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The Product Separability Of Groups

Shihong You

Department of Mathematics and Statistics

Carleton University

Ottawa, Ontario

Canada K1S 5B6

22 August 1994

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SHIHONG YOU, B. Sc., M. Sc.

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for the degree of Doctor of Philosophy

Chair, Department of Mathematics and Statistics

Thesis Supervisor

External Examiner

Carleton University
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Abstract

The topic of this dissertation is product separability of groups. Related to a conjecture of Pin and Reutenauer on free groups, the concept of product separability is introduced in the dissertation. A group G is said to be product separable if, for any finite collection $\{H_1, H_2, \dots, H_n\}$ of finitely generated subgroups of G , the product set $H_1 H_2 \cdots H_n$ is closed in the profinite topology of G . The definition of product separability is helpful for solving some decision problem relevant to the existence of certain algorithm for finite semigroups.

Using the theory of profinite groups acting on Boolean trees, it is proved in this thesis that $F \times Z$ is product separable where F is a free group and Z is an infinite cyclic group. This result leads to the product separability of the generalized free product of two cyclic groups. Meanwhile, a general projection method was developed that gives an alternative proof for the result of Ribes and Zalesskii.

It is also proved in this thesis that surface groups are product separable. This result generalizes a theorem of G.A. Niblo that states that surface groups are double coset separable. Using our result, we are able to show the product separability of finitely generated Fuchsian groups.

We study the separability properties of Abelian groups. We show that finitely generated Abelian groups are always product separable; however an infinitely generated Abelian group is not necessarily subgroup separable, even if it is residually finite, as we show by means of an example. Next we consider groups of form $G = F \times A$, where F is a free group and A is an Abelian group. We then prove that the product set $H_1 H_2 H_3 H_4$ of any four finitely generated subgroups of G is closed in the profinite topology of G . We also prove that $H_1 H_2 \cdots H_n$ is closed when all projections of H_i on F are cyclic subgroups of F .

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Chapter 1

Introduction

It is well known in the Galois theory, for a finite Galois extension K of a field F , that there is a one to one correspondence between the set of all intermediate fields of the extension and the set of all subgroups of the Galois group $G = \text{Gal}_F K$. In fact, this bijection is an anti-isomorphism of lattices. Unfortunately, this beautiful result is not valid for infinite Galois extensions. There are many subgroups of the Galois group that correspond to each intermediate field of the infinite Galois extension. To eliminate such a defects, Krull has defined a topology on G , called the Krull topology (cf. [Kr]), and modified the correspondence theorem so that it is true for both finite and infinite Galois extensions. The Krull topology on G has a basis of open neighborhoods of the identity consisting of the subgroups of G of the form $\text{Gal}(K/L)$ where $[L : F] < \infty$. Equipped with this topology, G becomes a Hausdorff,

compact and totally disconnected topological group. The modified theorem can be stated as follow: for any Galois extension there is an anti-isomorphism from the lattice of intermediate fields to the lattice of closed subgroups of the Galois group G with respect to the Krull topology of G .

Now we may define a profinite topology, which is a generalization of the Krull topology in the Galois group case, on any group G . The basis of open neighbourhoods of the identity consists of all subgroups of G with finite index. Endowed with this profinite topology, G is Hausdorff if and only if for any non-identity element g , there is a subgroup H of finite index in G such that $g \notin H$ (in other words, there is a neighbourhood H of the identity excluding g). This leads us to consider the residual finiteness of groups in which we may separate two distinct elements by their neighbourhoods. A group G is said to be residually finite if, with profinite topology, it is a Hausdorff one.

Residual finiteness provides a way of separating two distinct elements in a group G . Furthermore, we consider how to separate an element $g \in G$ from a finitely generated subgroup H of G which does not contain g . By this we mean that there exists a subgroup U of finite index in G such that $H \subset U$ and $g \notin U$. This is equivalent to saying that $H \subset U$ and $H \cap gU = \emptyset$. Observe that a finitely generated subgroup H

is closed in the profinite topology of G if and only if every element $g \in G \setminus H$ can be separated from H . Groups having this property are called subgroup separable.

It has been proved that the generalized word problem is soluble for finitely presented subgroup separable groups (cf. [M]). That is one of the reasons why subgroup separability has become the subject of intensive research now. The earliest result in this area was obtained by M. Hall in 1949. He proved that free groups are subgroup separable.

Based on Hall's result, J.E. Pin and C. Reutenauer made the following conjecture (cf.[P-R]): let $H_1, H_2, \dots, H_n$ be a finite sequence of finitely generated subgroups of a free group F , then the product subset $H_1 H_2 \cdots H_n$ is closed in the profinite topology of F . Now we define a group G to be product separable if for any finite set $\{H_1, H_2, \dots, H_n\}$ of finitely generated subgroups of G and for every element $g \in G$ which is not in the product set $H_1 H_2 \cdots H_n$, there is a subgroup U of finite index in G such that $gU \cap H_1 H_2 \cdots H_n = \emptyset$. This condition is equivalent to saying that the product set $H_1 H_2 \cdots H_n$ is closed in the profinite topology of G . With this terminology, the conjecture of Pin and Reutenauer can be stated as follows:

Conjecture 1.1 *Free groups are product separable.*

Pin and Reutenauer proved that this conjecture is equivalent to Rhodes's con-
 jec-

ture on the existence of certain algorithm for finite semigroups (cf.[P-R] and [Pin]). Since then, some mathematicians have worked on the product separability of groups. In 1991, G.A. Niblo proved that free groups are double coset separable (i.e. for every pair H, K of finitely generated subgroups of a free group F and every $g \in F$, the double coset HgK is closed). He also proved that surface groups are double coset separable. Note that the double coset separability of G is equivalent to asserting that the product set of two finitely generated subgroups of G is closed in G .

Ribes and Zalesskii found the positive answer to the conjecture of Pin and Reutenauer in 1991. In fact they proved a slightly more general version of that conjecture (cf.[R-Z], Theorem 2.1).

Theorem 1.2 (*Ribes - Zalesskii*) *Any finite extension of free groups is product separable.*

As a part of this thesis, we prove that surface groups are product separable. This result generalizes Niblo's result mentioned above.

In the polycyclic group case, J.S. Wilson proved that polycyclic groups are double coset separable. He also gave a counterexample to show that the product of three finitely generated cyclic subgroups may not be closed in a polycyclic group (cf. [L-W]). Wilson's result shows that product separability is a stronger property than subgroup

separability or double coset separability.

Throughout this thesis, we shall mainly focus on the product separability of groups and try to extend the result of Ribes and Zalesskii. Besides showing that surface groups are product separable, we shall also prove that $F \times \langle a \rangle$ is product separable where F is a free group and $\langle a \rangle$ is an infinite cyclic group. This result will lead to the product separability of the generalized free product of two cyclic groups.

In the final chapter of this thesis, we discuss the separability properties of $F \times A$ where F is a free group and A is a free Abelian group of finite rank. The complete description of separability properties of Abelian groups is obtained in Chapter 3. We shall prove the double coset separability of $F \times A$ and some generalized results of it.

Chapter 2

Preliminaries

In this chapter, we will provide all the material required to understand and prove our results.

2.1 Some basic concepts

We first state some definitions here.

Definition 2.1 *A group G is called residually finite if for any non-identity element g , there exists a subgroup U of G with finite index such that g is not contained in U .*

Definition 2.2 *Let G be a group. The profinite topology of G is defined by taking all subgroups of finite index in G as a basis of open neighbourhoods of the identity.*

When G is a residually finite group, the profinite topology of G is a Hausdorff topology, and any single element of G is a closed set in G .

Definition 2.3 Let G be any group, Λ the family of all normal subgroups of finite index in G . The family of quotients $\{G/U \mid U \in \Lambda\}$ forms an inverse system of finite groups, whose inverse limit is

$$\hat{G} = \varprojlim (G/U).$$

$\hat{G}$ is a profinite group (see [D-S-M-S], pp 21). We defined $\hat{G}$ as the profinite completion of group G .

If G is residually finite, then there is an embedding from G to $\hat{G}$ induced by projections $G \longrightarrow G/U$ for all subgroups U of finite index in G .

Definition 2.4 A group G is called subgroup separable if for any finitely generated subgroup H of G and an element g which is not included in H , there exists a subgroup U of finite index in G such that U contains H and g is not in U .

Here is an equivalent characterization of subgroup separability.

Let G be a residually finite group, $\hat{G}$ its profinite completion and $\phi : G \longrightarrow \hat{G}$ the embedding. Then G is subgroup separable if and only if whenever H is a finitely generated subgroup of G , $\phi^{-1}\overline{\phi(H)} \cap G = H$, where $\overline{\phi(H)}$ denotes the closure of $\phi(H)$ in $\hat{G}$. So, if one identifies G with its image in $\hat{G}$, the condition becomes that G is subgroup separable if and only if G is residually finite and whenever H is a finitely generated subgroup of G , $\bar{H} \cap G = H$ (where $\bar{H}$ denotes the closure of H in the

profinite completion $\hat{G}$ and G is viewed as a subgroup of $\hat{G}$). This is equivalent to saying that G is residually finite and every finitely generated subgroup H of G is closed in the profinite topology of G .

Definition 2.5 *A group G is called product separable if for any finite set $\{H_1, H_2, \dots, H_n\}$ of finitely generated subgroups of G , and any element g which is not included in the product set $H_1 H_2 \cdots H_n$, there exists a finite index subgroup U of G such that the intersection $gU \cap H_1 H_2 \cdots H_n$ is empty.*

The last two definitions tell us that in a subgroup separable group any finitely generated subgroup is closed and in a product separable group the product set of finitely generated subgroups $H_1 H_2 \cdots H_n$ is closed in the profinite topology. Clearly product separable groups must be subgroup separable.

2.2 Boolean graphs and Boolean trees

The concepts of Boolean graphs and Boolean trees were introduced by Gildenhuys and Ribes in the late seventies. They play an important role in the study of product separability of groups. The following concepts and results will be used in this thesis later. More detail can be found in [G-R] and [Z-M].

Definition 2.6 *A topological space T is called totally disconnected if each connected component of T consists of a single element.*

Definition 2.7 *A topological space is called a Boolean space if it is a Hausdorff, compact and totally disconnected topological space.*

Recall that a profinite group is a Hausdorff, compact and totally disconnected topological group. So for any profinite group G , its underlying topological space is a Boolean space. From the definition of Boolean space, it is easy to see that the direct product of Boolean spaces is also a Boolean space.

Definition 2.8 *A graph Γ is defined as an ordered quadruple $(V(\Gamma), E(\Gamma), d_0, d_1)$, where $V(\Gamma)$ is the vertex set of Γ , $E(\Gamma)$ is the edge set of Γ , and d_0, d_1 are mappings from $E(\Gamma)$ to $V(\Gamma)$. d_0 maps an edge to its initial vertex, d_1 maps an edge to its end vertex. If both $V(\Gamma)$ and $E(\Gamma)$ are Boolean spaces and d_0, d_1 are continuous, then the graph Γ is called a Boolean graph (or a profinite graph).*

Let G be a profinite group, and X a closed subset of G . Then both G and X are Boolean spaces. So is $G \times X$. One can construct a Boolean graph $\Gamma = \Gamma(G, X) = (V(\Gamma), E(\Gamma), d_0, d_1)$ as follows:

$$V(\Gamma) = G, \quad E(\Gamma) = G \times X$$

and

$$d_0(g, x) = g, \quad d_1(g, x) = gx$$

where $g \in G$ and $x \in X$. The graph Γ is called the *Cayley graph* of G with respect to the closed subset X of G .

Finite Boolean graphs are same as finite graphs in the usual sense.

Definition 2.9 *Let Γ and Δ be two Boolean graphs. A morphism α from Γ to Δ consists of a pair of continuous maps $\alpha : V(\Gamma) \longrightarrow V(\Delta)$ and $\alpha : E(\Gamma) \longrightarrow E(\Delta)$ such that*

$$d_i^\Delta \alpha(e) = \alpha d_i^\Gamma(e)$$

for $i = 0, 1$ and all $e \in E(\Gamma)$.

Here we use the same notation α to denote the two maps since it is easy to distinguish the edge map and the vertex map. So both $\alpha(e)$ and $\alpha(v)$ make sense to us. We say α is injective (surjective) if both edge map and vertex map are injective (surjective).

Definition 2.10 *Let Γ and Δ be two Boolean graphs such that $V(\Delta) \subseteq V(\Gamma)$ and $E(\Delta) \subseteq E(\Gamma)$. Furthermore, the maps d_i^Δ are the restrictions of d_i^Γ respectively. In this case, Δ is called a subgraph of Boolean graph Γ .*

When the Boolean graph Δ is a subgraph of Γ , the embedding of Δ into Γ is clearly an injective morphism of Boolean graphs.

Definition 2.11 *A quotient graph of a Boolean graph Γ is the image of Γ under a surjective morphism.*

We are interesting in the finite quotient graphs of a Boolean graph. Looking at the Cayley graph as an example, let G be a profinite group and X be a closed subset of G . One may construct the Cayley graph $\Gamma(G, X)$. Let K be any finite quotient group of G , which means there is a surjective homomorphism ϕ of profinite groups from G to K . Then it is obvious that the Cayley graph $\Gamma(K, \phi(X))$ is a quotient graph of $\Gamma(G, X)$. The following proposition gives a hint of why Boolean graphs are also called profinite graphs.

Proposition 2.12 *Every Boolean graph can be expressed as a projective limit of finite quotient graphs.*

The proof of this proposition can be found in [Z-M, pp.408]. In the case of a Cayley graph, if

$$G = \varprojlim (G/U)$$

where U runs over all subgroups of finite index in G and X is a closed subset of G , then

$$\Gamma(G, X) = \varprojlim \Gamma(G/U, X_U)$$

where X_U is the image of X in G/U .

Next we introduce the concept of profinite tree (or Boolean tree). Let $\hat{Z}$ denote the profinite completion of the group of integers Z (the free Abelian group of rank 1). Then $\hat{Z}$ is the free profinite group of rank one. When Y is a finite set, we use $\hat{Z}[Y]$ to denote the free $\hat{Z}$ -module over Y . $\hat{Z}[Y]$ is isomorphic to $|Y|$ copies of $\hat{Z}$.

Definition 2.13 *Let X be a Boolean space. The free profinite $\hat{Z}$ -module over X is defined by*

$$\hat{Z}[[X]] = \varprojlim \hat{Z}[X_U]$$

where $X = \varprojlim X_U$.

If Y is a finite Boolean space, then $\hat{Z}[[Y]]$ is the same as $\hat{Z}[Y]$.

Let Γ be a Boolean graph. Then

$$\Gamma = \varprojlim \Gamma_\lambda$$

where Γ_λ is a finite graph. Define $\delta_\lambda : E(\Gamma_\lambda) \longrightarrow V(\Gamma_\lambda)$ by $\delta_\lambda(e) = d_0(e) - d_1(e)$ for each $e \in E(\Gamma_\lambda)$ and $\varepsilon_\lambda : V(\Gamma_\lambda) \longrightarrow \hat{Z}$ by $\varepsilon_\lambda(v) = 1$ for each $v \in V(\Gamma_\lambda)$. One can

extend them to

$$\hat{\delta}_\lambda : \hat{Z}[E(\Gamma_\lambda)] \longrightarrow \hat{Z}[V(\Gamma_\lambda)]$$

and

$$\hat{\varepsilon}_\lambda : \hat{Z}[V(\Gamma_\lambda)] \longrightarrow \hat{Z}.$$

Since $\hat{Z}[[E(\Gamma)]] = \lim \hat{Z}[E(\Gamma_\lambda)]$ and $\hat{Z}[[V(\Gamma)]] = \lim \hat{Z}[V(\Gamma_\lambda)]$, $\hat{\delta}_\lambda$ and $\hat{\varepsilon}_\lambda$ can be extended to

$$\delta : \hat{Z}[[E(\Gamma)]] \longrightarrow \hat{Z}[[V(\Gamma)]]$$

and

$$\varepsilon : \hat{Z}[[V(\Gamma)]] \longrightarrow \hat{Z}.$$

So one can construct a sequence of $\hat{Z}$ -modules as follows:

$$0 \longrightarrow \hat{Z}[E(\Gamma)] \xrightarrow{\delta} \hat{Z}[V(\Gamma)] \xrightarrow{\varepsilon} \hat{Z} \longrightarrow 0(*).$$

We say the graph Γ is *connected* if the sequence $(*)$ is exact at $\hat{Z}[[E(\Gamma)]]$. We say the graph is *acyclic* if the sequence $(*)$ is exact at $\hat{Z}[[V(\Gamma)]]$. When the sequence $(*)$ is an exact sequence, the graph Γ is called a *profinite tree*.

The following results are proved by Zalesskii and Mel'nikov (see [Z-M]). Every connected subgraph of a profinite tree is a profinite tree. Projective limits of profinite trees are profinite trees. When Δ is a subtree of a tree Γ , the quotient graph Γ/Δ is also a tree.

We turn back to Cayley graphs. Gildenhuys and Ribes have proved the following proposition and theorem (see [G-R]).

Proposition 2.14 *Let G be a profinite group and X a closed subset of G . Then the Cayley graph $\Gamma = \Gamma(G, X)$ is connected if and only if X generates G topologically (i.e. there exists no proper closed subgroup of G containing X).*

Theorem 2.15 *The Cayley graph $\Gamma(F(X), X)$ of a free profinite group $F(X)$ on a Boolean space X is a profinite tree.*

The next proposition is helpful to define a path in a profinite tree. It is also used in the proof of product separability of free groups. Its proof can be found in [Z-M] (Prop. 1.18, pp.410).

Proposition 2.16 *Let $\Delta_i, i \in I$, be a system of subtrees of a tree T . Then:*

- (a) *The subgraph $\Delta = \cap_{i \in I} \Delta_i$ is a subtree of T .*
- (b) *If $\cap_{i \in I} V(\Delta_i) \neq \emptyset$, then $\Delta = \overline{\cup_{i \in I} \Delta_i}$ is a subtree of T (the bar denotes closure in the space T).*

Note if I is a finite index set, then $\Delta = \cup_{i \in I} \Delta_i$.

Based on this proposition, for any two vertices u, v , we define the *path* $[u, v]$ of these two vertices in a profinite tree T as the smallest subtree of T containing u and

v . Then $[u, v]$ is also equal to the intersection of all subtrees of T which contain both vertices u and v . As an example, in the Cayley graph $\Gamma(\hat{Z}, 1)$ (which is a tree by Gildenhuys-Ribes theorem, see [G-R]), the path $[u, v]$ of two vertices is either a finite path or the whole tree.

Definition 2.17 *If G is a profinite group, then a Boolean space X is called a G -space if G acts on X and the action is a continuous map from $G \times X$ to X . If X is a G -space, one also says G acts on the space X . For a profinite graph Γ , if G acts on both $V(\Gamma)$ and $E(\Gamma)$ in a compatible way, we say G acts on the profinite graph Γ .*

For any profinite group G , G acts on its Cayley graph with respect to any closed subset of G in a natural way.

