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MULTILEVEL BILINEAR SYSTEM OF
STOCHASTIC DIFFERENTIAL EQUATIONS

by

GENEVIEVE GAUTHIER

A thesis submitted to the faculty of Graduate
Studies and Research in partial fulfilment of the
requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

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Abstract

A multilevel bilinear system of stochastic differential equations is a multilevel mean field system in which the drift term is also linear. Two kinds of parameters coexist in this model: the rate of spatial mixing and the noise intensity. A phase transition occurs that is, the system exhibits qualitatively different behaviours in different parameter regimes. If the noise intensity is small compared to the rate of spatial mixing, we observe that the systems approach an equilibrium. This situation is termed subcritical. In contrast, the supercritical state is characterized by a large noise intensity and a small rate of spatial mixing. The system then exhibits intermittency, that is, very large spatial clumps develop as time increases which are separated by larger and larger distances. The border between these two subspaces of the parameters' space is defined as the critical case. No general behavior is known in this situation. We obtain a complete description of the subcritical case and, particularly, the limiting behavior of the process when we rescale the time. We develop a new technique involving fractional moments which allows us to describe partially the supercritical case. The critical case is a very difficult one and although there some open questions remains, we have obtained rigorous partial results.

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Introduction

Systems of bilinear stochastic differential equations appear in several fields such as linearizations of randomly perturbed nonlinear systems. For example, they arise as models of population growth in a random environment in ecology and as models of intermittent behaviour (called the nonstationary Anderson parabolic model) in chemical kinetics, oceanography and statistical physics.

Of particular interest is the fact that such systems can exhibit phase transition behaviour. To be more explicit about this notion for bilinear systems, we need to set up some notation. The class of mean field systems of stochastic differential equations is characterized by the special form of its drift term. They are expressed as follow:

$$dX_i^N(t) = c \left(\frac{1}{N} \sum_{j=1}^N X_j^N(t) - X_i^N(t) \right) dt + \gamma \left(X_i^N(t) \right) dW_i(t), \quad i \in \{1, 2, \dots, N\}$$

where $W = (W_0, W_1, \dots, W_{N-1})$ is a N -dimensional Brownian motion, c and γ are two constants and g is a function such that a solution to this system exists. Two kinds of parameters coexist in this model: the rate of spatial mixing and the noise intensity denoted respectively c and γ . At least for some particular functions g , a phase transition occurs that is, the system exhibits qualitatively different behaviours in different parameter regimes. If the noise intensity is small compared to the rate of spatial mixing, we observe that the systems approach an equilibrium. This situation is termed subcritical. In contrast, the supercritical state is characterized by a large noise intensity and a small rate of spatial mixing. The system then exhibits intermittency, that is, very large spatial clumps develop as time increases which are separated by larger and larger distances. The border between these two subspaces of the parameters' space is defined as the critical case. No general behavior is known in this situation.

This class can be enlarged if we consider multilevel systems. That is, we introduce a hierarchy in the system by defining a notion of distance between two particles. From the point of view of one tagged particle, the interaction it has with the whole system now occurs at several levels. For each stage, the drift term contains a mean field type component (we average over all the particles that are at distance k of this fixed particle $k = 0, 1, 2, \dots, L$) multiplied by a factor that will decrease as the distance k increases. The diffusion term remains unaffected.

The reason for studying the hierarchical mean field systems is that this method has many of the advantages of the mean field approximation but overcomes the principal limitations of the latter. Indeed, the qualitative behaviour of the multilevel process, which is simpler than the corresponding lattice system, closely reflects the behaviour of its analog.

In 1988, Ya. B. Zeldovich, S. A. Molchanov, A. A. Ruzmaikin and D. D. Sokoloff published (see [ZMRS]) a survey in which such systems are used to model physical situations. We also find in the literature studies of particular systems that belong to this broad class. For example, D. Dawson and A. Greven ([DG](a) and [DG](b)) considered the Fisher-Wright

process which is described by the multilevel mean-field system with $g(x) = \sqrt{x(1-x)}$. One property of such stochastic processes is that each of their components are $[0, 1]$ -valued. The authors obtained the McKean-Vlasov limit of the systems and also limiting results in a time rescaling context. They also studied, in [DG](c), the Feller-branching system in which $g(x) = \sqrt{x}$. Part of the work of T. Shiga is also related to this class of systems. In [Shi](a), he introduced a small perturbation into the drift term of a mean field system. Using time renormalization, he proved a limit theorem about the empirical distribution when g is bounded and $1/2$ -Hölder continuous.

A multilevel bilinear system of stochastic differential equations is a multilevel mean field system in which $g(x) = x$. This unbounded diffusion term implies that the stochastic process defined by the system is also unbounded. This forces us to be more cautious when proving results. One should note that this process behaves differently than the Feller-branching one. The phase transition is characterized by the convergence or the divergence of the second moments of our system when time is renormalized. Therefore, we study these moments which arise as the solution of an ordinary differential equations system. It is not possible in the general multilevel setting to obtain exact expressions for the moments but we can express them as

$$\sum_{i=1}^L k_i(N) \exp[r_i(N)t]$$

where the $k_i(N)$'s are function of the initial conditions and if λ_i denotes a constant determined by the parameters of the system, we then establish that as N increases to infinity,

$$r_i(N) = \frac{\lambda_i}{N^{i-1}} + o\left(\frac{1}{N^i}\right).$$

The convergence or the divergence of the moments then depends on the sign of the λ_i , and the rate of renormalization. We put the emphasis on the subcritical case and develop limit theorems when the time is properly rescaled.

The first chapter discusses the mean-field system of stochastic differential equations. We obtain a complete description of the subcritical case and, particularly, the limiting behavior of the process when we rescale the time. We develop a new technique involving fractional moments which allows us to describe partially the supercritical case. The critical case is a very difficult one and although there some open questions remains, we have obtained rigorous partial results. The last section contains a generalization of the mean field system, that is, we introduce into the model a small perturbation. We are able to adapt the main result of the chapter, which is the limiting theorem of the subcritical processes in the time rescaling context, to these new processes. It will be of great help when we come to work with higher level systems.

In chapter two, we study some aspects of the second-level bilinear system of stochastic differential equations. The great interest of this model is that it contains most of the complications associated with multilevel bilinear systems without the notational complexity of the general setting. All the results obtained in the first chapter find their analogs in this specific

context. Most of the proofs are direct adaptations of what has been done in the mean field case except when we deal with time renormalization. Therefore, we focus on this aspect (especially the subcritical case) and explain what are the solutions to the problems raised by the sophistication of the model.

The last chapter contains results about the most general system that we have considered: the multilevel bilinear system of stochastic differential equations with and without the small perturbation. Even when we work with the finite system, it is sometimes difficult to obtain exact results because the number of levels is arbitrary. However, we have generalized the main theorem about weak convergence of the average process in the subcritical case when the appropriate time scale is chosen. The proof is by induction on the number of levels. The critical and the supercritical cases are partially solved.

Chapter 1

Mean Field System of Stochastic Differential Equations

Introduction

First, let us define the stochastic process of interest. On a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$, we define a N -dimensional Brownian motion $W = (W_0, W_1, \dots, W_{N-1})$. We assume that $\mathcal{F}_0$ contains the collection of the $\mathbb{P}$ -null sets and that the filtration is right-continuous. The stochastic process that we will study, denoted $X^N = (X_0^N, \dots, X_{N-1}^N)$, is the unique strong solution of the system of stochastic differential equations

$$(*) \quad dX_i^N(t) = c(\bar{X}_N(t) - X_i^N(t))dt + \gamma X_i^N(t) dW_i(t), \quad i \in \{0, 1, \dots, N-1\}$$

where c and γ are positive constants and

$$\bar{X}_N(t) \equiv \frac{1}{N} \sum_{j=0}^{N-1} X_j^N(t).$$

Some results about the behaviour of this process are known. In particular, Christian Léonard (1986) obtained a law of large numbers for a family of systems of stochastic differential equations where, among others conditions, the drift and the diffusion coefficients depend on its empirical measure. Our system's coefficients satisfy the premises of his results and we restate here, in this particular context, one of his propositions:

THEOREM 1.1 (ref.: Proposition 6.5 of [Léo]): (*Propagation of chaos*)

If $\{X_0^N(0), X_1^N(0), \dots, X_{N-1}^N(0)\}$ is exchangeable for any $N \geq 1$, if $\mathcal{L}(\bar{X}_N(0))$ converges weakly (as $N \rightarrow \infty$) to a probability μ and if

$$\sup_N \mathbb{E}[\bar{X}_N^4(0)] < \infty$$

then for any $m \geq 0$, $\mathcal{L}(X_0^N, X_1^N, \dots, X_m^N) \xrightarrow{N \rightarrow \infty} P^{\otimes m}$ where P is the law of the one-dimensional diffusion process defined by

$$dX(t) = c(\mathbb{E}[X(0)] - X(t))dt + \gamma X(t) dW(t)$$

$$\mathcal{L}(X(0)) = \mu.$$

In 1991, Tokuzo Shiga obtained, as a particular case of the stochastic processes he considered, some results about the bilinear system on the lattice $\mathbb{Z}^d$. In particular, he proved that in dimensions $d \geq 3$ (under some hypothesis on the initial conditions) the process approaches an equilibrium as t tends to infinity if the constant c of the drift term is sufficiently large. On the other hand, he also established that if the constant c of the drift term is sufficiently small, then the process goes to extinction exponentially fast.

In section 3, we present the main result of this chapter. It is about the limiting behavior (as N goes to infinity) of $\bar{X}_N$ in a time rescaling context (we speed up the time by a factor of N). A phase transition is observed and we prove that there is weak convergence to a new diffusion if the constant c contained in the drift term is large enough to counteract the constant γ of the diffusion coefficient.

If the time is rescaled by a factor N^{1+p} , $p > 0$, then it is possible to show, using the fractional moment method, that $\bar{X}_N$ goes to extinction in probability, and this arises independently of the constants of the coefficients.

Even in this simple case, there are several open problems of great interest. From the perspective of time rescaling, we suspect that the limiting behavior of $\bar{X}_N$, (for the cases when the constants of the drift and diffusion coefficients place us on the boundary of the phase transition and the time is modified by a factor of $\sqrt{N}$), will be dramatically different than what we perceived.

Before everything else, we present in the next section some basic results about the finite system (nonnegativity of the strong solution, exact expressions for the first moments, etc.).

The Finite System

The next few results establish some properties of the finite system. It will become clear from the second moments that we will observe a phase transition in the limit as N goes to infinity if we choose the proper scale for the time. We will also prove that the finite system tends to extinction as the time increases. The first theorem of this section says that if we start with nonnegative initial conditions, then the process will remain nonnegative through time.

Nonnegativity of the Strong Solution

THEOREM 2.1: (*Nonnegativity of the strong solution*)

$$\text{If } \forall i \in \{0, \dots, N-1\}, X_i^N(0) \stackrel{\text{p.a.s.}}{\geq} 0 \text{ then } X_i^N(t) \stackrel{\text{p.a.s.}}{\geq} 0, \forall t \geq 0.$$

PROOF OF THEOREM 2.1: The idea is to define a random time τ as the first time for which one of the components of X^N is negative. We then prove that $\tau = \infty$ $\mathbb{P}$ -a.s. by defining a new process $Y^N(t) \equiv X^N(t + \tau)$ on the set $\{\omega \in \Omega : \tau(\omega) < \infty\}$. A contradiction between the

behavior of $Y^N(t)$ near the origin and that of $X^N(t)$ in the neighbourhood of τ will lead to the conclusion that $\{\omega \in \Omega : \tau(\omega) < \infty\}$ is a $\mathbb{P}$ -null set.

Define

$$\tau_i(\omega) \equiv \begin{cases} \inf \{t \geq 0 : X_i^N(\omega, t) < 0\} & \text{if } \{t \geq 0 : X_i^N(\omega, t) < 0\} \neq \emptyset \\ \infty & \text{if } \{t \geq 0 : X_i^N(\omega, t) < 0\} = \emptyset \end{cases}$$

$$\tau(\omega) \equiv \min \{\tau_i(\omega) : i \in \{0, \dots, N-1\}\}$$

and

$$A \equiv \{\omega \in \Omega : \tau(\omega) < \infty\}.$$

We want to show that A is a null set of $\mathbb{P}$. The continuity of X^N and the definition of τ give

$$(2.1.1) \quad A \stackrel{\mathbb{P}\text{-a.s.}}{\subseteq} \bigcap_{n \geq 1} \left\{ \omega \in \Omega : \exists i \in \{0, \dots, N-1\} \text{ such that } \exists t \in \left(\tau(\omega), \tau(\omega) + \frac{1}{n} \right) X_i^N(\omega, t) < 0 \right\}.$$

We define a new stochastic process by

$$Y(t, \omega) \equiv \begin{cases} X^N(t + \tau(\omega), \omega) & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c \end{cases}$$

We will study the behavior of $Y(t)$ for small t instead of the behavior of X^N near τ . First, note that Y satisfies

$$(2.1.2) \quad Y_i(t) \stackrel{\mathbb{P}\text{-a.s.}}{=} Y_i(0) + \int_0^t c(\bar{Y}(s) - Y_i(s)) ds + \int_0^t \gamma_i Y_i(s) dW_i^*(s), \quad i \in \{0, \dots, N-1\}$$

Now, let

$$C \equiv \{\omega \in A : \forall i \in \{0, \dots, N-1\}, X_i^N(\omega, \tau(\omega)) = 0\}$$

so

$$C^c = \{\omega \in A : \exists i \in \{0, \dots, N-1\} \text{ such that } X_i^N(\omega, \tau(\omega)) > 0\}.$$

On C , $Y(t) \stackrel{\mathbb{P}\text{-a.s.}}{=} 0$ for any t is one solution of (2.1.2) and from uniqueness of the strong solution of this system, it is the only one. What about the behavior of $Y(t)$ on C^c ? For any $\omega \in C^c$, we define

$$J(\omega) \equiv \{i \in \{0, \dots, N-1\} : X_i^N(\tau(\omega), \omega) = 0\}.$$

¹ W^* is the Brownian motion $W(t + \tau) - W(\tau)$. Note that Y is adapted to the new filtration $\{\mathcal{F}_t^* : t \geq 0\}$ under which W^* is a Brownian motion. We also require that the filtration is right-continuous, that $\mathcal{F}_0^*$ contains all the $\mathbb{P}$ -null sets and makes $X_i^N(\tau)$ measurable.

For any $\omega \in C^c$ and for any $i \in \mathcal{I}^c(\omega)$, $Y_i(0, \omega) > 0$ so $t \rightarrow Y_i(t, \omega)$ is nonnegative near $t = 0$. For $\mathbb{P}$ -almost every $\omega \in C^c$ and for any $i \in \mathcal{I}(\omega)$, we can use Anderson's theorem (ref.: [And], page 313, theorem 4.2. See the appendix for a statement of the result) to obtain

$$\lim_{t \downarrow 0} \frac{Y_i(t, \omega)}{t} = c \left(\frac{1}{N} \sum_{j=0}^{N-1} Y_j(0, \omega) - Y_i(0, \omega) \right) = \frac{c}{N} \sum_{j=0}^{N-1} Y_j(0, \omega) > 0$$

which implies that $t \rightarrow Y_i(t, \omega)$ is an increasing (and therefore nonnegative) function of t near $t = 0$. From this, we conclude that

$$(2.1.3) \quad A = C \cup C^c \stackrel{\mathbb{P}\text{-a.s.}}{\subseteq} \bigcup_{n \geq 1} \left\{ \omega \in A : \forall i \in \{0, \dots, N-1\}, \forall t \in \left[\tau(\omega), t + \frac{1}{n} \right), X_i^N(\omega, t) \geq 0 \right\}.$$

Equations (2.1.1) and (2.1.3) implies that $A \stackrel{\mathbb{P}\text{-a.s.}}{=} \emptyset$. ■

Martingale Property of the Average Process

LEMMA 2.2: (A bound for the k^{th} moment of $\bar{X}_N(t)$) Let $k \geq 1$. If we assume that the initial conditions are nonnegative random variables having finite k^{th} moment then

$$\mathbb{E}[\bar{X}_N^k(t)] \leq \mathbb{E}[\bar{X}_N^k(0)] \exp\left[\frac{k(k-1)}{2} \gamma^2 t\right].$$

PROOF OF LEMMA 2.2: Itô's formula gives

$$d\bar{X}_N^k(t) \stackrel{\mathbb{P}\text{-a.s.}}{=} k \bar{X}_N^{k-1}(t) d\bar{X}_N(t) + \frac{k(k-1)}{2} \bar{X}_N^{k-2}(t) d\langle \bar{X}_N \rangle(t).$$

Because

$$d\bar{X}_N(t) \stackrel{\mathbb{P}\text{-a.s.}}{=} \frac{\gamma}{N} \sum_{i=0}^{N-1} X_i^N(t) dW_i(t) \text{ and } d\langle \bar{X}_N \rangle(t) \stackrel{\mathbb{P}\text{-a.s.}}{=} \frac{\gamma^2}{N^2} \sum_{i=0}^{N-1} (X_i^N(t))^2 dt,$$

we have the following equality:

$$\bar{X}_N^k(t) \stackrel{\mathbb{P}\text{-a.s.}}{=} \bar{X}_N^k(0) + \frac{k\gamma}{N} \sum_{i=0}^{N-1} \int_0^t \bar{X}_N^{k-1}(s) X_i^N(s) dW_i(s) + \frac{k(k-1)}{2} \frac{\gamma^2}{N} \int_0^t \bar{X}_N^{k-2}(s) \left(\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(s))^2 \right) ds.$$

Taking expectation on each side of the equality, we get from Fubini-Tonelli's theorem and lemma A.1

$$\begin{aligned} \mathbb{E}[\bar{X}_N^k(t)] &\leq \mathbb{E}[\bar{X}_N^k(0)] + \frac{k(k-1)}{2} \gamma^2 \int_0^t \mathbb{E} \left[\bar{X}_N^{k-2}(s) \left(\frac{1}{N^2} \sum_{i=0}^{N-1} (X_i^N(s))^2 \right) \right] ds \\ &\leq \mathbb{E}[\bar{X}_N^k(0)] + \frac{k(k-1)}{2} \gamma^2 \int_0^t \mathbb{E}[\bar{X}_N^k(s)] ds. \end{aligned}$$

We conclude using Gronwall's inequality:

$$\mathbb{E}[\bar{X}_N^k(t)] \leq \mathbb{E}[\bar{X}_N^k(0)] \exp\left[\frac{k(k-1)}{2} \gamma^2 t\right].$$

■

PROPOSITION 2.3: Assuming the initial conditions are nonnegative square-integrable random variables, then for any finite T ,

$\{\bar{X}_N(t) : 0 \leq t \leq T\}$ is a continuous nonnegative martingale.

PROOF OF PROPOSITION 2.3: Since

$$\bar{X}_N(t) \stackrel{\text{p.a.s.}}{=} \bar{X}_N(0) + \frac{\gamma}{N} \sum_{i=0}^{N-1} \int_0^t X_i^N(s) dW_i(s),$$

$\{\bar{X}_N(t) : t \geq 0\}$ is a continuous local-martingale. Corollary 1.25, page 124 of [RY] says that if M is a continuous local-martingale, then the following conditions are equivalent:

- (i) $M_0 \in L^2$ and $\mathbb{E}[\langle M \rangle(t)] < \infty$;
- (ii) $\{M_s : 0 \leq s \leq t\}$ is an L^2 -bounded martingale.

But from lemma 2.2, we have for any $T \geq 0$,

$$\mathbb{E}[\langle \bar{X}_N \rangle(T)] = \mathbb{E}\left[\gamma^2 \int_0^T \frac{1}{N^2} \sum_{i=0}^{N-1} (X_i^N(s))^2 ds\right] \leq \gamma^2 \int_0^T \mathbb{E}[\bar{X}_N^2(s)] ds \leq \gamma^2 \mathbb{E}[\bar{X}_N^2(0)] \int_0^T \exp[\gamma^2 s] ds < \infty$$

so we have the desired result. ■

COROLLARY: Assuming the initial conditions are nonnegative square-integrable random variables, $\{\bar{X}_N(t) : t \geq 0\}$ is a continuous nonnegative square-integrable martingale.

Extinction of the Finite System over Time

To show the extinction of $X^N(t)$ as t goes to infinity, it suffices to prove the extinction of $\sum_{i=0}^{N-1} X_i^N(t)$ since each component of $X^N(t)$ is almost-surely nonnegative.

PROPOSITION 2.4: (*Extinction of the finite system in time*)

Assuming the initial conditions are nonnegative square-integrable random variables,

$$\sum_{i=0}^{N-1} X_i^N(t) \xrightarrow{t \rightarrow \infty} 0 \text{ } \mathbb{P}\text{-almost-surely.}$$

PROOF OF PROPOSITION 2.4: Because $\sum_{i=0}^{N-1} X_i^N(t)$ is a continuous nonnegative martingale, we know that the limit (as t goes to infinity) exists $\mathbb{P}$ -almost-surely. (ref. [RY], p.62, cor.2.11).

We will show that $\mathbb{P}(A) = 0$ where

$$A \equiv \left\{ \omega \in \Omega : \lim_{t \rightarrow \infty} \sum_{i=0}^{N-1} X_i^N(t) > 0 \right\}.$$

Using the random time substitution property (ref.: [McK], page 29, section 2.5 and/or [And], page 310), we have $\forall t \geq 0$,

$$0 \leq \sum_{i=0}^{N-1} X_i^N(t) \stackrel{\text{p.a.s.}}{=} \sum_{i=0}^{N-1} X_i^N(0) + \sum_{i=0}^{N-1} \int_0^t \gamma X_i^N(s) dW_i(s) = \sum_{i=0}^{N-1} X_i^N(0) + B(\tau(t))$$

where B is a new one-dimensional Brownian motion and τ is the following random time:

$$\tau(t) \equiv \gamma^2 \int_0^t \sum_{i=0}^{N-1} (X_i^N(s))^2 ds.$$

Therefore,

$$(2.4.1) \quad \infty > \sum_{i=0}^{N-1} X_i^N(0) \stackrel{\text{p.a.s.}}{\geq} -B(\tau(t)), \quad \forall t \geq 0.$$

Now define

$$C \equiv \left\{ \omega \in \Omega : \lim_{t \rightarrow \infty} \tau(t) = \infty \right\}.$$

We claim that $A \subseteq C$. Indeed, let $\omega \in A$ and note that from the definition of A , we have that

$$\lim_{t \rightarrow \infty} \sum_{i=0}^{N-1} X_i^N(t, \omega) > 0.$$

Therefore,

$$\lim_{t \rightarrow \infty} \tau(t, \omega) = \gamma^2 \int_0^\infty \sum_{i=0}^{N-1} (X_i^N(s, \omega))^2 ds \geq \frac{\gamma^2}{N} \int_0^\infty \left(\sum_{i=0}^{N-1} X_i^N(s, \omega) \right)^2 ds = \infty$$

which means that $\omega \in C$. To conclude the proof, it suffices to show that $\mathbb{P}(C) = 0$. To do this, we will prove that on C , equation (2.4.1) contradicts the law of iterated logarithm¹. Since on C , $\tau(t) \uparrow \infty$ almost-surely as $t \uparrow \infty$ and because $-B$ is a Brownian motion,

$$\limsup_{t \rightarrow \infty} \frac{-B(\tau(t))}{\sqrt{2\tau(t)\ln(\ln(\tau(t)))}} \stackrel{\text{p.a.s.}}{=} 1 \text{ on } C.$$

Therefore, there is a constant k for which

$$(2.4.2) \quad \forall \varepsilon > 0, \forall T \geq k, \exists t \geq T \text{ such that } -B(\tau(t)) \stackrel{\text{p.a.s.}}{>} (1-\varepsilon) \sqrt{2\tau(t)\ln(\ln(\tau(t)))}.$$

But (2.4.2) contradicts (2.4.1) since on C , $(1-\varepsilon) \sqrt{2\tau(t)\ln(\ln(\tau(t)))} \uparrow \infty$ as $t \uparrow \infty$. Therefore $\mathbb{P}(C)$ must be zero. ■

$$\limsup_{t \rightarrow \infty} \frac{B(t)}{\sqrt{2t\ln(\ln(t))}} \stackrel{\text{p.a.s.}}{=} 1$$

Computation of the First and Second Moments of the Finite System

We will need some control on the moments of the strong solution later on. The next results gives an expression for the first moment of the process.

LEMMA 2.5: (*First moment*)

$$\mathbb{E}[\bar{X}_N(t)] = \mathbb{E}[\bar{X}_N(0)].$$

PROOF OF LEMMA 2.5: $\bar{X}_N(t)$ is a $\{\bar{X}_t : t \geq 0\}$ -martingale. ■

LEMMA 2.6: (*Second moments*) Assume that the initial conditions are nonnegative and square-integrable. If we define $\alpha \equiv 2c - \gamma^2$ then

$$\begin{aligned} \mathbb{E}\left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t)\right] &= \frac{4c\mathbb{E}[\bar{X}_N^2(0)] + \left[\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} - \alpha\right] \mathbb{E}\left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0)\right]}{2\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}}} \exp\left[\frac{\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} - \alpha}{2} t\right] \\ (2.6.1) \quad &+ \frac{-4c\mathbb{E}[\bar{X}_N^2(0)] + \left[\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} + \alpha\right] \mathbb{E}\left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0)\right]}{2\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}}} \exp\left[-\frac{\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} + \alpha}{2} t\right] \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[\bar{X}_N^2(t)] &= \frac{\left[\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} + \alpha\right] \mathbb{E}[\bar{X}_N^2(0)] + \frac{2\gamma^2}{N} \mathbb{E}\left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0)\right]}{2\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}}} \exp\left[\frac{\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} - \alpha}{2} t\right] \\ (2.6.2) \quad &+ \frac{\left[\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} - \alpha\right] \mathbb{E}[\bar{X}_N^2(0)] - \frac{2\gamma^2}{N} \mathbb{E}\left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0)\right]}{2\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}}} \exp\left[-\frac{\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} + \alpha}{2} t\right] \end{aligned}$$

PROOF OF LEMMA 2.6: To simplify the notation, we will drop the superscript N . The strong solution of (*) will be denoted $X \equiv (X_0, \dots, X_{N-1})$.

Itô's formula gives us

$$d\bar{X}_N^2(t) \stackrel{p-a.s.}{=} 2\bar{X}_N(t) d\bar{X}_N(t) + d\langle \bar{X}_N \rangle(t).$$

Since

$$d\bar{X}_N(t) \stackrel{p-a.s.}{=} \frac{\gamma}{N} \sum_{i=0}^{N-1} X_i(t) dW_i(t),$$

then

$$d\langle \bar{X}_N \rangle(t) \stackrel{p-a.s.}{=} \frac{\gamma^2}{N^2} \sum_{i=0}^{N-1} X_i^2(t) dt.$$

Therefore

$$d\bar{X}_N^2(t) \stackrel{p-a.s.}{=} \frac{2\gamma}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} X_i(t) X_j(t) dW_i(t) + \frac{\gamma^2}{N^2} \sum_{i=0}^{N-1} X_i^2(t) dt.$$

Take the expectation on both sides of the equality and the use of the Fubini-Tonelli's theorem and lemma A.2 combined with lemma 2.2 to get

$$(2.6.3) \quad \frac{d}{dt} \mathbb{E} \left[\bar{X}_N^2(t) \right] = \frac{\gamma^2}{N} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t) \right].$$

We will use the same approach to determine a second equation. Itô's formula give us

$$dX_i^2(t) \stackrel{p-a.s.}{=} 2X_i(t) dX_i(t) + d\langle X_i \rangle(t).$$

Since,

$$dX_i(t) \stackrel{p-a.s.}{=} c(\bar{X}_N(t) - X_i(t)) dt + \gamma X_i(t) dW_i(t) \text{ and } d\langle X_i \rangle(t) \stackrel{p-a.s.}{=} \gamma^2 X_i^2(t) dt,$$

then

$$(2.6.4) \quad d \frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t) \stackrel{p-a.s.}{=} 2c \bar{X}_N^2(t) dt + (\gamma^2 - 2c) \frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t) dt + \frac{2\gamma}{N} \sum_{i=0}^{N-1} X_i^2(t) dW_i(t).$$

Therefore, taking the expectation on both sides of the equality, applying Fubini-Tonelli's theorem and lemma A.2 together with lemma 2.2, we have

$$(2.6.5) \quad \frac{d}{dt} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t) \right] = 2c \mathbb{E} \left[\bar{X}_N^2(t) \right] + (\gamma^2 - 2c) \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t) \right].$$

Equations (2.6.3) and (2.6.5) give us the following system of differential equations:

$$\begin{bmatrix} \frac{d}{dt} + 2c - \gamma^2 & -2c \\ \frac{\gamma^2}{N} & \frac{d}{dt} \end{bmatrix} \begin{bmatrix} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t) \right] \\ \mathbb{E} \left[\bar{X}_N^2(t) \right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

subject to the initial conditions $\mathbb{E} \left[\bar{X}_N^2(0) \right]$ and $\mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0) \right]$. Equations (2.6.1) and (2.6.2)

form the unique solution of this system. ■

If we assume the existence of

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\bar{X}_N^2(0) \right] \text{ and } \lim_{N \rightarrow \infty} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0) \right],$$

then¹

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t) \right] = \begin{cases} C(N^0) \exp \left[-|C(N^0)|t \right] + |C(N^0)| \exp \left[|C(N^0)| \frac{t}{N} \right] & \text{if } \gamma^2 - 2c < 0 \\ C(\sqrt{N}) \exp \left[-\gamma^2 \frac{t}{\sqrt{N}} \right] + |C(\sqrt{N})| \exp \left[\gamma^2 \frac{t}{\sqrt{N}} \right] & \text{if } \gamma^2 - 2c = 0 \\ C(N^0) \exp \left[-|C(N^0)| \frac{t}{N} \right] + |C(N^0)| \exp \left[|C(N^0)|t \right] & \text{if } \gamma^2 - 2c > 0 \end{cases}$$

and

$$\mathbb{E} \left[\bar{X}_N^2(t) \right] = \begin{cases} |C(N^0)| \exp \left[|C(N^0)| \frac{t}{N} \right] + C(N^0) \exp \left[-|C(N^0)|t \right] & \text{if } \gamma^2 - 2c < 0 \\ |C(N^0)| \exp \left[|C(N^0)| \frac{t}{\sqrt{N}} \right] + C(N^0) \exp \left[-|C(N^0)| \frac{t}{\sqrt{N}} \right] & \text{if } \gamma^2 - 2c = 0 \\ |C(N^0)| \exp \left[|C(N^0)|t \right] + C(N^0) \exp \left[-|C(N^0)| \frac{t}{N} \right] & \text{if } \gamma^2 - 2c > 0 \end{cases}$$

More precisely, if $\gamma^2 - 2c = 0$ then

$$\begin{aligned} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(t) \right] &= \left[\frac{\sqrt{N}}{2} \mathbb{E} \left[\bar{X}_N^2(0) \right] + \frac{1}{2} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0) \right] \right] \exp \left[\gamma^2 \frac{t}{\sqrt{N}} \right] \\ &+ \left[-\frac{\sqrt{N}}{2} \mathbb{E} \left[\bar{X}_N^2(0) \right] + \frac{1}{2} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0) \right] \right] \exp \left[-\gamma^2 \frac{t}{\sqrt{N}} \right] \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left[\bar{X}_N^2(t) \right] &= \left[\frac{1}{2} \mathbb{E} \left[\bar{X}_N^2(0) \right] + \frac{1}{2\sqrt{N}} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0) \right] \right] \exp \left[\gamma^2 \frac{t}{\sqrt{N}} \right] \\ &+ \left[\frac{1}{2} \mathbb{E} \left[\bar{X}_N^2(0) \right] - \frac{1}{2\sqrt{N}} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0) \right] \right] \exp \left[-\gamma^2 \frac{t}{\sqrt{N}} \right]. \end{aligned}$$

Our next goal is to see what happens to $\mathbb{E} \left[\bar{X}_N^2(N^p t) \right]$ when N goes to infinity for different values of p . If we assume that $\{X_i^N(0) : i \in \{0, 1, \dots, N-1\}, N \in \mathbb{N}\}$ are i.i.d., we have to consider three cases:

¹ A function $f(N) = O(N^{-p}) \Leftrightarrow \exists C$ such that $|f(N)| \leq CN^p$.

THE SUBCRITICAL CASE $\alpha = 2c - \gamma^2 > 0$

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\bar{X}_N^2(N^p t) \right] = \begin{cases} \mathbb{E} [X_0^2(0)] & \text{if } p < 1 \\ \infty & \text{if } p > 1 \end{cases}.$$

When $p = 1$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\bar{X}_N^2(Nt) \right] = \mathbb{E} [X_0^2(0)] \exp \left[\frac{2c\gamma^2 t}{2c - \gamma^2} \right].$$

THE SUPERCRITICAL CASE $\alpha = 2c - \gamma^2 < 0$

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\bar{X}_N^2(N^p t) \right] = \begin{cases} \mathbb{E} [X_0^2(0)] & \text{if } p \leq 0 \\ \infty & \text{if } p > 0 \end{cases}.$$

THE CRITICAL CASE $\alpha = 2c - \gamma^2 = 0$

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\bar{X}_N^2(N^p t) \right] = \begin{cases} \mathbb{E} [X_0^2(0)] & \text{if } p < \frac{1}{2} \\ \infty & \text{if } p > \frac{1}{2} \end{cases}.$$

When $p = \frac{1}{2}$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\bar{X}_N^2(\sqrt{N} t) \right] = \frac{1}{2} \mathbb{E} [X_0^2(0)] (\exp[2ct] + \exp[-2ct]).$$

LEMMA 2.7: (Third moments) If $2c - \gamma^2$, $5c - 4\gamma^2$ and $c - \gamma^2$ are different than zero¹ and

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^3 \right] < \infty$$

then

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(t))^3 \right] \\ &= \left(\frac{-6c^2}{(2c - \gamma^2)^2} \mathbb{E} [\bar{X}_N^3(0)] + \frac{3c}{(2c - \gamma^2)} \mathbb{E} \left[\bar{X}_N(0) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] + c(N^0) \right) \exp \left[\left(\gamma^2 - 2c + c \left(\frac{1}{N} \right) \right) t \right] \\ &+ \left(\frac{-3c}{2c - \gamma^2} \mathbb{E} \left[\bar{X}_N(0) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] + \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^3 \right] + c(N^0) \right) \exp \left[\left(3\gamma^2 - 3c + c \left(\frac{1}{N} \right) \right) t \right] \\ &+ \left(\frac{6c^2}{(2c - \gamma^2)^2} \mathbb{E} [\bar{X}_N^3(0)] + c(N^0) \right) \exp \left[\left(\frac{3\gamma^2 2c}{2c - \gamma^2} \frac{1}{N} + c \left(\frac{1}{N^2} \right) \right) t \right] \end{aligned}$$

¹ Note that if $c - \gamma^2 > 0$ then $5c - 4\gamma^2 > 0$, $(2c - \gamma^2)(3c - 3\gamma^2) > 0$ and $-18c\gamma^2(c - \gamma^2) < 0$.

Remember that the second moments blow up when $2c - \gamma^2 > 0$ while the third moments already explode whenever $c - \gamma^2 > 0$. This could be related to the work of S. A. Molchanov and others (see [CM] and [ZMRS]) in the study of intermittency. We will not explore the subject further in this thesis.

PROOF OF LEMMA 2.7: Exactly as we did when we compute the second moments, we obtain the following system of differential equations for the third moments:

$$\begin{bmatrix} \frac{d}{dt} & \frac{-3\gamma^2}{N} & 0 \\ -2c & \frac{d}{dt} + 2c - \gamma^2 & \frac{-2\gamma^2}{N} \\ 0 & -3c & \frac{d}{dt} + 3c - 3\gamma^2 \end{bmatrix} \begin{bmatrix} \mathbb{E}[\bar{X}_N^3(t)] \\ \mathbb{E}\left[\bar{X}_N(t) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(t))^2\right] \\ \mathbb{E}\left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(t))^3\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the system is homogeneous, we know that

$$(2.7.1) \quad \mathbb{E}[\bar{X}_N^3(t)] = \sum_{i=1}^3 k_i(N) \exp[r_i(N)t]$$

where $r_1(N), r_2(N), r_3(N)$ are the roots of

$$p(D) \equiv \det \begin{bmatrix} D & \frac{-3\gamma^2}{N} & 0 \\ -2c & D + 2c - \gamma^2 & \frac{-2\gamma^2}{N} \\ 0 & -3c & D + 3c - 3\gamma^2 \end{bmatrix}$$

and $k_1(N), k_2(N), k_3(N)$ are constants determined by the initial conditions. We will now determine those constants. From equation (1) we have

$$(2.7.2) \quad \mathbb{E}[\bar{X}_N^3(0)] = \sum_{i=1}^3 k_i(N).$$

The first line of our system tells us that

$$(2.7.3) \quad \mathbb{E}\left[\bar{X}_N(t) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(t))^2\right] = \frac{N}{3\gamma^2} \frac{d}{dt} \mathbb{E}[\bar{X}_N^3(t)] = \frac{N}{3\gamma^2} \sum_{i=1}^3 k_i(N) r_i(N) \exp[r_i(N)t]$$

so,

$$(2.7.4) \quad \mathbb{E}\left[\bar{X}_N(0) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2\right] = \frac{N}{3\gamma^2} \sum_{i=1}^3 k_i(N) r_i(N).$$

From the second line of the system, we obtain

$$(2.7.5) \quad \frac{2\gamma^2}{N} \mathbb{E}\left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(t))^3\right]$$

$$\begin{aligned}
&= -2c \mathbb{E} \left[\bar{X}_N^3(t) \right] + \frac{d}{dt} \mathbb{E} \left[\bar{X}_N(t) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(t))^2 \right] + (2c - \gamma^2) \mathbb{E} \left[\bar{X}_N(t) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(t))^2 \right] \\
&= -2c \sum_{i=1}^3 k_i(N) \exp[r_i(N)t] + \frac{N}{3\gamma^2} \sum_{i=1}^3 k_i(N) (r_i(N))^2 \exp[r_i(N)t] \\
&\quad + (2c - \gamma^2) \frac{N}{3\gamma^2} \sum_{i=1}^3 k_i(N) r_i(N) \exp[r_i(N)t]
\end{aligned}$$

Therefore,

$$(2.7.6) \quad \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^3 \right] = \sum_{i=1}^3 \frac{N}{2\gamma^2} \frac{N}{3\gamma^2} \left((r_i(N))^2 + (2c - \gamma^2) r_i(N) - 2c \frac{3\gamma^2}{N} \right) k_i(N)$$

Solving (2.7.2), (2.7.4) and (2.7.6) for $k_1(N)$, $k_2(N)$, $k_3(N)$, we obtain

$$\begin{aligned}
k_1(N) &= \frac{r_2(N)r_3(N)N^2 + 6c\gamma^2 N}{N^2(r_1(N) - r_2(N))(r_1(N) - r_3(N))} \mathbb{E} \left[\bar{X}_N^3(0) \right] \\
&\quad + \frac{-3\gamma^2 N (r_2(N) + r_3(N) + 2c - \gamma^2)}{N^2(r_1(N) - r_2(N))(r_1(N) - r_3(N))} \mathbb{E} \left[\bar{X}_N(0) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] \\
&\quad + \frac{6\gamma^4}{N^2(r_1(N) - r_2(N))(r_1(N) - r_3(N))} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^3 \right] \\
k_2(N) &= \frac{r_1(N)r_3(N)N^2 + 6c\gamma^2 N}{N^2(r_2(N) - r_1(N))(r_2(N) - r_3(N))} \mathbb{E} \left[\bar{X}_N^3(0) \right] \\
(2.7.7) \quad &\quad + \frac{-3\gamma^2 N (r_1(N) + r_3(N) + 2c - \gamma^2)}{N^2(r_2(N) - r_1(N))(r_2(N) - r_3(N))} \mathbb{E} \left[\bar{X}_N(0) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] \\
&\quad + \frac{6\gamma^4}{N^2(r_2(N) - r_1(N))(r_2(N) - r_3(N))} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^3 \right] \\
k_3(N) &= \frac{r_1(N)r_2(N)N^2 + 6c\gamma^2 N}{N^2(r_3(N) - r_1(N))(r_3(N) - r_2(N))} \mathbb{E} \left[\bar{X}_N^3(0) \right] \\
&\quad + \frac{-3\gamma^2 N (r_1(N) + r_2(N) + 2c - \gamma^2)}{N^2(r_3(N) - r_1(N))(r_3(N) - r_2(N))} \mathbb{E} \left[\bar{X}_N(0) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] \\
&\quad + \frac{6\gamma^4}{N^2(r_3(N) - r_1(N))(r_3(N) - r_2(N))} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^3 \right]
\end{aligned}$$

We will now determine the roots.

$$p(D) = D^3 + (5c - 4\gamma^2)D^2 + \left((2c - \gamma^2)(3c - 3\gamma^2) - \frac{12c\gamma^2}{N} \right) D - \frac{18c\gamma^2(c - \gamma^2)}{N}.$$

If ρ_1, ρ_2, ρ_3 are the roots of $p(x) \equiv x^3 + a_1 x^2 + a_2 x + a_3$ then $-a_1 = \rho_1 + \rho_2 + \rho_3$, $a_2 = \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_1 \rho_3$ and $-a_3 = \rho_1 \rho_2 \rho_3$. Therefore, it is possible to show that:

If $2c - \gamma^2$, $5c - 4\gamma^2$ and $c - \gamma^2$ are different than zero then it is possible to rename the roots such that

$$(2.7.8) \quad r_1(N) = \gamma^2 - 2c + \mathcal{O}\left(\frac{1}{N}\right), \quad r_2(N) = 3\gamma^2 - 3c + \mathcal{O}\left(\frac{1}{N}\right) \text{ and } r_3(N) = \frac{3\gamma^2 2c}{2c - \gamma^2} \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

Replacing (2.7.8) in (2.7.7),

$$k_1(N) = \left[\frac{3\gamma^2 2c}{(2c - \gamma^2)^2} \mathbb{E} \left[\bar{X}_N^3(0) \right] - \frac{3\gamma^2 (2c - \gamma^2)}{(2c - \gamma^2)^2} \mathbb{E} \left[\bar{X}_N(0) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] \right] \frac{1}{N} + \mathcal{O}\left(\frac{1}{N}\right)$$

$$k_2(N) = \frac{1}{N^2} \frac{-6c\gamma^4}{(c - 2\gamma^2)(2c - \gamma^2)(c - \gamma^2)} \mathbb{E} \left[\bar{X}_N(0) \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] \\ + \frac{1}{N^2} \frac{2(2c - \gamma^2)\gamma^4}{(c - 2\gamma^2)(2c - \gamma^2)(c - \gamma^2)} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^3 \right] + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$k_3(N) = \mathbb{E} \left[\bar{X}_N^3(0) \right] + \mathcal{O}\left(N^0\right)$$

We have all we need to complete the proof. Replace the constants and the roots in (2.7.5) to find the desired expression. ■

Time Rescaling

Behavior of $\{\bar{X}_N(Nt): t \in \mathbb{R}_+\}$ as N goes to Infinity in the Subcritical Case ($2c - \gamma^2 > 0$)

In this section, we will assume that the initial conditions are nonnegative square-integrable random variables. For any N , let P_N be the probability on $C(\mathbb{R}_+, \mathbb{R}_+)$ induced by the process $\{\bar{X}_N(Nt): t \in \mathbb{R}_+\}$.

