are generated by the finite-dimensional cylinders. We can also express this in terms of the truncation maps T_N .

Theorem 5.2.3: (i) For each N we have $C_N = \sigma(T_N)$.

(ii) $B(C) = \sigma(T_N, N=0,1,2,\dots)$.

Proof: Both assertions are obvious from earlier discussions. $\square$

As an immediate consequence of Theorem 5.2.3 we have:

Corollary 5.2.3.1: A map f from a measurable space E into C is measurable iff $T_N \circ f$ is measurable for each N . $\square$

3. Construction Of The Process

In this section we explicitly construct the Cantor set process

$X(t)$. As before let $W = \{(i,j) | i=1,2,\dots, j=1,2,\dots, 2^{i-1}\}$, and set

$\Omega = \mathbb{R}^W$. Define probability measures $P_{(i_1,j_1)\dots(i_n,j_n)}$ on the Borel sets of $\mathbb{R}^{(i_1,j_1)} \times \dots \times \mathbb{R}^{(i_n,j_n)}$ by the multivariate distribution functions

$$F_{(i_1,j_1)\dots(i_n,j_n)}(x_1,\dots,x_n) = \begin{cases} \prod_{k=1}^n (1 - e^{-\lambda_{i_k,j_k} x_k}) & \text{if } x_k \geq 0 \text{ for every } k \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda_{i_k j_k} > 0$ for each (i_k, j_k) .

Clearly this is a consistent, symmetric family of distributions and so the Kolmogorov extension theorem implies there exists a unique probability measure P defined on A (where A is the σ -algebra on Ω generated by the projection maps $\pi_{(i_1 j_1) \dots (i_n j_n)}(\omega) = (\omega_{i_1 j_1}, \dots, \omega_{i_n j_n})$) having the finite-dimensional distributions

$$P(\pi_{(i_1 j_1) \dots (i_n j_n)}^{-1}(B)) = P_{(i_1 j_1) \dots (i_n j_n)}(B).$$

Define the random variables $\xi_{ij}: \Omega \rightarrow \mathbb{R}$ by $\xi_{ij}(\omega) = \pi_{(i,j)}(\omega) = \omega_{ij}$. Then $\sigma(\xi_{ij}, (i,j) \in W) = \sigma(\pi_{(i,j)}, (i,j) \in W) = A$. The ξ_{ij} 's are independent, each having an exponential distribution with parameter $\lambda_{ij} > 0$. These random variables will act as "alarm clocks", governing the order and times at which the intervals I_{ij} are removed.

Letting $I = [0, 1]$ and $T = [0, \infty)$ from this point onward, define the set-valued "indicator" function $X_{ij}: T \times T \rightarrow C$ by

$$X_{ij}(t, x) = \begin{cases} I_{ij}^c & x \leq t \\ I & x > t \end{cases}$$

Letting $A_t = \sigma(\{\xi_{ij} \leq s\}, 0 \leq s \leq t)$, the past up to time t , we obtain:

Lemma 3.1 (i) For each t , $X_{ij}(t, \xi_{ij}(\cdot))$ is a random closed set mapping (Ω, A, P) into $(C, B(C))$. Moreover, the map $X_{ij}(\cdot, \xi_{ij}(\cdot)): T \times \Omega \rightarrow C$ is progressively measurable with respect to the σ -fields $B_t \times A_t$ on $T \times \Omega$, where $B_t = B([0, t])$.

(ii) For each ω , $X_{ij}(\cdot, \xi_{ij}(\omega)): T \rightarrow C$ is right-continuous with left-hand limits (cadlag).

proof: (i) Since $A_t \subseteq A$ the fact that $X_{ij}(t, \xi_{ij}^*(\cdot))$ is a random closed set will follow if we verify the progressive measurability of $X_{ij}(\cdot, \xi_{ij}(\cdot))$. Let $X_{ijt}: [0, t] \times T \rightarrow C$ be the restriction of X_{ij} to $[0, t] \times T$. If $H \subseteq C$ then we have:

$$X_{ijt}^{-1}(H) = \begin{cases} [0, t] \times T & \text{if } I, I_{ij}^C \in H \\ \emptyset & \text{if } I, I_{ij}^C \notin H \\ \{(s, x) \mid 0 \leq s < x, s \leq t\} & \text{if } I \in H \text{ and } I_{ij}^C \notin H \\ \{(s, x) \mid x \leq s \leq t\} & \text{if } I \notin H \text{ and } I_{ij}^C \in H \end{cases}$$

Let B^t be the σ -algebra on T defined by $B^t = \sigma([0, s], 0 \leq s \leq t)$. Since $\{(s, x) \mid x \leq s \leq t\}$ is a closed subset of $[0, t] \times [0, t]$ it follows that it belongs to $B_t \times B^t$, and hence the complementary set $\{(s, x) \mid 0 \leq s < x, s \leq t\}$ in $[0, t] \times T$ also belongs to $B_t \times B^t$. This shows that X_{ijt} is $B_t \times B^t$ -measurable.

Now let $K_{ijt}: [0, t] \times \Omega \rightarrow [0, t] \times T$ be defined by $K_{ijt}(s, \omega) = (s, \xi_{ij}(\omega))$. If $B_1 \in B_t$ and $B_2 = [0, s]$ for some $s \leq t$ then

$$\begin{aligned} K_{ijt}^{-1}(B_1 \times B_2) &= \{(s, \omega) \mid (s, \xi_{ij}(\omega)) \in B_1 \times B_2\} \\ &= \{(s, \omega) \mid s \in B_1 \text{ and } \xi_{ij}(\omega) \leq s\} \\ &= B_1 \times \{\xi_{ij} \leq s\} \end{aligned}$$

which is a member of $B_t \times A_t$. Since the sets $B_1 \times B_2$, $B_1 \in B_t$, $B_2 = [0, s]$ for $s \leq t$, generate $B_t \times B^t$ it follows that K_{ijt} is $B_t \times A_t = B_t \times B^t$ measurable.

Now

$$X_{ij}(\cdot, \xi_{ij}(\cdot))|_{[0, t] \times \Omega} = X_{ijt} \circ K_{ijt}$$

and so we can conclude that $X_{ij}(\cdot, \xi_{ij}(\cdot))$ is progressively measurable

with respect to $\mathcal{B}_t \times \mathcal{A}_t$.

(ii) It is immediate from the definition that X_{ij} is a cadlag function. $\square$

We now define $X: T \times \Omega \rightarrow C$ by

$$X(t, \omega) = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{2^{i-1}} X_{ij}(t, \xi_{ij}(\omega))$$

Thus $X(t)$ is the complement of the union of the intervals I_{ij} which have been removed at time t . Note that once an interval is removed it is never replaced so the process is always moving "forward" in the sense that $X(t) \subseteq X(s)$ whenever $t \geq s$. The process terminates with value $X(t) = C$, the Cantor set, once all the intervals have been removed. The rate at which the intervals are removed (and whether or not the process reaches the Cantor set in a finite time period) depends on the parameters λ_{ij} which we will refer to as the removal rates.

Theorem 5.3.1: (i) $X: T \times \Omega \rightarrow C$ is progressively measurable with respect to the product σ -algebra $\mathcal{B}_t \times \mathcal{A}_t$ on $T \times \Omega$. As a consequence $X(t)$ is $\mathcal{A}_t$ -adapted and, for each $t \in T$, is a random closed set in C .

(ii) For each $\omega \in \Omega$, $X(\cdot, \omega)$ is a cadlag map of T into C .

(iii) $\mathcal{A}_t = \sigma(X(s), 0 \leq s \leq t)$, the past of the process.

proof: (i) From Corollary 5.1.1.1(ii) and Lemma 5.3.1(i) we can con-

clude that $X(\cdot, \cdot) = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{2^{i-1}} X_{ij}(\cdot, \xi_{ij}(\cdot))$, $(i, j) \in W$ is

progressively measurable with respect to $\mathcal{B}_t \times \mathcal{A}_t$ and hence $X(t)$ is $\mathcal{A}_t$ -adapted (that is, $\sigma(X(t)) \subseteq \mathcal{A}_t$ for each t). This also shows that $X(t): (\Omega, \mathcal{A}, P) \rightarrow C$ is a random closed set in C .

(ii) Consider the truncated process X_N defined by

$$X_N(\cdot, \omega) = T_N(X(\cdot, \omega)) = \bigcap_{i=1}^N \bigcap_{j=1}^{2^{i-1}} X_{ij}(\cdot, \xi_{ij}(\omega)). \quad X_N(\cdot, \omega) \text{ has at}$$

most $2^N - 1$ jumps or discontinuities (at the points $t = \xi_{ij}(\omega)$, $i=1, \dots, N$, $j=1, \dots, 2^{i-1}$) and is easily seen to be cadlag since each $X_{ij}(\cdot, \xi_{ij}(\omega))$ is cadlag. Now $X_N \rightarrow X$ uniformly in both t and ω since

$$d(X_N(t, \omega), X(t, \omega)) \leq \frac{3}{2}^{-N} \rightarrow 0. \quad \text{The uniform convergence in } t \text{ then implies that the limit } X(\cdot, \omega) \text{ is also cadlag.}$$

(iii) From (i) we see that $\sigma(X(s), 0 \leq s \leq t) \subseteq \mathcal{A}_t$. To show the opposite inclusion let $s \in [0, t]$. Then $[\xi_{ij} > s] = [X(s) \geq I_{ij}] = X(s)^{-1}(\{F|F \geq I_{ij}\})$. Since it is easily seen that $\{F|F \geq I_{ij}\}$ is closed in C (apply Theorem 4.2.4) then $\mathcal{A}_t = \sigma([\xi_{ij} > s], 0 \leq s \leq t) \subseteq \sigma(X(s), 0 \leq s \leq t)$ and the result follows. $\square$

Thus we have established that $X(t)$ is a cadlag random closed set process in C . We now wish to show that $X(t)$ is Markov. This will depend on the following lemmas:

Lemma 5.3.2: Let $(E, \mathcal{E}, \mu)$ be a measure space and suppose g, h are integrable functions on E . Let $\mathcal{E}_0$ be a π -system such that $E \in \mathcal{E}_0$ and $\sigma(\mathcal{E}_0) = \mathcal{E}$. Suppose $\int_A g d\mu = \int_A h d\mu$ for each $A \in \mathcal{E}_0$. Then $g=h$ μ -a.s.

proof: We will show that $E'_0 = \{A \in E \mid \int_A g d\mu = \int_A h d\mu\}$ is a d-system.

By hypothesis $E \in E'_0$. Let $A_n \in E'_0$ for every n and suppose $A_n \uparrow A$.

Since g and h are integrable the dominated convergence theorem implies

$$\int_A g d\mu = \lim_{n \rightarrow \infty} \int_{A_n} g d\mu = \lim_{n \rightarrow \infty} \int_{A_n} h d\mu = \int_A h d\mu. \text{ Hence } A \in E'_0. \text{ Now if}$$

$A_1, A_2 \in E'_0$ with $A_1 \subseteq A_2$ then we have

$$\int_{A_2 \setminus A_1} g d\mu = \int_{A_2} g d\mu - \int_{A_1} g d\mu = \int_{A_2} h d\mu - \int_{A_1} h d\mu = \int_{A_2 \setminus A_1} h d\mu \text{ and so}$$

$(A_2 \setminus A_1) \in E'_0$. Thus E'_0 is a d-system, and $E_0 \subseteq E'_0$, $\sigma(E_0) = E$

implies, by Dynkin's lemma, that $E'_0 = E$. Therefore $\int_A g d\mu = \int_A h d\mu$

for every $A \in E$ so we must have $g = h$ μ -a.s. $\square$

Lemma 5.3.3: Let $(\Omega_0, \mathcal{A}_0, P_0)$ be any probability space. Let Z be a random variable on Ω_0 and let U and V be sub- σ -algebras of $\mathcal{A}_0$ such that $\sigma(Z, U)$ is independent of V . Then $E(Z \mid \sigma(U, V)) = E(Z \mid U)$.

proof: It is sufficient to prove the result for indicator functions. $Z = I_A$ since the complete result will then follow from linearity and the conditional monotone convergence theorem.

$E(Z \mid \sigma(U, V))$ is that $\sigma(U, V)$ -measurable function (unique up to sets of P_0 -measure zero) satisfying $\int_B E(Z \mid \sigma(U, V)) dP_0 = \int_B Z dP_0$

for each $B \in \sigma(U, V)$. To prove the result for indicator functions

$Z = I_A$ we need only show

$$\int_B E(I_A \mid U) dP_0 = P_0(A \cap B) \quad (1)$$

for each $A \in \sigma(U, V)$ since $E(I_A \mid U)$ is clearly $\sigma(U, V)$ -measurable.

In view of Lemma 5.3.2 we need only verify relation (1) for sets B belonging to some π -system E_0 satisfying $\Omega_0 \in E_0$ and $\sigma(E_0) = A_0$.

Take $E_0 = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$. Then E_0 has the required properties and:

$$\begin{aligned} \int_{U \cap V} E(I_A | \mathcal{U}) dP_0 &= \int_V I_U E(I_A | \mathcal{U}) dP_0 \\ &= \int_V E(I_U \cdot I_A | \mathcal{U}) dP_0 \quad \text{since } I_U \text{ is } \mathcal{U}\text{-measurable} \\ &= \int_V E(E(I_U \cdot I_A | \mathcal{U}) | \mathcal{V}) dP_0 \quad \text{as } V \in \mathcal{V} \\ &= \int_V E(I_U \cdot I_A) dP_0 \quad \text{since } \mathcal{U} \text{ and } \mathcal{V} \text{ are independent} \\ &= P_0(V) P_0(U \cap A) \\ &= P_0((U \cap V) \cap A) \quad \text{since } U \cap A \in \sigma(I_A, \mathcal{U}), V \in \mathcal{V}, \\ &\quad \text{and by assumption these } \sigma\text{-algebras are independent.} \end{aligned}$$

This shows that (1) holds on E_0 as required. $\square$

Lemma 5.3.4: Let (Ω_0, A_0, P_0) be a probability space. Let $E \subseteq A_0$ be a σ -algebra such that $E = \sigma(A_1, \dots, A_n)$ where $A_1, \dots, A_n$ are independent σ -algebras. Let $B_1, \dots, B_n$ be sets in A_0 such that $\sigma(B_1, A_1), \dots, \sigma(B_n, A_n)$ are independent. Then we have

$$P_0\left(\bigcap_{i=1}^n B_i \mid E\right) = \prod_{i=1}^n P_0(B_i | A_i) \quad P_0\text{-a.s.}$$

proof: Since $\prod_{i=1}^n P_0(B_i | A_i)$ is E -measurable it follows, from Lemma

5.3.2, that it is sufficient to prove

$$\int_{A_1 \cap \dots \cap A_n} \prod_{i=1}^n P_0(B_i | A_i) dP_0 = P_0\left(\bigcap_{i=1}^n B_i \cap \bigcap_{i=1}^n A_i\right)$$

for each collection $A_1, \dots, A_n$ where $A_i \in \mathcal{A}_i$, $i=1, \dots, n$.

$$\begin{aligned} \int_{A_1 \cap \dots \cap A_n} \prod_{i=1}^n P_0(B_i | A_i) dP_0 &= \int \prod_{i=1}^n I_{A_i} \cdot \prod_{i=1}^n E(I_{B_i} | A_i) dP_0 \\ &= \int \prod_{i=1}^n E(I_{A_i \cap B_i} | A_i) dP_0 \quad \text{since } A_i \in \mathcal{A}_i \\ &= E\left(\prod_{i=1}^n E(I_{A_i \cap B_i} | A_i)\right) \\ &= \prod_{i=1}^n E(E(I_{A_i \cap B_i} | A_i)) \quad \text{since the } A_i \text{'s} \\ &\quad \text{are independent} \\ &= \prod_{i=1}^n P_0(A_i \cap B_i) \\ &= P_0\left(\bigcap_{i=1}^n (A_i \cap B_i)\right) \quad \text{since } \sigma(B_1, A_1), \dots, \\ &\quad \sigma(B_n, A_n) \text{ are} \\ &\quad \text{independent} \\ &= P_0\left(\bigcap_{i=1}^n A_i \cap \bigcap_{i=1}^n B_i\right) \quad \text{as required. } \square \end{aligned}$$

Recall that we defined $\mathcal{A} = \sigma(\xi_{ij}, (i,j) \in W)$ and

$\mathcal{A}_t = \sigma([\xi_{ij} \leq s], 0 \leq s \leq t, (i,j) \in W)$. We established that

$\mathcal{A}_t = \sigma(X(s), 0 \leq s \leq t)$ in Theorem 5.3.1(iii). We now show that

$$\sigma(X(t)) = \mathcal{A}_t^0 = \sigma([\xi_{ij} \leq t], (i,j) \in W).$$

Lemma 5.3.5: For the Cantor set process $X(t)$ we have

$$\sigma(X(t)) = A_t^0 = \sigma([\xi_{ij} \leq t], (i,j) \in W).$$

proof: $A_t^0 \subseteq \sigma(X(t))$ since $[\xi_{ij} > t] = [X(t) \geq I_{ij}]$
 $= [X(t) \in \{F | F \geq I_{ij}\}]$

and since $\{F | F \geq I_{ij}\}$ is closed in C it follows that $[\xi_{ij} > t] \in \sigma(X(t))$.

