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**EQUIVALENCE OF CATEGORICAL CALCULUS THEORIES BASED ON
SMOOTHNESS AND BOUNDEDNESS**

by

Erhard Siegl

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Master of Science

**Department of Mathematics and Statistics
Carleton University
Ottawa, Canada
July 1994**

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**"EQUIVALENCE OF CATEGORICAL CALCULUS THEORIES
BASED ON SMOOTHNESS AND BOUNDEDNESS"**

submitted by
Erhard Martin Siegl, Magister
in partial fulfilment of the requirements for
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Abstract

In this thesis we show relations between two differentiation theories. The first is based on certain topocosms; categories of sets structured with maps from finite dimensional spaces. They have to satisfy a localization property and must be compatible with smooth maps. We show relations between such categories. We consider linear spaces in these categories and find nice reflective subcategories with help of factorization systems.

The second differentiation theory is based on bornological spaces. We prove isomorphisms between linear bornological spaces and linear spaces in topocosms. Finally we find a category in a topocosm which is isomorphic to the category of convenient vector spaces, a subcategory of bornological spaces. For these categories also their intrinsic differential calculus is equivalent.

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Introduction

The main goal of this thesis is to show equivalence between differentiation theories developed by Frölicher and Kriegl [7, 14, 15, 8] on one hand and by Nel [19, 20] on the other hand. While in the first theory the differentiable curves into linear spaces are described in terms of bounded sets (bornologies), the second uses families of maps from finite dimensional spaces to define differentiability. In order to create an appropriate setting for linear spaces, variable categories of nonlinear spaces structured with families of maps are considered.

To give a first idea of the methods used to construct these nonlinear categories we introduce a category of spaces with holomorphic structure in chapter 1. Sets will be structured with families of maps, called (holomorphic) imprints, which satisfy a localization property. Morphisms in this category are maps that map imprints to imprints. The classical Hartogs theorem shows that the morphisms between open subsets of $\mathbb{C}^n$ are exactly the usual holomorphic maps. The cartesian closedness property of this category leads to toponomes [17] in chapter 2, roughly cartesian closed topological constructs (concrete categories), where the right adjoint to the cartesian functor is given by structured hom-sets. As special case of toponomes we obtain topocosms. Similar concepts appear in the literature as topological universes [21] or concrete quasitopoi [5].

By generalizing the example in chapter 1 we construct topocosms out of base categories. We structure sets with prefinal maps which have domains in the base category; see also [5, 21, 29]. Localization is obtained by defining covering families

in the base category reminiscent of Grothendieck topologies; see e.g. [12]. Different base categories give us variable examples of categories that are useful for differentiation theory. They form the foundation of Nel's differentiation theory.

Sometimes one base category lies inside another. The natural question arises of how the resulting topocosms relate. We study this question in chapter 3 and apply the results to the examples in differentiation theory, thus making progress toward the above mentioned goal. Furthermore we give a relation of topocosms to categories introduced by Frölicher [7].

In chapter 4 we consider linear spaces in toponomes and how to build nice reflective subcategories with help of factorization systems [17, 18]. More general definitions of factorization systems (right bicategories) and relations between them are given in [6, 13, 24, 25, 26]. Especially so called embeddable spaces or $\mathbf{eLC}$ -spaces will play an important role in the following. Embeddable spaces are spaces that allow a linear embedding into a hom-space $C(X, \mathbb{R})$ where X is an object of the topocosm (or toponome) C .

Next we introduce bornological spaces (see also Hogbe-Nlend [10]) and the fundamental definitions and properties of the Frölicher-Kriegl calculus, and we prove isomorphisms between embeddable spaces in the toponome of bornological spaces and the embeddable spaces in our examples of topocosms. This prepares for the next step towards our goal. We construct a category $\mathbf{dLC}_{\mathbf{B}}$ of linear spaces and show that $\mathbf{dLC}_{\mathbf{B}}$ is isomorphic to the category of convenient vector spaces of Frölicher and Kriegl [14, 8]. Here the two approaches to differentiation theory finally come together. The morphisms between $\mathbf{dLC}_{\mathbf{B}}$ -spaces are exactly the smooth linear maps between the corresponding convenient vector spaces and the isomorphism is given by a restructuring functor.

In chapter 8 we give a further isomorphism between embeddable spaces in topocosms which arise in our examples. The essential part is given by an extension theorem of smooth maps defined on half-spaces, which was proved by Seeley [27].

The main sources for references will be [17, 18] for categorical questions and [14, 8] for the Frölicher-Kriegl calculus. Also most of the notation will be adapted

from [17, 18]. Further references will be taken from [20] and [27].

Chapter 1

Holological Spaces (An Example)

In this chapter we give a first example how to build a category out of a single base-space. All $\mathbb{C}^n$ will be objects in this category and the morphisms between them will be the usual holomorphic maps. This example is meant to show the basic ideas of the construction and will not play an important role in the rest of the thesis, where we will concentrate mostly on real spaces and smooth functions between them.

Definition 1.1 *Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc. By a covering family for $\mathbb{D}$ will be meant a surjective family $\iota_i : \mathbb{D} \rightarrow \mathbb{D}$ ($i \in I$) of holomorphic embeddings. A holological space means a set X structured with functions (more precisely, prefinal maps) $\mathbb{D} \rightarrow X$, to be called (holomorphic) imprints such that the following axioms are satisfied:*

1. *Every constant prefinal map $\mathbb{D} \rightarrow X$ is an imprint.*
2. *If $g : \mathbb{D} \rightarrow X$ is an imprint and $f : \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic map, then $g \circ f$ is an imprint.*
3. *If $\iota_i : \mathbb{D} \rightarrow \mathbb{D}$ ($i \in I$) is a covering family, and $g : \mathbb{D} \rightarrow X$ is a prefinal map such that every composite $g \circ \iota_i$ is an imprint, then g is an imprint.*

Holological maps $f : X \rightarrow Y$ between holological spaces are functions f such that for every imprint g into X the composite $f \circ g$ is an imprint into Y . We obtain the category $C_{\mathbb{D}}$ of holological spaces and holological maps, see [16].

We structure every open $U \subseteq \mathbb{C}^n$ by defining every holomorphic map $g : \mathbb{D} \rightarrow U$ to be an imprint. This defines a hological structure since all constant maps are holomorphic, the composite of holomorphic maps is holomorphic, and holomorphy depends only on a neighborhood of every point $x \in U$. Since holomorphy is a local property and every disc is biholomorphic to the unit disc we get that the holomorphic maps and the hological maps between open subsets of $\mathbb{C}$ are the same. That the hological maps and the holomorphic maps also coincide between arbitrary open sets $U \subseteq \mathbb{C}^n$ follows immediately from Hartogs' theorem.

Proposition 1.2 (Hartogs) *Let U_i ($i = 1 \dots n$) be open in $\mathbb{C}$. A map $f : U_1 \times U_2 \times \dots \times U_n \rightarrow \mathbb{C}^m$ is holomorphic iff it is holomorphic in each variable separately.*

For hological spaces X, Y one can define a hom-space $\mathcal{H}(X, Y)$ that has as underlying set all hological maps from X to Y and the imprints into $\mathcal{H}(X, Y)$ are of the following form: A map $g : \mathbb{D} \rightarrow \mathcal{H}(X, Y)$ is an imprint if and only if $g^\dagger : \mathbb{D} \times X \rightarrow Y$ is a hological map, where $g^\dagger(z, x) := g(z)(x)$. As we will see in chapter 2 we obtain a natural isomorphism

$$\mathcal{H}(X \times Y, Z) \simeq \mathcal{H}(X, \mathcal{H}(Y, Z))$$

which shows that the cartesian functor $(- \times Y)$ is a left adjoint to $\mathcal{H}(Y, -)$. Thus the category of hological spaces and hological maps is cartesian closed.

Chapter 2

Toponomes and Topocosms

The category C_D described above is not merely cartesian closed; it has the further important property that its power objects are structured hom-sets. This is made precise in the following definitions.

Definition 2.1 *A construct C (struct over Set) is a toponome if it has the following two properties:*

1. *C is a topological construct (see 11.a, 11.f in [17]).*
2. *Every C -space X induces a toponomous hom-functor $C(X, -) : C \rightarrow C$ i.e. a functor such that $\sqcup \circ C(X, -) = C(X, -)$ and there is a natural bijection $\dagger_{WY}^X : C(W \times X, Y) \rightarrow C(W, C(X, Y))$, $\dagger(f)(w)(x) = .^f(w, x)$.*

Here $\sqcup$ denotes the underlying functor of the construct, and $C(X, Y)$ denotes the set of all C -morphisms from X to Y .

Proposition 2.2 *A topological construct is a toponome iff*

1. *Every one-point space is discrete and*
2. *Every cartesian functor preserves final epifamilies.*

Proof: 18.f in [17]

□

Definition 2.3 *A topocosm is a topological construct in which one-point spaces and empty spaces are discrete and final epifamilies are preserved by pullbacks.*

Proposition 2.4 *Every topocosm is a toponome.*

Proof: We only have to show that in a topocosm final epifamilies are preserved by cartesian functors. Let $e_i : X_i \rightarrow Y$ be a final epifamily and $- \times W$ a cartesian functor. Since

$$\begin{array}{ccc} X_i \times W & \xrightarrow{\text{pr}_1} & X_i \\ e_i \times W \downarrow & & \downarrow e_i \\ Y \times W & \xrightarrow{\text{pr}_1} & Y \end{array}$$

are pullback diagrams and final epifamilies are preserved under pullbacks, we get that $e_i \times W : X_i \times W \rightarrow Y \times W$ is a final epifamily. $\square$

Definition 2.5 *Let $\mathbf{B}$ be a construct where every constant map is a morphism, and let $\mathcal{F}$ be a collection of families of morphisms in $\mathbf{B}$. We call the members of $\mathcal{F}$ covering families. The pair $(\mathbf{B}, \mathcal{F})$ is called a base for a topocosm, if $\mathcal{F}$ satisfies the following axioms:*

B1 *The identity map is a covering family.*

B2 *Covering families are closed under composition.*

B3 *Every covering family is a final surjective family in $\mathbf{B}$.*

B4 *If $\iota_i : \Omega_i \rightarrow \Omega$ ($i \in I$) is a covering family and $f : \Lambda \rightarrow \Omega$ a morphism in $\mathbf{B}$, then there exists a covering family $\tau_j : \Lambda_j \rightarrow \Lambda$ ($j \in J$) such that every $f \circ \tau_j : \Lambda_j \rightarrow \Omega$ factors through some Ω_i .*

$$\begin{array}{ccc} \Lambda_j & \xrightarrow{\tau_j} & \Lambda \\ k_j \downarrow & & \downarrow f \\ \Omega_{i(j)} & \xrightarrow{\iota_{i(j)}} & \Omega \end{array}$$

For the following examples of a bases for a topocosms we need a concept of smoothness between convex $\Omega \subseteq \mathbb{R}^n$ with nonempty interior. These are also called solid domains (see [20]). As in [20] we will use C_s^∞ -maps.

Definition 2.6 *By C_s -maps we mean sequentially continuous maps. Let $\Omega \subseteq \mathbb{R}^n$ be a solid domain and F a locally convex vector space. A C_s -map $f : \Omega \rightarrow F$ is called a C_s^1 -map, if there exists a C_s -map $\Phi : \Omega \times \Omega \rightarrow [\mathbb{R}^n, F]$ (a difference factorizer for f) such that $f(x) - f(y) = \Phi(x, y)(x - y)$ holds for all x, y . Here $[\mathbb{R}^n, F]$ denotes the space of linear C_s -maps with the compact-open topology. For such f the C_s -map $Df : \Omega \rightarrow [\mathbb{R}^n, F]$ is defined by $Df(x) = \Phi(x, x)$, the value being independent of the choice of Φ . For $r \in \mathbb{N}$ the concept of C_s^r -maps is defined inductively to be a C_s -map which has a difference factorizer which is a C_s^{r-1} -map. A map which is C_s^r for all r is called a C_s^∞ -map. For open $\Omega \subseteq \mathbb{R}^n$ the C_s^∞ -maps are in fact the usual smooth maps (all partial derivatives exist and are continuous). If Ω is not open, the partial derivatives have a continuous extension to the border of Ω .*

Example 2.7 As base-category take all convex $\Omega \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) with nonempty interior. We call this category Conv . As morphisms between them take C_s^∞ -maps. As covering families take families $\iota_i : \Omega_i \rightarrow \Omega$ ($i \in I$) of embeddings, such that the restrictions of ι_i to the interiors of Ω_i , denoted by $\text{int}\Omega_i$, build a surjective family of open maps. The properties B1 and B2 are clearly satisfied. For property B3 we take the factorization $\iota_i \circ \text{inc} : \text{int}\Omega_i \rightarrow \Omega_i \rightarrow \Omega$. By 2.i in [20] this surjective family is final for C_s^∞ -maps. Therefore the outer family ι_i is final. For property B4 take ι_i and f as in the definition. Since every C_s^∞ -map between finite dimensional spaces is continuous, we get that $U_i := f^{-1}(\iota_i(\text{int}\Omega_i))$ is open in Λ . Since $\Lambda \subseteq \mathbb{R}^n$ is convex we can find open (in Λ) convex $\Lambda_{ij} \subseteq U_i$ ($j \in J(i)$) which cover U_i and since the U_i ($i \in I$) cover Λ , we get the desired covering family Λ_{ij} ($i \in I, j \in J(i)$) for Λ . If Λ is open, we can choose the Λ_{ij} as n -dimensional balls or equivalently as $\mathbb{R}^n$ ($\mathbb{R}^n$ is diffeomorphic to every n -dimensional ball).

We obtain the base $(\text{Conv}, \text{Emb})$.

From a given base for a topocosm $(\mathbf{B}, \mathcal{F})$ we can form another base by taking a full subcategory $\mathbf{A}$ of $\mathbf{B}$ as base-category and as covering families we take all families in $\mathbf{A}$ which are covering families in $(\mathbf{B}, \mathcal{F})$. Then the properties B1-B3 are automatically satisfied and one only has to check property B4 for covering families. This leads to the following definition:

Definition 2.8 *Let $(\mathbf{B}, \mathcal{F})$ be a base for a topocosm and $\mathbf{A}$ a full subcategory of $\mathbf{B}$. We call $(\mathbf{A}, \mathcal{F}_1)$ an I-base in $(\mathbf{B}, \mathcal{F})$ if the following conditions are satisfied:*

IB1 *A family $\tau_j : \Lambda_j \rightarrow \Lambda$ in $\mathbf{A}$ lies $\mathcal{F}_1$ iff it lies in $\mathcal{F}$.*

IB2 *If $\iota_i : \Omega_i \rightarrow \Omega$ ($i \in I$) is a family in $\mathcal{F}$ and $f : \Lambda \rightarrow \Omega$ a morphism in $\mathbf{B}$ where Λ lies in $\mathbf{A}$, then there exists a family $\tau_j : \Lambda_j \rightarrow \Lambda$ ($j \in J$) in $\mathcal{F}_1$ such that every $f \circ \tau_j : \Lambda_j \rightarrow \Omega$ factors through some Ω_i .*

Notice that the second condition is slightly more than what we need to ensure that $(\mathbf{A}, \mathcal{F}_1)$ is a base for a topocosm. The following examples give I-bases in $(\mathbf{Conv}, \mathbf{Emb})$.