THEOREM 3.1: Let $2c - \gamma^2 > 0$ and assume that the following assumptions on the initial conditions are satisfied: there is a $\delta > 0$ for which

$$\sup_N \max_{0 \leq i \leq N-1} E \left[\left(X_i^N(0) \right)^{2+\delta} \right] < \infty$$

and $\mathcal{L}(\bar{X}_N(0))$ converges weakly with μ . Then the sequence $\{P_N: N \in \mathbb{N}\}$ converges weakly (as N goes to infinity) to the unique solution of the martingale problem associated to $\mathcal{A}$ with initial distribution μ where

$$(\mathcal{A}f)(x) \equiv \frac{c\gamma^2}{2c - \gamma^2} x^2 f''(x), \quad f \in C^2(\mathbb{R}_+, \mathbb{R}).$$

SKETCH OF THE PROOF OF THEOREM 3.1:

CLAIM¹: The sequence $\{P_N: N \in \mathbb{N}\}$ of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$ is tight.

Let $\{P_{N'}\}$ be an arbitrary convergent subsequence of $\{P_N: N \in \mathbb{N}\}$ and P_∞ denotes its limit point. We would like to show that P_∞ is a solution of the martingale problem associated with $\mathcal{A}$. To do so, it suffices to prove that

$$\left\{ f(\xi_t) - \int_0^t (\mathcal{A}f)(\xi_s) ds : t \in \mathbb{R}_+ \right\}$$

is a P_∞ -martingale for $f(x) \equiv x$ and $f(x) \equiv x^2$ where $\{\xi_t: t \in \mathbb{R}_+\}$ denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}_+)$ (ref.: [KS], page 315, proposition 4.6).

CLAIM²: $\{\xi_t: t \in \mathbb{R}_+\}$ is a P_∞ -martingale.

CLAIM³: $\left\{ \xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c - \gamma^2} \xi_s^2 ds : t \in \mathbb{R}_+ \right\}$ is a P_∞ -martingale.

The last two claims imply that P_∞ is a solution of the martingale problem associated with $\mathcal{A}$. Since this martingale problem has a unique solution then any convergent subsequence has P_∞ as limit point and therefore, $\{P_N: N \in \mathbb{N}\}$ converges weakly to P_∞ as N goes to infinity. ■

¹ proved at proposition 3.4.

² proved at proposition 3.5.

³ proved at proposition 3.6.

To prove these claims, it is helpful to have some control on the moments of the processes involved. The next two lemmas give us the tools we need.

LEMMA 3.2: If $2c - \gamma^2 > 0$ then there is some constant C such that

$$\mathbb{E} \left[\bar{X}_N^2(Nt) \right] \leq C \mathbb{E} \left[\bar{X}_N^2(0) \right] \exp \left[\frac{2c\gamma^2}{2c - \gamma^2} t \right]$$

and

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nt))^2 \right] \leq C \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] \exp \left[\frac{2c\gamma^2}{2c - \gamma^2} t \right].$$

PROOF OF LEMMA 3.2:

The right hand sides are bounds for the expressions (2.6.1)

and (2.6.2). Note that

$$-\left(\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} + \alpha \right) \frac{t}{2} < 0 \text{ and } 0 < \left(\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} - \alpha \right) \frac{t}{2} \leq \frac{2c\gamma^2}{\alpha N} t.$$

Therefore

$$\begin{aligned} & C_1 \exp \left[-\left(\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} + \alpha \right) \frac{t}{2} \right] + C_2 \exp \left[\left(\sqrt{\alpha^2 + \frac{8c\gamma^2}{N}} - \alpha \right) \frac{t}{2} \right] \\ & \leq 2 \max\{0, C_1, C_2\} \exp \left[\frac{2c\gamma^2}{\alpha N} t \right]. \end{aligned}$$

In equation (2.6.2),

$$0 \leq \max\{C_1, C_2\} \leq \frac{2c}{\alpha} \mathbb{E} \left[\bar{X}_N^2(0) \right]$$

and in equation (2.6.1)

$$0 \leq \max\{C_1, C_2\} \leq \left(1 + \frac{2c}{\alpha} \right) \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right]$$

LEMMA 3.3: If $2c - \gamma^2 > 0$ then there exists a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i \leq N-1} \mathbb{E} \left[(X_i^N(0))^{2+\delta} \right] < \infty \text{ implies } \sup_N \left(\sup_{0 \leq u \leq Nt} \max_{0 \leq j \leq N-1} \mathbb{E} \left[(X_j^N(u))^{2+\delta} \right] \right) \leq C_t$$

for some constant C_t . Note that C_t depends on $\sup_N \max_{0 \leq i \leq N-1} \mathbb{E} \left[(X_i^N(0))^{2+\delta} \right] < \infty$.

COROLLARY: Under the assumptions of lemma 3.3,

$$\sup_N \mathbb{E} \left[(\bar{X}_N(Nt))^{2+\delta} \right] \leq \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nt))^{2+\delta} \right] < \infty$$

and $\{\bar{X}_N^2(Nt) : N \in \mathbb{N}\}$ and $\{(X_i^N(Nt))^2 : i \in \{0, 1, \dots, N-1\}, N \in \mathbb{N}\}$ are uniformly integrable.

PROOF OF LEMMA 3.3¹: For $i, j \in \{0, 1, \dots, N-1\}$ define

$$a_t(i, j) \equiv \exp[-ct] \delta_i(j) + \frac{1}{N} (1 - \exp[-ct])$$

where $\delta_i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$ and note that:

$$\sum_{i=0}^{N-1} a_t(i, j) = 1 = \sum_{j=0}^{N-1} a_t(i, j),$$

$$\frac{d}{dr} a_{t-r}(i, j) = c \exp[-c(t-r)] \left(\delta_i(j) - \frac{1}{N} \right) = c a_{t-r}(i, j) - \frac{c}{N}$$

and

$$(3.3.1) \quad \sum_{j=0}^{N-1} a_t^2(i, j) = \exp[-2ct] + \frac{1}{N} (1 - \exp[-2ct]) = a_{2t}(0, 0) \geq 0.$$

Itô's formula gives, for any fix $t > 0$,

$$\begin{aligned} & \sum_{j=0}^{N-1} a_{t-s}(i, j) X_j^N(s) - \sum_{j=0}^{N-1} a_t(i, j) X_j^N(0) \\ &= \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) dX_j^N(r) + \sum_{j=0}^{N-1} \int_0^s X_j^N(r) da_{t-r}(i, j) \\ &= \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) c (\bar{X}_N(r) - X_j^N(r)) dr + \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) \gamma X_j^N(r) dW_j(r) \\ & \quad + \sum_{j=0}^{N-1} \int_0^s X_j^N(r) \left(c a_{t-r}(i, j) - \frac{c}{N} \right) dr \\ &= \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) \gamma X_j^N(r) dW_j(r). \end{aligned}$$

If we let $s = t$ then the last equation becomes

$$(3.3.2) \quad X_i^N(t) - \sum_{j=0}^{N-1} a_t(i, j) X_j^N(0) \stackrel{P-a.s.}{=} \sum_{j=0}^{N-1} \int_0^t a_{t-r}(i, j) \gamma X_j^N(r) dW_j(r) \equiv M_t.$$

Note that in the following, the constant C_p may not be the same from line to line. Using (3.3.2), we find for any $u > 0$ and for any $p > 2$,

$$\mathbb{E} \left[\left(X_i^N(u) \right)^p \right] \leq \mathbb{E} \left[\left(\left| X_i^N(u) - \sum_{j=0}^{N-1} a_u(i, j) X_j^N(0) \right| + \left| \sum_{j=0}^{N-1} a_u(i, j) X_j^N(0) \right| \right)^p \right]$$

¹ The idea of the proof comes from a paper of J.T Cox, Andreas Greven and Tokuzo Shiga (see [CGS]).

$$\begin{aligned}
&\leq (p-1)\mathbb{E}\left[|M_u|^p\right] + C_p \mathbb{E}\left[\left|\sum_{j=0}^{N-1} a_u(i,j) X_j^N(0)\right|^p\right] \\
&\leq (p-1)^{p+1} \mathbb{E}\left[\left(\langle M \rangle_u\right)^{\frac{p}{2}}\right] + C_p^2 \\
&= (p-1)^{p+1} \mathbb{E}\left[\left(\sum_{j=0}^{N-1} \int_0^u a_{u-r}^2(i,j) \gamma^2(X_j^N(r))^2 dr\right)^{\frac{p}{2}}\right] + C_p \\
&= (p-1)^{p+1} \mathbb{E}\left[\left(\sum_{j=0}^{N-1} \int_0^u a_{u-r}^\beta(i,j) a_{u-r}^{2-\beta}(i,j) \gamma^2(X_j^N(r))^2 dr\right)^{\frac{p}{2}}\right] + C_p
\end{aligned}$$

From Hölder's inequality, using $\beta \equiv 2 - \frac{4}{p}$ and $\frac{1}{q} + \frac{2}{p} = 1$ ³, we find that the expression above is

$$\begin{aligned}
&\leq (p-1)^{p+1} \mathbb{E}\left[\left(\left|\sum_{j=0}^{N-1} \int_0^u a_{u-r}^{\beta q}(i,j) dr\right|^{\frac{1}{q}} \left|\sum_{j=0}^{N-1} \int_0^u a_{u-r}^{(2-\beta)\frac{2}{p}}(i,j) \gamma^p(X_j^N(r))^p dr\right|^{\frac{2}{p}}\right)^{\frac{p}{2}}\right] + C_p \\
&= (p-1)^{p+1} \left(\sum_{j=0}^{N-1} \int_0^u a_{u-r}^2(i,j) dr\right)^{\frac{p}{2}-1} \sum_{j=0}^{N-1} \int_0^u a_{u-r}^2(i,j) \gamma^p(X_j^N(r))^p dr + C_p
\end{aligned}$$

¹ (ref. lemma A.4) Since for $p > 2$, there is a constant C_p such that

$$(\alpha + \beta)^p \leq (p-1)\alpha^p + C_p \beta^p; \alpha, \beta > 0.$$

² Because

$$\mathbb{E}\left[\left|\sum_{j=0}^{N-1} a_i(i,j) X_j^N(0)\right|^p\right] \leq \mathbb{E}\left[\left|X_i^N(0) + \bar{X}_N(0)\right|^p\right] \leq (p-1)\mathbb{E}\left[\left(X_i^N(0)\right)^p\right] + C_p \mathbb{E}\left[\left(\bar{X}_N(0)\right)^p\right] < \infty$$

and the Burkholder's inequality: (ref. [Bur], 4.1) Let M be a continuous L_p -martingale. If $p > 2$ then for all $0 \leq u \leq t$,

$$\mathbb{E}\left[|M_u|^p\right] \leq (p-1)\mathbb{E}\left[\left(|M|_t\right)^{\frac{p}{2}}\right].$$

³ $q = \frac{p}{p-2}$, $\beta q = 2 = (2-\beta)\frac{p}{2}$ and $\frac{1}{q}\frac{p}{2} = \frac{p}{2} - 1$.

$$\begin{aligned}
&= (p-1)^{p+1} \left(\sum_{j=0}^{N-1} \int_0^u a_s^2(i, j) ds \right)^{\frac{p}{2}-1} \sum_{j=0}^{N-1} \int_0^u a_s^2(i, j) \gamma^p \mathbb{E} \left[\left(X_j^N(u-s) \right)^p \right] ds + C_p \\
&\leq (p-1)^{p+1} \left(\int_0^u a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \int_0^u a_{2s}(0,0) \gamma^p \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(X_j^N(u-s) \right)^p \right] ds + C_p^1
\end{aligned}$$

Therefore,

(3.3.3)

$$\begin{aligned}
&\sup_{0 \leq u \leq t} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(X_j^N(u) \right)^p \right] \\
&\leq (p-1)^{p+1} \left(\int_0^t a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \int_0^t a_{2s}(0,0) \gamma^p \sup_{0 \leq u \leq t-s} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(X_j^N(u) \right)^p \right] ds + C_p.
\end{aligned}$$

Define

$$f(t) \equiv \sup_{0 \leq u \leq t} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(X_j^N(u) \right)^p \right] \text{ and } F(\theta, T) \equiv \int_0^{NT} \exp[-\theta t] f(t) dt; \theta > 0, T > 0.$$

By integrating (3.3.3), we find

$$\begin{aligned}
F(\theta, T) &\leq (p-1)^{p+1} \int_0^{NT} \exp[-\theta t] \left(\int_0^t a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \left(\int_0^t a_{2s}(0,0) \gamma^p f(t-s) ds \right) dt + \frac{C_p}{\theta} \\
&\leq (p-1)^{p+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \int_0^{NT} \exp[-\theta t] \left(\int_0^t a_{2s}(0,0) \gamma^p f(t-s) ds \right) dt + \frac{C_2}{\theta} \\
&= (p-1)^{p+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \int_0^{NT} \left(\int_0^t \exp[-\theta s] a_{2s}(0,0) \gamma^p \exp[-\theta(t-s)] f(t-s) ds \right) dt + \frac{C_p}{\theta} \\
&= (p-1)^{p+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \int_0^{NT} \exp[-\theta s] a_{2s}(0,0) \left(\int_s^{NT} \gamma^p \exp[-\theta(t-s)] f(t-s) dt \right) ds + \frac{C_p}{\theta} \\
&= (p-1)^{p+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \gamma^p \int_0^{NT} \exp[-\theta s] a_{2s}(0,0) F\left(\theta, T - \frac{s}{N}\right) ds + \frac{C_p}{\theta} \\
&\leq (p-1)^{p+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \gamma^p F(\theta, T) \int_0^\infty \exp[-\theta s] a_{2s}(0,0) ds + \frac{C_p}{\theta}
\end{aligned}$$

¹ From (3.3.1).

Hence,

$$(3.3.4) \quad \frac{\theta}{N} F\left(\frac{\theta}{N}, T\right) \left(1 - (\rho - 1)^{\rho+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{\rho}{2}-1} \gamma^\rho \int_0^\infty \exp\left[-\frac{\theta}{N}s\right] a_{2s}(0,0) ds \right) \leq C_\rho.$$

We now claim that for large N ,

$$(3.3.5) \quad (\rho - 1)^{\rho+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{\rho}{2}-1} \gamma^\rho \int_0^\infty \exp\left[-\frac{\theta}{N}s\right] a_{2s}(0,0) ds < 1.$$

Indeed, if we first note that

$$\int_0^\infty \exp\left[-\frac{\theta}{N}s\right] a_{2s}(0,0) ds \geq \int_0^{NT} \exp\left[-\frac{\theta}{N}s\right] a_{2s}(0,0) ds \geq \exp[-\theta T] \int_0^{NT} a_{2s}(0,0) ds$$

then

$$\begin{aligned} & (\rho - 1)^{\rho+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{\rho}{2}-1} \gamma^\rho \int_0^\infty \exp\left[-\frac{\theta}{N}s\right] a_{2s}(0,0) ds \\ & \leq (\rho - 1)^{\rho+1} \exp\left[\theta T \left(\frac{\rho}{2} - 1\right)\right] \left(\int_0^\infty \exp\left[-\frac{\theta}{N}s\right] a_{2s}(0,0) ds \right)^{\frac{\rho}{2}} \gamma^\rho \\ & = (\rho - 1)^{\rho+1} \exp\left[\theta T \left(\frac{\rho}{2} - 1\right)\right] \left(\left(1 - \frac{1}{N}\right) \frac{1}{2c + \frac{\theta}{N}} + \frac{1}{\theta} \right)^{\frac{\rho}{2}} \gamma^\rho \\ (3.3.6) \quad & \xrightarrow{N \rightarrow \infty} (\rho - 1)^{\rho+1} \exp\left[\theta T \left(\frac{\rho}{2} - 1\right)\right] \left(\frac{\gamma^2}{2c} + \frac{\gamma^2}{\theta} \right)^{\frac{\rho}{2}}. \end{aligned}$$

Because

$$\alpha = 2c - \gamma^2 > 0 \Rightarrow \frac{\gamma^2}{2c} < 1,$$

$$\theta > \frac{2c\gamma^2}{2c - \gamma^2} \Rightarrow \frac{\gamma^2}{2c} + \frac{\gamma^2}{\theta} < 1 \Rightarrow \left(\frac{\gamma^2}{2c} + \frac{\gamma^2}{\theta} \right)^{\frac{\rho}{2}} < \frac{\gamma^2}{2c} + \frac{\gamma^2}{\theta} < 1,$$

then it is possible to make

$$(\rho - 1)^{\rho+1} \exp\left[\theta T \left(\frac{\rho}{2} - 1\right)\right] \left(\frac{\gamma^2}{2c} + \frac{\gamma^2}{\theta} \right)^{\frac{\rho}{2}} < 1$$

by choosing p close enough to 2. Equation (3.3.5) is therefore satisfied. Going back to equation (3.3.4), we can find, using (3.3.5) and (3.3.6), a constant K_p and a $N_0 \in \mathbb{N}$ for which

$$\sup_{N \geq N_0} \frac{\theta}{N} F\left(\frac{\theta}{N}, T\right) \leq \sup_{N \geq N_0} C_p \left(1 - (p-1)^{p+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \gamma^p \int_0^\infty \exp\left[-\frac{\theta}{N}s\right] a_{2s}(0,0) ds \right)^{-1} \leq K_p.$$

Since

$$\sup_{N \geq N_0} \frac{\theta}{N} F\left(\frac{\theta}{N}, T\right) = \sup_{N \geq N_0} \frac{\theta}{N} \int_0^{NT} \exp\left[-\frac{\theta}{N}t\right] f(t) dt = \sup_{N \geq N_0} \theta \int_0^T \exp[-\theta s] f(Ns) ds,$$

then

$$\begin{aligned} \frac{K_p}{\theta} &\geq \sup_{N \geq N_0} \int_0^T \exp[-\theta s] f(Ns) ds \\ &= \sup_{N \geq N_0} \int_0^T \exp[-\theta s] \sup_{0 \leq u \leq Ns} \max_{0 \leq j \leq N-1} \mathbb{E} \left[(X_j^N(u))^p \right] ds \\ &\geq \exp[-\theta T] \sup_{N \geq N_0} \int_0^T \sup_{0 \leq u \leq Ns} \max_{0 \leq j \leq N-1} \mathbb{E} \left[(X_j^N(u))^p \right] ds \\ &\geq \exp[-\theta T] \sup_{N \geq N_0} \left(\int_{\frac{T}{2}}^T \sup_{0 \leq u \leq Ns} \max_{0 \leq j \leq N-1} \mathbb{E} \left[(X_j^N(u))^p \right] ds \right) \\ &\geq \exp[-\theta T] \sup_{N \geq N_0} \left(\frac{T}{2} \sup_{0 \leq u \leq N \frac{T}{2}} \max_{0 \leq j \leq N-1} \mathbb{E} \left[(X_j^N(u))^p \right] \right). \end{aligned}$$

Therefore,

$$\sup_{N \geq N_0} \left(\sup_{0 \leq u \leq N \frac{T}{2}} \max_{0 \leq j \leq N-1} \mathbb{E} \left[(X_j^N(u))^p \right] \right) \leq \frac{2 K_p}{T \theta} \exp[\theta T].$$

PROPOSITION 3.4:

If $2c - \gamma^2 > 0$ and $\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^2(0) \right] < \infty$ then the sequence $\{P_N : N \in \mathbb{N}\}$ of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$ is tight.

PROOF OF PROPOSITION 3.4:

The proof is based on the Aldous' theorem and the Rebolledo's theorem (ref.: [JM], theorem p.34, Aldous' theorem, p.35 and Rebolledo's theorem, p.40). We state here the result in our particular context:

Let $\{M_n : n \in \mathbb{N}\}$ be a sequence of real-valued processes which are right continuous locally square integrable martingales defined on their own probability spaces $(\Omega_n, \{\mathcal{F}_t^n : t \in \mathbb{R}_+\}, \mathbb{P}_n)$. Let $\langle M_n \rangle$ be the associated Meyer increasing process and let P_n denotes the law of $\{M_n(t) : t \in \mathbb{R}_+\}$. Then $\{P_n : n \in \mathbb{N}\}$ is tight if and only if the two following conditions hold:

[T₁] For every t in some dense subset of $\mathbb{R}_+$, the laws of the random variables $\{M_n(t) : n \in \mathbb{N}\}$ form a tight sequence of laws in $\mathbb{R}$.

[T₂] For every $K > 0$, $\eta > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ with the property that whatever be the family of stopping times $\{\tau_n : n \in \mathbb{N}\}$ (τ_n being a $\{\mathcal{F}_t^n : t \in \mathbb{R}_+\}$ -stopping time on Ω_n) with $\tau_n \leq K$

$$\sup_{n \geq n_0} \sup_{\theta \leq \delta} \mathbb{P}_n [|\langle M_n \rangle(\tau_n + \theta) - \langle M_n \rangle(\tau_n)| \geq \eta] \leq \varepsilon.$$

VERIFICATION OF [T₁]: It suffices to show that for a fixed but arbitrary t in $\mathbb{R}_+$,

$$\forall \varepsilon > 0, \exists K(\varepsilon) \text{ such that } \inf_N \mathbb{P} [|\bar{X}_N(Nt)| \leq K(\varepsilon)] > 1 - \varepsilon.$$

But from Markov's inequality,

$$\mathbb{P} \left[|\bar{X}_N(Nt)| > \frac{1}{\varepsilon} \sup_N \mathbb{E} [\bar{X}_N(0)] \right] \leq \varepsilon \frac{\mathbb{E} [\bar{X}_N(0)]}{\sup_N \mathbb{E} [\bar{X}_N(0)]} \leq \varepsilon.$$

VERIFICATION OF [T₂]: From Markov's inequality,

$$\begin{aligned} & \sup_{N \geq N_0} \sup_{\theta \leq \delta} \mathbb{P} [|\langle \bar{X}_N \rangle(N\tau_N) - \langle \bar{X}_N \rangle(N(\tau_N + \theta))| \geq \eta] \\ & \leq \sup_{N \geq N_0} \sup_{\theta \leq \delta} \frac{1}{\eta} \mathbb{E} [|\langle \bar{X}_N \rangle(N\tau_N) - \langle \bar{X}_N \rangle(N(\tau_N + \theta))|] \\ & = \sup_{N \geq N_0} \sup_{\theta \leq \delta} \frac{1}{\eta} \mathbb{E} \left[\frac{\gamma^2}{N} \int_{N\tau_N}^{N(\tau_N + \theta)} \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(s))^2 ds \right] \\ & = \sup_{N \geq N_0} \sup_{\theta \leq \delta} \frac{\gamma^2}{\eta} \mathbb{E} \left[\int_{\tau_N}^{\tau_N + \theta} \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Ns))^2 ds \right] \\ & = \sup_{N \geq N_0} \sup_{\theta \leq \delta} \frac{\gamma^2}{\eta} \mathbb{E} \left[\int_0^\theta \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Ns))^2 ds \middle| \mathcal{X}^N(0) = \mathcal{X}^N(\tau_N) \right] \\ & \leq \sup_{N \geq N_0} \frac{\gamma^2}{\eta} \int_0^\delta \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Ns))^2 \middle| \mathcal{X}^N(0) = \mathcal{X}^N(\tau_N) \right] ds \end{aligned}$$

¹ From the strong Markov's property.

$$\begin{aligned} &\leq \sup_{N \geq N_0} \frac{\gamma^2}{\eta} \int_0^\delta C \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(\tau_N))^2 \right] \exp \left[\frac{2c\gamma^2}{2c-\gamma^2} s \right] ds \\ &\leq \sup_{N \geq N_0} \frac{C\gamma^2}{\eta} \delta \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(\tau_N))^2 \right] \exp \left[\frac{2c\gamma^2}{2c-\gamma^2} \delta \right] \end{aligned}$$

and we can choose δ and N_0 to make the last expression as small as we want provided

$$\sup_{N \geq N_0} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(\tau_N))^2 \right] < \infty.$$

Now, since τ_N is bounded by K ,

$$\begin{aligned} \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(\tau_N))^2 &\stackrel{P-a.s.}{=} \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 + 2c \int_0^{\tau_N} \bar{X}_N^2(s) ds \\ &\quad + (\gamma^2 - 2c) \int_0^{\tau_N} \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(s))^2 ds + \frac{2\gamma}{N} \sum_{i=0}^{N-1} \int_0^{\tau_N} (X_i^N(s))^2 dW_i(s) \\ (3.4.1) \quad &\stackrel{P-a.s.}{\leq} \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 + 2c \int_0^K \bar{X}_N^2(s) ds + \frac{2\gamma}{N} \sum_{i=0}^{N-1} \int_0^{\tau_N \wedge K} (X_i^N(s))^2 dW_i(s) \end{aligned}$$

Therefore, taking expectation in (3.4.1),

$$\begin{aligned} \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(\tau_N))^2 \right] &\leq \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] + \sup_N 2c \int_0^K \mathbb{E} [\bar{X}_N^2(s)] ds \\ &\leq \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] + \sup_N 2c K C \mathbb{E} [\bar{X}_N^2(0)] \exp \left[\frac{2c\gamma^2}{2c-\gamma^2} \frac{K}{N} \right]^5 \end{aligned}$$

and the last expression is finite provided that $\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] < \infty$. ■

PROPOSITION 3.5: If $2c - \gamma^2 > 0$ and $\sup_N \mathbb{E} [\bar{X}_N^2(0)] < \infty$ then $\{\xi_t : t \in \mathbb{R}_+\}$ is a P_∞ -martingale.

PROOF OF PROPOSITION 3.5: It suffices to show that for any real numbers s and t with $0 \leq s \leq t$ and for any $\sigma\{\xi_r : 0 \leq r \leq s\}$ -measurable $\Phi \in C_b(C(\mathbb{R}_+, \mathbb{R}), \mathbb{R})$,

$$E_\infty[\Phi \xi_t] = E_\infty[\Phi \xi_s].$$

Since $\{\bar{X}_N(Nt) : t \geq 0\}$ is a continuous nonnegative $(\{\bar{X}_{Nt} : t \geq 0\}, \mathbb{P})$ -martingale, $\{\xi_t : t \in \mathbb{R}_+\}$ is a P_N -martingale and

$$E_N[\Phi \xi_t] = E_N[\Phi \xi_s].$$

1 From lemma 3.2.

2 This is the integral form of equation (2.6.4).

5 From lemma 3.2.

We would like to use the generalization of the continuous mapping theorem (see lemma A.4) which will give

$$E_{\infty}[\Phi \xi_t] = \lim_{N' \rightarrow \infty} E_{N'}[\Phi \xi_t] = \lim_{N' \rightarrow \infty} E_{N'}[\Phi \xi_s] = E_{\infty}[\Phi \xi_s].$$

Since $P_{N'}$ converges weakly to P_{∞} and $\Phi \xi_t$ is continuous (as a function on $C(\mathbb{R}_+, \mathbb{R})$), we only need to check that there is a δ such that $\sup_N E_N \left[|\Phi \xi_t|^{1+\delta} \right] < \infty$. Taking $\delta = 1$, we have

$$\sup_N E_N \left[\Phi^2 \xi_t^2 \right] \leq \|\Phi^2\|_{\infty} \sup_N E_N \left[\xi_t^2 \right] = \|\Phi^2\|_{\infty} \sup_N E \left[\bar{X}_N^2(Nt) \right] < \infty$$

where the last inequality comes from lemma 3.2. ■

PROPOSITION 3.6: If $2c - \gamma^2 > 0$ and if there is a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i \leq N-1} E \left[\left(X_i^N(0) \right)^{2+\delta} \right] < \infty$$

then

$$\left\{ \xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c - \gamma^2} \xi_s^2 ds : t \in \mathbb{R}_+ \right\} \text{ is a } P_{\infty}\text{-martingale}$$

provided

$$\lim_{N \rightarrow \infty} E \left[\left| \int_s^t \frac{1}{N} \sum_{i=0}^{N-1} \left(X_i^N(Nr) \right)^2 dr - \int_s^t \frac{2c}{2c - \gamma^2} \bar{X}_N^2(Nr) dr \right| \right] = 0.$$

This last condition is verified when some hypothesis on the initial conditions are respected. It is the object of propositions 3.10 and 3.17.

PROOF OF PROPOSITION 3.6: Again, it suffices to show that for any real numbers s and t with $0 \leq s \leq t$ and for any $\sigma\{\xi_r : 0 \leq r \leq s\}$ -measurable $\Phi \in C_b(C(\mathbb{R}_+, \mathbb{R}), \mathbb{R})$,

$$E_{\infty} \left[\Phi \left(\xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c - \gamma^2} \xi_r^2 dr \right) \right] = E_{\infty} \left[\Phi \left(\xi_s^2 - \int_0^s \frac{2c\gamma^2}{2c - \gamma^2} \xi_r^2 dr \right) \right].$$

From a property of the Meyer increasing process¹,

$$\left\{ \bar{X}_N^2(Nt) - \langle \bar{X}_N \rangle(Nt) : t \geq 0 \right\} = \left\{ \bar{X}_N^2(Nt) - \gamma^2 \int_0^t \frac{1}{N} \sum_{i=0}^{N-1} \left(X_i^N(Nr) \right)^2 dr : t \geq 0 \right\}$$

is a $\mathbb{P}$ -martingale for any N . Therefore,

$$E_N \left[\Phi \left(\xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c - \gamma^2} \xi_r^2 dr \right) \right] = E_N \left[\Phi \left(\xi_s^2 - \int_0^s \frac{2c\gamma^2}{2c - \gamma^2} \xi_r^2 dr \right) \right] + \epsilon(N^0)^2$$

¹ If M is a continuous square-integrable martingale then $M^2 - \langle M \rangle$ is also a martingale. (ref.: [IW], page 53, proposition 2.1).

² A function $f(N) = \epsilon(N^p) \Leftrightarrow f(N)N^{-p} \xrightarrow{N \rightarrow \infty} 0$.

since, as it will be shown in proposition 3.17,

$$(3.6.1) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_s^t \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nr))^2 dr - \int_s^t \frac{2c}{2c-\gamma^2} \bar{X}_N^2(Nr) dr \right| \right] = 0.$$

We would like to use the generalization of the continuous mapping theorem to get

$$\begin{aligned} E_\infty \left[\Phi \left(\xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr \right) \right] &= \lim_{N \rightarrow \infty} E_N \left[\Phi \left(\xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr \right) \right] \\ &= \lim_{N \rightarrow \infty} E_N \left[\Phi \left(\xi_s^2 - \int_0^s \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr \right) \right] + c(N^0) = E_\infty \left[\Phi \left(\xi_s^2 - \int_0^s \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr \right) \right] \end{aligned}$$

To do so, we need the following result: there is a $\delta > 0$ for which

$$\begin{aligned} \sup_N E_N \left[|\Phi|^{1+\delta} \left| \xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr \right|^{1+\delta} \right] &\leq \|\Phi\|_\infty^{1+\delta} \sup_N E_N \left[\left| \xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr \right|^{1+\delta} \right] \\ &= \|\Phi\|_\infty^{1+\delta} \sup_N \mathbb{E} \left[\left| \bar{X}_N^2(Nt) - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \bar{X}_N^2(Nr) dr \right|^{1+\delta} \right] \\ &\leq \|\Phi\|_\infty^{1+\delta} \left(1 + \frac{2c\gamma^2}{2c-\gamma^2} t \right)^{1+\delta} \sup_N \mathbb{E} \left[\left(\sup_{0 \leq r \leq t} \bar{X}_N^2(Nr) \right)^{1+\delta} \right] \\ &\leq \|\Phi\|_\infty^{1+\delta} \left(1 + \frac{2c\gamma^2}{2c-\gamma^2} t \right)^{1+\delta} \left(\frac{1+\delta}{\delta} \right)^{1+\delta} \sup_N \sup_{0 \leq r \leq t} \mathbb{E} \left[\left(\bar{X}_N^2(Nr) \right)^{1+\delta} \right] < \infty \end{aligned}$$

where the last inequality come from the corollary of lemma 3.3. ■

Let $P_t^\theta(x, dy)$ and ν_θ be respectively the transition probability and the equilibrium³ of the one-dimensional diffusion process described by the stochastic differential equation

$$(3.7.1) \quad dX(t) = c(\theta - X(t)) dt + \gamma X(t) dW(t), \quad \theta \geq 0.$$

In the following, X_θ and Y_θ denote two strong solutions of this stochastic differential equation with respective initial conditions $X_\theta(0) \equiv x$ and $\mathcal{L}(Y_\theta(0)) = \nu_\theta$.

PROPOSITION 3.7: Let $\theta \equiv \mathbb{E}[X(0)]$. The stationary density of (3.7.1) is

$$\psi_\theta(x) = \begin{cases} \frac{(\lambda\theta)^{\lambda+1}}{\Gamma(\lambda+1)} x^{-(\lambda+2)} \exp\left[-\frac{\lambda\theta}{x}\right] & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}, \text{ where } \lambda \equiv \frac{2c}{\gamma^2}$$

¹ From Doob's inequality.

² When there is a risk of confusion with the parameters of the system, we write $P_t^{\theta, c, \gamma}(x, dy)$ and $\nu_{\theta, c, \gamma}$.

³ See proposition 3.7.

PROOF OF PROPOSITION 3.7: (ref.: [KT], pages 220-221).

If there exist a stationary density ψ_θ of X then it has to be zero on $(-\infty, 0)$ and must satisfy

$$\psi_\theta(y) = \frac{C_1 S(y) + C_2}{s(y) \sigma^2(y)}, \quad \forall y \geq 0$$

where

$$\mu(y) \equiv \lim_{h \downarrow 0} \frac{1}{h} E[X(h) - X(0) | X(0) = y] = c(\theta - y), \quad \forall y \geq 0,$$

$$\sigma^2(y) \equiv \lim_{h \downarrow 0} \frac{1}{h} E[(X(h) - X(0))^2 | X(0) = y] = \gamma^2 y^2, \quad \forall y \geq 0,$$

$$s(y) \equiv \exp\left[-\int^y \frac{2\mu(\xi)}{\sigma^2(\xi)} d\xi\right] = y^\lambda \exp\left[\frac{\lambda\theta}{y}\right], \quad \forall y \geq 0,$$

$$S(y) \equiv \int^y s(z) dz$$

and C_1, C_2 are chosen such that $\psi_\theta(y) \geq 0, \forall y > 0$ and $\int_0^\infty \psi_\theta(y) dy = 1$.

First, note that C_1 must be zero since

$$S(y) = \int^y z^\lambda \exp\left[\frac{\lambda\theta}{z}\right] dz \geq \int^y z^\lambda dz = \frac{y^{\lambda+1}}{\lambda+1}$$

implies that

$$\int_0^\infty \frac{S(y)}{s(y) \sigma^2(y)} dy = \frac{1}{\gamma^2} \int_0^\infty y^{-(\lambda+2)} \exp\left[-\frac{\lambda\theta}{y}\right] S(y) dy \geq \frac{1}{\gamma^2} \frac{1}{\lambda+1} \int_0^\infty \frac{1}{y} \exp\left[-\frac{\lambda\theta}{y}\right] dy.$$

If $z \equiv \frac{\lambda\theta}{y}$ then the last expression is equivalent to

$$\frac{1}{\gamma^2} \frac{1}{\lambda+1} \int_0^\infty \frac{1}{z} \exp[-z] dz = \frac{1}{\gamma^2} \frac{1}{\lambda+1} \Gamma(0) = \infty.$$

The next step will be to determine C_2 .

$$\begin{aligned} I &\equiv \int_0^\infty \frac{1}{s(y) \sigma^2(y)} dy = \frac{1}{\gamma^2} \int_0^\infty y^{-(\lambda+2)} \exp\left[-\frac{\lambda\theta}{y}\right] dy \\ &= \frac{1}{\gamma^2} \frac{1}{(\lambda\theta)^{\lambda+1}} \int_0^\infty z^\lambda \exp[-z] dz = \frac{1}{\gamma^2} \frac{1}{(\lambda\theta)^{\lambda+1}} \Gamma(\lambda+1). \end{aligned}$$

Therefore,

$$C_2 = \frac{\gamma^2 (\lambda\theta)^{\lambda+1}}{\Gamma(\lambda+1)}.$$

Hence,

$$\psi_\theta(y) = \frac{C_2}{s(y) \sigma^2(y)} = \frac{(\lambda\theta)^{\lambda+1}}{\Gamma(\lambda+1)} y^{-(\lambda+2)} \exp\left[-\frac{\lambda\theta}{y}\right], \quad \forall y \geq 0$$

COROLLARY 3.8:

$$\int_{\mathbf{R}} y v_{\theta}(dy) = \theta \text{ and } \int_{\mathbf{R}} y^2 v_{\theta}(dy) = \begin{cases} \frac{2c}{2c - \gamma^2} \theta^2 & \text{if } \gamma^2 - 2c < 0 \\ \infty & \text{if } \gamma^2 - 2c \geq 0 \end{cases}.$$

PROOF OF COROLLARY 3.8: If $\lambda - p + 1 > 0$ then

$$\begin{aligned} \int_{\mathbf{R}} y^p v_{\theta}(dy) &= \int_0^{\infty} y^p \frac{(\lambda\theta)^{\lambda+1}}{\Gamma(\lambda+1)} y^{-(\lambda+2)} \exp\left[-\frac{\lambda\theta}{y}\right] dy \\ &= \int_0^{\infty} \frac{(\lambda\theta)^{\lambda+1}}{\Gamma(\lambda+1)} \left(\frac{\lambda\theta}{x}\right)^{-(\lambda+2)+p} \exp[-x] \frac{\lambda\theta}{x^2} dx \\ &= \frac{(\lambda\theta)^p}{\Gamma(\lambda+1)} \int_0^{\infty} x^{\lambda-p} \exp[-x] dx = \frac{(\lambda\theta)^p}{\Gamma(\lambda+1)} \Gamma(\lambda - p + 1). \end{aligned}$$

When $p = 1$,
$$\int_{\mathbf{R}} y^1 v_{\theta}(dy) = \frac{\lambda\theta}{\Gamma(\lambda+1)} \Gamma(\lambda) = \frac{\lambda\theta}{\lambda\Gamma(\lambda)} \Gamma(\lambda) = \theta.$$

When $p = 2$, then $\lambda - p + 1 = \lambda - 1 > 0 \Leftrightarrow \gamma^2 - 2c < 0$ and in this case,

$$\int_{\mathbf{R}} y^2 v_{\theta}(dy) = \frac{(\lambda\theta)^2}{\Gamma(\lambda+1)} \Gamma(\lambda-1) = \frac{(\lambda\theta)^2}{\lambda(\lambda-1)\Gamma(\lambda-1)} \Gamma(\lambda-1) = \frac{2c}{2c - \gamma^2} \theta^2. \quad \blacksquare$$

COROLLARY 3.9: If $2c - \gamma^2 > 0$ then

$$\int_{\alpha}^{\infty} y^2 v_{\theta}(dy) \leq \frac{\lambda^{\lambda+1}}{\Gamma(\lambda+1)} \frac{1}{\lambda-1} \alpha^{1-\lambda} \theta^{\lambda+1}.$$

PROOF OF COROLLARY 3.9:

$$\begin{aligned} \int_{\alpha}^{\infty} y^2 v_{\theta}(dy) &= \int_{\alpha}^{\infty} y^2 \frac{(\lambda\theta)^{\lambda+1}}{\Gamma(\lambda+1)} y^{-(\lambda+2)} \exp\left[-\frac{\lambda\theta}{y}\right] dy = \int_0^{\frac{\lambda\theta}{\alpha}} \frac{(\lambda\theta)^2}{\Gamma(\lambda+1)} x^{\lambda-2} \exp[-x] dx \\ &\leq \int_0^{\frac{\lambda\theta}{\alpha}} \frac{(\lambda\theta)^2}{\Gamma(\lambda+1)} x^{\lambda-2} dx = \frac{(\lambda\theta)^{\lambda+1}}{\Gamma(\lambda+1)} \frac{\alpha^{1-\lambda}}{\lambda-1} \quad \blacksquare \end{aligned}$$

The purpose of proposition 3.10 will be clarified in proposition 3.17. In brief, the idea is that we will construct a sequence of approximations of $\varphi(x) = x^2$ and show that the property of proposition 3.10 is preserved in the limit. Note that if we replace $\varphi(x) = x^2$ in the equation of

¹ Let $x \equiv \frac{\lambda\theta}{y}$.

the following proposition, we obtain an equation similar to (3.6.1) which is what we want to verify.

PROPOSITION 3.10¹: Assume that $2c - \gamma^2 > 0$ and

$$\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] < \infty.$$

For any positive real numbers t_1 and t_2 such that $t_1 \leq t_2$ and for any bounded continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$\lim_{N \rightarrow \infty} \sup_{t_1 \leq t \leq t_2} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) - \int_{\mathbb{R}} \varphi(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] = 0.$$

PROOF OF PROPOSITION 3.10: Let $\{\alpha_N: N \in \mathbb{N}\}$ be a positive increasing sequence tending to infinity ($\alpha_N \leq N$). For any fixed $t \geq 0$, we introduce a N -dimensional modified process, denoted $X^{N,t}$ as follows:

If $0 \leq s \leq (N - \alpha_N)t$ then

$$X_i^{N,t}(s) \equiv X_i^N(s)$$

and if $s > (N - \alpha_N)t$ then

$$X_i^{N,t}(s) \equiv X_i^N((N - \alpha_N)t) + c \int_{(N - \alpha_N)t}^s (\bar{X}_N((N - \alpha_N)t) - X_i^{N,t}(r)) dr + \gamma \int_{(N - \alpha_N)t}^s X_i^{N,t}(r) dW_i(r).$$

To prove the lemma, it suffices to show the four following inequalities:

there exists a constant C and, for any positive real number T , there is a constant C_T such that

$$(3.10.1) \quad \sup_{0 < t \leq T} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^{N,t}(Nt)) \right| \right] \leq C_T \|\varphi'\|_{\infty} \sqrt{\frac{\alpha_N^3}{N}}$$

$$(3.10.2) \quad \sup_{t > 0} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^{N,t}(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \middle| \tilde{\nu}_{(N - \alpha_N)t} \right] \right| \right] \leq 2 \frac{\|\varphi\|_{\infty}^2}{\sqrt{N}}$$

¹ This proof is inspired by Tokuzo Shiga's proof (ref. [Shi]).

