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Tolerance to Arbitrage:
Inclusion of fractional Brownian motion
to model stock price fluctuations

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

Department of Mathematics and Statistics
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Tolerance to Arbitrage:

Inclusion of fractional Brownian motion
to model stock price fluctuations

submitted by Donna Mary Salopek in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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Abstract

We propose a continuous-time model of stock price fluctuations. This model includes fractional Brownian motion (fBm) as a possible representation of stock price changes. We provide mathematical definitions of trading strategies, self-financing strategies, and attainability of contingent claims in this setting. Under certain conditions, we construct arbitrage opportunities using properties of continuous functions with bounded p -variation, with $1 \leq p < 2$. It is shown that almost all sample paths of fBm with $1/2 < H < 1$ have bounded p -variation with $1/H < p < 2$. Therefore, in a financial market where stock price changes behave like geometric fBm with $1/2 < H < 1$, there are arbitrage opportunities.

In addition, we show that functions of bounded p -variation with $p \geq 1$ may have the property of infinite crossing of zero. This is used to show that stop-loss start-gain strategy may not be self-financing for stock price processes having bounded p -variation with $1 < p < 2$. Thus, by definition, the stop-loss start-gain strategy may not guarantee arbitrage under the assumption of geometric fBm. This complements the result of Carr and Jarrow (1990) concerning standard geometric Brownian motion.

Simulations and statistical analyses are also included. In particular, we adapt certain modifications of quadratic variation to test the index H under the assumption that stock price process having a distribution of geometric fBm.

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Chapter 0. Introduction

Continuous-time stochastic models for stock price changes have been prevalent in financial mathematics since they were popularized by the introduction of the Black-Scholes (1973) pricing model (also cf. Merton, 1973). It is generally assumed that continuous analysis will provide a reasonable approximation to discrete-time solutions. In addition, it is assumed that, as real world financial markets become more efficient, the continuous-time model's predictions about actual financial prices, products, and institutions will become increasingly accurate. In short, "reality will eventually imitate theory". However, it is becoming increasingly obvious from empirical data that the traditional stock price model using semimartingales may not be sufficiently accurate to model stock price fluctuations.

For example, Huang (1985) claims that in a continuous trading economy, if information is revealed continuously and if agents' preferences are continuous in a certain topology, then equilibrium asset price processes must have continuous sample paths and these sample paths will be of unbounded variation. Based on these arguments, there is a school of thought that stock price changes (must) follow geometric Brownian motion (BM). More generally, the class of semimartingales is considered sufficiently rich to model the real stock price behaviour. However, empirical data of stock prices show the "Joseph effect" which is a long range dependence of the price movements (eg. Greene and Fielitz (1977), Fama and French (1988), Kunitomo (1993), Cutland, Kopp and Willinger (1995)). Intuitively, stock price changes exhibit the Joseph effect if their autocorrelations decay very slowly (eg. like a power); so slowly in fact, that the sum of the autocorrelations (over all lags) diverges. In the geometric BM model, autocorrelations decay exponentially fast and their sums converge (ie. short range dependence).

An additional reason for introducing an alternative to the predominant semimartingale model is the following: Trading strategies defined via the Itô integral do not make sense for a given trajectory of price process. The reason for this is that Itô's integral is not defined pathwise (cf. Bick and Willinger (1994) and Föllmer

(1981)).

Finally, it is desirable to have a more general model which will capture the long-range dependence which is empirically observed in financial markets, and would apply to problems other than option pricing theory (where the martingale property is desirable).

In this thesis, we propose and investigate a continuous-time model of stock price fluctuations, which includes the case of fractional Brownian motion (fBm) (known to have long-range dependence) as a possible representation of stock price changes. Our model is more in accordance with the observed market phenomena. Hitherto, a basic principle in financial modeling has been that the markets will not allow for prices which violate the absence of arbitrage assumption of continuous-time models. However, the derived price relations in the continuous-time model are sensitive to the assumed stochastic price process for the underlying security and on the opportunity to trade continuously. Hence, there are no assurances in the real world of the convergence of actual prices to their theoretical model values so the possibility of arbitrage may (and in fact does) arise.