To show the opposite inclusion note that $\sigma(X(t)) = \sigma([X(t) \in B], B$ an open ball). If B is an open ball of C then $B = C_{N,A}$ for some N and some $A \in L_N$.

Therefore

$$\begin{aligned} [X(t) \in B] &= [X(t) \in C_{N,A}] \\ &= [T_N(X(t)) = A] \\ &= \bigcap_{i=1}^N \bigcap_{j=1}^{2^{i-1}} A_{ijt} \quad \text{where } A_{ijt} = \begin{cases} [\xi_{ij} \leq t] & \text{if } (i,j) \in \langle A \rangle \\ [\xi_{ij} > t] & \text{if } (i,j) \notin \langle A \rangle. \end{cases} \end{aligned}$$

This shows that $[X(t) \in B] \in A_t^0$ which completes the proof. $\square$

Theorem 5.3.2: The Cantor set process $X(t)$ is Markov.

proof: We need to show that, for every future event $A \in \sigma(X(t), t \geq s)$, we have the relation $P(A|A_s) = P(A|A_s^0)$. We first consider future events of the form $[X(t) \in B]$ where $t \geq s$ and B is an open ball in C . Then there exists N and $A \in L_N$ such that $B = C_{N,A}$.

Therefore $P([X(t) \in B] | A_s)$

$$\begin{aligned} &= P([X(t) \in C_{N,A}] | A_s) = P([T_N(X(t)) = A] | A_s) \\ &= P\left(\bigcap_{(i,j) \in W_N} A_{ijt} \mid A_s\right) \\ &= E(Z | A_s) \end{aligned}$$

where

$$W_N = \{(i, j) \mid i=1, \dots, N, j=1, \dots, 2^{i-1}\}$$

$$A_{ij,t} = \begin{cases} [\xi_{ij} \leq t] & \text{if } (i, j) \in \langle A \rangle \\ [\xi_{ij} > t] & \text{if } (i, j) \notin \langle A \rangle \end{cases}$$

$$\text{and } Z = \prod_{(i,j) \in W_N} A_{ij,t} = \prod_{(i,j) \in W_N} I_{A_{ij,t}}.$$

Now define $A_s(N^-) = \sigma([\xi_{ij} \leq u], 0 \leq u \leq s, (i, j) \in W_N)$ and

$A_s(N^+) = \sigma([\xi_{ij} \leq u], 0 \leq u \leq s, (i, j) \in (W \setminus W_N))$.

Then $A_s = \sigma(A_s(N^-), A_s(N^+))$. Now clearly $\sigma(Z, A_s(N^-)) \subseteq \sigma(\xi_{ij}, (i, j) \in W_N)$ while $A_s(N^+) \subseteq \sigma(\xi_{ij}, (i, j) \in W \setminus W_N)$. From the independence of the ξ_{ij} 's it follows that $\sigma(Z, A_s(N^-))$ and $A_s(N^+)$ are independent σ -algebras. Applying Lemma 5.3.3 we obtain

$$P([X(t) \in B] | A_s) = E(Z | A_s) = E(Z | A_s(N^-))$$

$$= P\left(\prod_{(i,j) \in W_N} A_{ij,t} \mid A_s(N^-)\right).$$

claim: $P\left(\prod_{(i,j) \in W_N} A_{ij,t} \mid A_s(N^-)\right) = \prod_{(i,j) \in W_N} P(A_{ij,t} | A_{ijs})$

where $A_{ijs} = \sigma([\xi_{ij} \leq u], 0 \leq u \leq s)$.

proof of claim: Note that $A_s(N^-) = \sigma(A_{ijs}, (i, j) \in W_N)$ and the

σ -algebras $\sigma(A_{ij,t}, A_{ijs}), (i, j) \in W_N$, are independent. Thus the

result follows by applying Lemma 5.3.4. Therefore we obtain:

$$P([X(t) \in B] | \bar{A}_s) = P([X(t) \in \bar{C}_{N,A}] | A_s(N^-)) = \prod_{(i,j) \in W_N} P(A_{ijt} | A_{ijs}). \quad (1)$$

We can obtain an analogous representation for $P([X(t) \in B] | A_s^0)$:
 Letting $A_s^0(N^-) = \sigma([\xi_{ij} \leq s], (i,j) \in W_N)$, $A_s^0(N^+) = \sigma([\xi_{ij} \leq s], (i,j) \in W \setminus W_N)$, and $A_{ijs}^0 = \sigma([\xi_{ij} \leq s])$ then, applying Lemmas 5.3.3 and 5.3.4, we obtain

$$P([X(t) \in B] | A_s^0) = P([X(t) \in \bar{C}_{N,A}] | A_s^0(N^-)) = \prod_{(i,j) \in W_N} P(A_{ijt} | A_{ijs}^0). \quad (2)$$

In view of (1) and (2) we need only show $P(A_{ijt} | A_{ijs}) = P(A_{ijt} | A_{ijs}^0)$ P-a.s. for each (i,j) . Since $A_{ijs}^0 = \sigma([\xi_{ij} \leq s]) = \{\emptyset, \Omega, [\xi_{ij} \leq s], [\xi_{ij} > s]\}$ we must have

$$P(A_{ijt} | A_{ijs}^0) = P(A_{ijt} | [\xi_{ij} \leq s]) I_{[\xi_{ij} \leq s]} + P(A_{ijt} | [\xi_{ij} > s]) I_{[\xi_{ij} > s]} \quad \text{P-a.s.} \quad (3)$$

where we define $P(A_{ijt} | [\xi_{ij} \leq s])$ arbitrarily if $P([\xi_{ij} \leq s]) = 0$.

(Note this will only happen if $s=0$ under the assumption that ξ_{ij} is exponentially-distributed but in fact this proof of the Markov

property is valid without any distributional assumptions on the

ξ_{ij} 's; with this in mind we also define $P(A_{ijt} | [\xi_{ij} > s])$ arbitrarily

if $P([\xi_{ij} > s]) = 0$ where the ξ_{ij} 's are any nonnegative, independently-distributed random variables.)

Since $P(A_{ijt} | A_{ijs}^0)$ is A_{ijs} -measurable it is enough to show

$$\int_B P(A_{ijt} | A_{ijs}^0) dP = P(A_{ijt} \cap B)$$

for each B belonging to a π -system $\mathcal{E}_0$ generating A_{ijs} and satisfying

$\Omega \in \mathcal{E}_0$. Take $\mathcal{E}_0 = \{[\xi_{ij} \leq u], 0 \leq u \leq s\} \cup \{\Omega\}$. Let

$B = [\xi_{ij} \leq u]$ for some $0 \leq u \leq s$ and first suppose $A_{ijt} = [\xi_{ij} \leq t]$.

(Using (3) we obtain

$$\begin{aligned} \int_{[\xi_{ij} \leq u]} P(A_{ijt} | A_{ijs}^0) dP &= \int_{[\xi_{ij} \leq u]} P([\xi_{ij} \leq t] | [\xi_{ij} \leq s]) dP \\ &\quad \text{since } 0 \leq u \leq s \\ &= P([\xi_{ij} \leq u]) \quad \text{since } 0 \leq u \leq s \leq t \\ &= P([\xi_{ij} \leq t] \cap [\xi_{ij} \leq u]) \quad \text{since } 0 \leq u \leq s \leq t \\ &= P(A_{ijt} \cap [\xi_{ij} \leq u]). \end{aligned}$$

Now suppose $A_{ijt} = [\xi_{ij} > t]$. Since $s \leq t$ it follows that

$P([\xi_{ij} > t] | [\xi_{ij} \leq s]) = 0$ (or $P([\xi_{ij} \leq s]) = 0$) and as

$[\xi_{ij} \leq u] \subseteq [\xi_{ij} \leq s]$ we must have

$$\begin{aligned} \int_{[\xi_{ij} \leq u]} P([\xi_{ij} > t] | A_{ijs}^0) dP &= 0 = P([\xi_{ij} > t] \cap [\xi_{ij} \leq u]) \quad \text{as } u \leq t \\ &= P(A_{ijt} \cap [\xi_{ij} \leq u]). \end{aligned}$$

Since $\Omega \in A_{ijs}^0$ we immediately have (for all A_{ijt})

$$\int_{\Omega} P(A_{ijt} | A_{ijs}^0) dP = P(A_{ijt}) = P(A_{ijt} \cap \Omega).$$

Therefore $P(A_{ijt} | A_{ijs}^0)$ is a version of $P(A_{ijt} | A_{ijs})$. We can

thus conclude that for open balls B we have $P([X(t) \in B] | A_s)$

$= P([X(t) \in B] | A_s^0)$. We now note that the collection of open balls

of C is a π -system; in particular, two open balls of C are either

disjoint or one of them is a subset of the other (in which case their

intersection is the smaller of the two). It is easily verified (using

linearity and the conditional monotone convergence theorem) that the collection of Borel sets H for which the property

$$P([X(t) \in H] | A_s) = P([X(t) \in H] | A_s^0) \text{ holds for } t \geq s \text{ is a d-system.}$$

Thus Dynkin's lemma implies $P([X(t) \in H] | A_s) = P([X(t) \in H] | A_s^0)$ for all $H \in \mathcal{B}(C)$. This is sufficient to ensure $P(A | A_s) = P(A | A_s^0)$

P -a.s. for all $A \in \sigma(X(t), t \geq s)$. Therefore the Cantor set process $X(t)$ is Markov. $\square$

As noted in the text of the proof of Theorem 5.3.2 the Markov property holds without any distributional assumptions (other than nonnegativity and independence) on the ξ_{ij} 's. In particular, it is not necessary that the ξ_{ij} 's be exponential. The distribution of the ξ_{ij} 's becomes important when we examine the transition functions of the process. $X(t)$ will be time-homogenous (that is, have stationary transitions) iff each ξ_{ij} is exponentially-distributed. We will now develop the transition functions of the process under the assumption that ξ_{ij} is exponential with parameter $\lambda_{ij} > 0$.

As before, for each $A \in C$ we will use the notation

$$A_{ijt} = \begin{cases} [\xi_{ij} > t] & \text{if } (i,j) \in \langle A \rangle \\ [\xi_{ij} \leq t] & \text{if } (i,j) \notin \langle A \rangle \end{cases}$$

For each $F \in C$ and $t \geq 0$ we will define a probability measure

$P_t^{(N)}(F, \cdot)$ on C_N , the σ -algebra of N -cylinders, in the following way:

Let $s > 0$. For each atom $C_{N,A}$ of C_N define $P_t^{(N)}(F, C_{N,A})$ by

$$P_t^{(N)}(F, C_{N,A}) = P([T_N(X(t+s)) = A] | [T_N(X(s)) = T_N(F)]) \quad (4)$$

$$= P\left(\bigcap_{(i,j) \in W_N} A_{ijt+s} \mid \bigcap_{(i,j) \in W_N} F_{ijs}\right) \quad (5)$$

$$= \prod_{(i,j) \in W_N} P(A_{ijt+s} | F_{ijs}) \quad (6)$$

This last equality follows from the independence of the ξ_{ij} 's. These are well-defined conditional probabilities since for $s > 0$ the events $[\xi_{ij} > s]$, $[\xi_{ij} \leq s]$ have positive probability. The definition of $P_t^{(N)}(F, C_{N,A})$ is also independent of the value of $s > 0$ due to the no-memory property of the exponential distribution. In particular we have the following cases:

- (i) Suppose $(i,j) \in \langle F \rangle$. (Equivalently, $I_{ij} \subseteq F^C$). If $(i,j) \notin \langle A \rangle$ (that is, $I_{ij} \subseteq A$) then $P(A_{ijt+s} | F_{ijs}) = P([\xi_{ij} > t+s] | [\xi_{ij} \leq s]) = 0$. This simply illustrates the fact that once an interval is removed it is never replaced. If $(i,j) \in \langle A \rangle$ then $P(A_{ijt+s} | F_{ijs}) = P([\xi_{ij} \leq t+s] | [\xi_{ij} \leq s]) = 1$.
- (ii) Suppose $(i,j) \notin \langle F \rangle$, so $I_{ij} \subseteq F$. If $(i,j) \notin \langle A \rangle$ then $P(A_{ijt+s} | F_{ijs}) = P([\xi_{ij} > t+s] | [\xi_{ij} > s]) = P([\xi_{ij} > t]) = e^{-\lambda_{ij}t}$ since $\xi_{ij} \sim \exp(\lambda_{ij})$. If $(i,j) \in \langle A \rangle$ then $P(A_{ijt+s} | F_{ijs}) = P([\xi_{ij} \leq t+s] | [\xi_{ij} > s]) = P([\xi_{ij} \leq t]) = 1 - e^{-\lambda_{ij}t}$.

Note we can express $P_t^{(N)}(F, C_{N,A})$ as:

$$P_t^{(N)}(F, C_{N,A}) = \prod_{(i,j) \in W_N \setminus \langle F \rangle} P(A_{ijt+s} | [\xi_{ij} > s]) \prod_{(i,j) \in W_N \cap \langle F \rangle} P(A_{ijt+s} | [\xi_{ij} \leq s]) \quad (7)$$

and the last product on the right is clearly 0 or 1 according as

$A \not\subseteq T_N(F)$ or $A \subseteq T_N(F)$. This leads to the following alternate

representation of $P_t^{(N)}(F, C_{N,A})$:

$$P_t^{(N)}(F, C_{N,A}) = \begin{cases} 0 & \text{if } A \not\subseteq T_N(F) \\ \prod_{(i,j) \in W_N \setminus \langle F \rangle} P(A_{ij} | t+s | [\xi_{ij} > s]) & \text{if } A \subseteq T_N(F) \end{cases}$$

where this last product is taken to be 1 if $A \subseteq T_N(F)$ and $W_N \setminus \langle F \rangle = \emptyset$

(in which case $A = T_N(F) = \bigcap_{(i,j) \in W_N} I_{ij}^C$.)

Then, for $H \in C_N$, define $P_t^{(N)}(F, H) = \sum_{\substack{C_{N,A} \subseteq H \\ A \in L_N}} P_t^{(N)}(F, C_{N,A})$,

summing over the atoms of H . It follows that $P_t^{(N)}(F, \cdot)$ is a probability measure on C_N after noting that

$$\begin{aligned} P_t^{(N)}(F, C) &= \sum_{A \in L_N} P_t^{(N)}(F, C_{N,A}) \\ &= \sum_{A \in L_N} P([T_N(X(t+s)) = A] | [T_N(X(s)) = T_N(F)]) \\ &= P(\bigcup_{A \in L_N} [T_N(X(t+s)) = A] | [T_N(X(s)) = T_N(F)]) \\ &= P(\Omega | [T_N(X(s)) = T_N(F)]) \\ &= 1. \end{aligned}$$

Also for $t=0$ we have $P_0^{(N)}(F, \cdot) = \delta_F(\cdot)$ where

$$\delta_F(H) = \begin{cases} 1 & F \in H \\ 0 & F \notin H \end{cases} \quad \text{the unit mass at } F.$$

Lemma 5.3.6: $P_t^{(N)}(\cdot, H)$ is a measurable (in fact continuous) function on C for each fixed $t \geq 0$ and $H \in C_N$.

proof: $P_t^{(N)}(\cdot, C_{N,A}) = P([T_N(X(t+s)) = A] | [T_N(X(s)) = T_N(\cdot)])$. Thus

$P_t^{(N)}(\cdot, C_{N,A})$ is a function of $T_N(F)$ and therefore is a truncated (hence continuous) function (see Theorem 5.2.2 and the discussion following it.) For $H \in C_N$, $P_t^{(N)}(\cdot, H)$ is a finite sum of certain $P_t^{(N)}(\cdot, C_{N,A})$ and therefore is also continuous. $\square$

Lemma 5.3.7: The probabilities $P_t^{(N)}(F, \cdot)$ are consistent in N . That is, if $H \in C_N$ then $P_t^{(N)}(F, H) = P_t^{(N+1)}(F, H)$.

proof: It is enough to show that $P_t^{(N)}(F, C_{N,A}) = P_t^{(N+1)}(F, C_{N,A})$ for each $A \in L_N$. Now $C_{N,A} = \bigcup_{A' \in L_{N+1}} C_{N+1,A'}$ and so by definition we have

$$T_N(A') = A$$

$$\begin{aligned}
 P_t^{(N+1)}(F, C_{N,A}) &= \sum_{\substack{A' \in L_{N+1} \\ T_N(A') = A}} P_t^{(N+1)}(F, C_{N+1,A'}) \\
 &= \sum_{\substack{A' \in L_{N+1} \\ T_N(A') = A}} P([T_{N+1}(X(t+s)) = A'] | [T_{N+1}(X(s)) = T_{N+1}(F)]) \\
 &= P(\bigcup_{\substack{A' \in L_{N+1} \\ T_N(A') = A}} [T_{N+1}(X(t+s)) = A'] | [T_{N+1}(X(s)) = T_{N+1}(F)]) \\
 &= P([T_N(X(t+s)) = A] | [T_{N+1}(X(s)) = T_{N+1}(F)]) \\
 &= P(\bigcap_{(i,j) \in W_N} A_{ij,t+s} | \bigcap_{(i,j) \in W_{N+1}} F_{ijs}) \\
 &= P(\bigcap_{(i,j) \in W_N} A_{ij,t+s} | \bigcap_{(i,j) \in W_N} F_{ijs}) \text{ from independence of the } F_{ij}'\text{'s} \\
 &= P_t^{(N)}(F, C_{N,A}) \text{ as required. } \square
 \end{aligned}$$

We now define $P_t(F, \cdot)$ on C_F , the algebra of finite-dimensional

cylinders, by

$$P_t(F, H) = P_t^{(N)}(F, H) \quad \text{if } H \in C_N.$$

In view of Lemma 5.3.7 there is no ambiguity in this definition.

