Facial Nonrepetitive Graph Colourings

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Abstract

Given a plane graph G, a *facial path* is a path of consecutive vertices on the boundary walk of a face of G. A nonrepetitive facial colouring of G is a vertex colouring such that the sequence of colours of any facial path of G is nonrepetitive, and the minimum number of colours required for such a colouring is the facial Thue chromatic number of G.

Using a new *blocking set* technique, we show that the facial Thue chromatic number of an outerplane graph is bounded by 11, and by 7 for outerplane graphs that contain at most one 2-connected component. Furthermore, we show that the facial Thue chromatic number of plane graphs is bounded by twice the facial Thue chromatic number of outerplane graphs, which results in an upper bound of 22 for this parameter, an improvement over the previous bound of 24 by Barat and Czap (Journal of Graph Theory, 2013).

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Chapter 1

Introduction

A sequence $S = s_1, s_2, \ldots, s_{2r}$ is a *repetition* if $s_i = s_{r+i}$ for each $i \in \{1, \ldots, r\}$. For example, 1212 is a repetition while 1213 is not. Repetitions are also referred to as *squares* since they can be written as $aa = a^2$. A *block* of a sequence S is a subsequence of consecutive terms in S. A sequence is *nonrepetitive (square free)* if for every block B of S, B is not a repetition. Otherwise S is *repetitive*. For example, 1312124 is repetitive, as it contains the block 1212 which is a repetition, while 123213 is nonrepetitive as it contains no such block. A *palindrome* is a sequence $s_1, s_2, \ldots, s_r$ such that $s_1, s_2, \ldots, s_r = s_r, s_{r-1}, \ldots, s_1$. A sequence is *palindrome-free* if for every block B of S, B is not a palindrome.

Notice that every sequence of length at least 4 that uses only two symbols must be repetitive: without loss of generality, start with 1. The next symbols must be 2 followed by 1 to avoid repetitions of size 2. But, the next symbol cannot be 1 nor 2, since this would create a repetition in both cases. Therefore, three symbols are necessary.

A groundbreaking result by Axel Thue in 1906 shows that three symbols are also sufficient for nonrepetitive sequences of any length [50]. The proof is based on a substitution technique: start from a nonrepetitive sequence S on $\{1, 2, 3\}$ (for example, start with 1), and replace every term using the following rule:

$$\begin{aligned} 1 &\to 12312 \\ 2 &\to 131232 \\ 3 &\to 1323132. \end{aligned} \tag{1.1}$$

Thue showed that the resulting sequence is nonrepetitive. Thus, using this construction, arbitrarily long sequences can be built (and can be reduced to any specific size if required). Since Thue's original paper, many variations of the basic problem have been considered. One famous problem proposed by Erdös [19] and Brown [12] asks if there exists an infinite sequence on four symbols such that no two adjacent blocks are a permutation of each other. Such a sequence is a called a *strong nonrepetitive sequence*. Adjacent blocks that are a permutation of each other are also referred to as *abelian squares*. The existence of strong nonrepetitive sequences on 25 symbols was first showed by Evdokimov [22], then the number of required symbols was lowered to 5 by Pleasants [44], and finally to 4 by Keränen [33] with the help of a computer. The latter is best possible.

In this thesis, we consider a graph-theoretic variant of nonrepetitive sequences: *nonrepetitive colourings*, which are also referred to as *Thue colourings*. This variant was proposed by Alon, Grytczuk, Hałuszczak and Riordan in 2002 [3] and has received considerable attention since [5, 6, 7, 10, 15, 18, 17, 24, 25, 28, 27, 29, 30, 35, 36, 43, 48, 47].

Although few — if any — practical applications of nonrepetitive colourings have been found yet, graph colouring in general enjoys many practical applications such as scheduling, pattern matching and register allocation. The famous *Sudoku* puzzle game can be seen as the problem of finding a 9-colouring of a specified graph on 81 vertices. Perhaps a nonrepetitive version of the game could be interesting!

1.1 Problem Statement

We first introduce the concepts of vertex and edge colourings in their traditional form. A *(proper) vertex colouring* of a graph G is a function $c : V(G) \to \mathbb{N}$ that satisfies:

$$c(u) = c(v) \Rightarrow \{u, v\} \notin E(G) \tag{1.2}$$

for all $u, v \in V(G)$. For $k \in \mathbb{N}$, a *vertex k-colouring* is a proper vertex colouring $c : V(G) \to \{1, \dots, k\}$. G is *vertex k-colourable* if there exists a proper vertex k-colouring of G. The least k for which G is vertex k-colourable is called the *chromatic number* of G, denoted $\chi(G)$. A function $c : V(G) \to \mathbb{N}$ in which Equation 1.2 does not holds is a *nonproper vertex colouring* of G.

A *(proper) edge colouring* of a graph G is a function $c : E(G) \to \mathbb{N}$ that satisfies:

$$c(e_1) = c(e_2) \Rightarrow e_1 \cap e_2 = \varnothing \tag{1.3}$$

for all $e_1, e_2 \in E(G)$ such that $e_1 \neq e_2$. For $k \in \mathbb{N}$, an *edge k-colouring* is a proper edge colouring $c : E(G) \to \{1, \ldots, k\}$. G is *edge k-colourable* if there exists a proper edge k-colouring of G. The least k for which G is edge k-colourable is called the *chromatic index* of G, denoted $\chi'(G)$. A function $c : E(G) \to \mathbb{N}$ in which Equation 1.3 does not holds is a *nonproper edge colouring* of G. When no ambiguity is possible, a *colouring* means a vertex colouring.

A *nonrepetitive vertex k-colouring* of a graph G is a vertex k-colouring c in which, for every vertex path $v_1, \ldots, v_r$ in G, the sequence $c(p_1), \ldots, c(p_r)$ is nonrepetitive. The *Thue chromatic number* of a graph G, denoted $\pi(G)$, is the least k for which a nonrepetitive vertex k-colouring of G exists. We will later define *nonrepetitive edge colourings*, but for the moment, when no ambiguity is possible, a *nonrepetitive colouring* means a nonrepetitive vertex colouring.

We now arrive at the topic studied in this thesis: *facial nonrepetitive colourings*. For a plane graph G, a *facial walk* is a walk of consecutive vertices on the boundary of a face of G. A *facial path* is a facial walk in which every vertex is unique (see Figure 2.3). A nonrepetitive facial k-colouring is a vertex k-colouring in which, for every facial path $v_1, \ldots, v_r$ in G, the sequence $c(v_1), \ldots, c(v_r)$ is nonrepetitive. The *facial Thue chromatic number* of a plane graph G, denoted $\pi_f(G)$, is the least k for which a nonrepetitive facial k-colouring of G exists.

Notice that if a graph is disconnected, its chromatic number is the maximum of the chromatic number of each of its connected components. Similarly, the chromatic number of a multiple graph (a graph that contains parallel edges) is unchanged if all the parallel edges are replaced by single edges. Furthermore, notice that there is no valid colouring of a graph that contain loops. This also true of the Thue chromatic number and the facial Thue chromatic number. For these reasons, unless otherwise noted, all graphs considered in this thesis are simple, loopless and connected.

1.2 Contributions

In this thesis, we prove several upper bounds on the facial Thue chromatic number of various families of plane graphs. In particular, we show that the facial Thue chromatic number of an outerplane graph is bounded by 11, and by 7 if restricted to outerplane graphs that contain at most one 2-connected component.Our proof is based on a new blocking set technique and improves the existing bound of 12 due to Barát and Varjú [6] and Kündgen and Pelsmajer [36].[1] We also present a proof inspired by Barát and Czap [5] that shows that the facial Thue chromatic number

[1] $\pi(G) \leq 12$ for an outerplanar graph G.

of plane graphs is bounded by twice the facial Thue chromatic number of outerplane graphs. This directly improves the existing upper bound for plane graphs to 22 from the current best of 24 due to Barát and Czap [5], and is directly applicable in future work if a tighter bound for outerplane graphs is discovered.

1.3 Thesis Organization

The remainder of the thesis is organized as follows: Chapter 2 presents a literature review of the major results concerning nonrepetitive sequences and nonrepetitive graph colourings. In particular, we discuss vertex and edge nonrepetitive colourings, star colourings (from which many lower bounds can be derived) and facial vertex and edge nonrepetitive colourings. We also briefly mention other variants of the basic problem.

In Chapter 3, we present our results concerning nonrepetitive facial vertex colourings. In particular, we present our strategy and proof for the tighter upper bounds on the facial Thue chromatic number of outerplane and plane graphs.

Finally, we conclude and discuss possible avenues for future research in Chapter 4.

Chapter 2

Background

In this chapter, we present a survey of the literature concerning nonrepetitive colourings. In particular, Section 2.1 introduces the main definitions which will be used throughout the text. Section 2.2 presents a useful technique to generate nonrepetitive sequences on three symbols. Sections 2.3, 2.4 and 2.5 cover the main results pertaining to nonrepetitive colourings. Section 2.7 discusses facial nonrepetitive colourings, the specific topic we study in this thesis. Finally, we end by briefly discussing the algorithmic complexity of the problem in Section 2.8, and some variants of the nonrepetitive colouring problem in Section 2.9.

2.1 Definitions

In this text, a graph is always simple and loopless unless specified otherwise. Let G be a graph and H be a subgraph of G with vertex set S. $G[S]$ is the subgraph of G induced by S. $G \setminus H = G[V(G) \setminus V(H)]$. For $u \in V(G)$, $\deg_G(u)$ (or $\deg(u)$ when there is no danger of ambiguity) denotes the degree of u in G. $\Delta(G)$ is the maximum degree of any vertex in G, that is $\Delta(G) = \max_{u \in V(G)} \deg(u)$. A *clique* is a subgraph of G that is complete. The *clique number* of G, denoted $\omega(G)$, is the number of vertices in a maximum clique of G.

A *walk* is a sequence $v_1, v_2, \ldots, v_k$ of vertices such that for all $i \in \{1, \ldots, k-1\}$, $\{v_i, v_{i+1}\} \in E(G)$. A *path* is a walk in which all the vertices are distinct. An edge-walk is a sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}$ such that $\{v_i, v_{i+1}\} \in E(G)$ for all $i \in \{1, \ldots, k\}$. A *trail* is an edge-walk in which all the edges are distinct. A walk is *closed* if the initial vertex and the final vertex are the same. A *cycle* is a closed walk in which all the vertices are distinct except the first and last vertices which are the same.

For two vertices $u, v \in V(G)$, the *distance* between u and v in a graph G, denoted $d_G(u, v)$ (or $d(u, v)$ when there is no danger of ambiguity), is the number of edges in a shortest path between u and v. If there is no path between u and v, then $d(u, v) = \infty$. The k^{th} *iterated neighbourhood* of a vertex $v \in V(G)$ is the set

$$N^k(v) = \{u \in V(G) \mid d(u, v) = k\}.$$

A *levelling* of a graph G is a function $\lambda : V(G) \to \mathbb{N}$ such that for each $\{u, v\} \in E(G)$, $|\lambda(u) - \lambda(v)| \leq 1$. The level pattern of a path $v_1, v_2, \ldots, v_k$ is the sequence $\lambda(v_1), \lambda(v_2), \ldots, \lambda(v_k)$.

A *chord* of a cycle C is an edge connecting two vertices that are not neighbours in C. A graph G is a *chordal graph* if and only if for every cycle C of size greater than three in G, $G[V(C)]$ has a chord. Equivalently, each induced cycle of G has at most three vertices. A *chordal completion* of G is a chordal graph that contains G as a subgraph. The *treewidth* of G is one less than the number of vertices in a maximum clique of a chordal completion chosen to minimize this clique size.

The minimum number of vertices in a graph G whose removal disconnects G or creates a graph with a single vertex is called the *connectivity* of G and is denoted by $\kappa(G)$. A graph is k-connected if $k \leq \kappa(G)$. Note that by this definition, $\kappa(K_n) = n-1$ (K_n is the complete graph on n vertices). In particular, K_2 is not 2-connected. A k-connected component of G is a maximal induced subgraph H of G that is k-connected. A 1-connected component is usually referred to as a *connected component* or simply as a *component*. A cycle of G that includes every vertex of G is a *Hamiltonian* cycle. G is a *Hamiltonian graph* if it contains a Hamiltonian cycle.

A subdivision of an edge $\{u, v\}$ of a graph G is the operation of adding a new vertex w to G and replacing the edge $\{u, v\}$ by $\{u, w\}$ and $\{w, v\}$. A subdivision of a graph G is a graph H resulting from a (possibly empty) sequence of edge subdivisions applied to G. A tree is a connected graph with no cycle as a subgraph. A forest is a graph in which every component is a tree. P_n is the *path* graph on n vertices, a connected tree with maximum degree 2. The cycle graph C_n is a graph that consists of n vertices in a closed chain. The *girth* of a graph is the length of its shortest cycle. If a graph is acyclic (contains no cycle), its girth is defined to be infinity.

The following definitions are taken from Bondy and Murty [9]. A *planar graph* is a graph that can be embedded in the plane $\mathbb{R}^2$ such that its edges intersect only at their endpoints. Such an embedding is called a *planar embedding.* A *plane graph* $\tilde{G}$ is a planar embedding of a graph G in which $V(\tilde{G})$ is the set of points representing vertices of G, $E(\tilde{G})$ is the set of lines representing edges in G and a vertex of $\tilde{G}$ is incident to all the edges of $\tilde{G}$ that contain it. A plane graph G partitions $\mathbb{R}^2$ into regions $F_1, F_2, \ldots, F_f$ called *faces*. $F(G)$ denotes the set of faces of a plane graph G. A plane graph contains exactly one unbounded face, which is called the *outside* or *outer* face of G. The other faces are called the *inner faces* of G.

An *outerplanar graph* is a graph that has a planar embedding such that all its vertices are adjacent to its outside face. Such an embedding is called an *outerplanar embedding*. An *outerplane* graph $\tilde{G}$ is an outerplanar embedding of a graph G.

The *dual* of a plane graph G (G is the *primal* graph) is a plane graph G^* that has for vertex set the faces of G and has an edge $\{u, v\}$ if and only if the faces u and v share an edge in G. The *weak dual* of G, denoted G°, is the subgraph of the dual whose vertices correspond to the bounded faces of the primal graph. The weak dual of an outerplanar graph is a forest and the weak dual of a 2-connected outerplanar graph is a tree. A *Halin* graph is a plane graph constructed by embedding in the plane a tree that has at least four vertices, none of which have degree 2, then connecting all its leaves in a cycle such that no edge cross. Halin graphs have treewidth at most 3 [8]. The weak dual of a Halin graph is a 2-connected outerplanar graph.

2.2 Generating Nonrepetitive Sequences

In addition to the 1906 proof that used substitutions, Thue also showed a different technique to generate nonrepetitive sequences in a 1912 article [51]. This construction is based on the Prouhet–Thue-Morse sequence, which was independently discovered in different contexts ranging from chess to number theory [28]. It is named after the three people who independently discovered it: Eugène Prouhot, who first used the sequence in 1851 in his work on number theory [45], Axel Thue who was the first to explicitly mention it in 1906 [50], and Marston Morse who made it famous in 1921 with his work on differential geometry [40]. It is an infinite sequence t defined as $t = \lim_{n\to\infty} h^n$, where $h^0 = 0$, $h^{k+1} = h^k\overline{h^k}$.[1] Thus,

$$h^0 = 0$$
$$h^1 = 01$$
$$h^2 = 0110$$
$$h^3 = 01101001$$

[1] $\overline{a}$ is the binary complement of a: $\overline{10} = 01$.

$$t = 01101001100101101001011001101001\ldots$$

Thue discovered several interesting properties of this sequence [51]. It contains no *cubes*: blocks of the form aaa. It can also be built using substitutions: substitute 10 for each 1 and 01 for each 0 in h^k, the result is h^{k+1}. Also, t does not contain blocks of the form $0X0X0$ or $1X1X1$ where X is a possibly empty word. This is key to the following construction: Let S be the sequence S formed by counting the number of ones between successive zeros in t. Then, S is a nonrepetitive sequence! Suppose that this is not true, thus there is a block aa in S. This corresponds to a block of the form $0X0X0$ where a corresponds to the number of ones between consecutive zeros in $0X0$, but this is a contradiction. This provides a quick way to generate a nonrepetitive sequence "on-the-fly".