² Since $\{X_i^{N,t}(Nt): i \in \{0, 1, \dots, N-1\}\}$ are conditionally independent with respect to $\mathbb{P}[\bullet | \tilde{\nu}_{(N - \alpha_N)t}]$,

$$\begin{aligned} & \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^{N,t}(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \middle| \tilde{\nu}_{(N - \alpha_N)t} \right] \right|^2 \right] \\ &= \frac{1}{N^2} \sum_{i=0}^{N-1} \mathbb{E} \left[\left(\varphi(X_i^{N,t}(Nt)) - \mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \middle| \tilde{\nu}_{(N - \alpha_N)t} \right] \right)^2 \right] \leq \frac{4 \|\varphi\|_{\infty}^2}{N}. \end{aligned}$$

Therefore, Jensen's inequality gives

$$(3.10.3) \quad \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \right] \tilde{\gamma}_{(N-\alpha_N)t} - \int_{\mathbb{R}_+} \varphi(y) \nu_{\bar{X}_N((N-\alpha_N)t)}(dy) \right| \right] \leq C \|\varphi'\|_{\infty} \exp[-c\alpha_N t], \forall t > 0$$

$$(3.10.4) \quad \sup_{0 < t \leq T} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) \nu_{\bar{X}_N((N-\alpha_N)t)}(dy) - \int_{\mathbb{R}_+} \varphi(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] \leq C_T \|\varphi'\|_{\infty} \sqrt{\frac{\alpha_N}{N}}$$

PROOF OF THE INEQUALITY (3.10.1)

LEMMA 3.11: Assume that $2c - \gamma^2 > 0$ and $\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] < \infty$. For any positive real number T , there is a constant C_T such that for any real numbers s_1 and s_2 with $0 \leq s_1 \leq s_2 \leq NT$,

$$\mathbb{E} \left[|\bar{X}_N(s_2) - \bar{X}_N(s_1)| \right] \leq C_T \sqrt{\frac{s_2 - s_1}{N}}.$$

PROOF OF LEMMA 3.11: In the following, the constant C may not be the same at each step.

$$\begin{aligned} \mathbb{E} \left[(\bar{X}_N(s_2) - \bar{X}_N(s_1))^2 \right] &= \mathbb{E} \left[\bar{X}_N^2(s_2) - \bar{X}_N^2(s_1) \right] \\ &= \mathbb{E} \left[\langle \bar{X}_N \rangle(s_2) - \langle \bar{X}_N \rangle(s_1) \right] = \frac{\gamma^2}{N} \int_{s_1}^{s_2} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(r))^2 \right] dr \\ &\leq \frac{\gamma^2}{N} \int_{s_1}^{s_2} C \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] \exp \left[\frac{2c\gamma^2}{2c - \gamma^2} \frac{r}{N} \right] dr \\ &\leq \frac{C}{N} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] \exp \left[\frac{2c\gamma^2}{2c - \gamma^2} \frac{NT}{N} \right] (s_2 - s_1) \end{aligned}$$

$$\mathbb{E} \left[|\bar{X}_N(s_2) - \bar{X}_N(s_1)| \right] \leq \sqrt{\mathbb{E} \left[(\bar{X}_N(s_2) - \bar{X}_N(s_1))^2 \right]} \leq C_T \sqrt{\frac{s_2 - s_1}{N}} \quad \blacksquare$$

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^{N,t}(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \right] \tilde{\gamma}_{(N-\alpha_N)t} \right| \right] \leq \frac{2\|\varphi\|_{\infty}}{\sqrt{N}}.$$

¹ From lemma 3.2.

LEMMA 3.12: Assume that $2c - \gamma^2 > 0$ and $\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] < \infty$. For any positive real number T , there is a constant C_T such that for any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$\sup_{0 < t \leq T} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^{N,t}(Nt)) \right| \right] \leq C_T \|\varphi'\|_\infty \sqrt{\frac{\alpha_N^3}{N}}.$$

PROOF OF LEMMA 3.12: Because of the law of the mean it suffices to show that

$$\sup_{0 < t \leq T} \mathbb{E} \left[|X_i^N(Nt) - X_i^{N,t}(Nt)| \right] \leq C_T \sqrt{\frac{\alpha_N^3}{N}}.$$

since

$$\begin{aligned} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^{N,t}(Nt)) \right| \right] &\leq \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[|\varphi(X_i^N(Nt)) - \varphi(X_i^{N,t}(Nt))| \right] \\ &\leq \frac{1}{N} \sum_{i=0}^{N-1} \|\varphi'\|_\infty \mathbb{E} \left[|X_i^N(Nt) - X_i^{N,t}(Nt)| \right]. \end{aligned}$$

Using coupling and Tanaka's formula¹, we find

$$\begin{aligned} |X_i^N(Nt) - X_i^{N,t}(Nt)| &\stackrel{p-a.s.}{=} \int_{(N-\alpha_N)t}^{Nt} \text{sgn}(X_i^N(s) - X_i^{N,t}(s)) d(X_i^N(s) - X_i^{N,t}(s)) \\ &\quad + L_{Nt}^0(X_i^N - X_i^{N,t}) - L_{(N-\alpha_N)t}^0(X_i^N - X_i^{N,t}). \end{aligned}$$

But since for any $s > (N - \alpha_N)t$,

$$\begin{aligned} d(X_i^N(s) - X_i^{N,t}(s)) &\stackrel{p-a.s.}{=} c(\bar{X}_N(s) - \bar{X}_N((N - \alpha_N)t)) ds - c(X_i^N(s) - X_i^{N,t}(s)) ds \\ &\quad + \gamma(X_i^N(s) - X_i^{N,t}(s)) dW_i(s) \end{aligned}$$

then

$$d(X_i^N - X_i^{N,t})(s) \stackrel{p-a.s.}{=} \gamma^2(X_i^N(s) - X_i^{N,t}(s))^2 ds$$

¹ TANAKA'S FORMULA (ref.: [RY], page 207, theorem 1.2):

If X is a continuous semimartingale then for any real number a , there exists an increasing continuous process $L^a(X)$ called the local time of X at a such that

$$|X_t - a| \stackrel{a.s.}{=} |X_0 - a| + \int_0^t \text{sgn}(X_s - a) dX_s + L_t^a(X).$$

where $\text{sgn}(x) = 1$ if $x > 0$ and $\text{sgn}(x) = -1$ if $x \leq 0$.

and ¹

$$L_s^0(X_i^N - X_i^{N,t}) - L_{(N-\alpha_N)t}^0(X_i^N - X_i^{N,t})$$

$$\stackrel{p.a.s.}{=} \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{(N-\alpha_N)t}^s \mathbf{1}_{\{|X_i^N(r) - X_i^{N,t}(r)| < \epsilon\}} \gamma^2 (X_i^N(r) - X_i^{N,t}(r))^2 dr \leq \lim_{\epsilon \downarrow 0} \frac{\epsilon \gamma^2 s}{2} = 0.$$

Therefore

$$\begin{aligned} |X_i^N(Nt) - X_i^{N,t}(Nt)| &\stackrel{p.a.s.}{=} c \int_{(N-\alpha_N)t}^{Nt} \operatorname{sgn}(X_i^N(s) - X_i^{N,t}(s)) (\bar{X}_N(s) - \bar{X}_N((N-\alpha_N)t)) ds \\ &\quad - c \int_{(N-\alpha_N)t}^{Nt} |X_i^N(s) - X_i^{N,t}(s)| ds + \gamma \int_{(N-\alpha_N)t}^{Nt} |X_i^N(s) - X_i^{N,t}(s)| dW_i(s) \\ (3.12.1) \quad &\leq c \int_{(N-\alpha_N)t}^{Nt} |\bar{X}_N(s) - \bar{X}_N((N-\alpha_N)t)| ds + \gamma \int_{(N-\alpha_N)t}^{Nt} |X_i^N(s) - X_i^{N,t}(s)| dW_i(s). \end{aligned}$$

In the following, the constant C_T may not be the same from step to step. If we take expectation on both sides of the inequality then we find from lemma 3.11 that for any $0 < t \leq T$,

$$\begin{aligned} \mathbb{E}[|X_i^N(Nt) - X_i^{N,t}(Nt)|] &\leq c \int_{(N-\alpha_N)t}^{Nt} \mathbb{E}[|\bar{X}_N(s) - \bar{X}_N((N-\alpha_N)t)|] ds^2 \\ &\leq C_T \int_{(N-\alpha_N)t}^{Nt} \sqrt{\frac{s - (N-\alpha_N)t}{N}} ds = C_T \sqrt{\frac{\alpha_N^3}{N}} \quad \blacksquare \end{aligned}$$

PROOF OF THE INEQUALITY (3.10.3)

LEMMA 3.13: For any $s \geq 0$,

$$\mathbb{E}[|X_\theta(s) - Y_\theta(s)|] \leq (x + \theta) \exp[-cs].$$

where X_θ and Y_θ are two strong solutions of the one-dimensional diffusion

$$dX(t) \stackrel{p.a.s.}{=} c(\theta - X(t)) dt + \gamma X(t) dW(t), \quad \theta \geq 0.$$

PROOF OF LEMMA 3.13: Tanaka's formula gives

$$\begin{aligned} |X_\theta(s) - Y_\theta(s)| &\stackrel{p.a.s.}{=} |X_\theta(0) - Y_\theta(0)| \\ &\quad + \int_0^s \operatorname{sgn}(X_\theta(r) - Y_\theta(r)) d(X_\theta(r) - Y_\theta(r)) + L_s^0(X_\theta - Y_\theta). \end{aligned}$$

¹ (ref.: [RY], page 212, corollary 1.9) If X is a continuous local-martingale then

$$L_s^0(X) \stackrel{a.s.}{=} \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^s \mathbf{1}_{\{|X_r| < \epsilon\}} d\langle X \rangle_r$$

² From the equation (3.12.1) and lemma A.1.

From coupling

$$d(X_\theta(r) - Y_\theta(r)) \stackrel{P-as}{=} -c(X_\theta(r) - Y_\theta(r)) dr + \gamma(X_\theta(r) - Y_\theta(r)) dW(r).$$

Therefore,

$$d\langle X_\theta - Y_\theta \rangle(r) \stackrel{P-as}{=} \gamma^2 (X_\theta(r) - Y_\theta(r))^2 dr$$

and

$$L_s^0(X_\theta - Y_\theta) \stackrel{P-as}{=} \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s \mathbf{1}_{\{|X_\theta(r) - Y_\theta(r)| \leq \varepsilon\}} \gamma^2 (X_\theta(r) - Y_\theta(r))^2 dr \leq \lim_{\varepsilon \downarrow 0} \frac{\varepsilon \gamma^2 s}{2} = 0.$$

So

$$(3.13.1) \quad |X_\theta(s) - Y_\theta(s)| \stackrel{P-as}{=} |X_\theta(0) - Y_\theta(0)| - c \int_0^s |X_\theta(r) - Y_\theta(r)| dr + \gamma \int_0^s |X_\theta(r) - Y_\theta(r)| dW(r).$$

If we take expectation on both sides of the equality then from Fubini-Tonelli's theorem, we obtain

$$\mathbb{E}[|X_\theta(s) - Y_\theta(s)|] = \mathbb{E}[|X_\theta(0) - Y_\theta(0)|] - c \int_0^s \mathbb{E}[|X_\theta(r) - Y_\theta(r)|] dr.$$

Solving this integral equation, we get

$$\begin{aligned} \mathbb{E}[|X_\theta(s) - Y_\theta(s)|] &= \mathbb{E}[|X_\theta(0) - Y_\theta(0)|] \exp[-cs] \\ &\leq (\mathbb{E}[|X_\theta(0)|] + \mathbb{E}[|Y_\theta(0)|]) \exp[-cs] = (x + \theta) \exp[-cs] \end{aligned}$$

where the last equality ensues from corollary 3.8. ■

LEMMA 3.14: For any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative and for any $t > 0$,

$$\left| \int_{\mathbb{R}_+} \varphi(y) P_{\alpha_N t}^\theta(x, dy) - \int_{\mathbb{R}_+} \varphi(y) \nu_\theta(dy) \right| \leq \|\varphi'\|_\infty (x + \theta) \exp[-c \alpha_N t].$$

PROOF OF LEMMA 3.14: Using the law of the mean and lemma 3.13.

$$\begin{aligned} \left| \int_{\mathbb{R}_+} \varphi(y) P_{\alpha_N t}^\theta(x, dy) - \int_{\mathbb{R}_+} \varphi(y) \nu_\theta(dy) \right| &= \left| \mathbb{E}[\varphi(X_\theta(\alpha_N t))] - \mathbb{E}[\varphi(Y_\theta(\alpha_N t))] \right| \\ &\leq \|\varphi'\|_\infty \mathbb{E}[|X_\theta(\alpha_N t) - Y_\theta(\alpha_N t)|] \leq \|\varphi'\|_\infty (x + \theta) \exp[-c \alpha_N t] \end{aligned} \quad \blacksquare$$

LEMMA 3.15: If $\sup_N \mathbb{E}[\bar{X}_N(0)] < \infty$ then there is a constant C such that for any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative and for any $t > 0$,

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \right] \tilde{v}_{(N-\alpha_N)t} - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_N((N-\alpha_N)t)}(dy) \right| \right] \leq C \|\varphi'\|_{\infty} \exp[-c \alpha_N t]$$

PROOF OF LEMMA 3.15: First, note that

$$\mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \right] \tilde{v}_{(N-\alpha_N)t} = \int_{\mathbb{R}_+} \varphi(y) P_{\alpha_N t}^{\bar{X}_N((N-\alpha_N)t)}(X_i^N((N-\alpha_N)t), dy).$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \right] \tilde{v}_{(N-\alpha_N)t} - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_N((N-\alpha_N)t)}(dy) \right| \right] \\ & \leq \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\left| \mathbb{E} \left[\varphi(X_i^{N,t}(Nt)) \right] \tilde{v}_{(N-\alpha_N)t} - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_N((N-\alpha_N)t)}(dy) \right| \right] \\ & = \frac{1}{N} \sum_{i=0}^{N-1} \int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+} \varphi(y) P_{\alpha_N t}^{\theta}(x, dy) - \int_{\mathbb{R}_+} \varphi(y) v_{\theta}(dy) \right| \mathbb{P}[\bar{X}_N((N-\alpha_N)t) \in d\theta, X_i^N((N-\alpha_N)t) \in dx] \\ & \leq \frac{1}{N} \sum_{i=0}^{N-1} \int_{\mathbb{R}_+^2} \|\varphi'\|_{\infty} (x + \theta) \exp[-c \alpha_N t] \mathbb{P}[\bar{X}_N((N-\alpha_N)t) \in d\theta, X_i^N((N-\alpha_N)t) \in dx] \\ & = \|\varphi'\|_{\infty} \exp[-c \alpha_N t] \frac{1}{N} \sum_{i=0}^{N-1} \left(\mathbb{E}[\bar{X}_N((N-\alpha_N)t)] + \mathbb{E}[X_i^N((N-\alpha_N)t)] \right) \\ & = \|\varphi'\|_{\infty} \exp[-c \alpha_N t] 2 \mathbb{E}[\bar{X}_N(0)] \quad \blacksquare \end{aligned}$$

PROOF OF THE INEQUALITY (3.10.4)

LEMMA 3.16: Assume that $2c - \gamma^2 > 0$ and

$$\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(0))^2 \right] < \infty.$$

For any positive real number T , there is a constant C_T such that for any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$\sup_{0 < t \leq T} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_N((N-\alpha_N)t)}(dy) - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_N(Nt)}(dy) \right| \right] \leq C_T \|\varphi'\|_{\infty} \sqrt{\frac{\alpha_N}{N}}$$

¹ From lemma 3.14.

PROOF OF LEMMA 3.16: First note that from coupling, one can establish that

$$\mathbb{E} \left[\left| Y_{\theta_1}(0) - Y_{\theta_2}(0) \right| \right] = \left| \theta_1 - \theta_2 \right|^1.$$

Indeed, define $Z(t) \equiv Y_{\theta_1}(t) - Y_{\theta_2}(t)$ and $\theta \equiv \theta_1 - \theta_2$ and assume that $\theta_1 \geq \theta_2$. From the coupling method, we get

$$dZ(t) = c(\theta - Z(t)) dt + \gamma Z(t) dW(t).$$

Since $Z(t)$ is $\mathbb{P}$ -a.s. nonnegative²,

$$\mathbb{E} \left[\left| Y_{\theta_1}(0) - Y_{\theta_2}(0) \right| \right] = \mathbb{E} \left[\left| Z(t) \right| \right] = \mathbb{E} \left[Z(t) \right] = \theta_1 - \theta_2 = \left| \theta_1 - \theta_2 \right|.$$

$$\begin{aligned} & \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_N((N-\alpha_N)t)}(dy) - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_N(Nt)}(dy) \right| \right] \\ &= \int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+} \varphi(y) v_{\theta_1}(dy) - \int_{\mathbb{R}_+} \varphi(y) v_{\theta_2}(dy) \right| \mathbb{P} \left[\bar{X}_N((N-\alpha_N)t) \in d\theta_1, \bar{X}_N(Nt) \in d\theta_2 \right] \\ &= \int_{\mathbb{R}_+^2} \left| \mathbb{E} \left[\varphi(Y_{\theta_1}(0)) \right] - \mathbb{E} \left[\varphi(Y_{\theta_2}(0)) \right] \right| \mathbb{P} \left[\bar{X}_N((N-\alpha_N)t) \in d\theta_1, \bar{X}_N(Nt) \in d\theta_2 \right] \\ &\leq \int_{\mathbb{R}_+^2} \|\varphi'\|_{\infty} \mathbb{E} \left[\left| Y_{\theta_1}(0) - Y_{\theta_2}(0) \right| \right] \mathbb{P} \left[\bar{X}_N((N-\alpha_N)t) \in d\theta_1, \bar{X}_N(Nt) \in d\theta_2 \right]^3 \\ &= \int_{\mathbb{R}_+^2} \|\varphi'\|_{\infty} \left| \theta_1 - \theta_2 \right| \mathbb{P} \left[\bar{X}_N((N-\alpha_N)t) \in d\theta_1, \bar{X}_N(Nt) \in d\theta_2 \right] \\ &= \|\varphi'\|_{\infty} \mathbb{E} \left[\left| \bar{X}_N((N-\alpha_N)t) - \bar{X}_N(Nt) \right| \right] \leq C_T \|\varphi\|_{\infty} \sqrt{\frac{\alpha_N}{N}}^4 \end{aligned}$$

The proof of proposition 3.10 is complete. ■

PROPOSITION 3.17: If $2c - \gamma^2 > 0$ and if there is a $\delta > 0$ for which

$$\sup_N \max_{0 \leq i \leq N-1} \mathbb{E} \left[\left(X_i^N(0) \right)^{2+\delta} \right] < \infty,$$

then $\lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_s^t \frac{1}{N} \sum_{i=0}^{N-1} \left(X_i^N(Nr) \right)^2 dr - \int_s^t \frac{2c}{2c - \gamma^2} \bar{X}_N^2(Nr) dr \right| \right] = 0$, for any $0 < s \leq t < \infty$.

-
- ¹ Recall that Y_{θ} is the strong solution of (3.7.1) with $\mathcal{J}(Y_{\theta}(t)) = v_{\theta}$.
 - ² See the arguments that justify theorem 2.1.
 - ³ From the law of the mean.
 - ⁴ From lemma 3.11.

PROOF OF PROPOSITION 3.17: Let $\varphi(x) \equiv \gamma^2 x^2$ and note that

$$\frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) = \frac{\gamma^2}{N} \sum_{i=0}^{N-1} (X_i^N(Nt))^2 \text{ and } \int_{\mathbb{R}} \varphi(y) \nu_{\bar{X}_N(Nt)}(dy) = \frac{2c\gamma^2}{2c-\gamma^2} \bar{X}_N^2(Nt)^1.$$

Construct a sequence $\{\varphi_M : M \in \mathbb{N}\}$ of nonnegative bounded continuous differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ with bounded first derivative such that $\varphi_M(x) = \varphi(x)$, $\forall x \in [0, M]$ and $\varphi_M \uparrow \varphi$ as $M \uparrow \infty$. From proposition 3.10, we know that for any M ,

$$(3.17.1) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi_M(X_i^N(Nt)) - \int_{\mathbb{R}_+} \varphi_M(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] = 0.$$

Define $\phi_M \equiv \varphi - \varphi_M$.

$$\begin{aligned} \sup_N \left\{ \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) - \int_{\mathbb{R}_+} \varphi(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] - \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi_M(X_i^N(Nt)) - \int_{\mathbb{R}_+} \varphi_M(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] \right\} \\ \leq \sup_N \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \phi_M(X_i^N(Nt)) - \int_{\mathbb{R}_+} \phi_M(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] \\ \leq \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \phi_M(X_i^N(Nt)) \right] + \sup_N \mathbb{E} \left[\int_{\mathbb{R}_+} \phi_M(y) \nu_{\bar{X}_N(Nt)}(dy) \right] \\ (3.17.2) \quad \leq \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \gamma^2 (X_i^N(Nt))^2 \mathbf{1}_{\{X_i^N(Nt) > M\}} \right] + \sup_N \mathbb{E} \left[\int_{\mathbb{R}_+} \gamma^2 y^2 \mathbf{1}_{\{y > M\}} \nu_{\bar{X}_N(Nt)}(dy) \right] \end{aligned}$$

We will study the behavior of those two terms in the limit. From the corollary of lemma 3.3, we have the uniform integrability of $\{(X_i^N(Nt))^2 : N \in \mathbb{N}, 0 \leq i \leq N-1\}$ which implies that

$$\begin{aligned} (3.17.3) \quad \lim_{M \rightarrow \infty} \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \gamma^2 (X_i^N(Nt))^2 \mathbf{1}_{\{X_i^N(Nt) > M\}} \right] \\ \leq \lim_{M \rightarrow \infty} \sup_N \max_{0 \leq i \leq N-1} \mathbb{E} \left[\gamma^2 (X_i^N(Nt))^2 \mathbf{1}_{\{X_i^N(Nt) > M\}} \right] = 0. \end{aligned}$$

$$\begin{aligned} (3.17.4) \quad \lim_{M \rightarrow \infty} \sup_N \mathbb{E} \left[\int_{\mathbb{R}_+} \gamma^2 y^2 \mathbf{1}_{\{y > M\}} \nu_{\bar{X}_N(Nt)}(dy) \right] \\ = \lim_{M \rightarrow \infty} \sup_N \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \gamma^2 y^2 \mathbf{1}_{\{y > M\}} \nu_{\theta}(dy) \right) \mathbb{P}(\bar{X}_N(Nt) \in d\theta) \end{aligned}$$

¹ From corollary 3.8.

$$\begin{aligned}
&= \lim_{M \rightarrow \infty} \sup_N \int_{\mathbb{R}_+} \left(\int_M^\infty \gamma^2 \frac{(\lambda \theta)^{\lambda+1}}{\Gamma(\lambda+1)} y^{-\lambda} \exp\left[-\frac{\lambda \theta}{y}\right] dy \right) \mathbb{P}(\bar{X}_N(Nt) \in d\theta) \quad 1 \\
&= \lim_{M \rightarrow \infty} \sup_N \int_{\mathbb{R}_+} \left(\int_0^{\lambda \theta M^{-1}} \gamma^2 \frac{(\lambda \theta)^2}{\Gamma(\lambda+1)} y^{\lambda-2} \exp[-y] dy \right) \mathbb{P}(\bar{X}_N(Nt) \in d\theta) \\
&= \lim_{M \rightarrow \infty} \sup_N \int_0^\infty \left(\int_{M y^{\lambda-1}}^\infty \gamma^2 \frac{(\lambda \theta)^2}{\Gamma(\lambda+1)} y^{\lambda-2} \exp[-y] \mathbb{P}(\bar{X}_N(Nt) \in d\theta) \right) dy \\
&= \lim_{M \rightarrow \infty} \sup_N \int_0^\infty \left(\int_{M y^{\lambda-1}}^\infty \theta^2 \mathbb{P}(\bar{X}_N(Nt) \in d\theta) \right) \gamma^2 \frac{\lambda^2}{\Gamma(\lambda+1)} y^{\lambda-2} \exp[-y] dy \\
&= \lim_{M \rightarrow \infty} \sup_N \int_0^\infty \mathbb{E} \left[\bar{X}_N^2(Nt) \mathbf{1}_{\{\bar{X}_N(Nt) > \frac{M y}{\lambda}\}} \right] \gamma^2 \frac{\lambda^2}{\Gamma(\lambda+1)} y^{\lambda-2} \exp[-y] dy \\
&\leq \lim_{M \rightarrow \infty} \int_0^\infty \sup_N \mathbb{E} \left[\bar{X}_N^2(Nt) \mathbf{1}_{\{\bar{X}_N(Nt) > \frac{M y}{\lambda}\}} \right] \gamma^2 \frac{\lambda^2}{\Gamma(\lambda+1)} y^{\lambda-2} \exp[-y] dy \\
&= \int_0^\infty \lim_{M \rightarrow \infty} \sup_N \mathbb{E} \left[\bar{X}_N^2(Nt) \mathbf{1}_{\{\bar{X}_N(Nt) > \frac{M y}{\lambda}\}} \right] \gamma^2 \frac{\lambda^2}{\Gamma(\lambda+1)} y^{\lambda-2} \exp[-y] dy \quad 2 \\
&= 0 \quad 3
\end{aligned}$$

(3.17.2), (3.17.3) and (3.17.4) implies that for any $\varepsilon > 0$ there exists a $M(\varepsilon) \in \mathbb{N}$ such that for any $M \geq M(\varepsilon)$ and any $N \in \mathbb{N}$,

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) - \int_{\mathbb{R}} \varphi(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] < \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi_M(X_i^N(Nt)) - \int_{\mathbb{R}} \varphi_M(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] + \frac{\varepsilon}{2}$$

1 From proposition 3.7 where $\lambda \equiv \frac{2c}{\gamma^2}$.

2 From the dominated convergence theorem since, using lemma 3.2, we can find a constant C for which

$$\begin{aligned}
&\sup_M \sup_N \mathbb{E} \left[\bar{X}_N^2(Nt) \mathbf{1}_{\{\bar{X}_N(Nt) > \frac{M y}{\lambda}\}} \right] \gamma^2 \frac{\lambda^2}{\Gamma(\lambda+1)} y^{\lambda-2} \exp[-y] \\
&\leq \sup_N \mathbb{E} \left[\bar{X}_N^2(Nt) \right] \gamma^2 \frac{\lambda^2}{\Gamma(\lambda+1)} y^{\lambda-2} \exp[-y] \leq C y^{\lambda-2} \exp[-y]
\end{aligned}$$

where the last expression is Lebesgue integrable on the interval $[0, \infty[$ whenever $\lambda > 1$ (which is the case when $2c - \gamma^2 > 0$). (see Gamma function).

3 Because $\{\bar{X}_N^2(Nt) : N \in \mathbb{N}\}$ is uniformly integrable (corollary of lemma 3.3).

In particular, $\forall \varepsilon > 0, \exists M(\varepsilon)$ such that $\forall N$,

$$\begin{aligned} & \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) - \int_{\mathbb{R}} \varphi(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] \\ & < \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi_{M(\varepsilon)}(X_i^N(Nt)) - \int_{\mathbb{R}} \varphi_{M(\varepsilon)}(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] + \frac{\varepsilon}{2} \end{aligned}$$

From (3.17.1) we deduce that $\forall \varepsilon > 0, \exists N(\varepsilon, M(\varepsilon))$ such that $\forall N \geq N(\varepsilon, M(\varepsilon))$,

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi_{M(\varepsilon)}(X_i^N(Nt)) - \int_{\mathbb{R}} \varphi_{M(\varepsilon)}(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] < \frac{\varepsilon}{2}.$$

Therefore $\forall \varepsilon > 0, \exists N(\varepsilon)$ such that $\forall N \geq N(\varepsilon)$,

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i^N(Nt)) - \int_{\mathbb{R}} \varphi(y) \nu_{\bar{X}_N(Nt)}(dy) \right| \right] < \varepsilon.$$

We now conclude the proof using the dominated convergence theorem¹:

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_s^t \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nr))^2 dr - \int_s^t \frac{2c}{2c-\gamma^2} \bar{X}_N^2(Nr) dr \right| \right] \\ & \leq \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_s^t \left(\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nr))^2 - \frac{2c}{2c-\gamma^2} \bar{X}_N^2(Nr) \right) dr \right| \right] \\ & = \lim_{N \rightarrow \infty} \int_s^t \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nr))^2 - \frac{2c}{2c-\gamma^2} \bar{X}_N^2(Nr) \right| \right] dr \\ & = \int_s^t \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nr))^2 - \frac{2c}{2c-\gamma^2} \bar{X}_N^2(Nr) \right| \right] dr = 0 \end{aligned}$$

¹ From lemma 3.2, we can find a constant C such that

$$\begin{aligned} & \sup_N \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nr))^2 - \frac{2c}{2c-\gamma^2} \bar{X}_N^2(Nr) \right| \right] \\ & \leq \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(Nr))^2 \right] + \frac{2c}{2c-\gamma^2} \sup_N \mathbb{E} \left[\bar{X}_N^2(Nr) \right] \leq C \exp[Cr] \end{aligned}$$

which is Lebesgue integrable on $[s, t]$.

Behavior of $\{\bar{X}_N(N^p t): t \in \mathbb{R}_+\}$ as N goes to Infinity
when $p > 1$

If we increase the speed by a multiple of N then the average of the components in the process goes to extinction in probability as N tends to infinity. That is, as N goes to infinity, the average process tends to be close to zero except on a set of small probability where it takes very large values.

PROPOSITION 4.1: Assume that the initial conditions are nonnegative square-integrable random variables. If $p > 1$ then for any $t > 0$,

$$\bar{X}_N(N^p t) \xrightarrow{N \rightarrow \infty} 0 \text{ in probability.}$$

PROOF OF PROPOSITION 4.1: From Markov's inequality, we have for any $\varepsilon > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[|\bar{X}_N(N^p t)| > \varepsilon \right] \leq \frac{1}{\sqrt{\varepsilon}} \lim_{N \rightarrow \infty} \mathbb{E} \left[\sqrt{\bar{X}_N(N^p t)} \right].$$

Therefore, it suffices to show that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sqrt{\bar{X}_N(N^p t)} \right] = 0.$$

Itô's formula gives

$$d\bar{X}_N^k(t) \stackrel{p-a.s.}{=} k \bar{X}_N^{k-1}(t) d\bar{X}_N(t) + \frac{k(k-1)}{2} \bar{X}_N^{k-2}(t) d\langle \bar{X}_N \rangle(t).$$

Because

$$d\bar{X}_N(t) \stackrel{p-a.s.}{=} \frac{\gamma}{N} \sum_{i=0}^{N-1} X_i^N(t) dW_i(t) \text{ and } d\langle \bar{X}_N \rangle(t) \stackrel{a.s.}{=} \frac{\gamma^2}{N^2} \sum_{i=0}^{N-1} (X_i^N(t))^2 dt,$$

we have the following equality:

$$\begin{aligned} \bar{X}_N^k(t) &\stackrel{p-a.s.}{=} \bar{X}_N^k(0) + \frac{k\gamma}{N} \sum_{i=0}^{N-1} \int_0^t \bar{X}_N^{k-1}(s) X_i^N(s) dW_i(s) \\ &\quad + \frac{k(k-1)}{2} \frac{\gamma^2}{N} \int_0^t \bar{X}_N^{k-2}(s) \left(\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(s))^2 \right) ds. \end{aligned}$$

Taking expectation and using lemma A.2 with lemma 2.2,

$$\mathbb{E} \left[\bar{X}_N^k(t) \right] = \mathbb{E} \left[\bar{X}_N^k(0) \right] + \frac{k(k-1)}{2} \frac{\gamma^2}{N} \int_0^t \mathbb{E} \left[\bar{X}_N^{k-2}(s) \left(\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(s))^2 \right) \right] ds.$$

Let $k = \frac{1}{2}$:

$$\begin{aligned} \mathbb{E} \left[\sqrt{\bar{X}_N(t)} \right] &= \mathbb{E} \left[\sqrt{\bar{X}_N(0)} \right] - \frac{\gamma^2}{8} \frac{1}{N} \mathbb{E} \left[\int_0^t \bar{X}_N^{-\frac{3}{2}}(s) \left(\frac{1}{N} \sum_{i=0}^{N-1} (X_i^N(s))^2 \right) ds \right] \\ &\leq \mathbb{E} \left[\sqrt{\bar{X}_N(0)} \right] - \frac{\gamma^2}{8N} \int_0^t \mathbb{E} \left[\sqrt{\bar{X}_N(s)} \right] ds. \end{aligned}$$

So

$$\mathbb{E} \left[\sqrt{\bar{X}_N(N^p t)} \right] \leq \mathbb{E} \left[\sqrt{\bar{X}_N(0)} \right] - \frac{\gamma^2 N^{p-1}}{8} \int_0^t \mathbb{E} \left[\sqrt{\bar{X}_N(N^p s)} \right] ds.$$

For any $N \in \mathbb{N}$,

$$\mathbb{E} \left[\sqrt{\bar{X}_N(N^p t)} \right] + \frac{\gamma^2 N^{p-1}}{8} \int_0^t \mathbb{E} \left[\sqrt{\bar{X}_N(N^p s)} \right] ds \leq \mathbb{E} \left[\sqrt{\bar{X}_N(0)} \right] \leq \sup_N \mathbb{E} \left[\sqrt{\bar{X}_N(0)} \right] < \infty$$

which implies that for $p > 1$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sqrt{\bar{X}_N(N^p t)} \right] = 0. \quad \blacksquare$$

CONJECTURE 4.2: If $\gamma^2 - 2c = 0$ and $\frac{1}{2} < p < 1$ or if $\gamma^2 - 2c > 0$ and $0 < p < 1$ then for any $t > 0$,

$$\bar{X}_N(N^p t) \xrightarrow{N \rightarrow \infty} 0 \text{ in probability.}$$

The limiting behavior of the process in the critical case ($\gamma^2 - 2c = 0$) when we rescale the time by the factor $\sqrt{N}$ is still an enigma. However, we have partial results as, for example, the limit of the second moment.

Small Perturbation of the System of Stochastic Differential Equations in the Time Rescaling Context

In this section, our aim is to develop a tool to study the second level process. By using the same methodology as in section 3, we establish an equivalent theorem to theorem 3.1 by perturbing the initial system of stochastic differential equations slightly. It is possible to reproduce for this new process analogs to most of the results obtained in section 1. Very little complication arises with introduction of a perturbation to the system. Therefore, we can assume that if the initial condition are nonnegative square-integrable random variables then the process remain nonnegative through time and its second moments exist for any finite time.

The new stochastic process is define to be the unique strong solution of

$$dY_i^N(t) = c(\bar{Y}_N(t) - Y_i^N(t))dt + \frac{c_2}{N}(\theta - Y_i^N(t))dt + \gamma Y_i^N(t)dW_i(t), \quad i \in \{0, 1, \dots, N-1\}.$$

where c and γ are positive constants and c_2 and θ are nonnegative constants.

THEOREM 5.1: For any N , let Q_N be the probability on $C(\mathbb{R}_+, \mathbb{R}_+)$ induced by the process $\{\bar{Y}_N(Nt) : t \in \mathbb{R}_+\}$. Let $2c - \gamma^2 > 0$ and assume that the following assumptions on the initial conditions are satisfied: there is a $\delta > 0$ for which

$$\sup_N \max_{0 \leq i \leq N-1} \mathbb{E} \left[\left(Y_i^N(0) \right)^{2+\delta} \right] < \infty$$

and $\mathcal{L}(\bar{Y}_N(0))$ converges weakly to the probability μ . Then the sequence $\{Q_N : N \in \mathbb{N}\}$ converges weakly (as N goes to infinity) to the unique solution of the martingale problem associated to $\mathcal{B}$ with initial distribution μ where

$$(\mathcal{B}f)(x) \equiv \frac{c\gamma^2}{2c - \gamma^2} x^2 f''(x) + c_2(\theta - x)f'(x), \quad f \in C^2(\mathbb{R}_+, \mathbb{R}).$$

SKETCH OF THE PROOF OF THEOREM 5.1:

CLAIM: The sequence $\{Q_N : N \in \mathbb{N}\}$ of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$ is tight.

Let $\{Q_{N_k}\}$ be an arbitrary convergent subsequence of $\{Q_N : N \in \mathbb{N}\}$ and Q_∞ denotes its limit point. We would like to show that Q_∞ is a solution of the martingale problem associated with $\mathcal{B}$. To do so, it suffices to prove that

$$\left\{ f(\xi_t) - \int_0^t (\mathcal{B}f)(\xi_s) ds : t \in \mathbb{R}_+ \right\}$$

is a Q_∞ -martingale for $f(x) \equiv x$ and $f(x) \equiv x^2$ where $\{\xi_t : t \in \mathbb{R}_+\}$ denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}_+)$.

CLAIM: $\left\{ \xi_t - \int_0^t c_2(\theta - \xi_s) ds : t \in \mathbb{R}_+ \right\}$ is a Q_∞ -martingale.

CLAIM: $\left\{ \xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \xi_s^2 ds - \int_0^t 2c_2(\theta - \xi_s) \xi_s ds : t \in \mathbb{R}_+ \right\}$ is a Q_∞ -martingale.

The last two claims imply that Q_∞ is a solution of the martingale problem associated with $\mathcal{B}$. Since this martingale problem has a unique solution then any convergent subsequence has Q_∞ as its limit point and therefore, $\{Q_N : N \in \mathbb{N}\}$ converges weakly to Q_∞ as N goes to infinity. ■

In the proof of these claims, it is essential to have some control over the moments of our processes. The following lemma will give us this control.

LEMMA 5.3: If $2c - \gamma^2 > 0$ then there exists a $\delta > 0$ such that, for some constant C_t ,

$$\sup_N \max_{0 \leq i \leq N-1} \mathbb{E} \left[\left(Y_i^N(0) \right)^{2+\delta} \right] < \infty \text{ implies } \sup_N \sup_{0 \leq u \leq Nt} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(Y_j^N(u) \right)^{2+\delta} \right] \leq C_t$$

and

$$\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \left(Y_i^N(0) \right)^{2+\delta} \right] < \infty \text{ implies } \sup_N \sup_{0 \leq u \leq Nt} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \left(Y_i^N(u) \right)^{2+\delta} \right] \leq C_t.$$

Note that C_t depends on

$$\sup_N \sup_{0 \leq u \leq Nt} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(Y_j^N(u) \right)^{2+\delta} \right] \leq C_t \text{ or } \sup_N \sup_{0 \leq u \leq Nt} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \left(Y_i^N(u) \right)^{2+\delta} \right] \leq C_t.$$

PROOF OF LEMMA 5.3: For $i, j \in \{0, 1, \dots, N-1\}$ we use the same function as the one we built in lemma 3.3:

$$a_t(i, j) \equiv \exp[-ct] \delta_i(j) + \frac{1}{N} (1 - \exp[-ct]).$$

Itô's formula gives, for any fix $t > 0$,

$$\begin{aligned} & \sum_{j=0}^{N-1} a_{t-s}(i, j) Y_j^N(s) - \sum_{j=0}^{N-1} a_t(i, j) Y_j^N(0) \\ &= \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) dY_j^N(r) + \sum_{j=0}^{N-1} \int_0^s Y_j^N(r) da_{t-r}(i, j) \\ &= \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) c(\bar{Y}_N(r) - Y_j^N(r)) dr + \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) \frac{c_2}{N} (\theta - Y_j^N(r)) dr \\ & \quad + \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) \gamma Y_j^N(r) dW_j(r) + \sum_{j=0}^{N-1} \int_0^s Y_j^N(r) \left(c a_{t-r}(i, j) - \frac{c}{N} \right) dr \\ &= \sum_{j=0}^{N-1} \int_0^s a_{t-r}(i, j) \gamma Y_j^N(r) dW_j(r) + \frac{c_2}{N} \theta s - \frac{c_2}{N} \int_0^s \sum_{j=0}^{N-1} a_{t-r}(i, j) Y_j^N(r) dr. \end{aligned}$$

If we let $s = t$ then the last equation becomes

$$Y_i^N(t) + \frac{c_2}{N} \int_0^t \sum_{j=0}^{N-1} a_{t-r}(i, j) Y_j^N(r) dr - \frac{c_2}{N} \theta t - \sum_{j=0}^{N-1} a_t(i, j) Y_j^N(0) \stackrel{p.a.s.}{=} \sum_{j=0}^{N-1} \int_0^t a_{t-r}(i, j) \gamma Y_j^N(r) dW_j(r) \equiv M_t.$$

It should be noted that the constant $C_{p,T}$ may not be the same from line to line. Using the last equation, we find for any $T > 0$, any $0 < u \leq NT$ and for any $p > 2$, that

$$\begin{aligned} (5.3.1) \quad \mathbb{E} \left[\left(Y_i^N(u) \right)^p \right] &\leq \mathbb{E} \left[\left(Y_i^N(u) + \frac{c_2}{N} \int_0^u \sum_{j=0}^{N-1} a_{u-r}(i, j) Y_j^N(r) dr \right)^p \right] \\ &\leq \mathbb{E} \left[\left(\left| Y_i^N(u) + \frac{c_2}{N} \int_0^u \sum_{j=0}^{N-1} a_{u-r}(i, j) Y_j^N(r) dr - \frac{c_2}{N} \theta u - \sum_{j=0}^{N-1} a_u(i, j) Y_j^N(0) \right| + \left| \frac{c_2}{N} \theta u + \sum_{j=0}^{N-1} a_u(i, j) Y_j^N(0) \right| \right)^p \right] \\ &\leq (p-1) \mathbb{E} \left[|M_u|^p \right] + C_p \mathbb{E} \left[\left(c_2 \theta T + \sum_{j=0}^{N-1} a_u(i, j) Y_j^N(0) \right)^p \right] \\ &\leq (p-1)^{p+1} \mathbb{E} \left[\left(\langle M \rangle_u \right)^{\frac{p}{2}} \right] + C_{p,T} \\ &\leq (p-1)^{p+1} \left(\sum_{j=0}^{N-1} \int_0^u a_s^2(i, j) ds \right)^{\frac{p}{2}-1} \sum_{j=0}^{N-1} \int_0^u a_s^2(i, j) \gamma^p \mathbb{E} \left[\left(Y_j^N(u-s) \right)^p \right] ds + C_{p,T} \end{aligned}$$

The rest of the arguments are as in the proof of lemma 3.3 giving the equivalent to equation (3.3.3), that is: for any $0 < t \leq NT$,

$$\begin{aligned} &\sup_{0 \leq u \leq t} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(Y_j^N(u) \right)^p \right] \\ &\leq (p-1)^{p+1} \left(\int_0^t a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \int_0^t a_{2s}(0,0) \gamma^p \sup_{0 \leq u \leq t-s} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(Y_j^N(u) \right)^p \right] ds + C_{p,T} \cdot 1 \end{aligned}$$

1 If we sum over i in (5.3.1), we also obtain for any $0 < t \leq NT$,

$$\begin{aligned} &\sup_{0 \leq u \leq t} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} \left(Y_j^N(u) \right)^p \right] \\ &\leq (p-1)^{p+1} \left(\int_0^t a_{2s}(0,0) ds \right)^{\frac{p}{2}-1} \int_0^t a_{2s}(0,0) \gamma^p \sup_{0 \leq u \leq t-s} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} \left(Y_j^N(u) \right)^p \right] ds + C_{p,T}. \end{aligned}$$

In the following, we will use exactly the approach as the original proof so we give here only the main lines: define

$$f(t) \equiv \sup_{0 \leq u \leq t} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(Y_j^N(u) \right)^p \right]^{-1} \text{ and } F(\rho, T) \equiv \int_0^{NT} \exp[-\rho t] f(t) dt; \rho > 0, T > 0.$$

Therefore

$$F(\rho, T) \leq (\rho - 1)^{\rho+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{\rho}{2}-1} \gamma^\rho F(\rho, T) \int_0^\infty \exp[-\rho s] a_{2s}(0,0) ds + \frac{C_{\rho,T}}{\rho}$$

which implies that

$$\frac{\rho}{N} F\left(\frac{\rho}{N}, T\right) \left(1 - (\rho - 1)^{\rho+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{\rho}{2}-1} \gamma^\rho \int_0^\infty \exp\left[-\frac{\rho}{N} s\right] a_{2s}(0,0) ds \right) \leq C_{\rho,T}.$$

If $\alpha = 2c - \gamma^2 > 0$, $\rho > \frac{2c\gamma^2}{2c - \gamma^2}$ and if ρ is close enough to 2 then for any large N ,

$$(\rho - 1)^{\rho+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{\rho}{2}-1} \gamma^\rho \int_0^\infty \exp\left[-\frac{\rho}{N} s\right] a_{2s}(0,0) ds < 1.$$

Therefore, we can find a constant $K_{\rho,T}$ and a $N_0 \in \mathbb{N}$ for which

$$\begin{aligned} & \sup_{N \geq N_0} \frac{\rho}{N} F\left(\frac{\rho}{N}, T\right) \\ & \leq \sup_{N \geq N_0} C_{\rho,T} \left(1 - (\rho - 1)^{\rho+1} \left(\int_0^{NT} a_{2s}(0,0) ds \right)^{\frac{\rho}{2}-1} \gamma^\rho \int_0^\infty \exp\left[-\frac{\rho}{N} s\right] a_{2s}(0,0) ds \right)^{-1} \leq K_{\rho,T}. \end{aligned}$$

which implies that

$$\frac{K_{\rho,T}}{\rho} \geq \exp[-\rho T] \sup_{N \geq N_0} \left(\frac{T}{2} \sup_{0 \leq u \leq N \frac{T}{2}} \max_{0 \leq j \leq N-1} \mathbb{E} \left[\left(Y_j^N(u) \right)^p \right] \right)^{-2}.$$

1 The following argumentation also works with $f(t) \equiv \sup_{0 \leq u \leq t} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} \left(Y_j^N(u) \right)^p \right]$.