We wish to show $P_t(F, \cdot)$ has a unique extension to a probability measure on the Borel sets of C .

Lemma 5.3.8: $P_t(F, \cdot)$ is finitely-additive on C_F .

proof: Suppose $H = \bigcup_{k=1}^n H_k$ where $H_1, \dots, H_k \in C_F$. Then there exists

N such that $H_1, \dots, H_n \in C_N$ and therefore:

$$\begin{aligned} P_t(F, H) &= P_t^{(N)}(F, H) \\ &= P_t^{(N)}\left(F, \bigcup_{k=1}^n H_k\right) \\ &= \sum_{k=1}^n P_t^{(N)}(F, H_k) \quad \text{since } P_t^{(N)}(F, \cdot) \text{ is a probability measure} \\ &\quad \text{on } C_N \\ &= \sum_{k=1}^n P_t(F, H_k) \quad \text{as required. } \square \end{aligned}$$

Theorem 5.3.3: (i) $P_t(F, \cdot)$ has a unique extension to a probability measure (also denoted by $P_t(F, \cdot)$) on the Borel sets of C .

(ii) The extension $P_t(\cdot, H)$ is a measurable function on C for each fixed $H \in B(C)$.

proof: (i). In view of Lemma 5.3.8 we need only show that if $H_n \in C_F$ for every n and $H_n \rightarrow \emptyset$ then $\lim_{n \rightarrow \infty} P_t(F, H_n) = 0$. (We can then apply the Carathéodory extension theorem for measures to obtain $P_t(F, \cdot)$ on $B(C)$.) But $H_n \in C_F$ implies H_n is compact in C since H_n is a union of finitely-many closed (hence compact) balls $C_{N,A}$ for some N .

Therefore $H_n \neq \emptyset$, $H_n \supseteq H_{n+1}$ for every n implies $\bigcap_n H_n \neq \emptyset$. Thus

the necessary condition is vacuously true.

(ii) If $H \in C_F$ then $P_t(\cdot, H) = P_t^{(N)}(\cdot, H)$ for some N so $P_t(\cdot, H)$ is measurable by Lemma 5.3.6. Therefore the collection $N = \{H \in B(C) \mid P_t(\cdot, H) \text{ is measurable}\}$ contains the algebra C_F and to show $N = B(C)$ we need only prove that N is a monotone class.

Let $H_n \uparrow H$ where $H_n \in N$ for every n . Then we have $\lim_n P_t(F, H_n) = P_t(F, H)$ for each $F \in C$ so $P_t(\cdot, H_n) \rightarrow P_t(\cdot, H)$ pointwise and hence $P_t(\cdot, H)$ is measurable (that is, $H \in N$). An analogous argument holds for the case $H_n \downarrow H$. Therefore $N = B(C)$. $\square$

The only remaining step in showing that the family $P_t(F, H)$ is a system of transition functions for some Markov process in C is to verify the Chapman-Kolmogorov equations.

Theorem 5.3.4: The family $P_t(F, H)$ satisfies the Chapman-Kolmogorov equations.

That is, we have

$$P_{t+u}(F, H) = \int_C P_u(G, H) P_t(F, dG) \quad (8)$$

whenever $t, u \geq 0$, $F \in C$, and $H \in B(C)$.

proof: The RHS of (8) is clearly a probability measure for fixed t, u , and F so it is enough to verify (8) for Borel sets H which are open balls. (As noted in the proof of Theorem 5.3.2 the open balls of C form a π -system containing C and generating $B(C)$. Hence they constitute a determining class for probability measures on $B(C)$.)

Since $P_0(F, H) = \delta_F(H)$ relation (8) is easily verified if $t=0$ or $u=0$. Therefore we can assume $t>0, u>0$. Consider the open ball $C_{N,B}$

for some N and $B \in L_N$.

Note that we can write

$$P_u(G, C_{N,B}) = P([T_N(X(u+s)) = B] | [T_N(X(s)) = T_N(G)])$$

$$= \sum_{A \in L_N} P([T_N(X(u+s)) = B] | [T_N(X(s)) = A]) I_{C_{N,A}}(G)$$

$$\text{where } I_{C_{N,A}}(G) = \begin{cases} 1 & \text{if } G \in C_{N,A} \text{ (equivalently, } T_N(G) = A) \\ 0 & \text{if } G \notin C_{N,A} \text{ (equivalently, } T_N(G) \neq A) \end{cases}$$

Thus we obtain

$$\begin{aligned} & \int P_u(G, C_{N,B}) P_t(F, dG) \\ &= \int \sum_{A \in L_N} P([T_N(X(u+s)) = B] | [T_N(X(s)) = A]) I_{C_{N,A}}(G) P_t(F, dG) \\ &= \sum_{A \in L_N} P([T_N(X(u+s)) = B] | [T_N(X(s)) = A]) \int_{C_{N,A}} P_t(F, dG) \\ &= \sum_{A \in L_N} P([T_N(X(u+s)) = B] | [T_N(X(s)) = A]) P_t(F, C_{N,A}) \\ &= \sum_{A \in L_N} P([T_N(X(u+t+s)) = B] | [T_N(X(t+s)) = A]) \\ &\quad \cdot P([T_N(X(t+s)) = A] | [T_N(X(s)) = T_N(F)]) \quad (9) \end{aligned}$$

At this point we note that the truncated process $T_N(X(t))$ is also Markov. This follows immediately from Theorem 5.3.2, choosing $\xi_{ij} \equiv \infty$ for $i \geq N+1$. Thus (9) becomes

$$\begin{aligned} &= \sum_{A \in L_N} P([T_N(X(u+t+s)) = B] | [T_N(X(t+s)) = A, T_N(X(s)) = T_N(F)]) \\ &\quad \cdot P([T_N(X(t+s)) = A] | [T_N(X(s)) = T_N(F)]) \quad \text{from Markov property} \\ &= \sum_{A \in L_N} P([T_N(X(u+t+s)) = B, T_N(X(t+s)) = A] | [T_N(X(s)) = T_N(F)]) \\ &\quad \text{using } P(A \cap B | C) = P(A | C) P(B | C) \\ &= P([T_N(X(t+u+s)) = B] | [T_N(X(s)) = T_N(F)]) \\ &= P_{t+u}(F, C_{N,B}) \quad \text{as required. } \square \end{aligned}$$

Note that the Chapman-Kolmogorov equations for open balls can be expressed as

$$\begin{aligned}
 & P_u(G, C_{N,B}) P_t(F, dG) \\
 &= \int \sum_{A \in L_N} P_u(A, C_{N,B}) I_{C_{N,A}}(G) P_t(F, dG) \\
 &= \sum_{A \in L_N} P_u(A, C_{N,B}) P_t(F, C_{N,A}) \\
 &= \sum_{A \in L_N} P_t(F, C_{N,A}) P_u(A, C_{N,B}) \\
 &= \sum_{A \in L_N} Q_t^{(N)}(F', A) Q_u^{(N)}(A, B) \\
 &= Q_{t+u}^{(N)}(F', B)
 \end{aligned}$$

where $F' = T_N(F)$ and $Q_t^{(N)}(A, B)$ is the discrete transition probability corresponding to the truncated process $T_N(X(t))$. This leads to an alternate method of constructing the Cantor set process. We could have begun with a family of consistent discrete Markov transition functions $Q_t^{(N)}(A, B)$ on C_N for each N and defined $P_t(F, \cdot)$ on the finite-dimensional cylinders by $P_t(F, C_{N,A}) = Q_t^{(N)}(T_N(F), A)$. $P_t(F, \cdot)$ is then easily extended to a probability measure on the Borel sets of C , as in Theorem 5.3.3(i). Since C is a compact metric space the Kolmogorov extension theorem applies and gives the existence of a time-homogenous Markov process $\{Y(t)\}$ with state space C and transition functions P_t . In fact this approach also establishes the existence of a cadlag version of $Y(t)$; in the following section we show that the transition functions P_t are normal (that is, they are Feller and uniformly stochastically continuous). We can then apply Theorem 1 on page 189 of Lamperti [13] to conclude the existence of a cadlag version.

We make note of the fact that we have established that the P_t are Markov transition functions with the following theorem:

Theorem 5.3.5: The collection $\{P_t\}_{t \geq 0}$ are Markov transition functions on C . That is, they possess the following properties:

- (i) $P_t(F, \cdot)$ is a probability measure on $B(C)$ for each fixed $F \in C$.
- (ii) $P_t(\cdot, H)$ is a measurable function on C for each fixed $H \in B(C)$.
- (iii) $P_0(F, H) = \delta_F(H)$ the unit mass at F .
- (iv) $P_{t+u}(F, H) = \int_C P_u(G, H) P_t(F, dG)$ for all $u, t \geq 0$, $F \in C$, and $H \in B(C)$

(Chapman-Kolmogorov equations hold). $\square$

We now verify that the transition functions P_t are in fact the transition functions of the original process $X(t)$. This requires that we establish the relation

$$\begin{aligned} & P([X(t_1) \in H_1, \dots, X(t_n) \in H_n]) \\ &= \int_{F_0 \in C} \int_{F_1 \in H_1} \dots \int_{F_{n-2} \in H_{n-2}} \int_{F_{n-1} \in H_{n-1}} \mu_0(dF_0) P_{t_1}(F_0, dF_1) \\ & \quad P_{t_2-t_1}(F_1, dF_2) \dots P_{t_{n-1}-t_{n-2}}(F_{n-2}, dF_{n-1}) P_{t_n-t_{n-1}}(F_{n-1}, H_n) \quad (10) \end{aligned}$$

whenever $0 \leq t_1 \leq \dots \leq t_n$, $H_1, \dots, H_n \in B(C)$. μ_0 denotes the initial distribution of the process. That is, $\mu_0 = PX(0)^{-1}$. In this case $\mu_0 = \delta_I$, the unit mass at $I = [0, 1]$, since $X(0) = I$ P-a.s. under the assumption that the ξ_{ij} 's are exponentially distributed.

Lemma 5.3.9: For all $t, s \geq 0$ and $H \in B(C)$, $P_t(F, H)$ is a version of the conditional probability $P([X(t+s) \in H] | [X(s) = F])$.

proof: Let $\mu_s = PX(s)^{-1}$, the marginal distribution of $X(s)$. We need to verify the relation

$$\int_S P_t(F, H) \mu_s(dF) = P([X(t+s) \in H, X(s) \in S]) \text{ for each Borel set } S. \quad (11)$$

Equivalently, the joint distribution $\mu_{s+t, s}$ can be factored as

$$\mu_{s, s+t}(dF_0, dF_1) = P_t(F_0, dF_1) \mu_s(dF_0) \quad (12)$$

By definition $P_t(F, C_{N, B}) = P([X(t+s) \in C_{N, B}] | [X(s) \in C_{N, T_N(F)}])$ for all $s > 0$. By finite additivity this relation extends to sets

$H \in C_F$. That is, if H is a finite-dimensional cylinder we obtain

$$P_t(F, H) = P([X(t+s) \in H] | [X(s) \in C_{N, T_N(F)}]) \text{ when } H \in C_N. \quad (13)$$

The conditional probability in (13) is well-defined for $s > 0$.

In the case $s = 0$ we can still use expression (13) (even though the conditional probability is undefined for $F \neq I$) since the expression holds μ_0 -a.s. (We have $X(0) = I$ P-a.s.)

We will verify relation (11) when H is a finite-dimensional cylinder. As the open balls of C are a π -system generating $B(C)$ it is sufficient to verify (11) for Borel sets S which are open balls.

Therefore, let S be an open ball; $H \in C_F$. Then there exists N such that $H \in C_N$ and $S \in C_N$ so we can write

$$\begin{aligned} & \int P_t(F, H) \mu_s(dF) \\ &= \int_S P([X(t+s) \in H] | [X(s) \in C_{N, T_N(F)}]) \mu_s(dF) \quad \text{using (13)} \\ &= \int \sum_{A \in L_N} P([X(t+s) \in H] | [X(s) \in C_{N, A}]) I_{C_{N, A}}(F) \mu_s(dF) \\ &= \sum_{A \in L_N} P([X(t+s) \in H] | [X(s) \in C_{N, A}]) \mu_s(S \cap C_{N, A}) \\ &= \sum_{A \in L_N} P([X(t+s) \in H] | [X(s) \in C_{N, A}]) \mu_s(C_{N, A}) \quad \text{since } S \text{ is a finite union of atoms of } C_N \\ & \quad C_{N, A} \subseteq S. \end{aligned}$$

$$\begin{aligned}
&= \sum_{A \in L_N} P([X(t+s) \in H] | [X(s) \in C_{N,A}]) P([X(s) \in C_{N,A}]) \\
&\quad C_{N,A} \subseteq S \\
&= \sum_{A \in L_N} P([X(t+s) \in H] \cap [X(s) \in C_{N,A}]) \\
&\quad C_{N,A} \subseteq S \\
&= P([X(t+s) \in H, X(s) \in S]).
\end{aligned}$$

Thus the lemma holds for finite-dimensional cylinders H . It is easy to show that the collection of sets for which the lemma holds is a monotone class and hence coincides with the Borel sets of C . This completes the proof. $\square$

Theorem 5.3.6: The transition functions $\{P_t\}$, $t \geq 0$, are the transition functions of $X(t)$.

proof: For $n=1$ we obtain relation (10) by applying the preceding lemma (in particular use (12) with $s=0$). That is, we have

$$\begin{aligned}
P([X(t_1) \in H_1]) &= P([X(t_1) \in H_1, X(0) \in C]) \\
&= \int_{F_0 \in C} \int_{F_1 \in H_1} \mu_{0,t_1}(dF_0, dF_1) \\
&= \int_{F_0 \in C} \int_{F_1 \in H_1} P_{t_1}(F_0, dF_1) \mu_0(dF_0) = \int_C P_{t_1}(F_0, H_1) \mu_0(dF_0)
\end{aligned}$$

For $n=2$, $t_1 < t_2$, we obtain,

$$\begin{aligned}
P([X(t_2) \in H_2, X(t_1) \in H_1]) &= P([X(t_2) \in H_2, X(t_1) \in H_1, X(0) \in C]) \\
&= \int_{F_0 \in C} \int_{F_1 \in H_1} P([X(t_2) \in H_2] | [X(t_1) = F_1, X(0) = F_0]) \mu_{0,t_1}(dF_0, dF_1) \\
&= \int_{F_0 \in C} \int_{F_1 \in H_1} P([X(t_2) \in H_2] | [X(t_1) = F_1]) \mu_{0,t_1}(dF_0, dF_1)
\end{aligned}$$

using Markov property

$$= \int_{F_0 \in C} \int_{F_1 \in H_1} P_{t_2-t_1}(F_1, H_2) P_{t_1}(F_0, dF_1) \mu_0(dF_0)$$

using Lemma 5.3.9 and $\mu_{0,t_1}(dF_0, dF_1) = P_{t_1}(F_0, dF_1) \mu_0(dF_0)$.

Noting that this shows $\mu_{0,t_1,t_2}(dF_0, dF_1, dF_2) = P_{t_2-t_1}(F_1, dF_2)$

$P_{t_1}(F_0, dF_1) \mu_0(dF_0)$ the general result follows by induction on n . $\square$

4. Semigroup Analysis And Behavior Of The Process

In this section we use semigroup theory, (see Chapter 7 of Lamperti [13] for a presentation of this topic) to describe the Cantor set process $X(t)$ in terms of its infinitesimal generator. To do this we classify $X(t)$ into one of two (disjoint) categories (1) the slow Cantor process (corresponding to $\sum \lambda_{ij} < \infty$), and (2) the fast Cantor process (occurring when $\sum \lambda_{ij} = \infty$). The second category can be subdivided further into two groups to produce an ultrafast Cantor process (although it is possible for the process to switch from simply fast to ultrafast) by putting further restrictions on the convergence behavior of $\sum \lambda_{ij}$. Therefore we begin by determining the effect of the convergence behavior of $\sum \lambda_{ij}$ on the behavior of $X(t)$. We assume $\lambda_{ij} > 0$ for all i, j .

As before we will let L_T denote the collection of terminating points of C . We introduce the notation L_∞ to denote the complement of L_T ; that is, L_∞ is the collection of points of C which do not terminate (equivalently, infinitely-many intervals are removed).

We will let L_∞^+ be the class of points of C from which all but finitely-many intervals have been removed. That is, $L_\infty^+ = \{F | W \setminus \langle F \rangle \text{ is finite}\}$ where $W = \{(i, j) | i=1, 2, \dots, j=1, \dots, 2^{i-1}\}$ as before. Note that $L_\infty^+ \subseteq L_\infty$.

and both L_T and L_∞^+ are countable sets while L_∞ is not.

We now show that the process is always in L_T (the slow Cantor process) or always in L_∞ (the fast Cantor process).