2.3 Star Chromatic Number

We digress from our main topic for this section to discuss a related graph colouring variant called *star colouring*, from which lower bounds on nonrepetitive colourings can be derived. The *star chromatic number* of a graph G, denoted $\chi_{st}(G)$, is the smallest k for which there exists a k-colouring of G in which every path on four vertices uses at least three colours (there is no 2-coloured P_4) [18]. This name comes from an equivalent definition of the problem: the subgraph induced by any two colour classes must form a forest of stars.[2] As we discussed earlier, any quaternary sequence on two colours must be repetitive. This implies that the Thue chromatic number of a graph is bounded by below by its star chromatic number. Furthermore, every star colouring is also a proper vertex colouring. Therefore,

$$\chi(G) \leq \chi_{st}(G) \leq \pi(G). \tag{2.1}$$

Thus, any lower bound for $\chi_{st}(G)$ also apply to $\pi(G)$. Given a graph G, the square graph of G, denoted G^2, is a (simple) graph with vertex set $V(G)$ and with an edge between any two vertices at distance 2 or less in G. An interesting observation that is due to Fertin, Gaspaur and Reed [23] is that $\chi_{st}(G) \leq \chi(G^2)$. To see why this observation is true, notice that in G^2, every vertex v has a colour different than any vertices in its first and second neighbourhoods. Thus, any path on 4 vertices that includes v in G^2 contains at least three different colours, which satisfies the

[2] A star is a graph S_k such that $S_k = K_{1,k}$ for some $k \in \mathbb{N}$.

requirement for a valid star colouring of G. Since $\chi(G) \leq \Delta(G) + 1$, we get that[3]

$$\chi_{st}(G) \leq \Delta(G)^2 + 1. \tag{2.2}$$

This bound can be largely improved for many families of graphs, including a better bound for graphs of bounded degree. Table 2.1 summarizes these results. Note that some results we present are for the *star choice number*, denoted $\chi_{st,ch}(G)$, the least l such that for any l-list assignment $\mathcal{L}$ of G, a star $\mathcal{L}$-colouring of G exists.[4]

2.4 Thue Chromatic Number

In this section, we discuss the main results and problems on the Thue chromatic number for various families of graphs.

2.4.1 Thue Chromatic Number of Paths and Cycles

Thue's 1906 result about the existence of infinitely long nonrepetitive sequences is easily translated into a tight upper bound for the Thue chromatic number of paths, we have that $\pi(P_n) = \min\{n, 3\}$ (recall that $\pi(G)$ is the Thue chromatic number of G). Also, one can easily get an upper bound of four for cycles: let C be a cycle, assign a first colour to one vertex v. $C \setminus \{v\}$ is a path, colour it with a nonrepetitive sequence on three other colours. This colouring is clearly nonrepetitive, but can we do better? Alon et al. [3] conjectured that this was the case after observing that $\pi(C_n) = 3$ for all values of $n \leq 2001$, except for $n \in \{5, 7, 9, 10, 14, 17\}$ where four colours are required. Their conjecture stated that $\pi(C_n) = 3$ for $n \geq 18$. It was later shown true by Currie [15] using a construction based on the Prouhet–Thue-Morse sequence for all values of $n \geq 180$. Thus, the following holds:

Theorem 2.1 (Alon et al. [3] and Currie [15])**.** *Let C_n be a cycle on $n \geq 3$ vertices. We have that*

$$\pi(C_n) = \begin{cases} 4 & \text{if } n \in \{5, 7, 9, 10, 14, 17\} \\ 3 & \text{otherwise.} \end{cases}$$

[3]Every vertex $v \in V(G)$ is adjacent to at most Δ distinct vertices, so it is easy to see that the second neighbourhood of v has size at most Δ^2, hence $\Delta(G^2) \leq \Delta(G)^2$.

[4]List assignments are defined on page 10.

Family $\mathcal{F}$	Upper/Lower Bound	Reference
Max degree Δ	$c\frac{d^{3/2}}{(\log d)^{1/2}} \leq \chi_{st}(\mathcal{F}) \leq \lceil 20\Delta^{3/2} \rceil$	[23]
Trewidth t	$\chi_{st}(\mathcal{F}) \leq t(t+3)/2 + 1$	[23]
$K_{n,m}$	$\chi_{st}(\mathcal{F}) = \min\{n,m\} + 1$	[23]
C_n	$\chi_{st}(\mathcal{F}) = \begin{cases} 4 & \text{if } n = 5 \\ 3 & \text{otherwise.} \end{cases}$	[23]
$2 \times m$ grid	$\chi_{st}(\mathcal{F}) = \begin{cases} 2 & \text{if } m = 2 \\ 3 & \text{otherwise.} \end{cases}$	[23]
$n \times m$ grid, $n, m \geq 3$	$\chi_{st}(\mathcal{F}) = \begin{cases} 4 & \text{if } n = 3 \text{ or } m = 3 \\ 5 & \text{otherwise.} \end{cases}$	[23]
d-dimensional grid	$2 + \left\lfloor d - \sum_{i=1}^{d} 1/n_i \right\rfloor \leq \chi_{st}(\mathcal{F}) \leq 2d + 1$	[23]
Cubic	$6 \leq \chi_{st}(\mathcal{F}), \quad \chi_{st,ch}(\mathcal{F}) \leq 7$	[1, 23]
Outerplanar	$\chi_{st}(\mathcal{F}) = 6$	[23]
Planar, girth g	$\chi_{st}(\mathcal{F}) \leq \begin{cases} 4 & \text{if } g \geq 13 \\ 5 & \text{if } 9 \leq g \leq 12 \\ 6 & \text{if } g = 8 \\ 9 & \text{if } g = 7 \\ 14 & \text{if } 5 \leq g \leq 6 \\ 18 & \text{if } g = 4 \\ 20 & \text{otherwise.} \end{cases}$	[1, 13, 23, 42]
Planar bipartite	$8 \leq \chi_{st}(\mathcal{F}), \quad \chi_{st,ch}(\mathcal{F}) \leq 14$	[34]

Table 2.1: Lower and upper bounds on $\chi_{st}(\mathcal{F})$ where $\mathcal{F}$ is a family of graphs. Note that $\chi_{st}(\mathcal{F}) = \max_{G \in \mathcal{F}}\{\chi_{st}(F)\}$

2.4.2 List Colourings

A common generalization of the colouring problem first studied by Vizing [52] consists of restricting the colour of each vertex to a list of colours specific to that vertex. Lists of different vertices do not have to be the same, but they all have the same size. The problem is then to find the smallest list size such that a proper colouring of the graph exists for any possible list assignment. More formally, a *l-list assignment* of a graph G is a function $\mathcal{L}$ which assigns to each vertex $v \in V(G)$ a set $\mathcal{L}(v)$ of l allowed colours.

A *vertex $\mathcal{L}$-colouring* of G is a (proper) vertex colouring c in which $c(v) \in \mathcal{L}(v)$ for every vertex $v \in V(G)$. We say a graph is l-choosable if for any l-list assignment $\mathcal{L}$ of G, a vertex $\mathcal{L}$-colouring of G exists. The choice number of G, $\mathrm{ch}(G)$, is the least l for which G is l-choosable.

The problem is also well-studied in the nonrepetitive setting [18, 24, 25, 29, 35, 46, 47, 48]. A *nonrepetitive vertex $\mathcal{L}$-colouring* of G is a nonrepetitive vertex colouring c of G in which $c(v) \in \mathcal{L}(v)$ for every vertex $v \in V(G)$. A graph is nonrepetitively l-choosable if for any l-list assignment $\mathcal{L}$, a proper nonrepetitive vertex $\mathcal{L}$-colouring of G exists. The Thue choice number of G, $\pi_{ch}(G)$, is the least l for which G is nonrepetitively l-choosable. Notice that the Thue chromatic number is a special case of this problem in which $\mathcal{L}(u) = \mathcal{L}(v)$ for all $u, v \in V(G)$. Thus, $\pi(G) \leq \pi_{ch}(G)$ for any graph G.

The choice number of a graph is generally larger than the chromatic number. For example, there exists planar graphs that require lists of size 5 while there exists a four-colouring of any planar graph [53]. Even more extreme, the choice number of some bipartite graphs can be arbitrary large while the chromatic number of any bipartite graph is at most 2 [21]. For the nonrepetitive colouring problem, we do not know if this is the case. As we will see later, a conjecture states that the chromatic and choice nonrepetitive numbers are the same for some variant of nonrepetitive colourings.

The *Lovász Local Lemma* is a very famous technique introduced by Lovász [20] that can be used to prove the existence of some objects given a set of constraints. Many upper bound proofs on the chromatic number or choice number of graphs are based on this technique [3, 4, 23, 26, 31, 38, 39]. Recently, Moser and Tardos [41] developed a new algorithmic procedure that simplifies the use of the Lovász Local Lemma. The method is usually applicable anytime the Lovász Local Lemma is, and generally gives tighter bounds for the list colouring problem. Using this technique in a groundbreaking extension of Thue's original result, Grytczuk et al. [29] showed that $\pi_{ch}(P_n) \leq 4$, and conjectured that the bound is actually 3. The proof is based on a remarkably simple construction which can be summarized as follows: Let $\mathcal{L}$ be any 4-list assignment of P_n. For each vertex v starting at one end of the path, pick its colour at random in $\mathcal{L}(v)$ and check if this creates a repetition. If yes, erase the second half of the repetition and continue from the first uncoloured vertex, otherwise go to the next vertex and repeat until the graph is fully coloured. It can be shown that this algorithm terminates (in an expected linear number of steps), which is enough to prove the existence of a nonrepetitive colouring for any list assignment. This implies that for cycles, $\pi_{ch}(C_n) \leq 5$. However, Dujmović et al. asked the following:

Open Problem 2.1 (Dujmović et al. [18])**.** *Is* $\pi_{ch}(C_n) \leq 4$ *for every* $n \geq 1$*? For which* n *is* $\pi_{ch}(C_n) = 3$*?*

2.4.3 Maximum Degree

The most generic upper bound for the Thue chromatic number is a function of the maximum degree of the graph. It was first proved by Alon et al. [3] in 2002 using the Lovász Local Lemma, thus giving a result for the Thue choice number, and states that that there exists a constant c such that $\pi_{ch}(G) \leq c\Delta(G)^2$. The bound is almost tight: in the same paper, Alon et al. also prove that there exists graphs with Thue chromatic number at least $c\Delta^2/\log\Delta$ for some constant c, again with a probabilistic argument. The value of c in the upper bound was originally $2e^{16}$, but was improved successively to 36, 16, 12.92 and finally to $1 + o(1)$, respectively by Grytczuk [28], Grytczuk [27], Harant and Jendrol [30] and Dujmović et al.[18].[5] Thus, we have the following result:

Theorem 2.2 (Dujmović et al.[18])**.** *Let* G *be a graph. Then,*

$$\pi_{ch}(G) \leq (1 + o(1))\Delta(G)^2.$$

A related problem posed by Alon et al. [3] is the following: Let

$$\pi(k) = \max\{\pi(G) \mid G \text{ is a graph and } \Delta(G) \leq k\}.$$

Open Problem 2.2 (Alon et al. [3])**.** *The upper bounds for cycles and paths imply that* $\pi(2) \leq 4$*. For* $k \geq 3$*, Theorem 2.2 implies that* $\pi(k) \leq (1+o(1))k^2$*. Can we find a tighter bound for* $\pi(3)$*, or for any other value of* k*?*

2.4.4 Planar Graphs

One of the most famous unsolved problem in nonrepetitive colourings, if not the most, concerns the Thue chromatic number of planar graphs. While bounds based on the maximum degree also apply to planar graphs, there is no indication that the Thue chromatic number is a function of the maximum degree for these graphs. In fact,

[5]For 12.92, with a sightly different formula of $12.92(\Delta(G) - 1)^2$.

the best lower bound is 11 [17], compared to $c\Delta^2/\log\Delta$ (for some constant c) with general graphs [3].[6] This suggests the following conjecture:

Conjecture 2.1 (Alon et al. [3]). *There is a constant K such that $\pi(G) \leq K$ for any planar graph G.*

Although it not yet known if this conjecture is true, some alternative bounds based on the number of vertices n exist. Using a naive application of the Lipton-Tarjan planar separator theorem, one can get an upper bound of $O(\sqrt{n})$ for the Thue chromatic number of planar graphs [17]. An improvement over this using layered separators yields a better bound, which is currently the tightest known upper bound on the Thue chromatic number of planar graphs:

Theorem 2.3 (Dujmović et al. [17]). *Let G be a planar graph on n vertices. Then, $\pi(G) \leq 8(1 + \log_{3/2} n)$.*

In many cases, Conjecture 2.1 is true if restricted to some families of planar graphs or for some variants of the problem. Dujmović et al. also proved that if we only consider paths of size at most $2k$ for some constant k (thus allowing repetitive paths of size greater than $2k$), the conjecture is true:

Theorem 2.4 (Dujmović et al. [17]). *There exists a constant c such that, for every integer $k \geq 1$, every planar graph G can be coloured with at most c^{k^2} colours such that G does not contain a repetitive path of length at most $2k$.*

The case where $k = 2$ (paths of length 4) is equivalent to a star colouring, and as showed by Albertson et al, $\chi_{st}(G) \leq 20$ for any planar graph G [1]. In the next section, we will discuss a result concerning graphs of bounded treewidth, from which many constant upper bounds can be derived.

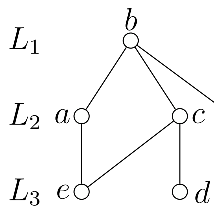
2.4.5 Graphs of Bounded Treewidth

For a graph G and a levelling λ of G, $G_{\lambda=k}$ is the graph induced by $\{v \in V(G) \mid \lambda(v) = k\}$ and $G_{\lambda>k}$ is the graph induced by $\{v \in V(G) \mid \lambda(v) > k\}$. Let H be a subgraph of G. The *k-shadow* of H is the set

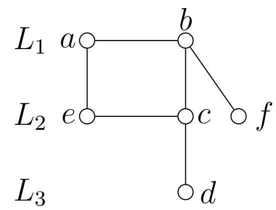
$$\{v \in V(G_{\lambda=k}) \mid u \in V(H) \text{ and } \{v, u\} \in E(G)\}.$$

[6]There is no known planar graph with Thue chromatic number greater than 11 while there exists non-planar graphs with Thue chromatic number of at least $c\Delta^2/\log\Delta$.

We say that G is *shadow-complete* with respect to λ if for every k, the k-shadow of every component of $G_{\lambda>k}$ induces a clique (see Figure 2.1). These definitions lead to the following theorem which is used to find an upper bound on various types of graphs, as will be seen later.



(a) For each component of $G_{\lambda>1}$, $\{a, c, d, e\}$ and $\{f\}$, the 1-shadow is $\{b\}$, which is a clique. The components of $G_{\lambda>2}$ are $\{e\}$ and $\{d\}$. The 2-shadow of $\{e\}$ is $\{a, c\}$, which is not a clique. Therefore G is not shadow complete with respect to λ_1.



(b) $G_{\lambda>1}$ has two components, $\{c, d, e\}$ and $\{f\}$. The 1-shadow of $\{c, d, e\}$ is $\{a, b\}$, which is a clique, and the 1-shadow of $\{f\}$ is $\{b\}$, also a clique. $G_{\lambda>2}$ has one component, $\{d\}$, and its 2-shadow is $\{c\}$, which is a clique. Therefore, G is shadow complete with respect to λ_2.

Figure 2.1: Two levellings λ_1 (a) and λ_2 (b) of a graph G. G is shadow complete with respect to λ_2 while it is not with respect to λ_1.

Theorem 2.5 (Kündgen and Pelsmajer [36])**.** *If a graph G is shadow-complete under a levelling λ and for every k, $\pi(G_{\lambda=k}) \leq b$, then $\pi(G) \leq 4b$.*

This proof is based on a nonrepetitive palindrome-free colouring of the levels, combined with a nonrepetitive colouring of each $G_{\lambda=k}$. A nonrepetitive palindrome-free sequence can be constructed from any ternary nonrepetitive sequence by adding a fourth symbol between blocks of size 2 [10]. The colour of each vertex is the cross product of its colour in $\pi(G_{\lambda=k})$ and of the colour associated to its level. The next lemma shows that chordal graphs are shadow complete with respect to the partition given by $N^k(x)$:

Lemma 2.6 (Kündgen and Pelsmajer [36])**.** *Let G be a connected chordal graph with clique number $\omega > 1$ and let x be any vertex. G is shadow-complete with respect to the levelling $\lambda(v) = k$ for $v \in N^k(x)$, and every $G_{\lambda=k}$ is a chordal graph with clique number $< \omega$.*

As maximal outerplanar graphs are chordal, this result extends to these graphs. K_4 is a forbidden minor in outerplanar graphs, so the clique number of an outerplanar graph is at most 3. This implies that for each k, $\omega(G_{\lambda=k}) \leq 2$, so each $G_{\lambda=k}$ is a forest of paths and has a nonrepetitive colouring with three colours, which gives a bound of $4 \times 3 = 12$. Removing edges cannot create repetitions, thus the bound also applies to any outerplanar graph, proving that Conjecture 2.1 is true for this family. Note that while it does not affect the Thue chromatic number, removing edges from a graph can affect some nonrepetitive colouring variants, as will be seen later. We mention that a similar proof for this result was found independently by Barát and Varjú [6].

