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EFFECTIVE ANALYTICAL TECHNIQUES FOR THE DYNAMIC ANALYSIS OF STRUCTURES

by

Hong Xia

B.Sc.(Hons.), Chengdu Univ. of Sci. & Tech., P.R. China

M.Sc., Chengdu Univ. of Sci. & Tech., P.R. China

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THE DYNAMIC ANALYSIS OF STRUCTURES**

submitted by Hong Xia
in partial fulfillment of the requirements of
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External Examiner

Thesis Supervisor

J.L. Humar

Chairman, Department of Civil and

Environmental Engineering

J.L. Humar

Faculty of Engineering
Carleton University
Ottawa, Ontario, Canada

Abstract

Analysis of structures subjected to dynamic loading is often a time consuming and complex task. Many methods have been developed for either simplifying the problems or making the analysis procedure more effective and efficient. This work presents several innovative procedures for improving the efficiency of dynamic analysis. These procedures focus on the application of Ritz vectors and on analysis in the frequency domain.

In order to reduce the size of problems involving analysis of response of structural systems, a transformation based on appropriately selected Ritz shapes is commonly employed. The lower mode shapes may at times serve as effective Ritz shapes. However, the computation of mode shapes is a time consuming task; in addition, the mode shapes may not form the best basis for representing the spatial distribution of loads. The recently developed load dependent vectors, which are derived from a static solution for the applied loads, address some of the problems inherent in the use of mode shapes. However, both the natural mode shapes and the load dependent vectors fail to account for the frequency content of the loading, a parameter that may influence strongly the response, particularly for loading with a high frequency content. A procedure is presented here for the generation of frequency dependent vectors. A combination of load dependent and frequency dependent vectors often forms a very efficient basis for the representation of the response, as shown by several examples presented. These examples also illustrate the application of cut-off criteria developed in this work.

Analysis in the frequency domain using discrete Fourier transforms is an efficient means of calculating the dynamic response of linear systems. The use of discrete transforms along with finite summation requires that the forcing function and the impulse function be converted into periodic forms. This modification may introduce unacceptable errors in the results of analysis, unless appropriate steps

are taken to avoid or minimize the effect of aliasing or overlapping. For single degree-of-freedom systems, procedures that will eliminate the effect of aliasing have been developed. However, problems related to frequency domain analysis still exist for multi-degree-of-freedom systems with non-proportional damping, in analysis through substructuring and in those cases where a continuum solution is involved. A new procedure which addresses these problems and is applicable to both single and multi-degree-of-freedom systems as well as to analysis through substructuring is presented here. The procedure is based on the introduction of artificial damping in the system. The introduction of damping is only an intermediate step in the process and the final solution is obtained for the undamped system.

The technique of component mode synthesis, in which a structure is divided into several components for the purpose of analysis, is often employed in the dynamic analysis of large structures. The method permits a substantial reduction in the volume of computation. In addition, it offers the advantage that characteristics of an individual component can be evaluated without any reference to other components in the structure. This is particularly significant in the development of a large system where different groups work on the design of different components. The existing substructure coupling methods rely on the determination of the mode shapes of the components. The present work explores the possibility of utilizing Ritz vectors as substitutes for mode shapes. In particular, the use of load dependent Ritz vectors is investigated. Ritz vectors are believed to be more efficient, and at times, more effective than the mode shapes which take considerable time to compute. The advantage of using these alternative vectors is illustrated by examples.

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List of Notations

a	imposed artificial damping
$a_g^h(t)$ and $a_g^v(t)$	ground accelerations in horizontal and vertical directions respectively
$\mathbf{A}$	a square matrix
A_g^h and A_g^v	Fourier transforms of the horizontal and vertical components of ground motion, $a_g^h(t)$ and $a_g^v(t)$ respectively
A_n	displacement amplitude ratio used in cut-off criterion
b_j	body force
$\mathbf{B}$	a square matrix
$\mathbf{B}_v$	domain boundary on which displacement is defined
$\mathbf{B}_t$	domain boundary on which stress is defined
c_1 and c_2	dilatational and shearing wave velocities respectively for soil
C	velocity of sound in water
$\mathbf{C}$	damping matrix of a structural system
$\mathbf{C}^*$	reduced damping matrix of a structural system
$\mathbf{C}^{**}$	reduced damping matrix of a structural system obtained by using two stage Ritz vectors
$\mathbf{C}^\alpha$	damping matrix for substructure α
$\mathbf{C}^\beta$	damping matrix for substructure β
$\mathbf{D}$	diagonal matrix obtained in factorization of a matrix
EEL_n	force amplitude ratio used in cut-off criterion
$\mathbf{E}^h$ and $\mathbf{E}^v$	load amplitude vectors for horizontal and vertical ground motions respectively
f_A and f_B	response amplitudes at peak points A and B respectively
f_{hi} and f_{vi}	nodal forces on the interface nodes between foundation soil and superstructures along horizontal and vertical directions respectively
$\mathbf{e}_{L1}$	error in Ritz vectors in representation of load vector $\mathbf{f}$
$\mathbf{f}$	vector of applied loads
$\mathbf{f}^\alpha$	vector of forces on substructure α
$\mathbf{f}^\beta$	vector of forces on substructure β
$\mathbf{f}_{L1}$	approximate load vector obtained by using $L1$ Ritz vectors
$\mathbf{F}_{sb}$	vector of interaction forces between foundation soil and superstructure
$\mathbf{F}^*$	reduced load vector
$\mathbf{F}^{**}$	reduced load vector obtained by using two stage Ritz vectors
$\mathbf{F}_g(\Omega)$	vector of frequency dependent forces on all nodes of the soil
$\mathbf{F}_f(\Omega)$	vector of frequency dependent interaction forces between soil and structure
$g(t)$	dynamic load

$g(l\Delta t)$	discretized version of dynamic load $g(t)$
$\mathbf{G}$	flexibility matrix for a substructure
$\mathbf{G}_e$	residual flexibility matrix for a substructure
$G(\Omega)$	continuous Fourier transform of $g(t)$
$G(k\Delta\Omega)$	discrete Fourier transforms of $g(l\Delta t)$
$h(t)$	unit impulse function for a single degree-of-freedom system
$h(l\Delta t)$	discretized version of unit impulse function $h(t)$
$\tilde{h}(k\Delta t)$	truncated unit impulse response function
$\hat{h}_j(t)$	modified unit impulse function for the j th D.O.F. obtained by introducing artificial damping
$\hat{\mathbf{h}}(t)$	modified unit impulse function vector obtained by introducing artificial damping
$H(\Omega)$	continuous complex frequency response function for a single degree-of-freedom system
$\mathbf{H}_b(\Omega)$	continuous frequency response function vector for foundation soil
$H_0^{(1)}, H_1^{(1)}, H_2^{(1)}$	Hankel functions of the first kind and of zero, first and second order respectively
$H(k\Delta\Omega)$	periodic extension of the continuous frequency response function $H(\Omega)$
$\tilde{H}(k\Delta\Omega)$	discrete Fourier transform of the truncated unit impulse function $\tilde{h}(k\Delta t)$
$\hat{H}_j(\Omega)$	modified frequency response function corresponding to $\hat{h}_j(t)$
$\hat{\mathbf{H}}(\Omega)$	modified frequency response function vector
i	an integer
$\mathbf{I}$	a unit matrix
j	an integer
J_0, J_1, J_2	Bessel functions of the first kind and of zero, first and second order respectively
k	an integer
$\mathbf{K}$	stiffness matrix of a structural system
$\tilde{\mathbf{K}}$	modified stiffness matrix of a structural system including foundation flexibility
$\mathbf{K}_{ij}$	submatrix of $\mathbf{K}$
$\mathbf{K}_{ss}$	stiffness matrix of the superstructure with all the supports fixed
$\mathbf{K}^*$	reduced stiffness matrix of a structural system
$\mathbf{K}^{**}$	reduced stiffness matrix of a structural system obtained by using two stage Ritz vectors
$\mathbf{K}^\alpha$	stiffness matrix for substructure α
$\mathbf{K}^\beta$	stiffness matrix for substructure β
K_0, K_1, K_2	modified Bessel functions of zero, first and second order respectively
l	an integer
L_1	number of Wilson's Ritz vectors used
L_2	total number of Ritz vectors used

L	lower triangular factor of a matrix
m	an integer
M	mass matrix of a structural system
M_i	submatrix of M
M*	reduced mass matrix of a structural system
M**	reduced mass matrix of a structural system obtained by using two stage Ritz vectors
n	an integer
N	total number of sampling intervals
n	unit normal vector on a boundary surface
N	matrix of shape functions
p	an integer
p(t)	dynamic exciting force
p̃(t)	unit impulse function for hydrodynamic pressure on the face of dam
p(x, y, t)	hydrodynamic pressure
p₀^h(y, t)	hydrodynamic pressure on a rigid dam face casued by horizontal ground motion
p̄₀^h(y, Ω)	complex frequency response function corresponding to p₀^h(y, t)
p₀^v(y, t)	hydrodynamic pressure on a rigid dam face casued by vertical ground motion
p̄₀^v(y, Ω)	complex frequency response function corresponding to p₀^v(y, t)
p_k^h(y, t)	hydrodynamic pressure on dam face due to the deformation of dam casued by horizontal ground motion
p̄_k^h(y, Ω)	complex frequency response function corresponding to p_k^h(y, t)
p_k^v(y, t)	hydrodynamic pressure on dam face due to the deformation of dam casued by vertical ground motion
p̄_k^v(y, Ω)	complex frequency response function corresponding to p_k^v(y, t)
p_c^h(y, t)	hydrodynamic pressure on dam face casued by horizontal ground motion
p̄_c^h(y, Ω)	complex frequency response function corresponding to p_c^h(y, t)
p_c^v(y, t)	hydrodynamic pressure on dam face casued by vertical ground motion
p̄_c^v(y, Ω)	complex frequency response function corresponding to p_c^v(y, t)
P_s	hydrostatic force on damface
P(Ω)	continuous frequency response function of the hydrodynamic pressure on the face of dam
q	an integer
q	transfomed coordinates for substructures
q^α	transfomed coordinates for substructure α
q^β	transfomed coordinates for substructure β
Q₁(Ω), Q₂(Ω)	interaction vectors obtained in analysis through substructuring
r	a vector
r_b	spatial distribution matrix of all nodal forces on the interface between foundation soil and superstructure

R_n	force response amplitude approximated by using Ritz vectors
R_p	applied force pulse
$\mathbf{R}(t)$	interaction forces between water body and the upstream face of the dam
$\mathbf{R}^h(t)$	interaction forces between water body and the upstream face of the dam caused by horizontal ground motion
$\bar{\mathbf{R}}^h(\Omega)$	representation of $\mathbf{R}^h(t)$ in frequency domain
$\mathbf{R}^{xJ}(t)$	vector of x -direction forces of $\mathbf{R}(t)$ on the upstream face of the dam
$\mathbf{R}^v(t)$	interaction forces between water body and the upstream face of the dam caused by vertical ground motion
$\mathbf{R}_{sb}$	vector of interaction forces between foundation soil and superstructure
$\mathbf{R}^v(\Omega)$	representation of $\mathbf{R}^v(t)$ in frequency domain
$\mathbf{S}$	coupling matrix for substructures
$\mathbf{S}_{ff}(\Omega)$	frequency dependent soil impedance matrix
$\mathbf{S}^h(\Omega)$	dynamic stiffness matrix for dam-reservoir system under horizontal ground motion excitation
$\mathbf{S}^v(\Omega)$	dynamic stiffness matrix for dam-reservoir system under vertical ground motion excitation
$\mathbf{S}(\Omega)$	dynamic stiffness matrix for superstructure
$\mathbf{S}_{bb}^f(\Omega)$	dynamic stiffness matrix for soil free field
$\mathbf{S}_{bb}^g(\Omega)$	dynamic stiffness matrix for soil with excavation
t_{AB}	time lapse between points A and B
$\mathbf{t}$	vector of nodal forces
$\tilde{\mathbf{t}}$	vector of surface tractions
T_h	duration of unit impulse function $h(t)$
T_g	duration of applied load $g(t)$
T_{ij}^*	fundamental solutions for surface tractions
$\mathbf{u}_i$	updated static vector
$v(t)$	displacement response of a single degree-of-freedom structural system
$v(l\Delta t)$	periodic displacement response for a single degree-of-freedom structural system
$\mathbf{v}(t)$	displacement vector of a structural system
$\mathbf{v}^\alpha$	displacement vector of substructure α
$\mathbf{v}^\beta$	displacement vector of substructure β
$\dot{\mathbf{v}}$	velocity vector of a structural system
$\ddot{\mathbf{v}}$	acceleration vector of a structural system
v_j	displacement along j th D.O.F.
$\dot{v}_j$	velocity along j th D.O.F.
$\ddot{v}_j$	acceleration along j th D.O.F.
$v(l\Delta t)$	periodical displacement response
$\tilde{\mathbf{v}}$	vector of element node displacements
V_{ij}^*	fundamental solution for displacements

$\mathbf{V}_g(\Omega)$	frequency dependent displacement vector for the soil nodes
$\mathbf{V}_f(\Omega)$	frequency dependent displacement vector for the interface nodes
$\mathbf{x}_i$	a Ritz vector
w	unit weight of water
$\mathbf{X}$	a set of Ritz vectors
$\mathbf{X}_0$	a revised set of Ritz vectors
$\mathbf{X}_2$	the second stage Ritz vectors
$\mathbf{X}_s$	partial Ritz vectors generated from the stiffness matrix $\mathbf{K}_{ss}$
$\mathbf{X}_c$	displacements at internal degrees of freedom of a substructure caused by a unit displacement applied in turn to each of the degrees of freedom at the soil-structure interface
$\mathbf{X}^{xf}$	x -components of Ritz vector set $\mathbf{X}$ at the nodal points on the upstream face of the dam
Y_0, Y_1, Y_2	Bessel functions of the second kind and of zero, first and second order respectively
$\mathbf{z}(t)$	generalized coordinates representing the approximate solutions of the reduced system
$\mathbf{Z}(\Omega)$	representation of the generalized coordinates, $\mathbf{z}(t)$, in frequency domain
$\mathbf{Z}^h(\Omega)$	complex frequency response functions for dam-reservoir system under horizontal ground motion excitation
$\mathbf{Z}^v(\Omega)$	complex frequency response functions for dam-reservoir system under vertical ground motion excitation
α	reflection coefficient of the reservoir bottom
β_n	ratio of excitation frequency to the n th natural frequency
δ	a constant used in the shifting of a matrix
$\delta(t)$	Dirac time function
Δt	time sampling interval
$\Delta \Omega$	frequency sampling interval
ϵ_{L1}	scalar expression of the error for Ritz vectors in load vector representation
$\mathbf{Z}$	matrix of eigenvectors
λ and μ	the Lamé constants of soil material
λ_j	j th eigenvalue
Λ	diagonal matrix of eigenvalues
ξ_n	damping ratio for the n th mode
$\Pi^m(\mathbf{r})$	Krylov matrix
$\Pi_s^m(\mathbf{r})$	Krylov subspace
ρ, ν and E	the mass density, Poisson's ratio, and Young's modulus of the soil material respectively
ρ_n	displacement response amplitude approximated by using the n th Ritz vector
ϕ_n	mode shape corresponding to ω_n
$\tilde{\Phi}_k$	matrix of a truncated set comprising k mode shapes of a structural system

ψ and χ	functions used in the fundamental solutions for soil domain
$\psi_{I,1}$	an error measure for Ritz vectors based on mass distribution
Ψ	Ritz vectors for substructures
Ψ_α	Ritz vectors for substructure α
Ψ_β	Ritz vectors for substructure β
Ψ_n	flexible Ritz vectors for a substructure
Ψ_a	attachment Ritz vectors for a substructure
Ψ_c	constraint Ritz vectors for a substructure
Ψ_r	rigid body Ritz vectors for a substructure
Ψ_f	free interface Ritz vectors for a substructure
ω_n	n th natural frequency of a structural system
Ω	frequency variable in frequency domain analysis

Chapter 1

INTRODUCTION

1.1 GENERAL

The material presented in this thesis deals with the development of efficient analytical techniques for dynamic analysis of structures. Several new techniques are presented for both time domain and frequency domain analysis. The application of these techniques is illustrated by numerous examples. Analysis of interaction problems such as dam-water and soil-structure interaction is investigated. A study is carried out of the substructure coupling methods and the use of Ritz vectors in place of mode shapes in several typical methods is explored.

1.2 OBJECTIVES OF THE THESIS

Problems encountered in structural dynamic analysis are usually characterized by large size (many degrees of freedom), and time-dependence. An accurate evaluation of displacement and stress response in such problems requires the solution of large systems of dynamic equilibrium equations. To minimize the effort needed in the solution of a large linear dynamic system, recourse must be had to procedures that reduce the size of the dynamic system, otherwise the computation would be very time-consuming and expensive. The reduction in the size of dynamic problems can often drastically improve the efficiency of the analysis. One way of achieving this is by using Ritz vectors. A considerable amount of research work has been directed in the past towards the development of efficient Ritz vectors. Traditionally, mode shapes of the undamped system have been used as Ritz vectors. Recently, load dependent Ritz vectors have been developed and have been shown to be more efficient. However, both the mode shapes and the load dependent vectors fail to take into account the frequency content of the loading. Additional research is therefore needed in this direction.

Another procedure that can improve the efficiency of the computation is the frequency domain analysis. With the development of fast Fourier transform, frequency domain procedure has become a very effective means of analysis of linear systems. Frequency domain analysis is often the primary choice in the analysis of structural interaction problems where a substructuring technique is used. This is so either because a particular numerical method employed in the solution of a substructure is best suited to the frequency domain analysis, or because formulation of the problem in the frequency domain is much simpler than in the time domain. One such case is the dam-reservoir interaction problem where either an analytical solution in the frequency domain (Chakrabarti and Chopra 1973b; 1974; Chopra and Chakrabarti 1981; and Chopra 1968) is employed, or the boundary element method with fundamental solutions formed in the frequency domain (Hanna and Humar 1982; and Humar and Jablonski 1988) is used. Another case is soil-structure interaction problem (Wolf 1985; 1989; and Karabalis and Beskos 1987) where the fundamental solutions for boundary element method are much simpler in the frequency domain than in the time domain. In the analysis of these structural interaction problems, the frequency domain procedure usually offers a more efficient solution than the time domain procedure. It is worth pointing out that some time domain procedures developed very recently for the solution of interaction problems can be as efficient as the frequency domain procedures. For example, in soil-structure interaction problems, the recursive technique proposed by Wolf and Motosaka (1989a; and 1989b) has made computation in the time domain as efficient as those in the frequency domain. However, the recursive technique provides only approximate results except in a few special cases where the exact response can be obtained.

The procedures used in the frequency domain analysis are based on rigorous mathematical theory and have been used for quite a long time. However, a number of problems that are inherent in the frequency domain analysis using discrete transforms need to be addressed. This is particularly so for multi-degree-of-freedom systems and for substructure analysis.

In structural interaction problems, such as dam-reservoir-foundation and soil-structure interaction as mentioned in the preceding paragraph, systems are often composed of two or more substructures with vastly differing properties. These substructures are analyzed separately often using different techniques. For instance, in soil-structure interaction problem, the structure part may be analyzed by using the Finite Element Method (FEM) while the soil foundation may be solved by using the Boundary Element Method (BEM). Efficient methods are needed for the representation of individual substructures and for coupling them in order to obtain the behaviour of the complete system. In fact, substructuring concepts are also often employed to facilitate the analysis of structural systems even with uniform properties. Such is the case for very large and complex structures such as, for example, aerospace and space structures (Craig and Chang 1976). The effectiveness and efficiency of substructure coupling method used in the analysis become very important in these cases. Traditionally, classic mode shapes of substructures have been used in the available substructure coupling procedures. In many cases, more effective and efficient substructure coupling procedures can be obtained by using Ritz vectors in place of mode shapes because the generation of mode shapes is more time consuming than that of Ritz vectors.

The research problems identified in the foregoing paragraphs have been addressed in the present work. An attempt has been made either to improve the existing procedures or to develop new ones. The objectives of the thesis can be summarized as follows

1. To develop a more efficient set of Ritz vectors to be used in a transformation meant to reduce the size of the dynamic analysis problem.
2. To develop new procedures for the frequency domain analysis of single and multi-degree-of-freedom systems with or without damping, with a view particular to address the problem of aliasing.
3. To discuss the problems encountered in the application of frequency domain procedures in analysis through substructuring and to develop new methods

that are suitable for such analysis.

4. To illustrate the application of the proposed methods by applying them to the solution of dam-reservoir and soil-structure interaction problems.
5. To study the possibility of using Ritz vectors in place of mode shapes in an analysis through substructure coupling, and to extend the application of proposed substructure coupling methods to the analysis of soil-structure interaction problems.

1.3 REVIEW OF PREVIOUS LITERATURE

1.3.1 Ritz Vectors

The exact response of the mathematical model of a system can be obtained by a direct analysis of the complete system including all its degrees of freedom. However, for a structural system with a large number of degrees of freedom, such a procedure can be very time consuming and expensive. Fortunately, in most cases, response of the system can be expressed in terms of a few generalized coordinates whose number is much smaller than the number of degrees of freedom in the system. Therefore, it is the usual practice to reduce the number of equations of motion to a much smaller number by an appropriate means before a dynamic response analysis is carried out. For many problems, an effective reduction in the size can be obtained by representing the response of the system by the superposition of a small number of lower order vibration modes. This method may not be very efficient in some cases (Vugts *et al* 1979). Several procedures to improve the efficiency of the mode superposition method, such as the modal acceleration method (Cornwell *et al* 1983; and Soriano and Filho 1988) and static correction (Hansteen and Bell 1979; and Léger and Wilson 1988), have therefore been developed. These procedures, in general, lead to better results with fewer coordinates.

The Rayleigh Ritz approximation is another way to reduce the size of the dynamic problems. In fact, the modal superposition method is a special case of Rayleigh Ritz method. The Rayleigh Ritz method can be interpreted as a kind

of coordinate transformation. The first step in the method is to choose a set of suitable linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_m$, of the same order as the number of degrees of freedom. The dynamic response $\mathbf{v}$ can now be expressed as

$$\mathbf{v} = \mathbf{X}\mathbf{z}(t) \quad (1.1)$$

where $\mathbf{z}(t)$ are the generalized coordinates representing the approximate solutions of the reduced system. The vectors in $\mathbf{X}$ should satisfy the geometric boundary conditions of the problem. By applying this coordinate transformation, the number of equations of motion can be significantly reduced. The dimension of the new set of equations of motion is equal to the number of independent Rayleigh Ritz vectors used. An appropriate number of mode shapes may be used as the transformation vectors in Eq. 1.1, but they may not be able to take into account the spatial distribution of the dynamic loads. The mode shapes are not therefore the most efficient set of transformation vectors (Vugts *et al* 1979; and Cornwell *et al* 1983). Unlike the mode shapes which must be solutions to an eigenvalue problem and must also satisfy geometric boundary conditions, the Rayleigh Ritz vectors only need to satisfy the geometric boundary conditions of the problem. This flexibility offers the possibility of generating Ritz vectors that are more effective in representing the original problem, and are at the same time computationally more efficient.

The Rayleigh Ritz vectors have been widely used in extraction of some of the frequencies and mode shapes of a system (Parlett 1980). Lanczos vectors, which are in fact Ritz vectors taken from a Krylov Space, were developed (Lanczos 1950; Parlett 1980) to provide an efficient way to achieve the same purpose. The application of Lanczos algorithm in the analysis of dynamic response was introduced by Nour-Omid and Clough (1984; and 1985). An alternative procedure of generating Ritz vectors that was almost identical to the Lanczos algorithm was proposed by Wilson *et al* (Wilson *et al* 1982; and Joo, Wilson and Léger 1989). This latter procedure uses full orthogonalization instead of the orthogonalization of the current vector against the preceding two vectors as is usually done in Lanczos algorithm. Both Lanczos and Wilson's vectors can give good representation of the spatial dis-

tribution of dynamic loads.

Recently, further modification has been made in Wilson's method (Léger and Wilson 1987). In the modified algorithm, the updated static load vector is used to generate the Ritz vector at each step instead of using the preceding Ritz vector. It is believed that this can consider the effect of spatial distribution of external load more efficiently.

In summary, there are three commonly used types of Rayleigh Ritz vectors, namely mode shapes, Lanczos vectors and the Ritz vectors obtained from Wilson's method. The first one uses a limited number of mode shapes corresponding to the lower frequencies. Such modal vectors are not very effective for those cases where the spatial distribution of the forcing function cannot be adequately represented by the lower modes. For example, when the external loads are distributed in such a way that they are orthogonal to some of the mode shapes, those mode shapes will not be excited at all by the external loads. If such modes happen to be among the lower frequency modes, the mode extraction method will still extract them. As a result, a larger number of modal Ritz vectors will have to be used in such cases. In Lanczos and Wilson's methods, the static force vector is taken into account in the generation of the Ritz vectors. In this way, the effect of spatial distribution of external loads is automatically included. This often means that fewer number of vectors are required for adequate representation of the original system than would be needed if the mode shapes of the system were used as the Ritz vectors. The recent modification of Wilson's method, in fact, leads to a more effective consideration of the spatial distribution of external load (Léger and Wilson 1987). The developments and improvements described above are very significant in dealing with large structural dynamic problems. However, it was realized early in the development that all available procedures for generating Ritz vectors still provide vectors that are affected by mode shapes in the neighborhood of the lowest frequencies. The modal extraction method actually takes the several lowest mode shapes. The algorithms for generating the Ritz vectors by using either Lanczos method or Wilson's

method are based on the Krylov space sequence:

$$[\mathbf{r}, \mathbf{K}^{-1}\mathbf{M}\mathbf{r}, (\mathbf{K}^{-1}\mathbf{M})^2\mathbf{r}, \dots, (\mathbf{K}^{-1}\mathbf{M})^m\mathbf{r}] \quad (1.2)$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix and $\mathbf{r}$ is an arbitrary vector. This sequence converges to the lowest mode shape. Therefore, both methods can still only produce a set of Ritz vectors which extract the lower modes. As a result, these methods are only effective when the predominant frequency content of the external loading lies in the vicinity of the lower frequencies of the system. It has been noted that the effectiveness of these methods is most apparent when the frequency of the applied loading or support excitation is lower than the fundamental natural frequency (Léger and Wilson 1987; and Léger 1988). However, some dynamic loads have high predominant frequencies. For such loads, the available Ritz vectors may lead to large errors even though the spatial distribution of the loads is well simulated. Therefore, a more efficient way of generating a different set of Ritz vectors which can consider both the spatial distribution and the frequency content of dynamic loads is necessary.

1.3.2 Frequency domain analysis

Numerous methods are available for dynamic response analysis in the time domain. For example, modal superposition, and numerical integration methods can efficiently and effectively be applied to common structural dynamic problems. A unique advantage of the numerical integration methods of time domain analysis is that it is possible to consider any nonlinearity in the dynamic system. For linear systems, frequency domain analysis offers an alternative to the analysis in the time domain. The use of discrete Fourier transforms and the Fast Fourier Transform (FFT) algorithm makes the frequency domain analysis procedures comparatively economical (Meek and Veletsos 1972; Veletsos and Kumar 1983; and Veletsos and Ventura 1984). For some structural problems such as soil-structure interaction problems, frequency domain analysis may be the most convenient way of solving

for the response, since the frequency response functions constitute the most directly available information in these cases. However, before frequency domain procedures become truly effective, a lot of work needs to be done on the implementation of such procedures for the analysis of real life structures. One of the recognized problems is that this method can not be used when the structural system does not possess any damping and the frequency response functions rather than time domain unit impulse functions are available for the system. The use of discrete Fourier transforms in the solution of undamped systems gives rise to spurious non-zero initial conditions, and the additional transient responses due to these spurious initial conditions reders the resultant response significantly in error.

A brief theoretical background to the frequency domain analysis procedure is as follows. If $H(\Omega)$ represents the response of a single degree-of-freedom system to a harmonic excitation $e^{i\Omega t}$, the response of the structural system to an external dynamic loading $g(t)$ is given by

$$v(t) = \frac{1}{2\pi} \int_0^\infty H(\Omega)G(\Omega)e^{i\Omega t} d\Omega \quad (1.3)$$

where $G(\Omega)$ is the continuous Fourier transform of $g(t)$.

One of the problems in frequency domain analysis arises because, in general, the discrete Fourier transform rather than the continuous Fourier transform shown above has to be used in the computation. In its discrete form, Eq. 1.3 becomes

$$v(l\Delta t) = \frac{1}{2\pi} \sum_{k=0}^{N-1} \bar{H}(k\Delta\Omega)\bar{G}(k\Delta\Omega)e^{2\pi ikl/N} \Delta\Omega \quad (1.4)$$

where $\bar{H}(k\Delta\Omega)$ is the periodic extension of continuous complex response function $H(\Omega)$, and $\bar{G}(k\Delta\Omega)$ the discrete Fourier transforms of $g(l\Delta t)$; Δt is the time sampling interval; and $\Delta\Omega$ is the frequency sampling interval. Equation 1.4 does not give the correct response but yields an erroneous response which is referred to as periodic response (Veletsos and Ventura 1984). The errors can be divided into two kinds. The first kind of errors is caused by insufficient range of summation, $N\Delta\Omega$, and the second by too coarse a value of interval $\Delta\Omega$. In some cases a

singularity may exist in the continuous complex frequency response function $H(\Omega)$. If there is some damping in the structural systems, singularity in the continuous complex frequency response function $H(\Omega)$ will disappear. In that case, the correct response can be obtained by the above summation provided $\Delta\Omega$ is small enough and the value of $N\Delta\Omega$ large enough (Humar 1990). However, if singularity exists in the continuous complex frequency response function $H(\Omega)$, the discrete convolution represented by Eq. 1.4 fails since no matter how small $\Delta\Omega$ is taken, the integration in Eq. 1.3 still cannot be correctly evaluated by the summation in Eq. 1.4. In the case where damping does exist but is very small, the structural system may behave almost like an undamped system and an extremely small value of the interval $\Delta\Omega$ is necessary to guarantee the right answer. A very small $\Delta\Omega$ means that a very long time period is needed, and this will obviously require a very large volume of computation, making the method very inefficient.

The problems outlined above have been solved for single degree-of-freedom systems by Meek and Veletsos (1972), Veletsos and Kumar (1983), and Veletsos and Ventura (1984). The basic premise of their work is that the use of Eq. 1.4 gives a periodic response rather than the real response. The difference between the two responses is equal to the free vibration response of the same structural system, corresponding to the differences in the initial conditions in the two responses. The method presented by Meek and Veletsos (1972) is based on the application of corrective force pulses toward the end of the time period of analysis. The number of force pulses used is equal to the number of initial conditions and their magnitudes are selected such as to make the initial conditions of the periodic response obtained from Eq. 1.4 to be the same as the initial conditions of the real response. In the work by Veletsos and Kumar (1983), the true transient free vibration response corresponding to the differences between the initial conditions in the periodic response and the real response is used to correct the periodic response. In a later work by Veletsos and Ventura (1984), a periodic free vibration response rather than a transient free vibration response is used to correct the periodic response. The methods referred to are all applicable to single degree-of-freedom systems or

to the modal equations of a decoupled multi-degree-of-freedom system. The force pulse corrective method has been extended by Humar (1990) for application to multi-degree-of-freedom systems without having to decouple the latter to modal coordinates. This method will be briefly introduced in this thesis. The method proposed by Humar (1990) becomes ineffective if too many degrees of freedom are included in the analysis, partly because the equations necessary to solve for the values of force pulses become ill-conditioned.

Since all of the above methods are based on the correction of the initial conditions of the entire system, they cannot be used to analyze problems in which a substructure approach is used in the formulation, and the response is to be obtained for only some of the substructures.

The task in the frequency domain analysis is therefore to develop new methods for systems without damping or with very light damping, especially when a substructure method of analysis is used.

1.3.3 Dynamic dam-reservoir interaction

The seismic response of a dam, including the effect of water body stored in the reservoir, is one example of a dynamic interaction problem. It is also one of the earliest problems studied where the effect of one substructure was taken into account in the analysis of another substructure which was of primary concern. Dam-reservoir interaction problem will be investigated in the later stage of the thesis as an illustration of the application of the proposed new methods of frequency domain analysis. It is therefore relevant to briefly review some of the previous work in this area.

The work done by Chopra and his colleagues provides a good representation of the currently available procedures for analyzing dam-reservoir interaction or dam-reservoir-rock foundation interaction problems. Chopra's work in this area dates back to 1960s when he studied the hydrodynamic pressures on rigid dams due to horizontal as well as vertical motions caused by earthquakes (Chopra 1967).

The work by Chopra (1968) marks the beginning of a series of publications by Chopra and his colleagues in the field of dam response. In the early stages of their work, consideration was limited to the hydrodynamic pressure exerted on rigid dams resting on rigid soil foundation. Then, they began to consider the effects of the flexibility of soil foundation on the hydrodynamic pressure due to vertical earthquake excitation and of the flexibility of the dam on the hydrodynamic pressure caused by both horizontal and vertical earthquake excitation. A formulation based on substructure concept was used (Chakrabarti and Chopra, 1973a; 1973b; and 1974) in all these stages. In subsequent stages of their work, Chopra and his co-workers studied the effects of interaction between the dam and the reservoir and the foundation (Chopra and Chakrabarti 1981; Chopra and Gupta 1981; Hall and Chopra 1982; Chopra and Gupta 1982; Fenves and Chopra 1984; 1985a; and 1985b) so that the dam was no longer treated as resting on a rigid foundation. The analytical studies in the references cited so far were restricted to gravity dams. However Chopra and his co-workers have extended the application of their methods to arch dams and the results of their studies related to such dams have been reported in a series of publications (Porter and Chopra 1982; Hall and Chopra 1982; Fok and Chopra 1986a; 1986b; and Fok and Chopra 1987).

In several studies by Humar *et al*, the boundary element method was successfully used to analyze the interaction problem in a dam-reservoir system (Hanna and Humar 1982; and Humar and Jablonski 1988). The proposed method was capable of analyzing reservoirs of infinite length and of arbitrary geometry of the bottom.

All of the above mentioned procedures for analyzing dam-water interaction problems generate sampled values of continuous frequency response functions corresponding to the coupled degrees of freedom of the dam structure. The final step in obtaining the time response history requires the application of inverse discrete Fourier transform to the products of frequency response functions and the Fourier transform of earthquake excitation. At present, discrete convolution procedure is the only method available for obtaining the inverse Fourier transform. The method

works only if there is sufficient damping in the system. However, usually the water in the reservoir is treated as an ideal fluid. No damping therefore exists in the water system when it is of finite extent. For reservoirs of infinite extent, radiation damping is present but the magnitude of such damping is very small and in many cases may not be large enough to guarantee that frequency domain analysis will yield correct results. It is therefore clear that the frequency domain analysis procedures may introduce undetected errors in the analysis of dam-reservoir interaction systems. This is one of the problems investigated in this thesis.

1.3.4 Dynamic soil-structure interaction

The response of a structure to an earthquake, or to other dynamic loading, may be strongly influenced by the characteristics of the foundation soil on which the structure is supported. Currently, the effect of the foundation soil on the response of the structure is considered by using one of the following methods: (1) the soil portion is modelled by equivalent Winkler's springs attached to the structure (Ghouse *et al* 1987); (2) finite element method is used to represent both the structure and the foundation soil (Hall and Chopra 1982); (3) boundary element method is used to model the foundation soil while the finite element method is used to model the structure (Karabalis and Beskos 1987). Boundary element method has become increasingly popular because it can more efficiently model the foundation soil of infinite extent in comparison to the finite element method. The obvious disadvantage of the finite element method in representing the soil is that an artificial boundary has to be introduced at the end of a finite mesh. This leads to reflection of waves at the boundary of the finite mesh, which can only be partly eliminated by using special finite elements or special boundary conditions. On the other hand, the boundary element method can implicitly take into account wave radiation to infinity (Wolf 1985; Beskos 1987; and Estorff and Kausel 1989).

When the boundary element method is used to model the foundation soil, the problem is most efficiently formulated in the frequency domain (Wolf 1985; Eilouch and Sandhu 1986; Beskos 1987; and Estorff and Kausel 1989), although recently,

increasing interest is being directed towards the formulation of soil-structure interaction problem in the time domain (Estorff and Kausel 1989; Wolf 1989; and Dangla 1988). The advantage of the formulation in the time domain is that the nonlinear properties of soil can be considered, but such a formulation is usually more complicated and computationally more expensive than a formulation in the frequency domain. Also the time domain formulation is difficult to apply to soil having irregular surface. Recently, Wolf and Motosaka (1989a; and 1989b) have proposed a recursive technique to be used in the evaluation of the Duhamel's integral (the convolution integral). Their method has made the time domain analysis of soil-structure interaction problems as efficient as an analysis in the frequency domain. However, the recursive method provides only an approximation to the true response except in some special cases. In summary, if the system can be treated as linear, a frequency domain analysis of soil-structure interaction problems still seems to be preferable to a time domain analysis.

When frequency domain analysis procedures are used in the analysis of soil-structure interaction, problems similar to those encountered in dam-reservoir interaction occur so that the discrete convolution method may not be useful in many cases. The investigation of the difficulties inherent in frequency domain analysis of soil-structure interaction problems is one of the objectives of the present work.

1.3.5 Substructure coupling methods

Substructure coupling using mode shapes or mode synthesis methods have been extensively used in the analysis of dynamic structural problems in aerospace engineering. Hurty (1965) was probably the first to initiate such work. Numerous methods have been developed since then. Benfield *et al* (1972), and Craig and Chang (1976) have reviewed the methods available.

The types of mode shapes and vectors used in the substructure coupling methods developed in the past can be summarized as follows:

1. Flexible normal modes. They can be subdivided into free-interface, fixed-

interface, loaded-interface, and hybrid-interface normal modes.

2. Constraint modes.
3. Attachment modes.
4. Rigid-body modes.

In the method proposed by Hurty (1965), fixed-interface normal modes, rigid-body modes, and redundant constraint modes were used to define substructure generalized coordinates. This method was modified in the work by Craig and Bampton (1968) where the rigid-body modes were shown to be a subset of constraint modes. McNeal (1971) developed a hybrid method which allowed some substructure interface coordinates to be constrained while others were free. The concept of "residual flexibility" was used by McNeal (1971) and was extended by Rubin (1975). The use of "residual flexibility" may improve the convergence of some methods and obviate the need to obtain the high frequency modes. The paper by Hintz (1975) thoroughly discussed the different class of substructure modes, and their possible combination. The method of "constraint modes" and the method of "attachment modes" were described. It was shown that the different modes were not necessarily linearly independent of each other. Benfield and Hruda (1971) used only the normal modes of the substructures in their free-free interface model, and the normal modes and constraint modes in their constraint model. A general procedure of substructure coupling was suggested by Craig and Chang (1977) and Chang (1977), and it was shown that a number of methods developed in the past could all be derived on the basis of a general procedure. This general procedure was also extended for damped systems (Craig and Chung, 1982). It is worth pointing out that the majority of work associated with modal synthesis has been carried out in aerospace engineering, and very few applications of such work can be found in civil engineering.

As stated earlier, the use of Ritz vectors in place of mode shapes in substructure coupling methods may lead to substantial improvement in the computational efficiency. The present work explores this possibility.

1.4 OUTLINE OF THE THESIS

Chapter 1 provides an introduction to the problems studied, reviews the existing literatures and outlines the scope of the work. A new method of generating Ritz vectors that are capable of considering both the spatial distribution and the frequency content of external loading is proposed in Chapter 2. Solutions are obtained for a few systems of small size in order to illustrate the effectiveness of the proposed method. The proposed method is also examined by considering a system of much larger size.

In Chapter 3, the complex frequency response functions which represent the properties of structural systems are studied in detail. A relationship is derived between the discrete Fourier transform of the truncated impulse function and the continuous complex frequency response function, both for a single degree-of-freedom system. Based on this relationship, a new method for frequency domain analysis is proposed which can be used as an alternative to the methods suggested by Veletsos *et al* (Meek and Veletsos 1972; Veletsos and Kumar 1983; and Veletsos and Ventura 1984). The presentation in this chapter also provides an understanding of the sources of errors that exist in the available methods. Frequency domain analysis procedures for multiple degree-of-freedom systems are also discussed in Chapter 3. The method proposed by Humar (1990) is introduced along with some suggested modifications. A method of obtaining the corrective response on the basis of direct integration of the equations of motion is also proposed. The relationship developed for a single degree of freedom system between the discrete Fourier transform of the truncated impulse function, and the continuous complex response function is extended to the multiple degree-of-freedom case. The use of the methods cited above is illustrated by some examples.

Chapter 4 discusses the difficulties inherent in the frequency domain analysis of structural interaction problems, and presents new methods developed specially for analyzing such problems. These methods can also be effectively used in solving systems with no damping or very little damping as well as multi degree-of-freedom

systems. Some simple examples are given in this chapter.

The dam-reservoir interaction problems are analyzed in Chapter 5. The use of an analytical solution in the frequency domain for infinite water body is presented. The analytical procedures serve as examples of the application of the methods developed in Chapter 4 to practical and complex systems.

Chapter 6 deals with the analysis of soil-structure interaction problems in the frequency domain using Boundary Element Method. The main focus of the work is on the application of methods proposed in Chapter 4 to the analysis of soil-structure interaction problems. Effective reduction of soil-structure interaction systems is also discussed. In addition, the formulation of the soil-structure interaction problems is discussed and a careful treatment of singularity in the boundary integrals is presented.

Use of Ritz vectors in place of mode shapes in the available substructure coupling methods is investigated in Chapter 7. Focus is placed on the generation of load dependent Ritz vectors for substructures, and on the coupling procedures.

Chapter 8 explores further the use of substructure coupling procedures employing Ritz vectors in the analysis of structural interaction problems, such as soil-structure interaction. Simplified forms of the Hintz's method, and the Benfield and Hruda's method are developed. Some soil-structure interaction examples, including one using BEM method are presented.

Since this thesis focuses on several different topics, each of the following chapters opens with an introduction to the subject matter contained in that chapter. The introduction is also meant to supplement the review of related literature contained in Chapter 1. At the end of each chapter, a short summary is provided. The figures and tables associated with a chapter are attached to the end of that chapter for convenience of reference.

Chapter 10 provides a summary of the whole thesis and presents some suggestions for possible future work. The references are listed in Appendix I.

Chapter 2

FREQUENCY DEPENDENT RITZ VECTORS

2.1 INTRODUCTION

In obtaining the response of a structure to dynamic loading, the first step is generally to establish a discretized model of the structural system. Through this process, what is usually an infinite number of degree-of-freedom problem is approximated by a finite number of degree-of-freedom problem. The equations of motion of a discretized model can be written in the following form

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{C}\dot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{f}. \quad (2.1)$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix of the system, $\mathbf{f}$ is the vector of applied loads, and $\mathbf{v}$ is the displacement vector. The purpose of dynamic response analysis is to solve Eq. 2.1 for the displacement vector $\mathbf{v}$. For classically damped linear structures with a relatively small number of degrees of freedom, the solution is easily obtained by using modal superposition method in which the response is expressed in terms of the undamped free vibration mode shapes. However, most practical problems in structural dynamics are complicated and may have over hundreds or even thousands of degrees of freedom. Even for a classically damped system, modal superposition method using all mode shapes is a very expensive means of obtaining the response. It is therefore common to apply a coordinate transformation to Eq. 2.1 to reduce the size of the problem. The transformation is based on a small number of mode shapes of the system, or on other appropriately selected Ritz vectors.

As already stated in Chapter 1, a set of properly selected Ritz vectors often gives better results than the same number of lowest mode shapes. However, the existing methods of generating the Ritz vectors, whether Lanczos's or Wilson's method, can

only take into account the spatial distribution of the external loading. On the other hand, the response is affected by both the spatial distribution of the load and its frequency content. In many situations, particularly for loads with high frequency content, Ritz vectors that only represent the spatial distribution of load may not be very effective. At the same time, a number of different kinds of loads possess high frequencies. Such loads include: loads due to machinery, construction loads, loads due to explosion, and blast loads. Their range of frequencies is generally quite wide, and in particular may contain some predominant high frequencies. Even earthquake loads may contain some high frequency components (Jenschke *et al* 1964). In some cases of high frequency loads, acceptable results can only be obtained when all of the mode shapes are used in the analysis. This is because such external loading tends to excite the modes corresponding to higher frequencies which are close to the predominant frequencies of the external loading. Since the available Rayleigh Ritz methods are not effective in extracting the modes of higher frequencies precisely, they fail to yield an efficient set of transformation vectors.

Considering the limitation of the available Rayleigh Ritz vector procedures in representing the effect of load components of high frequency, a new method is developed to generate a set of Ritz vectors which are especially useful in considering the predominant frequencies of external loads. The mathematical basis of the new method and the corresponding algorithm are described. Also, several examples are provided to show the effectiveness of the proposed method.

2.2 MATHEMATICAL BASIS FOR FREQUENCY DEPENDENT RITZ VECTORS

A forcing function affects the response of a structural system in two ways, first through its spatial distribution which determines the magnitude of the load components along the various degrees of freedom, and second through its frequency content. If the frequency of the forcing function is close to one of the natural frequencies of the structural system (which is not necessarily the fundamental frequency of the system), resonance can occur in the structural mode corresponding

to that natural frequency. Apparently, that mode and several other modes in the vicinity of that mode will contribute significantly to the response of the system and should not therefore be ignored in the analysis. In the currently available methods, the generation of Rayleigh Ritz vectors does not consider this. However, with a simple modification, some of the methods can be made capable of taking into account both the spatial distribution and the frequency contents of the forcing function.

As already mentioned, the Ritz vectors generated according to Lanczos method or Wilson's method are based on the Krylov space sequence. The new Ritz vectors proposed here are also related to Krylov space. For a matrix A , the Krylov matrix is defined as follows

$$\Pi^m(r) = [r \quad Ar \quad A^2r \quad \dots \quad A^{m-1}r] \quad (2.2)$$

where r can be any nonzero vector. Vectors $r \quad Ar, \quad A^2r, \quad \dots \quad A^{m-1}r$ are in fact the Ritz vectors used in either the Lanczos method or the Wilson's method. The linear independence of these vectors is not guaranteed. Therefore a suitable orthogonalization procedure should be implemented to make the Ritz vectors linearly independent.

The Krylov subspace is defined as the span of the Krylov matrix. It can be written as

$$\Pi^m(r) = \text{span}\{\Pi^m(r)\} \quad (2.3)$$

The dimension of space $\Pi^m(r)$ will be m if all the m vectors in Krylov matrix are linearly independent. The maximum dimension of $\Pi^m(r)$ is n which is the rank of matrix A . When an appropriate orthogonalization procedure is carried out during the generation of each vector in Eq. 2.2, all the m vectors can be guaranteed to be linearly independent.

Now consider the Krylov matrix and the Krylov space of $(A - \delta I)$, where δ is

any constant, and $\mathbf{I}$ is a unit matrix of the same order as matrix $\mathbf{A}$. The Krylov matrix is

$$\Pi^m(\delta, \mathbf{r}) = [\mathbf{r} \quad (\mathbf{A} - \delta\mathbf{I})\mathbf{r} \quad (\mathbf{A} - \delta\mathbf{I})^2\mathbf{r} \quad \dots \quad (\mathbf{A} - \delta\mathbf{I})^{m-1}\mathbf{r}] \quad (2.4)$$

and the Krylov space is

$$\Pi_r^m(\delta, \mathbf{r}) = \text{span}\{\Pi^m(\delta, \mathbf{r})\} \quad (2.5)$$

The two Krylov spaces in Eq. 2.3 and Eq. 2.5 will be the same as long as the number of linearly independent vectors is equal to n , the rank of the matrix $\mathbf{A}$. In other words, the maximum Krylov space of $\mathbf{A}$ and the maximum Krylov space of $\mathbf{A} - \delta\mathbf{I}$ are the same.

The vectors in Eq. 2.4 tend to converge to the vectors which extract the mode shapes with eigenvalues close to δ . Although, each individual vector from Eq. 2.4 will possibly be very much different from those in Eq. 2.2, $(\mathbf{A} - \delta\mathbf{I})^k\mathbf{r}$ in Eq. 2.4 is still one of the vectors taken from the maximum Krylov space of $\mathbf{A}$. As a result, a set of Ritz vectors comprising some vectors from Eq. 2.2 and others from Eq. 2.4 is also in the same Krylov space. Also, the vectors in the set satisfy the geometric boundary conditions defined by $\mathbf{A}$. These characteristics are very important to the present work, because the Ritz vectors taken from Eq. 2.2 will extract the lower modes of $\mathbf{A}$, which are very significant in the structural dynamic response, while the Ritz vectors taken from Eq. 2.4 will extract the modes of $\mathbf{A}$ that correspond to the natural frequencies close to a particular predominant excitation frequency of the external load, provided δ is intentionally chosen to represent that frequency.

In structural dynamic problems, the eigenvalue problem of $\mathbf{K}^{-1}\mathbf{M}$ is considered. Therefore, Eq. 2.2 should be written in the following form

$$\Pi^m(\mathbf{r}) = [\mathbf{r} \quad \mathbf{K}^{-1}\mathbf{M}\mathbf{r} \quad (\mathbf{K}^{-1}\mathbf{M})^2\mathbf{r} \quad \dots \quad (\mathbf{K}^{-1}\mathbf{M})^{m-1}\mathbf{r}]. \quad (2.6)$$

and Eq. 2.4 becomes

$$\Pi^m(\delta, \mathbf{r}) = [\mathbf{r} \quad (\mathbf{K} - \delta\mathbf{M})^{-1}\mathbf{M}\mathbf{r} \quad \{(\mathbf{K} - \delta\mathbf{M})^{-1}\mathbf{M}\}^2\mathbf{r} \quad \dots \quad \{(\mathbf{K} - \delta\mathbf{M})^{-1}\mathbf{M}\}^{m-1}\mathbf{r}] \quad (2.7)$$

The shifting strategy represented by Eqs. 2.4 and 2.7 has long been used in the extraction of mode shapes and frequencies (Humar 1990). Bathe and Ramaswamy (1980) have proposed the use of shifting to accelerate the convergence of subspace iteration. The use of shifts in context of Lanczos method has been suggested by Hughes (1987), Ericsson and Ruhe (1980) and Nour-Omid *et al* (1983). More recently, Ibrahimbegovic and Wilson (1990) have observed that the use of a shifted matrix in the Lanczos algorithm for generating load dependent Ritz vectors will speed up the convergence if the shift value δ is selected to be within the dominant range of frequencies.