2 $\frac{K_{\rho,T}}{\rho} \geq \exp[-\rho T] \sup_{N \geq N_0} \left(\frac{T}{2} \sup_{0 \leq u \leq N \frac{T}{2}-1} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} \left(Y_j^N(u) \right)^p \right] \right)$

Hence,

$$\sup_{N \geq N_0} \sup_{0 \leq u \leq T} \max_{\frac{1}{2} \leq j \leq N-1} \mathbb{E} \left[\left(Y_j^N(u) \right)^p \right] \leq \frac{2}{T} \frac{K_{p,T}}{\rho} \exp[\rho T]. \quad 1$$

PROPOSITION 5.4: If $2c - \gamma^2 > 0$ and if there is a positive δ such that

$$\sup_{N \geq N_0} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \left(Y_i^N(\tau_N) \right)^{2+\delta} \right] < \infty,$$

then the sequence $\{Q_N : N \in \mathbb{N}\}$ of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$ is tight.

PROOF OF PROPOSITION 5.4:

VERIFICATION OF $[T_1]$:

$$\begin{aligned} \mathbb{E}[\bar{Y}_N(t)] &= \mathbb{E}[\bar{Y}_N(0)] + \frac{c_2}{N} \int_0^t \theta - \mathbb{E}[\bar{Y}_N(s)] ds \\ \Rightarrow \mathbb{E}[\bar{Y}_N(t)] &= \mathbb{E}[\bar{Y}_N(0)] \exp\left[-\frac{c_2}{N} t\right] + \theta \left(1 - \exp\left[-\frac{c_2}{N} t\right]\right) \end{aligned}$$

From Markov's inequality,

$$\sup_N \mathbb{P} \left(\bar{Y}_N(t) > \left(\theta + \sup_N \mathbb{E}[\bar{Y}_N(0)] \right) \frac{1}{\varepsilon} \right) < \varepsilon.$$

VERIFICATION OF $[T_2]$:

$$\begin{aligned} &\sup_{N \geq N_0} \sup_{\theta \leq \delta} \mathbb{P} \left[\left| \langle \bar{Y}_N \rangle(N\tau_N) - \langle \bar{Y}_N \rangle(N(\tau_N + \theta)) \right| \geq \eta \right] \\ &\leq \sup_{N \geq N_0} \frac{\gamma^2}{\eta} \int_0^\delta \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \left(Y_i^N(Ns) \right)^2 \middle| Y^N(0) = Y^N(\tau_N) \right] ds \leq \frac{C \gamma^2}{\eta} \delta \end{aligned}$$

where the last inequality holds provided

$$\sup_{N \geq N_0} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \left(Y_i^N(\tau_N) \right)^{2+\delta} \right] < \infty.$$

$$\sup_{N \geq N_0} \sup_{0 \leq u \leq NT 2^{-1}} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} \left(Y_j^N(u) \right)^p \right] \leq \frac{2}{T} \frac{K_{p,T}}{\rho} \exp[\rho T].$$

Now, since τ_N is bounded by K ,

$$\begin{aligned}
\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(\tau_N))^{2+\delta} &\stackrel{p-a.s.}{=} \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(0))^{2+\delta} + (2+\delta)c \int_0^{\tau_N} \bar{Y}_N(s) \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(s))^{1+\delta} ds \\
&\quad + \left(\frac{(2+\delta)(1+\delta)}{2} \gamma^2 - (2+\delta) \left(c + \frac{c_2}{N} \right) \right) \int_0^{\tau_N} \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(s))^{2+\delta} ds \\
&\quad + (2+\delta) \frac{c_2}{N} \theta \int_0^{\tau_N} \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(s))^{1+\delta} ds + (2+\delta) \frac{\gamma}{N} \sum_{i=0}^{N-1} \int_0^{\tau_N} (Y_i^N(s))^{2+\delta} dW_i(s) \\
&\leq \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(0))^{2+\delta} + (2+\delta)c \int_0^K \bar{Y}_N(s) \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(s))^{1+\delta} ds \\
&\quad + (2+\delta) \frac{c_2}{N} \theta \int_0^K \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(s))^{1+\delta} ds + (2+\delta) \frac{\gamma}{N} \sum_{i=0}^{N-1} \int_0^{\tau_N \wedge K} (Y_i^N(s))^{2+\delta} dW_i(s)
\end{aligned}$$

if δ is small. Therefore,

$$\begin{aligned}
\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(\tau_N))^{2+\delta} \right] &\leq \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(0))^{2+\delta} \right] \\
&\quad + (2+\delta)c \sup_N \int_0^K \mathbb{E} \left[\bar{Y}_N(s) \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(s))^{1+\delta} \right] ds + (2+\delta) \frac{c_2}{N} \theta \sup_N \int_0^K \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(s))^{1+\delta} \right] ds \\
&\leq C \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(0))^{2+\delta} \right]
\end{aligned}$$

PROPOSITION 5.5: If $2c - \gamma^2 > 0$ and $\sup_N \mathbb{E} [\bar{Y}_N^2(0)] < \infty$ then

$$\left\{ \xi_t - \int_0^t c_2 (\theta - \xi_s) ds : t \in \mathbb{R}_+ \right\} \text{ is a } Q_\infty\text{-martingale.}$$

PROOF OF PROPOSITION 5.5: Note that

$$(5.5.1) \quad \bar{Y}_N(Nt) - c_2 \int_0^t (\theta - \bar{Y}_N(Ns)) ds \stackrel{p-a.s.}{=} \bar{Y}_N(0) + \frac{\gamma}{N} \sum_{i=0}^{N-1} \int_0^{Nt} Y_i^N(s) dW_i(s)$$

implies that $\left\{ \xi_t - \int_0^t c_2 (\theta - \xi_s) ds : t \in \mathbb{R}_+ \right\}$ is a Q_N -martingale. By the generalization of the continuous mapping theorem, the result follows. ■

PROPOSITION 5.6: If $2c - \gamma^2 > 0$ and if there is a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i \leq N-1} \mathbb{E} \left[(Y_i^N(0))^{2+\delta} \right] < \infty$$

then

$$\left\{ \xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \xi_s^2 ds - \int_0^t 2c_2 (\theta - \xi_s) \xi_s ds : t \in \mathbb{R}_+ \right\} \text{ is a } Q_\infty\text{-martingale}$$

provided

$$(5.6.1) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_s^t \frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(Nr))^2 dr - \int_s^t \frac{2c}{2c-\gamma^2} \bar{Y}_N^2(Nr) dr \right| \right] = 0.$$

PROOF OF PROPOSITION 5.6: Again, it suffices to show that for any real numbers s and t with $0 \leq s \leq t$ and for any $\sigma\{\xi_r : 0 \leq r \leq s\}$ -measurable $\Phi \in C_b(C(\mathbb{R}_+, \mathbb{R}), \mathbb{R})$,

$$\begin{aligned} & E_\infty \left[\left(\xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr - \int_0^t 2c_2 (\theta - \xi_r) \xi_r dr \right) \Phi \right] \\ &= E_\infty \left[\left(\xi_s^2 - \int_0^s \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr - \int_0^s 2c_2 (\theta - \xi_r) \xi_r dr \right) \Phi \right]. \end{aligned}$$

Equation (5.5.1) and Itô's formula gives

$$\begin{aligned} & \bar{Y}_N^2(Nt) - \bar{Y}_N^2(0) \\ &= c_2 \int_0^t (\theta - \bar{Y}_N(Ns)) \bar{Y}_N(Ns) ds + \gamma^2 \int_0^t \frac{1}{N} \sum_{i=0}^{N-1} Y_i^N(s) ds + \frac{\gamma}{N} \sum_{i=0}^{N-1} \int_0^t \bar{Y}_N(Ns) Y_i^N(s) dW_i(s) \end{aligned}$$

Therefore,

$$\begin{aligned} & \bar{Y}_N^2(Nt) - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \bar{Y}_N^2(Ns) ds - c_2 \int_0^t (\theta - \bar{Y}_N(Ns)) \bar{Y}_N(Ns) ds \\ &+ \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \bar{Y}_N^2(Ns) ds - \gamma^2 \int_0^t \frac{1}{N} \sum_{i=0}^{N-1} Y_i^N(s) ds \end{aligned}$$

is a $\mathbb{P}$ -martingale for any N . Hence,

$$\begin{aligned} & E_N \left[\left(\xi_t^2 - \int_0^t \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr - \int_0^t 2c_2 (\theta - \xi_r) \xi_r dr \right) \Phi \right] \\ &= E_N \left[\left(\xi_s^2 - \int_0^s \frac{2c\gamma^2}{2c-\gamma^2} \xi_r^2 dr - \int_0^s 2c_2 (\theta - \xi_r) \xi_r dr \right) \Phi \right] + (N^0)^{-1} \end{aligned}$$

Again, by the generalization of the continuous mapping theorem, the proof is completed. $\blacksquare$

¹ A function $f(N) = c(N^p) \Leftrightarrow f(N)N^{-p} \xrightarrow{N \rightarrow \infty} 0$.

Let $Q_t^\eta(x, dy)$ and ϑ_η be respectively the transition probability and the equilibrium of the one-dimensional diffusion process described by the stochastic differential equation

$$dX(t) = c(\eta - X(t)) dt + \frac{c_2}{N}(\theta - X(t)) dt + \gamma X(t) dW(t); \eta, \theta \geq 0$$

which can be rewritten as

$$dX(t) = \left(c + \frac{c_2}{N} \right) \left(\frac{c\eta + \frac{c_2}{N}\theta}{c + \frac{c_2}{N}} - X(t) \right) dt + \gamma X(t) dW(t); \eta, \theta \geq 0$$

In the following, X_η and Y_η denote two strong solutions of this stochastic differential equation with respective initial conditions $X_\eta(0) \equiv x$ and $\mathcal{L}(Y_\eta(0)) = \vartheta_\eta$. Note that $\vartheta_\eta = \nu_{\hat{\theta}}$ and $Q_t^\eta(x, dy) = P_t^{\hat{\theta}, \hat{c}, \hat{\gamma}}(x, dy)$ where

$$\hat{\theta} = \frac{c\eta + \frac{c_2}{N}\theta}{c + \frac{c_2}{N}}, \hat{c} = c + \frac{c_2}{N} \text{ and } \hat{\gamma} = \gamma$$

COROLLARY 5.8: If $2c - \gamma^2 > 0$ then

$$\int_{\mathbb{R}} y \vartheta_\eta(dy) = \frac{c\eta + \frac{c_2}{N}\theta}{c + \frac{c_2}{N}} \text{ and } \int_{\mathbb{R}} y^2 \vartheta_\eta(dy) = \frac{2\left(c\eta + \frac{c_2}{N}\theta\right)^2}{\left(2c + \frac{2c_2}{N} - \gamma^2\right)\left(c + \frac{c_2}{N}\right)}.$$

The purpose of proposition 5.10 will be clarified in proposition 5.17. In brief, the idea is that we will construct a sequence $\{\varphi_k : k \in \mathbb{N}\}$ of approximations of $\varphi(x) = x^2$ and show that the property of proposition 5.10 is preserved in the limit (as k tends to infinity). Note that if we replace $\varphi(x) = x^2$ in the equation of the following proposition, we obtain an equation similar to (5.6.1) which is what we want to verify.

PROPOSITION 5.10: Assume that $2c - \gamma^2 > 0$ and the existence of a positive δ for which

$$\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(0))^{2+\delta} \right] < \infty.$$

For any positive real numbers t_1 and t_2 such that $t_1 \leq t_2$ and for any bounded continuous differentiable function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$\lim_{N \rightarrow \infty} \sup_{t_1 \leq t \leq t_2} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(Y_i^N(Nt)) - \int_{\mathbb{R}} \varphi(y) \vartheta_{Y_N(Nt)}(dy) \right| \right] = 0.$$

PROOF OF PROPOSITION 5.10: Let $\{\alpha_N : N \in \mathbb{N}\}$ is a positive increasing sequence tending to infinity ($\alpha_N \leq N$). For any fixed $t \geq 0$, we introduce a N -dimensional modified process, denoted $Y^{N,t}$, as follows:

If $0 \leq s \leq (N - \alpha_N)t$ then

$$Y_i^{N,t}(s) \equiv Y_i^N(s)$$

and if $s > (N - \alpha_N)t$ then

$$\begin{aligned} Y_i^{N,t}(s) \equiv & Y_i^N((N - \alpha_N)t) + c \int_{(N - \alpha_N)t}^s (\bar{Y}_N((N - \alpha_N)t) - Y_i^{N,t}(r)) dr \\ & + \frac{c_2}{N} \int_{(N - \alpha_N)t}^s (\theta - Y_i^{N,t}(r)) dr + \gamma \int_{(N - \alpha_N)t}^s Y_i^{N,t}(r) dW_i(r). \end{aligned}$$

To prove the above, it suffices to show the four following inequalities:

there exists a constant C and, for any positive real number T , there is a constant C_T such that

$$(5.10.1) \quad \sup_{0 < t \leq T} \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(Y_i^N(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \varphi(Y_i^{N,t}(Nt)) \right\| \right] \leq C_T \|\varphi'\|_{\infty} \left(\sqrt{\frac{\alpha_N^3}{N}} + \frac{\alpha_N^2}{N} \right)$$

$$(5.10.2) \quad \sup_{t > 0} \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(Y_i^{N,t}(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(Y_i^{N,t}(Nt)) \middle| \tilde{\mathcal{N}}_{(N - \alpha_N)t} \right] \right\| \right] \leq 2 \frac{\|\varphi\|_{\infty}}{\sqrt{N}}$$

$$\begin{aligned} (5.10.3) \quad & \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(Y_i^{N,t}(Nt)) \middle| \tilde{\mathcal{N}}_{(N - \alpha_N)t} \right] - \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\bar{Y}_N((N - \alpha_N)t)}(dy) \right\| \right] \\ & \leq C \|\varphi'\|_{\infty} \exp \left[- \left(c + \frac{c_2}{N} \right) \alpha_N t \right], \forall t > 0 \end{aligned}$$

$$(5.10.4) \quad \sup_{0 < t \leq T} \mathbb{E} \left[\left\| \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\bar{Y}_N((N - \alpha_N)t)}(dy) - \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\bar{Y}_N(Nt)}(dy) \right\| \right] \leq C_T \|\varphi'\|_{\infty} \left(\sqrt{\frac{\alpha_N}{N}} + \frac{\alpha_N}{N} \right).$$

PROOF OF THE INEQUALITY (5.10.1)

LEMMA 5.11: Assume that $2c - \gamma^2 > 0$ and the existence of a positive δ for which

$$\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(0))^{2+\delta} \right] < \infty.$$

¹ Because $\{Y_i^{N,t}(Nt) : i \in \{0, 1, \dots, N-1\}\}$ are conditionally independent with respect to $\mathbb{P}[\cdot \mid \tilde{\mathcal{N}}_{(N - \alpha_N)t}]$.

For any positive real number T , there is a constant C_T such that for any real numbers s_1 and s_2 with $0 \leq s_1 \leq s_2 \leq NT$,

$$\mathbb{E} \left[\left| \bar{Y}_N(s_2) - \bar{Y}_N(s_1) \right| \right] \leq C_T \left(\sqrt{\frac{s_2 - s_1}{N}} + \frac{s_2 - s_1}{N} \right).$$

PROOF OF LEMMA 5.11: Since

$$\bar{Y}_N(t) - \bar{Y}_N(0) \stackrel{p.s.s.}{=} \frac{c_2}{N} \int_0^t (\theta - \bar{Y}_N(r)) dr + \frac{\gamma}{N} \sum_{i=0}^{N-1} \int_0^t Y_i^N(r) dW_i(r),$$

$$\bar{Y}_N(s_2) - \bar{Y}_N(s_1) - \frac{c_2}{N} \int_{s_1}^{s_2} (\theta - \bar{Y}_N(r)) dr \stackrel{p.s.s.}{=} \frac{\gamma}{N} \sum_{i=0}^{N-1} \int_{s_1}^{s_2} Y_i^N(r) dW_i(r).$$

Therefore

$$\begin{aligned} \mathbb{E} \left[\left(\bar{Y}_N(s_2) - \bar{Y}_N(s_1) - \frac{c_2}{N} \int_{s_1}^{s_2} (\theta - \bar{Y}_N(r)) dr \right)^2 \right] &= \frac{\gamma^2}{N} \int_{s_1}^{s_2} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(r))^2 \right] dr \leq C_T \frac{s_2 - s_1}{N} \\ &\mathbb{E} \left[\left| \bar{Y}_N(s_2) - \bar{Y}_N(s_1) \right| \right] \\ &\leq \mathbb{E} \left[\left| \bar{Y}_N(s_2) - \bar{Y}_N(s_1) - \frac{c_2}{N} \int_{s_1}^{s_2} (\theta - \bar{Y}_N(r)) dr \right| \right] + \frac{c_2}{N} \int_{s_1}^{s_2} \mathbb{E} \left[\left| \theta - \bar{Y}_N(r) \right| \right] dr \leq C_T \left(\sqrt{\frac{s_2 - s_1}{N}} + \frac{s_2 - s_1}{N} \right) \end{aligned}$$

■

The proof of the following results are omitted since they are mainly reproductions of their analogs which are shown in section 3.

LEMMA 5.12: Assume that $2c - \gamma^2 > 0$ and the existence of a positive δ for which

$$\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(0))^{2+\delta} \right] < \infty.$$

For any positive real number T , there is a constant C_T such that for any continuous differentiable function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \varphi(Y_i^N(Nt)) - \frac{1}{N} \sum_{i=0}^{N-1} \varphi(Y_i^{N,t}(Nt)) \right| \right] \leq C_T \|\varphi'\|_{\infty} \left(\sqrt{\frac{\alpha_N^3}{N}} + \frac{\alpha_N^2}{N} \right).$$

PROOF OF THE INEQUALITY (5.10.3)**LEMMA 5.13:** For any $s \geq 0$,

$$\mathbb{E} \left[|X_\eta(s) - Y_\eta(s)| \right] \leq \left(x + \frac{c\eta + \frac{c_2}{N}\theta}{c + \frac{c_2}{N}} \right) \exp \left[- \left(c + \frac{c_2}{N} \right) s \right].$$

LEMMA 5.14: For any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative and for any $t > 0$,

$$\left| \int_{\mathbb{R}_+} \varphi(y) Q_{\alpha_N t}^\eta(x, dy) - \int_{\mathbb{R}_+} \varphi(y) \vartheta_\eta(dy) \right| \leq \|\varphi'\|_\infty \left(x + \frac{c\eta + \frac{c_2}{N}\theta}{c + \frac{c_2}{N}} \right) \exp \left[- \left(c + \frac{c_2}{N} \right) \alpha_N t \right].$$

LEMMA 5.15: If $\sup_N \mathbb{E} \left[\bar{Y}_N(0) \right] < \infty$ then there is a constant C such that for any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative and for any $t > 0$,

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E} \left[\varphi(Y_i^{N,t}(Nt)) \middle| \tilde{Y}_{(N-\alpha_N)t} \right] - \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\tilde{Y}_N((N-\alpha_N)t)}(dy) \right| \right] \leq C \|\varphi'\|_\infty \exp \left[- \left(c + \frac{c_2}{N} \right) \alpha_N t \right]$$

PROOF OF THE INEQUALITY (5.10.4)**LEMMA 5.16:** Assume that $2c - \gamma^2 > 0$ and the existence of a positive δ for which

$$\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (Y_i^N(0))^{2+\delta} \right] < \infty.$$

For any positive real number T , there is a constant C_T such that for any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$\sup_{0 < t \leq T} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\tilde{Y}_N((N-\alpha_N)t)}(dy) - \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\tilde{Y}_N(Nt)}(dy) \right| \right] \leq C_T \|\varphi'\|_\infty \left(\sqrt{\frac{\alpha_N}{N}} + \frac{\alpha_N}{N} \right)$$

The proof of proposition 5.10 is complete. ■

¹ In the following, X_η and Y_η denote two strong solutions of

$$dX(t) = c(\eta - X(t)) dt + \frac{c_2}{N}(\theta - X(t)) dt + \gamma X(t) dW(t); \eta, \theta \geq 0$$

with respective initial conditions $X_\eta(0) \equiv x$ and $\mathcal{L}(Y_\eta(0)) = \vartheta_\eta$.

PROPOSITION 5.17: If $2c - \gamma^2 > 0$ and if there is a $\delta > 0$ for which

$$\sup_N \max_{0 \leq i \leq N-1} \mathbb{E} \left[\left(Y_i^N(0) \right)^{2+\delta} \right] < \infty,$$

then $\lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_s^t \frac{1}{N} \sum_{i=0}^{N-1} \left(Y_i^N(Nr) \right)^2 dr - \int_s^t \frac{2c}{2c - \gamma^2} \bar{Y}_N^2(Nr) dr \right| \right] = 0$, for any $0 < s \leq t < \infty$.

The proof of theorem 5.1 is completed. ■

The following result will be used only in the next chapter where we will prove the analog of theorem 3.1 for the two-levels system.

COROLLARY 5.18: Assume that $2c - \gamma^2 > 0$. Let X^N denote the N -dimensional process defined as the strong solution of

$$(5.18.1) \quad dY_i^N(t) = c(\bar{Y}_N(t) - Y_i^N(t))dt + \frac{c_2}{N}(\theta - Y_i^N(t))dt + \gamma Y_i^N(t)dW_i(t), \quad i \in \{0, 1, \dots, N-1\}$$

where θ is a nonnegative constant and Z^N is the one-dimensional strong solution of

$$(5.18.2) \quad dZ(t) = c_2(\theta - Z(t))dt + \sqrt{\frac{2c\gamma^2}{2c - \gamma^2}} Z(t)dW(t)$$

with initial condition $Z^N(0) \equiv \frac{1}{N} \sum_{i=0}^{N-1} X_i^N(0)$. If $\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^N(0) \right] < \infty$ then for any bounded continuous function φ with bounded first derivative,

$\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N}$ and $N(\varepsilon, K(\varepsilon)) \in \mathbb{N}$ such that $\forall K \geq K(\varepsilon)$ and $\forall N \geq N(\varepsilon, K(\varepsilon))$,

$$\mathbb{E} \left[\left| \varphi \left(\frac{1}{N} \sum_{i=0}^{N-1} X_i^N(NKt) \right) - \varphi(Z^N(Kt)) \right| \right] < \varepsilon$$

PROOF OF COROLLARY 5.18: The main steps of this proof are given by equations (5.18.3), (5.18.4) and (5.18.8).

Let Y^N and Z be respectively the strong solutions of (5.18.1) and (5.18.2) with the following initial conditions:

$$Y_i^N(0) \equiv y, \quad \forall i \in \{0, 1, \dots, N-1\}, \quad \forall N \in \mathbb{N} \text{ and } Z(0) \equiv y; \quad y \in \mathbb{R}_+.$$

From theorem 5.1, we know that $\mathcal{L}(\{\bar{Y}_N(Nt) : t \geq 0\})$ converges weakly (as N tends to infinity) to $\mathcal{L}(\{Z(t) : t \geq 0\})$. Therefore, for any $K > 0$ and any bounded continuous function φ ,

$$(5.18.3) \quad \mathbb{E}[\varphi(\bar{Y}_N(NKt))] \xrightarrow{N \rightarrow \infty} \mathbb{E}[\varphi(Z(Kt))].$$

We claim that if $\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^N(0) \right] < \infty$ then for any bounded continuous function φ with bounded first derivative,

$$(5.18.4) \quad \sup_N \left(\mathbb{E} \left[\varphi \left(\frac{1}{N} \sum_{i=0}^{N-1} X_i^N(NKt) \right) \right] - \mathbb{E} \left[\varphi \left(\frac{1}{N} \sum_{i=0}^{N-1} Y_i^N(NKt) \right) \right] \right) \xrightarrow{K \rightarrow \infty} 0^1$$

This means that the strong solution of (5.18.1) tends to forget its origin as time goes by.

Indeed, from coupling, we have

$$dX_i^N(t) - Y_i^N(t) \stackrel{p-a.s.}{=} c(\bar{X}_N(t) - \bar{Y}_N(t))dt - \left(c + \frac{c_2}{N}\right)(X_i^N(t) - Y_i^N(t))dt + \gamma(X_i^N(t) - Y_i^N(t))dW_i(t)$$

for any $i \in \{0, 1, \dots, N-1\}$. Therefore, for any t the local time $L_t^0(X_i^N - Y_i^N) = 0$ and Tanaka's formula gives, for any $0 \leq s \leq t$,

$$\begin{aligned} & \mathbb{E} \left[|X_i^N(t) - Y_i^N(t)| \right] \\ & \leq \mathbb{E} \left[|X_i^N(s) - Y_i^N(s)| \right] + c \int_s^t \mathbb{E} \left[|\bar{X}_N(r) - \bar{Y}_N(r)| \right] dr - \left(c + \frac{c_2}{N}\right) \int_s^t \mathbb{E} \left[|X_i^N(r) - Y_i^N(r)| \right] dr \end{aligned}$$

which implies that

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(t) - Y_i^N(t)| \right] \leq \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(s) - Y_i^N(s)| \right] - \frac{c_2}{N} \int_s^t \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(r) - Y_i^N(r)| \right] dr.$$

Therefore,

$$(5.18.5) \quad \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(t) - Y_i^N(t)| \right] \leq \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(s) - Y_i^N(s)| \right] \text{ for any } 0 \leq s \leq t$$

and, if we let $s = 0$,

$$(5.18.6) \quad \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(t) - Y_i^N(t)| \right] \leq \mathbb{E} [\bar{X}_N(0)] + \gamma - \frac{c_2}{N} \int_0^t \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(s) - Y_i^N(s)| \right] ds.$$

When we look at time NKt , the equation above becomes

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(NKt) - Y_i^N(NKt)| \right] + c_2 K \int_0^t \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(NKr) - Y_i^N(NKr)| \right] dr \leq \mathbb{E} [\bar{X}_N(0)] + \gamma.$$

Therefore

$$\begin{aligned} & c_2 K \sup_N \int_0^t \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(NKr) - Y_i^N(NKr)| \right] dr \\ & \leq c_2 K \sup_N \int_0^t \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(NKr) - Y_i^N(NKr)| \right] dr \leq \sup_N \mathbb{E} [\bar{X}_N(0)] + \gamma \end{aligned}$$

¹ Because of the law of the mean, it suffices to prove that $\mathbb{E} [|\bar{X}_N(NKt) - \bar{Y}_N(NKt)|]$ tends to zero as K tends to infinity uniformly in N .

From the law of the mean, we know there exists a $u \in (s, t)$ for which

$$\int_s^t \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(NK r) - Y_i^N(NK r)| \right] dr = \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(NK u) - Y_i^N(NK u)| \right] (t-s).$$

Replacing this in the previous inequality gives

$$\sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(NK u) - Y_i^N(NK u)| \right] \leq \frac{1}{c_2 K (t-s)} \left(\sup_N \mathbb{E} [\bar{X}_N(0)] + y \right) \xrightarrow{K \rightarrow \infty} 0.$$

Because of the inequality (5.18.5), we conclude that

$$(5.18.7) \quad \sup_N \mathbb{E} [|\bar{X}_N(NK t) - \bar{Y}_N(NK t)|] \leq \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} |X_i^N(NK t) - Y_i^N(NK t)| \right] \xrightarrow{K \rightarrow \infty} 0.$$

Lastly we need to prove that the strong solution of (5.18.2) also does not retain the past history! More precisely, we want to show that for any bounded continuous function φ with bounded first derivative,

$$(5.18.8) \quad \sup_N \mathbb{E} [\varphi(Z^N(K t)) - \varphi(Z(K t))] \xrightarrow{K \rightarrow \infty} 0.$$

In the proof of lemma 3.13, we established

$$\mathbb{E} [|Z(K t) - Z^N(K t)|] = \mathbb{E} \left[\left| y - \frac{1}{N} \sum_{i=0}^{N-1} X_i^N(0) \right| \right] \exp[-c_2 K t]$$

which implies that

$$\sup_N \mathbb{E} [|Z(K t) - Z^N(K t)|] \leq \left(y + \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} X_i^N(0) \right] \right) \exp[-c_2 K t] \xrightarrow{K \rightarrow \infty} 0.$$

We conclude the proof of (5.18.8) with the law of the mean. ■

Chapter 2

Second Level Bilinear System of Stochastic Differential Equations

Introduction

The novelty of this process is that the interaction between its components occurs at two levels whether they have been defined to be a close or distant neighbour. In fact, some multilevel processes have already been studied. For example, D. A. Dawson and A. Greven have considered Fisher-Wright and the Feller branching hierarchical models (see [DG](a), [DG](b) and [DG](c)) as well as the case in which the components follow a Feller continuous state branching. There is considerable literature written on the study of bilinear systems on the lattice $\mathbb{Z}^d$ but several problems remain open. See for examples [CGS], [Shi](a), [Shi](b) and [SS].

First, we define the stochastic process under study. Let $(\Omega, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t : t \geq 0\}, \mathbb{P})$ be a complete filtered probability space on which is defined a N^2 -dimensional Brownian motion $W = \{W_{i,j} : i, j \in \{0, 1, \dots, N-1\}\}$. We also assume that the filtration is right-continuous and that $\hat{\mathcal{F}}_0$ contains the collection of the $\mathbb{P}$ -null sets.

The stochastic process $X^N = (X_{0,0}^N, \dots, X_{0,N-1}^N, \dots, X_{N-1,0}^N, \dots, X_{N-1,N-1}^N)$ is defined as the unique strong solution of the second level bilinear system of stochastic differential equations:

$$(**) \quad dX_{i,j}^N(t) = c_1 (\bar{X}_{\bullet,j}^N(t) - X_{i,j}^N(t)) dt + \frac{c_2}{N} (\bar{X}_{\bullet\bullet}^N(t) - X_{i,j}^N(t)) dt + \gamma X_{i,j}^N(t) dW_{i,j}(t),$$
$$i, j \in \{0, 1, \dots, N-1\}$$

where

$$\bar{X}_{\bullet,j}^N(t) \equiv \frac{1}{N} \sum_{i=0}^{N-1} X_{i,j}^N(t), \quad \bar{X}_{\bullet\bullet}^N(t) \equiv \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} X_{i,j}^N(t)$$

and c_1, c_2 and γ are nonnegative constants (c_2 and γ are also assumed to be positive). $X_{i,j}^N(t)$ represents the "mass" of the i^{th} individual of the j^{th} colony at time t .

We will generalize some results we obtained in the mean-field case for this new process. In particular, we will focus on the time rescaling part.

The Finite System

We get for the two-level process equivalent results as the ones obtained in the mean-field case when we looked at the finite system. Since the proofs used the same ideas and there are no complications due to the sophistication of the system, we just state the properties of the strong solution without demonstrating it.

THEOREM 6.1¹: (Nonnegativity of the strong solution)

If $\forall i, j, X_{i,j}(0) \stackrel{\text{P-a.s.}}{\geq} 0$ then $X_{i,j}(t) \stackrel{\text{P-a.s.}}{\geq} 0, \forall t \geq 0$.

LEMMA 6.2²: (A bound for the k^{th} moment)

Let $k \geq 1$. Assuming that the initial conditions are nonnegative random variables with finite k^{th} moment,

$$\mathbb{E} \left[\left(\bar{X}_{..}^N(t) \right)^k \right] \leq \mathbb{E} \left[\left(\bar{X}_{..}^N(0) \right)^k \right] \exp \left[\frac{k(k-1)}{2} \gamma^2 t \right].$$

PROPOSITION 6.3³: Assuming the initial conditions are nonnegative square-integrable random variables,

$\{\bar{X}_{..}^N(t) : t \geq 0\}$ is a continuous nonnegative square-integrable $\{\tilde{W}_t : t \geq 0\}$ -martingale.

PROPOSITION 6.4⁴: (Extinction of the finite system through time)

Assuming the initial conditions are nonnegative square-integrable random variables,

$$\sum_{i,j=0}^{N-1} X_{i,j}^N(t) \xrightarrow{t \rightarrow \infty} 0 \text{ P-almost-surely.}$$

Computation of the Second Moments of the Finite System

We need to know the precise behaviour of the second moments of this process. The two next results will give us properties of these moments.

THEOREM 6.5: Assume that the initial conditions are nonnegative square-integrable random variables. If D represents the derivative operator $\left(D = \frac{d}{dt} \right)$, then the second moments satisfy the following system of differential equations:

$$\begin{bmatrix} D + 2c_1 + \frac{2c_2}{N} - \gamma^2 & -2c_1 & -\frac{2c_2}{N} \\ -\frac{\gamma^2}{N} & D + \frac{2c_2}{N} & -\frac{2c_2}{N} \\ -\frac{\gamma^2}{N^2} & 0 & D \end{bmatrix} \begin{bmatrix} \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(t))^2 \right] \\ \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{.j}^N(t))^2 \right] \\ \mathbb{E} \left[(\bar{X}_{..}^N(t))^2 \right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

PROOF OF THEOREM 6.5: This is a special case of theorem 8.5 but those three equations are easily obtained from Itô's formula. ■

-
- 1 This result is the equivalent of theorem 2.1 of the mean-field case.
 - 2 This result is the equivalent of lemma 2.2 of the mean-field case.
 - 3 This result is the equivalent the corollary of proposition 2.3 of the mean-field case.
 - 4 This result is the equivalent of proposition 2.4 of the mean-field case.

PROPOSITION 6.6: Assume that the initial conditions are nonnegative square-integrable random variables. If $\sup_N \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{ij}^N(0))^2 \right] < \infty$ and $\alpha_2 \equiv 2c_2(2c_1 - \gamma^2) - \gamma^2 2c_1 > 0$ then there exist a constant C such that

$$\mathbb{E} \left[\bar{X}_{..}^N(t) \right] \leq C \mathbb{E} \left[\bar{X}_{..}^N(0) \right] \exp \left[\frac{\gamma^2 2c_1 2c_2}{\alpha_2} \frac{t}{N^2} \right]$$

and
$$\mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{ij}^N(t))^2 \right] \leq C \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{ij}^N(0))^2 \right] \exp \left[\frac{\gamma^2 2c_1 2c_2}{\alpha_2} \frac{t}{N^2} \right].$$

REMARK: $\alpha_2 > 0 \Leftrightarrow \frac{1}{2c_1} + \frac{1}{2c_2} < \frac{1}{\gamma^2}.$

PROOF OF PROPOSITION 6.6: We will not restrict ourselves to the case $\alpha_2 > 0$ so a more general result will be obtain. Since the system in theorem 6.5 is homogeneous, we know that

$$(6.6.1) \quad \mathbb{E} \left[(\bar{X}_{..}^N(t))^2 \right] = \sum_{i=1}^3 k_i(N) \exp[r_i(N)t]$$

where $r_1(N), r_2(N), r_3(N)$ are the roots of

$$(6.6.2) \quad p(D) \equiv \det \begin{bmatrix} D + 2c_1 + \frac{2c_2}{N} - \gamma^2 & -2c_1 & -\frac{2c_2}{N} \\ -\frac{\gamma^2}{N} & D + \frac{2c_2}{N} & -\frac{2c_2}{N} \\ -\frac{\gamma^2}{N^2} & 0 & D \end{bmatrix}$$

and $k_1(N), k_2(N), k_3(N)$ are constants determined by the initial conditions. We will now determine those constants. From equation (6.6.1) we have

$$(6.6.3) \quad \mathbb{E} \left[(\bar{X}_{..}^N(0))^2 \right] = \sum_{i=1}^3 k_i(N).$$

From the third line of the system of 6.5 and equation (6.6.1), we obtain

$$(6.6.4) \quad \frac{\gamma^2}{N^2} \mathbb{E} \left[\frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (X_{ij}^N(t))^2 \right] = \frac{d}{dt} \mathbb{E} \left[(\bar{X}_{..}^N(t))^2 \right] = \sum_{i=1}^3 k_i(N) r_i(N) \exp[r_i(N)t]$$

which gives

$$(6.6.5) \quad \frac{\gamma^2}{N^2} \mathbb{E} \left[\frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (X_{ij}^N(0))^2 \right] = \sum_{i=1}^3 k_i(N) r_i(N)$$

We get the third equation we need to find the constants from the first line of the system of theorem 6.5:

$$\begin{aligned}
 (6.6.6) \quad & 2c_1 \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet j}^N(t))^2 \right] \\
 &= \frac{d}{dt} \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(t))^2 \right] + \left(2c_1 + \frac{2c_2}{N} - \gamma^2 \right) \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(t))^2 \right] - \frac{2c_2}{N} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(t))^2 \right] \\
 &= \frac{N^2}{\gamma^2} \sum_{i=1}^3 k_i(N) r_i^2(N) \exp[r_i(N)t] + \left(2c_1 + \frac{2c_2}{N} - \gamma^2 \right) \frac{N^2}{\gamma^2} \sum_{i=1}^3 k_i(N) r_i(N) \exp[r_i(N)t] \\
 &\quad - \frac{2c_2}{N} \sum_{i=1}^3 k_i(N) \exp[r_i(N)t]
 \end{aligned}$$

where the last equality comes from equations (6.6.1) and (6.6.4). Therefore

$$(6.6.7) \quad 2c_1 \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet j}^N(0))^2 \right] = \sum_{i=1}^3 \left[\frac{N^2}{\gamma^2} r_i^2(N) + \left(2c_1 + \frac{2c_2}{N} - \gamma^2 \right) \frac{N^2}{\gamma^2} r_i(N) - \frac{2c_2}{N} \right] k_i(N)$$

Solving the linear system (6.6.3), (6.6.5) and (6.6.7), we obtain

$$\begin{aligned}
 (6.6.8) \quad & k_1(N) = \frac{N^3 r_2(N) r_3(N) + 2c_2 \gamma^2}{N^3 (r_1(N) - r_3(N))(r_1(N) - r_2(N))} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right] \\
 & + \frac{2c_1 \gamma^2}{N^2 (r_1(N) - r_3(N))(r_1(N) - r_2(N))} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet j}^N(0))^2 \right] \\
 & + \frac{\gamma^2 (\gamma^2 - 2c_1 - 2c_2/N - r_2(N) - r_3(N))}{N^2 (r_1(N) - r_3(N))(r_1(N) - r_2(N))} \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right] \\
 & k_2(N) = \frac{N^3 r_1(N) r_3(N) + 2c_2 \gamma^2}{N^3 (r_2(N) - r_3(N))(r_2(N) - r_1(N))} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right] \\
 & + \frac{2c_1 \gamma^2}{N^2 (r_2(N) - r_3(N))(r_2(N) - r_1(N))} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet j}^N(0))^2 \right] \\
 & + \frac{\gamma^2 (\gamma^2 - 2c_1 - 2c_2/N - r_1(N) - r_3(N))}{N^2 (r_2(N) - r_3(N))(r_2(N) - r_1(N))} \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right]
 \end{aligned}$$

and

$$\begin{aligned}
 k_3(N) = & \frac{N^3 r_1(N) r_2(N) + 2c_2 \gamma^2}{N^3 (r_3(N) - r_2(N))(r_3(N) - r_1(N))} E \left[(\bar{X}_{**}^N(0))^2 \right] \\
 & + \frac{2c_1 \gamma^2}{N^2 (r_3(N) - r_2(N))(r_3(N) - r_1(N))} E \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{*j}^N(0))^2 \right] \\
 & + \frac{\gamma^2 (\gamma^2 - 2c_1 - \frac{2c_2}{N} - r_1(N) - r_2(N))}{N^2 (r_3(N) - r_2(N))(r_3(N) - r_1(N))} E \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right]
 \end{aligned}$$

Computing the roots give

$$\begin{aligned}
 p(D) = & D^3 + \left(2c_1 - \gamma^2 + 2\frac{2c_2}{N} \right) D^2 \\
 & + \left(\frac{2c_2(2c_1 - \gamma^2) - \gamma^2 2c_1}{N} + \frac{(2c_2)^2}{N^2} - \frac{\gamma^2 2c_2}{N^3} \right) D - \left(\frac{\gamma^2 2c_1 2c_2}{N^3} + \frac{\gamma^2 (2c_2)^2}{N^4} \right) \\
 = & D^3 + \left(\alpha_1 + 2\frac{2c_2}{N} \right) D^2 + \left(\frac{\alpha_2}{N} + \frac{(2c_2)^2}{N^2} - \frac{\gamma^2 2c_2}{N^3} \right) D - \left(\frac{\gamma^2 2c_1 2c_2}{N^3} + \frac{\gamma^2 (2c_2)^2}{N^4} \right)
 \end{aligned}$$

where $\alpha_1 \equiv 2c_1 - \gamma^2$ and $\alpha_2 \equiv 2c_2(2c_1 - \gamma^2) - \gamma^2 2c_1 = 2c_2 \alpha_1 - \gamma^2 2c_1$. Let p_1, p_2, p_3 be the roots of $p(x) \equiv x^3 + a_1 x^2 + a_2 x + a_3$, so that $-a_1 = p_1 + p_2 + p_3$, $a_2 = p_1 p_2 + p_2 p_3 + p_1 p_3$ and $-a_3 = p_1 p_2 p_3$, then it is possible to show that:

If $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ then

$$(6.6.9) \quad r_1(N) = -\alpha_1 + \mathcal{O}\left(\frac{1}{N}\right), \quad r_2(N) = -\frac{\alpha_2}{\alpha_1} \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \text{ and } r_3(N) = \frac{\gamma^2 2c_1 2c_2}{\alpha_2} \frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right);$$

if $\alpha_1 \neq 0$ and $\alpha_2 = 0$ then

$$(6.6.10) \quad r_1(N) = -\alpha_1 + \mathcal{O}\left(\frac{1}{N}\right), \quad r_2(N) = -\frac{\gamma^2 2c_1}{\alpha_1} \frac{1}{N^{3/2}} + \mathcal{O}\left(\frac{1}{N^2}\right) \text{ and } r_3(N) = \frac{\gamma^2 2c_1}{\alpha_1} \frac{1}{N^{3/2}} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

and if $\alpha_1 = 0$ and $\alpha_2 \neq 0$ then

$$(6.6.11) \quad r_1(N) = \frac{\gamma^2}{N} + \mathcal{O}\left(\frac{1}{N}\right), \quad r_2(N) = -\frac{\gamma^2}{N} + \mathcal{O}\left(\frac{1}{N}\right) \text{ and } r_3(N) = \frac{2c_2}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)$$

For the case $\alpha_1 = 0$ and $\alpha_2 = 0$, we return to the first level case because it implies that $c_2 = 0$.