Theorem 2.7 (Barát and Varjú [6], Kündgen and Pelsmajer [36]). *Let G be an outerplanar graph. Then, $\pi(G) \leq 12$.*

Since every graph of treewidth t is also a subgraph of a chordal graph with clique number $t + 1$, by using an argument based on Lemma 2.6 Kündgen and Pelsmajer also show that Conjecture 2.1 is true for graph families of bounded treewidth:

Theorem 2.8 (Kündgen and Pelsmajer [36]). *Let G be a graph of treewidth at most t. Then, $\pi(G) \leq 4^t$.*

Note that the treewidth of a planar graph can be arbitrary large, so this result does not confirm Conjecture 2.1 for all planar graphs. The bound of 4 for trees ($t = 1$) had already been shown by Alon et al. [3] and is best possible, as Brešar et al. showed that there exist trees with Thue chromatic number of 4 [10]. However, for $t \geq 2$, it is not known if the bound is tight. It is known that there exist graphs of treewidth t with $\chi_{st} = \binom{t+2}{2}$ [1], so by Equation 2.1, the upper bound for graphs of treewith t is at least $\binom{t+2}{2}$. Thus, if $\mathcal{F}_t$ is the family of graphs of treewidth t:

$$\binom{t+2}{2} \leq \pi(\mathcal{F}_t) \leq 4^t. \tag{2.3}$$

2.4.6 Trees

As we mentioned in the previous section, the Thue chromatic number of a tree (and by extension, of a forest) is bounded by 4, which is tight in many cases [10]. A remarkably simple proof goes as follows: Let T be a tree. Root T at some vertex r and construct a nonrepetitive palindrome-free sequence $S = s_0, s_1, \ldots, s_h$, where h is the height of the tree. Let c be a colouring defined as $c(v) = s_{d(r,v)}$. This colouring is nonrepetitive: take any maximal path P in T. Since S is nonrepetitive, the level pattern of P cannot be strictly monotone. Therefore, the colour sequence of P must contain a palindrome around the vertex with lowest depth on P. Since P is repetitive, this palindrome must be repeated somewhere else in the colour sequence of P. But, this corresponds to a palindrome in S, which is a contradiction.

Although the bound of 4 is tight in many cases, there exists many trees for which three colours suffice. Note that two colours are only sufficient if the tree is a star. The *eccentricity* $\epsilon_G(u)$ of a vertex u in a graph G is the distance from u to the vertex farthest from u, that is $\epsilon_G(u) = \max_{v \in V(G)} d_G(u, v)$. The *radius* of a graph G is the least eccentricity for all $u \in V(G)$, that is, $\text{radius}(G) = \min_{v \in V(G)} \epsilon_G(v)$. The set of vertices with minimum eccentricity is denoted as the *centre* of the graph:

$$\text{centre}(G) = \{v \in V(G) \mid \epsilon_G(v) = \text{radius}(G)\}.$$

Brešar et al. showed that if T is a tree with radius at most 4, then T has a nonrepetitive colouring with three colours [10]. The proof is almost the same as in the general case, but we select the root to be a vertex $r \in \text{centre}(G)$ and use the sequence $S = 21020$, which is nonrepetitive and palindrome-free. However, this is not a necessary condition, as trees with Thue chromatic number of 3 and with arbitrary large radius exist. Of course, a long path is such an example, but one interesting such family of trees is the set of all possible subdivisions of $K_{1,3}$ [10].

For the Thue choice number, Fiorenzi et al. [24] and Kozik and Micek [35] respectively give a lower bound of $\log \Delta / \log \log \Delta$ and an upper bound of $c\Delta^{1+\epsilon}$ on $\pi_{ch}(\mathcal{T})$ for the family $\mathcal{T}$ of trees. This contrasts to $\chi_{ch}(\mathcal{T})$ which is 3.

Family	Lower bound	Upper bound	Reference
Path	3		[50]
Cycle, n vertices	$\begin{cases} 4 & \text{if } n \in \{5,7,9,10,14,17\} \\ 3 & \text{otherwise.} \end{cases}$		[15]
Tree	4		[3]
Grid	16		[36]
Treewidth k	$\binom{t+2}{2}$	4^k	[36]
Outerplanar	7	12	[6, 36]
Planar, n vertices	11	$8(1 + \log_{3/2} n)$	[17]
Maximum degree Δ	$c\Delta^2 / \log \Delta$	$(1 + o(1))\Delta^2$	[18]

Table 2.2: Summary of upper and lower bounds on the Thue chromatic number for different families of graphs. The lower bound is the highest known value of the Thue chromatic number.

2.4.7 Subdivided Graphs

By subdividing edges, one can reduce the number of colours required for a nonrepetitive colouring. This is, however, at the expense of an increased size of the graph, depending on the number of subdivisions per edge. Let

$$\pi_S(G) = \min\{\pi(H) \mid H \text{ is a subdivision of } G\}.$$

For any cycle C, it is easy to see that $\pi_S(C) = 3$. For any tree T, there also exists a subdivision H of T with $\pi(H) = 3$, as shown in [10]. As mentioned by Grytczuk [27], one can get a subdivision H with $\pi(H) \leq 5$ by subdividing every edge of G with a distinct odd number of vertices. The original vertices of G all get the same colour, say 1, and the middle vertex of each subdivided edge gets colour 2. Then, each path between an original and a middle vertex is nonrepetitively coloured with the three remaining colours. Repetitions are avoided since the length of each path between two original vertices is distinct. This technique requires $\Omega(n)$ subdivisions per edge.

Improving on this result, Barát and Wood [7] found that for any graph G, $\pi_S(G) \leq 4$. In their proof, the vertices of G are ordered $v_0, v_1, \ldots, v_n$, and each edge $\{v_i v_j\}$, $i \leq j$ is subdivided $j - i - 1$ times. This forms a levelling λ of the graph, which allows them to use a colouring where every repetitive path $x_1, \ldots, x_t y_1, \ldots, y_t$ has $\lambda(x_i) = \lambda(y_i)$, which they use to come to the conclusion that no repetition can exist.

It was conjectured by Grytczuk [27] and Barát and Wood [7] that $\pi_S(G) \leq 3$. This conjecture was proven true by Pezarski and Zmarz [43]. The proof is based on the existence of families of strings $\{\alpha_i : 1 \leq i \leq m\}$ for every natural number m over $\{0, 1, 2\}$. These strings all have the same length, are nonrepetitive, are unique and not the reverse of any other string α_j, all begin with 1020 and end with 2021, and do not contain 010 or 212. In the subdivision H, every vertex of the original graph is assigned colour 1 and every edge is subdivided $2|\alpha_i| + 9$ times. Each edge e_i is assigned a sequence of colours based on a clever arrangement of its assigned sequence α_i along with some padding. Although original and subdivision vertices share the same colours, which was avoided by previous proofs since it makes it harder to avoid repetitions, the authors were able to prove that H is nonrepetitive by using the fact that every α_i can occur at most once in each path of the subdivision. This proof is a great achievement, in that it gives the best possible bound on the number of colours required in a subdivision (since most paths require at least three colours), while still keeping a linear number of subdivisions per edge. Table 2.3 summarizes these results.

As a complement of these results, Dujmovic et al. [18] showed an upper bound on the number of colours required for a nonrepetitively k-choosable subdivision. They

Family	Upper bound on π_S	Subdivisions per edge	Reference
Cycles	3	≤ 1	Obvious
Trees	3	$O(2^n)$	[10]
All	5	$O(n)$	[27]
All	4	$O(n)$	[7]
All	3	$O(n)$	[43]

Table 2.3: Results about the Thue chromatic index of subdivisions of different families of n-vertex graphs.

prove that if H is a subdivision of a graph G on n vertices with at least

$$\lceil 10^5 \log(\deg(u) + 1)\rceil + \lceil 10^5 \log(\deg(v) + 1)\rceil + 2 \tag{2.4}$$

subdivisions for each edge $\{u, v\} \in E(G)$ (which simplifies to $O(\log \Delta) \in O(\log n)$ subdivisions per edge), then $\pi_{ch}(H) \leq 5$. While more colours are needed than the subdivision of Pezarski and Zmarz, this result requires less subdivisions per edge and is applicable to the Thue choice number. It is also stronger in that *any* subdivision satisfying this condition is nonrepetitively five-choosable, compared to the existence of a subdivision in the other results. They also ask the following question:

Open Problem 2.3 (Dujmovic et al. [18]). *Does every graph have a nonrepetitively 3 or 4-choosable subdivision?*

We end with an interesting conjecture that relates to Conjecture 2.1:

Conjecture 2.2 (Grytczuk [28]). *There exists constants r and c such that every planar graph has a subdivision H with at most r subdivisions per edge such that $\pi(H) \leq c$.*

2.5 Thue Chromatic Index

A well studied nonrepetitive colouring variant consists in colouring the edges of the graph (instead of its vertices) in such a way that any trail must be nonrepetitive. In fact, the original paper on the application of nonrepetitive sequences on graphs by Alon et al. [3] discussed nonrepetitive edge colourings. A *nonrepetitive edge k-colouring* of a graph G is an edge k-colouring $c : E(G) \to \{1, \ldots, k\}$ in which for every trail $e_1, \ldots, e_k$ in G, the sequence $c(e_1), \ldots, c(e_k)$ is nonrepetitive. The *Thue chromatic index* of a graph G, denoted $\pi'(G)$, is the least k for which a nonrepetitive edge k-colouring of G exists. Note that in the original paper by Alon et al., the Thue chromatic index was referred to as the Thue chromatic number and was denoted by $\pi(G)$, but to stay consistent with the notation for the chromatic number (vertex colouring), denoted $\chi(G)$, and the chromatic index (edge colouring), denoted $\chi'(G)$, later literature has preferred the terms and symbols we use in this text.

Similarly to nonrepetitive vertex colourings, every proper nonrepetitive edge colouring is also a proper edge colouring. Thus, we have that

$$\Delta(G) \leq \chi'(G) \leq \pi'(G). \tag{2.5}$$

Thus, no constant bound can exist for graphs with unbounded degree. Furthermore, in the case of paths and cycles (except P_n for $n \leq 3$), the Thue chromatic index is equal to the Thue chromatic number. For trees, the tightest known upper bound is $4(\Delta - 1)$, which is due to Alon et al. [3]. Similarly to the proof for the Thue chromatic number of trees, this proof is also based on palindrome-free nonrepetitive sequences. However, it is not known if the bound is tight, as the best lower bound is currently $\frac{3}{2}\Delta$, due to Sudeep and Vishwanathan [49]. Brešar et al. [10] showed that every tree T has a subdivision H with $\pi'(H) \leq \Delta(T) + 1$. They also show that the Thue chromatic index of any subdivision of a star S_k is k, and it is easy to see that this is tight for $k \geq 3$ since $k = \Delta(S_k)$. Table 2.4 provides a summary of the tightest known upper bounds for the Thue chromatic index.

Family	Upper bound	Reference
d-dimension grid	$3d$	[3]
K_n	$2n - 3$	*Ibid.*
Tree with maximum degree Δ	$4(\Delta - 1)$	*Ibid.*
Maximum degree Δ	$c\Delta^2$	*Ibid.*
Subdivisions of S_k	k	[10]

Table 2.4: Upper bounds on the Thue chromatic index for various families of graphs.

2.6 Total Thue Chromatic Number

A *total colouring* is a colouring in which both edges and vertices of a graph are coloured such that no adjacent edges, and no edge and its endpoints, are assigned the same colour. Such colourings were also studied in the nonrepetitive setting by Schreyer and Škrabul'áková [48]. They defined two versions of the problem. A *weak total nonrepetitive colouring* of a graph G is a k-colouring in which the sequence of consecutive vertex and edge colours of every path in G is nonrepetitive. The least

k for which such a colouring exists is called the *weak total Thue chromatic number*, $\pi_{T_w}(G)$. If, in addition, the colouring is such that both the induced edge and vertex colourings are (proper) nonrepetitive colourings, the colouring is called a (strong) total nonrepetitive colouring and $\pi_T(G)$, the *total Thue chromatic number* is the least k for which such a colouring of G exists.

(a) Total strong colouring of a graph. This is also a valid nonrepetitive vertex colouring and edge colouring of the graph.

(b) Total weak colouring of a graph. Note that pairs of adjacent edges or pairs of adjacent vertices are allowed to share a colour.

Figure 2.2: Total nonrepetitive colourings. Labels inside the vertices and adjacent to the edges represent colours.

It is easy to see that a weak total nonrepetitive colouring of a graph G is equivalent to a nonrepetitive vertex colouring of a graph formed by subdividing every edge of G once. Such a colouring may also be formed by taking a vertex (or an edge) nonrepetitive colouring of G, and colouring every edge (vertex) with an additional colour. The resulting colouring is a weak total nonrepetitive colouring of G. This implies the following:

$$\pi_{T_w}(G) \leq \min\{\pi(G), \pi'(G)\} + 1. \tag{2.6}$$

Note that by definition, a total nonrepetitive colouring is also a proper vertex and edge nonrepetitive colouring, but the same is in general not true of a weak total nonrepetitive colouring, since adjacent edges and vertices may share the same colour. Also, we may always create a (strong) nonrepetitive colouring by colouring the vertices with a nonrepetitive colouring c_V and the edges with a nonrepetitive colouring c_E. If the intersection of the codomains of c_V and c_E is empty, then the union of c_V and c_E is a valid total nonrepetitive colouring. Thus,

$$\max\{\pi(G), \pi'(G)\} \leq \pi_T(G) \leq \pi(G) + \pi'(G). \tag{2.7}$$

The difference between $\pi_T(G)$ and $\pi_{T_w}(G)$ can be arbitrarily large. An easy example is the star graph. Schreyer and Škrabul'áková proved that $\pi_T(S_n) = n + 1$, which is tight. But, it is easy to see that three colours suffice for a weak total nonrepetitive colouring of any star graph: Assign a first colour to the middle vertex, a second common to all the edges and a third for all the leaves. No repetition can occur since the colour of the middle vertex is unique.

One interesting problem raised by Schreyer and Škrabul'áková is to find a tight bound for the total Thue chromatic number of paths. Note that the weak total Thue chromatic number of a path is the same as its Thue chromatic number if it has at least four vertices. However, for the strong version, four colours are necessary for paths of length at least 4: Suppose that three colours are sufficient. We need a first colour for the first vertex (1), a second for the first edge (2), a third for the second vertex (3). Colour 1 must be reused for the second edge, and colour (2) must be reused for the third vertex. Everything is good so far, but for the third edge, colours 1 and 2 are unavailable, and colour 3 would form a repetition. Therefore, four colours are necessary.

Schreyer and Škrabul'áková provided a constructive proof which shows that the total Thue chromatic number is bounded by 5 for paths of arbitrary length. Therefore,

$$4 \leq \pi_T(P_n) \leq 5 \text{ if } n \geq 4, \tag{2.8}$$

which implies that for cycles of at least four vertices, $4 \leq \pi_T(P_n) \leq 6$. In addition to these results, Schreyer and Škrabul'áková show that the total Thue chromatic is also bounded by a quadratic bound on the degree of the graph, similar to the Thue chromatic number and Thue chromatic index. In particular, they show the following theorem:

Theorem 2.9. *If G is a graph with $\Delta \geq 3$, then $\pi(G) \leq 14\Delta^2$.*

Furthermore, they also prove that a similar bound holds for the list version of the problem, with a constant of 17.9856.

2.7 Facial Nonrepetitive Colourings

In this section, we discuss previous work on facial nonrepetitive colourings, a relaxation of nonrepetitive colourings for plane graphs in which we only require facial paths — paths that lie on the boundary walk of a face — to be nonrepetitive (see Figure 2.3). Recall that the facial Thue chromatic number of a plane graph G, denoted $\pi_f(G)$, is the least k for which a facial nonrepetitive k-colouring of G exists. An edge colouring variant of this problem was first studied by Havet et al. [32], then Harant and Jendrol [30] followed up with results for the vertex colouring setting. We first discuss the vertex colouring problem, which is the main topic of this thesis.

2.7.1 Facial Vertex Colourings

Similarly to Conjecture 2.1 for the Thue chromatic number, it was also conjectured by Harant and Jendrol [30] that there exists a constant K such that $\pi_f(G) \leq K$ for any plane graph G. They showed that the conjecture is true for 2-connected cubic graphs, with an upper bound of 112, and for some other families of 2-connected plane graphs such as Hamiltonian plane graphs ($K = 16$). The conjecture was subsequently confirmed true for all plane graphs with an upper bound of 24 by Barat and Czap [5].