Theorem 5.4.1: For every $t > 0$ $P([X(t) \in L_T]) = 0$ or 1 . Furthermore $P([X(t) \in L_T \text{ for all } t > 0]) = 0$ or 1 according as $\sum \lambda_{ij} = \infty$ or $\sum \lambda_{ij} < \infty$.

proof: This follows from the Borel-Cantelli lemma. Note that

$\{X(t) \in L_T \text{ for all } t > 0\} = \bigcap_{t > 0} \{X(t) \in L_T\}$ is the intersection of a

decreasing ordered family of sets and $P([X(t) \in L_T \text{ for all } t > 0]) = \lim_{n \rightarrow \infty} P([X(t_n) \in L_T])$ where $t_n \uparrow \infty$.

Thus to prove the theorem it is sufficient to show that, for each $t > 0$, $P([X(t) \in L_T]) = 0$ iff $\sum \lambda_{ij} = \infty$ while $P([X(t) \in L_T]) = 1$ iff $\sum \lambda_{ij} < \infty$.

Note that $[X(t) \in L_T] = [A_{ij}(t) \text{ i.o.}]$ where $A_{ij}(t) = [\xi_{ij} \leq t]$. Since the ξ_{ij} 's are independent then, according to the Borel-Cantelli lemma, we have $P([X(t) \in L_T]) = 0$ if $\sum P([\xi_{ij} \leq t]) < \infty$ and $P([X(t) \in L_T]) = 1$ if $\sum P([\xi_{ij} \leq t]) = \infty$.

Now $P([\xi_{ij} \leq t]) = 1 - e^{-\lambda_{ij}t}$. If $0 < a_{ij} < 1$ then $\prod (1 - a_{ij})$ converges to a nonzero limit iff $\sum a_{ij} < \infty$. Letting

$a_{ij} = 1 - e^{-\lambda_{ij}t}$ then $\sum (1 - e^{-\lambda_{ij}t})$ converges iff $\prod e^{-\lambda_{ij}t} = e^{-\sum \lambda_{ij}t} > 0$ and clearly we have $e^{-\sum \lambda_{ij}t} > 0$ iff $\sum \lambda_{ij} < \infty$ (note this does not depend on t). Thus $P([X(t) \in L_T]) = 1$ if $\sum \lambda_{ij} < \infty$ and $P([X(t) \in L_T]) = 0$ if $\sum \lambda_{ij} = \infty$. $\square$

We can obtain a similar result which determines when $X(t)$ falls into L_{∞}^{+} . If $X(t) \in L_{\infty}^{+}$ we will refer to this as the ultrafast Cantor process. We will see that it is possible for $X(t)$ to switch from simply being fast (in this case meaning $X(t) \in (L_{\infty} \setminus L_{\infty}^{+})$) to being ultrafast ($X(t) \in L_{\infty}^{+}$) in a finite amount of time.

Theorem 5.4.2: If there exists $t_0 > 0$ such that $\sum e^{-\lambda_{ij} t_0} < \infty$ then $P([X(t) \in L_{\infty}^{+} \text{ for all } t \geq t_0]) = 1$. If $\sum e^{-\lambda_{ij} t_0} = \infty$ then $P([X(t) \in L_{\infty}^{+} \text{ for some } t \leq t_0]) = 0$.

proof: This also follows from the Borel-Cantelli lemma.

Note that $[X(t) \in L_{\infty}^{+} \text{ for all } t \geq t_0] = [X(t_0) \in L_{\infty}^{+}]$ since the process cannot travel backwards. Similarly $[X(t) \in L_{\infty}^{+} \text{ for some } t \leq t_0] = [X(t_0) \in L_{\infty}^{+}]$. Now $[X(t_0) \in L_{\infty}^{+}] = \liminf_{(i,j)} A_{ij}(t_0)$ where $A_{ij}(t_0) = [\xi_{ij} \leq t_0]$. Therefore $[X(t_0) \in L_{\infty}^{+}] = [A_{ij}^c(t_0) \text{ i.o.}]$ so $P([X(t_0) \in L_{\infty}^{+}]) = 0$ if $\sum P([\xi_{ij} > t_0]) < \infty$ and $P([X(t_0) \in L_{\infty}^{+}]) = 1$ if $\sum P([\xi_{ij} > t_0]) = \infty$ by Borel Cantelli lemma. Since $\sum P([\xi_{ij} > t_0]) = \sum e^{-\lambda_{ij} t_0}$ the result follows. $\square$

In general the convergence or divergence of the series $\sum e^{-\lambda_{ij} t}$ will depend on t as well as λ_{ij} . In order to obtain some practical criteria for convergence of the series we will consider the special case $\lambda_{ij} = \lambda_i > 0$ for all $j=1, \dots, 2^{i-1}$. The following theorem gives some convergence results in this case. Note we will have

$$\sum e^{-\lambda_{ij} t} = \sum_{i=1}^{\infty} 2^{i-1} e^{-\lambda_i t}$$

Theorem 5.4.3: (Convergence Results for $\sum_{i=1}^{\infty} 2^{i-1} e^{-\lambda_i t}$): Consider

the series $\sum_{i=1}^{\infty} 2^{i-1} e^{-\lambda_i t}$

(i) If $\lim_{i \rightarrow \infty} \frac{\lambda_i}{i} = 0$ then the series diverges for all $t > 0$.

(ii) If $\lim_{i \rightarrow \infty} \frac{\lambda_i}{i} = K > 0$ then the series converges for $t > \frac{\log 2}{K}$ and diverges for $t < \frac{\log 2}{K}$.

(iii) If $\lim_{i \rightarrow \infty} \frac{\lambda_i}{i \log i} = K > 0$ (K may be ∞) then the series converges for all $t > 0$.

proof: (i) $\sum_{i=1}^{\infty} 2^{i-1} e^{-\lambda_i t} = \frac{1}{2} \sum_{i=1}^{\infty} 2^i e^{-\frac{\lambda_i t}{i}} = \frac{1}{2} \sum_{i=1}^{\infty} (2e^{-\frac{\lambda_i t}{i}})^i$

Since $\lim_{i \rightarrow \infty} 2e^{-\frac{\lambda_i t}{i}} = 2$ there exists $i_0 = i_0(t)$ such that $i \geq i_0$ implies

$2e^{-\frac{\lambda_i t}{i}} > 1$ and so by comparison with the geometric series the series must diverge for each $t > 0$.

(ii) $\sum_{i=1}^{\infty} 2^{i-1} e^{-\lambda_i t} = \frac{1}{2} \sum_{i=1}^{\infty} (2e^{-\frac{\lambda_i t}{i}})^i$

If $t > \frac{\log 2}{K}$ then $\lim_{i \rightarrow \infty} 2e^{-\frac{\lambda_i t}{i}} = 2e^{-Kt} < 2e^{-K(\frac{\log 2}{K})} = 1$.

Therefore eventually $2e^{-\frac{\lambda_i t}{i}}$ is bounded below and away from 1 and by comparison with the geometric series the series must converge.

If $t < \frac{\log 2}{K}$ then $\lim_{i \rightarrow \infty} 2e^{-\frac{\lambda_i t}{i}} = 2e^{-Kt} > 2e^{-K(\frac{\log 2}{K})} = 1$

and so the series will diverge.

(iii) There exists K' such that $0 < K' < \infty$ and i_0 such that $i \geq i_0$ implies $\frac{\lambda_i}{i \log i} \geq K'$. Then $i \geq i_0$ implies $\lambda_i \geq K' i \log i$ which then implies

$$\frac{1}{2} \sum_{i=i_0}^{\infty} 2^i e^{-\lambda_i t} \leq \frac{1}{2} \sum_{i=i_0}^{\infty} 2^i e^{-(K' i \log 2) t} = \frac{1}{2} \sum_{i=i_0}^{\infty} \left(\frac{2}{i^{K'}} \right)^i.$$

Since $\lim_{i \rightarrow \infty} \frac{2}{i^{K'}} = 0$ it follows that the series converges. $\square$

As an application of Theorems 5.4.2 and 5.4.3 consider the choice of $\lambda_i = i^\alpha$ where $\alpha \geq 0$. If $0 \leq \alpha < 1$ then the series $\sum 2^{i-1} e^{-i^\alpha t}$ diverges for all $t > 0$ and hence the process never reaches L_∞^+ . If $\alpha > 1$ then the series converges for all $t > 0$ and so the process jumps immediately into L_∞^+ . The interesting case occurs when $\alpha = 1$. Since $\lim_{i \rightarrow \infty} \frac{1}{i} = 0$ the series converges for $t > \log 2$ and diverges for $t < \log 2$. Thus the process hits L_∞^+ after $t = \log 2$.

We can therefore determine the limiting behavior of the process. Define the hitting time $T_C = \inf\{t > 0 | X(t) = C\}$ where we take $T_C = \infty$ if $\{t > 0 | X(t) = C\} = \emptyset$. Then T_C is the length of time to reach the Cantor set. (Note that the right-continuity of $X(t)$ implies $X(T_C) = C$.) We can also express T_C as $T_C = \sup_{i,j} \xi_{ij}$. Noting that $\{T_C \leq t\} = \{X(t) = C\} \in A_t$ (where $A_t = \sigma(X(s), 0 \leq s \leq t)$) then T_C is a stopping time relative to the σ -fields A_t , $t > 0$.

Theorem 5.4.4: $\lim_{t \rightarrow \infty} X(t) = C$ P-a.s. (This does not depend on the rates λ_{ij} provided $\lambda_{ij} > 0$ for all i, j .)

proof: For each ω $\{X(t, \omega)\}_{t \geq 0}$ is a decreasing family of sets. Setting $Y(\omega) = \bigcap_{t \geq 0} X(t, \omega)$, it follows that $\lim_{t \rightarrow \infty} X(t, \omega) = Y(\omega)$ in the compact topology on C . We will show that $Y=C$ P-a.s. by showing that $X(t)$ converges to C in probability.

$$P([d(X(t), C) < \frac{3}{2}^{-N}]) = P(\bigcap_{(i,j) \in W_N} [\xi_{ij} \leq t])$$

$$= \prod_{i=1}^N \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij}t})$$

$$\text{Therefore } \lim_{t \rightarrow \infty} P([d(X(t), C) < \frac{3}{2}^{-N}]) = \lim_{t \rightarrow \infty} \prod_{i=1}^N \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij}t_0}) = 1.$$

Thus $X(t) \xrightarrow{P} C$ and the result follows. $\square$

We would now like to show that the ultrafast Cantor process reaches C in a finite amount of time with probability one. We use the following lemma.

Lemma 5.4.1: If $\prod_{k=1}^{\infty} a_k = a > 0$ where $0 < a_k < 1$ for every k , then

$$\lim_{n \rightarrow \infty} \prod_{k=n+1}^{\infty} a_k = 1.$$

proof: The proof is elementary and will be omitted. $\square$

Theorem 5.4.5: $P([T_C < \infty]) = 1$ iff there exists t_0 such that $\sum e^{-\lambda_{ij}t_0} < \infty$. Otherwise $P([T_C < \infty]) = 0$.

proof: If $\sum e^{-\lambda_{ij}t} = \infty$ for all $t > 0$ then the process never reaches L_{∞}^+ (see Theorem 5.4.2) and so $P([T_C < \infty]) = 0$.

Now suppose $\sum e^{-\lambda_{ij}t_0} < \infty$. Then $\prod_{i=1}^{\infty} \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij}t_0}) > 0$

and so by Lemma 5.4.1 $\lim_{N \rightarrow \infty} \prod_{i=N+1}^{\infty} \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij}t_0}) = 1. \quad (1)$

Let $t \geq t_0$. Then for every $N \in \mathbb{Z}^+$ we have

$$P([T_C \leq t]) = P([X(t) = C]) = P\left(\bigcap_{i=1}^{\infty} \bigcap_{j=1}^{2^{i-1}} [\xi_{ij} \leq t]\right)$$

$$= \prod_{i=1}^{\infty} \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij} t})$$

$$= \prod_{i=1}^N \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij} t}) \cdot \prod_{i=N+1}^{\infty} \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij} t})$$

$$\geq \prod_{i=1}^N \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij} t}) \cdot \prod_{i=N+1}^{\infty} \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij} t_0})$$

$$\text{Therefore } \lim_{t \rightarrow \infty} P([T_C \leq t]) \geq \lim_{N \rightarrow \infty} \prod_{i=1}^N \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij} t}) \cdot \prod_{i=N+1}^{\infty} \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij} t_0})$$

$$= \prod_{i=N+1}^{\infty} \prod_{j=1}^{2^{i-1}} (1 - e^{-\lambda_{ij} t_0})$$

Letting $N \rightarrow \infty$ then from (1) we obtain $\lim_{t \rightarrow \infty} P([T_C \leq t]) = 1$. But

$$\lim_{t \rightarrow \infty} P([T_C \leq t]) = P\left(\bigcup_{t \geq 0} [T_C \leq t]\right) = P([T_C < \infty]) \text{ and the result follows.}$$

We now examine $X(t)$ in the light of semigroup theory. The analysis here falls quite naturally into two categories, the slow Cantor process ($\sum \lambda_{ij} < \infty$) and the fast Cantor process ($\sum \lambda_{ij} = \infty$). We first review some basic terminology from semigroup theory.

If V is a Banach space a family $\{T_t\}_{t \geq 0}$ of linear operators $T_t: V \rightarrow V$ forms a contraction semigroup on V if

$$(i) \|T_t x\| \leq \|x\| \quad \text{for every } x \in V.$$

$$(ii) T_{t+s} = T_t \circ T_s \quad \text{for all } t, s \geq 0.$$

A contraction semigroup is strongly continuous if $\lim_{t \rightarrow 0^+} \|T_t x - x\| = 0$ for each $x \in V$. (Note this implies $T_0 = I$, the identity operator.)

If $\{T_t\}_{t \geq 0}$ is a strongly continuous contraction semigroup then the infinitesimal generator, or simply generator, of $\{T_t\}$ is given by

$$Ax = \lim_{t \rightarrow 0^+} \frac{T_t x - x}{t} \text{ for those } x \in V \text{ for which the limit (taken in the}$$

norm) exists. The subspace of V for which this limit exists is denoted by $D(A)$ (the domain of A) and $A: D(A) \rightarrow V$ is a linear

operator. We will sometimes refer to the pair $(A, D(A))$ as the generator of $\{T_t\}$. T_t and $(A, D(A))$ are related by the differential

equations $\frac{d}{dt} (T_t x) = A(T_t x) = T_t (Ax)$ for each $x \in D(A)$, $t > 0$. The generator $(A, D(A))$ uniquely determines the semigroup $\{T_t\}$ via these

equations (subject to certain mild conditions). For each $\lambda > 0$

we can define the resolvent operator $R_\lambda = (\lambda I - A)^{-1}: V \rightarrow D(A)$.

R_λ is a bounded, one-one map of V onto $D(A)$. Its inverse $\lambda I - A$ is a one-one map of $D(A)$ onto V but need not be bounded.

We begin with the analysis of the slow Cantor process. Since $P(\{X(t) \in L_T \text{ for all } t\}) = 1$ we can consider the countable set L_T (rather than C) as the state space of the process and the analysis reduces to that of a discrete Markov chain. The transition functions may be represented in an infinite-dimensional matrix

$P_t = (P_t(F, G))_{F, G \in L_T}$. Since L_T is countable we may identify the space of bounded measurable functions $f: L_T \rightarrow \mathbb{R}$ with ℓ_∞ , the Banach space of bounded real sequences under the supremum (sup) norm. Since

$\{P_t(\cdot, \cdot)\}_{t \geq 0}$ is a family of Markov transition functions it follows that $\{P_t\}_{t \geq 0}$ is a contraction semigroup on ℓ_∞ where for each

$f \in \ell_\infty$ we define $P_t f(F) = \sum_{G \in L_T} f(G) P_t(F, G)$. (In this case we use

the suggestive matrix notation P_t , rather than T_t , to denote the semigroup.)

Lemma 5.4.2: $\{P_t\}_{t \geq 0}$ is a strongly continuous contraction semigroup on ℓ_∞ .

proof: We need only verify the strong continuity. We wish to show

$\lim_{t \rightarrow 0^+} \|P_t f - f\| = 0$ for each $f \in \ell_\infty$. Note that for each $F \in L_T$, we have

$$P_t(F, F) = P\left(\bigcap_{(i,j) \in \langle F \rangle} [\xi_{ij} > t]\right) \geq P\left(\bigcap_{(i,j) \in W} [\xi_{ij} > t]\right) = P_t(I, I)$$

where $I = [0, 1]$ and so $0 \leq 1 - P_t(F, F) \leq 1 - P_t(I, I)$.