Replacing the roots in (6.6.8), we obtain:

If $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ then

$$\begin{aligned}
 k_1(N) &= \frac{\gamma^2}{N^2} \frac{2c_1 \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet,j}^N(0))^2 \right] - \alpha_1 \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right]}{\alpha_1^2} + c \left(\frac{1}{N^2} \right) \\
 (6.6.12) \quad k_2(N) &= \frac{1}{N} \frac{-\frac{\gamma^2 2c_1 2c_2}{\alpha_1 \alpha_2} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right] + \gamma^2 2c_1 \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet,j}^N(0))^2 \right]}{\alpha_2} + c \left(\frac{1}{N} \right) \\
 k_3(N) &= \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right] + c(N^0);
 \end{aligned}$$

if $\alpha_1 \neq 0$ and $\alpha_2 = 0$ then

$$\begin{aligned}
 k_1(N) &= \frac{\gamma^2}{N^2} \frac{2c_1 \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet,j}^N(0))^2 \right] - \alpha_1 \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right]}{\alpha_1^2} + c \left(\frac{1}{N^2} \right) \\
 (6.6.13) \quad k_2(N) &= \frac{1}{2} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right] + c(N^0) \\
 k_3(N) &= \frac{1}{2} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right] + c(N^0)
 \end{aligned}$$

and if $\alpha_1 = 0$ and $\alpha_2 \neq 0$ then

$$\begin{aligned}
 k_1(N) &= \frac{1}{N} \frac{2c_1}{\gamma^2} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet,j}^N(0))^2 \right] + c \left(\frac{1}{N} \right) \\
 (6.6.14) \quad k_2(N) &= \frac{1}{N} \frac{2c_1}{\gamma^2} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet,j}^N(0))^2 \right] + c \left(\frac{1}{N} \right) \\
 k_3(N) &= \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right] + c(N^0)
 \end{aligned}$$

We now have all we need to complete the proof.

Note that if

$$\sup_N \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right] < \infty \text{ and } \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^\infty(0))^2 \right] \equiv \lim_{N \rightarrow \infty} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right]$$

then we can observe different limiting behaviors of $\mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(N^p t))^2 \right]$. In particular

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(N^2 t))^2 \right] &= \exp \left[\frac{\gamma^2 2c_1 2c_2 t}{\alpha_2} \right] \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^\infty(0))^2 \right] \\
 \lim_{N \rightarrow \infty} \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(N^{3/2} t))^2 \right] &= \frac{1}{2} (\exp[-2c_2 t] + \exp[2c_2 t]) \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^\infty(0))^2 \right]
 \end{aligned}$$

■

Time Rescaling

Behavior of $\{\bar{X}_{\bullet\bullet}^N(N^2 t) : t \in \mathbb{R}_+\}$ as N goes to infinity in the subcritical case ($\alpha_2 > 0$)

Theorem 7.1 is the major result of this chapter. It gives us, under certain assumptions, a weak limit for the average process when we speed up the time at an appropriate rate.

In this section, we will assume that the initial conditions are square-integrable nonnegative random variables. For any N , let P_N be the probability on $C(\mathbb{R}_+, \mathbb{R}_+)$ induced by the average process $\{\bar{X}_{\bullet\bullet}^N(N^2 t) : t \in \mathbb{R}_+\}$.

THEOREM 7.1: Let $\alpha_2 > 0$ and assume that the following assumptions on the initial conditions are satisfied: there is a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i, j \leq N-1} \mathbb{E} \left[\left(X_{i,j}^N(0) \right)^{2+\delta} \right] < \infty$$

and $\mathcal{L}(\bar{X}_{\bullet\bullet}^N(0))$ converges weakly to a probability μ . Then the sequence $\{P_N : N \in \mathbb{N}\}$ converges weakly (as N goes to infinity) to the unique solution of the martingale problem associated with $\mathcal{A}$ with initial distribution μ where

$$(\mathcal{A}f)(x) \equiv \frac{1}{2} \frac{\gamma^2 2c_1 2c_2}{\alpha_2} x^2 f''(x), \quad f \in C^2(\mathbb{R}, \mathbb{R}).$$

SKETCH OF THE PROOF OF THEOREM 7.1:

CLAIM: The sequence $\{P_N : N \in \mathbb{N}\}$ of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$ is tight.

Let $\{P_{N'}\}$ be an arbitrary convergent subsequence of $\{P_N : N \in \mathbb{N}\}$ and P_∞ denotes its limit point. We would like to show that P_∞ is a solution of the martingale problem associated with $\mathcal{A}$. To do so, it suffices to prove that

$$\left\{ f(\xi_t) - \int_0^t (\mathcal{A}f)(\xi_s) ds : t \in \mathbb{R}_+ \right\}$$

is a P_∞ -martingale for $f(x) \equiv x$ and $f(x) \equiv x^2$ where $\{\xi_t : t \in \mathbb{R}_+\}$ denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}_+)$.

CLAIM: $\{\xi_t : t \in \mathbb{R}_+\}$ is a P_∞ -martingale.

CLAIM: $\left\{ \xi_t^2 - \int_0^t \frac{\gamma^2 2c_1 2c_2}{\alpha_2} \xi_s^2 ds : t \in \mathbb{R}_+ \right\}$ is a P_∞ -martingale

The last two claims imply that P_∞ is a solution of the martingale problem associated with $\mathcal{A}$. Since this martingale problem has a unique solution then any convergent subsequence has P_∞ as its limit point and therefore, $\{P_N : N \in \mathbb{N}\}$ converges weakly to P_∞ as N goes to infinity.

LEMMA 7.2: If $\alpha_2 > 0$ then there exists a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i, j \leq N-1} \mathbb{E} \left[\left(X_{i,j}^N(0) \right)^{2+\delta} \right] < \infty \text{ implies } \sup_N \left(\sup_{0 \leq u \leq N^2 t} \max_{0 \leq i, j \leq N-1} \mathbb{E} \left[\left(X_{i,j}^N(u) \right)^{2+\delta} \right] \right) \leq C_t$$

for some constant C_t . Note that C_t depends on $\sup_N \max_{0 \leq i, j \leq N-1} \mathbb{E} \left[\left(X_{i,j}^N(0) \right)^{2+\delta} \right] < \infty$.

PROOF OF LEMMA 7.2: The ideas involved here are the same as the ones used in chapter one. Therefore, we will emphasize the differences and refer to the proof of lemma 3.3 for the missing details. For $i, j, i', j' \in \{0, 1, \dots, N-1\}$ define

$$a_t(i, j, i', j') \equiv \exp \left[- \left(c_1 + \frac{c_2}{N} \right) t \right] \delta_i(i') \delta_j(j') \\ + \frac{1}{N} \delta_j(j') \left(\exp \left[- \frac{c_2}{N} t \right] - \exp \left[- \left(c_1 + \frac{c_2}{N} \right) t \right] \right) + \frac{1}{N^2} \left(1 - \exp \left[- \frac{c_2}{N} t \right] \right)$$

where $\delta_i(i') = \begin{cases} 1 & \text{if } i' = i \\ 0 & \text{if } i' \neq i \end{cases}$ and note that:

$$\sum_{i'=0}^{N-1} a_t(i, j, i', j') = \delta_j(j') \exp \left[- \frac{c_2}{N} t \right] + \frac{1}{N} \left(1 - \exp \left[- \frac{c_2}{N} t \right] \right),$$

$$\sum_{i, j=0}^{N-1} a_t(i, j, i', j') = 1 = \sum_{i', j'=0}^{N-1} a_t(i, j, i', j'),$$

$$\begin{aligned} & \frac{d}{dr} a_{t-r}(i, j, i', j') \\ &= \frac{d}{dr} \left\{ \exp \left[- \left(c_1 + \frac{c_2}{N} \right) (t-r) \right] \delta_j(j') \left(\delta_i(i') - \frac{1}{N} \right) + \exp \left[- \frac{c_2}{N} (t-r) \right] \frac{1}{N} \left(\delta_j(j') - \frac{1}{N} \right) + \frac{1}{N^2} \right\} \\ &= \left(c_1 + \frac{c_2}{N} \right) \exp \left[- \left(c_1 + \frac{c_2}{N} \right) (t-r) \right] \delta_j(j') \left(\delta_i(i') - \frac{1}{N} \right) + \frac{c_2}{N} \exp \left[- \frac{c_2}{N} (t-r) \right] \frac{1}{N} \left(\delta_j(j') - \frac{1}{N} \right) \\ &= \left(c_1 + \frac{c_2}{N} \right) a_{t-r}(i, j, i', j') - c_1 \exp \left[- \frac{c_2}{N} (t-r) \right] \frac{1}{N} \left(\delta_j(j') - \frac{1}{N} \right) - \left(c_1 + \frac{c_2}{N} \right) \frac{1}{N^2} \\ &= \left(c_1 + \frac{c_2}{N} \right) a_{t-r}(i, j, i', j') - \frac{c_1}{N} \sum_{i'=0}^{N-1} a_t(i, j, i', j') - \frac{c_2}{N} \frac{1}{N^2} \end{aligned}$$

and

$$\begin{aligned} \sum_{i', j'=0}^{N-1} a_t^2(i, j, i', j') &= \sum_{i', j'=0}^{N-1} \left\{ \exp \left[- \left(c_1 + \frac{c_2}{N} \right) t \right] \delta_j(j') \left(\delta_i(i') - \frac{1}{N} \right) + \exp \left[- \frac{c_2}{N} t \right] \frac{1}{N} \left(\delta_j(j') - \frac{1}{N} \right) + \frac{1}{N^2} \right\}^2 \\ &= \sum_{i', j'=0}^{N-1} \left\{ \exp \left[- 2 \left(c_1 + \frac{c_2}{N} \right) t \right] \delta_j(j') \left(\delta_i(i') - \frac{1}{N} \right)^2 + \exp \left[- 2 \frac{c_2}{N} t \right] \frac{1}{N^2} \left(\delta_j(j') - \frac{1}{N} \right)^2 + \frac{1}{N^4} \right\} \\ &= \exp \left[- \left(2c_1 + \frac{2c_2}{N} \right) t \right] \left(1 - \frac{1}{N} \right) + \exp \left[- \frac{2c_2}{N} t \right] \frac{1}{N} \left(1 - \frac{1}{N} \right) + \frac{1}{N^2} \\ &= a_{2t}(0, 0, 0, 0) = a_{2t}(\mathbf{0}) \geq 0. \end{aligned}$$

From Itô's formula, we obtain for any fixed $t > 0$,

$$\begin{aligned} & \sum_{i',j'=0}^{N-1} a_{t-s}(i,j,i',j') X_{i',j'}^N(s) - \sum_{i',j'=0}^{N-1} a_t(i,j,i',j') X_{i',j'}^N(0) \\ &= \sum_{i',j'=0}^{N-1} \int_0^s a_{t-r}(i,j,i',j') \gamma X_{i',j'}^N(r) dW_{i',j'}(r). \end{aligned}$$

If we let $s = t$ then last equation becomes

$$X_{i,j}^N(t) - \sum_{i',j'=0}^{N-1} a_t(i,j,i',j') X_{i',j'}^N(0) \stackrel{p-a.s.}{=} \sum_{i',j'=0}^{N-1} \int_0^t a_{t-r}(i,j,i',j') \gamma X_{i',j'}^N(r) dW_{i',j'}(r) \equiv M_t.$$

Note that in the following, the constant C_p may vary. For any $u > 0$ and for any $p > 2$,

$$\begin{aligned} & \mathbb{E} \left[\left(X_{i,j}^N(u) \right)^p \right] \\ & \leq (p-1)^{p+1} \left(\int_0^u a_{2s}(0) ds \right)^{\frac{p}{2}-1} \int_0^u a_{2s}(0) \gamma^p \max_{0 \leq i',j' \leq N-1} \mathbb{E} \left[\left(X_{i',j'}^N(u-s) \right)^p \right] ds + C_p \end{aligned}$$

Therefore,

$$\begin{aligned} (7.2.1) \quad & \sup_{0 \leq u \leq t} \max_{0 \leq i,j \leq N-1} \mathbb{E} \left[\left(X_{i,j}^N(u) \right)^p \right] \\ & \leq (p-1)^{p+1} \left(\int_0^t a_{2s}(0) ds \right)^{\frac{p}{2}-1} \int_0^t a_{2s}(0) \gamma^p \sup_{0 \leq u \leq t-s} \max_{0 \leq i,j \leq N-1} \mathbb{E} \left[\left(X_{i,j}^N(u) \right)^p \right] ds + C_p. \end{aligned}$$

Define

$$f(t) \equiv \sup_{0 \leq u \leq t} \max_{0 \leq i,j \leq N-1} \mathbb{E} \left[\left(X_{i,j}^N(u) \right)^p \right] \text{ and } F(\theta, T) \equiv \int_0^{N^2 T} \exp[-\theta t] f(t) dt; \theta > 0, T > 0.$$

By integrating (7.2.1), we find

$$\begin{aligned} & F(\theta, T) \\ & \leq (p-1)^{p+1} \left(\int_0^{N^2 T} a_{2s}(0) ds \right)^{\frac{p}{2}-1} \gamma^p F(\theta, T) \int_0^\infty \exp[-\theta s] a_{2s}(0) ds + \frac{C_p}{\theta} \end{aligned}$$

Hence,

$$(7.2.2) \quad \frac{\theta}{N^2} F\left(\frac{\theta}{N^2}, T\right) \left(1 - (p-1)^{p+1} \left(\int_0^{N^2 T} a_{2s}(0) ds \right)^{\frac{p}{2}-1} \gamma^p \int_0^\infty \exp\left[-\frac{\theta}{N^2} s\right] a_{2s}(0) ds \right) \leq C_p.$$

We now claim that for large N ,

$$(p-1)^{p+1} \left(\int_0^{N^2 T} a_{2s}(0) ds \right)^{\frac{p}{2}-1} \gamma^p \int_0^\infty \exp\left[-\frac{\theta}{N^2} s\right] a_{2s}(0) ds < 1.$$

Indeed, if we first note that

$$\int_0^\infty \exp\left[-\frac{\theta}{N^2}s\right] a_{2s}(\mathbf{0}) ds \geq \int_0^{N^2 T} \exp\left[-\frac{\theta}{N^2}s\right] a_{2s}(\mathbf{0}) ds \geq \exp[-\theta T] \int_0^{N^2 T} a_{2s}(\mathbf{0}) ds$$

then

$$\begin{aligned} & (p-1)^{p+1} \left(\int_0^{N^2 T} a_{2s}(\mathbf{0}) ds \right)^{\frac{p}{2}-1} \gamma^p \int_0^\infty \exp\left[-\frac{\theta}{N^2}s\right] a_{2s}(\mathbf{0}) ds \\ & \leq (p-1)^{p+1} \exp\left[\theta T \left(\frac{p}{2}-1\right)\right] \left(\int_0^\infty \exp\left[-\frac{\theta}{N^2}s\right] a_{2s}(\mathbf{0}) ds \right)^{\frac{p}{2}} \gamma^p \\ & = (p-1)^{p+1} \exp\left[\theta T \left(\frac{p}{2}-1\right)\right] \left[\left(1-\frac{1}{N}\right) \frac{1}{2c_1 + \frac{2c_2}{N} + \frac{\theta}{N^2}} + \left(1-\frac{1}{N}\right) \frac{1}{N} \frac{1}{\frac{2c_2}{N} + \frac{\theta}{N^2}} + \frac{1}{\theta} \right]^{\frac{p}{2}} \gamma^p \\ & \xrightarrow{N \rightarrow \infty} (p-1)^{p+1} \exp\left[\theta T \left(\frac{p}{2}-1\right)\right] \left(\frac{\gamma^2}{2c_1} + \frac{\gamma^2}{2c_2} + \frac{\gamma^2}{\theta} \right)^{\frac{p}{2}}. \end{aligned}$$

Because

$$\begin{aligned} \alpha_2 > 0 & \Rightarrow \frac{1}{2c_1} + \frac{1}{2c_2} < \frac{1}{\gamma^2}, \\ \theta > \frac{\gamma^2 2c_1 2c_2}{\alpha_2} & \Rightarrow \frac{\gamma^2}{2c_1} + \frac{\gamma^2}{2c_2} + \frac{\gamma^2}{\theta} < 1 \Rightarrow \left(\frac{\gamma^2}{2c_1} + \frac{\gamma^2}{2c_2} + \frac{\gamma^2}{\theta} \right)^{\frac{p}{2}} < 1, \end{aligned}$$

it is possible to make

$$(p-1)^{p+1} \exp\left[\theta T \left(\frac{p}{2}-1\right)\right] \left(\frac{\gamma^2}{2c_1} + \frac{\gamma^2}{2c_2} + \frac{\gamma^2}{\theta} \right)^{\frac{p}{2}} < 1$$

by choosing p close enough to 2. Going back to equation (7.2.2), we can find a constant K and a $N_0 \in \mathbb{N}$ for which

$$\sup_{N \geq N_0} \frac{\theta}{N^2} F\left(\frac{\theta}{N^2}, T\right) \leq \sup_{N \geq N_0} C_p \left[1 - (p-1)^{p+1} \left(\int_0^{N^2 T} a_{2s}(\mathbf{0}) ds \right)^{\frac{p}{2}-1} \gamma^p \int_0^\infty \exp\left[-\frac{\theta}{N^2}s\right] a_{2s}(\mathbf{0}) ds \right]^{-1} \leq K.$$

Therefore,

$$\sup_{N \geq N_0} \left(\sup_{0 \leq u \leq N^2 \frac{T}{2}} \max_{0 \leq i, j \leq N-1} \mathbb{E} \left[\left(X_{i,j}^N(u) \right)^p \right] \right) \leq \frac{2}{T} \frac{K}{\theta} \exp[\theta T]. \quad \blacksquare$$

The next proposition confirms our first claim. Its proof is omitted since one can easily adapt the corresponding one given in the mean field context. For the same reason, the proofs of propositions 7.4 and 7.5 are missing.

PROPOSITION 7.3: If $\alpha_2 > 0$ and

$$\sup_N \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right] < \infty$$

then the sequence $\{P_N : N \in \mathbb{N}\}$ of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$ is tight.

PROPOSITION 7.4: If $\alpha_2 > 0$ and $\sup_N \mathbb{E} \left[(\bar{X}_{\bullet\bullet}^N(0))^2 \right] < \infty$ then $\{\xi_t : t \in \mathbb{R}_+\}$ is a P_∞ -martingale.

PROPOSITION 7.5: If $\alpha_2 > 0$ and if there exists a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i,j \leq N-1} \mathbb{E} \left[(X_{i,j}^N(0))^{2+\delta} \right] < \infty$$

then $\left\{ \xi_t^2 - \int_0^t \frac{\gamma^2 2c_1 2c_2}{\alpha_2} \xi_s^2 ds : t \in \mathbb{R}_+ \right\}$ is a P_∞ -martingale

provided $\lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_s^t \frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(N^2 r))^2 dr - \int_s^t \frac{\gamma^2 2c_1 2c_2}{\alpha_2} (\bar{X}_{\bullet\bullet}^N(N^2 r))^2 dr \right| \right] = 0.$

This last condition is verified if some hypothesis on the initial conditions are respected. It is the object of proposition 7.7.

LEMMA 7.6: Assume that $\alpha_2 > 0$ and $\sup_N \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right] < \infty.$

For any positive real number T , there is a constant C_T such that for any real numbers s_1 and s_2 with $0 \leq s_1 \leq s_2 \leq N^2 T$,

$$(i) \quad \mathbb{E} \left[\left| \bar{X}_{\bullet\bullet}^N(s_2) - \bar{X}_{\bullet\bullet}^N(s_1) \right| \right] \leq C_T \sqrt{\frac{s_2 - s_1}{N^2}};$$

$$(ii) \quad \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[\left| \bar{X}_{\bullet j}^N(s_2) - \bar{X}_{\bullet j}^N(s_1) \right| \right] \leq C_T \left(\sqrt{\frac{s_2 - s_1}{N}} + \frac{s_2 - s_1}{N} \right).$$

PROOF OF LEMMA 7.6: This is a particular case of lemma 9.9. Equation (i) can be shown in the same way as lemma 3.11 by using the martingale property of the average process. Let us prove the equation (ii). Taking absolute value in the integral form of (**),

$$\left| \bar{X}_{\bullet j}^N(s_2) - \bar{X}_{\bullet j}^N(s_1) \right| \stackrel{p.a.s.}{\leq} \frac{c_2}{N} \int_{s_1}^{s_2} \bar{X}_{\bullet\bullet}^N(r) dr + \frac{c_2}{N} \int_{s_1}^{s_2} \bar{X}_{\bullet j}^N(r) dr + \left| \gamma \frac{1}{N} \sum_{i=0}^{N-1} \int_{s_1}^{s_2} X_{i,j}^N(r) dW_{i,j}(r) \right|$$

Therefore,

$$(7.6.1) \quad \begin{aligned} \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[\left| \bar{X}_{\bullet,j}^N(s_2) - \bar{X}_{\bullet,j}^N(s_1) \right| \right] &\leq \frac{2c_2}{N} \int_{s_1}^{s_2} \mathbb{E} \left[\bar{X}_{\bullet,\bullet}^N(r) \right] dr + \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[\left| \gamma \frac{1}{N} \sum_{i=0}^{N-1} \int_{s_1}^{s_2} X_{i,j}^N(r) dW_{i,j}(r) \right| \right] \\ &= 2c_2 \mathbb{E} \left[\bar{X}_{\bullet,\bullet}^N(0) \right] \frac{s_2 - s_1}{N} + \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[\left| \gamma \frac{1}{N} \sum_{i=0}^{N-1} \int_{s_1}^{s_2} X_{i,j}^N(r) dW_{i,j}(r) \right| \right]. \end{aligned}$$

Using Jensen's inequality,

$$(7.6.2) \quad \begin{aligned} \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[\left| \gamma \frac{1}{N} \sum_{i=0}^{N-1} \int_{s_1}^{s_2} X_{i,j}^N(r) dW_{i,j}(r) \right| \right] &\leq \frac{1}{N} \sum_{j=0}^{N-1} \sqrt{\mathbb{E} \left[\left| \gamma \frac{1}{N} \sum_{i=0}^{N-1} \int_{s_1}^{s_2} X_{i,j}^N(r) dW_{i,j}(r) \right|^2 \right]} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \sqrt{\mathbb{E} \left[\left\langle \gamma \frac{1}{N} \sum_{i=0}^{N-1} \int_{s_1}^{s_2} X_{i,j}^N(r) dW_{i,j}(r) \right\rangle (s_2) \right]} = \frac{1}{N} \sum_{j=0}^{N-1} \sqrt{\frac{\gamma^2}{N} \int_{s_1}^{s_2} \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} (X_{i,j}^N(r))^2 \right] dr} \\ &\leq \sqrt{\frac{\gamma^2}{N} \int_{s_1}^{s_2} \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(r))^2 \right] dr} \leq C_T \sqrt{\frac{s_2 - s_1}{N}} \quad \blacksquare \end{aligned}$$

PROPOSITION 7.7: Assume that $\alpha_2 > 0$. If there exists a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i,j \leq N-1} \mathbb{E} \left[(X_{i,j}^N(0))^{2+\delta} \right] < \infty,$$

then for any $0 \leq s \leq t < \infty$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \gamma^2 \int_s^t \frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(N^2 r))^2 dr - \frac{\gamma^2 2c_1 2c_2}{\alpha_2} \int_s^t (\bar{X}_{\bullet,\bullet}^N(N^2 r))^2 dr \right| \right] = 0.$$

PROOF OF PROPOSITION 7.7: To prove the equation above, we will show that

$$(7.7.1) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \gamma^2 \int_s^t \frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(N^2 r))^2 dr - \frac{\gamma^2 2c_1}{\alpha_1} \int_s^t \frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet,j}^N(N^2 r))^2 dr \right| \right] = 0$$

and

$$(7.7.2) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{\gamma^2 2c_1}{\alpha_1} \int_s^t \frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet,j}^N(N^2 r))^2 dr - \frac{\gamma^2 2c_1 2c_2}{\alpha_2} \int_s^t (\bar{X}_{\bullet,\bullet}^N(N^2 r))^2 dr \right| \right] = 0.$$

¹ For the first inequality, we use the following:

$$\begin{aligned} \forall i, a_i \geq 0 &\Rightarrow n \sum_{i=1}^n a_i - \sum_{i=1}^n \sum_{j=1}^n \sqrt{a_i} \sqrt{a_j} = \sum_{i=1}^n \sum_{j=i+1}^n (\sqrt{a_i} - \sqrt{a_j})^2 \geq 0 \\ &\Rightarrow n \sum_{i=1}^n a_i \geq \sum_{i=1}^n \sum_{j=1}^n \sqrt{a_i} \sqrt{a_j} \Rightarrow \sqrt{n \sum_{i=1}^n a_i} \geq \sum_{i=1}^n \sqrt{a_i}. \end{aligned}$$

It is also a consequence of Jensen's inequality since $-\sqrt{x}$ is a convex function. The second inequality is given by proposition 6.6.

We will prove equation (7.7.1) as we did in the mean field case. The differences will arise when we will work on the proof of equation (7.7.2).

PROOF OF EQUATION (7.7.1): The next proof is an adaptation of what we have done in the one-level case to the two-levels case. Let $\{\beta_N : N \in \mathbb{N}\}$ be an increasing sequence of positive real numbers ($\beta_N \leq N^2$). For any fixed $t > 0$, we introduce a modified process, denoted

$$\{Z_{i,j}^{N,t}(s) : i, j \in \{0, 1, \dots, N-1\}, s \geq 0\},$$

as follows: If $0 \leq s \leq (N^2 - \beta_N)t$ then

$$Z_{i,j}^{N,t}(s) \equiv X_{i,j}^N(s)$$

and if $s > (N^2 - \beta_N)t$ then

$$(7.7.3) \quad Z_{i,j}^{N,t}(s) \equiv X_{i,j}^N((N^2 - \beta_N)t) + \left(c_1 + \frac{c_2}{N}\right) \int_{(N^2 - \beta_N)t}^s \left(\bar{X}_{\bullet,j}^N((N^2 - \beta_N)t) - Z_{i,j}^{N,t}(r)\right) dr + \gamma \int_{(N^2 - \beta_N)t}^s Z_{i,j}^{N,t}(r) dW_{i,j}(r).$$

We will show that for any bounded continuous differentiable function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$(7.7.4) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N^2} \sum_{i,j=0}^{N-1} \left(\varphi(X_{i,j}^N(N^2 t)) - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,j}^N((N^2 - \beta_N)t), c_1 + c_2/N, \gamma}(dy) \right) \right| \right] = 0^1$$

by showing the next four inequalities:

$$(7.7.5) \quad \sup_{0 < t \leq T} \frac{1}{N^2} \sum_{i,j=0}^{N-1} \mathbb{E} \left[\left| \varphi(X_{i,j}^N(N^2 t)) - \varphi(Z_{i,j}^{N,t}(N^2 t)) \right| \right] \leq C_T \|\varphi'\|_{\infty} \left(\sqrt{\frac{\beta_N^3}{N}} + \frac{\beta_N^2}{N} + \frac{\beta_N}{N} \right)$$

$$(7.7.6) \quad \sup_{t > 0} \mathbb{E} \left[\left| \frac{1}{N^2} \sum_{i,j=0}^{N-1} \left(\varphi(Z_{i,j}^{N,t}(N^2 t)) - \mathbb{E} \left[\varphi(Z_{i,j}^{N,t}(N^2 t)) \middle| \tilde{\Lambda}_{(N^2 - \beta_N)t} \right] \right) \right| \right] \leq \frac{2 \|\varphi\|_{\infty}}{N}^2$$

$$(7.7.7) \quad \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} \left| \mathbb{E} \left[\varphi(Z_{i,j}^{N,t}(N^2 t)) \middle| \tilde{\Lambda}_{(N^2 - \beta_N)t} \right] - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,j}^N((N^2 - \beta_N)t), c_1 + c_2/N, \gamma}(dy) \right| \right] \leq C \|\varphi'\|_{\infty} \exp \left[- \left(2c_1 + \frac{2c_2}{N} \right) \beta_N t \right], \forall t > 0^3$$

¹ Recall that $P_t^{\theta, c, \gamma}(x, dy)$ and $v_{\theta, c, \gamma}$ are respectively the transition kernel and the equilibrium of the one-dimensional diffusion characterized by equation (3.7.1).

² Because $\{Z_{i,j}^{N,t}(N^2 t) : i, j \in \{0, 1, \dots, N-1\}\}$ are conditionally independent with respect to $\mathbb{P} \left[\bullet \middle| \tilde{\Lambda}_{(N^2 - \beta_N)t} \right]$. See the demonstration of inequality (3.10.2) for more details.

³ This inequality is a direct application of lemma 7.9 because

$$(7.7.8) \quad \sup_{0 < t \leq T} \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,j}^N((N^2 - \beta_N)t), c_1 + c_2/N, \gamma}(dy) - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,j}^N(N^2 t), c_1 + c_2/N, \gamma}(dy) \right| \right] \\ \leq \|\varphi'\|_{\infty} C_T \max \left\{ \sqrt{\frac{\beta_N}{N}}, \frac{\beta_N}{N} \right\},$$

When (7.7.4) is established, we apply the methodology used in the proof of proposition 3.17 to conclude that equation (7.7.1) is true. Recall that the original idea was to use a sequence $\{\varphi_M : M \in \mathbb{N}\}$ of bounded continuous differentiable functions with bounded first derivative that increase to $\varphi(y) \equiv \gamma^2 y^2$ and, from the uniform integrability of the random variables involved, to prove that the property (7.7.4) is preserved in the limit (as M tends to infinity) to give

$$(7.7.9) \quad 0 = \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N^2} \sum_{i,j=0}^{N-1} \left(\gamma^2 (X_{i,j}^N(N^2 t))^2 - \int_{\mathbb{R}_+} \gamma^2 y^2 v_{\bar{X}_{\bullet,j}^N(N^2 t), c_1 + c_2/N, \gamma}(dy) \right) \right| \right] \\ = \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N^2} \sum_{i,j=0}^{N-1} \left(\gamma^2 (X_{i,j}^N(N^2 t))^2 - \frac{2(c_1 + \frac{c_2}{N})\gamma^2}{2(c_1 + \frac{c_2}{N}) - \gamma^2} (\bar{X}_{\bullet,j}^N(N^2 t))^2 \right) \right| \right]$$

The proof of equation (7.7.1) is then complete since

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \left(\frac{2c_1\gamma^2}{2c_1 - \gamma^2} (\bar{X}_{\bullet,j}^N(N^2 t))^2 - \frac{2(c_1 + \frac{c_2}{N})\gamma^2}{2(c_1 + \frac{c_2}{N}) - \gamma^2} (\bar{X}_{\bullet,j}^N(N^2 t))^2 \right) \right| \right] \\ \leq \left| \frac{2c_1\gamma^2}{2c_1 - \gamma^2} - \frac{2(c_1 + \frac{c_2}{N})\gamma^2}{2(c_1 + \frac{c_2}{N}) - \gamma^2} \right| \sup_N \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} (\bar{X}_{\bullet,j}^N(N^2 t))^2 \right] \\ \leq \left| \frac{2c_1\gamma^2}{2c_1 - \gamma^2} - \frac{2(c_1 + \frac{c_2}{N})\gamma^2}{2(c_1 + \frac{c_2}{N}) - \gamma^2} \right| C \xrightarrow{N \rightarrow \infty} 0.$$

$$\mathbb{E} \left[\varphi(Z_{i,j}^{N,t}(N^2 t)) \right] \tilde{P}_{(N^2 - \beta_N)t} = \int_{\mathbb{R}_+} \varphi(y) P_{\beta_N t}^{\bar{X}_{\bullet,j}^N((N^2 - \beta_N)t), c_1 + c_2/N, \gamma}(X_{i,j}^N((N^2 - \beta_N)t), dy)$$

¹ From lemma 7.10.

PROOF OF INEQUALITY (7.7.5): For $s > (N^2 - \beta_N)t$,

$$\begin{aligned} & X_{i,j}^N(s) - Z_{i,j}^{N,t}(s) \\ &= c_1 \int_{(N^2 - \beta_N)t}^s \left(\bar{X}_{\bullet,j}^N(r) - \bar{X}_{\bullet,j}^N((N^2 - \beta_N)t) \right) dr + \frac{c_2}{N} \int_{(N^2 - \beta_N)t}^s \left(\bar{X}_{\bullet,\bullet}^N(r) - \bar{X}_{\bullet,\bullet}^N((N^2 - \beta_N)t) \right) dr \\ &\quad - \left(c_1 + \frac{c_2}{N} \right) \int_{(N^2 - \beta_N)t}^s \left(X_{i,j}^N(r) - Z_{i,j}^{N,t}(r) \right) dr + \gamma \int_{(N^2 - \beta_N)t}^s \left(X_{i,j}^N(r) - Z_{i,j}^{N,t}(r) \right) dW_{i,j}(r). \end{aligned}$$

The following is very similar to the proof of lemma 3.12 therefore, we will only give the main arguments. The difference of local times

$$L_{N^2 t}^0(X_{i,j}^N - Z_{i,j}^{N,t}) - L_{(N^2 - \beta_N)t}^0(X_{i,j}^N - Z_{i,j}^{N,t}) = 0.$$

From Tanaka's formula,

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} |X_{i,j}^N(N^2 t) - Z_{i,j}^{N,t}(N^2 t)| \right] \leq c_1 \int_{(N^2 - \beta_N)t}^{N^2 t} \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} |\bar{X}_{\bullet,j}^N(r) - \bar{X}_{\bullet,j}^N((N^2 - \beta_N)t)| \right] dr \\ & \quad + \frac{c_2}{N} \int_{(N^2 - \beta_N)t}^{N^2 t} \mathbb{E} \left[\frac{1}{N} \sum_{j=1}^{N-1} |\bar{X}_{\bullet,\bullet}^N(r) - \bar{X}_{\bullet,\bullet}^N((N^2 - \beta_N)t)| \right] dr \\ & \leq C_T \int_{(N^2 - \beta_N)t}^{N^2 t} \sqrt{\frac{r - (N^2 - \beta_N)t}{N} + \frac{r - (N^2 - \beta_N)t}{N}} dr + \frac{c_2}{N} \int_{(N^2 - \beta_N)t}^{N^2 t} 2 \mathbb{E} [\bar{X}_{\bullet,\bullet}^N(0)] dr \\ & \leq C_T \left(\sqrt{\frac{\beta_N^3}{N} + \frac{\beta_N^2}{N} + \frac{\beta_N}{N}} \right) \end{aligned}$$

We then use the law of the mean to conclude. ■

The proof of equation (7.7.1) is completed. ■

PROOF OF EQUATION (7.7.2): For fixed $t > 0$ and $K > 0$, we introduce a modified process, denoted

$$\{Z_{i,j}^{N,K,t}(s) : i, j \in \{0, 1, \dots, N-1\}, s \geq 0\},$$

as follows: If $0 \leq s \leq (N^2 - NK)t$ then

$$Z_{i,j}^{N,K,t}(s) \equiv X_{i,j}^N(s)$$

¹ From lemma 7.6 (ii).

and if $s > (N^2 - NK)t$ then

$$(7.7.10) \quad \begin{aligned} Z_{i,j}^{N,K,t}(s) &\equiv X_{i,j}^N((N^2 - NK)t) \\ &+ c_1 \int_{(N^2 - NK)t}^s (\bar{Z}_{\bullet,j}^{N,K,t}(r) - Z_{i,j}^{N,K,t}(r)) dr + \frac{c_2}{N} \int_{(N^2 - NK)t}^s (\bar{X}_{\bullet\bullet}^N((N^2 - NK)t) - Z_{i,j}^{N,K,t}(r)) dr \\ &+ \sqrt{\frac{2c_1\gamma^2}{\alpha_1}} \int_{(N^2 - NK)t}^s \bar{Z}_{\bullet,j}^{N,K,t}(r) dW_{i,j}(r). \end{aligned}$$

Therefore, if $0 \leq s \leq (N^2 - NK)t$ then $\bar{Z}_{\bullet,j}^{N,K,t}(s) = \bar{X}_{\bullet,j}^N(s)$ and if $s > (N^2 - NK)t$ then

$$(7.7.11) \quad \begin{aligned} \bar{Z}_{\bullet,j}^{N,K,t}(s) &= \bar{X}_{\bullet,j}^N((N^2 - NK)t) \\ &+ \frac{c_2}{N} \int_{(N^2 - NK)t}^s \bar{X}_{\bullet\bullet}^N((N^2 - NK)t) - \bar{Z}_{\bullet,j}^{N,K,t}(r) dr + \sqrt{\frac{2c_1\gamma^2}{\alpha_1}} \frac{1}{N} \sum_{i=0}^{N-1} \int_{(N^2 - NK)t}^s \bar{Z}_{\bullet,j}^{N,K,t}(r) dW_{i,j}(r) \end{aligned}$$

$$(7.7.12) \quad \begin{aligned} &= \bar{X}_{\bullet,j}^N((N^2 - NK)t) \\ &+ \frac{c_2}{N} \int_{(N^2 - NK)t}^s \bar{X}_{\bullet\bullet}^N((N^2 - NK)t) - \bar{Z}_{\bullet,j}^{N,K,t}(r) dr + \sqrt{\frac{2c_1\gamma^2}{N\alpha_1}} \int_{(N^2 - NK)t}^s \bar{Z}_{\bullet,j}^{N,K,t}(r) d\left(\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} W_{i,j}(r)\right). \end{aligned}$$

Note that $\left\{ \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} W_{i,j}(r) : r \geq 0 \right\}$ is a Brownian motion for any j .

We will show that for any bounded continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$(7.7.13) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \left(\varphi(\bar{X}_{\bullet,j}^N(N^2 t)) - \int_{\mathbb{R}_+} \varphi(y) \nu_{\bar{X}_{\bullet\bullet}^N(N^2 t)} \cdot \frac{c_2}{N} \cdot \sqrt{\frac{2c_1\gamma^2}{N\alpha_1}}(dy) \right) \right| \right] = 0$$

by showing the next four inequalities: There is a constant C and, for any positive real number T , there is a constant C_T such that

(7.7.14) for any $0 < t \leq T$ and for any $\varepsilon > 0$, there is $K(\varepsilon) \in \mathbb{N}$ and $N(\varepsilon, K(\varepsilon)) \in \mathbb{N}$ such that for any $K \geq K(\varepsilon)$ and any $N \geq N(\varepsilon, K(\varepsilon))$,

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \left(\varphi(\bar{X}_{\bullet,j}^N(N^2 t)) - \varphi(\bar{Z}_{\bullet,j}^{N,K,t}(N^2 t)) \right) \right| \right] \leq C_T \left(\frac{K^2}{N} + \sqrt{\frac{K^3}{N}} \right) + \sqrt{\frac{4\|\varphi\|_\infty^2}{N^2} + \varepsilon^2}$$

$$(7.7.15) \quad \sup_{t>0} \mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \left(\varphi(\bar{Z}_{\bullet,j}^{N,K,t}(N^2 t)) - \mathbb{E} \left[\varphi(\bar{Z}_{\bullet,j}^{N,K,t}(N^2 t)) \middle| \tilde{\mathcal{H}}_{(N^2-NK)t} \right] \right) \right| \right] \leq \frac{2\|\varphi\|_{\infty}}{\sqrt{N}}$$

$$(7.7.16) \quad \mathbb{E} \left[\frac{1}{N} \sum_{j=0}^{N-1} \left| \mathbb{E} \left[\varphi(\bar{Z}_{\bullet,j}^{N,K,t}(N^2 t)) \middle| \tilde{\mathcal{H}}_{(N^2-NK)t} \right] - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,\bullet}^N((N^2-NK)t), \frac{c_2}{N}, \sqrt{\frac{2c_1\gamma^2}{N\alpha_1}}} (dy) \right| \right] \leq C \|\varphi'\|_{\infty} \exp[-2c_2 K t], \quad \forall t > 0^2$$

$$(7.7.17) \quad \sup_{0 < t \leq T} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,\bullet}^N((N^2-NK)t), \frac{c_2}{N}, \sqrt{\frac{2c_1\gamma^2}{N\alpha_1}}} (dy) - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,\bullet}^N(N^2 t), \frac{c_2}{N}, \sqrt{\frac{2c_1\gamma^2}{N\alpha_1}}} (dy) \right| \right] \leq \|\varphi'\|_{\infty} C_T \max \left\{ \sqrt{\frac{K}{N}}, \frac{K}{N} \right\}^3$$

The last three inequalities are already shown. It remains to demonstrate the inequality (7.7.14) which hides the difficulty associated with the multilevel property.

Again, as we did previously in the proof of proposition 3.17, we use a sequence $\{\varphi_M : M \in \mathbb{N}\}$ of functions with the above properties that increase to $\frac{2c_1\gamma^2}{\alpha_1} y^2$ and conclude from the uniform integrability of the variables involved that the property (7.6.13) is preserved in the limit (as M tends to infinity) to give

$$(7.7.18) \quad 0 = \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \left(\frac{2c_1\gamma^2}{\alpha_1} (\bar{X}_{\bullet,j}^N(N^2 t))^2 - \int_{\mathbb{R}_+} \frac{2c_1\gamma^2}{\alpha_1} y^2 v_{\bar{X}_{\bullet,\bullet}^N(N^2 t), \frac{c_2}{N}, \sqrt{\frac{2c_1\gamma^2}{N\alpha_1}}} (dy) \right) \right| \right] \\ = \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \left(\frac{2c_1\gamma^2}{\alpha_1} (\bar{X}_{\bullet,j}^N(N^2 t))^2 - \frac{2c_2}{\alpha_2} \frac{2c_1\gamma^2}{\alpha_1} (\bar{X}_{\bullet,j}^N(N^2 t))^2 \right) \right| \right]$$

¹ Because $\{\bar{Z}_{\bullet,j}^{N,K,t}(N^2 t) : j \in \{0, 1, \dots, N-1\}\}$ are conditionally independent with respect to $\mathbb{P} \left[\bullet \middle| \tilde{\mathcal{H}}_{(N^2-NK)t} \right]$. See the demonstration of inequality (3.10.2) for more details.