By appropriately defining *trading strategies* via the Riemann-Stieltjes integral for functions having bounded p -variation with $1 \leq p < 2$, we have been able to prove the existence of arbitrage opportunities in our model. This was an open problem even for the special case of an fBm process, and accords much more closely with the real-life observations of the market-place. (Recall that arbitrage is a riskless plan to make money without investment!) This existence of an arbitrage opportunity is proven almost surely (unlike the semimartingale model, where an arbitrage opportunity is defined in the weaker sense of expectation). Recall that fBm is not a semimartingale (cf. Liptser and Shirayev (1986,1989, p 300).

We also show that functions of bounded p -variation (with $p \geq 1$) may have the property of infinite crossing of zero. This may be used to show that certain popular strategies (such as the stop-loss start-gain) may not be self-financing for certain stock prices processes (in the sense that the integral representation to define self-financing may not exist).

In order to prove these results, we propose a stochastic calculus for fractional Brownian motion without probability. The pathwise integral of fractional Brownian motion is based on the p -variation property. That is, almost all sample paths of fractional Brownian motion with index $H \in (1/2, 1)$ have bounded p -variation with $p \in (1/H, 2)$ (cf. Proposition 4 in §1.3). Our definition of p -variation is as follows.

Let f be a real valued function on a closed interval $[a, b]$ and let $0 < p < \infty$. Given a partition $\kappa = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$, let

$$v_p(f; \kappa) := \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p.$$

The p -variation of f is defined by

$$v_p(f; [a, b]) := \sup\{v_p(f; \kappa) : \text{all partitions } \kappa \text{ of } [a, b]\}.$$

We say that f has bounded p -variation if $v_p(f; [a, b]) < \infty$.

A related but different notion is used under the same name in stochastic analysis. Namely, given a sequence $\{\kappa_m : m \geq 1\}$ of partitions of $[a, b]$ such that the mesh (ie. $\max |t_k^m - t_{k-1}^m|$) tends to zero, the limit

$$v_p^r(f; [a, b]) := \lim_{m \rightarrow \infty} v_p(f; \kappa_m)$$

if it exists is also called p -variation. We use the superscript r (restrictive) to indicate that a fixed sequence of partitions is used in its definition rather than all partitions as defined above for p -variation. If $p = 2$ and f is a stochastic process then $v_2^r(f; [a, b])$ is also called a *quadratic variation* if it exists almost surely.

Being of unbounded 1-variation, it is not possible to define integrals of the form $(LS) \int Y_s(\omega) dX_s(\omega)$ for $\mathcal{P}$ -almost every $\omega \in \Omega$, where $(LS) \int$ denotes Lebesgue-Stieltjes integral for semimartingales (cf. Karatzas and Shreve (1991)). In contrast, the class of functions of bounded p -variation is strictly increasing in p for $p \geq 1$ (cf. Lemma 2a in §1.1), and constants are the only continuous functions which have bounded p -variation with $0 < p < 1$ (Dudley and Norvaiša (1997)). Furthermore, stochastic calculus using pathwise integrals based on the class of functions

of bounded p -variation can be easily adapted to handle other stochastic processes which have been proposed as a viable alternative to the study of stock price fluctuations (such as α -stable Lévy motions with index $\alpha \in (0, 2)$ (Mandelbrot (1963), Norvaiša (1997b)), hyperbolic Lévy motions, the normal inverse Levy process of Barndorff-Nielsen (cf. Eberlein and Keller (1995)), and subordinate processes (cf. Clark (1973), Madan and Milne (1991), Madan and Senata (1990)). In particular, the first two processes have bounded p -variation with $p > 1$ and the last process has bounded p -variation with $p > 0$. For a nice expository survey on modeling stock price changes and p -variation, we refer the reader to Norvaiša (1997b).

We define p -variation in §1 and state several properties of p -variation in §1.1. The required properties of fractional Brownian motion can be found in §1.2, and we introduce a calculus for fractional Brownian motion in §1.3, where we also establish the relationship between fractional Brownian motion and p -variation. In §2, we discuss and illustrate the possibility of arbitrage opportunities. In this section, we provide a detailed method of constructing arbitrage when functions are continuous, and have bounded p -variation, for $1 \leq p < 2$. We also define trading strategies, self-financing, and attainability of contingent claims. Finally, in this section, we clarify the circumstances under which the stop-loss start-gain strategy will yield an arbitrage opportunity for various stochastic processes. In §3, we adapt a modification of quadratic variation introduced by Gladyshev (1961) to test the index H under the assumption that the stock price process has the distribution of geometric fractional Brownian motion. In §4, we try to draw some conclusions and suggest future direction in research for fractional Brownian motion as a possible model of stock price fluctuations. The technical supplement contains background definitions, lemmas and theorems which are used in the previous sections.