Now $\|P_t f - f\| = \sup_{F \in L_T} |P_t f(F) - f(F)|$ and we have

$$\begin{aligned} |P_t f(F) - f(F)| &= \left| \sum_{G \in L_T} f(G) P_t(F, G) - f(F) \right| \\ &\leq \sum_{\substack{G \in L_T \\ G \neq F}} |f(G)| P_t(F, G) + |f(F)| |P_t(F, F) - 1| \\ &\leq \|f\| ((1 - P_t(F, F)) + (1 - P_t(F, F))) \\ &= 2\|f\| (1 - P_t(F, F)) \end{aligned}$$

Therefore $\sup_{F \in L_T} |P_t f(F) - f(F)| \leq 2\|f\| \sup_{F \in L_T} (1 - P_t(F, F))$

$$\leq 2\|f\| (1 - P_t(I, I)) \rightarrow 0 \text{ as } t \rightarrow 0^+$$

since $\lim_{t \rightarrow 0^+} P_t(I, I) = \lim_{t \rightarrow 0^+} e^{-\sum_{ij} t} = 1$. $\square$

Corollary 5.4.2.1: Let $(A, \mathcal{D}(A))$ be the generator of P_t . Then $\mathcal{D}(A)$ is dense in ℓ_∞ .

proof: This is a consequence of P_t being strongly continuous. See problem 2, page 139, of Lamperti [13]. $\square$

We can in fact show that A is a bounded operator with domain $\mathcal{D}(A) = \ell_\infty$. For $F, G \in L_T$ we will call G an immediate successor of F if $G = F \setminus I_{ij}$ for some $(i, j) \in \langle F \rangle$. For convenience we will denote

$F \setminus I_{ij}$ as $F + (i, j)$. The collection $\{F + (i, j) \mid (i, j) \in \langle F \rangle\}$ of immediate successors of F will be denoted $IS(F)$. $G \in L_T$ will be called an eventual successor of F if $G = F \setminus \bigcup_{(i, j) \in W'} I_{ij}$ for some finite set W' having at least two elements and satisfying $W' \cap \langle F \rangle = \emptyset$. The set of eventual successors of F will be denoted by $ES(F)$.

Note that if $G \in (IS(F) \cup ES(F) \cup \{F\})$ then $P_t(F, G) = 0$ for all $t \geq 0$. We will need the following lemma:

Lemma 5.4.3: For every fixed $t > 0$ the function $f(x) = x - \left(\frac{1-e^{-tx}}{t}\right)$

is a positive increasing function of x on $(0, \infty)$. For every fixed

$t > 0$ and $K > 0$ the function $g(x) = x - \frac{(1 - e^{-tx})e^{-(K-x)t}}{t}$ is also a positive increasing function of x on $(0, K)$.

proof: Note that $f(0) = 0$ and $f'(x) = 1 - e^{-tx} > 0$ for $x > 0$. Therefore f must be positive and increasing on $(0, \infty)$. Similarly $g(0) = 0$ and we have

$$g(x) = x - \frac{(1 - e^{-tx})e^{-(K-x)t}}{t} = x - \frac{e^{-Kt}e^{tx}}{t} + \frac{e^{-Kt}}{t}$$

and so $g'(x) = 1 - te^{-Kt}e^{tx} = 1 - e^{-(K-x)t} > 0$ for $0 \leq x \leq K$. $\square$

Theorem 5.4.6: The generator A of the slow Cantor process is defined by

$$Af(F) = -\lambda_F f(F) + \sum_{(i, j) \in \langle F \rangle} \lambda_{ij} f(F + (i, j))$$

where $\lambda_F = \sum_{(i, j) \in \langle F \rangle} \lambda_{ij}$. A is a bounded operator with $\mathcal{D}(A) = \mathcal{L}_\infty$

and $\|A\| = 2\lambda_I$. (Note $\lambda_I = \sum \lambda_{ij}$.)

proof: We will let A be defined as in the statement of the theorem

and show $\lim_{t \rightarrow 0^+} \sup_{F \in L_T} \left| Af(F) - \left(\frac{P_t f(F) - f(F)}{t} \right) \right| = 0$.

Since $P_t(F, G) = 0$ if $G \notin (IS(F) \cup ES(F) \cup \{F\})$ we can write

$$\begin{aligned} \frac{P_t f(F) - f(F)}{t} &= \frac{1}{t} \left(\sum_{G \in L_T} f(G) P_t(F, dG) - f(F) \right) \\ &= f(F) \frac{(P_t(F, F) - 1)}{t} + \sum_{G \in IS(F)} f(G) \frac{P_t(F, G)}{t} + \sum_{G \in ES(F)} f(G) \frac{P_t(F, G)}{t} \\ &= f(F) \frac{(P_t(F, F) - 1)}{t} + \sum_{(i, j) \in \langle F \rangle} f(F + (i, j)) \frac{P_t(F, F + (i, j))}{t} \\ &\quad + \sum_{G \in ES(F)} f(G) \frac{P_t(F, G)}{t} \end{aligned}$$

$$\begin{aligned} \text{Therefore } |Af(F) - \left(\frac{P_t f(F) - f(F)}{t} \right)| &= \left| -\lambda_F f(F) - (f(F)) \left(\frac{P_t(F, F) - 1}{t} \right) + \sum_{(i, j) \in \langle F \rangle} f(F + (i, j)) (\lambda_{ij} - \frac{P_t(F, F + (i, j))}{t}) \right. \\ &\quad \left. - \sum_{G \in ES(F)} f(G) \frac{P_t(F, G)}{t} \right| \\ &\leq ||f|| \left(\left| -\lambda_F - \frac{(P_t(F, F) - 1)}{t} \right| + \sum_{(i, j) \in \langle F \rangle} \left| \lambda_{ij} - \frac{P_t(F, F + (i, j))}{t} \right| + \sum_{G \in ES(F)} \frac{P_t(F, G)}{t} \right). \end{aligned}$$

(Note there is no problem with the rearranging of order of terms since all of the series involved are absolutely convergent; recall $\lambda_I = \sum \lambda_{ij} < \infty$ here.)

$$(1) \text{ We will show } \lim_{t \rightarrow 0} \sup_{F \in L_T} \left| -\lambda_F - \frac{(P_t(F, F) - 1)}{t} \right| = 0.$$

Note that we can express $P_t(F, F) = e^{-\lambda_F t}$. Therefore we obtain

$$\left| -\lambda_F - \frac{(P_t(F, F) - 1)}{t} \right| = \left| \lambda_F - \frac{(1 - P_t(F, F))}{t} \right|$$

$$= \lambda_F - \frac{(1 - e^{-\lambda_F t})}{t} \quad \text{by Lemma 5.4.3}$$

$$\leq \lambda_I - \frac{(1 - e^{-\lambda_I t})}{t} \quad \text{by Lemma 5.4.3}$$

$$\rightarrow 0 \text{ as } t \rightarrow 0^+ \text{ since } \lim_{t \rightarrow 0^+} \frac{1 - e^{-\lambda_I t}}{t} = \lambda_I.$$

Therefore

$$\lim_{t \rightarrow 0^+} \sup_{F \in L_T} \left| -\lambda_F - \frac{(P_t(F, F) - 1)}{t} \right| = 0.$$

$$(2) \text{ We will show } \lim_{t \rightarrow 0^+} \sup_{F \in L_T} \sum_{(i,j) \in \langle F \rangle} \left| \lambda_{ij} - \frac{P_t(F, F+(i,j))}{t} \right| = 0.$$

$$\begin{aligned} \text{Note that } P_t(F, F+(i,j)) &= P([\xi_{ij} \leq t]) \prod_{(k,l) \in \langle F+(i,j) \rangle} P([\xi_{kl} > t]) \\ &= (1 - e^{-\lambda_{ij} t}) e^{-(\lambda_F - \lambda_{ij})t} \end{aligned}$$

From Lemma 5.4.3 we obtain

$$\begin{aligned} \left| \lambda_{ij} - \frac{P_t(F, F+(i,j))}{t} \right| &= \lambda_{ij} - \frac{(1 - e^{-\lambda_{ij} t}) e^{-(\lambda_F - \lambda_{ij})t}}{t} \\ &\leq \lambda_{ij} - \frac{(1 - e^{-\lambda_{ij} t}) e^{-(\lambda_I - \lambda_{ij})t}}{t} \quad \text{as } \lambda_I \geq \lambda_F \end{aligned}$$

and so

$$\sum_{(i,j) \in \langle F \rangle} \left| \lambda_{ij} - \frac{P_t(F, F+(i,j))}{t} \right| \leq \sum_{(i,j) \in W} \left(\lambda_{ij} - \frac{(1 - e^{-\lambda_{ij} t}) e^{-(\lambda_I - \lambda_{ij})t}}{t} \right). \quad (2)$$

$$\text{Now } \lim_{t \rightarrow 0^+} \left(\lambda_{ij} - \frac{(1 - e^{-\lambda_{ij} t}) e^{-(\lambda_I - \lambda_{ij})t}}{t} \right) = 0 \text{ and so each term of the}$$

sum on the RHS of (2) tends to 0 as $t \rightarrow 0^+$. We can pull the limit

outside the summation by noting $0 \leq \lambda_{ij} - \frac{(1-e^{-\lambda_{ij}t})e^{-(\lambda_I - \lambda_{ij})t}}{t} \leq \lambda_{ij}$

for all $t \geq 0$ and hence the series converges uniformly in t on $[0, \infty]$

since $\sum \lambda_{ij} < \infty$. Therefore

$$\lim_{t \rightarrow 0^+} \sup_{F \in L_T} \left| \lambda_{ij} - \frac{P_t(F, F+(i,j))}{t} \right| \leq \lim_{t \rightarrow 0^+} \sum_{(i,j) \in W} \left(\lambda_{ij} - \frac{(1-e^{-\lambda_{ij}t})e^{-(\lambda_I - \lambda_{ij})t}}{t} \right) = 0 \quad \text{as required.}$$

$$(3) \quad \text{We now show } \lim_{t \rightarrow 0^+} \sup_{F \in L_T} \sum_{G \in ES(F)} \frac{P_t(F, G)}{t} = 0.$$

$$\begin{aligned} \sum_{G \in ES(F)} \frac{P_t(F, G)}{t} &= \frac{1 - P_t(F, F)}{t} - \sum_{(i,j) \notin \langle F \rangle} \frac{P_t(F, F+(i,j))}{t} \\ &= \frac{1 - P_t(F, F)}{t} - \lambda_F + \lambda_F - \sum_{(i,j) \notin \langle F \rangle} \frac{P_t(F, F+(i,j))}{t} \\ &\leq \left| \frac{1 - P_t(F, F)}{t} - \lambda_F \right| + \left| \lambda_F - \sum_{(i,j) \notin \langle F \rangle} \frac{P_t(F, F+(i,j))}{t} \right| \\ &= \left| \frac{1 - P_t(F, F)}{t} - \lambda_F \right| + \left| \sum_{(i,j) \notin \langle F \rangle} \left(\lambda_{ij} - \frac{P_t(F, F+(i,j))}{t} \right) \right| \\ &\rightarrow 0 \text{ uniformly in } F \text{ as } t \rightarrow 0^+ \text{ by (1) and (2).} \end{aligned}$$

This verifies that $Af(F) = -\lambda_F f(F) + \sum_{(i,j) \notin F} \lambda_{ij} f(F+(i,j))$ for

every $f \in \mathcal{L}_\infty$. It is easily seen that $\|A\| = 2\lambda_I$. $\square$

We now consider the fast Cantor process. Here the semigroup will be denoted by T_t where, for each bounded measurable function $f: C \rightarrow \mathbb{R}$, we have

$$T_t f(F) = \int_C f(G) P_t(F, dG).$$

It is easy to see that T_t is not strongly continuous on $F(C)$ (the space of bounded measurable real-valued functions on C) since if $F \in L_T$ we have $P_t(F, \{F\}) = e^{-\lambda_F t} = 0$ as $\lambda_F = \sum_{(i,j) \in \langle F \rangle} \lambda_{ij} = \infty$.

We can show, however, that T_t is a strongly continuous contraction semigroup on $C(C)$, the continuous functions.

Recall that a semigroup T_t is Feller if $T_t: C(C) \rightarrow C(C)$. That is T_t maps continuous functions to continuous functions. A system of Markov transition functions is uniformly stochastically continuous if

$$\lim_{t \rightarrow 0^+} \sup_{F \in C} |P_t(F, B(F, \epsilon)) - 1| = 0 \text{ for each } \epsilon > 0, \text{ where } B(F, \epsilon) \text{ is}$$

the open ball of radius ϵ centered

at F .

Since uniform stochastic continuity of the transition functions implies that a Feller semigroup T_t on $C(C)$ is strongly continuous (see the Corollary to Proposition 2 on page 158 of Lamperti [13] - this uses the compactness of C) we need only verify that T_t is Feller and that the transition functions are uniformly stochastically continuous.

Lemma 5.4.4: The semigroup $\{T_t\}_{t \geq 0}$ is Feller.

proof: First suppose $f \in T(C)$, the set of truncated functions. Then we can express f as $f = \sum_{A \in L_N} f(A) I_{C_{N,A}}$ for some N .

Therefore

$$\begin{aligned} T_t f(F) &= \int_C f(G) P_t(F, dG) \\ &= \int_C \sum_{A \in L_N} f(A) I_{C_{N,A}}(G) P_t(F, dG) \\ &= \sum_{A \in L_N} f(A) P_t(F, C_{N,A}). \end{aligned}$$

Now $P_t(F, C_{N,A}) = P_t^{(N)}(F, C_{N,A})$ by definition and so from Lemma 5.3.6 we can conclude $T_t f$ is continuous.

Now let $f \in C(C)$. Then $f_N \rightarrow f$ uniformly, where $f_N = f \circ T_N$ (see Theorem 5.2.2(i)) and since T_t is a contraction we obtain

$$\lim_{N \rightarrow \infty} \|T_t f_N - T_t f\| = \lim_{N \rightarrow \infty} \|T_t(f_N - f)\| \leq \lim_{N \rightarrow \infty} \|f_N - f\| = 0.$$

Since $T_t f_N$ is continuous for each N it follows that $T_t f$ is continuous.

Thus $\{T_t\}_{t \geq 0}$ is a Feller semigroup. $\square$

Lemma 5.4.5: The transition functions $\{P_t\}_{t \geq 0}$ are uniformly stochastically continuous.

proof: We need to show $\lim_{t \rightarrow 0^+} \sup_{F \in C} (1 - P_t(F, B(F, \epsilon))) = 0$ for every $\epsilon > 0$.

Now each open ball $B(F, \epsilon)$ is of the form $C_{N, T_N(F)}$ for some N , and we have

$$\begin{aligned} P_t(F, C_{N, T_N(F)}) &= \prod_{(i,j) \in (W_N \setminus \langle F \rangle)} P(\{\xi_{ij} > t\}) \\ &= e^{-\lambda_{F,N} t} \quad \text{where } \lambda_{F,N} = \sum_{(i,j) \in (W_N \setminus \langle F \rangle)} \lambda_{ij} \\ &\geq e^{-\lambda_{I,N} t} \end{aligned}$$

Therefore $\lim_{t \rightarrow 0^+} \sup_{F \in C} (1 - P_t(F, C_{N, T_N(F)})) \leq \lim_{t \rightarrow 0^+} (1 - e^{-\lambda_{I,N} t}) = 0$, as

required. $\square$

Theorem 5.4.7: $\{T_t\}_{t \geq 0}$ is a strongly continuous contraction semigroup on $C(C)$.

proof: This follows from Lemmas 5.4.4 and 5.4.5. $\square$

Corollary 5.4.7.1: Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of T_t . Then $\mathcal{D}(A)$ is dense in $C(C)$. $\square$

Theorem 5.4.8: The truncated functions $T(C)$ belong to $\mathcal{D}(A)$ and for each $f \in T(C)$ we have

$$Af(F) = \sum_{(i,j) \in W} \lambda_{ij} (f(F+(i,j)) - f(F))$$

proof: We can prove this using our knowledge of the finite state space truncated process $T_N(X(t))$. (This corresponds to $\lambda_{ij} = 0$ for $i \geq N+1$ and thus is a special case of the slow Cantor process.) We will denote the discrete transition probabilities of $T_N(X(t))$ by $Q_t^{(N)}(A,B)$, $A,B \in L_N$, and let $A^{(N)}$ denote the generator of $T_N(X(t))$. We will use the notation $Q_{\frac{t}{N}}^{(N)} = (Q_t^{(N)}(A,B))_{A,B \in L_N}$ for the associated (matrix) semigroup.

From Theorem 5.4.6 we have

$$A^{(N)} f(F) = -\lambda_{F,N} f(F) + \sum_{(i,j) \in (W_N \setminus \langle F \rangle)} \lambda_{ij} f(F+(i,j))$$

for each $F \in L_N$ and f defined on L_N .

