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CARLETON UNIVERSITY
ON PRO-SUPERSOLVABLE GROUPS

AND
SUBGROUPS OF FREE PRO-C PRODUCTS OF PRO-C GROUPS

by
BHALCHANDRA C. OLTIKAR

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES & RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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The undersigned recommend to the Faculty of Graduate
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Groups and Subgroups of Free Pro-C Products of Pro-C Groups"
submitted by Bhalchandra Oltikar, B.Sc., M.Sc., in partial fulfilment
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ABSTRACT

Let C be a class of finite groups, closed under finite products, subgroups and homomorphic images. Free pro- C products are studied in Neukirch⁽¹³⁾, Gildenhuys, D. and Ribes, L.⁽⁸⁾. Gildenhuys and Ribes proved a structure theorem for open subgroups of such free products along the lines of the Kurosch subgroup theorem for discrete groups. This motivated to ask about the structure of closed subgroups of free pro- C products. In Chapter II, we show that some closed normal subgroups of free pro- C product $A \amalg B$, of pro- C groups A and B may be expressed as free pro- C products of pro- C groups where C is assumed to be closed under extensions. The result we derived from the analogous situation in discrete free products (Gruenberg⁽¹⁰⁾). This leads to prove that the cartesian subgroup of $A \amalg B$ is a free pro- C group on a pointed topological space, the class C as before is closed under extensions. Chapter II is concluded in pointing out that the derived group of free pro-meta-abelian products of two pro-abelian groups is free pro-abelian.

We study pro-supersolvable groups in Chapter III. The class of finite supersolvable groups is not closed under extensions. The concept of Frattini subgroup plays an important role in the theory of pro- p groups and pro-nilpotent groups.

(iii)

We use the concept to fetch an information about pro-supersolvable groups. For class C which is closed under extensions and is not contained in the class of finite p -groups (p -prime) Sylow- p subgroups of the free pro- C group on more than one generator are infinitely generated. On the other hand, we show that for any prime p , Sylow- p subgroups of a finitely generated pro-supersolvable group are again finitely generated.

It is not known whether for a finitely generated pro-finite group, a subgroup of finite index is necessarily open. Serre has proved that for f.g. pro- p groups such subgroups are necessarily open. We show that this holds for f.g. pro-supersolvable groups. We conclude by showing that the Frattini subgroup of free pro- C groups on more than one generator is the identity where C is assumed to be closed under extensions and containing p -groups for infinitely many primes, p .

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NOTATIONS AND TERMINOLOGY

We list here the notations and terminology frequently used.

$\{S_\alpha, \phi_{\alpha\beta} | \alpha \geq \beta; \alpha, \beta \in \Lambda\}$ denotes a projective mapping family.

We mean by this that the following is given.

A directed set Λ .

A family $\{S_\alpha, \alpha \in \Lambda\}$ of objects S_α (of the category)

indexed by Λ .

For every $\alpha, \beta \in \Lambda$, $\alpha \geq \beta$ there are maps $\phi_{\alpha\beta} : S_\alpha \rightarrow S_\beta$ consistent (or compatible) in the sense for $\alpha \geq \beta \geq \gamma$ the diagram

commutes. (We insist $\phi_{\alpha\alpha} = \text{id}$.)

A family $\{S_\alpha, \phi_\alpha | \alpha \in \Lambda\}$ is a consistent (or compatible) family with above means that the diagrams

commute for $\alpha, \beta \in \Lambda$, $\alpha \geq \beta$.

(vi)

proj. limit S_α is a consistent system $\{S, \phi_\alpha \mid \alpha \in \Lambda\}$ with the property that given any other consistent system $\{T, \psi_\alpha \mid \alpha \in \Lambda\}$ there exists a unique map $\phi : T \rightarrow S$ so that the diagrams

commute for every $\alpha \in \Lambda$.

A pro-finite group is a projective limit of finite groups.

A pro-C group is a projective limit of groups in C.

$\hat{\mathbb{Z}}$ is a free pro-finite group on one generator.

$\hat{\mathbb{Z}}_C$ is a free pro-C group on one generator.

$A * B$ is a (discrete) free product of groups A and B.

$A \amalg B$ is a free pro-finite product of pro-finite groups A, B.

Also, when there is no confusion, $A \amalg B$ denotes a free pro-C product of pro-C groups A, B; sometimes, we also denote this pro-C product by $A \amalg_C B$.

H_G is the core of a subgroup H in G is a maximal normal subgroup of G contained in H, i.e. $H_G = \bigcap H^x$ where x runs over coset representatives of H in G.

G^* is the Frattini subgroup of G.

G' or (G, G) is the derived subgroup of G.

CHAPTER I

INTRODUCTION

Let C be a class of finite groups closed under the formations of i) subgroups, ii) homomorphic image, and iii) extensions. Let A, B be pro- C groups, i.e. pro-finite groups whose quotients by open normal subgroups are in the class C . We are interested in expressing closed subgroups of $A \amalg B$ -free pro- C product of A and B as free pro- C products (in non-trivial ways) of certain pro- C groups. Gildenhuys, D. and Ribes, L. ⁽⁸⁾ proved that all open subgroups of free pro- C products are again expressible as free pro- C products of certain pro- C groups (Kurosch Subgroup Theorem). This certainly can not be true for all closed subgroups. Because if C is a class of finite groups which contains the class of finite p -groups (p -prime), but is not contained in that class, then Sylow- p subgroups of free pro- C groups are not free pro- C groups. (However, it is a free pro- p .) Now, a free pro- C group on a set X is a free pro- C product of $|X|$ copies of $\hat{\mathbb{Z}}_C$ -free pro- C groups on one generator while a Sylow- p subgroup of this group is not expressible as a free pro- C product of pro- C groups. (However, in this case, it may be expressed as a free pro- p product of pro- p groups.)

Suppose (X, Z_0) is a pointed Hausdorff topological space (Y, Z_0) is a compact, Hausdorff pointed topological space. (Z, Z_0) is the co-product of (X, Z_0) and (Y, Z_0) in the category of pointed topological spaces, i.e. $Z = X \cup Y$, $X \cap Y = Z_0$, and open sets containing Z_0 are of the form $U \cup V$ where $Z_0 \in U$, $Z_0 \in V$ and U open in X , V open in Y .

Let $F(Z)$ be a free pro-C group on topological space Z (see for definition of $F(Z)$, Proposition 1.3, p. 235⁽⁷⁾). In (7), Gildenhuys, D. and Lim, C. showed that the closed normal subgroup of $F(Z)$, generated by the image of Y , is isomorphic to the free pro-C group generated by the topological space $\tilde{F}(X, Z_0) \rtimes Y$.

Now $F(Z)$ is a free pro-C product of free pro-C groups $\tilde{F}(X, Z_0)$ and $F(Y)$ (Lemma 3.8) and the situation described above turns out to be a particular case of more general situation (Theorem 3.4):

$$\coprod_{b \in B} A^b \xrightarrow{\alpha} A \amalg B \xrightarrow[\gamma]{\beta} B$$

where A, B are pro-C groups; for every $b \in B$, A^b denotes an isomorphic copy of A and $\coprod_{b \in B} A^b$ denotes free pro-C product of A^b 's and $b \in B$.

Now, if $A = F(Y)$ and $B = \tilde{F}(X, Z_0)$, then $F(Z) = A \amalg B$, and $\coprod_{b \in B} F(Y)^b$ is isomorphic to $F(\tilde{F}(X, Z_0) \rtimes Y)$ (Lemma 3.6).

It is well-known that the Cartesian subgroup (see for definition⁽¹⁰⁾, p. 35 or (12)) of $A*B$ - the free product of A, B is a (discrete) free group. In § 3, Chapter II, we obtain an analogue result for pro- C groups (Theorem 3.2). In § 4, Chapter II, we look at the class C where the condition iii) - C closed under extensions, is substituted by a weaker condition iii)' - C closed under finite direct products. However, we do not get the similar results as in § 3, Chapter II. In fact, it need not be any longer true that the Cartesian subgroup of $A \amalg B$ is a free pro- C group (Theorem 4.4). We end the chapter by pointing out that the derived subgroup of the free pro-meta-abelian group on two generators is free pro-abelian group on pointed topological space $(\hat{\mathbb{Z}} \circ \hat{\mathbb{Z}}, *)$ where $*$, the distinguished point, is an equivalence class of $(1,1)$.

In Chapter III, we look at the Frattini subgroup - the intersection of all maximal (proper) closed subgroups of a pro-finite group. The Frattini subgroup plays an important role in the theory of pro- p groups and pro-nilpotent groups. In the case of pro- p groups and pro-nilpotent groups, the structure of the Frattini factor is more known and enables us to get more information about the structures of these groups.

Let G be a pro-finite group. G is (topologically) finitely generated if algebraically finitely generated group is

dense in G . Let G^* denote the Frattini subgroup of G . If G^* is open in G , then G is clearly finitely generated. (A subgroup S (algebraically) generated by coset representatives of G^* in G is easily seen to be dense in G .) In case of (topologically) finitely generated pro- p groups, the converse is true, i.e. its Frattini subgroup is open (Theorem 1.12).

In the case of pro-nilpotent groups, the structure of the Frattini factor yields information about the structure of the group since, in this case, G is a direct product of its Sylow- p subgroups, and the Frattini factor $G/G^* \cong \prod_p G_p/G_p^*$ where G_p denotes Sylow- p subgroup of G . (See Theorem 2, p. 161⁽¹¹⁾.)

For finite supersolvable groups, the class which falls between the classes of finite nilpotent groups and finite solvable groups, the Frattini subgroup is somewhat useful.

In reference (7), Gildenhuys, D. and Lim, C. show that for an extension closed class C , Sylow- p subgroups of a free pro- C group on more than one generator are not finitely generated provided C is not contained in the class of finite p -groups (see also Smith⁽²¹⁾). For a free nilpotent group on X , Sylow- p groups are free pro- p groups on X (Gruenberg⁽¹¹⁾). In § 5, Chapter III, we show that Sylow- p groups of a free pro-supersolvable group on a finite set X are finitely generated free pro- p groups (Theorem 5.3).

For a pro-finite group G , G is said to be of type F if for every natural number n , G has finitely many open subgroups of index n . For pro- p groups, the two conditions i) of being of type F, ii) of being finitely generated, are equivalent. In fact, the same holds for pro-supersolvable groups when the number of primes dividing the order of the group is finite (Theorem 5.4). However, the two conditions are not equivalent even in the case of pro-abelian groups when the number of primes dividing the order of the group is not finite.

A subgroup H of a topological group G having finite index is called large in G . For a compact topological group, the open subgroup is always large. In § 6, Chapter III, we consider the question "For profinite groups, is every large subgroup open?" For infinitely generated pro-finite groups, the large non-open subgroups exist. (See H.L. Peterson⁽¹⁴⁾.) It is not known whether every large subgroup of a finitely generated pro-finite group is open. J.P. Serre has proved that a large subgroup of a finitely generated pro- p group is open. We can extend this result to finitely generated pro-supersolvable groups. It would be interesting to know whether a finitely generated pro-solvable group has a large non-open subgroup.

We conclude Chapter III in § 7 by pointing out that the Frattini subgroup of a free pro- C group is the identity where C

is a class of finite groups closed under formations of i) subgroups, ii) homomorphic images and iii) extensions and for infinitely many primes p , $\mathcal{C}$ contains p -groups. As a consequence of this, it follows that the Frattini subgroup of a free profinite group or a free pro-solvable group on a set X is the identity where $|X| > 1$.

CHAPTER II

FREE PRO-C PRODUCTS

1. PRELIMINARIES

In this section, we fix terminology and recall some results most of them from Gildenhuys D. and Ribes L.⁽⁸⁾, for future reference. We also prove some of the results when we need them in slightly different forms. Some of the definitions and terminology used here are from the different papers of the same authors viz. (9). Thus, the proofs given are slightly different.

Let C be a class of finite groups closed under the formations of i) subgroups, ii) homomorphic images, and iii) finite direct products.

The concept of restricted ('corestringierete') free pro- C product was introduced in Neukirch⁽¹³⁾. We recall the definitions.

Definition 1.1: Let $\{A_x | x \in X\}$ be a family of pro- C groups indexed by a set X . Let A be a pro- C group, and $\{\phi_x : A_x \rightarrow A | x \in X\}$ be a family of continuous homomorphisms. We say that $\{\phi_x | x \in X\}$ converges to 1, if every open subgroup of A contains all but a finite number of the images $\phi_x(A_x)$, $x \in X$.

Definition 1.2: A system (A, ϕ_x) where $\phi_x: A_x \rightarrow A$, $x \in X$, is a family of continuous homomorphisms converging to 1, is a restricted free pro-C product of the A_x 's, if whenever G is a pro-C group and $\psi_x: A_x \rightarrow G$ is a family of continuous homomorphisms converging to 1, there is a unique continuous homomorphism $\bar{\psi}: A \rightarrow G$ with $\bar{\psi} \phi_x = \psi_x$ for all $x \in X$.

A restricted free pro-C group F on a set X is easily seen to be the restricted free pro-C product of $|X|$ copies of $\hat{\mathbb{Z}}_C$, the free pro-C group on one generator (cf. Neukirch⁽¹³⁾).

The notion of free pro-C product was extended by Gildenhuys D. and Ribes L. in (8). We recall their definitions and terminology.

Let $(X, *)$ be a pointed Hausdorff topological space with distinguished point $*$. We shall suppose that to each $x \in X$ there corresponds a pro-C group G_x , in such a way that:

$$(1) \quad G_* = 1.$$

(2) The mapping $X - \{*\} \rightarrow \{G_x \mid x \in X\}$ is "locally constant", i.e. for each $x \in X$ there exists open neighbourhood V of x such that for each $y \in V$ there is an isomorphism $\sigma_{x,y}: G_x \rightarrow G_y$ of pro-C groups.

(3) For every pair $x, y \in X$ at most one isomorphism σ_{xy} is given. The class of all isomorphisms is closed under the composition

$$\sigma_{xz} = \sigma_{yz} \cdot \sigma_{xy}; \sigma_{xx} = 1_x, \sigma_{yx} = \sigma_{xy}^{-1}$$

We will refer to the neighbourhood in (2) as constant open neighbourhood.

Definition 1.3: We define the étale space E , associated with the groups G_x , to be the disjoint union $\coprod_{x \in X} G_x$, viewed as a pointed topological space by taking the (unique) element 1 (of G_*) as the distinguished point of E and defining a topology on E as follows. A basic open neighbourhood of an element $g \in G_x$, $x \neq *$, is of the form $\{\sigma_{xy}(g') \mid g' \in W, y \in V\}$ where W is an open neighbourhood of g in G_x and V is a constant open neighbourhood of x in X ; and a basic open neighbourhood of $1 \in G_*$ in E is of the form $\bigcup_{x \in U} G_x$, where U is an open neighbourhood of $*$ in X .

Definition 1.4: The free pro-C product $\coprod_{x \in X} G_x$ of the G_x 's is a pro-C group P together with a morphism $\eta: E \rightarrow P$ of pointed topological spaces (1 is the distinguished point of P) such that each restriction $\eta|_{G_x}$ is a morphism of pro-C groups, and satisfying the following universal property: whenever $\eta': E \rightarrow P'$ is another such morphism from E into a pro-C group P' , there exists a unique morphism $\psi: P \rightarrow P'$ of pro-C groups, such that $\psi \cdot \eta = \eta'$.

If the distinguished point $*$ of X is isolated, then $\coprod_{x \in X} G_x$ may also be referred to as the free pro-C product of the G_x 's indexed by the topological space $X - \{*\}$.

Although the concept of 'étale space' given above is different from the one in (8), the definitions of free pro-C product are equivalent. (To see this, cf. Gildenhuys D. and Ribes, L. (9).)

Proposition 1.5: The free pro-C product (P, η) of G_x 's always exists and is unique in the obvious sense. The subgroup of P , generated algebraically by the set $\eta(E)$, is dense in P .

Proof: Let $G^d = \coprod_{x \in X}^d G_x$ ($\ast_{x \in X} G_x$) be the co-product (or free product) of the underlying discrete groups G_x , in the category of discrete groups.

Let $f: E = \bigcup G_x \rightarrow G^d$ be such that $f|_{G_x} \rightarrow G^d$ is the natural inclusion.

Consider the topological group structure on G^d given by the following fundamental system of open neighbourhoods of 1:

$$\Phi = \{N \triangleleft G^d \mid G^d/N \in C, f^{-1}(bN) \text{ is open in } E, \text{ for all } b \in G^d\}.$$

Let $P = \text{proj. limit}_{N \in \Phi} G^d/N$, and $\bar{f}: G^d \rightarrow P$ be the natural continuous homomorphism.

$$\text{Put } \eta = \bar{f} \circ f$$

(P, η) is the free pro-C product of G_x 's.

For, let $\eta': E \rightarrow P'$ be the morphism of pointed topological spaces where P' is a pro-C group, and $\eta'|_{G_x} \rightarrow P'$ is the morphism of pro-C groups.

Let $\psi_0 : G^d \rightarrow P'$ be a unique homomorphism such that $\psi_0 f = \eta'$,
i.e. the diagram commutes.

The map ψ_0 is continuous since, if U is open normal subgroup in P' and $N = \psi_0^{-1}(U)$, then the natural map $G^d/N \rightarrow P'/U$ is injective; so $P'/U \in C$ implies $G^d/N \in C$.

Next, for $b \in G^d$, $bN = b\psi_0^{-1}(U) = \psi_0^{-1}(\psi_0(b)U)$ and $f^{-1}(bN) = f^{-1}\psi_0^{-1}(\psi_0(b)U) = \eta'^{-1}(\psi_0(b)U)$ is open in E . Thus, $N = \psi_0^{-1}(U)$ is open in G^d and ψ_0 is continuous.

Now, for each open normal subgroup U of P' , consider the continuous homomorphism $P \rightarrow G^d/\psi_0^{-1}(U) \rightarrow P'/U$.

This compatible family of maps defines a continuous homomorphism:

$$\psi : P \rightarrow \text{proj. limit } P'/U = P'$$

Clearly, $\psi \circ \bar{f} = \psi_0$.

Hence,

$$\psi \circ \eta = \psi_0(\bar{f} \circ f) = (\psi \circ \bar{f}) \circ f = \psi_0 \circ f = \eta'$$

Finally, if $\psi' : P \rightarrow P'$ is a continuous homomorphism satisfying

$\psi' \circ \eta = \eta'$, put $\psi'_0 = \psi' \circ \bar{f}$. Then,

$$\psi'_0 \circ f = (\psi'_0 \circ f) \circ f = \psi'_0 \circ (f \circ f) = \psi'_0 \circ \eta = \eta'$$

The uniqueness of ψ'_0 implies $\psi'_0 = \psi_0$. Since $\overline{f(G^d)}$ is dense in P , $\psi' = \psi$. The uniqueness of P is clear.