Let G be a profinite group that acts on a Boolean tree Γ . Suppose that the quotient graph $G \backslash \Gamma$ is finite, and let $\phi : \Gamma \rightarrow G \backslash \Gamma$ be the canonical epimorphism of graphs. Clearly $G \backslash \Gamma$ has a subtree T' whose vertices are those of $G \backslash \Gamma$. Choose a vertex v of Γ . Since T' is finite, there exists a connected lifting T of T' (that is, T is a subtree of Γ such that ϕ maps T isomorphically to T'), with v as one of its vertices. Moreover, each edge e' of $G \backslash \Gamma$ not in T' can be lifted to an edge e of Γ such that one of the vertices of e is in T . The finite structure Σ , consisting of T and the set of those e , is called a *connected transversal containing v* of the action G on Γ . Note that Σ

is not a graph since those edges which are not in T have only one vertex in Σ .

Throughout this thesis, we adopt the following notations. Let H be a group. Let T be a subgroup of H and x, y two elements of H . Then $x^y = y^{-1}xy$ and $T^y = \{t^y \mid t \in T\}$. If we say a subgroup H of G is closed in G , it always means that H is closed in the profinite topology of G . $\bar{H}$ always refers to the closure of H in the profinite topology of $\hat{G}$, the profinite completion of G .

2.3 The separability properties of free groups

Free groups, as a kind of special groups, have been studied more extensively than other groups. The separability properties of free groups is completely clear to us now, due to the contribution of Hall, Ribes and Zalesskii. Here we state their main results and provide an alternative proof in some case.

Theorem 2.18 (*M. Hall*) *Given a free group F with an arbitrary number of generators, and a finite number of elements $a_1, a_2, \dots, a_m$ of F . Suppose we are also given a finite number of elements $b_1, b_2, \dots, b_n$ such that no b_i belongs to the subgroup generated by $a_1, a_2, \dots, a_m$. Then we may construct a subgroup of finite index in F containing $a_1, a_2, \dots, a_m$ but no one of $b_1, b_2, \dots, b_n$.*

The proof of this theorem can be found in [Hal]. The proof is based on the coset representation in the free group F . The following two corollaries are derived immediately from Hall's theorem.

Corollary 2.19 *Free groups are residually finite groups.*

Proof: Let F be a free group, e the identity of F . Suppose that $g \in F$ and $g \neq e$, then $g \notin \langle e \rangle = \{e\}$. So there is a subgroup U of finite index in F such that $g \notin U$. Hence F is residually finite. ■

Corollary 2.20 *Free groups are subgroup separable.*

Proof: Let $H = \langle h_1, h_2, \dots, h_m \rangle$ be a finitely generated subgroup of a free group F . Let $g \in F$ and $g \notin H$. Then by Hall's theorem, there is a subgroup U of finite index in F such that $H \subset U$ and $g \notin U$. So $Ug \cap H = \emptyset$ and F is subgroup separable. ■

A more exciting result is obtained by Ribes and Zalesskii. By using the theory of profinite groups acting on Boolean graphs, they proved in 1991 that free groups are product separable (cf.[R-Z] Th 2.1). Actually they proved a slightly generalized version of this. Here we just focus on the result about free groups. We give an alternative proof by proving the following lemma.

Lemma 2.21 *Let F be a free group, and let $K_1, K_2, \dots, K_n$ be finitely generated subgroups of F where $n \geq 3$. Assume K_n is a free factor of F . If $f_1 f_2 \cdots f_n \in F$ where $f_i \in K_i$ (the closure of K_i in $\hat{F}$), then there exist $g_n \in \bar{K}_n, g_1 \in \bar{K}_1$ and $g_j \in \bar{K}_j$ for some j ($2 \leq j \leq n-1$) such that*

$$(1) \ g_n = g_1^t \text{ where } t = f_2 f_3 \cdots f_{n-1}$$

$$(2) \ f_1 g_1^{-1} f_2 \cdots f_j g_j^{-1} \in F$$

$$(3) \ g_j f_{j+1} f_{j+2} \cdots f_{n-1} g_n f_n \in F$$

Proof: First choose a basis X for the free group K_n . Since K_n is a free factor of F , one can extend X to a basis B of F . Then X and B are also bases of $\hat{K}_n$ and $\hat{F}$ respectively. We construct the Cayley graphs $\Gamma(K_n, X), \Gamma(\hat{K}_n, X), \Gamma(F, B)$ and $\Gamma(\hat{F}, B)$. Then $\Gamma(K_n, X), \Gamma(F, B)$ are trees as ordinary graphs (cf. [Se], pp. 17) and $\Gamma(\hat{K}_n, X), \Gamma(\hat{F}, B)$ are Boolean trees (cf. [G-R], Theorem 1.2). It is clear that $\Gamma(K_n, X) \subset \Gamma(\hat{K}_n, X) \subset \Gamma(\hat{F}, B)$ and $\Gamma(F, B) \subset \Gamma(\hat{F}, B)$ since K_n and F are embedded into $\hat{K}_n$ and $\hat{F}$ respectively.

Let $S_i = \{r_{i1}, \dots, r_{ik_i}\}$ be a set of the generators of K_i and 1 be the identity of group G . For $i = 2, \dots, n$, we construct

$$D_i = \cup K_i[1, r]$$

where r runs over the generator set S_i of K_i . Also we construct

$$D_1 = \cup \hat{K}_1[1, r]$$

where r runs over the union of the generator set S_1 of K_1 and $\{f_1 f_2 \cdots f_n\}$. Then D_i

is a $\bar{K}_i$ -invariant Boolean subtree of $\Gamma(\hat{F}, B)$ containing the vertex 1 for every i . So

$f_{n-1}^{-1} f_{n-2}^{-1} \cdots f_{i+1}^{-1} D_i$ is also a Boolean tree.

Now $f_{n-1}^{-1} \cdots f_{i+2}^{-1} D_{i+1} \cap f_{n-1}^{-1} \cdots f_{i+1}^{-1} D_i \neq \emptyset$ for all i , so

$$D = f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cup f_{n-1}^{-1} \cdots f_3^{-1} D_2 \cup \cdots \cup f_{n-1}^{-1} D_{n-2} \cup D_{n-1}$$

is also a Boolean tree (cf. [Z-M], pp 410, Proposition 1.18). From the definition of

D_1 , $f_1 f_2 \cdots f_n \in D_1$, so $f_n \in f_{n-1}^{-1} f_{n-2}^{-1} \cdots f_2^{-1} D_1$. Hence

$$[f_n, 1] \subset f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cup f_{n-1}^{-1} \cdots f_3^{-1} D_2 \cup \cdots \cup D_{n-1}.$$

Therefore, by Lemma 1.4 in [R-Z],

$$[f_n, 1] \cap f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cap (f_{n-1}^{-1} \cdots f_3^{-1} D_2 \cup \cdots \cup f_{n-1}^{-1} D_{n-2} \cup D_{n-1}) \neq \emptyset.$$

Let k' be a vertex in this intersection, $k' \in [f_n, 1] \cap f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cap f_{n-1}^{-1} \cdots f_{j+1}^{-1} D_j$

for some j ($2 \leq j \leq n-1$). Let $t = f_2 f_3 \cdots f_{n-1}$ and $(K_1)^t = t^{-1} K_1 t$. Then $(K_1)^t$ acts

on $f_{n-1}^{-1} \cdots f_2^{-1} D_1$ and $(\bar{K}_1)^t \cap \bar{K}_n$ acts on $T = f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cap \Gamma(\hat{K}_n, X)$ naturally.

Note that K_n is a free factor of F , so $K_n = \hat{K}_n$.

We claim that under such natural action, the quotient graph $((\bar{K}_1)^t \cap \bar{K}_n) \backslash T$ is a finite graph. First the quotient graph $(\bar{K}_1)^t \backslash f_{n-1}^{-1} \cdots f_2^{-1} D_1$ is finite since

$$f_{n-1}^{-1} \cdots f_2^{-1} D_1 = (\bar{K}_1)^t \backslash f_{n-1}^{-1} \cdots f_2^{-1} (\cup[1, r])$$

where r runs over the union of generator set S_1 of K_1 and $\{f_1 f_2 \cdots f_n\}$. Let

$$\pi : f_{n-1}^{-1} \cdots f_2^{-1} D_1 \longrightarrow (\bar{K}_1)^t \backslash f_{n-1}^{-1} \cdots f_2^{-1} D_1$$

be the canonical epimorphism of graphs. Then $\pi(T)$ is a finite graph also. The morphism π induces an epimorphism

$$\bar{\pi} : ((\bar{K}_1)^t \cap \bar{K}_n) \backslash T \longrightarrow \pi(T).$$

Furthermore, $\bar{\pi}$ is an isomorphism. To see this, let $s, t \in V(T)$ and $hs = t$ for some $h \in (\bar{K}_1)^t$. Then h must be in K_n since $T \subset \Gamma(\bar{K}_n, X)$ and $s, t \in \bar{K}_n$. So $((\bar{K}_1)^t \cap \bar{K}_n) \backslash T$ is finite.

Let $\phi : T \longrightarrow ((\bar{K}_1)^t \cap \bar{K}_n) \backslash T$ be the canonical morphism of graphs. There exists a connected transversal Σ of ϕ in T which contains the vertex k' . Since $f_n \in T$, there exists a $g_n \in (\bar{K}_1)^t \cap \bar{K}_n$ such that $g_n f_n \in \Sigma$. Now $[g_n f_n, k']$ is a finite path in Σ and hence in $T \subset \Gamma(\bar{K}_n)$. So there exists an element $h_n \in K_n$ such that $k' h_n = g_n f_n$. Because of $k' \in f_{n-1}^{-1} \cdots f_{j+1}^{-1} D_j$, one can find a $g_j \in \bar{K}_j$ such that

$g_j f_{j+1} \cdots f_{n-1} k' \in U[1, r]$. That means

$$g_j f_{j+1} \cdots f_{n-1} k' \in F.$$

Let $g_n = g_1^t$. We can rewrite the product as follows:

$$\begin{aligned} f_1 f_2 \cdots f_{n-1} f_n &= f_1 f_2 \cdots f_{n-1} g_n^{-1} g_n f_n \\ &= f_1 g_1^{-1} f_2 \cdots f_{n-1} g_n f_n \\ &= f_1 g_1^{-1} f_2 \cdots f_{n-1} k' h_n \\ &= f_1 g_1^{-1} f_2 \cdots f_j g_j^{-1} g_j f_{j+1} \cdots f_{n-1} k' h_n \\ &= f_1 g_1^{-1} f_2 \cdots f_j g_j^{-1} g_j f_{j+1} \cdots f_{n-1} g_n f_n \end{aligned}$$

Since $f_1 f_2 \cdots f_n \in F$ and $g_j f_{j+1} \cdots f_{n-1} g_n f_n \in F$, $f_1 g_1^{-1} f_2 \cdots f_j g_j^{-1} \in F$. The lemma is proved. ■

Now we prove the product separability of free groups.

Theorem 2.22 (*Ribes-Zaleskii*) *Free groups are product separable.*

Proof: Let F be a free group and $H_1, H_2, \dots, H_n$ be finitely generated subgroups of F . For any subgroup U of finite index in F , the profinite topology of U coincides with the one induced from the profinite topology of F . So the product separability of F is equivalent to that of U . Now since H_n is finitely generated, by a theorem of M. Hall, there exists a subgroup U of finite index in F such that H_n is a free factor

of U (cf. [H1]). Hence we may assume that H_n is a free factor of F to prove F is product separable.

Since F is residually finite, any subset K of F is closed if and only if $\bar{K} \cap F = K$ (cf. [R-Z], pp. 38). So we shall prove that $\bar{H}_1 \bar{H}_2 \cdots \bar{H}_n \cap F = H_1 H_2 \cdots H_n$ since $\overline{H_1 H_2 \cdots H_n} = \bar{H}_1 \bar{H}_2 \cdots \bar{H}_n$ due to compactness.

We use induction on the number n of subgroups. The case $n = 1$ is just the subgroup separability, which has been proved by M. Hall.

When $n = 2$, H_2 is a free factor of F . Let $f_1 \in \bar{H}_1$ and $f_2 \in \bar{H}_2$ such that $f_1 f_2 \in F$. We choose a basis of H_2 and extend it to a basis of F , then construct the Cayley graphs of H_2 , F , $\bar{H}_2$ and $\hat{F}$ with respect to the chosen bases and denote them as $\Gamma(H_2)$, $\Gamma(F)$, $\Gamma(\bar{H}_2)$ and $\Gamma(\hat{F})$ respectively. Similarly as in the above lemma, construct a Boolean tree as follows:

$$D = \cup \bar{H}_1[1, r]$$

where r runs over the union of the generator set of H_1 and $\{f_1 f_2\}$. $f_1 f_2 \in D$ implies that $f_2 \in D$. Consider the Boolean subtree $T = D \cap \Gamma(\bar{H}_2)$ and the natural action of $\bar{H}_1 \cap H_2$ on it. Let $\pi : D \longrightarrow \bar{H}_1 \setminus D$ be the canonical epimorphism of graphs. Then $\pi(T)$ is a finite graph. π induces an epimorphism $\bar{\pi} : (\bar{H}_1 \cap \bar{H}_2) \setminus T \longrightarrow \pi(T)$. Actually $\bar{\pi}$ is an isomorphism. So $(\bar{H}_1 \cap \bar{H}_2) \setminus T$ is finite. Let $\phi : T \longrightarrow (\bar{H}_1 \cap \bar{H}_2)$

be the canonical morphism of graphs. There exists a connected transversal Σ of ϕ in T which contains vertex 1. Hence one can find an element $g \in H_1 \cap \bar{H}_2$ such that $gf_2 \in \Sigma$. Since Σ is finite and contains vertex 1, $\Sigma \subset \Gamma(F) \cap \Gamma(\bar{H}_2) = \Gamma(H_2)$. Therefore $gf_2 \in H_2$ and $f_1g^{-1} \in \bar{H}_1 \cap F = H_1$. That proves $\bar{H}_1\bar{H}_2 \cap F = H_1H_2$.

Now let $n \geq 3$. We assume that $\bar{H}_1\bar{H}_2 \cdots \bar{H}_k \cap F = H_1H_2 \cdots H_k$ is true for all $k < n$. Let $f_1f_2 \cdots f_n \in \bar{H}_1\bar{H}_2 \cdots \bar{H}_n \cap F$. By Lemma 2.21, there exist $g_n \in \bar{H}_n, g_1 \in H_1$ and $g_j \in \bar{H}_j$ for some j ($2 \leq j \leq n-1$) such that

$$f_1g_1^{-1}f_2 \cdots f_jg_j^{-1} \in F, g_jf_{j+1} \cdots f_{n-1}g_nf_n \in F$$

and $g_n = g_1^t$ where $t = f_2f_3 \cdots f_{n-1}$. So by the induction assumption,

$f_1g_1^{-1}f_2 \cdots f_jg_j^{-1} \in H_1H_2 \cdots H_j$ and $g_jf_{j+1} \cdots f_{n-1}g_nf_n \in H_jH_{j+1} \cdots H_n$. Therefore

$$f_1f_2 \cdots f_n = f_1g_1^{-1}f_2 \cdots f_jg_j^{-1}g_jf_{j+1} \cdots f_{n-1}g_nf_n \in H_1H_2 \cdots H_jH_jH_{j+1} \cdots H_n.$$

The conclusion follows. ■

The proof above is based on the lemma which focuses on the presentation of a product of elements in a free group. Actually the proof of the $n = 2$ case is similar to the proof of Lemma 2.21. One may combine them into one with several cases but the complicated notations will cause confusion. Lemma 2.21 is useful later in the proof of the separability property of the direct product of a free group and an Abelian group.

Chapter 3

The Direct Product And Projection

In this chapter, we study the separability property of a direct product of groups and a projection property of the closure of a subset in the profinite completion of group G . We also get a complete description of the separability properties of Abelian groups.

3.1 The separability of Abelian groups

We first consider the separability properties of a direct product of groups. The following are some simple results.

Proposition 3.1 *The direct product G of groups G_α is residually finite if and only if each factor G_α is residually finite.*

Proof: Let $g = (g_\alpha)$ be a non-identity element of G . Then at least one g_α is not the identity of G_α . There is a subgroup U of finite index in G_α such that $g_\alpha \notin U$. Now $V = U \times \prod G_\beta$ has finite index in G where the product is taken over $\beta \neq \alpha$, and

it is easy to see that $g \notin V$. The conclusion follows.

Conversely, since any subgroup of a residually finite group is also residually finite, so the residually finiteness of G implies the residually finiteness of each factor group G_α . ■

From Proposition 3.1 and the fact that any cyclic group is residually finite, we deduce that finitely generated Abelian groups are residually finite since the structure of such groups is well known. In an Abelian group, the product set of subgroups is again a subgroup. So product separability of an Abelian group is equivalent to subgroup separability. The next proposition shows that finitely generated Abelian groups have nice separability properties.

Proposition 3.2 *Finitely generated Abelian groups are subgroup separable, and hence product separable.*

Proof: Let G be a finitely generated Abelian group. By the structure theorem of finitely generated Abelian groups, G can be denoted as

$$G \cong \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z} \times A$$

where A is a free Abelian group of finite rank. Subgroup separability of A implies subgroup separability of G since A is a subgroup of finite index in G . So, without loss in generality, we may assume that G is a free Abelian group of finite rank.

Let H be a subgroup of G ; then H is finitely generated. We write $G = \langle a \rangle \times A_1$ and $H = \langle a^{k_1} f_1, a^{k_2} f_2, \dots, a^{k_n} f_n \rangle$ where $f_i \in A_1$. Using $a^{k_i} f_i a^{-k_i} f_j^{-1}$ instead of $a^{k_i} f_i$ if necessary, we get

$$H = \langle a^k g_1, g_2, \dots, g_n \rangle$$

where $g_i \in A_1$. So H is a direct factor of a subgroup of finite index in $G_1 = \langle a^k \rangle \times A_1$ (to see this, one may choose a suitable generator set of G_1 , i.e. by using $a^k g_1$ instead of a^k). Hence H is closed in the profinite topology of G_1 . Note that G_1 is open, and hence closed, in the profinite topology of G . Also the profinite topology of G_1 is induced from that of G . Therefore H is closed in the profinite topology of G . The conclusion is proved. ■

The above proposition cannot be improved; if we drop the condition of "finitely generated", the conclusion is no longer true. There does exist an infinitely generated Abelian group which is not even residually finite. The simplest examples are divisible groups such as R (the additive group of real numbers) and Q (the additive group of rational numbers), and so on. They are not subgroup separable either.

One may ask the following question: is an infinitely generated residually finite Abelian group subgroup separable? Or whether residual finiteness is equivalent to subgroup separability in the category of Abelian groups? Let G be a residually finite

group. For any non-identity element g , there exists a open subgroup U_α such that $g \notin U_\alpha$. Then G can be embedded into $T = \prod G/U$ where U runs through the subgroups of finite index in G . Let H be a finitely generated subgroup of G and $g \notin H$. If T is subgroup separable, then there exists an open subgroup U of T such that $gU \cap H = \emptyset$. Let $V = U \cap G$. Note $|G : V| < \infty$. Then V is open in G and $gV \cap H \subset gU \cap H = \emptyset$. That implies that G is subgroup separable. Hence it is enough to consider the case that $G = \prod G_\alpha$ where each G_α is a finite group to consider the above problem. The next proposition shows that there does exist an infinitely generated Abelian group which is residually finite but not subgroup separable.