Examples 2.9

- Let $\mathcal{H}^n := \{(x_1 \dots x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$ be the n -dimensional half-space and take as base-spaces all convex subsets of an $\mathcal{H}^n$ with nonempty interior. Denote this category by $\mathcal{H}$. By the same proof as in the first example, IB2 is satisfied and we obtain the base $(\mathcal{H}, \mathbf{Emb})$.
- Take all open convex $U \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) as base-spaces and call this category $\mathbf{Op}$. We get the I-base $(\mathbf{Op}, \mathbf{Emb})$ in $(\mathbf{Conv}, \mathbf{Emb})$.
- Take the category $\mathbf{Rn}$ of all $\mathbb{R}^n$ ($n \in \mathbb{N}$) as base-category. By the remark about the form of the $\Lambda_{i,j}$ in example 2.7, we conclude that it is an I-base in $(\mathbf{Conv}, \mathbf{Emb})$ but also in $(\mathbf{Op}, \mathbf{Emb})$. We call this base $(\mathbf{Rn}, \mathbf{Emb})$.
- Take $\mathbb{R}$ as the only base-space. We call this I-base $(\mathbb{R}, \mathbf{Emb})$.

Definition 2.10 Let $(\mathbf{B}, \mathcal{F})$ be a base for a topocosm. We define $\mathbf{C}_B$ spaces to be sets X structured with a set of maps $g : \Omega \rightarrow X$ (Ω a $\mathbf{B}$ -space), called imprints, which satisfies the following axioms:

A1 Every constant prefinal map $\Omega \rightarrow X$ is an imprint.

A2 If $g : \Omega \rightarrow X$ is an imprint and $f : \Lambda \rightarrow \Omega$ is a $\mathbf{B}$ -map, then $g \circ f$ is an imprint.

A3 If $\iota_i : \Omega_i \rightarrow \Omega$ ($i \in I$) is a covering family and $g : \Omega \rightarrow X$ a prefinal map such that every composite $g \circ \iota_i$ is an imprint, then g is an imprint.

A map $f : X \rightarrow Y$ between $\mathbf{C}_B$ -spaces is called a $\mathbf{C}_B$ -morphism, if $f \circ g$ is an imprint for all imprints g into X . We obtain the construct $\mathbf{C}_B$ of $\mathbf{C}_B$ -spaces and $\mathbf{C}_B$ -morphisms.

Let $(\mathbf{B}, \mathcal{F})$ be a base for a topocosm. We structure every base space Ω as $\mathbf{C}_B$ -space by taking all $\mathbf{B}$ -morphisms into Ω as imprints. This is a $\mathbf{C}_B$ structure since all constant maps are morphisms, the composite of morphisms is a morphism and since covering families are final families in $\mathbf{B}$. With this structure $\mathbf{B}$ becomes a subcategory of $\mathbf{C}_B$. This subcategory is also full since the identity map is an imprint. This gives the first part of the following proposition.

Proposition 2.11 The inserter $\mathbf{B} \rightarrow \mathbf{C}_B$ is a full and faithful functor. Let X be a $\mathbf{C}_B$ -space. A map $g : \Omega \rightarrow X$ is an imprint iff it is a $\mathbf{C}_B$ -morphism. Hence $\{g \in \mathbf{C}_B(\Omega, X) \mid \Omega \text{ a base space}\}$ forms a final epifamily for X .

Proof: Let $g : \Omega \rightarrow X$ be an imprint. By axiom A2 of $\mathbf{C}_B$ -spaces and by the definition of the $\mathbf{C}_B$ -structure of Ω we get that g is a $\mathbf{C}_B$ -morphism. On the other hand let $g : \Omega \rightarrow X$ be a $\mathbf{C}_B$ -map. Since $\text{id} : \Omega \rightarrow \Omega$ is an imprint we get that $g = g \circ \text{id}$ is an imprint. $\square$

Proposition 2.12 The construct $\mathbf{C}_B$ has initial structures. For a preinitial family $f_i : X \rightarrow Y_i$ ($i \in I$) the initial $\mathbf{C}_B$ -structure on X is given by all maps g such that $f_i \circ g$ is an imprint for all $i \in I$.

Proof: The imprints as defined in the proposition form a C_B structure: Every constant function is an imprint since the composite with every f_i is a constant function. Let $h : \Lambda \rightarrow \Omega$ be a B -map and $g : \Omega \rightarrow X$ an imprint, then $f_i \circ g$ is an imprint in Y and therefore (A2) $f_i \circ g \circ h$ is an imprint. By definition of the imprints into X we get that $g \circ h$ is an imprint. To verify the third axiom we take a covering family $\iota_j : \Omega_j \rightarrow \Omega$ ($j \in J$) and a map $g : \Omega \rightarrow X$ such that $g \circ \iota_j$ is an imprint for all $j \in J$. Then $f_i \circ g \circ \iota_j$ ($j \in J, i \in I$) is an imprint and therefore also $f_i \circ g$. We get that g is an imprint.

It remains to show that this structure is initial. Clearly all f_i become morphisms. Let $h : W \rightarrow X$ be a map such that $f_i \circ h$ is a C_B -map for all $i \in I$ and $g : \Omega \rightarrow W$ an imprint. Since $f_i \circ h$ is a morphism we get that $f_i \circ h \circ g$ is an imprint and therefore $h \circ g$ is an imprint into X . $\square$

Corollary 2.13 *If f_i ($i \in I$) is an initial family in B then it is an initial family in C_B .*

By proposition 2.12 we get that C_B is a topological construct. Therefore C_B also has final structures. The next proposition will give an explicit description of the final structure for epifamilies.

Proposition 2.14 (cf. 2.7 in [21]) *Let $e_i : X_i \rightarrow Y$ ($i \in I$) be an epifamily in C_B . By Γ we denote the following family: A map $g : \Omega \rightarrow Y$ is in Γ iff there exists a covering family $\iota_j : \Omega_j \rightarrow \Omega$ ($j \in J$) such that every $g \circ \iota_j$ factors over some X_i , i.e. there exist imprints $k_j : \Omega_j \rightarrow X_{i(j)}$ such that every $g \circ \iota_j$ can be written as $e_{i(j)} \circ k_j$. Then the epifamily e_i is final iff Y has Γ as its C_B -structure.*

Proof: First we show that Γ forms a C_B -structure. Let $g : \Omega \rightarrow Y$ be a constant map with value y . Since the epifamilies in a topological construct are surjective families, we can find a map e_i and $x \in X_i$ such that $e_i(x_i) = y$. For the covering family id and the imprint $k : \Omega \rightarrow X_i, \omega \mapsto x_i$ we get $e_i \circ k = g \circ \text{id}$.

For axiom A2 take a map $g : \Omega \rightarrow Y$ in Γ and a B -map $f : \Lambda \rightarrow \Omega$. Take a covering family ι_j and imprints k_j such that $e_{i(j)} \circ k_j = g \circ \iota_j$. By property B4 of covering families there exists a covering family $\tau_l : \Lambda_l \rightarrow \Lambda$ ($l \in L$) and B -maps m_l such that

$\iota_{j(l)} \circ m_j = f \circ \tau_l$. Since k_j is an imprint and m_l is a $\mathbf{B}$ -map, we get that $k_j \circ m_l$ is an imprint and we obtain the factorization $e_{i(j(l))} \circ (k_{j(l)} \circ m_l) = (g \circ f) \circ \tau_l$.

$$\begin{array}{ccccc}
 \Lambda_l & \xrightarrow{m_l} & \Omega_{j(l)} & \xrightarrow{k_{j(l)}} & X_{i(j(l))} \\
 \tau_l \downarrow & & \downarrow \iota_{j(l)} & & \downarrow e_{i(j(l))} \\
 \Lambda & \xrightarrow{f} & \Omega & \xrightarrow{g} & Y
 \end{array}$$

For axiom A3 take a map g and a covering family ι_j ($j \in J$) such that $g \circ \iota_j$ is in Γ for all $j \in J$. Then there exist covering families τ_{jl} ($j \in J, l \in L(j)$) and imprints k_{jl} such that $e_{i(jl)} \circ k_{jl} = (g \circ \iota_j) \circ \tau_{jl} = g \circ (\iota_j \circ \tau_{jl})$. Since the composite of covering families is a covering family A3 is satisfied.

$$\begin{array}{ccc}
 \Lambda_{jl} & \xrightarrow{k_{jl}} & X_{i(jl)} \\
 \tau_{jl} \downarrow & & \downarrow e_{i(jl)} \\
 \Omega_j & \xrightarrow{g \circ \iota_j} & Y \\
 & \searrow \iota_j & \nearrow g \\
 & \Omega &
 \end{array}$$

It remains to show that the structure defined by Γ is final. Let $f : Y \rightarrow W$ be a map such that $f \circ e_i$ is a $\mathbf{C}_B$ -map for all $i \in I$ and let g be an imprint into Y . We have to show that $f \circ g$ is an imprint. By definition of the structure on Y there exists a covering family ι_j and imprints k_j such that $e_{i(j)} \circ k_j = g \circ \iota_j$. Therefore $f \circ g \circ \iota_j = f \circ e_{i(j)} \circ k_j$ which is an imprint since $f \circ e_{i(j)}$ is a $\mathbf{C}_B$ -map. By axiom A3 $f \circ g$ is an imprint. $\square$

Proposition 2.15 *Let $(\mathbf{B}, \mathcal{F})$ be a base for a topocosm. Then $\mathbf{C}_B$ is a topocosm.*

Proof: Since there are no imprints into the empty space, it is trivially discrete. Let P be an one-point space, X a $\mathbf{C}_B$ -space and $f : P \rightarrow X$ a map. Since $f \circ g$ is a constant map for all imprints g into P , we get that f is a $\mathbf{C}_B$ -map and therefore

one-point spaces are discrete. Next we show that final epifamilies are preserved by pullbacks. Let

$$\begin{array}{ccc} A_i & \xrightarrow{g_i} & B_i \\ f_i \downarrow & & \downarrow e_i \\ C & \xrightarrow{h} & D \end{array}$$

be pullback squares and e_i a final epifamily (=final surjective family). Since pullbacks in topological constructs are formed by initial lift of the pullback in **Set**, we can assume that $A_i = \{(c, b) \in C \times B_i \mid e_i(b) = h(c)\}$ with the initial structure of f_i and g_i . First we show that f_i ($i \in I$) is a surjective family. Let $c \in C$. Since e_i ($i \in I$) is a surjective family, we find an index $j \in I$ and $b \in B_j$ such that $e_j(b) = h(c)$. Therefore $(c, b) \in A_j$ and we get that f_i is a surjective family. To show that f_i is also a final family we have to show (by proposition 2.14) that for every imprint $g : \Omega \rightarrow C$ we can find a covering family $\iota_j : \Omega_j \rightarrow \Omega$ ($j \in J$) and imprints $k_j : \Omega_j \rightarrow A_{i(j)}$ such that $g \circ \iota_j = f_{i(j)} \circ k_j$. Since $h \circ g$ is an imprint and e_i is a final epifamily, we can find a covering family ι_j and imprints $m_j : \Omega_j \rightarrow B_{i(j)}$ such that $h \circ g \circ \iota_j = e_{i(j)} \circ m_j$. By the universal property of the pullback we find imprints $k_j : \Omega_j \rightarrow A_{i(j)}$ such that $f_i \circ k_j = g \circ m_j$ and $g_{i(j)} \circ k_j = m_j$.

$$\begin{array}{ccccc} \Omega_j & & & & \\ \downarrow \iota_j & \searrow k_j & & \searrow m_j & \\ \Omega & & A_{i(j)} & \xrightarrow{g_{i(j)}} & B_{i(j)} \\ & \searrow g & \downarrow f_{i(j)} & & \downarrow e_{i(j)} \\ & & C & \xrightarrow{h} & D \end{array}$$

□

From the examples of bases for topocosms (see example 2.7 and examples 2.9) we can now build topocosms. These will be the main objects of this theses and form the foundation for one of the two calculus theories under investigation.

Examples 2.16

C_d is the topocosm where we take $(\text{Conv}, \text{Emb})$ as base. In [20] the associated spaces and maps are called diffeomeric spaces and diffeomeric maps.

$C_{\mathcal{H}}$ is the topocosm where we take $(\mathcal{H}, \text{Emb})$ as base (see examples 2.9). It can be used to generalize manifolds with boundary.

$C_{\mathcal{O}}$ is the topocosm where we take (Op, Emb) as base.

$C_{\mathbb{R}^n}$ is the topocosm where we take $(\mathbb{R}^n, \text{Emb})$ as base.

$C_{\mathbb{R}}$ is the topocosm where we take $(\mathbb{R}, \text{Emb})$ as base. It is also called the topocosm of diffeological spaces and maps [19].

$C_{\mathbb{D}}$ the topocosm of holological maps and spaces (see chapter 1), the complex version of $C_{\mathbb{R}}$.

Let X and Y be C_B -spaces. From proposition 2.15 we know that C_B is a topocosm and therefore (proposition 2.4) that there exists a toponomous hom-functor $C_B(X, -)$ which lifts the hom-functor in **Set**. Now we give an explicit description of the structure of the hom-spaces $C_B(X, Y)$.

Proposition 2.17 *Let C_B be a topocosm built from a base $(\mathcal{B}, \mathcal{F})$ and X, Y C_B -spaces. Then $g : \Omega \rightarrow C_B(X, Y)$ is an imprint iff $g^\dagger : \Omega \times X \rightarrow Y$ is a C_B -map, where $g^\dagger(\omega, x) = g(\omega)(x)$.*

Proof: First we show that the structure given in the proposition is a C_B -structure. Let $g : \Omega \rightarrow C_B(X, Y)$, $\omega \mapsto f$ be a constant function. We have to show, that $g^\dagger : \Omega \times X \rightarrow Y$ is a C_B -map. We take an imprint $g_1 \times g_2 : \Lambda \rightarrow \Omega \times X$ where g_1 is a $\mathcal{B}$ -map and g_2 is an imprint into X . Then $g^\dagger \circ (g_1 \times g_2) = (f \circ g_2)^\dagger$ which is an imprint since f is a C_B -map. For axiom A2 take an imprint $g : \Omega \rightarrow C_B(X, Y)$ and a $\mathcal{B}$ -map $f : \Lambda \rightarrow \Omega$. Then $(g \circ f)^\dagger = g^\dagger \circ (f \times \text{id})$ is an morphism. For axiom A3 let $\iota_i : \Omega_i \rightarrow \Omega$ ($i \in I$) be a covering family and $g : \Omega \rightarrow C_B(X, Y)$ a function such that every composite $g \circ \iota_i$ is an imprint. Then $g^\dagger \circ (\iota_i \times \text{id}) = (g \circ \iota_i)^\dagger$ is a

C_B -map. Since ι_i ($i \in I$) is a final epifamily in C_B and final epifamilies are preserved by cartesian functors, we get that $(\iota_i \times \text{id})$ ($i \in I$) is a final family and therefore $g^\dagger$ is a morphism.

Next we show that this structure has the desired property; i.e. a map $h : W \rightarrow C_B(X, Y)$ is a morphism iff $h^\dagger : W \times X \rightarrow Y$ is a morphism. By uniqueness of such a structure this completes the proof.

$$\begin{array}{ccc}
 \Omega & \xrightarrow{h \circ g} & C_B(X, Y) \\
 g \downarrow & & \downarrow \text{id} \\
 W & \xrightarrow{h} & C_B(X, Y)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Omega \times X & \xrightarrow{(h \circ g)^\dagger} & Y \\
 g \times \text{id} \downarrow & & \downarrow \text{id} \\
 W \times X & \xrightarrow{h^\dagger} & Y
 \end{array}$$

A map h is a morphism iff $h \circ g$ is an imprint for all imprints g into W . This is equivalent to $h^\dagger \circ (g \times \text{id}) = (h \circ g)^\dagger$ being a morphism. Since the family of all imprints into W form a final epifamily which is preserved by cartesian functors, we are done. $\square$

Chapter 3

Relations Between Topocosms

From the examples 2.16 we see that often one base for a topocosm lies inside another, more precisely is an I-base in the bigger base category. How do the resulting topocosms relate? To answer this question we give the following definition.