The shifted stiffness method has also been applied in the eigensolution of semi-definite systems which exhibit rigid body modes and have a singular stiffness matrix (Humar 1990). The method is not, however, quite effective in obtaining Ritz vectors suitable for the dynamic analysis of such systems (Yiu 1990).

2.3 IMPROVEMENT TO WILSON'S METHOD

2.3.1 Algorithm for Generating the Ritz Vectors

The concept of using some vectors from the Krylov space of $(\mathbf{K} - \delta\mathbf{M})^{-1}\mathbf{M}$, outlined in the preceding section, can be easily incorporated in the algorithm devised by Wilson *et al* to generate the Ritz vectors. The algorithm presented by Joo *et al* (1989) is employed here. That algorithm uses updated static load vector rather than the preceding Ritz vector to generate the next vector. As a result, the vectors do not belong to the Krylov space described in the preceding section. They are actually taken from the following sequence

$$\Pi^m(\mathbf{r}) = [\mathbf{K}^{-1}\mathbf{M}\mathbf{u}_1 \quad (\mathbf{K}^{-1}\mathbf{M})^2\mathbf{u}_2 \quad \dots \quad (\mathbf{K}^{-1}\mathbf{M})^m\mathbf{u}_m] \quad (2.8)$$

in which each $\mathbf{u}_i$ is a updated static vector with the first one calculated from $\mathbf{K}\mathbf{u}_1 = \mathbf{f}$.

The vectors taken from Eq. 2.8 have been found quite effective in representing the spatial distribution of load and hence in obtaining the dynamic response. In the improved algorithm, the first L_1 vectors are generated by the Wilson's method. The remaining vectors are then generated by a procedure that is similar to the Wilson's method but uses the matrix $(K - \delta M)$ in place of K , with δ being selected to be close to the square of the predominant frequency of the exciting force. These additional vectors are orthogonalized, as they are generated, with respect to all previous vectors.

The algorithm described in the foregoing paragraph is summarized in Table 2.1.

If the exciting force has several predominant frequencies in a cluster, δ can be taken as a value in the middle of the cluster. On the other hand, if there are several predominant frequencies that are widely separated, the generation of frequency dependent vectors may be repeated with one or more additional shifts.

Occasionally, the selected shift may be too close to an eigenvalue of the system. The shifted matrix $K - \delta M$ will then be nearly singular and numerical instability may arise. If an LDL^T factorization is used for the matrix $K - \delta M$, singularity can be detected by observing a very large value on the diagonal of D . If this should happen, the process may have to be repeated with a new value for the shift. At the same time, if the selected shift that causes the singularity is equal to the excitation frequency, the presence of singularity serves as a warning of a possible resonance.

The coordinate transformation is given by

$$\mathbf{v} = \mathbf{X}\mathbf{z} \quad (2.9)$$

in which $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_{L_1} \ \mathbf{x}_{L_1+1} \ \dots \ \mathbf{x}_{L_2}]$ is the matrix of Ritz vectors obtained from the algorithm shown in Table 2.1.

2.3.2 Cut-Off Criteria

The cut-off criteria used in generating the Wilson's vectors, that is in determining the integer value L_1 , are adopted directly from the work by Joo *et al* (1989).

Since the number of vectors used in the transformation, M , is less than N , the number of degrees of freedom, errors are introduced in the response analysis. Obviously M should be selected to be sufficiently large so that the errors are acceptable. The vector describing the error in the representation of load $\mathbf{f}$ is given by

$$\mathbf{e}_{L1} = \mathbf{f} - \mathbf{f}_{L1}. \quad (2.10)$$

where

$$\mathbf{f}_{L1} = \sum_{i=1}^{L1} \mathbf{x}_i \mathbf{f}^T \mathbf{M} \mathbf{x}_i \quad (2.11)$$

A scalar representation of the error is obtained from

$$\epsilon_{L1} = \frac{|\mathbf{f}^T \mathbf{e}_{L1}|}{\mathbf{f}^T \mathbf{f}}. \quad (2.12)$$

Vector generation may be terminated when ϵ_{L1} is less than a specified value.

An error measure based on mass distribution is another criterion suggested in the work by Joo *et al* (1989)

$$\psi_{L1} = 1 - \frac{\sum_{i=1}^{L1} (\tilde{\mathbf{M}}^T \mathbf{x}_i)^2}{\sum_{i=1}^N \tilde{M}_i}. \quad (2.13)$$

where $\tilde{\mathbf{M}}$ is a vector whose i th term is calculated from

$$\tilde{M}_i = \sum_{j=1}^N m_{ij}$$

and m_{ij} is the i, j th element of matrix $\mathbf{M}$.

If the two measures are to be used simultaneously, vector generation is terminated when both ϵ_{L1} and ψ_{L1} are lower than certain specified values. Considering a diagonal mass matrix, it is apparent that if the exciting force is applied along a massless degree of freedom, $\mathbf{f}^T \mathbf{M}$ appearing in Eq. 2.11 will be a null vector. In this case, ϵ_{L1} will always be 1 and the error norm based on load representation will fail to indicate convergence. Therefore, if that happens, the error norm in Eq. 2.13 will be the only effective criterion.

The above cut off criteria take into consideration the following two characteristics

- (i) Spatial distribution of load vector; and
- (ii) Distribution of mass.

Criteria need to be established to determine the number of frequency dependent Ritz vectors needed from the space of $(\mathbf{K} - \delta\mathbf{M})^{-1}\mathbf{M}$. Such criteria have to consider the following factors

- (i) Spatial distribution of load vector;
- (ii) Distribution of mass;
- (iii) Frequency content of the external loads.

In order to illustrate the effect of frequency content of the exciting force on the system response, suppose that $\mathbf{p}(t) = \mathbf{f} \sin \Omega t$, in which $\sin \Omega t$ represents the time variation, Ω being, in this case, the sole frequency of the exciting force. Also assume that the mode shapes of the system are used as the Ritz vectors. As a result, the transformed equations are uncoupled and the solution to the n th uncoupled equation is given by

$$z_n = \frac{\phi_n^T \mathbf{f}}{\omega_n^2} \frac{1}{\sqrt{(1 - \beta_n^2)^2 + (2\xi_n \beta_n)^2}} \sin(\Omega t - \theta). \quad (2.14)$$

where ω_n is the n th natural frequency, ϕ_n is the corresponding mode shape, ξ_n is the damping ratio, $\beta_n = \Omega/\omega_n$ and $\tan \theta = 2\xi_n \beta_n / (1 - \beta_n^2)$.

The effect of spatial distribution of the exciting force is represented by the participation factor $\phi_n^T \mathbf{f}$, while the effect of exciting frequency is reflected in the term $A_D = 1/\sqrt{(1 - \beta_n^2)^2 + (2\xi_n \beta_n)^2}$. This term has the highest value when $\beta_n \approx 1$, that is when the exciting frequency is close to the natural frequency ω_n . Obviously, modes that have frequencies near Ω may make a large contribution to the response and should not be ignored in the cut-off criteria.

If the generated Ritz vectors are close to the mode shapes of the system, an estimate of the response corresponding to the n th Ritz vector can be obtained from Eq. 2.14, with the mode shape ϕ_n being substituted by the n th Ritz vector $\mathbf{x}_n$.

Except near resonance, the amplitude of this response is approximately given by

$$\rho_n = \frac{\mathbf{x}_n^T \mathbf{f}}{\omega_n^2} \frac{1}{1 - \beta_n^2} \quad (2.15)$$

where $\mathbf{x}_n$ is the n th mass-orthonormal Ritz vector, $\beta_n^2 = \delta/\omega_n^2$, δ is the square of predominant excitation frequency, and ω_n^2 may be taken as being equal to $\mathbf{x}_n^T \mathbf{K} \mathbf{x}_n$.

In order to define a cut-off criterion, amplitude ρ_n is obtained for each Ritz vector including those generated by the Wilson's procedure. An amplitude ratio is calculated for each frequency dependent Ritz vector by using the following expression

$$A_n = \frac{|\rho_n|}{\sum_{i=1}^{n-1} |\rho_i|} \quad (2.16)$$

where $\sum_{i=1}^{n-1} \rho_i$ is the sum of the amplitudes of the Ritz vectors previously extracted. The generation of the frequency dependent vectors is terminated when A_n is below a specified value, say ϵ_f , and at the same time $\omega_n^2 = \mathbf{x}_n^T \mathbf{K} \mathbf{x}_n$ is larger than δ , the square of the predominant exciting frequency.

Equation 2.16 is meant to estimate the errors for displacement response. A similar cut-off criterion can also be set up for estimating errors for nodal force response. In this case, the following equations are used.

$$R_n = \mathbf{x}_n^T \mathbf{f} \frac{1}{1 - \beta_n^2} \quad (2.17)$$

and

$$EEL_n = \frac{|R_n|}{\sum_{i=1}^{n-1} |R_i|} \quad (2.18)$$

A deficiency of the proposed cut-off criterion is that $\mathbf{x}_n^T \mathbf{K} \mathbf{x}_n$ provides a rather crude estimate of the frequency. In order to improve the criterion, it is essential to solve the reduced eigenvalue problem given by

$$\mathbf{K}^* \mathbf{Z} = \mathbf{M}^* \mathbf{Z} \mathbf{\Lambda} \quad (2.19)$$

where $\mathbf{K}^* = \mathbf{X}^T \mathbf{K} \mathbf{X}$, $\mathbf{M}^* = \mathbf{X}^T \mathbf{M} \mathbf{X}$, $\mathbf{Z}$ is the matrix of eigenvectors and $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues. Matrices $\mathbf{Z}$ and $\mathbf{\Lambda}$ provide good estimates of the system's mode shapes and eigenvalues, respectively.

A revised set of Ritz vectors is now obtained from

$$\mathbf{X}_0 = \mathbf{XZ} \quad (2.20)$$

Vectors in $\mathbf{X}_0$ are mass-orthonormal and stiffness orthogonal and their use yields a set of uncoupled equations of motion. The cut-off criteria defined in Eqs. 2.14 through 2.18 are used with $\mathbf{x}_{0n}$ in place of $\mathbf{x}_n$, and the frequency estimates obtained from the solution of Eq. 2.19. As before, the generation of the frequency dependent vectors is truncated when the cut-off criteria are satisfied and at the same time the highest estimated frequency is greater than the predominant excitation frequency. In general, the highest frequency obtained by the solution of the reduced eigenvalue problems may not provide a good estimate of the highest frequency spanned by the set of Ritz vectors. Therefore, in comparing the frequencies covered by the Ritz vectors with the predominant frequency of excitation, the highest frequency obtained from the reduced eigenvalue problem may be ignored and the next highest used instead.

The solution of the reduced eigenvalue problem will, of course, involve additional computations. For a problem of large size, this additional computation time will generally form a small proportion of the total time.

In the examples presented here, the improved cut-off criteria have been used.

2.4 APPLICATIONS

2.4.1 Example 1. Plate subjected to Harmonic Loading

The plate shown in Fig. 2.1 is identical to the one analyzed in the work by Joo *et al* (1989). It is modelled by six four-node plane stress elements, with 18 degrees of freedom, and as in the same reference, the lumped mass matrix is formed by summing each column of the consistent mass matrix. The loading is a sinusoidal pressure of frequency 15.0 rad/s applied to the upper edge of the plate.

In the work by Joo *et al* (1989), the plate was analyzed for its response to a load with frequency of 3.14 rad/s, which is lower than 4.94 rad/s, the calculated

fundamental frequency of the plate. It was observed that 6 Wilson's Ritz vectors were adequate for obtaining a reasonably accurate estimate of the response. In the present case the excitation frequency is much higher, and as seen from Fig. 2.2a, which shows the amplitude of the steady state displacement response along each of the 18 degrees of freedom, six Wilson's Ritz vectors give completely erroneous results. Even 17 Ritz vectors give results that are in considerable error. On the other hand, as evident from the results presented in Fig. 2.2b, a combination of 6 Wilson's Ritz vectors and 3 frequency dependent vectors obtained with $\delta = 15^2$ gives response values that are almost exact. The maximum stresses at the center of each element are also obtained and listed in Table 2.2. Again, 9 vectors (6 Wilson's and 3 frequency dependent vectors) provide very good estimate of the stress, while even 17 Wilson's vectors give stress estimate that are in significant error.

The eighteen natural frequencies of the plate are presented in Table 2.3. The natural frequencies of the reduced system obtained by using 17 load dependent Ritz vectors and those obtained from 9 vectors — 6 load dependent and 3 frequency dependent — are also shown. In addition, the values of the term $\sqrt{\mathbf{x}_i^T \mathbf{K} \mathbf{x}_i}$, that can be used in the proposed cut-off criteria as rough estimates of the natural frequencies, are given in the last column of the table. It is found that the 17 Wilson's Ritz vectors miss one of the natural frequencies close to the frequency of the exciting force. However, the 9 combined Ritz vectors give very good estimates of the three natural frequencies that are close to the exciting frequency while skipping some other frequencies that are farther away from the exciting frequency. The values of $\sqrt{\mathbf{x}_i^T \mathbf{K} \mathbf{x}_i}$ listed in the last column provide rather crude estimates of the frequencies.

Table 2.4 shows the convergence of the Ritz vectors. The second column is the error factor of the Wilson's load dependent Ritz vectors obtained by using Eq. 2.12. The third column shows the displacement errors estimated for the frequency dependent Ritz vectors by using Eq. 2.16 and column 4 is the nodal force errors estimated by using Eq. 2.18. In each case, the Ritz vectors and frequencies used are obtained by solving a reduced eigenvalue problem. Error norms are not

calculated for vector No. 1. In determining L_1 and L_2 , the following criteria are used: $r_{L1} < 0.1$, $A_n < 6.15$, and that the last but one frequency is higher than the predominant excitation frequency. The computer programs for the solution of this problem were implemented on an Apollo DN3000 work station. The computer CPU time for generating Wilson's load dependent Ritz vectors were 4.93 s for 9 vectors, and 8.68 s for 17 vectors; the CPU time for generating 6 Wilson's vectors and 3 frequency dependence vectors was 5.60 s. In addition, 5.15 s were used in solving the reduced eigenvalue problems required for measuring convergence.

In this small example, the time consumed in the solution of reduced eigenvalue problem off-sets savings realized by using a more efficient set of vectors. For larger size problems, this is not expected to be the case. This will be shown later on in the analysis of a 1900 degree-of-freedom system. At the same time, it is significant to note that even when a large number of Wilson's vectors, more than adequate in numbers to satisfy a reasonable cut-off criterion (Eqs. 2.12 or 2.13), are used, the accuracy of the results is not assured. This may be attributed to the fact that the cut-off criteria in Eqs. 2.12 and 2.13 are based on a good spatial representation of the load but do not take into account frequency content.

2.4.2 Example 2. A Shear frame model

This example is also taken from the work by Joo *et al* (1989) and deals with the response of a shear frame model with 40 degrees of freedom as shown in Fig. 2.3. The fundamental frequency of the frame is 1.128 rad/s. Degrees of freedom 37, 38 and 39 have a zero mass associated with them. The applied force acts along degree of freedom 37 and is given by $p(t) = 1000(\sin 14t + \sin 16t + \sin 18t)$.

Figures 2.4a through c show the amplitude of response along each of the 40 degrees of freedom for the three harmonic components of the load. In each case, the so called exact values are obtained by using 34 mode shapes as the Ritz vectors. It is seen that 8 Wilson load dependent vectors fail to give even an approximate estimate of the response. On the other hand, a combination of two load dependent Ritz vectors and six frequency dependent Ritz vectors obtained with $\delta = 16^2$ pro-

vides very good estimates of the response. Figures 2.5a through c show that the comparison of maximum nodal force responses leads to a similar conclusion.

Table 2.5 presents the first 15 exact natural frequencies of the system. The important natural frequencies are the 6th, 7th, 8th, and 9th because they are closest to the predominant frequencies of loading, namely 14.0 , 16.0 and 18.0 rad/s. The 8 Wilson's vectors cover frequencies that are close to the 6th and 8th missing the 7th and 9th. However, the combination vectors with 2 load dependent and 6 frequency dependent vectors cover the entire spectrum of the important frequencies, providing good estimates of their values. The values in the last column, provide very rough estimates of the natural frequencies.

Table 2.6 shows the convergence of the Ritz vectors. The second column is the error factor of the Wilson's load dependent Ritz vectors obtained by using Eq. 2.14. As mentioned earlier, Eq. 2.12 will fail to provide a criterion for convergence in this case because the loading is applied at a massless degree of freedom. The third column shows the displacement errors estimated for the frequency dependent Ritz vectors by using Eq. 2.16 and column 4 is the nodal force errors estimated by using Eq. 2.18, both obtained by using Ritz vectors and frequencies calculated from the solution of a reduced eigenvalue problem. Error norms were not calculated for the first vector. In this example, the following criteria were used to determine L_1 and L_2 : $\psi_{L1} < 0.1$, $A_n < 0.15$, and that the last but one frequency is greater than the highest excitation frequency. The CPU time required for generating 8 combined Ritz vectors was 23.20 s. The additional time required to solve the reduced eigenvalue problems was 11.23 s. It was observed that, in order to obtain equivalent amount of accuracy in the displacements, 15 Wilson's vectors had to be used; the CPU time required to generate these vectors was 30.58 s.

2.4.3 Example 3. Plate subjected to arbitrary loading

The previous two examples have shown that the proposed frequency dependent Ritz vectors are very effective in the analysis for loading that has high predominant frequencies. In this example the plate model of Example 1 is analyzed for its

response to a horizontal earthquake motion. Earthquake loading is regarded as being in the class of dynamic loading where low excitation frequencies dominate, occasionally accompanied by a few higher frequency components. Also, a major portion of the response is usually contained in the lower modes. The purpose of this example is to show that even for such a loading, a proper use of several frequency dependent Ritz vectors will produce a more efficient set of combined Ritz vectors in comparison to Wilson's Ritz vectors alone. The exciting support acceleration is assumed to vary in the same fashion as the Taft horizontal earthquake acceleration in SE direction. The acceleration history is shown in Fig. 2.6.

Figure 2.7 depicts the Fourier transform of the support acceleration shown in Fig. 2.6. It is obvious that there is an important range of frequencies below 5.0 rad/s for this loading. Since the fundamental frequency of the plate is also about 5 rad/s, load components with frequencies below 5 rad/s are adequately considered by Wilson's Ritz vectors. However, another important range of frequencies is around 8.0 rad/s, and the effect of force components with these frequencies may be significant. Therefore, the plate is analyzed first by using 4 Wilson's Ritz vectors and then by using a combined set of Ritz vectors — 2 Wilson's and 2 frequency dependent with $\delta = 64$. The results are compared with the exact values obtained by using all of the 18 modes. Figures 2.8a through d show these results for degrees of freedom 11, 12, 13, and 14. Average acceleration method is used to compute all transient responses with the time step taken as $\Delta t = 0.005$.

Figure 2.8 shows that the use of 4 Wilson's vectors gives displacement response results that, in some cases, are significantly in error. On the other hand, the use of 2 frequency dependent vectors in combination with 2 Wilson's vectors gives response values that are very close to the exact values. The use of 4 Wilson's vectors alone results in even more severe errors in the stresses. Figures 2.9a and 2.9b compare the stresses evaluated by using 4 Wilson's vectors and by using 2 Wilson's vectors in conjunction with 2 frequency dependent vectors against the exact values obtained from the superposition of all of the 18 modes. The horizontal normal stresses at the

centers of elements 1 and element 4 are presented in these figures. A comparison with results for other elements, not presented here, shows that the best results obtained from 4 Wilson's vectors are for element No. 1 (Fig. 2.9a). Even here the errors are significant. The corresponding errors for element No. 4 (Fig. 2.9b) are large and unacceptable. On the other hand, when 2 Wilson's vectors are used in conjunction with 2 frequency dependent vectors the estimates of stresses are in very good agreement with the exact values.

It is apparent that even for earthquake loading, which is commonly considered to be a dynamic loading that excites the lower modes of the system more severely, frequency dependent vectors used in conjunction with load dependent vectors may prove to be more efficient in comparison to the Ritz vectors that do not take any account of the exciting frequencies of the external loading.

2.5 ANALYSIS OF A LARGE PLANE FRAME STRUCTURE BY USING FREQUENCY DEPENDENT RITZ VECTORS

The examples provided in the foregoing sections have shown the effectiveness of the proposed frequency dependent Ritz vectors as well as the suggested cut-off criteria. However, the efficiency of frequency dependent Ritz vectors has not been demonstrated for the analysis of structural systems with a large number of degrees of freedom. For a system with a large number of degrees of freedom, the extra computation effort required in the following two operations involved in generating the frequency dependent Ritz vectors should be further investigated. In generating the frequency dependent vectors, the LDL^T decomposition has to be carried out twice, once for decomposing the original stiffness matrix of the system, K , and a second time for decomposing the shifted stiffness matrix, $K - \delta M$, which is used in generating the frequency dependent Ritz vectors. In the original algorithm proposed by Joo *et al* (1989), only one LDL^T decomposition of the stiffness matrix, K is required. Thus, the computing time required for the LDL^T decomposition of $K - \delta M$ in the algorithm for generating frequency dependent Ritz vectors will lead to

additional computation cost as compared to that required for the generation of Wilson's Ritz vectors. The LDL^T decomposition of $K - \delta M$ for a large size of structural system usually accounts for a significant amount of CPU time in the analysis. The cut-off criteria suggested for determining the number of frequency dependent Ritz vectors that must be generated in order to ensure adequate accuracy require the computation of the mode shapes for a reduced system each time a new frequency dependent Ritz vector is obtained. Some additional computing time is also needed in this part of the procedure. However, the promising aspect of using the frequency dependent Ritz vectors is that the total number of Ritz vectors needed for correctly representing the behavior of the structural system may be much smaller than the number of Wilson's Ritz vectors or mode shapes of the system and the time saved in using a smaller number of Ritz vectors may more than offset the time required in the two operations mentioned above.

The main purpose in this section is to show the effectiveness and efficiency of the frequency dependent Ritz vectors in the analysis of a large size structural system.

2.5.1 Description of problem

Figure 2.10 shows the structural system to be analyzed. A plane frame building structure with 95 storeys is considered. The height of each storey is taken as 2.5 m; and the spacing between columns is taken as 10.0 m. The material parameters are: Young's modulus of the material $E = 2 \times 10^4$ MPa, Poisson's ratio $\nu = 0.15$. The area of the cross section for both columns and beams is taken as 0.36 m^2 , and the moment inertia of both the columns and beams is $1.08 \times 10^{-2} \text{ m}^4$. Each node of the structure is assumed to have two degrees of freedom, a horizontal displacement and a vertical displacement. The total number of degrees of freedom is 1900. The SAP-IV finite element program is used to get the stiffness matrix and the diagonal mass matrix. The stiffness matrix has an order of 1900 with the half bandwidth being 22. The time variation of the exciting load is chosen as $p(t) = \sin 25.0t$. The amplitude vector for this load is taken as the negative value of the product of a spatial distribution vector $\mathbf{r} = \{1\}$ and the mass matrix $\mathbf{M}$, where vector $\mathbf{r}$ has all

unit elements and is of order 1900. The objective is to obtain the amplitudes of the steady state responses along each degree of freedom of the system considered to the specified harmonic exciting load.

2.5.2 Analysis of the system

The 1900 degree-of-freedom system described in the last section is analyzed by reducing it to a system of much smaller size. Both Wilson's Ritz vectors (Joo *et al* 1989) and frequency dependent Ritz vectors proposed in this work are used for this purpose. The cut-off criteria for the Wilson's Ritz vectors is taken as $\epsilon_{L1} < 0.1$, and for the frequency dependent Ritz vectors as $A_n < 0.15$. The shift frequency is taken as $\delta = 25.0^2 = 625.0$. With the given cut-off criteria, the computer program automatically generates 8 combined Ritz vectors, the first five being Wilson's Ritz vectors and the last three being frequency dependent Ritz vectors. It is found that the use of only five Wilson's Ritz vectors gives responses that are in severe error. Obviously, a significantly larger number of Ritz vectors are required in order to obtain acceptable response estimates. However, when frequency dependent Ritz vectors are used along with the five Wilson's Ritz vectors, a total of 8 Ritz vectors gives almost the same responses for each of the degrees of freedom as that obtained by using the first 40 mode shapes of the system. Further trials with Wilson's Ritz vectors disclose that even with 13 such vectors, the responses are in severe error. Comparison between the various estimates of the amplitudes of steady state responses along some typical degrees of freedom is presented in Fig. 2.11 to 2.13. It is found that the errors become acceptable only when 15 or more Wilson's Ritz vectors are used in the analysis.

Computations required for this analysis were carried out on an Apollo DN3000 work station. The computer CPU time required for generating 8 Wilson's Ritz vectors was a total of 325.0 seconds, which is less than 405.0 seconds, the CPU time spent in generating 5 Wilson's Ritz vectors plus another 3 frequency dependent Ritz vectors. Obviously, significant extra time is consumed in the latter case. As mentioned before, this extra time is required by two additional steps in the

computations required for generating frequency dependent Ritz vectors. One is the additional $\mathbf{LDL}^T$ decomposition carried on the shifted stiffness matrix, $\mathbf{K} - \delta\mathbf{M}$, which accounts for 59.3 seconds of computer CPU time. The other is the extra effort required in applying the convergence criteria for determining the required number of frequency dependent Ritz vectors, which accounts for 20.7 seconds of computer CPU time. Although, the generation of 8 Wilson's Ritz vectors requires much less computer CPU time than that of 8 combined Ritz vectors, the system behaviors can not be represented by using only 8 Wilson's Ritz vectors. From Figures 2.11b through 2.13b, it is seen that even with 13 Wilson's Ritz vectors, completely wrong responses are obtained. The generation of 13 Wilson's Ritz vectors needs 506.0 seconds of computer CPU time. If responses with acceptable accuracy are to be obtained by using only the Wilson's Ritz vectors, 15 vectors are required as evidenced by the results shown from Figs. 2.11b all though 2.13b. The computer CPU time spent in the process of generating these vectors is 564.7 seconds. The above comparison shows that the use of frequency dependent Ritz vectors can lead to considerable saving in computing time since a smaller number of vectors is required to achieve acceptable accuracy. The example given here compares only the computer CPU time consumed in the stage of generating these vectors. In a complete analysis, the much larger size of the reduced system produced by using only the Wilson's Ritz vectors means that the total analysis process will be much less efficient than that involved in using the combined set of Wilson's Ritz vectors and frequency dependent Ritz vectors. The latter can reduce the original problem to a much smaller size without sacrificing the accuracy of the final response.

Table 2.7 lists the natural frequencies of the first 13 modes for the system. The exact values are the first 13 of those obtained from a modal extraction of 40 modes. The results show that the fundamental and a few of the lowest natural frequencies of the system are well predicted by both a set of 13 Wilson's vectors and a set of 8 combined vectors. However, when Wilson's vectors are used, errors appear for higher values of natural frequencies. Although the set of 13 Wilson's vectors provides a range of natural frequencies covering the exciting frequency of

the external load, the frequencies in the vicinity of the exciting frequency are in considerable error. Therefore, the effect of the exciting frequency on response is not well represented by this set of Wilson's Ritz vectors. The set of 13 Wilson's Ritz vectors provides good estimates of the natural frequencies near the exciting frequency of external load. As a result, the responses obtained by using this set of Ritz vectors are quite close to the exact ones. The natural frequencies extracted by the combined Ritz vectors, i.e. 5 Wilson's Ritz vectors plus 3 frequency dependent Ritz vectors are given in Column 5. Estimates of frequencies located near the frequency of the external load are very accurate, even though only 8 vectors are used. Consequently, the set of 8 combined Ritz vectors yields fairly accurate estimates of the response of the structural system to the specified dynamic loading.

Table 2.8 shows the convergence of the Ritz vectors. All items in this table are similar to those in Table 2.6, and similar explanations apply.

2.6 CHAPTER SUMMARY

A transformation using appropriately selected Ritz vectors is often necessary in order to reduce the size of a problem involving dynamic response analysis. Load dependent vectors may prove to be very effective Ritz vectors in such a transformation because they are able to represent the spatial distribution of the load. However, these vectors fail to account for the frequency distribution of the applied loading. Since the response is affected both by the spatial distribution of the load and its frequency content, the limitation of load dependent vectors in representing the frequency content may reduce their effectiveness. This is particularly so for a loading with predominant high frequencies. In such cases, the use of frequency dependent Ritz vectors obtained by the procedure described in this chapter may greatly improve the efficiency of the computation. The load dependent vectors are often derived from the Krylov space of $\mathbf{K}^{-1}\mathbf{M}$, the starting vector in the space being equal to the static response of the system to the applied loading. On the other hand, the frequency dependent vectors are derived from the Krylov space of

$(\mathbf{K} - \delta\mathbf{M})^{-1}\mathbf{M}$ where δ is a shift whose value is selected to be close to the square of the predominant frequency in the loading.

In determining the required number of Wilson's load dependent Ritz vectors, L_1 , and frequency dependent Ritz vectors, L_2 , Eq. 2.12 or Eqs. 2.13 and 2.16 can be used to provide estimates of truncation errors. In the examples given in this chapter, a value of 0.1 is used for ϵ_{L1} and ψ_{L1} , and a value of 0.15 is used for A_n .

In practice, best results are obtained by using a combination of the load dependent and frequency dependent vectors. This requires the decomposition of at least two different matrices, $\mathbf{K}$ and $\mathbf{K} - \delta\mathbf{M}$. For the first three examples presented in this chapter, the extra computation involved in this was compensated by the ability to use a smaller number of Ritz vectors to obtain the same or better level of accuracy in the response results. However, the proposed cut-off criteria for frequency dependent vectors required the solution of a reduced eigenvalue problem. When the CPU time used in the eigensolution was included, the total computation time for the generation of frequency dependent vectors was higher than that required for the generation of load dependent vectors. This is not the case for a large system as shown in the last example where a 1900 degree of freedom system is analyzed. For a large size problem, the time required for the eigensolution of a reduced problem is just a small proportion of the total computation time.

Examples of both small and large systems presented in this chapter show that a high frequency content of external load may have a dominant effect on the dynamic responses of a structure. The effect of the high frequency content in the external load may not be easily taken into account if the commonly used Ritz vectors, such as those proposed by Joo *et al* (1989) are used to transform the problem to that of a smaller size. First, the number of Ritz vectors generated by following the algorithm devised by Wilson *et al* (1989) may not be enough to span the higher predominant frequency range of external load. This is because the effect of frequency content of the external load is not considered in the cut-off criteria used in Wilson's algorithm, and the cut-off criteria actually used with such vectors may terminate the generation

of Ritz vectors much before the desired higher frequency is adequately covered by the extracted natural frequencies. Second, even when a few more Ritz vectors are generated by disregarding the cut-off criteria, and the increased number of Ritz vectors apparently cover the high exciting frequencies, the frequency estimates obtained by using the Ritz vectors may not be accurate enough in the vicinity of the exciting frequencies. Only when a substantially large number of Wilson's Ritz vectors is generated giving a significant number of frequencies greater than the high frequencies of the load, will the response of the reduced system resemble that of the original system. On the other hand, the use of frequency dependent Ritz vectors combined with Wilson's Ritz vectors can effectively take into account the influence of the high frequency content of the external load. The use of frequency dependent Ritz vectors may drastically reduce the number of Ritz vectors required in the analysis, and this can significantly improve the efficiency of the computation.

Table 2.1
Algorithm for generating load and
frequency dependent Ritz vectors

(1)	
Solve	$\mathbf{K}\mathbf{u}_1 = \mathbf{f}$
Solve	$\mathbf{K}\mathbf{x}_1^* = \mathbf{M}\mathbf{u}_1$
M – Normalize	$\mathbf{x}_1 = \mathbf{x}_1^* / \ \mathbf{x}_1^*\ _{\mathbf{M}}$
Calculate	For $(j = 1, \dots, N)$: $\bar{M}_j = \sum_{i=1}^N m_{ji}$
	$\mathbf{f}_1 = \mathbf{x}_1 \mathbf{f}^T \mathbf{M} \mathbf{x}_1$
	$P_1 = (\bar{\mathbf{M}}^T \mathbf{x}_1)^2$
Calculate additional vectors	Starting with $i = 2$
(2)	
Update load vector	$a_{i-1} = \mathbf{x}_{i-1}^T \mathbf{M} \mathbf{u}_{i-1}$
	$\mathbf{u}_i = \mathbf{u}_{i-1} - a_{i-1} \mathbf{x}_{i-1}$
Solve	$\mathbf{K}\mathbf{x}_i^* = \mathbf{M}\mathbf{u}_i$
M – Orthogonalize :	For $(j = 1, 2, \dots, i-1)$
	$C_j = \mathbf{x}_j^T \mathbf{M} \mathbf{x}_i^*$
	$\mathbf{x}_i^{*(j+1)} = \mathbf{x}_i^{*(j)} - C_j \mathbf{x}_j$
M – Normalize	$\mathbf{x}_i = \mathbf{x}_i^{*(i)} / \ \mathbf{x}_i^{*(i)}\ _{\mathbf{M}}$
Convergence criterion 1 :	$\mathbf{f}_i = \mathbf{f}_{i-1} + \mathbf{x}_i \mathbf{f}^T \mathbf{M} \mathbf{x}_i$
	$\mathbf{e}_i = \mathbf{f} - \mathbf{f}_i$
	$\epsilon_i = \frac{ \mathbf{f}^T \mathbf{e}_i }{\mathbf{f}^T \mathbf{f}}$
If	$\epsilon_i \geq [\epsilon_{L1}]$ Then
	$i = i + 1$ Goto Step 2
Else	$L_1 = i$
	Endif
Convergence criterion 2 :	$P_i = P_{i-1} + (\bar{\mathbf{M}}^T \mathbf{x}_i)^2$
	$\psi_i = 1 - \frac{P_i}{\sum_{j=1}^N \bar{M}_j}$
If	$\psi_i \geq [\psi_{L1}]$ Then
	$i = i + 1$ Goto Step 2
Else	$L_1 = i$
	Endif
Obtain reduced mass and stiffness matrices	
	$\mathbf{M}^* = \mathbf{X}^T \mathbf{M} \mathbf{X}$
	$\mathbf{K}^* = \mathbf{X}^T \mathbf{K} \mathbf{X}$

(Table 2.1 continued):

Solve the reduced eigenproblem:

$$\mathbf{K}^* \mathbf{Z} = \mathbf{M}^* \mathbf{Z} \mathbf{\Lambda}$$

If $\omega_{i-1}^2 > \delta$

$$\mathbf{X}_0 = \mathbf{X} \mathbf{Z}$$

Terminate the program

Endif

Generate frequency dependent vectors, starting with $i = L_1 + 1$

(3)

$$\text{Solve } (\mathbf{K} - \delta \mathbf{M}) \mathbf{x}_i^* = \mathbf{M} \mathbf{x}_{i-1}$$

$\mathbf{M}$ – Orthogonalize : For $j = 1, 2, \dots, i-1$

$$C_j = \mathbf{x}_j^T \mathbf{M} \mathbf{x}_i^{*(j)}$$

$$\mathbf{x}_i^{*(j+1)} = \mathbf{x}_i^{*(j)} - C_j \mathbf{x}_j$$

$$\mathbf{M} \text{ – Normalize } \mathbf{x}_i = \mathbf{x}_i^{*(i)} / \|\mathbf{x}_i^{*(i)}\|_{\mathbf{M}}$$

Obtain reduced mass and stiffness matrices

$$\mathbf{M}^* = \mathbf{X}^T \mathbf{M} \mathbf{X}$$

$$\mathbf{K}^* = \mathbf{X}^T \mathbf{K} \mathbf{X}$$

Solve the reduced eigenproblem:

$$\mathbf{K}^* \mathbf{Z} = \mathbf{M}^* \mathbf{Z} \mathbf{\Lambda}$$

If $\omega_{i-1}^2 < \delta$ Then

$i = i + 1$ Goto Step 3

For $(j = 1, 2, \dots, i)$:

$$\mathbf{x}_{0j} = \mathbf{X} \mathbf{z}_j$$

$$\beta_j = \frac{\delta}{\omega_j^2}$$

$$R_j = \mathbf{x}_{0j}^T \mathbf{f} \frac{1}{1 - \beta_j^2}$$

$$\rho_j = R_j / \omega_j^2$$

$$A_i = \frac{|\rho_i|}{\sum_{j=1}^{i-1} |\rho_j|}$$

$$EEL_i = \frac{|R_i|}{\sum_{j=1}^{i-1} |R_j|}$$

If $A_i \geq [A_{L2}]$ Or $EEL_i \geq [EEL_{L2}]$ Then

$i = i + 1$ Goto Step 3

Else $L_2 = i - L_1$

Endif

(4)

Repeat Step 3 for additional values of the shift δ

Table 2.2. Stress Response at the Center of Elements of the Plate

Elem. No.	Stress	Exact Val.	17 Ritz Vec.	9 Comb. Vec.
1	σ_x	-3194.72	-1213.17	-3043.75
	σ_y	-116.45	-22.41	-67.73
	τ_{xy}	2703.45	2349.71	2710.86
2	σ_x	-135.68	1750.46	-36.99
	σ_y	2953.75	2628.45	2988.87
	τ_{xy}	-2618.11	-2051.30	-2637.60
3	σ_x	2496.75	761.97	2366.28
	σ_y	-2411.90	-2827.59	-2430.38
	τ_{xy}	2462.29	1468.28	2412.56
4	σ_x	1750.35	-343.11	1674.12
	σ_y	5102.58	5019.21	5051.15
	τ_{xy}	-161.45	488.95	-140.87
5	σ_x	3074.03	814.11	2932.96
	σ_y	-2332.39	-3605.39	-2420.35
	τ_{xy}	299.15	1105.88	311.27
6	σ_x	2220.52	176.30	2049.59
	σ_y	2834.21	2590.76	2844.71
	τ_{xy}	-3973.82	-5908.84	-4027.80

Table 2.3. Estimates of the Natural Frequencies of the Plate

Mode No.	Exact Val.	17 Ritz Vec.	9 Comb. Vec.	$\sqrt{\mathbf{x}_i^T \mathbf{K} \mathbf{x}_i}$
1	4.94	4.94	4.94	5.40
2	5.46	5.46	5.46	7.23
3	7.36	7.36	7.36	10.01
4	8.96	8.96	8.96	9.96
5	11.32	11.32		
6	12.32	12.32		
7	12.79	12.79	12.96	13.77
8	13.70	13.72		
9	13.83	13.83		
10	14.76	14.84	14.74	16.32
11	15.32	15.52	15.29	15.73
12	15.79		15.79	14.94
13	16.09	15.94		
14	16.79	16.78		
15	18.24	18.24		
16	19.60	19.60	19.80	15.55
17	20.51	20.51		
18	20.57	20.57		

Table 2.4. Error Measures for Ritz Vectors, Plane Stress Plate

Ritz Vec. No.	ϵ_{L1}	A_n^*	EEL_n^*
1			
2	0.50170300	2.28763250	5.00974720
3	0.34677820	0.94096050	2.18552187
4	0.22034588	1.87565133	6.86417234
5	0.10284152	1.86030345	9.17918368
6	0.00753320	0.34709541	1.16281058
7	0.00124684	0.16643831	0.36683833
8	0.00077249	0.14409938	0.30832943
9	0.00003815	0.11545139	0.23873292

* The first six vectors referred to in these columns are Wilson's vectors. The remaining are frequency dependent vectors

Table 2.5. Estimates of the Natural Frequencies of Shear Frame

Mode No.	Exact Val.	8 Ritz Vec.	8 Comb. Vec.	$\sqrt{\mathbf{x}_i^T \mathbf{K} \mathbf{x}_i}$
1	1.128	1.128	1.128	1.129
2	3.348	3.340	3.351	3.367
3	5.539	5.539	5.847	15.48
4	7.807	7.807		
5	10.20	10.21		
6	12.67	12.95	12.62	14.85
7	15.18		15.18	18.84
8	17.70	17.39	17.70	14.23
9	20.20		20.21	22.37
10	22.69		23.78	17.85
11	25.14			
12	27.56	28.04		
13	29.93			
14	32.24			
15	34.51			

Table 2.6. Error Estimate for the Ritz Vectors, Shear Frame

Ritz Vec. No.	ψ_{L1}	A_n^*	EEL_n^*
1			
2	0.0974605	0.88523813	7.88806380
3	0.0568712	0.77562775	24.7835620
4	0.0549009	3.76665680	30.9822790
5	0.0495943	0.39030512	0.79060173
6	0.0371970	0.21790953	0.37266196
7	0.0289251	0.12971437	0.29649907
8	0.0235745	0.13703609	0.29824795

* The first two vectors referred to in these columns are Wilson's vectors. The remaining are frequency dependent vectors

**Table 2.7 Comparison of Extracted Natural Frequencies,
1900 D.O.F. Frame**

Vect. No.	Exact Val.	13 Ritz Vec.	15 Ritz Vec.	8 Comb. Vec.
1	2.128	2.128	2.128	2.128
2	6.382	6.382	6.382	6.382
3	8.812	8.812	8.812	8.812
4	10.64	10.64	10.64	10.72
5	13.87	13.87	13.87	
6	14.88	14.88	14.88	15.90
7	19.13	19.13	19.13	
8	21.57	22.38	21.58	
9	23.37	24.77	23.38	23.33
10	25.90	26.38	25.92	25.90
11	27.63	33.25	28.74	28.00
12	28.92	43.88	33.84	
13	32.15	69.73	42.75	

Table 2.8. Error Estimate for the Ritz Vectors, 1900 D.O.F. Frame

Ritz Vec. No.	ψ_{L1}	A_n^*	EEL_n^*
1			
2	0.2449907	0.3539143	3.1845764
3	0.1187879	0.8353232	4.6357833
4	0.1022135	0.0990226	0.2673766
5	0.0781993	0.1536578	0.8767181
6	0.0506333	1.4119710	11.7180204
7	0.0458099	1.2080898	5.7211588
8	0.0374820	0.0737505	0.1340173

* The first five vectors referred to in these columns are Wilson's vectors. The remaining are frequency dependent vectors

Fig. 2.1 The plane stress plate

(a)

(b)

Fig. 2.2 Maximum steady state displacement response of the plate

Fig. 2.3 The shear frame

Fig. 2.4 Maximum steady state displacement response for the shear frame
(a) Response to component $1000 \sin 14 t$
(b) Response to component $1000 \sin 16 t$
(c) Response to component $1000 \sin 18 t$

Fig. 2.5 Maximum steady state nodal force response for the shear frame

(a) Response to component $1000 \sin 14 t$

(b) Response to component $1000 \sin 16 t$

(c) Response to component $1000 \sin 18 t$

Fig. 2.6 The support excitation (Acceleration)

Fig. 2.7 Fourier transform of the support excitation

2.8 (a)

2.8 (b)

2.8 (c)

2.8 (d)

Fig. 2.8 Reponse of the plate to support motion

(a) Degree of freedom 11

(b) Degree of freedom 12

(c) Degree of freedom 13

(d) Degree of freedom 14

2.9 (a)

2.9 (b)

Fig. 2.9 Horizontal normal stress response of the plate to support motion
 (a) Response at the center of element 1
 (b) Response at the center of element 4

Fig. 2.10 A frame system with 1900 degrees of freedom

Fig. 2.11 Displacement amplitude responses for D.O.F. 1 to 50

Fig. 2.12 Displacement amplitude responses for D.O.F. 901 to 950

Fig. 2.13 Displacement amplitude responses for D.O.F. 1851 to 1900

Chapter 3

DYNAMIC RESPONSE ANALYSIS METHODS IN FREQUENCY DOMAIN

3.1 INTRODUCTION

With the development of fast Fourier transform algorithm, discrete Fourier transform method has become an effective method of analyzing the response of linear structural systems subjected to any kind of dynamic loading. For many problems, such as those related to soil-structure interaction and water-dam interaction, the formulation is often in terms of frequency dependent structural properties. In such cases, frequency domain analysis offers a much more direct method of solution. However, a number of problems are inherent in frequency domain analysis procedures. One of these is the presence of spurious non-zero initial conditions which makes the calculated response inaccurate. Spurious initial conditions arise because of the use of discrete Fourier transforms in place of their continuous counterparts.

In a frequency domain analysis, if the continuous complex frequency response function is already known, the response of the structural system can be obtained by an evaluation of the following integral

$$v(t) = \frac{1}{2\pi} \int_0^{\infty} H(\Omega) G(\Omega) e^{i\Omega t} d\Omega. \quad (3.1)$$

where $H(\Omega)$ is the continuous complex response function, and $G(\Omega)$ is the continuous Fourier transform of the applied dynamic loading $g(t)$. The latter is obtained from the following

$$G(\Omega) = \int_0^{\infty} g(t) e^{-i\Omega t} dt. \quad (3.2)$$

In practical computation, Equation 3.1 is actually replaced by the following summation

$$v(t\Delta t) = \frac{1}{2\pi} \sum_{k=0}^{N-1} \bar{H}(k\Delta\Omega) \bar{G}(k\Delta\Omega) e^{2\pi i k l / N} \Delta\Omega. \quad (3.3)$$

in which $\bar{H}(k\Delta\Omega)$ is the periodic extension of the continuous version $H(\Omega)$, and $\bar{G}(k\Delta\Omega)$ the discrete Fourier transform of $g(t)$. The relationship between $H(k\Delta\Omega)$ and $H(\Omega)$ can be written as follows

$$\bar{H}(k\Delta\Omega) = \sum_{m=-\infty}^{m=+\infty} H(k\Delta\Omega + m\Omega_0). \quad (3.4)$$

where Ω_0 is the selected period of function $\bar{H}(k\Delta\Omega)$. If Ω_0 is large enough so that $H(k\Delta\Omega)$ becomes almost zero for $k\Delta\Omega > \Omega_0/2$, then $\bar{H}(k\Delta\Omega)$ takes the same values as $H(k\Delta\Omega)$ upto $\Omega_0/2$. The values from $\Omega_0/2$ to Ω_0 are taken such that the real part of $\bar{H}(k\Delta\Omega)$ is symmetric and the imaginary part is anti-symmetric about $\Omega_0/2$.

Transform $\bar{G}(k\Delta\Omega)$ is obtained from

$$\bar{G}(k\Delta\Omega) = \sum_{l=0}^{N-1} g(l\Delta\Omega) e^{-2\pi i k l / N \Delta t}. \quad (3.5)$$

The response obtained from Eq. 3.3, is usually different from the true response given by Eq. 3.1, and is called periodic response. However, if there is enough damping in the structural systems, the periodic response $\bar{v}(t)$ obtained from Eq. 3.3 tends to be the same as the real response obtained from Eq. 3.1. This characteristic is widely used in the frequency domain analysis. However in the dynamic analysis of lightly damped or undamped structural systems, the errors become unacceptable and methods must be found to improve the accuracy of the analysis procedure. Veletsos *et al* have suggested a series of corrective methods that address this problem (Meek and Veletsos 1972; Veletsos and Kumar 1983; and Veletsos and Ventura 1984). Although their methods are applicable to either damped or undamped systems, they are designed for single degree-of-freedom systems and are not effective for multi degree-of-freedom systems unless such systems can be uncoupled by modal transformation.

This chapter describes several new methods that can be used in the frequency domain analysis of single or multi degree-of-freedom systems. The relationship between the continuous frequency response function and the discrete Fourier transform of the truncated impulse function is derived in Section 3.2 for a single degree-of-freedom system. By studying this relationship, it is possible to realize that the

errors in the frequency domain analysis are caused by the use of an inappropriate frequency response function, which in turn gives rise to incorrect initial conditions in the final periodic response. The examples presented show that no correction is necessary if the right frequency response function is used. The procedure is effective for single-degree-of-freedom systems, but not so for multi-degree of freedom systems. However, the use of such a relationship for multi degree-of-freedom systems is shown in section 3.2. In any case, the theoretical basis is significant for an understanding of the frequency domain analysis procedure. The extension of Velet-sos's method presented by Humar (1990) is introduced briefly in Section 3.3. It is also pointed out that the current frequency domain analysis procedures (including the methods presented in this chapter) are not applicable to interaction problems such as soil-structural interaction and dam-water interaction. New procedures for such interaction problems need to be developed.

3.2 RELATIONSHIP BETWEEN THE CONTINUOUS FREQUENCY RESPONSE FUNCTION AND FOURIER TRANSFORM OF TRUNCATED IMPULSE FUNCTION

If the unit impulse response function for each degree of freedom of the system is given, the Fast Convolution Method (Humar 1990) can be used to obtain the responses, and the correct results can be assured. Suppose it is required to determine the response of a single degree-of-freedom system, whose unit impulse function, $h(t)$, is shown in Fig. 3.1. The exciting force, $g(t)$, is shown in Fig. 3.2. An analysis in the frequency domain using fast convolution procedure involves the following steps:

- (1) the unit impulse function $h(t)$ is truncated at the time $q\Delta t$ upto which the final response of the structural system is to be determined, and is augmented by a series of zeros upto $N\Delta t > (q + p + 1)\Delta t$, where $p\Delta t$ is the total time duration of the dynamic loading $g(t)$ (Fig. 3.3);
- (2) function $g(t)$ is also augmented upto $N\Delta t$ as shown in Fig. 3.4;
- (3) the discrete Fourier transform of the truncated unit impulse function, $\tilde{H}(k\Delta\Omega)$

and the discrete Fourier transform $\bar{G}(k\Delta\Omega)$ are obtained by applying forward discrete fast Fourier transform to the truncated unit impulse response function $\tilde{h}(k\Delta t)$, and the augmented external dynamic loading $g(k\Delta t)$ respectively;

- (4) To get the response of the system, discrete inverse fast Fourier transform is applied to the product of $\tilde{H}(k\Delta\Omega)$ and $\bar{G}(k\Delta\Omega)$.

The fast convolution method gives the correct response of the structural system to the specified dynamic loading for the time length of $q\Delta t$ provided $(q + p + 1)\Delta t \leq N\Delta t$, the time period of the Fourier transforms, and the time interval Δt is small enough to capture the functions $\tilde{h}(t)$ and $g(t)$ correctly, and to reduce the errors involved in representing an integral by a rectangular summation.