Proposition 3.3 *Let p be a prime number. Then $G = \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ is residually finite, but not subgroup separable.*

Proof: The residually finiteness of G is given by Proposition 3.1. Let $H = \langle a \rangle$ where $a = (1, 1, \dots, 1, \dots)$ ($a_{p^n} = 1$), and $g = (1, 1 + p, 1 + p + p^2, \dots)$ ($g_{p^n} = 1 + p + \dots + p^{n-1}$). For any integer m , $g_{p^n} \equiv m \pmod{p^n}$ fails when $p^{n-1} \geq m$. So $g \neq ma$ and $g \notin H$. We claim that $g \in H$.

Let U be any open subgroup of G . Then $[G : U] < \infty$. Let n_0 be the smallest integer satisfied that $p^{n_0} > [G : U]$. For any group $\mathbb{Z}/p^n\mathbb{Z}$ where $n \geq n_0$, there exists an element $p^r t \in U$ where $r \leq n_0$ and t and p are relatively prime. From

$g.c.d(t, p^{n-r}) = 1$, we get $tu + p^{n-r}v = 1$ for some u, v . So $p^r t \in U$ provides that

$$p^r = p^r tu + p^r p^{n-r} v = p^r tu \in U.$$

Hence $p^{n_0} = p^r p^{n_0-r} \in U$.

Now let $M = 1 + p + \cdots + p^{n_0}$. Then $Ma \in H = \langle a \rangle$ and $(Ma)_{p^n} = g_{p^n}$ when $n \leq n_0$. When $n \geq n_0$, $(Ma)_{p^n} - g_{p^n} \in U$. So $Ma \in gU$ and $gU \cap H \neq \emptyset$. The result follows. ■

This completes our discussion about the separability properties of Abelian groups.

In the last part of this thesis, we restrict our attention to direct products of groups in which one of the direct factors is a finitely generated Abelian group and the other is a free group.

3.2 Closure and profinite completion

In this section we investigate the closure of a set in the profinite completion $\hat{G}$ of G .

Proposition 3.4 *Let G be a residually finite group. Let H_i be a finitely generated subgroup of G , and $\bar{H}_i$ its closure in $\hat{G}$ ($i = 1, \dots, n$). Then $H_1 H_2 \cdots H_n$ is closed if and only if $\bar{H}_1 \bar{H}_2 \cdots \bar{H}_n \cap G = H_1 H_2 \cdots H_n$.*

Proof: It is obvious that $H_1 H_2 \cdots H_n$ is closed in G if and only if

$$\overline{H_1 H_2 \cdots H_n} \cap G = H_1 H_2 \cdots H_n.$$

Now $\bar{H}_i$ is closed in $\hat{G}$, so it is compact. Hence $\overline{H_1 H_2 \cdots H_n} = \bar{H}_1 \bar{H}_2 \cdots \bar{H}_n$ since the product of compact sets is compact and closed. Therefore,

$$\overline{H_1 H_2 \cdots H_n} \cap G = H_1 H_2 \cdots H_n \cap G.$$

The result follows. ■

Our interest is to generalize the result in Chapter 2 and prove the product separability of the direct product of a free group and an infinite cyclic group. To do so, we need more information about the profinite completion of this group and its subgroups.

Proposition 3.5 *Let F be a free group of finite rank and $\langle a \rangle$ an infinite cyclic group.*

Clearly the profinite completion of $G = F \times \langle a \rangle$ is $\hat{G} = \hat{F} \times \langle \hat{a} \rangle$. Let $P_{\hat{G}}$ be the projection from $\hat{G}$ to $\hat{F}$ and P_G be the projection from G to F . (Note that the restriction of $P_{\hat{G}}$ on G is P_G .) Then for any subgroup H of G , $P_{\hat{G}}(H) = \overline{P_G(H)}$.

Proof: Since $\bar{H}$ is compact, we get that $P_{\hat{G}}(H)$ is compact and hence closed.

Now $P_{\hat{G}}(\bar{H}) \supseteq P_G(H)$, which implies that $P_{\hat{G}}(H) \supseteq \overline{P_G(H)}$.

Conversely, for any $f \in P_{\hat{G}}(\bar{H}) \subset \hat{F}$, there exists an $f' \in H$ such that $P_{\hat{G}}(f') = f$.

So for any open subgroup $\hat{O}$ of $\hat{G}$, $\hat{O}f' \cap H \neq \emptyset$. We get $P_{\hat{G}}(\hat{O})f \cap P_{\hat{G}}(H) \neq \emptyset$ by applying $P_{\hat{G}}$. This is just: $P_{\hat{G}}(\hat{O})f \cap P_G(H) \neq \emptyset$. Now $P_{\hat{G}}(\hat{O})$ can be any open subgroup of $\hat{G}$, so $f \in \overline{P_G(H)}$ in $\hat{F}$. Therefore, $P_{\hat{G}}(H) = \overline{P_G(H)}$. ■

The following proposition plays an important role in our proof of product separability of the direct product of a free group and an infinite cyclic group.

Proposition 3.6 *Let H_1 and H_2 be finitely generated subgroups of $G = F \times \langle a \rangle$. Let Π be the projection homomorphism from G to $\langle a \rangle$. If $\Pi(H_1) = \langle a \rangle$ and $H_1 H_2 \cap \langle a \rangle = 1$, then $H_1 \bar{H}_2 \cap \overline{\langle a \rangle} = 1$.*

Proof: Let $H_1 = \langle ae_{11}, e_{12}, \dots, e_{1s} \rangle$ and $H_2 = \langle a^k e_{21}, e_{22}, \dots, e_{2t} \rangle$, where k is an integer and $e_{ij} \in F$. Suppose that $b \in \bar{H}_1 \bar{H}_2 \cap \overline{\langle a \rangle}$ and $b \neq 1$, then there is a number p such that $\rho(b) = a^s \neq 1$, where ρ is the projection from $\overline{\langle a \rangle}$ to $\langle a \rangle / \langle a^p \rangle$. Now $H_1 H_2 \cap \langle a \rangle = 1$, so $1 \notin e_{11}^s \langle e_{11}^p \rangle H_1 H_2$. Otherwise $a^{s+pt} \in (ae_{11})^s (ae_{11})^{pt} H_1 H_2 = H_1 H_2$. Since $\langle e_{11}^p \rangle H_1 H_2 = \langle a^p \rangle H_1 H_2$ is closed in G (the complete proof of this is in Theorem 4.8 of this thesis), $e_{11}^s \langle e_{11}^p \rangle H_1 H_2$ is closed also. So there is an open subgroup O of G satisfying $O \cap e_{11}^s \langle e_{11}^p \rangle H_1 H_2 = \emptyset$. $O \cap F$ is open in F , hence there exists an open subgroup Q of $\hat{F}$ such that $Q \cap F = O \cap F$. Now $Q \times \ker \rho$ is an open subgroup of $\hat{G} = \hat{F} \times \langle \hat{a} \rangle$ and $b(Q \times \ker \rho) \cap H_1 H_2 \neq \emptyset$ because of $b \in \bar{H}_1 \bar{H}_2$. Thus we can find $f \in Q$ and $c \in \ker \rho$ such that $bfc \in H_1 H_2 \subset G$. Since $bc \in \overline{\langle a \rangle}$ and $f \in \hat{F}$, then $bfc \in G$ implies that $f \in \hat{F} \cap G = F$, and $bc \in \overline{\langle a \rangle} \cap G = \langle a \rangle$. Now $\rho(bc) = \rho(b)\rho(c) = \rho(b) = a^s$. So $bc = a^s a^{pt}$ and $f \in Q \cap F = O \cap F$. That is: $fa^s a^{pt} \in H_1 H_2$. Therefore $f \in e_{11}^s e_{11}^{pt} H_1 H_2 \subset e_{11}^s \langle e_{11}^p \rangle H_1 H_2$. This contradicts that $O \cap e_{11}^s \langle e_{11}^p \rangle H_1 H_2 = \emptyset$. ■

The proposition leads to a more clear and simple conclusion.

Corollary 3.7 *Let G be $F \times \langle a \rangle$ and H a finitely generated subgroup of G . If $H \cap \langle a \rangle = 1$, then $\bar{H} \cap \overline{\langle a \rangle} = 1$.*

Proof: Set $H_2 = 1$ in above proposition. ■

Next we consider the closure of a finite union of subsets in a group. Following are some simple conclusions .

Proposition 3.8 *Let H, K be subgroups of G where K is a subgroup of finite index in H . Write $H = \cup b_j K$ (finite union) and let $\bar{H}, \bar{K}$ be their closures in $\hat{G}$. Then $\bar{H} = \cup b_j \bar{K}$*

Proof: Obvious. ■

Corollary 3.9 *Let k be any positive integer, then $\overline{\langle a \rangle} = \cup a^j \overline{\langle a^k \rangle}$.*

Proposition 3.10 *For any subgroup H of G and any element t in $\hat{G}$,*

$$\overline{H^t} = (\bar{H})^t$$

Proof: This is obvious since conjugation by t is a topological automorphism of $\hat{G}$. ■

Chapter 4

The Generalized Free Product Of Cyclic Groups

In this chapter, we will discuss the separability property of the direct product of a free group and an infinite cyclic group. Then we deduce the product separability of the generalized free product of cyclic groups.

4.1 Subgroups of $F \times Z$

Let F be a group, and let $\langle a \rangle$ be an infinite cyclic group. We begin our discussion by investigating the subgroups of $F \times \langle a \rangle$. In this section, we will treat Z as $\langle a \rangle$, an infinite cyclic group.

Proposition 4.1 *Let H be a finitely generated subgroup of $G = F \times Z$, where $Z = \langle a \rangle$, then $H = \langle a^k e_1, e_2, \dots, e_n \rangle$ where $e_i \in F$ ($i = 1, 2, \dots, n$).*

Proof: Let $\{r_1, r_2, \dots, r_m\}$ be a generator set of H , $r_i = a^{k_i} t_i$, where $t_i \in F$. We may assume that all k_i are non-negative integers, otherwise replace r_i by its inverse if k_i is negative. If $k_i \leq k_j$, use $r_i^{-1} r_j$ instead of r_j ; then

$$\{r_1, r_2, \dots, r_{j-1}, r_i^{-1} r_j, r_{j+1}, \dots, r_m\}$$

is also a generator set of H . Repeating this process if necessary, we may assume that at most one k_i is nonzero. So $H = \langle a^k e_1, e_2, \dots, e_n \rangle$ where $k \geq 0$ and $e_i \in F$. ■

Proposition 4.2 *Let $H = \langle a^k e_1, e_2, \dots, e_n \rangle$ be a finitely generated subgroup of $G = F \times \langle a \rangle$ and $k \neq 0$, where $e_1, \dots, e_n \in F$. Then for any integer $m \neq 0$, there exists a subgroup K satisfying: $K \triangleleft H, [H : K] < \infty$ and $K = \langle a^{mk} f_1, f_2, \dots, f_t \rangle$ where $f_j \in F$ ($j = 1, 2, \dots, t$). Moreover, if $H \cap \langle a \rangle \neq 1$, for any integer m , one may choose K so that $K = K_1 \times \langle a^{mk} \rangle$, where K_1 is a finitely generated subgroup of F .*

Proof: Let ϕ_1 be the projection from $F \times \langle a^k \rangle$ to $\langle a^k \rangle$, ϕ_2 be the homomorphism from $\langle a^k \rangle$ to Z defined by $\phi_2(a^{sk}) = s$. Let ϕ_3 be the projection from Z to Z/mZ . Then $\phi_3 \phi_2 \phi_1$ is a homomorphism from $F \times \langle a^k \rangle$ to Z/mZ . Let ρ be its restriction to H , i.e. $\rho = (\phi_3 \phi_2 \phi_1)|_H$. So $\rho(a^k e_1) = 1, \rho(e_i) = 0, i = 2, \dots, n$. Now let $K = \ker \rho$. Then $K \triangleleft H$ and $[H : K] < \infty$. Since $(a^k e_1)^m \in K, e_j^{(a^k e_1)^i} \in K$, so $K \supset \langle (a^k e_1)^m, e_j^{(a^k e_1)^i} \mid j = 2, \dots, n; i = 0, \dots, m-1 \rangle$. For any $h \in K \subset H$,

$h = \dots e_i (a^k e_1)^t e_j \dots = \dots (a^k e_1)^q e_i^{(a^k e_1)^q} (a^k e_1)^{ml} e_j \dots$, where $t = ml + q$ and $0 \leq q < m$.

So $h \in K$ implies that $h \in \langle (a^k e_1)^m, e_j^{(a^k e_1)^i} \mid j = 2, \dots, n; i = 0, \dots, m-1 \rangle$. That

provides $K = \langle (a^k e_1)^m, e_j^{(a^k e_1)^i} \mid j = 2, \dots, n; i = 0, \dots, m-1 \rangle$. Now $(a^k e_1)^m = a^{mk} e_1^m$

and $e_j^{(a^k e_1)^i} = (a^k e_1)^{-i} e_j (a^k e_1)^i = e_1^{-i} e_j e_1^i \in F$. So $K = \langle a^{mk} f_1, f_2, \dots, f_t \rangle$ where

$f_i \in F$. ■

These two propositions give explicit generator sets of a finitely generated subgroup H of $F \times Z$ and a subgroup of finite index in H . Using them, it is not difficult to prove the subgroup separability of $F \times Z$ where F is a free group.

Proposition 4.3 *If H is a finitely generated subgroup of $G = F \times \langle a \rangle$, where F is a subgroup separable group, and $H \cap \langle a \rangle \neq 1$, then H is closed under the profinite topology of G .*

Proof: Let $H = \langle a^k e_1, e_2, \dots, e_n \rangle$, and $H \cap \langle a \rangle \neq 1$. We may assume that $H \cap \langle a \rangle = \langle a^{mk} \rangle$. From Proposition 4.2, there exists a subgroup $K = \langle a^{mk} f_1, \dots, f_t \rangle$ so that $K = \ker \rho$ where $\rho : H \longrightarrow Z/nZ$ and $[H : K] < \infty$. Now $\rho(a^{mk}) = \phi_3 \phi_2 \phi_1(a^{mk}) = \phi_3 \phi_2(a^{mk}) = \phi_3(a^m) = 0$. That provides $a^{mk} \in K$. Thus $K = \langle a^{mk}, a^{mk} f_1, \dots, f_t \rangle$, $K = \langle a^{mk}, f_1, \dots, f_t \rangle$ and $K = \langle a^{mk} \rangle \langle f_1, f_2, \dots, f_t \rangle$. Since $\langle a^{mk} \rangle$ is closed in $\langle a \rangle$ and $\langle f_1, \dots, f_t \rangle$ is closed in F (see Chapter 2 and Chapter 3), and K is the direct product of $\langle f_1, f_2, \dots, f_t \rangle$ and $\langle a^{mk} \rangle$. Hence K is closed in $G = F \times \langle a \rangle$

and so are the cosets of K . H is finite union of cosets of K and therefore is closed.

■

Corollary 4.4 *If H is a finitely generated subgroup of $G = F \times \langle a \rangle$, then for any positive integer n , $\langle a^n \rangle H$ is closed in G .*

Proof: Since a is in the centre of $G = F \times \langle a \rangle$, $\langle a^n \rangle H$ is a subgroup of G .

Obviously, $\langle a^n \rangle H \cap \langle a \rangle \supset \langle a^n \rangle \neq 1$. Apply Proposition 4.3 to get the conclusion. ■

Corollary 4.5 *$G = F \times \langle a \rangle$ is subgroup separable.*

Proof: Let H be a finitely generated subgroup of $G = F \times \langle a \rangle$. Then for any positive n , $\langle a^n \rangle H$ is closed. The intersection $\cap \langle a^n \rangle H$ is also closed in G . Clearly $H \subset \cap \langle a^n \rangle H$. If $H = \cap \langle a^n \rangle H$, then H is closed in G . If $H \neq \cap \langle a^n \rangle H$, there is an $a^k h \in \cap \langle a^n \rangle H$ with $k \neq 0$ and $h \in H$. We can find an m such that $|m| > |k|$. So $a^k h \in \cap \langle a^n \rangle H \subset \langle a^m \rangle H$ implies that $a^k h = a^{ml} h_1$ for some l and h_1 . Therefore, $a^{ml-k} = h h_1^{-1} \in H \cap \langle a \rangle$, hence this intersection is non-trivial. From Proposition 4.3, it follows that H is closed in G . ■

The following proposition will be useful to prove our main result later in this chapter.

Proposition 4.6 *Let $H_1, H_2, \dots, H_n$ be subgroups of any group G . For a fixed i , let $K \triangleleft H_i$ and $[H_i : K] < \infty$. Write $H_i = \cup b_j K$ (disjoint union) where each b_j is in H_i . Then $H_1^{b_1} H_2^{b_2} \dots H_{i-1}^{b_{i-1}} K H_{i+1} \dots H_n$ is closed for each b_j implies that $H_1 H_2 \dots H_n$ is closed.*

Proof: Note that $H_1 H_2 \dots H_n = \cup H_1 \dots H_{i-1} b_j K H_{i+1} \dots H_n = \cup b_j H_1^{b_1} \dots H_{i-1}^{b_{i-1}} K H_{i+1} \dots H_n$ (all unions are disjoint). So $H_1 H_2 \dots H_n$ is a finite union of closed sets and hence is closed. ■

Recall that any finite extension of a product separable group is also product separable (cf. [R-Z], proof of Theorem 2.1). Hence to ensure our result is not a simple consequence of Ribes and Zalesskii's theorem, we need to prove the following proposition.

Proposition 4.7 *If $\text{rank } F \geq 1$, then $F \times Z$ has no free subgroup of finite index.*

Proof: If H is a free subgroup of $G = F \times Z$ with finite index, then $Z \cap H$ is a non-trivial infinite subgroup which is contained in the centre of H , since Z is the centre of G . Now H is free, so H is cyclic. $Z \cap H$ being infinite implies that $H \subset Z$. Therefore, H is not of finite index in G . ■

Now we turn to prove that $F \times Z$ is product separable.

4.2 Some simple cases

We start our proof with some simple cases. Throughout this section, we assume that

$G = F \times \langle a \rangle$ is endowed with its profinite topology.

Theorem 4.8 *Let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of G . If there is some subgroup H_i satisfying $H_i \cap \langle a \rangle \neq 1$, then $H_1 H_2 \cdots H_n$ is closed in G .*

Proof: Let t be the number of subgroups H_i which are not contained in F . Then $t \geq 1$. We prove the conclusion by induction on t .