Definition 3.1 *Let $(B, \mathcal{F})$ be a base for a topocosm and $(A, \mathcal{F}_1)$ an I-base in $(B, \mathcal{F})$. We define a restructuring functor $R : C_B \rightarrow C_A$ by restricting the imprints in B to A , i.e.: For Λ an A -space, we define a map $c : \Lambda \rightarrow RX$ to be an imprint, if $c : \Lambda \rightarrow X$ is an imprint in C_B .*

In the other direction we define a restructuring functor $V : C_A \rightarrow C_B$ in the following way: A map $g : \Omega \rightarrow VW$ is an imprint iff $g : R\Omega \rightarrow W$ is a C_A -map, i.e. iff for every A -space Λ and every B -map $c : \Lambda \rightarrow \Omega$ we get that $g \circ c$ is an imprint in W .

Remark 3.2 If we take an A -space Λ and regard it as C_B -space, it has as imprints all B -maps $g : \Omega \rightarrow \Lambda$, where Ω is a B -space. If we apply the functor R to this space, the imprints to $R\Lambda$ will be all B -maps from $\Gamma \rightarrow \Lambda$, where Γ is an A -space. This is exactly the structure of Λ regarded as a C_A -space. Therefore we will always write Λ instead of $R\Lambda$.

Before we describe some properties of the functors R and V , we have to show that they are well defined. First we show that the structure on RX is a C_A -structure if X is a C_B -space. This is clear since A is a subcategory of B and every covering family in A is also a covering family in B . Let $f : X \rightarrow Y$ be a C_B -map. We have to show that $f : RX \rightarrow RY$ is a C_A -map. Let $c : \Lambda \rightarrow RX$ be an imprint. Then

$c : \Lambda \rightarrow X$ is an imprint and therefore $f \circ c : \Lambda \rightarrow Y$ is an imprint. By definition of the structure on $\mathbf{R}Y$ we get that $f \circ c : \Lambda \rightarrow \mathbf{R}Y$ is an imprint.

Next we show that the structure on $\mathbf{V}W$ is well defined for every $\mathbf{C}_A$ -space W . Every constant map is an imprint since every constant map is a $\mathbf{C}_A$ -map. Let $g : \Omega \rightarrow \mathbf{V}W$ be an imprint and $f : \Lambda \rightarrow \Omega$ a $\mathbf{B}$ -map. Then $f : \mathbf{R}\Lambda \rightarrow \mathbf{R}\Omega$ and $g : \mathbf{R}\Omega \rightarrow W$ are $\mathbf{C}_A$ -maps. Therefore $g \circ f : \Lambda \rightarrow \mathbf{V}W$ is an imprint. For the axiom A3 take a covering family $\iota_i : \Omega_i \rightarrow \Omega$ ($i \in I$) and a map $g : \Omega \rightarrow \mathbf{V}W$ such that $g \circ \iota_i$ is an imprint into $\mathbf{V}W$ for all $i \in I$. We have to show that $g : \mathbf{R}\Omega \rightarrow W$ is a $\mathbf{C}_A$ -map. Take an imprint $c : \Lambda \rightarrow \mathbf{R}\Omega$. Since the imprints into $\mathbf{R}\Omega$ are exactly the $\mathbf{B}$ -maps and $(\mathbf{A}, \mathcal{F}_1)$ is an I-base in $(\mathbf{B}, \mathcal{F})$ we get a covering family $\tau_j : \Lambda_j \rightarrow \Lambda$ in $\mathbf{A}$ and imprints ($\mathbf{B}$ -maps) $k_j : \Lambda_j \rightarrow \mathbf{R}\Omega_{i(j)}$ such that $\iota_{i(j)} \circ k_j = c \circ \tau_j$. Since $(g \circ \iota_i)$ is a $\mathbf{C}_A$ -map we get that $(g \circ \iota_i) \circ k_j = (g \circ c) \circ \tau_j$ is an imprint in W and therefore $g \circ c$ is an imprint.

$$\begin{array}{ccc}
 \Omega_i & \xrightarrow{\iota_i} & \Omega \\
 & \searrow & \vdots \\
 & & \mathbf{V}W
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & \Lambda_j & \xrightarrow{\tau_j} & \Lambda \\
 & & \downarrow k_j & & \downarrow c \\
 \Omega_i & \xrightarrow{\iota_i} & \mathbf{R}\Omega_i & \xrightarrow{\iota_i} & \mathbf{R}\Omega \\
 & \searrow & \downarrow & & \downarrow g \\
 & & & & W
 \end{array}$$

It remains to show that $h : \mathbf{V}W \rightarrow \mathbf{V}Z$ is a $\mathbf{C}_B$ -map, if $h : W \rightarrow Z$ is a $\mathbf{C}_A$ -map. Take an imprint $g : \Omega \rightarrow \mathbf{V}W$. Then $g : \mathbf{R}\Omega \rightarrow W$ is a $\mathbf{C}_A$ -map and therefore the composite $h \circ g$ is an $\mathbf{C}_A$ -map.

Proposition 3.3 *The functor $\mathbf{V} : \mathbf{C}_A \rightarrow \mathbf{C}_B$ is a full and faithful right adjoint. The associated left adjoint is given by $\mathbf{R}$. The counit is the identity and $\mathbf{R} \circ \mathbf{V}$ is the identity functor.*

Proof: First we show that the rigid map $u : X \rightarrow \mathbf{V}\mathbf{R}X$ is a $\mathbf{C}_B$ -map for every $\mathbf{C}_B$ -space X . Let $g : \Omega \rightarrow X$ be an imprint. Since $\mathbf{R}$ is a functor, we get that $g : \mathbf{R}\Omega \rightarrow \mathbf{R}X$ is a $\mathbf{C}_A$ -map and therefore $g : \Omega \rightarrow \mathbf{V}\mathbf{R}X$ is an imprint. To check the

universal property for adjunctions, take a C_B -map $f : X \rightarrow VW$. We have to show that $f : RX \rightarrow W$ is a C_A -map. Take an imprint $c : \Lambda \rightarrow RX$. Then $c : \Lambda \rightarrow X$ is an imprint and since f is a C_B -map, we get that $f \circ c : \Lambda \rightarrow VW$ is an imprint. Since Λ is an A -space, this is equivalent to $f \circ c : \Lambda \rightarrow W$ is an imprint. To show that the counit $v : RVW \rightarrow W$ is the identity, we take an imprint $g : \Lambda \rightarrow W$. Then g is also an imprint in VW and therefore in RVW . Since the counit is rigid (see e.g. 14.g in [17]), we get that it is the identity and $RV = \text{Id}$. From the adjunction we get

$$C_A(RX, W) \simeq C_B(X, VW)$$

and substituting X by VZ we get

$$C_A(Z, W) \simeq C_B(VZ, VW)$$

which shows that the functor V is full and faithful (see also 6.e in [17]). $\square$

Proposition 3.4 *The topocosms $C_{\mathbb{R}^n}$ and C_O are isomorphic (see examples 2.16).*

Proof: By proposition 3.3 we only have to show, that $V \circ R = \text{Id}$. Let $g : O \rightarrow VRX$ be an imprint where X is a C_O -space and O open in $\mathbb{R}^n$. We can find a covering family $\iota_i : \mathbb{R}^n \rightarrow O$ ($i \in I$). Then all $g \circ \iota_i : \mathbb{R}^n \rightarrow RX$ are $C_{\mathbb{R}^n}$ -maps, therefore imprints in RX and therefore imprints in X . Because ι_i ($i \in I$) is a covering family we conclude that $g : O \rightarrow X$ is an imprint. $\square$

Proposition 3.5 *The functors R and V of definition 3.1 preserve initial families.*

Proof: Let $f_i : X \rightarrow Y_i$ ($i \in I$) be an initial family in C_B . Take a map $c : \Lambda \rightarrow RX$ such that $f_i \circ c$ is an imprint in RY_i for all $i \in I$. Then $f_i \circ c : \Lambda \rightarrow Y_i$ is an imprint for all $i \in I$ and therefore c is an imprint in X and hence in RX .

Let $f_i : W \rightarrow Z_i$ ($i \in I$) be an initial family in C_A . Take a map $g : \Omega \rightarrow VW$ such that $f_i \circ g$ is an imprint in VZ_i for all $i \in I$. By the adjunction we get that $f_i \circ g : R\Omega \rightarrow Z_i$ is a C_A -map for all $i \in I$. Since the f_i form an initial family in C_A , we get that $g : R\Omega \rightarrow W$ is a C_A -map. Again by the adjunction we conclude, that $g : \Omega \rightarrow VW$ is an C_B -map and therefore an imprint.

$$\begin{array}{ccc}
 VW & \xrightarrow{f_i} & VZ_i \\
 \uparrow g & \nearrow & \\
 \Omega & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 W & \xrightarrow{f_i} & Z_i \\
 \uparrow q & \nearrow & \\
 R\Omega & &
 \end{array}$$

□

While proposition 3.3 shows that the functor V is a full and faithful right adjoint, we will show now that it also preserves the hom-spaces.

Proposition 3.6 *Let $(\mathcal{B}, \mathcal{F})$ be a base for a topocosm, $(A, \mathcal{F}_1)$ an I -base in $(\mathcal{B}, \mathcal{F})$ and let Z, W be C_A -spaces. Then*

$$VC_A(Z, W) = C_B(VZ, VW)$$

Proof: It is clear from the definition of V and from proposition 3.3 that the underlying sets are the same. To prove that also the structures are the same, take an imprint $g : \Omega \rightarrow VC_A(Z, W)$. We show that it is also an imprint in $C_B(VZ, VW)$. By the adjunction we get that $g : R\Omega \rightarrow C_A(Z, W)$ is an C_A -map and therefore $g^\dagger : R\Omega \times Z \rightarrow W$ (C_A is a toponome). Since V preserves initial families (proposition 3.5) it also preserves products. Therefore we can apply V and obtain that $g^\dagger : VR\Omega \times VZ \rightarrow VW$ is a C_B -map. Hence $g : VR\Omega \rightarrow C_B(VZ, VW)$ is a C_B -map. Since the unit of the adjunction $u_\Omega : \Omega \rightarrow VR\Omega$ is a rigid map, we get that $g : \Omega \rightarrow VR\Omega \rightarrow C_B(VZ, VW)$ is a C_B -map and therefore an imprint.

On the other hand take an imprint $g : \Omega \rightarrow C_B(VZ, VW)$. Then $g^\dagger : \Omega \times VZ \rightarrow VW$ is a C_B -map. Since R preserves products we get that $g^\dagger : R\Omega \times RVZ \rightarrow RVW$ is a C_A -map. Since $R \circ V = \text{Id}$ and since C_A is a toponome we get that $g : R\Omega \rightarrow C_A(Z, W)$ is a C_A -map. By the adjunction we get that $g : \Omega \rightarrow VC_A(Z, W)$ is an imprint. □

Lemma 3.7 *Let $(A, \mathcal{F}_1)$ be an I -base in $(\mathcal{B}, \mathcal{F})$ and take an A -space Λ . If B equals the final hull of A in $\mathcal{B}$, then $V\Lambda = \Lambda$.*

Proof: A map $g : \Omega \rightarrow \mathbf{V}\Lambda$ is an imprint iff $g \circ c$ is a $\mathbf{B}$ -map for all $\mathbf{B}$ -maps (imprints) $c : \Gamma \rightarrow \Omega$ where Γ is an $\mathbf{A}$ -space. By assumption this family is final and therefore g is a $\mathbf{B}$ -map. $\square$

Lemma 3.8 *The functor $\mathbf{R}$ maps the initial hull of $\mathbf{A}$ in $\mathbf{C}_B$, denoted by $\text{ih}_{\mathbf{C}_B}(\mathbf{A})$, into $\text{ih}_{\mathbf{C}_A}(\mathbf{A})$ and if $\mathbf{B}$ equals the final hull of $\mathbf{A}$ in $\mathbf{B}$, then $\mathbf{V}$ maps $\text{ih}_{\mathbf{C}_A}(\mathbf{A})$ into $\text{ih}_{\mathbf{C}_B}(\mathbf{A})$.*

Proof: Let W be in $\text{ih}_{\mathbf{C}_A}(\mathbf{A})$. Then there exists an initial family $f_i : W \rightarrow \Lambda_i$ ($i \in I$) with Λ_i $\mathbf{A}$ -spaces. By proposition 3.5 we get that $f_i : \mathbf{V}W \rightarrow \mathbf{V}\Lambda_i$ ($i \in I$) is an initial family in $\mathbf{C}_B$. By lemma 3.7 $\mathbf{V}W$ lies in $\text{ih}_{\mathbf{C}_B}(\mathbf{A})$. The other direction is analogous (see remark 3.2). $\square$

Proposition 3.9 *Let $(\mathbf{B}, \mathcal{F})$ be a base for a topocosm and $(\mathbf{A}, \mathcal{F}_1)$ an I -base in $(\mathbf{B}, \mathcal{F})$ such that $\mathbf{B}$ is the final hull of $\mathbf{A}$ in $\mathbf{B}$. Then $\text{ih}_{\mathbf{C}_A}(\mathbf{A})$ and $\text{ih}_{\mathbf{C}_B}(\mathbf{A})$ are isomorphic constructs. The isomorphism is given by the restructuring functors $\mathbf{R}$ and $\mathbf{V}$.*

Proof: By lemma 3.8 and by proposition 3.3 the only thing to be shown is that $\mathbf{V} \circ \mathbf{R}$ is the identity functor on $\text{ih}_{\mathbf{C}_B}(\mathbf{A})$. Let $g : \Omega \rightarrow \mathbf{V}\mathbf{R}X$ be an imprint, which is equivalent to $g : \mathbf{R}\Omega \rightarrow \mathbf{R}X$ being a $\mathbf{C}_A$ -map. We have to show that $g : \Omega \rightarrow X$ is a $\mathbf{C}_B$ -map. Since X lies in $\text{ih}_{\mathbf{C}_B}(\mathbf{A})$, we can find an initial family $f_i : X \rightarrow \Lambda_i$ ($i \in I$) with Λ_i being $\mathbf{A}$ -spaces. Therefore it suffices to show that $f_i \circ g$ is a $\mathbf{B}$ -map for all $i \in I$. Since $\mathbf{B}$ is the final hull of $\mathbf{A}$ in $\mathbf{B}$, we find a final family $c_j : \Lambda_j \rightarrow \Omega$ ($j \in J$) and we only have to show that $f_i \circ g \circ c_j$ is a $\mathbf{B}$ -map. But since the $f_i \circ g : \mathbf{R}\Omega \rightarrow \mathbf{R}X \rightarrow \Lambda_i$ are $\mathbf{C}_A$ -maps, they map $\mathbf{B}$ -maps to $\mathbf{B}$ -maps and therefore $f_i \circ g \circ c_j$ is a $\mathbf{B}$ -map for all $i \in I$ and $j \in J$. $\square$

Proposition 3.10 (Boman 67) *Let P be an open subset of $\mathbb{R}^n$. A map $g : P \rightarrow \mathbb{R}$ is smooth iff its composites with smooth curves are smooth.*

Proof: 4.3.27 in [8] $\square$

Corollary 3.11 *The functors $\mathbf{R}$ and $\mathbf{V}$ restrict to mutually inverse functors between the constructs $\mathbf{ih}_{C_{\mathbb{R}^n}}(\mathbb{R})$ and $\mathbf{ih}_{C_{\mathbb{R}}}(\mathbb{R})$ (see examples 2.16).*

Proof: Proposition 3.10 tells us that every $\mathbb{R}^n$ is in the final hull of $\mathbb{R}$. Therefore we can apply proposition 3.9 to obtain the result. $\square$

The next example shows that we can't expand the isomorphism to $C_{\mathbb{R}^n}$.