Chapter 1. p - Variation Calculus

Norbert Wiener (1924) was the first to introduce (a variant of) p -variation, and for $p = 2$, he called it *quadratic variation*. It was Lawrence C Young (1936) who developed a theory of p -variation for any $0 < p < \infty$, and applied it to the theory of Fourier series. However, unexpectedly, we found that Józef Marcinkiewicz (1934) also discussed and proved several deep theorems using p -variation before the work of Young. We will begin this section with the definitions of subdivisions, refinements, and p -variation, $p > 0$. Then we will establish results of p -variation which are applicable to the latter part of this chapter.

Let f and g denote real-valued functions defined on a closed interval $[a, b]$ of the real line. A *subdivision*, κ , of $[a, b]$ is a finite subset of real numbers $(y_0, y_1, \dots, y_n)$ such that

$$a = y_0 < y_1 < \dots < y_n = b$$

and such that the subintervals occurring in the subdivision κ are the intervals $[y_{i-1}, y_i]$, $i = 1, \dots, n$. We will write $\kappa = \{y_i : i = 1, \dots, n\}$. From hereon, we will use *partition* and *subdivision* interchangeably (even though the proper definition of partition is slightly different).

If κ and κ' are partitions of $[a, b]$, we say that κ' is a *refinement* of κ or that κ' is *finer than* κ when every subinterval in κ' is contained in some subinterval in κ . In other words, every partition point in κ is also a partition point in κ' . Then if κ' is a refinement of κ , we write $\kappa \subseteq \kappa'$.

Definition 1. Let $0 < p < \infty$. A function $f: [a, b] \rightarrow \mathbb{R}$ has *p -variation* defined by

$$v_p(f; [a, b]) := \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p : x_i \in \kappa, \text{ all partitions } \kappa \text{ of } [a, b] \right\},$$

which may be infinite. Write $v_p(f) := v_p(f; [a, b])$. We say that f has *bounded p -variation* if $v_p(f; [a, b]) < \infty$.

Denote by $\mathcal{W}_p([a, b])$ the class of functions defined on the interval $[a, b]$ with

bounded p -variation, that is

$$\mathcal{W}_p([a, b]) := \{f: [a, b] \rightarrow \mathbb{R}: v_p(f; [a, b]) < \infty\}.$$

Let $\mathcal{CW}_p([a, b]) := \mathcal{C}([a, b]) \cap \mathcal{W}_p([a, b])$ denote the class of functions that have bounded p -variation and are continuous.

Let $V_p(f) := V_p(f; [a, b]) := (v_p(f; [a, b]))^{1/p}$. Note that bounded 1-variation is the same as bounded variation. In other words, $\mathcal{W}_1([a, b])$ is the well known class of functions with bounded variation.

1.1 Properties of p -variation.

Lemma 1. *Let $a < c < b$ and let $f \in \mathcal{W}_p([a, b])$ with $0 < p < \infty$. Then*

$$v_p(f; [a, c]) + v_p(f; [c, b]) \leq v_p(f; [a, b]).$$

Proof. Let $\{y_m : m = 1, \dots, n\} = \{a = y_0 < y_1 < \dots < y_n = c\}$ be an arbitrary partition of $[a, c]$ and $\{z_k : k = 1, \dots, l\} = \{c = z_0 < z_1 < \dots < z_l = b\}$ be an arbitrary partition of $[c, b]$. Then $\{w_i : i = 1, \dots, j\} := \{y_m\} \cup \{z_k\}$ is a partition of $[a, b]$. If we choose this partition of $[a, b]$, the following inequality is true by definition

$$\sum_{m=1}^n |f(y_m) - f(y_{m-1})|^p + \sum_{k=1}^l |f(z_k) - f(z_{k-1})|^p \leq v_p(f; [a, b]). \quad (1.1.1)$$

If the result is not true, then

$$v_p(f; [a, c]) + v_p(f; [c, b]) > v_p(f; [a, b]).$$

Hence there exist partitions $\{y'_m : m = 1, \dots, n'\}$ of $[a, c]$ and $\{z'_k : k = 1, \dots, l'\}$ of $[c, b]$ such that

$$\sum_{m=1}^{n'} |f(y'_m) - f(y'_{m-1})|^p + \sum_{k=1}^{l'} |f(z'_k) - f(z'_{k-1})|^p > v_p(f; [a, b]).$$

This contradicts (1.1.1) which is valid for all partitions of $[a, c]$ and $[c, b]$. So the proof is complete. $\square$

Lemma 2. $V_p(f; [a, b])$ is a decreasing function of $p \geq 1$.