When $t = 1$, say H_i is not contained in F , then $H_i \cap \langle a \rangle \neq 1$ (since the other H_j 's are contained in F). Let $H_i = \langle a^k e_1, e_2, \dots, e_s \rangle$ and $\langle a^{mk} \rangle = H_i \cap$

$\langle a \rangle$. By Proposition 4.2, there exists $K = \langle a^{mk} f_1, f_2, \dots, f_p \rangle$ with $[H : K]$

$< \infty$. As we saw in the proof of Proposition 4.3, we may choose K such that $K =$

$\langle a^{mk} \rangle \langle f_1, f_2, \dots, f_p \rangle$. Now let $H_i = \cup b_j K$ (finite union), then $H_1 H_2 \cdots H_n = \cup b_j H_1^{b_j} \cdot$

$\cdots H_{i-1}^{b_j} K H_{i+1} \cdots H_n$ (finite union). Since the subgroups $H_1^{b_j}, \dots, H_{i-1}^{b_j},$

$\langle f_1, f_2, \dots, f_p \rangle, H_{i+1}, \dots, H_n$ are all finitely generated and contained in F because of

$t = 1$ (note that if $H \subseteq F$, then $H_j^q \subseteq F$ for any $q \in G$ since $\langle a \rangle$ is central in G),

$H_1^{b_j} \cdots H_{i-1}^{b_j} \langle f_1, f_2, \dots, f_p \rangle H_{i+1} \cdots H_n$ is closed for any b_j (cf. Chapter 2), hence

$$H_1^{b_j} \cdots H_{i-1}^{b_j} K H_{i+1} \cdots H_n$$

$$= H_1^{b_j} \cdots H_{i-1}^{b_j} \langle a^{mk} \rangle \langle f_1, f_2, \dots, f_p \rangle H_{i+1} \cdots H_n$$

$$= \langle a^{mk} \rangle H_1^{b_1} \cdots H_{i-1}^{b_{i-1}} \langle f_1, f_2, \dots, f_p \rangle H_{i+1} \cdots H_n$$

and it is closed for any b_j . Therefore $H_1 H_2 \cdots H_n$ is a finite union of closed sets of G and so it is closed also.

When $t > 1$, pick an H_i such that $H_i \cap \langle a \rangle \neq 1$. Let $\langle a^{mk} \rangle = H_i \cap \langle a \rangle$. Using the same notations as above, we get a subgroup $K = \langle a^{mk} \rangle \langle f_1, \dots, f_p \rangle$ of H_i such that $[H_i : K] < \infty$ and $H_1 H_2 \cdots H_n = \cup b_j H_1^{b_1} \cdots H_{i-1}^{b_{i-1}} K H_{i+1} \cdots H_n$ (finite union). Let H_q be another subgroup which is not contained in F , then

$$\begin{aligned} H_1^{b_1} \cdots H_{i-1}^{b_{i-1}} K H_{i+1} \cdots H_n &= H_1^{b_1} \cdots H_{i-1}^{b_{i-1}} \langle a^{mk} \rangle \langle f_1, \dots, f_p \rangle H_{i+1} \cdots H_n \\ &= H_1^{b_1} \cdots H_{i-1}^{b_{i-1}} \langle f_1, \dots, f_p \rangle H_{i+1} \cdots H_{q-1} (\langle a^{mk} \rangle H_q) H_{q+1} \cdots H_n. \end{aligned}$$

Note that $\langle a^{mk} \rangle H_q \cap \langle a \rangle \neq 1$, and so by induction ($t-1$ case),

$$H_1^{b_1} \cdots H_{i-1}^{b_{i-1}} K H_{i+1} \cdots H_n$$

is closed. Therefore $H_1 H_2 \cdots H_n$ is closed. ■

Theorem 4.9 *Let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of $G = F \times \langle a \rangle$. If $H_i H_{i+1} \cap \langle a \rangle \neq 1$ for some i , then $H_1 H_2 \cdots H_n$ is closed in G .*

Proof: If $H_i H_{i+1} \cap \langle a \rangle \neq 1$, we claim that $H_i H_{i+1} \cap \langle a \rangle$ is a subgroup of $\langle a \rangle$.

Take $a^n = f_1 f_2$ and $a^m = g_1 g_2$ in $H_i H_{i+1} \cap \langle a \rangle$, where $f_1, g_1 \in H_i$ and $f_2, g_2 \in H_{i+1}$.

Then $g_1^{-1} g_2^{-1} g_1 g_2 = g_1^{-1} g_1 g_2 g_2^{-1} = 1$ shows that $g_1^{-1} g_2^{-1} = (g_1 g_2)^{-1}$. So $f_1 f_2 (g_1 g_2)^{-1}$

$= f_1 f_2 g_1^{-1} g_2^{-1} = f_1 g_1^{-1} g_2^{-1} f_2$ is in $H_i H_{i+1}$ and hence in $H_i H_{i+1} \cap \langle a \rangle$. That means $H_i H_{i+1} \cap \langle a \rangle$ is a subgroup.

Let $H_i H_{i+1} \cap \langle a \rangle = \langle a^t \rangle$, then $H_1 H_2 \cdots H_n = H_1 H_2 \cdots H_i \langle a^t \rangle H_{i+1} \cdots H_n$. By previous theorem, this product is closed. ■

Theorem 4.10 *Let H_1, H_2 be finitely generated subgroups of $G = F \times \langle a \rangle$, then $H_1 H_2$ is closed.*

Proof: If $H_1 H_2 \cap \langle a \rangle \neq 1$, the result follows from the last theorem. Let $H_1 H_2 \cap \langle a \rangle = 1$. Since $\langle a^n \rangle H_1 H_2$ is closed in G for any integer n , it is sufficient to prove that $\bigcap_{n=1}^{\infty} \langle a^n \rangle H_1 H_2 = H_1 H_2$. It is obvious that $\bigcap_{n=1}^{\infty} \langle a^n \rangle H_1 H_2 \supseteq H_1 H_2$. Let $a^k h_1 h_2 \in \bigcap_{n=1}^{\infty} \langle a^n \rangle H_1 H_2$. If $k \neq 0$, then for any $|m| > |k|$, $a^k h_1 h_2 \in \bigcap_{n=1}^{\infty} \langle a^n \rangle H_1 H_2 \subset \langle a^m \rangle H_1 H_2$. This implies that $a^k h_1 h_2 = a^{ml} g_1 g_2$, where $g_i \in H_i$ ($i = 1, 2$). So $a^{ml-k} = g_1^{-1} h_1 h_2 g_2^{-1} \in H_1 H_2 \cap \langle a \rangle = 1$. But $a^{ml-k} \neq 1$. Which is a contradiction. Hence k must be 0. Therefore $H_1 H_2 = \bigcap_{n=1}^{\infty} \langle a^n \rangle H_1 H_2$ and is closed. ■

So we have proved that $F \times Z$ is double coset separable. The definition of double coset separability can be found in Niblo's paper (cf. [Ni]) or in Chapter 1.

4.3 The product separability of $F \times Z$

Let F be a free group, and $\langle a \rangle$ an infinite cyclic group. Let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of $F \times \langle a \rangle$. According to the results in the last section, to prove that $H_1 H_2 \cdots H_n$ is closed in $F \times Z$, one may assume that $H_i H_{i+1} \cap \langle a \rangle = 1$ for all $i = 1, 2, \dots, n-1$. We shall see that the results of Chapter 3 can be applied here to get our main theorem of this chapter.

Theorem 4.11 *Let F be a free group, and let $\langle a \rangle$ be an infinite cyclic group, then $G = F \times \langle a \rangle$ is a product separable group.*

Proof: We start with a series of reductions.

(1) Let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of $F \times \langle a \rangle$. Then there exists a finitely generated subgroup F_0 of F such that $P_G(H_i) \subseteq F_0$ for all i where P_G is the projection from G to F . Since F_0 is closed in F (cf. Chapter 2, Theorem 2.22), so $F_0 \times \langle a \rangle$ is closed in $F \times \langle a \rangle$. Hence $H_1 H_2 \cdots H_n$ is closed in $F \times \langle a \rangle$ if and only if it is closed in $F_0 \times \langle a \rangle$. So we may assume that F is a finitely generated free group.

(2) Let $H_i = \langle a^{k_i} e_{i1}, e_{i2}, \dots, e_{in_i} \rangle$, where $k_i \in Z, e_{ij} \in F$ and $i = 1, 2, \dots, n$ (see the first section of this chapter). Let k be the least common multiple of the set consisting

of all k_i which are nonzero. Then from Proposition 4.2, there is a subgroup T_n in H_n of finite index such that $T_n = \langle a^k f_1, f_2, \dots, f_{t_n} \rangle$. By Proposition 4.6, $H_1 H_2 \dots H_n$ is closed if for every b_j , $H_1^{b_j} H_2^{b_j} \dots H_{n-1}^{b_j} T_n$ is closed, where $H_n = \cup b_j T_n$ (finite union). So we turn to prove that $H_1^{b_j} H_2^{b_j} \dots H_{n-1}^{b_j} T_n$ is closed. Note that $H_i^{b_j} = \langle a^{k_i} e_{i1}^{b_j}, e_{i2}^{b_j}, \dots, e_{in_i}^{b_j} \rangle$. Repeating the process for $H_{n-1}^{b_j}$, there is a subgroup T_{n-1} in $H_{n-1}^{b_j}$ of finite index and $H_1^{c_1} H_2^{c_1} \dots H_{n-2}^{c_1} T_{n-1} T_n$ is closed in G for all c_l . This implies that $H_1 H_2 \dots H_n$ is closed where $T_{n-1} = \langle a^k g_1, g_2, \dots, g_s \rangle$, $H_{n-1}^{b_j} = \cup d_l T_{n-1}$ (finite union) and $c_l = b_j d_l$. Applying this procedure for each subgroup H_i , we see that $H_1 H_2 \dots H_n$ is closed if $T_1 T_2 \dots T_n$ is closed in G and $T_i = \langle a^k f_{i1}, f_{i2}, \dots, f_{in_i} \rangle$. So it remains to consider the case that $H_i = \langle a^k e_{i1}, e_{i2}, \dots, e_{in_i} \rangle$ ($k \neq 0$) or $H_i \subset F$. Now $H_i \subset F \times \langle a^k \rangle$ and $F \times \langle a^k \rangle$ has finite index in $F \times \langle a \rangle$, So $H_1 H_2 \dots H_n$ is closed $F \times \langle a^k \rangle$ implies that it is closed in $F \times \langle a \rangle$. Therefore, we have reduced the proof of the theorem to the case that each $H_i = \langle a e_{i1}, e_{i2}, \dots, e_{in_i} \rangle$ or $H_i \subset F$.

(3) By Theorem 4.9, if $H_i H_{i+1} \cap \langle a \rangle \neq 1$ for some i , then $H_1 H_2 \dots H_n$ is closed in G . So we may assume that $H_i H_{i+1} \cap \langle a \rangle = 1$ for all i . By Proposition 3.7, we get that $\overline{H_i H_{i+1}} \cap \overline{\langle a \rangle} = 1$.

(4) Since H_n is finitely generated, so is $P_G(H_n) = K_n$. By a theorem of M. Hall (cf. [Hal1]), there is a subgroup U of F such that K_n is a free factor of U and $[F : U] < \infty$. Then $U \times \langle a \rangle$ has finite index in $F \times \langle a \rangle$. So $H_i \cap (U \times \langle a \rangle)$ is finitely generated and of finite index in H_i . Let $H_i = \cup b_j (H_i \cap (U \times \langle a \rangle))$, then $H_1 H_2 \cdots H_n = \cup b_j H_1^{b_j} H_2^{b_j} \cdots H_{i-1}^{b_j} (H_i \cap (U \times \langle a \rangle)) H_{i+1} \cdots H_n$. Repeating this process from $i = n-1$ to $i = 1$, we get that $H_1 H_2 \cdots H_n$ is closed in G if and only if $H_1^* H_2^* \cdots H_{n-1}^* H_n$ is closed in $U \times \langle a \rangle$, where each H_i^* is also finitely generated. Hence we may assume that $H_1, H_2, \dots, H_n$ are finitely generated subgroups of $F \times \langle a \rangle$ and $P_G(H_n) = K_n$ is a free factor of F .

(5) We choose a basis for $K_n = P_G(H_n)$ and expand it to a basis for F . These bases are also bases of the free profinite groups $\hat{K}_n$ and $\hat{F}$. Let $\Gamma(K_n), \Gamma(F)$ be the Cayley graphs of K_n and F with respect to the chosen bases. Let $\Gamma(\hat{K}_n), \Gamma(\hat{F})$ be the profinite Cayley graphs of $\hat{K}_n$ and $\hat{F}$. Then $\Gamma(K_n)$ and $\Gamma(F)$ are trees (cf. [Se], pp 17), and $\Gamma(\hat{K}_n)$ and $\Gamma(\hat{F})$ are boolean trees (cf. [G-R], Theorem 1.4). Note that $\Gamma(K_n) \subset \Gamma(F)$, $\Gamma(\hat{K}_n) \subset \Gamma(\hat{F})$ and $\Gamma(K_n) \subset \Gamma(\hat{K}_n)$, $\Gamma(F) \subset \Gamma(\hat{F})$.

(6) We shall now prove $\bar{H}_1 \bar{H}_2 \cdots \bar{H}_n \cap G = H_1 H_2 \cdots H_n$ by induction on n . The case $n = 1$ has been proved already (cf. Cor. 4.5).

Let $h_1 h_2 \cdots h_n \in \bar{H}_1 \bar{H}_2 \cdots \bar{H}_n \cap G$, where $h_i \in \bar{H}_i$. Then $P_{\hat{G}}(h_1) P_{\hat{G}}(h_2) \cdots P_{\hat{G}}(h_n) \in \bar{K}_1 \bar{K}_2 \cdots \bar{K}_n \cap F$, where $\bar{K}_i = \overline{P_G(H_i)} = P_{\hat{G}}(\bar{H}_i)$. Setting $P_{\hat{G}}(h_i) = f_i \in K_i$, then $f_1 f_2 \cdots f_n \in \bar{K}_1 \bar{K}_2 \cdots \bar{K}_n \cap F$. We know $[1, f_1 f_2 \cdots f_n]$ is a finite path in $\Gamma(\hat{F})$ and hence so is $[f_{n-1}^{-1} \cdots f_1^{-1}, f_n]$.

(7) Let $D_i = \cup \bar{K}_i[1, r_j]$ where the r_j 's constitute a generator set of K_i (when $H_i = \langle a e_{i1}, e_{i2}, \cdots, e_{in_i} \rangle$, r_j runs over $\{e_{i1}, e_{i2}, \cdots, e_{in_i}\}$). Then D_i is the minimal $\bar{K}_i$ -invariant boolean subtree of $\Gamma(\hat{F})$ containing the vertex 1. Let

$$D = f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cup f_{n-1}^{-1} \cdots f_3^{-1} D_2 \cup \cdots \cup f_{n-1}^{-1} D_{n-2} \cup D_{n-1}.$$

Since $f_{n-1}^{-1} \cdots f_{i+1}^{-1} D_i$ is a boolean tree, and

$$f_{n-1}^{-1} \cdots f_{i+1}^{-1} D_i \cap f_{n-1}^{-1} \cdots f_i^{-1} D_{i-1} \neq \emptyset$$

($i = 2, \cdots, n-1$), so D is also a boolean tree (cf. [R-Z], Lemma 1.3). Now

$f_{n-1}^{-1} \cdots f_2^{-1} f_1^{-1} \in f_{n-1}^{-1} \cdots f_2^{-1} D_1$ and $1 \in D_{n-1}$ implies that $[f_{n-1}^{-1} \cdots f_1^{-1}, 1] \subset D$. So

the finite path $[f_{n-1}^{-1} \cdots f_1^{-1}, f_n]$ is contained in $D \cup \Gamma(K_n)$.

(8) If $f_{n-1}^{-1} \cdots f_1^{-1} \in \Gamma(\bar{K}_n)$, then $f_1 f_2 \cdots f_n \in \bar{K}_n \cap F = K_n$. Let $f_1 f_2 \cdots f_n = k_n \in K_n = P_G(H_n)$, then there exists $g_n \in H_n$ such that $P_G(g_n) = k_n$ (g_n is unique since $H_n \cap \langle a \rangle = 1$). Next we consider two cases depending on whether $K_1 \subset K_n$.

(9) For the case $K_1 \not\subset K_n$, there is a $k_1 \in K_1$ such that $k_1 \notin K_n$. Lift k_1 to $g_1 \in H_1$ such that $P_G(g_1) = k_1$. Then $g_1 h_1 \cdots h_n \in \bar{H}_1 \cdots \bar{H}_n \cap G$ and its projection is $k_1 f_1 \cdots f_n$ with the property $f_{n-1}^{-1} \cdots f_1^{-1} k_1^{-1} \notin \Gamma(\bar{K}_n)$. Now $h_1 \cdots h_n \in H_1 H_2 \cdots H_n$ if and only if $g_1 h_1 \cdots h_n \in H_1 H_2 \cdots H_n$. So one is reduced to the case that $f_{n-1}^{-1} \cdots f_1^{-1} \notin \Gamma(\bar{K}_n)$ (see (11) below).

(10) If $K_1 \subseteq K_n$, then $f_1 \in \bar{K}_1 \subseteq \bar{K}_n = P_G(\bar{H}_n)$. There is an $h'_n \in \bar{H}_n$ such that $P_G(h'_n) = f_1$. If $h'_n = h_1$, then $h_1 h_2 \cdots h_n g_n^{-1} = h_1^{-1} h_1 (h_1 h_2 \cdots h_n g_n^{-1}) = h_1^{-1} (h_1 h_2 \cdots h_n g_n^{-1}) h_1 = h_2 \cdots h_n g_n^{-1} h'_n$ because $h_1 h_2 \cdots h_n g_n^{-1} \in \overline{\langle a \rangle}$. So

$$h_1 h_2 \cdots h_n g_n^{-1} \in \bar{H}_2 \cdots \bar{H}_n \cap G = H_2 \cdots H_n \subset H_1 H_2 \cdots H_n$$

by the induction hypothesis. Therefore $h_1 h_2 \cdots h_n \in H_2 \cdots H_n \subset H_1 H_2 \cdots H_n$.

If $h'_n \neq h_1$, then $H_1 H_n \cap \langle \hat{a} \rangle \neq 1$. By Proposition 3.7, $H_1 H_n \cap \langle a \rangle \neq 1$. Set $H_1 H_n \cap \langle a \rangle = \langle a^t \rangle$, where $t \neq 0$. Then there are $l_1 \in H_1, l_n \in H_n$ such that $l_1 l_n = a^t$.

Now

$$l_1 h_1 \cdots h_n g_n^{-1} l_n = h_1 \cdots h_n g_n^{-1} l_1 l_n = h_1 \cdots h_n g_n^{-1} a^t.$$

and so

$$l_1^s h_1 \cdots h_n g_n^{-1} l_n^s = h_1 \cdots h_n g_n^{-1} l_1^s l_n^s = h_1 \cdots h_n g_n^{-1} a^{ts}$$

for any integer s . We know that $h_1 \cdots h_n g_n^{-1} \in H_1 H_2 \cdots H_n$ if and only if $l_1^s h_1 \cdots h_n g_n^{-1} l_n^s \in H_1 H_2 \cdots H_n$ for some s . This is equivalent to saying that $h_1 \cdots h_n g_n^{-1} a^{ts} \in H_1 H_2 \cdots H_n$ for some s . Since $H_1 H_2 \cdots H_n \langle a^t \rangle$ is closed (Theorem 4.8), $h_1 h_2 \cdots h_n g_n^{-1} \in H_1 \cdots H_n \cap G \subset H_1 \cdots H_n \overline{\langle a^t \rangle} \cap G = H_1 H_2 \cdots H_n \langle a^t \rangle$. Hence $h_1 h_2 \cdots h_n g_n^{-1} = t_1 t_2 \cdots t_n a^{ts}$ where $t_i \in H_i$, and s is some integer. So $h_1 h_2 \cdots h_n g_n^{-1} a^{-ts} \in H_1 H_2 \cdots H_n$. Therefore $h_1 h_2 \cdots h_n g_n^{-1} \in H_1 H_2 \cdots H_n$ as explained above and hence $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$.