If $f \in T(C)$ then $f = f \circ T_N$ for some N and can be regarded as a function on L_N . Expressing f as $f = \sum_{A \in L_N} f(A) I_{C_{N,A}}$ we obtain

$$T_t f(F) = \sum_{A \in L_N} f(A) P_t(F, C_{N,A})$$

as noted in the proof of

Lemma 5.4.4

$$\begin{aligned} &= \sum_{A \in L_N} f(A) Q_t^{(N)}(T_N(F), A) \\ &= Q_{\frac{t}{N}}^{(N)} f(T_N(F)) \end{aligned}$$

and therefore

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{T_t f(F) - f(F)}{t} &= \lim_{t \rightarrow 0^+} \frac{Q_{\frac{t}{N}}^{(N)} f(T_N(F)) - f(T_N(F))}{t} \\ &= A^{(N)} f(T_N(F)). \end{aligned} \quad (3)$$

Furthermore the convergence on the LHS of (3) occurs in the norm since it occurs in the norm on the RHS of (1). Therefore we have

$Af(F) = A^{(N)}f(T_N(F))$. Now note that $T_N(F) + (i,j) = T_N(F + (i,j))$ for $(i,j) \in W_N$, and since $\lambda_{F,N} = \lambda_{T_N(F),N}$ and $f = f \circ T_N$ we have

$$\begin{aligned}
 Af(F) &= A^{(N)}f(T_N(F)) \\
 &= -\lambda_{T_N(F),N} f(T_N(F)) + \sum_{(i,j) \in (W_N \setminus \langle T_N(F) \rangle)} \lambda_{ij} f(T_N(F) + (i,j)) \\
 &= -\lambda_{F,N} f(T_N(F)) + \sum_{(i,j) \in (W_N \setminus \langle F \rangle)} \lambda_{ij} f(T_N(F + (i,j))) \\
 &= -\lambda_{F,N} f(F) + \sum_{(i,j) \in (W_N \setminus \langle F \rangle)} \lambda_{ij} f(F + (i,j)) \\
 &= \sum_{(i,j) \in (W_N \setminus \langle F \rangle)} \lambda_{ij} (f(F + (i,j)) - f(F)) \\
 &= \sum_{(i,j) \in W} \lambda_{ij} (f(F + (i,j)) - f(F)) \quad \text{since } (i,j) \in (W \setminus (W_N \setminus \langle F \rangle)) \\
 &\quad \text{implies } T_N(F + (i,j)) = T_N(F) \\
 &\quad \text{and hence } f(F + (i,j)) = f(F).
 \end{aligned}$$

Therefore $Af(F) = \sum_{(i,j) \in W} \lambda_{ij} (f(F + (i,j)) - f(F))$. $\square$

Theorem 5.4.9: The generator A of the fast Cantor process is not bounded. (As a consequence we have $\mathcal{D}(A) \neq C(C)$.)

proof: Let $N \in \mathbb{Z}^+$ and define the function $f: C \rightarrow \mathbb{R}$ by

$$f(F) = \begin{cases} -1 & \text{if } T_N(F) = I \\ 1 & \text{if } T_N(F) \neq I \end{cases}$$

Then $f = f \circ T_N$ and Af is defined as in Theorem 5.4.8.

$$\begin{aligned}
 \text{Note } Af(I) &= \sum_{(i,j) \in W} \lambda_{ij} (f(I + (i,j)) - f(I)) \\
 &= \sum_{(i,j) \in W_N} \lambda_{ij} (f(I + (i,j)) - f(I))
 \end{aligned}$$

$$= \sum_{(i,j) \in W_N} 2 \lambda_{ij} = 2 \lambda_{I,N}$$

Therefore $\|Af\| = \sup_{F \in C} |Af(F)| \geq |Af(I)| = 2 \lambda_{I,N}$, and so

$$\|A\| = \sup_{g \neq 0} \frac{\|Ag\|}{\|g\|} \geq \frac{\|Af\|}{\|f\|} = \|Af\| \geq 2 \lambda_{I,N} \rightarrow \infty \text{ as } N \rightarrow \infty.$$

Therefore A is not bounded. $\square$

Note: Up until this point we have been using the notation

$$\sum_{(i,j) \in W} \lambda_{ij} (f(F+(i,j)) - f(F)) \text{ rather than } \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(F+(i,j)) - f(F))$$

because it is more convenient and the order of terms in the summation is irrelevant for truncated functions f (as noted in the proof of Theorem 5.4.8 this is really a finite sum when $f = f \circ T_N$). As we will now be considering more general functions $f \in C(C)$ we will use the double summation notation to indicate that the order of the terms may be important.

Theorem 5.4.10: Let $\mathcal{D}_0 = \{f \in C(C) \mid \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(F+(i,j)) - f(F))$

converges uniformly in $F\}$. Then $\mathcal{D}_0 \subseteq \mathcal{D}(A)$ and $Af(F) = \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij}$

$(f(F+(i,j)) - f(F))$ for every $f \in \mathcal{D}_0$.

proof: We will show that if $f \in \mathcal{D}_0$ then there exists a sequence

$\{f_N\}_{N=1}^{\infty}$ from $T(C)$ such that $\lim_{N \rightarrow \infty} \|f_N - f\| = 0$ and $\lim_{N \rightarrow \infty} \|Af_N - g\| = 0$,

where g is defined by $g(F) = \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(F+(i,j)) - f(F))$. It

then follows that $f \in \mathcal{D}(A)$ since a generator A is always a closed operator (see page 191 of Wentzell [22]).

Therefore let $f \in \mathcal{D}_0$ and let $f_N = f \circ T_N$. We immediately have

$f_N \in T(C)$, $\|f_N - f\| \rightarrow 0$, and can write

$$\sup_{F \in C} |Af_N(F) - g(F)| \leq \sup_{F \in C} |Af_N(F) - g(T_N(F))| + \sup_{F \in C} |g(T_N(F)) - g(F)|.$$

Since g is continuous (as a consequence of the uniform convergence of the series) we have $\lim_{N \rightarrow \infty} \sup_{F \in C} |g(T_N(F)) - g(F)| = 0$ using Theorem 5.2.2(i).

Therefore we need only show $\lim_{N \rightarrow \infty} \sup_{F \in C} |Af_N(F) - g(T_N(F))| = 0$.

$$\text{Since } f_N = f \circ T_N \text{ we have } Af_N(F) = \sum_{i=1}^N \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f_N(F+(i,j)) - f_N(F))$$

$$= \sum_{i=1}^N \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(T_N(F+(i,j))) - f(T_N(F)))$$

$$= \sum_{i=1}^N \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(T_N(F)+(i,j)) - f(T_N(F))).$$

(4)

$$\text{Now } g(T_N(F)) = \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(T_N(F)+(i,j)) - f(T_N(F)))$$

$$= \sum_{i=1}^N \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(T_N(F)+(i,j)) - f(T_N(F)))$$

$$+ \sum_{i=N+1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(T_N(F)+(i,j)) - f(T_N(F)))$$

$$= Af_N(F) + \sum_{i=N+1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(T_N(F)+(i,j)) - f(T_N(F)))$$

from (4).

Therefore

$$\sup_{F \in C} |Af_N(F) - g(T_N(F))| = \sup_{F \in C} \left| \sum_{i=N+1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(T_N(F)+(i,j)) - f(T_N(F))) \right|$$

$$\leq \sup_{G \in C} \left| \sum_{i=N+1}^{\infty} \sum_{j=1}^{2^{i-1}} \lambda_{ij} (f(G+(i,j)) - f(G)) \right|$$

$\rightarrow 0$ as $N \rightarrow \infty$ from uniform convergence.

Therefore $\limsup_{N \rightarrow \infty} \sup_{F \in C} |Af_N(F) - g(F)| = 0$ and the result follows. $\square$

A core of a generator $(A_0, \mathcal{D}(A_0))$ is a linear subspace L of $\mathcal{D}(A_0)$ such that $(A_0, \mathcal{D}(A_0))$ is the closure of $(A_0|_L, L)$. (That is, the graph of $(A_0, \mathcal{D}(A_0))$ is the closure of the graph of $(A_0|_L, L)$.) Thus a core uniquely determines the generator. We will show that $T(C)$ is a core of $(A, \mathcal{D}(A))$.

Lemma 5.4.5: Let $(A_0, \mathcal{D}(A_0))$ be a generator of a strongly continuous contraction semigroup on a Banach space V . Let L be a dense linear subspace of V such that for all $\lambda > 0$ and for every $g \in L$ the equation

$$\lambda f - A_0 f = g$$

has a solution $f \in L$. Then L is a core of $(A_0, \mathcal{D}(A_0))$.

proof: We need to show that for every $g \in V$ and $\lambda > 0$ the equation $\lambda f - A_0 f = g$ has a solution f belonging to the closure of $A_0|_L$. (This is sufficient since $\lambda I - A_0$ maps $\mathcal{D}(A_0)$ onto V in a one-one fashion.) Since $\mathcal{D}(A_0)$ is dense in V and L is dense in $\mathcal{D}(A_0)$ it follows that L is dense in V . Therefore, letting $g \in V$, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ from L such that $g_n \rightarrow g$ in the norm. By hypothesis there exists $f_n \in L$ (that $\lambda f_n - A_0 f_n = g_n$ and so $(\lambda f_n - A_0 f_n) \rightarrow g$ in the norm. Now, since $\lambda I - A_0$ maps $\mathcal{D}(A_0)$ one-one, onto V there exists $f \in \mathcal{D}(A_0)$ such that $\lambda f - A_0 f = g$. Therefore $(\lambda f_n - A_0 f_n) \rightarrow (\lambda f - A_0 f) = g$ in the norm. We now apply the resolvent $R_\lambda = (\lambda I - A_0)^{-1}$. Since R_λ is bounded we obtain $f_n = R_\lambda(\lambda f_n - A_0 f_n) \rightarrow R_\lambda(\lambda f - A_0 f) = f$. Thus $f_n \rightarrow f$ and since $(\lambda f_n - A_0 f_n) \rightarrow (\lambda f - A_0 f)$ it follows that $A_0 f_n \rightarrow A_0 f$ and thus $(f, A_0 f)$ belongs to the closure of

the graph of $(A_0|L, L)$ as required. $\square$

Lemma 5.4.6: For every $\lambda > 0$ and $g \in T(C)$, the equation $\lambda f - Af = g$ has a solution $f \in T(C)$.

proof: If $g \in T(C)$ then $g = g_0 \circ T_N$ for some N . Let g_0 denote the restriction of g to L_N . (Then, for any $F \in C$, we have

$g(F) = g(T_N(F)) = g_0(T_N(F))$.) Let $A^{(N)}$ be the generator of the truncated process $T_N(X(t))$. Since $T_N(X(t))$ has a finite state space we know that $A^{(N)}$ is bounded with domain $F(L_N) = \{f|f: L_N \rightarrow \mathbb{R}\}$,

and $\lambda 1_N - A^{(N)}$ maps $F(L_N)$ to $F(L_N)$ in a one-one, onto fashion (1_N indicates the identity operator on $F(L_N)$). Therefore there exists

$f_0 \in F(L_N)$ such that $\lambda f_0 - A^{(N)} f_0 = g_0$. Define $f: C \rightarrow \mathbb{R}$ by $f(F) = f_0(T_N(F))$. Then $f = f_0 \circ T_N$ so $f \in T(C)$. Since $Af(F) = A^{(N)} f_0(T_N(F))$

(see the proof of Theorem 5.4.8) we have

$$\begin{aligned} \lambda f(F) - Af(F) &= \lambda f_0(T_N(F)) - A^{(N)} f_0(T_N(F)) \\ &= \lambda f_0(T_N(F)) - A^{(N)} f_0(T_N(F)) \\ &= g_0(T_N(F)) \\ &= g(F). \end{aligned}$$

This completes the proof. $\square$

Theorem 5.4.11: $T(C)$ is a core of $(A, \mathcal{D}(A))$.

proof: Since $T(C)$ is a dense linear subspace of $\mathcal{D}(A)$ the result follows from Lemmas 5.4.5 and 5.4.6. $\square$

Thus the truncated functions uniquely determine the generator of the fast Cantor process. An interesting problem would be to determine the actual domain of the generator; this would enable us to use the generator to distinguish between the ultrafast Cantor process and the somewhat slower process which remains in $L_\infty \setminus L_\infty^+$.

5. A Measure-Valued Cantor Process

In this section we begin by constructing a measure-valued map $v: C \rightarrow M_1([0,1])$ where $M_1([0,1])$ is the space of probability measures on $[0,1]$ under the weak topology. We then obtain a measure-valued Cantor process by taking the composition $Z(t) = (v \circ Y)(t)$ where $Y(t)$ is the left-continuous (with RHS limits) version of the Cantor set process. We choose the left-continuous version because the nature of v results in $Z(t)$ then being left-continuous (with RHS limits) but right-continuity will not be preserved by v .

λ will denote Lebesgue measure on $[0,1]$ and v will denote the Cantor measure on $[0,1]$ (representing the uniform distribution over the Cantor set). For convenience we will let $\lambda(x)$ and $v(x)$ denote their respective distribution functions, so $\lambda(x) = \lambda([0,x])$ and $v(x) = v([0,x])$.

If $F \in C$ then $T_N(F)$ is the result of a finite number of intervals (say) n being removed ($0 \leq n \leq 2^N - 1$) and can be expressed uniquely as $T_N(F) = J_1 \cup \dots \cup J_{n+1}$ where the J_i 's are disjoint closed intervals. Letting $J_i = [a_i, b_i]$ we can w.l.o.g. order the J_i 's so that $a_1 < b_1 < a_2 < b_2 < \dots < a_{n+1} < b_{n+1}$. (For convenience we can express this as $J_1 < \dots < J_{n+1}$.)

For each N define the map $V_N: C \rightarrow M_1([0,1])$ by

$$V_N(F, \cdot) = V_N(T_N(F), \cdot) = \sum_{k=1}^{n+1} \frac{v(J_k)}{\lambda(J_k)} \lambda(\cdot \cap J_k)$$

where $T_N(F) = J_1 \cup \dots \cup J_{n+1}$ as described above.

Note that the mass $v(J_k)$ is spread uniformly over each J_k and no mass is assigned to the removed intervals. (The support of $V_N(F, \cdot)$ is precisely $T_N(F)$.) It is clear that $V_N(F, \cdot)$ is a measure,

being a finite linear combination of measures, and in fact $V_N(F, \cdot)$ is a probability measure on $[0,1]$ since

$$V_N(F, [0,1]) = \sum_{k=1}^{n+1} v(J_k) = v\left(\bigcup_{k=1}^{n+1} J_k\right) = 1 \quad \text{as } C \subseteq \bigcup_{k=1}^{n+1} J_k.$$

Note also $V_N(F, [0, b_k]) = V_N(F, [0, a_{k+1}]) = v(J_1 \cup \dots \cup J_k) = v(b_k) = v(a_{k+1})$ and if $M \geq N$ then $V_M(F, [0, b_k]) = V_M(F, [0, a_{k+1}]) = v(b_k) = v(a_{k+1})$. This is perhaps easiest to understand in terms of the distribution functions. The distribution function $V_N(F, x)$ assigns the value of the Cantor function $v(x)$ to all points x belonging to $(T_N(F))^c$. Thus $V_N(F, x)$ is constant over each removed interval. $V_N(F, x)$ is then obtained completely by joining the endpoints $v(a_k)$ and $v(b_k)$ by a straight line.

Since $V_N = V_N \circ T_N$ it follows that V_N is a continuous map of C into $M_1([0,1])$. We will show that, for each $F \in C$, $V_N(F, \cdot)$ converges weakly to a probability measure so that the pointwise limit $V(F, \cdot) = \lim_{N \rightarrow \infty} V_N(F, \cdot)$ exists and is a measurable mapping of C into $M_1([0,1])$.

Let $F \in \mathcal{C}$. To show that $V_N(F, \cdot)$ converges weakly we will show that the distribution functions $V_N(F, x)$ converge pointwise for each $x \in [0, 1]$. Let $P_1(F) = \{x \in [0, 1] \mid \text{there exists a closed interval } K \text{ (of positive length) such that } K \subseteq F \text{ and } x \in K\}$ and let $P_2(F) = (P_1(F))^c$.

Lemma 5.5.1: If $x \in P_2(F)$ then $\lim_{N \rightarrow \infty} V_N(F, x) = v(x)$.

proof: Let $x \in P_2(F)$.

If $x = 0$ then $V_N(F, 0) = v(0) = 0$ and $x = 1$ implies $V_N(F, 1) = v(1) = 1$, so there is nothing to prove here. If $x \in \bar{I}_{ij}$ for some $(i, j) \in \langle F \rangle$ (where $\bar{I}_{ij}$ denotes the closure of I_{ij}) it follows that $V_N(F, x) = v(x)$ for all $N \geq i$, so $\lim_{N \rightarrow \infty} V_N(F, x) = v(x)$.

We now consider the case where $0 < x < 1$ and x does not belong to the closure of an interval I_{ij} removed from F . For each N let a_N be the RHS endpoint of the removed interval of length at least 3^{-N} lying closest to x on the LHS (if no such interval exists set $a_N = 0$). We must have $a_N \leq a_{N+1}$ for each N , so let $a = \lim_{N \rightarrow \infty} a_N$. Similarly define b_N to be the LHS endpoint of the removed interval of length at least 3^{-N} lying closest to x on the RHS (if no such interval exists let $b_N = 1$). It follows that $b_{N+1} \leq b_N$ for each N , so let $b = \lim_{N \rightarrow \infty} b_N$. Since $x \notin P_1(F)$ it follows that every nbd. $(x-\epsilon, x+\epsilon)$ contains points from F^c lying to the left of x and to the right of x so we must have $a = x = b$.

Now $N \geq K$ implies $V_N(F, a_K) = V_K(F, a_K) = v(a_K)$ and, since $V_N(F, \cdot)$ is increasing in x (being a d.f.), we have

$$v(a_K) = V_N(F, a_K) \leq V_N(F, x) \quad \text{for all } N \geq K.$$

Therefore $v(a_K) \leq \liminf_{N \rightarrow \infty} V_N(F, x)$.

Letting $K \rightarrow \infty$ we obtain $v(x) \leq \liminf_N V_N(F, x)$.