Proof. Let $\kappa := \{a = y_0 < y_1 < \cdots < y_n = b\}$ be a partition of $[a, b]$ and $a_i = |f(y_i) - f(y_{i-1})|$. It is sufficient to prove for $0 < q < p$,

$$\left(\sum_{i=1}^n a_i^p\right)^{1/p} \leq \left(\sum_{i=1}^n a_i^q\right)^{1/q}. \quad (1)$$

Suppose that $\sum_{i=1}^n a_i^q = 1$ (since both sides of (1) are homogeneous of degree 1 in the a_i). So $a_i^q \leq 1$ and $a_i^p = (a_i^q)^{p/q} \leq a_i^q$. Then $\sum_i a_i^p \leq \sum_i a_i^q$. Hence,

$$\begin{aligned} \frac{(\sum_i a_i^p)^{1/p}}{(\sum_i a_i^q)^{1/q}} &= \left\{ \sum_i \frac{a_i^p}{(\sum_k a_k^q)^{p/q}} \right\}^{1/p} = \left\{ \sum_i \left(\frac{a_i^q}{\sum_k a_k^q} \right)^{p/q} \right\}^{1/p} \\ &\leq \left\{ \sum_i \frac{a_i^q}{\sum_k a_k^q} \right\}^{1/p} = 1. \end{aligned}$$

So, $(\sum_{i=1}^n a_i^p)^{1/p} \leq (\sum_{i=1}^n a_i^q)^{1/q}$. This implies that $V_p(f) \leq V_q(f)$ for $q < p$. $\square$

It should be noted that constants are the only continuous functions which have bounded p -variation with $0 < p < 1$ (cf. Theorem 2.11, Dudley and Norvaiša, 1997 in T.5).

By Lemma 2, it follows that $\mathcal{W}_{p_1} \subseteq \mathcal{W}_{p_2}$ if $1 \leq p_1 < p_2 < \infty$. The next lemma shows that this inclusion is strict.

Lemma 2a. $\mathcal{W}_p$ is strictly increasing in p for $p \geq 1$.

Proof. For $p \geq 1$, define a continuous function, f , on $[0, 1]$ by

$$f(x) := \begin{cases} x^{1/p} \cos[\pi/(2x)] & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0. \end{cases}$$

Choose the following partition

$$x_0 = 0 < x_1 = \frac{1}{2n} < x_2 = \frac{1}{2n-1} < x_3 = \frac{1}{2n-2} < \cdots < x_{2n} = 1.$$

Note that $|\cos[(2n-k)\pi/2]| = 1$ if $k = 0, 2, 4, 6, \dots, 2n$ and $\cos[(2n-k)\pi/2] = 0$ if $k = 1, 3, 5, \dots, 2n-1$. Then,

$$\begin{aligned} \sum_{i=1}^{2n} |f(x_i) - f(x_{i-1})|^p &= \sum_{i=1}^{2n} |x_i^{1/p} \cos[\pi/(2x_i)] - x_{i-1}^{1/p} \cos[\pi/(2x_{i-1})]|^p \\ &= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \dots + \frac{1}{2} + \frac{1}{2} = \frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{2} + 1. \end{aligned}$$

Since n is arbitrary, $V_p(f; [0, 1]) = +\infty$.

However, for $q > p$ (or $q/p > 1$), we have

$$\begin{aligned} \sum_{i=1}^{2n} |f(x_i) - f(x_{i-1})|^q &= \sum_{i=1}^{2n} |x_i^{1/p} \cos[\pi/(2x_i)] - x_{i-1}^{1/p} \cos[\pi/(2x_{i-1})]|^q \\ &= \frac{2^{1-q/p}}{n^{q/p}} + \frac{2^{1-q/p}}{(n-1)^{q/p}} + \dots + 2^{1-q/p} = 2^{1-q/p} \left[\frac{1}{n^{q/p}} + \frac{1}{(n-1)^{q/p}} + \dots + \frac{1}{2} + 1 \right]. \end{aligned}$$