(11) Now we assume that $f_{n-1}^{-1} \cdots f_1^{-1} \notin \Gamma(K_n)$. Since $[f_{n-1}^{-1} \cdots f_1^{-1}, f_n]$ is a finite path and it is in $D \cup \Gamma(K_n)$, there exists a first vertex v of $[f_{n-1}^{-1} \cdots f_1^{-1}, f_n]$ which is in $\Gamma(\bar{K}_n)$. Then $[f_{n-1}^{-1} \cdots f_1^{-1}, v]$ is a finite subpath of $[f_{n-1}^{-1} \cdots f_1^{-1}, 1]$ and $[v, f_n]$ is a finite path in $\Gamma(\bar{K}_n)$. There must exist $t \in K_n$ such that $vt = f_n$. Lift t to g in H_n such that $P_G(g_n) = t$ and note that $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$ if and only if $h_1 h_2 \cdots h_n g^{-1} \in H_1 H_2 \cdots H_n$. Hence we may use $h_n g^{-1}$ instead of h_n . This allows

us to assume that $f_n \in [f_{n-1}^{-1} \cdots f_1^{-1}, 1]$. That means that

$$f_n \in D = f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cup \cdots \cup f_{n-1}^{-1} D_{n-2} \cup D_{n-1}.$$

(12) Suppose $f_n \in D_{n-1}$. Then there exist $f \in \overline{K_{n-1}}$ and $r \in \cup[1, r_j] \subset F$ such that $f_n = fr$. So $f^{-1}f_n = r \in F \cap \overline{K_{n-1}}\bar{K}_n$. Since F is product separable, we can find $k \in K_{n-1}$, $k_n \in K_n$ such that $f^{-1}f_n = kk_n$. Now lift f, k, k_n to h, g, g_n respectively, satisfying $h \in \overline{H_{n-1}}$, $g \in H_{n-1}$, $g_n \in H_n$ and $P_{\hat{G}}(h) = f$, $P_G(g) = k$, $P_G(g_n) = k_n$. Notice that

$$P_{\hat{G}}(h^{-1}h_n) = f^{-1}f_n = kk_n = P_G(gg_n),$$

and $\overline{H_{n-1}}\bar{H}_n \cap \overline{\langle a \rangle} = 1$ (see (3)), so $h^{-1}h_n = gg_n$. This shows that

$$h_1h_2 \cdots h_n = h_1h_2 \cdots h_{n-1}hh^{-1}h_n = h_1h_2 \cdots h_{n-1}hgg_n.$$

So $h_1h_2 \cdots h_{n-1}h \in \bar{H}_1\bar{H}_2 \cdots \overline{H_{n-1}} \cap G = H_1H_2 \cdots H_{n-1}$ (by induction). Therefore, $h_1h_2 \cdots h_n \in H_1H_2 \cdots H_n$.

(13) Suppose $f_n \in f_{n-1}^{-1} \cdots f_2^{-1} D_1$ ($n > 2$). Observe that $f_{n-1}^{-1} \cdots f_2^{-1} D_1$ and $f_{n-1}^{-1} \cdots f_3^{-1} D_2 \cup \cdots \cup D_{n-1}$ are boolean trees, $f_n \in f_{n-1}^{-1} \cdots f_2^{-1} D_1$ and $1 \in D_{n-1}$, and

$f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cap (f_{n-1}^{-1} \cdots f_3^{-1} D_2 \cup \cdots \cup D_{n-1}) \neq \emptyset$. Then we get (cf. [R-Z], Lemma 1.4)

$$[f_n, 1] \cap f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cap (f_{n-1}^{-1} \cdots f_3^{-1} D_2 \cup \cdots \cup D_{n-1}) \neq \emptyset.$$

Hence there must be an i ($2 \leq i \leq n-1$) such that

$$[f_n, 1] \cap f_{n-1}^{-1} \cdots f_2^{-1} D_1 \cap f_{n-1}^{-1} \cdots f_{i+1}^{-1} D_i \neq \emptyset.$$

Let k' be a vertex in this intersection. Note $k' \in \bar{K}_n$ because of $[f_n, 1] \subset \Gamma(\bar{K}_n)$.

Observe that $\bar{K} = f_{n-1}^{-1} \cdots f_2^{-1} \bar{K}_1 f_2 \cdots f_{n-1}$, $\bar{K}$ acts on $f_{n-1}^{-1} \cdots f_2^{-1} D_1$ and $\bar{K} \cap \bar{K}_n$ acts on $T = f_{n-1}^{-1} \cdots f_2 D_1 \cap \Gamma(\bar{K}_n)$ naturally. Thus the quotient graph $\bar{K} \backslash f_{n-1}^{-1} \cdots f_2^{-1} D_1$ is finite because $f_{n-1}^{-1} \cdots f_2^{-1} D_1 = \cup f_{n-1}^{-1} \cdots f_2^{-1} \bar{K}_1 [1, r_j] = \cup \bar{K} f_{n-1}^{-1} \cdots f_2^{-1} [1, r_j]$, and each $[1, r_j]$ is a finite path. Define

$$\pi : f_{n-1}^{-1} \cdots f_2^{-1} D_1 \longrightarrow \bar{K} \backslash f_{n-1}^{-1} \cdots f_2^{-1} D_1$$

as the canonical epimorphism. We claim that $(\bar{K} \cap \bar{K}_n) \backslash T \cong \pi(T) \subset \bar{K} \backslash f_{n-1}^{-1} \cdots f_2^{-1} D_1$.

First, clearly the morphism $(\bar{K} \cap \bar{K}_n) \backslash T \longrightarrow \pi(T)$ induced by π is epic since $\bar{K} \cap \bar{K}_n \subset \bar{K}$. Second, suppose $t, t' \in T$ and $xt = t'$ with some $x \in \bar{K}_n$. If t is a vertex, then $t, t' \in \bar{K}_n$ and hence $x \in \bar{K}_n$. If t is an edge, its vertices are in $\bar{K}_n$, and also $x \in \bar{K}_n$. So $(\bar{K} \cap \bar{K}_n) \backslash T \cong \pi(T)$, and hence $(\bar{K} \cap \bar{K}_n) \backslash T$ is finite.

Let $\rho : T \longrightarrow (\bar{K} \cap \bar{K}_n) \backslash T$ be the canonical epimorphism. Then there exists a connected transversal Σ of ρ in T which contains the vertex k' (cf. Chapter 2). Now

f_n is a vertex of T , so there is an $f \in \bar{K} \cap \bar{K}_n$ such that $ff_n = k'' \in \Sigma$. Since $[k', k'']$ is a finite path in Σ , we can find some $r \in K_n$ satisfying $k'r = k''$. Then we get

$$f_1 f_2 \cdots f_n = f_1 f_2 \cdots f_{n-1} f^{-1} f f_n = f_1 f_2 \cdots f^{-1} k'' = f_1 f_2 \cdots f_{n-1} f^{-1} k' r.$$

Now note that $f \in \bar{K} \cap \bar{K}_n = f_{n-1}^{-1} \cdots f_2^{-1} \bar{K}_1 f_2 \cdots f_{n-1} \cap \bar{K}_n$ and

$P_{\hat{G}}(h_{n-1}^{-1} \cdots h_2^{-1} \bar{H}_1 h_2 \cdots h_{n-1}) = f_{n-1}^{-1} \cdots f_2^{-1} \bar{K}_1 f_2 \cdots f_{n-1}$, $P_{\hat{G}}(\bar{H}_n) = \bar{K}_n$, so there are $h \in \bar{H}_n$ and $h' \in h_{n-1}^{-1} \cdots h_2^{-1} \bar{H}_1 h_2 \cdots h_{n-1}$ such that $P_{\hat{G}}(h) = P_{\hat{G}}(h') = f$.

If $h = h'$, then $h_1 h_2 \cdots h_n = h_1 \cdots h_{n-1} h^{-1} h h_n$. Let $s \in H_n$ such that $P_G(s) = r \in K_n$. One knows $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$ if and only if $h_1 h_2 \cdots h_n s^{-1} \in H_1 H_2 \cdots H_n$. Now $h_1 h_2 \cdots h_n s^{-1} = h_1 h_2 \cdots h_{n-1} h^{-1} h h_n s^{-1} = h'_1 h_2 \cdots h_{n-1} h h_n s^{-1}$ for some $h'_1 \in \bar{H}_1$, since $h = h' \in h_{n-1}^{-1} \cdots h_2^{-1} \bar{H}_1 h_2 \cdots h_{n-1}$. So $P_{\hat{G}}(h_1 h_2 \cdots h_n s^{-1}) = f'_1 f_2 \cdots f_{n-1} f f_n r^{-1} = f'_1 f_2 \cdots f_{n-1} k'$, where $k' \in f_{n-1}^{-1} \cdots f_{i+1}^{-1} D_i$ and $f'_1 = P_{\hat{G}}(h'_1)$. Therefore it is sufficient to prove the conclusion in the case : $f_n \in f_{n-1}^{-1} \cdots f_{i+1}^{-1} D_i$ (see below (13)).

If $h \neq h'$, then $\bar{H}_1^t \bar{H}_n \cap \overline{\langle a \rangle} \neq 1$ where $t = h_2 \cdots h_{n-1}$. That implies

$H_1^t H_n \cap \langle a \rangle \neq 1$ by Proposition 3.7. Let $H_1^t H_n \cap \langle a \rangle = \langle a^q \rangle$ where $q \neq 0$. Then there are $l_1 \in H_1$, $l_n \in H_n$ such that $l_1^t l_n = a^q$. So $h_2 \cdots h_{n-1} a^q = h_2 \cdots h_{n-1} l_1^t l_n = l_1 h_2 \cdots h_{n-1} l_n$. Let $t_1 = h_1 l_1 h_1^{-1} \in H_1$, $t_n = h_n^{-1} l_n h_n \in H_n$. Then $t_1 h_1 h_2 \cdots h_n t_n = h_1 h_2 \cdots h_{n-1} l_1^t l_n h_n = h_1 h_2 \cdots h_n a^q$, and $t_1^s h_1 \cdots h_n t_n^s = h_1 h_2 \cdots h_n a^{qs}$ for any $s \in \mathbb{Z}$ (it works for negative integer s because of that $t_1^t l_n \in \langle a \rangle$). Note that $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$

if and only if $t_1^s h_1 \cdots h_n t_n^s \in H_1 H_2 \cdots H_n$ for some s , and hence if and only if $h_1 h_2 \cdots h_n a^{qs} \in H_1 H_2 \cdots H_n$. We know that $H_1 H_2 \cdots H_n \langle a^q \rangle$ is closed in G , so $h_1 h_2 \cdots h_n \in \bar{H}_1 \bar{H}_2 \cdots \bar{H}_n \cap G \subset \bar{H}_1 \bar{H}_2 \cdots \bar{H}_n \overline{\langle a^q \rangle} \cap G = H_1 H_2 \cdots H_n \langle a^q \rangle$. That means $h_1 h_2 \cdots h_n = g_1 g_2 \cdots g_n a^{qs}$ for some $s \in \mathbb{Z}$ and $g_i \in H_i$ ($i = 1, 2, \dots, n$). Therefore $h_1 h_2 \cdots h_n a^{-qs} \in H_1 H_2 \cdots H_n$, and hence $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$.

(14) Let $f_n \in f_{n-1}^{-1} \cdots f_{i+1}^{-1} D_i$ where $i \neq 1, n-1$. We use induction on $n-i$. If $n-i=1$, the result holds as we have seen in (12). Now we assume that our result is true when $f_n \in f_{n-1}^{-1} \cdots f_{j+1}^{-1} D_j$ where $j > i$. Let $h_1 h_2 \cdots h_n \in \bar{H}_1 \cdots \bar{H}_n \cap G$. Its projection $f_1 f_2 \cdots f_n \in \bar{K}_1 \cdots \bar{K}_n \cap F$. As in (13), there is $f_0 \in \bar{K}_n \cap f_{n-1}^{-1} \cdots f_{i+1}^{-1} \bar{K}_i f_{i+1} \cdots f_{n-1}$ such that $f_0 f_n = k' r$ where $r \in K_n$, $k' \in \bar{K}_n$ and $k' \in f_{n-1}^{-1} \cdots f_{j+1}^{-1} D_j$ with $j > i$. By lifting f_0 , there are $h_0 \in \bar{H}_n$ and $h'_0 \in h_{n-1}^{-1} \cdots h_{i+1}^{-1} \bar{H}_i h_{i+1} \cdots h_{n-1}$ such that $P_{\bar{G}}(h_0) = P_{\bar{G}}(h'_0) = f_0 = f_{n-1}^{-1} \cdots f_{i+1}^{-1} f'_i f_{i+1} \cdots f_{n-1}$ where $f'_i \in \bar{K}_i$.

If $h_0 = h'_0$, then $h_1 h_2 \cdots h_n = h_1 \cdots h_{n-1} (h_0)^{-1} h_0 h_n$. Let $s \in H_n$ be such that $P_G(s) = r$. Then

$$\begin{aligned} P_{\bar{G}}(h_1 \cdots h_n s^{-1}) &= P_{\bar{G}}(h_1 \cdots h_{n-1} (h_0)^{-1} h_0 h_n s^{-1}) = f_1 \cdots f_{n-1} f_0^{-1} f_0 f_n r^{-1} \\ &= f_1 \cdots f_{n-1} f_0^{-1} k' = f_1 \cdots f_i (f'_i)^{-1} f_{i+1} \cdots f_{n-1} k' \end{aligned}$$

Since $k' \in f_{n-1}^{-1} \cdots f_{j+1}^{-1} D_j$, by induction, $h_1 h_2 \cdots h_n s^{-1} \in H_1 H_2 \cdots H_n$. Hence $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$.

If $h_0 \neq h'_0$, then $\bar{H}_i^{t'} \bar{H}_n \cap \overline{\langle a \rangle} \neq 1$ for $t' = h_{i+1} \cdots h_{n-1}$. Since $f_n \in f_{n-1}^{-1} \cdots f_{i+1}^{-1} D_i$, $f_1 f_2 \cdots f_n = f_1 \cdots f_i f_i^{-1} f f_{i+1} \cdots f_n$ where $f \in \bar{K}_i = P_{\bar{G}}(\bar{H}_i)$ and $f f_{i+1} \cdots f_n \in F$. Lift f to h in $\bar{H}_i$, we note that $\bar{H}_i^t \bar{H}_n \cap \overline{\langle a \rangle} \neq 1$ is also true for $t = h h_{i+1} \cdots h_n$. So we get $H_i^t H_n \cap \langle a \rangle \neq 1$ and $H_i H_n^{t^{-1}} \cap \langle a \rangle \neq 1$.

Let $H_i H_n^{t^{-1}} \cap \langle a \rangle = \langle a^k \rangle \neq 1$, then $\bar{H}_i \bar{H}_n^{t^{-1}} \cap \overline{\langle a \rangle} \supset \overline{\langle a^k \rangle}$. From $f f_{i+1} \cdots f_n = r \in F$, $h h_{i+1} \cdots h_n = r a_1$ where $a_1 \in \overline{\langle a \rangle}$. Note $\overline{\langle a \rangle} = \cup a^j \overline{\langle a^k \rangle}$ (finite union), so $h h_{i+1} \cdots h_n = r a^p a_2$ where $a_2 \in \overline{\langle a^k \rangle}$ and $0 \leq p \leq k-1$. Let $a_2^{-1} = t_i t_n^{t^{-1}}$ where $t_i \in \bar{H}_i$, $t_n \in \bar{H}_n$ because of $\overline{\langle a^k \rangle} \subset \bar{H}_i \bar{H}_n^{t^{-1}}$. We get that $t_i t_n^{t^{-1}} h h_{i+1} \cdots h_n = r a^p \in G$ and also $t_i t_n^{t^{-1}} h h_{i+1} \cdots h_n = t_i h h_{i+1} \cdots h_n t_n \in \bar{H}_i \cdots \bar{H}_n$, so $t_i t_n^{t^{-1}} h h_{i+1} \cdots h_n \in H_i \cdots H_n$ by induction. We rewrite $t_i t_n^{t^{-1}} h h_{i+1} \cdots h_n = g'_i g'_{i+1} \cdots g'_{n-1} g'_n$, where $g'_i \in H_i, g'_n \in H_n$ and $g_j \in H_j (j = i+1, \cdots, n-1)$.

Now $h_1 h_2 \cdots h_n = h_1 \cdots h_i h^{-1} t_i^{-1} (t_n^{t^{-1}})^{-1} t_i t_n^{t^{-1}} h h_{i+1} \cdots h_n$. Since $h_1 \cdots h_i h^{-1} t_i^{-1} (t_n^{t^{-1}})^{-1} \in \bar{H}_1 \cdots \bar{H}_i \bar{H}_n^{t^{-1}} \cap G = H_1 \cdots H_i H_n^{t^{-1}}$, we write $h_1 \cdots h_i h^{-1} t_i^{-1} (t_n^{t^{-1}})^{-1} = g_1 \cdots g_i g_n^{t^{-1}}$ where $g_j \in H_j (j = 1, \cdots, i, n)$.

So $h_1 \cdots h_n$

$$= h_1 \cdots h_i h^{-1} t_i^{-1} (t_n^{t^{-1}})^{-1} \cdot t_i t_n^{t^{-1}} h h_{i+1} \cdots h_n$$

$$= g_1 \cdots g_i g_n^{t_i^{-1}} \cdot t_i t_n^{t_i^{-1}} h h_{i+1} \cdots h_n$$

$$= g_1 \cdots g_i t_i t_n^{t_i^{-1}} g_n^{t_i^{-1}} h h_{i+1} \cdots h_n$$

$$= g_1 \cdots g_i t_i t_n^{t_i^{-1}} h h_{i+1} \cdots h_n g_n$$

$$= g_1 \cdots g_i g'_i g_{i+1} \cdots g_{n-1} g'_n g_n.$$

Therefore $h_1 h_2 \cdots h_n$ is in $H_1 H_2 \cdots H_n$.

Now we have completed our proof for the main result of this chapter. ■

4.4 The generalized free product of cyclic groups

Here we provide an application of our main result.

Let G be the generalized free product of two cyclic groups $\langle a \rangle$ and $\langle b \rangle$ with a subgroup $\langle c \rangle$ amalgamated. Brunner, Burns and Solitar have proved its subgroup separability (cf. [B-B-S]). We shall use the theorem in last section to prove that G is product separable. We consider the case that $\langle a \rangle \cong \langle b \rangle \cong Z$ and $c = a^n = b^m$ with $n \neq 1, m \neq 1$ (the other cases are just simple exercises).

Theorem 4.12 *Let $G = \langle a \rangle *_{\langle c \rangle} \langle b \rangle$ where $\langle a \rangle \cong \langle b \rangle \cong Z$ and $c = a^n = b^m$. Then G is product separable.*

Proof: We are going to prove that G has a subgroup of finite index which is isomorphic to $F \times Z$. Hence our result will follow from Theorem 4.11 and the fact

that a finite extension of a product separable group is product separable.