Steps 3 and 4 can be represented by the following equations

$$\bar{G}(k\Delta\Omega) = \sum_{l=0}^{N-1} g(l\Delta t) e^{-2\pi i k l / N \Delta t}. \quad (3.6)$$

$$\tilde{H}(k\Delta\Omega) = \sum_{l=0}^{N-1} \tilde{h}(l\Delta t) e^{-2\pi i k l / N \Delta t}. \quad (3.7)$$

$$v(l\Delta t) = \frac{1}{2\pi} \sum_{k=0}^{N-1} \tilde{H}(\Delta\Omega) \bar{G}(k\Delta\Omega) e^{2\pi i k l / N \Delta\Omega}. \quad (3.8)$$

As stated earlier, in many problems $H(\Omega)$ is given so that $\tilde{H}(\Omega)$ can be obtained from Eq. 3.4. The discrete transform of $\bar{G}(\Omega)$ is calculated from Eq. 3.6. The implementation of the discrete convolution procedure then begins with Eq. 3.3. This is similar to starting the procedure from Step 4 above, except that $\tilde{H}(k\Delta\Omega)$ is not the same as $\tilde{H}(k\Delta\Omega)$ but is obtained from the continuous Fourier transform of $h(t)$. This procedure that uses $\tilde{H}(k\Delta\Omega)$ in place of $\tilde{H}(k\Delta\Omega)$ will be referred to as Discrete Fourier Transform (DFT) method in this work. Aliasing is introduced in the process of obtaining $\tilde{H}(k\Delta\Omega)$. The errors involved in the periodic response $v(t)$ obtained from Eq. 3.3 arise from two sources. The first source is the use of discrete periodic version of continuous complex response function $\tilde{H}(k\Delta\Omega)$, whose inverse discrete Fourier transform does not give the true unit impulse response function $h(t)$. The other source of the errors is that even if the true unit impulse response

function $h(t)$ could be obtained from the discrete Fourier transform, the nonzero values of $h(t)$ from the time $q\Delta t$ to time $N\Delta t$ cause aliasing (or overlapping). Only if considerable damping exists in the system so that $h(t)$ becomes negligible after time $q\Delta t$, will Eq. 3.3 give the correct response.

The errors introduced into the response by the two sources referred above when the damping in the system is very small or zero can be avoided if the complex frequency response function, $\tilde{H}(k\Delta\Omega)$, corresponding to the truncated unit impulse function, $\tilde{h}(t)$, can be obtained. The method described in the following paragraph as well as certain other procedures discussed in Chapter 4 are based on the foregoing concept.

The discussion presented above makes it clear that the periodic response, $\bar{v}(t)$, obtained from Eq. 3.3 is not the true response $v(t)$ because the periodic version of continuous frequency response function, $\bar{H}(\Omega)$, instead of the discrete transform of a truncated impulse function, $\tilde{H}(k\Delta\Omega)$, is used in that equation. The main purpose in this section is to establish a relationship between the two so that the discrete function $\tilde{H}(k\Delta\Omega)$ can be derived from $H(\Omega)$, and the true response can therefore be obtained from Equation. 3.3 using $\tilde{H}(k\Delta\Omega)$ in place of $\bar{H}(k\Delta\Omega)$.

The use of discrete transforms in place of continuous transforms implies that the functions involved are converted into periodic form and the summation cut off at finite limits. This gives rise to aliasing so that the inverse transform of $\bar{H}(\Omega)$, given by

$$\frac{1}{2\pi} \sum_{k=0}^{N-1} \bar{H}(k\Delta\Omega) e^{2\pi i k l / N \Delta\Omega} \quad (3.9)$$

does not give the continuous impulse response function $h(t\Delta t)$ but another function, $h(l\Delta t)$. If the continuous impulse response function $h(t\Delta t)$ and the forcing function of $g(t)$ were both known, the discrete Fourier transforms could be used to obtain the exact response by following the fast convolution procedure. This suggests that if the discrete transform $\tilde{H}(k\Delta\Omega)$ can be obtained, the exact response to a forcing function $g(t)$ can be calculated according to the Eq. 3.8. It is worth pointing out that

such a response is the transient response corresponding to zero initial conditions. The transient response due to non zero initial conditions should be superimposed on $v(t\Delta t)$ from Eq. 3.8 to obtain the true response.

Commonly, the continuous complex frequency response function $H(\Omega)$ can be directly obtained from the properties of the structural system. The relationship between the continuous complex frequency response function $H(\Omega)$ and the discrete Fourier transform of the truncated impulse function, $\tilde{H}(\Omega)$, is investigated here in order to get $\tilde{H}(\Omega)$ from $H(\Omega)$.

For a single degree-of-freedom structural system, the force pulse correction method and other corrective response methods suggested by Veletsos *et al* provide very effective means of solving for the response (Meek and Veletsos 1972; Veletsos and Kumar 1983; Veletsos and Ventura 1984; and Humar 1990). The method presented here is another alternative and may prove to be more convenient in comparison with the methods currently available. The continuous complex frequency response function $H(\Omega)$ for a single degree-of-freedom system is given by

$$H(\Omega) = \frac{1}{m(\omega^2 - \Omega^2)}. \quad (3.10)$$

It can be compared with its discrete counterpart represented by Eq. 3.7. If the summation in Eq. 3.7 is replaced by finite integration, Equation 3.7 is actually of the form

$$\tilde{H}(\Omega) = \int_0^{T_k} h(t)e^{-i\Omega t} dt. \quad (3.11)$$

where, for a single degree-of-freedom system without damping, the continuous impulse response function $h(t)$ is given by

$$h(t) = \frac{1}{m\omega} \sin \omega t. \quad (3.12)$$

Substitution of Eq. 3.12 into Eq. 3.11 and integration by parts twice gives

$$\begin{aligned}
\tilde{H}(\Omega) &= \int_0^{T_h} \frac{1}{m\omega} \sin(\omega t) e^{-i\Omega t} dt \\
&= -\frac{1}{m\omega^2} \cos \omega t e^{-i\Omega t} \Big|_0^{T_h} - \frac{i\Omega}{m\omega^2} \int_0^{T_h} \cos(\omega t) e^{-i\Omega t} dt \\
&= -\frac{1}{m\omega^2} \cos \omega t e^{-i\Omega t} \Big|_0^{T_h} - \frac{i\Omega}{m\omega^3} \sin(\omega t) e^{-i\Omega t} \Big|_0^{T_h} \\
&\quad + \frac{\Omega^2}{m\omega^3} \int_0^{T_h} \sin(\omega t) e^{-i\Omega t} dt \\
&= -\frac{1}{m\omega^2} \cos \omega t e^{-i\Omega t} \Big|_0^{T_h} - \frac{i\Omega}{m\omega^3} \sin(\omega t) e^{-i\Omega t} \Big|_0^{T_h} + \frac{\Omega^2}{\omega^2} \tilde{H}(\Omega)
\end{aligned} \tag{3.13}$$

Hence

$$\tilde{H}(\Omega) = -\frac{1}{m(\omega^2 - \Omega^2)} \left[\cos \omega t e^{-i\Omega t} - \frac{i\Omega}{\omega} \sin(\omega t) e^{-i\Omega t} \right] \Big|_0^{T_h}. \tag{3.14}$$

or

$$\tilde{H}(\Omega) = H(\Omega)(A + iB). \tag{3.15}$$

where

$$A = 1 - \left\{ \cos(\omega T_h) \cos(\Omega T_h) + \frac{\Omega}{\omega} \sin(\omega T_h) \sin(\Omega T_h) \right\}. \tag{3.16}$$

$$B = \left\{ \cos(\omega T_h) \sin(\Omega T_h) - \frac{\Omega}{\omega} \sin(\omega T_h) \cos(\Omega T_h) \right\}. \tag{3.17}$$

The overlapping or aliasing effect of the Fourier transform can be avoided if T_h is set to be less than $T_0 - T_p - \Delta t$, where $T_0 = N\Delta t$ is the period used in the time, $\Delta\Omega = 2\pi/T_0$, Δt is the sampling interval, T_p is the time duration of the external loading, and T_h represents the time duration of the truncated continuous unit response function $h(t)$. Function $\tilde{H}(\Omega)$ obtained from Eq. 3.15 is sampled and modified so that the real part is symmetric and the imaginary part anti-symmetric about the folding frequency $\pi/\Delta t$. This modified function is now used in Eq. 3.8. The response obtained from that equation will be correct for the length of time T_h .

Example 3.1

A single degree of freedom system with mass of 5.0 kg, stiffness of 100 N/m, and zero damping is considered. The applied loading is given in Fig. 3.5. The time interval of $\Delta t = 0.05$ s is used to sample the loading history. Number of samples is taken as $N = 210$. The responses obtained through frequency domain analysis, using the periodic version of continuous frequency response function, $H(\Omega)$, without any correction, and the discrete Fourier transform of the truncated impulse function, $\tilde{H}(\Omega)$, derived here, are both shown in Fig. 3.6 together with the exact solution obtained by mode superposition.

It is seen that by using the discrete Fourier transform of the truncated impulse function, $\tilde{H}(\Omega)$, the exact response can be obtained without any further correction.

For the case of a single degree-of-freedom system with damping, Equations. 3.16 and 3.17 take the following form

$$A = 1 - \left[\cos(\omega_d T_h) \cos(\Omega T_h) + \frac{\xi \omega}{\omega_d} \sin(\omega_d T_h) \cos(\Omega T_h) + \frac{\Omega}{\omega_d} \sin(\omega_d T_h) \sin(\Omega T_h) \right] e^{-\xi \omega T_h} \quad (3.18)$$

and

$$B = \left[\cos(\omega_d T_h) \sin(\Omega T_h) + \frac{\xi \omega}{\omega_d} \sin(\omega_d T_h) \sin(\Omega T_h) - \frac{\Omega}{\omega_d} \sin(\omega_d T_h) \cos(\Omega T_h) \right] e^{-\xi \omega T_h} \quad (3.19)$$

where ξ is the damping ratio of the system, and $\omega_d = \omega \sqrt{1 - \xi^2}$ is the damped natural circular frequency.

In order to make a comparison of the computation efficiencies of the method using $\tilde{H}(\Omega)$ and the force pulse method proposed by Meek and Veletsos (1972), let Eqs. 3.17 and 3.19 be combined and expressed as

$$A + iB = 1 - \left[\left\{ \cos(\omega_d T_h) + \frac{\xi \omega}{\omega_d} \sin(\omega_d T_h) \right\} e^{-\xi \omega T_h} + i \Omega \frac{\sin(\omega_d T_h)}{\omega_d} e^{-\xi \omega T_h} \right] e^{-i \Omega T_h} \quad (3.20a)$$

or

$$A + iB = 1 - \{C + i\Omega D\} e^{-i\Omega T_h} \quad (3.20b)$$

in which

$$C = \left[\cos(\omega_d T_h) + \frac{\xi \omega}{\omega_d} \sin(\omega_d T_h) \right] e^{-\xi \omega T_h} \quad (3.21)$$

and

$$D = \frac{\sin(\omega_d T_h)}{\omega_d} e^{-\xi \omega T_h} \quad (3.22)$$

The number of arithmetic operations needed in the evaluation of C and D is small and can be neglected because they are constants. The evaluation of Eq. 3.20b needs $N_1/2$ real multiplications and $N_1/2$ complex multiplications, where $N_1 \Delta t$ is the selected period. Conservatively, N_1 complex multiplications are counted for Eq. 3.20b. The evaluation of Eq. 3.15 needs another $N_1/2$ complex multiplications. The rest of the analysis is the same as the standard convolution method, which needs $(N_1 + 3N_1/2 \log_2 N_1)$ complex multiplications if $H(\Omega)$ is to be calculated, or $(N_1 + N_1 \log_2 N_1)$ if $H(\Omega)$ is given. Therefore, the total computation operations required for this method are $(3N_1/2 + N_1 + N_1 \log_2 N_1)$ if $H(\Omega)$ is given.

The force pulse method requires that the inverse Fourier transform of the periodic version of the continuous frequency function be obtained so that the response resulting from the application of two force pulses be calculated. The discrete inverse transform is given by,

$$\bar{h} = \frac{1}{2\pi} \sum_{k=0}^{N-1} \bar{H}(k\Delta\Omega) e^{2\pi i k l / N} \Delta\Omega. \quad (3.23)$$

Assuming that N is equal to a power of 2, $(N/2) \log_2 N$ complex multiplications are required for the evaluation of Eq. 3.23 through fast Fourier transform algorithm. The computation effort spent in the calculation of initial conditions needs $3N/2$ real multiplications. The operations needed to superpose the corrective response on the

periodic response comprise $2N$ real multiplications. In all $7N/2$ real multiplications or equivalently $7N/8$ complex multiplications are required in obtaining and superposing the corrective response. Therefore, the total number of complex multiplications involved in the force pulse method is $(15N/8 + (3N/2)\log_2 N)$.

The difference between the computations in the method using $\hat{H}(\Omega)$ and the force pulse method is $\{(3N_1/2 + N_1 + N_1\log_2 N_1) - (15N/8 + (3N/2)\log_2 N)\}$ complex multiplications. If $N_1 = N$, the former method may take less computation time than the latter. Usually, of course, $N_1 > N$, and the method using $\hat{H}(\Omega)$ will not be more efficient than the force pulse method.

The relationships between the discrete Fourier transforms of the truncated impulse functions and the continuous frequency response functions can be theoretically derived for multi-degree-of-freedom systems by considering the expressions for the continuous Fourier transform and the discrete Fourier transform of the truncated impulse function presented earlier for a single degree-of-freedom system. For multi degree-of-freedom systems, the complex frequency response function $H(\Omega)$ becomes a vector $\mathbf{H}(\Omega)$. In a similar manner, the discrete Fourier transform of truncated impulse function, $\tilde{H}(\Omega)$, becomes a vector $\tilde{\mathbf{H}}(\Omega)$. The relationship between $\mathbf{H}(\Omega)$ and $\tilde{\mathbf{H}}(\Omega)$ can be easily derived for classically damped system and is given by

$$\tilde{\mathbf{H}}(\Omega) = \tilde{\mathbf{A}}\mathbf{H}(\Omega) + i\tilde{\mathbf{B}}\mathbf{H}(\Omega). \quad (3.24)$$

where

$$\tilde{\mathbf{A}} = \Phi\mathbf{A}\Phi^{-1}. \quad (3.25)$$

$$\tilde{\mathbf{B}} = \Phi\mathbf{B}\Phi^{-1}. \quad (3.26)$$

Matrices $\mathbf{A}$ and $\mathbf{B}$ are both diagonal with the diagonal elements given by

$$A_i = 1 - \left[\cos(\omega_{id}T_h) \cos(\Omega T_h) + \frac{\xi_i \omega_i}{\omega_{id}} \sin(\omega_{id}T_h) \cos(\Omega T_h) + \frac{\Omega}{\omega_{id}} \sin(\omega_{id}T_h) \sin(\Omega T_h) \right] e^{-\xi_i \omega_i T_h} \quad (3.27)$$

and

$$B_i = \left[\cos(\omega_{id}T_h) \sin(\Omega T_h) + \frac{\xi_i \omega_i}{\omega_{id}} \sin(\omega_{id}T_h) \sin(\Omega T_h) - \frac{\Omega}{\omega_{id}} \sin(\omega_{id}T_h) \cos(\Omega T_h) \right] e^{-\xi_i \omega_i T_h} \quad (3.28)$$

In order to use the above method, the mode shapes of the system represented by Φ , as well as the natural circular frequencies ω_{id} and ω_i must be obtained first. The inverse of the modal matrix can be calculated from

$$\Phi^{-1} = (\mathbf{M}^*)^{-1} \Phi^T \mathbf{M} \quad (3.29)$$

in which $\mathbf{M}^* = \Phi^T \mathbf{M} \Phi$ is the diagonalized mass matrix.

Example 3.2

A two degrees of freedom undamped system is used to show the effectiveness of this method. The mass matrix, stiffness matrix and the load distribution vector are as follows

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 192 & -64 \\ -64 & 64 \end{bmatrix}$$

and

$$\mathbf{f} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$$

The forcing function $g(t)$ is the same as in Fig. 3.5. Again, a time interval of $\Delta t = 0.05$ and the total number of time steps of $N = 210$ are used in this example. The responses along the two degrees of freedom obtained by the method presented here are compared with the exact values obtained by using mode superposition method, and are shown in Figs. 3.7a, and 3.7b. As evident from these figures, correct values of response are obtained by using the method presented.

The method presented above may not be very effective for a multi degree-of-freedom system because major computational effort is needed in obtaining the mode shapes of the system. However, it does give an insight into the type of adjustment that must be made in the frequency domain analysis procedure.

3.3 FORCE PULSE METHOD FOR MULTI DEGREE-OF-FREEDOM SYSTEMS

Force pulse method was developed by Meek and Veletsos (1972) for the frequency domain analysis of single degree-of-freedom systems. The first step in the procedure is to select a period for the forcing function equal to the duration of time for which the response is desired. The period is now extended by enough zero values. Corrective force pulses are applied at two of these zeros. The response due to these force pulses is superimposed on the periodic response to the augmented forcing function. The magnitudes of the force pulses are selected so that the initial conditions in the total response are equal to the specified initial conditions. Humar (1990) has described an extension of this procedure to the analysis of multi degree-of-freedom systems. The extended version is described here. The limitations of the method are also discussed.

The equations of motion for a multi-degree-of-freedom system are given by

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{C}\dot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{f}g(t). \quad (3.30)$$

where it has been assumed that the forcing function can be expressed by the product of vector $\mathbf{f}$ representing the spatial variation, and $g(t)$, a scalar function of time. As usual, frequency domain analysis requires the decomposition of $g(t)$ into its harmonic components through a discrete Fourier transformation. In addition, the complex frequency response functions corresponding to each degree of freedom in the system are needed. If the scalar function $g(t)$ in Eq. 3.30 is replaced by a Dirac function $\delta(t)$ applied at $t = 0.0$ and a forward Fourier transform is applied to the equation, the following is obtained

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$$(-\Omega^2 \mathbf{M} + i\Omega \mathbf{C} + \mathbf{K})\mathbf{H}(\Omega) = \mathbf{f}. \quad (3.31)$$

For a given Ω , Equation 3.31 can be solved for the complex-valued vector $\mathbf{H}$. Element H_j of vector $\mathbf{H}(\Omega)$ represents the complex frequency response function corresponding to the degree of freedom j . It would therefore appear that the response along a physical coordinate can be obtained by taking the inverse Fourier transform of the product of the periodic version $\bar{H}_j$, and G , the discrete Fourier transform of $g(t)$, so that

$$\bar{v}_j(l\Delta t) = \frac{1}{2\pi} \sum_{k=0}^{N-1} \bar{H}_j(k\Delta\Omega) G(k\Delta\Omega) e^{2\pi i k l / N} \Delta\Omega. \quad (3.32)$$

where $N\Delta t$ represents the period of the periodic extension of forcing function, and Δt is the sampling time interval. However, $\bar{v}_j$ is not the true response of the j th degree of freedom of the system. It must be suitably corrected in order to obtain the true responses. In the present case, an effective means of doing this is to use the same method of force pulse correction as used in a single degree-of-freedom system (Veletsos and Ventura 1984). The number and magnitude of these force pulses should be selected such that the initial conditions along each of the N_d degrees of freedom are satisfied. Since for an N_d degrees of freedom system, there are $2N_d$ initial conditions (N_d displacement and N_d velocity initial conditions), $2N_d$ number of force pulses must be applied. The first step in the procedure is thus to augment the period of the forcing function by a sufficient number of zeros at the end. The force pulses are then applied backward from the end of the period as shown in the Fig. 3.8

As shown in Fig. 3.8, the p th force pulse of the magnitude R_p has been applied at a distance t_p from the end of the period. The initial displacement along degree of freedom j produced by the p th force pulse is given by

$$\bar{v}_p(t_p) = R_p \bar{h}_j(t_p). \quad (3.33)$$

where the unit impulse function h_j is obtained from $\bar{H}_j$ by using the discrete inverse transform

$$h_j = \frac{1}{2\pi} \sum_{k=0}^{N-1} \bar{H}_j(l\Delta\Omega) e^{2\pi i k l / N \Delta\Omega}. \quad (3.34)$$

The next step in this method as given in the book by Humar (1990) is to set up the following N_d equations to satisfy the initial displacement conditions

$$\dot{v}_j(0) + \sum_{p=1}^{2N_d} R_p \dot{h}_j(t_p) = v_j(0). \quad j = 1, 2, \dots, N_d \quad (3.35)$$

where N_d is the number of degrees of freedom. Another N_d number of equations come from the initial velocity conditions

$$\dot{v}_j(0) + \sum_{p=1}^{2N_d} R_p \dot{h}_j(t_p) = \dot{v}_j(0). \quad j = 1, 2, \dots, N_d \quad (3.36)$$

where $v_j(0)$ and $\dot{v}_j(0)$ are the initial conditions obtained in the periodic response v_j and $\dot{v}_j(0)$ and $\dot{v}_j(0)$ are the specified initial conditions along the j th degree of freedom in the system. Often the latter two have zero values.

The value of $\dot{h}_j(t_p)$ in Eq. 3.36 can be obtained by using the following central difference expression

$$\dot{h}_j(t_p) = \frac{h_j(t_p + \Delta t) - h_j(t_p - \Delta t)}{2\Delta t}. \quad (3.37)$$

The initial displacement in the periodic response is obtained from Eq. 3.32 by substituting $t = 0$ and the initial velocity $\dot{v}_j(0)$ is obtained by taking the time derivative of Eq. 3.32 which gives

$$\dot{v}_j(0) = -\frac{4\pi}{(T_0)^2} \sum_{n=0}^{N/2} n \text{Im}\{V_j(n\Delta\Omega)\}. \quad (3.38)$$

where $V_j(n\Delta\Omega) = H_j(n\Delta\Omega)\dot{G}(n\Delta\Omega)$

The magnitudes of force pulses are obtained by solving Eqs. 3.35 and 3.36. The true response is then given by

$$v_j(k\Delta t) = \bar{v}_j(k\Delta t) + \sum_{p=1}^{2N_d} R_p \bar{h}_j(k\Delta t + t_p). \quad (3.39)$$

Instead of using Eq. 3.38 to obtain $\dot{\bar{v}}_j(0)$, the following alternative equation can be used.

$$\dot{\bar{v}}_j(0) = \frac{\bar{v}_j(2\Delta t) - \bar{v}_j(0)}{2\Delta t} - \frac{v_j(2\Delta t) - 2v_j(\Delta t) + v_j(0)}{\Delta t}. \quad (3.40)$$

Example 3.3

The same two degrees of freedom undamped system as used in Example 3.2 is analyzed by using the force pulse method here. The forcing function $g(t)$ is shown in Fig. 3.5. A time interval of $\Delta t = 0.05$ and the total number of time steps of $N = 128$ are used with two force pulses being placed at the locations with $N = 126$ and $N = 124$ respectively.

Figures 3.9a and 3.9b present comparisons of the dynamic displacement values obtained by force pulse correction method and those obtained by using mode superposition method. The results obtained from mode superposition method are regarded as the exact response of the system. The response values obtained by the force pulse correction method are valid up to the time at which the first corrective force pulse is applied. Also the results obtained by using the usual discrete periodical function $\bar{\mathbf{H}}(\Omega)$ (or DFT method) are shown in the figures for comparison.

As stated in the work by Humar (1990), occasionally, the coefficient matrix in the equation for the unknown force pulses R_p may become ill-conditioned. If such a case arises, it can be corrected either by using double precision arithmetic or by spacing the pulses farther apart rather than next to each other. However, this problem can not be easily solved for a system with a large number of degrees of freedom, where many force pulses have to be applied and they are usually located close to each other. The number of the force pulses required is $2N_d$. The com-

putation involved in the solution of these force pulses becomes substantial when a large number of degrees of freedom are included in the analysis. In addition, the length of the augmented forcing function and consequently the time required for the evaluation of discrete Fourier transforms increases substantially.

3.4 AVERAGE ACCELERATION CORRECTION METHOD

An alternative method for the frequency domain analysis of multi degree-of-freedom systems is presented here. In this method, the corrective response corresponding to each degree of freedom in the system is calculated by using the average acceleration numerical integration method. As discussed earlier, the spurious initial displacements obtained in a frequency domain analysis are given by Eq. 3.32, while the spurious initial velocities along each degree freedom can be obtained from Eq. 3.38.

If the true initial conditions are $v_j(0)$ and $\dot{v}_j(0)$, the corrective response should have the following initial conditions

$$v_{Rj}(0) = v_j(0) - \bar{v}_j(0). \quad (3.41)$$

$$\dot{v}_{Rj}(0) = \dot{v}_j(0) - \dot{\bar{v}}_j(0). \quad (3.42)$$

The suggested method is based on the idea that the corrective response must equal the free vibration response of the same structural system with the initial conditions as given in Eqs. 3.41 and 3.42. A number of different time-domain methods are available for calculating such corrective response. Here the average acceleration method is used. This method is based on the assumption that the acceleration along each coordinate of the system remains constant over the small time interval Δt and its value is the average of those at the beginning and the end of the interval. The relationship between the response parameters at time points n and $n + 1$ are as follows

$$\ddot{\mathbf{v}}_{n+1} = -\ddot{\mathbf{v}}_n + \frac{4}{(\Delta t)^2}(\mathbf{v}_{n+1} - \mathbf{v}_n - \Delta t \dot{\mathbf{v}}_n). \quad (3.43)$$

$$\dot{\mathbf{v}}_{n+1} = -\dot{\mathbf{v}}_n + \frac{2}{\Delta t}(\mathbf{v}_{n+1} - \mathbf{v}_n). \quad (3.44)$$

A third equation is obtained from the equations of motion for free vibration written at time point $n + 1$:

$$\mathbf{M}\ddot{\mathbf{v}}_{n+1} + \mathbf{C}\dot{\mathbf{v}}_{n+1} + \mathbf{K}\mathbf{v}_{n+1} = \mathbf{0}. \quad (3.45)$$

Substitution for $\dot{\mathbf{v}}_{n+1}$ and $\ddot{\mathbf{v}}_{n+1}$ from Eqs. 3.43 and 3.44 into Eq. 3.45 gives

$$\left(\frac{4}{(\Delta t)^2} \mathbf{M} + \frac{2}{\Delta t} \mathbf{C} + \mathbf{K} \right) \mathbf{v}_{n+1} = \mathbf{M} \left(\frac{4}{(\Delta t)^2} \mathbf{v}_n + \frac{4}{\Delta t} \dot{\mathbf{v}}_n + \ddot{\mathbf{v}}_n \right) + \mathbf{C} \left(\frac{2}{\Delta t} \mathbf{v}_n + \dot{\mathbf{v}}_n \right). \quad (3.46)$$

Equation 3.46 is solved for $\mathbf{v}_{n+1}$ and the resulting value is substituted into Eqs. 3.43 and 3.44 respectively to get the values of $\ddot{\mathbf{v}}_{n+1}$ and $\dot{\mathbf{v}}_{n+1}$. If the notation v_{Rj} is used to represent the corrective response obtained in this way for the j th degree of freedom, the exact response corresponding to this degree of freedom is given by

$$v_j(k\Delta t) = \bar{v}_j(k\Delta t) + v_{Rj}(k\Delta t). \quad (3.47)$$

This method is also applicable to nonclassically damped systems. It must be noted that this method requires the property matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$. Also the method is of utility only when the corrective free vibration response due to the spurious initial conditions is significant for only a short duration of time after which damping makes the correction small. If the correction had to be applied for a long duration, the direct time integration may as well be used to obtain the entire response and there is no need to apply the frequency domain analysis procedure. Thus the method described here may be looked upon more as a verification of the frequency domain analysis procedure developed in the preceding sections.

Example 3.4

The same two degrees of freedom system as in Example 3.2 is analyzed by the method described in this section. Values of $\Delta t = 0.05$ and $N = 128$ are used in the analysis. Results obtained by using the present method are shown in Fig. 3.10a and Fig. 3.10b. For comparison, the exact values obtained by mode superposition method, and the results obtained by using the DFT method without correction are also shown in these figures.

Before closing this chapter, it may be worth noting that the particular significance of the frequency domain analysis procedure is in the solution of certain problems which can not easily be solved in the time domain. Such problems arise when the structural properties are frequency dependent. Often, these problems are formulated by using the substructuring technique. Soil-structure interaction, or dam-water interaction problems are of this kind. However, the current methods of frequency domain analysis, including those presented in this chapter, are not effective in the solution of substructure problems. The dynamic response in such cases can be obtained by the frequency analysis procedure only when there is enough damping in the system so that the aliasing problem can be avoided by using a sufficiently long time period. For very lightly damped and undamped systems, frequency analysis procedure is not able to provide acceptable approximations. In fact, the method is not applicable to undamped systems. These problems are discussed in detail in Chapter 4.

3.5 CHAPTER SUMMARY

The relationship between the discrete Fourier transforms of the truncated impulse functions and the continuous frequency response functions derived in Section 3.2 is useful in understanding the source of errors in the frequency domain analysis procedure that uses discrete transforms. Such relationship for a single-degree-of-freedom system can be used in developing an alternative method of solving undamped as well as classically damped multi-degree-of-freedom systems. For

multi-degree-of-freedom systems, the Force Pulse method presented in Section 3.3 can be used in many cases without the need for modal uncoupling. However, a number of problems are involved in using the Force Pulse Method to analyze a structural system with a large number of degrees of freedom.

Fig. 3.1 Unit Impulse Response Function

Fig. 3.2 External Dynamic Loading

Fig. 3.3 The Truncated Unit Impulse Response Function $\tilde{h}(k\Delta t)$

Fig. 3.4 The Augmented External Dynamic Loading $g(k\Delta t)$

Fig. 3.5 External Dynamic Loading

Fig. 3.6 Response of a Single Degree-of-Freedom System

(a) Response along the D.O.F. 1

(b) Response along the D.O.F. 2

Fig. 3.7 Responses of the Two Degree of Freedom System Obtained by Using $\tilde{H}(\Omega)$

Fig. 3.8 Force Pulses

(a) Response Along the D.O.F. 1

(b) Response along the D.O.F. 2

Fig. 3.9 Responses of the Two Degree of Freedom System Obtained by Using Force Pulse Method

(a) Response along the D.O.F. 1

(b) Response along the D.O.F. 2

Fig. 3.10 Responses of the Two Degree of Freedom System Obtained by Using Average Acceleration Correction Method

Chapter 4

SOLUTION OF DYNAMIC INTERACTION PROBLEMS IN FREQUENCY DOMAIN

4.1 INTRODUCTION

Interaction problems usually arise when a substructuring technique is used in the formulation of a structural problem. The technique is an effective way of solving complex problems of large size. Often, the system being analyzed consists of a substructure with all its boundaries clearly defined, for example, a building, a dam, or a bridge, and another substructure with infinite boundaries such as a water body, or a soil field. The response of the first type of substructures, a building, a dam or a bridge, is generally of primary concern. The response of the water body or the soil field is not of direct interest. The analysis of the substructure of primary interest must take into account the external loading as well as the interaction forces on the boundaries in contact with the other substructures. These interaction forces are unknowns and are dependent on the external loading. Many interaction problems such as dam-water, and soil-structure are more conveniently solved in the frequency domain. This is because such interaction problems involve the solution of dynamic differential equations for the infinite substructures, and these solutions, obtained either analytically or by other methods such as boundary element methods etc., are commonly expressed in terms of the exciting frequency. As a result, the properties of the interacting structural system are frequency dependent.

Discrete Fourier transforms are commonly used for analysis in the frequency domain. However, as stated earlier, for those cases where the structural systems possess very small damping or no damping, discrete Fourier transforms lead to wrong results. Veletsos *et al* have proposed several corrective methods for single degree-of-freedom systems (Meek and Veletsos 1972; Veletsos and Kumar 1983;

and Veletsos and Ventura 1984). These methods are based on the application of suitable corrections for the spurious initial conditions caused by the use of discrete transforms. Before these methods can be applied to a multi degree-of-freedom system, the mode shapes of the system must be obtained so that the equations can be uncoupled into independent single degree-of-freedom systems. Therefore, these methods are only suitable for those cases where classical mode shapes exist, and are inapplicable even for the simplest interaction problems, as already mentioned in Chapter 3. The methods presented in Chapter 3 for analysis of multi degree-of-freedom systems are also not applicable to systems involving interaction, as will be shown here.

Methods of analyzing an interaction problem in the frequency domain are presented in this Chapter. These methods are believed to provide the only effective means of frequency domain analysis of an undamped or partially damped interaction problem. It is also worth noting that the methods presented in this chapter are equally effective in the analysis of any kind of multi-degree-of-freedom systems.

4.2 INTERACTION PROBLEMS AND THEIR FORMULATION

The general procedure for the formulation of an interaction problem is presented in this section. Also, the unsuitability of the methods presented in Chapter 3 in solving such interaction problems is discussed.

The formulation of an interaction problem by using substructuring technique is conceptually quite simple and the basic procedure can be shown by partitioning the equations of motion of the complete structural system.

The equations of motion for a structure system with no damping can be written as

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{f}_g(t). \quad (4.1)$$

If the system is divided into two substructures, the mass and stiffness matrices can

be partitioned as in the following

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}$$

The load amplitude vector is also separated into two parts corresponding to the two substructures as shown below,

$$\mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}$$

The equations of motion for individual substructures are as follows.

For substructure 1

$$\mathbf{M}_1 \ddot{\mathbf{v}}_1 + \mathbf{K}_{11} \mathbf{v}_1 = \mathbf{f}_1 g(t) - \mathbf{K}_{12} \mathbf{v}_2. \quad (4.2)$$

For substructure 2

$$\mathbf{M}_2 \ddot{\mathbf{v}}_2 + \mathbf{K}_{22} \mathbf{v}_2 = \mathbf{f}_2 g(t) - \mathbf{K}_{21} \mathbf{v}_1. \quad (4.3)$$

The terms $\mathbf{K}_{12} \mathbf{v}_2$ and $\mathbf{K}_{21} \mathbf{v}_1$ represent the interaction effects between substructures.

Suppose that it is required to analyze in the frequency domain the first substructure considering the interaction effects of the second substructures. A unit impulse forcing function $\delta(t)$ applied at $t = 0$ is used to replace the forcing function $g(t)$ in order to get the frequency response functions. Equations 4.2 and 4.3 become

$$\mathbf{M}_1 \ddot{\mathbf{h}}_1 + \mathbf{K}_{11} \mathbf{h}_1 = \mathbf{f}_1 \delta(t) - \mathbf{K}_{12} \mathbf{h}_2. \quad (4.4)$$

$$\mathbf{M}_2 \ddot{\mathbf{h}}_2 + \mathbf{K}_{22} \mathbf{h}_2 = \mathbf{f}_2 \delta(t) - \mathbf{K}_{21} \mathbf{h}_1. \quad (4.5)$$

Application of forward Fourier transform to the above two equations yields

$$(-\Omega^2 \mathbf{M}_1 + \mathbf{K}_{11}) \mathbf{H}_1(\Omega) = \mathbf{f}_1 - \mathbf{K}_{12} \mathbf{H}_2(\Omega). \quad (4.6)$$

and

$$(-\Omega^2 \mathbf{M}_2 + \mathbf{K}_{22}) \mathbf{H}_2(\Omega) = \mathbf{f}_2 - \mathbf{K}_{21} \mathbf{H}_1(\Omega). \quad (4.7)$$

Solving for the frequency response functions of $\mathbf{H}_2(\Omega)$ from Eq. 4.7 and substituting it into Eq. 4.6 gives

$$\mathbf{H}_1(\Omega) = [(-\Omega^2 \mathbf{M}_1 + \mathbf{K}_{11}) - \mathbf{Q}_2(\Omega)]^{-1}(\mathbf{f}_1 - \mathbf{Q}_1(\Omega)). \quad (4.8)$$

where

$$\mathbf{Q}_1(\Omega) = \mathbf{K}_{12}(-\Omega^2 \mathbf{M}_2 + \mathbf{K}_{22})^{-1} \mathbf{f}_2. \quad (4.9)$$

$$\mathbf{Q}_2(\Omega) = \mathbf{K}_{12}(-\Omega^2 \mathbf{M}_2 + \mathbf{K}_{22})^{-1} \mathbf{K}_{21}. \quad (4.10)$$

which represent the interaction effects of the second substructure.

Equation 4.8 is the continuous frequency response function for this simple interaction problem. The formulation is typical for interaction problems except that the interaction effects are usually not obtained from Equations such as Eqs. 4.9 and 4.10. Instead, they are generally obtained by other analytical methods or discrete element methods. If the standard discrete convolution method is used to get the response of $\mathbf{v}_1$ from the calculated values of $\mathbf{H}_1(\Omega)$, the continuous frequency function for substructure 1, very large errors may result. In addition, since the frequency response function is obtained from Eq. 4.8, with the $\mathbf{Q}_1(\Omega)$ and $\mathbf{Q}_2(\Omega)$ derived by other methods, classic mode shapes of the system cannot be obtained and the equations cannot be uncoupled. The methods described in Chapter 3 cannot therefore be used for the kind of interaction problems described here. The methods based on the correction of spurious initial conditions also fail because the initial conditions for the second substructure are totally missed in the process of formulation. Such lost spurious initial conditions cannot be corrected. The following example illustrates the difficulties encountered in obtaining the true response by correcting for spurious initial conditions.

Example 4.1

The two degree-of-freedom system of Example 3.2 is considered again. It is divided into two substructures as shown in Fig. 4.1

Let the load distribution vector and the time variation of the load be the same as before. The three methods presented in Chapter 3 for multi-degree-of-freedom

systems are applied to the solution of this interaction problem. The time interval $\Delta t = 0.05$ is used for all the methods. However, the total number of steps N used varies for different methods. A value of $N = 210$ is used for the the method using $\tilde{H}(\Omega)$, and $N = 128$ is used for rest of the methods.

Figure 4.2 compares the response of substructure obtained by the force pulse method with the exact response. In applying the force pulse method, no reference is made to substructure 2, so that the initial conditions are corrected only for substructure 1. As would be evident, this gives rise to errors in the response which are reflected by the results presented in Fig. 4.2.

Large errors also arise when either the average acceleration correction method or the method using $\tilde{H}(\Omega)$ are used with the reference being made to substructure 1 only. These results are given in Figs. 4.3 and 4.4 respectively. In fact, none of these methods improves the response and at times the corrected response is even more inaccurate than the uncorrected response.

As discussed in earlier chapters, it is apparent that the errors in the response obtained from the current frequency domain analysis technique result because of using the continuous frequency response functions $H_1(\Omega)$ or its periodical form, rather than the discrete transform of a truncated impulse function, $\tilde{H}_1(\Omega)$, and the difference between the true response and the erroneous response is equal to the free vibration response of the analyzed structural system due to the incorrect initial conditions. Therefore, there are two possible ways of correctly solving the interaction problems. One is to obtain the correct free vibration response due to the false initial conditions. The correct response in this case can be obtained by superimposing such free vibration response on the calculated periodic response. Another possible way is to obtain the functions $\tilde{H}_1(\Omega)$, which is the discrete Fourier transforms of the truncated impulse functions $h_1(t)$. This implies that $h_1(t)$ should be obtained first by some means.

4.3 ANALYSIS METHOD BASED ON SUPERPOSITION OF CORRECTIVE FREE VIBRATION RESPONSE

4.3.1 Formulation of The Method

The theory provided here is intended to prepare a basis for the calculation of the true response of a structure problem through frequency domain analysis. The procedure developed should be applicable to multi-degree-of-freedom systems as well as to interaction problems. The method should also be effective and efficient.

In the case of interaction problems, the frequency analysis procedure must provide the correct response for any one degree of freedom individually. In other words, if only one continuous response function $H_j(\Omega)$ corresponding to any j th degree of freedom of the structural system is given, and all other degrees of freedom of the system are treated as substructures, the analysis procedure should still be valid. The free vibration response along this degree of freedom due to the incorrect initial conditions along all degrees of freedom of which only two can be determined has to be obtained for the procedure to work. The idea of the procedure presented here is to use the free vibration response of the j th degree of freedom after the external loading has ceased to act to obtain the corrective response.

The procedure and implementation of the new method can be summarized as follows.

Let the external loading $g(t)$ be as shown in Fig. 4.5 and be of duration T_g . Also, let the total response of the j th degree of freedom be required for time T_r . The first step is to obtain the periodic response due to $g(t)$ using the following Fourier transforms.

$$\bar{G}(k\Delta\Omega) = \sum_{l=0}^{N-1} g(l\Delta t) e^{-2\pi i k l / N \Delta t}. \quad (4.11)$$

$$\bar{v}_j(l\Delta t) = \sum_{k=0}^{N-1} \bar{H}_j(k\Delta\Omega) \bar{G}(k\Delta\Omega) e^{2\pi i k l / N \Delta\Omega}. \quad (4.12)$$

Figure 4.6 shows one period of the response $\bar{v}_j$. The response for $t > T_g$ is a free vibration response due to the initial conditions that remain at $t = T_g$.

Let A denote the first peak in this free vibration response. If the system is undamped, this free vibration response would be of the form

$$\bar{v}_j(t) = A_1 \sin(\omega_1 t + \theta_1) + A_2 \sin(\omega_2 t + \theta_2) + A_3 \sin(\omega_3 t + \theta_3) \dots \quad \text{for } t > T_g \quad (4.13)$$

where $\omega_1, \omega_2, \omega_3 \dots$ are the natural frequencies of the system. It can be assumed that the above response will repeat itself after a time $2n\pi$. This would be precisely true if $n\omega_1, n\omega_2 \dots$ were all integers. As an example, if n is taken as 2, the above assumption would imply that the frequencies are approximated to within 0.5; $n = 10$ will imply approximation to within 0.1.

Let B represent another peak in the free vibration where the time between A and B is approximately equal to $2n\pi$. For zero damping the free vibration response beyond B will be the same as that between A and B . For a damped system this response can be taken as that from A to B scaled in the ratio f_B/f_A , where f_B and f_A are the response amplitudes at B and A respectively.

Since the response is periodic with a period of T_0 , the free vibration component of the response towards the beginning of the period will be as shown in Fig. 4.6b. The free vibration between points A' and B' is identical to that between A and B but with a scaling factor of f_B/f_A . Also point A' is located the same distance to the left of time $t = 0$ as B is located to the left of time $t = T_0$.

Finally the corrected response can be expressed as

$$v(t) = \bar{v}(t) - \frac{f_B}{f_A} \bar{v}(T_1 + T_2 + t) \quad 0 < t < (t_{AB} - T_2) \quad (4.14)$$

Obviously t_{AB} should be selected to be larger than $T_r + T_2$ and T_0 greater than $t_{AB} + T_g$.

4.3.2 Example 4.2

The same two degree-of-freedom system of Example 3.2 is used here. The division of this system into two substructures is shown in Fig. 4.1. The time variation of forcing function $g(t)$ is the same as shown in Fig. 3.5. Again, a time interval

$\Delta t = 0.05$ is used in the analysis. In order for the total computation period to be close to the value of $2n\pi$, a total number of time steps $N = 252$ is used, which give a period $T_0 = 12.6$.

The response of substructure 1 is obtained by using the method proposed here and is shown in Fig. 4.7a.

Comparison is also made with the response obtained by using the mode superposition method for the two degree-of-freedom system without dividing into substructures. The results obtained by the method presented here are indistinguishable from the results obtained by mode superposition. The results of mode superposition method can be considered to be exact because both modes have been used in the analysis.

The analysis can also be conducted on substructure 2 with the consideration of the interaction effects of substructure 1. These results are presented in Fig. 4.7b.

4.4 METHOD OF ARTIFICIAL DAMPING

4.4.1 Formulation of The Method

The frequency domain analysis of a multi-degree-of-freedom system with or without substructuring begins with the calculation of complex frequency response function $H(\Omega)$. A discretized version of $H(\Omega)$ is used in the subsequent analysis using discrete Fourier transforms. If there is sufficient damping in the system, accuracy of results can be assured by selecting a comparatively large value for the time period T_0 used in the discrete analysis. For undamped or lightly damped systems, the procedure developed in the preceding section can be used for the frequency domain analysis of a substructure.

The method presented in the preceding section is dependent on the use of free vibration response of the structural systems due to the spurious initial conditions. Another method, developed as a part of the present work and which uses a modified complex frequency response function to obtain the true impulse fraction $h(t)$, is described here. The essential steps in the method are summarized below.

A unit impulse function $h_j(t)$ for an undamped multi-degree-of-freedom system is shown in Fig. 4.8. The function does not, of course, decay with time. Rigorously speaking, the continuous Fourier transform of such a function does not exist (Weaver 1989). This is reflected in the singularity of the continuous complex response function $H_j(\Omega)$ for an undamped structural system. A consequence of this singularity is that the inverse discrete transform of $H_j(\Omega)$ does not yield $h_j(t)$ but the periodic version $\bar{h}_j(t)$. On the other hand, if an impulse function $\hat{h}(t)$ decays with time, as shown in Fig. 4.9, its continuous Fourier transform $\hat{H}(\Omega)$ exists and does not contain any singularity. At the same time the inverse discrete transform of the periodic version of $\hat{H}_j(\Omega)$ is nearly equal to $\hat{h}_j(t)$ provided the period selected for the transform is sufficiently long. It is because of this fact that the sufficiently damped structural system can be analyzed by the application of discrete transforms. The method proposed in this section relies on the introduction of artificial damping in the system. It is to be emphasized that the introduction of the damping is only an intermediate step in the process and the final solution is obtained for the unaltered system. We define first a modified impulse function given by

$$\hat{h}_j(t) = h_j(t)e^{-at} \quad (4.15)$$

where a is an arbitrary real positive constant. The Fourier transform of the periodic version of $\hat{h}_j(t)$ does not have any singularity and is denoted by $\hat{H}_j(\Omega)$.

For a structural system represented by the following equations of motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}g(t). \quad (4.16)$$

the unit impulse functions are the responses of the structural systems when the external loading $g(t)$ is given by the Dirac function $\delta(t)$ applied at the time $t = 0$, i.e.

$$\mathbf{M}\ddot{\mathbf{h}} + \mathbf{C}\dot{\mathbf{h}} + \mathbf{K}\mathbf{h} = \mathbf{f}\delta(t). \quad (4.17)$$

Now from Eq. 4.15

$$\mathbf{h}(t) = \hat{\mathbf{h}}(t)e^{at} \quad (4.18)$$

The first and second derivatives of $h(t)$ are given by

$$\frac{dh(t)}{dt} = \frac{d\hat{h}(t)}{dt}e^{at} + a\hat{h}(t)e^{at} \quad (4.19)$$

and

$$\frac{d^2h(t)}{dt^2} = \frac{d^2\hat{h}(t)}{dt^2}e^{at} + 2a\frac{d\hat{h}(t)}{dt}e^{at} + a^2\hat{h}(t)e^{at} \quad (4.20)$$

Substitution of Eqs. 4.18, 4.19, and 4.20 into Eq. 4.17 gives the following expression

$$\mathbf{M}\frac{d^2\hat{\mathbf{h}}}{dt^2} + 2a\mathbf{M}\frac{d\hat{\mathbf{h}}}{dt} + a^2\mathbf{M}\hat{\mathbf{h}} + \mathbf{C}\frac{d\hat{\mathbf{h}}}{dt} + a\mathbf{C}\hat{\mathbf{h}} + \mathbf{K}\hat{\mathbf{h}} = \mathbf{f}\delta(t)e^{-at}. \quad (4.21)$$

By applying the continuous Forward Fourier transform to Eq. 4.21, the complex frequency response functions $\hat{\mathbf{H}}(\Omega)$ are derived as

$$\hat{\mathbf{H}}(\Omega) = \left[(a^2 - \Omega^2)\mathbf{M} + \mathbf{K} + a\mathbf{C} + 2a\Omega i\mathbf{M} + \Omega i\mathbf{C} \right]^{-1} \mathbf{f}. \quad (4.22)$$

The modified frequency response functions $\hat{\mathbf{H}}(\Omega)$ can also be obtained from $\mathbf{H}(\Omega)$ by directly applying the frequency shifting theorem (Weaver 1989), which is given in the following

$$\begin{aligned} \hat{\mathbf{H}}(\Omega) &= \int_0^{+\infty} \hat{h}(t)e^{-i\Omega t} dt \\ &= \int_0^{+\infty} h(t)e^{-i(\Omega - ia)t} dt = \mathbf{H}(\Omega - ia) \end{aligned} \quad (4.23)$$

It can be shown by substituting $\Omega - ia$ for Ω in the expression for $\mathbf{H}$ that Eqs. 4.22 and 4.23 are equivalent.

The procedure based on the introduction of artificial damping begins by selecting an appropriate value of a so that there is sufficient damping in the system represented by $\hat{h}_j(t)$. Now $\hat{H}_j(\Omega)$ is obtained from the calculated value of $H_j(\Omega)$ by using Eq. 4.23. The inverse discrete transform of $\hat{H}(\Omega)$ gives a close approximation to $\hat{h}(t)$. Because of the damping represented by the factor e^{-at} , aliasing errors are small. The true impulse function $h(t)$ is now obtained from $\hat{h}(t)$ by using the expression

$$h_j(t) = \hat{h}_j(t)e^{at} \quad (4.24)$$

Once $h_j(t)$ has been calculated as above, the standard Fast convolution technique can be used to find the response along degree-of-freedom j .

As stated earlier, the Fast convolution technique works whether or not there is damping in the system, but it requires that $h(t)$ be known. This presents a difficulty in those cases where $H(\Omega)$, rather than $h(t)$, is given or can be obtained. The difficulty is associated with the fact that for very low damping, $h(t)$ cannot be obtained by taking an inverse discrete transform of the known frequency response function, $H(\Omega)$, because of aliasing. The use of artificial damping circumvents this difficulty.

The question that still remains is related to what is an appropriate value for the arbitrary constant a . This parameter should be sufficiently large to minimize aliasing. However, a very large value of a causes numerical difficulties so that the values of $h(t)$ towards the end of the period T_0 become grossly inaccurate. A criterion for selecting a is to ensure that the value of factor e^{-at} at the end of a period T_0 is, say, $1/m$ of the value at the beginning, so that

$$e^{-aT_0} = \frac{1}{m} \quad (4.25a)$$

$$a = \frac{\ln m}{T_0} \quad (4.25b)$$

where m is a positive number.