² It is a direct application of lemma 7.9 because, from equation (7.7.12), we have

$$\mathbb{E} \left[\varphi(\bar{Z}_{\bullet,j}^{N,K,t}(N^2 t)) \middle| \tilde{\mathcal{H}}_{(N^2-NK)t} \right] = \int_{\mathbb{R}_+} \varphi(y) P_{NKt}^{\bar{X}_{\bullet,\bullet}^N((N^2-NK)t), \frac{c_2}{N}, \sqrt{\frac{2c_1\gamma^2}{N\alpha_1}}} (\bar{X}_{\bullet,j}^N((N^2-NK)t), dy).$$

³ From lemma 7.10.

PROOF OF INEQUALITY (7.7.14):

LEMMA 7.8: For any positive real number T there is a constant C_T such that for any bounded continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative and for any $\varepsilon > 0$, there is $K(\varepsilon) \in \mathbb{N}$ and $N(\varepsilon, K(\varepsilon)) \in \mathbb{N}$ such that for any $K \geq K(\varepsilon)$ and any $N \geq N(\varepsilon, K(\varepsilon))$

$$\mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \left(\varphi(\bar{X}_{\bullet,j}^N(N^2 t)) - \varphi(\bar{Y}_{\bullet,j}^{N,K,t}(N^2 t)) \right) \right| \right] \leq C_T \sqrt{\frac{K^3}{N}} + \sqrt{\frac{4 \|\varphi\|_\infty^2}{N}} + \varepsilon^2$$

provided that $\sup_N \mathbb{E}[\bar{X}_{\bullet,\bullet}^N(0)] < \infty$.

PROOF OF LEMMA 7.8: For fixed $t > 0$ and $K > 0$, we define another process, denoted

$$\{Y_{i,j}^{N,K,t}(s) : i, j \in \{0, 1, \dots, N-1\}, s \geq 0\},$$

as following: if $0 \leq s \leq (N^2 - NK)t$ then

$$Y_{i,j}^{N,K,t}(s) \equiv X_{i,j}^N(s)$$

and if $s > (N^2 - NK)t$ then

$$(7.8.1) \quad \begin{aligned} Y_{i,j}^{N,K,t}(s) &\equiv X_{i,j}^N((N^2 - NK)t) \\ &+ c_1 \int_{(N^2 - NK)t}^s (\bar{Y}_{\bullet,j}^{N,K,t}(r) - Y_{i,j}^{N,K,t}(r)) dr + \frac{c_2}{N} \int_{(N^2 - NK)t}^s (\bar{X}_{\bullet,\bullet}^N((N^2 - NK)t) - Y_{i,j}^{N,K,t}(r)) dr \\ &+ \gamma \int_{(N^2 - NK)t}^s Y_{i,j}^{N,K,t}(r) dW_{i,j}(r). \end{aligned}$$

We claim that for any $T > 0$ there exists a constant C_T for which

$$(7.8.2) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left[\left| \varphi(\bar{X}_{\bullet,\bullet}^N(N^2 t)) - \varphi(\bar{Y}_{\bullet,\bullet}^{N,t}(N^2 t)) \right| \right] \leq C_T \sqrt{\frac{K^3}{N}}.$$

The main steps in the proof of the inequality above are the following: from coupling,

$$\begin{aligned} &X_{i,j}^N(N^2 t) - Y_{i,j}^{N,K,t}(N^2 t) \\ &\stackrel{\text{as}}{=} c_1 \int_{(N^2 - NK)t}^{N^2 t} \bar{X}_{\bullet,j}^N(r) - \bar{Y}_{\bullet,j}^{N,K,t}(r) dr + \frac{c_2}{N} \int_{(N^2 - NK)t}^{N^2 t} \bar{X}_{\bullet,\bullet}^N(r) - \bar{X}_{\bullet,\bullet}^N((N^2 - NK)t) dr \\ &- \left(c_1 + \frac{c_2}{N} \right) \int_{(N^2 - NK)t}^{N^2 t} X_{i,j}^N(r) - Y_{i,j}^{N,K,t}(r) dr + \gamma \int_{(N^2 - NK)t}^{N^2 t} X_{i,j}^N(r) - Y_{i,j}^{N,K,t}(r) dW_{i,j}(r). \end{aligned}$$

Therefore, the difference of local times

$$L_{N^2 t}^0(X_{i,j}^N - Y_{i,j}^{N,K,t}) - L_{(N^2 - NK)t}^0(X_{i,j}^N - Y_{i,j}^{N,K,t}) = 0.$$

Tanaka's formula leads to

$$\begin{aligned} & \left| X_{i,j}^N(N^2 t) - Y_{i,j}^{N,K,t}(N^2 t) \right| \\ & \stackrel{p-ss}{\leq} c_1 \int_{(N^2-NK)t}^{N^2 t} \frac{1}{N} \sum_{i=0}^{N-1} \left| X_{i,j}^N(r) - Y_{i,j}^{N,K,t}(r) \right| dr + \frac{c_2}{N} \int_{(N^2-NK)t}^{N^2 t} \left| \bar{X}_{\bullet\bullet}^N(r) - \bar{X}_{\bullet\bullet}^N((N^2-NK)t) \right| dr \\ & - \left(c_1 + \frac{c_2}{N} \right) \int_{(N^2-NK)t}^{N^2 t} \left| X_{i,j}^N(r) - Y_{i,j}^{N,K,t}(r) \right| dr + \gamma \int_{(N^2-NK)t}^{N^2 t} \left| X_{i,j}^N(r) - Y_{i,j}^{N,K,t}(r) \right| dW_{i,j}(r) \end{aligned}$$

The last inequality implies that

$$\begin{aligned} \mathbb{E} \left[\left| \bar{X}_{\bullet j}^N(N^2 t) - \bar{Y}_{\bullet j}^{N,K,t}(N^2 t) \right| \right] & \leq \mathbb{E} \left[\frac{1}{N} \sum_{i=0}^{N-1} \left| X_{i,j}^N(N^2 t) - Y_{i,j}^{N,K,t}(N^2 t) \right| \right] \\ & \leq \frac{c_2}{N} \int_{(N^2-NK)t}^{N^2 t} \mathbb{E} \left[\left| \bar{X}_{\bullet\bullet}^N(r) - \bar{X}_{\bullet\bullet}^N((N^2-NK)t) \right| \right] dr. \end{aligned}$$

We use lemma 7.6 and the law of the mean to complete the argument which establish (7.8.2).

Now, from Jensen's inequality,

$$\begin{aligned} (7.8.3) \quad & \mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \varphi(\bar{Y}_{\bullet j}^{N,K,t}(N^2 t)) - \frac{1}{N} \sum_{j=0}^{N-1} \varphi(\bar{Z}_{\bullet j}^{N,K,t}(N^2 t)) \right| \right] \\ & \leq \sqrt{\mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \varphi(\bar{Y}_{\bullet j}^{N,K,t}(N^2 t)) - \frac{1}{N} \sum_{j=0}^{N-1} \varphi(\bar{Z}_{\bullet j}^{N,K,t}(N^2 t)) \right|^2 \right]} \\ & \leq \sqrt{\mathbb{E} \left[\mathbb{E} \left[\left| \frac{1}{N} \sum_{j=0}^{N-1} \varphi(\bar{Y}_{\bullet j}^{N,K,t}(N^2 t)) - \frac{1}{N} \sum_{j=0}^{N-1} \varphi(\bar{Z}_{\bullet j}^{N,K,t}(N^2 t)) \right|^2 \middle| \mathcal{F}_{(N^2-NK)t} \right] \right]} \\ & = \sqrt{\mathbb{E} \left[\mathbb{E} \left[\frac{1}{N^2} \sum_{j,j'=0}^{N-1} \left(\varphi(\bar{Y}_{\bullet j}^{N,K,t}(N^2 t)) - \varphi(\bar{Z}_{\bullet j}^{N,K,t}(N^2 t)) \right) \left(\varphi(\bar{Y}_{\bullet j'}^{N,K,t}(N^2 t)) - \varphi(\bar{Z}_{\bullet j'}^{N,K,t}(N^2 t)) \right) \middle| \mathcal{F}_{(N^2-NK)t} \right] \right]} \\ & = \left(\mathbb{E} \left[\mathbb{E} \left[\frac{1}{N^2} \sum_{j=0}^{N-1} \left(\varphi(\bar{Y}_{\bullet j}^{N,K,t}(N^2 t)) - \varphi(\bar{Z}_{\bullet j}^{N,K,t}(N^2 t)) \right)^2 \middle| \mathcal{F}_{(N^2-NK)t} \right] \right] \right)^{\frac{1}{2}} \\ & + \mathbb{E} \left[\frac{1}{N^2} \sum_{\substack{j,j'=0 \\ j \neq j'}}^{N-1} \left[\left(\varphi(\bar{Y}_{\bullet j}^{N,K,t}(N^2 t)) - \varphi(\bar{Z}_{\bullet j}^{N,K,t}(N^2 t)) \right) \left(\varphi(\bar{Y}_{\bullet j'}^{N,K,t}(N^2 t)) - \varphi(\bar{Z}_{\bullet j'}^{N,K,t}(N^2 t)) \right) \middle| \mathcal{F}_{(N^2-NK)t} \right] \right]^{\frac{1}{2}} \end{aligned}$$

$$\leq \sqrt{\frac{4\|\varphi\|_\infty^2}{N} + \mathbb{E} \left[\left(\mathbb{E} \left[\varphi(\bar{Y}_{\bullet,0}^{N,K,t}(N^2 t)) - \varphi(\bar{Z}_{\bullet,0}^{N,K,t}(N^2 t)) \middle| \mathcal{F}_{(N^2 - NK)t} \right] \right)^2 \right]}.$$

From corollary 5.18, we can prove that for any $\varepsilon > 0$, there is $K(\varepsilon) \in \mathbb{N}$ and $N(\varepsilon, K(\varepsilon)) \in \mathbb{N}$ such that for any $K \geq K(\varepsilon)$ and any $N \geq N(\varepsilon, K(\varepsilon))$, the expression above is bounded by

$$\sqrt{\frac{4\|\varphi\|_\infty^2}{N} + \varepsilon^2}$$

since
$$\sup_N \mathbb{E} \left[\bar{X}_{\bullet,\bullet}^N((N^2 - NK)t) \right] = \sup_N \mathbb{E} \left[\bar{X}_{\bullet,\bullet}^N(0) \right] < \infty$$
 ■

The proof of equation (7.7.2) is completed. ■

LEMMA 7.9: If $\sup_N \mathbb{E} \left[(\bar{X}_{\bullet,\bullet}^N(0))^2 \right] < \infty$ then there is a constant C such that for any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative and any $t > 0$,

$$\begin{aligned} & \frac{1}{N^2} \sum_{i,j=0}^{N-1} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) P_{\beta_N t}^{\bar{X}_{\bullet,j}^N((N^2 - \beta_N)t), c_1 + c_2/N, \gamma} \left(\bar{X}_{i,j}^N((N^2 - \beta_N)t) \right) dy \right. \right. \\ & \left. \left. - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,j}^N((N^2 - \beta_N)t), c_1 + c_2/N, \gamma} (dy) \right| \right] \leq C \|\varphi'\|_\infty \exp \left[- \left(c_1 + \frac{c_2}{N} \right) \beta_N t \right] \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) P_{NKt}^{\bar{X}_{\bullet,j}^N((N^2 - NK)t), \frac{c_2}{N}, \frac{2c_1\gamma^2}{N\alpha_1}} \left(\bar{X}_{\bullet,j}^N((N^2 - NK)t) \right) dy \right. \right. \\ & \left. \left. - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,j}^N((N^2 - NK)t), \frac{c_2}{N}, \frac{2c_1\gamma^2}{N\alpha_1}} (dy) \right| \right] \leq C \|\varphi'\|_\infty \exp[-c_2 K t] \end{aligned}$$

PROOF OF LEMMA 7.9: We obtain this proof from reproducing the one of lemma 3.15. This is a special case of lemma 9.10. ■

¹ Because $\left\{ \varphi(\bar{Y}_{\bullet,j}^{N,K,t}(N^2 t)), \varphi(\bar{Z}_{\bullet,j}^{N,K,t}(N^2 t)) : j \in \{0, 1, \dots, N-1\} \right\}$ are conditionally independent with respect to $\mathbb{P} \left[\bullet \middle| \mathcal{F}_{(N^2 - NK)t} \right]$.

LEMMA 7.10: If $\alpha_2 > 0$ and $\sup_N \mathbb{E} \left[\frac{1}{N^2} \sum_{i,j=0}^{N-1} (X_{i,j}^N(0))^2 \right] < \infty$ then for any positive real number T , there is a constant C_T such that for any continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$\sup_{0 < t \leq T} \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,j}^N((N^2 - \beta_N)t), c_1 + c_2/N, \gamma}(dy) - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,j}^N(N^2 t), c_1 + c_2/N, \gamma}(dy) \right| \right] \\ \leq \|\varphi'\|_{\infty} C_T \max \left\{ \sqrt{\frac{\beta_N}{N}}, \frac{\beta_N}{N} \right\}$$

and

$$\sup_{0 < t \leq T} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,\bullet}^N((N^2 - NK)t), \frac{c_2}{N}, \frac{2c_1\gamma^2}{Na_1}}(dy) - \int_{\mathbb{R}_+} \varphi(y) v_{\bar{X}_{\bullet,\bullet}^N(N^2 t), \frac{c_2}{N}, \frac{2c_1\gamma^2}{Na_1}}(dy) \right| \right] \\ \leq \|\varphi'\|_{\infty} C_T \max \left\{ \sqrt{\frac{K}{N}}, \frac{K}{N} \right\}$$

PROOF OF LEMMA 7.10: The same arguments as the ones of the demonstration of lemma 3.16 hold. For a generalization of this result, see lemma 9.11. ■

Chapter 3

Multilevel Bilinear System of Stochastic Differential Equations

Introduction

The special feature of the process characterized by the multilevel bilinear system of stochastic differential equations is that the interaction between its components occurs at different orders of magnitude. This time, the number of levels is arbitrary (but finite). The impact that one component has on another depends on how far apart they are in this hierarchical setting; the closer they are, the more influence they will have on each other.

Let us define what is this N^L -dimensional stochastic process. On a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$, we define a N^L -dimensional Brownian motion

$$W = \{W_i : i = (i_1, i_2, \dots, i_L), i_j \in \{0, 1, \dots, N-1\}, \forall j \in \{1, 2, \dots, L\}\}.$$

We assume that $\mathcal{F}_0$ contains the collection of the $\mathbb{P}$ -null sets and that the filtration is right-continuous. The process is defined as the unique strong solution of the system of stochastic differential equations

$$(***) \quad dX_i^N(t) = \sum_{k=1}^L \frac{c_k}{N^{k-1}} \left(\left(\frac{1}{N^k} \sum_{i_1, \dots, i_k=0}^{N-1} X_i^N(t) \right) - X_i^N(t) \right) dt + \gamma X_i^N(t) dW_i(t),$$

$$i = (i_1, i_2, \dots, i_L), i_j \in \{0, 1, \dots, N-1\} \forall j \in \{1, 2, \dots, L\}$$

where $c_1, c_2, \dots, c_L$ and γ are nonnegative constants (γ and c_L are also assumed to be strictly positive). $X_i^N(t)$ can be viewed as the "mass" of the i_1^{th} individual of the i_2^{th} "village" of the i_3^{th} "province" ... at time t .

There are two sections to this chapter. The first is about the finite system. The second section treats of time rescaling.

Note that in the following,

$$\bar{X}_{N^L}(t) \equiv \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(t)$$

and we will adopt the following conventions:

$$\sum_{i_k=1}^{N-1} a_{i_k} \equiv a_{i_k}, \quad \sum_{k=m+1}^m b_k \equiv 0 \quad \text{and} \quad \prod_{k=m+1}^m b_k \equiv 1.$$

The Finite System

Our aim in this section is the determination of the second moments. The complication arises with the resolution of a general system of differential equations.

THEOREM 8.1¹: *(Nonnegativity of the strong solution)*

$$\text{If } \forall i, X_i(0) \stackrel{\mathcal{P}\text{-a.s.}}{\geq} 0 \text{ then } X_i(t) \stackrel{\mathcal{P}\text{-a.s.}}{\geq} 0, \forall t \geq 0.$$

LEMMA 8.2²: *(A bound for the k^{th} moment)*

Let $k \geq 1$. Assuming that the initial conditions are nonnegative random variables with finite k^{th} moment,

$$\mathbb{E}[\bar{X}_{N^L}^k(t)] \leq \mathbb{E}[\bar{X}_{N^L}^k(0)] \exp\left[\frac{k(k-1)}{2} \gamma^2 t\right] \text{ where } \bar{X}_{N^L}(t) \equiv \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_{i_1 \dots i_L}^N(t).$$

PROPOSITION 8.3³: Assuming the initial conditions are nonnegative square-integrable random variables,

$$\{\bar{X}_{N^L}(t) : t \geq 0\} \text{ is a continuous nonnegative } \{\mathcal{F}_t^N : t \geq 0\}\text{-martingale.}$$

PROPOSITION 8.4⁴: *(Extinction of the finite system through time)*

Assuming the initial conditions are nonnegative square-integrable random variables,

$$\sum_{i_1, \dots, i_L=0}^{N-1} X_{i_1 \dots i_L}^N(t) \xrightarrow{t \rightarrow \infty} 0 \text{ } \mathcal{P}\text{-almost-surely}$$

The proofs of these results are similar to the corresponding ones in the mean-field case. For this reason, we will not reproduce the proofs here.

- 1 This result is the equivalent of theorem 2.1 of the mean-field case.
- 2 This result is the equivalent of lemma 2.2 of the mean-field case.
- 3 This result is the equivalent of corollary of proposition 2.3 of the mean-field case.
- 4 This result is the equivalent of proposition 2.4 of the mean-field case.

Computation of the Second Moments of the Finite System

THEOREM 8.5: If D represent the derivative operator $D = \frac{d}{dt}$, then the second moments satisfy the following system of differential equations:

$$M_{(L+1) \times (L+1)} \begin{bmatrix} \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(t))^2 \right] \\ \mathbb{E} \left[\frac{1}{N^{L-1}} \sum_{i_2, \dots, i_L=0}^{N-1} \left(\frac{1}{N} \sum_{i_1=0}^{N-1} X_i^N(t) \right)^2 \right] \\ \vdots \\ \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right)^2 \right] \\ \vdots \\ \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(t) \right)^2 \right] \end{bmatrix}_{(L+1) \times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(L+1) \times 1}$$

where

$$M_{(L+1) \times (L+1)} \equiv \begin{bmatrix} D + \sum_{k=1}^L \frac{2c_k}{N^{k-1}} - \gamma^2 & -2c_1 & -\frac{2c_2}{N} & -\frac{2c_3}{N^2} & \dots & -\frac{2c_L}{N^{L-1}} \\ -\frac{\gamma^2}{N} & D + \sum_{k=2}^L \frac{2c_k}{N^{k-1}} & -\frac{2c_2}{N} & -\frac{2c_3}{N^2} & \dots & -\frac{2c_L}{N^{L-1}} \\ -\frac{\gamma^2}{N^2} & 0 & D + \sum_{k=3}^L \frac{2c_k}{N^{k-1}} & -\frac{2c_3}{N^2} & \dots & -\frac{2c_L}{N^{L-1}} \\ -\frac{\gamma^2}{N^3} & 0 & 0 & D + \sum_{k=4}^L \frac{2c_k}{N^{k-1}} & \dots & -\frac{2c_L}{N^{L-1}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\gamma^2}{N^L} & 0 & 0 & 0 & \dots & D \end{bmatrix}$$

PROOF OF THEOREM 8.5: For any integer M such that $0 \leq M \leq L$,

$$(8.5.1) \quad \frac{d}{dt} \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \\ = \sum_{k=M+1}^L \frac{c_k}{N^{k-1}} \left(\left(\frac{1}{N^k} \sum_{i_1, \dots, i_k=0}^{N-1} X_i^N(t) \right) - \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right) dt + \frac{\gamma}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) dW_i(t)$$

and

$$\frac{d}{dt} \left\langle \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right\rangle = \gamma^2 \frac{1}{N^{2M}} \sum_{i_1, \dots, i_M=0}^{N-1} (X_i^N(t))^2 dt$$

Therefore, we get from Itô's formula

$$\begin{aligned}
 d \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right)^2 &\stackrel{\text{p-as}}{=} 2 \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right) d \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right) + d \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right) \\
 &\stackrel{\text{p-as}}{=} 2 \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right) \left(\sum_{k=M+1}^L \frac{c_k}{N^{k-1}} \left(\left(\frac{1}{N^k} \sum_{i_1, \dots, i_k=0}^{N-1} X_i^N(t) \right) - \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right) dt \right) \\
 &\quad + 2\gamma \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right) \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) dW_i(t) + \gamma^2 \frac{1}{N^{2M}} \sum_{i_1, \dots, i_M=0}^{N-1} (X_i^N(t))^2 dt \\
 (8.5.2) \quad &d \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right)^2 \\
 &\stackrel{\text{p-as}}{=} \sum_{k=M+1}^L \frac{2c_k}{N^{k-1}} \left(\frac{1}{N^{L-k}} \sum_{i_{k+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^k} \sum_{i_1, \dots, i_k=0}^{N-1} X_i^N(t) \right)^2 \right) dt \\
 &\quad - \left(\sum_{k=M+1}^L \frac{2c_k}{N^{k-1}} \right) \left(\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right)^2 \right) dt \\
 &\quad + \frac{2\gamma}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right) X_i^N(t) dW_i(t) + \frac{\gamma^2}{N^M} \left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(t))^2 \right) dt
 \end{aligned}$$

Using Fubini-Tonelli's theorem, lemma 8.2 and A.2,

$$\begin{aligned}
 (8.5.3) \quad &\frac{d}{dt} \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right)^2 \right] \\
 &= \sum_{k=M+1}^L \frac{2c_k}{N^{k-1}} \mathbb{E} \left[\frac{1}{N^{L-k}} \sum_{i_{k+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^k} \sum_{i_1, \dots, i_k=0}^{N-1} X_i^N(t) \right)^2 \right] \\
 &\quad - \left(\sum_{k=M+1}^L \frac{2c_k}{N^{k-1}} \right) \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right)^2 \right] + \frac{\gamma^2}{N^M} \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(t))^2 \right] \quad \blacksquare
 \end{aligned}$$

Since the system is homogeneous, we know that

$$(8.5.4) \quad \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_{i_1 \dots i_L}^N(t) \right)^2 \right] = \sum_{i=1}^{L+1} k_i(N) \exp[r_i(N)t]$$

where $r_1(N), r_2(N), \dots, r_{L+1}(N)$ are the roots of

$$p(D) \equiv \det(\mathbf{M}_{(L+1) \times (L+1)})$$

and $k_1(N), k_2(N), \dots, k_{L+1}(N)$ are constants determined by the initial conditions.

In the quest of those roots, we need an expression for the determinant:

THEOREM 8.6:

$$\det(\mathbf{M}_{(L+1) \times (L+1)}) = D^{L+1} + \sum_{i=1}^L \left(\frac{\alpha_i}{N^{i(i-1)/2}} + c \left(\frac{1}{N^{i(i-1)/2+1}} \right) \right) D^{L+1-i} - \frac{\gamma^2}{N^{L(L+1)/2}} \prod_{i=1}^L 2c_i + c \left(\frac{1}{N^{L(L+1)/2+1}} \right)$$

where $\alpha_1 \equiv 2c_1 - \gamma^2$ and $\alpha_i \equiv 2c_i - \gamma^2 \prod_{j=1}^{i-1} 2c_j$.

NOTE: If $\alpha_L > 0$ then for any $i \in \{0, 1, \dots, L\}$, $c_i > 0$ ¹.

If $\alpha_L > 0$ then for any $i \in \{0, 1, \dots, L-1\}$, $\alpha_i > 0$ ².

REMARK 8.7:

$$\alpha_L > 0 \Leftrightarrow \sum_{k=1}^L \frac{1}{2c_k} < \frac{1}{\gamma^2} \text{ }^3.$$

¹ **PROOF:** By contradiction. Assume that there is an $i \in \{0, 1, \dots, L\}$ such that $c_i = 0$ (recall that the c_i 's are assumed to be non negative) and let $j \equiv \min\{i \in \{0, 1, \dots, L\} : c_i = 0\}$. Then

$$\alpha_j \equiv 2c_j - \gamma^2 \prod_{k=1}^{j-1} 2c_k = -\gamma^2 \prod_{k=1}^{j-1} 2c_k < 0.$$

But $\alpha_j \leq 0 \Rightarrow \alpha_{j+1} \equiv 2c_{j+1} - \gamma^2 \prod_{k=1}^j 2c_k \leq 0$ therefore $\alpha_L \leq 0$. $\Rightarrow \Leftarrow$ ■

² **PROOF:** By contradiction. $\alpha_j \leq 0 \Rightarrow \alpha_{j+1} \equiv 2c_{j+1} - \gamma^2 \prod_{k=1}^j 2c_k \leq 0$. $\Rightarrow \Leftarrow$ ■

³ **PROOF:** One can show by induction on i that for any $1 \leq i \leq L$

$$\frac{\alpha_i}{\gamma^2} \left(\prod_{k=1}^i 2c_k \right)^{-1} = \frac{1}{\gamma^2} - \sum_{k=1}^i \frac{1}{2c_k}.$$

We then use the definition of α_L to conclude the proof. ■

PROOF OF THEOREM 8.6: To be able to compute the determinant of $\mathbf{M}_{(L+1) \times (L+1)}$, we need to find the determinant of specific submatrices of $\mathbf{M}_{(L+1) \times (L+1)}$. Therefore, denoted respectively by $\mathbf{P}_{(k+1) \times (k+1)}$ and $\mathbf{Q}_{(k+1) \times (k+1)}$ the submatrices of $\mathbf{M}_{(L+1) \times (L+1)}$

$$\begin{bmatrix} m_{1,1} & \cdots & m_{1,k+1} \\ \vdots & \ddots & \vdots \\ m_{k+1,1} & \cdots & m_{k+1,k+1} \end{bmatrix} \text{ and } \begin{bmatrix} m_{1,2} & \cdots & m_{1,k+2} \\ \vdots & \ddots & \vdots \\ m_{k+1,2} & \cdots & m_{k+1,k+2} \end{bmatrix}.$$

Define $\mathbf{R}_{(k+1) \times (k+1)}$ as

$$\begin{bmatrix} m_{1,2} & \cdots & m_{1,k+1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ m_{k+1,2} & \cdots & m_{k+1,k+1} & 1 \end{bmatrix}.$$

We will first show by induction on k that

$$(8.6.1) \quad \det(\mathbf{R}_{(k+1) \times (k+1)}) = \sum_{i=0}^{k-1} D^{k-i} \left(\frac{1}{N^{i(i-1)/2}} \right) + \frac{1}{N^{k(k-1)/2}} \prod_{i=1}^k -2c_i + \left(\frac{1}{N^{k(k-1)/2+1}} \right)$$

for any integer $1 \leq k \leq L$. When $k = 1$,

$$\det(\mathbf{R}_{2 \times 2}) = -2c_1 - D - \sum_{i=2}^L \frac{2c_i}{N^{i-1}} = D \left(N^0 \right) - 2c_1 + \left(\frac{1}{N} \right)$$

Now, assume that the result is verified for some positive integer $k < L - 1$. Then

$$\begin{aligned} \det(\mathbf{R}_{(k+2) \times (k+2)}) &= \det(\mathbf{Q}_{(k+1) \times (k+1)}) - \left(D + \sum_{i=k+2}^L \frac{2c_i}{N^{i-1}} \right) \det(\mathbf{R}_{(k+1) \times (k+1)}) \\ &= -\frac{2c_{k+1}}{N^k} \det(\mathbf{R}_{(k+1) \times (k+1)}) - \left(D + \sum_{i=k+2}^L \frac{2c_i}{N^{i-1}} \right) \det(\mathbf{R}_{(k+1) \times (k+1)}) \\ &= \left(-D - \frac{2c_{k+1}}{N^k} + \left(\frac{1}{N^{k+1}} \right) \right) \left(\sum_{i=0}^{k-1} D^{k-i} \left(\frac{1}{N^{i(i-1)/2}} \right) + \frac{1}{N^{k(k-1)/2}} \prod_{i=1}^k -2c_i + \left(\frac{1}{N^{k(k-1)/2+1}} \right) \right) \\ &= \sum_{i=0}^k D^{k+1-i} \left(\frac{1}{N^{i(i-1)/2}} \right) + \sum_{i=0}^{k-1} D^{k-i} \left(\frac{1}{N^{i(i-1)/2+k}} \right) + \frac{1}{N^{(k+1)k/2}} \prod_{i=1}^{k+1} -2c_i + \left(\frac{1}{N^{(k+1)k/2+1}} \right) \\ &= \sum_{i=0}^k D^{k+1-i} \left(\frac{1}{N^{i(i-1)/2}} \right) + \sum_{i=1}^k D^{k+1-i} \left(\frac{1}{N^{i(i-1)/2+2+k}} \right) + \frac{1}{N^{(k+1)k/2}} \prod_{i=1}^{k+1} -2c_i + \left(\frac{1}{N^{(k+1)k/2+1}} \right) \\ &= \sum_{i=0}^k D^{k+1-i} \left(\frac{1}{N^{i(i-1)/2}} \right) + \frac{1}{N^{(k+1)k/2}} \prod_{i=1}^{k+1} -2c_i + \left(\frac{1}{N^{(k+1)k/2+1}} \right) \end{aligned}$$

which completes the induction.

¹ From induction hypothesis.

We will again use induction on k to prove that for $1 \leq k < L$,

$$(8.6.2) \quad \det(\mathbf{P}_{(k+1) \times (k+1)}) = D^{k+1} + \sum_{i=1}^{k+1} \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \mathcal{O}\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{k+1-i}$$

where $\alpha_1 \equiv 2c_1 - \gamma^2$ and $\alpha_i \equiv 2c_i \alpha_{i-1} - \gamma^2 \prod_{j=1}^{i-1} 2c_j$. When $k = 1$,

$$\begin{aligned} \det(\mathbf{P}_{2 \times 2}) &= \left(D + \sum_{l=1}^L \frac{2c_l}{N^{l-1}} - \gamma^2 \right) \left(D + \sum_{l=2}^L \frac{2c_l}{N^{l-1}} \right) - \frac{\gamma^2}{N} 2c_1 \\ &= D^2 + \left[2c_1 - \gamma^2 + \mathcal{O}\left(\frac{1}{N}\right) \right] D + \frac{2c_2(2c_1 - \gamma^2) - \gamma^2 2c_1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \\ &= D^2 + \left[\alpha_1 + \mathcal{O}\left(\frac{1}{N}\right) \right] D + \frac{\alpha_2}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \end{aligned}$$

Assume that the result is true for some positive integer k strictly smaller than $L-2$. Then

$$(8.6.3) \quad \begin{aligned} &\det(\mathbf{P}_{(k+2) \times (k+2)}) \\ &= \left(D + \sum_{l=k+2}^L \frac{2c_l}{N^{l-1}} \right) \det(\mathbf{P}_{(k+1) \times (k+1)}) + (-1)^{k+3} \frac{-\gamma^2}{N^{k+1}} \det(\mathbf{Q}_{(k+1) \times (k+1)}) \\ &= \left(D + \sum_{l=k+2}^L \frac{2c_l}{N^{l-1}} \right) \det(\mathbf{P}_{(k+1) \times (k+1)}) + (-1)^{k+3} \frac{-\gamma^2}{N^{k+1}} \frac{-2c_{k+1}}{N^k} \det(\mathbf{R}_{(k+1) \times (k+1)}) \end{aligned}$$

$$(8.6.4) \quad \begin{aligned} &\left(D + \sum_{l=k+2}^L \frac{2c_l}{N^{l-1}} \right) \det(\mathbf{P}_{(k+1) \times (k+1)}) \\ &= \left(D + \sum_{l=k+2}^L \frac{2c_l}{N^{l-1}} \right) \left(D^{k+1} + \sum_{i=1}^{k+1} \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \mathcal{O}\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{k+1-i} \right)^2 \\ &= D^{k+2} + \sum_{i=1}^{k+1} \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \mathcal{O}\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{k+2-i} \\ &\quad + D^{k+1} \mathcal{O}\left(\frac{1}{N^{k+1}}\right) + \sum_{i=1}^k \left(\frac{2c_{k+2} \alpha_i}{N^{i(i-1)/2+k+1}} + \mathcal{O}\left(\frac{1}{N^{i(i-1)/2+k+2}}\right) \right) D^{k+1-i} \\ &\quad + \frac{2c_{k+2} \alpha_{k+1}}{N^{(k+1)(k+2)/2}} + \mathcal{O}\left(\frac{1}{N^{(k+1)(k+2)/2+1}}\right) \\ &= D^{k+2} + \sum_{i=1}^{k+1} \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \mathcal{O}\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{k+2-i} + \frac{2c_{k+2} \alpha_{k+1}}{N^{(k+1)(k+2)/2}} + \mathcal{O}\left(\frac{1}{N^{(k+1)(k+2)/2+1}}\right) \end{aligned}$$

1 From the definitions of α_1 and α_2 .

2 From the induction hypothesis.

$$\begin{aligned}
 (8.6.5) \quad & (-1)^{k+3} \frac{-\gamma^2}{N^{k+1}} \frac{-2c_{k+1}}{N^k} \det(R_{(k+1) \times (k+1)}) \\
 &= (-1)^{k+3} \frac{-\gamma^2}{N^{k+1}} \frac{-2c_{k+1}}{N^k} \left[\sum_{i=0}^{k-1} D^{k-i} \zeta\left(\frac{1}{N^{i(i-1)/2}}\right) + \frac{1}{N^{k(k-1)/2}} \prod_{i=1}^k -2c_i + \zeta\left(\frac{1}{N^{k(k-1)/2+1}}\right) \right] \\
 &= \sum_{i=0}^{k-1} D^{k-i} \zeta\left(\frac{1}{N^{i(i-1)/2+2k+1}}\right) - \frac{\gamma^2}{N^{(k+2)(k+1)/2}} \prod_{i=1}^{k+1} 2c_i + \zeta\left(\frac{1}{N^{(k+2)(k+1)/2+1}}\right) \\
 &= \sum_{i=1}^k D^{k+1-i} \zeta\left(\frac{1}{N^{(i-2)(i-1)/2+2k+1}}\right) - \frac{\gamma^2}{N^{(k+2)(k+1)/2}} \prod_{i=1}^{k+1} 2c_i + \zeta\left(\frac{1}{N^{(k+2)(k+1)/2+1}}\right)
 \end{aligned}$$

Therefore, replacing (8.6.4) and (8.6.5) in (8.6.3),

$$\begin{aligned}
 \det(P_{(k+2) \times (k+2)}) &= D^{k+2} + \sum_{i=1}^{k+1} \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \zeta\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{k+2-i} + \frac{2c_{k+2}\alpha_{k+1}}{N^{(k+1)(k+2)/2}} + \zeta\left(\frac{1}{N^{(k+1)(k+2)/2+1}}\right) \\
 &\quad + \sum_{i=1}^k D^{k+1-i} \zeta\left(\frac{1}{N^{(i-2)(i-1)/2+2k+1}}\right) - \frac{\gamma^2}{N^{(k+2)(k+1)/2}} \prod_{i=1}^{k+1} 2c_i + \zeta\left(\frac{1}{N^{(k+2)(k+1)/2+1}}\right) \\
 &= D^{k+2} + \sum_{i=1}^{k+1} \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \zeta\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{k+2-i} \\
 &\quad + \frac{2c_{k+2}\alpha_{k+1}}{N^{(k+1)(k+2)/2}} - \frac{\gamma^2}{N^{(k+2)(k+1)/2}} \prod_{i=1}^{k+1} 2c_i + \zeta\left(\frac{1}{N^{(k+2)(k+1)/2+1}}\right) \\
 &= D^{k+2} + \sum_{i=1}^{k+2} \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \zeta\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{k+2-i} \quad 2
 \end{aligned}$$

which complete the induction. We complete the proof by this last computation:

$$\begin{aligned}
 \det(M_{(L+1) \times (L+1)}) &= D \det(P_{L \times L}) + (-1)^{L+2} \frac{-\gamma^2}{N^L} \det(Q_{L \times L}) \\
 &= D \det(P_{L \times L}) + (-1)^{L+2} \frac{-\gamma^2}{N^L} \frac{-2c_L}{N^{L-1}} \det(R_{L \times L}) \\
 &= D^{L+1} + \sum_{i=1}^L \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \zeta\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{L+1-i} \\
 &\quad + (-1)^{L+2} \frac{-\gamma^2}{N^L} \frac{-2c_L}{N^{L-1}} \left(\sum_{i=0}^{L-2} D^{L-1-i} \zeta\left(\frac{1}{N^{i(i-1)/2}}\right) + \frac{1}{N^{L-1(L-2)/2}} \prod_{i=1}^{L-1} -2c_i + \zeta\left(\frac{1}{N^{L-1(L-2)/2+1}}\right) \right) \quad 3 \\
 &= D^{L+1} + \sum_{i=1}^L \left(\frac{\alpha_i}{N^{i(i-1)/2}} + \zeta\left(\frac{1}{N^{i(i-1)/2+1}}\right) \right) D^{L+1-i} - \frac{\gamma^2}{N^{L(L+1)/2}} \prod_{i=1}^L 2c_i + \zeta\left(\frac{1}{N^{L(L+1)/2+1}}\right) \quad \blacksquare
 \end{aligned}$$

- 1 From equation (8.6.1)
- 2 From the definition of α_{k+2} .
- 3 From equations (8.6.1) and (8.6.2).

COROLLARY 8.7: If $\alpha_L > 0$ then the $L+1$ roots of $p(D) \equiv \det(\mathbf{M}_{(L+1) \times (L+1)})$ are

$$-\frac{\alpha_k}{\alpha_{k-1}} \frac{1}{N^{k-1}} + O\left(\frac{1}{N^k}\right) \text{ for } 1 \leq k \leq L \text{ and } \frac{\gamma^2 \prod_{i=1}^L 2c_i}{\alpha_L} \frac{1}{N^L} + O\left(\frac{1}{N^{L+1}}\right)$$

where $\alpha_0 \equiv 1$ and $\alpha_i \equiv 2c_i \alpha_{i-1} - \gamma^2 \prod_{j=1}^{i-1} 2c_j$.

REMARK: The assumption on α_L ensures that the α_i 's are different than zero. In fact, only this last condition is needed to apply the lemma 8.8. Therefore, the corollary can be extended to all the cases where the α_i 's are non null.

PROOF OF COROLLARY 8.7: This is a consequence of the following result. ■

LEMMA 8.8: Let $\rho_1(N), \rho_2(N), \dots, \rho_n(N)$ be the roots of

$$p(x) \equiv x^n + a_1(N)x^{n-1} + a_2(N)x^{n-2} + \dots + a_n(N).$$

If $\forall k \in \{1, 2, \dots, n\}, a_k(N) = \frac{\beta_k}{N^{k(k-1)/2}} + O\left(\frac{1}{N^{(k(k-1)/2)+1}}\right)$ where $\beta_k \neq 0$

then we can rename the roots such that $\forall k \in \{1, 2, \dots, n\},$

$$\rho_k(N) = -\frac{\beta_k}{\beta_{k-1}} \frac{1}{N^{k-1}} + O\left(\frac{1}{N^k}\right) \text{ where } \beta_0 \equiv 1.$$

PROOF OF LEMMA 8.8: The proof is in the appendix. ■

To get an exact expression for the second moments when the α_i 's are non null, we have to find the constants $k_1(N), k_2(N), \dots, k_{L+1}(N)$ which are determined by the initial conditions. In fact, we have to solve a system of $L+1$ equations which arises from (8.5.3) and (8.5.4) evaluated at $t = 0$. Unlike the first and second level cases, this system could be quite hard to solve in this general setting. However, knowing the behavior of the roots, we can determine when the second moments have some chances to converge as N increases to infinity when the time is rescaled. From the previous results, it seems that the subcritical case is characterized by the condition $\alpha_L > 0$ and the appropriate rate for the renormalization of the time in order to get a nice weak limit for the average process is N^L . Since $\alpha_L < 0$ implies that $\alpha_k < 0$ for some $k \in \{1, 2, \dots, L-1\}$, then one could probably guess that $\alpha_L < 0$ will be the criterion for the supercritical case. Therefore, $\alpha_L = 0$ will determine the critical case and from a similar argumentation as in lemma 8.8, we can conjecture that the proper rate for renormalization will be $N^{L-1/2}$.

LEMMA 8.9: If $\alpha_L > 0$ then there exists a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i_1, \dots, i_L \leq N-1} \mathbb{E} \left[\left(X_i^N(0) \right)^{2+\delta} \right] < \infty \text{ implies } \sup_N \sup_{0 \leq u \leq Nt} \max_{0 \leq i_1, \dots, i_L \leq N-1} \mathbb{E} \left[\left(X_i^N(u) \right)^{2+\delta} \right] \leq C_t$$

for some constant C_t . Note that C_t depends on $\sup_N \max_{0 \leq i_1, \dots, i_L \leq N-1} \mathbb{E} \left[\left(X_i^N(0) \right)^{2+\delta} \right] < \infty$.