Since $\langle c \rangle$ is contained in the center of G , then $\langle c \rangle$ is a normal subgroup of G .

Hence $G/\langle c \rangle \cong (Z/nZ) * (Z/mZ)$. We define a homomorphism

$$\rho : (Z/nZ) * (Z/mZ) \longrightarrow (Z/nZ) \times (Z/mZ)$$

where its restriction on each factor is the identity map. Its kernel $\ker \rho$ is a free group with basis $\{[s, t] \mid s \in (Z/nZ) \setminus \{0\}, t \in (Z/mZ) \setminus \{0\}\}$ (see [Se] p. 6). Selecting representatives i and j for all cosets in $(Z/nZ) \setminus \{0\}$ and $(Z/mZ) \setminus \{0\}$ such that $1 \leq i \leq n-1$ and $1 \leq j \leq m-1$, we may define a homomorphism η from $\ker \rho$ to G by

$$\eta([i, j]) = [a^i, b^j]$$

where $\bar{i} \in (Z/nZ) \setminus \{0\}$ and $\bar{j} \in (Z/mZ) \setminus \{0\}$. Observe that the set $S = \{[a^i, b^j] \mid 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$ is a basis of the group generated by S . For if $r \geq 1, \varepsilon_l = 1$ or -1 , and whenever $\varepsilon_l = -\varepsilon_{l+1}$ one has $i_l \neq i_{l+1}$ or $j_l \neq j_{l+1}$ for any l , then clearly $[a^{i_1}, b^{j_1}]^{\varepsilon_1} \cdots [a^{i_r}, b^{j_r}]^{\varepsilon_r} \neq 1$. Hence η is an injection. So η is an isomorphism from $\ker \rho$ to $\text{Im } \eta$ which is a free subgroup of G . We use ζ to denote the homomorphism:

$$\zeta : G \longrightarrow G/\langle c \rangle \cong (Z/nZ) * (Z/mZ).$$

Then the composition $\zeta\eta$ is the identity map on $\ker \rho$. The homomorphism ρ induces

a homomorphism ϕ from G to $(Z/nZ) \times (Z/mZ)$ by

$$\phi : G \longrightarrow G/\langle c \rangle \cong (Z/nZ) * (Z/mZ) \longrightarrow (Z/nZ) \times (Z/mZ)$$

Let $K = \ker \phi$. Then K is a normal subgroup of G and $[G : K] = nm$. If K is product separable, so is G (cf. [R-Z], the proof of Th.2.1). Considering the structure of K , we claim that $K \cong \langle c \rangle \times \ker \rho$. Define a homomorphism π from $\langle c \rangle \times \ker \rho$ to G by $\pi(c^t, s) = c^t \eta(s)$. Note that $c^t s$ is in normal form now. Since $\phi \pi(c^t, s) = \phi(c^t \eta(s)) = \phi(c^t) \phi(\eta(s)) = \rho(s) = 1$, so actually

$$\pi : \langle c \rangle \times \ker \rho \longrightarrow K$$

For any element in K , write it in normal form $c^t s$. Then s is the product of some a^i and b^j with $-n < i < n$ and $-m < j < m$. So s can be identified as an element in $(Z/nZ) * (Z/mZ)$. Hence π is surjective. Suppose $(c^t, s) \in \ker \pi$, then $c^t \eta(s) = 1$. Because the normal form of element in amalgamating product is unique, so $t = 0$ and $\eta(s) = 1$, i.e. $s = 1$. Therefore π is injective and hence an isomorphism. Now $\ker \rho$ is free, $K \cong F \times Z$. By our main result, K is product separable. The conclusion is proved. ■

Therefore all generalized free products of any two cyclic groups are product separable.

Chapter 5

The Separability Of Surface Groups

An important kind of groups from topology are surface groups. To explain what surface groups are, we need first to introduce some preliminary notations.

Definition 5.1 *An n -manifold is a Hausdorff space M in which every point has an open neighbourhood homeomorphic to an open subset of $R_+^n = \{(x_1, x_2, \dots, x_n) \in R^n \mid x_1 \geq 0\}$. The points of M mapped into $\partial R_+^n = \{(x_1, x_2, \dots, x_n) \in R^n \mid x_1 = 0\}$ comprise the boundary ∂M of M .*

The definition of manifolds is independent of the choice of open neighbourhood and of homeomorphism due to Brouwer's theorem on Invariance of Domain (see [HW], pp 95). We restrict our attention to the manifolds that are compact and connected. It is customary to do so since any arbitrary manifold can be compactified by adding a single point and a compact n -manifold has only finitely many components which are still Hausdorff and compact. In the case of dimension 2, we have the following

definition.

Definition 5.2 *A surface is a compact and connected 2-manifold with empty boundary set.*

In any topological space M , the homotopic classes of loops at a given base point x form a group under the composition of representative paths. This group is called the *fundamental group* of the space M at x . When M is path-connected, the fundamental group is independent of the choice of base point. In such a case, we simply call it the fundamental group of topological space M .

Definition 5.3 *A surface group G is the fundamental group of a surface S . When S has constant negative curvature, G is called a hyperbolic surface group.*

In general we require a lot of geometric terminology for explaining what curvature means (cf. [L-S] pp 138-139). However for the compact closed surface groups, it is easy to indicate which surface groups are hyperbolic. By the classification of closed compact surfaces, every such surface is homotopic to a sphere with some handles. If the number of handles is greater than 1, then the surface has negative constant curvature and its fundamental group is a hyperbolic surface group.

In this chapter, our object is to prove the product separability of surface groups. Here we state a lemma proved by G.A.Niblo (cf. [Ni], Prop. 4.1).

Lemma 5.4 *Let G be a hyperbolic surface group, and let H, K be finitely generated subgroups of finite index in G . Then there are finite-index subgroups $H_0 < H$ and $K_0 < K$ such that $\langle H_0, K_0 \rangle$ is free.*

This lemma provides a connection between hyperbolic surface groups and free groups. It also gives a potential way to study the product separability of surface groups, and hence the separability property of surface groups. The next lemma is required for proving the main theorem of this chapter.

Lemma 5.5 *Let G be a subgroup separable group, H a finitely generated subgroup of G . Then the profinite topology of H is induced from that of G .*

Proof: We prove that any open subgroup of H is of form $O_G \cap H$ where O_G is an open subgroup of G .

Let O_H be an open subgroup of H . Let $\overline{O_H}$ and $\bar{H}$ be the closures of O_H and H in the profinite completion $\hat{G}$ of G . Then $\overline{O_H}$ is open in $\bar{H}$. Since the profinite topology of $\bar{H}$ is induced from that of G , there is an open subgroup U of $\bar{G}$ such that $\overline{O_H} = U \cap \bar{H}$. Set $O_G = U \cap G$, then O_G is open in G . We claim that $O_G \cap H = O_H$.

Indeed

$$O_G \cap H = U \cap G \cap H = U \cap G \cap G \cap \bar{H} = U \cap \bar{H} \cap G = \overline{O_H} \cap G = O_H$$

since H and O_H are finitely generated and G is subgroup separable. The lemma is proved. ■

Now we prove the following theorem

Theorem 5.6 *Surface groups are product separable.*

Proof: Let G be the fundamental group of a surface S . If S is not a closed surface (i.e. S is not finite compact with empty boundary), then G is free. So G is product separable (see Chapter 2, Theorem 2.9). If S is closed, S has a double covering space which is orientable. So G has a subgroup H of index at most 2 such that H is the fundamental group of a closed orientable surface. G is product separable if and only if H is product separable (cf.[R-Z], proof of Theorem 2.1). Moreover, H is either $Z \times Z$ (the fundamental group of a torus) or a hyperbolic surface group except the trivial group. Since any finitely generated Abelian group is product separable (see Chapter 3, Proposition 3.3), it is sufficient to prove that hyperbolic surface groups are product separable.

Let G be a hyperbolic surface group, and let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of G . If one of these subgroups is open (i.e. of finite index in G), then the product set $H_1 H_2 \cdots H_n$ is a finite union of the cosets of that open subgroup in G . So it is open and closed.

Now we assume that all subgroups $H_1, H_2, \dots, H_n$ have infinite index in G . By Lemma 5.4, there exist subgroups $K_1 \langle H_1, K_2 < H_2$ such that $[H_1 : K_1] < \infty$, $[H_2 : K_2] < \infty$ and $L_2 = \langle K_1, K_2 \rangle$ is free. Thus

$$H_1 H_2 \cdots H_n = (\cup a_i K_1)(\cup K_2 b_j) H_3 \cdots H_n = \cup a_i K_1 K_2 H_3^{b_j^{-1}} \cdots H_n^{b_j^{-1}} b_j$$

(finite union). So $H_1 H_2 \cdots H_n$ is closed if and only if $K_1 K_2 H_3^{b_j^{-1}} \cdots H_n^{b_j^{-1}}$ is closed.

Hence we just need to prove that $H_1 H_2 \cdots H_n$ is closed where $L_2 = \langle H_1, H_2 \rangle$ is free.

If L_2 is finite index in G , then G is free-by-finite, and hence it is product separable (cf.[R-Z], Theorem 2.1).

If L_2 is infinite index in G , then applying Lemma 5.4 again, there is $L < L_2$, $K_3 < H_3$ such that $[L_2 : L] < \infty$, $[H_3 : K_3] < \infty$ and $\langle L, K_3 \rangle$ is free. Setting $K_2 = L \cap H_2$, $[H_2 : K_2] < \infty$. Write $H_2 = \cup a_i K_2$, $H_3 = \cup K_3 b_j$, then $H_1 H_2 \cdots H_n = \cup a_i H_1^{a_i^{-1}} K_2 K_3 H_4^{b_j^{-1}} \cdots H_n^{b_j^{-1}} b_j$. We have to prove that $H_1^{a_i^{-1}} K_2 K_3 H_4^{b_j^{-1}} \cdots H_n^{b_j^{-1}}$ is closed for each pair a_i and b_j . For a fixed pair a_i and b_j , set $K_1 = L \cap H_1^{a_i^{-1}}$. Since $[H_1^{a_i^{-1}} : K_1] < \infty$, $H_1^{a_i^{-1}} K_2 K_3 H_4^{b_j^{-1}} \cdots H_n^{b_j^{-1}}$ is closed if and only if $K_1 K_2 K_3 H_4^{b_j^{-1}} \cdots H_n^{b_j^{-1}}$ is closed. Now $\langle K_1, K_2, K_3 \rangle \subseteq \langle L, K_3 \rangle$, so $L_3 = \langle K_1, K_2, K_3 \rangle$ is also free. Therefore, we have reduced the proof of the theorem to the case that $L_3 = \langle H_1, H_2, H_3 \rangle$ is free.

Repeating the above reducing process, it is sufficient to prove that $H_1 H_2 \cdots H_n$ is closed where $L_n = \langle H_1, H_2, \dots, H_n \rangle$ is free, unless G is free-by-finite whose product

separability has been proved. Now L_n is finitely generated, and so L_n is closed in G because of the subgroup separability of surface groups. So by Lemma 5.5, the profinite topology of L_n is induced from the topology of G . Since L_n is free, $H_1 H_2 \cdots H_n$ is closed in L_n with respect to the profinite topology of L_n . It is also closed in L_n with respect to the profinite topology of G . Therefore $H_1 H_2 \cdots H_n$ is closed in G . ■

A *Fuchsian group* is a discontinuous subgroup of the hyperbolic group H . We say a subgroup G of H is *discontinuous* if, given a point P in H and a disc D in H , there are only finitely many images Pa of P , for a in G , in the disc D . The presentation of Fuchsian groups can be found in [Ly], pp 205. Those Fuchsian groups which are not free product of cyclic groups are called *F-groups*. Hoare, Karrass and Solitar proved that if G is an F-group, then every subgroup K of finite index is an F-group, while every subgroup H of infinite index is a free product of cyclic groups (cf. [L-S], Chapter III Proposition 7.4 or [H-K-S1] and [H-K-S2]). In that book, Lyndon and Schupp proved that every finitely generated F-group contains a torsion free subgroup (by the Hoare-Karrass-Solitar result mentioned above, which is also a F-group, and hence a surface group) of finite index (cf. [L-S], Chapter III, Proposition 7.12).

Any finitely generated Fuchsian group is either an F-group or the free product of finitely cyclic groups. As above mentioned, an F-group contains a surface subgroup of

finite index. When G is a free product of finite cyclic groups, there is a projection epimorphism from G to the direct product of those finite cyclic groups. The kernel of this projection, which is called *Cartesian subgroup* of G , is a free group (cf. [Ri2]). Hence G has a free subgroup of finite index. So we get the following result immediately.

Corollary 5.7 *Finitely generated Fuchsian groups are product separable.*

Chapter 6

The Separability Properties Of $F \times A$

As we saw in Chapter 3, Proposition 3.4, an infinitely generated Abelian group may not be residually finite. So we just consider the separability of $F \times A$ where F is free and A is a finitely generated Abelian group. In Chapter 4, we have considered the case that A is an infinite cyclic group. In this chapter, we consider the case when A is a finitely generated Abelian group whose rank is more than 1. Throughout this chapter, F is always a free group and A is an Abelian group of torsion free rank q where $q \geq 2$.

6.1 The subgroups of $F \times A$

By the structure theorem of finitely generated Abelian groups, one can write $A = \prod \mathbb{Z} \times \prod \mathbb{Z}/k_i\mathbb{Z}$. Since $\prod \mathbb{Z}/k_i\mathbb{Z}$ is finite, we may assume that $A = \prod \mathbb{Z}$.

We need some results about the structure of finitely generated subgroups of $F \times A$.

These results generalize the analogues in Chapter 4.

Proposition 6.1 *Let H be a finitely generated subgroup of $G = F \times A$. Then there exists a subgroup A_0 of finite index in A such that $A_0 = \langle a_1, a_2, \dots, a_l \rangle$ and $H = \langle a_1 f_1, a_2 f_2, \dots, a_k f_k, f_{k+1}, \dots, f_n \rangle$ where $f_i \in F$ for $1 \leq i \leq n$.*

Proof: We use induction on the rank of the Abelian group A . Proposition 4.1 shows that this is true when the rank of A is 1. So we assume the rank of A is greater than 1.

Let $\{r_1, r_2, \dots, r_m\}$ be a generating set of H , $r_i = a^{k_i} t_i$ where $t_i \in S = F \times B$ and $A = B \times \langle a \rangle$. We may assume that all k_i are non-negative integers, otherwise replace r_i by its inverse if k_i is negative. If $k_i \leq k_j$, and we use $r_i^{-1} r_j$ instead of r_j , then $\{r_1, r_2, \dots, r_{j-1}, r_i^{-1} r_j, r_{j+1}, \dots, r_m\}$ is also a generator set of H . Repeating this process if necessary, we may assume that at most one k_i is nonzero. So $H = \langle a^k e_1, e_2, \dots, e_n \rangle$ where $k \geq 0$, $e_i \in S = F \times B$ and $A = B \times \langle a \rangle$.

If $k = 0$, then H is a subgroup of $F \times B$. The conclusion is true by induction.

If $k \neq 0$, we set $T = B \times \langle a^k \rangle$. Then T is a subgroup of finite index in A and $H \subset F \times T$. One may choose a suitable generator element a_1 instead of a such that $T = B \times \langle a_1 \rangle$ and $H = \langle a_1 f_1, e_2, \dots, e_m \rangle$ where $f_1 \in F$ and $e_i \in S = F \times B$ ($2 \leq i \leq m$). Now $\langle e_2, \dots, e_m \rangle$ is a finitely generated subgroup of $F \times B$. By induction, there exists

a subgroup B_0 of finite index in B such that $B_0 = \langle a_2, \dots, a_m \rangle$ and $\langle e_2, \dots, e_n \rangle$ can be written as $\langle a_2 f_2, \dots, a_k f_k, f_{k+1}, \dots, f_n \rangle$ where $f_i \in F$. Then $A_0 = B_0 \times \langle a_1 \rangle$ is a subgroup of finite index in A and the result follows. ■

The next proposition will be used to prove the subgroup separability of $G = F \times A$.

Proposition 6.2 *Let $H = \langle a_1 f_1, a_2 f_2, \dots, a_k f_k, f_{k+1}, \dots, f_n \rangle$ be a finitely generated subgroup of $G = F \times A$ where $A = \langle a_1, a_2, \dots, a_k, \dots, a_s \rangle$ and $f_i \in F$. If $\langle a_1^m \rangle = H \cap \langle a_1 \rangle \neq \{1\}$, then there exists a subgroup K satisfying : $K \triangleleft H, [H : K] < \infty$ and $K = \langle a_1^m \rangle \times \langle e_1, e_2, \dots, e_t \rangle$ where $e_j \in F \times \langle a_2, \dots, a_k \rangle$.*

Proof: Let ϕ_1 be the projection from $G = T \times \langle a_1 \rangle$ to $\langle a_1 \rangle$ where $T = F \times \langle a_2, \dots, a_s \rangle$, and let ϕ_2 be the isomorphism from $\langle a_1 \rangle$ to Z defined by $\phi_2(a_1^s) = s$. Let ϕ_3 be the projection from Z to Z/mZ . Then $\phi_3 \phi_2 \phi_1$ is a homomorphism from $T \times \langle a_1 \rangle$ to Z/mZ . Let ρ be its restriction to H , i.e. $\rho = (\phi_3 \phi_2 \phi_1) |_H$. So $\rho(a_1 f_1) = 1, \rho(a_i f_i) = \rho(f_j) = 0$ where $i = 2, \dots, k$ and $j = k+1, \dots, n$. Now let $K = \ker \rho$. Then $K \triangleleft H$ and $[H : K] < \infty$. Since $(a_1 f_1)^m \in K, f_j^{(a_1 f_1)^s} \in K$, so $K \supset \langle (a_1 f_1)^m, f_j^{(a_1 f_1)^s} \mid j = 2, \dots, n; i = 0, \dots, m-1 \rangle$. For any $h \in K \subset H$, $h = \dots f_i (a_1 f_1)^t f_j \dots = \dots (a_1 f_1)^q f_i^{(a_1 f_1)^q} (a_1 f_1)^{ml} f_j \dots$, where $t = ml + q$ and $0 \leq q < m$. So $h \in K$ implies that $h \in \langle (a_1 f_1)^m, f_j^{(a_1 f_1)^s} \mid j = 2, \dots, n; i = 0, \dots, m-1 \rangle$. That is $K = \langle (a_1 f_1)^m, f_j^{(a_1 f_1)^s} \mid j = 2, \dots, n; i = 0, \dots, m-1 \rangle$. Now $(a_1 f_1)^m = a_1^m f_1^m$ and

$f_j^{(a_1 f_1)^i} = (a_1 f_1)^{-i} f_j (a_1 f_1)^i = f_1^{-i} f_j f_1^i \in F$. So $K = \langle a_1^m e_1, e_2, \dots, e_t \rangle$ where $e_i \in F$.

Since $a_1^m \in \ker \rho$, $K = \langle a_1^m, a_1^m e_1, e_2, \dots, e_t \rangle = \langle a_1^m \rangle \times \langle e_1, \dots, e_t \rangle$. ■

Using Proposition 6.2, we prove that $F \times A$ is subgroup separable.