Parametric studies carried out as a part of this work disclosed that if the value of m in Eq. 4.25b is selected to lie in range from 50 to 100, numerical instability is avoided and the aliasing errors are small. Any value of m which is much below 50 may be too low to introduce sufficient damping in the analyzed system and will give rise to overlapping errors. On the other hand, any value of m which is much greater than 100 may cause numerical instability in the computation process because values of $\hat{h}(t)$ at many sampled points close to the end of the duration may be treated as all zeros by the computer, and large errors may therefore be produced when the inverse Fourier transform is applied to obtain the final responses.

4.4.2 Example 4.3

The same two degree-of-freedom systems as in Example 4.2 is used here. The time interval $\Delta t = 0.05$, and the total number of time steps $N = 210$. The results obtained by using the proposed method of artificial damping are shown in Figs. 4.10 and 4.11. For comparison, the exact responses are also shown. The two sets of responses agree very well.

4.4.3 Example 4.4

To illustrate the application of this procedure to a system in which substructure analysis would be a logical choice, consider the gravity dam and the impounded reservoir shown in Fig. 4.11. For simplicity both the foundation and the dam would be considered as rigid. The system is excited by an earthquake motion along the x axis, that is, in the upstream-downstream direction. The ground acceleration is shown in Fig. 4.12. Assuming that the ground motion does not vary along the length of the dam and the dam is long in comparison to its cross-sectional dimension, the system can be represented by the two-dimensional model shown in Fig. 4.11. It can be shown that when the ground motion is $e^{i\Omega t}$ the total pressure on the face of the dam, expressed as a fraction of the hydrostatic force $P_s = \frac{1}{2}wH^2$, is given by (Chopra 1967)

$$P(\Omega) = \frac{\tilde{P}(\Omega)}{P_s} = \frac{32}{\pi^3 g} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 \sqrt{(2n-1)^2 - (\Omega/\omega_1)^2}} \quad (4.25)$$

where w is the unit weight of water, $\omega_1 = \pi c/2H$ is the fundamental natural frequency of the reservoir, and H is the height of the reservoir.

Equation 4.25 gives the complex frequency response function for the normalized hydrodynamic force on the dam. In the present case, the expression for $P(\Omega)$ is simple enough so that its continuous inverse Fourier transform, the unit impulse function, can be easily evaluated and can be shown to be (Chopra 1967)

$$p(t) = \frac{32\omega_1}{\pi^3 g} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n c t)}{(2n-1)^2} \quad (4.26)$$

where J_0 is the Bessel function of zero order and first kind, and $\lambda_n = (2n - 1)\pi/2t$. The Bessel function J_0 and hence the impulse function $p(t)$ decays with time. In physical terms, the decay represents loss of energy in waves moving towards infinity. The energy loss can be viewed as the result of a damping in the system.

The damping introduced by wave radiation is quite small so that $p(t)$ does not die out for a long period of time. As a result, the impulse function $p(t)$ obtained by taking the inverse discrete Fourier transform of $P(\Omega)$ is significantly different from $p(t)$ because of aliasing effect, unless a very large value is selected for the period T_0 .

As an illustration of the foregoing statement, consider the example in which the height of water reservoir $H = 100$ m and the velocity of sound in water $c = 1440$ m/s. With these values, the fundamental frequency of the reservoir $\omega_1 = \pi c/2H = 22.2$ rad/s. For an analysis in the frequency domain, the frequency response function obtained from Eq. 4.25 is converted into a discrete form. The time sampling interval is selected as 0.025 s with a period consisting of 1000 samples so that $T_0 = 25$ s. The frequency sampling interval is $\Delta\Omega = 2\pi/T_0 = 0.2513$ rad/s and the period of the frequency function $T_\Omega = 2\pi/\Delta\Omega = 251.3$ rad/s. The discretization of $P(\Omega)$ is completed by truncating it at $\pi/\Delta t = 125.7$ rad/s and taking the sample values for $\Omega > \pi/\Delta t$ as complex conjugates of values for frequencies located symmetrically about $\pi/\Delta t$.

Frequency domain analysis is now carried out with the discrete version of $P(\Omega)$. The analysis results are compared in Fig. 4.13a with discrete convolution of the impulse function and the excitation function. The significant difference between the two sets of results is because of the aliasing effect. This factor does not seem to have been appreciated in many of the frequency domain substructure analyses reported in the literature. A very large value of T_0 will have to be selected to avoid the aliasing effect. This is apparent from Figs. 4.13b which compares the results of discrete convolution and frequency domain analysis for $T_0 = 240$ s.

In using the method based on the introduction of artificial damping, let response values be required for the first 12.5 s. The impulse function $p(t)$ must be truncated

at $T_f = 12.5$ s and T_0 selected to be greater than $T_g + T_f$, that is $8 + 12.5 = 20.5$ s. Let us choose $T_0 = 25$ s. Parameter a is selected to be equal to $\ln 100/25 = 0.184$. Frequency function $\hat{P}(\Omega) = P(\Omega - ia)$ is obtained by substituting $(\Omega - ia)$ for Ω in Eq. 4.25. The rest of the analysis follows the steps already outlined. The results obtained are compared with those obtained by discrete convolution in time domain in Fig. 4.13c and are found to match very closely. It is of interest to note that, in the present problem, the CPU time required on an Apollo DN3000 node for computing the response by standard discrete Fourier transform with $T_0 = 240$ s was 2.373 minutes in comparison to a CPU time of 0.397 minutes required for calculating the same response using the concept of artificial damping in conjunction with the standard Fast convolution analysis, although both methods gave the same order of accuracy.

4.5 CHAPTER SUMMARY

Two new methods are developed in this chapter for obtaining accurate estimates of the response through a frequency domain analysis using discrete Fourier transforms. One is the free vibration corrective method, and the other is the method of artificial damping. The application of the first method depends on the periodicity of the free vibration response which occurs when the applied load has ceased to act. The second method is based on the introduction of a sufficient amount of temporary damping into the structural system analyzed in order that the true impulse function may be obtained by taking the discrete inverse transform of frequency response function. Once the impulse function has been obtained, the standard fast convolution technique may be used. The method of artificial damping is found to be more accurate and more dependable than the free vibration corrective method. In subsequent chapters of this thesis, focus is therefore placed on the method of artificial damping. However, in some special cases where the modified frequency response functions $\hat{H}(\Omega)$ can not be obtained, which may be the case for example when the continuous frequency response functions $H(\Omega)$ is given in numerical form, the use of free vibration corrective method may be the only effective choice for

obtaining a reasonable estimate of the response.

It should be noted here that almost at the same time when the method of artificial damping was developed, Kausel and Rosset (1992) reported the development of a procedure of frequency domain analysis, which they termed the exponential window method and which was based on a very similar idea. Both methods have appeared in the literature at about the same time (Humar and Xia 1992).

Fig. 4.1 The Substructure System

Fig. 4.2 Response History for Substructure 1 Obtained by Using Force Pulse Corrective Method

Fig. 4.3 Response History for Substructure 1 Obtained by Using $\tilde{H}(\Omega)$

Fig. 4.4 Response History for Substructure 1 Obtained by Using Ave. Accel. Correct. Method

Fig. 4.5 External Loading

(a) The Periodic Response to $g(t)$

(b) The Free Vibration Response after The Applied Load $g(t)$

Fig. 4.6 Displacement Responses and Its Free Vibration Part

(a) Response of Substructure 1

(b) Response of Substructure 2

Fig. 4.7 Responses Obtained by Using Free Vibration Correction Method

Fig. 4.8 Undecayed Unit Impulse Function

Fig. 4.9 The Modified Unit Impulse Function

(a) Response of Substructure 1

(b) Response of Substructure 2

Fig. 4.10 Responses obtained by Using the Method of Artificial Damping

Fig. 4.11 Rigid Gravity Dam Impounding Water Reservoir

Fig. 4.12 Ground Acceleration

(a)

(b)

(c)

Fig. 4.13 Analysis of the Rigid Gravity Dam In Time and Frequency Domain, Response to Horizontal Ground Motion

(a) $T = 25$ s; (b) $T = 240$ s

(c) Frequency Domain Results With the Introduction of Artificial Damping

Chapter 5

ANALYSIS OF DAM-RESERVOIR INTER-ACTION IN FREQUENCY DOMAIN

5.1 INTRODUCTION

Structural interaction problems are often encountered in practice, especially when a substructure technique is used in the analysis. Dynamic response of a dam with consideration of the effects of the water in the reservoir is one example of an interaction problem. Another example is dynamic soil-structure interaction which will be dealt with in the next chapter.

In this chapter, the method of artificial damping proposed in Chapter 4 is applied to the analysis of a two dimensional dam-reservoir system. A brief review of the previous work in the field of dam response analysis with consideration given to reservoir interaction has been presented in Chapter 1. A great deal of contribution to this field has been made by Chopra and his colleagues (Chakrabarti and Chopra 1973a; 1973b; 1974; Chopra and Chakrabarti 1981; and Chopra and Gupta 1981), and Humar *et al* (1988). The previous work focuses on the formulation of the problem and consideration of environment surrounding the dam in order to simulate the dynamics of dam more properly. In all cited references, analysis in the frequency domain has been used irrespective of whether an analytical or a boundary element solution of wave propagation in the water body is employed. As already shown clearly in Chapter 4, the correct implementation of discrete Fourier transform (DFT) method in the frequency domain analysis requires that sufficient amount of damping exist in the analyzed system. In the studies of dam-reservoir interaction problems, the water body is, in general, treated as an ideal fluid with no viscosity. Therefore, this part of the structure is actually undamped. Radiation damping due to outgoing waves is present when the excitation is horizontal and

the water domain is of infinite length. However, such damping may not be large enough for correct implementation of the DFT method. This point may not have been appreciated in the previous studies. In addition to radiation damping, a certain amount of damping is also assumed to exist in the dam structure. In some cases, such damping may be too small for a proper use of the DFT method. As a result, there may possibly be some errors inherent in the frequency domain analysis of dam-reservoir interaction problems. The objective of the work presented here is to investigate these errors and the application of the proposed method of artificial damping in order to minimize or eliminate such errors. Comparison will be made between the correct results obtained here with those obtained by the standard DFT method.

In the analytical procedure presented here, the basic formulation of Chakrabarti and Chopra (1973) is followed for obtaining the frequency response functions for dam-reservoir system. It is assumed that the dam rests on a rigid base; as a result, the interaction effect between dam and foundation is not considered. The formulation of dam-reservoir interaction is similar to that presented by Chopra *et al* except for the introduction of Ritz vectors in place of mode shapes as in the original formulation. This will make the analysis more efficient. The important difference is in the final step in converting response in the frequency domain to that in the time domain. The method of artificial damping outlined in Chapter 4 is used here for this purpose.

Although the main work in this chapter is related to the use of analytical solutions for the reservoir, the application of discrete Fourier transform with artificial damping in both a Finite Element (FE) and a Boundary Element (BE) formulation of the reservoir is also briefly mentioned. The last two methods have attracted increasing interest of researchers since a proper representation of the reservoir and its surrounding regions can be obtained by these methods (Hanna and Humar 1982; and Humar and Jablonski 1988).

5.2 FORMULATION OF DAM-RESERVOIR INTER-ACTION BY USING ANALYTICAL SOLUTIONS FOR WATER BODY

The basic formulation of the seismic response of a two dimensional dam-reservoir system with the use of analytical solution of the water body of infinite length has been introduced in a series of publications authored by Chopra and coworkers (Chakrabarti and Chopra, 1973a; 1973b; 1974; Chopra and Chakrabarti 1981; and Chopra and Gupta 1981). Such analysis procedures are summarized in this section. As shown in Fig. 5.1, it is assumed that the dam has a vertical upstream face, the water body in the reservoir can be treated as continuum of an infinite length with the same height, and the dam rests on a rigid foundation.

With the use of Ritz vectors to reduce the size of problems, the formulation given by Chakrabarti and Chopra (1973b) for a dam-reservoir system can be described in the following form. If a dam is to be analyzed with the inclusion of the effect of the water body, the equations of motion can be written as

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{C}\dot{\mathbf{v}} + \mathbf{K}\mathbf{v} = -\mathbf{E}^h a_g^h(t) - \mathbf{E}^v a_g^v(t) + \mathbf{R}(t). \quad (5.1)$$

in which, $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, damping and stiffness matrices respectively for the dam only; $\mathbf{v}$, is the displacement vector relative to the ground motion; $\mathbf{E}^h$ and $\mathbf{E}^v$ are the load amplitude vectors of horizontal and vertical motions respectively; $a_g^h(t)$ and $a_g^v(t)$ are the ground motion accelerations in horizontal and vertical directions respectively; and term $\mathbf{R}(t)$ represents the interaction forces between the water body and the upstream face of the dam.

Equation 5.1 is commonly reduced in size by using the mode shapes of the dam. It can also be effectively reduced by using Ritz vectors. The Ritz vector coordinate transformation can be expressed as follows

$$\mathbf{v} = \mathbf{X}\mathbf{z} \quad (5.2)$$

where $\mathbf{X}$ is the matrix of Ritz vectors and $\mathbf{z}$ the vector of Ritz coordinates. The

reduced equations of motion are then given by

$$\mathbf{M}^* \ddot{\mathbf{z}} + \mathbf{C}^* \dot{\mathbf{z}} + \mathbf{K}^* \mathbf{z} = \mathbf{P}^* \quad (5.3)$$

where

$$\mathbf{M}^* = \mathbf{X}^T \mathbf{M} \mathbf{X}$$

$$\mathbf{C}^* = \mathbf{X}^T \mathbf{C} \mathbf{X}$$

$$\mathbf{K}^* = \mathbf{X}^T \mathbf{K} \mathbf{X}$$

$$\mathbf{P}^* = -\mathbf{X}^T \mathbf{E}^u a_g^h(t) - \mathbf{X}^T \mathbf{E}^v a_g^v(t) + \mathbf{X}^T \mathbf{R}(t)$$

are the reduced mass, damping, and stiffness matrices and the load vector respectively.

Noting that only the x -components of $\mathbf{R}(t)$ on the upstream face of a dam are non-zero, the following relationship holds

$$\mathbf{X}^T \mathbf{R}(t) = (\mathbf{X}^{xJ})^T \mathbf{R}^{xJ}(t)$$

in which, $\mathbf{R}^{xJ}(t)$ are the x -components of $\mathbf{R}(t)$ on the upstream face of the dam. $\mathbf{X}^{xJ}$ represents the x -components of $\mathbf{X}$ on the nodal points on the upstream face of the dam.

The response of the water body in the reservoir is obtained by solving the wave equation of a 2-dimensional problem given by

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{C^2} \frac{\partial^2 p}{\partial t^2} \quad (5.4)$$

where $p(x, y, t)$ is the hydrodynamic pressure, and C the velocity of sound in water. The hydrodynamic pressure acting on the upstream face of the dam is $p(y, t) = p(0, y, t)$.

5.2.1 RESPONSE TO HARMONIC HORIZONTAL GROUND MOTION

The response of the dam to a harmonic horizontal ground motion is addressed first.

The equations of motion in the Ritz coordinates and for a horizontal excitation $a_g^h(t)$ are given by

$$\mathbf{M}^* \ddot{\mathbf{z}}^h + \mathbf{C}^* \dot{\mathbf{z}}^h + \mathbf{K}^* \mathbf{z}^h = -\mathbf{X}^T \mathbf{E}^h a_g^h(t) + (\mathbf{X}^{xf})^T \mathbf{R}^{hxf}(t) \quad (5.5)$$

in which the interaction forces $\mathbf{R}^{hxf}$ are associated with hydrodynamic pressures in the reservoir. These pressures are governed by the wave equation, Eq. 5.4, and the following boundary conditions

- (i) Pressure at the free surface is zero;
- (ii) Vertical motion at the base of the reservoir is zero; and
- (iii) The horizontal component of motion of the boundary $x = 0.0$ is the same as the horizontal motion of the upstream face of the dam.

If the ground motion is assumed to be $a_g^h(t) = e^{i\Omega t}$, the corresponding response can be assumed to take the following form

$$\mathbf{z}^h(t) = \mathbf{Z}^h(\Omega) e^{i\Omega t} \quad (5.6)$$

Also the interaction force vector can be assumed to be of the form

$$\mathbf{R}^h(t) = \bar{\mathbf{R}}^h(\Omega) e^{i\Omega t} \quad (5.7)$$

Substitution of Eq. 5.6 and 5.7 into Eq. 5.5 gives

$$(-\Omega^2 \mathbf{M}^* + i\Omega \mathbf{C}^* + \mathbf{K}^*) \mathbf{Z}^h(\Omega) = -\mathbf{X}^T \mathbf{E}^h + (\mathbf{X}^{xf})^T \bar{\mathbf{R}}^{hxf}(\Omega) \quad (5.8)$$

Equation 5.8 represents a formulation in the frequency domain.

The absolute horizontal accelerations of the upstream face are

$$\ddot{\mathbf{v}}^t(t) = \mathbf{1} e^{i\Omega t} + \mathbf{X}^{xf} \ddot{\mathbf{z}}^h(t) = \left\{ \mathbf{1} + \sum_{k=1}^M \mathbf{x}_k^{xf} \ddot{Z}_k^h(\Omega) \right\} e^{i\Omega t}. \quad (5.9)$$

in which $\ddot{Z}_k^h(\Omega) = -\Omega^2 Z_k^h(\Omega)$.

The hydrodynamic pressures due to the assumed harmonic ground motion take the following form

$$p_c^h(y, t) = \bar{p}_c^h(y, \Omega) e^{i\Omega t} \quad (5.10)$$

In Eq. 5.9, if $\mathbf{x}_k^{zf}$ is replaced by its continuous version $f_k(y)$, then

$$\ddot{v}^t(0, y, t) = \left\{ 1 + \sum_{k=1}^M f_k(y) \ddot{Z}_k^h(\Omega) \right\} e^{i\Omega t} \quad (5.11)$$

The boundary conditions for wave equations in the time domain can be summarized as follows

$$\begin{aligned} p(x, H, t) &= 0 \\ \frac{\partial p}{\partial y}(x, 0, t) &= 0 \\ \frac{\partial p}{\partial x}(0, y, t) &= -\frac{w}{g} \left\{ 1 + \sum_{k=1}^M f_k(y) \ddot{Z}_k^h(\Omega) \right\} e^{i\Omega t} \end{aligned} \quad (5.12)$$

Parameter $\bar{p}_c^h(y, \Omega)$ in Eq. 5.10 is the complex frequency response function for $p_c^h(y, t)$, and can be separated into two parts as in the following

$$\bar{p}_c^h(y, \Omega) = \bar{p}_0^h(y, \Omega) + \sum_{k=1}^M \ddot{Z}_k^h(\Omega) \bar{p}_k^h(y, \Omega). \quad (5.13)$$

where $p_0^h(y, t) = \bar{p}_0^h(y, \Omega) e^{i\Omega t}$ is the solution of the wave equation at $x = 0.0$ with the boundary conditions given by

$$\begin{aligned} p(x, H, t) &= 0 \\ \frac{\partial p}{\partial y}(x, 0, t) &= 0 \\ \frac{\partial p}{\partial x}(0, y, t) &= -\frac{w}{g} e^{i\Omega t} \end{aligned} \quad (5.14)$$

and $p_k^h(y, t) = \bar{p}_k^h(y, \Omega) e^{i\Omega t}$ is the solution of the wave equation at $x = 0$ with the following boundary conditions

$$\begin{aligned} p(x, H, t) &= 0 \\ \frac{\partial p}{\partial y}(x, 0, t) &= 0 \\ \frac{\partial p}{\partial x}(0, y, t) &= -\frac{w}{g} f_k(y) e^{i\Omega t} \end{aligned} \quad (5.15)$$

The analytical solutions in the frequency domain are given by

$$\bar{p}_0^h(y, \Omega) = \frac{4w}{\pi g} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1) \sqrt{\lambda_n^2 - \frac{\Omega^2}{C^2}}} \cos \lambda_n y \quad (5.16)$$

and

$$\bar{p}_k^h(y, \Omega) = -\frac{2w}{gH} \sum_{n=1}^{\infty} \frac{I_{kn}}{\sqrt{\lambda_n^2 - \frac{\Omega^2}{C^2}}} \cos \lambda_n y \quad (5.17)$$

in which

$$\lambda_n = (2n - 1)\pi/2H \quad (5.18a)$$

and

$$I_{kn} = \int_0^H f_k(y) \cos \lambda_n y dy \quad (5.18b)$$

The above integration is done by summation with the Ritz vectors $\mathbf{x}_k^{xf}$ in place of $f_k(y)$ in the actual computation.

The frequency response vector $\bar{\mathbf{R}}^{hxf}(\Omega)$ of the interaction force vector $\mathbf{R}^{hxf}(t)$ can be expressed as

$$\bar{\mathbf{R}}^{hxf}(\Omega) = \bar{\mathbf{R}}_0^h(\Omega) + \sum_{k=1}^M \bar{\mathbf{Z}}_k^h(\Omega) \bar{\mathbf{R}}_k^h(\Omega) \quad (5.19)$$

where $\bar{\mathbf{R}}_0^h(\Omega)$ and $\bar{\mathbf{R}}_k^h(\Omega)$ are the static equivalents of the corresponding pressures $\bar{p}_0^h(y, \Omega)$ and $\bar{p}_k^h(y, \Omega)$ respectively. The nodal force vectors are $\mathbf{R}_0^h(t) = \bar{\mathbf{R}}_0^h(\Omega)e^{i\Omega t}$ and $\mathbf{R}_k^h(t) = \bar{\mathbf{R}}_k^h(\Omega)e^{i\Omega t}$ respectively.

Equations of motion are finally given by

$$(-\Omega^2 \mathbf{M}^* + i\Omega \mathbf{C}^* + \mathbf{K}^*) \mathbf{Z}^h(\Omega) = -\mathbf{X}^T \mathbf{E}^h + (\mathbf{X}^{xf})^T \left\{ \bar{\mathbf{R}}_0^h(\Omega) - \Omega^2 \sum_{k=1}^M \bar{\mathbf{Z}}_k^h(\Omega) \bar{\mathbf{R}}_k^h(\Omega) \right\} \quad (5.20)$$

Equation 5.20 can be expressed by

$$\mathbf{S}^h(\Omega) \mathbf{Z}^h(\Omega) = \mathbf{L}^h(\Omega) \quad (5.21)$$

where

$$\mathbf{S}^h(\Omega) = (-\Omega^2 \mathbf{M}^* + i\Omega \mathbf{C}^* + \mathbf{K}^* + \Omega^2 (\mathbf{X}^{xf})^T \bar{\mathbf{Q}}^h(\Omega))$$

$$\mathbf{L}^h(\Omega) = -\mathbf{X}^T \mathbf{E}^h + (\mathbf{X}^{xf})^T (\bar{\mathbf{R}}_0^h)(\Omega)$$

in which, $\bar{\mathbf{Q}}^h(\Omega)$ is a matrix with its k th column being the vector $\bar{\mathbf{R}}_k^h(\Omega)$.

5.2.2 RESPONSE TO HARMONIC VERTICAL GROUND MOTION

In this case, the equations of motion are given by

$$\mathbf{M}^* \ddot{\mathbf{z}}^v + \mathbf{C}^* \dot{\mathbf{z}}^v + \mathbf{K}^* \mathbf{z}^v = -\mathbf{X}^T \mathbf{E}^v a_g^v(t) + (\mathbf{X}^{xJ})^T \mathbf{R}^{vxJ}(t) \quad (5.22)$$

in which the interaction forces $\mathbf{R}^{vxJ}$ are associated with hydrodynamic pressures in the reservoir, corresponding to the following boundary conditions

- (i) Pressure at the free surface is zero;
- (ii) Vertical motion at the base of the reservoir ($y = 0$) is prescribed by the vertical component of ground acceleration; and
- (iii) the horizontal component of motion of the boundary $x = 0.0$ is the same as the horizontal motion of the upstream face of the dam.

If the vertical component of the ground motion is assumed to be $a_g^v(t) = e^{i\Omega t}$, the corresponding response can be assumed to take the following form

$$\mathbf{z}^v(t) = \mathbf{Z}^v(\Omega) e^{i\Omega t} \quad (5.23)$$

Also the interaction force vector can be assumed to be of the following form

$$\mathbf{R}^v(t) = \mathbf{R}^v(\Omega) e^{i\Omega t} \quad (5.24)$$

By introducing Eqs. 5.23 and 5.24 into Eq. 5.22, the following equations of motion are obtained

$$(-\Omega^2 \mathbf{M}^* + i\Omega \mathbf{C}^* + \mathbf{K}^*) \mathbf{Z}^v(\Omega) = -\mathbf{X}^T \mathbf{E}^v + (\mathbf{X}^{xJ})^T \mathbf{R}^{vxJ}(\Omega) \quad (5.25)$$

The absolute horizontal accelerations of the upstream face are

$$\ddot{\mathbf{v}}^t(t) = \left\{ \sum_{k=1}^M \mathbf{x}_k^{xJ} \ddot{Z}_k^v(\Omega) \right\} e^{i\Omega t} \quad (5.26)$$

in which $\ddot{Z}_k^v(\Omega) = -\Omega^2 Z_k^v(\Omega)$

The hydrodynamic pressures due to the assumed harmonic ground motion take the following form

$$p_c^v(y, t) = \bar{p}_c^v(y, \Omega) e^{i\Omega t} \quad (5.27)$$

In Eq. 5.26, if x^{zj} is replaced by its continuous version $f_k(y)$, then

$$\ddot{v}^t(0, y, t) = \left\{ \sum_{k=1}^M f_k(y) \ddot{Z}_k^v(\Omega) \right\} e^{i\Omega t} \quad (5.28)$$

The boundary conditions for wave equations in the time domain can then be summarized by

$$\begin{aligned} p(x, H, t) &= 0 \\ \frac{\partial p}{\partial y}(x, 0, t) &= -\frac{w}{g} e^{i\Omega t} \\ \frac{\partial p}{\partial x}(0, y, t) &= -\frac{w}{g} \left\{ \sum_{k=1}^M f_k(y) \ddot{Z}_k^v(\Omega) \right\} e^{i\Omega t} \end{aligned} \quad (5.29)$$

$\bar{p}_c^v(y, \Omega)$ in Eq. 5.27 is the complex frequency response function for $p_c^v(y, t)$, and can be separated into two parts as

$$\bar{p}_c^v(y, \Omega) = \bar{p}_0^v(y, \Omega) + \sum_{k=1}^M \ddot{Z}_k^v(\Omega) \bar{p}_k^v(y, \Omega) \quad (5.30)$$

where $p_0^v(y, t) = \bar{p}_0^v(y, \Omega) e^{i\Omega t}$ is the solution of the wave equation at $x = 0.0$ with the boundary conditions given by

$$\begin{aligned} p(x, H, t) &= 0 \\ \frac{\partial p}{\partial y}(x, 0, t) &= -\frac{w}{g} e^{i\Omega t} \\ \frac{\partial p}{\partial x}(0, y, t) &= 0 \end{aligned} \quad (5.31)$$

and $p_k^v(y, t) = \bar{p}_k^v(y, \Omega) e^{i\Omega t}$ is the solution of the wave equation at $x = 0$ with the following boundary conditions

$$\begin{aligned} p(x, H, t) &= 0 \\ \frac{\partial p}{\partial y}(x, 0, t) &= 0 \\ \frac{\partial p}{\partial x}(0, y, t) &= -\frac{w}{g} f_k(y) e^{i\Omega t} \end{aligned} \quad (5.32)$$

The analytical solution for $\bar{p}_0^v$ is given by (Chopra, 1967)

$$\bar{p}_0^v(y, \Omega) = \frac{wc}{g\Omega} \frac{\sin \frac{\Omega}{c}(H-y)}{\cos \frac{\Omega H}{c}} \quad (5.33)$$

The above expression derived on the basis of a rigid bottom gives pressure values that are unrealistically high (Chopra 1967). Rosenbluth (1968) has suggested an expression to account for foundation damping. When the foundation flexibility is taken into account, $\bar{p}_0^v(y, \Omega)$ becomes

$$\bar{p}_0^v(y, \Omega) = \frac{wc}{g\Omega} \frac{(1+\alpha)\sin \frac{\Omega}{c}(H-y)}{(1+\alpha)\cos \frac{\Omega H}{c} + i(1-\alpha)\sin \frac{\Omega H}{c}} \quad (5.34)$$

where $\alpha = (\eta - 1)/(\eta + 1)$ is called the reflection coefficient,

$$\eta = \frac{c_r w_r}{c_w}, \quad (5.35)$$

w_r is the unit weight of foundation material; c_r is the P -wave velocity in the foundation material; and c_w is the P -wave velocity in water, with $c_w = c$. In addition, $\bar{p}_k^v(y, \Omega)$ is the same as $\bar{p}_k^h(y, \Omega)$ which is given in Eq. 5.17.

The complex response function $\bar{\mathbf{R}}^{vxf}(\Omega)$ for the interaction force vector $\mathbf{R}^{vxf}(t)$ can be expressed as

$$\bar{\mathbf{R}}^{vxf}(\Omega) = \bar{\mathbf{R}}_0^v(\Omega) + \sum_{k=1}^M \ddot{\mathbf{Z}}_k^v(\Omega) \bar{\mathbf{R}}_k^v(\Omega) \quad (5.36)$$

where $\bar{\mathbf{R}}_0^v(\Omega)$ and $\bar{\mathbf{R}}_k^v(\Omega)$ are the static equivalents of the corresponding pressures $\bar{p}_0^v(y, \Omega)$ and $\bar{p}_k^v(y, \Omega)$ respectively. The nodal force vectors are $\mathbf{R}_0^v(t) = \bar{\mathbf{R}}_0^v(\Omega)e^{i\Omega t}$ and $\mathbf{R}_k^v(t) = \bar{\mathbf{R}}_k^v(\Omega)e^{i\Omega t}$ respectively.

Equations of motion for vertical component of ground motion can now be written as follows

$$(-\Omega^2 \mathbf{M}^* + i\Omega \mathbf{C}^* + \mathbf{K}^*) \mathbf{Z}^v(\Omega) = -\mathbf{X}^T \mathbf{E}^v + (\mathbf{X}^{xf})^T \left\{ \bar{\mathbf{R}}_0^v(\Omega) - \Omega^2 \sum_{k=1}^M \mathbf{Z}_k^v(\Omega) \bar{\mathbf{R}}_k^v(\Omega) \right\} \quad (5.37)$$

Equation 5.37 can be expressed as

$$\mathbf{S}^v(\Omega) \mathbf{Z}^v(\Omega) = \mathbf{L}^v(\Omega) \quad (5.38)$$

where

$$\mathbf{S}^v(\Omega) = \left(-\Omega^2 \mathbf{M}^* + i\Omega \mathbf{C}^* + \mathbf{K}^* + \Omega^2 (\mathbf{X}^{zf})^T \bar{\mathbf{Q}}^v(\Omega) \right)$$

$$\mathbf{L}^v(\Omega) = -\mathbf{X}^T \mathbf{E}^v + (\mathbf{X}^{zf})^T \bar{\mathbf{R}}_0^v(\Omega)$$

in which, $\bar{\mathbf{Q}}^v(\Omega)$ is a matrix with its k th column being the vector $\bar{\mathbf{R}}_k^v(\Omega)$.

5.2.3 RESPONSE TO AN ARBITRARY GROUND MOTION

For a plane problem, an arbitrary ground motion can be expressed by its vertical and horizontal components. Let $A_g^h(\Omega)$ and $A_g^v(\Omega)$ respectively be the Fourier transforms of the horizontal and vertical components of ground motion, $a_g^h(t)$ and $a_g^v(t)$. Next, let the frequency response functions be obtained for a range of values of Ω by solving for $\mathbf{Z}^h(\Omega)$ in Eq. 5.21 and $\mathbf{Z}^v(\Omega)$ in Eq. 5.38. The responses in Ritz coordinates are then given by:

Due to the horizontal excitation

$$\mathbf{z}^h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{Z}^h(\Omega) A_g^h(\Omega) e^{i\Omega t} d\Omega \quad (5.39)$$

Due to the vertical excitation

$$\mathbf{z}^v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{Z}^v(\Omega) A_g^v(\Omega) e^{i\Omega t} d\Omega \quad (5.40)$$

The total response in Ritz coordinates is given by

$$\mathbf{z}(t) = \mathbf{z}^h(t) + \mathbf{z}^v(t) \quad (5.41)$$

Finally, the displacements in physical coordinates can be obtained from

$$\mathbf{v}(t) = \mathbf{X} \mathbf{z}(t). \quad (5.42)$$

In Eq. 5.39 and 5.40, the inverse Fourier transform is used in converting the responses in the frequency domain into the responses in the time domain. However, the integration of the inverse Fourier transform can only be approximated by finite number of summations. If the damping involved in the system is zero or too small,

the integrands in Eq. 5.39 or 5.40 become singular or almost singular at specific values of Ω . Under the circumstances, accurate evaluation of the integrals by numerical summation becomes difficult. For this reason, errors may be introduced in implementing the DFT method. The following section deals with the application of the method of artificial damping in the analysis of dam-reservoir interaction so that a correct evaluation of the integrals in Eq. 5.39 and 5.40 is assured even when there is no damping in the system.

5.3 INTRODUCTION OF ARTIFICIAL DAMPING IN DAM-RESERVOIR SYSTEM

In Chapter 4, it has been clearly shown that the response of an undamped structural system cannot be obtained by the usual DFT (discrete Fourier transform) method, since the unit impulse function of the undamped system, $h(t)$, does not decay with time. However, by the introduction of a certain kind of artificial damping, the unit impulse function can be forced to decay with time. The modified impulse function is of the form $h(t)e^{-at}$, where a is a positive real number selected appropriately. An analysis procedure similar to the fast convolution method can be applied once $h(t)e^{-at}$ and hence $h(t)$ is obtained, and the problem inherent in the frequency domain analysis for undamped structures is successfully solved. This method requires the modified frequency response function, $\hat{H}(\Omega)$ which is the Fourier transform of $h(t)e^{-at}$. Let $\Omega^* = (\Omega - ia)$, then the relationship between the modified frequency response function, $\hat{H}(\Omega)$, and the original frequency response function, $H(\Omega)$ is as follows

$$\hat{H}(\Omega) = H(\Omega^*) \quad (5.43)$$

By applying Eq. 5.43 directly to the frequency response functions of dam-reservoir system, the following relationships are derived

Equation 5.21 becomes

$$S^h(\Omega^*)\hat{Z}^h(\Omega^*) = L^h(\Omega^*) \quad (5.44)$$

and Eq. 5.38 becomes

$$\mathbf{S}^v(\Omega^*)\hat{\mathbf{Z}}^v(\Omega^*) = \mathbf{L}^v(\Omega^*) \quad (5.45)$$

The foregoing equations imply that in order to obtain the modified frequency response functions, that is either $\hat{\mathbf{Z}}^h(\Omega^*)$ due to a harmonic horizontal ground motion or $\hat{\mathbf{Z}}^v(\Omega^*)$ due to a harmonic vertical ground motion, one can simply substitute Ω^* for Ω in the expression for $\mathbf{Z}^h(\Omega)$ or $\mathbf{Z}^v(\Omega)$. After $\hat{\mathbf{Z}}^h(\Omega^*)$ and $\hat{\mathbf{Z}}^v(\Omega^*)$ are obtained, the artificial damping method as described in Chapter 4 can be directly applied.

5.4 NODAL FORCES ON A RIGID DAM DUE TO HYDRODYNAMIC PRESSURE

As the first example of frequency domain analysis of a dam-reservoir system, the nodal forces on the upstream face of a rigid dam subjected to a horizontal and a vertical ground motion are examined. A rigid dam with a height of 100.0 m and resting on a rigid foundation is considered. The dam is shown in Fig. 5.1. The velocity of sound in water is taken as $c = 1440$ m/s. The time sampling interval is selected as 0.025 s with 1000 time steps so that the period is $T_0 = 25.0$ s. The frequency sampling interval is $\Delta\Omega = 2\pi/T_0 = 0.2513$ rad/s and the period of the frequency function is $T_\Omega = 2\pi/\Delta\Omega = 259.3$ rad/s. For simplicity, three nodes are considered on the upstream face of the dam as in Fig. 5.1. It should be noted that the use of only three nodes may not be adequate for obtaining a good estimate of hydrodynamic pressure on the dam. However, the present concern is to demonstrate the method of evaluating the nodal forces correctly once a discretized model has been selected. The approximation involved in selecting a small number of nodes does not affect the conclusions drawn in this section.

Total hydrodynamic pressures on the rigid dam caused by the horizontal ground excitation have been studied in Example 4.4 in Chapter 4. The results here are similar to those presented in Chapter 4 except that the hydrodynamic pressure on each node rather than the total pressure are obtained. The results for horizontal excitation are therefore not presented. The hydrodynamic pressures on the dam

caused by the vertical excitation are discussed in the following section.

5.4.1 VERTICAL GROUND MOTION

The dam-reservoir system as shown in Fig. 5.1 is used to examine the response to a vertical ground acceleration. The variation of vertical ground acceleration with time is taken to be the one shown in Fig. 5.2. When the foundation is considered to be rigid, the hydrodynamic pressure is given by Eq. 5.33. Unlike in the case of horizontal ground excitation, radiation damping does not exist in this case. Therefore, the system is completely undamped. Figure 5.3 shows the results obtained by using the standard DFT method as well as those obtained by using the artificial damping method. It is found that large difference exists between the two sets of results. The results obtained by artificial damping method are compared with the analytical solution in Fig. 5.4. The latter is obtained by directly using the unit impulse response function given by Chopra (1967), which is

$$p^v(x, y, t) = \frac{4wc}{\pi g} \sum_{n=1}^{\infty} \frac{1}{2n-1} \cos \lambda_n y \sin(\lambda_n ct) \quad (5.46)$$

A time interval of 0.025 s is taken in sampling the unit impulse function which is truncated at time $T_h = 20.7$ s. The sampled and truncated unit impulse function is then augmented to the full time period $T_0 = 25.0$ s by adding a series of zero values at every sampling points after time $T_h = 20.7$ s.

Figure 5.4 shows that the two sets of results are almost the same. On the other hand, large errors are present in this case in the responses obtained by using the standard DFT method. The use of a long time period does not at all help in improving the results. It should be noted that this error has been recognized by Chakrabarti and Chopra (1973a; and 1973b). Consequently, they make the recommendation that foundation flexibility be included in the formulation as in Eq. 5.34.

If Eq. 5.34 is used instead of Eq. 5.33 for computing the hydrodynamic pressure due to a vertical ground motion, there is a certain amount of damping in the system caused by the flexibility of the reservoir foundation. Because of the presence of this

damping, the standard DFT method without any modification may be able to predict the true response. To illustrate this, the foregoing example is repeated with the reflection coefficient taken as $\alpha = 0.90$. The nodal force on node 2 obtained by both the DFT method and the artificial damping method are shown in Fig. 5.5. A time period of $T_0 = 25.0$ s is used in both methods. It is found that the DFT method gives correct response in this case.

5.5 DYNAMIC RESPONSE OF PINE FLAT DAM

The Pine Flat dam used in the work by Chakrabarti and Chopra (1973b) is analyzed for its response to a ground motion by using the method of artificial damping. The finite element mesh for the dam is shown in Fig. 5.6. It has 136 quadrilateral isoparametric finite elements and 162 nodal points. The number of degrees of freedom is 306. For the dam material, $E = 34520$ MPa, Poisson's ratio = 0.20, and unit weight = 2485.0 kg/m³ are used. The mode shapes of the dam without reservoir and resting on a rigid foundation are used to reduce the system. The use of Ritz vectors will be considered later. The first four modes of the dam are extracted and used in the analysis. The corresponding natural circular frequencies are 25.06, 52.71, 69.20, and 91.08 rad/s. The equivalent periods are 0.2507, 0.1192, 0.09079, 0.06898 seconds respectively, which are somewhat different from the values given in the work by Chakrabarti and Chopra (1973b) due to a slightly different finite element mesh. The water level is considered to be El.951; this corresponds to a height of the water of 116.15 m from the base of the reservoir.

Records from the Taft California earthquake of 21st July 1952 are used as the ground motion input. The horizontal and vertical accelerations of the ground motion are shown in Figs. 5.7 and Fig. 5.8 respectively.

5.5.1 RESPONSE TO HORIZONTAL GROUND MOTION

The displacement responses to a horizontal ground acceleration are computed by using the usual DFT method as well as by using the method of artificial damping. At first, the material damping in the dam is assumed to be negligible. The

results presented in Figs. 5.9a and 5.9b show respectively the horizontal and vertical displacement responses at node 1 due to the horizontal excitation indicated in Fig. 5.7. In both approaches, the time interval Δt is taken as 0.02 s and the number of time steps as 2048. Therefore, the total period is 40.96 seconds. In the application of the method of artificial damping, parameter a is taken as 0.1. It is found that the responses obtained by using the DFT method are far different from the results obtained by using the method of artificial damping. The results obtained by using the method of artificial damping can be regarded as representing the correct responses.

For a 2% modal damping in the dam, the responses at node 1 obtained by using the DFT method and by the method of artificial damping are as shown in Figs. 5.10a and 5.10b for the horizontal and vertical directions respectively. The existence of 2% damping in the dam is found to improve the responses obtained by using DFT method remarkably. However, obvious differences still exist between the responses obtained by using the DFT method and those obtained by using the method of artificial damping. If a 5% modal damping is assumed to exist in the dam, the results are as shown in Fig. 5.11a for horizontal motion and Fig. 5.11b for vertical motion. In this case, the responses obtained by using the DFT method and the method of artificial damping are almost the same.

5.5.2 RESPONSE TO VERTICAL GROUND MOTION

The displacement responses of the Pine Flat dam to the vertical ground acceleration shown in Fig. 5.7 are also investigated. The curves given in Figs. 5.12a and 5.12b are the displacement responses in horizontal and vertical directions respectively at node 1 for the case of undamped dam and with the foundation reflection coefficient α being taken as 0.75. The dashed lines represent the responses obtained by using the DFT method, while the solid lines show those obtained by using the method of artificial damping. The latter may be regarded as the correct results. Significant difference is found to exist between the two sets of responses. Figures 5.13a and 5.13b show the response results when the dam has 2 % modal damping and

$\alpha = 1.0$, which implies that the reservoir bottom is rigid. A discernible difference exists between the results obtained by using the standard DFT method and those obtained by using the method of artificial damping. When $\alpha = 0.75$ is used along with a material damping of 2%, differences between the two sets of results still exist as shown in Fig. 5.14a and 5.14b. Figures. 5.15a and 5.15b show the displacement responses respectively in horizontal and vertical directions at node 1 with 5% modal damping in the dam and the reservoir bottom considered as rigid. The match is now better. In all these figures, dashed lines represent the responses obtained by using the DFT method and solid lines the responses obtained by using the method of artificial damping. These figures further confirm the effectiveness of the method of artificial damping

In considering the responses to both the horizontal and vertical excitations, two points are worth mentioning. First, if the analysis is conducted on dam structure alone, without giving consideration to reservoir effect, the existence of only 0.5% modal damping will be large enough to obtain correct results by using the DFT method provided a time period of 40.96 seconds or longer is used. These results are shown in Figs. 5.16a and 5.16b. Second, the discrepancy between the results obtained by the DFT method and those obtained by the method of artificial damping will diminish if a longer time period is used in the DFT method, as long as there is some damping in the dam. Figures. 5.17a and 5.17b provide a comparison of the two sets of results for the dam with an assumed 2% modal damping and with a time period $T_0 = 81.92$ used in the DFT method. These two sets of results are almost the same now. However, if there is no damping in the dam, correct results can never be obtained by using the DFT method no matter how long the selected time period is.

5.6 USE OF RITZ VECTORS

The Pine Flat dam example is also analyzed by using load dependent Ritz vectors instead of mode shapes. In generating these Ritz vectors, a load distribution vector with individual elements equal to the diagonal elements of the mass matrix

is used. Four load-dependent ritz vectors are used. Since the four Ritz vectors have already covered the important range of the frequencies for excitation, there is no necessity of including any frequency-dependent Ritz vectors in this case. The four Ritz vectors extract the natural frequencies of 25.06, 52.71, 69.90, and 97.37 rad/s. Slightly larger frequency range is covered by Ritz vectors than the first four mode shapes. The total displacement responses due to the horizontal and the vertical excitations shown in Figs. 5.7 and 5.8 respectively, are given in Figs. 5.18a, 5.18b, 5.19a, and 5.19b. No damping is assumed to exist in the dam and the method of artificial damping is used in the analysis. Comparison with the responses obtained by using the first four mode shapes also presented in these figures, shows that the use of Ritz vectors gives almost the same results as those obtained by using mode shapes. Because the computation of Ritz vectors is faster than the computation of mode shapes, the use of Ritz vectors leads to greater efficiency.

5.7 APPLICATION OF THE METHOD OF ARTIFICIAL DAMPING IN OTHER NUMERICAL PROCEDURES

The foregoing sections described the analysis of dam-reservoir interaction based on the use of analytical solutions for hydrodynamic pressures and the application of the method of artificial damping in the frequency domain analysis. Such analysis procedure is only applicable to those reservoirs which can be idealized as being rectangular. For reservoirs of arbitrary geometry, the analysis of dam-reservoir interaction is commonly formulated by using Finite Element Method (FEM) or Boundary Element Method (BEM). It is usual for the frequency domain formulation to be utilized in these procedures too. The method of artificial damping proposed in Chapter 4 can also be used in cases where the reservoirs are modelled by either FEM or BEM. The FEM method will be examined here first. When using the FEM formulation for dam-reservoir interaction as proposed by Hall *et al* (1982), the application of the method of artificial damping in the finite element analysis of the dam-reservoir system depends on the evaluation of the frequency response

function of hydrodynamic pressure, $\mathbf{P}(\Omega^*)$, given by the following expressions

For finite fluid domains of irregular geometry:

$$\left[\mathbf{H} + i\Omega^* q \mathbf{B} - \frac{(\Omega^*)^2}{c^2} \mathbf{G} \right] \mathbf{P}(\Omega^*) = \frac{w}{g} \mathbf{D} \quad (5.47)$$

and for infinite fluid domains of irregular geometry:

$$\begin{aligned} & \left[\begin{array}{c} \left[\mathbf{H}_{11} + i\Omega^* q \mathbf{B}_{11} - \frac{(\Omega^*)^2}{c^2} \mathbf{G}_{11} \right] \\ \mathbf{X}^T \left[\mathbf{H}_{21} + i\Omega^* q \mathbf{B}_{21} - \frac{(\Omega^*)^2}{c^2} \mathbf{G}_{21} \right] \end{array} \quad \begin{array}{c} \left[\mathbf{H}_{12} + i\Omega^* q \mathbf{B}_{12} - \frac{(\Omega^*)^2}{c^2} \mathbf{G}_{12} \right] \mathbf{X} \\ \mathbf{X}^T \left[\mathbf{H}_{22} + i\Omega^* q \mathbf{B}_{22} - \frac{(\Omega^*)^2}{c^2} \mathbf{G}_{22} \right] \mathbf{X} + \mathbf{K} \end{array} \right] \\ & \times \left\{ \frac{\mathbf{P}_1(\Omega^*)}{\eta_2(\Omega^*)} \right\} = \frac{w}{g} \left\{ \frac{\mathbf{D}_1}{\mathbf{X}^T \mathbf{D}_2 + \mathbf{K}^{-1} \mathbf{X}^T \mathbf{D}_1} \right\} \end{aligned} \quad (5.48)$$

Equations 5.47 and 5.48 are obtained by substituting $\Omega^* = (\Omega - ia)$ for ω in Eqs. 37 and 70 in the work by Hall and Chopra (1982). The notations used in these two equations are as follows

B	finite element 'damping' matrix for a finite fluid domain
B'	finite element 'damping' matrix for the constant depth portion of an infinite fluid domain
c	velocity of compression waves in water
c_r	velocity of compression waves in the foundation medium
D	vector of nodal accelerations for fluid domain
D'	vector of nodal accelerations for the cross-section of the constant depth portion of an infinite fluid domain
g	acceleration due to gravity
G	finite element 'mass' matrix for a finite fluid domain
H	finite element 'stiffness' matrix for a finite fluid domain
P	vector of nodal pressures in the fluid domain
$q = w/c_r w_r$	the damping coefficient along the fluid-foundation boundary
w	unit weight of water

w_r	unit weight of foundation medium
η_n	amplitude of X
$\bar{\eta}_n$	complex frequency response of η_n
$\kappa_n = \sqrt{(\lambda_n^2 - \frac{\Omega^2}{c^2})}$	, with positive real and imaginary parts
K	an $N \times N$ diagonal matrix with n th diagonal term $= \kappa_n$
X	a matrix with the N eigenvectors $\mathbf{x}_n$ as columns.
$\mathbf{x}_n$	n th complex valued eigenvectors of the cross section of the constant depth portion of an infinite fluid domain.

The method of artificial damping can also be directly applied in BEM formulation of the dam-reservoir interaction. The critical point is to evaluate the Green's functions with complex variables rather than with real variables. For dynamic water pressure governed by the Helmholtz equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + k^2 p = 0 \quad (5.49)$$

in which, $k = \Omega^2/c^2$, the Green's function is the solution of the equation

$$\frac{\partial^2 p^*}{\partial x^2} + \frac{\partial^2 p^*}{\partial y^2} + k^2 p^* = \delta(x - x_i, y - y_i) \quad (5.50)$$

in which $\delta(x - x_i, y - y_i)$ is a Dirac Delta function centered at point i having coordinates (x_i, y_i) , and p^* the solution of the equation with a unit body force acting at point (x_i, y_i) . Solution p^* is referred to as the fundamental solution or the Green's function.

The Green's function is given by

$$p^* = \frac{i}{4} H_0^{(2)}(kr) = \frac{i}{4} [J_0(kr) - iY_0(kr)] \quad (5.51)$$

Another Green's function, q^* , which represents the pressure derivative, can be derived from

$$q^* = \frac{dp^*}{dn} \quad (5.52)$$

The detailed formulation of BEM method in the analysis of dam-reservoir interaction can be found in the work by Hanna and Humar (1982), and Humar and

Jablonski (1988). In applying the method of artificial damping, $k = \Omega^2/c^2$ is substituted by $k^* = (\Omega^*)^2/c^2$, where $\Omega^* = (\Omega - ia)$.