Any nonrepetitive colouring is also a facial nonrepetitive colouring, so upper bounds for $\pi(G)$ also apply to $\pi_f(G)$, while the converse is certainly not true — except for very simple graphs such as paths and cycles — as Figure 2.3 demonstrates. However, since there exists a facial path between any two adjacent vertices, such a colouring is certainly a proper vertex colouring. Thus,

$$\chi(G) \leq \pi_f(G) \leq \pi(G). \tag{2.9}$$

It is not known how tight the upper bound of 24 for the facial Thue chromatic number of plane graphs is, but no plane graph G with $\pi_f(G) > 5$ is known, which suggests than the constant can almost certainly be reduced. As noted by Barat and Czap [5], a plane graph G with $\pi_f(G) = 5$ is the 5-wheel:[7] the outer cycle needs four colours, and the inner vertex is adjacent to at least one vertex of each colour, thus five colours are required. Similarly, for outerplane graphs, no bound specific to the facial Thue chromatic number is known other than Theorem 2.7, which gives an upper bound of 12, while no outerplane graph with facial Thue chromatic number greater than four — $\pi_f(C_5) = 4$ — is known. Table 2.5 summarizes the current upper bounds on the facial Thue chromatic number.

An interesting (and simple looking!) problem is the to find a tight upper bound for the facial Thue chromatic number of trees. For the classical Thue chromatic number, we mentioned earlier that some trees require at least four colours. However, the exact same examples have a facial nonrepetitive colouring with three colours. This suggests the following problem:

Open Problem 2.4. *What is the facial Thue chromatic number of trees. It must be*

[7] A k-wheel W_k (for $k \geq 3$) is a graph formed by connecting a single vertex to all vertices of an k-cycle.

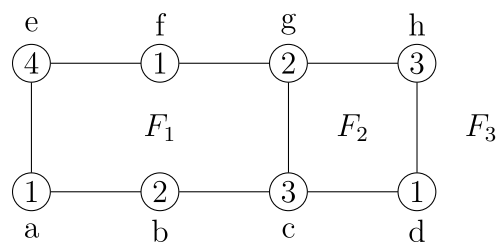


Figure 2.3: A valid facial nonrepetitive colouring (labels inside the vertices represent colours) of the vertices of this graph (notice that this is not a valid "classical" nonrepetitive colouring). a, b, d, c is a facial path on F_3, b, c, g, f is another one on F_1. b, c, g, h is not a facial path, since even if all its vertices lay on the same face (F_3), they are not consecutive on a boundary walk of F_3.

at least 3, and is bounded by 4. Is there an example of a tree that requires 4 colours in a facial colouring?

Family	Current	This thesis	Lower Bound	Reference
Plane	24	22	5	[5]
Hamiltonian	16		4	[30]
$n \times n$ grid	4		3	[5]
Outerplane	12	11	4	[6, 36]
Outerplane, 2-connected		7	4	

Table 2.5: Upper bounds on the facial Thue chromatic number for different families of plane graphs. Lower bounds for plane Hamiltonian and outerplane graphs are derived from a 5-cycle.

2.7.2 Facial Edge Colourings

We now discuss a problem related to facial nonrepetitive colourings in which we wish to colour edges instead of vertices. A *facial trail* of G is a trail on G that consists of consecutive edges on the boundary walk of a face of G. A facial nonrepetitive edge colouring of a plane graph G is a colouring such that the sequence of colours on any facial trail of G is nonrepetitive. The *facial Thue chromatic index* of a graph G, denoted $\pi'_f(G)$, is the least k for which a nonrepetitive facial edge k-colouring of G exists.

Although they may look similar to facial vertex nonrepetitive colourings, the facial edge nonrepetitive colouring problem is quite different. One major difference is that a facial edge nonrepetitive colouring is not necessarily a proper edge colouring of the graph, since edges not on a same facial trail can have the same colours even if they are adjacent to the same vertex. For example, a proper edge facial nonrepetitive colouring of a star S_k can be done with at most four colours, while a proper edge colouring of the same star requires $k+1$ colours. However, we do have that every nonrepetitive edge colouring of a plane graph is also a valid facial edge nonrepetitive colouring of the same graph. Furthermore, facial nonrepetitive edge colourings are in a way easier to find than facial nonrepetitive vertex colourings, since each edge belongs to at most two faces, while vertices can be adjacent to Δ other vertices. Using this observation, Havet et al. showed that the facial Thue chromatic index is bounded by a constant for planar graphs. In their proof, they first colour a spanning tree of the graph with four colours. They then show that the edges not in the spanning tree can be coloured using a set of four additional colours without creating repetitive facial trails, which gives a bound of 8. The bound can be further reduced for some families of plane graphs (see Table 2.6). However, Havet et al. believe that most of these bounds are not tight, and in particular they believe that the facial Thue chromatic index for 3-connected plane graphs is at most 6. They ask the following:

Open Problem 2.5 (Havet et al. [32]). *What is the smallest K such that $\pi'_f(G) \leq K$ for all plane graphs G?*

Using the Moser and Tardos technique, Schreyer and Škrabul'áková [47] proved that a constant bound also exists for the facial Thue choice index — the smallest l such that in any l-list assignment of the edges of a graph G, a facial nonrepetitive edge colouring of G exists, which is denoted $\pi'_{f,ch}(G)$. They found an upper bound of 291 for this parameter, which was later improved to 12 by Przybyło [46], and most recently to 10 by Gonçalves et al. [25]. Observing that no known graph has a facial Thue chromatic index different than its facial Thue choice index, Schreyer and Škrabul'áková conjectured the following:

Conjecture 2.3 (Schreyer and Škrabul'áková [47]). *For any plane graph G, $\pi'_f(G) = \pi'_{f,ch}(G)$.*

2.8 Complexity

For sake of completeness, we now take a look at the algorithmic complexity of the nonrepetitive colouring problem. In the generic case, a result by Marx and Schaefer

[8] An *almost even* tree is a tree in which at most one vertex of degree greater than 1 has odd degree. Equivalently, all non-leaves, except one, have even degree.

Family $\mathcal{F}$	$\pi'_f(\mathcal{F})$
Plane	8
3-connected	7
Triangulation	3
Outerplane, 2-connected	7
Plane, dual contains Hamiltonian cycle	6
Halin	6
Tree	4
Almost even tree[8]	3

Table 2.6: Upper bounds on the facial Thue chromatic index for different families of plane graphs. All results from Havet et al. [32]

[37] states that it is coNP-complete to decide if a colouring of a graph is nonrepetitive, even if the colouring is on at most four colours. Even for star colourings, Coleman and Moré [14] showed that is NP-complete to decide if a graph G has a star-free colouring on three colours, even if G is bipartite. However, the problem of deciding if a graph contains a repetitive path of size $2k$ is polynomial for a given k:

Theorem 2.10 (Marx and Schaefer [37])**.** *Given a graph G on n vertices and a colouring C of G, it can be checked in $k^{O(k)} \cdot n^5 \log n$ whether G has a repetitive sequence of length $2k$.*

2.9 Other Variants

We finish this chapter by discussing several other variants of the nonrepetitive colouring problem.

2.9.1 Thue Threshold

The Thue threshold is a generalization of Thue chromatic number. A κ-power is a sequence $S = w^\kappa$. For example, w^2 is a 2-power (square), w^3 is a 3-power (cube), etc. A colouring of a graph is κ-power free if the colour sequence on no path is a κ-power. The least k for which such a k-colouring exists is denoted by $\pi_\kappa(G)$. Notice that a 2-power free colouring is equivalent to a nonrepetitive colouring, thus $\pi(G) = \pi_2(G)$. Since a κ-power free sequence is also $(\kappa+1)$-power free, we have that $\pi_k(G) \geq \pi_{\kappa+1}(G)$. Note that no 1-power colouring can exist. Similarly to $\pi(G)$, it was proved by Alon and Grytczuk [2] that $\pi_\kappa(G)$ is bounded by $\Delta(G)$. Let $\pi_\kappa(\Delta)$ be the maximum of $\pi_\kappa(G)$ for all graphs G of maximum degree Δ. Alon and Grytczuk proved that

$$\frac{c_1}{\kappa} \frac{\Delta^{\kappa/(\kappa-1)}}{(\log \Delta)^{1/(\kappa-1)}} \leq \pi_\kappa(\Delta) \leq c_2 \Delta^{\kappa/(\kappa-1)}. \tag{2.10}$$

Let $\mathcal{F}$ be a family of graphs. $\pi_\kappa(\mathcal{F}) = \sup_{G \in \mathcal{F}} \pi_\kappa(G)$. Let $t(\mathcal{F}) = \inf_{\kappa \in \mathbb{N}} \pi_\kappa(\mathcal{F})$. $t(\mathcal{F})$ is called the *Thue threshold* (or rhythm threshold) of $\mathcal{F}$. Note that $t(\mathcal{F})$ is infinite if and only if $\pi_\kappa(\mathcal{F})$ is not bounded by a constant for all κ. For example, if $\mathcal{P}$ is the family of the paths, $t(\mathcal{P}) = 2$ since $\pi_3(G) = \pi_4(G) = \cdots = 2$. A result of Currie and Fitzpatrick [16] shows that this is also true for cycles (thus for all graphs with $\Delta \leq 2$). Let $t(\Delta)$ be the maximum value of the Thue threshold for the family of graphs of maximum degree Δ. Currie and Fitzpatrick's result imply that $t(2) = 2$. No other values of $t(\Delta)$ is known, but the following was showed by Alon and Grytczuk [2]:

$$\frac{1}{2}(\Delta + 1) \leq t(\Delta) \leq \Delta + 1$$

A problem related to Conjecture 2.1 is the following:

Open Problem 2.6. *What is the Thue threshold of planar graphs?*

Note that proving Conjecture 2.1 would imply an upper bound on the Thue threshold of planar graphs as well, but the opposite would not necessarily be true. A lower bound for the Thue threshold of planar graphs can be established by the following construction [28]: Let $G_1 = K_3$ and G_n be a plane graph obtained from G_{n-1} by inserting a vertex in each inner face of G_{n-1} and connecting it to each of the three vertices on that face. The graph is clearly planar, and it can be showed that for each k, there exists n such that G_n contains a monochromatic path of length k. Thus, the Thue threshold for planar graphs is at least 3.

2.9.2 Nonrepetitive Walks

Some variants of the nonrepetitive colouring problem require that walks, instead of paths, be nonrepetitively coloured. One first version, introduced by Brešar and Klavžar [11], defines a *walk nonrepetitive edge colouring* to be an edge colouring in which the colour sequence of any acyclic edge-walk W is nonrepetitive. The least k for which such a k-colouring of a graph G exists is the *walk Thue chromatic number* of G, denoted $\pi_w(G)$. Note that $\pi'(G) \leq \pi_w(G)$.

A second version, this time defined for the vertex colouring setting, requires that the colour sequence of any non-boring walk[9] be nonrepetitive and was introduced by Barát and Wood [7]. The least k for which such a k-colouring of a graph G exists is denoted $\sigma(G)$.

Note that the restrictions on the walks (acyclic for the edge version and non-boring for the vertex version) are necessary for each case, otherwise no colouring, even one in which all edges/vertices have distinct colours, would satisfy the constraints.

2.9.3 Induced Paths

Another variant of nonrepetitive colourings was proposed by Brešar and Klavžar [28]. An *induced path* is a path that is an induced subgraph of a graph G. An *induced nonrepetitive colouring* is a colouring in which the colour sequence on any induced path is nonrepetitive. The *induced Thue chromatic number*, denoted $\pi_{ind}(G)$ is the least k for which such a k-colouring of a graph G exists. Clearly, every induced nonrepetitive colouring is also a valid vertex colouring, but not every induced nonrepetitive colouring is a nonrepetitive colouring. An important observation about the induced Thue chromatic number is that it behaves similarly to the chromatic number. We say a graph G is π_{ind}-perfect if $\pi_{ind}(G) = \omega(H)$ for each induced subgraph H of G. Brešar showed that the π_{ind}-perfect graphs are exactly the P_4-free graphs, the graphs that do not contain P_4 as an induced subgraph [28].

2.9.4 Strong Square-Free Colouring

A stronger variant of the nonrepetitive colouring problem asks that no sequence be followed by a permutation of itself. Thus, no sequence has the form ww', where w' is a permutation of w. For example, the sequence 123132 does not satisfies this condition since 132 is a permutation of 123. If no path (or trail) has a colour sequence of this form, the colouring is called a *strong square-free vertex (edge) colouring.* A result by Keränen [33] shows that such a 4-colouring exists for P_n. A slight variation of the previous problems yields an easier problem: instead of requiring that no sequence be followed by a permutation of itself, we require that no sequence be followed by a *cyclic permutation* of itself [49].

[9] A *non-boring walk* is a walk W in which the sequence of vertices on W is nonrepetitive.

Chapter 3

Facial Nonrepetitive Colourings

In this chapter, we present our results concerning nonrepetitive facial colourings of plane graphs. In Section 3.1, we discuss a new technique for the facial nonrepetitive colouring of outerplane graphs. This technique consists in finding a *blocking set*, from which we construct a *blocking graph*. The blocking set is such that its removal from the outerplane graph results in a forest, for which a nonrepetitive 4-colouring exists. We show that a facial nonrepetitive colouring of any outerplane graph can be obtained by combining this colouring of a forest and a facial nonrepetitive colouring of the blocking graph. We then proceed to show how to find a nonrepetitive colouring on at most 7 colours of the blocking graph. Although it seems hard — if not impossible — to find a such a colouring if the blocking graph contains odd cycles, we show that this is feasible if all cycles are even, and that it is possible to select the blocking set as such. This results in the following theorem:

Theorem 3.9. *Let G be an outerplane graph. Then, $\pi_f(G) \leq 11$.*

Furthermore, we show that for outerplane graphs that contain at most one 2-connected component, we can carefully select the blocking set such that the blocking graph can be coloured using at most three colours, which gives the following result:

Theorem 3.10. *Let G be an outerplane graph that contains at most one 2-connected component. Then, $\pi_f(G) \leq 7$.*

Finally, in Section 3.2, we show that a similar argument to Barát and Czap's proof for the facial Thue chromatic number of plane graphs [5] can be used to show that the facial Thue chromatic number of plane graphs is bounded by twice the upper bound of the facial Thue chromatic number of outerplane graphs:

Theorem 3.11. *Let $r = \max\{\pi_f(G) \mid G \text{ is outerplane}\}$ and let G be a plane graph. Then, $\pi_f(G) \leq 2r$.*

This directly implies by Theorem 3.9 that $\pi_f(G) \leq 22$ for any plane graph G.

We now introduce a lemma which will be used throughout the proofs. It is due to Havet et al. [32] and provides a way to interlace nonrepetitive sequences.

Lemma 3.1. *Let $A = A_1, A_2, \ldots, A_k$ be a nonrepetitive sequence over an alphabet $\mathcal{A}$ in which each A_i has size at least 1. For each $i \in \{0, \ldots, k\}$, let B_i be a (possibly empty) nonrepetitive sequence over an alphabet $\mathcal{B}$. If $\mathcal{A} \cap \mathcal{B} = \varnothing$, then $S = B_0, A_1, B_1, \ldots, A_k, B_k$ is a nonrepetitive sequence.*

3.1 Outerplane Graphs

We are now ready to present our results concerning the facial Thue chromatic number of outerplane graphs. In order to do this, we will first study *blocking sets* of outerplane graphs. Blocking sets play a crucial role in isolating subsets of vertices from each other in a graph. This splits the graph in various parts which are easier to colour, and the specific properties of the blocking set makes it possible to reassemble the graph together without creating repetitions.

3.1.1 Blocking Sets

Let G be an outerplane graph. A *blocking set* of G is a set of vertices $B \subseteq V(G)$ such that for each 2-connected component H of G, $H \setminus B$ is a tree and for each inner face F, $F \setminus B \neq \varnothing$. Thus, $G \setminus B$ is a forest. See Figure 3.1 for an example of a blocking set.

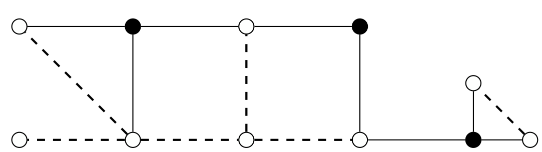


Figure 3.1: Blocking set B of an outerplane graph G, with blocking set vertices denoted in black. In dashed, edges of $G \setminus B$. Notice that it is a tree for each 2-connected component of G.

Let us now show that such a blocking set exists for all outerplane graphs. In fact, we will show something stronger: we can pick any single vertex of G and require it to be in the blocking set. This constraint will be crucial later.