Theorem 6.3 *The group $G = F \times A$ is subgroup separable.*

Proof: We use induction on the rank of A . In Chapter 4, the result has been proved when the rank of A is 1. Now suppose the rank of A is greater than 1. Let $H < G$. By Proposition 6.1, there is a subgroup A_0 of finite index in A such that $A_0 = \langle a_1, a_2, \dots, a_m \rangle$ and $H = \langle a_1 f_1, a_2 f_2, \dots, a_k f_k, f_{k+1}, \dots, f_n \rangle$ where $f_i \in F$. Since $F \times A_0$ has finite index in $G = F \times A$, H is closed in G if and only if H is closed in $F \times A_0$. So one may assume that $A = \langle a_1, a_2, \dots, a_m \rangle$ and write $H = \langle a_1 f_1, a_2 f_2, \dots, a_k f_k, f_{k+1}, \dots, f_n \rangle$ where $f_i \in F$.

If $H \cap \langle a_1 \rangle \neq 1$, say $H \cap \langle a_1 \rangle = \langle a_1^t \rangle$, then by Proposition 6.2 there exists $K \triangleleft H$, $[H : K] < \infty$ such that $K = \langle a_1^t \rangle \times \langle e_1, e_2, \dots, e_s \rangle$ where $e_i \in F \times \langle a_2, \dots, a_m \rangle$. Since $\langle a_1^t \rangle$ is closed in $\langle a_1 \rangle$ and, by induction, $\langle e_1, e_2, \dots, e_s \rangle$ is closed in $F \times \langle a_2, \dots, a_m \rangle$, K is closed in G and so is H because of $[H : K] < \infty$.

If $H \cap \langle a_1 \rangle = 1$, then $\langle a_1^n \rangle H$ is closed for any $n = 1, 2, \dots$, by the previous case.

Hence $H = \bigcap \langle a_1^n \rangle H$ implies that H is closed in G . ■

Up to here, we have proved that $G = F \times A$ is subgroup separable. We turn to

consider the product separability of $G = F \times A$. The following lemma will be applied several times later.

Lemma 6.4 *Let A be a finitely generated free Abelian group, H a subgroup of A . Then there exists an $A_0 < A$, $[A : A_0] < \infty$ such that $A_0 = H \times T$ for some subgroup T .*

Proof: Let $H < A$, then $A/H = \prod Z \prod Z/k_i Z$. Assume that $\prod Z = \langle t_1 \rangle \times \langle \bar{t}_2 \rangle \times \cdots \times \langle \bar{t}_n \rangle$, and $H = \langle s_1 \rangle \times \langle s_2 \rangle \times \cdots \times \langle s_p \rangle$ (note that H is torsion-free). Then set $A_0 = \langle t_1, \dots, t_n, s_1, \dots, s_p \rangle$. It is easy to verify that $A_0 = H \times T$ where $T = \langle t_1, t_2, \dots, t_n \rangle$ and $[A : A_0] < \infty$. ■

Similarly to Proposition 4.7, it is also true that when the rank of F is greater than 1, $F \times A$ has no free subgroup of finite index.

6.2 Some simple results

Let $G = F \times A$. Denote $r(A)$ as the rank of the group A (i.e. the minimal number of the elements required to generate the group A). We prove the product separability of G by induction on $r(A)$. The case $r(A) = 1$ is proved in Chapter 4. We deal with some simple cases first for $r(A) > 1$.

Theorem 6.5 *Let $G = F \times A$ and let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of G . If $H_i \cap A \neq 1$ for some i , then $H_1 H_2 \cdots H_n$ is closed in G .*

Proof: Put $H_i \cap A = A_0 \neq 1$. By Lemma 6.4, there exists a subgroup B of finite index in A such that $B = A_0 \times L$ for some $L \subset A$. Since $F \times B$ has finite index in $F \times A$, $K_i = H_i \cap (F \times B)$ has finite index in H_i . We may write $H_i = \cup b_j K_i$ where $\{b_j \mid j \in J\}$ are the coset representatives. By Proposition 4.6, it is sufficient to prove that $H_1^{b_j} \cdots H_{i-1}^{b_j} K_i H_{i+1} \cdots H_n$ is closed for each b_j . Note that $H_i^{b_j}$ is still finitely generated and $H_i^{b_j} \subset F$ if and only if $H_i \subset F$. So we may safely assume that A_0 is a direct factor of A .

Let $A_0 = \langle a_1 \rangle \times B_0$ with $a_1 \neq 1$. Then $H_i = \langle a_1 \rangle \times T_i$ for some $T_i \subset H_i$ since $H_i = \langle a_1, H_i \rangle$. Let $A = \langle a_1 \rangle \times B$ and t be the number of subgroups H_j which are not contained in $F \times B$. We use induction on t . Note that $T_i \subset F \times B$.

When $t = 1$, then the other subgroups H_j are contained in $F \times B$. Then

$$H_1 H_2 \cdots H_n = \langle a_1 \rangle \times H_1 \cdots H_{i-1} T_i H_{i+1} \cdots H_n.$$

By induction (on the rank of the Abelian group, where the $r(A) = 1$ case is the content of Theorem 4.11), $H_1 \cdots H_{i-1} T_i H_{i+1} \cdots H_n$ is closed in $F \times B$. Hence $H_1 H_2 \cdots H_n$ is closed in $F \times A$.

When $t > 1$, assume that H_q is another subgroup which is not contained in $F \times B$.

Then

$$\begin{aligned} & H_1 H_2 \cdots H_n \\ &= H_1 H_2 \cdots H_{i-1} \langle a_1 \rangle T_i H_{i+1} \cdots H_n \\ &= H_1 \cdots H_{i-1} T_i H_{i+1} \cdots (\langle a_1 \rangle H_q) \cdots H_n. \end{aligned}$$

By induction ($t - 1$ case), it is closed in $G = F \times A$.

Therefore, $H_1 H_2 \cdots H_n$ is closed. ■

Theorem 6.6 *Let $G = F \times A$ and $H_1, H_2, \dots, H_n$ be finitely generated subgroups of G . If $H_i H_{i+1} \cap A \neq 1$ for some i , then $H_1 H_2 \cdots H_n$ is closed in G .*

Proof: Let $A_0 = H_i H_{i+1} \cap A$ where $A_0 \neq \{1\}$. Then $H_1 H_2 \cdots H_n$
 $= H_1 \cdots H_i A_0 H_{i+1} \cdots H_n$. So by Theorem 6.5, $H_1 H_2 \cdots H_n$ is closed in G . ■

As a first step to prove the product separability of $G = F \times A$, we prove that G is double coset separable.

Theorem 6.7 *Let H_1, H_2 be finitely generated subgroups of $G = F \times A$, then $H_1 H_2$ is closed.*

Proof: If $H_1 H_2 \cap A \neq \{1\}$, then $H_1 H_2$ is closed by Theorem 6.6. Assume that $H_1 H_2 \cap A = \{1\}$. Let $a \in A$. Since $\langle a^n \rangle H_1 H_2$ is closed in G (by Theorem 6.6), it is

sufficient to prove that $\cap \langle a^n \rangle H_1 H_2 = H_1 H_2$. It is obvious that $\cap \langle a^n \rangle H_1 H_2 \supseteq H_1 H_2$ (since $\langle a^n \rangle H_1 H_2 \supseteq H_1 H_2$). Let $a^k h_1 h_2 \in \cap \langle a^n \rangle H_1 H_2$. If $k \neq 0$, then for any $|m| > |k|$, $a^k h_1 h_2 \in \cap \langle a^n \rangle H_1 H_2 \subset \langle a^m \rangle H_1 H_2$. This implies that $a^k h_1 h_2 = a^m g_1 g_2$ where $g_i \in H_i$. So $a^{m-k} = g_1^{-1} h_1 h_2 g_2^{-1} \in H_1 H_2 \cap \langle a \rangle = \{1\}$. But $a^{m-k} \neq 1$ leads to a contradiction. Hence k must be 0. Therefore $H_1 H_2 = \cap \langle a^n \rangle H_1 H_2$ and is closed. ■

6.3 Closure and profinite completion

In this section, we investigate the closure of some sets in the profinite completion $\hat{G}$ of $G = F \times A$.

Proposition 6.8 *Let $G = F \times A$, its profinite completion $\hat{G} = \hat{F} \times \hat{A}$. Let $P_{\hat{G}}$ be the projection from $\hat{G}$ to $\hat{F}$ and P_G be the projection from G to F . (Note that the restriction of $P_{\hat{G}}$ on G is P_G .) Then for any subgroup H of G , $P_{\hat{G}}(\bar{H}) = \overline{P_G(H)}$.*

Proof: Since $\bar{H}$ is compact, we claim $P_{\hat{G}}(\bar{H})$ is compact. Let $\{O_\alpha \mid \alpha \in I\}$ be an open covering of $P_{\hat{G}}(\bar{H})$, i.e. $\cup O_\alpha \supseteq P_{\hat{G}}(\bar{H})$, where the O_α 's are open subgroups of F . Then $O_\alpha \times \hat{A}$ is an open subgroup of $\hat{G}$ and $\cup O_\alpha \times \hat{A} \supseteq \bar{H}$. So there is a finite subcovering: $\cup O_i \times \hat{A} \supseteq \bar{H}$. This implies that $\cup O_i \supseteq P_G(\bar{H})$. Then we get $P_{\hat{G}}(\bar{H})$ is compact and hence is closed. Now $P_{\hat{G}}(H) \supseteq P_G(H)$, it provides $P_{\hat{G}}(\bar{H}) \supseteq \overline{P_G(H)}$.

Conversely, for any $f \in P_{\hat{G}}(\bar{H}) \subset \hat{F}$, there exists an $f' \in H$ such that $P_{\hat{G}}(f') = f$.

So for any $\tilde{O} <_o \hat{G}$ (which means that $\tilde{O}$ has finite index in $\hat{G}$), $\tilde{O}f' \cap H \neq \emptyset$. We get

$P_{\hat{G}}(\tilde{O})f \cap P_{\hat{G}}(H) \neq \emptyset$ by applying $P_{\hat{G}}$. That is just: $P_{\hat{G}}(\tilde{O})f \cap P_G(H) \neq \emptyset$. Now $P_{\hat{G}}(\tilde{O})$

can be any open subgroup of $\hat{G}$, so $f \in \overline{P_G(H)}$ in $\hat{F}$. Therefore, $P_{\hat{G}}(H) = \overline{P_G(H)}$.

■

The following proposition is a generalization of Proposition 3.7.

Proposition 6.9 *Let H_1 and H_2 be finitely generated subgroups of $G = F \times A$ and assume that $\pi(H_1)$ is a direct factor of A , where π is the projection from $G = F \times A$ to A . We also assume that $\pi'(H_2)$ is a direct factor of $A/\pi(H_1)$, where π' is the projection from G to $A/\pi(H_1)$. If $H_1H_2 \cap A = \{1\}$, then $H_1H_2 \cap \hat{A} = \{1\}$ where $\bar{H}_1, \bar{H}_2$ are the closure of H_1, H_2 respectively in the profinite completion $\hat{G}$ of G .*

Proof: By Proposition 6.1, we may pick one $H_1 = \langle a_1e_1, \dots, a_ke_k, e_{k+1}, \dots, e_n \rangle$ where $e_i \in F$ and $A = \langle a_1 \rangle \times \dots \times \langle a_q \rangle$ with $q \geq k$. Applying the same method as in the proof of Proposition 4.1, one may assume that

$$H_2 = \langle b_1g_1, \dots, b_kg_k, a_{k+1}g_{k+1}, \dots, a_sg_s, g_{s+1}, \dots, g_m \rangle$$

where $g_j \in F$ and $b_j \in \langle a_1, a_2, \dots, a_k \rangle$. Since H_1 and H_2 are contained in $F \times$

$\langle a_1, a_2, \dots, a_s \rangle$, by the induction on the rank of A , we may assume that $A =$

$\langle a_1, a_2, \dots, a_s \rangle$.

Suppose that $b \in H_1 H_2 \cap \hat{A}$ and $b \neq 1$. Then $b = (b_i) \in A \cong \prod \hat{Z}$. Let I be the index set such that $b_i \neq 1$ when $i \in I$. For each $j \in I$, there exists a number p_j such that $\rho_j(b_j) = a_j^{s_j} \neq 1$, where ρ_j is the projection from $\overline{\langle a_j \rangle}$ to $\langle a_j \rangle / \langle a_j^{p_j} \rangle$. For $j \notin I$, define ρ_j as identity automorphism of $\langle a_j \rangle$. Denote $\prod \rho_j$ as ρ . Then $\ker \rho$ is an open subgroup of A . Set $T = H_1 \prod a_j^{-s_j} \langle a_j^{p_j} \rangle H_2$. Then T is closed in $G = F \times A$ by Proposition 6.5. One may write that

$$\begin{aligned} T &= H_1 \prod a_j^{-s_j} \langle a_j^{p_j} \rangle H_2 \\ &= H_1 \prod a_j^{-s_j} \langle a_j^{p_j} \rangle \prod a_j^{-s_j} \langle a_j^{p_j} \rangle H_2 \\ &= H_1 \prod e_j^{s_j} \langle e_j^{-p_j} \rangle \prod g_j^{s_j} \langle g_j^{-p_j} \rangle H_2. \end{aligned}$$

Now $H_1 H_2 \cap A = 1$, so $1 \notin T$ (otherwise $\prod a_j^{s_j + p_j t} \in H_1 \prod (a_j e_j)^{s_j} (a_j e_j)^{p_j t} H_2 = H_1 H_2$).

By the residual finiteness of G , there is an open subgroup O of G satisfying $O \cap T = \emptyset$.

We know $O \cap F$ is open in F , hence there exists an open subgroup Q of $\hat{F}$ such that $Q \cap F = O \cap F$. Now $Q \times \ker \rho$ is an open subgroup of $\hat{G} = \hat{F} \times \hat{A}$ and $b(Q \times \ker \rho) \cap H_1 H_2 \neq \emptyset$ because of $b \in H_1 H_2$. Thus we can find $f \in Q$ and $c \in \ker \rho$ such that $bfc \in H_1 H_2 \subset G$. While $bc \in \hat{A}$ and $f \in \hat{F}$, so $bfc \in G$ implies that $f \in \hat{F} \cap G = F$, and $bc \in \hat{A} \cap G = A$. Hence $f \in Q \cap F = O \cap F$. Now

$$\rho(bc) = \rho(b)\rho(c) = \prod \rho_j(b_j) = \prod a_j^{s_j}.$$

So $bc = \prod a_j^{s_j} a_j^{p_j t}$ and $f \in Q \cap F = O \cap F$. That is: $f \prod a_j^{s_j} a_j^{p_j t} \in H_1 H_2$. Therefore

$$f \in H_1 \prod a_j^{-s_j} a_j^{-p_j t} H_2 \subset H_1 \prod a_j^{-s_j} \langle a_j^{p_j} \rangle H_2.$$

That is just $f \in T$, and hence $f \in O \cap T$. It contradicts that $O \cap T = \emptyset$. ■

Corollary 6.10 *If H is a finitely generated subgroup of $G = F \times A$ and $H \cap A = 1$, then $\bar{H} \cap \hat{A} = 1$.*

Proof: Set $H_2 = 1$ in above proposition; we get it immediately. ■

We may go further to prove a more general result. To do this, we need the following lemma.

Lemma 6.11 *Let $G = F \times A \times B$ where F is a free group, A and B are in the center of G . Let H be a subgroup of G that contains B as a subgroup. Let $\bar{H}$, $\bar{B}$ and $\overline{(H/B)}$ be the closures of H , B and H/B in $\hat{G}$ and $\hat{G}/\bar{B}$ respectively. Then $\bar{H}/\bar{B}$ can be embedded into $\overline{(H/B)}$, i.e.*

$$\bar{H}/\bar{B} \hookrightarrow \overline{(H/B)}.$$

Proof: Let $[h] \in \bar{H}/\bar{B}$, $h \in \bar{H}$ and $h\bar{B} \subset \bar{H}$. So $hB \subset H$. For any open subgroup O of G , $hO \cap H \neq \emptyset$. We can find $g \in O$ such that $hg \in H$. Hence $hOB/B \cap H/B \neq \emptyset$ since hgB is in the intersection. Therefore $hB \in \overline{(H/B)}$. Now

$$\eta : h\bar{B} \longrightarrow hB$$

defines a homomorphism from $H/\bar{B}$ to $\overline{(H/B)}$. If $h_1B = h_2B$, then $h_1h_2^{-1} \in B \subset \bar{B}$ and hence $h_1B = h_2B$. So η is injective. ■

Proposition 6.12 *Let $G = F \times A$ where $A = A_0 \times B$ is a free Abelian group of finite rank. Let H_1, H_2 be finitely generated subgroups of G . If $H_1H_2 \cap A = A_0$, then*

$$\bar{H}_1\bar{H}_2 \cap \bar{A} = \bar{A}_0.$$

Proof: Since $H_1H_2 \cap A = A_0$, $H_1A_0H_2 \cap A = A_0$. Let $K_1 = H_1A_0$ and $K_2 = H_2A_0$. Then K_1, K_2 are also finitely generated subgroups of G satisfying $K_1K_2 \cap A = A_0$. So $(K_1K_2 \cap A)/A_0 = 1$ (note that A_0 is normal in any supergroup), and hence $(K_1/A_0K_2/A_0) \cap A/A_0 = 1$. By Proposition 6.9, we get

$$\overline{(K_1/A_0)(K_2/A_0)} \cap \overline{(A/A_0)} = 1.$$

We claim that $\bar{K}_1\bar{K}_2 \cap \bar{A} = \bar{A}_0$. Suppose $a \in \bar{K}_1\bar{K}_2 \cap \bar{A}$. Write $a = k_1k_2$ where $k_i \in \bar{K}_i (i = 1, 2)$. Let ρ be the projection from $\hat{G}$ to $\hat{G}/\bar{A}_0$. Then $\rho(k_i) \in \bar{K}_i/\bar{A}_0$ and, Lemma 6.11, $\rho(k_i) \in \overline{(K_i/A_0)}$. Hence

$$\rho(a) = \rho(k_1)\rho(k_2) \in \overline{(K_1/A_0)(K_2/A_0)} \cap \overline{(A/A_0)}.$$

Therefore, we know that $\bar{K}_1\bar{K}_2 \cap \bar{A} = \bar{A}_0$.

Now $K_i = \overline{(H_iA_0)} = H_iA_0$. So

$$A_0 = K_1K_2 \cap A = H_1H_2A_0 \cap A = \bar{H}_1\bar{H}_2 \cap \bar{A}.$$

■

The conclusion of Proposition 3.11 is also true for $G = F \times A$. We give a proof as follows.