Example 3.12 Take $(\mathbb{R}, \text{Emb})$ as I-base in $(\mathbb{R}^n, \text{Emb})$. Define a $C_{\mathbb{R}^n}$ -space $\mathcal{R}^2$ by giving $\mathbb{R}^2$ the final $C_{\mathbb{R}^n}$ -structure of the smooth maps $\mathbb{R} \rightarrow \mathbb{R}^2$. By the form of final structures in $C_{\mathbb{R}^n}$, all imprints have one dimensional image. Therefore the rigid map $h : \mathbb{R}^2 \rightarrow \mathcal{R}^2$ can't be an imprint in $C_{\mathbb{R}^n}$. But $h : \mathbb{R}\mathbb{R}^2 \rightarrow \mathbb{R}\mathcal{R}^2$ is a $C_{\mathbb{R}}$ -map since all smooth curves are imprints in $\mathcal{R}^2$, and therefore $h : \mathbb{R}^2 \rightarrow \mathbf{VR}\mathcal{R}^2$ is an imprint in $C_{\mathbb{R}^n}$. We conclude that $\mathbf{V} \circ \mathbf{R}$ is not the identity functor.

In [8] the following definition is given to generate categories by sets of maps.

Definition 3.13 *Let $\mathcal{M}$ be a set of maps from Ω to R , where Ω and R are fixed sets. An $\mathcal{M}$ -space is a set X together with a set $\mathcal{C}_X \subseteq X^\Omega$ called curves and a set $\mathcal{F}_X \subseteq R^X$ called functions, which determine each other in the way that:*

$c \in \mathcal{C}_X \Leftrightarrow f \circ c : \Omega \rightarrow R$ is in $\mathcal{M}$ for all $f \in \mathcal{F}_X$ and

$f \in \mathcal{F}_X \Leftrightarrow f \circ c : \Omega \rightarrow R$ is in $\mathcal{M}$ for all $c \in \mathcal{C}_X$

A map $h : X \rightarrow Y$ between $\mathcal{M}$ -spaces X and Y is called an $\mathcal{M}$ -map, if $h \circ c \in \mathcal{C}_Y$ for all $c \in \mathcal{C}_X$. Equivalently one could demand

$f \circ h \circ c = f_(c^*(h)) : \Omega \rightarrow R$ is in $\mathcal{M}$ for all $f \in \mathcal{F}_Y$ and $c \in \mathcal{C}_X$, or*

$f \circ h \in \mathcal{F}_X$ for all $f \in \mathcal{F}_Y$.

We obtain the category $\underline{\mathcal{M}}$ of $\mathcal{M}$ -spaces and $\mathcal{M}$ -maps.

We give a few examples of categories defined in this way.

Examples 3.14

- If one takes $\Omega = R = \mathbb{R}$ and for $\mathcal{M}$ all smooth maps, one gets the category of smooth spaces and maps (see [7] or [8]). Analogously Lipschitz spaces are defined by taking Lipschitz maps instead of smooth maps.

- In [8] a certain class of bornological spaces were defined by taking $\Omega = \mathbb{N}$, $R = \mathbb{R}$ and for $\mathcal{M}$ the set of bounded sequences of real numbers was taken.
- In [28] holomorphic spaces were defined by choosing $\Omega = \mathbb{D}$ (the unit disc), $R = \mathbb{C}$ and for $\mathcal{M}$ the set of all holomorphic maps from $\mathbb{D}$ to $\mathbb{C}$.

Proposition 3.15 *Let $(B, \mathcal{I})$ be a base for a topos where B has only one object Ω and let R be a C_B -space. Denote the set of all imprints $g : \Omega \rightarrow R$ by $\mathcal{M}$. Then $\underline{\mathcal{M}}$ is isomorphic to $\text{ih}_{C_B}(R)$.*

Proof: We define a functor $\mathbf{M} : \text{ih}_{C_B}(R) \rightarrow \underline{\mathcal{M}}$ in the following way: For an $\text{ih}_{C_B}(R)$ -space X we take the same underlying set, as C_{MX} we take all imprints $c : \Omega \rightarrow X$ and for $\mathcal{F}_{MX}$ we take all C_B -maps $f : X \rightarrow R$. By definition of the C_B -maps it is clear that C_{MX} determines $\mathcal{F}_{MX}$, and since X lies in the initial hull of R , we get that $\mathcal{F}_{MX}$ determines C_{MX} . Since the morphisms are defined in the same way, the functor $\mathbf{M}$ is well defined.

In the other direction we define a functor $\mathbf{I} : \underline{\mathcal{M}} \rightarrow \text{ih}_{C_B}(R)$ by taking all $c \in C_W$ as imprints in IW , where W is an $\underline{\mathcal{M}}$ -space. We show that this gives a C_B -structure. All constant maps are imprints, since the composite with maps in $\mathcal{F}$ are constant maps, and these are in $\mathcal{M}$. Let $c : \Omega \rightarrow X$ be an imprint and $h : \Omega \rightarrow \Omega$ a B -map, then $f \circ c$ is an imprint in R for all $f \in \mathcal{F}$ and therefore $f \circ c \circ h$ is an imprint for all $f \in \mathcal{F}$. We conclude that $c \circ h$ is an imprint in IW . For the third axiom take a covering family $\iota_i : \Omega \rightarrow \Omega$ ($i \in I$) and $c : \Omega \rightarrow IW$ a function such that $c \circ \iota_i$ is an imprint in IW for all $i \in I$. Then $f \circ c \circ \iota_i$ is in $\mathcal{M}$ and therefore an imprint in R for all $i \in I$ and $f \in \mathcal{F}$. We get that $f \circ c$ is an imprint in R and therefore c is an imprint. Since $\mathcal{F}$ is an initial family in C_B with codomain R , we get that IW lies in $\text{ih}_{C_B}(R)$. Clearly the functors are inverse to each other. $\square$

Proposition 3.15 shows for example that $\text{ih}_{C_{\mathbb{R}}}(\mathbb{R})$ equals the smooth spaces and $\text{ih}_{C_{\mathbb{C}}}(\mathbb{C})$ equals the holomorphic spaces (see examples 2.16, 3.14).

Chapter 4

Linear Spaces

Linear spaces play an important role in calculus. But linear spaces in topocosms may have rather pathological behavior. For example they need not be separated. Therefore we will find subcategories which are better suited for our purposes. The main tool for this will be factorization systems.

Definition 4.1 *A real toponome ($\mathbb{R}$ -toponome) is a toponome $\mathcal{C}$ in which a $\mathcal{C}$ -structure is specified for the field of reals $\mathbb{R}$ so that the arithmetical operations $+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\text{scale} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ (addition and multiplication) are $\mathcal{C}$ -maps. If we replace the real field $\mathbb{R}$ by the complex field $\mathbb{C}$, we obtain a complex toponome ($\mathbb{C}$ -toponome).*

All examples of toponomes which are mentioned in this text are real or complex toponomes with the usual arithmetical structure. From now on we will assume all toponomes to be $\mathbb{K}$ -toponomes, where $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.2 *A linear $\mathcal{C}$ -space, or $\mathcal{LC}$ -space, is a triple (E, ξ, λ) such that the pair (E, ξ) is a $\mathcal{C}$ -space and λ is a pair of $\mathcal{C}$ -maps, the linear operations, namely*

$$+_E : E \times E \rightarrow E \text{ (addition),}$$

$$\text{scale}_E : \mathbb{K} \times E \rightarrow E \text{ (scalar multiplication),}$$

such that the linear space axioms are satisfied. For $\mathcal{LC}$ -spaces E, F we denote by $[E, F]$ the set of all linear $\mathcal{C}$ -maps ($\mathcal{LC}$ -maps) with the pointwise linear structure and with the initial $\mathcal{C}$ -structure of the inclusion map $[E, F] \rightarrow \mathcal{C}(E, F)$. By 24.a in [18] this is a $\mathcal{LC}$ -space. We define $E^ := [E, \mathbb{K}]$.*

Definition 4.3 A factorization system in a category $\mathbf{A}$ means a pair (E, M) such that

1. E is a class of epimorphisms closed under composition;
2. M is a class of monomorphisms closed under composition;
3. Every isomorphism lies in E and in M ;
4. (Diagonal Property) if $m \circ f = n \circ e$ holds with $e \in E$ and $m \in M$ then there exists a diagonal morphism d such that $d \circ e = f$ and $m \circ d = n$.

When such a factorization system exists, we say $\mathbf{A}$ has (E, M) factorization.

Example 4.4 Since $\mathbf{LC}$ is a topological struct over $\mathbf{Lin}$ (see e.g. 23.h in [18]), we get by 13.g in [17] that

- $\mathbf{LC}$ has (epi, embedding)-factorization.
- $\mathbf{LC}$ has (quotient, mono)-factorization.

Definition 4.5 Let (E, M) be a factorization system for $\mathbf{A}$. Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{V}$ be a functor with right adjoint $\mathbf{G}$ and unit of adjunction $u : \mathbf{Id} \rightarrow \mathbf{G} \circ \mathbf{F}$. We denote by $\mathbf{MsubG}$ the full subcategory determined by all $\mathbf{A}$ -objects A such that there exists a $\mathbf{V}$ -object V and a morphism $m : A \rightarrow \mathbf{G}V$ in the class M .

Proposition 4.6 Let $\mathbf{G} : \mathbf{V} \rightarrow \mathbf{A}$ be a right adjoint, where $\mathbf{A}$ has (E, M) factorization. Then

1. $\mathbf{MsubG}$ is a replete reflective subcategory of $\mathbf{A}$.
2. Every reflection map $r_A : A \rightarrow !_{\tau}A$ is obtained as the inner factor in the (E, M) factorization of the unit component $u_A : A \rightarrow \mathbf{GFA}$.
3. A lies in $\mathbf{MsubG}$ iff u_A lies in M .

Proof: 6.i in [17]

□

The main application of this proposition is shown in

Example 4.7 For every toponome $\mathcal{C}$ we get (see definition 2.1) natural bijective maps $\mathcal{C}(E, \mathcal{C}(X, F)) \rightarrow \mathcal{C}(E \times X, F) \simeq \mathcal{C}(X \times E, F) \rightarrow \mathcal{C}(X, \mathcal{C}(E, F))$. The composite is given by the flip-operator $\S_{EF}^X(g)(x)(e) = g(e)(x)$. This bijection is in fact a natural isomorphism (see 18.j in [17]). If E and F are LC-spaces this isomorphism restricts to an isomorphism

$$[E, \mathcal{C}(X, F)] \simeq \mathcal{C}(X, [E, F]).$$

From

$$[E, \mathcal{C}(X, F)] \simeq \mathcal{C}^{op}([E, F], X)$$

we get that $\mathcal{C}(-, \mathbb{R}) : \mathcal{C}^{op} \rightarrow \text{LC}$ is a right adjoint to $\text{LC}(-, \mathbb{R})$.

By proposition 4.6 we build the replete reflective subcategory mLC of LC by taking (quotient, mono) as factorization system; see example 4.4. It is formed by all LC-spaces E such that there exists a monomorphism $m : E \rightarrow \mathcal{C}(X, \mathbb{R})$ with X a $\mathcal{C}$ -space.

Definition 4.8 *An analyte (in LC) is a subcategory aLC such that:*

1. *aLC is replete and reflective;*
2. *aLC is preserved by all toponomial functors $\mathcal{C}(X, -)$;*
3. *aLC has $\mathbb{K}$ among its spaces.*

Proposition 4.9 *The category mLC of example 4.7 is an analyte.*

Proof: Property 1 follows from the construction (proposition 4.6). Because the functors $\mathcal{C}(X, -)$ are right adjoint (with left adjoint $(- \times X)$), they preserve monomorphisms (see 13.b in [17]). So if $m : F \rightarrow \mathcal{C}(Y, \mathbb{K})$ is a monomorphism, the same holds for $\mathcal{C}(X, m) : \mathcal{C}(X, F) \rightarrow \mathcal{C}(X, \mathcal{C}(Y, \mathbb{K})) \simeq \mathcal{C}(X \times Y, \mathbb{K})$. Therefore property 2 is satisfied. Since the map $K : \mathbb{K} \rightarrow \mathcal{C}(\mathbb{K}, \mathbb{K})$, which maps every $t \in \mathbb{K}$ to the constant map with value t , is a monomorphism, property 3 is also satisfied. □

Definition 4.10 M is called an outer class in a category K if there exists a class of epimorphisms E such that K has (E, M) factorization. Let D be a class of epimorphisms in the category K . We define $D_{\nearrow}$ to consist of all monomorphisms m such that for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ u \downarrow & & \downarrow v \\ C & \xrightarrow{m} & D \end{array}$$

with $e \in D$ there exists a diagonal morphism $d : B \rightarrow C$ such that $d \circ e = u$ and $m \circ d = v$. A D -strong monomorphism means a member of $D_{\nearrow}$.

Proposition 4.11 Let R be a replete reflective subcategory of a topological Set or Lin struct. For a class M of monomorphisms in R the following statements are equivalent.

1. M is an outer class in R .
2. $M = D_{\nearrow}$ for some class D of epimorphisms in R .
3. M contains every regular mono in R and is closed under compositions, pull-backs and intersections.

Proof: 26.e in [18]

□

Definition 4.12 A toponomial outer class in an analyte $\mathbf{aLC}$ is an outer class preserved by all functors $C(X, -) : \mathbf{LC} \rightarrow \mathbf{LC}$.

Proposition 4.13 Let D be a class of epimorphisms in an analyte $\mathbf{aLC}$. If D contains all surjective $\mathbf{aLC}$ -maps, then $D_{\nearrow}$ is a toponomial outer class.

Proof: 28.c in [18]

□

Proposition 4.14 *Let M be a toponomial outer class in an analyte $\mathbf{aLC}$ and $\mathbf{slC}$ the replete reflective subcategory induced by M and the right adjoint $C(-, \mathbb{K})$, see example 4.7. Then $\mathbf{slC}$ is an analyte in $\mathbf{aLC}$.*

Proof: 28.b in [18] □

In the following examples we construct the categories which will be used most frequently in the rest of this thesis.

Examples 4.15 Take $(\textit{epi}, \textit{embedding})$ -factorization in $\mathbf{LC}$; see example 4.4. Since the epimorphisms in $\mathbf{LC}$ are the surjective maps, we get by proposition 4.13 and proposition 4.14 an analyte that we call $\mathbf{eLC}$. It is formed by all $\mathbf{LC}$ -spaces E such that there exists an embedding $m : E \rightarrow C(X, \mathbb{R})$ with X a $\mathbf{C}$ -space. These spaces are called embeddable spaces.

Let D° be the class of all epimorphisms in $\mathbf{mLC}$. Since this class contains all surjective maps, D° is an toponomial outer class. Again we get an analyte which will be denoted by $\mathbf{oLC}$ (optimal $\mathbf{LC}$ -spaces).

Chapter 5

Linear Bornological Spaces

As we will see in chapter 6, bounded sets play a central role in the Frölicher-Kriegel calculus. In this chapter we will introduce the toponome of bornological spaces and maps. Its structure is described in terms of bounded sets. Furthermore we will show, that the eLC-spaces of most toponomes (topocosms) in our examples carry the initial structure of its linear functionals. This will be used to prove isomorphisms between categories of embeddable spaces.

Definition 5.1 *A bornological space is a pair (X, ξ) formed by a set X and its structure ξ . The latter is a bornology; i.e. a set of specified bounded subsets of X , subject to the axioms:*

Cb1 *Every singleton $\{x\}$ is bounded.*

Cb2 *Every subset of a bounded set is bounded.*

Cb3 *Every finite union of bounded sets is bounded.*

A bornological map between bornological spaces is a function that preserves bounded subsets. We denote the construct of bornological spaces and bornological maps by C_b .