Examples are not given for the use of the method of artificial damping in the analysis of dam-reservoir interaction systems employing FEM or BEM as mentioned above. However, the use of the method of artificial damping in BEM solution of soil-structure interaction problems will be thoroughly discussed in the next chapter. Such an application will show that in BEM analysis using the method of artificial damping, the Bessel functions involved in the Green's function must be evaluated for a complex argument; this however poses no special difficulty.

5.8 CHAPTER SUMMARY

A formulation for the seismic response of dam-reservoir system is described in this chapter. The application of the method of artificial damping in frequency domain analysis of the system is illustrated. The use of Ritz vectors in the analysis of dam-reservoir interaction problem is also investigated. The responses obtained by using the standard DFT method in the analysis procedures are found to give unacceptable errors in the cases where the damping in the dam-reservoir system is small. On the other hand, use of artificial damping provides good estimates of the response. The use of the method of artificial damping often greatly improves the efficiency of the computation in the analysis of lightly damped systems, since a much shorter time period may be used in the computation. The method of artificial damping can also be applied in the dam-reservoir interaction analysis procedures formulated by using either the finite element method or boundary element method. Such applications are briefly mentioned in this chapter without examples. The analytical results presented in this Chapter also show that load dependent Ritz vectors provide a more efficient substitute for mode shapes in the analysis of dam-reservoir system in the transformed domain.

Fig. 5.1 Rigid dam impounding an infinite reservoir

Fig. 5.2 Ground acceleration

Fig. 5.3 Hydrodynamic force on rigid dam caused by vertical excitation

Fig. 5.4 Hydrodynamic force on rigid dam caused by vertical excitation (comparison with analytical results)

Fig. 5.5 Hydrodynamic force on rigid dam caused by vertical excitation (flexible reservoir bottom)

Fig. 5.6 Pine Flat dam and its finite element mesh

**Fig. 5.7 Horizontal component of ground acceleration
along S69E direction, Taft earthquake**

**Fig. 5.8 Vertical component of ground acceleration,
Taft earthquake**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.9 Responses to the Horizontal Ground Acceleration
(undamped dam, rigid reservoir bottom)**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

Fig. 5.10 Responses to the Horizontal Ground Acceleration
 (2 % modal damping in dam, rigid reservoir bottom)

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.11 Responses to the Horizontal Ground Acceleration
 (5% modal damping in dam, rigid reservoir bottom)**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.12 Responses to the Vertical Ground Acceleration
 (undamped dam, flexible reservoir bottom)**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.13 Responses to the Vertical Ground Acceleration
 (2% modal damping in dam, rigid reservoir bottom)**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.14 Responses to the Vertical Ground Acceleration
 (2% modal damping in dam, flexible reservoir bottom)**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.15 Responses to the Vertical Ground Acceleration
(5% modal damping in dam, rigid reservoir bottom)**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.16 Responses of Dam to the Horizontal Ground Acceleration
 (0.5% modal damping in dam, empty reservoir)**

(a) Response to Horizontal Ground Acceleration

(b) Response to Vertical Ground Acceleration

**Fig. 5.17 Displacement Responses at Node 1 along Vertical Direction
 (2% modal damping in dam, rigid reservoir bottom)**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.18 Responses to the Horizontal Ground Acceleration
 By Using DFT with Art. Damp. (undamped dam,
 rigid reservoir bottom)**

(a) Displacement Response at Node 1 along Horizontal Direction

(b) Displacement Response at Node 1 along Vertical Direction

**Fig. 5.19 Responses to the Vertical Ground Acceleration
 by Using DFT with Art. Damp. (undamped dam,
 flexible reservoir bottom)**

Chapter 6

ANALYSIS OF SOIL-STRUCTURE INTERACTION IN FREQUENCY DOMAIN

6.1 INTRODUCTION

Dynamic analysis of soil-structure interaction problems based on substructure method is usually carried out in the frequency domain, for the following reasons. First, the analysis of soil part which often involves the solution of wave propagation equations in a semi-infinite regular or irregular domain is best carried out in the frequency domain. Second, by employing the Fast Fourier Transform (FFT) technique, a very efficient analysis procedure can be formulated in the frequency domain. Finally, if the Boundary Element Method (BEM) is used in the analysis, the fundamental solution for the soil in the frequency domain is much simpler than that in the time domain. Frequency domain analysis using BEM can lead to substantial saving in computing time. This advantage of BEM can also be realized in some time domain formulations of the soil-structure interaction problem as described in the work by Wolf (1985; and 1989). However, compared with the frequency domain analysis, the time domain analysis is computationally inefficient since much effort is required in evaluating the Duhamel's convolution integral directly in the latter analysis. Recently, Wolf and Motosaka have proposed some new methods to approximate the evaluation of the Duhamel's convolution integral. In their papers (Wolf and Motosaka 1989a; and 1989b), it has been shown that the time domain analysis can be made computationally as competitive as the frequency domain analysis. In the present study, concern is restricted to the frequency domain analysis procedures, and linear elastic properties are assumed for all the materials involved.

Dynamic soil-structure interaction analysis is a comparatively new topic and has received increasing attention of both researchers and engineers in the last two decades. Indisputably, a number of problems still exists in the analysis procedures. The objective of this chapter is to examine a few of these problems which have either been ignored or not adequately treated in the earlier work. The work presented can be roughly summarized as follows:

- (1). BEM method is used to calculate the dynamic stiffness matrix of soil. Singularity in the boundary integrals is carefully treated.
- (2). The ability of Discrete Fourier Transform (DFT) technique to provide an accurate estimate of the dynamic response of soil-structure interaction system is studied. The method of artificial damping proposed in Chapter 4 is incorporated into the analysis procedures so that the accuracy of the calculated response is improved.
- (3). An idea of using two stage Ritz vector technique is suggested in order to achieve more efficient reduction of soil structure interaction system.

6.2 BOUNDARY ELEMENT ANALYSIS OF THE INFINITE ELASTIC SOIL MEDIUM

In the analysis of soil-structure interaction problems, soil is commonly treated as an infinite solid elastic medium. The dynamic theory for solid elastic continuum is then used for the foundation. By following the elastic theory, the differential equations governing the motion of an elastic medium can be written as

$$(c_1^2 - c_2^2)v_{i,ij} + c_2^2 v_{j,ii} + b_j = \partial^2 v_j / \partial t^2 \quad (6.1)$$

where

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}$$

$$c_2^2 = \frac{\mu}{\rho}$$

$$\lambda = \frac{E\nu}{(1-\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

c_1 and c_2 are respectively the dilatational and shearing wave velocities; λ and μ are the Lamé constants of the material; ρ , ν and E are the mass density, Poisson's ratio, and Young's modulus of the material respectively; b_i is the body force; u represents the displacement, t the time variable and indicial notations are employed.

Equation 6.1 is the wave equations for solid elastic continuum. It has been commonly used in the analysis of soil-structure interaction problems.

The following two kinds of boundary conditions are generally considered in the analysis.

Displacement boundary conditions:

$$v_i = q_i(\mathbf{r}, t), \quad \text{on } \mathbf{B}_v \quad (6.2)$$

Surface traction boundary conditions:

$$t_{(\mathbf{n})i} = t_{ij}n_j = p_i(\mathbf{r}, t), \quad \text{on } \mathbf{B}_t \quad (6.3)$$

where $\mathbf{r}$ stands for the position vector and t for time; $\mathbf{n}$ is the unit normal vector on a differential element on the surface $\mathbf{B}$; $t_{(\mathbf{n})i}$ and t_{ij} are the traction vector and tensor, respectively, and $\mathbf{B}_t + \mathbf{B}_v = \mathbf{B}$ represents the boundary. For convenience, zero initial conditions are assumed as this is usually the case in foundation problems, i.e.

$$v_i(\mathbf{r}, 0^+) = \dot{v}_i(\mathbf{r}, 0^+) = 0, \quad \text{on } \mathbf{R} + \mathbf{B} \quad (6.4)$$

where $\mathbf{R}$ is the domain with the boundary $\mathbf{B}$. The constitutive relationship for linear elastic medium (or Hooke's law) is of the form

$$t_{ij} = \rho(c_1^2 - 2c_2^2)v_{m,m}\delta_{ij} + \rho c_2^2(v_{i,j} + v_{j,i}), \quad (6.5)$$

where ρ is the mass density, and δ_{ij} is the Kronecker delta.

Equation 6.1 can be solved by the BEM method. The basic BEM formulation is omitted here, and may be found in the work by Brebbia (1984).

The fundamental solutions (or the Green's functions) are the kernels in the BEM method. For a plane strain soil space, the fundamental solutions in the frequency domain are given by (Beskos 1987; and Brabbia 1987)

$$V_{ij}^* = \frac{1}{2\pi\rho c_2^2}(\psi\delta_{ij} - \chi r_{,i}r_{,j}) \quad (6.6)$$

$$T_{ij}^* = \frac{1}{2\pi} \left[\left(\frac{d\psi}{dr} - \frac{1}{r}\chi \right) \left(\frac{\partial r}{\partial n}\delta_{ij} + r_{,j}n_i \right) - \frac{2}{r}\chi \left(n_j r_{,i} - 2r_{,i}r_{,j} \frac{\partial r}{\partial n} \right) - 2\frac{d\chi}{dr}r_{,i}r_{,j} \frac{\partial r}{\partial n} + \left(\frac{c_1^2}{c_2^2} - 2 \right) \left(\frac{d\psi}{dr} - \frac{d\chi}{dr} - \frac{1}{r}\chi \right) r_{,i}n_j \right] \quad (6.7)$$

in which, c_1 and c_2 are the velocities in soil for P and S-waves respectively; ρ is the mass density of soil; n is the unit normal vector of the element; r is the distance between the source point and the integration point, with $r = |\mathbf{r} - \mathbf{r}_0|$; V_{ij}^* and T_{ij}^* are the displacement and surface traction tensors under harmonic excitation of $e^{i\Omega t}$ at the source point, $\mathbf{r}_0$; and functions ψ and χ are related to Bessel functions. The last two can be expressed in forms of Hankel functions of the first kind (Kobayashi 1987) as follows

$$\psi = \frac{\pi}{2}i \left\{ H_0^{(1)} \left(-\frac{\Omega r}{c_2} \right) - \frac{c_2}{\Omega r} \left[H_1^{(1)} \left(-\frac{\Omega r}{c_2} \right) - \frac{c_2}{c_1} H_1^{(1)} \left(-\frac{\Omega r}{c_1} \right) \right] \right\} \quad (6.8)$$

$$\chi = -\frac{\pi}{2}i \left[H_2^{(1)} \left(-\frac{\Omega r}{c_2} \right) - \frac{c_2^2}{c_1^2} H_2^{(1)} \left(-\frac{\Omega r}{c_1} \right) \right] \quad (6.9)$$

in which $i = \sqrt{-1}$.

Once the type of boundary elements is chosen, the basic equations for BEM solution can be set up. If constant boundary elements are chosen with a uniform variation of traction and displacements being assumed over each one of these elements, the following equation is obtained from the basic theory of BEM (Karabalis and Beskos 1987)

$$0.5\tilde{\mathbf{v}}^k = \sum_{m=1}^M (\bar{\mathbf{V}}_{km}\tilde{\mathbf{t}}^m - \bar{\mathbf{T}}_{km}\tilde{\mathbf{v}}^m) \quad \text{for } k = 1, 2, 3, \dots, M \quad (6.10)$$

where $\tilde{\mathbf{v}}^k$ is the displacement vector of the “receiver” or source element k , $\tilde{\mathbf{t}}^m$ the traction vector of the “field” element m , and M the number of boundary elements. Matrices $\bar{\mathbf{V}}_{km}$ and $\bar{\mathbf{T}}_{km}$ are the two influence matrices and can be obtained by integrating the fundamental solutions in Eqs. 6.8 and 6.9 over element m . They are given by

$$\bar{\mathbf{V}}_{km} = \int_{L_m} \mathbf{V}^*(m, k, \Omega) dL_m \quad (6.11a)$$

$$\bar{\mathbf{T}}_{km} = \int_{L_m} \mathbf{T}^*(m, k, \Omega) dL_m \quad (6.11b)$$

where $\mathbf{V}^*$ and $\mathbf{T}^*$ are 2×2 matrices with their elements being fundamental solutions V_{ij}^* and T_{ij}^* respectively as given by Eqs. 6.6 and 6.7.

The evaluation of the integrals in Eqs. 6.11a and 6.11b can be carried out by using the normal four point Gaussian Quadrature method (Stroud and Secrest 1966), provided the element on which integration is carried out is not the same element as the one on which the source point is located, i.e. for $m \neq k$. When the source point lies on the element on which the integration is being carried out, attention should be paid to the singularity of the fundamental solutions in Eqs. 6.8 and 6.9. The method used in the current work to treat the problems of singularity is presented in the next section.

Equation 6.10 can be expressed in a matrix form with the complete displacement and traction vectors designated by $\tilde{\mathbf{v}} = [\tilde{v}^1, \tilde{v}^2, \dots, \tilde{v}^M]^T$ and $\tilde{\mathbf{t}} = [\tilde{t}^1, \tilde{t}^2, \dots, \tilde{t}^M]^T$ respectively. Such a form is given by

$$\bar{\mathbf{T}}'\tilde{\mathbf{v}} = \bar{\mathbf{V}}\tilde{\mathbf{t}} \quad (6.12)$$

where matrix $\bar{\mathbf{V}}$ has submatrices $\bar{\mathbf{V}}_{km}$ as its elements and the elements of matrix $\bar{\mathbf{T}}'$ are submatrices $\bar{\mathbf{T}}'_{km}$ that are obtained from

$$\mathbf{T}'_{km} = \begin{cases} \mathbf{T}_{km}, & \text{when } m \neq k \\ \bar{\mathbf{T}}_{km} + 0.5 \times \mathbf{I}, & \text{when } m = k \text{ and the source point is on the boundary} \end{cases}$$

and $\mathbf{I}$ is a 2×2 unit matrix.

Equation 6.12 gives the displacement and traction vectors over each boundary element. In order to incorporate these results for foundation soil in the analysis of the superstructure, equivalent nodal displacements and nodal forces need to be determined on each interface node and used as input in the finite element analysis of the superstructure. These equivalent nodal displacements and nodal forces can be calculated directly from the displacement and traction vectors $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{t}}$, obtained from Eq. 6.12. Relationship between the equivalent nodal displacements and displacements $\tilde{\mathbf{v}}$, as well as that between the equivalent nodal forces and tractions $\tilde{\mathbf{t}}$ can be found in literature related to the boundary element analysis of foundation soil (Karabalis and Beskos, 1987). Such relationship is different for rigid body foundation and flexible foundation. In the present work, a simple approach is used. The plane frame structure shown in Fig. 6.1 is used to illustrate the typical soil-structure interaction problems. The columns of the frame are assumed to rest directly on the soil. The lower end of each column coincides with an interface node between the soil and the structure. At these interface nodes, only the translation degrees of freedom along the vertical and horizontal directions are taken into account in the computation. When the boundary elements used in the analysis of foundation soil are constant elements, the displacements and tractions obtained from Eq. 6.12 are assumed to be uniformly distributed over the boundary elements. With the

above assumptions, the equivalent nodal displacements and forces on the interface nodes can be easily obtained from the displacement and traction vectors $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{t}}$ respectively. Let $\mathbf{v}^m = [u_x^m, u_y^m]^T$ and $\mathbf{t}^m = [F_x^m, F_y^m]^T$ be the nodal displacement and nodal force vectors respectively for element m , and $\mathbf{v}$ and $\mathbf{t}$ the nodal displacement vector and the complete nodal force vector respectively. Thus, $\mathbf{v}$ and $\mathbf{t}$ are given by

$$\mathbf{v} = [\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^M]^T \quad (6.13a)$$

$$\mathbf{t} = [\mathbf{t}^1, \mathbf{t}^2, \dots, \mathbf{t}^M]^T \quad (6.13b)$$

It is obvious that the nodal displacements on the interface nodes of the soil and the structure are identical. Therefore, the relationship between $\tilde{\mathbf{v}}$ and $\mathbf{v}$ can be expressed as

$$\mathbf{v} = \tilde{\mathbf{v}} \quad (6.14a)$$

The equivalent nodal forces are derived from the resultants of the tractions on the corresponding boundary elements. Since constant boundary elements are used, these tractions are uniformly distributed over the element. Thus the relationship between $\tilde{\mathbf{t}}$ and $\mathbf{t}$ is given by

$$\mathbf{t} = \mathbf{L}\tilde{\mathbf{t}} \quad (6.14b)$$

where $\mathbf{L}$ is a diagonal matrix, whose elements are equal to the lengths of the corresponding boundary elements. If all boundary elements are assumed to be of a constant length l , Eq. 6.14b can be simplified as

$$\mathbf{t} = l\tilde{\mathbf{t}} \quad (6.15)$$

Substitution of Eqs. 6.14a and 6.15 into Eq. 6.12 gives

$$\mathbf{T}\mathbf{v} = \mathbf{V}\mathbf{t} \quad (6.16)$$

where

$$\mathbf{T} = \bar{\mathbf{T}}'/l$$

and

$$\mathbf{V} = \bar{\mathbf{V}}$$

6.2.1 Treatment of Singular Boundary Integrals

As stated above, singularity occurs in the fundamental solutions when the source point is on the same element on which the boundary integration is being carried out. In this case, the normal Gaussian Quadrature method fails to give a correct evaluation of the boundary integrals in Eqs. 6.11a and 6.11b, and some other method has to be used. The singularities in Eqs. 6.6 and 6.7 are caused by the singularity of the Bessel functions. It actually stems from Bessel functions of the second kind, such as Y_0 , Y_1 and Y_2 , since Bessel functions of the first kind are not singular. To elaborate, let Eqs. 6.8 and 6.9 be examined. The relationship between the Hankel function of the first kind and the Bessel functions is given by

$$H_n^{(1)}(x) = J_n(x) + iY_n(x) \quad (6.17)$$

Bessel function Y_0 has $\ln(x)$ order of singularity and it can be expressed in the following form

$$Y_0 = g_0(x) + f_0(x)\ln(x) \quad (6.18)$$

Bessel function Y_1 has a singularity of order $(\frac{1}{x})$ as well as of order $\ln(x)$, and it can be separated into three parts as follows

$$Y_1 = g_1(x) + f_1(x)\ln(x) + C1\frac{1}{x} \quad (6.19)$$

Bessel function Y_2 has a singularity of order $(\frac{1}{x^2})$ as well as of order $\ln(x)$, and it can be also separated into three parts as follows

$$Y_2 = g_2(x) + f_2(x)\ln(x) + C2\frac{1}{x^2} \quad (6.20)$$

where $C1, C2$ are constants. Furthermore, it can be easily proven that the following relationships hold

$$\begin{aligned} f_0(x) &= \frac{2}{\pi}J_0(x) \\ f_1(x) &= \frac{2}{\pi}J_1(x) \\ f_2(x) &= \frac{2}{\pi}J_2(x) \end{aligned} \quad (6.21)$$

After substitution of these relationship into Eqs. 6.8 and 6.9, the singularities of order $(\frac{1}{x})$ in Eq. 6.8 cancel each other since

$$C1\frac{1}{(-\frac{\Omega r}{c_2})} - C1\frac{c_2}{c_1}\frac{1}{(-\frac{\Omega r}{c_1})} = 0$$

The singularities of order $(\frac{1}{x^2})$ in Eq. 6.9 also cancel each other since

$$C2\frac{1}{(-\frac{\Omega r}{c_2})^2} - C2\frac{c_2^2}{c_1^2}\frac{1}{(-\frac{\Omega r}{c_1})^2} = 0$$

As a result, the singularities of orders $(\frac{1}{x})$ and $(\frac{1}{x^2})$ disappear from functions ψ and χ . The singularities retained in both ψ and χ are thus of order, $\ln(x)$. The two functions can then be expressed in the following forms

$$\psi = g_\psi + f_\psi\ln(r) \quad (6.23)$$

and

$$\chi = g_\chi + f_\chi\ln(r) \quad (6.24)$$

in which

$$f_\psi = -J_0\left(-\frac{\Omega r}{c_2}\right) - \frac{c_2}{\Omega r} \left[J_1\left(-\frac{\Omega r}{c_2}\right) - \frac{c_2}{c_1} J_1\left(-\frac{\Omega r}{c_1}\right) \right] \quad (6.24)$$

and

$$f_x = \left[J_2\left(-\frac{\Omega r}{c_2}\right) - \frac{c_2^2}{c_1^2} J_2\left(-\frac{\Omega r}{c_1}\right) \right] \quad (6.25)$$

Noting that the boundary integrals in Eq. 6.11a are always zero when the source point is taken on the same element on which the integration is carried out (Beskos 1987), only the singular integration in Eq. 6.11b needs to be evaluated. The fundamental solution V_{ij}^* can also be separated into two parts as follows

$$V_{ij}^* = V_{gij}^* + V_{fij}^* \ln(r) \quad (6.26)$$

where V_{gij}^* and V_{fij}^* are regular and the singularity is represented by $\ln(r)$. Also,

$$V_{fij}^* = \frac{1}{2\pi\rho c_2^2} (f_\psi \delta_{ij} - f_x r_{,i} r_{,j}) \quad (6.27)$$

Therefore, the boundary integrals of Eq. 6.16 for $m = k$ can be written as follows

$$\mathbf{V}_{mm} = \int_{L_m} \mathbf{V}_g^*(m, m, \Omega) dL_m + \int_{L_m} \mathbf{V}_f^*(m, m, \Omega) \ln(r) dL_m \quad (6.28)$$

where the following relationship has been used

$$\mathbf{V}^* = \mathbf{V}_g^* + \mathbf{V}_f^* \ln(r) \quad (6.29)$$

Matrices $\mathbf{V}_g^*$ and $\mathbf{V}_f^*$ have the elements of V_{gij}^* and V_{fij}^* respectively. The first part of integral on the right in Eq. 6.23 can still be evaluated by using the normal four point Gaussian Quadrature method, but the second part of the integral has to be evaluated by using the four point Logarithmic Gaussian Quadrature method.

The treatment of singular integrals here does not depend on any simplification in the basic fundamental solutions. This allows more accurate evaluation of those

integrals, which is an important consideration in a correct implementation of the BEM, since the values of the singular integrals may contribute much more to the singular points in the final frequency response functions of soil than the regular integrals.

6.3 EXAMPLE SOIL-STRUCTURE INTERACTION PROBLEM

To illustrate the concepts and the methodology involved in the frequency domain analysis of soil-structure problem, a multi-storey frame resting on a flexible foundation is analyzed for earthquake motion. Figure 6.1 shows the superstructure and the soil foundation. One dimensional constant boundary elements are used here. This system as well as the size and the number of the one dimensional boundary elements given in this figure are used in subsequent analysis. For simplicity, only the translation degrees of freedom are considered in this example. The rotational displacement at every joint is locked. A complete system with all rotation degrees of freedom considered is analyzed in Chapter 8.

6.3.1 Frequency Response Functions for Soil

For each value of Ω , the matrices $\mathbf{T}$ and $\mathbf{V}$ are obtained from Eq. 6.16. These two matrices together represent the dynamic stiffness for foundation soil. In order to incorporate the effect of foundation soil into the response of the superstructure, a soil impedance matrix is required with reference to the degrees of freedom along the interface between the superstructure and the soil. The derivation of this matrix can be illustrated with reference to the frame shown in Fig. 6.2, where the soil and the structure are in contact at nodal points 2, 3, 4 and 5. Interaction forces exist at these contact points. Forces on rest of the nodal points on the soil (nodes 1 and 6 in this case) are zero since these points are located on a free boundary. The interaction forces on the interface are indicated by f_h and f_v respectively in Fig. 6.2. Let vector $\mathbf{F}_f(\Omega)$ denote the frequency dependent interaction forces. The

vector of forces on all nodes of the soil $\mathbf{F}_g(\Omega)$ is related to $\mathbf{F}_f(\Omega)$ by the following equation

$$\mathbf{F}_g(\Omega) = \mathbf{r}_b \mathbf{F}_f(\Omega) \quad (6.30a)$$

$$\mathbf{F}_g(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{h2} \\ f_{v2} \\ f_{h3} \\ f_{v3} \\ f_{h4} \\ f_{v4} \\ f_{v5} \\ f_{v5} \end{bmatrix} \quad (6.30b)$$

where $\mathbf{r}_b$ is a spatial distribution matrix.

The displacements on the soil nodes, $\mathbf{V}_g(\Omega)$, resulting from the application of forces $\mathbf{F}_g(\Omega)$ can be derived by using Eq. 6.16 which yields

$$\mathbf{V}_g(\Omega) = \mathbf{T}^{-1} \mathbf{V} \mathbf{r}_b \mathbf{F}_f(\Omega) \quad (6.31)$$

The displacements on the interface nodes are now given by

$$\begin{aligned} \mathbf{V}_f(\Omega) &= \mathbf{r}_b^T \mathbf{V}_g(\Omega) \\ &= (\mathbf{r}_b^T \mathbf{T}^{-1} \mathbf{V} \mathbf{r}_b) \mathbf{F}_f(\Omega) \end{aligned} \quad (6.32a)$$

or

$$\begin{aligned} \mathbf{F}_f(\Omega) &= (\mathbf{r}_b^T \mathbf{T}^{-1} \mathbf{V} \mathbf{r}_b)^{-1} \mathbf{V}_f(\Omega) \\ &= \mathbf{S}_{ff}(\Omega) \mathbf{V}_f(\Omega) \end{aligned} \quad (6.32b)$$

where $\mathbf{T}$ and $\mathbf{V}$ are, of course, functions of Ω and $\mathbf{S}_{ff}(\Omega)$ is the frequency dependent soil impedance matrix.

6.3.2 Finite Element Representation of Superstructure

The superstructure is a five storeys high, two-dimensional frame as shown in Fig. 6.3 and represented by 35 finite elements with 48 degrees of freedom. The 'SAP IV' finite element program is used in the analysis. The moment of inertia I for all beams is taken as $I = 0.0108 \text{ m}^4$, the area of each beams $A = 0.36 \text{ m}^2$, the Young's modulus $E_s = 2.5 \times 10^{10} \text{ Pa}$, and the mass density $\rho = 2400 \text{ kg/m}^3$. In the example, since the rotational displacement at every joint is locked for simplicity, a 48×48 stiffness matrix and a 48×48 diagonal mass matrix are obtained when node 1, 2, 3, and 4 are released. It should be noted that the stiffness matrix obtained at this stage is singular and includes rigid body motions.

6.4 FORMULATION OF SOIL STRUCTURE INTERACTION

The formulation of the equations of motion for a soil-structure system has been well described in the literature (Vaish and Chopra, 1973; and Wolf, 1985). However, for the sake of completeness, a brief review is presented below.

For an earthquake excitation, the equations of motion for a structure resting on a flexible foundation are given by

$$\begin{aligned} & \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sb} \\ \mathbf{M}_{bs} & \mathbf{M}_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{v}}_s^t \\ \ddot{\mathbf{v}}_b^t \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sb} \\ \mathbf{C}_{bs} & \mathbf{C}_{bb} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}}_s^t \\ \dot{\mathbf{v}}_b^t \end{Bmatrix} \\ & + \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sb} \\ \mathbf{K}_{bs} & \mathbf{K}_{bb} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_s^t \\ \mathbf{v}_b^t \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{R}_b \end{Bmatrix} \end{aligned} \quad (6.33)$$

where $\mathbf{v}^T = [(\mathbf{v}_s^t)^T (\mathbf{v}_b^t)^T]$ is the vector of absolute displacements of the structure; $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the mass, damping, stiffness matrices expressed in their partition form and $\mathbf{R} = [\mathbf{0} \ \mathbf{R}_b^T]$ is the vector of applied forces. Subscript **b** refers to structure degrees of freedom along the interface between the structure and the soil, and **s** refers to the remaining structure degrees of freedom. For an earthquake excitation, the forces along only the interface degrees of freedom are nonzeros. Thus

$\mathbf{R}_b$ represents the forces applied by the soil foundation on the structure along the contact degrees of freedom.

The free-field motion is defined as the motion of the soil along the interface degrees of freedom caused by the earthquake excitation when the structure is not present. The free-field displacements along the interface degrees of freedom are represented by $\mathbf{v}_{f0}$. In reality, the free field displacements are different at different degrees of freedom. In other words, the elements of $\mathbf{v}_{f0}$ are all distinct. In practice, because spatial variation of free-field motion is not well known, it is usual to take the free field motion as being uniform at all interface degrees of freedom.

When the free-field motion is applied statically to the interface degrees of freedom, the structure undergoes a set of displacements called pseudo-static displacements denoted by

$$\mathbf{v}_0 = \begin{bmatrix} \mathbf{v}_{s0} \\ \mathbf{v}_{b0} \end{bmatrix} \quad (6.34)$$

The total displacements can be expressed as the sum of pseudo-static displacements caused by free-field motion and displacements relative to the latter. Denoting the relative displacements by $\mathbf{v}$, we have

$$\mathbf{v}^t = \mathbf{v} + \mathbf{v}_0$$

or

$$\begin{bmatrix} \mathbf{v}_s^t \\ \mathbf{v}_b^t \end{bmatrix} = \begin{bmatrix} \mathbf{v}_s \\ \mathbf{v}_b \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{s0} \\ \mathbf{v}_{b0} \end{bmatrix} \quad (6.35)$$

Substitution of Eq. 6.35 into Eq. 6.33 gives

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{v}} + \mathbf{C}\dot{\mathbf{v}} + \mathbf{K}\mathbf{v} \\ = -(\mathbf{M}\ddot{\mathbf{v}}_0 + \mathbf{C}\dot{\mathbf{v}}_0 + \mathbf{K}\mathbf{v}_0) + \mathbf{R} \end{aligned} \quad (6.36)$$

When the free field displacements are the same at all interface degrees of freedom, vector $\mathbf{v}_0$ represents rigid body motion and $\mathbf{K}\mathbf{v}_0 = 0$. Also, if $\mathbf{C}$ is assumed to

be proportional to $\mathbf{K}$, $\mathbf{C}\mathbf{v}_0 = 0$. Equation 6.36 then reduces to

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sb} \\ \mathbf{M}_{bs} & \mathbf{M}_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{v}}_s \\ \ddot{\mathbf{v}}_b \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sb} \\ \mathbf{C}_{bs} & \mathbf{C}_{bb} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}}_s \\ \dot{\mathbf{v}}_b \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sb} \\ \mathbf{K}_{bs} & \mathbf{K}_{bb} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_s \\ \mathbf{v}_b \end{Bmatrix} \\ = - \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sb} \\ \mathbf{M}_{bs} & \mathbf{M}_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{v}}_{s0} \\ \ddot{\mathbf{v}}_{b0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{R}_b \end{Bmatrix} \end{aligned} \quad (6.37)$$

In the frequency domain, that is when the free field excitation is taken as $\mathbf{v}_{s0} = \mathbf{V}_{s0}(\Omega)e^{i\Omega t}$, Eq. 6.37 reduces to

$$\begin{aligned} \left(-\Omega^2 \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sb} \\ \mathbf{M}_{bs} & \mathbf{M}_{bb} \end{bmatrix} + i\Omega \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sb} \\ \mathbf{C}_{bs} & \mathbf{C}_{bb} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sb} \\ \mathbf{K}_{bs} & \mathbf{K}_{bb} \end{bmatrix} \right) \begin{bmatrix} \mathbf{V}_s(\Omega) \\ \mathbf{V}_b(\Omega) \end{bmatrix} \\ = - \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sb} \\ \mathbf{M}_{bs} & \mathbf{M}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{V}}_{s0}(\Omega) \\ \ddot{\mathbf{V}}_{b0}(\Omega) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{R}_b(\Omega) \end{bmatrix} \end{aligned} \quad (6.38)$$

where

$$\mathbf{R}_b = \bar{\mathbf{R}}_b(\Omega)e^{i\Omega t} \quad (6.39)$$

Equilibrium and compatibility require

$$\begin{aligned} \mathbf{V}_{b0}(\Omega) &= \mathbf{V}_{f0}(\Omega) \\ \mathbf{V}_b(\Omega) &= \mathbf{V}_f(\Omega) \\ \bar{\mathbf{R}}_b(\Omega) - \mathbf{F}_f(\Omega) &= 0 \end{aligned} \quad (6.40)$$

Substitution of Eqs 6.32b and 6.40 in Eq. 6.38 gives

$$\begin{aligned} \left(-\Omega^2 \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sb} \\ \mathbf{M}_{bs} & \mathbf{M}_{bb} \end{bmatrix} + i\Omega \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sb} \\ \mathbf{C}_{bs} & \mathbf{C}_{bb} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sb} \\ \mathbf{K}_{bs} & \mathbf{K}_{bb} + \mathbf{S}_{ff}(\Omega) \end{bmatrix} \right) \begin{bmatrix} \mathbf{V}_s(\Omega) \\ \mathbf{V}_b(\Omega) \end{bmatrix} \\ = - \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sb} \\ \mathbf{M}_{bs} & \mathbf{M}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{V}}_{s0}(\Omega) \\ \ddot{\mathbf{V}}_{b0}(\Omega) \end{bmatrix} \end{aligned} \quad (6.41)$$

For identical free field motion at all interface degrees of freedom, the pseudo static displacements can be expressed as

$$\ddot{\mathbf{V}}_0 = \mathbf{r}_h A_h + \mathbf{r}_v A_v \quad (6.42)$$

where A_h and A_v are respectively amplitudes of horizontal and vertical free field accelerations and $\mathbf{r}_h$ and $\mathbf{r}_v$ are distribution vectors. For the frame of Fig. 6.2, they are given by

$$\begin{aligned}\mathbf{r}_h &= [1 \ 0 \ 1 \ 0 \ \dots \ 1 \ 0]^T \\ \mathbf{r}_v &= [1 \ 0 \ 1 \ 0 \ \dots \ 1 \ 0]^T\end{aligned}\tag{6.43}$$

The dynamic problem represented by Eq. 6.41 is of a large size, and its solution therefore involves a large volume of computation. It is possible to reduce the size of the problem by a Ritz transformation given by

$$\mathbf{V}(\Omega) = \mathbf{X}\mathbf{Z}(\Omega)\tag{6.44}$$

in which $\mathbf{X}$ is the matrix of Ritz vectors and $\mathbf{Z}(\Omega)$ are the generalized coordinates. The size of $\mathbf{X}$ is $N \times M$, where N is the total number of structure degrees of freedom and M is the number of Ritz vectors.

The transformed equations are given by

$$\begin{aligned}\left[-\Omega^2 \mathbf{M}^* + i\Omega \mathbf{C}^* + \mathbf{K}^* + \mathbf{L}(\Omega)\right] \mathbf{Z}(\Omega) \\ = -\mathbf{X}^T \mathbf{M} \mathbf{r}_h A_h - \mathbf{X}^T \mathbf{M} \mathbf{r}_v A_v\end{aligned}\tag{6.45}$$

where $\mathbf{M}^* = \mathbf{X}^T \mathbf{M} \mathbf{X}$, $\mathbf{C}^* = \mathbf{X}^T \mathbf{C} \mathbf{X}$, $\mathbf{K}^* = \mathbf{X}^T \mathbf{K} \mathbf{X}$, and

$$\begin{aligned}\mathbf{L}(\Omega) &= \mathbf{X}^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{ff} \end{bmatrix} \mathbf{X} \\ &= \mathbf{X}_{sb}^T \mathbf{S}_{ff} \mathbf{X}_{sb}\end{aligned}\tag{6.46}$$

where $\mathbf{X}_{sb}$ is obtained from $\mathbf{X}$ by taking the rows corresponding to the interface degrees of freedom.

When only the horizontal excitation is taken $A_v = 0$ and when only the vertical excitation is taken $A_h = 0$. For the special case when the excitation has a unit amplitude, the generalized coordinates $\mathbf{Z}(\Omega)$ are referred to as the complex frequency response function and denoted by $\mathbf{H}(\Omega)$. The specialized form of Eq. 6.45 is

$$\left[-\Omega^2 \mathbf{M}^* + i\Omega \mathbf{C}^* + \mathbf{K}^* + \mathbf{L}(\Omega)\right] \mathbf{H}(\Omega) = -\mathbf{X}^T \mathbf{F} \quad (6.47)$$

where $\mathbf{F}$ is either $\mathbf{F}_h = \mathbf{M} \mathbf{r}_h$ or $\mathbf{F}_v = \mathbf{M} \mathbf{r}_v$.

The Ritz vectors to be used in the transformation of Eq. 6.44 can be obtained by using the following modified stiffness matrix

$$\tilde{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sb} \\ \mathbf{K}_{bs} & \mathbf{K}_{bb} + \mathbf{S}_{ff}(\epsilon(\Omega)) \end{bmatrix} \quad (6.48)$$

where $\epsilon(\Omega)$ is a very small value of Ω . In the example being considered, $\epsilon(\Omega)$ is taken as 0.001 and the Ritz vectors are generated by considering the eigenvalue problem given by

$$(-\omega^2 \mathbf{M} + \tilde{\mathbf{K}}) \mathbf{Z} = 0 \quad (6.49)$$

Other types of Ritz vectors may be used and will be described in subsequent sections of this chapter.

6.5 TWO STAGE RITZ VECTOR CONCEPT

The Ritz vectors used in the preceding section are obtained by using the modified stiffness matrix given by Eq. 6.48. The modified stiffness matrix $\tilde{\mathbf{K}}$ is an unsymmetric matrix. Therefore, a half bandwidth storage of $\tilde{\mathbf{K}}$ is not possible, neither is the $\mathbf{LDL}^T$ decomposition. Symmetry of stiffness matrix can lead to considerable saving in computing time in generating Ritz vectors. It is a drawback of using $\tilde{\mathbf{K}}$ that such economy can not be realized. Some other disadvantages of Ritz vectors generated by using $\tilde{\mathbf{K}}$ are mentioned in the introduction. However, these disadvantages can be avoided if the Ritz reduction is only applied to part of the structure system. The idea of using mode shapes of part of the structure system can be found in the work by Vaish and Chopra (1973), and Gutierrez and Chopra (1978). In their work, Gutierrez and Chopra (1978) considered the base to be rigid and obtained good results by including several rigid deformation modes

for the foundation. However, when a flexible foundation was investigated as in the work by Vaish and Chopra (1973), significant errors were produced when partial mode shapes of the substructure on a rigid base were used in the reduction. The author believes that those errors were not actually caused by the use of partial mode shapes of the system, but because of the assumption of zero displacements for all degrees of freedom of the foundation. A new technique is used here to avoid such an assumption and a set of effective partial Ritz vectors is presented here.

The mass and stiffness matrices of the superstructure including the interface degrees of freedom are shown in partitioned form in Eq. 6.33. The complete stiffness matrix $\mathbf{K}$ in Eq. 6.33 is singular with rigid body motion included. However, the submatrix $\mathbf{K}_{ss}$ is not singular.

Let $\mathbf{X}_s$ be the Ritz vectors generated by using the stiffness matrix of the superstructure with all the supports fixed. The Ritz vectors $\mathbf{X}_s$ are then equivalent to those generated based on the following eigenvalue problem

$$(-\omega^2 \mathbf{M}_{ss} + \mathbf{K}_{ss}) \mathbf{Z}_s = 0 \quad (6.50)$$

Matrix $\mathbf{K}_{ss}$ is usually a positive definite, symmetric, sparse matrix. These properties allow the use of half bandwidth storage and $\mathbf{LDL}^T$ decomposition, and thus greatly enhance the computing efficiency when the order of matrix $\mathbf{K}_{ss}$ is very large.

A set of Ritz vectors that contains the partial Ritz vectors $\mathbf{X}_s$ may be expressed as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_s & \mathbf{X}_c \\ \mathbf{0} & \mathbf{I}_b \end{bmatrix} \quad (6.51)$$

in which $\begin{bmatrix} \mathbf{X}_c \\ \mathbf{I}_b \end{bmatrix}$ represents the displacements at each degree of freedom caused by a unit displacement applied in turn to each of the fixed degrees of freedom at the soil-structure interface. Matrix $\begin{bmatrix} \mathbf{X}_c \\ \mathbf{I}_b \end{bmatrix}$ is, in fact, the so-called constraint Ritz vector matrix discussed in Chapter 8.

$$\begin{bmatrix} \mathbf{X}_c \\ \mathbf{I}_b \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{ss}^{-1} \mathbf{K}_{sb} \\ \mathbf{I}_b \end{bmatrix} \quad (6.52)$$

Transformation using the Ritz vectors given in Eq. 6.51 yields following reduced mass, stiffness and damping matrices

$$\begin{aligned} \mathbf{M}^* &= \mathbf{X}^T \mathbf{M} \mathbf{X} \\ &= \begin{bmatrix} \mathbf{X}_s^T \mathbf{M}_{ss} \mathbf{X}_s & \mathbf{X}_s^T \mathbf{M}_{ss} \mathbf{X}_c + \mathbf{X}_s^T \mathbf{M}_{sb} \\ \mathbf{X}_c^T \mathbf{M}_{ss} \mathbf{X}_s + \mathbf{M}_{bs} \mathbf{X}_s & \mathbf{X}_c^T \mathbf{M}_{ss} \mathbf{X}_c + \mathbf{M}_{bs} \mathbf{X}_c + \mathbf{X}_c^T \mathbf{M}_{sb} + \mathbf{M}_{bb} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{M}_{ss}^* & \mathbf{M}_{sb}^* \\ \mathbf{M}_{bs}^* & \mathbf{M}_{bb}' + \mathbf{M}_{bb} \end{bmatrix} \end{aligned} \quad (6.53a)$$

$$\begin{aligned} \mathbf{K}^* &= \mathbf{X}^T \mathbf{K} \mathbf{X} \\ &= \begin{bmatrix} \mathbf{X}_s^T \mathbf{K}_{ss} \mathbf{X}_s & \mathbf{X}_s^T \mathbf{K}_{ss} \mathbf{X}_c + \mathbf{X}_s^T \mathbf{K}_{sb} \\ \mathbf{X}_c^T \mathbf{K}_{ss} \mathbf{X}_s + \mathbf{K}_{bs} \mathbf{X}_s & \mathbf{X}_c^T \mathbf{K}_{ss} \mathbf{X}_c + \mathbf{K}_{bs} \mathbf{X}_c + \mathbf{X}_c^T \mathbf{K}_{sb} + \mathbf{K}_{bb} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{K}_{ss}^* & \mathbf{K}_{sb}^* \\ \mathbf{K}_{bs}^* & \mathbf{K}_{bb}' + \mathbf{K}_{bb} \end{bmatrix} \end{aligned} \quad (6.53b)$$

$$\begin{aligned} \mathbf{C}^* &= \mathbf{X}^T \mathbf{C} \mathbf{X} \\ &= \begin{bmatrix} \mathbf{X}_s^T \mathbf{C}_{ss} \mathbf{X}_s & \mathbf{X}_s^T \mathbf{C}_{ss} \mathbf{X}_c + \mathbf{X}_s^T \mathbf{C}_{sb} \\ \mathbf{X}_c^T \mathbf{C}_{ss} \mathbf{X}_s + \mathbf{C}_{bs} \mathbf{X}_s & \mathbf{X}_c^T \mathbf{C}_{ss} \mathbf{X}_c + \mathbf{C}_{bs} \mathbf{X}_c + \mathbf{X}_c^T \mathbf{C}_{sb} + \mathbf{C}_{bb} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{C}_{ss}^* & \mathbf{C}_{sb}^* \\ \mathbf{C}_{bs}^* & \mathbf{C}_{bb}' + \mathbf{C}_{bb} \end{bmatrix} \end{aligned} \quad (6.53c)$$

Equation 6.47 can thus be expressed as

$$\begin{aligned} &\left\{ -\Omega^2 \begin{bmatrix} \mathbf{M}_{ss}^* & \mathbf{M}_{sb}^* \\ \mathbf{M}_{bs}^* & \mathbf{M}_{bb}' + \mathbf{M}_{bb} \end{bmatrix} + i\Omega \begin{bmatrix} \mathbf{C}_{ss}^* & \mathbf{C}_{sb}^* \\ \mathbf{C}_{bs}^* & \mathbf{C}_{bb}' + \mathbf{C}_{bb} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ss}^* & \mathbf{K}_{sb}^* \\ \mathbf{K}_{bs}^* & \mathbf{K}_{bb}' + \mathbf{K}_{bb} \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{ff}(\Omega) \end{bmatrix} \right\} \begin{Bmatrix} \mathbf{H}_s^* \\ \mathbf{H}_b^* \end{Bmatrix} = - \begin{Bmatrix} \mathbf{X}_s^T \mathbf{F}_s \\ \mathbf{F}_b \end{Bmatrix} \end{aligned} \quad (6.54)$$

The transformation described above is different from the one used by Vaish and Chopra (1973), but is that obtained in substructure coupling method presented by Benfield and Hrudá (1971). It is found that if an adequate number of Ritz vectors are used, the transformation described by Eq. 6.54 gives quite an accurate representation of the original system provided the interface displacements, or

their counterpart in frequency domain H_b , are not taken as zeros. This will be demonstrated by examples presented later.

It is apparent that the Ritz vectors in Eq. 6.51 only reduce the size of one substructure keeping the number of degrees of freedom of another substructure (soil foundation) intact, and if the number of Ritz vectors used in Eq. 6.54 equals the total number of degrees of freedom of the superstructure, Eq. 6.54 is completely equivalent to the equations of motion of the original system without transformation. Compared with the transformed problem obtained by using Ritz vectors described in the preceding section, which reduce the size of both substructures, the problem obtained by using the Ritz vectors in Eq. 6.51 is somewhat larger in size. Therefore, more computing time is required in solving the reduced equations in the latter analysis. However, the saving in computing time realized in the process of generating Ritz vectors of Eq. 6.51 will offset part of the extra time required in the complete analysis process. If the number of degrees of freedom on the interaction boundary is small, it may even be possible to achieve a net saving in computing time by using the Ritz vectors of Eq. 6.51.

When the number of interface nodes is comparatively large, the idea of using two stage Ritz vector proposed here to further reduce the size of Eq. 6.54 may be useful. The reduced equations take the form

$$[-\Omega^2 \mathbf{M}^{**} + i\Omega \mathbf{C}^{**} + \mathbf{K}^{**} + \mathbf{L}^{**}(\Omega)] \mathbf{H}^{**}(\Omega) = -\mathbf{F}^{**} \quad (6.55)$$

in which

$$\mathbf{M}^{**} = \mathbf{X}_2^T \begin{bmatrix} \mathbf{M}_{ss}^{*} & \mathbf{M}_{sb}^{*} \\ \mathbf{M}_{bs}^{*} & \mathbf{M}_{bb}' + \mathbf{M}_{bb} \end{bmatrix} \mathbf{X}_2 \quad (6.56a)$$

$$\mathbf{C}^{**} = \mathbf{X}_2^T \begin{bmatrix} \mathbf{C}_{ss}^{*} & \mathbf{C}_{sb}^{*} \\ \mathbf{C}_{bs}^{*} & \mathbf{C}_{bb}' + \mathbf{C}_{bb} \end{bmatrix} \mathbf{X}_2 \quad (6.56b)$$

$$\mathbf{K}^{**} = \mathbf{X}_2^T \begin{bmatrix} \mathbf{K}_{ss}^{*} & \mathbf{K}_{sb}^{*} \\ \mathbf{K}_{bs}^{*} & \mathbf{K}_{bb}' + \mathbf{K}_{bb} \end{bmatrix} \mathbf{X}_2 \quad (6.56c)$$

$$\mathbf{L}^{**}(\Omega) = \mathbf{X}_2^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{ff}(\Omega) \end{bmatrix} \mathbf{X}_2 \quad (6.56d)$$

and

$$\mathbf{F}^{**} = \mathbf{X}_2^T \begin{Bmatrix} \mathbf{X}_s^T \mathbf{F}_s \\ \mathbf{F}_b \end{Bmatrix} \quad (6.56f)$$

where $\mathbf{X}_2$ are the second stage Ritz vectors obtained from the mass matrix $\mathbf{M}^*$ and the following modified stiffness matrix

$$\tilde{\mathbf{K}}^* = \begin{bmatrix} \mathbf{K}_{ss}^* & \mathbf{K}_{sb}^* \\ \mathbf{K}_{bs}^* & \mathbf{K}_{bb}' + \mathbf{K}_{bb} + \mathbf{S}_{ff}(\epsilon(\Omega)) \end{bmatrix} \quad (6.57)$$

In Eq. 6.57, $\epsilon(\Omega)$ indicates a small value of Ω . In this chapter, it is taken as 0.001. The Ritz vectors are generated by considering the eigenvalue problem given by

$$(-\omega^2 \mathbf{M}^* + \tilde{\mathbf{K}}^*) \mathbf{Z}^* = \mathbf{0} \quad (6.58)$$

The generation of Ritz vectors, $\mathbf{X}_s$ in Eq. 6.51 requires information related to the superstructure only and is not dependent on a knowledge of the properties of foundation soil. Matrix $\mathbf{X}_s$ thus needs to be obtained only once from $\mathbf{M}_{ss}$ and $\mathbf{K}_{ss}$, respectively the mass and stiffness matrices for the superstructure with the interface nodes fixed. The matrix $\tilde{\mathbf{K}}^*$ used in the generation of the second stage Ritz vectors, $\mathbf{X}_2$, is unsymmetric and neither half band storage nor $\mathbf{LDL}^T$ decomposition is possible. Also, changes in the properties of foundation soil requires the regeneration of $\mathbf{X}_2$ by considering the eigenvalue problem of Eq. 6.58. However, the eigenvalue problem represented by Eq. 6.58 is of quite a small size, and the generation of Ritz vectors based on it will need only a very small fraction of total computation time. As stated earlier, an efficient procedure is used in the generation of Ritz vectors $\mathbf{X}_s$. The use of two stage Ritz vector technique described can further improve the efficiency of computation. The Ritz vectors used in Eq. 6.47 are based on a nonsymmetric modified stiffness matrix $\tilde{\mathbf{K}}$ as given in Eq. 6.48. That matrix is of a size equal to the number of degrees of freedom of the complete soil-structure

system. Matrix $\tilde{\mathbf{K}}$ also depends on the properties of foundation soil. Therefore, the generation of the Ritz vectors used in Eq. 6.47 has to be repeated each time the properties of foundation soil change. In addition, significantly greater computation effort is required in generating these Ritz vectors since $\mathbf{LDL}^T$ decomposition is not applicable.