Lemma 3.2. *Let G be an outerplane graph with at least $f \geq 2$ inner faces and v be a vertex of G incident to some inner face F_1. There exists a blocking set B of G such that for each inner face F of G, $|V(F) \cap B| = 1$ and $v \in B$.*

Proof. Let $F_1, F_2, \ldots, F_f$ be an ordering of the inner faces of G such that $v \in V(F_1)$ and for each face F_i, F_i shares an edge with at most one face in $\{F_1, \ldots, F_{i-1}\}$. Note that at most one edge can be shared by any two faces of an outerplane graph, otherwise a vertex would not be adjacent to the outer face. Such an ordering exists since G° is cycle-free. For each i, let G_i be the graph induced by

$$\bigcup_{1 \leq j \leq i} V(F_j). \tag{3.1}$$

We construct B as follows: For F_1, set $B_1 = \{v\}$. For each face F_i, $2 \leq i \leq f$, if $V(F_i) \cap B_{i-1} \neq \varnothing$, let $B_i = B_{i-1}$. Otherwise, select any vertex $v \in V(F_i) \setminus V(G_{i-1})$ and let $B_i = B_{i-1} \cup \{v\}$. Such a vertex exists since $|V(F_i) \cap V(G_{i-1})| \leq 2$ and each face is incident to at least three vertices. We refer to the vertices in $V(F_i) \cap V(G_{i-1})$ as the *anchors* of F_i. Recall that B_f is the blocking set after considering all faces of G. Let $B = B_f$.

Claim 3.2.1. *For each inner face F_i of G, $|V(F_i) \cap B| = 1$. Thus, $F_i \setminus B$ is a non-empty path.*

Proof. We prove both sides of the equality:

- $|V(F_i) \cap B| \geq 1$. This is true if $F_i = F_1$, and for $i \in \{2, \ldots, f\}$, we add a vertex of F_i to B_i if $V(F_i) \cap B_{i-1} = \varnothing$, so $|V(F) \cap B| \geq 1$.
- $|V(F_i) \cap B| \leq 1$. For each face F_j, $i < j \leq f$ subsequently visited, if $V(F_j) \cap B_{j-1} \neq \varnothing$, no other vertex is added, otherwise, a vertex not in $G_{j-1} \supseteq F_i$ is selected. Thus, $|V(F_i) \cap B| \leq 1$.

This completes the proof. □

Claim 3.2.2. *For each 2-connected component H of G, $H \setminus B$ is connected.*

Proof. Let $F_1^H, \ldots, F_{f_H}^H$ be the inner faces of H in the order considered by the procedure. and let G_i^H be the graph induced by

$$\bigcup_{1 \leq j \leq i} V(F_j^H). \tag{3.2}$$

We prove this by induction on i. Base case: $G_1^H = F_1^H$ is a cycle and by Claim 3.2.1 $|F_1^H \cap B| = 1$, so $G_1^H \setminus B$ is a path, which is a connected graph. Induction step: suppose this holds for G_{i-1}^H. Let B_{k-1} be the subset of B that corresponds to all the faces visited prior to F_i^H. There are two cases:

1. $|B_{k-1} \cap V(F_i^H)| = 1$. In this case, one of the anchors of F_i^H is in B_{k-1}. Thus, since F_i^H is a cycle, $F_i^H \setminus B$ is a path. This path is connected by the other anchor of F_i^H to G_{i-1}^H, which is connected. Thus, G_i^H is also connected.

2. $B_{k-1} \cap V(F_i^H) = \varnothing$. In this case, none of the anchors a_1, a_2 of F_i^H are in B_{k-1} and since $a_1, a_2 \in G_{i-1}^H$, $a_1, a_2 \notin B_k$. F_i^H is a cycle, so by Claim 3.2.1, $F_i^H \setminus B$ is a path which contains both a_1 and a_2. This path is connected to G_{i-1}^H, which is connected by the induction hypothesis, therefore G_i^H is also connected.

This completes the proof. □

Claim 3.2.3. *For each 2-connected component H of G, $H \setminus B$ is a tree.*

Proof. Suppose $H \setminus B$ is not a tree. By Claim 3.2.2, $H \setminus B$, is connected, therefore, $H \setminus B$ must contain a cycle. Let C be the smallest such cycle. C cannot correspond to some inner face F of H, as by Claim 3.2.1, $F \setminus B$ is a path. Therefore, since H is outerplane, $H[C]$ must contain a chord. Therefore, $H[C]$ contains at least two simple cycles C_1, C_2 such that $C_1, C_2 \subset C \subseteq V(H) \setminus B$, which implies that both C_1 and C_2 are smaller than C. But, we chose C to be the smallest such cycle. Contradiction. □

This completes the proof of the lemma. □

Corollary 3.3. *Let G be a 2-connected outerplane graph that contains at least two inner faces. There exists a blocking set B of G such that for each inner face F of G, $|V(F) \cap B| = 1$, and there exists an inner face F_1 of G with $\deg_{G^\circ}(F_1) = 1$ such that the vertex $v \in V(F_1) \cap B$ has $\deg_G(v) \geq 3$.*

Proof. Pick a face F_1 such that $\deg_{G^\circ}(F_1) = 1$ and a vertex $v \in V(F_1)$ such that $\deg_G(v) \geq 3$. Such a vertex exists since G is 2-connected and contains at least two inner faces. Lemma 3.2 completes the proof. □

3.1.2 Colouring Outerplane Graphs

We will now show how to use a blocking set to get a facial nonrepetitive colouring of an outerplane graph. Let G be an outerplane graph and B be a blocking set of G.

A *blocking graph* of G is a (simple) graph denoted by $\text{block}_B(G)$ with vertex set B and with an edge set defined as follows: Let $W = w_1, w_2, \ldots, w_k$ be a closed facial walk around the outer face of G such that $w_1 = w_k \in B$. For each subsequence $w_i, w_{i+1}, \ldots, w_{i+r}$ such that

a) $w_i, w_{i+1}, \ldots, w_{i+r}$ is a facial path,

b) $r > 0$,

c) $w_i, w_{i+r} \in B$ and

d) $w_{i+j} \notin B$ for all $j \in \{1, \ldots, r-1\}$,

add an edge $\{w_i, w_{i+r}\}$ to $E(\text{block}_B(G))$. Thus, every pair of vertices of B connected by a facial path on the outside face of G are connected by an edge in $\text{block}_B(G)$ (see Figure 3.2). If we remove condition (a) (thus also adding edges between vertices connected together via a facial walk that contains repeated vertices), the resulting graph is called the *augmented blocking graph*, and denoted $\text{block}^+_B(G)$.

$\text{block}_B(G)$ is a minor of G, thus it is also an outerplane graph. Furthermore, since edges only link successive vertices on W, $\text{block}_B(G)$ is chordless. A graph where every cycle is chordless is called a *cactus graph*.[1]

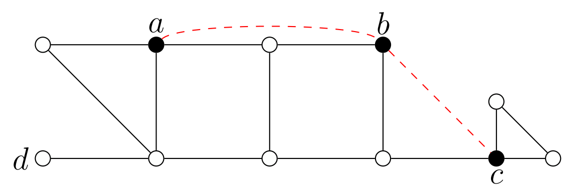


Figure 3.2: The blocking graph has the blocking set as vertex set, and an edge between each pair of vertices that are connected through a facial path (dashed). There is no $\{a, c\}$ edge since there is no facial path between them that does not include another blocking set vertex. However, were d not in the graph, the $\{a, c\}$ edge would exist.

Lemma 3.4. *Let G be an outerplane graph and B be a blocking set of G. Suppose that there exists a facial nonrepetitive k-colouring of* $\text{block}_B(G)$. *Then, there exists a facial nonrepetitive* $(4 + k)$*-colouring of G.*

We will need the following claim in our proof. Note that this claim is different from Claim 3.2.1 since a blocking set may contain multiple vertices of a same inner face (which will happen in a later proof), while Lemma 3.2 shows the existence of a blocking set with exactly *one* vertex per inner face.

[1]The most common definition says that a cactus graph is a graph in which any two simple cycles have at most one vertex is common. This is equivalent to a chordless graph.

Claim 3.4.1. *For every inner face F of G, $F \setminus B$ is a non-empty path.*

Proof. Suppose that this is not the case, thus since $F \setminus B \neq \varnothing$, there exists an inner face F where $F \setminus B$ consists at least two connected components. Let H be the 2-connected component of G that contains F. Let $u, v \in V(F) \cap B$ such that $u \neq v$ and $\{u, v\} \notin E(F)$. Two such vertices certainly exist since $F \setminus B$ consists at least two connected components. Let $H' = H \setminus \{u\}$. v is a cut vertex of H' since F is a cycle, H' is an outerplane graph and u and v are not adjacent. Thus, since $\{u, v\} \subseteq B$, $H \setminus B$ is not connected. But, by definition of a blocking set, $H \setminus B$ must be connected. Contradiction. □

Proof. Since B is a blocking set of G, $G' = G \setminus B$ is a forest and by Theorem 2.8, there exists a nonrepetitive 4-colouring of each connected component of G', which translates to a nonrepetitive 4-colouring of G'. Let $c_{G'} : V(G') \to \{1, 2, 3, 4\}$ be such a colouring, and let $c_B : B \to \{5, \ldots, 4 + k\}$ be a facial nonrepetitive k-colouring of $\text{block}_B(G)$. Such a colouring exists by our hypothesis. Let $c : V(G) \to \{1, \ldots, 4 + k\}$ be a $(4 + k)$-colouring of G defined as

$$c(v) = \begin{cases} c_B(v) \text{ if } v \in B \\ c_{G'}(v) \text{ otherwise.} \end{cases} \tag{3.3}$$

We now show that c is a facial nonrepetitive colouring of G. Suppose that this is not the case. Thus, there exists a path P on some face of G such that the colour sequence of consecutive vertices of P is a repetition. If P is also a path on G', then its colour sequence cannot be repetitive since G' is nonrepetitively coloured under c_G, and by extension, under c. Thus, P is not a path on G'. By the same reasoning, P cannot be a path on $\text{block}_B(G)$. Thus, P alternates between vertices in G' and vertices in B.

If P is on an inner face F of a 2-connected component H of G, then by Claim 3.4.1, $P' = F \setminus B$ is a non-empty path on F. This implies that $P'' = F \cap B$ is also a path — on at least one vertex — which lays on a face of $\text{block}_B(G)$ by definition of the blocking graph since P' is a path. Furthermore, by the respective colourings of $\text{block}_B(G)$ and G', P and P' will be nonrepetitive. Since P is repetitive and the intersection of the domains of c_G and c_B is empty, P must contain a block of the form X_1, Y_1, X_2, Y_2 where X_1, X_2 are vertices of $V(G')$ (or B) and Y_1, Y_2 are vertices of B (or $V(G')$ respectively) and none of X_1, X_2, Y_1, Y_2 are empty. But, observe that this contradicts our claim that P' and P'' are paths and, joined together, form a cycle — F.

Therefore, P is on the outer face of G. Let P' be the sequence of vertices of $P \cap B$ in the same order as in P. Observe that P' will be a path in $\text{block}_B(G)$. Thus, the colour sequence of P' is nonrepetitive. Furthermore, each subsequence of consecutive vertices on G' of $P \cap G'$ will also be nonrepetitive by the colouring of G'. Hence, by Lemma 3.1, P is nonrepetitive. But this contradicts our assumption that P is repetitive. This completes the proof. □

Notice that the blocking graph of an outerplane graph with exactly one 2-connected component is a cycle if $|B| \geq 3$ (otherwise it is a path on 1 or 2 vertices). Since every cycle or path has a nonrepetitive colouring with at most four colours, this implies the existence of a facial nonrepetitive 8-colouring of any such outerplane graph. We will later show how to tighten this bound to 7.

3.1.3 Colouring Blocking Graphs

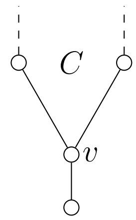
We now show how to colour the blocking graph — a cactus graph — of an outerplane graph with at least two 2-connected components. By Lemma 3.4, if we can find a facial nonrepetitive k-colouring of any cactus graph, we can get a facial nonrepetitive $k + 4$-colouring of any outerplane graph.

The tightest upper bound for the facial Thue chromatic number of outerplane graphs is 12, which is the bound for the Thue chromatic number [6, 36]. Thus, to improve this bound, we need to find a facial nonrepetitive 7-colouring of the blocking graph. It seems difficult to do this unless all cycles of the cactus graph are even. This is partly caused by two problems which will be explained in the the proof: a levelling of an odd cycle always has two vertices on the last level, and three-cycles in cactus graphs have at most two vertices of degree 2 (assuming the cactus graph is not a cycle and is connected).

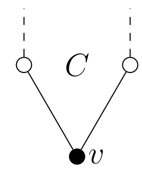
We will address these difficulties by choosing a blocking set such that the blocking graph has no odd cycle. We will show how to achieve this in Lemma 3.7. Let us now show how to colour these cactus graphs. The following lemma is crucial in our proof:

Lemma 3.5 (Kündgen and Pelsmajer [36])**.** *Let G be a graph and λ be a levelling of G. Let $S = s_0, s_1, \ldots, s_m$ be a nonrepetitive palindrome-free sequence on an alphabet $\mathcal{A}$ with $m = \max\{\lambda(v) \mid v \in V(G)\}$ and $c : V(G) \to \mathcal{A}$ be a colouring of G defined as $c(v) = s_{\lambda(v)}$. If a path $P = P_1, P_2$ with $|P_1| = |P_2|$ in G is repetitively coloured under c, then P_1 and P_2 have the same level pattern.*

Lemma 3.6. *Let G be a plane cactus graph such that every cycle of G is even. Then, $\pi_f(G) \leq 7$.*



(a) v has degree greater than 2, so it is not in H



(b) v has degree 2, so it is in H

Figure 3.3: Adding vertices to H.

Proof. We may assume that G is connected as this does not affect its facial Thue chromatic number. Also, assume that G is neither a cycle nor a tree since $\pi_f(G) \leq 4 < 7$ for both these classes of graphs. If there exists a vertex v of G such that $\deg_G(v) = 1$, then let the *root* r of G be v. Otherwise, let r be any vertex of G. Let λ be a levelling of G defined as $\lambda(v) = d(r, v)$. Let H be a graph that contains all vertices $v \in V(G)$ such that

a) v is on a cycle C of G,

b) $\lambda(v) = \max_{u \in C} \lambda(u)$ and

c) $\deg_G(v) = 2$.

In other words, H contains the vertices of degree 2 that are on the deepest level of a cycle (see Figure 3.3). Notice that since every cycle of G is even, there is at most one vertex of H in each cycle of G. If $\deg_G(r) \neq 1$, there must exist at least one face F_T of G such that exactly one vertex v of F_T has degree greater than two (otherwise, since G is neither a cycle nor a tree and is connected, there would be infinitely many faces in G). Since all faces of G are even and G does not contain double edges, F_T contains at least three vertices of degree 2, say a, b, c. Without loss of generality, assume that $a \in V(H)$. If $|V(H)| \in \{5, 7, 10, 14, 17\}$, add b to $V(H)$. If $|V(H)| = 9$, add both b and c to $V(H)$. Notice that now, either $\deg_G(r) = 1$ or $|V(H)| \notin \{5, 7, 9, 10, 14, 17\}$.

We now define the edge set $E(H)$ of H. Let W be a closed facial walk around the outer face of G. Note that since they have degree two and are on cycles, vertices in H appear exactly once in W. Add an edge between any two vertices $u, v \in V(H)$ if and only if

a) $u, v_1, v_2, \ldots, v_k, v$ is a walk in W,

b) for each v_i, $\deg_G(v_i) > 1$ and $v_i \notin V(H)$

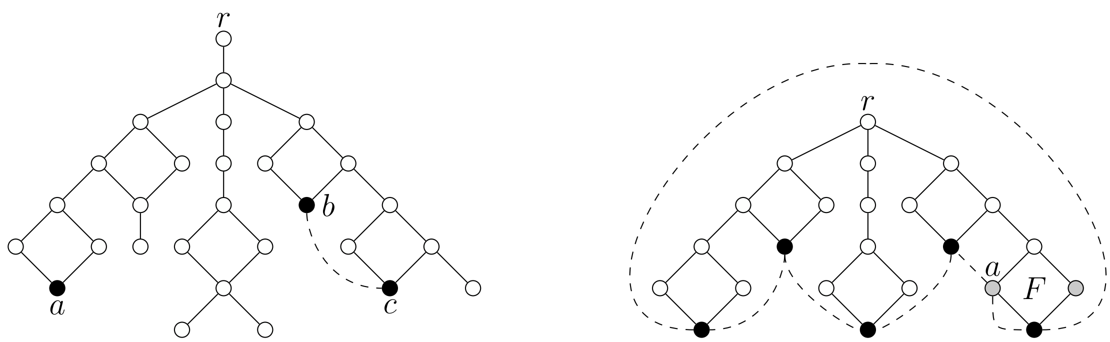


Figure 3.4: Two examples of cactus graphs and their levellings (r is at level 0). In black: vertices of H, with the edges of H represented in black. Left: There is at least one vertex of degree 1, which becomes the root. Notice there is no edge between a and b since the facial walk between them contains vertices of degree 1, but this is not the case between b and c. Right: No such vertex exists, but there is at least one face (F in this example) in which two additional vertices can be added. Here, since $|V(H)| = 5$, one vertex (a, in gray), was added to H.

c) $u \neq v$.