Therefore, $V_q(f; [0, 1]) < +\infty$ because $q/p > 1$. Hence, $f \in \mathcal{W}_q$ but $f \notin \mathcal{W}_p$. That is, $\mathcal{W}_p$ is strictly increasing in p for $p \geq 1$. $\square$

Lemma 3. *Let $f \in C\mathcal{W}_p([a, b])$, $1 \leq p < \infty$. Then for any $x \in (a, b]$ and $y \in [a, b)$*

$$\lim_{\eta \uparrow x} v_p(f; [\eta, x]) = \lim_{\xi \downarrow y} v_p(f; [y, \xi]) = 0. \quad (1.1.2)$$

Proof. It is enough to show the first statement since the proof of the second statement is symmetrical. By Lemma 1, the function $\eta \rightarrow v_p(f; [\eta, x])$ is decreasing to some limit as $\eta \uparrow x$. Suppose the limit is δ for some $\delta > 0$. Hence one can find $z < x$ and $z \in [a, b]$ such that

$$v_p(f; [z, x]) < \frac{5\delta}{4} \quad (1.1.3)$$

and (due to the continuity of f), for all $y \in [z, x]$,

$$|f(y) - f(x)|^p < \frac{\delta}{8}. \quad (1.1.4)$$

We can choose a partition $\{z = x_0 < x_1 < \cdots < x_n = x\}$, such that

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p > \frac{7\delta}{8}.$$

Then by (1.1.4), we have

$$\sum_{i=1}^{n-1} |f(x_i) - f(x_{i-1})|^p > \frac{3\delta}{4}. \quad (1.1.5)$$

Choose another partition $\{x_{n-1} = y_0 < \cdots < y_m = x\}$, such that

$$\sum_{j=1}^{m-1} |f(y_j) - f(y_{j-1})|^p > \frac{3\delta}{4}. \quad (1.1.6)$$

Combining (1.1.5) and (1.1.6) (where we are summing over intervals $[z, x_{n-1}]$ and $[x_{n-1}, x]$), we get

$$v_p(f; [z, x]) > \frac{6\delta}{4}$$

which contradicts (1.1.3). $\square$

Lemma 4. *Let $f \in CW_p([a, b])$ with $1 \leq p < \infty$. Then, for every $\epsilon > 0$, there exists a partition $\kappa = \{a = t_0 < \cdots < t_n = b\}$ of $[a, b]$ such that*

$$\max_{1 \leq i \leq n} v_p(f; [t_{i-1}, t_i]) < \epsilon.$$

Proof. It follows from Lemma 3 that given $x \in (a, b)$, there exist $\eta_x < x$ and $\xi_x > x$ such that $v_p(f; [\eta_x, x]) \vee v_p(f; [x, \xi_x]) < \epsilon$. Therefore, $\{\Delta_x := (\eta_x, \xi_x); x \in (a, b)\}$ is an open covering of $[a, b]$. Then by Heine-Borel theorem (cf. Theorem A.14 in T.5), there exists a finite subcovering $\{\Delta_{x_j} : j = 1, \dots, m\}$. Take any point $y_j \in (\eta_{x_{j+1}}, \xi_{x_j})$ for $j = 1, \dots, m-1$ and add these points to $\{x_j : j = 0, 1, \dots, n\}$ to get the desired partition $\{a = x_0 < x_1 < y_1 < x_2 < y_2 < \cdots < y_{m-1} < x_m = b\}$. So

$$v_p(f; [a, x_1]) \vee \max_{1 \leq j \leq m-1} v_p(f; [x_j, y_j]) \vee \max_{1 \leq j \leq m-1} v_p(f; [y_j, x_{j+1}]) < \epsilon.$$

$\square$

Lemma 5. For $a, b \in \mathbb{R}^+$ and $p \geq 1$ we have

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p). \quad (1.1.9)$$

Proof. By Proposition A.8 in T.3, the composition of two convex function is convex. Since $|t|$ and t^p with $p \geq 1$ are both convex functions, $|t|^p$ is convex. Then the convexity of the function $|t|^p$ shows that

$$\left| \frac{a+b}{2} \right|^p \leq \frac{1}{2}(|a|^p + |b|^p),$$

or equivalently $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$. $\square$

Lemma 6. The $\log v_p(f)$ is convex in $p \in (0, \infty)$.

Proof. Let $\kappa = \{a = y_0 < y_1 < \dots < y_n = b\}$ and let $a_i = |f(y_i) - f(y_{i-1})|$.