Proposition 1.6: Let $(X, *)$ and $(Y, *)$ be pointed Hausdorff topological spaces, with Y compact (by abuse of notation, $*$ denotes the distinguished point of both X and Y). Let $X \times Y$ be the quotient $X \times Y / \sim$ of the product of X and Y , where $(x, y) \sim (x', y')$ if $x = *$ and $y' = *$, or $x' = *$ and $y = *$. The equivalence class $[(*, *)]$ of $(*, *)$ is taken as the distinguished point of $X \times Y$ and is also denoted by $*$. Suppose now that $A_x = \tilde{F}(Y, *)$ for all $x \in X - \{*\}$ and $A_* = 1$. Then,

$$\coprod_{x \in X} A_x = \tilde{F}(X \times Y, *).$$

For the proof, see Gildenhuys and Ribes⁽⁸⁾. For the definition of $\tilde{F}(X, *)$, see Gildenhuys and Lim⁽⁷⁾.

Now, in proposition 1.6, let Y be the singleton space.

Then, $A_x = \tilde{F}(Y, *) = \hat{Z}_C$ the free pro- C group on one generator, $x \in X - \{*\}$ and $A_* = 1$. $X \times Y$ is homeomorphic to X . Thus,

$$\tilde{F}(X, *) = \coprod_{x \in X} A_x$$

Proposition 1.7: Let $\{A_x, x \in X\}$ be a family of pro- C groups indexed by a set X . Let $\bar{X} = X \cup \{*\}$ be the one point compactification of X . Put $A_* = 1$. Then, the free pro- C product $\coprod_{x \in X} A_x$ coincides with the restricted free pro- C product of the groups A_x , $x \in X$ (Definition 1.2).

(cf. Gildenhuys and Ribes⁽⁸⁾ Proposition 1.5, p. 314).

The structure of free pro-C product over compact, Hausdorff, totally disconnected spaces:

Let X be a compact, Hausdorff, totally disconnected pointed space. Let $G = \prod_{x \in X} G_x$ be the free pro-C product of $\{G_x | x \in X\}$. Our purpose is to express G as a projective limit of free pro-C product over finite discrete spaces.

First, we introduce some notations. Let R be an equivalence relation on a set X .

$xR, \{y \in X | yR_x\}$ - an equivalence class of x , $x \in X$.

Let $\{G_x | x \in X\}$ be a family of pro-C groups indexed by X . We assume the conditions 1) to 3) on page 8.

Let R be an equivalence relation on X such that i) X/R is a finite discrete space with the quotient topology, and ii) there exists $\sigma_{xy} : G_x \rightarrow G_y$ whenever xRy and $x \notin *R$.

For each equivalence class xR , we choose a representative, say, c , and we put $G_{xR} = G_c$ (so that $G_c = G_{xR} = G_{yR}$ if xRy) $x \notin *R$, $G_{xR} = 1$ if $x \in *R$. Let $E = \bigcup_{x \in X} G_x$ be as in the definition 1.3, and define a function

$$\rho_R : E \longrightarrow \prod_{xR \in X/R} G_{xR} = G_R \text{ (say)}$$

by sending G_x onto $G_c = G_{xR}$ by means of σ_{xc} if $x \notin *R$ and onto

$1 = G_{xR}$ if $x \in *R$.

ρ_R is continuous: To see this, let $\alpha \in G_x$ and N be an open neighbourhood of $\rho_R(\alpha)$ in G_R . Let W be an open neighbourhood of α in G_x such that $W \subseteq \sigma_{xc}^{-1}(N \cap G_c)$. We also choose a constant open neighbourhood V of x in X such that $* \cap V$ and $V \subseteq f_R^{-1}(xR)$ (where $f_R : X \rightarrow X/R$ is a canonical map.)

Now consider $M = \{\sigma_{xy}(g') \mid g' \in W, y \in V\}$. M is an open neighbourhood of α in E such that $\rho_R(M) \subseteq N$. Thus, ρ_R is continuous at $\alpha \in G_x$ such that $x \neq *$.

For $\alpha = 1 \in G_*$, let U be an open neighbourhood of $*$ in X such that $U \subseteq f_R^{-1}(xR)$. Then, $\rho_R(\bigcup_{x \in U} G_x) \subseteq N$ where N is an open neighbourhood of $\rho_R(\alpha)$ in G_R .

Thus, ρ_R is also continuous at $\alpha = 1 \in G_*$. This establishes that $\rho_R : E \rightarrow G_R$ is a continuous morphism of pointed topological spaces.

By the universal property of $\eta : E \rightarrow G = \coprod_{x \in X} G_x$, there exists a unique continuous homomorphism: $\pi_R : G \rightarrow G_R = \coprod_{x \in X/R} G_{xR}$ such that $\pi_R \circ \eta = \rho_R$.

R, R' are equivalence relations on X , satisfying i) and ii) above and $R \subseteq R'$, then define:

$$\pi_{R,R'} : G_R = \coprod_{x \in X/R} G_{xR} \rightarrow G_{R'} = \coprod_{x \in X/R} G_{xR'}$$

to be a continuous homomorphism induced by the maps:

$$G_c = G_{xR} \xrightarrow{\sigma_{cc'}} G_{c'} = G_{xR'} \xrightarrow{\phi_{xR'}} \coprod_{xR' \in X/R} G_{xR'} = G_{R'}$$

where c, c' are the chosen representatives of xR and xR' , respectively, if $x \neq *$ then $\sigma_{cc'}$ exists (note $cR'c'$) otherwise, the map $G_c \rightarrow G_{c'}$ is a trivial map, and $\phi_{xR'}$ are canonical monomorphisms.

Proposition 1.8: Let $(X, *)$, $\{G_x | x \in X\}$ and $G = \coprod_{x \in X} G_x$ be as above. Consider the set $\underline{R}$ of equivalence relations on X satisfying the conditions i) and ii) above. Then,

$$G = \text{proj. limit}_{R \in \underline{R}} (G_R = \coprod_{xR \in X/R} G_{xR})$$

Proof: Let V be an open normal subgroup of G , and $q_V : G \rightarrow G/V$ be the canonical homomorphism. Consider the equivalence relation R_V on X characterized by: if $x \in X$, then xR_V* when $q_V(G_x) = 1$; if $x, y \in X$ and $x \neq * \neq y$, we have $xR_V y$ iff there is $\sigma_{xy} : G_x \rightarrow G_y$ and $q_V(a_x) = q_V(a_y)$ where $a_x \in G_x$ and $a_y = \sigma_{xy}(a_x)$.

We proceed to show that R_V satisfies the conditions i) and ii). Since X is compact, (i) is satisfied if R_V is open, i.e. if every xR_V ($x \in X$) is open.

Assume $x \neq *$, then for every $\alpha \in G/V$ consider the closed open set $O_\alpha = q_V^{-1}(q_V(\alpha))$. For every $a \in G_x \cap O_\alpha$, let $U_{x,\alpha}^a$ be a constant open neighbourhood of x in X , missing $*$ and let $D_{x,\alpha}^a$ be an open neighbourhood of a in G_x with $\{\sigma_{xy}(g') | g' \in D_{x,\alpha}^a, y \in U_{x,\alpha}^a\} \subseteq O_\alpha$.

Let $\{D_{x,\alpha}^{a_i}\}_{i=1}^{n_\alpha}$ be a finite cover of $G_x \cap O_\alpha$, and set $U_x = \bigcap_{\alpha} \bigcap_{i=1}^{n_\alpha} U_{x,\alpha}^{a_i}$

where α runs through those elements of G/V for which $G_x \cap O_\alpha \neq \emptyset$.

Clearly, U_x is a constant open neighbourhood of x . Moreover, if $y \in U_x$ and $a_y \in G_y$, $a_x \in G_x$ and $a_y = \sigma_{xy}(a_x)$, then $a_x \in D_{x,\alpha}^{a_i}$ for some a_i and

$$a_y \in \{\sigma_{xy}(g') \mid g' \in D_{x,\alpha}^{a_i}, y \in U_{x,\alpha}^{a_i}\}$$

Hence, $q_V(a_y) = q_V(a_x) = \alpha$. Therefore, $y R_V x$, and $U_x \subseteq x R_V$.

Thus, $x R_V$ ($x \neq *$) is open.

Assume now that $x = *$. Since $q_V^{-1}(1)$ is an open neighbourhood of 1 in G , $\eta^{-1}(q_V^{-1}(1))$ is an open neighbourhood of $1 \in G_*$.

Thus, there is an open neighbourhood U_* of $*$ in X such that

$$\bigcup_{x \in U_*} G_x \subset \eta^{-1}(q_V^{-1}(1)).$$

Then, $x \in U_*$ implies $x R_V *$. Thus, $*$ is an interior point of $* R_V$.

This establishes the condition i), obviously R_V satisfies the condition ii).

Now consider the diagram:

where ψ^V is the continuous homomorphism induced by the homomorphisms:

$\psi_x : G_{xR_V} \rightarrow G/V$, given by

$$\psi_x(a) = q_V(a) \quad (a \in G_{xR_V}, x \in *R_V)$$

Now the maps

$$\{\psi^V \mid V \text{ open normal subgroup of } G\}$$

are easily seen to induce a continuous homomorphism:

$$\text{proj. limit}_{R_V} G_{R_V} \longrightarrow \text{proj. limit}_V G/V = G$$

and hence a continuous homomorphism

$$\psi : \text{proj. limit}_{R \in \underline{R}} G_R \longrightarrow G$$

Now, it is easily checked that ψ and the homomorphism $G \rightarrow \text{proj.}$

$\text{limit}_{R \in \underline{R}} G_R$ are inverse to each other.

Corollary 1.9: Let X be a compact, Hausdorff, pointed topological space. Let $\underline{R}$ be the family of all equivalence relations on X such that $R \in \underline{R}$ implies X/R is a finite discrete space with quotient topology.

Then,

$$F(X) \cong \text{proj. limit}_{R \in \underline{R}} F(X/R),$$

$$\tilde{F}(X, *) \cong \text{proj. limit}_{R \in R} F(X_R)$$

where X_R is obtained from X/R by removing equivalence class containing $*$.

Proof: In Proposition 1.8, take $G_x = \hat{Z}_C$, free pro-C group on one generator, $x \in X$, $x \neq *$. Then,

$$\coprod_{x \in X} G_x \cong \tilde{F}(X, *)$$

$$\coprod_{xR \in X/R} G_{xR} \cong F(X_R)$$

The other isomorphism is obtained similarly.

2. Subgroups of Free Pro-C Product of Pro-C Groups

Throughout this section, we assume class C of finite groups is closed under formations of i) subgroups, ii) homomorphic images, and iii) extensions.

In the category of (discrete) groups, we have a Kurosch subgroup theorem. If

$$G = \ast_{i \in I} A_i$$

then every subgroup U of G is a free product of subgroups conjugate in G to certain subgroups of the free factors A_i , and of a certain free subgroup. (In order to get more explicit formulation

of this theorem, see Magnus, Karrass, and Solitar⁽¹²⁾). In (8), Gildenhuys, D. and Ribes, L. gave a version of Kurosch subgroup theorem for open subgroups of free pro-C products of pro-C groups. In general, for closed subgroups, the theorem is not true.

To see this, consider a class C of finite groups which is not contained in a class of finite p-groups (p-prime). Let F be a free pro-C group on a set X ($|X| > 1$). Then, $F \cong \coprod_{x \in X} (\hat{Z}_C)_x$ where $(\hat{Z}_C)_x$ is an isomorphic copy of $\hat{Z}_C$ - the free pro-C group on one generator. Let U be a closed subgroup of F. If the Kurosch subgroup would have been true for closed subgroups, using the similar arguments as in Theorem 5.2 (Gildenhuys, D. and Ribes, L.⁽⁸⁾, p. 327), we could have proved that U is a free pro-C group on a certain set Y. But a closed subgroup of a free pro-C group need not be free pro-C, as for prime p where C contains finite p-groups, F_p - Sylow -p subgroup of F is not free pro-C.

In this section, we show that the certain closed subgroups of free pro-C products of pro-C groups are again free pro-C products of certain pro-C groups. We need a lemma due to Gruenberg.

First, we recall the definition of a transversal of a subgroup H of a group G. A subset of G, that is a complete set of right coset representatives of H in G, where the representative of coset H is taken to be 1, is called a transversal of H in G.

Lemma 2.1: (Gruenberg⁽¹⁰⁾, Lemma 5.5, p. 47)

Let $G = A * B$ (the free product in the category of discrete groups, of groups A and B). Let H, K be normal subgroups of A, B with transversals S, T in A, B respectively. The natural homomorphisms of A, B into $A/H \times B/K$ extend to a homomorphism of G onto $A/H \times B/K$ and the kernel L of this homomorphism has the form:

$$L = F * U * V$$

where i) F is freely generated by all (s, t) (commutators of s, t) with $l \neq s$ in S and $l \neq t$ in T , and

$$\text{ii) } U = *_{t \in T} t^{-1} H t$$

and

$$V = *_{s \in S} s^{-1} K s$$

In order to give a version of Lemma 2.1 in pro- C groups, we consider the following.

Let G be a pro- C group, H a closed subgroup of G . Let π be the natural projection: $G \rightarrow G/H$. A continuous map $\phi : G/H \rightarrow G$ is called a section if $\pi\phi = \text{identity map on } G/H$. For every closed subgroup H of G , a continuous section $\phi : G/H \rightarrow G$ exists (see Ribes, L.⁽¹⁵⁾, p. 31 Proposition 3.5). For a pro- C group G and a closed subgroup H of G , we consider a (continuous) section $\phi : G/H \rightarrow G$ such that $\phi(H) = 1$ in G . Then, we define $S = \phi(G/H)$.

as a transversal of H in G . Note that as a subspace of G , S , is a compact, Hausdorff, totally disconnected topological space. If H is open in G , then S is a finite discrete space.

Let A, B be pro- C groups, $A \amalg B$ their free pro- C product. Let H, K be closed subgroups of A, B with transversals S, T in A, B respectively. Now, for every $t \in T$, consider an isomorphic copy $t^{-1}Ht$ of H in $A \amalg B$. Next, consider $* \notin T$, put $\bar{T} = T \cup \{*\}$ and $\bar{T}$ as a pointed topological space (co-product topology) with the distinguished point $*$. Note that $*$ is an isolated point of $\bar{T}$. Now, the family $\{H^t | t \in \bar{T}\}$ (where $H^* = 1$) of pro- C groups satisfies the conditions 1)-3) on page 8. Thus, it makes sense to talk of $\coprod_{t \in \bar{T}} H^t$. By the remark following the Definition 1.4, we can write:

$$\coprod_{t \in \bar{T}} H^t = \coprod_{t \in T} H^t$$

For the transversals S, T above, consider the product space $S \times T$. Define an equivalence relation $\sim$ on $S \times T$ as follows:

$$(s, t) \sim (s', t') \text{ if } \begin{aligned} &s = 1 \text{ and } t' = 1, \text{ or} \\ &s' = 1 \text{ and } t = 1 \text{ or} \\ &s = s' \text{ and } t = t'. \end{aligned}$$

Let SoT be the quotient space $S \times T / \sim$. Next, suppose S, T are finite discrete spaces. The cardinality of SoT is $((|S|-1) \times (|T|-1)) + 1$. Let $X_{H, K}$ be the space obtained from SoT by dropping the distinguished point, i.e. the equivalence class of $(1, 1)$, $1 \in S$, $1 \in T$.

Now, we are in a position to give a version of Lemma 2.1 for pro-C groups.

Proposition 2.2: Let A, B be pro-C groups; H, K be open normal subgroups of A, B respectively. Let S be a transversal of H in A , and T that of K in B . Consider the continuous homomorphism π of $A \amalg B$ onto $A/H \times B/K$ that extends natural homomorphisms of A, B into $A/H \times B/K$. Then the kernel M of this homomorphism has the form:

$$M = F(X_{H,K}) \amalg \left(\amalg_{t \in T} H^t \right) \amalg \left(\amalg_{s \in S} K^s \right)$$

with the notations explained above.

Proof: Let $M = \text{Ker}(A \amalg B \xrightarrow{\pi} A/H \times B/K)$. Let $A * B$ be the (discrete) free product of groups A, B . We shall identify $A * B$ with its canonical image in $A \amalg B$ (see Gildenhuys and Ribes, L. (8), Remark 2.4). Consider a homomorphism f (as in Lemma 2.1) of $A * B$ onto $A/H \times B/K$ which extends natural homomorphisms of A, B into $A/H \times B/K$. Let L be the kernel of f . Now π restricted to $A * B$ is f . Thus, $L = M \cap (A * B)$.

Now, by Lemma 2.1, $L = F * U * V$ where F is free group on the generator (s, t) , $1 \neq s \in S$, $1 \neq t \in T$.

$$U = *_{t \in T} t^{-1} H t, \quad V = *_{s \in S} s^{-1} K s$$

We consider two topological structures on L .

1) τ^i induced by the topology on $A*B$; i.e. the topology on $A*B$ determined by the fundamental system of open neighbourhoods of 1 in $A*B$:

$$\underline{N} : \{ N \triangleleft A*B \mid A*B/N \in C, N \cap A \text{ is open in } A, \text{ and } N \cap B \text{ is open in } B. \}$$

2) τ^n determined by the following system of open neighbourhoods of 1 in L:

$N \triangleleft L$ such that:

i) $L/N \in C$

ii) $N \cap H^t$ is open in H^t for every $t \in T$.

$N \cap K^S$ is open in K^S for every $s \in S$.

iii) $N \cap F$ is open in F with natural pro-C topology on F .

Now, the completion of L with regard to τ^i is M .

Thus, if we can show that two topologies τ^i, τ^n of L coincide, then from the proof of Proposition 1.5, it follows that M is (topologically) isomorphic to

$$F(X_{H,K}) \coprod (\coprod_{t \in T} H^t) \coprod (\coprod_{s \in S} K^S)$$

and the proof of the proposition would be complete.

Now we need to show that $\tau^i = \tau^n$.

First, $\tau^i \leq \tau^n$:

Let U be an open normal subgroup of $A*B$ with regard to τ^i and $N = U \cap L$.

Now,

$$L/N = L/U \cap L = LU/U \subseteq A*B/U \in C$$

Hence, (i) holds.

Next, $N \cap H^t = (U \cap L) \cap H^t = U \cap H^t$ is open in H^t , $t \in T$.

Similarly, $N \cap K^s$ is open in K^s , $s \in S$.

Thus (ii) holds.

Finally, $N \cap F$ is open in F , hence (iii).

Thus,

$$\tau^i \subseteq \tau^n$$

Conversely,

$$\tau^n \subseteq \tau^i.$$

Let N be a normal subgroup of L satisfying i), ii) and iii).