Note that H is either a cycle or a forest of paths (the latter possibly empty or a single connected component), see Figure 3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint colour sets. Let $h = \max_{v \in V(G)} \lambda(v)$ and $S = s_0, s_1, \ldots, s_h$ be a palindrome-free nonrepetitive sequence on $\mathcal{A}$. Let $c_H : V(H) \to \mathcal{B}$ be a nonrepetitive colouring of H. We define a colouring $c : V(G) \to \mathcal{A} \cup \mathcal{B}$ as follows:

$$c(v) = \begin{cases} c_H(v) & \text{if } v \in V(H) \\ s_{\lambda(v)} & \text{otherwise.} \end{cases} \tag{3.4}$$

We will now show that c is a valid nonrepetitive colouring of G. Suppose that this is not the case. Thus, there exists a path $P = P_1, P_2$ such that the colour sequence S corresponding to vertices of P is a repetition. Let us first suppose that P is on the outer face of G. We will need the following claim:

Claim 3.6.1. *Let P be a path on the outer face of G such that $P \cap V(H) = \varnothing$. The level sequence L corresponding to vertices of P must be strictly decreasing, strictly increasing, or strictly decreasing then strictly increasing.*

Proof. Suppose that this is not the case. L cannot contain a block of the form i, i as this can only correspond to an odd cycle of G, but all cycles of G are even. Thus, L must contain a block of the form $i, i+1, i$. Since P is on the outer face, we must have that the vertex v corresponding to $i+1$ is the lowest vertex on some cycle C

and that $\deg_G(v) = 2$. But in this case, v must be in H, which is a contradiction. □

By Lemma 3.5, P_1 and P_2 have the same level pattern. However, if $P \cap V(H) = \varnothing$ this is incompatible with Claim 3.6.1 which states that the level sequence of P must be strictly decreasing, strictly increasing, or strictly decreasing then strictly increasing. Thus, P must contain vertices of H. Let $P_H = p_1, p_2, \ldots, p_k$ be the sequence of vertices of $P \cap V(H)$ in the same order as in P. Notice that we must have $\{p_i, p_{i+1}\} \in E(H)$ for each $1 \leq i < k$ as this corresponds to the definition of an edge in H. Therefore, the colour sequence corresponding to P_H must be nonrepetitive. Since vertices on $G \setminus H$ and vertices on H are coloured with $\mathcal{A}$ and $\mathcal{B}$ respectively, and $\mathcal{A} \cap \mathcal{B} = \varnothing$, then by Lemma 3.1, the colour sequence of vertices in P is nonrepetitive as well.

Thus, P must be on some inner face F of G. With the exception of F_T (if it exists), all inner faces of G contain at most one vertex in H. If $P \cap V(H) \neq \varnothing$, then since $\mathcal{A} \cap \mathcal{B} = \varnothing$, we must have that $F = F_T$. Let $P_H = p_1, \ldots, p_k$ be the sequence of vertices of $P \cap V(H)$ in the same order as in P. Since all the vertices of P_H are on the same inner face and F_T has exactly one vertex of degree greater than three, $\{p_i, p_{i+1}\} \in E(H)$ for each $1 \leq i < k$. Thus, using the same argument as in the previous case, the colour sequence of vertices in P must be nonrepetitive.

Therefore, $P \cap V(H) = \varnothing$. Again, by Lemma 3.5, P_1 and P_2 have the same level pattern. But, since P is constrained to F, whose level sequence is circular, P_1 and P_2 cannot have the same level pattern unless P contains duplicate vertices. Therefore, c is a valid nonrepetitive facial colouring of G.

It now remains to show that $|\mathcal{A}| + |\mathcal{B}| = 7$. A nonrepetitive palindrome-free sequence can be constructed from any ternary nonrepetitive sequence by adding a fourth symbol between blocks of size 2 [10]. Thus, four symbols are sufficient for $\mathcal{A}$. Notice that H is a cycle if and only if $\deg_G(r) \neq 1$. In this case, we made sure that $|V(H)| \notin \{5, 7, 9, 10, 14, 17\}$. Such a cycle has a nonrepetitive 3-colouring by Theorem 2.1. Otherwise, it is a forest of paths, for which three symbols are also sufficient by Thue's original result. This completes the proof. □

3.1.4 Even Cycles

Let us now show how to ensure that every cycle of the blocking graph is even. A *spider graph* is a connected outerplane graph which contains exactly one 2-connected component. The following lemma shows how to ensure the blocking graph of a spider graph is an even cycle. In a later proof, we will show how to carefully split an outerplane graph into spider graphs so that this even cycle property is preserved throughout the graph. Each spider graph will be connected to a previously considered spider graph by exactly one vertex that is already coloured, which is why we need to specify the $\{b, w\}$ edge in the definition of the lemma.

We will need to use the extended version of the blocking graph — the augmented blocking graph — for this proof. Recall that this is the same as the blocking graph, with the addition of edges between blocking set vertices connected together via a facial walk on the outer face of the graph that contain repeated vertices. This is necessary since we will be looking at each 2-connected component separately and that some cycles that might be in the blocking graph of the entire graph might only be paths in a spider subgraph. Once we show that all cycles of the augmented blocking graph are even, all what remains to do is to remove the extra edges to get back the (regular) blocking graph, which preserves the even cycle property.

For clarity purposes, in the next proofs, a graph that consists of a single vertex is an odd cycle.

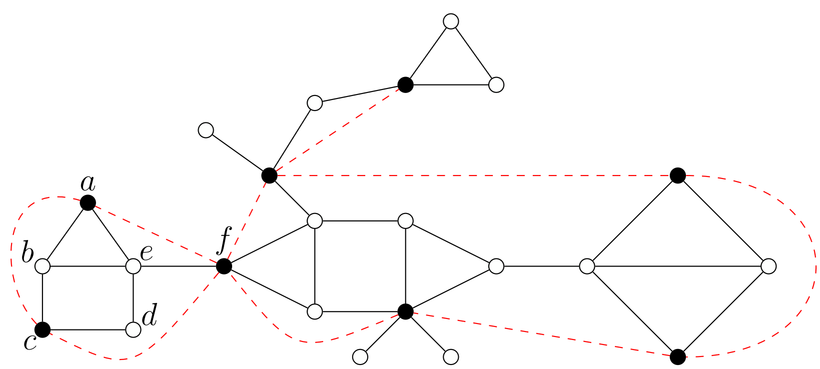


Figure 3.5: An outerplane graph G with blocking set B (denoted in black). The edges of $\text{block}_B(G)$ are denoted in dashed. Vertices a, b, c, d, e, f induce a spider graph H

Lemma 3.7. *Let G be a spider graph that contains an edge $\{b, w\}$ which is incident to the outer face and to some inner face of G. There exists a blocking set B of G such that $b \in B$, $w \notin B$ and $\text{block}_B^+(G)$ is an even cycle or a single edge.*

Proof. Let H be the 2-connected component of G. Let F be the inner face of G adjacent to $\{b, w\}$. Let B be a blocking set of G such that $b \in B$. Such a set exists by Lemma 3.2. Note that $\text{block}_B(G)$ is a cycle or a single edge since $B \subseteq V(H)$. If $\text{block}_B(G)$ is an even cycle or a single edge, we are done. Thus, suppose $\text{block}_B^+(G)$ is an odd cycle. Let us say a vertex $v \in V(G)$ is *black* if $v \in B$ and *white* otherwise. In the subsequent figures, we denote black vertices in black, white vertices in white and, if unknown, in gray. We will first show two claims which will be used throughout the proof.

Claim 3.7.1. *If there exists an edge $\{u, v\}$ such that*

a) $u \in V(H) \setminus B$,

b) $u \neq w$ *and*

c) $v \in V(G \setminus H)$,

then $B' = \{v\} \cup B$ *is a blocking set of* G *and* $\text{block}^+_{B'}(G)$ *is a cycle with* $|B'| = |B| + 1$.

Proof. Since no vertex of $G \setminus H$ is black, $|B'| = |B| + 1$. Also, since $u \in V(H) \setminus B$, $\text{block}^+_{B'}(G)$ is a cycle (see Figure 3.6). Furthermore, since $v \notin V(H)$, $H \setminus B'$ is a tree. This completes the proof. □

Figure 3.6: Illustration of Claim 3.7.1. Dashed lines represent the edges of the blocking graph. Left: B, right: B'. Since $\text{block}^+_B(G)$ is a cycle and $u, v \notin B$, then $\text{block}^+_{B'}(G)$ is also a cycle with $|B'| = |B| + 1$.

Claim 3.7.2. *If there exists an edge $\{c, d\} \in E(H)$ such that*

a) $d \neq w$,

b) $\deg_G(d) = 2$,

c) $\{c, d\}$ *is incident to the outer face and*

d) c *is black and* d *is white,*

then $B' = \{d\} \cup B$ *is a blocking set of* G *and* $\text{block}^+_{B'}(G)$ *is a cycle with* $|B'| = |B| + 1$.

Proof. Suppose that B' is not a blocking set of G. Recall that in this setting, B' is a blocking set if and only if $H \setminus B'$ is a tree. $H \setminus B$ contains no cycles, so if $H \setminus B'$ is not a blocking set, then $H \setminus B'$ is disconnected. Since $\deg_G(d) = 2$, $\{c, d\} \in E(G)$

and c is black, we must have $\deg_{H\setminus B}(d) = 1$. Thus, d is not a cut vertex of $H \setminus B$ and its removal does not disconnect $H \setminus B'$. Since $d \in V(H)$, adding it to B' will result in B' having one more vertex than B. Furthermore, since $d \in V(H)$, $\text{block}^+_{B'}(G)$ must be a cycle (see Figure 3.7). This completes the proof. □

Figure 3.7: Illustration of Claim 3.7.2. Dashed lines represent the edges of the blocking graph. Note that we may have $b = c$. Left: B, right: B'. Since $\text{block}^+_B(G)$ is a cycle and $d \notin B$, then $\text{block}^+_{B'}(G)$ is also a cycle with $|B'| = |B| + 1$.

Let x be the neighbour of b on the outside face of H such that $x \neq w$. Such a neighbour exists since H must have at least three vertices. x is white, as otherwise the inner face adjacent to the edge $\{b, x\}$ would have two black vertices, but each inner face has exactly one such vertex by Lemma 3.2. If x is adjacent to a vertex in $V(G) \setminus V(H)$, then there exists a blocking set B' such that $\text{block}^+_{B'}(G)$ is even by Claim 3.7.1. Thus, we may assume x is not adjacent to any vertex in $V(G) \setminus V(H)$.

Let $F_{\{b,x\}}$ be the inner face of H that is adjacent to the edge $\{b, x\}$. If x is not adjacent to any other internal face than $F_{\{b,x\}}$, it implies that $\deg_G(x) = 2$, so there exists a blocking set B' such that $\text{block}^+_{B'}(G)$ is even by Claim 3.7.2. Thus, assume x is adjacent to at least one other internal face. x must be connected via a chord to at least one other vertex of H. Let y be the first vertex connected via a chord c to x when doing a walk counterclockwise around G from x. c splits H into two subgraphs H' and H'', and without loss of generality, b is incident to H' (see Figure 3.8).

Figure 3.8: x is adjacent to at least two internal faces. y is the first vertex connected with a chord to x that is found when doing a walk counterclockwise from x. We may have $w \neq y$ (left) or $w = y$ (right).

Since H'' is outerplanar, it contains an ear, which is a face of degree at most 1 in the weak dual. Let F_e be an ear of H'' that is incident to a vertex of degree 2 in H'' other than x or y. Such an ear exists because if F_e is the only face of H'', then it is adjacent to at least three vertices, two of them being x and y. Otherwise, it must contain an ear that is not adjacent to at most one of x, y. This ear is adjacent to at least one vertex of degree 2 that is neither x not y. Let the anchors of F_e be the vertices of F_e in $\{z \in V(H'') \mid \deg_{H''}(z) \geq 3\} \cup \{x, y\}$. There are exactly two such vertices (Figure 3.9).

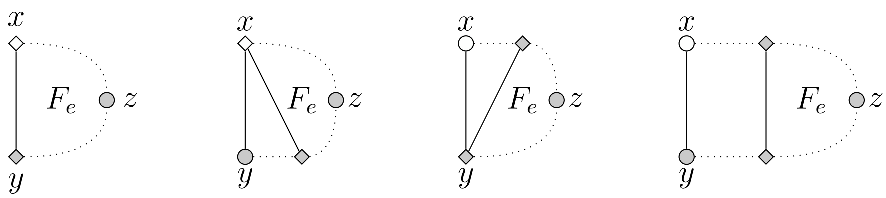


Figure 3.9: Anchors of F_e (anchors represented with diamonds). There are four cases.

If one of the anchors is black, then its non-anchor neighbour on F_e has degree 2 on H and is white, therefore there exists a blocking set B' such that $\text{block}^+_{B'}(G)$ is even by Claim 3.7.1 or by Claim 3.7.2. Thus, assume no anchor of F_e is black. Then, one of the non-anchor vertices of F_e, say z, must be in black. If there is more than one non-anchor vertex on F_e, then one of these vertices is adjacent to z and has degree 2 in H, and there exists a blocking set B' such that $\text{block}^+_{B'}(G)$ is even by Claim 3.7.1 or by Claim 3.7.2. Thus, assume there is only one non-anchor vertex on F_e. We may then assume every ear of H'' is a cycle on three vertices with the non-anchor vertex black (Figure 3.10), as we could otherwise find a vertex for which one of Claim 3.7.1 or Claim 3.7.2 could be applied to get a blocking set B' such that $\text{block}^+_{B'}(G)$ is an even cycle.

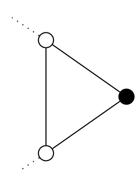


Figure 3.10: Every ear of H'' is a cycle on three vertices with the non-anchor vertex in B and the anchor vertices not in B.

If H'' has only one face, then by the choice of y, x is adjacent to exactly two inner faces of H, one of them adjacent to b and the other is an ear of H'' (Figure 3.11). Let z be the vertex of degree 2 on H incident to this ear. z is black. Thus, $B' = \{x\} \cup B$ is a blocking set since $\deg_{H \setminus B}(x) = 1$, so it is not a cut vertex of $H \setminus B$ and adding it to B' does not disconnect $H \setminus B'$, so we have that $\text{block}^+_{B'}(G)$ is an even cycle. It remains to consider the case where H'' has more than one face.

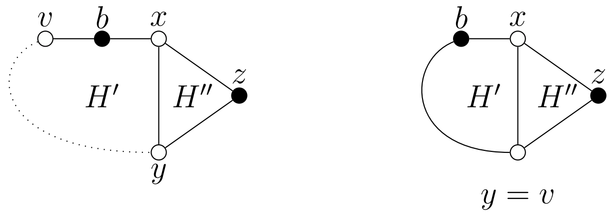


Figure 3.11: H'' has only one face. We constructed H'' such that all chords from x are in H'', thus x only has one chord, $\{x, y\}$. We can have $v \neq y$ (left) or $v = y$ (right).

Let F be a face of H' that is adjacent to the edge $\{x, y\}$. Let T be the weak dual of $H'' \cup F$. T is a tree since $H'' \cup F$ is outerplanar. Root T at F and do a breadth-first search on T. The height[2] h of T must be at least 2 since F has degree 1 in T and H'' has at least two faces, only one of which is adjacent to F. Let $F^\dagger$ be a face of T that has depth $h - 1$ in T (it is on the second layer from the bottom in the breadth-first search layering of T). All the children of $F^\dagger$ in T are leaves, thus they must be ears of H''. Let $\{l, k\}$ be the edge that divides $F^\dagger$ and its parent in T. Without loss of generality, let k be white (they cannot both be black).

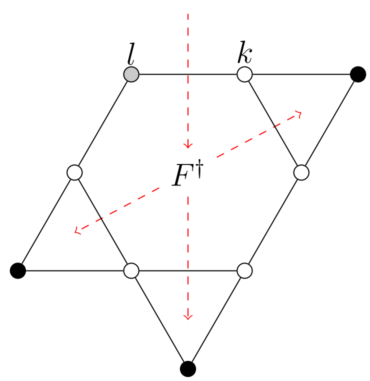
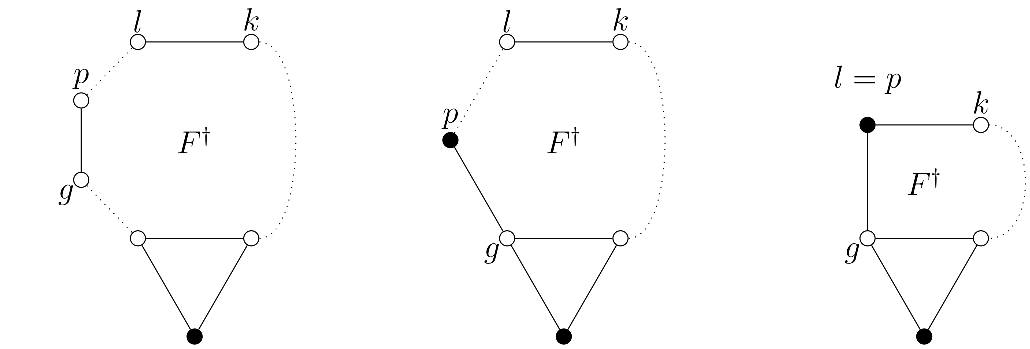


Figure 3.12: A possible configuration for $F^\dagger$ and its children. In dashed: the weak dual corresponding to these faces, with arrows pointing away from F.