If $p_1 < p_2 < p_3$, then by Theorem 18 of Hardy, Littlewood, and Polya (1959, p.28) we have

$$\sum_{i=1}^n a_i^{p_2} \leq \left(\sum_{i=1}^n a_i^{p_1} \right)^{\frac{p_3-p_2}{p_3-p_1}} \left(\sum_{i=1}^n a_i^{p_3} \right)^{\frac{p_2-p_1}{p_3-p_1}}.$$

By taking the supremum, we get

$$v_{p_2}(f) \leq (v_{p_1}(f))^{\frac{p_3-p_2}{p_3-p_1}} (v_{p_3}(f))^{\frac{p_2-p_1}{p_3-p_1}}.$$

If now we take the $p_3 - p_1$ power and then take the logarithm on both sides, we get

$$(p_3 - p_1) \log v_{p_2}(f) \leq (p_3 - p_2) \log v_{p_1}(f) + (p_2 - p_1) \log v_{p_3}(f).$$

Then by Proposition A.7 (ii) (in T.3), $\log v_p(f)$ is convex in p . $\square$

Lemma 7. $v_p(f)$ is continuous in $p \in (0, \infty)$.

Proof. By Lemma 6, we know that $\log v_p(f)$ is convex in p . Therefore, $v_p(f)$ is convex in p by Proposition A.9 in T.3. Then by Theorem A.10 (in T.3), $v_p(f)$ is continuous in p on (a, b) . $\square$

Lemma 8. *For $f \in C\mathcal{W}_p([a, b])$ with $p \geq 1$, the function $[a, b] \ni x \mapsto v_p(f; [a, x])$ is continuous.*

Proof. See Young (1937, Lemma 4.3, p.454) or Marcinkiewicz (1934, Lemma 3, p.39). $\square$

Lemma 9. *If $f_1, f_2 \in \mathcal{W}_p$, then $f_1 + f_2 \in \mathcal{W}_p$.*

Proof. This follows from Minkowski's inequality, and a more general statement is proven by Dudley and Norvaiša (1997). $\square$

A function f on $[a, b]$ satisfies a Lipschitz condition of order $\alpha \in (0, 1]$ if there exists a constant K such that $|f(x) - f(y)| \leq K|x - y|^\alpha$ for all $x, y \in [a, b]$. Recall that a Lipschitz condition is also referred to Hölder's condition. We will interchangeably use Lipschitz and Hölder conditions throughout the paper.

Lemma 10. *If f satisfies a Lipschitz condition of order α , then $f \in \mathcal{W}_{1/\alpha}$.*

Proof. See Marcinkiewicz (1934, Theorem 4). Let $\{x_i : i = 1, \dots, n\}$ be a partition of $[a, b]$. Then we have

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^{1/\alpha} \leq K^{1/\alpha} \sum_{i=1}^n |x_i - x_{i-1}| = K^{1/\alpha}(b - a).$$

Thus $v_{1/\alpha}(f; [a, b]) \leq K^{1/\alpha}(b - a) < \infty$. $\square$

Lemma 11. *If $f \in \mathcal{W}_p$, then f is bounded and $f \in \mathcal{W}_{p_1}$ for every $p_1 > p$.*

Proof. It follows immediately from Jensen's inequality and Lemma 2. Also see Marcinkiewicz (1934). $\square$

Recall that a function ϕ on $(-\infty, \infty)$ is *locally Lipschitz*, if for every $M > 0$, there exists $K = K(M)$ such that

$$|\phi(u) - \phi(v)| \leq K|u - v| \quad \text{for every } u, v \in [-M, M].$$

In addition, the composite function $\phi \circ f$ on $[a, b]$ is defined by $(\phi \circ f)(x) := \phi(f(x))$ for $x \in [a, b]$.

Lemma 12. *Let ϕ be a locally Lipschitz function and let $f \in \mathcal{W}_p$. Then the composite $\phi \circ f$ has bounded p -variation, that is, $\phi \circ f \in \mathcal{W}_p$.*

Proof. Let ϕ be locally Lipschitz and let $f \in \mathcal{W}_p$. Then, for each $a \leq x_1 < x_2 \leq b$, we have

$$|(\phi \circ f)(x_2) - (\phi \circ f)(x_1)| \leq K|f(x_2) - f(x_1)|.$$

Thus, for any partition $\kappa = \{x_i : i = 1, \dots, n\}$, it follows that

$$v_p(\phi \circ f; \kappa) := \sum_{i=1}^n |(\phi \circ f)(x_i) - (\phi \circ f)(x_{i-1})|^p \leq K^p v_p(f; \kappa) \leq K^p v_p(f; [a, b])$$

where $v_p(f; \kappa) := \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p$. Since κ is an arbitrary partition, we get

$$v_p(\phi \circ f; [a, b]) \leq K^p v_p(f; [a, b]).$$

If $v_p(f; [a, b]) < \infty$, then $v_p(\phi \circ f; [a, b]) < \infty$. $\square$

The converse to Lemma 12 is also true (cf. Ciernoczołowski and Orlicz (1986)).