It suffices to show that $\bar{N} = \text{core of } N \text{ in } A*B$ (maximal normal subgroup of $A*B$ contained in N) is open with regard to τ^i .

That is to show that $A*B/\bar{N} \in C$, $\bar{N} \cap A$ is open in A , and $\bar{N} \cap B$ is open in B .

First, $A*B/L \cong A/H \times B/K \in C$.

Let $n = [A*B:N]$ and $\{g_1, \dots, g_n\}$ be the complete set of coset representatives of N in $A*B$. Denote $N^i = N^{g_i} = g_i^{-1} N g_i$,

$i=1, 2, \dots, n$. For every i , $N^i \triangleleft L$ and $L/N^i \cong L/N^1 = L/N \in C$.

Now, $\bar{N} = \bigcap_{i=1}^n N^i$. Thus, $L/\bar{N}$ is isomorphic to a subgroup of $\prod_{i=1}^n L/N^i$.

Since

$$\prod_{i=1}^n L/N^i \in C, \quad L/\bar{N} \in C.$$

Now, $A*B/L \in C$, $L/\bar{N} \in C$ and C is closed under extensions, implying that $A*B/\bar{N} \in C$.

Now, in order to see that $\bar{N} \cap A$ is open in A , it is enough to see that $N^x \cap A$ is open in A for every $x \in A*B$. (Note that N has finite index in $A*B$).

$$\text{Now, } N^x \cap A = (N \cap A^{x^{-1}x}).$$

For x^{-1} , there exists $s \in S$, $t \in T$ and $y \in L$ such that $x^{-1} = sty$. Thus, $A^{x^{-1}} = A^{sty} = A^{ty}$ and $N \cap A^{x^{-1}} = N \cap A^{ty} = (N \cap A^t)^y$.

Now, $N \cap H^t \subset N \cap A^t$ and $N \cap H^t$ is open in A^t by assumption. This in turn shows that $N \cap A^t$ is open in A^t .

Thus, $(N \cap A^t)^y$ is open in $A^{ty} = A^{x^{-1}}$.
i.e. $N \cap A^{x^{-1}}$ is open in $A^{x^{-1}}$.

$$(N \cap A^{x^{-1}x}) \text{ is open in } A$$

$$N^x \cap A \text{ is open in } A.$$

Thus, $\bar{N} \cap A$ is open in A .

Similarly, $\bar{N} \cap B$ is open in B .

This proves that $\tau^i = \tau^n$, and the proof of the Proposition is complete.

Next, we extend the result of Proposition 2.2 for closed normal subgroups satisfying some conditions on transversals. We begin by proving some lemmas that we need.

Lemma 2.3: Let A, B be pro- C groups. ϕ_A, ϕ_B are the families of open normal subgroups of A, B respectively such that:

$$A = \text{proj. limit}_{U \in \phi_A} A/U$$

$$B = \text{proj. limit}_{V \in \phi_B} B/V$$

Then,

$$A \times B = \text{proj. limit}_{(U, V) \in \phi_A \times \phi_B} (A/U \times B/V)$$

Proof: Define a partial order on $\phi_A \times \phi_B$ as:

$$(U, V) \geq (U', V') \iff U \subseteq U' \text{ and } V \subseteq V'.$$

$$(I) \quad \left\{ (A/U \times B/V), (\phi_{U, U'}, \psi_{V, V'}) \mid \begin{array}{l} U, U' \in \phi_A, U \subseteq U' \\ V, V' \in \phi_B, V \subseteq V' \end{array} \right\}$$

is clearly a projective mapping family where

$$\{A/U, \phi_{U, U'} \mid U, U' \in \phi_A, U \subseteq U'\}$$

and

$$\{B/V, \psi_{V, V'} \mid V, V' \in \phi_B, V \subseteq V'\}$$

are given projective mapping families.

Now the family

$$(II) \quad \{A \times B, (\phi_U, \psi_V) \mid U \in \Phi_A, V \in \Phi_B\}$$

is a consistent family, i.e. the diagrams

$$\begin{array}{ccc} A \times B & \xrightarrow{(\phi_U, \psi_V)} & A/U \times B/V \\ & \searrow (\phi_{U'}, \psi_{V'}) & \downarrow (\phi_{U, U'}, \psi_{V, V'}) \\ & & A/U' \times B/V' \end{array}$$

commute where

$$\begin{array}{ccc} A & \xrightarrow{\phi_U} & A/U \\ & \searrow \psi_V & \\ B & \xrightarrow{\psi_V} & B/V \end{array} \text{ are canonical homomorphisms.}$$

We claim that (II) is a projective limit of (I).

Let $\{C, \alpha_{U, V} : C \rightarrow A/U \times B/V\}$ be any other consistent family, where C is a pro- C group. Thus, the diagrams:

$$\begin{array}{ccc} C & \xrightarrow{\alpha_{U, V}} & A/U \times B/V \\ & \searrow \alpha_{U', V'} & \downarrow (\phi_{U, U'}, \psi_{V, V'}) \\ & & A/U' \times B/V' \end{array}$$

commute.

Let $P_U : A/U \times B/V \rightarrow A/U$ be a projection. Now the diagrams

commute.

By the universal property of projective limit $A/U = A$, we have a unique continuous homomorphism $P_A : C \rightarrow A$ such that the diagrams

commute.

Similarly, we get a unique continuous homomorphism $P_B : C \rightarrow B$ such that corresponding diagrams in III commute.

Now, it follows that:

$$\begin{aligned}
 (P_A; P_B) : C &\longrightarrow A \times B \\
 x &\longrightarrow (P_A(x), P_B(x))
 \end{aligned}$$

is a unique continuous homomorphism such that the diagrams

commute.

Thus,

$$A \times B = \text{proj. limit}_{\substack{U \in \Phi_A \\ V \in \Phi_B}} (A/U \times B/V)$$

Lemma 2.4: With the notations as in Lemma 2.3,

$$A \amalg B = \text{proj. limit}_{U \in \Phi_A, V \in \Phi_B} (A/U \amalg B/V)$$

Proof: First, note that for $U, U' \in \Phi_A$ and $V, V' \in \Phi_B$ such that $U \subseteq U', V \subseteq V'$, we have a continuous homomorphism:

$$\phi_{U, U'} \amalg \psi_{V, V'} : A/U \amalg B/V \longrightarrow A/U' \amalg B/V'$$

that extends $\phi_{U, U'}$ and $\psi_{V, V'}$.

Thus,

$$(I) \quad \left\{ A/U \amalg B/V; \phi_{U, U'} \amalg \psi_{V, V'} \mid \begin{array}{l} U, U' \in \Phi_A, U \subseteq U' \\ V, V' \in \Phi_B, V \subseteq V' \end{array} \right\}$$

is a projective mapping family, and it makes sense to talk of proj limit $(A/U \amalg B/V)$; it is a pro-C group since U, V it is projective limit of pro-C groups (see Ribes, L. (15) p. 30).

Next, consider the consistent family.

$$(ii) \quad \{A \amalg B, \phi_U \amalg \psi_V \mid U \in \Phi_A, V \in \Phi_B\}$$

where $\phi_U \perp \psi_V : A \perp B \longrightarrow A/U \perp B/V$ is a unique homomorphism that extends

$$\phi_U : A \rightarrow A/U \text{ and } \psi_V : B \rightarrow B/V$$

We claim that II is a projective limit of I.

If C is any pro-C group with morphisms $\alpha_{U,V} : C \rightarrow A/U \perp B/V$ such that

$$\{C; \alpha_{U,V} \mid U \in \Phi_A, V \in \Phi_B\}$$

is a consistent family. We will show that there is a unique homomorphism:

$P : C \rightarrow A \perp B$ such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{P} & A \perp B \\ & \searrow \alpha_{U,V} & \downarrow \phi_U \perp \psi_V \\ & & A/U \perp B/V \end{array}$$

commute.

Let W be open normal subgroup of $A \perp B$. Identifying the images of A, B in $A \perp B$, let $U \subseteq U_W = W \cap A$; $V \subseteq V_W = W \cap B$, s.t.

$$U \in \Phi_A, V \in \Phi_B.$$

Now we have a morphism:

$$C \xrightarrow{\alpha_{U,V}} A/U \perp B/V \xrightarrow{\pi_W} A \perp B/W$$

where π_W is that morphism which extends canonical morphisms

$$A/U \rightarrow (A \amalg B)/W \text{ and}$$

$$B/V \rightarrow (A \amalg B)/W.$$

Since $A \amalg B = \text{proj. limit}_W A \amalg B/W$, there is a unique continuous homomorphism $P : C \rightarrow A \amalg B$ such that the diagrams

$$\begin{array}{ccccc} C & \xrightarrow{\alpha_{U,V}} & A/U \amalg B/V & \xrightarrow{\pi_W} & A \amalg B/W \\ & \searrow P & \uparrow \phi_U \amalg \psi_V & \nearrow P_W & \\ & & A \amalg B & & \end{array}$$

commute (P_W - canonical homomorphism), i.e.

$$P_W P = \pi_W \alpha_{U,V}$$

and

$$\pi_W (\phi_U \amalg \psi_V) = P_W$$

Thus,

$$(\pi_W (\phi_U \amalg \psi_V) P = \pi_W \alpha_{U,V}$$

$$\text{i.e. } \pi_W ((\phi_U \amalg \psi_V) P) = \pi_W \alpha_{U,V}$$

Since π_W is epimorphism, $(\phi_U \amalg \psi_V) P = \alpha_{U,V}$.

Now, if $P' : C \rightarrow A \amalg B$ is any other homomorphism such that

$$(\phi_U \amalg \psi_V) P' = \alpha_{U,V} \text{ then } \pi_W ((\phi_U \amalg \psi_V) P') = \pi_W \alpha_{U,V}. \text{ Thus,}$$

$$P_W P' = \pi_W \alpha_{U,V} \quad (*)$$

Since P is unique such that $(*)$ holds, $P = P'$. Thus,

$$A \perp B = \text{proj. limit}_{\substack{U \in \Phi_A \\ V \in \Phi_B}} A/U \perp B/V$$

Proposition 2.5: Let A, B be pro- C groups. H, K closed normal subgroups of A, B respectively such that there exist families Φ_A of open normal subgroups of A, B respectively such that:

$$i) \quad A = \text{proj. limit}_{U \in \Phi_A} A/U; \quad B = \text{proj. limit}_{V \in \Phi_B} B/V$$

ii) there exist transversals $\phi : A/H \rightarrow A; \psi : B/K \rightarrow B$ such that there are transversals:

$$\phi_U : A/UH \rightarrow A/U, \quad U \in \Phi_A$$

$$\psi_V : B/VK \rightarrow B/V, \quad V \in \Phi_B$$

such that

$$\phi = \text{proj. limit}_U \phi_U; \quad \psi = \text{proj. limit}_V \psi_V$$

Then, the kernel L of the continuous homomorphism $A \perp B$ onto $A/H \times B/K$ that extends canonical homomorphisms A, B into $A/H \times B/K$ is (topologically) isomorphic to

$$\tilde{F}(\text{SoT}, *) \perp \bigcap_{t \in T} (1H^t) \perp \bigcap_{s \in S} (1K^s)$$

where $S = \phi(A/H), T = \psi(B/K)$.

Proof: For every $U \in \Phi_A$, $V \in \Phi_B$, HU/V groups of A/U and B/V respectively.

By Proposition 2.2, we have exa

$$L_{U,V} \longrightarrow A/U \amalg B/V \longrightarrow A/HU \times B/KV$$

where

$$L_{U,V} \approx F(X_{HU/U, KV/V}) \coprod_{t \in T_V} (\coprod (HU/U)^t) \coprod_{s \in S_U} (\coprod (KV/V)^s)$$

where $X_{HU/U, KV/V}$ is (as explained) a finite set of cardinality $(|S_U|-1)(|T_V|-1)$ and S_U is a transversal of HU/U in A/U and T_V is a transversal of KV/V in B/V i.e. $S_U = \phi_U(A/HU)$; $T_V = \psi_V(B/KV)$.

Now, for $U' \in \Phi_A$, $V' \in \Phi_B$ such that $U \subseteq U'$ and $V \subseteq V'$; we have commutative diagrams.

Consider a unique homomorphism:

$$p_{U,U'} \parallel q_{V,V'} : A/U \parallel B/V \longrightarrow A/U' \parallel B/V'$$

that extends $P_{U,U'}$ and $q_{V,V'}$

Next, consider the diagram:

$$\begin{array}{ccc}
 A/HU \times B/KV & \xrightarrow{\phi_U \times \psi_V} & S_U \times T_V \\
 \downarrow p'_{U,U'} \times q'_{V,V'} & & \downarrow \alpha = \alpha_{UU',VV'} \\
 A/HU' \times B/KV' & \xrightarrow{\phi_{U'} \times \psi_{V'}} & S_{U'} \times T_{V'}
 \end{array}$$

Note $\phi_U(A/HU) = S_U$, $\psi_V(B/KV) = T_V$

$$\phi_{U'}(A/HU') = S_{U'}, \quad \psi_{V'}(B/KV') = T_{V'}.$$

Thus, there is $\alpha : S_U \times T_V \rightarrow S_U \times T_V$, such that the diagram commutes.

Next, $s, s' \in S_U$ $t, t' \in T_V$ such that $(s, t) \sim (s', t')$. Then, $\alpha(s, t) \sim \alpha(s', t')$. Thus, we have a map: $\alpha : S_U \circ T_V \rightarrow S_{U'} \circ T_{V'}$.

This map induces a continuous homomorphism also denoted by α :

$$\alpha : F(X_{HU/U, KV/V}) \rightarrow F(X_{HU'/U', KV'/V'})$$

Let $t \in T_V = \psi_V(B/KV)$. We have a canonical homomorphism:

$$(HU/U)^t \longrightarrow (HU'/U')^{t'}$$

where

$$t' = q_{V, V'}(t) = \psi_{V'}(q'_{V, V'}(t))$$

i.e. $t' \in T_{V'}$:

These homomorphisms induce a continuous homomorphism:

$$\beta_1 : \coprod_{t \in T_V} (HU/U)^t \longrightarrow \coprod_{t' \in T_{V'}} (HU'/U')^{t'}$$

Similarly, we construct a continuous homomorphism:

$$\beta_2 : \coprod_{s \in S_U} (KV/V)^s \longrightarrow \coprod_{s' \in S_{U'}} (KV'/V')^{s'}$$

Next, note

$$L_{U, V} = F(X_{HU/U, KV/V}) \coprod \left(\coprod_{t \in T_V} (HU/U)^t \right) \coprod \left(\coprod_{s \in S_U} (KV/V)^s \right)$$

Thus, homomorphisms α , β_1 and β_2 induce a homomorphism

$$\phi_{UU', VV'} = \phi : L_{U, V} \rightarrow L_{U', V'}.$$

Now, consider the diagram

$$\begin{array}{ccccc}
 L_{U,V} & \xrightarrow{i_{U,V}} & A/U \amalg B/V & \xrightarrow[U]{\pi_V} & A/HU \times B/KV \\
 \downarrow \phi & & \downarrow p_{U,U'} \amalg q_{V,V'} & & \downarrow p'_{U,U'} \times q'_{V,V'} \\
 L_{U',V'} & \xrightarrow{i_{U',V'}} & A/U' \amalg B/V' & \xrightarrow[U']{\pi_{V'}} & A/HU' \times B/KV
 \end{array} \quad (**)$$

where $i_{U,V} = i$ (say) is defined by sending generators of $L_{U,V}$ as follows:

$$i(x) = s^{-1} t^{-1} s t \text{ where } x = \overline{(s,t)} \in X_{HU/U, KV/V}$$

$$1 \neq s \in S_U, 1 \neq t \in T_V$$

$$i(\overline{a}^{-t}) = t^{-1} \overline{a}^{-1} t \text{ where } \overline{a} \in HU/U, t \in T_V$$

$$i(\overline{b}^{-s}) = s^{-1} \overline{b}^{-1} s \text{ where } \overline{b} \in KV/V, s \in S_U.$$

The right square in (**) commutes is clear from the diagram (*).

We check the commutativity of left square in (**). It is enough to check the commutativity on the generators of $L_{U,V}$.

Since

$$L_{U,V} = F(X_{HU/U, KV/V}) \amalg \left(\bigsqcup_{t \in T_V} (HU/U)^t \right) \amalg \left(\bigsqcup_{s \in S_U} (KV/V)^s \right)$$

we have to check on elements of $X_{HU/U, KV/V}$: $(HU/U)^t$, $t \in T_V$; $(KV/V)^s$, $s \in S_U$.

Let $x \in X_{HU/U, KV/V} = S_U \circ T_V - \{(1,1)\}$. Then,

$$x = \overline{(s,t)} \quad 1 \neq s \in S_U, 1 \neq t \in T_V$$

and

$$\begin{aligned}
 & P_{U,U} \parallel q_{V,V}(i(x)) \\
 &= P_{U,U} \parallel q_{V,V}(s^{-1}t^{-1}st) \\
 &= P_{UU}(s^{-1})q_{VV}(t^{-1})P_{UU}(s)q_{VV}(t) \\
 &= i(s', t') \quad \text{where } s' = P_{UU}(s) \quad t' = q_{VV}(t) \\
 &= i\alpha(s, t) \quad (\text{see definition of } \alpha) \\
 &= i\theta(x)
 \end{aligned}$$

Next, let $x \in (HU/U)^t$, then $x = (\bar{a})^t$ where $\bar{a} = aU$, $a \in A$

$$\begin{aligned}
 P_{UU} \parallel q_{VV}(i(x)) &= P_{UU} \parallel q_{VV}(t^{-1}\bar{a}t) \\
 &= q_{VV}(t)^{-1}P_{UU}(\bar{a})q_{VV}(t) \\
 &= t'^{-1}\bar{a}'t'
 \end{aligned}$$

where $\bar{a}' = P_{UU}(\bar{a})$ and $t' = q_{VV}(t)$

$$\begin{aligned}
 &= i(\bar{a}', t') \\
 &= i(\theta(\bar{a}^t)) \quad (II)
 \end{aligned}$$

(Note $\bar{a}', t' = \theta(\bar{a}^t)$)

Finally, for $x \in (KV/V)^S$, $x = (\bar{b})^S$ where $\bar{b} = bV$, $b \in B$, $s \in S_U$,

we have similarly:

$$P_{UU} \parallel q_{VV}(i(x)) = i\theta(x) \quad (III)$$

From I, II and III, it follows that the left square in ** commutes.