Let p be the black vertex incident to $F^\dagger$. If p is incident to a child of F', or if any child of $F^\dagger$ has more than three vertices, then we can apply one of the previous arguments. Let g be the neighbour of p along $F^\dagger$ other than l or k. Such a vertex must exist otherwise there would be two black vertices on an ear of $F^\dagger$. Then, the edge $\{p, g\}$ is adjacent to the outside face since p cannot be incident to any children

[2] The *height* of a tree is the eccentricity of its root.



(a) if $\deg_H(g) = 2$, then we use Claim 3.7.1 or Claim 3.7.2. **(b)** $p \neq l$ **(c)** $p = l$

Figure 3.13: p and g on $F^\dagger$. $F^\dagger$ must be adjacent to at least one ear of H'' by choice of $F^\dagger$.

of $F^\dagger$. We may assume that g is not adjacent to any vertex in $V(G) \setminus V(H)$ and that $\deg_G(g) \neq 2$, since otherwise there exists a blocking set B' such that $\text{block}^+_{B'}(G)$ is even by Claim 3.7.1 or by Claim 3.7.2 (Figure 3.13a). Thus, we may assume that $\deg_G(g) \geq 3$ and is not adjacent to a vertex in $V(G) \setminus V(H)$. Then, $F^\dagger$ has the configuration of Figure 3.13b or Figure 3.13c.

Note that since $g \in V(H'') \setminus \{l, k\}$, this implies that $g \neq w$. Thus, $B' = \{g\} \cup B$ is a blocking set as $\deg_{H \setminus B}(g) = 1$, so g is not a cut vertex of $H \setminus B$, and adding it to B' does not disconnect $H \setminus B'$, and we have that $|\text{block}^+_{B'}(G)|$ is even. This completes the proof. □

The following lemma shows how to use Lemma 3.7 to create a blocking graph of any outerplane graph such that every cycle of the blocking graph is even.

Lemma 3.8. *There exists a blocking set B of G such that every cycle in* $\text{block}_B(G)$ *is even.*

Proof. Assume that G is not a tree since in this case, an empty blocking set satisfies the condition. Also, assume that G is connected, otherwise we may apply the lemma for each of its connected components. Let $S_1, S_2, \ldots, S_k$ be subgraphs of G such that: 1) each S_i is a spider graph and 2) for each $i \in \{2, \ldots, k\}$, if H_i is the 2-connected component of S_i, then

a) $V(S_i) \cap V(S_{i-1}) = \{v\}$ such that $v \in V(H_i)$,

b) $S_{i-1} \setminus S_i$ is connected,

c) $V(G) = V(S_1) \cup V(S_2) \cup \cdots \cup V(S_k)$ and

d) $E(G) = E(S_1) \cup E(S_2) \cup \cdots \cup E(S_k)$

(see Figure 3.14). Let B_1 be a blocking set of S_1 such that $\text{block}^+_{B_1}(S_1)$ is an even cycle. Such a set exists by Lemma 3.7. For each spider graph S_i for $i \in \{2, \ldots, k\}$, let v be the vertex $v \in V(S_i) \cap V(S_{i-1})$. Let u be a neighbour of v on H_i. If $v \in B_{i-1}$, there exists a blocking set B_i of S_i in which $v \in B_i$ and $u \notin B_i$ by Lemma 3.7. Similarly, if $v \notin B_{i-1}$, there exists a blocking set B_i in which $v \notin B_i$ and $u \in B_i$. In both cases, $\text{block}^+_{B_i}(S_i)$ is an even cycle.

Let $B = \bigcup_{1 \le i \le k} B_i$. Observe that B is a blocking set of G. For each cycle C of $\text{block}^+_B(G)$, if $V(C) \subseteq V(S_i)$ for some S_i, then C is even. Otherwise, C includes vertices in multiple spider graphs $S'_1, \ldots, S'_l$ with blocking sets $B'_1, \ldots, B'_l$. Let $\mathcal{S}_C = \{S'_1, \ldots, S'_l\}$. Notice that

$$\bigcup_{1 \le j_1 < j_2 \le l} (B'_{j_1} \cap B'_{j_2}) = \varnothing, \tag{3.5}$$

otherwise $\text{block}^+_B(\mathcal{S}_C)$ would contain at least two simple cycles. Thus,

$$\left|V(\text{block}^+_B(\mathcal{S}_C))\right| = \sum_{1 \le j \le l} |V(\text{block}^+_B(S'_j))| \tag{3.6}$$

and since each of $|V(\text{block}^+_B(S'_j))|$ is even, $|V(\text{block}^+_B(\mathcal{S}_C))|$ is also even. Therefore, all cycles of $\text{block}^+_B(G)$ are even, which implies that all cycles of $\text{block}_B(G)$ are also even since $E(\text{block}_B(G)) \subseteq E(\text{block}^+_B(G))$. This completes the proof. □

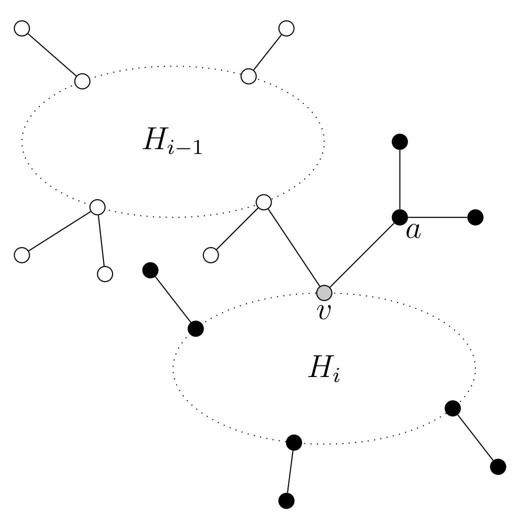


Figure 3.14: Two spider graphs S_{i-1}, S_i of some outerplane graph. Vertices in white are part of S_{i-1}, vertices in black are part of S_i, and v, in gray, is part of both. Notice that a must be part of S_i, otherwise $S_{i-1} \setminus S_i$ would not be connected.

3.1.5 Main Results

We are now ready to prove the main results for this section.

Theorem 3.9. *Let G be an outerplane graph. Then, $\pi_f(G) \leq 11$.*

Proof. Let B be a blocking set of G such that every cycle in $\text{block}_B(G)$ is even. Such a set exists by Lemma 3.8. By Lemma 3.6, there exists a facial nonrepetitive 7-colouring of $\text{block}_B(G)$. Thus, by Lemma 3.4, there exists a facial nonrepetitive 11-colouring of G. This completes the proof. □

As mentioned earlier, we can improve this bound for outerplane graphs that contain at most one 2-connected component.

Theorem 3.10. *Let G be an outerplane graph that contains at most one 2-connected component. Then, $\pi_f(G) \leq 7$.*

We will need the following claim:

Claim 3.10.1. *Let G be a 2-connected outerplane graph. There exists a blocking set B of G such that $|B| \notin \{5, 7, 9, 10, 14, 17\}$.*

Proof. If G is a cycle, then any set that contains one vertex of G respects this constraint. Thus, assume G has at least two inner faces. By Corollary 3.3, there exists a blocking set B of G such that F_1 is a face of degree 1 in G° and $v \in B \cap V(F_1)$ such that $\deg_G(v) \geq 3$. If $|B| \notin \{5, 7, 9, 10, 14, 17\}$, then we are done. Thus, suppose $|B| \in \{5, 7, 9, 10, 14, 17\}$. There must be at least 5 inner faces in G, as there is at most one vertex in B for each inner face of G. Since $v \in V(F_1) \cap B$ has degree at least three and F_1 has degree 1 in G°, there must be a vertex $u \in V(F_1) \setminus B$ such that $\deg_G(u) = 2$ and $\{u, v\} \in E(G)$ (Figure 3.15). Thus, u is not a cut vertex of $G \setminus B$ which implies that $B' = \{u\} \cup B$ is a blocking set of G. Furthermore, $|B'| = |B+1|$, so if $|B| \in \{5, 7, 10, 14, 17\}$, then $|B'| \notin \{5, 7, 9, 10, 14, 17\}$ and we are done. Therefore, assume $|B| = 9$, thus $|B'| = 10$.

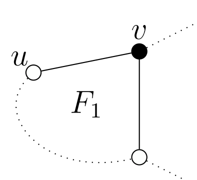


Figure 3.15: $B' = \{u\} \cup B$ is a blocking set of G.

We will now show how to add another vertex to the blocking set. Let us again say that a vertex $u \in V(G)$ is *black* if $u \in B'$ and is *white* otherwise. If there exists an edge $\{b, w\} \in E(G)$ that is incident to the outer face and such that b is black, w is white and $\deg_G(w) = 2$, let $B'' = \{w\} \cup B'$. We must have $\deg_{G \setminus B'}(w) = 1$, so w is not a cut vertex of $G \setminus B'$, so B'' is a blocking set of G with $|B''| = |B'| + 1$, thus $|B''| = 11 \notin \{5, 7, 9, 10, 14, 17\}$ and we are done. Thus, we may assume that no such edge exists.

G° is a tree since G is outerplanar. Root G° at F_1 and do a breadth-first search on G°. The height h of G° must be at least 3 since G is 2-connected and contains at least five faces. Let F be a face of G° that has height $h - 1$ in G° (thus it is on the second layer from the bottom in the breadth-first search layering of G°). All the children of F in G° are leaves, and none of them are F_1. Let x be the black vertex incident to F. Note that x cannot be incident to any child of F in G°, as in this case we could find a $\{b, w\}$ edge on this child, but that is a contradiction. For the same reason, no neighbour of x incident to F can have degree 2, so all neighbours of x must have degree at least 3. Also, by choice of F, at least one such neighbour, say y, must be incident to exactly two faces: F and some face F' that is a child of F in G°. Let z be the black vertex on F'. By our reasoning, z must be adjacent to y (see Figure 3.16). Let $B'' = \{y\} \cup B'$. Since $\deg_{G \setminus B'}(y) = 1$, u is not a cut vertex of $G \setminus B'$, so B'' is a blocking set of G with $|B''| = |B'| + 1$, thus $|B''| = 11 \notin \{5, 7, 9, 10, 14, 17\}$ and we are done. □

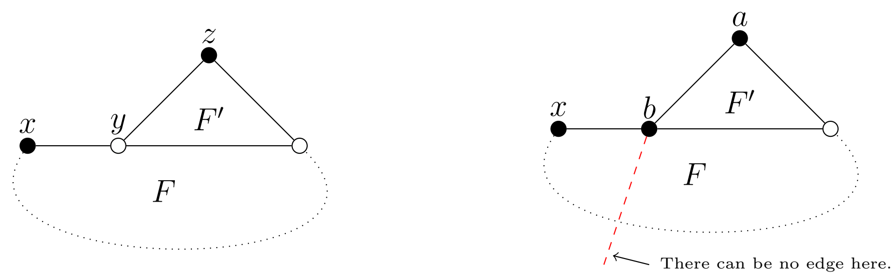


Figure 3.16: $B'' = \{y\} \cup B'$ is a blocking set of G.

Proof of Theorem 3.10. If G does not contain a 2-connected component, then it is a tree and has a nonrepetitive 4-colouring. Thus, suppose G contains exactly one 2-connected component H. By Claim 3.10.1, there exists a blocking set B of H such that $|B| \notin \{5, 7, 9, 10, 14, 17\}$. Note that B is also a blocking set of G. If $\text{block}_B(G)$ is a cycle, then there exists a nonrepetitive 3-colouring of $\text{block}_B(G)$ by Theorem 2.1. Otherwise, it is a forest of paths, for which a nonrepetitive 3-colouring also exists. Thus, by Lemma 3.4, there exists a facial nonrepetitive 7-colouring of G. This completes the proof. □

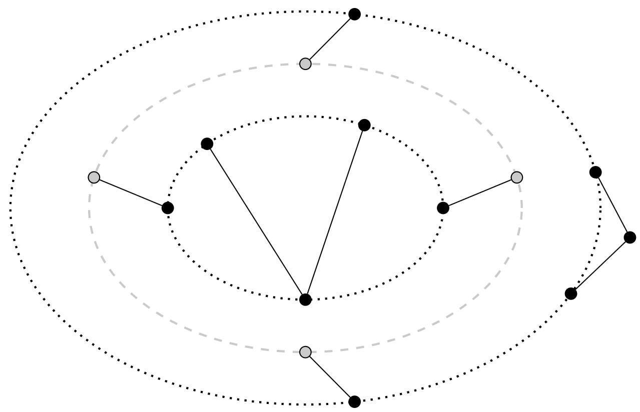


Figure 3.17: Visualization of the decomposition of a plane graph into outerplane layers. Each dotted/dashed ellipse corresponds to a layer of vertices, forming an outerplane graph. Dotted layers denoted get coloured with one colour set and vertices on the dashed layer get coloured with a second colour set.

3.2 Plane Graphs

We are now ready to apply Theorem 3.9 to plane graphs. For this, we use a modified version of Barát and Czap's proof for the facial Thue chromatic number of plane graphs [5]. Recall that their proof partitions a plane graph in layers, each of which is an outerplanar graph, which can then be nonrepetitively 12-coloured. By using two 12-colour sets, one for odd-numbered layers and another to even-numbered layers, they are able to show that this construction avoids repetitive paths (see Figure 3.17). However, we may not use Theorem 3.9 directly with their proof to tighten the upper bound, as it requires a nonrepetitive colouring of an outerplanar graph, not a *facial* nonrepetitive colouring of an *outerplane* graph. In the following proof, we show that their approach still works when using these restrictions, that is; each layer is not only an outerplanar graph but is also an outerplane graph, and using a facial nonrepetitive colouring of each layer, again using two colour sets for odd and even numbered layers, is enough to prevent repetitions on facial paths.

Let H be a plane graph. We define $\delta(H)$ to be the set of vertices in H adjacent to the outer face, and $[\delta(H)]$ to be the subgraph of H induced by $\delta(H)$.

Theorem 3.11. *Let* $r = \max\{\pi_f(G) \mid G \text{ is outerplane}\}$ *and let* G *be a plane graph. Then,* $\pi_f(G) \leq 2r$.

Proof. First, we label the vertices of G black or white as follows:

a) Let $G_1 = G$ and $i = 1$

b) While $G_i \neq [\delta(G_i)]$, let $G_{i+1} = G_i \setminus \delta(G_i)$ and $i \leftarrow i + 1$

c) Label every $u \in \delta(G_i)$ black if i is odd or white if i is even.

Notice that $[\delta(G_i)]$ is outerplane for every i. Let F be a face of G. $B(F)$ and $W(F)$ are the set of vertices adjacent to F that are respectively labelled black and white. We create an augmented graph G^+ as follows: for each facial path $a, v_1, v_2, \ldots, v_k, b$ such that

a) $\{a, b\} \notin E(G)$,

b) $a, b \in \delta(G_i)$ for some i, and

c) $v_1, v_2, \ldots, v_k \in \delta(G_{i+1})$,

add an edge $\{a, b\}$ to G^+ with its curve following the edges of P. We denote such $\{a, b\}$ edges as *correction edges*.

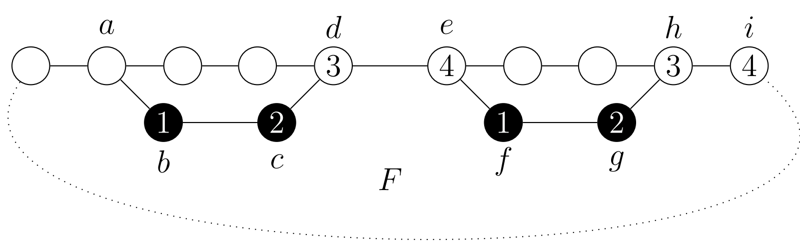


Figure 3.18: Correction edges are necessary to prevent repetitions. In this example, the white vertices are part of $\delta(G_i)$ and the black vertices are part of $\delta(G_i + 1)$. Labels inside the vertices represent colours. This colouring induces a repetitive path b, c, d, e, f, g, h, i. This repetition would be avoided in our construction since there would be a correction edge $\{e, h\}$ in $[\delta(G_i)]^+$, which would prevent the sequence of vertices d, e, h, i to be repetitive.