By an analogous argument on the RHS we obtain $v(b_K) \geq \limsup_N V_N(F, x)$ and so $v(x) \geq \limsup_N V_N(F, x)$.

Thus the limit exists with $\lim_{N \rightarrow \infty} V_N(F, x) = v(x)$. $\square$

Lemma 5.5.2: If $x \in P_1(F)$ there exists a unique maximal closed interval $K \subseteq F$ such that $x \in K$. (That is, if K' is a closed interval such that $K \subset K'$ then $K' \cap F^C \neq \emptyset$.)

proof: Since $x \in P_1(F)$ there exists a closed interval $[z, y] \subseteq F$ such that $x \in [z, y]$. (Note that x may be an endpoint of this interval.)

Let $h_1 = \sup\{h \geq 0 \mid [z-h, z] \subseteq F\}$

and let $h_2 = \sup\{h \geq 0 \mid [y, y+h] \subseteq F\}$.

Let $a = z - h_1$ and $b = y + h_2$.

We will show $[a, b] \subseteq F$. From the definition of h_1 and h_2 it is clear that $(a, b) \subseteq F$ so we need only consider the endpoints. If $b \notin F$ then there exists a nbd. $(b-\epsilon, b+\epsilon) \subseteq F^C$ since F^C is open but this immediately contradicts the fact that $(a, b) \subseteq F$. Therefore $b \in F$ and by an analogous argument we can show $a \in F$. Thus $[a, b] \subseteq F$ and it is clear that $[a, b]$ is unique and maximal. $\square$

Lemma 5.5.3: Let $x \in P_1(F)$ and let $[a, b]$ be the maximal closed interval in F containing x . Then $\lim_{N \rightarrow \infty} V_N(F, x) = v(a) + \frac{(v(b) - v(a))}{b-a}(x-a)$.

proof: Since $[a, b]$ is maximal then either $a=0$ or every interval $[a-h, a]$, $h>0$, meets F^C , and either $b=1$ or every interval $[b, b+h]$, $h>0$, meets F^C . Therefore by the technique used in Lemma 5.5.1 we can construct a sequence $\{a_N\}$ of endpoints of removed intervals such that $a_N \uparrow a$ and a sequence $\{b_N\}$ of endpoints of removed intervals such that $b_N \downarrow b$. (If $a=0$ then $a_N = 0$ for every N and if $b=1$ then $b_N = 1$

for every N .) Recall that a_N is the RHS endpoint of the removed interval of length at least 3^{-N} which is closest to a on the LHS (note it is possible that a itself may be the endpoint of a removed interval, in which case $a_N = a$ for sufficiently large N .) Similarly b_N is the LHS endpoint of the removed interval of length at least 3^{-N} which lies closest to b on the RHS (b may also be the endpoint of a removed interval, in which case $b_N = b$ for sufficiently large N).

It follows then that $T_N(F) \supset [a_N, b_N]$ and for all $z \in [a_N, b_N]$

we have

$$V_N(F, z) = v(a_N) + \frac{(v(b_N) - v(a_N))}{b_N - a_N} (z - a_N).$$

Therefore $\lim_{N \rightarrow \infty} V_N(F, z) = v(a) + \frac{(v(b) - v(a))}{b-a} (z-a)$ for all $z \in [a, b]$.

□

Theorem 5.5.1: (i) For each $F \in \mathcal{C}$, the distribution functions

$V_N(F, x)$ converge pointwise to a function $V(F, x)$ where

$$V(F, x) = \begin{cases} v(x) & \text{if } x \in P_2(F) \\ v(a) + \frac{(v(b) - v(a))}{b-a} (x-a) & \text{if } x \in P_1(F), \text{ where } [a, b] \\ & \text{is the maximal closed} \\ & \text{interval in } F \text{ containing } x. \end{cases}$$

The function $V(F, x)$ is continuous in x and is therefore the distribution function of a diffuse probability measure $V(F, \cdot)$ on $\mathcal{B}([0, 1])$.

We have $V_N(F, \cdot) \xrightarrow{w} V(F, \cdot)$ for each $F \in \mathcal{C}$.

(ii) The mapping $V: \mathcal{C} \rightarrow M_1([0, 1])$ which takes $F \rightarrow V(F, \cdot)$ is measurable.

proof: (i) In view of Lemmas 5.5.1 and 5.5.3 we need only prove the

continuity of $V(F, x)$. Since $V(F, x)$ is increasing in x this reduces to showing $V(F, x^-) = V(F, x) = V(F, x^+)$ for each $x \in [0, 1]$. To show $V(F, x^-) = V(F, x)$ we classify x according to the behavior of F immediately to the left of x . (Note that the special case $x=0$ requires no proof.) For $x>0$ this results in three cases:

- (1) x is the RHS endpoint of a removed interval.

In this case clearly $V(F, x^-) = v(x) = V(F, x)$.

- (2) x is the limit point of a sequence $\{a_n\}$ of RHS endpoints of a sequence of removed intervals lying to the left of x .

In this case $V(F, x) = v(x)$ since x cannot belong to the interior of an interval $[a, b] \subseteq F$ and so $V(F, a_n) = v(a_n) \rightarrow v(x) = V(F, x)$. Therefore $V(F, x^-) = V(F, x)$.

- (3) There exists $h>0$ such that $[x-h, x] \subseteq F$.

In this case $x \in P_1(F)$ and there exists a maximal closed interval $[a, b] \subseteq F$ such that $x \in [a, b]$ (note $a \leq x-h$) and

$$V(F, a) = v(a) + \frac{(v(b) - v(a))}{b-a} (z-a) \quad \text{for all } z \in [a, b].$$

$$\begin{aligned} \text{Thus } \lim_{z \rightarrow x} V(F, z) &= \lim_{z \rightarrow x} \left(v(a) + \frac{(v(b) - v(a))}{b-a} (z-a) \right) \\ &= v(a) + \frac{(v(b) - v(a))}{b-a} (x-a) = V(F, x) \end{aligned}$$

and so $V(F, x^-) = V(F, x)$.

The proof that $V(F, x) = V(F, x^+)$ is similar; classify x into one of three cases based on the behavior of F to the right of x .

Thus we can conclude that $V(F, x)$ is continuous so the associated probability measure $V(F, \cdot)$ on $\mathcal{B}([0, 1])$ has no atoms.

(ii) Since $V(F, \cdot) = \lim_{N \rightarrow \infty} V_N(F, \cdot)$ for each $F \in \mathcal{C}$ it follows that V is a measurable map of $\mathcal{C}$ into $M_1([0, 1])$. $\square$

Note: V is not a continuous function on C . We will exhibit a sequence $\{F_K\}_{K=1}^{\infty}$ from C such that $d(F_K, I) \rightarrow 0$ but $V(F_K, \cdot) \not\xrightarrow{w} V(I, \cdot)$. (Note that $V(I, \cdot) \equiv \lambda(\cdot)$.) Take $x = 2/3$ and let I_N be the interval of length 3^{-N} ($N \geq 2$) lying closest to x on the RHS (of course I_N must be chosen from the collection $\{I_{N,j} \mid j=1, \dots, 2^{N-1}\}$). Let

$$F_K = \left(\bigcup_{N=K}^{\infty} I_N \right)^c. \text{ Since the } I_N \text{'s are disjoint it follows that}$$

$F_K \uparrow I$ so $d(F_K, I) \rightarrow 0$. Since for each K $x = 2/3$ is the limit point of a sequence of intervals removed from F_K it follows that

$$V(F_K, [0, 2/3]) = v(2/3) = \frac{1}{2} \text{ for each } K. \text{ However } V(I, [0, 2/3]) = \lambda([0, 2/3]) = 2/3 \text{ so } \lim_{K \rightarrow \infty} V(F_K, [0, 2/3]) \neq V(I, [0, 2/3]). \text{ Since}$$

$[0, 2/3]$ is a continuity set of $V(I, \cdot)$ this shows that $V(F_K, \cdot) \not\xrightarrow{w} V(I, \cdot)$. Thus V is not continuous.

However V does have certain continuity properties which we establish in the next theorem.

Theorem 5.5.2: (i) If $F_n \uparrow F$ then $\lim_{n \rightarrow \infty} V(F_n, \cdot) = V(F, \cdot)$.

(ii) If $F_n \uparrow F$ then $\lim_{n \rightarrow \infty} V(F_n, \cdot)$ exists and is a

diffuse probability measure on $[0, 1]$.

This may be expressed as " V is continuous from above and has limits from below."

proof: (i) Let $F_n \uparrow F$ and for convenience denote $V(F_n, x)$ by $V_n(x)$ and $V(F, x)$ by $V(x)$. We will show $V_n(x) \rightarrow V(x)$ for each $x \in [0, 1]$. Let $x \in [0, 1]$. There are two cases to be considered.

(1) Suppose there exists n such that every nbd. of x meets F_n^c . Since the F_k 's are decreasing then every nbd. of x meets F_k^c for all $k \geq n$. It follows that $V_k(x) = v(x)$ for all $k \geq n$ so $\lim_{k \rightarrow \infty} V_k(x) = v(x)$

and clearly also $V(x) = v(x)$.

(2) Suppose for every n there exists a nbd. of x which is contained in F_n . Then $x \in P_1(F_n)$ for each n . Let $[a_n, b_n]$ be the maximal closed interval in F_n which contains x . Since $F_{n+1} \subseteq F_n$ it follows that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for each n . Therefore the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing and the sequence $\{b_n\}_{n=1}^{\infty}$ is decreasing. Let $a = \lim_{n \rightarrow \infty} a_n$ and $b = \lim_{n \rightarrow \infty} b_n$. (Note we must have $a \leq x \leq b$.)

$$\text{Then } V_n(x) = v(a_n) + \frac{(v(b_n) - v(a_n))}{b_n - a_n} (x - a_n)$$

$$\text{so } \lim_{n \rightarrow \infty} V_n(x) = v(a) + \frac{(v(b) - v(a))}{b - a} (x - a)$$

Now $[a, b] \subseteq [a_n, b_n] \subseteq F_n$ for every n implies $[a, b] \subseteq F$, and furthermore $[a, b]$ must be the maximal closed interval in F containing x (since $a_n \rightarrow a$ and $b_n \rightarrow b$ any larger interval would contradict the maximality of $[a_n, b_n]$ for some n).

$$\text{Therefore it follows that } V(x) = v(a) + \frac{(v(b) - v(a))}{b - a} (x - a) \\ = \lim_{n \rightarrow \infty} V_n(x) \quad (\text{Note if } a=b=x \text{ this reduces to } v(x).)$$

Thus (i) is proved.

(ii) The proof here is similar to that of (i). By classifying each point $x \in [0, 1]$ into either case (1) or case (2) as in (i) the existence of the limit $\lim_{n \rightarrow \infty} V_n(x) = U(x)$ can be established. The resulting distribution function $U(x)$ can be shown to be continuous so $V(F_n, \cdot) \xrightarrow{w} U(\cdot)$ where $U(\cdot)$ is a diffuse probability measure on $[0, 1]$. $\square$

We will let $K_F = \{[a, b] \mid [a, b] \text{ is a maximal closed interval in } F\}$ and let $S_F = \{x \in F \mid B \cap F^c \neq \emptyset \text{ for every nbd. } B \text{ of } x\}$. It is easily seen that $S_F \subseteq C$.

Theorem 5.5.3: (i) $V_{ac}(F, \cdot) = \sum_{[a,b] \in K_F} \frac{v([a,b])}{\lambda([a,b])} \lambda(\cdot \cap [a,b])$

and $\text{Supp } V_{ac}(F, \cdot) \subseteq \overline{\bigcup_{[a,b] \in K_F} [a,b]}$.

(ii) $V_s(F, \cdot) = v(\cdot \cap S_F)$ and $\text{Supp } V_s(F, \cdot) \subseteq S_F \subseteq C$.

proof: These representations of the absolutely continuous and singular components of $V(F, \cdot)$ follow easily from the distribution function $V(F, x)$ obtained in Theorem 5.5.1. $\square$

Theorem 5.5.4: (i) $V(F, \cdot)$ has a nontrivial singular part iff

$v(S_F) > 0$ and is purely singular iff $S_F = C$ (in which case $V(F, \cdot) = v(\cdot)$).

$$(ii) \dim(\text{Supp } V_s(F, \cdot)) = \begin{cases} 0 & \text{if } v(S_F) = 0 \\ \frac{\log 2}{\log 3} & \text{if } v(S_F) > 0 \end{cases}$$

and $V(F, \cdot) = v(S_F) \delta_{\frac{\log 2}{\log 3}}$ where $\hat{V}(F, \cdot)$ is the dimension measure

associated with $V(F, \cdot)$ (see Chapter 3).

proof: (i) $V_s(F, \cdot) = v(\cdot \cap S_F)$ so clearly $V_s(F, \cdot) \neq 0$ iff $v(S_F) > 0$.

If $S_F = C$ then $V_s(F, S_F) = v(C) = 1$ which implies $V(F, \cdot) = V_s(F, \cdot) = v$.

Conversely, if $V_s(F, [0,1]) = 1$ then $v(S_F) = 1$ so S_F supports v . But $S_F \subseteq C$ and S_F is closed which therefore implies $S_F = C$ as $C = \text{Supp } v$.

(ii) If $v(S_F) = 0$ then $V_s(F, \cdot) = 0$ so $\text{Supp } V_s(F, \cdot) = \emptyset$

and $\dim(\emptyset) = 0$. Now suppose $v(S_F) > 0$ and let $S = \text{Supp } V_s(F, \cdot)$.

Then $S \subseteq S_F \subseteq C$ so $\dim(S) \leq \frac{\log 2}{\log 3}$. (This follows since $\dim(C) = \frac{\log 2}{\log 3}$.) But $v(S) > 0$ implies $\dim_v(S) = 1$ where $\dim_v(S)$ is the

v -Hausdorff dimension of S determined by the outer measures

$$H_v^\alpha(S) = \lim_{\delta \rightarrow 0^+} \inf_{\substack{UB_i \supseteq S \\ \int v(B_i) < \delta}} \sum_i (v(B_i))^\alpha.$$

$$\int v(B_i) < \delta$$

B_i an open ball

(See page 141 of Billingsley [2] for a discussion of v -Hausdorff dimension; this is a generalization of the Hausdorff-Besicovitch dimension discussed in Chapter 3 and has similar properties. In particular, note that in the case of Lebesgue measure $\dim_\lambda(E) = \dim(E)$, the usual Hausdorff-Besicovitch dimension of E .)

We now state the following result (Theorem 14.1, page 141, of Billingsley [2]):

If μ and γ are two probability measures on $[0,1]$ and

$$M \subseteq \{x \in [0,1] \mid \lim_{n \rightarrow \infty} \frac{\log \gamma(I_n(x))}{\log \mu(I_n(x))} = \delta\} \quad \text{where } I_n(x) \text{ is the triadic interval of length } 3^{-n} \text{ containing } x$$

then $\dim_\mu(M) = \delta \dim_\gamma(M)$.

This result is in fact used to prove $\dim(C) = \frac{\log 2}{\log 3}$ by taking

$\mu = \lambda$, $\gamma = \nu$, and $\delta = \frac{\log 2}{\log 3}$. We use the same technique here for S .

$$\text{Since } S \subseteq C \subseteq \{x \in [0,1] \mid \lim_{n \rightarrow \infty} \frac{\log \nu(I_n(x))}{\log \lambda(I_n(x))} = \frac{\log 2}{\log 3}\}$$

then $\dim(S) = \dim_\lambda(S) = \frac{\log 2}{\log 3} \dim_\nu(S)$. As $\dim_\nu(S) = 1$ (noted

earlier) we obtain $\dim(S) = \frac{\log 2}{\log 3}$.

$$\text{This gives } \dim(\text{Supp } \nu_S(F, \cdot)) = \begin{cases} 0 & \nu(S_F) = 0 \\ \frac{\log 2}{\log 3} & \nu(S_F) > 0 \end{cases}$$

Now if $\nu_S(F, A) > 0$ then $\nu(A \cap S_F) > 0$ and the preceding argument used for S applies to $A \cap S_F$ to give $\dim(A \cap S_F) = \frac{\log 2}{\log 3}$ and hence

$\dim(A) \geq \frac{\log 2}{\log 3}$. This shows that $V_s(F, \cdot)$ assigns no mass to sets having dimension less than $\frac{\log 2}{\log 3}$ and since $V_s(F, \cdot)$ assigns all its mass to S_F (and $\dim(S_F) = \frac{\log 2}{\log 3}$) we have

$$(V_s(F, \cdot))_\alpha = \begin{cases} 0 & 0 \leq \alpha < \frac{\log 2}{\log 3} \\ V_s(F, \cdot) & \alpha \geq \frac{\log 2}{\log 3} \end{cases}$$

Thus we obtain $\hat{V}(F, \cdot) = V_s(F, [0, 1]) \delta_{\frac{\log 2}{\log 3}} = v(S_F) \delta_{\frac{\log 2}{\log 3}}$ as claimed. $\square$

It might appear initially that $\text{Supp } V(F, \cdot) = F$ (which would then imply V is a one-one map of C into $M_1([0, 1])$) but in fact this is generally not true. Note that we can split K_F , the collection of maximal closed intervals of F , into two disjoint sets

$$K_F^+ = \{[a, b] \in K_F \mid v([a, b]) > 0\} \text{ and } K_F^0 = \{[a, b] \in K_F \mid v([a, b]) = 0\}.$$

If $[a, b] \in K_F^0$ then $[a, b] = \bar{I}_{ij}$ for some $I_{ij} \subseteq F$ (any larger interval would meet the Cantor set in a set of positive v -measure).