Thus, we have projective mapping families:

$$(i) \left\{ L_{U,V} ; \phi = \phi_{UU',VV'} \mid \begin{array}{l} U, U' \in \phi_A, U \subseteq U' \\ V, V' \in \phi_B, V \subseteq V' \end{array} \right\}$$

$$(ii) \left\{ A/U \amalg B/V ; p_{UU'}, q_{VV'} \mid \begin{array}{l} U, U' \in \phi_A, U \subseteq U' \\ V, V' \in \phi_B, V \subseteq V' \end{array} \right\}$$

$$(iii) \left\{ A/HU \times B/KV ; p'_{UU'} \times q'_{VV'} \mid \begin{array}{l} U, U' \in \phi_A, U \subseteq U' \\ V, V' \in \phi_B, V \subseteq V' \end{array} \right\}$$

and exact sequence

$$(iv) \quad 1 \longrightarrow (i) \longrightarrow (ii) \longrightarrow (iii) \longrightarrow 1$$

Since the projective limit functor is exact on pro-C groups, we have from (iv)

$$\begin{aligned} \text{proj. limit } L_{U,V} &\longrightarrow \text{proj. limit } (A/U \amalg B/V) \longrightarrow \\ \text{proj. limit } (A/HU \times B/KV) \end{aligned}$$

By Lemma 2.4,

$$A \amalg B = \text{proj. limit } (A/U \amalg B/V).$$

By Lemma 2.3,

$$A/H \times B/K = \text{proj. limit } (A/HU \times B/KV).$$

Let $L = \text{proj. limit } L_{U,V}$. Thus, we have the exact sequence:

$$L \longrightarrow A \amalg B \longrightarrow A/H \times B/K.$$

Now, $L = \text{proj. limit } L_{U,V}$

$$= \text{proj. limit } (F(X_{HU/U, KV/V}) \amalg (\amalg_{t \in T_V} (HU/U)^t) \amalg (\amalg_{s \in S_U} (KV/V)^s))$$

Since proj. limit commutes with coproducts (Lemma 2.4), we have

$$\begin{aligned} & \text{proj. limit } F(X_{HU/U, KV/V}) \amalg (\text{proj. limit } \amalg_{t \in T_V} (HU/U)^t) \\ & (\text{proj. limit } \amalg_{s \in S_U} (KV/V)^s) \end{aligned}$$

By Corollary 1.9,

$$\tilde{F}(SoT, *) \cong \text{proj. limit } F(X_{HU/U, KV/V})$$

$$(\text{Note } X_{HU/U, KV/V} = S_U \circ T_V - \{(1,1)\})$$

Secondly,

$$\text{proj. limit}_{U,V} \amalg_{t \in T_V} (HU/U)^t$$

$$= \text{proj. limit}_V \amalg_{t \in T_V} (\text{proj. limit}_U (HU/U)^t)$$

(Lemma 2.4)

$$= \text{proj. limit}_V \amalg_{t \in T_V} H^t$$

$$= \amalg_{t \in T} H^t$$

(Proposition 1.8)

Similarly,

$$\text{proj. limit}_{U,V} \coprod_{S \in S_U} (KV/V)^S = \coprod_{S \in S} K^S$$

Thus,

$$L = \tilde{F}(SoT, *) \coprod_{t \in T} (\coprod_{t \in T} H^t) \coprod_{s \in S} (\coprod_{s \in S} K^s)$$

(For notations $\coprod_{t \in T} H^t$, $\coprod_{s \in S} K^s$, see p.21)

This completes the proof of the Proposition 2.5.

Definition 2.6: Let X be a topological space. A base for the neighbourhood system of a point $x \in X$, or a local base at x , is a family of neighbourhoods of x such that every neighbourhood of x contains a member of the family. A topological space satisfies the first axiom of countability if the neighbourhood system at every point has a countable local base.

Lemma 2.7: Let G be a pro-C group and H a closed normal subgroup of G . Let $U_i \supset H$ ($i \in I$) be open normal subgroups of G such that U_i/H is a basis of neighbourhoods of 1 in G/H . Let $\{U_{ij} | j \in J_i\}$ be a collection of open normal subgroups of G , s.t. $U_{ij}H = U_i$. Then, $\{U_{ij} | i \in I, j \in J_i\}$ is a basis of neighbourhoods of 1 in G .

Proof : Let U be open normal in G . Then, UH/H is open in G/H and there exists $i \in I$ s.t. $UH \supset U_i$. Clearly, $U \cap U_i \subseteq U$ and $(U \cap U_i)H = UH \cap U_i = U_i$ (Note $H \subseteq U_i$).

Lemma 2.8: Let G be a pro- C group, and H be a closed normal subgroup of G such that G/H satisfies the first axiom of countability. Let ϕ be any continuous section $G \xrightleftharpoons[\phi]{\pi} G/H$. Then, there is a proj. family $\{\phi_V | V \in \Phi\}$ of continuous sections $G/V \xrightleftharpoons[\phi_V]{\pi_V} G/HV$, $V \in \Phi$ such that $\phi = \text{proj. limit } \phi_V$ where Φ is a basis for the neighbourhood system at 1 in G .

Proof: Since G/H is the first axiom countable space, there is a countable family $\{U_i, i=0,1,\dots\}$ of open normal subgroups of G such that $G = U_0 \supset U_1 \supset U_2 \dots$, $U_i \supset H$ for every i , and U_i/H is a basis for the neighbourhood system of 1 in G/H .

Let Φ be the family of open normal subgroups V of G such that $VH = U_iH = U_i$.

By Lemma 2.7, Φ is a basis for the neighbourhood system of 1 in G .

Using induction, construct sections $\psi_i : G/U_i \rightarrow G/H$ in such a way that the diagram

commutes for every i , i.e. $\psi_{i+1} = \psi_i \circ P'_{i+1}$.

Let $V \in \phi$, then there is i such that $VH = U_i H = U_i$. Define

$\phi_V = P_V \phi \psi_i$. Thus,

$$\begin{array}{ccc} G/VH = G/U_i & \xrightarrow{\psi_i} & G/H \\ \downarrow \phi_V & & \downarrow \phi \\ G/V & \xleftarrow{P_V} & G \end{array}$$

ϕ_V is a section:

We need to show that $\pi_V \phi_V(x) = x$. Consider the diagram

$$\begin{array}{ccc} G/VH = G/U_i & \xrightarrow{\psi_i} & G/H \\ \uparrow \pi_V & \xleftarrow{P'_{U_i}} & \uparrow \pi \\ \downarrow \phi_V & & \downarrow \phi \\ G/V & \xleftarrow{P_V} & G \end{array}$$

We have $\pi_V P_V = P'_{U_i} \pi$. Since ϕ and ψ_i are sections, $P'_{U_i} \pi(\phi \psi_i) = 1_{G/VH}$

i.e. $\pi_V P_V(\phi \psi_i) = 1_{G/VH}$

Thus, $\pi_V \phi_V = 1_{G/VH}$, ($\phi_V = P_V(\phi \psi_i)$)

Thus, ϕ_V is a section.

Moreover,

$$\phi_V(VH) = P_V \phi \psi_i(VH) = P_V \phi(H) = P_V(1) = V$$

i.e. ϕ_V sends VH to identity in G/V .

Now, if $V, V' \in \phi$ and $V \subseteq V'$, then the diagram

$$\begin{array}{ccccc}
 & & \phi_{V'} & & \\
 & G/V' & \longleftarrow & G/HV' & = & G/U_j \\
 \uparrow P_{V,V'} & & & \uparrow & & \uparrow P'_{i,j} \\
 G/V & \longleftarrow & \phi_V & G/HV & = & G/U_i
 \end{array} \quad (*)$$

commutes.

Since ϕ is a basis for the system of neighbourhoods at 1 in G ,

$$G = \text{proj. limit}_{V'} G/V, \quad G/H = \text{proj. limit}_V G/HV.$$

From the commutative diagram (*), $\text{proj. limit}_V \phi_V$ is a unique continuous map : $G/H \rightarrow G$ such that the diagram

$$\begin{array}{ccc}
 G/H & \xrightarrow{\text{proj. limit}_V \phi_V} & G \\
 \downarrow P'_{i_0} & & \downarrow P_V \\
 G/HV & \xrightarrow{\phi_V} & G/V
 \end{array}$$

commutes for every $V \in \phi$. The diagram

$$\begin{array}{ccc}
 G/H & \xrightarrow{\phi} & G \\
 \downarrow P'_{i_0} & & \downarrow P_V \\
 G/HV & \xrightarrow{\phi_V} & G/V
 \end{array}$$

commutes for every $V \in \phi$ by definition of ϕ_V .

Thus, by uniqueness, $\phi = \text{proj. limit}_V \phi_V$.

Theorem 2.9: Let A, B be pro- C groups; H, K be closed normal subgroups of A, B respectively such that $A/H, B/K$ are first axiom countable spaces. Then, with the notations as before, we have the exact sequence:

$$\tilde{F}(\text{SoT}, *) \amalg \left(\amalg_{t \in T} H^t \right) \amalg \left(\amalg_{s \in S} K^s \right) \longrightarrow A \amalg B \longrightarrow A/H \times B/K$$

Proof: Clear.

3. THE CARTESIAN SUBGROUP OF $A \amalg B$

Throughout this section, we assume that class C of finite groups is closed under the formations of i) subgroups, ii) homomorphic images and iii) extensions.

Let A, B be (discrete) groups, $A*B$ denote their free product (in the category of discrete groups). The cartesian subgroup is, by definition, the kernel of the homomorphism $A*B \rightarrow A \times B$. It is well-known that the cartesian subgroup of $A*B$ is a free group on the set of all mixed commutators (a, b) , $1 \neq a \in A$, $1 \neq b \in B$. (See Magnus, Karrass, Solitar⁽¹²⁾.)

Let A, B be pro- C groups. We define the cartesian subgroup of $A \amalg B$ as the kernel of the continuous homomorphism: $A \amalg B \rightarrow A \times B$, and show that it is a free pro- C group on a pointed topological space $A \circ B$ (see p. 21) with the distinguished point $*$ as the equivalence class of $(1, 1)$, $1 \in A$, $1 \in B$.

First, we shall prove the following Lemma.

Lemma 3.1. Let A, B be groups in C , then the cartesian subgroup of $A \amalg B$ is a free pro- C group on a finite discrete space X , where X is obtained from $A \circ B$ by dropping the equivalence class of $(1, 1)$. (Note $|X| = (|A| - 1)(|B| - 1)$)

Proof: We have the exact sequence:

$$F^d(x) \longrightarrow A*B \longrightarrow A \times B$$

$F^d(x)$ denotes the free discrete group on X . Let

$$\phi = \{ U \triangleleft A*B \mid A*B/U \in C, \text{ } A \cap U \text{ is open in } A, \text{ } B \cap U \text{ is open in } B \}$$

$A*B$ is a topological group with a topology determined by ϕ as the fundamental system of open neighbourhoods at 1 in $A*B$.

Let L be the cartesian subgroup of $A \amalg B$, i.e. L is the kernel of $A \amalg B \longrightarrow A \times B$.

L is closure of $F^d(x)$ in $A \amalg B$.

Since $F^d(x)$ has finite index in $A*B$, L is pro- C completion of $F^d(x)$. Thus, L is a free pro- C group on X .

Theorem 3.2: Let A, B be pro- C groups, then the cartesian subgroup of $A \amalg B$ is isomorphic to $\tilde{F}(A*B, *)$ where $A*B$ is compact, Hausdorff, totally disconnected space (see p. 21), $*$ is the distinguished point of $A*B$.

Proof: There are projective mapping families:

$$\{A_i; \phi_{i', i}; i \leq i'; i, i' \in I\}$$

and

$$\{B_j; \psi_{j', j}; j \leq j'; j, j' \in J\}$$

of groups in C such that

$$A = \text{proj. limit}_{i \in I} A_i; \quad B = \text{proj. limit}_{j \in J} B_j$$

Now consider a projective mapping family:

$$\{A_i \times B_j; \phi_{i', i} \times \psi_{j', j}; (i, j) \leq (i', j'); i, i' \in I, j, j' \in J\}$$

Then

$$A \times B = \text{proj. limit}_{i \in I, j \in J} A_i \times B_j \quad (\text{see Lemma 2.4})$$

For $i \in I, j \in J$, we have the exact sequence by Lemma 3.1:

$$F(X_{ij}) \longrightarrow A_i \times B_j \longrightarrow A_i \times B_j$$

where X_{ij} is a finite set of cardinality $(|A_i| - 1)(|B_j| - 1)$, i.e.

X_{ij} is obtained from $A_i \times B_j$ by removing equivalence class $(1, 1)$.

Applying the projective limit functor which is exact on pro- C groups (see Ribes, L. ⁽¹⁵⁾, Proposition 3.6, p. 35), we have:

$$\text{proj. limit}_{i \in I, j \in J} F(X_{ij}) \longrightarrow \text{proj. limit}_{i \in I, j \in J} (A_i \times B_j) \longrightarrow \text{proj. limit}_{i \in I, j \in J} A_i \times B_j$$

Now,

$$\text{proj. limit}_{i \in I, j \in J} F(X_{ij}) = F(A \times B, *) \quad (\text{Corollary 1.5})$$

$$\text{proj. limit}_{i \in I, j \in J} A_i \times B_j = A \times B \quad (\text{Lemma 2.3})$$

Thus, we have the exact sequence:

$$\tilde{F}(A \cdot B, *) \longrightarrow A \cdot B \longrightarrow A \times B$$

This proves the theorem.

Suppose that a pointed topological space (Z, Z_0) is the co-product in the category of pointed topological spaces, of a pointed topological space (X, Z_0) and a compact, Hausdorff pointed topological space (Y, Z_0) . In Reference (7), Gildenhuys, D. and Lim, C. obtain a split exact sequence of pro-C groups:

$$F(\tilde{F}(X, Z_0) \times Y) \xrightarrow{\alpha} F(Z) \xrightleftharpoons[\gamma]{\beta} \tilde{F}(X, Z_0)$$

where β is induced by

$$\beta' : Z \rightarrow X$$

$$\beta'(Z) = \begin{cases} Z_0 & \text{if } Z \in Y \\ Z & \text{if } Z \in X. \end{cases}$$

The map γ is induced by:

$$X \rightarrow F(Z)$$

$$x \rightarrow [Z_0]^{-1} [x]$$

and α is induced by:

$$\tilde{F}(X, Z_0) \times Y \rightarrow F(Z)$$

$$(a, y) \rightarrow \gamma(a)^{-1} [y] \gamma(a)$$

where the bracket $[]$ denotes the canonical map:

$$\eta_Z : Z \rightarrow F(Z). \quad (\text{for various } Z\text{'s})$$

$$Z \rightarrow [Z]$$

We have a similar type of situation in the general case of free pro-C products.

Lemma 3.3: Let A be a pro-C group and B be a group in class C; then we have the following split exact sequence:

$$\coprod_{b \in B} A^b \xrightarrow{\alpha} A \amalg B \xrightleftharpoons[\gamma]{\beta} B$$

where β is induced by:

$$\begin{array}{ccc} \text{the trivial morphism } A \rightarrow B & \text{and } l_B : B \rightarrow B \\ a \mapsto 1 & & \end{array}$$

$\gamma : B \rightarrow A \amalg B$ is a canonical injection and α is that homomorphism which extends

$$\begin{array}{l} \alpha_b : A^b \rightarrow A \amalg B \\ a^b \mapsto \gamma(b)^{-1} a \gamma(b) \end{array}$$

Notation: Note that B is a finite discrete space.

For every $b \in B$, A^b is an isomorphic copy of A. The distinguished point $*$ is an isolated point of $\bar{B} = \bar{B} \cup \{*\}$ (coproduct topology).

Thus,

$$\coprod_{b \in B} A^b = \coprod_{b \in \bar{B}} A^b$$

(Of course, $A^* = 1$, see remark on p. 9.)

Proof: In Proposition 2.2, let $H=A$, $K=1$. Note 1 is open in B .

Clearly, $X_{H,K} = \emptyset$. Thus, $F(X_{H,K}) = 1$, and we have the exact sequence:

$$\coprod_{b \in B} A^b \xrightarrow{\alpha} A \amalg B \xrightarrow{\beta} B$$

The sequence splits by canonical injection:

$$\gamma : B \rightarrow A \amalg B.$$

Theorem 3.4: Let A, B be pro- C groups, then with notations as in

Lemma 3.3, we have a split exact sequence:

$$\coprod_{b \in B} A^b \xrightarrow{\alpha} A \amalg B \xrightleftharpoons[\gamma]{\beta} B$$

Proof: Let $\{B_j; \psi_{j',j}; j \leq j'; j, j' \in J\}$ be a projective mapping family such that $B = \text{proj. limit } B_j$, where $B_j \in C$.

For every $j \in J$, we have (by Lemma 3.3) a split exact sequence:

$$\coprod_{b_j \in B_j} A^{b_j} \xrightarrow{\alpha_j} A \amalg B_j \xrightleftharpoons[\gamma_j]{\beta_j} B_j$$

For $j, j' \in J$ such that $j \leq j'$, we have a commutative diagram:

$$\begin{array}{ccccc} \coprod_{b_{j'} \in B_{j'}} A^{b_{j'}} & \xrightarrow{\alpha_{j'}} & A \amalg B_{j'} & \xrightleftharpoons[\gamma_{j'}]{\beta_{j'}} & B_{j'} \\ \downarrow \phi_{j',j} & & \downarrow 1_A \amalg \psi_{j',j} & & \downarrow \psi_{j',j} \\ \coprod_{b_j \in B_j} A^{b_j} & \xrightarrow{\alpha_j} & A \amalg B_j & \xrightleftharpoons[\gamma_j]{\beta_j} & B_j \end{array}$$

where $\phi_{j',j}$ is induced by the homomorphisms:

$$A_{j'}^{b_{j'}} \rightarrow A_j^{b_j} \quad (\text{where } b_j = \psi_{j',j}(b_{j'}))$$

$$a_{j'}^{b_{j'}} \rightarrow a_j^{b_j}.$$

Applying the projective limit functor which is exact on pro-C groups (see Ribes, L. ⁽¹⁵⁾, Proposition 3.6, p. 35),

$$\text{proj. limit}_{j \in J} A_{j'}^{b_{j'}} \xrightarrow{\alpha} \text{proj. limit}_{j \in J} A \amalg B_j \xrightleftharpoons[\gamma]{\beta} \text{proj. limit}_{j \in J} B_j$$

i.e.

$$\amalg_{b \in B} A^b \xrightarrow{\alpha} A \amalg B \xrightleftharpoons[\gamma]{\beta} B$$

(Note

$$\amalg_{b \in B} A^b = \text{proj. limit}_{j \in J} \amalg_{b_{j'} \in B_{j'}} A_{j'}^{b_{j'}}$$

by Proposition 1.4).