Proposition 5.2 *C_b is a toponome. The toponomous mapping spaces $C_b(X, Y)$ are structured as follows: H is bounded in $C_b(X, Y)$ iff for every bounded A in X the image $H(A)$ is bounded in Y .*

Proof: 21.d in [17]

□

Let $\mathcal{C}$ be a toponome. We say a space $E \in \mathbf{LC}$ upholds the categorical Banach-Steinhaus principle, if every bilinear function $b : E \times F \rightarrow G \in \mathbf{eLC}$ which is a $\mathcal{C}$ -map in each variable separately, is a $\mathcal{C}$ -map (jointly). A class Q of $\mathcal{C}$ -spaces is called finally generating when every $\mathcal{C}$ -space X admits a final epimorphic family $f_i : \Omega_i \rightarrow X$ ($i \in I$) with all $\Omega_i \in Q$. We will say $[E, G]$ carries the pointwise $\mathcal{C}$ -structure if it carries the initial $\mathcal{C}$ -structure induced by the family $@(e) : [E, G] \rightarrow G$ ($e \in E$), where $@_{EG} : E \rightarrow [[E, G], G]$ is the natural map $@(e)(u) = u(e)$.

Proposition 5.3 *Let $\mathbf{aLC}$ be an analyte in $\mathbf{mLC}$. Then the following statements are equivalent.*

1. *Every $E \in \mathbf{aLC}$ upholds the Banach-Steinhaus principle.*
2. *Every $[E, G]$ with $E \in \mathbf{aLC}$ and $G \in \mathbf{eLC}$ carries the pointwise $\mathcal{C}$ -structure.*
3. *Every $[\mathbf{F}_u\Omega, \mathbf{K}]$ ($\Omega \in Q$) carries the pointwise $\mathcal{C}$ -structure, where Q is some finally generating class and $\mathbf{F}_u\Omega$ denotes the free $\mathbf{aLC}$ -space of Ω (see 28.h in [18]).*

Proof: 1.a in [20]

□

Lemma 5.4 *Let $\mathcal{C}$ be a toponome, where every $\mathbf{oLC}$ -space upholds the categorical Banach-Steinhaus principle. Then for $F := C(X, \mathbb{R})$ the linear functionals $l \in [F, \mathbb{R}]$ form an initial monofamily.*

Proof: Since F is an $\mathbf{eLC}$ -space, the map $@_F : F \rightarrow [[F, \mathbb{R}], \mathbb{R}]$ is an embedding, and because $[F, \mathbb{R}]$ is an $\mathbf{oLC}$ -space, it upholds the BS-principle. Therefore the maps $@(l) : [[F, \mathbb{R}], \mathbb{R}] \rightarrow \mathbb{R}$ form an initial monofamily, where $l \in [F, \mathbb{R}]$. Since $@(l) \circ @_F = l$, the linear functionals form a composite of initial monofamilies. □

Proposition 5.5 *Let $\mathcal{C}$ be a toponome, where every $\mathbf{oLC}$ -space upholds the categorical Banach-Steinhaus principle. Then a $\mathbf{LC}$ -space E is an $\mathbf{eLC}$ -space iff its linear functionals $l \in [E, \mathbb{R}]$ form an initial monofamily.*

Proof: Let E be an $\mathbf{eLC}$ -space. Then there exists a $\mathbf{C}$ -space X and an embedding $m : E \rightarrow \mathbf{C}(X, \mathbb{R})$. Since $\mathbf{C}(X, \mathbb{R})$ is an $\mathbf{oLC}$ space, we get the initial monofamily $\{l \circ m \mid l \in [\mathbf{C}(X, \mathbb{R}), \mathbb{R}]\}$. If $\{l \in E^*\}$ is an initial monofamily, then we get that $@_E : E \rightarrow [E^*, \mathbb{R}]$ is an embedding, since it is the inner factor of the initial monofamily $\{@_{E^*}(l) \circ @_E = l \mid l \in E^*\}$. Therefore E is an $\mathbf{eLC}$ -space. $\square$

Lemma 5.6 *Let $(\mathbf{B}, \mathcal{F})$ be a base for a topocosm and $(\mathbf{A}, \mathcal{F}_1)$ an $\mathbf{I}$ -base in $(\mathbf{B}, \mathcal{F})$*

Proposition 5.7 *Toponomes where all $\mathbf{oLC}$ -spaces uphold the categorical Banach-Steinhaus principle include $\mathbf{C}_b$, $\mathbf{C}_d$ and all topocosms of the examples 2.16 which are built out of bases which lie in $\mathbf{Conv}$.*

Proof: See 1.b in [20] for $\mathbf{C}_b$ and 4.j in [20] for $\mathbf{C}_d$. The third part follows analogously to the second. $\square$

Corollary 5.8 *The functors $\mathbf{R}$ and $\mathbf{V}$ of definition 3.1 restrict to mutually inverse functors between $\mathbf{eLC}_{\mathbf{B}}$ and $\mathbf{eLC}_{\mathbf{B}^*}$.*

Proof: Since $\mathbf{R}$ and $\mathbf{V}$ preserve initial monofamilies the restrictions are well defined; see proposition 3.5 and proposition 5.5. By corollary 3.11 we get that $\mathbf{R}$ and $\mathbf{V}$ are inverse. $\square$

The propositions 3.5 and 5.5 also serve as motivation for the following definition.

Definition 5.9 *We define a restructuring functor $(-)_b : \mathbf{eLC}_{\mathbf{B}} \rightarrow \mathbf{eLC}_b$ in the following way. An $\mathbf{eLC}_{\mathbf{B}}$ space E becomes a bornological space E_b by structuring its underlying set with the initial bornology of the linear diffeomeric functionals $l \in E^*$. Similarly we define a restructuring functor $(-)_d : \mathbf{eLC}_b \rightarrow \mathbf{eLC}_{\mathbf{B}}$. An $\mathbf{eLC}_b$ space G becomes a diffeomeric space G_d by structuring its underlying set with the initial diffeomeric structure of the linear bornological functionals $l \in G^*$.*

The following definition will be used only to show corollary 5.15.

Definition 5.10 *Let E be an $\mathbf{eLC}_{\mathbf{B}}$ -space. We say the derivative $c^{(1)}$ exists for an imprint $c : \mathbb{R} \rightarrow E$, if there exist an imprint $c^{(1)}$ such that $\text{loc}^{(1)} = (\text{loc})'$ for all l in an initial monofamily of linear functionals. We say that $c =: c^{(0)}$ has all derivatives, if there exists the derivative $c^{(i+1)}$ for every $c^{(i)}$ ($i \in \mathbb{N}_0$).*

Lemma 5.11 *Let E be an $\text{elC}_\mathbf{R}$ -space. If the derivative $c^{(1)}$ for an imprint c exists, then there exists an $\text{C}_\mathbf{R}$ -map $S : \mathbf{R} \times \mathbf{R} \rightarrow E$ such that $c(\alpha) - c(\beta) = (\alpha - \beta)S(\alpha, \beta)$ i.e. the difference quotient S*

$$(\alpha, \beta) \mapsto \begin{cases} \frac{c(\alpha) - c(\beta)}{\alpha - \beta} & \text{for } \alpha \neq \beta \\ c^{(1)}(\alpha) & \text{for } \alpha = \beta \end{cases}$$

is an $\text{C}_\mathbf{R}$ -map.

Proof: Let $l \in E^*$. Then

$$(l \circ S)(\alpha, \beta) = \int_0^1 (l \circ c)'(\beta + r(\alpha - \beta)) dr$$

holds also for $\alpha = \beta$ since $(l \circ c)' = l \circ c^{(1)}$. The right hand side is smooth in (α, β) since $l \circ c$ is smooth and therefore S is an $\text{C}_\mathbf{R}$ -map. $\square$

Lemma 5.12 *Let $c : \mathbf{R} \rightarrow \mathbf{R}$ be differentiable on $\mathbf{R} \setminus \{0\}$ and assume that the derivative has a continuous extension to $\mathbf{R}$. Then c is differentiable at 0 and $c'(0) = \lim_{t \rightarrow 0} c'(t)$.*

Proof: Let $a := \lim_{t \rightarrow 0} c'(t)$. By the mean value theorem we have that $\frac{c(t) - c(0)}{t} \in \{c'(s) : 0 < |s| < |t|\}$. Since c' is assumed to be continuously extendable to 0, we have that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $c'(t) \in a + [-\epsilon, +\epsilon]$ for all $|t| < \delta$. Hence $\frac{c(t) - c(0)}{t} - a \in [-\epsilon, +\epsilon]$, i.e. $c'(0) = a$. $\square$

Definition 5.13 *A sequence $(a_k)_{k \in \mathbf{N}}$ in an elC_b -space G converges fast to a_∞ , if $k^n(a_k - a_\infty)$ is bounded in k for every $n \in \mathbf{N}$.*

Proposition 5.14 (See also 4.2.15 in [8].) *Let N be a positive integer, and let $(a_k^i)_{k \in \mathbf{N}_0}$ ($i \in \{0, \dots, N\}$) be sequences in an elC_b -space G , which converge fast to a_∞^i for $k \rightarrow \infty$. Then there exists an imprint $c : \mathbf{R} \rightarrow G_d$, such that $c(t) \in \langle a_k^i \mid i \in \{0, \dots, N\}, k \in \mathbf{N}_0 \rangle$ for all $t \in \mathbf{R} \setminus \{0\}$, all derivatives exist and $c^{(i)}(0) = a_\infty^i$ for $i \in \mathbf{N}$, where $a_\infty^i := 0$ for $i > N$. Here $\langle a_k^i \rangle$ denotes the vector space spanned by the a_k^i .*

Proof: Let $\phi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function with support in $[-1, 1]$, such that $\phi(0) = 1$ and $\sum_{k \in \mathbb{Z}} \phi(t - k) = 1$ for all $t \in \mathbb{R}$. (For the existence take a positive smooth bump-function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ with support $[-1, 1]$ and define $\phi(t) := \frac{\psi(t)}{\sum_{k \in \mathbb{Z}} \psi(t - k)}$.) Define $a_k^i := a_{-k}^i$ for $k < 0$, $a_\infty^i := 0$ for $i > N$, and $\phi_k : \mathbb{R} \setminus \{0\} \rightarrow [0, 1]$ by

$$\phi_k(t) = \phi\left(\frac{1}{t} - k\right)$$

and $g_k : \mathbb{R} \rightarrow G$ as

$$g_k(t) := \sum_{i=0}^N \frac{a_k^i}{i!} t^i.$$

Clearly all derivatives exist for all g_k and are given by

$$g_k^{(j)}(t) := \sum_{i=j}^N \frac{a_k^i}{(i-j)!} t^{i-j}.$$

Finally we define $c : \mathbb{R} \rightarrow G$ as

$$c(t) = \begin{cases} \sum_{k \in \mathbb{Z}} \phi_k(t) g_k(t) & \text{for } t \neq 0 \\ a_\infty^0 & \text{for } t = 0 \end{cases}$$

Let us show that (see definition 5.10)

$$c^{(n)}(t) = \begin{cases} \sum_{k \in \mathbb{Z}} \sum_{j=0}^n \binom{n}{j} \phi_k^{(n-j)}(t) g_k^{(j)}(t) & \text{for } t \neq 0 \\ a_\infty^n & \text{for } t = 0 \end{cases}$$

For $t \neq 0$ the sums have only finitely many summands and therefore for these t we get that $(l \circ c)^{(n)}(t) = (l \circ c^{(n)})(t)$ for all $n \in \mathbb{N}$. By Lemma 5.12 it remains to show that $\lim_{t \rightarrow 0} (l \circ c)^{(n)}(t) = (l \circ c^{(n)})(0) = l(a_\infty^n)$ for all $n \in \mathbb{N}_0$ and $l \in G^*$. For $t \in [\frac{1}{k+1}, \frac{1}{k}]$ we get

$$\begin{aligned} c(t) &= \phi_k(t) g_k(t) + \phi_{k+1}(t) g_{k+1}(t) \\ &= \phi_k(t) g_k(t) + (1 - \phi_k(t)) g_{k+1}(t) \\ &= g_{k+1}(t) + \phi_k(t) (g_k(t) - g_{k+1}(t)) \end{aligned}$$

and therefore for $t \in [\frac{1}{k+1}, \frac{1}{k}]$

$$(l \circ c)^{(n)}(t) - l(a_\infty^n) =$$

$$\begin{aligned}
 &= (l \circ g_{k+1}^{(n)})(t) - l(a_\infty^n) + \sum_{j=0}^n \binom{n}{j} \phi_k^{(n-j)}(t) (l \circ g_k^{(j)} - l \circ g_{k+1}^{(j)})(t) \\
 &= \sum_{i \geq n} l(a_{k+1}^i) \frac{t^{i-n}}{(i-n)!} - \sum_{i \geq n} l(a_\infty^i) \frac{t^{i-n}}{(i-n)!} + \sum_{i \geq n} l(a_\infty^i) \frac{t^{i-n}}{(i-n)!} - l(a_\infty^n) \\
 &\quad + \sum_{j=0}^n \binom{n}{j} \phi_k^{(n-j)}(t) ((l \circ g_k)^{(j)} - (l \circ g_{k+1})^{(j)})(t) \\
 &= \sum_{i \geq n} l(a_{k+1}^i - a_\infty^i) \frac{t^{i-n}}{(i-n)!} + t \sum_{i \geq n+1} l(a_\infty^i) \frac{t^{i-(n+1)}}{(i-n)!} \\
 &\quad + \sum_{j=0}^n \binom{n}{j} \phi_k^{(n-j)}(t) ((l \circ g_k)^{(j)} - (l \circ g_{k+1})^{(j)})(t)
 \end{aligned}$$

The first term converges to 0 for $k \rightarrow \infty$ and the second term converges to 0 for $t \rightarrow 0$. Since

$$((l \circ g_k) - (l \circ g_{k+1}))^{(n)}(t) = \sum_{i \geq n} l(a_k^i - a_{k+1}^i) \frac{t^{i-n}}{(i-n)!}$$

converges fast to 0 for $k \rightarrow \infty$ and since $\phi_k^{(n)}(t)$ is bounded by a polynomial in k , we get that the third term converges to 0 for $k \rightarrow \infty$ and t in a bounded set. Because the k go to infinity for $t \rightarrow 0$ ($t \in [\frac{1}{k+1}, \frac{1}{k}]$) we get that $\lim_{t \rightarrow 0} (l \circ c)^{(n)}(t) = l(a_\infty^n)$. $\square$

Corollary 5.15 (Special curve lemma) (See also 2.3.4 in [8].) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence which converges fast to a_∞ in an $\text{el}C_b$ -space G . Then there exists an imprint $c : \mathbb{R} \rightarrow G_d$ such that $c(t) \in \langle a_n : n \in \mathbb{N} \rangle$ for $t \neq 0$, $c(\frac{1}{n}) = a_n$ and $c(0) = a_\infty$. Furthermore there exists an $C_\mathbb{R}$ -map $S : \mathbb{R} \times \mathbb{R} \rightarrow E$ such that $c(\alpha) - c(\beta) = (\alpha - \beta)S(\alpha, \beta)$

Proof: If one takes in the proof of proposition 5.14 for g_n the constant functions $g_n(t) = a_n$ one gets the desired curve. The second part follows from lemma 5.11 $\square$

Proposition 5.16 For every $\text{el}C_b$ -space G , a linear functional $l : G_d \rightarrow \mathbb{R}$ is an $\text{el}C_\mathbb{R}$ -map iff $l : G \rightarrow \mathbb{R}$ is bornological.