6.6 APPLICATION OF ARTIFICIAL DAMPING METHOD

In Chapter 5, it was shown that the DFT method can not always provide accurate response results for dam-reservoir systems. If enough damping does not exist in such a system, the responses obtained by using the DFT method may be in severe error. Soil-structure interaction systems are similar to the dam-reservoir systems in the sense there is interaction between substructures. To ensure that the correct response of the soil-structure system is obtained, the method of artificial damping proposed in Chapter 4 could be used instead of the standard DFT method. To use the method of artificial damping, the modified frequency response functions $\hat{\mathbf{H}}(\Omega)$ are required in place of $\mathbf{H}(\Omega)$. When the BEM method is used to obtain the frequency functions all that required is to evaluate the Bessel functions involved in the analysis for $(\Omega - ia)$ in place of Ω . Since the infinite domain of soil does provide a certain amount of damping in the system through radiation of waves to infinity, the standard DFT method can also be used provided a sufficiently long period is selected. The results obtained by the method of artificial damping can thus be compared with those obtained by standard DFT method. This is illustrated in the examples presented later in this chapter.

6.7 RESULTS OF RESPONSE ANALYSIS

6.7.1 Validity of The Method of Artificial Damping

The structure shown in Fig. 6.1 is analyzed by following the formulation leading to Eq. 6.47. The system is reduced to a five degree-of-freedom system by using 5 Ritz vectors of the type described in Section 6.4. The properties of soil are

taken as $c_1 = 1124.0$ m/s, $c_2 = 569.3$ m/s, mass density $\rho = 2160$ kg/m³. Six constant boundary elements, each of them 10.0 m in length, are selected. The ground acceleration in Fig. 6.4 is used for horizontal excitation; vertical ground acceleration is not considered. In this analysis, no damping is assumed to exist in the building frame. Therefore, the only damping in the system arises from radiation damping due to the waves propagating to infinity in soil foundation. Figures. 6.5a, 6.5b, and 6.6c, which display the displacement responses along degrees of freedom 1, 23, and 45 respectively, show that the results obtained by using DFT method are in error compared with the results obtained by using the method of artificial damping. It will be shown later that the method of artificial damping gives accurate results. The results referred to above are obtained with the time period taken as $T_0 = 40.96$ seconds, and the total number of time steps N selected to be 2048. The sampling interval for frequency is $\Delta\Omega = 0.1534$ rad/s and the folding frequency is $\Omega_0/2 = 156.08$ rad/s which well covers the natural frequency range of the structure analyzed. In the method of artificial damping, parameter a is taken as 0.1.

The errors caused by using the standard DFT method become greater as the time period selected for computation becomes shorter. Figure 6.6a, 6.6b and 6.6c show comparisons between the results obtained by using the DFT method and those obtained by using the method of artificial damping when the total time period is $T_0 = 20.48$ s. These figures also display displacement responses along degrees of freedom 1, 23, and 45 respectively. In this case, $N = 1024$ and $\Delta\Omega = 0.3068$ rad/second, and parameter value $a = 0.2$ is used in the method of artificial damping. It is found that the errors in the results obtained by using the DFT method are quite unacceptable.

In order for the DFT method to lead to better results, a longer time period is required so that the spurious initial conditions are minimized. As discussed earlier, these nonzero spurious initial conditions are caused by the overlapping of the responses in different periods in the DFT method. In this case, since the damping due to the soil foundation is not large enough, the time period required in a DFT

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analysis is extremely long. This will be evident from the following. Figures 6.7a, 6.7b, and 6.7c show comparisons between the results obtained by using the DFT method with a time period of $T_0 = 81.92$ seconds and those obtained by using the method of artificial damping with a time period of $T_0 = 40.96$ seconds. The results obtained by DFT method are closer to the results obtained by using the method of artificial damping than those as shown in Fig 6.5s. However, discernible differences still exist between the two sets of displacement responses. The DFT method is able to give accurate results only when the computing time period T_0 is taken as 163.84 seconds or longer. Results obtained with such a value of T_0 are presented in Figs. 6.8a, 6.8b and 6.8c for the displacement responses along degrees of freedom 1, 23, and 45 respectively. They are now very close to those obtained by using the method of artificial damping. The computer CPU time required for the DFT method with $T_0 = 81.92$ s is 1056.48 minutes, while that with $T_0 = 163.84$ s is 2230.0 minutes on an Apollo DN3000 workstation. The computer CPU time spent in the method of artificial damping with $T_0 = 40.96$ is 274.31 minutes, and for the method of artificial damping with $T_0 = 20.48$ seconds 73.57 minutes. Drastic improvement in computing efficiency is accomplished in this case by using the method of artificial damping.

The results in Fig. 6.8 further prove the validity of the method of artificial damping proposed in Chapter 4 for its application in structural interaction problems. These results also imply that there is no difficulty in the use of the method of artificial damping in the frequency domain analysis of structural systems, in which the BEM method is used, and specifically, when a set of Bessel functions are contained in the fundamental solutions of the BEM method. For a structural system with low damping such as in the case dealt with in the above example, the use of method of artificial damping leads to efficiency and accuracy in computing procedure. On the other hand, the use of standard DFT may require the selection of a very long time period in order to obtain accurate responses. Apparently, such longer time period will lead to very large computation costs for the analysis of soil structure interaction problems, because the BEM solution to the soil part has to

be repeated at many frequency steps, and this portion of computation is the most time consuming part in the whole procedure of analysis. Furthermore, in the DFT method, the time period is usually selected by trial and error method. It is difficult to make a good guess for a period before an analysis is carried out. As a result, in practice, when the DFT method is used, the analysis must be repeated a second time or more before acceptable results are obtained. On the other hand, by using the method of artificial damping in the analysis, accurate responses can usually be obtained in the first trial irrespective of the length of time period selected.

6.7.2 Partial Ritz Vector And Two Stage Ritz Vector Technique

The application of the Ritz vectors given by Eq. 6.51 is illustrated in this section by means of an example. The same soil-structure system as used in the last section is analyzed, again by using the formulation given by Eq. 6.47. However, this time the Ritz vectors suggested in Eq. 6.51 are used. Four Ritz vectors are generated for the superstructure alone, with all its supports fixed. Thus, $\mathbf{X}_r$ in Eq. 6.51 is a 40×4 matrix. The unit matrix $\mathbf{I}_r$ in this equation is an 8×8 matrix, since there are 8 degrees of freedom on the interface of soil and superstructure. Figures 6.9a, 6.9b, and 6.9c, which display the displacement responses along degrees of freedom 1, 23, and 45 respectively, show that the results obtained by using this set of Ritz vectors are in very good agreement with the exact results. The results referred to as exact here are obtained by using the original full order 48×48 mass and stiffness matrices, with no Ritz transformation. The exact results are shown by the solid lines. The responses obtained by using the Ritz vectors given in Eq. 6.51 are represented by dashed lines in the same figures. However, these dashed lines are indiscernible because they almost coincide with the solid lines. All responses are obtained by using the method of artificial damping, since this method has already been confirmed as being reliable in the previous section. The reduced soil-structure system here has an order of 12. The reduced size for the same soil-structure system in the example presented in the preceding section was smaller, of order 5. It still

gave the same order of accuracy. In both examples, the time spent in the BEM solution for the dynamic stiffness matrix is exactly the same. The differences in computing time lie in the generation of Ritz vectors and in solving Eq. 6.47. The example in the preceding section required more computing time in generating the 5 Ritz vectors, first because the complete BEM analysis had to be carried out for one time point to get the initial values of the dynamic stiffness matrix for soil, and second because a full 48×48 nonsymmetric stiffness matrix $\tilde{\mathbf{K}}$ was used in generating Ritz vectors. The generation of the 4 Ritz vectors in the example presented here takes much less time since the stiffness matrix used is symmetric and can be stored in half bandwidth form. The process of generating the Ritz vectors in the example of this section needed computer CPU time of 0.052 minutes, while the corresponding time required for the example of preceding section was 1.02 minutes. The process of solving Eq. 6.47 in the present example, where the reduced problem is of order 12×12 , consumes a computer CPU time of 281.0 minutes, the corresponding time for the example of preceding section was 273.39 minutes. In total, an additional 6.642 minutes are required in using the Ritz vectors in Eq. 6.51. Evidently the Ritz vectors of Eq. 6.51 are no more efficient than those of Section 6.4 in the present problem.

The efficiency of the Ritz vectors in Eq. 6.51 will be evident in those cases where the degrees of freedom for the superstructure are much larger in number than the interface degrees of freedom. In other words, the use of superstructure Ritz vectors, will not lead to efficiency in the computation unless the number of degrees of freedom on the interaction boundary are small in number compared to the number of degrees of freedom of the superstructures. When the number of interface degrees of freedom is significant, the two stage Ritz vector technique described in this chapter can be used, giving an analysis procedure that is more efficient in comparison to one that uses the Ritz vectors described in Eq. 6.47. The example presented in the following illustrates the application of two stage Ritz vector technique.

The same soil-structure system as in the preceding examples is analyzed by using the two stage Ritz vector technique. The original problem of order 48×48 is reduced to one of order 12×12 first, by using the Ritz vectors of Eq. 6.51. Five second stage Ritz vectors are then generated by using the already reduced system, and the system is further reduced to that of order 5×5 . The computer CPU time spent in the entire process of generating the two sets of Ritz vectors is 0.385 minute. A saving of approximately 62.25 % is realized in this part of the process compared with the computer time needed in the generation of the Ritz vectors of Eq. 6.47, which is 1.02 minutes. In both procedures, the reduced number of equations is exactly the same. Therefore, the remaining part of the analysis consumes almost the same amount of CPU time. Figures 6.10a, 6.10b and 6.10c compare the exact responses with the responses obtained by using the two stage Ritz vectors. Once again, the exact responses are those obtained by using the original system without Ritz reduction.

These figures display the responses along degrees of freedom 1, 23 and 45 respectively. It is seen that the two sets of responses are in very good agreement.

The time saved by using the two stage Ritz vector technique illustrated in the above example of a small size is a small fraction of the total computation time. In practical terms it doesn't seem to matter what type of Ritz vectors is used in the analysis. However, for a large size interaction problem also involving a large number of interface nodes, use of the two stage Ritz vector technique proposed here is expected to lead to significant savings in the computation cost.

6.8 CHAPTER SUMMARY

In this chapter, BEM is used in the analysis of the soil foundation. Special procedures are proposed for the numerical evaluation of the boundary integrals around the singular points in the fundamental solutions. The formulation of soil-structure problems in the frequency domain is discussed in detail. An alternative set of Ritz vectors is used. In order to further improve the efficiency of computation,

the use of two stage Ritz vector technique is suggested.

The examples provided show that the use of the method of artificial damping can significantly improve the efficiency of the frequency domain analysis procedures in soil-structure interaction problems. The use of a set of Ritz vectors based on the mode shapes of the superstructure on fixed base may result in more efficient analysis if the number of degrees of freedom on the soil-structure interface is very small, and the whole analysis procedure needs to be repeated for different properties of soil foundation. However, the set of Ritz vectors in Eq. 6.51 are not as effective if the number of degrees of freedom on the soil-structure interface is substantial in comparison with the total number of degrees of freedom of the system. In the latter cases, the suggested two stage Ritz vector technique may be the best choice, because such vectors will always lead to a more efficient analytical process than that based on the Ritz vectors described in Section 6.4.

Since a large amount of time is required in the BEM analysis of soil foundation, in practice the time saved in the use of the set of Ritz vectors in Eq. 6.51 as well as the two stage Ritz vectors is not significant in comparison to the total computing time required in the examples given in this chapter. However, when only the generation of Ritz vectors is considered, the time saving indicates that the set of Ritz vectors, particularly the two stage Ritz vectors are considerably more effective in the analysis of soil-structure interaction problems.

Fig. 6.1 Soil-structure system and the selected constant boundary elements for soil foundation

Fig. 6.2 Interacting forces on the common contact nodes between superstructure and soil

Fig. 6.3 Finite elements and degrees of freedom for the superstructure with released supports

**Fig. 6.4 Horizontal component of ground acceleration
along S69E direction, Taft Earthquake**

(a) Response along degree of freedom 1

(b) Response along degree of freedom 23

(c) Response along degree of freedom 45

----- Standard DFT ——— DFT with Art. Damp.

Fig. 6.5 Comparison of the responses obtained by using the method of artificial damping with the responses obtained by using the standard DFT with a time period of 40.96 seconds

(a) Response along degree of freedom 1

(b) Response along degree of freedom 23

(c) Response along degree of freedom 45

----- Standard DFT ——— DFT with Art. Damp.

Fig. 6.6 Comparison of the responses obtained by using the method of artificial damping with the responses obtained by using the standard DFT with a time period of 20.48 seconds

(a) Response along degree of freedom 1

(b) Response along degree of freedom 23

(c) Response along degree of freedom 45

----- Standard DFT ——— DFT with Art. Damp.

Fig. 6.7 Comparison of the responses obtained by using the method of artificial damping with the responses obtained by using the standard DFT with a time period of 81.92 seconds

(a) Response along degree of freedom 1

(b) Response along degree of freedom 23

(c) Response along degree of freedom 45

----- Standard DFT ——— DFT with Art. Damp.

Fig. 6.8 Comparison of the responses obtained by using the method of artificial damping with the responses obtained by using the standard DFT with a time period of 163.84 seconds

(a) Response along degree of freedom 1

(b) Response along degree of freedom 23

(c) Response along degree of freedom 45

----- 12 Indep. Ritz Vectors ——— Exact Response

Fig. 6.9 Comparison of the responses obtained by using the whole system with the responses obtained by using 12 Ritz vectors independent of soil foundation

(a) Response along degree of freedom 1

(b) Response along degree of freedom 23

(c) Response along degree of freedom 45

----- 5 Second Stage Ritz Vectors ——— Exact Response

Fig. 6.10 Comparison of the responses obtained by the whole system with the responses obtained by using 5 second stage Ritz vectors

Chapter 7

SUBSTRUCTURE COUPLING USING RITZ VECTORS

7.1 INTRODUCTION

The dynamic analysis of very large structures such as, for example, aircraft and space structures is a formidable computational task. To reduce the task to that of a manageable size, a method of analysis known as component mode synthesis or substructure coupling has been developed. This method basically consist of subdividing the structure into several components. The dynamic characteristics of individual components are normally represented in terms of their normal modes and frequencies, which can be evaluated either from the analysis of a finite element model of the individual component or determined experimentally. It is also usually possible to include only a truncated set of mode shapes in the analysis without introducing large errors. The individual components are coupled to obtain a representation of the total structure. This representation allows the determination of mode shapes and frequencies of the structure, as well as its response to a specified load.

The component mode synthesis method is quite effective in reducing computing time. Another advantage of the method is that when different groups work on the design of different components of a structure, it is useful for them to be able to derive the dynamic properties of individual components without reference to the properties of other components in the assembly. The dynamic properties of individual components along with a knowledge of interface details permits evaluation of the characteristics of the assembled structure.

As stated earlier, mode shapes and frequencies have traditionally been used to represent the characteristics of individual components. It is shown here that

Ritz vectors may serve as efficient alternatives to mode shapes in many cases. The evaluation of Ritz vectors is considerably more efficient and computationally less intensive than that of mode shapes; and, in fact, the use of Ritz vectors may at times lead to better accuracy than the use of mode shapes.

The first paper published on substructure coupling using mode shapes was probably that by Hurty (1965). Since then, numerous methods for coupling substructures for dynamic analysis have been presented. A survey of these methods can be found in the work by (Benfield, *et al* 1972; and Craig and Chang 1976). A brief review of some previous studies has been presented in Chapter 1.

The generation of normal mode shapes is much more time consuming and expensive than that of Ritz vectors. Each mode shape is usually generated by an iteration process that has to be repeated many times until convergence to an eigenvalue is achieved. On the other hand, in generating Ritz vectors, such iterations are not necessary. The use of Ritz vectors instead of mode shapes will, therefore, often improve the efficiency of the analysis method. However, the reduced stiffness matrix obtained by using Ritz vectors is not diagonal while the reduced stiffness matrix obtained by using mode shapes is. This may cause some difficulty in using Ritz vectors in substructure coupling. Therefore, the feasibility of using Ritz vectors should be further investigated. Such an investigation is made here by referring to the procedures outlined in the work by Craig and Chang (1977) and Chang (1977). In the current work, Ritz vectors based on the use of a singular stiffness matrix are studied in detail and several methods are suggested for considering the effect of load distribution as well as for generating the Ritz vectors. The application of Ritz vectors is also studied in context of the various substructure coupling methods described in the literature. Examples are given to illustrate the effectiveness of using Ritz vectors. The natural frequencies of the system extracted by typical coupling procedures are compared with the exact values obtained by treating the structure system as a whole. It should be noted that a majority of dynamic problems in civil engineering are in the class of low frequency vibrations. Therefore, the natural

frequencies of interest in this work are those of low values. The size of the coupled system using Ritz vectors is also a major concern in civil engineering, because it can often determine the efficiency of the later analysis, especially that carried out in frequency domain.

7.2 DEFINITION OF RITZ VECTORS

As in the application of normal modes in substructure coupling, definition of some of the terminology related to the use of Ritz vectors is necessary for clarity in describing the use of Ritz vectors. Reference is made to the structure shown in Fig. 7.1 when the different types of Ritz vectors are defined.

The equations of motion for a substructure without damping can be written in the matrix form

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{f} \quad (7.1)$$

Equation 7.1 can be expressed in a partitioned form as

$$\begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ij} \\ \mathbf{M}_{ji} & \mathbf{M}_{jj} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{v}}_i \\ \ddot{\mathbf{v}}_j \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_i \\ \mathbf{v}_j \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \end{Bmatrix} \quad (7.2)$$

in which, symbols 'i' and 'j' represent respectively the interior degrees of freedom and the junction of degrees of freedom. The types of Ritz vectors are defined in the following with the reference to Eqs. 7.1 and 7.2.

7.2.1 Fixed Interface Ritz Vectors

The fixed interface Ritz vectors are defined by the eigenvalue problem obtained from Eq. 7.2 with all the junction degrees of freedom fixed. Thus $\mathbf{v}_j = 0$ and Eq. 7.2 with the right hand side set to zero leads to

$$(-\omega^2 \mathbf{M}_{ii} + \mathbf{K}_{ii})\mathbf{Z}_i = 0 \quad (7.3)$$

It should be noted that the mode shapes of the substructure are also obtained from Eq. 7.3. The difference between mode shapes and Ritz vectors can be explained

by considering the inverse iteration method derived from Eq. 7.3. The iteration procedure can be expressed as

$$\mathbf{X}_k = \mathbf{K}_{ii}^{-1} \mathbf{M}_{ii} \hat{\mathbf{X}}_{k-1} \quad (7.4)$$

where $\hat{\mathbf{X}}_{k-1}$ is the vector obtained from step $k-1$ in the iteration, and $\mathbf{X}_k$ the vector computed at the current step. Vector $\mathbf{X}_k$ is usually normalized with respect to mass matrix before being used as vector $\hat{\mathbf{X}}_{k-1}$ in the next step. The iteration process is repeated until each element of the normalized vector $\hat{\mathbf{X}}_k$, is almost equal to the corresponding element in vector $\hat{\mathbf{X}}_{k-1}$. This converged vector is a mode shape of the structure. Iteration can be commenced with an arbitrary starting vector and will converge to the fundamental mode shape. To obtain a mode shape other than those already extracted, the starting vector is made orthogonal to all the mode shapes already extracted. This starting vector is then used as vector $\hat{\mathbf{X}}_0$ to start the iteration.

In obtaining a Ritz vector, vector $\mathbf{X}_k$ obtained at each step in the iteration process represented by Eq. 7.4 is used as a Ritz vector. Therefore, generation of Ritz vector is more economic. The extracted Ritz vector is orthogonalized with respect to all previous Ritz vectors and normalized with respect to the mass matrix. The set of Ritz vectors obtained in this manner can be expressed as

$$\Psi_n^i = \{\psi_1^i, \psi_2^i, \dots, \psi_n^i\} \quad (7.5a)$$

or

$$\Psi_n = \left\{ \begin{matrix} \psi_1^i & \psi_2^i & \dots & \psi_n^i \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{matrix} \right\} \quad (7.5b)$$

Since Ritz vectors Ψ_n have been normalized with respect to the mass matrix,

$$\Psi_n^T \mathbf{M} \Psi_n = \mathbf{I}_n \quad (7.6)$$

However, $\Psi_n^T \mathbf{K} \Psi_n$ is not a diagonal matrix.

7.2.2 Free Interface Ritz Vectors

The free interface Ritz vectors are those obtained with all interface degrees of freedom free. This set of Ritz vectors can be derived by setting both $\mathbf{f}_i$ and $\mathbf{f}_j = 0$

in Eq. 7.2 and then considering the corresponding eigenvalue problem as follows,

$$(-\omega^2 \mathbf{M} + \mathbf{K})\mathbf{Z} = 0 \quad (7.7)$$

in which, matrices $\mathbf{M}$, and $\mathbf{K}$ are respectively the mass and stiffness matrices for the entire substructure. The corresponding Ritz vectors may be expressed as

$$\Psi_n = \left\{ \begin{matrix} \psi_1^i, \psi_2^i, \dots, \psi_n^i \\ \psi_1^j, \psi_2^j, \dots, \psi_n^j \end{matrix} \right\} \quad (7.8)$$

If the substructure is unconstrained, the stiffness matrix $\mathbf{K}$ is singular, and the above free interface Ritz vectors will include rigid body Ritz vectors (or modes). Let the free Ritz vectors in this case be partitioned as

$$\Psi_n = [\Psi_r \quad \Psi_f] \quad (7.9)$$

in which, Ψ_r represent the orthonormalized rigid body Ritz vectors, and Ψ_f the flexible Ritz vectors. The following relationships exist between them

$$\Psi_r^T \mathbf{M} \Psi_f = 0 \quad (7.10a)$$

$$\Psi_n^T \mathbf{M} \Psi_n = \mathbf{I} \quad (7.10b)$$

$$\Psi_r^T \mathbf{K} \Psi_r = 0 \quad (7.10c)$$

However, unlike the case of normal mode shapes, $\Psi_n^T \mathbf{K} \Psi_n$ is again not a diagonal matrix.

7.2.3 Loaded Interface Ritz Vectors

This set of Ritz vectors is defined by considering the eigenvalue problem in Eq. 7.2 with all interface degrees of freedom loaded by the interaction forces from the adjoining substructure, i.e.

$$\left(-\omega^2 \begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ij} \\ \mathbf{M}_{ji} & \mathbf{M}_{jj} + \bar{\mathbf{M}}_{jj} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} + \bar{\mathbf{K}}_{jj} \end{bmatrix} \right) \mathbf{Z} = 0 \quad (7.11)$$

where, $\bar{M}_{jj}$ and $\bar{K}_{jj}$ are the interface mass and stiffness contributions from the adjoining substructures. Their values can be assigned by the analyst on a rational basis.

7.2.4 Constraint Ritz Vectors

Constraint Ritz vectors are defined as being identical to the constraint modes described in the work by Craig and Chang (1977) and Chang (1977). They are the static deformation shapes of the substructure obtained when a unit displacement is imposed on one physical coordinate and zero displacement on the remainder of a specified subset of the substructure physical coordinates. Usually, the specified subset consists of the set of junction coordinates. Let the degrees of freedom in the specified subset, that are at least enough in number to prevent the substructure from rigid body motion, be designated by subscript 'c', and the remainder by 'v'. The constraint Ritz vectors can be then derived from the following

$$\begin{bmatrix} \mathbf{K}_{vv} & \mathbf{K}_{vc} \\ \mathbf{K}_{cv} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \Psi_v \\ \mathbf{I}_c \end{bmatrix} = \begin{Bmatrix} \mathbf{0}_v \\ \mathbf{R}_c \end{Bmatrix} \quad (7.12)$$

in which $\mathbf{I}_c$ is a unit matrix and $\mathbf{K}_{vv}$ is nonsingular. It follows that

$$\Psi_v = -\mathbf{K}_{vv}^{-1} \mathbf{K}_{vc} \quad (7.13)$$

Therefore, the constraint Ritz vectors are

$$\Psi_c = \begin{bmatrix} \Psi_{vc} \\ \mathbf{I}_{cc} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{vv}^{-1} \mathbf{K}_{vc} \\ \mathbf{I}_{cc} \end{bmatrix} \quad (7.14)$$

If the considered substructure is unrestrained, the constraint Ritz vectors will contain rigid body Ritz vectors (or modes). If the specified constraint subset consists of all the junction degrees of freedom, it can be shown that the constraint Ritz vectors and the fixed interface Ritz vectors are orthogonal with respect to the stiffness matrix as follows

$$\Psi_n^T \mathbf{K} \Psi_c = [\Psi_n^T \mathbf{0}] \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} \end{bmatrix} \begin{bmatrix} \Psi_{ic} \\ \mathbf{I}_{jc} \end{bmatrix} = \mathbf{0} \quad (7.15)$$

7.2.5 Rigid Body Ritz Vectors

For the purpose of consistency, the rigid body modes are also called rigid body Ritz vectors in this work. They are defined in the same way as the constraint Ritz vectors. Let r be the smallest number of degrees of freedom which can prevent rigid body motions, and u be rest of the degrees of freedom for the substructure. Rigid body Ritz vectors can then be defined by

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{ur} \\ \mathbf{K}_{ru} & \mathbf{K}_{rr} \end{bmatrix} \begin{bmatrix} \Psi_{ur} \\ \mathbf{I}_{rr} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (7.16)$$

The rigid body Ritz vectors are thus of the form

$$\Psi_r = \begin{bmatrix} \Psi_{ur} \\ \mathbf{I}_{rr} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{uu}^{-1} \mathbf{K}_{ur} \\ \mathbf{I}_{rr} \end{bmatrix} \quad (7.17)$$

It is obvious the rigid body Ritz vectors are a subset of constraint Ritz vectors.

7.2.6 Attachment Ritz Vectors and Inertia-Relief Ritz Vectors

Again, the definitions in the following are identical to those used for mode shapes in the work by Craig and Chang (1977) and Chang (1977). The change of name from mode shapes to Ritz vectors is for the purpose of consistency with the terminology used in this work.

Let $\bar{A}$ be a subset of degrees of freedom of the substructure. Usually $\bar{A}$ consists of all junction degrees of freedom. An attachment Ritz vector is defined as the "static deflection" of the substructure when a unit force is applied at one coordinate of the $\bar{A}$ set while the remaining coordinates in $\bar{A}$ are force free. Let the whole set of degrees of freedom be divided into three sets: $\bar{R}$, $\bar{A}$, and $\bar{W}$, where $\bar{R}$ set of degrees of freedom provide restraint against rigid body motion. Then attachment Ritz vectors are defined by

$$\begin{bmatrix} \mathbf{K}_{ww} & \mathbf{K}_{wa} & \mathbf{K}_{wr} \\ \mathbf{K}_{aw} & \mathbf{K}_{aa} & \mathbf{K}_{ar} \\ \mathbf{K}_{rw} & \mathbf{K}_{ra} & \mathbf{K}_{rr} \end{bmatrix} \begin{bmatrix} \Psi_{wa} \\ \Psi_{aa} \\ \mathbf{0}_{ra} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{wa} \\ \mathbf{I}_{aa} \\ \mathbf{R}_{ra} \end{bmatrix} \quad (7.18)$$

From the top two row-partitions of Eq. 7.18

$$\begin{bmatrix} \Psi_{wa} \\ \Psi_{aa} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ww} & \mathbf{K}_{wa} \\ \mathbf{K}_{aw} & \mathbf{K}_{aa} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0}_{wa} \\ \mathbf{I}_{aa} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{ww} & \mathbf{G}_{wa} \\ \mathbf{G}_{aw} & \mathbf{G}_{aa} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{wa} \\ \mathbf{I}_{aa} \end{bmatrix} \quad (7.19)$$

Thus, the attachment Ritz vectors are of the form

$$\Psi_a = \begin{bmatrix} \Psi_{wa} \\ \Psi_{aa} \\ \mathbf{0}_{ra} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{wa} \\ \mathbf{G}_{aa} \\ \mathbf{0}_{ra} \end{bmatrix} \quad (7.20)$$

An alternative manner of defining attachment Ritz vectors for unrestrained substructure leads to inertia-relief Ritz vectors which are identical to the inertia-relief modes described in the work by Craig and Chang (1977) and Chang (1977). The details of derivation are given in those references. The attachment Ritz vectors (modes) are given by

$$\tilde{\Psi}_a = \mathbf{G}\mathbf{P}\mathbf{f}_a \quad (7.21)$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{ww} & \mathbf{G}_{wa} & \mathbf{0} \\ \mathbf{G}_{aw} & \mathbf{G}_{aa} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (7.22)$$

$$\mathbf{P} = \mathbf{I} - \mathbf{M}\Psi_r\Psi_r^T \quad (7.23)$$

and

$$\mathbf{f}_a = \begin{bmatrix} \mathbf{0}_{wa} \\ \mathbf{I}_{aa} \\ \mathbf{0}_{ra} \end{bmatrix} \quad (7.24)$$

The attachment Ritz vectors obtained from Eq. 7.21 can be made orthogonal to the rigid body modes. It can be shown that the orthogonal vectors $\hat{\Psi}_a$ are given by

$$\hat{\Psi}_a = \mathbf{P}^T \mathbf{G}\mathbf{P}\mathbf{f}_a = \mathbf{G}_e \mathbf{f}_a \quad (7.25)$$

It is worth noting that the attachment Ritz vectors and the constraint Ritz vectors are linearly dependent on each other. Also, if a complete set of free-interface Ritz vectors (that include rigid body Ritz vectors) is used with the attachment Ritz vectors, the latter are linearly dependent on the former. The truncation of Ritz vectors is necessary in such a case. As shown in the work by Craig and Chang (1977), if 'k' is the number of kept normal modes, a set of attachment modes that are independent of the rigid-body modes and also of the kept elastic modes can

be obtained by substituting $\mathbf{G}_e$ for $\bar{\mathbf{G}}_e$ in Eq. 7.25 where $\mathbf{G}_e$, called the residual flexibility matrix, is given by

$$\mathbf{G}_e = \mathbf{P}^T \mathbf{G} \mathbf{P} - \Phi_k \Lambda_k^{-1} \Phi_k^T \quad (7.26)$$

When Ritz vectors are used in place of elastic modes and k denotes the number of several Ritz vectors kept, the attachment Ritz vectors that are independent of the k Ritz vectors are again obtained from Eq. 7.25 with $\bar{\mathbf{G}}_e$ replaced by $\mathbf{G}_e$. However, $\mathbf{G}_e$ is now given by

$$\mathbf{G}_e = \mathbf{P}^T \mathbf{G} \mathbf{P} - \Psi_k (\Psi_k^T \mathbf{K} \Psi_k)^{-1} \Psi_k^T \quad (7.27)$$

where Ψ_k are the kept Ritz vectors.

To prove that Eq. 7.27 is equivalent to Eq. 7.26, let the original equations of motion in Eq. 7.1 be reduced by the coordinate transformation

$$\mathbf{v} = \Psi_k \mathbf{q} \quad (7.28)$$

The reduced form of Eq. 7.1 can be written as follows

$$\ddot{\mathbf{q}} + \mathbf{K}^* \mathbf{q} = \Psi_k^T \mathbf{f} \quad (7.29)$$

in which

$$\Psi_k^T \mathbf{M} \Psi_k = \mathbf{I}_k \quad (7.30a)$$

and

$$\Psi_k^T \mathbf{K} \Psi_k = \mathbf{K}^* \quad (7.30b)$$

It should be noted that although the Ritz vectors are orthogonal with respect to the mass matrix, they are not so with respect to the stiffness matrix. Therefore, the reduced stiffness matrix $\mathbf{K}^*$ is not diagonal. By considering the eigenvalue problem in Eq. 7.29, a further coordinate transformation can be set up by using the normal modes of the reduced system. This coordinate transformation is expressed by

$$\mathbf{q} = \tilde{\Phi}_k \mathbf{z} \quad (7.31)$$

in which $\tilde{\Phi}_k$ is the matrix of untruncated normal modes for the reduced system in Eq. 7.29, i.e.

$$\tilde{\Phi}_k^T \mathbf{K} \tilde{\Phi}_k = \Lambda_k \quad (7.32)$$

By combining the two coordinate transformation given by Eq. 7.28 and 7.31, the following is obtained

$$\mathbf{v} = \Phi_k \mathbf{q} = \Psi_k \tilde{\Phi}_k \mathbf{z} \quad (7.33)$$

By applying the transformation in Eq. 7.33 to the original equations of motion in Eq. 7.1, the reduced stiffness matrix becomes

$$\Phi_k^T \mathbf{K} \Phi_k = \tilde{\Phi}_k^T \Psi_k^T \mathbf{K} \Psi_k \tilde{\Phi}_k = \Lambda_k \quad (7.34)$$

In Eq. 7.26, the last term on the right side can be expressed as

$$\begin{aligned} \Phi_k \Lambda_k^{-1} \Phi_k^T &= \Phi_k (\Phi_k^T \mathbf{K} \Phi_k)^{-1} \Phi_k^T \\ &= \Psi_k \tilde{\Phi}_k \left(\tilde{\Phi}_k^T \Psi_k^T \mathbf{K} \Psi_k \tilde{\Phi}_k \right)^{-1} \tilde{\Phi}_k^T \Psi_k^T \\ &= \Psi_k \tilde{\Phi}_k \tilde{\Phi}_k^{-1} (\Psi_k^T \mathbf{K} \Psi_k)^{-1} (\tilde{\Phi}_k^T)^{-1} \tilde{\Phi}_k \Psi_k^T \\ &= \Psi_k (\Psi_k^T \mathbf{K} \Psi_k)^{-1} \Psi_k^T \end{aligned} \quad (7.35)$$

This proves that Eq. 7.26 and 7.27 are equivalent.

The attachment Ritz vectors that are independent of the rigid body motion as well as of the truncated flexible Ritz vectors Ψ_k can thus be obtained from

$$\Psi_a = \mathbf{G}_c \mathbf{f}_a \quad (7.36)$$

The Ritz vectors obtained from Eq. 7.36 can be termed as residual inertia relief Ritz vectors in analogy with the mode shapes of similar type (Craig and Chang 1977).

7.3 COUPLING OF SUBSTRUCTURES

The coupling procedures described in the work by Craig and Chang (1977), and Chang (1977) are investigated by using the Ritz vectors defined in the foregoing sections in place of mode shapes of the original formulations.

Let the system be composed of two substructures indicated by superscripts ' α ' and ' β ' respectively. The two substructures share a common set of interface degrees of freedom. Let the physical coordinate vector, mass and stiffness matrices, load vector, Ritz vector matrix, and the transformed coordinates for the two substructures be expressed as

$$\begin{aligned}\mathbf{v} &= \begin{Bmatrix} \mathbf{v}^\alpha \\ \mathbf{v}^\beta \end{Bmatrix}, \\ \mathbf{M} &= \begin{bmatrix} \mathbf{M}^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^\beta \end{bmatrix}, \\ \mathbf{K} &= \begin{bmatrix} \mathbf{K}^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^\beta \end{bmatrix}, \\ \mathbf{f} &= \begin{Bmatrix} \mathbf{f}^\alpha \\ \mathbf{f}^\beta \end{Bmatrix}, \\ \Psi &= \begin{bmatrix} \Psi^\alpha & \mathbf{0} \\ \mathbf{0} & \Psi^\beta \end{bmatrix},\end{aligned}$$

and

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}^\alpha \\ \mathbf{q}^\beta \end{Bmatrix}$$

The expressions for kinetic and potential energies of the system can be written as

$$\mathbf{T} = \frac{1}{2} \dot{\mathbf{v}}^T \mathbf{M} \dot{\mathbf{v}} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}^* \dot{\mathbf{q}} \quad (7.37a)$$

and

$$\mathbf{U} = \frac{1}{2} \mathbf{v}^T \mathbf{K} \mathbf{v} = \frac{1}{2} \mathbf{q}^T \mathbf{K}^* \mathbf{q} \quad (7.37b)$$

where

$$\mathbf{M}^* = \Psi^T \mathbf{M} \Psi$$

and

$$\mathbf{K}^* = \Psi^T \mathbf{K} \Psi$$

The coordinates $\mathbf{q}$ are not linearly independent. They are related by force equilibrium and displacement compatibility at the interface. The displacement compatibility conditions at the interface is given by

$$\mathbf{v}_j^a = \mathbf{v}_j^i \quad (7.38)$$

which leads to a constraint equation of the form

$$\mathbf{B}\mathbf{q} = \mathbf{0} \quad (7.39)$$

When attachment Ritz vectors (or inertia relief Ritz vectors) are employed, the force equilibrium condition at the attachment degrees of freedom are given by

$$\mathbf{f}_j^a + \mathbf{f}_j^i = \mathbf{0} \quad (7.40)$$

which leads to the constraint equation

$$\mathbf{A}\mathbf{q} = \mathbf{0} \quad (7.41)$$

The Lagrangian function can be written as

$$\mathbf{L} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^T \mathbf{K} \mathbf{q} + \boldsymbol{\eta}^T \mathbf{A}\mathbf{q} + \boldsymbol{\nu}^T \mathbf{B}\mathbf{q} \quad (7.42)$$

where $\boldsymbol{\eta}$, and $\boldsymbol{\nu}$ are Lagrangian multiplier vectors. To simplify the expression of Eq. 7.42, let

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \quad (7.43a)$$

and

$$\boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\nu} \end{bmatrix} \quad (7.43b)$$

Equation 7.42 now becomes

$$\mathbf{L} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^T \mathbf{K} \mathbf{q} + \boldsymbol{\sigma}^T \mathbf{C}\mathbf{q} \quad (7.44)$$

The system equations of motion can be obtained by using Lagrange's equation given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_s} \right) - \frac{\partial L}{\partial \theta_s} = Q_s \quad (7.45)$$

where Q_s is the generalized force associated with θ_s , which can refer to either q_s or σ_s . For the substructure coupling problem with forces exerted only at the interface, Q_s is determined from the virtual work expression

$$\delta W = (\delta \mathbf{v}_j^\alpha)^T \mathbf{f}_j^\alpha + (\delta \mathbf{v}_j^\beta)^T \mathbf{f}_j^\beta = \delta \mathbf{q}_j^{\alpha T} \Psi_j^{\alpha T} \mathbf{f}_j^\alpha + \delta \mathbf{q}_j^{\beta T} \Psi_j^{\beta T} \mathbf{f}_j^\beta = \delta \mathbf{q}^T \Psi^T \mathbf{f} \quad (7.46)$$

Substitution of Eq. 7.44 and 7.46 into Eq. 7.45 gives

$$\mathbf{M}^* \ddot{\mathbf{q}} + \mathbf{K}^* \mathbf{q} = \Psi^T \mathbf{f} + \mathbf{C}^T \sigma \quad (7.47)$$

The constraint equations are

$$\mathbf{C} \mathbf{q} = \mathbf{0} \quad (7.48)$$

The dependent coordinates can be transformed to a set of linearly independent coordinates by a transformation of the form

$$\mathbf{q} = \mathbf{S} \mathbf{u} \quad (7.49)$$

Let $\mathbf{q}$ be partitioned into dependent coordinates $\mathbf{q}_d$, and linearly independent coordinates, $\mathbf{q}_l$, and let Eq. 7.48 be partitioned accordingly, i.e.

$$[\mathbf{C}_{dd}, \mathbf{C}_{dl}] \begin{Bmatrix} \mathbf{q}_d \\ \mathbf{q}_l \end{Bmatrix} = \mathbf{0} \quad (7.50)$$

where $\mathbf{C}_{dd}$ is a nonsingular square submatrix of $\mathbf{C}$. Then

$$\begin{Bmatrix} \mathbf{q}_d \\ \mathbf{q}_l \end{Bmatrix} = \begin{bmatrix} -\mathbf{C}_{dd}^{-1} \mathbf{C}_{dl} \\ \mathbf{I}_l \end{bmatrix} \mathbf{u} \quad (7.51)$$

where

$$\mathbf{u} = \mathbf{q}_I \quad (7.52)$$

Thus

$$\mathbf{S} = \begin{bmatrix} -\mathbf{C}_{dd}^{-1} \mathbf{C}_{dl} \\ \mathbf{I}_I \end{bmatrix} \quad (7.53)$$

When Eq. 7.51 is substituted into Eq. 7.50, the following results

$$\mathbf{C}\mathbf{S} = \mathbf{0} \quad (7.54)$$

By using the transformation in Eq. 7.49, Eq. 7.47 can be written in terms of independent coordinates $\mathbf{u}$ as follows

$$\mathbf{M}'\ddot{\mathbf{u}} + \mathbf{K}'\mathbf{u} = \mathbf{S}^T \Psi^T \mathbf{f} + \mathbf{S}^T \mathbf{C}^T \sigma = \mathbf{S}^T \Psi^T \mathbf{f} \quad (7.55)$$

where

$$\mathbf{M}' = \mathbf{S}^T \mathbf{M} \mathbf{S} \quad (7.56)$$

$$\mathbf{K}' = \mathbf{S}^T \mathbf{K} \mathbf{S} \quad (7.57)$$

It can further be shown that, since $\mathbf{f}_j^\alpha + \mathbf{f}_j^\beta = \mathbf{0}$, $\mathbf{S}^T \Psi^T \mathbf{f} = \mathbf{0}$ and Eq. 7.55 reduces to

$$\mathbf{M}'\ddot{\mathbf{u}} + \mathbf{K}'\mathbf{u} = \mathbf{0} \quad (7.58)$$

The foregoing development shows that there is no fundamental problem in using Ritz vectors in the general substructure coupling procedure described by Craig and Chang (1977), and Chang (1977). Further study also proves that Ritz vectors can be easily incorporated into all formulations that are based on the general coupling procedure described above. Only slight modification in the reduced stiffness matrix

needs to be made since it is no longer of a diagonal form. The formulations referred to include the Benfield and Hrudá's method, Hintz's method, Rubin's method, McNeal's method, and the hybrid interface method. All of these methods have been described in the work by Craig and Chang (1977) with reference to the use of mode shapes.

In the work by Chang (1977), a substructure coupling procedure is described for a system with modal damping and external loading. A similar procedure can be developed when Ritz vectors are used in place of mode shapes.

In general, the original system equations have the form

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{C}\dot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{f} + \mathbf{F} \quad (7.59)$$

in which $\mathbf{f}$ and $\mathbf{F}$ are the interacting forces on the interface and the external loads respectively, and

$$\mathbf{v} = \begin{Bmatrix} \mathbf{v}^\alpha \\ \mathbf{v}^\beta \end{Bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^\beta \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^\beta \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^\beta \end{bmatrix},$$

$$\mathbf{f} = \begin{Bmatrix} \mathbf{f}^\alpha \\ \mathbf{f}^\beta \end{Bmatrix},$$

and

$$\mathbf{F} = \begin{Bmatrix} \mathbf{F}^\alpha \\ \mathbf{F}^\beta \end{Bmatrix},$$

Let the Ritz vectors be of the form

$$\Psi = \begin{bmatrix} \Psi^\alpha & \mathbf{0} \\ \mathbf{0} & \Psi^\beta \end{bmatrix},$$

and the reduced coordinates be of the form

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}^\alpha \\ \mathbf{q}^\beta \end{Bmatrix}$$

The reduced equations of motion before coupling can then be expressed as

$$\mathbf{M}^* \ddot{\mathbf{v}} + \mathbf{C}^* \dot{\mathbf{v}} + \mathbf{K}^* \mathbf{v} = \mathbf{f}^* + \mathbf{F}^* \quad (7.60)$$

where

$$\mathbf{M}^* = \Psi^T \mathbf{M} \Psi$$

$$\mathbf{C}^* = \Psi^T \mathbf{C} \Psi$$

$$\mathbf{K}^* = \Psi^T \mathbf{K} \Psi$$

$$\mathbf{f}^* = \Psi^T \mathbf{f}$$

and

$$\mathbf{F}^* = \Psi^T \mathbf{F}$$

After carrying out the coupling procedure similar to that described in this section for the undamped system subjected to no external loads, the following equations are obtained

$$\mathbf{M}' \ddot{\mathbf{u}} + \mathbf{C}' \dot{\mathbf{u}} + \mathbf{K}' \mathbf{u} = \mathbf{f}' + \mathbf{F}' \quad (7.61)$$

where

$$\mathbf{M}' = \mathbf{S}^T \mathbf{M} \mathbf{S} \quad (7.62a)$$

$$\mathbf{C}' = \mathbf{S}^T \mathbf{C} \mathbf{S} \quad (7.62b)$$

$$\mathbf{K}' = \mathbf{S}^T \mathbf{K} \mathbf{S} \quad (7.62c)$$

$$\mathbf{f}' = \mathbf{S}^T \mathbf{f} \quad (7.62d)$$

and

$$\mathbf{F}' = \mathbf{S}^T \mathbf{F} \quad (7.62e)$$

It can also be shown that

$$\mathbf{f}' = \mathbf{S}^T \mathbf{f} = \mathbf{0} \quad (7.63)$$

since

$$\mathbf{f}_j^\alpha + \mathbf{f}_j^\beta = \mathbf{0} \quad (7.64)$$

Therefore, the coupled equations of motion for the damped system subjected to external loads reduce to

$$\mathbf{M}' \ddot{\mathbf{u}} + \mathbf{C}' \dot{\mathbf{u}} + \mathbf{K}' \mathbf{u} = \mathbf{F}' \quad (7.65)$$

where $\mathbf{u}$ are the independent coordinates.

7.4 GENERATION OF RITZ VECTORS

The general substructure coupling procedure using Ritz vectors is described in the last section. The procedure has earlier been used effectively with mode

shapes (Craig and Chang 1977; Chang 1977; and Wang 1992). As observed in the previous sections the same procedure applies with only a few modifications when Ritz vectors are used in place of mode shapes. This section deals with the procedures for the generation of Ritz vectors, in particular, the free interface flexible Ritz vectors. When a free interface is used for a substructure and no constraint exists, the substructure has rigid body motions, and the stiffness matrix is singular. In such a case, the procedure outlined in Chapter 2 for generating Ritz vectors is not applicable since it involves the factorization of stiffness matrix. Further, as already noted, load dependent Ritz vectors are usually more efficient. Hence, the procedures to be developed should allow the generation of such vectors. In the following, the effect of load distribution is addressed first.

7.4.1 Effect of Load Distribution

As already stated, load dependent Ritz vectors generated by considering the effect of load distribution provide a more efficient and effective means of reducing the structural system. When load dependent Ritz vectors are used, no static correction is necessary in order to take into account the contribution of the higher modes. The effect of load distribution is included while generating Ritz vectors by taking a starting vector that is equal to the static deformation of the structural system. This static deformation vector is updated each time an additional Ritz vector is generated. This procedure was described in detail in Chapter 2 already. The static deformation vector of a structure is obtained from

$$\mathbf{K}\mathbf{v}_s = \mathbf{f} \quad (7.66)$$

in which, $\mathbf{K}$ is the stiffness matrix, $\mathbf{f}$ the distribution vector of loads, and $\mathbf{v}_s$ the static displacement vector. However, if the stiffness matrix $\mathbf{K}$ is singular, the solution to the above equation does not give a unique answer. In this case, a reasonable way is to use the pseudo-static deformation vector as the starting vector in generating the Ritz vectors, updating it each time an additional Ritz vector is to be generated. This

pseudo-static deformation vector of a structural system (usually a substructure) can be obtained as follows. Consider the undamped equations of motion of the system,

$$M\ddot{\mathbf{v}} + K\mathbf{v} = \mathbf{f} \quad (7.67a)$$

Let the displacement vector be separated into two parts as follows

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_f \quad (7.67b)$$

where $\mathbf{v}_r$ represents the rigid body motions, and $\mathbf{v}_f$ the elastic deformation of the system relative to the rigid body motions. From the definition of the rigid body Ritz vectors, $\mathbf{v}_r$ can be expressed as

$$\mathbf{v}_r = \Psi_r \mathbf{q}_r \quad (7.67c)$$

Similarly, supposing that the flexible Ritz vectors have already been obtained, $\mathbf{v}_f$ can be expressed as

$$\mathbf{v}_f = \Psi_f \mathbf{q}_f \quad (7.68)$$

Vector $\mathbf{v}$ can thus be placed in the form

$$\mathbf{v} = [\Psi_r \quad \Psi_f] \begin{Bmatrix} \mathbf{q}_r \\ \mathbf{q}_f \end{Bmatrix} \quad (7.69)$$

Substitution of Eq. 7.67c into 7.67a gives

$$\ddot{\mathbf{q}}_r = \Psi_r^T \mathbf{f} \quad (7.70)$$

The inertia forces due to the rigid body motions are given by

$$\mathbf{f}_i = -M\ddot{\mathbf{v}}_r \quad (7.71)$$

Using Eqs. 7.67c and 7.70, $\ddot{\mathbf{v}}_r$ can be expressed as

$$\ddot{\mathbf{v}}_r = \Psi_r \ddot{\mathbf{q}}_r = \Psi_r \Psi_r^T \mathbf{f} \quad (7.72)$$

Therefore, the inertia forces due to rigid body motions are given by

$$\mathbf{f}_i = -\mathbf{M} \Psi_r \Psi_r^T \mathbf{f} \quad (7.73)$$

In order to obtain the pseudo-static deformation of the system, the inertia force $\mathbf{f}_i$ is added to the load vector $\mathbf{f}$ in Eq. 7.67a. The system is then said to be in pseudo-static equilibrium. Now, arbitrary statically-determinate constraints can be added to the system, and the flexibility matrix $\mathbf{G}$ can be obtained. Matrix $\mathbf{G}$ may also be obtained by taking $\mathbf{K}$ (which is singular), deleting rows and columns corresponding to the constraints, inverting the resulting matrix, and expanding it back to the original size by adding zeros. The resulting augmented matrix is called $\mathbf{G}$. The pseudo-static deformation vector of the system can now be obtained from

$$\mathbf{v}_s = \mathbf{G}(\mathbf{f} + \mathbf{f}_i) = \mathbf{G}(\mathbf{I} - \mathbf{M} \Psi_r \Psi_r^T) \mathbf{f} = \mathbf{G} \mathbf{P} \mathbf{f} \quad (7.74)$$

The pseudo-static deformation vector $\mathbf{v}_s$, obtained from the above can thus be used as the starting vector in generating Ritz vectors. This vector can also be updated in the same way as in the algorithm in Chapter 2 each time an additional Ritz vector is to be generated.