Lemma 3.5: Let X, Y be two topological spaces, $Z = XUY$ (coproduct in the category of topological spaces). Then

$$F(Z) = F(X) \amalg F(Y).$$

Proof: Consider canonical maps:

$$\alpha : Z \rightarrow F(Z)$$

$$\beta : X \rightarrow F(X)$$

$$\gamma : Y \rightarrow F(Y)$$

$$i : F(X) \rightarrow F(X) \amalg F(Y)$$

$$j : F(Y) \rightarrow F(X) \amalg F(Y)$$

Define:

$$f : Z \rightarrow F(X) \amalg F(Y)$$

$$z \mapsto i\beta(z) \quad z \in X$$

$$j\gamma(z) \quad z \in Y$$

f is continuous since Z has a coproduct topology.

By the universal property of $\alpha : Z \rightarrow F(Z)$, there is a unique continuous homomorphism:

$$\phi : F(Z) \rightarrow F(X) \amalg F(Y)$$

such that $\phi\alpha = f$.

Also continuous maps:

$$X \rightarrow F(Z); Y \rightarrow F(Z)$$

$$x \mapsto \alpha(x); y \mapsto \alpha(y)$$

induce continuous homomorphisms:

$$F(X) \rightarrow F(Z); F(Y) \rightarrow F(Z)$$

These homomorphisms extend to a unique continuous homomorphism:

$$\psi : F(X) \amalg F(Y) \rightarrow F(Z)$$

Claim: ϕ, ψ are inverse to each other.

First, we show that $\psi\phi = 1_{F(Z)}$. Consider $(\psi\phi)\alpha, z \in X$.

$$((\psi\phi)\alpha)(z) = (\psi(\phi\alpha))(z)$$

$$= \psi((\phi\alpha)(z))$$

$$= \psi(f(z))$$

$$= \psi(i_\beta(z))$$

$$= (\psi i)(\beta(z))$$

$$= \alpha(z)$$

Similarly,

$$((\psi\phi)\alpha)(z) = \alpha(z), \quad z \in Y$$

$$\text{i.e. } ((\psi\phi)\alpha)(z) = \alpha(z) \quad z \in Z$$

By uniqueness, $\psi\phi = 1_{F(Z)}$.

Finally,

$$\phi\psi = 1_{F(X)} \amalg 1_{F(Y)}$$

for

$$x \in X, ((\phi\psi)i)\beta(x) = (\phi(\psi i))\beta(x)$$

$$= \phi((\psi i)\beta(x))$$

$$= \phi(\alpha(x))$$

$$= \phi\alpha(x)$$

$$= f(x)$$

$$= i\beta(x)$$

$$= i(\beta(x))$$

Thus, $((\phi\psi)i)\beta = i\beta$.

By uniqueness, $(\phi\psi)i = i$.

Similarly, $(\phi\psi)j = j$.

Thus, $\phi\psi = 1_{F(X) \amalg F(Y)}$.

Hence, the claim; thus: $F(Z) = F(X) \amalg F(Y)$.

Lemma 3.6: Let X be a Hausdorff topological space and Y a compact, Hausdorff, totally disconnected topological space. Then,

$$\coprod_{g \in F(X)} F(Y)^g \cong F(F(X) \times Y)$$

Proof: Let $*$ be any point which does not belong to $F(X) \cup Y$, and $\bar{Y} = Y \cup \{*\}$, $S = F(X) \cup \{*\}$. $\bar{Y}, S$ having coproduct topology, i.e. open sets in $\bar{Y}$ (resp. S) containing $*$, are $U \cup \{*\}$ where U is open in Y (resp. open in $F(X)$).

Now, $F(Y)$ is isomorphic to $F(\bar{Y}, *)$ and $\coprod_{g \in F(X)} F(Y)^g$ is isomorphic to $\coprod_{g \in S} \tilde{F}(\bar{Y}, *)^g$ where $\tilde{F}(\bar{Y}, *)^g$ is an isomorphic copy of $\tilde{F}(\bar{Y}, *)$ for $g \neq *$ and $\tilde{F}(\bar{Y}, *)^g = 1$ for $g = *$. Thus,

$$\coprod_{g \in F(X)} F(Y)^g \cong \coprod_{g \in S} \tilde{F}(\bar{Y}, *)^g \cong \tilde{F}(S \times \bar{Y}, *) \quad (\text{Proposition 1.2})$$

Further, $S \times \bar{Y}$ is homeomorphic to $(F(X) \times Y) \cup \{*\}$ under the map

$$(s, y) \mapsto (s, y), \quad s \neq *, y, \\ * \quad \text{otherwise}.$$

(Note Y is compact).

Thus,

$$\begin{aligned} \coprod_{g \in F(X)} F(Y)^g &\approx \tilde{F}(S \cdot \bar{Y}, *) \\ &\approx \tilde{F}((F(X) \times Y) \cup \{*\}, *) \\ &\approx F(F(X) \times Y) \end{aligned}$$

Corollary 3.7 (Gildenhuys, and Lim⁽⁷⁾).

If X is a Hausdorff topological space, Y is a compact, Hausdorff, totally disconnected topological space, and $Z = X \dot{\cup} Y$ (coproduct). Then, we have a split exact sequence:

$$F(F(X) \times Y) \xrightarrow{\alpha} F(Z) \xrightleftharpoons[\gamma]{\beta} F(X)$$

with canonical maps:

$$\begin{aligned} \alpha_1 &: Z \rightarrow F(Z) \\ \beta_1 &: X \rightarrow F(X) \\ \gamma_1 &: Y \rightarrow F(Y) \end{aligned}$$

and α induced by

$$\alpha[(a, y)] = \gamma(a)^{-1} \alpha_1(y) \gamma(a), \quad (a, y) \in F(X) \times Y$$

β induced by

$$\beta(z) = \begin{cases} 1 & \text{if } z \in Y \\ \beta_1(z) & \text{if } z \in X \end{cases}$$

γ induced by

$$\gamma(x) = \alpha_1(x) \quad \text{for all } x \in X.$$

Proof: In Theorem 3.4, let $A=F(Y)$, $B=F(X)$; then $F(Z)=F(X)\amalg F(Y)=A\amalg B$ by Lemma 3.5.

By Theorem 3.4, we have a split exact sequence:

$$\coprod_{g \in F(X)} F(Y)^g \xrightarrow{\alpha} F(Z) \xrightleftharpoons[\gamma]{\beta} F(X)$$

By Lemma 3.6,

$$\coprod_{g \in F(X)} F(Y)^g \cong F(F(X) \times Y)$$

From Theorem 3.4, we see that:

$$\beta(\alpha_1(z)) = \begin{cases} 1 & \text{if } z \in Y \\ \beta_1(z) & \text{if } z \in X \end{cases}$$

$$\gamma(\beta_1(x)) = \alpha_1(x) \quad \text{for all } x \in X$$

α induced by

$$\alpha[(a,y)] = \gamma(a)^{-1} \alpha_1(y) \gamma(a), \quad (a,y) \in F(X) \times Y.$$

Let (X, z_0) be a pointed topological space and (Y, z_0) be a compact, Hausdorff and totally disconnected pointed topological space. Let (Z, z_0) be the coproduct in the category of pointed topological spaces, i.e., $Z=X \cup Y$ and $X \cap Y = z_0$ and open sets containing z_0 are UV where $z_0 \in U$ and $z_0 \in V$, U open in X , V open in Y .

Lemma 3.8: With the notations as above, we have:

$$F(Z) = \tilde{F}(X, z_0) \amalg F(Y)$$

Proof: Consider canonical maps:

$$\mu : Z \rightarrow F(Z)$$

$$\alpha : X \rightarrow \tilde{F}(X, z_0)$$

$$\beta : Y \rightarrow F(Y)$$

Define: $\sigma : Z \rightarrow G = \tilde{F}(X, z_0) \amalg F(Y)$ as follows:

$$\begin{aligned} \sigma(z) &= \beta(z_0)\alpha(z) && \text{if } z \in X \setminus \{z_0\} \\ &= \beta(z) && \text{if } z \in Y. \end{aligned}$$

The only difficulty is the continuity at z_0 .

Let U be a neighbourhood of $\beta(z_0)$ in G . Let V_1 be a neighbourhood of 1 in G with $z_0 V_1 \subset U$. Let $V = \alpha^{-1}(V_1)$ and $W = \beta^{-1}(U)$. Then V is a neighbourhood of z_0 in X , and W is a neighbourhood of z_0 in Y . Hence, VW is a neighbourhood of z_0 in Z . On the other hand, $\sigma(VW) \subset U$. Therefore, σ is continuous at z_0 .

Now, by the universal property of $\mu : Z \rightarrow F(Z)$, there exists a unique continuous homomorphism $\phi : F(Z) \rightarrow \tilde{F}(X, z_0) \amalg F(Y)$ such that $\phi\mu = \sigma$.

Next, consider $\gamma_1 : \tilde{F}(X, z_0) \rightarrow F(Z)$ induced by the map:

$$X \rightarrow F(Z) \text{ defined by}$$

$$x \mapsto \mu(z_0)^{-1}\mu(x)$$

Let $\gamma_2 : F(Y) \rightarrow F(Z)$ be induced by the map

$Y \rightarrow F(Z)$ defined by

$$y \mapsto \mu(y).$$

Now, let $\gamma : \tilde{F}(X, z_0) \rightarrow F(Y) \rightarrow F(Z)$ be the homomorphism that extends γ_1 and γ_2 .

Claim: γ and ϕ are inverse to each other.

First, $\phi\gamma = 1_{\tilde{F}(X, z_0) \rightarrow F(Y)}$

Let $x \in X, x \neq z_0$.

$$\begin{aligned} \phi\gamma(\alpha(x)) &= \phi\gamma_1(\alpha(x)) \\ &= \phi(\gamma_1\alpha(x)) \\ &= \phi(\mu(z_0)^{-1}\mu(x)) \\ &= (\phi\mu(z_0))^{-1}(\phi\mu(x)) \\ &= \sigma(z_0)^{-1}\sigma(x) \\ &= \beta(z_0)^{-1}\beta(z_0)\alpha(x) \\ &= \alpha(x). \end{aligned}$$

Now, for $y \in Y$,

$$\begin{aligned} \phi\gamma(\beta(y)) &= \phi\gamma_2(\beta(y)) \\ &= \phi(\gamma_2\beta(y)) \\ &= \phi(\mu(y)) \end{aligned}$$

$$= \phi\mu(y)$$

$$= \sigma(y)$$

$$= \beta(y).$$

By uniqueness,

$$\phi\gamma = 1_{\tilde{F}(X, z_0)} \perp\!\!\!\perp F(Y)$$

On the other hand,

$$(\gamma\phi)\mu = \gamma(\phi\mu) = \gamma\sigma$$

Now, if $x \in X$, $x \neq z_0$,

$$\gamma\sigma(x) = \gamma(\beta(z_0)\alpha(x))$$

$$= \gamma\beta(z_0)\gamma\alpha(x)$$

$$= \gamma_2\beta(z_0)\gamma_1\alpha(x)$$

$$= \mu(z_0)\mu(z_0)^{-1}\mu(x)$$

$$= \mu(x).$$

For $y \in Y$,

$$\gamma\sigma(y) = \gamma\beta(y)$$

$$= \gamma_2(\beta(y))$$

$$= \mu(y).$$

Thus, $(\gamma\phi)\mu = \mu$.

Again, by uniqueness, $\gamma\phi = 1_{F(Z)}$.

This proves the claim and we have $F(Z) \approx \tilde{F}(X, z_0) \amalg F(Y)$.

Corollary 3.9: With the notations as above, we have a split exact sequence:

$$F(\tilde{F}(X, z_0) \times Y) \xrightarrow{\alpha_1} F(Z) \xrightleftharpoons[\gamma_1]{\beta_1} \tilde{F}(X, z_0)$$

where β_1 is induced by:

$$\beta' : Z \rightarrow X$$

$$\beta'(z) = \begin{cases} z_0 & \text{if } z \in Y \\ z & \text{if } z \in X. \end{cases}$$

γ_1 is as in Lemma 3.8, and α_1 is induced by:

$$\tilde{F}(X, z_0) \times Y \rightarrow F(Z)$$

$$(a, y) \mapsto \gamma_1(a)^{-1} \mu(y) \gamma_1(a), \quad (a, y) \in \tilde{F}(X, z_0) \times Y$$

Proof: In Theorem 3.4, let $A = F(Y)$, $B = \tilde{F}(X, z_0)$. Now, by Lemma 3.8, $A \amalg B = F(Z)$.

From the split exact sequence of Theorem 3.4, we have:

$$\amalg_{g \in \tilde{F}(X, z_0)} F(Y)^g \xrightarrow{\alpha_1} F(Z) \xrightleftharpoons[\gamma_1]{\beta_1} \tilde{F}(X, z_0)$$

By Lemma 3.6,

$$\amalg_{g \in \tilde{F}(X, z_0)} F(Y)^g \approx F(\tilde{F}(X, z_0) \times Y)$$

Thus,

$$F(\tilde{F}(X, z_0) \times Y) \xrightarrow{\alpha_1} F(Z) \xrightleftharpoons[\gamma_1]{\beta_1} \tilde{F}(X, z_0)$$

with morphisms as explained above.

4. PRO-META-ABELIAN GROUPS

In this section, we turn our attention to a class C which is not closed under extensions. In § 3, we showed that the cartesian subgroup of ALL (free pro- C product) is free pro- C group. This need not be any longer true when C is not extension closed. We assume that C is closed under formations of i) subgroups, ii) homomorphic images and iii) finite direct products. A few interesting examples of such classes are: 1) finite abelian groups, 2) finite nilpotent groups, 3) finite meta-abelian groups and 4) finite supersolvable groups.

In this section, we will show that the cartesian subgroup of free pro-meta-abelian product of two pro-abelian groups is free pro-abelian. In the next chapter, we give some information about pro-supersolvable groups.

Lemma 4.1: Let G be a discrete group such that G/G' is finite abelian group. (G' denotes the derived group of G .) Then, pro-abelian completion of G' is contained in the pro-meta-abelian completion of G .

Proof: Let $\Phi = \{N \trianglelefteq G \mid G/N \text{ is finite meta-abelian group}\}$

$$\Phi \cap G' = \{N \cap G' \mid N \in \Phi\}$$

$$\psi = \{M \cap G' \mid G'/M \text{ is finite abelian group}\}$$

Let us denote pro-abelian completion of G' by G'^{ab} and pro-meta-abelian completion of G by G^{ma} . Thus,

$$G'^{\text{ab}} = \text{proj. limit}_{M \in \psi} G/M$$

$$G^{\text{ma}} = \text{proj. limit}_{N \in \Phi} G/N$$

We wish to show that G'^{ab} is contained in G^{ma} ,

For every $N_1, N_2 \in \Phi$ such that $N_1 \subseteq N_2$, we have a commutative diagram:

$$\begin{array}{ccc} G'/N_1 \cap G' & \xrightarrow{\quad} & G/N_1 \\ \downarrow & & \downarrow \\ G'/N_2 \cap G' & \xrightarrow{\quad} & G/N_2 \end{array}$$

with all natural homomorphisms.

Thus,

$$\text{proj. limit}_{N \in \Phi} G'/N \cap G'$$

is contained in

$$\text{proj. limit}_{N \in \Phi} G/N = G^{\text{ma}}.$$

We claim that

$$G'^{ab} = \text{proj. limit}_{N \in \Phi} G'/N \cap G'$$

In order to prove our claim, it is enough to show that $\Phi \cap G'$ is cofinal subfamily of ψ , i.e. for every $M \in \psi$, there is $N \in \Phi$ such that $N \cap G' \subseteq M$.

First, $\Phi \cap G' \subseteq \psi$, for $N \in \Phi$ and $M = N \cap G'$, $G'/M = G'/N/N = (G/N)'$ since G/N is a finite meta-abelian group, $(G/N)'$ is a finite abelian group. Thus, $M \in \psi$.

Next, let $M \in \psi$ so G'/M is a finite abelian group. Let M_G denote the core of M in G (maximal normal subgroup of G contained in M). Thus,

$$M_G = \bigcap_{i=1}^n M^i$$

where $M^i = x_i^{-1} M x_i$, $\{x_1, \dots, x_n\}$ are coset representatives of M in G , and $n = [G:M]$.

Note each $M^i \trianglelefteq G'$ and G'/M^i is finite abelian group for each i , $i=1, 2, \dots, n$.

Now consider homomorphism $G' \rightarrow \prod_{i=1}^n G'/M^i$ defined by $g \mapsto (gM^i)_{i=1}^n$. Then, the kernel of this homomorphism is M_G . Thus, G'/M_G is isomorphic to a subgroup of $\prod_{i=1}^n G'/M^i$, and hence is a finite abelian group. Thus, G'/M_G is finite meta-abelian group, i.e. $M_G \in \Phi$. Clearly, $M_G = M_G \cap G' \subseteq M$.

This proves the claim, and proof of the lemma is complete.

Proposition 4.2: If G_1, G_2 are finite abelian groups, then we have an exact sequence:

$$F_{Ab}(X_{m,n}) \rightarrow G_1 \underset{ma}{\amalg} G_2 \rightarrow G_1 \times G_2$$

where $F_{Ab}(X_{m,n})$ is a free pro-abelian group on the set $X_{m,n}$ of cardinality $(m-1)(n-1)$, $m=|G_1|$; $n=|G_2|$ and $G_1 \underset{ma}{\amalg} G_2$ denotes the pro-meta-abelian product of G_1 and G_2 .

Proof: Let $G=G_1 * G_2$ be the free product of G_1 and G_2 , G' be the derived group of G . G' is free group on $X_{m,n}$. Denote G' by $F(X_{m,n})=F'$ (say).

Let $\Phi = \{N \trianglelefteq G \mid G/N \text{ is finite meta-abelian group}\}$

For every $N \in \Phi$, we have the exact sequence:

$$F/N \cap F \rightarrow G/N \rightarrow G/NF$$

with obvious natural maps.

Again, by the arguments used several times before, we have the exact sequence of projective mapping families:

$$\{F/N \cap F \mid N \in \Phi\} \twoheadrightarrow \{G/N \mid N \in \Phi\} \twoheadrightarrow \{G/NF \mid N \in \Phi\} \quad (*)$$

Applying the projective limit functor, we have:

$$\text{proj. limit } F/N \cap F \twoheadrightarrow \text{proj. limit } G/N \twoheadrightarrow \text{proj. limit } G/NF$$

Since $F=G'$ and G/F is a finite abelian group, by Lemma 4.1,
 $F_{Ab}(X_{m,n_x}) = \text{proj. limit } F/N_\alpha F$. Clearly, $\text{proj. limit } G/N = G_1 \underset{ma}{\amalg} G_2$
 (Ref. definition of pro-C product) and $\text{proj. limit } G/NF = G/F =$
 $G_1 \times G_2$. Thus, from (*) we have the exact sequence:

$$F_{Ab}(X_{m,n}) \twoheadrightarrow G_1 \underset{ma}{\amalg} G_2 \twoheadrightarrow G_1 \times G_2$$

Let G_1, G_2 be pro-C groups. Let $\{G_{1i} | i \in I\}; \{G_{2j} | j \in J\}$ be
 projective mapping families of groups in C such that

$$G_1 = \text{proj. limit}_{i \in I} G_{1i}; \quad G_2 = \text{proj. limit}_{j \in J} G_{2j}$$

The set $I \times J$ can be partially ordered by defining $(i, j) \leq (i', j')$ iff
 $i \leq i'$ and $j \leq j'$, $i, i' \in I; j, j' \in J$.