Proof: Clearly every bornological linear functional $l : G \rightarrow \mathbb{R}$ corresponds to an $\text{el}C_\mathbb{R}$ -map $l : G_d \rightarrow \mathbb{R}$ (initial structure). Suppose $l : G \rightarrow \mathbb{R}$ is not bornological,

but $l : G_d \rightarrow \mathbb{R}$ is an $\text{eLC}_{\mathbb{R}}$ -map. Then there exists a bounded absolute convex set B and a sequence $(x_n)_{n \in \mathbb{N}} \subseteq B$ such that $(l(x_n))_{n \in \mathbb{N}}$ is unbounded. By taking a subsequence, we can assume, that $l(n) > ng(n)$, where g goes faster to infinity than any polynomial. So we get a fast falling sequence $y_n := \frac{1}{g(n)}x_n$ such that $l(y_n)$ is unbounded in n . By corollary 5.15, we can find an imprint c through these points, such that $y_n \in c([0, 1])$. By assumption, however, $l(c([0, 1]))$ is bounded since $l \circ c$ is smooth and $[0, 1]$ is compact. Contradiction. $\square$

Definition 5.17 Let D be an arbitrary subset of $\mathbb{R}$, $f : D \rightarrow E$ a function with values in a vector space E and $k \in \mathbb{N}_0$. Let $D^{(k)} := \{(t_0, \dots, t_k) \in D^{k+1} \mid t_i \neq t_j \text{ for } i \neq j\}$. On $D^{(k)}$ we consider the bornology induced by the compact bornology of D^{k+1} . We define the difference quotient $\delta^k f : D^{(k)} \rightarrow E$ of order k of f recursively by:

$$\begin{aligned} \delta^0 f &= f \\ \delta^k f(t_0, \dots, t_k) &:= \frac{k}{t_0 - t_k} (\delta^{k-1} f(t_0, \dots, t_{k-1}) - \delta^{k-1} f(t_1, \dots, t_k)) \end{aligned}$$

Lemma 5.18 The difference quotient can be written as

$$\delta^k f(t_0, \dots, t_k) = k! \sum_{i=0}^k \beta_i f(t_i)$$

where the coefficients

$$\beta_i = \beta_i(t_0, \dots, t_k) := \prod_{\substack{0 \leq r \leq k \\ r \neq i}} \frac{1}{t_i - t_r}$$

are independent of f .

Proof: Induction on k ; see 1.3.2 in [8] $\square$

Lemma 5.19 For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the following are equivalent.

1. f is smooth.

2. Every point in $\mathbb{R}$ has a neighborhood V such that $\delta^k f$ is bounded on $V^{(k)}$. (V doesn't depend on f)

Proof: 1.3.24 in [8] □

Proposition 5.20 *For every $\text{eLC}_{\mathbb{R}}$ -space E , a linear functional $l : E_b \rightarrow \mathbb{R}$ is bornological iff $l : E \rightarrow \mathbb{R}$ is diffeomorphic.*

Proof: Clearly every $\text{eLC}_{\mathbb{R}}$ -map $l : E \rightarrow \mathbb{R}$ corresponds to a bornological $l : E_b \rightarrow \mathbb{R}$ (initial structure). Let $l : E_b \rightarrow \mathbb{R}$ be a bornological linear functional. We have to show that $l \circ c : \mathbb{R} \rightarrow \mathbb{R}$ is smooth for every diffeomorphic imprint $c : \mathbb{R} \rightarrow E$. By lemma 5.19 we have to find for every point in $\mathbb{R}$ a neighborhood V such that $\delta^k(l \circ c)$ is bounded on $V^{(k)}$. Therefore take a neighborhood V of a point in $\mathbb{R}$ such that $\delta^k f(V^{(k)})$ is bounded for all smooth $f : \mathbb{R} \rightarrow \mathbb{R}$ (lemma 5.19). Since l is bornological and $\delta^k(l \circ c) = l \circ \delta^k c$ (see lemma 5.18) it suffice, to show that $\delta^k c(V^{(k)})$ is bounded. But since $g(\delta^k c(V^{(k)})) = \delta^k(g \circ c)(V^{(k)})$ is bounded for all $g \in E^*$ ($g \circ c$ is smooth) and these g form an initial family for the bornology of E_b , we get that $\delta^k(l \circ c)(V^{(k)})$ is bounded. □

By the propositions 5.16 and 5.20 we get that E^* and $(E_b)^*$ respectively G^* and $(G_d)^*$ have the same underlying vector space. Therefore $((E_b)_d)^* = (E_b)^* = E^*$ as vector-space and hence $(E_b)_d = E$. Analogously $(G_d)_b = G$ and we get the following proposition.

Proposition 5.21 *The functors $(-)_d : \text{eLC}_b \rightarrow \text{eLC}_{\mathbb{R}}$ and $(-)_b$ are mutual inverses.*

As we will see in chapter 6 the category eLC_b builds the foundation for the Frölicher-Kriegl calculus. Therefore this shows that the categories of linear spaces of the two calculus theories under investigation are isomorphic.

Chapter 6

The Frölicher-Kriegl Calculus

In this chapter we introduce the foundations of the Frölicher-Kriegl calculus and explain its relations to the category $\mathbf{eLC}_b$. We show that for certain spaces (convenient vector spaces) the smooth maps in this theory coincide with the C_∞ -maps. By using these results we will strengthen proposition 5.21.

Definition 6.1 *Let E be a locally convex vector space. A curve $c : \mathbb{R} \rightarrow E$ is called differentiable, if the derivative $c'(t) := \lim_{s \rightarrow 0} \frac{1}{s}(c(t+s) - c(t))$ at t exists for all t . A curve $c : \mathbb{R} \rightarrow E$ is called smooth or C^∞ , if all iterated derivatives exist. A map $f : E \rightarrow F$ between locally convex spaces is called smooth, iff $c \circ f$ is smooth for all smooth curves c .*

This definition uses a locally convex vector space E , which we always assume to be separated. One can show, however, (see [14]) that the smooth curves only depend on the bounded sets of the locally convex space. Therefore we can as well use the bornologification (bornological coreflection) of E instead of E itself, if we are interested only in the smooth structure. The bornologification E_{born} of a locally convex space E is the finest locally convex structure having the same bounded sets as E . The bornologification E_{born} has the following properties:

1. Its continuous seminorms are exactly the bornological seminorms of E .
2. An absolutely convex set is a 0-neighborhood in E_{born} , iff it absorbs bounded sets.

The spaces in the image of the bornologification functor $E \mapsto E_{born}$ are called the *bornological locally convex spaces*.

On the other hand there are *topological convex bornological spaces*. These are linear bornological spaces G whose structure is determined by their duals G^* (see 2.1.20 and 2.4.3 in [8]). By proposition 5.5 these are exactly the $\mathbf{eLC}_b$ -spaces. Furthermore this category is equivalent to the category of bornological locally convex spaces (see e.g. 2.1.13 in [8]) and is also called the category of prevenient spaces. Therefore we can use all definitions and propositions of the Frölicher Kriegl calculus, which are given for locally convex spaces, in the setting of $\mathbf{eLC}_b$ -spaces.

Proposition 6.2 *Let G be an $\mathbf{eLC}_b$ -space. A linear functional $l : G \rightarrow \mathbb{R}$ is smooth iff it is bornological.*

Proof: See [14] or 2.3.5 in [8] □

Therefore the smooth linear functionals $l : G \rightarrow \mathbb{R}$ are exactly those that are $\mathbf{eLC}_b$ -maps from G to $\mathbb{R}$. But the smooth structure on the $\mathbf{eLC}_b$ -space G need not be the initial smooth structure induced by these functionals. There is a certain class of $\mathbf{eLC}_b$ -spaces, however, in which the spaces have this property.

Definition 6.3 *A sequence (x_n) in an $\mathbf{eLC}_b$ -space E is called Mackey convergent to $x \in E$, if there exists a bounded absolutely convex $B \subseteq E$, such that one of the following equivalent conditions are satisfied (see e.g. 2.2.18 in [8])*

1. (x_n) converges in E_B , where E_B denotes the linear span of B normed with the Minkowsky-functional.
2. There exists a sequence $\mu_n \rightarrow 0$ in $\mathbb{R}$, such that $x_n - x \in \mu_n B$

A sequence (x_n) is called a *Mackey Cauchy sequence*, if it is a Cauchy sequence in E_B for a bounded absolutely convex B . An $\mathbf{eLC}_b$ -space is called *Mackey complete* (locally complete or c^∞ -complete), if every Mackey Cauchy sequence converges Mackey; cf. [11, page 196]. For further characterizations see [14] or 2.6.2 in [8]. A Mackey complete $\mathbf{eLC}_b$ -space is also called a *convenient vector space*.

Proposition 6.4 *Let G be a convenient vector space. Then a curve $c : \mathbb{R} \rightarrow G$ is smooth iff $l \circ c$ is smooth for all bornological linear functionals $l \in G^*$.*

Proof: See [14] □

Corollary 6.5 *Let G and H be convenient vector spaces. Then a map $f : G \rightarrow H$ is smooth iff $f : G_d \rightarrow H_d$ is a $C_{\mathbb{R}}$ -map.*

We denote the classical Fréchet space of smooth functions from $\mathbb{R}$ to $\mathbb{R}$ by $C^\infty(\mathbb{R}, \mathbb{R})$. It has the topology of uniform convergence on compact sets in each derivative separately. Therefore a set $B \subseteq C^\infty(\mathbb{R}, \mathbb{R})$ is bounded iff $d^{(n)}(B) := \{f^{(n)} \mid f \in B\}$ is bounded in $C_b(\mathbb{R}, \mathbb{R})$ for each $n \in \mathbb{N}$. Since $C^\infty(\mathbb{R}, \mathbb{R})$ is a Fréchet space it is a convenient vector space and has the same underlying vector space as $C_{\mathbb{R}}(\mathbb{R}, \mathbb{R})$.

It can be shown, (see e.g. 1.4.3 in [8]) that $C^\infty(\mathbb{R}, \mathbb{R})$ fulfills the exponential law, i.e.: a curve $c : \mathbb{R} \rightarrow C^\infty(\mathbb{R}, \mathbb{R})$ is smooth iff $c^\dagger : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is smooth. By corollary 6.5 we get that a map $c : \mathbb{R} \rightarrow C^\infty(\mathbb{R}, \mathbb{R})_d$ is an imprint iff $c : \mathbb{R} \rightarrow C^\infty(\mathbb{R}, \mathbb{R})$ is smooth which is equivalent to $c^\dagger : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is smooth ($C_{\mathbb{R}}$ -map). Finally this is equivalent to $c : \mathbb{R} \rightarrow C_{\mathbb{R}}(\mathbb{R}, \mathbb{R})$ is an imprint. We conclude that $C^\infty(\mathbb{R}, \mathbb{R})_d = C_{\mathbb{R}}(\mathbb{R}, \mathbb{R})$ and $C^\infty(\mathbb{R}, \mathbb{R}) = C_{\mathbb{R}}(\mathbb{R}, \mathbb{R})_b$. Restricted to the linear spaces we get that this bornology is the same as the bornology induced by $C_b(\mathbb{R}, \mathbb{R})$. A set $B \subseteq [\mathbb{R}, \mathbb{R}]_b$ is bounded, iff it is bounded on bounded (or compact) sets.

In the following we show, that the functors $(-)_b$ and $(-)_d$ commute with the Hom-functor $[-, -]$.

$[E, F]_b$ and $[E_b, F_b]$ have the same underlying vector space for E and F in $C_{\mathbb{R}}$ spaces. Since the linear functionals $l \in F^*$ form an initial family, which is preserved by the functors $(-)_b$ and $[E, -]$ (see proposition 5.21 and 18.i in [17]) we can assume $F = \mathbb{R}$.

Since the imprints $c : \mathbb{R} \rightarrow E$ form a final family for the $C_{\mathbb{R}}$ -structure on E , we get that $\{C_{\mathbb{R}}(c, \mathbb{R}) : [E, \mathbb{R}] \rightarrow [\mathbb{R}, \mathbb{R}] \mid c \in C_{\mathbb{R}}(\mathbb{R}, E)\}$ is an initial family for $[E, \mathbb{R}]$ (see 18.i in [17]) and therefore for $[E, \mathbb{R}]_b$ (The functor $(-)_b$ preserves initial families). A set $B \subseteq [E, \mathbb{R}]_b$ is bounded iff $B \circ c$ is bornological in $[\mathbb{R}, \mathbb{R}]$ or equivalently in $C_{\mathbb{R}}(\mathbb{R}, \mathbb{R})_b$. By identifying F and F_b we can say that $B \subseteq [E, F]_b$ is bounded iff

it is bounded on images of compact sets under imprints. On the other hand a set $B \in [E_b, F_b]$ is bounded iff B is bounded on bounded sets of E_b .

Proposition 6.6 *For $\mathbf{eLC}_\mathbf{a}$ -spaces E, F and $\mathbf{eLC}_b$ -spaces G, H we get: $[E_b, F_b] = [E, F]_b$ and $[G_d, H_d] = [G, H]_d$.*

Proof: Since $l \circ c$ is smooth for all $l \in E^*$ and all imprints c , we get $l(c(K))$ and therefore $c(K)$ is bounded for every compact K . Therefore if B is bounded in $[E_b, F_b]$ it is also bounded in $[E, F]_b$. On the other hand let $B \subseteq [E, F]$ be bounded on images of compact sets under imprints. Suppose there is a bounded set $P \subseteq E$ such that $B(P)$ is not bounded. Similar to the proof of proposition 5.16 we can find an imprint $c : \mathbb{R} \rightarrow E$ such that $B(c([0, 1]))$ is not bounded. Therefore $[E_b, F_b] = [E, F]_b$. By applying the functor $(-)_d$ on the equation $[G_d, H_d]_b = [G_{db}, H_{db}] = [G, H]_{db}$ we get $[G_d, H_d] = [G, H]_d$. $\square$

This proposition shows that the categories $\mathbf{eLC}_b$ and $\mathbf{eLC}_\mathbf{a}$ are not only isomorphic (see proposition 5.21) but that this isomorphism is also an enriched one.

Chapter 7

Derivative Closure and Mackey Closure

In chapter 6 we saw that $\mathbf{eLC}_b$ has an important subcategory, the convenient vector spaces. These are the spaces where the two differentiation theories define the same smooth maps; see corollary 6.5. In this chapter we will describe the corresponding subcategory of $\mathbf{eLC}_{\mathbf{R}}$ and therefore reach the main goal mentioned in the introduction.

Definition 7.1 *An embedding $m : E \rightarrow F$ between $\mathbf{eLC}_{\mathbf{R}}$ -spaces E and F is called derivative closed, if it has the following property: For every $y \in F$ such that there exist an imprint $\phi : \mathbf{R} \rightarrow E$ and an $\mathbf{eLC}_{\mathbf{R}}$ -map $S : \mathbf{R} \times \mathbf{R} \rightarrow F$ with*

$$(m \circ \phi)(\alpha) - (m \circ \phi)(\beta) = (\alpha - \beta)S(\alpha, \beta) \quad (7.1)$$

and $S(\lambda, \lambda) = y$, there exists $x \in E$ with $m(x) = y$.

An $\mathbf{eLC}_{\mathbf{R}}$ -map $u : E \rightarrow F$ is called derivative dense, if it has the following property: For every $y \in F$ there exists an imprint $\phi : \mathbf{R} \rightarrow E$ and an $\mathbf{eLC}_{\mathbf{R}}$ -map $S : \mathbf{R} \times \mathbf{R} \rightarrow F$ such that equation 7.1 is satisfied and $y = S(\lambda, \lambda)$ for some $\lambda \in \mathbf{R}$. Define the class DD to consist of all derivative dense $\mathbf{eLC}_{\mathbf{R}}$ maps.

An embedding $m : E \rightarrow F$ between $\mathbf{eLC}_b$ -spaces E and F is called Mackey closed, if it has the following property: If (x_n) is a sequence in E such that $(m(x_n))$ is Mackey convergent to $y \in F$, then there exists an $x \in E$ with $m(x) = y$.

A subspace $E \subseteq F$ is called derivative (Mackey) closed in F , if the canonical insertion map is a derivative (Mackey) closed embedding. A subspace $E \subseteq F$ is

called derivative dense in F , if the canonical insertion map is a derivative dense map.