Proposition 6.13 *For any subgroup H of $G = F \times A$ and any element t in G ,*

$$\overline{H^t} = (H)^t$$

Proof: It is easy to see that $(\bar{H})^t \supset \overline{H^t}$ since $(H)^t \supset H^t$ and $(H)^t$ is closed. For any $g \in (\bar{H})^t$, $g^{t^{-1}} \in \bar{H}$. For any open subgroup O of G , $O^{t^{-1}}$ is also open in G . Then $g^{t^{-1}} O^{t^{-1}} \cap H \neq \emptyset$, that is just $gO \cap H^t \neq \emptyset$. Hence $g \in \overline{H^t}$. ■

6.4 The product separability

Theorem 6.14 *Let F be a free group and A be a free Abelian group of finite rank.*

Let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of $G = F \times A$. Then the product set $H_1 H_2 \cdots H_n$ is closed when $n \leq 4$ or when all projections $P_G(H_i)$ are cyclic subgroups of F .

Proof: Let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of $G = F \times A$. We are going to prove that $H_1 H_2 \cdots H_n \cap G = H_1 H_2 \cdots H_n$ (see Proposition 3.5).

(1) Let $H_1, H_2, \dots, H_n$ be finitely generated subgroups of $F \times A$. Then there exists a finitely generated subgroup F_0 of F such that $P_G(H_i) \subseteq F_0$ for all i , where P_G is the projection from G to F . The profinite topology of F_0 is induced from that of F (see Lemma 5.5). Since F_0 is closed in F , so $F_0 \times A$ is closed in $F \times A$. Hence $H_1 H_2 \cdots H_n$ is closed in $F \times A$ if and only if it is closed in $F_0 \times A$. So we may assume that F is a finitely generated free group.

(2) Let π be the projection homomorphism from G to A . By induction on $s = \text{rank}(A)$, we may assume $\langle \pi(H_1), \dots, \pi(H_n) \rangle = A$. Otherwise, set $A_0 = \langle \pi(H_1), \dots, \pi(H_n) \rangle$. Then by Lemma 6.4 and Lemma 5.5, $F \times A_0$ is closed in $G = F \times A$ and the profinite topology of $F \times A_0$ is induced from that of $F \times A$. So $H_1 H_2 \cdots H_n$ is closed in $F \times A$ if and only if it is closed in $F \times A_0$.

(3) Let $A = \langle a_1 \rangle \times B$ where $B = \langle a_2 \rangle \times \cdots \times \langle a_s \rangle$. Apply the same method in the proof of Proposition 4.1, we may assume that $H_i = \langle a_1^{k_i} e_{i1}, e_{i2}, \dots, e_{in_i} \rangle$ where $i = 1, 2, \dots, n$ and $e_{ij} \in F \times B$. Let $k = \text{l.c.m}\{k_1, k_2, \dots, k_n\}$ (the least common multiple). Then from Proposition 4.2, there is a subgroup T_n in H_n of finite index such that $T_n = \langle a_1^k f_1, f_2, \dots, f_{t_n} \rangle$ where $f_l \in F \times B$ ($l = 1, \dots, t_n$). By Proposition

4.6, $H_1 H_2 \cdots H_n$ is closed if for any b_j , $H_1^{b_j} H_2^{b_j} \cdots H_{n-1}^{b_j} T_n$ is closed where $H_n = \cup b_j T_n$. So we turn to prove the latter is closed. Note that $H_i^{b_j} = \langle a_1^k e_{i1}^{b_j}, e_{i2}^{b_j}, \dots, e_{im_i}^{b_j} \rangle$. Repeating the process for $H_{n-1}^{b_j}$, there is a subgroup T_{n-1} in $H_{n-1}^{b_j}$ of finite index and $H_1^{c_1} H_2^{c_1} \cdots H_{n-2}^{c_1} T_{n-1} T_n$ is closed in G for all c_l implies that $H_1 H_2 \cdots H_n$ is closed, where $T_{n-1} = \langle a_1^k g_1, g_2, \dots, g_{t_{n-1}} \rangle$, $H_{n-1}^{b_j} = \cup d_l T_{n-1}$ and $c_l = b_j d_l$. Applying this procedure for each subgroup, we deduce that $H_1 H_2 \cdots H_n$ is closed if $T_1 T_2 \cdots T_n$ is closed in G and $T_i = \langle a_1^k f_{i1}, f_{i2}, \dots, f_{im_i} \rangle$. So we just consider the case that $H_i = \langle a_1^k e_{i1}, e_{i2}, \dots, e_{im_i} \rangle$ where $(k \neq 0)$ or $H_i \subset F \times B$. Now $H_i \subset F \times B \times \langle a_1^k \rangle$ and $F \times B \times \langle a_1^k \rangle$ has finite index in $F \times A$, So $H_1 H_2 \cdots H_n$ is closed $F \times B \times \langle a_1^k \rangle$ implies that it is closed in $F \times A$. Therefore, we reduce the theorem to the case that each $H_i = \langle a_1 e_{i1}, e_{i2}, \dots, e_{im_i} \rangle$ or $H_i \subset F \times B$.

Let $C = \langle a_3 \rangle \times \cdots \times \langle a_s \rangle$. So $B = \langle a_2 \rangle \times C$. For subgroups $L_i = \langle e_{i2}, \dots, e_{im_i} \rangle (2 \leq i \leq n)$, apply same procedure as above, we get that $L_i = \langle a_2 f_{i2}, f_{i3}, \dots, f_{im_i} \rangle$ where $f_{ij} \in F \times C$. Hence $H_i = \langle a_1 e_{i1}, a_2 f_{i2}, f_{i3}, \dots, f_{im_i} \rangle$. Choose a suitable l , replace $a_1 e_{i1}$ by $a_1 e_{i1} (a_2 f_{i2})^{-l}$. We finally get that $H_i = \langle a_1 f_{i1}, a_2 f_{i2}, f_{i3}, \dots, f_{im_i} \rangle$. Repeating this procedure s times, we may assume that each $H_i = \langle a_1 e_{i1}, \dots, a_s e_{is}, e_{i(s+1)}, \dots, e_{im_i} \rangle$ (note: some $a_j e_{ij}$ may be missing).

(4) From Theorem 6.6, we may assume that $H_i H_{i+1} \cap A = 1$ (otherwise, the product set $H_1 H_2 \cdots H_n$ is closed). So $\bar{H}_i \bar{H}_{i+1} \cap \bar{A} = 1$ by Proposition 6.9. Denote $P_G(H_i) = K_i$ for $i = 1, \cdots, n$. Similarly to (4) in the proof of Theorem 4.11, we may assume that $P_G(H_n) = K_n$ is a free factor of F where P_G is the projection from $G = F \times A$ to F .

(5) Choose a basis for F which is expanded from a basis of K_n . We construct the Cayley graphs $\Gamma(K_n), \Gamma(F), \Gamma(\hat{K}_n), \Gamma(\hat{F})$ with respect to the selected bases. Note that $\Gamma(K_n) \subset \Gamma(F)$, $\Gamma(\hat{K}_n) \subset \Gamma(\hat{F})$ and $\Gamma(K_n) \subset \Gamma(\hat{K}_n)$, $\Gamma(F) \subset \Gamma(\hat{F})$.

(6) We use induction on n (the number of subgroups) and s (the rank of A). The case $n = 1$ is just subgroup separability which has been proved in Theorem 6.3. The case $s = 1$ has been proved in Chapter 4 (see Theorem 4.11).

Let $h_1 h_2 \cdots h_n \in \bar{H}_1 \bar{H}_2 \cdots \bar{H}_n \cap G$, where $h_i \in \bar{H}_i$. Then $P_{\hat{G}}(h_1) P_{\hat{G}}(h_2) \cdots P_{\hat{G}}(h_n) \in K_1 K_2 \cdots K_n \cap F$, where $\bar{K}_i = \overline{P_{\hat{G}}(\bar{H}_i)} = P_{\hat{G}}(\bar{H}_i)$. Setting $P_{\hat{G}}(h_i) = f_i \in \bar{K}_i$, so $f_1 f_2 \cdots f_n \in \bar{K}_1 \bar{K}_2 \cdots \bar{K}_n \cap F$.

(7) By Lemma 2.8, there exist $t_n \in \bar{K}_n$, $t_1 \in \bar{K}_1$ and $t_j \in \bar{K}_j$ for some j

($2 \leq j \leq n-1$) such that

(i) $t_n = t_1^t$ where $t = f_2 f_3 \cdots f_{n-1}$

(ii) $f_1 t_1^{-1} f_2 f_3 \cdots f_j t_j^{-1} \in F$

(iii) $t_j f_{j+1} f_{j+2} \cdots f_{n-1} t_n f_n \in F$.

So $f_1 f_2 \cdots f_n$ can be rewritten as

$$f_1 f_2 \cdots f_n = f_1 t_1^{-1} f_2 f_3 \cdots f_j t_j^{-1} \cdot t_j f_{j+1} f_{j+2} \cdots f_{n-1} t_n f_n.$$

(8) Lifting t_n , t_1 and t_j to l_n , l_1 and l_j respectively (from $\bar{K}_i$ to H_i where $i = 1, n, j$).

Since $H_i \cap A = 1$ and $\bar{H}_i \cap \bar{A} = 1$, the lifting is unique. Setting $s = h_2 h_3 \cdots h_{n-1}$

(that means $P_{\bar{G}}(s) = t$). It is easy to see that $l_1^t = l_1^s$ because A is the center of

$G = F \times A$. While l_n may not be equal to l_1^s , but they are in the same coset of A .

(9) If $l_n \neq l_1^s$, then $l_1^{-s} l_n \in \bar{A}$. So $(\bar{H}_1)^s \bar{H}_n \cap A \neq 1$ and also $(H_1)^r H_n \cap A \neq 1$ for $r = h_1 h_2 \cdots h_n$. By Proposition 6.9 and Proposition 6.13,

$$H_1^r H_n \cap A \neq 1.$$

Let $\langle a \rangle \subset H_1^r H_n \cap A$. Then $h_1 h_2 \cdots h_n \in \overline{\langle a \rangle} H_1 H_2 \cdots H_n \cap G$. So we may write that

$$h_1 h_2 \cdots h_n = a^m g_1 g_2 \cdots g_n$$

for some $g_i \in H_i$ and some integer m . Then

$$h_1 h_2 \cdots h_n a^{-m} = g_1 g_2 \cdots g_n$$

and

$$h_1 h_2 \cdots h_n k_1^r k_n = g_1 g_2 \cdots g_n$$

where $k_1 \in H_1$ and $k_n \in H_n$. Hence we get

$$h_1 h_2 \cdots h_n = k_1^{-1} g_1 g_2 \cdots g_n k_n^{-1}.$$

This gives that $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$.

(10) If $l_n = l_1^s$, then $l_1^{-s} l_n = 1$. Hence we get

$$h_1 h_2 \cdots h_n = h_1 h_2 \cdots h_{n-1} l_1^{-s} l_n h_n = h_1 l_1^{-1} h_2 \cdots h_{n-1} l_n h_n.$$

Since the projection of $h_1 h_2 \cdots h_n$ is

$$f_1 f_2 \cdots f_n = f_1 t_1^{-1} f_2 f_3 \cdots f_j t_j^{-1} \cdot t_j f_{j+1} f_{j+2} \cdots f_{n-1} t_n f_n.$$

Lifting t_j to the unique $x_j \in \bar{H}_j$, we can write

$$h_1 h_2 \cdots h_n = h_1 l_1^{-1} h_2 \cdots h_j x_j^{-1} \cdot x_j h_{j+1} h_{j+2} \cdots h_{n-1} l_n h_n.$$

(11) When $n \leq 4$, there must be either $j = 2$ or $j = n - 1$. In the case of $j = 2$,

let $f_1 t_1^{-1} f_2 x_2^{-1} = y_1 y_2 \in K_1 K_2$ since $f_1 t_1^{-1} f_2 x_2 \in F$. Lifting $y_i \in K_i$ to $g_i \in H_i$ where

$i = 1, 2$. Then $h_1 l_1^{-1} h_2 x_2^{-1}$ and $g_1 g_2$ have same projection on F . So $H_1 H_2 \cap A = 1$ forces that

$$h_1 l_1^{-1} h_2 x_2^{-1} = g_1 g_2.$$

Hence

$$h_1 h_2 \cdots h_n = g_1 g_2 \cdot x_2 h_3 \cdots h_n.$$

By induction on n , $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$.

(12) Now we consider the case that $P_G(H_1), \dots, P_G(H_n)$ are cyclic groups. Then $H_i \cap A = 1$ implies that H_i is cyclic itself. By (9), we may assume that $(H_1)^s H_n \cap A = 1$. From (10), we get a splitting of the product $h_1 h_2 \cdots h_n$ as follows:

$$h_1 h_2 \cdots h_n = h_1 l_1^{-1} \cdots h_j x_j^{-1} \cdot x_j h_{j+1} \cdots l_n h_n.$$

(13) Since $P_G(H_j)$ is finitely generated, there is a subgroup F_0 of finite index in F such that $P_G(H_j)$ is a free factor of F_0 . Let $G_0 = F_0 \times A$. For each $i = 1, \dots, n$, $K_i = H_i \cap G_0$ is a finite index subgroup of H_i . Note $K_j = H_j$. Let $H_i = \cup b_{iq} K_i$ for $i < j$ and $H_i = \cup K_i b_{iq}$ for $i > j$. Then $H_i = \cup b_{iq} K_i$ or $H_i = \cup K_i b_{iq}$ by Proposition 3.9.

Hence

$$\begin{aligned}
& h_1 l_1^{-1} \cdots h_j x_j^{-1} \cdot x_j h_{j+1} \cdots l_n h_n \\
&= b_{1q_1} \cdots b_{jq_j} y_1^{b_{2q_2} \cdots b_{(j-1)q_{j-1}}} \cdots h_j x_j^{-1} \cdot x_j y_{j+1} \cdots y_n^{(b_{(j+1)q_{j+1}} \cdots b_{(n-1)q_{n-1}})^{-1}} b_{(j+1)q_{j+1}} \cdots b_{nq_n}
\end{aligned}$$

where $h_1 l_1^{-1} = b_{1q_1} y_1$ and so on. If we can prove that

$$\begin{aligned}
& y_1^{b_{2q_2} \cdots b_{(j-1)q_{j-1}}} \cdots h_j x_j^{-1} \cdot x_j y_{j+1} \cdots y_n^{(b_{(j+1)q_{j+1}} \cdots b_{(n-1)q_{n-1}})^{-1}} \\
& \in H_1^{b_{2q_2} \cdots b_{(j-1)q_{j-1}}} \cdots H_j \cdots H_n^{(b_{(j+1)q_{j+1}} \cdots b_{(n-1)q_{n-1}})^{-1}}
\end{aligned}$$

then $h_1 h_2 \cdots h_n \in H_1 H_2 \cdots H_n$. Hence we may assume that $P_G(H_j)$ is a free factor of F to deal with the splitting of product:

$$h_1 h_2 \cdots h_n = h_1 l_1^{-1} \cdots h_j x_j^{-1} \cdot x_j h_{j+1} \cdots l_n h_n.$$

Note the projection images of partial products $h_1 l_1^{-1} \cdots h_j x_j^{-1}$ and $x_j h_{j+1} \cdots h_{n-1} l_n h_n$ are in F .

(14) Let $r_{1j} = h_2 \cdots h_{j-1}$ and $r_{jn} = h_j \cdots h_{n-1}$. If both $H_1^{r_{1j}} H_j \cap A \neq 1$ and $H_j^{r_{jn}} H_n \cap A \neq 1$, then there are $u_1 \in H_1, u_j \in H_j$ and $u_n \in H_n$ such that $u_1^{r_{1j}} u_j \in A$ and $u_j^{-1} u_n^{r_{jn}^{-1}} \in A$. So

$$u_1^{r_{1j}} u_j u_j^{-1} u_n^{r_{jn}^{-1}} = r_{1j}^{-1} u_1 r_{1j} r_{jn} u_n r_{jn}^{-1} \in A.$$

That implies $u_1^s u_n \in A$ where $s = h_2 \cdots h_{n-1}$.

If $u_1^s u_n \neq 1$, then $H_1^s H_n \cap A \neq 1$. By (9), $H_1 H_2 \cdots H_n$ is closed in G .

If $u_1^* u_n = 1$, then there is a finite index subgroup V of H_1 such that V^* is a subgroup of H_n (since H_1 is cyclic). So $\bar{V}$ is finite index in $\bar{H}_1$ and $(V)^*$ is a subgroup of $\bar{H}_n$. Hence, setting $h_1 = bv_1$ where $b \in H_1$ and $v_1 \in \bar{V}$,

$$h_1 h_2 \cdots h_n = b h_2 \cdots h_{n-1} v_1 h_n.$$

By induction on the number of subgroups, $h_1 h_2 \cdots h_n$ is in $H_1 H_2 \cdots H_n$.

(15) Now we may assume that $H_1^{r_1} H_j \cap A = 1$ or $H_j^{r_j} H_n \cap A = 1$. Steps (12), (13), (14) provide a splitting procedure. We may continue this splitting procedure for one of two partial products. As the next splitting, if $H_1^{r_1} H_j \cap A = 1$, then in F

$$f_1 f_2 \cdots f_n = f_1 t_1^{-1} s_1^{-1} f_2 \cdots f_i s_i^{-1} \cdot s_i f_{i+1} \cdots f_{j-1} s_j f_j t_j^{-1} \cdot t_j f_{j+1} \cdots f_{n-1} t_n f_n$$

by Lemma 2.21. Lifting to G , we get

$$h_1 h_2 \cdots h_n = h_1 l_1^{-1} k_1^{-1} h_2 \cdots h_i k_i^{-1} \cdot k_i h_{i+1} \cdots h_{j-1} k_j h_j x_j^{-1} \cdot x_j h_{j+1} \cdots h_{n-1} l_n h_n.$$

Note again, let $r_{1i} = h_2 \cdots h_{i-1}$ and $r_{ij} = h_i \cdots h_{j-1}$, we may assume that

$$H_1^{r_{1i}} H_i \cap A = 1 \text{ or } H_i^{r_{ij}} H_j \cap A = 1.$$

(16) Repeat the splitting procedure above until some partial product is in the product set of two subgroups. Then

$$h_1 h_2 \cdots h_n = \cdots l_k l_{k+1} \cdots$$

and its projection on F is

$$f_1 f_2 \cdots f_n = \cdots q_k q_{k+1} \cdots$$

where $q_k q_{k+1} \in F$. So $q_k q_{k+1} = p_k p_{k+1}$ where $p_k \in P_G(H_k)$ and $p_{k+1} \in P_G(H_{k+1})$.

Lifting p_k, p_{k+1} to g_k, g_{k+1} in H_k, H_{k+1} respectively, we know $l_k l_{k+1} = g_k g_{k+1}$ due to Proposition 6.9.

(17) Step (16) is to replace a partial product of two elements in $\hat{G}$ by a product of two elements in G . Repeating the splitting procedure and replacing procedure, we will reach the situation where there are two successive partial products of two elements being replaced. Then the conclusion is proved by induction on the number of subgroups. This completes the proof. ■

Note that Theorem 6.14 implies the double coset separability of $F \times A$. In general, it is still open whether $F \times A$ is product separable. Theorem 6.14 shows that it may be quite hard to find a counterexample if such an example exists. Another question related to this one, is whether, for a finitely generated group G and an element g in

the profinite completion $\hat{G}$ of G but not in G , is there an element $h \in G$ such that $g \in \overline{\langle h \rangle}$ in $\hat{G}$. For other subgroup separable groups, the separability properties of free product of two free groups with a cyclic subgroup amalgamated are quite attractive.

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