Lemma 13. *If $f_1, f_2 \in \mathcal{W}_p$, then $f_1 \cdot f_2 \in \mathcal{W}_p$.*

Proof. See Krabbe (1961) or the proof of Proposition 1, p.18. $\square$

Lemma 14. *If $f, g \in \mathcal{W}_p$, and $f(x) \geq c > 0$ for every $x \in [a, b]$, then $\frac{g}{f} \in \mathcal{W}_p$.*

Proof. We first show that $f^{-1} \in \mathcal{W}_p$. Then, for each $a \leq x_1 < x_2 \leq b$,

$$\left| \frac{1}{f(x_2)} - \frac{1}{f(x_1)} \right| = \frac{|f(x_1) - f(x_2)|}{f(x_1)f(x_2)} \leq \frac{1}{c^2} |f(x_1) - f(x_2)|$$

since, for every x , $f(x) \geq c > 0$. Thus, for any partition $\kappa = \{x_i\}$, it follows that

$$v_p(f^{-1}; \kappa) := \sum_{i=1}^n |f^{-1}(x_i) - f^{-1}(x_{i-1})|^p \leq \frac{1}{c^{2p}} v_p(f; \kappa) \leq \frac{1}{c^{2p}} v_p(f; [a, b]).$$

Since κ is an arbitrary partition, we get

$$v_p(f^{-1}; [a, b]) \leq \frac{1}{c^{2p}} v_p(f; [a, b]).$$

If $v_p(f; [a, b]) < \infty$, then $v_p(f^{-1}; [a, b]) < \infty$. Write $\frac{g}{f} = gf^{-1}$. The general case now follows from Lemma 13. $\square$

Theorem 2. (*Young's Stieltjes integrability theorem*). Let $f \in \mathcal{W}_p([a, b])$ and $g \in \mathcal{W}_q([a, b])$ with $p, q > 0$, $1/p + 1/q > 1$. If f and g have no common discontinuities then the Riemann-Stieltjes integral $\int_a^b g df$ exists and, for any $\xi \in [a, b]$, the following inequality holds:

$$\left| \int_a^b g df - g(\xi)[f(b) - f(a)] \right| \leq (1 + \zeta(1/p + 1/q)) V_p(f; [a, b]) V_q(g; [a, b]), \quad (1.1.10)$$

where ζ is the Riemann zeta function $\zeta(s) := \sum_{n \geq 1} n^{-s}$.

This theorem is proven by Young (1936). In addition, we will need Young's (1936) version of Hölder's inequality

$$\sum_{i=1}^n |u_i v_i| \leq \left(\sum_{i=1}^n |u_i|^p \right)^{1/p} \left(\sum_{i=1}^n |v_i|^q \right)^{1/q} \quad (1.1.11)$$

which is valid for any $p, q > 0$ such that $1/p + 1/q \geq 1$. This inequality follows from a combination of the usual Hölder inequality (cf. Hardy, Littlewood, and Pólya (1959, pp.21-26) and Jensen's inequality (cf. Theorem A.11 in T.3)).

Theorem 3. Let $f = (f_1, \dots, f_d): [a, b] \mapsto \mathbb{R}^d$ be a function such that, for each $l = 1, \dots, d$, $f_l \in CW_p([a, b])$ with $1 \leq p < 2$. Let $g: \mathbb{R}^d \mapsto \mathbb{R}$ be a differentiable function with locally Lipschitz partial derivatives g'_l , $l = 1, \dots, d$. Then each $g'_l \circ f$ is Riemann-Stieltjes integrable with respect to f_l and

$$(g \circ f)(b) - (g \circ f)(a) = \sum_{l=1}^d \int_a^b (g'_l \circ f) df_l. \quad (1.1.12)$$