Let $[\delta(G_i)]^+$ be the subgraph of G^+ induced by $\delta(G_i)$. Notice that each $[\delta(G_i)]^+$ is outerplane as all of its vertices lay on the outside face by construction, and correction edges were added on top of edges of G, which was plane, and such edges are not included in $[\delta(G_i)]^+$. Let c_i be a facial nonrepetitive r-colouring of $[\delta(G_i)]^+$ on colours $\{1, \ldots, r\}$ if i is odd, and colours $\{r + 1, \ldots, 2r\}$ is i is even. Such a colouring exists by our hypothesis. The union of all these colourings induce a $2r$-colouring c of G. We will now show that c is a facial nonrepetitive colouring. Suppose that this is not the case. Thus, there exists a facial path P on some face F such that the colour sequence S of vertices in P is repetitive. There are two cases:

1. P is composed of vertices of one of $B(F)$ or $W(F)$ (thus we have $P \setminus B(F) = P$ or $P \setminus W(F) = P$). Notice that P must be contained in exactly one subgraph

of G, say $[\delta(G_i)]$. If P is also a facial path of $[\delta(G_i)]^+$, then P cannot be repetitive. Thus, P must not be a facial path in $[\delta(G_i)]^+$. However, the only edges in $[\delta(G_i)]^+$ that are not in $[\delta(G_i)]$ are correction edges, which are located on top of paths in G. Therefore, the vertices of P must not be consecutive on a single face, which is a contradiction.

2. P is composed of vertices of both $B(F)$ and $W(F)$. It must lay on exactly two subgraphs of G, say $[\delta(G_i)]$ and $[\delta(G_{i+1})]$, as otherwise it would not be a facial path. Since the colours classes of vertices in $\delta(G_i)$ and of vertices in $\delta(G_{i+1})$ are disjoint and P is repetitive, P must be alternating between vertices in $\delta(G_i)$ and vertices in $\delta(G_{i+1})$, with at least three "spanning" edges transitioning between the two subgraphs. Let P' be the sequence of vertices of $P \cap \delta(G_i)$ in the same order as P. Note that P' must be repetitive by Lemma 3.1, otherwise P would not be repetitive. Thus, there must be at least two vertices u, v that are subsequent in P' but not adjacent in $[\delta(G_i)]^+$, otherwise P' would be a facial path of $[\delta(G_i)]^+$ and would not be repetitive. However, all the vertices between u and v in P are in $\delta(G_{i+1})$ (Figure 3.19). This implies there must be an edge $\{u, v\}$ in $[\delta(G_i)]^+$. Contradiction.

This completes the proof. □

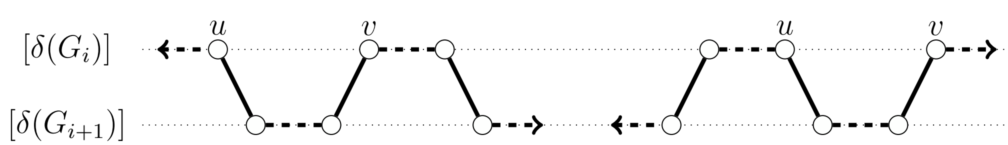


Figure 3.19: Repetitive path P composed of vertices of both $B(F)$ and $W(F)$ (in bold). Independently of the start/end layer of the path (left: $[\delta(G_{i+1})]$, right: $[\delta(G_i)]$), it will contain at least three edges between the layers as the colour classes of vertices on both layers are distinct and P is repetitive.

By Theorems 3.9 and 3.11, we get the following corollary:

Corollary 3.12. *Let G be a plane graph.* $\pi_f(G) \leq 22$.

Chapter 4

Conclusion and Future Work

4.1 Summary of Results

We showed that the facial Thue chromatic number of outerplane graphs is bounded by 11, and by 7 for outerplane graphs that contain at most one 2-connected component. These results improve the previous upper bound of 12 by Barát and Varjú, and Kündgen and Pelsmajer, which was for the "classical" nonrepetitive colouring problem. We also show that the facial Thue chromatic number of plane graphs is bounded by twice the upper bound of the facial Thue chromatic number of outerplane graphs, which results in a bound of 22, an improvement over the previous upper bound of 24 obtained by Barát and Czap [5]. However, as we already mentioned, there is no known outerplane graph with facial Thue chromatic number greater than 4, and no plane graph with facial Thue chromatic number greater than 5. This suggests the following conjecture:

Conjecture 4.1. *Let G be an outerplane graph.* $\pi_f(G) < 11$.

4.2 Future Work

We finish by proposing several ideas for future research.

- Can we tighten the upper bound on the facial Thue chromatic number of cactus graphs? This would directly tighten the upper bounds for outerplane and plane graphs.
- Can we apply our blocking graph structure to other types of plane graphs, such as series-parallel graphs, to get tighter bounds in these cases? We could also

study other variants of outerplane graphs.

- It would be interesting to study a stronger version of facial nonrepetitive colourings: colourings of planar graphs such that for any embedding of the graph, no facial path is repetitive. This may be the same as the nonrepetitive colouring for some planar graphs (it is obviously the same for cycles and paths), but for most graphs, we believe that it is different. Can an easy example be found?

- It is unknown whether the facial Thue chromatic number of trees is four or three. It would be interesting to find either a tree that requires four colours for a facial nonrepetitive colouring, or a proof that all trees have facial Thue chromatic number at most three. In the latter case, this would improve all our bounds for outerplane graphs by 1, and improve the bound for plane graphs to 20.

- Many upper and lower bounds remain unknown or not tight in the total nonrepetitive colouring variant. In particular, we mentioned earlier that for paths, a lower bound of 4 and an upper bound of 5 exist. Using brute force tests, we found that the total Thue chromatic number is bounded by 4 for all paths of length at most 2000 (larger paths were not tested due to the increase in computing power required). Is 4 sufficient for paths of all sizes? In any case, it would be also interesting to find tight bounds for cycles. Is there a finite set of cycles that require 5 (or 6) colours, while all other cycles have a total nonrepetitive colouring on 4 (or 5) colours, similar to the Thue chromatic number?

- One can also define the total nonrepetitive colouring problem on facial paths, which opens up many possible avenues.

- Finally, it would be very interesting to prove (or disprove) Conjecture 2.1, which states that the Thue chromatic number of any planar graph is bounded by a constant K.

Notation

$\chi'(G)$	Chromatic index of G
$\chi(G)$	Chromatic number of G
$\deg_G(u)$	Degree of u in G
$\Delta(G)$	Maximum degree of any vertex in G
$d_G(u, v)$	Minimum distance between u and v in G
$\epsilon_G(u)$	Eccentricity of a vertex U in G
$\kappa(G)$	Connectivity of G
$\mathrm{ch}(G)$	Choice number of G
$\omega(G)$	Clique number of G
$\pi'(G)$	Thue chromatic index of G
$\pi(G)$	Thue chromatic number of G
$\pi'_f(G)$	Facial Thue chromatic index of G
$\pi_f(G)$	Facial Thue chromatic number of G
$\pi_S(G)$	Minimum Thue chromatic number of a subdivision of G
$\pi_{ch}(G)$	Thue choice number of G
$\pi'_{f,ch}(G)$	Facial Thue choice index of G
$\pi_{T_w}(G)$	Weak total Thue chromatic number of G
$\pi_T(G)$	Total Thue chromatic number of G

$\chi_{st}(G)$	Star chromatic number of G
G°	Weak dual of a plane graph G
C_n	Cycle graph on n vertices
$G[S]$	Subgraph of G induced by $S \subseteq S$
$G \setminus H$	$G[V(G) \setminus V(H)]$
G^*	Dual of a plane graph G
G^2	Square graph of G
K_n	Complete graph on n vertices
$N^k(v)$	k^{th} iterated neighbourhood from v
P_n	Path graph on n vertices
S_k	Star graph on $k+1$ vertices

Bibliography

[1] Michael O Albertson, Glenn G Chappell, Hal A Kierstead, André Kündgen, and Radhika Ramamurthi. Coloring with no 2-colored P_4's. *Electronic Journal of Combinatorics*, 11(1):R26, 2004.

[2] Noga Alon and Jarosław Grytczuk. Breaking the rhythm on graphs. *Discrete Mathematics*, 308(8):1375–1380, 2008.

[3] Noga Alon, Jarosław Grytczuk, Mariusz Hałuszczak, and Oliver Riordan. Nonrepetitive colorings of graphs. *Random Structures & Algorithms*, 21(3-4):336–346, 2002.

[4] Noga Alon, Benny Sudakov, and Ayal Zaks. Acyclic edge colorings of graphs. *Journal of Graph Theory*, 37(3):157–167, 2001.

[5] János Barát and Július Czap. Facial nonrepetitive vertex coloring of plane graphs. *Journal of Graph Theory*, 74(1):115–121, 2013.

[6] János Barát and Péter P Varjú. On square-free vertex colorings of graphs. *Studia Scientiarum Mathematicarum Hungarica*, 44(3):411–422, 2007.

[7] János Barát and David R. Wood. Notes on nonrepetitive graph colouring. *Electronic Journal of Combinatorics*, 15(1):Research Paper 99, 13, 2008.

[8] Hans L. Bodlaender. Planar graphs with bounded treewidth. Technical Report RUU-CS-88-14, University of Utrecht - Department of Computer Science, March 1988.

[9] John Adrian Bondy and Uppaluri Siva Ramachandra Murty. *Graph theory with applications*, volume 290. Macmillan London, 1976.

[10] Boštjan Brešar, Jaroslaw Grytczuk, Sandi Klavžar, Staszek Niwczyk, and Iztok Peterin. Nonrepetitive colorings of trees. *Discrete Mathematics*, 307(2):163–172, 2007.

[11] Boštjan Brešar and Sandi Klavzar. Square-free colorings of graphs. *Ars Combin*, 70:3–13, 2004.

[12] T. C. Brown. Is there a sequence on four symbols in which no two adjacent segments are permutations of one another? *American Mathematical Monthly*, pages 886–888, 1971.

[13] Yuehua Bu, Daniel W Cranston, Mickaël Montassier, André Raspaud, and Weifan Wang. Star coloring of sparse graphs. *Journal of Graph Theory*, 62(3):201–219, 2009.

[14] Thomas F Coleman and Jorge J Moré. Estimation of sparse hessian matrices and graph coloring problems. *Mathematical Programming*, 28(3):243–270, 1984.

[15] James D Currie. There are ternary circular square-free words of length n for $n \geq 18$. *Electronic Journal Combinatorics*, 9(1):N10, 2002.

[16] James D Currie and D Sean Fitzpatrick. Circular words avoiding patterns. In *Developments in Language Theory*, volume 2450, pages 319–325. 2003.

[17] Vida Dujmović, Fabrizio Frati, Gwenaël Joret, and David R Wood. Nonrepetitive Colourings of Planar Graphs with $O(\log n)$ Colours. *The Electronic Journal of Combinatorics*, 20(1):P51, 2013.

[18] Vida Dujmović, Gwenaël Joret, Jakub Kozik, and David R Wood. Nonrepetitive Colouring via Entropy Compression. *Combinatorica*, (in press) 2013.

[19] Paul Erdös. Some unsolved problems. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 6:221–254, 1961.

[20] Paul Erdos and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. *Infinite and finite sets, Coll. Math. Soc. J. on*, 10(2):609–627, 1975.

[21] Paul Erdös, Arthur L Rubin, and Herbert Taylor. Choosability in graphs. *Congr. Numer*, 26:125–157, 1979.

[22] A. A. Evdokimov. Strongly asymmetric sequences generated by a finite number of symbols. *Soviet Math. Dokl*, (9):536–539, 1968.

[23] Guillaume Fertin, André Raspaud, and Bruce Reed. Star coloring of graphs. *Journal of Graph Theory*, 47(3):163–182, 2004.

[24] Francesca Fiorenzi, Pascal Ochem, Patrice Ossona de Mendez, and Xuding Zhu. Thue choosability of trees. *Discrete Applied Mathematics*, 159(17):2045–2049, 2011.

[25] Daniel Gonçalves, Mickaël Montassier, and Alexandre Pinlou. Entropy compression method applied to graph colorings. In *9th International colloquium on graph theory and combinatorics*, 2014.

[26] Catherine Greenhill and Oleg Pikhurko. Bounds on the generalised acyclic chromatic numbers of bounded degree graphs. *Graphs and Combinatorics*, 21(4):407–419, 2005.

[27] Jarosław Grytczuk. Nonrepetitive colorings of graphs—a survey. *International Journal of Mathematics and Mathematical Sciences*, 2007 Article 74639, 2007.

[28] Jarosław Grytczuk. Nonrepetitive Graph Coloring. In *Graph Theory in Paris*, pages 209–218. Springer, 2007.

[29] Jarosław Grytczuk, Jakub Kozik, and Piotr Micek. New approach to nonrepetitive sequences. *Random Structures & Algorithms*, 42(2):214–225, 2013.

[30] Jochen Harant and Stanislav Jendrol. Nonrepetitive vertex colorings of graphs. *Discrete Mathematics*, 312(2):374–380, 2012.

[31] Hamed Hatami. $\delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number. *Journal of Combinatorial Theory, Series B*, 95(2):246–256, 2005.

[32] Frédéric Havet, Stanislav Jendrol, Roman Soták, and Erika Škrabul'áková. Facial non-repetitive edge-coloring of plane graphs. *Journal of Graph Theory*, 66(1):38–48, 2011.

[33] Veikko Keränen. Abelian squares are avoidable on 4 letters. In *Automata, languages and programming*, volume 623, pages 41–52. Springer, 1992.

[34] Hal A Kierstead, André Kündgen, and Craig Timmons. Star coloring bipartite planar graphs. *Journal of Graph Theory*, 60(1):1–10, 2009.

[35] Jakub Kozik and Piotr Micek. Nonrepetitive choice number of trees. *SIAM Journal on Discrete Mathematics*, 27(1):436–446, 2013.

[36] Andre Kündgen and Michael J Pelsmajer. Nonrepetitive colorings of graphs of bounded tree-width. *Discrete Mathematics*, 308(19):4473–4478, 2008.

[37] Dániel Marx and Marcus Schaefer. The complexity of nonrepetitive coloring. *Discrete Applied Mathematics*, 157(1):13–18, 2009.

[38] Michael Molloy and Bruce Reed. A bound on the strong chromatic index of a graph. *Journal of Combinatorial Theory, Series B*, 69(2):103–109, 1997.

[39] Michael Molloy and Bruce Reed. A bound on the total chromatic number. *Combinatorica*, 18(2):241–280, 1998.

[40] Marston Morse. Recurrent geodesies on a surface of negative curvatue. *Transactions of the AMS*, 22:84–110, 1921.

[41] Robin A Moser and Gábor Tardos. A constructive proof of the general lovász local lemma. *Journal of the ACM*, 57(2):11, 2010.

[42] Jaroslav Nešetril and Patrice Ossona de Mendez. Colorings and homomorphisms of minor closed classes. *Discrete and Computational Geometry*, 25:651–664, 2003.

[43] Andrzej Pezarski and Michał Zmarz. Non-repetitive 3-coloring of subdivided graphs. *Electronic Journal of Combinatorics*, 16(1):N15, 2009.

[44] Peter AB Pleasants. Non-repetitive sequences. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 68, pages 267–274. Cambridge Univ Press, 1970.

[45] Eugène Prouhet. Mémoire sur les relations entre les puissances des nombres. *Comptes-Rendus de l'Académie des Sciences de Paris*, 33:225, 1851.

[46] Jakub Przybyło. On the facial Thue choice index via entropy compression. *Journal of Graph Theory*, 77(3):180–189, 2014.

[47] Jens Schreyer and Erika Škrabul'áková. On the facial Thue choice index of plane graphs. *Discrete Mathematics*, 312(10):1713–1721, 2012.

[48] Jens Schreyer and Erika Škrabul'áková. Total Thue colourings of graphs. *European Journal of Mathematics*, 1(1):186–197, 2015.

[49] K.S. Sudeep and Sundar Vishwanathan. Some results in square-free and strong square-free edge-colorings of graphs. *Discrete mathematics*, 307(14):1818–1824, 2007.

[50] Axel Thue. Über Unendliche Zeichenreichen. *Norske Vid Selsk. Skr. I. Mat. Nat. Kl. Christiana*, (7):1–22, 1906.

[51] Axel Thue. Über die gegenseitige lage gleicher teile gewisser zeichenreihen. *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiana*, (1):1–67, 1912.

[52] VG Vizing. Vertex colorings with given colors. *Metody Diskret. Analiz*, 29:3–10, 1976.

[53] Margit Voigt. List colourings of planar graphs. *Discrete Mathematics*, 120(1):215–219, 1993.