Now consider the homomorphisms:

$$\phi_{i', i} : G_{1i'} \rightarrow G_{1i}, \quad \psi_{j', j} : G_{2j'} \rightarrow G_{2j}$$

These homomorphisms induce a continuous homomorphism:

$$\phi_{i', i} \amalg \psi_{j', j} : G_{1i'} \amalg G_{2j'} \rightarrow G_{1i} \amalg G_{2j}$$

With these homomorphisms, the family of pro-C groups
 $\{G_{1i} \amalg G_{2j} | i \in I, j \in J\}$ is a projective mapping family.

Lemma 4.3: With the notations as above,

$$G_1 \amalg G_2 = \text{proj. limit}_{i \in I, j \in J} (G_{1i} \amalg G_{2j})$$

Proof: Since

$$G_1 = \text{proj. limit}_{i \in I} G_{1i}$$

and

$$G_2 = \text{proj. limit}_{j \in J} G_{2j}$$

there are continuous homomorphisms:

$$\phi_i : G_1 \rightarrow G_{1i}, \psi_j : G_2 \rightarrow G_{2j}$$

for every $i \in I, j \in J$.

$$\text{Let } U_i = \ker \phi_i, V_j = \ker \psi_j. \text{ Then, } G_{1i} = G_1/U_i, G_{2j} = G_2/V_j$$

$$\text{and } G_1 = \text{proj. limit}_{i \in I} G_1/U_i; G_2 = \text{proj. limit}_{j \in J} G_2/V_j.$$

Now, using the proof of Lemma 2.4 (Note: we never used the condition C extension closed in the proof), we have:

$$\begin{aligned} G_1 \amalg G_2 &= \text{proj. limit}_{i \in I} G_1/U_i \amalg G_2/V_j \\ &= \text{proj. limit}_{i \in I} G_{1i} \amalg G_{2j}. \end{aligned}$$

Now, we are in a position to prove the main theorem of the section.

Theorem 4.4: Let G_1, G_2 be pro-abelian groups. Then, we have an exact sequence:

$$F_{\text{Ab}}(G_1 \circ G_2, *) \rightarrow G_1 \amalg_{\text{ma}} G_2 \rightarrow G_1 \times G_2$$

Proof: Let $G_1 = \text{proj. limit}_{i \in I} G_{1i}$; $G_2 = \text{proj. limit}_{j \in J} G_{2j}$

where $\{G_{1i} | i \in I\}$ and $\{G_{2j} | j \in J\}$ are the families of finite abelian groups.

By Proposition 4.2, we have the exact sequences:

$$F_{\text{Ab}}(X_{m_i, n_j}) \xrightarrow{\alpha_{ij}} G_{1i} \underset{\text{ma}}{\amalg} G_{2j} \longrightarrow G_{1i} \times G_{2j}$$

where $m_i = |G_{1i}|$ and $n_j = |G_{2j}|$, $i \in I$, $j \in J$ and α_{ij} is induced by the map:

$$X_{m_i, n_j} \rightarrow G_{1i} \underset{\text{ma}}{\amalg} G_{2j}$$

defined by

$$[x, y] \rightarrow x^{-1} y^{-1} xy$$

where $l \neq x \in G_{1i}$, $l \neq y \in G_{2j}$.

Now, for $i, i' \in I$, $j, j' \in J$ such that $i \leq i'$, $j \leq j'$, consider

the diagram:

$$\begin{array}{ccccc} F_{\text{Ab}}(X_{m_{i'}, n_{j'}}) & \xrightarrow{\alpha_{i'j'}} & G_{1i'} \underset{\text{ma}}{\amalg} G_{2j'} & \longrightarrow & G_{1i'} \times G_{2j'} \\ \downarrow \alpha & & \downarrow \phi_{i', i} \psi_{j', j} & & \downarrow \phi_{i', i} \times \psi_{j', j} \\ F_{\text{Ab}}(X_{m_i, n_j}) & \xrightarrow{\alpha_{ij}} & G_{1i} \underset{\text{ma}}{\amalg} G_{2j} & \longrightarrow & G_{1i} \times G_{2j} \end{array} \quad (*)$$

where α is induced by

$$[x, y] \longrightarrow [\phi_{i', i}(x), \psi_{j', j}(y)]$$

$1 \neq x \in G_{1i}$, $1 \neq y \in G_{2j}$ are such that $x \in \ker \phi_{i,i}$, $y \in \ker \psi_{j,j}$; otherwise,

$$[x, y] \longrightarrow 1 \text{ in } F_{Ab}(X_{m_i, n_j})$$

We check the commutativity of (*).

For commutativity of the left square in (*), it is enough to check it on elements of X_{m_i, n_j} , which is the generating set of $F_{Ab}(X_{m_i, n_j})$.

Let $1 \neq x \in G_{1i}$, and $1 \neq y \in G_{2j}$.

$$\alpha_{i,j}([x, y]) = x^{-1} y^{-1} xy$$

If $x \in \ker \phi_{i,i}$ or $y \in \ker \psi_{j,j}$, then

$$\phi_{i,i} \downarrow \psi_{j,j} (\alpha_{i,j}([x, y])) = 1 = \alpha_{ij}(\overline{[x, y]})$$

and the square commutes in this case. So, assume $x \notin \ker \phi_{i,i}$, $y \notin \ker \psi_{j,j}$. Now,

$$\phi_{i,i} \downarrow \psi_{j,j} (\alpha_{i,j}([x, y])) = \phi_{i,i}(x)^{-1} \psi_{j,j}(y)^{-1} \phi_{i,i}(x) \psi_{j,j}(y)$$

On the other hand,

$$\begin{aligned} \alpha_{ij}(\overline{[x, y]}) &= \alpha_{ij}(\overline{[\phi_{i,i}(x), \psi_{j,j}(y)]}) \\ &= \phi_{i,i}(x)^{-1} \psi_{j,j}(y)^{-1} \phi_{i,i}(x) \psi_{j,j}(y) \end{aligned}$$

Thus, in this case also, the left square in (*) commutes.

Again, to check commutativity of the right square in (*), we do it on elements of G_{1i} and G_{2j} , which are (topological) generators of $G_{1i} \amalg G_{2j}$.

Thus, for $x \in G_{1i}$, x is mapped onto (x, e) in $G_{1i} \times G_{2j}$, and $\phi_{i,i} \times \psi_{j,j}$ maps (x, e) to $(\phi_{i,i}(x), e)$. On the other hand, $\phi_{i,i} \amalg \psi_{j,j}$ maps x to $\phi_{i,i}(x)$ and this is mapped to $(\phi_{i,i}(x), e)$ in $G_{1i} \times G_{2j}$.

Thus, the right square in (*) commutes on elements of G_{1i} . Similarly, it commutes on elements of G_{2j} .

Thus, for $i, i' \in I, j, j' \in J$ such that $i \leq i', j \leq j'$, the diagram (*) commutes.

Applying the inverse limit functor which is exact on pro-C groups (Ribes, L. (15), Proposition 3.6, p. 35), we have:

$$\text{proj. limit}_{i \in I, j \in J} F_{\text{Ab}}(X_{m_i n_j}) \rightarrow \text{proj. limit}_{i \in I, j \in J} G_{1i} \amalg_{\text{ma}} G_{2j} \rightarrow$$

$$\text{proj. limit}_{i \in I, j \in J} G_{1i} \times G_{2j}$$

$$\text{proj. limit}_{i \in I, j \in J} F_{\text{Ab}}(X_{m_i n_j}) = \tilde{F}(G_1 \times G_2, *) \quad (\text{Corollary 1.5})$$

$$\text{proj. limit}_{i \in I, j \in J} G_{1i} \amalg_{\text{ma}} G_{2j} = G_1 \amalg_{\text{ma}} G_2 \quad (\text{Lemma 4.3})$$

$$\text{proj. limit}_{i \in I, j \in J} G_{1i} \times G_{2j} = G_1 \times G_2 \quad (\text{Lemma 2.3})$$

Thus, we have:

$$\tilde{F}(G_1 * G_2, *) \rightarrow G_1 \underset{ma}{\downarrow} G_2 \rightarrow G_1 \times G_2.$$

Corollary 4.5: If $G_1 = G_2 = \hat{Z}$ free pro-abelian group on one generator,

then we have an exact sequence:

$$\tilde{F}_{Ab}(\hat{Z} * \hat{Z}, *) \rightarrow G \rightarrow \hat{Z} * \hat{Z}$$

i.e. if G is a free pro-meta-abelian group on two generators,

$$G' = \tilde{F}_{Ab}(\hat{Z} * \hat{Z}, *).$$

CHAPTER III

PRO-SUPERSOLVABLE GROUPS

1. PRELIMINARIES

We collect here a few definitions and results, some of them well-known, for future use.

Definition 1.1: Let G be a pro-finite group. The Frattini subgroup G^* of G is defined to be the intersection of all (proper) maximal closed subgroups of G .

Lemma 1.2: Let G be a pro-finite group. Every (proper) maximal closed subgroup M of G is open in G .

Proof: Suppose M is not open. Then, for every open normal subgroup U of G , $MU = G$. For, if $MU \neq G$, then $MU = M$ by maximality of M , but then M is open.

Now, $MU = G$ for every open normal subgroup U of G implies that M is dense in G . Since M is closed, $M = G$. This contradicts that M is proper. Thus, M is open.

Definition: If S is a subset of G , we shall denote the smallest closed subgroup of G containing S by $G_p\{S\}$. We define an element x to be a non-generator of G if $G \neq G_p\{S; x\}$ always implies $G = G_p\{S\}$.

Lemma 1.3 (Gruenberg): G^* is the set of all non-generators of G .

Proof: If $x \in G^*$, there exists a maximal subgroup M not containing x . If U is the core of M (maximal closed normal subgroup of G contained in M) in G , then $xU \notin M/U$ and $xU \notin (G/U)^*$. But, then xU fails to be a non-generator of a finite group G/U and so there is a set containing U such that $G/U = \text{Gp}\{S/U, xU\}$ but $G/U \neq \text{Gp}\{S/U\}$. Hence, $G = \text{Gp}\{S, x\}$ but $G \neq \text{Gp}\{S\}$. It follows that x fails to be a non-generator of G . Therefore, G^* contains all non-generators of G . Conversely, let $x \in G^*$; then we show that $G = \text{Gp}\{S, x\}$ implies $G = \text{Gp}\{S\}$.

For every open normal subgroup U of G , we have $G/U = \text{Gp}\{SU/U, xU\}$, $xU \notin (G/U)^*$. Thus, $G/U = \text{Gp}\{SU/U\}$, i.e. $G = \text{Gp}\{S, U\} = \text{Gp}\{S\}U$.

It follows that $\text{Gp}\{S\}$ is dense in G . Hence equals G .

Corollary 1.4: If $HG^* = G$, then $H = G$, where H is a closed subgroup of G .

Corollary 1.5: If $\theta : G_1 \rightarrow G_2$ then $\theta(G_1^*) \subseteq G_2^*$. In particular, G_1^* is the characteristic subgroup of G_1 .

Corollary 1.6: Let ψ be the morphism: $G_1 \rightarrow G_2$. Then, ψ is surjective if, and only if, the induced morphism: $G_1/G_1^* \rightarrow G_2/G_2^*$ is surjective.

Corollary 1.7: Let $G = \text{proj. limit } G_\tau$, G pro-finite group.

Then $G^* = \text{proj. limit } G_\tau^*$.

Proof: We have a commutative diagram:

$$\begin{array}{ccccc} G^* & \longrightarrow & G & \twoheadrightarrow & G/G^* \\ \downarrow & & \downarrow & & \downarrow \\ G_\tau^* & \longrightarrow & G_\tau & \twoheadrightarrow & G_\tau/G_\tau^* \end{array}$$

Applying the projective limit functor to the bottom row, we have a commutative diagram:

$$\begin{array}{ccccc} G^* & \longrightarrow & G & \twoheadrightarrow & G/G^* \\ \downarrow & & \downarrow 1_G & & \downarrow \\ \text{proj. limit } G_\tau^* & \longrightarrow & \text{proj. limit } G_\tau & \longrightarrow & \text{proj. limit } G_\tau/G_\tau^* \end{array}$$

Thus,

$$G^* \subseteq \text{proj. limit } G_\tau^*.$$

Next, we claim that $\text{proj. limit } G_\tau^* \subseteq G^*$.

Let $x = (x_\tau) \in \text{proj. limit } G_\tau^*$ and $x \notin G^*$ then there exists maximal open subgroup M of G such that $x \notin M$. Now there exists τ such that M_τ is a maximal subgroup of G_τ such that $x_\tau \notin M_\tau$.

But $x \in \text{proj. limit } G_\tau^*$, i.e. $x_\tau \in G_\tau^* \subseteq M_\tau$ contradiction.

This contradiction proves the claim and thus $G^* = \text{proj. limit } G_\tau^*$.

Corollary 1.8: G^* is pro-nilpotent, i.e. projective limit of finite nilpotent groups.

Lemma 1.9: Let G be a pro-finite group, H a closed normal subgroup of G . Then,

$$H^* \subseteq G^*.$$

Proof: Let $G = \text{proj. limit } G/U$, U runs over open normal subgroups of G . Then,

$$G^* = \text{proj. limit } (G/U)^*.$$

Since HU/U is normal subgroup of G/U ,

$$(HU/U)^* \subseteq (G/U)^* \quad (\text{see Reference (17)})$$

Hence,

$$\begin{aligned} H^* &= \text{proj. limit } (HU/U)^* \subseteq \text{proj. limit } (G/U)^* \\ &= G^*. \end{aligned}$$

Theorem 1.10: (Brauer, W. (5))

$$\text{Let } G = \prod_{\alpha \in I} G_{\alpha}$$

where $\{G_{\alpha} | \alpha \in I\}$ is a family of pro-finite groups. Then,

$$G^* = \prod_{\alpha \in I} G_{\alpha}^*.$$

Proof: For every $\alpha \in I$, G_{α} is closed normal subgroup of G . Thus,

by Lemma 1.9, $G_{\alpha}^* \subseteq G^*$ for every α , $\alpha \in I$. Therefore,

$$\prod_{\alpha \in I} G_{\alpha}^* \subseteq G^*$$

For the reverse inclusion, consider M_α maximal (open) subgroup of G_α and put $G^\alpha = \prod_{\beta \neq \alpha} G_\beta$. Then, $M_\alpha G^\alpha$ is a maximal open subgroup of G .

Hence, $G^* \subseteq M_\alpha G^\alpha$ for every maximal open subgroup M_α of G_α .

Therefore, $G^* \subseteq G_\alpha^* G^\alpha$ for every α , $\alpha \in I$, i.e.

$$G^* \subseteq \bigcap_{\alpha \in I} G_\alpha^* G^\alpha = \prod_{\alpha \in I} G_\alpha^*$$

Thus, $G^* = \prod_{\alpha \in I} G_\alpha^*$

Theorem 1.11: Let G be a pro-nilpotent group, G_p denote the Sylow- p subgroup of G . (Note G_p is unique.) Clearly, $G = \prod_p G_p$. Then, $G^* = \prod_{p \in P} G_p^*$ where P denotes the set of primes dividing the order G .

Theorem 1.12: (Reference (20), p. 70)

Let G be a pro- P group (p -prime). Then,

- i) G/G^* is Pontrajagin dual of $H^1(G, \mathbb{Z}/p\mathbb{Z})$
- ii) $G^* = G^P[G, G]$
- iii) G is (topologically) finitely generated if and only if G^* is open in G .

2. PRO-SUPERSOLVABLE GROUPS

In this section and the following sections, we study pro-supersolvable groups. Since the class of all finite supersolvable groups is not closed under extensions (for example, the alternating group A_4 which is the extension of the Klein 4 group by the cyclic group of order 3 is not supersolvable), the results about pro-supersolvable groups are different from those of pro-C groups where class C is closed under extensions.

Definition 2.1: A discrete group G is said to be supersolvable if G contains an invariant series of finite length having the property that each factor group of the series is cyclic, i.e. there exists series $G = G_0 \supset G_1 \supset \dots \supset G_n = e$ with $G_i \triangleleft G$ and G_i/G_{i+1} is a cyclic group, for each i , $0 \leq i \leq n-1$.

Definition 2.2: A pro-finite group G is said to be pro-supersolvable if G is projective limit of finite supersolvable groups.

We have an analogue of a well-known result of Huppert.

Theorem 2.3: A pro-finite group G is pro-supersolvable if and only if every maximal (open) subgroup of G has a prime index in G .

Proof: Let G be a pro-supersolvable group and U be a maximal open subgroup of G . Let C be the core of U in G (i.e. the closed maximal normal subgroup of G contained in U). Then, G/C is a

finite supersolvable group. Thus, by the theorem in Reference (16) (Theorem VII.7.c, p. 236), U/C has a prime index in G/C . Hence, $(G:U) = (G/C:U/C)$ is prime. Conversely, let G be a pro-finite group in which every maximal open subgroup has a prime index. It is enough to show that for every open normal subgroup U of G , G/U is finite supersolvable group.

Let $\bar{G} = G/U$ and $\bar{M}$ be any maximal subgroup of $\bar{G}$. Now, there is a subgroup M of G such that $U \subset M$ and $M/U = \bar{M}$. Clearly, M is an open subgroup of G . Moreover, M is maximal open subgroup of G . Thus, by our assumption $(G:M)$ is prime, but then $(\bar{G}:\bar{M}) = (G:M)$ is prime too.

Thus, every maximal subgroup of $\bar{G}$ has a prime index in $\bar{G}$; hence, $\bar{G}$ is supersolvable by the theorem in Reference (16).

Theorem 2.4: Let G be a pro-supersolvable group and G^* the Frattini subgroup of G . Then, G/G^* is meta-abelian.

Proof: Let $G = \text{proj. limit}_{t \in I} G_t$,

where G_t , $t \in I$ are finite supersolvable groups. Then,

$$G/G^* = \text{proj. limit}_{t \in I} G_t/G_t^*$$

Now, G_t/G_t^* is meta-abelian (G finite supersolvable group $\Rightarrow G/G^*$ is meta-abelian).

It is enough to show that $G'' \subseteq G^*$. Since G is supersolvable, G' is nilpotent. Now G' nilpotent $\Rightarrow G'' \subseteq (G')^*$ but $(G')^* \subseteq G^*$ as $G' \triangleleft G$. Thus, $G'' \subseteq G^*$.