Lemma 7.2 *Let $\mathbf{R}$ be a replete reflective subcategory of a topological $\mathbf{Lin}\text{-struct}$ $\mathbf{A}$. If D is a class of epimorphisms in $\mathbf{R}$ larger than the class of $\mathbf{A}$ -epimorphisms in $\mathbf{R}$, then every D -strong mono is an $\mathbf{A}$ -embedding.*

Proof: 26.k in [18] □

Lemma 7.3 *Every surjective $\mathbf{eLC}_{\mathbf{A}}$ -map lies in DD , every derivative dense map is an epimorphism in $\mathbf{eLC}_{\mathbf{A}}$ and every DD -strong mono is an embedding.*

Proof:

Every linear map $l : \mathbf{R} \rightarrow E$ is an imprint since its composite with every linear functional is linear from $\mathbf{R}$ to $\mathbf{R}$ and therefore smooth. Let $u : E \rightarrow F$ be a surjective $\mathbf{eLC}_{\mathbf{A}}$ -map and $y \in F$. Take an $x \in E$ with $u(x) = y$ and define $\phi : \mathbf{R} \rightarrow E$ as $\alpha \mapsto \alpha x$ and $S : \mathbf{R} \times \mathbf{R} \rightarrow F$ as $(\alpha, \beta) \mapsto y$. Then $u(\phi(\alpha)) - u(\phi(\beta)) = \alpha u(x) - \beta u(x) = (\alpha - \beta)y = (\alpha - \beta)S(\alpha, \beta)$ and $S(\lambda, \lambda) = y$.

For the second part let $u : E \rightarrow F$ be a derivative dense map and let $r, w : F \rightarrow G$ be $\mathbf{eLC}_{\mathbf{A}}$ -maps such that $r \circ u = w \circ u$. Take $x \in F$, $\phi : \mathbf{R} \rightarrow E$ an imprint and $S : \mathbf{R} \times \mathbf{R} \rightarrow F$ an $\mathbf{eLC}_{\mathbf{A}}$ -map such that $S(\lambda, \lambda) = x$ for a $\lambda \in \mathbf{R}$ and

$$(u \circ \phi)(\alpha) - (u \circ \phi)(\beta) = (\alpha - \beta)S(\alpha, \beta)$$

$$\begin{array}{ccccc} E & \xrightarrow{u} & F & \xrightleftharpoons[w]{r} & G \\ \phi \uparrow & & S \uparrow & & \downarrow l \\ \mathbf{R} & & \mathbf{R} \times \mathbf{R} & & \mathbf{R} \end{array}$$

Then

$$\begin{aligned} (\alpha - \beta)(r \circ S)(\alpha, \beta) &= (r \circ u \circ \phi)(\alpha) - (r \circ u \circ \phi)(\beta) \\ &= (w \circ u \circ \phi)(\alpha) - (w \circ u \circ \phi)(\beta) \\ &= (\alpha - \beta)(w \circ S)(\alpha, \beta) \end{aligned}$$

and for $l \in G^*$ we get

$$(\alpha - \beta) \underbrace{(l \circ r \circ S)}_{\text{continuous}}(\alpha, \beta) = (\alpha - \beta) \underbrace{(l \circ w \circ S)}_{\text{continuous}}(\alpha, \beta)$$

Therefore we get $l \circ r \circ S = l \circ w \circ S$, since the equation holds on the dense subset $\{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha \neq \beta\} \subseteq \mathbb{R} \times \mathbb{R}$. By canceling the monofamily $l \in G^*$ we get $r \circ S = w \circ S$ and therefore $r(x) = (r \circ S)(\lambda, \lambda) = (w \circ S)(\lambda, \lambda) = w(x)$.

The third part follows from the first by lemma 7.2 since $\mathbf{eLC}_{\mathbf{A}}$ is a replete reflective subcategory of $\mathbf{LC}_{\mathbf{A}}$ and the epimorphisms in $\mathbf{LC}_{\mathbf{A}}$ are the surjective maps. $\square$

Lemma 7.4 *$DD_{\nearrow}$ equals the class of derivative closed embeddings ($=DC$)*

Proof: Every derivative closed embedding is a DD -strong monomorphism: Let

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ u \downarrow & & \downarrow v \\ C & \xrightarrow{m} & D \end{array}$$

be a commutative square with $e \in DD$, $m \in DC$. We have to find a diagonal map $d : B \rightarrow C$. We show that $v(B) \subseteq m(C)$: Let $v(b) \in v(B)$. Since e is derivative dense, we can find $\phi : \mathbb{R} \rightarrow A$, a map $S : \mathbb{R} \times \mathbb{R} \rightarrow B$ and $\lambda \in \mathbb{R}$ such that $S(\lambda, \lambda) = b$ and ϕ, S satisfy

$$(e \circ \phi)(\alpha) - (e \circ \phi)(\beta) = (\alpha - \beta)S(\alpha, \beta)$$

Then $\bar{\phi} := u \circ \phi$, $\bar{S} := v \circ S$ satisfies

$$(m \circ \bar{\phi})(\alpha) - (m \circ \bar{\phi})(\beta) = (\alpha - \beta)\bar{S}(\alpha, \beta)$$

and $\bar{S}(\lambda, \lambda) = v(b)$. Since m is derivative closed, we find a necessarily unique $c \in C$ with $m(c) = v(b)$. We define the diagonal map $d : B \rightarrow C$ to map $b \in B$ to $c \in C$ according to the procedure above. The map d is a morphism since $m \circ d = v$ and m is initial, and $d \circ e = u$ since m is a monomorphism.

On the other hand let $m : E \rightarrow F$ be a DD -strong monomorphism. Lemma 7.3

shows that m is an embedding. It remains to verify that m is derivative closed. Let $y \in F$ such that there exist an imprint $\phi : \mathbb{R} \rightarrow E$ and an $\mathbf{eLC}_{\mathbb{R}}$ -map $S : \mathbb{R} \times \mathbb{R} \rightarrow F$ which satisfy equation 7.1 and such that $S(\lambda, \lambda) = y$. Suppose $y \notin m(E)$. Form the $\mathbf{eLC}_{\mathbb{R}}$ -subspace $W := m(E) \oplus \langle y \rangle$ of F . Then the corestriction $m^! : E \rightarrow W$ is derivative dense and $m \circ \text{id} = \text{inc}_W \circ m^!$.

$$\begin{array}{ccc} E & \xrightarrow{m^!} & W \\ \text{id} \downarrow & & \downarrow \text{inc}_W \\ E & \xrightarrow{m} & F \end{array}$$

Since m is a DD-strong mono, there exists a diagonal map d such that $m \circ d = \text{inc}_W$. Therefore $m(d(y)) = y$ and $y \in m(E)$. Contradiction. $\square$

By proposition 4.13 we get that the derivative closed maps form a toponomial outer class. We denote the induced analyte (proposition 4.14) by $\mathbf{dLC}_{\mathbb{R}}$. It consists of all $\mathbf{eLC}_{\mathbb{R}}$ -spaces E for which an derivative closed embedding into a space of the form $\mathbf{C}_{\mathbb{R}}(X, \mathbb{R})$ exists where X is a $\mathbf{C}_{\mathbb{R}}$ -space.

Proposition 7.5 *An embedding $m : E \rightarrow F$ in $\mathbf{eLC}_{\mathbb{R}}$ is derivative closed iff $m : E_b \rightarrow F_b$ is Mackey closed.*

Proof: Mackey closed implies derivative closed: Let $m : E_b \rightarrow F_b$ be Mackey closed, $\phi : \mathbb{R} \rightarrow E$ an imprint, $S : \mathbb{R} \times \mathbb{R} \rightarrow F$ a $\mathbf{C}_{\mathbb{R}}$ -map and $\lambda \in \mathbb{R}$ such that they satisfy equation 7.1 and $S(\lambda, \lambda) = y$. We can assume that $\lambda = 0$. In order to verify that $y \in m(E)$ it suffices to show that $(S(\frac{1}{n}, 0))_{n \in \mathbb{N}}$ converges Mackey to y . (We have $S(\frac{1}{n}, 0) = m(n(\phi(\frac{1}{n}) - \phi(0))) \in m(E)$ and m is Mackey closed.) But since S is an $\mathbf{C}_{\mathbb{R}}$ -map we have for all linear bounded functionals $l : F \rightarrow \mathbb{R}$ that $l(\frac{1}{t}(S(t, 0) - S(0, 0))) = \frac{1}{t}((l \circ S)(t, 0) - (l \circ S)(0, 0))$ is bounded on $\mathbb{R} \setminus \{0\}$. Hence $n(S(\frac{1}{n}, 0) - y)$ is bounded.

Derivative closed implies Mackey closed: Let $m : E \rightarrow F$ be derivative closed and $(x_n)_{n \in \mathbb{N}}$ a sequence in E_b which converges Mackey to $y \in F_b$ (We identify E_b with $m(E_b)$). By the special curve lemma we can find an imprint $c : \mathbb{R} \rightarrow F$

such that $c(0) = y$ and $c(t) \in E$ for $t \neq 0$ and a $C_{\mathbf{R}}$ -map $S_c : \mathbf{R} \times \mathbf{R} \rightarrow F$ with $c(\alpha) - c(\beta) = (\alpha - \beta)S_c(\alpha, \beta)$. We define $\phi : \mathbf{R} \rightarrow E$ as $t \mapsto tc(t)$ and $S_\phi : \mathbf{R} \times \mathbf{R} \rightarrow F$ as $S_\phi(\alpha, \beta) = \alpha S_c(\alpha, \beta) + c(\beta)$. Then

$$\begin{aligned} (\alpha - \beta)S_\phi(\alpha, \beta) &= \alpha(\alpha - \beta)S_c(\alpha, \beta) - (\alpha - \beta)c(\beta) \\ &= \alpha c(\alpha) - \alpha c(\beta) + \alpha c(\beta) - \beta c(\beta) \\ &= \phi(\alpha) - \phi(\beta) \end{aligned}$$

and $S_\phi(0, 0) = y$. By derivative closedness of E , we get that $y \in E$. $\square$

Lemma 7.6 *Let E be an eLC_b -space and $m : E \rightarrow F$ a Mackey closed embedding into a convenient vector space F . Then E is a convenient vector space.*

Proof: Let x_n be a Mackey-Cauchy sequence in E . Then $m(x_n)$ is a Mackey-Cauchy sequence in F and therefore Mackey-convergent to a $y_\infty \in F$. Since m is a Mackey-closed embedding, we can find a $x_\infty \in E$ such that x_n is Mackey convergent to x_∞ . $\square$

Proposition 7.7 *Let X be an $C_{\mathbf{R}}$ -space. Then $C_{\mathbf{R}}(X, \mathbf{R})_b$ is a convenient vector space.*

Proof: Let $(f_n)_{n \in \mathbf{N}}$ be a Mackey-Cauchy sequence in $C_{\mathbf{R}}(X, \mathbf{R})_b$. Since the evaluation $@(x) : C_{\mathbf{R}}(X, \mathbf{R})_b \rightarrow \mathbf{R}$ is bounded for all $x \in X$ and $\mathbf{R}$ is complete, we get a function $f : X \rightarrow \mathbf{R}$. Since $g_n^c : f_n \circ c = C_{\mathbf{R}}(c, \mathbf{R})(f_n)$ is a Mackey-Cauchy sequence for all imprints c , and $C_{\mathbf{R}}(\mathbf{R}, \mathbf{R})_b$ is complete, we get that g_n^c converges Mackey to a $C_{\mathbf{R}}$ -map g^c . Since $C_{\mathbf{R}}(c, \mathbf{R})(f)(t) = @(c(t))(f) = g^c(t)$, we have that $C_{\mathbf{R}}(c, \mathbf{R})(f) = g^c = f \circ c$ for all imprints c and therefore f is a $C_{\mathbf{R}}$ -map. $\square$

Corollary 7.8 *Let E be an eLC_b -space. Then $[E_d, \mathbf{R}]_b$ is a convenient space.*

Proof: Since $\text{inc} : [E_d, \mathbf{R}] \rightarrow C_{\mathbf{R}}(E_d, \mathbf{R})$ is a regular monomorphism (see 24.f in [18]) it is a derivative closed embedding (proposition 4.11). By proposition 7.5 we get that $\text{inc} : [E_d, \mathbf{R}]_b \rightarrow C_{\mathbf{R}}(E_d, \mathbf{R})_b$ is a Mackey closed embedding and therefore that $[E_d, \mathbf{R}]_b$ is convenient (lemma 7.6). $\square$

Proposition 7.9 *For an eLC_b -space E the following is equivalent:*

1. E is convenient.
2. There exists a Mackey closed embedding $m : E \rightarrow F$ into a convenient vector space F .
3. The map $@ : E \rightarrow E^{**}$ is a Mackey closed embedding.
4. Every embedding $m : E \rightarrow G$ into another eLC_b -space G is Mackey closed.

Proof: $(4 \Rightarrow 3)$ is trivial, $(3 \Rightarrow 2)$ follows by corollary 7.8, and $(2 \Rightarrow 1)$ is lemma 7.6. It remains to show $(1 \Rightarrow 4)$: Let $m : E \rightarrow G$ be an embedding into an eLC_b -space G and (x_n) a sequence in E such that $(m(x_n))$ converges Mackey to $y \in G$. Then (x_n) is a Mackey-Cauchy sequence in E since m is an embedding and $(m(x_n))$ is a Mackey-Cauchy sequence in G . Therefore (x_n) converges to an $x \in E$ and by uniqueness of limits in separated spaces we get that $m(x) = y$. $\square$

Proposition 7.10 *For an $\text{eLC}_{\mathbb{R}}$ -space E the following is equivalent:*

1. E is a $\text{dLC}_{\mathbb{R}}$ -space.
2. The map $@_E : E \rightarrow C_{\mathbb{R}}([E, \mathbb{R}], \mathbb{R})$ is a derivative dense embedding.
3. The map $@_E : E \rightarrow E^{**}$ is a derivative dense embedding.
4. There exists a derivative dense embedding from E to some dual space F^* .

Proof: Special case of 28.k in [18]. $\square$

Proposition 7.11 *The functors $(-)_b : \text{eLC}_{\mathbb{R}} \rightarrow \text{eLC}_b$ and $(-)_d : \text{eLC}_b \rightarrow \text{eLC}_{\mathbb{R}}$ restrict to isomorphisms between $\text{dLC}_{\mathbb{R}}$ and the convenient vector spaces.*

Proof: Since $(-)_b$ and $(-)_d$ are inverse to each other, we only have to show that the restrictions are well defined. Let E be an $\text{dLC}_{\mathbb{R}}$ -space. Then $@ : E \rightarrow E^{**}$ is derivative closed and therefore $@ : E_b \rightarrow (E^{**})_b = E_b^{**}$ is a Mackey closed embedding

into a convenient vector space. Therefore E_b is a convenient vector space. The other direction is analogous. $\square$

This proposition describes the intersection of the two differentiation theories under investigation. In $\mathbf{C}_{\mathbf{R}}$ we described linear spaces embeddable spaces and finally $\mathbf{dLC}_{\mathbf{R}}$ -spaces in terms of imprints. On the other hand we started with bornological spaces and described convenient spaces and smooth curves in terms of bounded sets (bornologies). While proposition 7.11 shows that $\mathbf{dLC}_{\mathbf{R}}$ and the category of convenient spaces and bornological linear maps are isomorphic, corollary 6.5 tells us that also the $\mathbf{C}_{\mathbf{R}}$ -maps and the smooth maps, defined between the corresponding spaces, coincide.

Chapter 8

The Analyte $eLC_{\mathcal{H}}$

In the previous chapters we proved several isomorphisms where subcategories of the topocosms $C_{\mathbb{R}}$, $C_{\mathbb{R}^n}$ and $C_{\mathcal{O}}$ were involved; see examples 2.16. Especially all categories of embeddable spaces in these examples are isomorphic. In this chapter we will prove, that also $eLC_{\mathcal{H}}$ (see examples 2.16) is isomorphic to these categories.