IDEA OF THE PROOF OF LEMMA 8.9: For $i, i' \in \{0, 1, \dots, N-1\}^{\otimes L}$ define

$$\begin{aligned} a_t(i, i') &\equiv \frac{1}{N^{k-1}} + \sum_{k=1}^L \frac{1}{N^{k-1}} \left(\prod_{j=k+1}^L \delta_{i_j}(i'_j) \right) \left(\delta_{i_k}(i'_k) - \frac{1}{N} \right) \exp \left[-t \sum_{j=k}^L \frac{c_j}{N^{j-1}} \right] \\ &= \exp \left[-t \sum_{j=1}^L \frac{c_j}{N^{j-1}} \right] \prod_{j=1}^L \delta_{i_j}(i'_j) + \sum_{k=1}^L \frac{1}{N^k} \left(\prod_{j=k+1}^L \delta_{i_j}(i'_j) \right) \left(\exp \left[-t \sum_{j=k+1}^L \frac{c_j}{N^{j-1}} \right] - \exp \left[-t \sum_{j=k}^L \frac{c_j}{N^{j-1}} \right] \right) \end{aligned}$$

where $\delta_i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$ and reproduce the proof of lemma 3.3. ■

LEMMA 8.10: If $\alpha_L > 0$ then there exists a constant C such that

$$\mathbb{E} \left[\bar{X}_{N^L}^2(t) \right] \leq C \mathbb{E} \left[\bar{X}_{N^L}^2(0) \right] \exp \left[\frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \frac{t}{N^L} \right]$$

PROOF OF LEMMA 8.10: The integral form of equation (8.5.3) is, for any integer M such that $0 \leq M \leq L$,

$$\begin{aligned} (8.10.1) \quad & \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(t) \right)^2 \right] \\ &= \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(0) \right)^2 \right] + \frac{\gamma^2}{N^M} \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left(X_i^N(s) \right)^2 \right] ds \\ &\quad + \sum_{k=M+1}^L \frac{2c_k}{N^{k-1}} \int_0^t \mathbb{E} \left[\frac{1}{N^{L-k}} \sum_{i_{k+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^k} \sum_{i_1, \dots, i_k=0}^{N-1} X_i^N(s) \right)^2 \right] ds \\ &\quad - \left(\sum_{k=M+1}^L \frac{2c_k}{N^{k-1}} \right) \int_0^t \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(s) \right)^2 \right] ds \end{aligned}$$

which implies that

$$(8.10.2) \quad -\frac{\gamma^2}{N^M} \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(t))^2 \right] ds \leq \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(0) \right)^2 \right] \\ + \frac{2c_{M+1}}{N^M} \int_0^t \mathbb{E} \left[\frac{1}{N^{L-M-1}} \sum_{i_{M+2}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{M+1}} \sum_{i_1, \dots, i_{M+1}=0}^{N-1} X_i^N(s) \right)^2 \right] ds \\ - \frac{2c_{M+1}}{N^M} \int_0^t \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} X_i^N(s) \right)^2 \right] ds$$

because when $L \geq M_1 \geq M_2 \geq 0$ (see lemma A.5)

$$\frac{1}{N^{L-M_1}} \sum_{i_{M_1+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{M_1}} \sum_{i_1, \dots, i_{M_1}=0}^{N-1} X_i^N(s) \right)^2 \leq \frac{1}{N^{L-M_2}} \sum_{i_{M_2+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{M_2}} \sum_{i_1, \dots, i_{M_2}=0}^{N-1} X_i^N(s) \right)^2.$$

When $M = 0$, (8.10.2) becomes

$$(8.10.3) \quad \alpha_1 \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(s))^2 \right] ds \\ \leq \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(0))^2 \right] + 2c_1 \int_0^t \mathbb{E} \left[\frac{1}{N^{L-1}} \sum_{i_2, \dots, i_L=0}^{N-1} \left(\frac{1}{N} \sum_{i_1=0}^{N-1} X_i^N(s) \right)^2 \right] ds$$

We will show by induction that there is a constant C such that

$$(8.10.4) \quad \frac{\alpha_L}{N^{L-1}} \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(s))^2 \right] ds \leq C \mathbb{E} \left[\frac{1}{N} \sum_{i_L=0}^{N-1} \left(\frac{1}{N^{L-1}} \sum_{i_1, \dots, i_{L-1}=0}^{N-1} X_i^N(0) \right)^2 \right] \\ + \frac{1}{N^{L-1}} \left(\prod_{k=1}^L 2c_k \right) \int_0^t \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(s) \right)^2 \right] ds.$$

In the following, the constant C may not be the same from step to step. Assume that there is an integer K such that $0 \leq K \leq L-2$ and

$$(8.10.5) \quad \frac{\alpha_{K+1}}{N^K} \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(s))^2 \right] ds \leq C \mathbb{E} \left[\frac{1}{N^{L-K}} \sum_{i_{K+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^K} \sum_{i_1, \dots, i_K=0}^{N-1} X_i^N(0) \right)^2 \right] \\ + \frac{1}{N^K} \left(\prod_{k=1}^{K+1} 2c_k \right) \int_0^t \mathbb{E} \left[\frac{1}{N^{L-K-1}} \sum_{i_{K+2}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{K+1}} \sum_{i_1, \dots, i_{K+1}=0}^{N-1} X_i^N(s) \right)^2 \right] ds.$$

The inequality (8.10.3) shows that (6.10.5) is verified when $K = 0$. Adding $\frac{2c_{K+2}}{N}$ time (8.10.5) and $\prod_{k=1}^{K+1} 2c_k$ time (8.10.2) with $M = K + 1$ gives

$$\begin{aligned}
 (8.10.6) \quad & \frac{\alpha_{K+2}}{N^{K+1}} \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(s))^2 \right] ds \\
 & \leq C \frac{1}{N} \mathbb{E} \left[\frac{1}{N^{L-K}} \sum_{i_{K+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^K} \sum_{i_1, \dots, i_K=0}^{N-1} X_i^N(0) \right)^2 \right] + C \mathbb{E} \left[\frac{1}{N^{L-K-1}} \sum_{i_{K+2}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{K+1}} \sum_{i_1, \dots, i_{K+1}=0}^{N-1} X_i^N(0) \right)^2 \right] \\
 & \quad + \frac{1}{N^{K+1}} \left(\prod_{k=1}^{K+2} 2c_k \right) \int_0^t \mathbb{E} \left[\frac{1}{N^{L-K-2}} \sum_{i_{K+3}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{K+2}} \sum_{i_1, \dots, i_{K+2}=0}^{N-1} X_i^N(s) \right)^2 \right] ds^1 \\
 & \leq C \mathbb{E} \left[\frac{1}{N^{L-K-1}} \sum_{i_{K+2}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{K+1}} \sum_{i_1, \dots, i_{K+1}=0}^{N-1} X_i^N(0) \right)^2 \right] \\
 & \quad + \frac{1}{N^{K+1}} \left(\prod_{k=1}^{K+2} 2c_k \right) \int_0^t \mathbb{E} \left[\frac{1}{N^{L-K-2}} \sum_{i_{K+3}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{K+2}} \sum_{i_1, \dots, i_{K+2}=0}^{N-1} X_i^N(s) \right)^2 \right] ds^2
 \end{aligned}$$

The inequality (8.10.4) is obtain from (8.10.6) when $K = L - 2$.

The equation (8.10.1) with $M = L$ gives

$$\begin{aligned}
 \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(t) \right)^2 \right] &= \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(0) \right)^2 \right] + \frac{\gamma^2}{N^L} \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (X_i^N(s))^2 \right] ds \\
 &\leq \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(0) \right)^2 \right] + C \frac{1}{N} \mathbb{E} \left[\frac{1}{N} \sum_{i_1, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{L-1}} \sum_{i_1, \dots, i_{L-1}=0}^{N-1} X_i^N(0) \right)^2 \right] \\
 &\quad + \frac{1}{N^L} \frac{\gamma^2}{\alpha_L} \left(\prod_{k=1}^L 2c_k \right) \int_0^t \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(s) \right)^2 \right] ds^3
 \end{aligned}$$

¹ From the induction hypothesis and the definition of the α 's.

² Because if $\forall i, a_i \geq 0$ then

$$\begin{aligned}
 \frac{1}{N} \frac{1}{N^{L-K}} \sum_{i_{K+1}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^K} \sum_{i_1, \dots, i_K=0}^{N-1} a_i \right)^2 &= \frac{1}{N^{L-K+1}} \sum_{i_{K+1}, \dots, i_L=0}^{N-1} \left(\sum_{i_1, \dots, i_K=0}^{N-1} a_i \right)^2 \\
 &\leq \frac{1}{N^{L-K+1}} \sum_{i_{K+2}, \dots, i_L=0}^{N-1} \left(\sum_{i_1, \dots, i_{K+1}=0}^{N-1} a_i \right)^2 = \frac{1}{N^{L-K-1}} \sum_{i_{K+2}, \dots, i_L=0}^{N-1} \left(\frac{1}{N^{K+1}} \sum_{i_1, \dots, i_{K+1}=0}^{N-1} a_i \right)^2.
 \end{aligned}$$

³ From (8.10.4) provided $\alpha_L > 0$.

$$\leq C \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(0) \right)^2 \right] + \frac{1}{N^L} \frac{\gamma^2}{\alpha_L} \left(\prod_{k=1}^L 2c_k \right) \int_0^t \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(s) \right)^2 \right] ds$$

Gronwall's inequality gives

$$(8.10.7) \quad \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(t) \right)^2 \right] \leq C \mathbb{E} \left[\left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} X_i^N(0) \right)^2 \right] \exp \left[\frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \frac{t}{N^L} \right]$$

which is our first inequality. ■

Time Rescaling

In this section, we consider a system of stochastic differential equations that is a little more general than the multilevel bilinear one. We introduce a small perturbation into the system as in section 5. With this new process, we are able to use induction on the number L of levels to prove the main theorem.

More precisely, this N^L -dimensional stochastic process is defined on a complete filtered probability space $(\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t : t \geq 0\}, \mathbb{P})$ where lives a N^L -dimensional Brownian motion

$$W = \{W_i : i = (i_1, i_2, \dots, i_L), i_j \in \{0, 1, \dots, N-1\}, \forall j \in \{1, 2, \dots, L\}\}.$$

We assume that $\tilde{\mathcal{F}}_0$ contains the collection of the $\mathbb{P}$ -null sets and that the filtration is right-continuous. The process Z^N is defined as the unique strong solution of the system of stochastic differential equations

$$dZ_i^N(t) = \sum_{k=1}^L \frac{c_k}{N^{k-1}} (\bar{Z}_{i,k}^N(t) - Z_i^N(t)) dt + \frac{c_{L+1}}{N^L} (\theta - Z_i^N(t)) dt + \gamma Z_i^N(t) dW_i(t),$$

$$i = (i_1, i_2, \dots, i_L), i_j \in \{0, 1, \dots, N-1\} \forall j \in \{1, 2, \dots, L\}$$

where

$$\bar{Z}_{i,k}^N(s) \equiv \frac{1}{N^k} \sum_{i_1, \dots, i_k=0}^{N-1} Z_i^N(s), \quad \bar{Z}_{i,0}^N(s) \equiv Z_i^N(s),$$

$c_1, c_2, \dots, c_L, \gamma$ and θ are nonnegative constants (γ and c_L are also assumed to be strictly positive). Note that, despite the notation, $\bar{Z}_{i,M}^N(s)$ does not depend on $i_1, \dots, i_M$. For this reason, we also use

$$\bar{Z}_{N^L}(s) \equiv \bar{Z}_{i_L}^N(s) = \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} Z_i^N(s)$$

for the average process.

Behavior of $\{\bar{Z}_{N^L}(N^L t), t \in \mathbb{R}_+\}$ as N goes to infinity in the subcritical case ($\alpha_L > 0$)

For any N , let Q_N be the probability on $C(\mathbb{R}_+, \mathbb{R}_+)$ induced by the process $\{\bar{Z}_{N^L}(N^L t), t \in \mathbb{R}_+\}$. In this section, we will assume that the initial conditions are $\mathbb{P}$ -a.s. nonnegative.

THEOREM 9.1: Let $\alpha_L > 0$ and assume that the following assumptions on the initial conditions are satisfied: there is a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i_1, \dots, i_L \leq N-1} \mathbb{E} \left[(Z_i^N(0))^{2+\delta} \right] < \infty$$

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and $\mathcal{L}(\bar{Z}_N^L(0))$ converges weakly to μ . Then the sequence $\{Q_N : N \in \mathbb{N}\}$ converges weakly (as N goes to infinity) to the unique solution of the martingale problem associated with $\mathcal{B}$ with initial distribution μ where

$$(\mathcal{B}f)(z) \equiv \frac{1}{2\alpha_L} \gamma^2 \prod_{k=1}^L 2c_k x^2 f''(z) + 2c_{L+1}(\theta - z) f'(z), \quad f \in C^2(\mathbb{R}, \mathbb{R}).$$

SKETCH OF THE PROOF OF THEOREM 9.1:

CLAIM: The sequence $\{Q_N : N \in \mathbb{N}\}$ of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$ is tight. (Proposition 9.1).

Let $\{Q_{N'} : N' \in \mathbb{N}\}$ be an arbitrary convergent subsequence of $\{Q_N : N \in \mathbb{N}\}$ and Q_∞ denotes its limit point. We would like to show that Q_∞ is a solution of the martingale problem associated with $\mathcal{B}$. To do so, it suffices to prove that

$$\left\{ f(\xi_t) - \int_0^t (\mathcal{B}f)(\xi_s) ds : t \in \mathbb{R}_+ \right\}$$

is a Q_∞ -martingale for $f(x) \equiv x$ and $f(x) \equiv x^2$ where $\{\xi_t : t \in \mathbb{R}_+\}$ denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}_+)$.

CLAIM: $\left\{ \xi_t - c_{L+1} \int_0^t (\theta - \xi_s) ds : t \in \mathbb{R}_+ \right\}$ is a Q_∞ -martingale. (Proposition 9.5).

CLAIM: $\left\{ \xi_t^2 - \int_0^t \frac{1}{\alpha_L} \gamma^2 \prod_{k=1}^L 2c_k \xi_s^2 ds - 2c_{L+1} \int_0^t (\theta - \xi_s) \xi_s ds : t \in \mathbb{R}_+ \right\}$ is a Q_∞ -martingale. (Proposition 9.6 and following).

The last two claims imply that Q_∞ is a solution of the martingale problem associated with $\mathcal{B}$. Since this martingale problem has a unique solution then any convergent subsequence has Q_∞ as its limit point and therefore, $\{Q_N : N \in \mathbb{N}\}$ converges weakly to Q_∞ as N goes to infinity.

LEMMA 9.3: If $\alpha_L > 0$ then there exists a $\delta > 0$ such that, for some constant C_t

$$\sup_N \max_{0 \leq t_1, \dots, t_L \leq N-1} \mathbb{E} \left[\left(Z_t^N(0) \right)^{2+\delta} \right] < \infty \text{ implies } \sup_N \sup_{0 \leq u \leq Nt} \max_{0 \leq t_1, \dots, t_L \leq N-1} \mathbb{E} \left[\left(Z_t^N(u) \right)^{2+\delta} \right] \leq C_t$$

and

$$\sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{t_1, \dots, t_L=0}^{N-1} \left(Z_t^N(0) \right)^{2+\delta} \right] < \infty \text{ implies } \sup_N \sup_{0 \leq u \leq Nt} \mathbb{E} \left[\frac{1}{N^L} \sum_{t_1, \dots, t_L=0}^{N-1} \left(Z_t^N(u) \right)^{2+\delta} \right] \leq C_t$$

Note that C_t depends on

$$\sup_N \max_{0 \leq t_1, \dots, t_L \leq N-1} \mathbb{E} \left[\left(Z_t^N(0) \right)^{2+\delta} \right] < \infty \text{ or } \sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{t_1, \dots, t_L=0}^{N-1} \left(Z_t^N(0) \right)^{2+\delta} \right] < \infty.$$

IDEA OF THE PROOF OF LEMMA 9.3:

For $i, i' \in \{0, 1, \dots, N-1\}^{\otimes L}$ define

$$\begin{aligned} a_t(i, i') &\equiv \frac{1}{N^L} + \sum_{k=1}^L \frac{1}{N^{k-1}} \left(\prod_{j=k+1}^L \delta_{i_j}(i'_j) \right) \left(\delta_{i_k}(i'_k) - \frac{1}{N} \right) \exp \left[-t \sum_{j=k}^L \frac{c_j}{N^{j-1}} \right] \\ &= \exp \left[-t \sum_{j=1}^L \frac{c_j}{N^{j-1}} \right] \prod_{j=1}^L \delta_{i_j}(i'_j) + \sum_{k=1}^L \frac{1}{N^k} \left(\prod_{j=k+1}^L \delta_{i_j}(i'_j) \right) \left(\exp \left[-t \sum_{j=k+1}^L \frac{c_j}{N^{j-1}} \right] - \exp \left[-t \sum_{j=k}^L \frac{c_j}{N^{j-1}} \right] \right) \end{aligned}$$

where $\delta_i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$ and reproduce the proof of lemma 5.3. ■

PROPOSITION 9.4:

If $\alpha_L > 0$ and if there is a positive δ for which

$$\sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(0))^{2+\delta} \right] < \infty$$

then the sequence $\{Q_N : N \in \mathbb{N}\}$ of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$ is tight.

PROOF OF PROPOSITION 9.4: We need to check the two conditions of proposition 3.4. The first one is straight forward. Let us verify the second one.

We want to prove that for every $K > 0$, $\eta > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ and $N_0 \in \mathbb{N}$ with the property that whatever be the family of stopping times $\{\tau_n : n \in \mathbb{N}\}$ with $\tau_n \leq K$,

$$\sup_{N \geq N_0} \sup_{\theta \leq \delta} \mathbb{P} \left[\left| \langle \bar{Z}_{N^L} \rangle (N^L \tau_N) - \langle \bar{Z}_{N^L} \rangle (N^L (\tau_N + \theta)) \right| \geq \eta \right] < \varepsilon.$$

But

$$\begin{aligned} &\sup_{N \geq N_0} \sup_{\theta \leq \delta} \mathbb{P} \left[\left| \langle \bar{Z}_{N^L} \rangle (N^L \tau_N) - \langle \bar{Z}_{N^L} \rangle (N^L (\tau_N + \theta)) \right| \geq \eta \right] \\ &= \sup_{N \geq N_0} \frac{\gamma^2}{\eta} \int_0^\delta \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(N^L s))^2 \left\| Z^N(0) - Z^N(\tau_N) \right\| \right] ds \leq \frac{C \gamma^2}{\eta} \delta \end{aligned}$$

where the last inequality holds if

$$\sup_{N \geq N_0} \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(\tau_N))^2 \right] < \infty.$$

However,

$$\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(t))^2 \leq \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(0))^2 + 2c_{L+1} \theta \int_0^{\tau_N} \bar{Z}_{N^L}(s) ds + \gamma^2 \int_0^t \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(s))^2 ds + M(t).$$

where the process M is a martingale with null expectation. Since τ_N is bounded by K ,

$$\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(\tau_N))^2$$

$$\begin{aligned}
&\leq \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(0))^{2+\delta} + 2c_{L+1}\theta \int_0^{\tau_N} \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(s))^{1+\delta} ds \\
&\quad + \gamma^2 \int_0^{\tau_N} \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(s))^{2+\delta} ds + M(\tau_N). \\
&\leq \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(0))^{2+\delta} + 2c_{L+1}\theta \int_0^K \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(s))^{1+\delta} ds \\
&\quad + \gamma^2 \int_0^K \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(s))^{2+\delta} ds + M(K \wedge \tau_N).
\end{aligned}$$

Therefore, if $\sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(0))^{2+\delta} \right] < \infty$ then

$$\begin{aligned}
&\sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(\tau_N))^{2+\delta} \right] \\
&\leq \sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(0))^{2+\delta} \right] + 2c_{L+1}\theta \int_0^K \sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(s))^{1+\delta} \right] ds \\
&\quad + \gamma^2 \int_0^K \sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(s))^{2+\delta} \right] ds \\
&\leq C \sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(0))^{2+\delta} \right].
\end{aligned}$$

PROPOSITION 9.5: If $\alpha_L > 0$ and $\sup_N \mathbb{E} \left[\bar{X}_{N^L}^2(0) \right] < \infty$ then

$$\left\{ \xi_t - c_{L+1} \int_0^t (\theta - \xi_s) ds : t \in \mathbb{R}_+ \right\} \text{ is a } Q_- \text{-martingale.}$$

PROOF OF PROPOSITION 9.5: Note that

$$(9.5.1) \quad \bar{Z}_{N^L}(N^L t) - c_{L+1} \int_0^t (\theta - \bar{Z}_{N^L}(N^L s)) ds \stackrel{\text{p-as}}{=} \bar{Z}_{N^L}(0) + \gamma \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \int_0^{N^L t} Z_i^N(s) dW_i(s)$$

implies that $\left\{ \xi_t - c_{L+1} \int_0^t (\theta - \xi_s) ds : t \in \mathbb{R}_+ \right\}$ is a Q_N -martingale. We use the generalization of the continuous mapping theorem to conclude the proof. ■

PROPOSITION 9.6: If $\alpha_L > 0$ and if there is a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i_1, \dots, i_L \leq N-1} \mathbb{E} \left[(Z_i^N(0))^{2+\delta} \right] < \infty$$

then

$$\left\{ \xi_t^2 - \int_0^t \frac{1}{\alpha_L} \gamma^2 \prod_{k=1}^L 2c_k \xi_s^2 ds - 2c_{L+1} \int_0^t (\theta - \xi_s) \xi_s ds : t \in \mathbb{R}_+ \right\} \text{ is a } Q_\infty\text{-martingale}$$

provided that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_s^t \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(N^L r))^2 dr - \int_s^t \frac{1}{\alpha_L} \prod_{k=1}^L 2c_k \bar{Z}_{N^L}^2(N^L r) dr \right| \right] = 0.$$

PROOF OF PROPOSITION 9.6: It suffices to show that for any real numbers s and t with $0 \leq s \leq t$ and for any $\sigma\{\xi_r : 0 \leq r \leq s\}$ -measurable $\Phi \in C_b(C(\mathbb{R}_+, \mathbb{R}), \mathbb{R})$,

$$\begin{aligned} & E_\infty \left[\Phi \left(\xi_t^2 - \int_0^t \frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \xi_r^2 dr - 2c_{L+1} \int_0^t (\theta - \xi_r) \xi_r dr \right) \right] \\ &= E_\infty \left[\Phi \left(\xi_s^2 - \int_0^s \frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \xi_r^2 dr - 2c_{L+1} \int_0^s (\theta - \xi_r) \xi_r dr \right) \right]. \end{aligned}$$

From equation (9.5.1) and Itô's formula,

$$\begin{aligned} & \bar{Z}_{N^L}^2(N^L t) - \bar{Z}_{N^L}^2(0) \\ & \stackrel{\mathbb{P}\text{-a.s.}}{=} 2c_{L+1} \int_0^t (\theta - \bar{Z}_{N^L}(N^L r)) \bar{Z}_{N^L}(N^L r) dr + \gamma^2 \int_0^t \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(N^L r))^2 dr \\ & \quad + 2\gamma \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \int_0^{N^L t} \bar{Z}_{N^L}(r) Z_i^N(r) dW_i(r). \end{aligned}$$

Therefore

$$\begin{aligned} & \bar{Z}_{N^L}^2(N^L t) - \int_0^t \frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \bar{Z}_{N^L}^2(N^L r) dr - 2c_{L+1} \int_0^t (\theta - \bar{Z}_{N^L}(N^L r)) \bar{Z}_{N^L}(N^L r) dr \\ & \quad + \int_0^t \frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \bar{Z}_{N^L}^2(N^L r) dr - \gamma^2 \int_0^t \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(N^L r))^2 dr \end{aligned}$$

is a $\mathbb{P}$ -martingale. Hence,

$$\begin{aligned} & E_N \left[\Phi \left(\xi_t^2 - \int_0^t \frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \xi_r^2 dr - 2c_{L+1} \int_0^t (\theta - \xi_r) \xi_r dr \right) \right] \\ &= E_N \left[\Phi \left(\xi_s^2 - \int_0^s \frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \xi_r^2 dr - 2c_{L+1} \int_0^s (\theta - \xi_r) \xi_r dr \right) \right] + c(N^0), \end{aligned}$$

We apply the generalization of the continuous mapping theorem to complete the proof.

¹ A function $f(N) = c(N^p) \Leftrightarrow f(N)N^{-p} \xrightarrow{N \rightarrow \infty} 0$.

PROPOSITION 9.8: If $\alpha_L > 0$ and if there is a $\delta > 0$ such that

$$\sup_N \max_{0 \leq i_1, \dots, i_L \leq N-1} \mathbb{E} \left[\left(Z_i^N(0) \right)^{2+\delta} \right] < \infty$$

then for any $0 \leq s \leq t < \infty$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \gamma^2 \int_s^t \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left(Z_i^N(N^L r) \right)^2 dr - \frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k \int_s^t \bar{Z}_{N^L}^2(N^L r) dr \right| \right] = 0.$$

PROOF OF PROPOSITION 9.8: We will show that for any $1 \leq M \leq L$,

$$(9.8.1) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{\gamma^2}{\alpha_{M-1}} \prod_{k=1}^{M-1} 2c_k \int_s^t \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(\bar{Z}_{i, M-1}^N(N^L r) \right)^2 dr - \frac{\gamma^2}{\alpha_M} \prod_{k=1}^M 2c_k \int_s^t \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \left(\bar{Z}_{i, M}^N(N^L r) \right)^2 dr \right| \right] = 0.$$

Let

$$\hat{\gamma}_M \equiv \sqrt{\frac{1}{N^M} \frac{\gamma^2}{\alpha_M} \prod_{m=1}^M 2c_m} \text{ and } \hat{c}_M \equiv \sum_{k=M+1}^L \frac{c_k}{N^{k-1}}.$$

For any fixed integer M such that $1 \leq M \leq L$, and any fixed positive real numbers K and t , we introduce a modified process, denoted

$$\{Z_i^{N, M, t}(s) : i = (i_1, \dots, i_L), i_k \in \{0, 1, \dots, N-1\}, s \geq 0\},$$

as follows: If $0 \leq s \leq (N^L - N^{M-1} K)t$ then

$$Z_i^{N, M, t}(s) \equiv Z_i^N(s)$$

and if $s > (N^L - N^{M-1} K)t$ then

$$(9.8.2) \quad \begin{aligned} Z_i^{N, M, t}(s) \equiv & Z_i^N((N^L - N^{M-1} K)t) \\ & + \sum_{k=1}^{M-1} \frac{c_k}{N^{k-1}} \int_{(N^L - N^{M-1} K)t}^s \left(\bar{Z}_{i, k}^{N, M, t}(r) - Z_i^{N, M, t}(r) \right) dr \\ & + \hat{c}_{M-1} \int_{(N^L - N^{M-1} K)t}^s \left(\bar{Z}_{i, M}^N((N^L - N^{M-1} K)t) - Z_i^{N, M, t}(r) \right) dr \\ & + \frac{c_{L+1}}{N^L} \int_{(N^L - N^{M-1} K)t}^s \left(\theta - Z_i^{N, M, t}(r) \right) dr + \sqrt{N^{(M-1)}} \hat{\gamma}_{M-1} \int_{(N^L - N^{M-1} K)t}^s \bar{Z}_{i, M-1}^{N, M, t}(r) dW_i(r). \end{aligned}$$

Therefore, if $s > (N^L - N^{M-1}K)t$ then

$$\begin{aligned}
 \bar{Z}_{i,M-1}^{N,M,t}(s) &= \bar{Z}_{i,M-1}^N((N^L - N^{M-1}K)t) \\
 &+ \hat{c}_{M-1} \int_{(N^L - N^{M-1}K)t}^s \left(\bar{Z}_{i,M}^N((N^L - N^{M-1}K)t) - \bar{Z}_{i,M-1}^{N,M,t}(r) \right) dr + \frac{c_{L+1}}{N^L} \int_{(N^L - N^{M-1}K)t}^s \left(\theta - \bar{Z}_{i,M-1}^{N,M,t}(r) \right) dr \\
 &+ \frac{\hat{\gamma}_{M-1}}{\sqrt{N^{(M-1)}}} \sum_{i_1, \dots, i_{M-1}=0}^{N-1} \int_{(N^L - N^{M-1}K)t}^s \bar{Z}_{i,M-1}^{N,M,t}(r) dW_{i_1}(r) \\
 (9.8.3) \quad &= \bar{Z}_{i,M-1}^N((N^L - N^{M-1}K)t) \\
 &+ \hat{c}_{M-1} \int_{(N^L - N^{M-1}K)t}^s \left(\bar{Z}_{i,M}^N((N^L - N^{M-1}K)t) - \bar{Z}_{i,M-1}^{N,M,t}(r) \right) dr + \frac{c_{L+1}}{N^L} \int_{(N^L - N^{M-1}K)t}^s \left(\theta - \bar{Z}_{i,M-1}^{N,M,t}(r) \right) dr \\
 &+ \hat{\gamma}_{M-1} \int_{(N^L - N^{M-1}K)t}^s \bar{Z}_{i,M-1}^{N,M,t}(r) d \left(\frac{1}{\sqrt{N^{(M-1)}}} \sum_{i_1, \dots, i_{M-1}=0}^{N-1} W_{i_1}(r) \right).
 \end{aligned}$$

We will show that for any bounded continuous differentiable function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}$ with bounded first derivative,

$$(9.8.4) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(\varphi(\bar{Z}_{i,M-1}^N(N^L t)) - \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\bar{Z}_{i,M}^N(N^L t), \hat{c}_{M-1}, \hat{\gamma}_{M-1}}(dy) \right) \right| \right] = 0$$

by showing the next four inequalities: There is a constant C and, for any positive real number T , there is a constant C_T such that

$$(9.8.5) \quad \forall t \in]0, T] \text{ and } \forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N} \text{ such that } \forall K \geq K(\varepsilon), \exists N(\varepsilon, K) \text{ such that } \forall N \geq N(\varepsilon, K),$$

$$\mathbb{E} \left[\left| \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(\varphi(\bar{Z}_{i,M-1}^N(N^L t)) - \varphi(\bar{Z}_{i,M-1}^{N,M,t}(N^L t)) \right) \right| \right] \leq C_T \left(\sqrt{\frac{K^3}{N}} + \frac{K^2}{N} + \varepsilon \right)$$

¹ Note that $\bar{Z}_{i,M-1}^{N,M,t}$ depends on K .

$$(9.8.6) \quad \sup_{t>0} \mathbb{E} \left[\left| \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(\varphi(\bar{Z}_{i, M-1}^{N, M, t}(N^L t)) - \mathbb{E} \left[\varphi(\bar{Z}_{i, M-1}^{N, M, t}(N^L t)) \middle| \tilde{\lambda}_{(N^L - N^{M-1} K)t} \right] \right) \right| \right] \\ \leq \frac{2 \|\varphi\|_\infty}{N^{(L-M+1)/2}} 1$$

$$(9.8.7) \quad \mathbb{E} \left[\frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left| \mathbb{E} \left[\varphi(\bar{Z}_{i, M-1}^{N, M, t}(N^L t)) \middle| \tilde{\lambda}_{(N^L - N^{M-1} K)t} \right] \right. \right. \\ \left. \left. - \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\bar{Z}_{i, M-1}^N((N^L - N^{M-1} K)t), \hat{c}_{M-1}, \hat{\gamma}_{M-1}}(dy) \right| \right] \leq C \exp \left[- \left(\hat{c}_{M-1} + \frac{C_{L+1}}{N^L} \right) N^{M-1} K t \right] \quad \forall t > 0^2$$

$$(9.8.8) \quad \sup_{0 < t \leq T} \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \mathbb{E} \left[\left| \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\bar{Z}_{i, M}^N((N^L - N^{M-1} K)t), \hat{c}_{M-1}, \hat{\gamma}_{M-1}}(dy) \right. \right. \\ \left. \left. - \int_{\mathbb{R}_+} \varphi(y) \vartheta_{\bar{Z}_{i, M}^N(N^L t), \hat{c}_{M-1}, \hat{\gamma}_{M-1}}(dy) \right| \right] \leq C_T \left(\sqrt{\frac{K}{N}} + \frac{K}{N} \right)^3$$

¹ Because $\{\bar{Z}_{i, M-1}^{N, M, t}(N^L t) : M \leq k \leq L, i_k \in \{0, 1, \dots, N-1\}\}$ are conditionally independent with respect to $\mathbb{P} \left[\bullet \middle| \tilde{\lambda}_{(N^L - N^{M-1} K)t} \right]$. See the demonstration of inequality (10.7.2) for more details.

² This is an application of lemma 3.15 because, from equation (9.8.3), we have

$$\mathbb{E} \left[\varphi(\bar{Z}_{i, M-1}^{N, M, t}(N^L t)) \middle| \tilde{\lambda}_{(N^L - N^{M-1} K)t} \right] \\ = \int_{\mathbb{R}_+} \varphi(y) Q_{N^{M-1} K t}^{\bar{Z}_{i, M-1}^N((N^L - N^{M-1} K)t), \hat{c}_{M-1}, \hat{\gamma}_{M-1}} \left(\bar{Z}_{i, M-1}^N((N^L - N^{M-1} K)t), dy \right).$$

Recall that, as it has been defined before in corollary 5.8, $Q_t^{\eta, c, \gamma}(x, dy)$ and $\vartheta_{\eta, c, \gamma}$ are respectively the transition probability and the equilibrium of the one-dimensional diffusion process described by the stochastic differential equation

$$dX(t) = c(\eta - X(t)) dt + \frac{b}{N}(\theta - X(t)) dt + \gamma X(t) dW(t), \quad \eta, \theta \geq 0.$$

³ From lemma 3.16.

When (9.8.4) is established, we apply the arguments of proposition 3.17 to conclude that (9.8.1) is true. (The idea was to build a sequence $\{\varphi_k : k \in \mathbb{N}\}$ of bounded continuous differentiable functions with bounded first derivative that increase to $\varphi(y) \equiv N^{M-1} \hat{\gamma}_{M-1}^2 y^2$ and use the uniform integrability of the random variables involved to show that the property (9.8.4) is preserved in the limit (as k tends to infinity) to gives (9.8.1)). Therefore

$$(9.8.9) \quad 0 \xleftarrow{N \rightarrow \infty} \mathbb{E} \left[\left| \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(N^{M-1} \hat{\gamma}_{M-1}^2 (\bar{Z}_{i,M-1}^N(N^L t))^2 - \int_{\mathbb{R}_+} N^{M-1} \hat{\gamma}_{M-1}^2 y^2 \vartheta_{\bar{Z}_{i,M}^N(N^L t), \hat{c}_{M-1}, \hat{\gamma}_{M-1}}(dy) \right) \right| \right]$$

$$= \mathbb{E} \left[\left| \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(N^{M-1} \hat{\gamma}_{M-1}^2 (\bar{Z}_{i,M-1}^N(N^L t))^2 - N^{M-1} \frac{2 \hat{c}_{M-1} \hat{\gamma}_{M-1}^2}{2 \hat{c}_{M-1} - \hat{\gamma}_{M-1}^2} (\bar{Z}_{i,M}^N(N^L t))^2 \right) \right| \right]$$

$$= \mathbb{E} \left[\left| \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(N^{M-1} \hat{\gamma}_{M-1}^2 (\bar{Z}_{i,M-1}^N(N^L t))^2 - N^M \hat{\gamma}_M^2 (\bar{Z}_{i,M}^N(N^L t))^2 \right) \right| \right] + o(N^0)$$

because

$$N^{M-1} \frac{2 \hat{c}_{M-1} \hat{\gamma}_{M-1}^2}{2 \hat{c}_{M-1} - \hat{\gamma}_{M-1}^2} = \frac{\sum_{k=M}^L \frac{2 c_k}{N^{k-M}} \frac{\gamma^2}{\alpha_{M-1}} \prod_{m=1}^{M-1} 2 c_m}{\sum_{k=M}^L \frac{2 c_k}{N^{k-M}} - \frac{\gamma^2}{\alpha_{M-1}} \prod_{m=1}^{M-1} 2 c_m}$$

$$= \frac{\gamma^2 \prod_{m=1}^M 2 c_m + \sum_{k=M+1}^L \frac{2 c_k}{N^{k-M}} \gamma^2 \prod_{m=1}^{M-1} 2 c_m}{\alpha_M + \alpha_{M-1} \sum_{k=M+1}^L \frac{2 c_k}{N^{k-M}}} \xrightarrow{N \rightarrow \infty} \frac{\gamma^2}{\alpha_M} \prod_{m=1}^M 2 c_m = N^M \hat{\gamma}_M^2$$

so

$$\mathbb{E} \left[\left| \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(N^{M-1} \frac{2 \hat{c}_{M-1} \hat{\gamma}_{M-1}^2}{2 \hat{c}_{M-1} - \hat{\gamma}_{M-1}^2} (\bar{Z}_{i,M-1}^N(N^L t))^2 - N^M \hat{\gamma}_M^2 (\bar{Z}_{i,M}^N(N^L t))^2 \right) \right| \right]$$

$$= \left| N^{M-1} \frac{2 \hat{c}_{M-1} \hat{\gamma}_{M-1}^2}{2 \hat{c}_{M-1} - \hat{\gamma}_{M-1}^2} - N^M \hat{\gamma}_M^2 \right| \mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_M, \dots, i_L=0}^{N-1} (\bar{Z}_{i,M}^N(N^L t))^2 \right]$$

$$\leq \left| N^{M-1} \frac{2 \hat{c}_{M-1} \hat{\gamma}_{M-1}^2}{2 \hat{c}_{M-1} - \hat{\gamma}_{M-1}^2} - N^M \hat{\gamma}_M^2 \right| \sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(N^L t))^2 \right] \xrightarrow{N \rightarrow \infty} 0.$$

provided

$$\sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(N^L t))^2 \right] < \infty.$$

LEMMA 9.9: Assume that $\alpha_L > 0$ and the existence of a positive δ for which

$$\sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_{i_1}^N(0))^{2+\delta} \right] < \infty.$$

For any integer M such that $0 \leq M \leq L$ and any positive real number T , there is a constant C_T such that for any real numbers s_1 and s_2 with $0 \leq s_1 \leq s_2 \leq N^L T$,

$$\frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \mathbb{E} \left[\left| \bar{Z}_{i,M}^N(s_2) - \bar{Z}_{i,M}^N(s_1) \right| \right] \leq C_T \left(\sqrt{\frac{s_2 - s_1}{N^M}} + \frac{s_2 - s_1}{N^M} \right).$$

PROOF OF LEMMA 9.9: (Note that the constant C_T may vary from step to step). Taking absolute value in the integral form of (8.5.1) (without forgetting to add the small perturbation term),

$$\begin{aligned} & \left| \bar{Z}_{i,M}^N(s_2) - \bar{Z}_{i,M}^N(s_1) \right| \\ & \leq \sum_{k=M+1}^L \frac{C_k}{N^{k-1}} \int_{s_1}^{s_2} \bar{Z}_{i,k}^N(r) dr + \left(\sum_{k=M+1}^L \frac{C_k}{N^{k-1}} \right) \int_{s_1}^{s_2} \bar{Z}_{i,M}^N(r) dr \\ & \quad + \frac{C_{L+1}}{N^L} \int_{s_1}^{s_2} |\theta - \bar{Z}_{i,M}^N(r)| dr + \left| \gamma \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} \int_{s_1}^{s_2} Z_{i_1}^N(r) dW_{i_1}(r) \right| \end{aligned}$$

Therefore,

$$\begin{aligned} (9.9.1) \quad & \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \mathbb{E} \left[\left| \bar{Z}_{i,M}^N(s_2) - \bar{Z}_{i,M}^N(s_1) \right| \right] \\ & \leq \left(\sum_{k=M+1}^{L+1} \frac{2C_k}{N^{k-1}} \right) \int_{s_1}^{s_2} \mathbb{E} \left[\bar{Z}_{i,M}^N(r) \right] dr + C_{L+1} \theta \frac{s_2 - s_1}{N^L} \\ & \quad + \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \mathbb{E} \left[\left| \gamma \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} \int_{s_1}^{s_2} Z_{i_1}^N(r) dW_{i_1}(r) \right| \right] \\ & \leq C \frac{s_2 - s_1}{N^M} + \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \mathbb{E} \left[\left| \gamma \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} \int_{s_1}^{s_2} Z_{i_1}^N(r) dW_{i_1}(r) \right| \right] \end{aligned}$$

$$\begin{aligned} (9.9.3) \quad & \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \mathbb{E} \left[\left| \gamma \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} \int_{s_1}^{s_2} Z_{i_1}^N(r) dW_{i_1}(r) \right| \right] \\ & \leq \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \mathbb{E} \left[\left| \gamma \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} \int_{s_1}^{s_2} Z_{i_1}^N(r) dW_{i_1}(r) \right|^2 \right]^{1/2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \sqrt{\mathbb{E} \left[\left\langle \gamma \frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} \int_{s_1}^{s_2} Z_i^N(r) dW_i(r) \right\rangle (s_2) \right]} \\
&= \frac{1}{N^{L-M}} \sum_{i_{M+1}, \dots, i_L=0}^{N-1} \sqrt{\frac{\gamma^2}{N^M} \int_{s_1}^{s_2} \mathbb{E} \left[\frac{1}{N^M} \sum_{i_1, \dots, i_M=0}^{N-1} (Z_i^N(r))^2 \right] dr} \\
&\leq \sqrt{\frac{\gamma^2}{N^M} \int_{s_1}^{s_2} \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} (Z_i^N(r))^2 \right] dr} \\
&\leq C_T \sqrt{\frac{s_2 - s_1}{N^M}}
\end{aligned}$$

It remains to show the inequality (9.8.5). The proof uses the ideas of lemma 7.8 and an induction argument on the number L of levels.

LEMMA 9.12: Assume that $\alpha_L > 0$, $1 \leq M \leq L$ and $\sup_N \mathbb{E} [\bar{Z}_{i,L}^N(0)] < \infty$.

$\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N}$ such that $\forall K \geq K(\varepsilon), \exists N(\varepsilon, K)$ such that $\forall N \geq N(\varepsilon, K)$,

$$\begin{aligned}
&\mathbb{E} \left[\frac{1}{N^{L-M}} \sum_{i_M, \dots, i_L=0}^{N-1} \left| \varphi(\bar{Z}_{i,M-1}^N(N^L t)) - \varphi(\bar{Z}_{i,M-1}^{N,M,t}(N^L t)) \right| \right] \\
&\leq C_T \left(\sqrt{\frac{K^3}{N} + \frac{K^2}{N}} \right) + \frac{C_T K}{N} + \sqrt{\frac{4 \|\varphi\|_\infty^2}{N^{L-M}}} + \varepsilon.
\end{aligned}$$

PROOF OF LEMMA 9.12: For any fixed $t > 0$ and $K > 0$, we introduce a modified process, denoted by

$$\{Y_i^{N,M,t}(s) : i = (i_1, \dots, i_L), i_k \in \{0, 1, \dots, N-1\}, s \geq 0\}$$

as follows: If $0 \leq s \leq (N^L - N^{M-1} K)t$ then

$$Y_i^{N,M,t}(s) \equiv Z_i^N(s)$$

$$\forall i, a_i \geq 0 \Rightarrow n \sum_{i=1}^n a_i - \sum_{i=1}^n \sum_{j=1}^n \sqrt{a_i} \sqrt{a_j} = \sum_{i=1}^n \sum_{j=i+1}^n (\sqrt{a_i} - \sqrt{a_j})^2 \geq 0 \Rightarrow \sum_{i=1}^n \sqrt{a_i} \leq \sqrt{n} \sqrt{\sum_{i=1}^n a_i}$$

and if $s > (N^L - N^{M-1}K)t$ then

$$\begin{aligned} Y_i^{N,M,t}(s) &\equiv Z_i^N((N^L - N^{M-1}K)t) + \sum_{k=1}^{M-1} \frac{C_k}{N^{k-1}} \int_{(N^L - N^{M-1}K)t}^s (\bar{Y}_{i,k}^{N,M,t}(r) - Y_i^{N,M,t}(r)) dr \\ &\quad + \hat{C}_{M-1} \int_{(N^L - N^{M-1}K)t}^s (\bar{Z}_{i,M}^N((N^L - N^{M-1}K)t) - Y_i^{N,M,t}(r)) dr \\ &\quad + \frac{C_{L+1}}{N^L} \int_{(N^L - N^{M-1}K)t}^s (\theta - Y_i^{N,M,t}(r)) dr + \gamma \int_{(N^L - N^{M-1}K)t}^s Y_i^{N,M,t}(r) dW_i(r). \end{aligned}$$

Note that if $s > (N^L - N^{M-1}K)t$ and if we condition with respect to $\mathcal{F}_{(N^L - N^{M-1}K)t}$ then $Y^{N,M,t}(s)$ behaves like N^{L-M+1} independent $(M-1)$ -levels bilinear processes with perturbation.