Thus, G/G^* is pro-meta-abelian.

Now the result follows from

Lemma 2.5: G pro-meta-abelian $\Rightarrow G$ meta-abelian.

Proof: Let $G' = \overline{[G, G]} = \text{proj. limit } G'U/U$ where U runs over the open normal subgroups of G . Now, $G'U/U = (G/U)'$. On the other hand, $(G/U)'$ is abelian as G/U is meta-abelian.

Thus, G' is abelian and G is meta-abelian.

3. SYLOW- π SUBGROUPS

Definition 3.1: Let π be a set of primes and δ a supernatural number (see Ribes, L. ⁽¹⁵⁾, p. 37). We say:

δ and π are coprime when $p \in \pi \rightarrow p \nmid \delta$
(i.e. p does not divide δ)

$\pi \mid \delta$ (π divides δ) when $p \in \pi \rightarrow p \mid \delta$.

$\delta \mid \pi$ when $p \mid \delta \rightarrow p \in \pi$.

Let G be a pro-finite group. G is a π -group if $|G| \mid \pi$.

A closed subgroup S is called a Sylow- π subgroup of G if π and $(G:S)$ are coprime, and S is a π -group (sometimes this is called in the literature a Hall- π -subgroup (see Schenkman ⁽¹⁶⁾, p. 129) Hall- π -subgroup and Sylow- π -subgroup are the concepts which coincide for finite solvable groups. (See Schenkman ⁽¹⁶⁾, VII, 3.d., p. 226)).

Corollary 3.2: (E.D. Bolker ⁽⁴⁾) A closed subgroup S is a Sylow- π subgroup of G for some π if and only if its order and index are coprime.

Corollary 3.3: (E.D. Bolker ⁽⁴⁾) One Sylow- π subgroup cannot strictly contain another.

Theorem 3.4: (E.D. Bolker⁽⁴⁾) Let G be a pro-solvable group. For each π dividing $|G|$ (order G), G has Sylow- π -subgroups, any two are conjugate and any π -subgroup of G is contained in one of them.

Notations: P denotes the set of all primes.

$$S_p = \{q \in P \mid q \leq p\} p \in P$$

$$\pi_p = P - S_p$$

Theorem 3.5: Let G be a pro-supersolvable group. Then G has a unique Sylow- π_p -subgroup K_p for every $p \in P$.

Proof: First, note that each Sylow- π subgroup S may be written as the proj. limit of groups $\{S_N\}$ where S_N is a Sylow- π subgroup of G/N and ϕ_N^M maps S_M onto S_N . (N runs over the open normal subgroups of G).

Conversely, if $\{S_N\}$ is a family of subgroups of G/N with these properties then S the projective limit of $\{S_N\}$ is a Sylow- π subgroup of G (cf. E.D. Bolker⁽⁴⁾).

Now, let G be a pro-supersolvable group and N an open normal subgroup of G . By Theorem in Reference (16) (VII.5.9, p. 232) G/N has a unique Sylow- π_p subgroup S_N for every $p \in P$.

Then, S , the projective limit of S_N , is the unique Sylow- π_p subgroup of G , for every $p \in P$.

Let $\underline{S}_p$ denote the class of all finite supersolvable groups G such that $|G| \mid S_p$.

Theorem 3.6: Let $F(X)$ be the free pro-supersolvable group on a finite set X , $|X| > 1$. For $p \in P$, let K_p be its unique Sylow- π_p subgroup. Then, $F(X)/K_p$ is a free pro- $\underline{S}_p$ group on X .

Proof: Consider $X \xrightarrow{\phi} F(X) \xrightarrow{j_p} H_p$ where $H_p = \text{coker}(K_p \xrightarrow{i_p} F(X))$ put $\phi_p = j_p \phi$.

We show that (H_p, ϕ_p) has the required universal property in the category of pro- $\underline{S}_p$ groups.

Now, let G be any pro- $\underline{S}_p$ group and $\sigma : X \rightarrow G$ be a (continuous) map. Since G is also a pro-supersolvable group, there exists unique ψ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & F(X) \\ & \searrow \sigma & \downarrow \psi \\ & & G \end{array}$$

commutes, i.e. $\psi\phi = \sigma$.

For any prime $q \in \pi_p$ let F_q be a Sylow- q subgroup of $F(X)$. Then $\psi(F_q)$ is contained in one of the Sylow- q subgroups of G . Since $q \in \pi_p$, q does not divide $|G|$; thus, $\psi(F_q) = 1$ for every $q \in \pi_p$.

Since K_p is generated by F_q 's, $q \in \pi_p$ and $\psi(F_q) = 1$, $q \in \pi_p$, we have $\psi(K_p) = 1$.

Thus ψ induces a homomorphism:

$$\bar{\psi} : H_p \rightarrow G \text{ such that}$$

$$\bar{\psi}\phi_p = \bar{\psi}(j_p\phi) = (\bar{\psi}j_p)\phi = \psi\phi = \sigma.$$

Claim: $\bar{\psi}$ is unique such that

$$\bar{\psi}\phi_p = \sigma.$$

If $\psi_1 : H_p \rightarrow G$ is another continuous homomorphism such that

$\psi_1\phi_p = \sigma$, then

$$\psi_1\phi_p = \psi_1(j_p\phi) = (\psi_1j_p)\phi = \sigma = \psi\phi$$

By uniqueness of $\bar{\psi}$, $\psi = \psi_1j_p$. Thus,

$$\psi_1j_p = \psi = \bar{\psi}j_p.$$

Since j_p is an epimorphism, $\psi_1 = \bar{\psi}$.

This proves the claim and completes the proof of the theorem.

4. THE FRATTINI SUBGROUP OF A PRO-SUPERSOLVABLE GROUP

We begin this section by recalling some results of Gaschutz about the Frattini subgroup of a finite group.

Definition 4.1: A complement of a normal subgroup N in a group G is a subgroup of $A \subseteq G$ such that $G=NA$ and $N \cap A=1$. If a normal subgroup N has a complement in a group G , then we say that G splits over N .

Clearly, if G splits over N , then G is a semidirect product of N by A and a splitting extension can be formed.

Theorem 4.2: (Gaschutz)

Consider a subgroup B in a group G such that B contains an abelian subgroup N , normal in G and having $((G:B), |N|)=1$.

If B splits over N , then G splits over N . (See Reference (2), p. 38).

Proposition 4.3: Let G be a finite group. Let p be a prime dividing the order of G . Further, suppose that P , the Sylow- p subgroup, is normal in G . Then, $P \cap G^* = P^*$.

Proof: Since $P \trianglelefteq G$, $P^* \subseteq G^*$ (cf. (17)). Thus, $P^* \subseteq P \cap G^*$.

Conversely, we need $P \cap G^* \subseteq P^*$.

Since $P^* \trianglelefteq G$, for P^* is characteristic subgroup of P and P normal in G , in order to prove that $P \cap G^* \subseteq P^*$, it is enough

to prove that $P^* = 1 \Rightarrow P \cap G^* = 1$.

Otherwise, we work with $\bar{G} = G/P^*$, Sylow-p subgroup $\bar{P}$ of $\bar{G}$ is P/P^* , and $\bar{P}^* = 1$. This would imply $P \cap G^* = 1$ in $\bar{G}$, i.e.

$$P/P^* \cap G^*/P^* = P^* \Rightarrow P \cap G^* \subseteq P^*.$$

Next, suppose $P^* = 1$ and $N = P \cap G^*$. Now P splits over N .

(Note P is elementary abelian p-group.)

By Theorem 4.3, G splits over N .

Since G splits over N , we have $G = NH$ for some H , $H \cap N = 1$.

If $H \neq G$, let M be a maximal subgroup of G with $H \subseteq M$. Then,

$G = NH = NM$. But, $N \subseteq G^* \subseteq M$. So, $G = NM = M \neq G$. Therefore, $H = G$, i.e.

$N = 1$.

Corollary 4.4: Let G be a finite supersolvable group. Let p be the largest prime dividing the order G . Let P be the Sylow-p subgroup of G . (Note there is a unique Sylow-p subgroup by Theorem 3.6.).

Then, $P \cap G^* = P^*$.

Theorem 4.5: Let G be a pro-supersolvable group. Suppose there is a largest prime p dividing the order G ; then, $P \cap G^* = P^*$ where P is the unique Sylow-p subgroup of G .

Proof: Let Φ be the family of all open normal subgroups of G .

Then, $G = \varprojlim_{U \in \Phi} G/U$.

$$P = \text{proj. limit}_{U \in \Phi} PU/U$$

$$\text{Then, } G^* = \text{proj. limit}_{U \in \Phi} (G/U)^* \quad (\text{cf. Lemma 1.7})$$

$$\text{and } P^* = \text{proj. limit}_{U \in \Phi} (PU/U)^*$$

$$\begin{aligned} \text{Now, } P \cap G^* &= [\text{proj. limit}_{U \in \Phi} (PU/U)] \cap [\text{proj. limit}_{U \in \Phi} (G/U)^*] \\ &= \text{proj. limit}_{U \in \Phi} (PU/U \cap (G/U)^*) \\ &= \text{proj. limit}_{U \in \Phi} (PU)^* \quad (\text{Corollary 4.4}) \\ &= P^*. \end{aligned}$$

Definition 4.6: A topological group G is said to be (topologically) generated by $\{x_\alpha\}_{\alpha \in A}$ if $\{x_\alpha\}_{\alpha \in A}$ generates a dense subgroup of G . If $|A|$ is finite, then G is said to be (topologically) finitely generated.

Lemma 4.7: Let G be a (topologically) finitely generated pro-finite group. Then, for every natural number n , there are only finitely many open subgroups of index n .

Proof: Let n be a natural number. Now there exist only finitely many non-isomorphic groups of order n , and finitely many homomorphisms into each of those groups, since G is finitely generated. Thus, there are only finitely many open normal subgroups of G of index n .

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Suppose there is an infinite family $\{U_\alpha\}_{\alpha \in A}$ of open subgroups of G of index n . Now, every U_α contains an open normal subgroup of index less than or equal to $k=n$ (viz. the core of U_α in G). Since there are only finitely many open normal subgroups of index less than or equal to k , there is one open normal subgroup C (of index less than or equal to k) contained in infinitely many U_α 's, $\alpha \in A$.

But then this means that the finite group G/C has infinitely many distinct subgroups U_α/C , $\alpha \in A$. This is impossible. Thus, there are only finitely many open subgroups of index n .

Theorem 4.8: Let G be a (topologically) finitely generated pro-supersolvable group such that $|G|$ is divisible by only finitely many primes. Then, G^* is open in G .

Proof: Let $\{p_1 \dots p_r\}$ be the precise set of primes which divide the order G .

Let n_{p_i} be the number of open subgroups of G of index p_i , $1 \leq i \leq r$.

For each i , $n_{p_i} < \infty$ by Lemma 4.7.

Set $n = \sum_{i=1}^r n_{p_i}$.

Since every maximal open subgroup of G has a prime index (cf. Theorem 2,3), G has precisely n maximal open subgroups. Thus, G^* being the finite intersection of open subgroups of G , is itself open in G .

5. FREE PRO-SUPERSOLVABLE GROUPS

Let $S=S_p$ be as before (as in § 3) and $F_S(X)$ denote a free pro- S group on a set X . When the class C of finite groups is closed under extensions and C is not contained in the class of finite p -groups (p -prime), Gildenhuys and Lim⁽⁷⁾ show that Sylow- p subgroups of a free pro- C group on a finite set X ($|X|>1$) are not finitely generated. (See also J. Smith⁽²¹⁾.)

The corresponding statement is not true for free nilpotent groups, as Sylow- p subgroups of free pro-nilpotent groups on X are free pro- p on the same set X (see Gruenberg⁽¹¹⁾ or Gildenhuys and Lim⁽⁷⁾). In this section, we show that Sylow- p subgroups of a free pro-supersolvable group on a finite set X , are again finitely generated. First, we have the following theorem.

Theorem 5.1: For any prime $q \leq p$, a Sylow- q subgroup of $F_S(X)$ is isomorphic to a Sylow- q subgroup of $F(X)$ - the free pro-supersolvable group on X .

Proof: Consider the exact sequence:

$$\begin{array}{ccccc} & i_p & & j_p & \\ & \downarrow & & \downarrow & \\ K_p & \xrightarrow{\quad} & F(X) & \xrightarrow{\quad} & F_S(X) \end{array}$$

Let T be a Sylow- q subgroup of $F(X)$ where q is a prime; $q \leq p$.

Since K_p is the Sylow- p subgroup of $F(X)$, $T \cap K_p = 1$. Thus, j_p restricted to T is an isomorphism of T onto Sylow- q subgroup of $F_S(X)$.

Theorem 5.2: Let P be the Sylow- p subgroup of $F_S(X)$, (Note, $S = \underline{S}_p$, thus P is unique.) where X is a finite set. Then, P is a finitely generated free pro- p group.

Proof: By Theorem 5.1, P is isomorphic to a Sylow- p subgroup of $F(X)$ - the free pro-supersolvable group on X . Since the class of all finite supersolvable groups is saturated, a Sylow- p subgroup of $F(X)$ is free pro- p . (cf. Gruenberg⁽¹¹⁾, p. 164).

Thus P is free pro- p . It remains to show that P is finitely-generated.

Now, F_S^* is open in F_S (where $F_S = F_S(X)$) by Theorem 4.7 and $P^* = P \cap F_S^*$. Thus, P^* is open in P .

Hence, P is finitely generated. (cf. Theorem 1.12).

Theorem 5.3: Let $F(X)$ be a free pro-supersolvable group on a finite set X . Then, for every prime p , each Sylow- p subgroup of $F(X)$ is a finitely generated free pro- p group.

Proof: Let $\underline{S}_p$ be the class as before, F_p be a Sylow- p subgroup of $F(X)$. By Theorem 5.1, F_p is isomorphic to the Sylow- p subgroup of $F_{\underline{S}_p}(X)$. Since the Sylow- p subgroup of $F_{\underline{S}_p}(X)$ is

finitely generated free pro- p (Theorem 5.2), F_p is a finitely generated free pro- p .

Corollary 5.4: If G is a finitely generated pro-supersolvable group, then so are its Sylow- p subgroups.

Definition 5.5: A pro-finite group G is of type F if for every natural number n , G has finitely many open subgroups of index n .

As we have seen (Lemma 4.8), any finitely generated pro-finite group is of type F ; the next theorem shows for pro-supersolvable groups where the order of the group involves finitely many primes we have the converse.

Theorem 5.6: Let G be a pro-supersolvable group such that the order $|G|$ is divisible by only finitely many primes. Then G is finitely generated if and only if G is of type F .

Proof: Let G be a pro-supersolvable group in the theorem. It is enough to show that if G is of type F then G is finitely generated.

Let $\{p_1 \dots p_r\}$ be the precise set of primes that divide the order $|G|$. Note that every maximal open subgroup of G has a prime index. Thus, the index of a maximal open subgroup is one of the primes, $p_1 \dots p_r$. Since G is of type F , there are only finitely many open subgroups of index p_i , $1 \leq i \leq r$. Thus there are

only finitely many maximal open subgroups of G , i.e. G^* is open in G . Thus, G/G^* is a finite group. Let $x_1, \dots, x_n$ be the coset representatives of G^* in G . Now, $Gp\{x_1 \dots x_n\}$ is G , i.e. G is finitely generated.

Even in pro-abelian cases, the two conditions being of type F and finitely generated are not equivalent if the number of primes dividing the order is not finite.

Example: Consider

$$G = \mathbb{Z}/2 \times (\mathbb{Z}/3 \times \mathbb{Z}/3) \times (\mathbb{Z}/5 \times \mathbb{Z}/5 \times \mathbb{Z}/5) \times \dots$$

where for p_r th prime there are r copies of $\mathbb{Z}/p_r$.

Clearly, G is not finitely generated (note $G^*=1$) but G is of type F. For, if k is any natural number, let p be the largest prime dividing k . Let U be any open subgroup of G of index k .

Now for $q > p$, G_q - a Sylow- q subgroup of G is contained in U .

$$\text{Thus, } H_p = \bigcap_{q > p} G_q \subseteq U.$$

Now, G/H_p is a finite group and hence the number of subgroups of G/H_p is finite. But then the number of open subgroups of G of index k is finite, i.e. G is of type F.

6. LARGE SUBGROUPS OF PRO-FINITE GROUPS

Definition 6.1: Let G be a pro-finite group. A subgroup H (not necessarily closed) is called large if H has a finite index in G .

Clearly, every open subgroup of G is large (G -pro-finite group). The problem we consider in this section is "Is every large subgroup of a pro-finite group necessarily open?" It is well-known that when G is infinitely generated, G has large non-open subgroups (see H.L. Peterson⁽¹⁴⁾). When G is finitely generated, the answer is not known in the general case. In this section, we provide some partial answers to the above problem.

Theorem 6.2: Let G be a finitely generated pro-abelian group. Then every large subgroup of G is open.

Proof: Let H be a large subgroup of G having index n . Let $G^n = \{x^n \mid x \in G\}$.

G^n is a closed subgroup of G since it is the image of G under the continuous homomorphism: $G \rightarrow G$ sending $x \rightarrow x^n$.

Clearly, $G^n \subseteq H$.

Next, let D be an algebraically finitely generated subgroup which is dense in G .

Consider the natural continuous homomorphism $f : G \rightarrow G/G^n$. Then, $f(D) = DG^n/G^n$ is also dense in G/G^n , but DG^n/G^n is finitely generated torsion abelian group; hence, it is finite.

Thus, $DG^n/G^n = \text{closure of } DG^n/G^n = G/G^n$, i.e. G/G^n is finite.

Now, since G^n is a closed subgroup of finite index, G^n is open in G . Since $G^n \subseteq H$, we deduce that H is open in G .

The following result, due to J.P. Serre, has not been published. We shall give here a sketch of a proof for the benefit of the reader.

Theorem 6.3: (J.P. Serre) Let G be a finitely generated pro- p group (p -prime). Then every large subgroup of G is open.

Before we prove this theorem, we need some lemmas.

Lemma 6.4: Let G be a pro-finite group and n be a natural number such that $|G|$ and n are co-prime. Then $G^n = G$ where G^n is the group generated by $\{x^n \mid x \in G\}$.

Proof: Let $G = \text{proj. limit}_{i \in I} G_i$ where G_i are finite groups. Now $|G_i|$ and n are co-prime which implies $G_i^n = G_i$ for every $i \in I$.

Thus, $\text{proj. limit}_{i \in I} G_i^n = \text{proj. limit}_{i \in I} G_i = G$

We claim that $G^n = \text{proj. limit}_{i \in I} G_i^n$

Let $\sigma \in G^n$ then $\sigma = \langle \sigma_i \rangle = \langle \tau_i^n \rangle$ where $\sigma_i = \tau_i^n \in G_i^n$. Thus,

$\sigma \in \text{proj. limit}_{i \in I} G_i^n$.