Conversely, if $\bar{I}_{ij} \in K_F$ then $\bar{I}_{ij} \in K_F^0$ since $v(\bar{I}_{ij}) = 0$. Note that

$\bar{I}_{ij} = [a, b] \in K_F^0$ implies for each $\varepsilon > 0$ the intervals $[a - \varepsilon, a]$ and $[b, b + \varepsilon]$ meet F^c . Since $V(F, a) = v(a) = v(b) = V(F, b)$ when

$[a, b] \in K_F^0$ this indicates that $V(F, \cdot)$ behaves exactly as if $(a, b) = I_{ij}$ had been removed (that is, $V(F, \cdot) = V(F \setminus I_{ij}, \cdot)$). A rigorous proof

will be presented in the next theorem.

Define $G_F = \bigcup_{[a, b] \in K_F^0} (a, b)$. We can put an equivalence relation

on C by defining $F_1 \sim F_2$ if $F_1 \setminus G_{F_1} = F_2 \setminus G_{F_2}$.

Theorem 5.5.5: $\text{Supp } V(F, \cdot) = F \setminus G_F$. Furthermore $F_1 \sim F_2$ iff $V(F_1, \cdot) = V(F_2, \cdot)$.

proof: Let $F \in C$. Then $F^c = \bigcup_{(i, j) \in \langle F \rangle} I_{ij}$. Since $V(F, I_{ij}) = 0$ for

$I_{ij} \subseteq F^C$, then $V(F, F^C) = 0$ and since F^C is open this shows $\text{Supp } V(F, \cdot) \subseteq F$. Clearly we then have $\text{Supp } V(F, \cdot) \subseteq (F \setminus G_F)$ since $V(F, G_F) = 0$ and G_F is open. Now let $x \in (F \setminus G_F)$. If $x \in [a, b] \in K_F^+$ then $x \in \text{Supp } V(F, \cdot)$ since the (positive) mass on $[a, b]$ is spread uniformly across $[a, b]$. If x does not belong to such an $[a, b]$ then there exists a strictly decreasing (or strictly increasing) sequence $\{y_n\}_{n=1}^\infty$ of endpoints of removed intervals such that $y_n \rightarrow x$. W.l.o.g. assume $y_n \uparrow x$. It follows that $V(F, x) = v(x) < v(y_n) = V(F, y_n)$ for each n , so for every nbd. B of x we have $V(F, B) > 0$. Thus $x \in \text{Supp } V(F, \cdot)$. This shows $\text{Supp } V(F, \cdot) = F \setminus G_F$.

Now note that $K_{F \setminus G_F}^O = \emptyset$ and $K_{F \setminus G_F} = K_{F \setminus G_F}^+ = K_F^+$. For any Borel set A we have $V(F, A) = V_{ac}(F, A) + V_s(F, A)$ and $V(F \setminus G_F, A) = V_{ac}(F \setminus G_F, A) + V_s(F \setminus G_F, A)$.

$$\text{Now } V_{ac}(F, A) = \sum_{[a,b] \in K_F^+} \frac{v([a,b])}{\lambda([a,b])} \lambda(A \cap [a,b]) =$$

$$\sum_{[a,b] \in K_{F \setminus G_F}^+} \frac{v([a,b])}{\lambda([a,b])} \lambda(A \cap [a,b]) = V_{ac}(F \setminus G_F, A) \text{ so the absolutely}$$

continuous components agree. Setting $E = \left(\bigcup_{[a,b] \in K_F^+} [a,b] \right) \cup F^C$ then

clearly both $V_s(F, \cdot)$ and $V_s(F \setminus G_F, \cdot)$ are supported on E^C , $E^C \subseteq S_F^\circ$, and $E^C \subseteq S_{F \setminus G_F}^\circ$. Therefore $V_s(F, \cdot) = v(\cdot \cap E^C) = V_s(F \setminus G_F, \cdot)$ so the singular components coincide. Thus $V(F, \cdot) = V(F \setminus G_F, \cdot)$. Therefore $F_1 \sim F_2$ implies $V(F_1, \cdot) = V(F_2, \cdot)$.

Conversely, suppose $F_1 \not\sim F_2$. Then $F_1 \setminus G_{F_1} \neq F_2 \setminus G_{F_2}$ and so w.l.o.g. there exists $(i, j) \in \langle F_1 \setminus G_{F_1} \rangle \setminus \langle F_2 \setminus G_{F_2} \rangle$. Therefore $I_{ij} \subseteq (F_1 \setminus G_{F_1})^C$ while $\bar{I}_{ij} \subseteq (F_2 \setminus G_{F_2})$. Now $\bar{I}_{ij} \subseteq (F_2 \setminus G_{F_2})$ implies $\bar{I}_{ij} \in K_{F_2}^O$ and so there exists a maximal interval $[a, b] \in K_{F_2}^+$ such that $\bar{I}_{ij} \subseteq [a, b]$.

Since the mass is spread uniformly across $[a, b]$ this shows

$$V(F_2, I_{ij}) > 0. \text{ But } I_{ij} \subseteq (F_1 \setminus G_{F_1})^c \text{ implies } V(F_1, I_{ij}) = V(F_1 \setminus G_{F_1}, I_{ij}) = 0.$$

Therefore $V(F_2, \cdot) \neq V(F_1, \cdot)$.

This shows $F_1 \sim F_2$ iff $V(F_1, \cdot) = V(F_2, \cdot)$. $\square$

We can define a measure-valued Cantor process $Z(t)$ by the composition $Z(t) = (V \circ Y)(t)$ where $Y(t)$ is the left-continuous version of the Cantor set process. We can explicitly define $Y(t)$ by

$$Y(t, \omega) = \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} X'_{ij}(t, \xi_{ij}(\omega)) \text{ where}$$

$X'_{ij} : T \times T \rightarrow C$ is given by

$$X'_{ij}(t, x) = \begin{cases} I_{ij}^c & x < t \\ I & x \geq t \end{cases}$$

In this case the past of the process $Y(t)$ is given by $A_t = \sigma(Y(s), s \leq t) = \sigma([\xi_{ij} < s], 0 \leq s \leq t)$. The left-continuous version is chosen as Theorem 5.5.2 then implies that $Z(t)$ is left-continuous with righthand limits. Since $Y(t)$ is progressively measurable with respect to $\mathcal{B}([0, t]) \times A_t$ it follows that $Z(t)$ is progressively measurable with respect to $\mathcal{B}([0, t]) \times A_t$.

From Theorem 5.5.5 we have $\text{Supp } Z(t) = Y^*(t)$ where $Y^*(t) = Y(t) \setminus G_{Y(t)}$ and so $Y^*(t)$ is also progressively measurable. From Theorem 5.5.5 we furthermore obtain $Z(t) = (V \circ Y^*)(t)$ and since $Z(t)$ and $Y^*(t)$ are in one-one correspondence the Markov property of $Z(t)$ can easily be shown by verifying the Markov property for $Y^*(t)$. In order to do this we introduce the following notation:

Consider an interval $I_{ij} = (a_{ij}, b_{ij})$.

$$\text{Let } G^{ij}(t) = \bigcup_{n=1}^{\infty} I_{i,j} \subseteq [a_{ij} - \frac{1}{n}, a_{ij}] \cap [\xi_{i,j} < t] \cap [\xi_{i''j''} < t] \\ I_{i''j''} \subseteq [b_{ij}, b_{ij} + \frac{1}{n}]$$

Note that $\omega \in G^{ij}(t)$ implies $I_{ij} \subseteq Y^*(t, \omega)$ and that for any $k \in \mathbb{Z}^+$ we have

$$G^{ij}(t) = \bigcup_{n=k}^{\infty} I_{i,j} \subseteq [a_{ij} - \frac{1}{n}, a_{ij}] \cap [\xi_{i,j} < t] \cap [\xi_{i''j''} < t] \\ I_{i''j''} \subseteq [b_{ij}, b_{ij} + \frac{1}{n}] \quad (1)$$

We will let $E^{ij}(t) = [\xi_{ij} < t] \cup G^{ij}(t)$. Equivalently $E^{ij}(t) = [I_{ij} \subseteq Y^*(t)]$. This follows since there are two ways of removing an interval $I_{ij} = (a_{ij}, b_{ij})$ from $Y^*(t)$; (1) I_{ij} is removed if $\xi_{ij} < t$ and (2) I_{ij} is removed if there exists a sequence of removed intervals to the left of I_{ij} and to the right of I_{ij} whose endpoints converge to a_{ij} and to b_{ij} respectively.

Lemma 5.5.4: The σ -algebras $\sigma(E^{ij}(t), t \geq 0), (i,j) \in W$, are independent.

proof: We will show that the events $E^{i_1 j_1}(t_1), \dots, E^{i_n j_n}(t_n)$ are independent for any distinct collection $\{I_{i_k j_k}\}_{k=1}^n$ (where $(i_k, j_k) = (i_l, j_l)$ iff $k = l$) and $t_1, \dots, t_n \in (0, \infty)$.

Since the collection $\{I_{i_k j_k}\}_{k=1}^n$ is finite there exists a certain minimum positive distance by which any two of these intervals is separated. Hence there exists $k_0 \in \mathbb{Z}^+$ such that the sets

$$A_k = \{I_{i,j}, I_{i''j''} \mid I_{i,j} \subseteq [a_{i_k j_k} - \frac{1}{k_0}, a_{i_k j_k}], I_{i''j''} \subseteq [b_{i_k j_k}, b_{i_k j_k} + \frac{1}{k_0}]\}$$

for $k=1, \dots, n$ are disjoint. This plus expression (1) and the independence of the ξ_{ij} 's implies $E^{i_1 j_1}(t_1), \dots, E^{i_n j_n}(t_n)$ are independent.

Noting that $\{E^{ij}(t), t \geq 0\}$ is a π -system generating $\sigma(E^{ij}(t), t \geq 0)$ it then follows easily that the σ -algebras $\sigma(E^{ij}(t), t \geq 0)$, $(i,j) \in W$, are independent. $\square$

Theorem 5.5.6: $Y^*(t)$ is a Markov process with $A_t^{*0} = \sigma(Y^*(t)) = \sigma(E^{ij}(t), (i,j) \in W)$ and past $A_t^* = \sigma(E^{ij}(s), 0 \leq s \leq t, (i,j) \in W)$.

proof: It is easy to see that $\sigma(Y^*(t)) = \sigma(E^{ij}(t), (i,j) \in W)$.

The proof of the Markov property for $Y^*(t)$ is then analogous to the proof of the Markov property for $X(t)$. (The events $E^{ij}(t)$ play the role of the events $\{\xi_{ij} \leq t\}$ and the necessary independence is provided by Lemma 5.5.4) $\square$

Corollary 5.5.6.1: $Z(t)$ is a Markov process with $\sigma(Z(t)) = A_t^{*0}$.

proof: This follows from the one-one correspondence between $Z(t)$ and $Y^*(t)$.

Let $C^* = \{F | F = F \setminus G_F\}$. Then C^* is a measurable subset of C since we can write

$$C^* = \bigcup_{(i,j) \in W} \{F | F \cap I_{ij} = \emptyset\} \cup \bigcup_{n=1}^{\infty} \{F | F \supseteq (I_{ij} - \frac{1}{n})\} \cup \bigcup_{n=1}^{\infty} \{F | F \supseteq (I_{ij} + \frac{1}{n})\}$$

where, putting $I_{ij} = (a_{ij}, b_{ij})$, we define $I_{ij} - \frac{1}{n}$ by

$$I_{ij} - \frac{1}{n} = [a_{ij} - \frac{1}{n}, a_{ij}] \text{ and } I_{ij} + \frac{1}{n} \text{ by } I_{ij} + \frac{1}{n} = [b_{ij}, b_{ij} + \frac{1}{n}].$$

Let $U = v|C^*$. Then $U: C^* \rightarrow M_1([0,1])$ is a one-one measurable map into $M_1([0,1])$ and from the Kuratowski mapping theorem (see Theorem 2.1.4) the inverse U^{-1} is also measurable. We can write $Z(t) = (U \circ Y^*)(t)$ and $Y^*(t) = (U^{-1} \circ Z)(t)$.

Therefore if B is a measurable set in $M_1([0,1])$ we have

$$\begin{aligned} [Z(t) \in B] &= [(U \circ Y^*)(t) \in B] \\ &= [Y^*(t) \in U^{-1}(B)] \quad \text{which belongs to } A_t^{*0} = \sigma(Y^*(t)). \end{aligned}$$

Conversely, if B is a measurable set in C we have

$$\begin{aligned} [Y^*(t) \in B] &= [(U^{-1} \circ Z)(t) \in B] \\ &= [Z(t) \in U(B)] \quad \text{which belongs to } \sigma(Z(t)), \text{ using} \\ &\quad \text{the measurability of } U^{-1}. \end{aligned}$$

Thus $\sigma(Z(t)) = A_t^{*0}$ and it then follows that $\sigma(Z(s), 0 \leq s \leq t) = A_s^*$. Therefore we have for $t \geq s$:

$$\begin{aligned} P([Z(t) \in B] | A_s^*) &= P([Y^*(t) \in U^{-1}(B)] | A_s^*) \\ &= P([Y^*(t) \in U^{-1}(B)] | A_s^{*0}) \text{ as } Y^*(t) \text{ is Markov} \\ &= P([Z(t) \in B] | A_s^{*0}) \end{aligned}$$

Thus $Z(t)$ is Markov. $\square$

However, while the map V is quite interesting, the independence of the ξ_{ij} 's results in the behavior of $Z(t)$ being quite unexciting. This can be seen in the following theorem. A more sophisticated removal mechanism for the intervals might produce more interesting results.

Theorem 5.5.7: (i) If $\sum \lambda_{ij} < \infty$ then $Y = Y^*$ P-a.s. (that is, $P([Y(t) = Y^*(t) \text{ for all } t]) = 1$). We have $Z(t) \ll \lambda$ P-a.s. and $\hat{Z}(t) = \delta_1$ P-a.s.

(ii) Suppose $\lambda_{ij} = \lambda_i > 0$ for all $j=1, \dots, 2^{i-1}$. If $\sum_{i=1}^{\infty} \lambda_i = \infty$ then $Y^* = C$ P-a.s. (that is, $P([Y^*(t) = C \text{ for all } t > 0]) = 1$). As a consequence $Z = v$ P-a.s. and so $\hat{Z}(t) = \delta_{\frac{\log 2}{\log 3}}$ P-a.s. ($t > 0$).

proof: (i) $\sum \lambda_{ij} < \infty$ implies $P(\cap_{t \geq 0} [Y(t) \in L_T]) = 1$.

Now $Y(t) \in L_T$ implies $\langle Y(t) \rangle$ is finite so $K_{Y(t)}^0 = \emptyset$ and thus

$G_{Y(t)} = \emptyset$. Hence $[Y(t) \in L_T] \subseteq [Y(t) = Y^*(t)]$ and so $Y = Y^*$ P-a.s.

In this case we then have $\text{Supp } Y^* = Y(t)$ and since $\langle Y(t) \rangle$ is finite

it follows that $Z(t) \ll \lambda$ P-a.s.

(ii) Recall that we can express the Cantor set as $C = \bigcap_{N=1}^{\infty} C_N$

where each $C_N = \bigcup_{k=1}^{2^N} C_{NK}$, a union of 2^N disjoint closed intervals.

(The collection $\{C_{NK} | K=1, \dots, 2^N\}$ consists of the closed intervals remaining after the intervals I_{ij} , $(i,j) \in W_N$, are removed.) Since $\lambda_{ij} = \lambda_i > 0$ By assumption and $\sum 2^{i-1} \lambda_i = \infty$ the independence of the ξ_{ij} 's results in the process behaving in a similar fashion over each interval C_{NK} , $K=1, \dots, 2^N$. If $Y^{NK}(t)$ denotes the process which only removes intervals from C_{NK} then $P(\bigcap_{t>0} \{Y^{NK}(t) \in L_{\infty}\}) = 1$ by the usual Borel-Cantelli argument.

Thus $P(\bigcap_{N=1}^{\infty} \bigcap_{K=1}^{\infty} \bigcap_{t>0} \{Y^{NK}(t) \in L_{\infty}\}) = 1$.

If $Y^{NK}(t) \in L_{\infty}$ for all N and K and $I_{ij} \subseteq Y(t)$ it then follows that $I_{ij} \in K_{Y(t)}^0$ since the endpoints of I_{ij} must be limit points of removed intervals. This implies $Y^*(t) = {}^{\circ}C$.

Therefore $\bigcap_{N=1}^{\infty} \bigcap_{K=1}^{\infty} \{Y^{NK}(t) \in L_{\infty}\} \subseteq \{Y^*(t) = C\}$ and so it

follows that $P(\bigcap_{t>0} \{Y^*(t) = C\}) = 1$. Since $\text{Supp } Z(t) = Y^*(t)$ we must then have $Z = \nu$ P-a.s. $\square$

An interesting problem would be to study more sophisticated removal mechanisms for the intervals.

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