The force amplitude vector $\mathbf{f}$ in the above consists of two parts. One is the amplitude of externally applied forces, and another is the amplitude of internal forces on the interface due to the interaction between substructures. Generally, the former is given, and the latter are unknowns. Expressing $\mathbf{f}$ in terms of its two components

$$\mathbf{f} = \mathbf{f}_{ex} + \mathbf{f}_{in} \quad (7.75)$$

where $\mathbf{f}_{ex}$ is for external loads, and $\mathbf{f}_{in}$ for interface loads. To consider the latter, an arbitrary load distribution vector $\mathbf{r}_i$ can be assumed to represent the interface

forces. Vector $\mathbf{f}_{in}$ can then be taken as

$$\mathbf{f}_{in} = -\mathbf{M}\mathbf{r}_i \quad (7.76)$$

where $\mathbf{M}$ is the mass matrix of the substructure concerned. In some cases, the substructure is not subjected to any external loadings, the force vector $\mathbf{f}$ then consists of $\mathbf{f}_{in}$ only.

7.4.2 Ritz Vectors for Singular Stiffness Matrix

This section describes the methods that can be used to generate Ritz vectors in those cases where the stiffness matrix of the substructure is singular. The generation of free-free interface Ritz vectors is of major interest in the current work. As mentioned earlier, both mode shapes and Ritz vectors are derived from the following eigenvalue problem

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{Z} = 0 \quad (7.77)$$

except that in generating Ritz vectors, a complete iteration is not required for individual vectors. If the stiffness matrix $\mathbf{K}$ in Eq. 7.77 is singular (or in other words, if Eq. 7.77 contains several zero eigenvalues), the factorization or inverse of the matrix $\mathbf{K}$ will be impossible. In this case, one of the obvious solution to the eigenvalue problem is to develop a procedure for generating Ritz vectors by using the direct iteration method rather than the inverse iteration method (Humar 1990). The former involves the inversion (or factorization) of mass matrix. However, the direct iteration method will extract modes in sequence from the highest to the lowest. Therefore, a straightforward use of direct iteration method will require the generation of all the independent Ritz vectors down to the lowest, even if only a few of the lowest modes are actually used in the analysis. A modified eigenvalue problem given by

$$(-\omega^2\mathbf{M} + \mathbf{K} - \mu\mathbf{M})\mathbf{Z} = 0 \quad (7.78)$$

may assist in extracting the lowest modes first even when the direct iteration method is used. Equation 7.78 has the same set of eigenvectors as Eq. 7.77 and, in addition,

the direct iteration method when applied to Eq. 7.78 will generate the mode with maximum absolute value of $(\lambda_i - \mu)$ first (Parlett 1980). A proper selection of μ will lead to the generation of Ritz vectors in the order of low λ_i to high λ_i .

The method referred to in the previous paragraph may be useful, but is not very effective. Several other methods of extracting the mode shapes for a singular stiffness matrix are described in standard texts on structural dynamics (Clough and Penzien 1975; Humar 1990). The standard methods include the inverse iteration method with shift, the method of sweeping of the rigid body modes, and the method which relies on the introduction of low stiffness springs to restrain rigid body motions. All of these methods may use the inverse iteration procedure, and generally better results are obtained by using them than by using the direct iteration method. The same methods with little modification can be used in generating Ritz vectors. Again, the main difference is that a complete iteration procedure is not required for individual Ritz vectors.

An alternative to using the Ritz vectors of a free-free structure is to use the Ritz vectors of a structure with an adequate number of degrees of freedom restrained to prevent rigid body motions of the structure. In such a case, the stiffness matrix obtained will no longer be singular and the Ritz vectors can be generated easily, for example, by using the algorithm described in Chapter 2. Such a set of Ritz vectors is similar to the fixed interface Ritz vectors when the restrained degrees of freedom are on the interface, but not otherwise. The degrees of freedom to be fixed can be chosen arbitrarily as long as the rigid body motions are prevented. In the current work, the Ritz vectors obtained by restraining rigid body motions are termed Ritz vectors with rigid body motions restrained. Their use is further explored in Chapter 8.

7.5 SUBSTRUCTURE COUPLING METHODS

Several typical substructure coupling methods based on the use of Ritz vectors are studied here. These methods include the Hintz's free interface method (Hintz

1975), the Rubin's Method derived by Craig and Chang (1977), and the Benfield and Hruda's free interface method (Benfield and Hruda 1971).

7.5.1 Hintz's Free Interface Method

In this method, free interface normal modes along with attachment modes are used. Further, it is assumed that the set of interface nodes between substructures are also chosen to generate the attachment modes. In the development presented here, Ritz vectors are used in place of normal modes. Let a system consist of two substructures, indicated by ' α ', and ' β ' respectively. The physical coordinates, transformed coordinates, and the Ritz vectors of these two substructures can be written in the forms

$$\mathbf{v} = \begin{Bmatrix} \mathbf{v}_i^\alpha \\ \mathbf{v}_j^\alpha \\ \mathbf{v}_i^\beta \\ \mathbf{v}_j^\beta \end{Bmatrix} \quad (7.79)$$

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_n^\alpha \\ \mathbf{q}_a^\beta \\ \mathbf{q}_n^\beta \end{Bmatrix} \quad (7.80)$$

and

$$\Psi_\alpha = \begin{bmatrix} \Psi_{ia}^\alpha & \Psi_{in}^\alpha \\ \Psi_{ja}^\alpha & \Psi_{jn}^\alpha \end{bmatrix} \quad (7.81)$$

and

$$\Psi_\beta = \begin{bmatrix} \Psi_{ia}^\beta & \Psi_{in}^\beta \\ \Psi_{ja}^\beta & \Psi_{jn}^\beta \end{bmatrix} \quad (7.82)$$

The independent coordinates for the coupled system will be

$$\mathbf{u} = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_n^\alpha \\ \mathbf{q}_n^\beta \end{Bmatrix} \quad (7.83)$$

The compatibility on the interface gives

$$\mathbf{v}_j^\alpha = \mathbf{v}_j^\beta \quad (7.84)$$

i.e.

$$\Psi_{ja}^\alpha \mathbf{q}_a^\alpha + \Psi_{jn}^\alpha \mathbf{q}_n^\alpha = \Psi_{ja}^\beta \mathbf{q}_a^\beta + \Psi_{jn}^\beta \mathbf{q}_n^\beta \quad (7.85)$$

Equation 7.85 can be expressed as

$$\mathbf{C}\mathbf{q} = 0 \quad (7.86)$$

where

$$\mathbf{C} = \begin{bmatrix} \Psi_{ja}^\alpha & \Psi_{jn}^\alpha & -\Psi_{ja}^\beta & -\Psi_{jn}^\beta \end{bmatrix} \quad (7.87)$$

Following the coupling procedures, $\mathbf{q}$ is partitioned into linearly independent part $\mathbf{q}_l$ and dependent part $\mathbf{q}_d$ in the form

$$\mathbf{q}_d = \mathbf{q}_a^\beta \quad (7.88)$$

and

$$\mathbf{q}_l = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_n^\alpha \\ \mathbf{q}_n^\beta \end{Bmatrix} = \mathbf{u} \quad (7.89)$$

The corresponding partitioned matrix $\mathbf{C}_{dd}$ and $\mathbf{C}_{dl}$ are therefore as follows

$$\mathbf{C}_{dd} = -\Psi_{ja}^\beta \quad (7.90a)$$

$$\mathbf{C}_{dl} = \begin{bmatrix} \Psi_{ja}^\alpha \Psi_{jn}^\alpha & -\Psi_{jn}^\beta \end{bmatrix} \quad (7.89b)$$

Thus

$$\mathbf{S} = \begin{bmatrix} -\mathbf{C}_{dd}^{-1} \mathbf{C}_{dl} \\ \mathbf{I}_l \end{bmatrix} \quad (7.91)$$

Corresponding to the order of the coordinates in Eq. 7.89, $\mathbf{S}$ matrix is given by

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_{aa}^\alpha & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{nn}^\alpha & \mathbf{0} \\ (\Psi_{ja}^\beta)^{-1} \Psi_{ja}^\alpha & (\Psi_{ja}^\beta)^{-1} \Psi_{jn}^\alpha & -(\Psi_{ja}^\beta)^{-1} \Psi_{jn}^\beta \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{nn}^\beta \end{bmatrix} \quad (7.92)$$

and the coupled coordinates are $\mathbf{u}$.

Use of this method will result in a coupled system with the number of degrees of freedom equal to $N_j + N_n^\alpha + N_n^\beta$, where N_j is the number of degrees of freedom

at the interface nodes, N_n^α the number of generalized coordinates corresponding to Ritz vectors for substructure α , and N_n^β the number of generalized coordinates corresponding to Ritz vectors for substructure β .

7.5.2 Rubin's Method as Derived by Craig and Chang

One special aspect of the methods used by MacNeal (1971), Rubin (1975) and their forms derived by Craig and Chang (1977) is that all junction degrees of freedom (i.e. the degrees of freedom on the interface) can be eliminated in the final form of the coupled equations of motion. This characteristic leads to a more efficient reduction of the structure system compared with the Hintz's method. The Rubin's method in the form derived by Craig and Chang (1977) is discussed here in relation to the use of Ritz vectors defined in the present work. The method utilizes a combination of interior Ritz vectors and attachment modes.

The attachment modes used are inertia relief Ritz and the set of attachment nodes is chosen as the same as the set of junction nodes. The physical coordinates, transformed coordinates, and the Ritz vectors of two substructures can be written in the following forms

$$\mathbf{v} = \begin{Bmatrix} \mathbf{v}_i^\alpha \\ \mathbf{v}_j^\alpha \\ \mathbf{v}_i^\beta \\ \mathbf{v}_j^\beta \end{Bmatrix} \quad (7.93)$$

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_n^\alpha \\ \mathbf{q}_a^\beta \\ \mathbf{q}_n^\beta \end{Bmatrix} \quad (7.94)$$

and

$$\Psi_\alpha = \begin{bmatrix} \Psi_{ia}^\alpha & \Psi_{in}^\alpha \\ \Psi_{ja}^\alpha & \Psi_{jn}^\alpha \end{bmatrix} \quad (7.95)$$

and

$$\Psi_\beta = \begin{bmatrix} \Psi_{ia}^\beta & \Psi_{in}^\beta \\ \Psi_{ja}^\beta & \Psi_{jn}^\beta \end{bmatrix} \quad (7.96)$$

However, unlike in the Hintz's method, the independent coordinates for the coupled system will be

$$\mathbf{u} = \begin{Bmatrix} \mathbf{q}_n^\alpha \\ \mathbf{q}_n^\beta \end{Bmatrix} \quad (7.97)$$

The compatibility on the interface gives

$$\mathbf{v}_j^a = \mathbf{v}_j^b \quad (7.98)$$

i.e.

$$\Psi_{ja}^a \mathbf{q}_a^a + \Psi_{jn}^a \mathbf{q}_n^a = \Psi_{ja}^b \mathbf{q}_a^b + \Psi_{jn}^b \mathbf{q}_n^b \quad (7.99)$$

In eliminating the junction degrees of freedom from the final coupled equations of motion, the same relationship as discussed by Craig and Chang (1977) is used here. This relationship states that when the residual flexibility matrix defined in Eq. 7.27 is used, the attachment mode coordinates may be identified with junction forces by equations

$$\mathbf{q}_a^a = \mathbf{f}_j^a \quad (7.100)$$

and

$$\mathbf{q}_a^b = \mathbf{f}_j^b \quad (7.101)$$

The following relationship is used for the junction forces

$$\mathbf{f}_j^a + \mathbf{f}_j^b = \mathbf{0} \quad (7.102)$$

Equations 7.99 and 7.102 can be grouped in the form

$$\mathbf{C}\mathbf{q} = \mathbf{0} \quad (7.103)$$

where

$$\mathbf{C} = \begin{bmatrix} \Psi_{ja}^a & \Psi_{jn}^a & -\Psi_{ja}^b & -\Psi_{jn}^b \\ \mathbf{I}_{aa}^a & \mathbf{0}_{na}^a & \mathbf{I}_{aa}^b & \mathbf{0}_{na}^b \end{bmatrix} \quad (7.104)$$

Following the coupling procedures, $\mathbf{q}$ is partitioned into linearly independent part $\mathbf{q}_i$ and dependent part $\mathbf{q}_d$ in the form

$$\mathbf{q}_d = \begin{Bmatrix} \mathbf{q}_a^a \\ \mathbf{q}_a^b \end{Bmatrix} \quad (7.105)$$

and

$$\mathbf{q}_l = \begin{Bmatrix} \mathbf{q}_n^\alpha \\ \mathbf{q}_n^\beta \end{Bmatrix} = \mathbf{u} \quad (7.106)$$

The corresponding partitioned matrix $\mathbf{C}_{dd}$ and $\mathbf{C}_{dl}$ are given by

$$\mathbf{C}_{dd} = \begin{bmatrix} \Psi_{ja}^\alpha & -\Psi_{ja}^\beta \\ \mathbf{I}_{aa}^\alpha & \mathbf{I}_{aa}^\beta \end{bmatrix} \quad (7.107)$$

and

$$\mathbf{C}_{dl} = \begin{bmatrix} \Psi_{jn}^\alpha & -\Psi_{jn}^\beta \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (7.108)$$

The inverse of $\mathbf{C}_{dd}$ takes the form

$$\mathbf{C}_{dd}^{-1} = \begin{bmatrix} \kappa' & \kappa' \Psi_{ja}^\beta \\ -\kappa' & \kappa' \Psi_{ja}^\alpha \end{bmatrix} \quad (7.109)$$

where

$$\kappa' = (\Psi_{ja}^\alpha + \Psi_{ja}^\beta)^{-1} \quad (7.110)$$

Thus

$$\mathbf{S} = \begin{bmatrix} -\mathbf{C}_{dd}^{-1} \mathbf{C}_{dl} \\ \mathbf{I}_n \end{bmatrix} \quad (7.111)$$

Substitution of Eqs. 7.108 and 7.109 into Eq. 7.111, and rearranging the elements to conform to the vector $\mathbf{q}$ of Eq. 7.97 gives

$$\mathbf{S} = \begin{bmatrix} -\kappa' \Psi_{jn}^\alpha & \kappa' \Psi_{jn}^\beta \\ \mathbf{I}_{nn}^\alpha & \mathbf{0} \\ \kappa' \Psi_{jn}^\alpha & -\kappa' \Psi_{jn}^\beta \\ \mathbf{0} & \mathbf{I}_{nn}^\beta \end{bmatrix} \quad (7.112)$$

Then

$$\mathbf{q} = \mathbf{S} \mathbf{u} \quad (7.113)$$

To obtain the reduced equations of motion for the coupled system, the following properties described by Craig and Chang (1977) have to be used.

$$\Phi_n^T \mathbf{M} \Phi_a = \Phi_a^T \mathbf{M} \Phi_n = \mathbf{0} \quad (7.114)$$

$$\Phi_n^T \mathbf{K} \Phi_a = \Phi_a^T \mathbf{K} \Phi_n = \mathbf{0} \quad (7.115)$$

$$\Phi_a^T \mathbf{K} \Phi_a = \Phi_{ja}^T \quad (7.116)$$

$$\Phi_{ja} = \Phi_{ja}^T \quad (7.117)$$

where Φ_n are the kept normal modes, Φ_a the attachment modes based on the residual flexibility matrix, and Φ_{ja} the partitioned part of the attachment modes corresponding to the junction degrees of freedom. It is also imperative that the set of attachment nodes on the interface is the same as the set of junction nodes. The properties described by Eqs. 7.114 through 7.117 can be easily proved to be valid for the Ritz vectors too. By using the Ritz vectors defined in the present work, they can be expressed by

$$\Psi_n^T M \Psi_a = \Psi_a^T M \Psi_n = 0 \quad (7.118)$$

$$\Psi_n^T K \Psi_a = \Psi_a^T K \Psi_n = 0 \quad (7.119)$$

$$\Psi_a^T K \Psi_a = \Psi_{ja}^T \quad (7.120)$$

$$\Psi_{ja} = \Psi_{ja}^T \quad (7.121)$$

where Ψ_n and Ψ_a come from the partition

$$\Psi = [\Psi_a \quad \Psi_n] \quad (7.122)$$

Let M_α and K_α be the mass and stiffness matrices respectively for substructure α , and M_β and K_β the mass and stiffness matrices respectively for substructure β . The reduced matrices obtained by Ritz vector transformation for substructure α are

$$M_\alpha^* = \begin{Bmatrix} \Psi_a^{\alpha T} \\ \Psi_n^{\alpha T} \end{Bmatrix} M_\alpha [\Psi_a^\alpha \quad \Psi_n^\alpha] = \begin{bmatrix} M_a^\alpha & 0 \\ 0 & I_{nn} \end{bmatrix} \quad (7.123)$$

and

$$K_\alpha^* = \begin{Bmatrix} \Psi_a^{\alpha T} \\ \Psi_n^{\alpha T} \end{Bmatrix} K_\alpha [\Psi_a^\alpha \quad \Psi_n^\alpha] = \begin{bmatrix} K_a^\alpha & 0 \\ 0 & K_n^\alpha \end{bmatrix} \quad (7.124)$$

in which

$$M_a^\alpha = \Psi_a^{\alpha T} M_\alpha \Psi_a^\alpha \quad (7.125)$$

$$K_a^\alpha = \Psi_a^{\alpha T} K_\alpha \Psi_a^\alpha \quad (7.126)$$

and

$$\mathbf{K}_n^\alpha = \Psi_n^{\alpha T} \mathbf{K}_\alpha \Psi_n^\alpha \quad (7.127)$$

Similarly for substructure β

$$\mathbf{M}_\beta^* = \left\{ \begin{array}{c} \Psi_a^{\beta T} \\ \Psi_n^{\beta T} \end{array} \right\} \mathbf{M}_\beta [\Psi_a^\beta \Psi_n^\beta] = \left[\begin{array}{cc} \mathbf{M}_a^\beta & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{nn} \end{array} \right] \quad (7.128)$$

and

$$\mathbf{K}_\beta^* = \left\{ \begin{array}{c} \Psi_a^{\beta T} \\ \Psi_n^{\beta T} \end{array} \right\} \mathbf{K}_\beta [\Psi_a^\beta \Psi_n^\beta] = \left[\begin{array}{cc} \mathbf{K}_a^\beta & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_n^\beta \end{array} \right] \quad (7.129)$$

in which

$$\mathbf{M}_a^\beta = \Psi_a^{\beta T} \mathbf{M}_\beta \Psi_a^\beta \quad (7.130)$$

$$\mathbf{K}_a^\beta = \Psi_a^{\beta T} \mathbf{K}_\beta \Psi_a^\beta \quad (7.131)$$

and

$$\mathbf{K}_n^\beta = \Psi_n^{\beta T} \mathbf{K}_\beta \Psi_n^\beta \quad (7.132)$$

The coupling of two substructures further reduces the system into the form

$$\mathbf{M}^{**} = \left[\begin{array}{cc} \mathbf{M}^{\alpha\alpha} & \mathbf{M}^{\alpha\beta} \\ \mathbf{M}^{\beta\alpha} & \mathbf{M}^{\beta\beta} \end{array} \right] \quad (7.133)$$

$$\mathbf{K}^{**} = \left[\begin{array}{cc} \mathbf{K}^{\alpha\alpha} & \mathbf{K}^{\alpha\beta} \\ \mathbf{K}^{\beta\alpha} & \mathbf{K}^{\beta\beta} \end{array} \right] \quad (7.134)$$

in which

$$\mathbf{M}^{\alpha\alpha} = \mathbf{I} + \Psi_{jn}^{\alpha T} \mathbf{M}' \Psi_{jn}^\alpha \quad (7.135)$$

$$\mathbf{M}^{\alpha\beta} = (\mathbf{M}^{\beta\alpha})^T = -\Psi_{jn}^{\alpha T} \mathbf{M}' \Psi_{jn}^\beta \quad (7.136)$$

$$\mathbf{M}^{\beta\beta} = \mathbf{I} + \Psi_{jn}^{\beta T} \mathbf{M}' \Psi_{jn}^\beta \quad (7.137)$$

$$\mathbf{K}^{\alpha\alpha} = \mathbf{K}_n^\alpha + \Psi_{jn}^{\alpha T} \mathbf{K}' \Psi_{jn}^\alpha \quad (7.138)$$

$$\mathbf{K}^{\alpha\beta} = (\mathbf{K}^{\beta\alpha})^T = -\Psi_{jn}^{\alpha T} \mathbf{K}' \Psi_{jn}^\beta \quad (7.139)$$

$$\mathbf{K}^{\beta\beta} = \mathbf{K}_n^\beta + \Psi_{jn}^{\beta T} \mathbf{K}' \Psi_{jn}^\beta \quad (7.140)$$

where

$$\mathbf{M}' = \kappa'^T (\mathbf{M}_a^\alpha + \mathbf{M}_a^\beta) \kappa' = \kappa'^T (\Psi_a^{\alpha T} \mathbf{M}_\alpha \Psi_a^\alpha + \Psi_a^{\beta T} \mathbf{M}_\beta \Psi_a^\beta) \kappa' \quad (7.141)$$

and

$$\mathbf{K}' = \kappa'^T (\mathbf{K}_a^\alpha + \mathbf{K}_a^\beta) \kappa' = \kappa'^T (\Psi_a^{\alpha T} \mathbf{K}_\alpha \Psi_a^\alpha + \Psi_a^{\beta T} \mathbf{K}_\beta \Psi_a^\beta) \kappa' \quad (7.142)$$

Noticing the properties in Eqs. 7.118, 7.119, 7.120 and 7.121, Eq. 7.142 is simplified in the following form

$$\mathbf{K}' = \kappa'^T (\Psi_{ja}^\alpha + \Psi_{ja}^\beta) \kappa' = \kappa' \quad (7.143)$$

Therefore,

$$\mathbf{K}^{\alpha\alpha} = \mathbf{K}_n^\alpha + \Psi_{jn}^{\alpha T} \kappa' \Psi_{jn}^\alpha \quad (7.144)$$

$$\mathbf{K}^{\alpha\beta} = (\mathbf{K}^{\beta\alpha})^T = -\Psi_{jn}^{\alpha T} \kappa' \Psi_{jn}^\beta \quad (7.145)$$

$$\mathbf{K}^{\beta\beta} = \mathbf{K}_n^\beta + \Psi_{jn}^{\beta T} \kappa' \Psi_{jn}^\beta \quad (7.146)$$

In comparison to the coupled equations of motion obtained by using normal mode shapes described by Chang (1977), the results obtained by using Ritz vectors look slightly different.

It should be noted that the relationship in Eqs. 7.100 and 7.101 used in Rubin's method are not valid when external forces act on the substructure.

7.5.3 Benfield and Hruda's Free Interface Method

In this method, only the free interface normal modes are used for both substructures. The coupled equations of motion for this method can also be derived by using the general substructure coupling procedure suggested by Craig and Chang (1977). However, the coupling procedure used by Benfield and Hruda is quite straightforward and is applicable even when Ritz vectors are used in place of mode shapes. Let the flexible Ritz vectors for the two substructures α and β be partitioned in the form

$$\Psi_\alpha = \begin{bmatrix} \Psi_{in}^\alpha \\ \Psi_{jn}^\alpha \end{bmatrix} \quad (7.147a)$$

and

$$\Psi_{\beta} = \begin{bmatrix} \Psi_{jn}^{\beta} \\ \Psi_{in}^{\beta} \end{bmatrix} \quad (7.147b)$$

respectively. Symbols i and j respectively represent the internal and junction degrees of freedom for each substructures. The relationship between the physical coordinates and the transformed coordinates can be expressed as

$$\begin{Bmatrix} \mathbf{v}_i^{\alpha} \\ \mathbf{v}_j^{\alpha} \end{Bmatrix} = \begin{bmatrix} \Psi_{in}^{\alpha} \\ \Psi_{jn}^{\alpha} \end{bmatrix} \mathbf{q}_{\alpha} \quad (7.148a)$$

for substructure α , and

$$\begin{Bmatrix} \mathbf{v}_j^{\beta} \\ \mathbf{v}_i^{\beta} \end{Bmatrix} = \begin{bmatrix} \Psi_{jn}^{\beta} \\ \Psi_{in}^{\beta} \end{bmatrix} \mathbf{q}_{\beta} \quad (7.148b)$$

for substructure β . The displacements at interface between the two substructures should be compatible. This leads to

$$\mathbf{v}_j^{\alpha} = \mathbf{v}_j^{\beta} \quad (7.149)$$

or

$$\Psi_{jn}^{\alpha} \mathbf{q}_{\alpha} = \Psi_{jn}^{\beta} \mathbf{q}_{\beta} \quad (7.150)$$

The application of this relationship couples the physical coordinates of the two substructures in the following form

$$\begin{Bmatrix} \mathbf{v}_i^{\alpha} \\ \mathbf{v}_j^{\alpha} \\ \mathbf{v}_j^{\beta} \\ \mathbf{v}_i^{\beta} \end{Bmatrix} = \begin{bmatrix} \Psi_{in}^{\alpha} & 0 \\ \Psi_{jn}^{\alpha} & 0 \\ \Psi_{jn}^{\alpha} & 0 \\ 0 & \Psi_{in}^{\beta} \end{bmatrix} \begin{Bmatrix} \mathbf{q}^{\alpha} \\ \mathbf{q}^{\beta} \end{Bmatrix} \quad (7.151)$$

Therefore, the Ritz vectors that can be used in coupling of two substructures as well as in reducing the size of the whole system may be expressed as

$$\Psi = \begin{bmatrix} \Psi_{in}^{\alpha} & 0 \\ \Psi_{jn}^{\alpha} & 0 \\ \Psi_{jn}^{\alpha} & 0 \\ 0 & \Psi_{in}^{\beta} \end{bmatrix} \quad (7.152)$$

The size of the reduced system obtained by using Benfield and Hruda's free interface method is $N_n^{\alpha} + N_n^{\beta}$, since no attachment Ritz vectors are used in this method.

7.6 EXAMPLES

7.6.1 Application of Ritz Vectors

In this section, examples are given to show the application of Ritz vectors in substructure coupling analysis. The natural frequencies of the complete system obtained by these substructure coupling methods are compared with the exact values of natural frequencies which are obtained by using the well known program SAP-IV in an eigenvalue solution of the complete structural system. Illustrations here are restricted to methods that use free interface Ritz vectors. In particular, the Hintz's free interface method and Benfield and Hrudá's free interface method are investigated.

Consider first the plane frame structure system shown in Fig. 7.2. Substructure α is a 36 degrees of freedom system with no constraints against rigid body motions. Substructure β is a 30 degrees of freedom with two fixed supports. There are six interface degrees of freedom in this case. Although the total number of degrees of freedom for the complete system is 60, only 40 independent modes exist because the 20 rotation degrees of freedom have zero masses. All members in the frame have the same cross section and material properties. The Young's modulus is taken as 25000 MPa, the area of cross section 0.36 m^2 , and the in-plane moment of inertia 0.0108 m^4 . In generating Ritz vectors, unless specified, the diagonal elements of the mass matrices are used as the load amplitude vectors and the pseudo-static deformation vector is used as the starting vector.

Table 7.1 compares the natural frequencies of the system extracted by using Hintz's free interface method. The values listed in column entitled 'Item 1' are the exact natural frequencies of the complete system obtained by using SAP-IV program; those in columns entitled 'Item 2' and 'Item 3' are the natural frequencies obtained by using Hintz's free interface method with 3 flexible Ritz vectors for each substructure plus another 6 attachment Ritz vectors, and with 5 flexible Ritz vectors for each substructure plus another 6 attachment Ritz vectors, respectively.

Three rigid body Ritz vectors are calculated statically, i.e. by using Eq. 7.17 for substructure α . These rigid body Ritz vectors are normalized with respect to the mass matrix and included in the flexible Ritz vectors of substructure. The remaining flexible Ritz vectors for this substructure are generated by orthogonalizing the starting vector with respect to the rigid body Ritz vectors. Six attachment Ritz vectors are obtained based on the residual flexibility matrix in Eq. 7.27. In obtaining the residual flexibility matrix, temporary supports are introduced as shown in Fig. 7.2. Item 4 in Table 7.1 shows the results obtained by using 3 normal modes for each substructures plus six attachment modes. For substructure α , the 3 normal modes are simply the 3 rigid body modes. Compared with the exact values in Item 1, the first four values in Items 2, 3, and 4 all give good estimates of the first four natural frequencies. The results obtained with normal modes are slightly better than those obtained by using Ritz vectors. Increase in the number of Ritz vectors used for each substructure produces a better results as in Item 3.

It should be noted in relation to the example presented above that if 5 mode shapes are used for each substructure with residual attachment modes, computation becomes unstable. The same phenomenon is observed when 6 or more Ritz vectors are used for each substructure. This is because, in sweeping the lower mode shapes or Ritz vectors from the attachment vector, the residual may become almost null. In other words, the elements of flexibility matrix G_e in Eq. 7.27 may become very small, causing numerical instability in the subsequent analysis.

In Hintz's method, the use of attachment modes is shown to give quite accurate results as in Table 7.1. However, as mentioned in the last paragraph, it may also result in numerical instability when many Ritz vectors (or normal modes) are used in the analysis. Also, if the number of degrees of freedom corresponding to the attachment modes is substantial, the reduced system may still be quite large and optimum efficiency may not be achieved in the analysis. Benfield and Hruda's free interface method, which does not rely on the use of attachment modes or constraint modes therefore needs serious consideration. This method has been regarded as less

accurate than many other methods (Craig and Chang 1977; and Benfield 1972). However, since the method does not use any attachment modes or Ritz vectors, the problems related to the use of attachment modes do not exist. Another merit of this method is that the reduced coupled system may be of a smaller size since the interface degrees of freedom do not appear in the coupled equations.

Table 7.2 lists the results obtained by using the Benfield and Hruda's free interface method for the same frame as shown in Fig. 7.2. Again, Item 1 lists the exact natural frequencies obtained by considering the complete structural system. Items 2, 3, 4 are the coupled natural frequencies extracted by the Benfield and Hruda's method using 3+3, 6+6, and 8+8 Ritz vectors respectively. As before, the first three Ritz vectors for substructure α are the rigid body Ritz vectors. Item 5 is the coupled natural frequencies obtained from the Benfield and Hruda's method using 6+6 mode shapes. It is seen that the natural frequencies of the coupled system estimated by using the Benfield and Hruda's free interface method are less accurate than those obtained by using Hintz's method as in Table 7.1. However, the estimate of the fundamental mode by using only 3+3 number of Ritz vectors as listed in Item 2 is fairly accurate. The first four values in Item 3, which are obtained by using 6+6 Ritz vectors in the Benfield and Hruda's method, are quite satisfactory compared with the corresponding exact values. Much better estimates can be obtained when a larger number of Ritz vectors is used as in Item 4 which lists values obtained by using the Benfield and Hruda's method with 8+8 Ritz vectors. Item 5 gives the results obtained by using 6+6 normal modes in the Benfield and Hruda's method. By comparing the results in Item 3 and Item 5 with the exact ones in Item 1, it is found the use of Ritz vectors gives better values than the use of mode shapes.

As a second example, let the Benfield truss structure shown in Fig. 7.3 be examined. The geometry of this truss is taken as the same as described in the work by Benfield (Benfield 1972). It consists of two truss substructures having uniform bay sections. All members have a constant area and uniform mass properties.

Substructure α has 30 degrees of freedom including 3 rigid body degrees of freedom, and substructure β has 36 degrees of freedom, also including 3 rigid body degrees of freedom. The number of interface degrees of freedom is 6. The lengths of members are indicated in Fig. 7.3. Neither Benfield (1972) nor Chang (1977), both of whom use the same truss, give any information on the elastic modulus and the mass density. When the ratio between the elastic modulus and the mass density, $\frac{E}{\rho g}$ is taken as 10,000 (It should be noted that MPa is used as the unit for elastic modulus, and kg/m^3 as the unit for mass density, and $g = 9.81 \text{ m/s}^2$), the frequency results match those given in the thesis by Chang (1977).

Table 7.3 shows the square of natural frequencies for the assembled truss analyzed by using the Hintz's method. Values in Item 1 of this table are the so-called exact values obtained by eigenvalue solution of the complete structure using SAP-IV. Item 2 are the results obtained by using Hintz's method with 3+3 flexible Ritz vectors for substructure α and β respectively, plus 6 attachment Ritz vectors. The 3+3 flexible Ritz vectors are simply 3+3 rigid body Ritz vectors for substructure α and β . Results in Item 3 are obtained by using the Hintz's method with 7+7 flexible Ritz vectors for substructure α and β respectively, plus 6 attachment Ritz vectors. Among these 7+7 flexible Ritz vectors, there are 3+3 rigid body Ritz vectors. Items 4 and 5 list the results for the same cases as in Item 2 and 3 respectively except that normal modes are used in place of Ritz vectors. These results are taken directly from the work by Chang (1977) for comparison. It is seen that for lower modes the results obtained by using the Hintz's method with Ritz vectors give almost as good results as those obtained by using the Hintz's method with mode shapes. Results obtained with the use of Ritz vectors tend to span greater range of eigenvalues.

The Benfield truss is next examined by using the Benfield and Hruda's method. Table 7.4 presents the results obtained. Exact values of the square of the natural frequencies are still listed in Item 1. Items 2, 3, and 4 present the results obtained by using the Benfield and Hruda's method with 3+3, 6+6, and 10+10 Ritz vectors

for substructure α and β respectively. Item 5 gives the results obtained by using the Benfield and Hruda's method with 6+6 mode shapes for substructure α and β respectively. It is shown that there is not too large a difference between the results obtained by using Ritz vectors and those obtained by using mode shapes. By comparing the results presented in Tables 7.3 and 7.4, it becomes clear that for equal number of total vectors, the results of Benfield and Hruda's method are much poorer than those obtained by using Hintz's method.

7.6.2 Effect of The Number of Ritz Vectors Used for Substructures

The number of Ritz vectors used to represent each substructures can have marked influence on the behavior of the coupled system. It is not true that the use of an equal number of Ritz vectors for each substructure will produce good coupling between two substructures. A general study of this problem could be very substantial. The discussion here is restricted to the use of Benfield and Hruda's free interface method and to the structural system shown in Fig. 7.2. The success of coupling procedure depends on how well the interface displacements are represented in each of the two substructures. Therefore, for substructure β which has fixed supports at the bottom and interface degrees of freedom at the top, a good representation of the interface displacements may need just a few Ritz vectors that extract a small number of lower natural frequencies including the fundamental frequency of the substructure. This is so because interface displacements which are the tip displacements are well represented by the lower modes. For substructure α , the interface degrees of freedom are at the bottom. As a result, a good representation of these interface displacements needs the inclusion of some higher frequencies of the substructure. On the other hand, the tip displacements of substructure α play an important role in extracting the lower natural frequencies of the coupled system. Therefore, in order to achieve a good coupling effect, some Ritz vectors that extract lowest natural frequencies as well as some others which extract high natural frequencies should be included. As a result, a large number of Ritz vectors should be used for substructure α . The foregoing reasoning is explained by using 3

Ritz vectors for substructure β and various number of Ritz vectors for substructure α in the Benfield and Hruda's method to analyze the structural system in Fig. 7.2.

Table 7.5 shows the results obtained. Item 1 still refers to the exact natural frequencies of the system. Items 2, 3, and 4 presents the results obtained by using 4, 6, 8 Ritz vectors respectively for substructure α . The results tend to be closer to the accurate values with increasing number of Ritz vectors used to represent substructure α . An interesting observation can be made by comparing Items 3 and 4 in this table with Items 3 and 4 respectively in Table 7.2. In each case, items being compared are based on the use of same number of Ritz vectors are used for substructure α . However, only 3 Ritz vectors are used for substructure β in the results shown in Table 7.5. Values in Item 3 of Table 7.2 are the results obtained with the use of 6+6 Ritz vectors while those in Item 3 in Table 7.5 are the results obtained with 6+3 Ritz vectors. Both results are almost the same for the first several elements which are close to the exact values. Similarly, values in Item 4 in Table 7.2 are the results obtained with the use of 8+8 Ritz vectors while those in Item 3 in Table 7.5 are the results obtained with 8+3 Ritz vectors. Both results are also almost the same for a number of elements, which are also close to the exact ones.

7.6.3 Effect of Spatial Distribution of The Applied Loads

In all of the previous examples, the diagonal elements of the mass matrices are used as the amplitudes of the starting load vector in generating the flexible Ritz vectors. One advantage of using load dependent Ritz vectors rather than mode shapes, is the possibility of taking into consideration the effect of spatial distribution of the applied loads and thus generating a set of Ritz vectors that are most effective in representing the specified load distribution. When the problem of interest is to find the natural frequencies of the coupled system, no external loads are present. The process of selecting an initial load vector for such a structure needs to be examined. These issues are investigated in the present section. The structural system shown in fig. 7.2 is treated as an example. Discussion is restricted to the

Benfield and Hruda's free interface method. Table 7.6 presents the frequencies obtained by using this method with several different spatial distributions for the starting load vector. The exact natural frequencies of the complete system are given under Item 1 for comparison. Results under Item 2 of this table are obtained by using 6+3 Ritz vectors for a load of unit amplitude applied horizontally at each node of the system, including the interface nodes. Results under Item 3 are obtained by using 6+3 Ritz vectors in the generation of which the starting load vector consists of unit amplitude vertical loads at each node of the system, including the interface nodes. Item 4 lists the results obtained by using 6+3 Ritz vectors but with a starting load vector consisting of unit amplitude moments applied in a clockwise direction at each node of the system, including the interface nodes. These results show that different spatial distributions of the starting load vector have a very significant effect on the generating Ritz vectors leading to considerable differences in the calculated coupled natural frequencies for the system. In considering the distribution of loads in the starting vector, care should be exercised to see that the loads are not restricted to degrees of freedom that will correspond to higher frequency modes. If such a distribution of loads is used in generating Ritz vectors, the lower natural frequencies for the coupled system may be in large error. In the present example, the distribution consisting of loads along vertical direction corresponds to the higher frequency modes, and the frequency results in Item 3 are therefore in significant error.

When the spatial distribution of external loads is unknown, as is common when the problem of interest relates to the calculation of natural frequencies, the load distribution used in generating Ritz vectors can be taken as consisting of loads along each degree of freedom with a magnitude equal to the lumped mass along that degree of freedom. Frequencies obtained for the frame of Fig. 7.2 by using such a distribution have already been presented under Item 4 in Table 7.5. In the example being considered here, a distribution consisting of loads applied at the tip nodes for substructure β and at both the interface and tip nodes for substructure α , along both the horizontal and vertical directions also gives good results, because

this type of load distribution is likely to provide a good representation of the tip and interface displacements. Results obtained by using such a distribution of loads with unit load amplitudes are given in Item 5 of Table 7.6.

It is recognized that a more comprehensive study of this part may be required. In the example presented here, all load amplitudes are taken as being equal. Use of different load amplitudes needs further study. Also, the examples are limited to the determination of natural frequencies. Forced vibration response under applied loads has not been considered. In the later case, the effect of the frequency contents of applied loads may also be worth considering as shown in Chapter 2.

7.7 CHAPTER SUMMARY

The use of Ritz vectors in place of mode shapes in substructure coupling methods is explored. Necessary modifications are made in the coupling procedure for the assembly of substructures to be used in association with Ritz vectors. The methods for generating Ritz vectors for the case of a singular stiffness matrix are discussed. Considerations that apply in the selection of a starting load vector are described. The use of Ritz vectors in several different methods of substructure coupling is studied. Examples are given of the application of Ritz vectors in the Hintz's free interface method and the Benfield and Hruda's free interface method. It is found that the Hintz's method which uses the attachment Ritz vectors generally gives better results than the Benfield and Hruda's method. However, as the number of Ritz vectors used in the Hintz's method is increased, numerical instability may arise in computing residual attachment modes or Ritz vectors. This phenomenon is less apparent when the number of interface degrees of freedom of a structure system is small relative to the total number of degrees of freedom of the system. The merit of the Benfield and Hruda's method is that, in some cases the coupled system can be reduced to that of a smaller size compared to the one in Hintz's. This can be of some significance in the analysis of interaction problems such as soil-structure interaction. The examples also show that results obtained by using Ritz vectors

tends to span a greater range of eigenvalues than those obtained by using mode shapes.

Investigation is also carried out on the effect of the number of Ritz vectors used, and the effect of the spatial distribution of loads in the starting vector. It is shown that the accuracy of frequency results depends on how well the displacements at the interfaces between substructures are represented by the extracted substructure Ritz vectors as well as on whether the lowest modes are included in the Ritz vector set. Through studies of the effect of different spatial distribution of loads in the starting vector, it is shown that a judicious choice of spatial distribution of loads in the starting vector could result in greater accuracy in the calculated frequencies.

Table 7.1 Frame Example using Hintz Method

Mode No.	Item 1	Item 2	Item 3	Item 4
	ω	ω	ω	ω
1	1.406455	1.406483	1.406482	1.406455
2	4.327304	4.328979	4.331760	4.328107
3	7.620147	7.640272	7.632326	7.633447
4	11.06380	11.54007	11.08052	11.49522
5	14.77408	17.07807	15.05803	16.88903
6	18.61945	22.43064	19.68165	23.93399
7	22.46817	40.57298	24.05345	40.80829
8	26.03101	48.24455	26.37311	48.25295
9	28.97921	80.65146	30.43530	82.57626
10	30.93517	139.7709	40.01042	142.6007
11	40.67386	373.1362	52.30614	373.1362
12	44.71999	11138.49	101.0794	11290.03
13	121.0245		239.2144	
14	122.3974		373.1456	
15	198.4095		396.2309	
16	199.2279		8119.990	
17	270.9372			
18	271.5138			
19	336.8433			
20	337.2825			
21	372.6841			
22	372.7061			
23	372.7537			
24	372.8416			
25	372.9848			
26	373.1922			
27	373.4536			
28	373.7263			
29	373.9375			
30	394.5441			
31	394.8932			
32	442.6954			
33	442.9808			
34	456.5796			
35	480.2739			
36	480.5138			
37	506.7108			
38	506.9205			
39	522.0778			
40	522.2717			

*notes: ω is in radians/sec in this and all the following tables.

Table 7.2 Frame Example using Benfield and Hruda's methods

Mode No.	Item 1	Item 2	Item 3	Item 4	Item 5
	ω	ω	ω	ω	ω
1	1.406455	1.622364	1.461460	1.442215	1.497750
2	4.327304	7.332834	5.006352	4.631580	5.529416
3	7.620147	15.53388	7.978880	7.785854	9.338306
4	11.06380	28.68455	12.46783	11.65031	13.55733
5	14.77408	34.34515	19.29728	14.85948	20.25385
6	18.61945	110.7349	26.25514	20.43129	27.07080
7	22.46817		30.70427	22.91150	31.51862
8	26.03101		45.37606	28.48788	45.27485
9	28.97921		61.29224	39.64245	54.83010
10	30.93517		92.81530	44.72654	224.5874
11	40.67386		223.8492	90.24224	373.8043
12	44.71999		429.3208	177.2387	427.0908
13	121.0245			286.2294	
14	122.3974			338.1309	
15	198.4095			373.8033	
16	199.2279			701.8227	
17	270.9372				
18	271.5138				
19	336.8433				
20	337.2825				
21	372.6841				
22	372.7061				
23	372.7537				
24	372.8416				
25	372.9848				
26	373.1922				
27	373.4536				
28	373.7263				
29	373.9375				
30	394.5441				
31	394.8932				
32	442.6954				
33	442.9808				
34	456.5796				
35	480.2739				
36	480.5138				
37	506.7108				
38	506.9205				
39	522.0778				
40	522.2717				

Table 7.3 Benfield Truss by using Hintz's method

Mode No.	Item 1	Item 2	Item 3	Item 4	Item 5
	$\omega^2 \times 10^{-2}$	$\omega^2 \times 10^{-2}$	$\omega^2 \times 10^{-2}$	$\omega^2 \times 10^{-2}$	$\omega^2 \times 10^{-2}$
1	0.043905	0.043945	0.043921	0.043944	0.043906
2	0.183289	0.193376	0.183700	0.193168	0.183296
3	0.305077	0.307001	0.305335	0.309609	0.305172
4	0.415887	0.619888	0.405721	0.611660	0.415993
5	0.685847	1.666020	0.671821	1.646189	0.686420
6	0.987703	2.136665	1.027936	2.121762	0.990516
7	1.139594	3.653706	1.111390	3.617815	1.141742
8	1.223620	5.166250	1.289201	5.150533	1.242987
9	1.507012	8.217200	1.702246	8.220058	1.570687
10	1.551455		2.089932		1.620250
11	1.740518		2.661196		2.182797
12	2.009789		3.067846		2.376230
13	2.151047		3.577868		3.604723
14	2.533974		4.435438		4.555345
15	2.595440		6.550472		4.679652
16	2.913301		7.807967		6.125424
17	3.199521		16.38807		10.67890
18	3.574607				
19	3.878755				
20	4.293037				
21	4.417111				
22	4.501776				
23	4.588603				
24	4.654412				
25	4.765337				
26	5.708008				
27	6.235887				
28	6.282200				
29	6.507787				
30	6.849990				
31	7.021691				
32	7.926300				
33	8.020393				
34	8.182599				
35	9.005579				
36	9.262202				
37	9.523869				

Table 7.4 Benfield Truss by using the Benfield and Hruda's method

Mode No.	Item 1	Item 2	Item 3	Item 4	Item 5
	$\omega^2 \times 10^{-2}$	$\omega^2 \times 10^{-2}$	$\omega^2 \times 10^{-2}$	$\omega^2 \times 10^{-2}$	$\omega^2 \times 10^{-2}$
1	0.043905	0.194252	0.067447	0.058883	0.064664
2	0.183289	0.767928	0.192366	0.188442	0.190320
3	0.305077	1.526457	0.364558	0.323572	0.359973
4	0.415887		0.465536	0.438663	0.449912
5	0.685847		0.794570	0.692237	0.701167
6	0.987703		1.300812	1.091916	1.157365
7	1.139594		1.622404	1.129416	1.573924
8	1.223620		2.316496	1.365364	2.238865
9	1.507012		4.757480	1.901398	4.706253
10	1.551455			2.169977	
11	1.740518			2.445986	
12	2.009789			3.300448	
13	2.151047			3.417625	
14	2.533974			4.713524	
15	2.595440			5.133895	
16	2.913301			7.269089	
17	3.199521			8.469556	
18	3.574607				
19	3.878755				
20	4.293037				
21	4.417111				
22	4.501776				
23	4.588603				
24	4.654412				
25	4.765337				
26	5.708008				
27	6.235887				
28	6.282200				
29	6.507787				
30	6.849990				
31	7.021691				
32	7.926300				
33	8.020393				
34	8.182599				
35	9.005579				
36	9.262202				
37	9.523809				
38					
39					
40					

Table 7.5 Effect of the number of Ritz vectors

Mode No.	Item 1	Item 2	Item 3	Item 4
	ω	ω	ω	ω
1	1.406455	1.547279	1.468138	1.449056
2	4.327304	6.375925	5.068622	4.680198
3	7.620147	13.04619	7.982524	7.799588
4	11.06380	19.99111	12.77476	11.73153
5	14.77408	33.41450	19.68145	15.17629
6	18.61945	34.79489	31.90244	20.92769
7	22.46817	110.7349	61.49604	24.62003
8	26.03101		105.3467	41.80427
9	28.97921		110.7350	110.7346
10	30.93517			113.7771
11	40.67386			373.8033
12	44.71999			
13	121.0245			
14	122.3974			
15	198.4095			
16	199.2279			
17	270.9372			
18	271.5138			
19	336.8433			
20	337.2825			
21	372.6841			
22	372.7061			
23	372.7537			
24	372.8416			
25	372.9848			
26	373.1922			
27	373.4536			
28	373.7263			
29	373.9375			
30	394.5441			
31	394.8932			
32	442.6954			
33	442.9808			
34	456.5796			
35	480.2739			
36	480.5138			
37	506.7108			
38	506.9205			
39	522.0778			
40	522.2717			

Table 7.6 Effect of the Space Distribution of Applied Load

Mode No.	Item 1	Item 2	Item 3	Item 4	Item 5
	ω	ω	ω	ω	ω
1	1.406455	1.523416	2.993912	1.501199	1.501734
2	4.327304	6.087279	16.07328	5.586168	5.686918
3	7.620147	11.68943	41.67563	9.570791	10.48112
4	11.06380	14.67451	59.25758	13.96703	14.17879
5	14.77408	21.69045	107.1606	20.50628	20.84875
6	18.61945	25.04595	130.0780	35.80098	24.14607
7	22.46817	40.99814	218.4734	50.39274	41.15667
8	26.03101	69.55118	334.8261	64.72480	74.71170
9	28.97921	110.7023	439.1208	110.7029	110.7024
10	30.93517				
11	40.67386				
12	44.71999				
13	121.0245				
14	122.3974				
15	198.4095				
16	199.2279				
17	270.9372				
18	271.5138				
19	336.8433				
20	337.2825				
21	372.6841				
22	372.7061				
23	372.7537				
24	372.8416				
25	372.9848				
26	373.1922				
27	373.4536				
28	373.7263				
29	373.9375				
30	394.5441				
31	394.8932				
32	442.6954				
33	442.9808				
34	456.5796				
35	480.2739				
36	480.5138				
37	506.7108				
38	506.9205				
39	522.0778				
40	522.2717

Fig. 7.1 Coupling between two substructures

Fig. 7.2 A simple structure coupling example

Substructure β

Substructure α

Fig. 7.3 Benfield truss structure

Chapter 8

SOIL-SUBSTRUCTURE INTERACTION USING RITZ VECTOR COUPLING METHODS

8.1 INTRODUCTION

A possible application of the substructure coupling methods described in Chapter 7 is in the analysis of interaction problems in structures. Real problems involving interaction include soil-structure and dam-foundation-reservoir analyses. These systems are characterized by coupling between substructures with different physical properties. As a result, the analytical modelling approach used for each substructure is usually different. In many cases, the Boundary Element Method(BEM) is found to be useful for a substructure which consists of an infinite domain(soil) or a continuous medium(water). When BEM method is used for a substructure, that substructure is usually represented by only the interface degrees of freedom, and the analysis is often carried out in the frequency domain. Hence, the coupling procedures to be used for such systems should be applicable to analysis in the frequency domain, and to cases where one of the substructures is represented by only the interface degrees of freedom. These characteristics may, on the one hand, rule out the use of some of the available methods, while on the other, they may simplify the application of some methods. Studies are carried out here on the application of some of the substructure coupling methods in the analysis of soil-structure interaction and, where possible, simplified forms of the coupling procedures are derived. Several examples are provided to show the effectiveness of the simplified methods developed in this chapter. These examples include a soil-structure analysis using BEM method and with earthquake input. Results are compared with the exact solutions where available or with the those obtained by alternative methods of analysis.