Thus $G^n \subseteq \text{proj. limit}_{i \in I} G_i^n$.

Conversely, let $\sigma \in \text{proj. limit } G_i^n$.

Then, $\sigma = \langle \sigma_i \rangle$ where $\sigma_i \in G_i^n$. Thus there exists $\tau_i \in G_i$ such that $\sigma_i = \tau_i^n$.

Now for compatible homomorphisms $\phi = \phi_{i,j} : G_i \rightarrow G_j$ ($i \geq j$), we have

$$\phi(\sigma_i) = \sigma_j$$

$$\text{Thus, } \phi(\sigma_i) = \phi(\tau_i^n) = [\phi(\tau_i)]^n = \sigma_j = \tau_j^n$$

Since the map $G_j \rightarrow G_j$ that sends x to x^n is one to one, $\phi(\tau_i) = \tau_j$

i.e. $\langle \tau_i \rangle \in G$.

Thus, $\sigma = \langle \tau_i \rangle^n \in G^n$.

This proves that

$$G = \text{proj. limit } G_i^n \subseteq G^n$$

Thus

$$G^n = G$$

Lemma 6.5: Let G be a finitely generated pro- p group. Then,

every element of (G, G) can be written as $(X_1, U_1)(X_2, U_2) \dots$

(X_n, U_n) with $U_i \in G$, where $\{X_1, \dots, X_n\}$ are topological generators of G .

Proof (Sketch): It is sufficient to prove the lemma for free pro- p groups on a finite set. Consider the descending central series of G . We construct U_i by successive approximations. Note that $\text{gr. } G$ is then free Lie algebra on variables X_i corresponding

to the x_i 's, and that every element of $\text{gr}^m G$, $m \geq 2$ may be written as

$$\Sigma [x_i, y_i]$$

with $y_i \in \text{gr}^{m-1} G$.

Lemma 6.6: Let G be a finitely generated pro- p group. Then, (G, G) is closed.

Proof: Consider the following continuous map:

$$f : \underbrace{G \times G \times \dots \times G}_{n \text{ copies}} \longrightarrow G$$

$$(u_1, \dots, u_n) \longmapsto (x_1, u_1) \dots (x_n, u_n)$$

Then, (G, G) is the image of $G \times \dots \times G$ under f by Lemma 6.5.

Thus, (G, G) is closed.

Proof of Theorem 6.3 Let H be any large subgroup of G , then

H_G - the core of H in G (maximal normal subgroup of G contained in H) - is also the large subgroup of G . If H_G were open, then $H \supset H_G$ would be open. So we may and do assume that H is normal in G . Thus, $\bar{G} = G/H$ is a finite group.

If q is a prime other than p , then by Lemma 6.4, $G^q = G$ and hence $\bar{G}^q = \bar{G}$. Thus, q does not divide the order of $\bar{G}$, i.e. $\bar{G}$ is a p -group.

Now there is a normal subgroup H_1 of G such that $H \subseteq H_1 \subseteq G$ and $|G/H_1| = p$.

Thus, $(G, G)G^p \subseteq H_1$.

Since (G, G) is closed, the Frattini subgroup $G^* = (G, G)G^p$, i.e.

$G^* \subseteq H_1$.

Since G is finitely generated, G^* is open in G . Thus, $H_1 \supset G^*$ is open in G .

Now we induct on $(G:H)$.

Since H_1 is finitely generated pro- p group and $(H_1:H) < (G:H)$, H is open in H_1 by induction. Thus, H is open in G .

Theorem 6.7: Let G be a finitely generated pro-nilpotent group. Then every large subgroup of G is open.

More generally, we shall prove the theorem for finitely generated pro-supersolvable groups. First, this lemma.

Lemma 6.8: Let G be a pro-supersolvable group such that $|G|$ is divisible by only primes $p_1 > p_2 > \dots > p_n$; and for each i , let S_i be a Sylow- p_i subgroup of G .

Then, $S_1, S_1S_2, \dots, S_1S_2 \dots S_n = G$ are closed normal subgroups of G . Furthermore, S_i 's can be chosen such that

$S_i S_j = S_j S_i$ for $1 \leq i, j \leq n$.

Proof: Since p_1 is the largest prime dividing the order G , it is clear that S_1 - the (unique) Sylow- p_1 subgroup is a normal subgroup of G .

Now, since S_1 is closed normal subgroup of G , $S_1 S_2$ is a closed subgroup of G . Further, $|S_1 S_2|$ and $(G : S_1 S_2)$ are co-prime. Thus, $S_1 S_2$ is a Sylow- π_{p_3} group of G . (Note $\pi_{p_3} = \pi_G - S_{p_3} = \{p_1, p_2\}$ where π_G denotes the set of primes dividing order G and S_{p_3} denotes the set of all primes smaller or equal to p_3).

Since G has a unique Sylow- π_{p_3} subgroup (Theorem 3.5), $S_1 S_2$ is a closed normal subgroup of G .

Now for $k \leq n$, by repeating the above argument, we get $S_1 S_2 \dots S_k$ is a closed normal subgroup of G .

Clearly, $S_1 S_2 \dots S_n = G$.

For the latter part of the lemma, we choose C_{p_i} - a Sylow- p_i subgroup of G where $p_i = \pi_G - \{p_i\}$, $1 \leq i \leq n$.

Let

$$S_i = \bigcap_{j \neq i} C_{p_j}$$

Then, for every i , $1 \leq i \leq n$; S_i is a Sylow- p_i subgroup of G .

Thus,

$$S_i S_j = \bigcap_{\substack{k \neq i \\ k \neq j}} C_{p_k} = S_j S_i; \quad 1 \leq i, j \leq n.$$

Theorem 6.9: Let G be a finitely generated pro-supersolvable group such that $|G|$ is divisible by only finitely many primes. Then every large subgroup of G is open.

Proof: Let $\{p_1 > p_2 > \dots > p_n\}$ be the precise set of primes that divide $|G|$.

For each i , choose S_i - a Sylow- p_i subgroup of G as in Lemma 6.8, so that $S_i S_j \leq S_j S_i$, $1 \leq i, j \leq n$.

Since S_1 is normal in G , we have an exact sequence:

$$S_1 \rightarrow G \xrightarrow{\pi} G/S_1 \quad (*)$$

The exact sequence (*) splits as $G = S_1 (S_2 \dots S_n)$ and $S_1 \cap (S_2 \dots S_n) = 1$, for if

$$S_1 \cap (S_2 \dots S_n) \neq 1$$

then for some finite quotient $\underline{G}$, let $\underline{S}_i$ be the image of S_i , $1 \leq i \leq n$; $\underline{S}_i$'s are still Sylow subgroups of $\underline{G}$; $\underline{G} = \underline{S}_1 \underline{S}_2 \dots \underline{S}_n$, and $\underline{S}_1 \cap (\underline{S}_2 \dots \underline{S}_n) \neq 1$ which is impossible.

Now we use induction on n - the number of primes dividing order G .

For $n=1$, G is a finitely generated pro- p group and the result holds by Theorem 6.3.

Assume the result for $n-1$ and we will show that it is true when $|G|$ is divisible by the primes $p_1 > p_2 > \dots > p_n$.

Case (i): Let U be a large subgroup of G such that p_1 does not divide $(G:U)$.

Now, since $S_1 \cap U$ is large in S_1 and since S_1 is a finitely generated pro- p_1 group (Corollary 5.4), we have by

Theorem 6.3 that $S_1 \cap U$ is open in S_1 and so $(S_1 : S_1 \cap U) = p_1^\alpha$; α is non-negative integer, i.e. $(S_1 U : U) = p_1^\alpha$.

Since $(S_1 U : U)$ divides $(G : U)$ and p_1 does not divide $(G : U)$, $\alpha = 0$, i.e. $S_1 \subseteq S_1 U = U$.

Now, $S_1 \subseteq U \Rightarrow U = \pi^{-1}(\pi(U))$

Since $\pi(U)$ is open in $S_2 \dots S_n$ by the induction hypothesis, we conclude that $U = \pi^{-1}(\pi(U))$ is open in G .

Case (ii): U is a normal subgroup of G such that $|G/U| = p_1^\alpha$, α is a non-negative integer.

Let $S = S_2 \dots S_n$. Since SU/U is a subgroup of G/U , $|SU/U|$ is a power of p_1 say p_1^β where β is a non-negative integer. But, $SU/U = S/S \cap U$.

Since p_1 does not divide $|S|$, $\beta = 0$. Thus, $U = SU \supset S$.

Now we claim that $U = (S_1 \cap U)S$. Clearly, $(S_1 \cap U)S \subseteq U$.

For the converse, let $x \in U$; then $x = st$ where $s \in S_1$ and $t \in S$. But now $s = xt^{-1} \in U$, since $t \in S \subseteq U$. Thus, $s \in S_1 \cap U$ and $x = st \in (S_1 \cap U)S$ i.e. $U \subseteq (S_1 \cap U)S$.

It follows that $U = (S_1 \cap U)S$. Since $S_1 \cap U$ is open in S_1 by Theorem 6.3 and hence closed in G , we have that $U = (S_1 \cap U)S$ is closed in G .

Thus, U is open in G in this case also.

Case (iii): Let U be a normal subgroup of G having an arbitrary finite index.

If p_1 does not divide $|G/U|$ then we are done by case (i). So assume that p_1 divides $|G/U|$.

Let $\bar{X}$ be a Sylow- p_1 subgroup of G/U . Then there is a subgroup X of G such that $U \subset X$ and $\bar{X} = X/U$.

Now the prime p_1 does not divide $(G:X)$; therefore, X is open in G by case (i). Now X is finitely generated pro-supersolvable group. Since U is a normal subgroup of X such that $|X/U|$ is a power of p_1 , U is open in X by case (ii).

It follows that U is open in G .

Case (iv): Let U be any large subgroup of G . Let C be the core of U in G . Then C is a large normal subgroup of G and is open in G by case (iii). We conclude that $U \supset C$ is open in G .

Theorem 6.10: Let F be a free pro-supersolvable group on a finite set X . Then, every large subgroup U of F is open in F .

Proof: We may and do assume that U is normal in F .

Let p be the largest prime dividing $|F/U|$ and put $\underline{S} = \underline{S}_p$ the class of all finite supersolvable groups whose orders are divisible by only primes that are less than or equal to p .

Consider the exact sequence:

$$K_p \xrightarrow{\quad} F \xrightarrow{\quad \pi \quad} F_S$$

where F_S is a free pro- S group on X .

Now, $n = |K_p / K_p \cap U|$ divides $|F/U|$.

Thus, n and $|K_p|$ are co-prime and by Lemma 6.4, $K_p^n = K_p$.

Thus,

$$K_p = K_p^n \subseteq K_p \cap U \subseteq U$$

and

$$U = \pi^{-1}(\pi(U))$$

Now, $\pi(U)$ a large subgroup of F_S is open in F_S by Theorem 6.9.

Thus, U is open in F .

Corollary 6.11: Let G be a finitely generated pro-supersolvable group. Then every large subgroup of G is open.

7. THE FRATTINI SUBGROUP OF FREE PRO-C GROUPS

Let C be a class of finite groups closed under the formations of i) subgroups, ii) homomorphic images, and iii) extensions. Further, suppose that C contains p -groups for infinitely many primes p . In this section, we show that the Frattini subgroup of free pro- C groups on more than one generator is the identity.

Definition 7.1: Let G be a finite group. A - the abelian socle of G is a subgroup of G generated by all minimal normal abelian subgroups of G .

Lemma 7.2: Let G be a finite group, A the abelian socle of G . Then $G^* = 1$ if and only if G splits over A .

Proof: If $G^* = 1$, then G splits over A (Theorem 5.4.10, p. 66 in Reference (2)).

For the converse, suppose G splits over A and $G^* \neq 1$.

Let N be a minimal normal (non-identity) subgroup of G contained in G^* .

Since G^* is nilpotent, N is abelian. Thus, $N \subseteq A$.

Now since A is a direct product of minimal normal abelian subgroups of G , there is a normal subgroup H of G such that $A = HN$ and $H \cap N = 1$. Since G splits over A , there is a subgroup K of G such that $AK = G$; $A \cap K = 1$. Now, $G = AK = (NH)K = N(HK)$.

Since $H \neq G$, HK is a subgroup of G . HK is a proper subgroup of G as $N \neq 1$ and $N \cap HK = 1$.

Let M be a (proper) maximal subgroup of G such that $HK \subseteq M$.

Now, $N \subseteq G^* \subseteq M$.

Thus, $G = AK = N(HK) \subseteq M$ contradiction.

This contradiction proves that $G^* = 1$.

Lemma 7.3: Let F be a (discrete) free group of finite rank greater than 1, R be a normal subgroup of F such that F/R is a finite group. Let $x \in F$ such that $x \notin R$ then $[x, R] = \{x^{-1}r^{-1}xr \mid r \in R\}$ is not contained in the derived group R' of R .

Proof: Suppose $[x, R] \subseteq R'$ and consider $G = \text{gp.}\{x, R\}$. Then G is a free group of finite rank since G is a subgroup of F having finite index.

Since R is a proper subgroup of G , $\text{rank } R > \text{rank } G$.

Now, $[x, R] \subseteq R' \Rightarrow [x^2, R] = [x, R]^x [x, R] \subseteq R'$.

Thus, $[x^n, R] \subseteq R'$ for any integer n .

For $r \in R$, $[x^n r, R] = [x^n, R]^r [r, R] \subseteq R'$.

Thus, $[g, R] \subseteq R'$ for every $g \in G$.

Let $g = x^n r \in G$ and m be any integer. Then,

$$\begin{aligned} [g, x^m] &= [x^n r, x^m] = r^{-1} x^{-n} x^{-m} x^n r x^m \\ &= r^{-1} x^{-m} r x^m = [r, x^m] \\ &= [x^m, r]^{-1} \in R' \end{aligned}$$

Further for $g, h \in G$, $g = x^n r$, $h = x^m s$, $r, s \in R$ we have:

$$[g, h] = [g, x^m s] = [g, s] [g, x^m s] \in R'$$

Thus, $G' \subseteq R'$.

On the other hand, $R' \subseteq G'$.

Thus, $G' = R'$.

It follows that a free abelian group G/G' of rank G contains a free abelian group R/R' of rank R greater than rank G . This is impossible.

This contradiction establishes that:

$$[x, R] \not\subseteq R', \text{ for } x \in F \setminus R.$$

Lemma 7.4: Let G be a free abelian group of finite rank and P any infinite set of primes. Then, $\bigcap_{p \in P} G^p = 0$.

Proof: $G \cong \prod_{s \in S} Z_s$ $|S| < \infty$

where Z_s is a group of integers under addition.

Let $x = (x_s)_{s \in S} \in \bigcap_{p \in P} G^p$

Then for every $p \in P$, $p | x_s$, $s \in S$. Since infinitely many primes divide x_s , $x_s = 0$ for every $s \in S$.

Thus, $x = 0$, i.e. $\bigcap_{p \in P} G^p = 0$.

Theorem 7.5: Let $\hat{F}$ be a free pro-C group on a finite set X.

Then, $(\hat{F})^* = 1$.

Proof: Let F be a (discrete) free group on a finite set X. In order to prove the theorem, it is sufficient to show that for every $R \trianglelefteq F$ such that $F/R \in C$ there is a normal subgroup S of F such that $S \subseteq R$, $F/S \in C$ and $(F/S)^* = 1$. (Note $\hat{F} = \text{proj. limit } F/R$ and $(\hat{F})^* = \text{proj. limit } (F/R)^*$). Let P be a set of all primes such that $p \in P$ if and only if C contains p-groups. By our assumption, P is an infinite set. Let P_1 be a set of all primes in P that do not divide $|F/R|$. P_1 is still an infinite set. Now, by Lemma 7.4,

$$\bigcap_{p \in P_1} (R/R')^p = R'$$

Let $\{x_1, \dots, x_k\}$ be the set of coset representatives of R in F with $x_1 \in R$.

Now, for every x_i there is a prime p_i in P_1 such that $[x_i, R] \not\subseteq R^{p_i} R'$, $2 \leq i \leq k$.

Thus there is a finite subset $\{p_2, \dots, p_r\}$ of P_1 of distinct primes such that $[x_i, R] \not\subseteq S = \bigcap_{i=2}^r R^{p_i} R'$ for $2 \leq i \leq k$; $r \leq k$.

Clearly, $S \trianglelefteq F$ such that $S \subseteq R$ and $F/S \in C$.

We claim that $(F/S)^* = 1$. Now R/S is an abelian normal subgroup of F/S and by construction R/S is a direct product of minimal normal abelian subgroups of F/S .

Thus, $R/S \subseteq A$ - the abelian sockle of F/S .

Now, $A \subseteq C = C_{F/S}(R/S)$.

Thus, $R/S \subseteq A \subseteq C$.

If we show that $R/S=C$ then it will follow that $R/S=A$.

Let $x \in F \setminus R$; then there is x_i for $2 \leq i \leq k$ such that $xx_i^{-1} \in R$, i.e. $xx_i^{-1}S \in R/S \subseteq C$.

Let us denote xS by $\bar{x}$.

Thus, $\bar{x} \bar{x}_i^{-1} \in C$.

Now since $[x_i, R] \subseteq S$ (by construction) there is $r \in R$ such that $(x_i, r) = x_i^{-1} r^{-1} x_i r \in S$, i.e. $\bar{r} = (\bar{r})^{x_i}$, i.e. $\bar{x}_i \in C$. But then $\bar{x} \in C$.

This shows that $R/S=C$ which in turn proves that R/S is an abelian sockle of F/S .

Now since R/S is an abelian normal subgroup of F/S such that $|R/S|$ and $(F:R)$ are co-prime, F/S splits over R/S (Theorem 4.2.3, p. 40, Reference (2)).

Since F/S splits over R/S which is an abelian sockle of F/S , $(F/S)^* = 1$ by Lemma 7.2. This completes the proof of the theorem.

Theorem 7.7: Let X be a compact, Hausdorff, pointed topological space. Then

$$F(X)^* = 1 \text{ and } [\tilde{F}(X, *)]^* = 1,$$

where $F(X)$, $\tilde{F}(X, *)$ are free pro- C groups.

Proof: With the notations as in Corollary 1.9, Chapter 2,

$$F(X) = \text{proj. limit}_{R \in \underline{R}} F(X/R)$$

and

$$F(X, *) = \text{proj. limit}_{R \in \underline{R}} F(X_R)$$

The theorem now follows from Lemma 1.7 and Theorem 7.6.

Corollary 7.8: Let F be a free pro-finite (or free pro-solvable) group on generators more than 1, then $F^* = 1$.

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