We will show that

$$(9.12.1) \quad \mathbb{E} \left[\frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left| \phi(\bar{Z}_{i,M-1}^N(N^L t)) - \phi(\bar{Y}_{i,M-1}^{N,M,t}(N^L t)) \right| \right] \leq C_T \left(\sqrt{\frac{K^3}{N} + \frac{K^2}{N}} \right) + \frac{C_T K}{N}$$

and $\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N}$ such that $\forall K \geq K(\varepsilon), \exists N(\varepsilon, K)$ such that $\forall N \geq N(\varepsilon, K)$,

$$(9.12.2) \quad \mathbb{E} \left[\frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left| \phi(\bar{Y}_{i,M-1}^{N,M,t}(N^L t)) - \phi(\bar{Z}_{i,M-1}^N(N^L t)) \right| \right] \leq \sqrt{\frac{4 \|\phi\|_\infty^2}{N^{L-M}} + \varepsilon}.$$

(PROOF OF (9.12.1)): Using coupling and Tanaka's formula, we get

$$\begin{aligned} \left| Z_i^N(N^L t) - Y_i^{N,M,t}(N^L t) \right| &\leq \sum_{k=1}^{M-1} \frac{C_k}{N^{k-1}} \int_{(N^L - N^{M-1}K)t}^{N^L t} \left| \bar{Z}_{i,k}^N(r) - \bar{Y}_{i,k}^{N,M,t}(r) \right| dr \\ &\quad + \sum_{k=M}^L \frac{C_k}{N^{k-1}} \int_{(N^L - N^{M-1}K)t}^{N^L t} \left| \bar{Z}_{i,k}^N(r) - \bar{Z}_{i,M}^N((N^L - N^{M-1}K)t) \right| dr \\ &\quad - \sum_{k=1}^{L+1} \frac{C_k}{N^{k-1}} \int_{(N^L - N^{M-1}K)t}^{N^L t} \left| Z_i^N(r) - Y_i^{N,M,t}(r) \right| dr + \gamma \int_{(N^L - N^{M-1}K)t}^{N^L t} \left| Z_i^N(r) - Y_i^{N,M,t}(r) \right| dW_i(r) \end{aligned}$$

Hence,

$$\mathbb{E} \left[\left| \bar{Z}_{i,M-1}^N(N^L t) - \bar{Y}_{i,M-1}^{N,M,t}(N^L t) \right| \right] \leq \mathbb{E} \left[\frac{1}{N^{M-1}} \sum_{i_1, \dots, i_{M-1}=0}^{N-1} \left| Z_i^N(N^L t) - Y_i^{N,M,t}(N^L t) \right| \right]$$

¹ Note that $Y^{N,1,t} = Z^{N,1,t}$.

$$\leq \sum_{k=M}^L \frac{C_k}{N^{k-1}} \int_{(N^L - N^{M-1}K)t}^{N^L t} \mathbb{E} \left[\left| \bar{Z}_{i,k}^N(r) - \bar{Z}_{i,M}^N((N^L - N^{M-1}K)t) \right| \right] dr$$

which implies that

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left| \bar{Z}_{i,M-1}^N(N^L t) - \bar{Y}_{i,M-1}^{N,M,t}(N^L t) \right| \right] \\ & \leq \sum_{k=M}^L \frac{C_k}{N^{k-1}} \int_{(N^L - N^{M-1}K)t}^{N^L t} \mathbb{E} \left[\frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left| \bar{Z}_{i,k}^N(r) - \bar{Z}_{i,M}^N((N^L - N^{M-1}K)t) \right| \right] dr \\ & \leq \frac{C_M}{N^{M-1}} \int_{(N^L - N^{M-1}K)t}^{N^L t} \mathbb{E} \left[\frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left| \bar{Z}_{i,M}^N(r) - \bar{Z}_{i,M}^N((N^L - N^{M-1}K)t) \right| \right] dr \\ & \quad + \frac{C}{N^M} \int_{(N^L - N^{M-1}K)t}^{N^L t} \mathbb{E} \left[\bar{Z}_{i,L}^N(r) + \bar{Z}_{i,L}^N((N^L - N^{M-1}K)t) \right] dr \\ & \leq \frac{C_M}{N^{M-1}} \int_{(N^L - N^{M-1}K)t}^{N^L t} C_T \left(\sqrt{\frac{r - (N^L - N^{M-1}K)t}{N^M}} + \frac{r - (N^L - N^{M-1}K)t}{N^M} \right) dr + \frac{C_T K}{N} \\ & \leq C_T \left(\sqrt{\frac{K^3}{N}} + \frac{K^2}{N} \right) + \frac{C_T K}{N} \end{aligned}$$

The proof of inequality (9.12.1) is completed by using the law of the mean.

(PROOF OF (9.12.2)): As we did for (7.8.3),

$$\begin{aligned} (9.12.3) \quad & \mathbb{E} \left[\left| \frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(\bar{Y}_{i,M-1}^{N,M,t}(N^L t) - \bar{Z}_{i,M-1}^{N,M,t}(N^L t) \right) \right| \right] \\ & \leq \sqrt{\mathbb{E} \left[\left(\frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(\bar{Y}_{i,M-1}^{N,M,t}(N^L t) - \bar{Z}_{i,M-1}^{N,M,t}(N^L t) \right) \right)^2 \right]} \\ & = \sqrt{\mathbb{E} \left[\mathbb{E} \left[\left(\frac{1}{N^{L-M+1}} \sum_{i_M, \dots, i_L=0}^{N-1} \left(\bar{Y}_{i,M-1}^{N,M,t}(N^L t) - \bar{Z}_{i,M-1}^{N,M,t}(N^L t) \right) \right)^2 \middle| \mathcal{F}_{(N^L - N^{M-1}K)t} \right] \right]} \\ & \leq \sqrt{\frac{4 \|\varphi\|_\infty^2}{N^{L-M+1}} + \mathbb{E} \left[\left(\mathbb{E} \left[\left(\bar{Y}_{i,M-1}^{N,M,t}(N^L t) - \bar{Z}_{i,M-1}^{N,M,t}(N^L t) \right) \middle| \mathcal{F}_{(N^L - N^{M-1}K)t} \right] \right)^2 \right]} \end{aligned}$$

¹ From lemma 9.9.

² The constant C_T may vary from line to line.

and, $\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N}$ such that $\forall K \geq K(\varepsilon), \exists N(\varepsilon, K)$ such that $\forall N \geq N(\varepsilon, K)$, the expression above is bounded by

$$\sqrt{\frac{\|\phi\|_{\infty}^2}{N^{L-M}} + \varepsilon}.$$

Indeed, when we condition with respect to $\mathcal{F}_{(N^L - N^{M-1}K)t}$,

$$\{Y^{N,M,t}(s) : s \geq (N^L - N^{M-1}K)t\}$$

behaves like a $(M-1)$ -levels bilinear processes with perturbation¹ and

$$\{\bar{Z}_{M-1}^{N,M,t}(N^{M-1}s) : s \geq (N^{L-M+1} - K)t\}$$

satisfies

$$\begin{aligned} \bar{Z}_{i,M-1}^{N,M,t}(N^{M-1}s) &\equiv \frac{1}{N^{M-1}} \sum_{i_1, \dots, i_{M-1}=0}^{N-1} z_i \\ &+ \left(N^{M-1} \hat{c}_{M-1} + \frac{c_{L+1}}{N^{L-M+1}} \right) \int_{(N^{L-M+1}-K)t}^s \left(\frac{\hat{c}_{M-1}\eta + \frac{c_{L+1}}{N^L}\theta}{\hat{c}_{M-1} + \frac{c_{L+1}}{N^L}} - Z_{i,M-1}^{N,M,t}(N^{M-1}r) \right) dr \\ &+ \sqrt{N^{M-1}} \hat{\gamma}_{M-1} \int_{(N^{L-M+1}-K)t}^s \bar{Z}_{i,M-1}^{N,M,t}(N^{M-1}r) d \left(\frac{1}{N^{M-1}} \sum_{i_1, \dots, i_{M-1}=0}^{N-1} W_i(N^{M-1}r) \right). \end{aligned}$$

Therefore, because $M-1 < L$, we use the induction hypothesis which allows us to apply corollary 9.14. ■

LEMMA 9.13: Let U^N and V^N be two strong solutions of

$$(9.13.1) \quad dU_i^N(t) = \sum_{k=1}^L \frac{c_k}{N^{k-1}} (\bar{U}_{i,k}^N(t) - U_i^N(t)) dt + \frac{c_{L+1}}{N^L} (\theta - U_i^N(t)) dt + \gamma U_i^N(t) dW_i(t),$$

$$i = (i_1, i_2, \dots, i_L), i_j \in \{0, 1, \dots, N-1\} \quad \forall j \in \{1, 2, \dots, L\}.$$

If the initial conditions of both processes are nonnegative and square-integrable, and if

$$\sup_N \mathbb{E} [\bar{U}_{N^L}(0)] < \infty \text{ and } \sup_N \mathbb{E} [\bar{V}_{N^L}(0)] < \infty$$

¹ That is

$$\begin{aligned} Y_i^{N,M,t}(s) &\equiv z_i + \sum_{k=1}^{M-1} \frac{c_k}{N^{k-1}} \int_{(N^L - N^{M-1}K)t}^s (\bar{Y}_{i,k}^{N,M,t}(r) - Y_i^{N,M,t}(r)) dr \\ &+ \left(\hat{c}_{M-1} + \frac{c_{L+1}}{N^L} \right) \int_{(N^L - N^{M-1}K)t}^s \left(\frac{\hat{c}_{M-1}\eta + \frac{c_{L+1}}{N^L}\theta}{\hat{c}_{M-1} + \frac{c_{L+1}}{N^L}} - Y_i^{N,M,t}(r) \right) dr + \gamma \int_{(N^L - N^{M-1}K)t}^s Y_i^{N,M,t}(r) dW_i(r). \end{aligned}$$

then for any bounded continuous function ϕ with bounded first derivative,

$$(9.13.2) \quad \sup_N \left(\mathbb{E} \left[\phi \left(\bar{U}_{N^L}(N^L K t) \right) \right] - \mathbb{E} \left[\phi \left(\bar{V}_{N^L}(N^L K t) \right) \right] \right) \xrightarrow{K \rightarrow \infty} 0.$$

This means that the average process of the strong solution of (9.13.1) tends to forget its origin as time goes by.

PROOF OF LEMMA 9.13: Because of the law of the mean, it suffices to prove that $\mathbb{E} \left[\left| \bar{U}_{N^L}(N^L K t) - \bar{V}_{N^L}(N^L K t) \right| \right]$ tends to zero as K tends to infinity uniformly in N .

From coupling, we have

$$\begin{aligned} & dU_i^N(t) - V_i^N(t) \\ &= \sum_{k=1}^L \frac{c_k}{N^{k-1}} (\bar{U}_{i,k}^N(t) - \bar{V}_{i,k}^N(t)) dt - \sum_{k=1}^{L+1} \frac{c_k}{N^{k-1}} (U_i^N(t) - V_i^N(t)) dt + \gamma (U_i^N(t) - V_i^N(t)) dW_i(t). \end{aligned}$$

Therefore, for any t , the local time $L_t^0(U_i^N - V_i^N) = 0$ and Tanaka's formula gives, for any $0 \leq s \leq t$,

$$\begin{aligned} & \mathbb{E} \left[\left| U_i^N(t) - V_i^N(t) \right| \right] - \mathbb{E} \left[\left| U_i^N(s) - V_i^N(s) \right| \right] \\ & \leq \sum_{k=1}^L \frac{c_k}{N^{k-1}} \int_s^t \mathbb{E} \left[\left| \bar{U}_{i,k}^N(r) - \bar{V}_{i,k}^N(r) \right| \right] dr - \sum_{k=1}^{L+1} \frac{c_k}{N^{k-1}} \int_s^t \mathbb{E} \left[\left| U_i^N(r) - V_i^N(r) \right| \right] dr \end{aligned}$$

which implies that

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left| U_{i_1}^N(t) - V_{i_1}^N(t) \right| \right] - \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left| U_{i_1}^N(s) - V_{i_1}^N(s) \right| \right] \\ & \leq -\frac{c_{L+1}}{N^L} \int_s^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left| U_{i_1}^N(r) - V_{i_1}^N(r) \right| \right] dr. \end{aligned}$$

Therefore,

$$(9.13.3) \quad \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left| U_{i_1}^N(t) - V_{i_1}^N(t) \right| \right] \leq \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left| U_{i_1}^N(s) - V_{i_1}^N(s) \right| \right] \text{ for any } 0 \leq s \leq t$$

and, if we let $s = 0$,

$$\begin{aligned} (9.14.4) \quad & \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left| U_{i_1}^N(t) - V_{i_1}^N(t) \right| \right] \\ & \leq \mathbb{E} \left[\bar{U}_{N^L}(0) \right] + \mathbb{E} \left[\bar{V}_{N^L}(0) \right] - \frac{c_{L+1}}{N^L} \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} \left| U_{i_1}^N(r) - V_{i_1}^N(r) \right| \right] dr. \end{aligned}$$

When we look at time $N^L K t$, the equation above becomes

$$\mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} |U_i^N(N^L K t) - V_i^N(N^L K t)| \right] \\ + c_{L+1} K \int_0^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} |U_i^N(N^L K r) - V_i^N(N^L K r)| \right] dr \leq \mathbb{E} [\bar{U}_{N^L}(0)] + \mathbb{E} [\bar{V}_{N^L}(0)].$$

Therefore

$$c_{L+1} K \int_s^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} |U_i^N(N^L K r) - V_i^N(N^L K r)| \right] dr \leq \mathbb{E} [\bar{U}_{N^L}(0)] + \mathbb{E} [\bar{V}_{N^L}(0)]$$

From the law of the mean, we know that there is a $u \in (s, t)$ for which

$$\int_s^t \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} |U_i^N(N^L K r) - V_i^N(N^L K r)| \right] dr \\ = \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} |U_i^N(N^L K u) - V_i^N(N^L K u)| \right] (t-s).$$

Replacing in the previous inequality,

$$\mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} |U_i^N(N^L K u) - V_i^N(N^L K u)| \right] \leq \frac{1}{c_{L+1} K (t-s)} (\mathbb{E} [\bar{U}_{N^L}(0)] + \mathbb{E} [\bar{V}_{N^L}(0)])$$

which implies that

$$\sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} |U_i^N(N^L K u) - V_i^N(N^L K u)| \right] \\ \leq \frac{1}{c_{L+1} K (t-s)} \left(\sup_N \mathbb{E} [\bar{U}_{N^L}(0)] + \sup_N \mathbb{E} [\bar{V}_{N^L}(0)] \right) \xrightarrow{K \rightarrow \infty} 0.$$

Because of the inequality (9.13.3), we conclude that

$$\sup_N \mathbb{E} [|\bar{U}_{N^L}(N^L K t) - \bar{V}_{N^L}(N^L K t)|] \leq \sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} |U_i^N(N^L K t) - V_i^N(N^L K t)| \right] \xrightarrow{K \rightarrow \infty} 0.$$

■

COROLLARY 9.14: Let U^N denotes the N -dimensional process defined as the strong solution of

$$(9.14.1) \quad dU_i^N(t) = \sum_{k=1}^L \frac{c_k}{N^{k-1}} (\bar{U}_{i,k}^N(t) - U_i^N(t)) dt + \frac{c_{L+1}}{N^L} (\theta - U_i^N(t)) dt + \gamma U_i^N(t) dW_i(t),$$

$$i = (i_1, i_2, \dots, i_L), i_j \in \{0, 1, \dots, N-1\} \forall j \in \{1, 2, \dots, L\}.$$

where θ is a nonnegative constant and V^N is the one-dimensional strong solution of

$$(9.14.2) \quad dV(t) = c_{L+1}(\theta - V(t))dt + \sqrt{\frac{\gamma^2}{\alpha_L} \prod_{k=1}^L 2c_k} V(t) dW(t)$$

with initial condition $V^N(0) \equiv \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} U_{i_1}^N(0)$. If $\alpha_L > 0$ and $\sup_N \mathbb{E} \left[\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} U_{i_1}^N(0) \right] < \infty$ then

for any bounded continuous function φ with bounded first derivative,

$\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N}$ such that $\forall K \geq K(\varepsilon), \exists N(\varepsilon, K) \in \mathbb{N}$ such that $\forall N \geq N(\varepsilon, K)$,

$$\mathbb{E} \left[\varphi \left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} U_{i_1}^N(N^L K t) \right) - \varphi(V^N(K t)) \right] < \varepsilon$$

PROOF OF COROLLARY 9.14: If the initial conditions are constant (that is if $U_i^N(0) = u, \forall i$) then, from theorem 9.1, we know that

$$\mathcal{L} \left(\left\{ \frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} U_{i_1}^N(N^L t) : t > 0 \right\} \right)$$

converges weakly (as N tends to infinity) to $\mathcal{L}(\{V(t) : t > 0\})$ where V is the unique strong solution of (9.14.2) with the initial condition $V(0) = u$. Therefore, for any $K > 0$ and any bounded continuous function φ ,

$$\mathbb{E} \left[\varphi \left(\frac{1}{N^L} \sum_{i_1, \dots, i_L=0}^{N-1} U_{i_1}^N(N^L K t) \right) \right] \xrightarrow{N \rightarrow \infty} \mathbb{E}[\varphi(V(K t))].$$

The previous lemma (9.14) allows us to reproduce the proof of lemma 5.18 to conclude this demonstration. That is, we use the fact that both processes involved here tend to forget their origins to put back the desired initial conditions. ■

Behavior of $\{\bar{X}_{N^L}(N^p t) : t \in \mathbb{R}_+\}$ as N goes to infinity

PROPOSITION 10.1: Assume that the initial conditions are nonnegative square-integrable random variables. If $p > L$ then for any $t > 0$,

$$\bar{X}_{N^L}(N^p t) \xrightarrow{N \rightarrow \infty} 0 \text{ in probability.}$$

See proposition 4.1 for the idea of the proof.

Conclusion and Direction for Further Research

Until now, we have concentrated our effort on the subcritical case. A simple necessary and sufficient criterion¹ based on the parameters of the system determines whether the system is in this state or not. In this case, complete limiting results have been obtained even for the most general model considered in this thesis. Indeed, we have proven a weak limit theorem for the average process in a particular time renormalization. We have also shown that the whole process goes to extinction in probability if the rate at which the time is rescaled is larger than a critical value. In fact, this last result is generalized to the critical and the supercritical states and we suspect that the theorem remains true for smaller rates of renormalization (the new critical values being dependent upon which critical or supercritical states the system is in and the number of levels).

The partial results we have obtained for the critical case open up an interesting field for investigation. This suggests the speed at which we should rescale the time so that a non trivial limit could be reached. The first natural question that comes to mind is what this limit could be? Indeed, the limit we found for the second moments indicates that something different than before happens here and that new techniques will be needed to identify the limiting dynamics.

Another way to generalize the models is to allow the sizes of each level to be different. That is, in the systems we have studied, we assumed that each "village" contains exactly N individuals, each "province" has N "villages", etc. If we let the sizes of each level be different then several aspects of the resulting stochastic process can be studied and some questions arise:

Are we restricted in the choice of the groups' sizes? That is, do we have to impose at least some order of magnitude to these sizes in order to get an interesting limiting behaviour?

Do we still observe phase transition? If yes, what are the conditions on the parameters of the model that determines the subcritical, critical and supercritical states?

What would be the appropriate time rescaling? When does the process get to extinction? Is it possible that the system explodes? etc.

Well, a lot can be done and this thesis is just the beginning.

$$\sum_{k=1}^L \frac{1}{2c_k} < \frac{1}{\gamma^2}.$$

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Appendix

The following theorem is a slight modification of a result due to William J. Anderson (ref.: [And], page 313, theorem 4.2.).

THEOREM: Let W be a N -dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$. We shall assume that X is a $\mathbb{R}^N$ -valued continuous $\{\mathcal{F}_t : t \geq 0\}$ -adapted function with

$$\mathbb{P}\left[\int_0^t X_i^2(s) ds < \infty, \forall t \geq 0\right] = 1$$

for any $i = 1, \dots, N$ and such that the stochastic equation

$$X_i(t) \stackrel{p.a.s.}{=} X_i(0) + \int_0^t b_i(X(s)) ds + \sum_{k=1}^N \int_0^t \sigma_{ki}(X(s)) dW_k(s), \quad i = 1, \dots, N, \quad t \geq 0$$

is satisfied. Here, the functions $b: \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $\sigma: \mathbb{R}^N \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N$ are continuous¹. Let us define the function $a: \mathbb{R}^N \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N$ by $a(\bullet) \equiv \sigma(\bullet) \sigma^T(\bullet)$ where $\sigma^T(\bullet)$ is the transpose of the matrix $\sigma(\bullet)$.

Assume that $X(0) \equiv x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and fix i between 1 and n . Suppose that the function $a_{ii}(\bullet)$ satisfies the following conditions:

$$(1) \quad a_{ii}(x) = 0,$$

and (2) there is $K \geq 0$, $\delta > 0$, and $\alpha > \frac{1}{2}$ such that $a_{ii}(y) \leq K |y_i - x_i|^{2\alpha}$ whenever $|y_i - x_i| < \delta$.

Then
$$\lim_{t \downarrow 0} \frac{X_i(t) - x_i}{t} \stackrel{p.a.s.}{=} b_i(x).$$

We restate the result in the following way:

If (1*) $X(0)$ is a $\mathcal{F}_0$ -measurable random variable such that $X_i(0) \stackrel{p.a.s.}{=} 0$,

$$(2^*) \quad a_{ii}(X(0)) \stackrel{p.a.s.}{=} 0,$$

(3*) there is $K \geq 0$, $\delta > 0$, and $\alpha > \frac{1}{2}$ such that $a_{ii}(y) \leq K |y_i|^{2\alpha}$ whenever $|y_i| < \delta$,

and (4*) $|b_i(X(0))| \stackrel{p.a.s.}{<} \infty$.

Then
$$\lim_{t \downarrow 0} \frac{X_i(t)}{t} \stackrel{p.a.s.}{=} b_i(X(0)).$$

Since the proof is almost the same as the one in the original theorem, we refer to the author's publication. ■

¹ This means that each component $b_k(\bullet)$ of $b(\bullet)$ and $\sigma_{ik}(\bullet)$ of $\sigma(\bullet)$, $i, k = 1, \dots, N$ is continuous.

LEMMA A.1: If X is a continuous non negative real-valued stochastic process and $\{M_i(t) : t \geq 0\} : i = 1, 2, \dots, k\}$ are a local-martingales such that

$$\sum_{i=1}^k M_i(t) + \int_0^t X(s) ds \geq 0$$

then

$$\mathbb{E} \left[\sum_{i=1}^k M_i(t) + \int_0^t X(s) ds \right] \leq \mathbb{E} \left[\sum_{i=1}^k M_i(0) \right] + \mathbb{E} \left[\int_0^t X(s) ds \right].$$

Moreover, if X is integrable¹ then

$$\mathbb{E} \left[\sum_{i=1}^k M_i(t) \right] \leq \mathbb{E} \left[\sum_{i=1}^k M_i(0) \right].$$

PROOF OF LEMMA A.1: From the definition of local-martingale, we have for any i the existence of a sequence $\{\tau_n^i : n \geq 1\}$ of stopping times such that

- 1) $\tau_n^i \xrightarrow{\mathbb{P}-a.s.} \infty$
- 2) $\tau_n^i \uparrow \infty$ $\mathbb{P}$ -almost- surely as $n \uparrow \infty$
- 3) $\{M_i(t \wedge \tau_n^i) : t \geq 0\}$ is a martingale for any $n \geq 1$.

Define $T_n \equiv \min\{\tau_n^i : i \in \{0, \dots, N-1\}\}$ and note that it is a stopping time which implies that

$$\{M_i(t \wedge \tau_n^i \wedge T_n) : t \geq 0\}$$

is a martingale². Therefore, using successively Fatou's lemma and the monotone convergence theorem,

$$\begin{aligned} \text{(A.1.1)} \quad \mathbb{E} \left[\sum_{i=1}^k M_i(t) + \int_0^t X(s) ds \right] &\leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\sum_{i=1}^k M_i(t \wedge T_n) + \int_0^{t \wedge T_n} X(s) ds \right] \\ &= \mathbb{E} \left[\sum_{i=1}^k M_i(0) \right] + \liminf_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{t \wedge T_n} X(s) ds \right] = \mathbb{E} \left[\sum_{i=1}^k M_i(0) \right] + \mathbb{E} \left[\int_0^t X(s) ds \right]. \end{aligned}$$

We use the integrability of X to conclude. ■

¹ That is $X(t)$ is integrable for any t .

² If M is a càdlàg martingale and T a stopping time then the stopped process M^T is a martingale. (ref.: [RY], page 67, corollary 3.6.)

LEMMA A.2: If $\int_0^t \mathbb{E} [X^2(s)] ds < \infty$ and $t \rightarrow \int_0^t X(s) dW(s)$ is continuous then

$$\mathbb{E} \left[\int_0^t X(s) dW(s) \right] = 0.$$

PROOF OF LEMMA A.2: Since

$$\left\langle \int_0^\cdot X(s) dW(s) \right\rangle(t) = \int_0^t X^2(s) ds,$$

then corollary 1.25, page 124 of [RY]¹ gives that

$$\left\{ \int_0^s X(r) dW(r) : 0 \leq s \leq t \right\}$$

is a martingale. Therefore,

$$\mathbb{E} \left[\int_0^t X(s) dW(s) \right] = 0. \quad \blacksquare$$

GRONWALL'S INEQUALITY:

Suppose f and g Lebesgue integrable in $[0, T]$ and there is a constant $c > 0$ such that

$$(A.3.1) \quad f(t) \leq g(t) + c \int_0^t f(s) ds, \quad \forall t \in [0, T]$$

then

$$f(t) \leq g(t) + c \int_0^t \exp[c(t-s)] g(s) ds, \quad \forall t \in [0, T].$$

Moreover, if g is differentiable in $[0, T]$, then

$$f(t) \leq g(0) \exp[ct] + \int_0^t \exp[c(t-s)] g'(s) ds, \quad \forall t \in [0, T].$$

When g is constant, we have as a particular case of the inequality above,

$$f(t) \leq g(0) \exp[ct], \quad \forall t \in [0, T].$$

¹ if M is a continuous local-martingale, then the following conditions are equivalent:

- (i) $M_0 \in L^2$ and $\mathbb{E} [\langle M \rangle(t)] < \infty$;
- (ii) $\{M_s : 0 \leq s \leq t\}$ is an L^2 -bounded martingale.

PROOF OF GRONWALL'S INEQUALITY:

Define $F(t) \equiv \int_0^t f(s) ds$. Equation (A.3.1) becomes

$$(A.3.2) \quad F'(t) \leq g(t) + cF(t), \quad \forall t \in [0, T].$$

Using integration by parts and equation (A.3.2),

$$e^{-ct}F(t) + c \int_0^t e^{-cs}F(s) ds = \int_0^t e^{-cs}F'(s) ds \leq \int_0^t e^{-cs}g(s) ds + \int_0^t e^{-cs}cF(s) ds, \quad \forall t \in [0, T]$$

which implies that

$$(A.3.3) \quad e^{-ct}F(t) \leq \int_0^t e^{-cs}g(s) ds, \quad \forall t \in [0, T].$$

Therefore,

$$f(t) = F'(t) \leq g(t) + cF(t) \leq g(t) + ce^{ct} \int_0^t e^{-cs}g(s) ds, \quad \forall t \in [0, T]$$

which prove the first inequality. If g is differentiable then, applying integration by parts in the last inequality, we obtain

$$f(t) \leq g(t) + e^{ct} \left(\int_0^t e^{-cs}g'(s) ds - e^{-ct}g(t) + g(0) \right) = e^{ct} \int_0^t e^{-cs}g'(s) ds + g(0)e^{ct}$$

which gives the two last inequalities. ■

GENERALIZATION OF THE CONTINUOUS MAPPING THEOREM.

In the specific context where we use this theorem, we can state it as follow:

Let (μ_n) be a sequence of probabilities on $C(\mathbb{R}_+, \mathbb{R})$ which converge weakly to a probability μ and let f be a real measurable function on $C(\mathbb{R}_+, \mathbb{R})$. We assume that the set of discontinuities of f is a null set for μ and we also suppose that there is a $\delta > 0$ such that $\sup_n \int |f|^{1+\delta} d\mu_n < \infty$. In such conditions, the sequence $\int f d\mu_n$ converges to $\int f d\mu$.

L'énoncé plus général est le suivant:

Soient E un espace lusinien, (μ_n) une suite de mesures de probabilité sur E convergeant étroitement vers une probabilité μ et f une fonction réelle mesurable sur E . On suppose que l'ensemble des points de discontinuité de f est de mesure nulle pour μ et qu'il existe un $\delta > 0$ tel que $\sup_n \int |f|^{1+\delta} d\mu_n < \infty < \infty$. Dans ces conditions, la suite $\int f d\mu_n$ converge vers $\int f d\mu$.

PREUVE DU THÉORÈME DES APPLICATIONS CONTINUES GÉNÉRALISÉ:

Sans perte de généralité, on peut supposer que f est non négative. Pour $M > 0$, la fonction $f_M(x) \equiv f(x) \wedge M$ est mesurable, bornée et continue en tout point où f l'est. On peut donc lui appliquer le théorème des applications continues, c'est-à-dire que

$$\lim_{n \rightarrow \infty} \int f_M d\mu_n = \int f_M d\mu.$$

On en déduit les inégalités suivantes:

$$\begin{aligned} \int f_M d\mu &\leq \sup_n \int f_M d\mu_n = \sup_n \left[\int f_M \mathbf{1}_{\{f_M \leq 1\}} d\mu_n + \int f_M \mathbf{1}_{\{f_M > 1\}} d\mu_n \right] \\ &\leq \sup_n \left[1 + \int f_M^{1+\delta} \mathbf{1}_{\{f_M > 1\}} d\mu_n \right] \leq \sup_n \left[1 + \int f^{1+\delta} d\mu_n \right] = 1 + \sup_n \int f^{1+\delta} d\mu_n < \infty. \end{aligned}$$

En prenant la limite lorsque M croît vers l'infini, on trouve

$$\int f d\mu_n = \lim_{M \rightarrow \infty} \int f_M d\mu_n \leq 1 + \sup_n \int f^{1+\delta} d\mu_n < \infty$$

ce qui montre que $\int f d\mu$ et $\int f d\mu_n$ existent. D'autre part, on a

$$\begin{aligned} \int f d\mu_n &\leq \int f_M d\mu_n + \int f \mathbf{1}_{\{f > 1\}} d\mu_n \leq \int f_M d\mu_n + \int \left(\frac{f}{M} \right)^\delta \mathbf{1}_{\{f > 1\}} d\mu_n \\ &\leq \int f_M d\mu_n + \frac{1}{M} \int f^{1+\delta} d\mu_n \leq \int f_M d\mu_n + \frac{\text{const.}}{M}. \end{aligned}$$

Il s'ensuit que

$$\limsup_{n \rightarrow \infty} \int f d\mu_n \leq \limsup_{n \rightarrow \infty} \int f_M d\mu_n + \frac{\text{const.}}{M} = \int f_M d\mu + \frac{\text{const.}}{M}$$

et laissant tendre M vers l'infini,

$$\limsup_{n \rightarrow \infty} \int f d\mu_n \leq \lim_{M \rightarrow \infty} \int f_M d\mu + \frac{\text{const.}}{M} = \int f d\mu.$$

Finalement,

$$\int f_M d\mu = \liminf_{n \rightarrow \infty} \int f_M d\mu_n \leq \liminf_{n \rightarrow \infty} \int f d\mu_n$$

si bien qu'en laissant M tendre encore une fois vers l'infini, nous obtenons

$$\int f d\mu = \lim_{M \rightarrow \infty} \int f_M d\mu \leq \liminf_{n \rightarrow \infty} \int f d\mu_n$$

Ainsi,

$$\int f d\mu \leq \liminf_{n \rightarrow \infty} \int f d\mu_n \leq \limsup_{n \rightarrow \infty} \int f d\mu_n \leq \int f d\mu.$$

La preuve est complète. ■

LEMMA A.4: If $p > 2$ then there is a constant C_p such that

$$(\alpha + \beta)^p \leq (p-1)\alpha^p + C_p \beta^p; \alpha, \beta > 0.$$

PROOF A.4:

$$(\alpha + \beta)^p \leq (p-1)\alpha^p + C_p \beta^p \Leftrightarrow (p-1) \left(\frac{\alpha}{\alpha + \beta} \right)^p + C_p \left(\frac{\beta}{\alpha + \beta} \right)^p \geq 1.$$

Define

$$f(x) \equiv (p-1)(x)^p + C_p(1-x)^p.$$

Can we find a constant C_p such that

$$(A.4.1) \quad \inf_{0 < x < 1} f(x) \geq 1 ?$$

Note that $f(0) \equiv C_p$ and $f(1) \equiv (p-1) \geq 1$ so we need that $C_p \geq 1$. Since

$$f'(x) = p(p-1)(x)^{p-1} - pC_p(1-x)^{p-1},$$

$$f'(x) = 0 \Leftrightarrow x = \frac{C_p^{\frac{1}{p-1}}}{(p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}}.$$

$$f''(x) = p(p-1)^2(x)^{p-2} + p(p-1)C_p(1-x)^{p-2}$$

$$f''\left(\frac{C_p^{\frac{1}{p-1}}}{(p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}}\right) = p(p-1)^2 \left(\frac{C_p^{\frac{1}{p-1}}}{(p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}}\right)^{p-2} + p(p-1)C_p \left(\frac{(p-1)^{\frac{1}{p-1}}}{(p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}}\right)^{p-2} \geq 0$$

Therefore,

$$\begin{aligned} \inf_{0 < x < 1} f(x) &= f\left(\frac{C_p^{\frac{1}{p-1}}}{(p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}}\right) \\ &= (p-1) \left(\frac{C_p^{\frac{1}{p-1}}}{(p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}}\right)^p + C_p \left(\frac{(p-1)^{\frac{1}{p-1}}}{(p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}}\right)^p = (p-1)C_p \left((p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}\right)^{1-p} \\ (p-1)C_p \left((p-1)^{\frac{1}{p-1}} + C_p^{\frac{1}{p-1}}\right)^{1-p} &\geq 1 \Leftrightarrow C_p^{\frac{1}{p-1}} \left((p-1)^{\frac{1}{p-1}} - 1\right) \geq (p-1)^{\frac{1}{p-1}} \end{aligned}$$

and since $p > 2 \Leftrightarrow (p-1)^{\frac{1}{p-1}} - 1 > 0$ then (A.4.1) is true if

$$C_p^{\frac{1}{p-1}} \geq \frac{(p-1)^{\frac{1}{p-1}}}{(p-1)^{\frac{1}{p-1}} - 1}.$$

■

LEMMA A.8.8: Let $p_1(N), p_2(N), \dots, p_n(N)$ be the roots of

$$p(x) \equiv x^n + a_1(N)x^{n-1} + a_2(N)x^{n-2} + \dots + a_n(N).$$

If

$$\forall k \in \{1, 2, \dots, n\}, a_k(N) = \frac{\beta_k}{N^{k(k-1)/2}} + C \left(\frac{1}{N^{(k(k-1)/2)+1}} \right) \text{ where } \beta_k \neq 0$$

then we can rename the roots such that $\forall k \in \{1, 2, \dots, n\}$,

$$\rho_k(N) = -\frac{\beta_k}{\beta_{k-1}} \frac{1}{N^{k-1}} + O\left(\frac{1}{N^k}\right) \text{ where } \beta_0 \equiv 1.$$

PROOF OF LEMMA A.8.8:

We will use frequently the following result:

If $\rho_1, \rho_2, \dots, \rho_n$ are the roots of $p(x) \equiv x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$ then

$$(A.8.8.1) \quad (-1)^k a_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} \rho_{i_1} \rho_{i_2} \dots \rho_{i_k}.$$

Using (A.8.8.1), it is easy to show that if

$$\rho_k(N) = -\frac{\beta_k}{\beta_{k-1}} \frac{1}{N^{k-1}} + O\left(\frac{1}{N^k}\right), \quad \forall k \in \{1, 2, \dots, n\}$$

then

$$a_k(N) = \frac{\beta_k}{N^{k(k-1)/2}} + O\left(\frac{1}{N^{(k(k-1)/2)+1}}\right), \quad \forall k \in \{1, 2, \dots, n\}.$$

Now, we want to prove the other direction. Define

$$\mathcal{C}_k \equiv \left\{ f(N) : f(N) = O\left(\frac{1}{N^k}\right), f(N) \neq o\left(\frac{1}{N^k}\right) \right\}.$$

Note that if $\forall k \in \{1, 2, \dots, n\}$, $\rho_k(N) \in \mathcal{C}_{k-1}$ then using (A.8.8.1) and the fact that,

$$a_k(N) = \frac{\beta_k}{N^{k(k-1)/2}} + O\left(\frac{1}{N^{(k(k-1)/2)+1}}\right), \quad \forall k \in \{1, 2, \dots, n\}$$

one can easily prove that

$$\rho_k(N) = -\frac{\beta_k}{\beta_{k-1}} \frac{1}{N^{k-1}} + O\left(\frac{1}{N^k}\right), \quad \forall k \in \{1, 2, \dots, n\}.$$

Therefore, it will be sufficient to prove that if $\forall k \in \{1, 2, \dots, n\}$, $a_k(N) \in \mathcal{C}_{k(k-1)/2}$ then $\forall k \in \{1, 2, \dots, n\}$, $\rho_k(N) \in \mathcal{C}_{k-1}$. It will be easier to show the contrapositive, that is:

$$(A.8.8.2) \quad \begin{aligned} &\text{if } \exists k \in \{1, 2, \dots, n\} \text{ for which } \rho_k(N) \notin \mathcal{C}_{k-1} \\ &\text{then } \exists k \in \{1, 2, \dots, n\} \text{ such that } a_k(N) \notin \mathcal{C}_{k(k-1)/2}. \end{aligned}$$

It will be helpful to fix the following convention: $\forall k \in \{1, 2, \dots, n\}$, $\rho_k(N) \in \mathcal{C}_{m_k}$ so we can rename the roots so that $i \leq j \Rightarrow m_i \leq m_j$.

The proof of (A.8.8.2) is based on an induction on n . When $n = 1$,

$$p(x) = x + a_1(N) = x - \rho_1(N) \Rightarrow a_1(N) = -\rho_1(N).$$

Hence, $\rho_1(N) \notin \mathcal{C}_0 \Rightarrow a_1(N) \notin \mathcal{C}_0$.

Now, assume that (A.8.8.2) is true for any polynomial of degree $n-1$ that satisfies the premises of (A.8.8.2). Note that if $\forall k \in \{1, 2, \dots, n-1\}$, $\rho_k(N) \in \mathcal{C}_{k-1}$ then, from (8.8.1), $\rho_n(N) \in \mathcal{C}_{n-1}$. Therefore, we assume that $\exists k \in \{1, 2, \dots, n-1\}$ for which $\rho_k(N) \notin \mathcal{C}_{k-1}$

$$p(x) = \prod_{k=1}^n (x - \rho_k(N)) = (x^{n-1} + b_1(N)x^{n-2} + \dots + b_{n-1}(N))(x - \rho_n(N))$$

$$= x^n + \sum_{k=1}^{n-1} (b_k(N) - b_{k-1}(N) \rho_n(N)) x^k + b_{n-1}(N) \rho_n(N).$$

Therefore,

$$(A.8.8.3) \quad \forall k \in \{1, 2, \dots, n-1\}, a_k(N) = b_k(N) - b_{k-1}(N) \rho_n(N) \text{ and } a_n(N) = b_{n-1}(N) \rho_n(N).$$

Using the induction hypothesis, the existence of a $k \in \{1, 2, \dots, n-1\}$ for which $\rho_k(N) \notin \mathcal{C}_{k-1}$ implies that $\exists k \in \{1, 2, \dots, n-1\}$ such that $b_k(N) \notin \mathcal{C}_{k(k-1)/2}$. Define

$$k_0 \equiv \min \{k \in \{1, 2, \dots, n-1\} : b_k(N) \notin \mathcal{C}_{k(k-1)/2}\}.$$

We assume that $\forall k \in \{1, 2, \dots, n\}$, $a_k(N) \in \mathcal{C}_{k(k-1)/2}$ and will prove that this assumption leads to a contradiction. Using (A.8.8.1), one can show that $\forall k \in \{1, 2, \dots, n-1\}$,

$$(A.8.8.4) \quad \begin{aligned} a_k(N) \in \mathcal{C}_{k(k-1)/2} \text{ and } a_k(N) &= b_k(N) - b_{k-1}(N) \rho_n(N) \\ \Rightarrow \frac{k(k-1)}{2} &\geq \min \left\{ \sum_{i=1}^k m_i; \sum_{i=1}^{k-1} m_i + m_n \right\} = \sum_{i=1}^k m_i. \end{aligned}$$

and

$$(A.8.8.5) \quad \begin{aligned} a_{k_0}(N) \in \mathcal{C}_{k_0(k_0-1)/2}, b_{k_0}(N) \notin \mathcal{C}_{k_0(k_0-1)/2} \text{ and } a_{k_0}(N) &= b_{k_0}(N) - b_{k_0-1}(N) \rho_n(N) \\ \Rightarrow \sum_{i=1}^{k_0} m_i &= \sum_{i=1}^{k_0-1} m_i + m_n \Leftrightarrow m_{k_0} = m_n \Rightarrow \forall k \in \{k_0, k_0+1, \dots, n\}, m_k = m_n \end{aligned}$$

$$\Rightarrow \forall k \in \{k_0, k_0+1, \dots, n\},$$

$$\sum_{i=1}^k m_i = \sum_{i=1}^{k_0-1} m_i + \sum_{i=k_0}^k m_i = \sum_{i=1}^{k_0-1} i - 1 + \sum_{i=k_0}^k m_n = \frac{(k_0-2)(k_0-1)}{2} + (k-k_0+1)m_n.$$

Replacing the above in (A.8.8.4), we get

$$(A.8.8.6) \quad \forall k \in \{k_0, k_0+1, \dots, n\}, m_n \leq \frac{k(k-1) - (k_0-2)(k_0-1)}{2(k-k_0+1)} = (k_0-2) + \frac{(k-k_0+2)}{2},$$

Still using (A.8.8.1),

$$\begin{aligned} a_n(N) &= b_{n-1}(N) \rho_n(N) \text{ and } a_n(N) \in \mathcal{C}_{n(n-1)/2} \\ \Rightarrow \frac{n(n-1)}{2} &= \sum_{i=1}^n m_i = \frac{(k_0-2)(k_0-1)}{2} + (n-k_0+1)m_n \end{aligned}$$

1 If $k, n \in \mathbb{N}$, $k < n$ then

$$\begin{aligned} \frac{n(n-1)}{2} - \frac{k(k-1)}{2} &= k + (k+1) + \dots + (n-1) = [(k-1)+1] + [(k-1)+2] + \dots + [(k-1)+n-k] \\ &= (k-1)(n-k) + \frac{(n-k)(n-k+1)}{2} \Rightarrow \frac{n(n-1) - k(k-1)}{2(n-k)} = (k-1) + \frac{(n-k+1)}{2}. \end{aligned}$$

$$\Rightarrow m_n = \frac{n(n-1) - (k_0-2)(k_0-1)}{2(n-k_0+1)} = (k_0-2) + \frac{(n-k_0+2)}{2}.$$

Replacing back in (A.8.8.6), we get the contradiction since

$$\forall k \in \{k_0, k_0+1, \dots, n\}, (k_0-2) + \frac{(n-k_0+2)}{2} \leq (k_0-2) + \frac{(k-k_0+2)}{2}$$

$$\Leftrightarrow \forall k \in \{k_0, k_0+1, \dots, n\}, n=k \Leftrightarrow k_0=n.$$

■

END

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