8.2 HINTZ'S FREE INTERFACE METHOD

The Hintz's free interface method using Ritz vectors has been thoroughly discussed in the last chapter. In the following a simplified form of the procedure is developed for use in the analysis of soil-structure interaction problems. The structural system shown in Fig. 8.1 is used to simulate a soil-structure interaction problem in the development in this chapter. Substructure α in this figure represents a structure built on a soil foundation. The stiffness and mass matrices of the structure are usually obtained by the Finite Element Method. Substructure β contains only the interface degrees of freedom, and is used to represent the soil part.

8.2.1 Simplified Hintz's Free Interface Method

In this method, it is assumed that the set of interface nodes between substructures are also chosen as the attachment nodes. Let the system under consideration consist of two substructures, indicated by ' α ', and ' β ' respectively, and let the substructure β have only the interface degrees of freedom. Since substructure β involves only junction degrees of freedom, Ritz vectors need not be generated for the substructure. The physical coordinates, transformed coordinates, and the Ritz vectors of these two substructures can be written in the forms

$$\mathbf{v} = \begin{Bmatrix} \mathbf{v}_i^\alpha \\ \mathbf{v}_j^\alpha \\ \mathbf{v}_j^\beta \end{Bmatrix} \quad (8.1)$$

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_n^\alpha \\ \mathbf{v}_j^\beta \end{Bmatrix} \quad (8.2)$$

$$\Psi^\alpha = \begin{bmatrix} \Psi_{ia}^\alpha & \Psi_{in}^\alpha \\ \Psi_{ja}^\alpha & \Psi_{jn}^\alpha \end{bmatrix} \quad (8.3)$$

The independent coordinates for the coupled system will be

$$\mathbf{u} = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_n^\alpha \end{Bmatrix} \quad (8.4)$$

The compatibility on the interface gives

$$\mathbf{v}_j^\alpha = \mathbf{v}_j^\beta \quad (8.5)$$

that is,

$$\Psi_{ja}^{\alpha} \mathbf{q}_a^{\alpha} + \Psi_{jn}^{\alpha} \mathbf{q}_n^{\alpha} = \mathbf{v}_j^{\beta} \quad (8.6)$$

The transformation between coordinates $\mathbf{q}$ and $\mathbf{u}$ can now be written as

$$\mathbf{q} = \mathbf{S} \mathbf{u} \quad (8.7)$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_{aa}^{\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{nn}^{\alpha} \\ \Psi_{ja}^{\alpha} & \Psi_{jn}^{\alpha} \end{bmatrix} \quad (8.8)$$

Let the Ritz vectors for substructure α be partitioned as

$$\Psi^{\alpha} = [\Psi_a^{\alpha} \quad \Psi_n^{\alpha}] \quad (8.9)$$

Also let $\mathbf{M}_{\alpha}$ and $\mathbf{K}_{\alpha}$ be the mass and stiffness matrices respectively for substructure α , and $\mathbf{M}_{\beta}$ and $\mathbf{K}_{\beta}$ be the mass and stiffness matrices respectively for substructure β . The reduced matrices obtained by a transformation involving Ritz vectors for substructure α are given by

$$\mathbf{M}_{\alpha}^* = \begin{Bmatrix} \Psi_a^{\alpha T} \\ \Psi_n^{\alpha T} \end{Bmatrix} \mathbf{M}_{\alpha} [\Psi_a^{\alpha} \quad \Psi_n^{\alpha}] = \begin{bmatrix} \mathbf{M}_a^{\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{nn} \end{bmatrix} \quad (8.10)$$

and

$$\mathbf{K}_{\alpha}^* = \begin{Bmatrix} \Psi_a^{\alpha T} \\ \Psi_n^{\alpha T} \end{Bmatrix} \mathbf{K}_{\alpha} [\Psi_a^{\alpha} \quad \Psi_n^{\alpha}] = \begin{bmatrix} \mathbf{K}_a^{\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_n^{\alpha} \end{bmatrix} \quad (8.11)$$

in which

$$\mathbf{M}_a^{\alpha} = \Psi_a^{\alpha T} \mathbf{M}_{\alpha} \Psi_a^{\alpha} \quad (8.12)$$

$$\mathbf{K}_a^{\alpha} = \Psi_a^{\alpha T} \mathbf{K}_{\alpha} \Psi_a^{\alpha} \quad (8.13)$$

and

$$\mathbf{K}_n^{\alpha} = \Psi_n^{\alpha T} \mathbf{K}_{\alpha} \Psi_n^{\alpha} \quad (8.14)$$

The coupling of two structures using the transformation $\mathbf{q} = \mathbf{S} \mathbf{u}$ gives

$$\mathbf{M}^{**} = \begin{bmatrix} \mathbf{M}^{\alpha\alpha} & \mathbf{M}^{\alpha\beta} \\ \mathbf{M}^{\beta\alpha} & \mathbf{M}^{\beta\beta} \end{bmatrix} \quad (8.15)$$

$$\mathbf{K}^{**} = \begin{bmatrix} \mathbf{K}^{\alpha\alpha} & \mathbf{K}^{\alpha\beta} \\ \mathbf{K}^{\beta\alpha} & \mathbf{K}^{\beta\beta} \end{bmatrix} \quad (8.16)$$

where

$$\mathbf{M}^{\alpha\alpha} = \mathbf{M}_a^\alpha + \Psi_{ja}^{\alpha T} \mathbf{M}_\beta \Psi_{ja}^\alpha \quad (8.17)$$

$$\mathbf{M}^{\alpha\beta} = (\mathbf{M}^{\beta\alpha})^T = \Psi_{ja}^{\alpha T} \mathbf{M}_\beta \Psi_{jn}^\alpha \quad (8.18)$$

$$\mathbf{M}^{\beta\beta} = \mathbf{I}_{nn} + \Psi_{jn}^{\alpha T} \mathbf{M}_\beta \Psi_{jn}^\alpha \quad (8.19)$$

$$\mathbf{K}^{\alpha\alpha} = \mathbf{K}_a^\alpha + \Psi_{ja}^{\alpha T} \mathbf{K}_\beta \Psi_{ja}^\alpha \quad (8.20)$$

$$\mathbf{K}^{\alpha\beta} = (\mathbf{K}^{\beta\alpha})^T = \Psi_{ja}^{\alpha T} \mathbf{K}_\beta \Psi_{jn}^\alpha \quad (8.21)$$

$$\mathbf{K}^{\beta\beta} = \mathbf{K}_n^\alpha + \Psi_{jn}^{\alpha T} \mathbf{K}_\beta \Psi_{jn}^\alpha \quad (8.22)$$

Let the coupled equations of motion be expressed as

$$\mathbf{M}^{**} \ddot{\mathbf{u}} + \mathbf{C}^{**} \dot{\mathbf{u}} + \mathbf{K}^{**} \mathbf{u} = \mathbf{S}^T \Psi^T \mathbf{f}_p(t) \quad (8.23)$$

where

$$\mathbf{u} = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_n^\alpha \end{Bmatrix} \quad (8.24)$$

$$\Psi = \begin{bmatrix} \Psi_{ia}^\alpha & \Psi_{in}^\alpha & \mathbf{0} \\ \Psi_{ja}^\alpha & \Psi_{jn}^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}^\beta \end{bmatrix} \quad (8.25)$$

and the external load amplitude vector

$$\mathbf{f} = \begin{Bmatrix} \mathbf{f}_i^\alpha \\ \mathbf{f}_j^\alpha \\ \mathbf{f}_j^\beta \end{Bmatrix} \quad (8.26)$$

Usually the external loads on the interface nodes are taken to be zeros, then Eq. 8.26 becomes

$$\mathbf{f} = \begin{Bmatrix} \mathbf{f}_i^\alpha \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \quad (8.27)$$

Substitution of Eqs. 8.25, 8.27 and 8.8 into the right side of Eq. 8.23 gives

$$\mathbf{S}^T \Psi^T \mathbf{f}_p(t) = \begin{Bmatrix} \Psi_{ia}^{\alpha T} \\ \Psi_{in}^{\alpha T} \end{Bmatrix} \mathbf{f}_i^\alpha(t) \quad (8.28)$$

8.2.2 Simplified Hintz's Free Interface Method for Analysis In Frequency Domain

The simplified Hintz's free interface method described in the last section leads to some interesting results when applied in the frequency domain analysis. In order to place the equation in a form suitable for frequency domain analysis, let the coupled mass and stiffness matrices be denoted by

$$\mathbf{M}^{**} = \mathbf{M}_s + \mathbf{M}_b \quad (8.29)$$

and

$$\mathbf{K}^{**} = \mathbf{K}_s + \mathbf{K}_b \quad (8.30)$$

in which,

$$\mathbf{M}_s = \begin{bmatrix} \mathbf{M}_a^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{nn} \end{bmatrix} \quad (8.31)$$

$$\mathbf{M}_b = \begin{bmatrix} \Psi_{ja}^{\alpha T} \mathbf{M}_\beta \Psi_{ja}^\alpha & \Psi_{ja}^{\alpha T} \mathbf{M}_\beta \Psi_{jn}^\alpha \\ \Psi_{jn}^{\alpha T} \mathbf{M}_\beta \Psi_{ja}^\alpha & \Psi_{jn}^{\alpha T} \mathbf{M}_\beta \Psi_{jn}^\alpha \end{bmatrix} = \left\{ \begin{matrix} \Psi_{ja}^{\alpha T} \\ \Psi_{jn}^{\alpha T} \end{matrix} \right\} \mathbf{M}_\beta [\Psi_{ja}^\alpha \ \Psi_{jn}^\alpha] \quad (8.32)$$

$$\mathbf{K}_s = \begin{bmatrix} \mathbf{K}_a^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_n^\alpha \end{bmatrix} \quad (8.33)$$

and

$$\mathbf{K}_b = \begin{bmatrix} \Psi_{ja}^{\alpha T} \mathbf{K}_\beta \Psi_{ja}^\alpha & \Psi_{ja}^{\alpha T} \mathbf{K}_\beta \Psi_{jn}^\alpha \\ \Psi_{jn}^{\alpha T} \mathbf{K}_\beta \Psi_{ja}^\alpha & \Psi_{jn}^{\alpha T} \mathbf{K}_\beta \Psi_{jn}^\alpha \end{bmatrix} = \left\{ \begin{matrix} \Psi_{ja}^{\alpha T} \\ \Psi_{jn}^{\alpha T} \end{matrix} \right\} \mathbf{K}_\beta [\Psi_{ja}^\alpha \ \Psi_{jn}^\alpha] \quad (8.34)$$

If proportional damping exists for the two substructures, it is easy to prove that the coupled damping matrix will take the form

$$\mathbf{C}^{**} = \mathbf{C}_s + \mathbf{C}_b \quad (8.35)$$

where

$$\mathbf{C}_s = \begin{bmatrix} \mathbf{C}_a^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_n^\alpha \end{bmatrix} \quad (8.36)$$

and

$$\mathbf{C}_b = \begin{bmatrix} \Psi_{ja}^{\alpha T} \mathbf{C}_\beta \Psi_{ja}^\alpha & \Psi_{ja}^{\alpha T} \mathbf{C}_\beta \Psi_{jn}^\alpha \\ \Psi_{jn}^{\alpha T} \mathbf{C}_\beta \Psi_{ja}^\alpha & \Psi_{jn}^{\alpha T} \mathbf{C}_\beta \Psi_{jn}^\alpha \end{bmatrix} = \left\{ \begin{matrix} \Psi_{ja}^{\alpha T} \\ \Psi_{jn}^{\alpha T} \end{matrix} \right\} \mathbf{C}_\beta [\Psi_{ja}^\alpha \ \Psi_{jn}^\alpha] \quad (8.37)$$

The coupled equations of motion can therefore be written as

$$[\mathbf{M}_s + \mathbf{M}_b] \ddot{\mathbf{u}} + [\mathbf{C}_s + \mathbf{C}_b] \dot{\mathbf{u}} + [\mathbf{K}_s + \mathbf{K}_b] \mathbf{u} = \left\{ \begin{matrix} \Psi_{ia}^{\alpha T} \\ \Psi_{in}^{\alpha T} \end{matrix} \right\} \mathbf{f}_i^\alpha p(t) \quad (8.38)$$

where, the external loads on the interface nodes are taken to be zeros. Replacing the load function $p(t)$ with the delta function $\delta(t)$ in Eq. 8.38 and applying the forward Fourier transform yields

$$[\mathbf{S}_s(\Omega) + \mathbf{S}_b(\Omega)] \mathbf{H}(\Omega) = \left\{ \begin{matrix} \Psi_{ia}^{\alpha T} \\ \Psi_{in}^{\alpha T} \end{matrix} \right\} \mathbf{f}_i^{\alpha} \quad (8.39)$$

where

$$\mathbf{S}_s(\Omega) = -\Omega^2 \mathbf{M}_s + i\Omega \mathbf{C}_s + \mathbf{K}_s \quad (8.40)$$

is the dynamic stiffness matrix for substructure α . The computations for $\mathbf{S}_s$ are associated with the characteristics of substructure α only. Also,

$$\mathbf{S}_b(\Omega) = -\Omega^2 \mathbf{M}_b + i\Omega \mathbf{C}_b + \mathbf{K}_b = \left\{ \begin{matrix} \Psi_{ja}^{\alpha T} \\ \Psi_{jn}^{\alpha T} \end{matrix} \right\} \mathbf{S}_{\beta}(\Omega) [\Psi_{ja}^{\alpha} \quad \Psi_{jn}^{\alpha}] \quad (8.41)$$

in which

$$\mathbf{S}_{\beta}(\Omega) = -\Omega^2 \mathbf{M}_{\beta} + i\Omega \mathbf{C}_{\beta} + \mathbf{K}_{\beta} \quad (8.42)$$

is the dynamic stiffness matrix of substructure β . The computations for $\mathbf{S}_{\beta}$ do not involve substructure α . If BEM method is used for foundation soil, the dynamic stiffness matrix $\mathbf{S}_{\beta}(\Omega)$ can often be obtained directly for the soil domain as discussed in detail in Chapter 6. In that case,

$$\mathbf{S}_{\beta}(\Omega) = \mathbf{S}_{ff}^f(\Omega) \quad (8.43)$$

where $\mathbf{S}_{ff}^f(\Omega)$ is defined as the free field dynamic stiffness matrix of foundation soil.

The final coupled equations of motion obtained by using the simplified Hintz's free interface method are of the same size as the reduced equations of motion obtained by the first stage Ritz vectors described in Chapter 6. However, in that method, the Ritz vectors used are, in fact, based on a constrained interface.

8.3 SIMPLIFIED RUBIN'S METHOD AS DERIVED BY CRAIG AND CHANG

In this method, the inertia relief Ritz vectors defined in Eq. 7.27 are used and the set of attachment nodes is chosen as the same as the set of junction nodes. The

coupled equations of motion for Rubin's method, as presented by Craig and Chang but using Ritz vectors, are given in the last chapter. For the special case where only the junction degrees of freedom exists for substructure β , the physical coordinates, transformed coordinates, and the Ritz vectors of two substructures can be written in the forms

$$\mathbf{v} = \begin{Bmatrix} \mathbf{v}_i^\alpha \\ \mathbf{v}_j^\alpha \\ \mathbf{v}_j^\beta \end{Bmatrix} \quad (8.44)$$

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_n^\alpha \\ \mathbf{q}_a^\beta \end{Bmatrix} \quad (8.45)$$

and

$$\Psi_\alpha = \begin{bmatrix} \Psi_{ia}^\alpha & \Psi_{in}^\alpha \\ \Psi_{ja}^\alpha & \Psi_{jn}^\alpha \end{bmatrix} \quad (8.46)$$

Also

$$\Psi_\beta = \Psi_{ja}^\beta \quad (8.47)$$

and the independent coordinates for the coupled system are

$$\mathbf{u} = \{\mathbf{q}_n^\alpha\} \quad (8.48)$$

The compatibility on the interface gives

$$\mathbf{v}_j^\alpha = \mathbf{v}_j^\beta \quad (8.49)$$

i.e.

$$\Psi_{ja}^\alpha \mathbf{q}_a^\alpha + \Psi_{jn}^\alpha \mathbf{q}_n^\alpha - \Psi_{ja}^\beta \mathbf{q}_a^\beta = \mathbf{0} \quad (8.50)$$

If attachment Ritz vectors based on the residual flexibility matrix defined in Eq. 7.27 are used

$$\mathbf{q}_a^\alpha = \mathbf{f}_j^\alpha \quad (8.51)$$

and

$$\mathbf{q}_a^\beta = \mathbf{f}_j^\beta \quad (8.52)$$

The following relationship is assumed for the junction forces

$$\mathbf{f}_j^\alpha + \mathbf{f}_j^\beta = \mathbf{0} \quad (8.53)$$

It should be noted again that Eq. 8.51 is not strictly valid when external forces act on the structure.

Equations 8.49 and 8.53 can be grouped in the form

$$\mathbf{C}\mathbf{q} = 0 \quad (8.54)$$

Noting that the attachment Ritz vectors for substructure β are simply the components of residual flexibility matrix of this substructure, which is equal to the inverse of its stiffness matrix, $\mathbf{C}$ can be written in the form

$$\mathbf{C} = \begin{bmatrix} \Psi_{ja}^\alpha & \Psi_{jn}^\alpha & -\mathbf{K}_\beta^{-1} \\ \mathbf{I}_{aa}^\alpha & \mathbf{0}_{na} & \mathbf{I}_{aa}^\beta \end{bmatrix} \quad (8.55)$$

in which $\mathbf{K}_\beta^{-1}$ is the inverse of the stiffness matrix $\mathbf{K}_\beta$ for substructure β .

Vector $\mathbf{q}$ can be partitioned into linearly independent part $\mathbf{q}_l$ and dependent part $\mathbf{q}_d$ in the form

$$\mathbf{q}_d = \begin{Bmatrix} \mathbf{q}_a^\alpha \\ \mathbf{q}_a^\beta \end{Bmatrix} \quad (8.56)$$

and

$$\mathbf{q}_l = \{\mathbf{q}_n^\alpha\} = \mathbf{u} \quad (8.57)$$

The corresponding partitioned matrix $\mathbf{C}_{dd}$ and $\mathbf{C}_{dl}$ are given by

$$\mathbf{C}_{dd} = \begin{bmatrix} \Psi_{ja}^\alpha & -\mathbf{K}_\beta^{-1} \\ \mathbf{I}_{aa}^\alpha & \mathbf{I}_{aa}^\beta \end{bmatrix} \quad (8.58)$$

and

$$\mathbf{C}_{dl} = \begin{bmatrix} \Psi_{jn}^\alpha \\ \mathbf{0}_{na} \end{bmatrix} \quad (8.59)$$

After some manipulation, matrix $\mathbf{S}$ is obtained as

$$\mathbf{S} = \begin{bmatrix} -\kappa' \Psi_{jn}^\alpha \\ \mathbf{I}_{nn}^\alpha \\ \kappa' \Psi_{jn}^\alpha \end{bmatrix} \quad (8.60)$$

where

$$\kappa' = (\Psi_{ja}^\alpha + \mathbf{K}_\beta^{-1})^{-1} \quad (8.61)$$

Then

$$\mathbf{q} = \mathbf{S}\mathbf{u} \quad (8.62)$$

Now let $\mathbf{M}_\alpha$ and $\mathbf{K}_\alpha$ be the mass and stiffness matrices respectively for substructure α , and $\mathbf{M}_\beta$ and $\mathbf{K}_\beta$ the mass and stiffness matrices respectively for substructure β . The reduced matrices by for substructure α obtained by a Ritz transformation are of the form

$$\mathbf{M}_\alpha^* = \left\{ \begin{matrix} \Psi_\alpha^{\alpha T} \\ \Psi_n^{\alpha T} \end{matrix} \right\} \mathbf{M}_\alpha [\Psi_\alpha^\alpha \Psi_n^\alpha] = \begin{bmatrix} \mathbf{M}_\alpha^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{nn} \end{bmatrix} \quad (8.63)$$

and

$$\mathbf{K}_\alpha^* = \left\{ \begin{matrix} \Psi_\alpha^{\alpha T} \\ \Psi_n^{\alpha T} \end{matrix} \right\} \mathbf{K}_\alpha [\Psi_\alpha^\alpha \Psi_n^\alpha] = \begin{bmatrix} \mathbf{K}_\alpha^\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_n^\alpha \end{bmatrix} \quad (8.64)$$

in which

$$\mathbf{M}_\alpha^\alpha = \Psi_\alpha^{\alpha T} \mathbf{M}_\alpha \Psi_\alpha^\alpha \quad (8.65)$$

$$\mathbf{K}_\alpha^\alpha = \Psi_\alpha^{\alpha T} \mathbf{K}_\alpha \Psi_\alpha^\alpha \quad (8.66)$$

and

$$\mathbf{K}_n^\alpha = \Psi_n^{\alpha T} \mathbf{K}_\alpha \Psi_n^\alpha \quad (8.67)$$

For substructure β

$$\mathbf{M}_\beta^* = \mathbf{K}_\beta^{-T} \mathbf{M}_\beta \mathbf{K}_\beta^{-1} \quad (8.68)$$

and

$$\mathbf{K}_\beta^* = \mathbf{K}_\beta^{-T} \quad (8.69)$$

The coupling of two substructures leads to the following mass and stiffness matrices for the coupled system

$$\mathbf{M}^{**} = \mathbf{I} + \Psi_{jn}^{\alpha T} \mathbf{M}' \Psi_{jn}^\alpha \quad (8.70)$$

$$\mathbf{K}^{**} = \mathbf{K}_n^\alpha + \Psi_{jn}^{\alpha T} \mathbf{K}' \Psi_{jn}^\alpha \quad (8.71)$$

where

$$\mathbf{M}' = \kappa'^T (\mathbf{M}_\alpha^\alpha + \mathbf{K}_\beta^{-T} \mathbf{M}_\beta \mathbf{K}_\beta^{-1}) \kappa' \quad (8.72)$$

and

$$\mathbf{K}' = \kappa' \quad (8.73)$$

In deriving Eq. 8.63, the properties of the attachment Ritz vectors based on the residual flexibility matrix are used. These properties were listed in the previous chapter.

By comparing the coupled mass and stiffness matrices given in this section with those obtained by using the simplified Hintz's free interface method as given in the preceding section, it is obvious that the coupled equations of motion obtained by using the simplified Rubin's method are for a system of smaller size. Only N_n^a degrees of freedom appear in the final form of the matrices. Unfortunately, it is not possible to transform these equations into a form which can be used for frequency domain analysis where the dynamic stiffness matrix of the foundation soil is obtained from a BEM analysis. This is also true for the original form of Rubin's method (1975).

8.4 BENFIELD AND HRUDA'S FREE INTERFACE METHOD

In the last two sections, it has been shown that the interface degrees of freedom appear in the coupled system when using the simplified form of Hintz's free interface method. These degrees of freedom are absent from the equations derived by Rubin's method. The simplified form of Rubin's method however, cannot be used in the frequency domain analysis. In this section, the Benfield and Hruda's method is studied. In the previous chapter, it has already been shown that this method takes the simplest form among all the methods discussed. Further simplification is possible in the analysis of soil-structure interaction problem making the method attractive for use in such an analysis. Examples presented later show that this method gives quite good results.

8.4.1 Simplified Benfield and Hruda's Free Interface Method

In this method, only free interface normal modes or Ritz vectors are used and there are no attachment modes or Ritz vectors. When one of the substructure has only the interface degrees of freedom, it can be shown by following the procedure

used in work by Benfield and Hruda (1971), that the final form of Ritz vectors to be used in coupling of the soil and the structure as well as in reducing the size of such a system are given by

$$\Psi = \begin{bmatrix} \Psi_{in}^\alpha \\ \Psi_{jn}^\alpha \\ \Psi_{jn}^\alpha \end{bmatrix} \quad (8.74)$$

where

$$\begin{bmatrix} \Psi_{in}^\alpha \\ \Psi_{jn}^\alpha \end{bmatrix} = \Psi_\alpha \quad (8.75)$$

are the Ritz vectors for substructure α .

The reduced size of the system obtained by using Benfield and Hruda's free interface method is N_n^α , since no attachment Ritz vectors are used in this method.

8.4.2 Simplified Benfield and Hruda's Free Interface Method for Analysis in Frequency domain

Even when the analysis of substructure β is carried out in the frequency domain, and its frequency dependent stiffness and mass matrices are obtained from BEM method or some other method, use of the Benfield and Hruda's method leads to a very simple form as described in the following paragraphs.

The coupled equations in the frequency domain are given by

$$[S_s(\Omega) + \Psi_{jn}^{\alpha T} S_\beta(\Omega) \Psi_{jn}^\alpha] H(\Omega) = \begin{Bmatrix} \Psi_{ia}^{\alpha T} \\ \Psi_{jn}^{\alpha T} \end{Bmatrix} f_i^\alpha \quad (8.76)$$

where

$$S_s(\Omega) = -\Omega^2 \Psi_\alpha^T M_\alpha \Psi_\alpha + i\Omega \Psi_\alpha^T C_\alpha \Psi_\alpha + \Psi_\alpha^T K_\alpha \Psi_\alpha \quad (8.77)$$

is the dynamic stiffness matrix for substructure α , dependent only on the characteristics of that substructure.

$$S_\beta(\Omega) = -\Omega^2 M_\beta + i\Omega C_\beta + K_\beta \quad (8.78)$$

is the dynamic stiffness matrix of substructure β . The computations for S_β do not involve substructure α can thus be obtained independently for a soil domain by BEM or other methods.

8.5 METHODS USING RITZ VECTORS BASED ON MIXED INTERFACE CONDITIONS

The methods discussed in the previous sections use free interface Ritz vectors. As a result, Ritz vectors for substructures that are not adequately supported to restrain rigid body motions are generated by using a singular stiffness matrix. In this work, a shift is applied to the stiffness matrix in order to get over the singularity problem. However, since all the rigid body modes have a zero eigenvalue, the procedures for generating Ritz vectors discussed here are not capable of extracting repeated eigenvalues. Rigid body Ritz vectors obtained statically have been used in the last chapter to overcome this problem. An alternative technique using the Ritz vectors obtained from a stiffness matrix with rigid body motions restrained is proposed in this section. The concepts of such Ritz vectors has already been described in the last chapter. To illustrate the generation of such Ritz vectors, let the structural system shown in Fig. 8.2 be considered. In substructure α , enough interface degrees of freedoms have been restrained to prevent rigid body motion. Let the Ritz vectors for the restrained substructure α be denoted by Ψ_s . These Ritz vectors do not contain any rigid body Ritz vectors. Matrix Ψ_s can be expanded to have the same number of rows as the Ritz vectors for substructure α with no restraints by adding rows of zeros corresponding to the restrained degrees of freedom. Let the Ritz vectors so expanded be denoted by $\Psi_{\alpha s}$. Thus $\Psi_{\alpha s}$ can be expressed as

$$\Psi_{\alpha s} = \begin{bmatrix} \Psi_s \\ \mathbf{0} \end{bmatrix} \quad (8.79)$$

A Ritz transformation using Ritz vectors in Eq. 8.79 gives only the elastic displacements for the restrained substructure α . Let $\mathbf{v}_{\alpha e}$ denote these elastic displacements and be written as

$$\mathbf{v}_{\alpha e} = \begin{Bmatrix} \mathbf{v}_{se} \\ \mathbf{0} \end{Bmatrix} \quad (8.80)$$

in which $\mathbf{v}_{se}$ corresponds to the elastic displacements for unrestrained degrees of freedom and $\mathbf{0}$ the zero elastic displacements for the restrained degrees of freedom. The Ritz transformation is of the form

$$\mathbf{v}_{ae} = \begin{Bmatrix} \mathbf{v}_{se} \\ \mathbf{0} \end{Bmatrix} = \begin{bmatrix} \Psi_s \\ \mathbf{0} \end{bmatrix} \mathbf{q}_s \quad (8.81)$$

where $\mathbf{q}_s$ are the generalized coordinates in the transformed domain.

The total displacements for unrestrained substructure α are given by

$$\mathbf{v}_\alpha = \mathbf{v}_{ae} + \mathbf{v}_{ac} \quad (8.82)$$

in which $\mathbf{v}_{ac}$ is the vector of displacements of substructure α due to the displacements at the restrained degrees of freedom. Vector $\mathbf{v}_{ac}$ includes the displacements of substructure α caused by rigid body motions. By its definition, $\mathbf{v}_{ac}$ can be expressed in terms of the constraint Ritz vectors described in Eq. 7.14 in Chapter 7 as follows

$$\mathbf{v}_{ac} = \Psi_{ac} \mathbf{q}_c = \begin{bmatrix} -\mathbf{K}_{vv}^{-1} \mathbf{K}_{vc} \\ \mathbf{I}_{cc} \end{bmatrix} \mathbf{q}_c \quad (8.83)$$

in which Ψ_{ac} is the constraint Ritz vectors.

Substitution of Eq. 8.81 and 8.83 into Eq. 8.82 gives

$$\begin{aligned} \mathbf{v}_\alpha &= \begin{bmatrix} \Psi_s \\ \mathbf{0} \end{bmatrix} \mathbf{q}_s + \begin{bmatrix} -\mathbf{K}_{vv}^{-1} \mathbf{K}_{vc} \\ \mathbf{I}_{cc} \end{bmatrix} \mathbf{q}_c \\ &= \begin{bmatrix} \Psi_s & -\mathbf{K}_{vv}^{-1} \mathbf{K}_{vc} \\ \mathbf{0} & \mathbf{I}_{cc} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{q}_c \end{Bmatrix} = [\Psi_{as} \quad \Psi_{ac}] \mathbf{q}_\alpha \end{aligned} \quad (8.84)$$

where $\mathbf{q}_\alpha = \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{q}_c \end{Bmatrix}$ are the transformed coordinates. Let $[\Psi_{as} \quad \Psi_{ac}]$ be denoted by Ψ_α so that

$$\Psi_\alpha = [\Psi_{as} \quad \Psi_{ac}] \quad (8.85)$$

In this Ritz set, Ψ_{as} have been normalized with respect to the mass matrix, but Ψ_{ac} have not been. Although Ψ_{ac} can be easily made normal with respect to the mass matrix, this is not necessary in most cases.

The Ritz vectors in Eq. 8.85 can be used to substitute for the Ritz vector set in all the methods mentioned earlier, and generally speaking, lead to a more accurate

and more efficient procedure. In this chapter, use of Ritz vector set in Eq. 8.85 is illustrated through examples that employ Benfield and Hruda's method.

The use of Ritz vector set in Eq. 8.85 implies the use of a mixed type of interface and offers many advantages. The interface of substructure α is treated as partly free and partly fixed. Since rigid body motion is prevented by the fixed interface degrees, the stiffness matrix for substructure α is no longer singular and a shifting is not necessary. Also, considerable saving can be achieved in the computation time required for the generation of the Ritz vectors. In all methods that use free interface Ritz vectors, the factorization (or inverse) of a largest rank nonsingular-submatrix of the stiffness matrix for substructure α is required for obtaining the pseudo-static deformation vector of the substructure. This pseudo-static deformation vector is needed to take into consideration the load distribution. Further, the factorization (or inverse) of the shifted stiffness matrix is needed in order to generate the Ritz vectors. However, when Ritz vectors based on restrained rigid body motion are used, only the factorization (or inverse) of a nonsingular stiffness matrix is needed, since the same matrix can be used in obtaining both the pseudo-static deformation vector and the Ritz vectors as well as the constraint Ritz vectors. The size of this nonsingular matrix is equal to or smaller than the size of the largest rank nonsingular-submatrix of the stiffness matrix for substructure α with all interface degrees of freedom free. Finally, as will be shown in the examples provided later, use of Ritz vectors in Eq. 8.85 gives a better coupling between the two substructures.

It is worth noting that the so-called pseudo-static deformation vector used in generating the Ritz vectors with mixed conditions at interface is obtained corresponding to the restrained degrees of freedom which can be more than the minimum required to restrain the rigid body motion. The number of restrained degrees of freedom on the interface can be selected randomly as long as they are enough to prevent substructure α from rigid body motion. In the case where all interface degrees are fixed, the Ritz vectors in Eq. 8.85 become the first stage Ritz vectors described in Chapter 6. On the other hand, if only enough number of interface de-

degrees of freedom capable of preventing substructure α from rigid body motion are restrained, the constraint Ritz vectors used in all equations in this section become equal to the rigid body Ritz vectors.

8.6 EXAMPLES

Several examples are given in this section to illustrate the application of substructure coupling methods in the analysis of soil-structure interaction problems.

In the first example, the structural system shown in Fig. 8.1 is examined by using the simplified form of Hintz's free interface method. For simplicity, the rotations at interface are restrained. Substructure α has 56 degrees of freedom, with interface degrees of freedom being 8. All members in the structure have the same cross section and material properties. The Young's modulus is taken as 25000 MPa, the area of cross section 0.36 m^2 , and the in-plane moment of inertia 0.0108 m^4 .

In Hintz's method, four flexible Ritz vectors and 8 residual attachment Ritz vectors are taken for substructure α . Among the four flexible Ritz vectors, there are two rigid body Ritz vectors. They are generated statically and swept from the starting vector used in generating the remaining 2 Ritz vectors. The spatial distribution of loads used in obtaining the starting vector for substructure α consists of horizontal forces applied at the interface nodes and the nodes at the top of the substructure. The magnitudes of these forces are taken as the lumped masses at the corresponding nodes. The remaining nodes are force free. The natural frequencies extracted by using the simplified Hintz's free interface method are given in Item 2 in Table 8.1. For comparison, the exact natural frequencies of the same structural system obtained without substructuring are shown in Item 1. In all, 12 natural frequencies have been extracted in this case, because the coupled system is of order 12 by 12. It is found that only the first two natural frequencies of the coupled system obtained by substructure coupling agree with the exact values fairly well.

The simplified Benfield and Hrudá's method is also applied to the same structural system. In this case, the computational work can be reduced to some extent

since the total number of Ritz vector to be generated is less than that required in Hintz's method. Eight Ritz vectors including 2 rigid body Ritz vectors are generated for substructure α . Again, the 2 rigid body Ritz vectors are generated statically. Use of these 8 Ritz vectors in coupling the two substructures results in a coupled system of order 8 by 8. Therefore, in all, 8 natural frequencies are extracted by the Benfield and Hruda's simplified method. It is worth mentioning that the same spatial distribution of loads as used in Hintz's method is also used here. The eight natural frequencies of the coupled system are listed in Item 3 for comparison. It is shown that the first 3 natural frequencies given in this case are close to the corresponding exact values. The computations for this problem were carried out on Apollo workstation DN3000. The CPU time needed for the Hintz's method was 59.0 seconds. A CPU time of 40.7 seconds was required in using the Benfield and Hruda's method. The latter method thus leads to saving in the computer time. Furthermore, the smaller size of the coupled system obtained by the Benfield and Hruda's method can be very beneficial to an analysis in the frequency domain, especially when BEM method is used. It is also worth mentioning that the simplified Hintz's free interface method leads to a singular coupled stiffness matrix if 5 or more flexible Ritz vectors are used.

Obviously, the accuracy of the results obtained by using Benfield and Hruda's method can be improved if a larger number of Ritz vectors is used for substructure α . Item 4 in Table 8.1 gives the natural frequencies obtained by using 10 free interface Ritz vectors for substructure α . It is seen that the first 5 natural frequencies listed in this column are almost identical to those given in Item 1, which are regarded as the exact values. A CPU time of 43.5 seconds is required in this case, which still is more economical in comparison to the Hintz's method.

The structural system shown in Fig. 8.1 is also examined by using the simplified Benfield and Hruda's method employing the Ritz vectors with restrained interface nodes. Item 5 gives the results obtained by fixing all the interface degrees of freedom in substructure α as shown in Fig. 8.3. The number of flexible Ritz

vectors generated for substructure α is 4. Since there are 8 constraint Ritz vectors, the coupled structure is of order 12 by 12. The computer CPU time consumed in this computation is 32.4 seconds, which is even less than the computer CPU time consumed by using the Benfield and Hruda's method with the 8 free interface Ritz vectors the results for which are listed in Item 3 in Table 8.1. When two interface degrees of freedom are fixed, for example, as shown in Fig. 8.4, the results obtained by Benfield and Hruda's method using Ritz vectors with restrained rigid body motion are listed in Item 4 in Table 8.1. In this case, 8 flexible Ritz vectors plus 2 constraint Ritz vectors (rigid body Ritz vectors) are used. The coupled system is of order 10 by 10. Therefore, 10 natural frequencies are given in Item 4. The computer CPU time consumed in this case is 37.7 seconds. These results are better than those obtained by using 8 free interface Ritz vectors (Item 3). Much of the saving in computer time in using Ritz vectors with restrained rigid body motions comes from the fact that the factorization of stiffness matrix is carried out only once and the size of the stiffness matrix is smaller. Since the use of 8 constraint Ritz vectors as shown in Fig. 8.3 results in a stiffness matrix of the smallest size, the least computer time is consumed even though in all 12 Ritz vectors are generated.

Another example is provided by considering the soil-structure interaction problem shown in Fig. 6.2. The rotation degrees of freedom at the interface are fixed for the purpose of simplicity. Therefore, the structural part has 68 degrees of freedom and the interface degrees of freedom are 8. In the computations, the soil domain is analyzed by using BEM method, and the whole analysis is carried out in the frequency domain. The displacement responses to the horizontal earthquake excitation shown in Fig. 6.5 are obtained by using the method of artificial damping as described in Chapter 5.

It should be noted that the formulation in the frequency domain for either the simplified Hintz's method or the simplified Benfield and Hruda's method given in this Chapter is for the case of externally applied loads not earthquake excitation. In the later case, these methods should be reformulated by following the proce-

dures outlined in Chapter 6. Details of such formulations are not provided but are straightforward. For example, in using the Benfield and Hruda's method, the formulation in Eq. 6.47 can be directly employed with Ritz vectors $\mathbf{X}$ being replaced by the free interface Ritz vectors, $\Psi_\alpha = \begin{bmatrix} \Psi_{in}^\alpha \\ \Psi_{jn}^\alpha \end{bmatrix}$, for substructure α . The partitioned part of the Ritz vectors, $\mathbf{X}^{sb}$ is simply equal to Ψ_{jn}^α in this case.

In the analysis, ten flexible Ritz vectors are generated for the structural part. Among these 10 Ritz vectors, there are 2 rigid body Ritz vectors generated statically. These ten Ritz vectors are used for the substructure coupling using the simplified Benfield and Hruda's free interface method. Comparison is made between the results obtained here with the exact displacement responses obtained for the complete structure without Ritz reduction.

The horizontal displacement, vertical displacement, as well as rotation responses of the soil-structure system at node 23 are shown in Fig. 8.5a, b, and c respectively. It is found that the results obtained by substructure coupling agree quite well with the so called exact values.

The same soil-structure interaction system is analyzed by using the Ritz vector set given in Eq. 8.85. Substructure α is restricted from rigid body motion in the same way as substructure α shown in Fig. 8.4. Eight Ritz vectors are generated for structural part. These 8 Ritz vectors do not include the rigid body modes. Another 2 constraint Ritz vectors (rigid body Ritz vectors) are obtained statically and combined with the 8 Ritz vectors to form a set of 10 Ritz vectors. The Benfield and Hruda's method is used to couple the two substructures. The horizontal displacement, vertical displacement, as well as rotation responses of the soil-structure system at node 23 are shown in Fig. 8.6a, b, and c respectively. For comparison, the exact responses that are obtained for the complete system without Ritz reduction are also shown. The two sets of results agree quite well with each other.

8.7 CHAPTER SUMMARY

Substructure coupling methods using Ritz vectors are studied for their possible

application in the analysis of soil-structure interaction problems. Attention is focused on the use of the Hintz's, the Rubin's and the Benfield and Hruda's methods. It is shown that the Rubin's method does not lend itself to analysis in the frequency domain. Examples are given of the use of the Hintz's free interface method and the Benfield and Hruda's method, both being simplified for the case where one of the substructures has only the interface degrees of freedom. A soil-structure interaction example using BEM method is solved by using the simplified Benfield and Hruda's method, and satisfactory results are obtained. The use of the Ritz vectors obtained for mixed interface condition, as substitutes for the free interface Ritz vectors, appears to be more efficient in the analysis of soil-structure interaction problems.

Table 8.1 Soil-structure interaction example

Mode No.	Item 1	Item 2	Item 3	Item 4	Item 5
	ω	ω	ω	ω	ω
1	3.439393	3.377287	3.816890	3.444563	3.442822
2	10.08556	9.371686	10.26260	10.09413	10.09178
3	16.19472	12.43179	16.42343	16.19525	16.19487
4	22.47043	13.23552	26.80355	22.47827	22.47206
5	27.62048	21.57463	45.31555	27.62215	27.62096
6	68.86288	111.0256	112.3666	93.59510	83.10571
7	71.52642	115.3570	119.0547	109.4015	92.05197
8	80.98054	127.9545	328.0953	112.3666	112.2862
9	81.17559	138.2742		234.3456	190.1472
10	196.8024	197.7316		349.1628	235.6919
11	197.1596	349.2301			311.1440
12	197.8572	432.8512			349.1642
13	197.9622				
14	198.7371				
15	198.8984				
16	232.0506				
17	232.0767				
18	238.1350				
19	308.7944				
20	309.4167				
21	348.6748				
22	348.8693				
23	349.2083				
24	349.6996				
25	361.4424				
26	361.4549				
27	391.4866				
28	391.9632				
29	416.8474				
30	432.4972				
31	432.6349				
32	432.8773				
33	433.2389				
34	440.6712				
35	441.0726				
36	456.6198				
37	456.6308				
38	506.0672				
39	510.9183				
40	510.9312

Fig. 8.1 A system simulating soil-structure interaction

Fig. 8.2 Soil-structure Interaction using constrained interface

**Fig. 8.3 Soil-structure Interaction with
all interface d.o.f constrained**

Fig. 8.4 Soil-structure Interaction with minimum number of interface d.o.f constrained

(a) Response along horizontal direction at node 23

(b) Response along vertical direction at node 23

(c) Response of rotation at node 23

— Exact response
 - - - By using 10 free interface Ritz vectors

Fig. 8.5 Displacement response obtained by using the simplified Benfield and Hruda's method with free interface Ritz vectors

(a) Response along horizontal direction at node 23

(b) Response along vertical direction at node 23

(c) Response of rotation at node 23

— Exact response
 - - - By using 8+2 mixed interface Ritz vectors

Fig. 8.6 Displacement response obtained by using the simplified Benfield and Hrudá's method with mixed interface Ritz vectors

Chapter 9

SUMMARY OF THE THESIS

The objective of the present work is to develop certain effective analytical techniques for the dynamic analysis of structures. Major contributions are in the areas of frequency dependent Ritz vectors, frequency domain analysis and the application of Ritz vectors in substructure coupling. An algorithm is developed for generating Ritz vectors which can take into account both the spatial distribution and the frequency content of loading. Efficient cut-off criteria are presented for determining the required number of frequency dependent Ritz vectors. Several examples of relatively small size are given in order to illustrate the application of the proposed frequency dependent Ritz vectors. The application of frequency dependent Ritz vectors in the analysis of a large size structure system is shown by analyzing a 1900 degree-of-freedom plane stress frame. It is shown that the use of frequency dependent Ritz vectors provides a more effective and efficient reduction in the size of the structural systems subjected to external loadings of high excitation frequencies.

In the work on frequency domain analysis, a relationship is described between the continuous complex frequency response function and the discrete Fourier transform of truncated impulse function for a single degree-of-freedom system as well as for a multi degree-of-freedom system. The difficulties inherent in solving structural dynamic interaction problems through the frequency domain analysis are discussed, and a few simple examples are given for the purpose of illustration. Two new methods of analysis in the frequency domain are developed. These methods are applicable to multi degree-of-freedom systems as well as to substructure analysis for both damped and undamped systems. The first of these methods uses the free vibration component of the response to determine the correction for spurious initial conditions. The second method deals with the introduction of temporary damping to avoid aliasing errors when impulse function is obtained from frequency response

function by using discrete Fourier transforms. The application of the method of artificial damping in the analysis of dam reservoir interaction problem is studied. The studies show that the standard DFT method may not be capable of giving the correct response of the dam-reservoir system if the dam itself is undamped or lightly damped. Even for the damped case where the DFT method may give the correct response provided the time period used in the analysis is long enough for the errors to become negligible, the use of method of artificial damping may be more effective and more efficient.

As another application of Ritz vectors and the method of artificial damping, studies are carried out on the dynamic response of soil-structure system. The boundary element solution of soil foundation in frequency domain is discussed briefly with emphasis on the treatment of singularity in the boundary integrals. A set of Ritz vectors that are independent of soil properties are applied in the solution. In addition, a two stage Ritz vector technique is developed and is shown to lead to greater efficiency in the analysis of soil-structure interaction problems. Examples are presented to illustrate the different stages of analysis. The studies show that in the analysis of soil-structure interaction problems where the boundary element method with fundamental solutions containing some Bessel functions is used, the method of artificial damping poses no special problems and is, at the same time, more effective and more efficient as compared to the DFT method. The use of two stage Ritz vector technique can further improve the efficiency of the analysis.

In another phase of the work, substructure coupling methods are studied. Load dependent Ritz vectors are successfully incorporated into several available substructure coupling methods as replacements for mode shapes of the original formulations. Several examples are presented and the results obtained by analyzing a reduced structure compared with those obtained from an analysis of the complete structure. Extension of the use of substructure coupling methods to soil-structure interaction problems is presented. Discussion is focused on the simplification of

several available methods when applied to this particular case, and on their potential use in frequency domain analysis procedures. Benfield and Hruda's method using Ritz vectors based on mixed interface conditions is shown to be an effective procedure.

The limitations and advantages of each of the proposed techniques are clearly outlined. The frequency dependent Ritz vectors proposed in this work have a sound theoretical basis. However the procedure for generating them is based on the assumption that the exciting load has one peak frequency. For an arbitrary loading with several frequency peaks, the efficiency of the new algorithm is affected by the choice of shift frequency used in the calculation of Ritz vectors. The relationship between the continuous frequency response functions and the discrete Fourier transform of the truncated impulse function can be effectively used for only a single degree-of-freedom-system. For multi degree-of-freedom systems, the relationship is still theoretically correct but its application becomes impractical because of the additional effort needed to solve for the mode shapes of the entire system.

The free vibration correction method proposed in this work for analysis in the frequency domain can offer a very effective means of solving both multi degree-of-freedom systems and interaction problems with or without damping. However, the results are approximate, particularly for damped systems because the exponential decay of displacement with time is not exactly simulated in the method. Much better accuracy can be achieved by using the method which relies on the introduction of artificial damping, but new complex frequency response functions are needed.

The use of Ritz vectors that are independent of soil properties offers several advantages including the possibility of utilizing half bandwidth storage and LDL^T factorization. However, these Ritz vectors can only reduce the size of part of the soil-structure system. The two stage Ritz vector technique overcomes this deficiency.

It is shown that the use of Ritz vectors in substructure coupling is quite feasible. Additional studies is, however, required on the selection of initial load vector. The methods using attachment Ritz vectors may at times lead to instability in the

computations. Benfield and Hruda's method is simple in form, but the results obtained by using this method are often less accurate than those obtained by using the Hintz's method.

Based on the work presented in this thesis, several conclusions can be drawn as follows.

1. Use of frequency dependent Ritz vectors makes it possible to include not only the effect of the spatial distribution, but also the effect of frequency content of the exciting load on the final response. The examples presented in this work confirm the effectiveness and efficiency of these vectors when the external loading contains some high frequency content.
2. The relationship between continuous frequency response function and discrete Fourier transform of the truncated impulse function obtained in this work can be used in the analysis of single degree-of-freedom systems effectively and efficiently. Similar relationship for multi degree-of-freedom systems is not as useful in practice as that for single degree-of-freedom systems.
3. The existing methods of frequency domain analysis are not effective for undamped or lightly damped structural systems, where substructuring technique is used in the analysis. The free vibration correction method and the method using frequency response functions with artificial damping are effective in such cases.
4. The method of artificial damping is quite effective in the frequency domain analysis of dam-reservoir interaction problems where the standard discrete Fourier transform method may at times lead to significant errors in the response analysis.
5. The method of artificial damping is also effective in the analysis of soil-structure interaction in the frequency domain using a boundary element solution of the soil foundation. The DFT method, on the other hand, leads to significant errors unless there is enough damping in the whole system. The use of newly proposed Ritz vectors that are independent of the characteristics of soil foundation and

of the two stage Ritz vector technique can make the analysis of soil-structure interaction problems even more efficient and effective.

6. Load dependent Ritz vectors can be effectively incorporated into several methods for substructure coupling as replacements for normal modes. This is also true soil-structure interaction problems. For the latter case, the Ritz vectors based on mixed interface conditions offer better accuracy in analysis as compared to free interface Ritz vectors.

SUGGESTED FUTURE WORK

In the course of this work, several areas of study where additional work is required have been identified. These can be summarized as follows

1. Studies are required on the solution of an appropriate shift for the stiffness matrix in the generation of frequency dependent Ritz vectors, particularly when the applied load contains several high frequency components.
2. More work is required in order to determine the suitability of the proposed cut-off criteria and the error tolerance used to terminate the generation of frequency dependent Ritz vectors.
3. Additional studies are required to confirm that the proposed value of damping parameter in the method of artificial damping is suitable.
4. Additional work is required on the use of Ritz vectors in substructure coupling, particularly in the selection of initial load vector.

APPENDIX I

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