Lemma 8.1 *There are sequences (a_k) , (b_k) such that*

1. $b_k < 0$
2. $\sum_k |a_k| |b_k|^n < \infty$ for $n = 0, 1, 2, \dots$
3. $\sum_k a_k (b_k)^n = 1$ for $n = 0, 1, 2, \dots$
4. $b_k \rightarrow -\infty$

Proof: [27] □

The definition of the base for $C_{\mathcal{H}}$ uses open subsets of $\mathcal{H}^n$ ($n \in \mathbb{N}$), but we can assume, that it consists only of $\mathcal{H}^n$ ($n \in \mathbb{N}$) (see proposition 3.4).

Definition 8.2 *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ a smooth function with $\phi(t) = 1$ for $0 \leq t \leq 1$ and $\phi(t) = 0$ for $t \geq 2$, and take sequences (a_k) , (b_k) as in lemma 8.1. For every map $f : \mathcal{H}^n \rightarrow E$ with values in a vector space E define an extension $\mathcal{E}f : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow E$ as*

$$(x, t) \mapsto \begin{cases} f(x, t) & \text{for } t \geq 0 \\ \sum_k a_k \phi(b_k t) f(x, b_k t) & \text{for } t < 0 \end{cases}$$

Because $b_k \rightarrow -\infty$, the sum is finite for each $t < 0$.

Proposition 8.3 (Seeley 1964) *Let $f : \mathcal{H}^n \rightarrow \mathbb{R}$ be a C_s^∞ -map. Then the extension $\mathcal{E}f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C_s^∞ -map.*

Proof: We can assume that $f(x, t) = 0$ for $t \geq 3$. (Use a partition of unity $\{\psi_1, \psi_2\}$ which underlies the open covering $\{U_1, U_2\}$ of $\mathcal{H}^n$, where $U_1 := \{(x, t) \in \mathcal{H}^n \mid t < 3\}$ and $U_2 := \{(x, t) \in \mathcal{H}^n \mid t > 2\}$. Then $f = \psi_1 f + \psi_2 f$ and since $\psi_2 f(x, t) = 0$ for $t < 2$ we get $\mathcal{E}f(x, t) = \mathcal{E}\psi_1 f(x, t) + \mathcal{E}\psi_2 f(x, t) = \mathcal{E}\psi_1 f(x, t)$ for $t < 0$ with $\psi_1 f(x, t) = 0$ for $t \geq 3$.) Therefore we can assume that all derivatives of f are uniformly bounded in t for x in a bounded set. For $t < 0$ we get

$$\frac{\partial^n}{\partial x^n} \mathcal{E}f(x, t) = \sum_k a_k \phi(b_k t) \frac{\partial^n}{\partial x^n} f(x, b_k t)$$

and

$$\begin{aligned} \frac{\partial^n}{\partial t^n} \mathcal{E}f(x, t) &= \sum_k \sum_{i=0}^n \binom{n}{i} a_k \frac{\partial^i}{\partial t^i} \phi(b_k t) b_k^i \frac{\partial^{n-i}}{\partial t^{n-i}} f(x, b_k t) b_k^{n-i} \\ &= \sum_k a_k b_k^n \sum_{i=0}^n \binom{n}{i} \frac{\partial^i}{\partial t^i} \phi(b_k t) \frac{\partial^{n-i}}{\partial t^{n-i}} f(x, b_k t) \end{aligned}$$

Since $\sum_k |a_k| |b_k|^n < \infty$ these sums are uniformly bounded in t for x in a bounded set and therefore the limits of all derivatives exist for $t \rightarrow 0-$. Because $\frac{\partial^i}{\partial t^i} \phi(b_k t) = 0$ for $b_k t < 1$ and $i \neq 0$, and because $\sum_k a_k b_k^n = 1$ for all $n = 0, 1, 2, \dots$, we get that $\lim_{t \rightarrow 0} \frac{\partial^n}{\partial t^n} \mathcal{E}f(x, t) = f(x, 0)$. Therefore all partial derivatives exist and are continuous and $\mathcal{E}f$ is smooth. $\square$

Definition 8.4 *We define a restructuring functor $(-)_n : \text{eLC}_{\mathcal{H}} \rightarrow \text{eLC}_{\mathbb{R}^n}$ in the following way: For an $\text{eLC}_{\mathcal{H}}$ -space G define G_n to be the $\text{eLC}_{\mathbb{R}^n}$ -space which has the same underlying set as G and the initial $\text{eLC}_{\mathbb{R}^n}$ -structure of the linear functionals $l \in G^*$.*

Analogously we define a restructuring functor $(-)_h : \text{eLC}_{\mathbb{R}^n} \rightarrow \text{eLC}_{\mathcal{H}}$. For an $\text{eLC}_{\mathbb{R}^n}$ -space E define E_h to be the $\text{eLC}_{\mathcal{H}}$ -space which has the same underlying set as E and the initial $\text{eLC}_{\mathcal{H}}$ -structure of the linear functionals $l \in E^$.*

Proposition 8.5 *For an $\text{eLC}_{\mathbb{R}^n}$ -space E the imprints $c : \mathcal{H}^n \rightarrow E_h$ are the restrictions of the imprints $g : \mathbb{R}^n \rightarrow E$ to H^n ; i.e. $c = g \circ \iota_{\mathcal{H}}$ where $\iota_{\mathcal{H}}$ is the inclusion map of $\mathcal{H}^n$ into $\mathbb{R}^n$.*

Proof: Let $c = g \circ \iota_{\mathcal{H}}$ where g is an imprint into E . Then $l \circ g \circ \iota_{\mathcal{H}}$ is an C_s^∞ -map for every $l \in E^*$ since $\iota_{\mathcal{H}}$ and $l \circ g$ are C_s^∞ . Because E_h has the initial structure of the $l \in E^*$, we get that c is an imprint in E_h . On the other hand let $c : \mathcal{H}^n \rightarrow E_h$ be an imprint. Then $\mathcal{E}c : \mathbb{R}^n \rightarrow E$ is an imprint since $l \circ \mathcal{E}c = \mathcal{E}(l \circ c)$ is C_s^∞ for all $l \in E^*$. Clearly the restriction of $\mathcal{E}c$ to $\mathcal{H}^n$ is c . $\square$

Lemma 8.6 *For every $\text{eLC}_{\mathcal{H}}$ -space G , a linear functional $l : G_n \rightarrow \mathbb{R}$ is an $\text{eLC}_{\mathbb{R}^n}$ -map iff $l : G \rightarrow \mathbb{R}$ is an $\text{eLC}_{\mathcal{H}}$ -map.*

Proof: If $l \in G^*$ then it is also in $(G_n)^*$ (initial structure). On the other hand let $l : G_n \rightarrow \mathbb{R}$ be an $\text{eLC}_{\mathbb{R}^n}$ -map. Let $c : \mathcal{H}^n \rightarrow G$ be an imprint. We have to show that $l \circ c$ is a C_s^∞ -map. Define $\mathcal{E}c : \mathbb{R}^n \rightarrow G$ to be the extension of c . The map $\mathcal{E}c : \mathbb{R}^n \rightarrow G_n$ is an imprint since $l \circ \mathcal{E}c = \mathcal{E}(l \circ c)$ is smooth for all $l \in G^*$. Therefore $l \circ \mathcal{E}c$ is C_s^∞ and so is $l \circ c = l \circ \mathcal{E}c \circ \iota_{\mathcal{H}}$. $\square$

Lemma 8.7 *For every $\text{eLC}_{\mathbb{R}^n}$ -space E , a linear functional $l : E \rightarrow \mathbb{R}$ is an $\text{eLC}_{\mathbb{R}^n}$ -map iff $l : E_h \rightarrow \mathbb{R}$ is an $\text{eLC}_{\mathcal{H}}$ -map.*

Proof: If $l \in E^*$ then it is also in $(E_n)^*$ (initial structure). On the other hand let $l : E_h \rightarrow \mathbb{R}$ be an $\text{eLC}_{\mathcal{H}}$ -map and $g : \mathbb{R} \rightarrow E$ an imprint. We have to show that $l \circ g$ is smooth. It is smooth on $\{(x, t) \in (\mathbb{R} \times \mathbb{R}) : t > 0\}$ since $l \circ g \circ \iota_{\mathcal{H}}$ is C_s^∞ and it is also smooth on $\{(x, t) \in (\mathbb{R} \times \mathbb{R}) : t < 1\}$ since $l \circ g \circ h \circ \iota_{\mathcal{H}}$ is C_s^∞ , where $h(x, t) = (x, 1 - t)$. $\square$

Proposition 8.8 *The structs $\text{eLC}_{\mathcal{H}}$ and $\text{eLC}_{\mathbb{R}^n}$ are isomorphic.*

Proof: Let E be an $\text{eLC}_{\mathbb{R}^n}$ -space. By lemma 8.6 and lemma 8.7 we get that E^* and $(E_{hn})^*$ have the same underlying set. Since E and E_{hn} have the initial structure of this set, they are equal. Analogously $G = G_{nh}$ for every $\text{eLC}_{\mathcal{H}}$ -space G . $\square$

Open Questions

In chapter 8 we showed that $\mathbf{eLC}_{\mathcal{H}} \simeq \mathbf{eLC}_{\mathbb{R}}$. Can we extend this isomorphism to hold between larger categories? One way to find such an isomorphism could be along the lines of proposition 3.9 and corollary 3.11. In order to apply proposition 3.9 we have to show that every $\mathcal{H}^n$ has the final $\mathbf{C}_s^\infty$ -structure induced by the $\mathbf{C}_s^\infty$ -maps from $\mathbb{R}^n$ to $\mathcal{H}^n$. Let us consider the one-dimensional case. Take a map $f : \mathcal{H}^1 \rightarrow \mathbb{R}$ such that the composite $f \circ c$ is smooth for all smooth $c : \mathbb{R} \rightarrow \mathcal{H}^1$. Clearly f is smooth in the interior of $\mathcal{H}^1$. Since the interesting point is $0 \in \mathcal{H}^1$, we assume that $c(0) = 0$. The curve c has to stay in $\mathcal{H}^1$ and since it is smooth we must have $\lim c'(t_n) = 0$ for $t_n \rightarrow 0$. We want to show that $f'(x)$ is bounded for $x \rightarrow 0$. Suppose f' is not bounded. Can we find a smooth curve c such that $(f \circ c)'(t_n) = f'(c(t_n))c'(t_n)$ is unbounded for $t_n \rightarrow 0$ and $c(0) = 0$, or roughly a c such that c' goes slow enough to 0?

The second question is related to the topocosm $\mathbf{C}_d$ of diffeomorphic spaces; see examples 2.16. How is it related to $\mathbf{C}_{\mathbb{R}}$? If we can't find an answer to the first question, we can't expect to find an answer to the analogous question for diffeomorphic spaces, but we might find one to the following: Are $\mathbf{eLC}_d$ and $\mathbf{eLC}_{\mathbb{R}}$ isomorphic? We would have to find an extension theorem analogous to proposition 8.3. Whitney [31] proved an extension theorem for Whitney-jets ("derivatives" are given) from closed sets. This extension, however, is not linear in the smooth case. Therefore a generalization of Seeley's approach, maybe with an intermediate step over manifolds with boundary, could be more promising.

List of Categories

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Bibliography

- [1] J. Adámek; H. Herrlich, *Cartesian closed categories, quasitopoi and topological universes*, Comment. Math. Univ. Carol. 27 (1986) 235-257
- [2] J. Adámek; H. Herrlich; G.E. Strecker, *Abstract and concrete categories*, (John Wiley 1990)
- [3] J. Adámek; J. Reiterman; F. Schwarz, *On universally topological hulls and quasitopos hulls*, Houston J. Math 17 (1991) 375-383
- [4] J. Boman, *Differentiability of a function and of its compositions with functions of one variable*, Math Scand. 20 (1967) 249-268
- [5] E.J. Dubuc, *Concrete Quasitopoi*, Applications of Sheaves, Lecture Notes in Math. (Springer) 753 (1979) 239-254
- [6] P.J. Freyd; G.M. Kelly, *Categories of continuous functors I.*, J. Pure Appl. Algebra 2 (1972) 169-191
- [7] A. Frölicher, *Catégories cartésiennement fermées engendrées par des monoides*, Cahiers Top. Géom. Diff., 21 (1980) 367-375
- [8] A. Frölicher; A. Kriegl, *Linear spaces and differentiation theory*, Pure and Applied Mathematics, J. Wiley, Chichester (1988)
- [9] A. Frölicher; A. Kriegl, *Differentiable extensions of functions*, Differential Geometry and its Applications 3 (1993) 71-90
- [10] H. Hogbe-Nlend, *Bornologies and functional analysis*, North-Holland Math. St. 26, Amsterdam (1977)

- [11] H. Jarchow, *Locally convex spaces*, Teubner Stuttgart (1981)
- [12] P.T. Johnstone, *Topos Theory*, Academic Press, London (1977)
- [13] J.F. Kennison, *Full reflective subcategories and generalized covering spaces*, Illinois J. Math. 12 (1968) 353-365
- [14] A. Kriegl, *Die richtigen Räume für Analysis im Unendlich-Dimensionalen*, Monatsh. Math. 94 (1982) 108-124
- [15] A. Kriegl, *Eine kartesisch abgeschlossene Kategorie glatter Abbildungen zwischen beliebigen lokalkonvexen Vektorräumen*, Monatsh. Math. 95 (1983) 287-309
- [16] A. Kriegl; L.D. Nel, *A convenient setting for holomorphy*, Cahiers Top. Géo. Diff. 26 (1985) 273-309
- [17] L.D. Nel, *Introduction to Categorical Methods (Parts One and Two)*, Carleton-Ottawa Mathematical Lecture notes Series 11, Carleton Univ. (1991)
- [18] L.D. Nel, *Introduction to Categorical Methods (Part Three)*, Carleton-Ottawa Mathematical Lecture notes Series 12, Carleton Univ. (1992)
- [19] L.D. Nel, *Enriched locally convex structures, differential calculus and Riesz Representation*, J. Pure Appl. Algebra 42 (1986) 165-184 North-Holland
- [20] L.D. Nel, *Categorical differential calculus and Banach-Steinhaus*, Categorical Topology (Prague Conference proceedings) (World Scientific) (1989) 149-162
- [21] L.D. Nel, *Topological universes and smooth Gelfand Naimark duality*, Mathematical Applications of Category Theory, Contemporary Math. 30 (1984) 224-276.
- [22] L.D. Nel, *Categorical differential calculus for infinite dimensional spaces* Cahiers Top. Géo. Diff. 29 (1988) 257-286
- [23] L.D. Nel, *Infinite Dimensional Calculus Allowing Nonconvex Domains with Empty Interior*, Mh. Math. 110, (1990) 145-166

- [24] D.G. Quillen, *Homotopical Algebra*, Lecture Notes in Math. 43 (Springer) (1967)
- [25] C.M. Ringel, *Diagonalisierungspaare I*, Math. Z. 117 (1970) 249-266
- [26] C.M. Ringel, *Diagonalisierungspaare II*, Math. Z. 122 (1971) 10-32
- [27] R.T. Seeley, *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. 15 (1964) 625-626
- [28] E. Siegl, *A free convenient vector space for holomorphic spaces*, Mh. Math. (199?) (preprint)
- [29] E.H. Spanier, *Quasi-Topologies*, Duke Math. J. 30 (1963) 1-14
- [30] J.C. Tougeron, *Idéaux de Fonctions Differentiables*, Ergebnisse d. Math., Band 71 (Springer, Berlin, 1972)
- [31] H. Whitney, *Analytical extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934) 63-89

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