Proof. Since each f_l is continuous, there exists a finite constant M such that $|f_l(x)| \leq M$ for all $x \in [a, b]$ and $l = 1, \dots, d$. Since each g'_l is locally Lipschitz, there exists a finite constant K such that, for all $u = (u_l), v = (v_l) \in [-M, M]^d$,

$$\max_{1 \leq l \leq d} |g'_l(u) - g'_l(v)| \leq K \sum_{l=1}^d |u_l - v_l|. \quad (1.1.13)$$

It then follows that, for all $l = 1, \dots, d$, $g'_l \circ f \in \mathcal{W}_p([a, b])$ with $1 \leq p < 2$ since the composition of continuous functions is continuous. Therefore by Young's theorem of Stieltjes integrability, all d -integrals on the right hand side of (1.1.12) exist.

To prove equality (1.1.12) let $\kappa := \{a = x_0 < x_1 < \dots < x_n = b\}$ be a partition of $[a, b]$. By writing a telescoping sum, and adding and subtracting $\sum_{i=1}^d g'(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})]$, we have

$$\begin{aligned} (g \circ f)(b) - (g \circ f)(a) &= \sum_{i=1}^n [g(f(x_i)) - g(f(x_{i-1}))] \\ &= \sum_{l=1}^d \sum_{i=1}^n g'_l(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})] \\ &\quad + \sum_{i=1}^n \left\{ g(f(x_i)) - g(f(x_{i-1})) - \sum_{l=1}^d g'_l(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})] \right\}. \end{aligned} \quad (1.1.14)$$

For each $l = 1, \dots, d$, let

$$I_l(\kappa) := \sum_{i=1}^n g'_l(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})]$$

and let

$$R(\kappa) := \sum_{i=1}^n [g(f(x_i)) - g(f(x_{i-1})) - \sum_{l=1}^d g'_l(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})]].$$

Since each $I_l(\kappa)$ is a Riemann-Stieltjes sum based on the partition κ , we have

$$\lim_{|\kappa| \rightarrow 0} I_l(\kappa) = \lim_{|\kappa| \rightarrow 0} \sum_{i=1}^n g'_l(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})] = \int_a^b (g'_l \circ f) df_l,$$

where $|\kappa| = \max_i(x_i - x_{i-1})$ is the mesh of κ . To estimate $R(\kappa)$ we use the multidimensional mean value theorem $g(f(b)) - g(f(a)) = \sum_{l=1}^d g'_l(\theta)[f_l(b) - f_l(a)]$ where $\theta \in S := \{f(a) + \alpha(f(b) - f(a)) : 0 < \alpha < 1\}$ (cf. Stromberg (1981), p 373). This will give

$$\begin{aligned}
|R(\kappa)| &= \left| \sum_{i=1}^n [g(f(x_i)) - g(f(x_{i-1})) - \sum_{l=1}^d g'_l(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})]] \right| \\
&\leq \sum_{i=1}^n \left| [g(f(x_i)) - g(f(x_{i-1})) - \sum_{l=1}^d g'_l(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})]] \right| \\
&\leq \sum_{i=1}^n \left| \left[\sum_{l=1}^d g'_l(\theta_i)[f_l(x_i) - f_l(x_{i-1})] - \sum_{l=1}^d g'_l(f(x_{i-1}))[f_l(x_i) - f_l(x_{i-1})] \right] \right| \\
&\leq \sum_{l=1}^d \sum_{i=1}^n |g'_l(\theta_i) - g'_l(f(x_{i-1}))| |f_l(x_i) - f_l(x_{i-1})| \\
&\leq K d^2 \max_{1 \leq l \leq d} \sum_{i=1}^n |\theta_{l,i} - f_l(x_{i-1})| |f_l(x_i) - f_l(x_{i-1})| \\
&\leq K d^2 \max_{1 \leq l \leq d} \sum_{i=1}^n [f_l(x_i) - f_l(x_{i-1})]^2 \\
&\leq K v_p(f) d^2 \max_{1 \leq l \leq d} \max_{1 \leq i \leq n} |f_l(x_i) - f_l(x_{i-1})|^{2-p} \tag{1.1.15}
\end{aligned}$$

where $\theta_i = (\theta_{1,i}, \dots, \theta_{d,i})$ with $\theta_{l,i} \in [f_l(x_{i-1}) \wedge f_l(x_i), f_l(x_{i-1}) \vee f_l(x_i)]$ for $i = 1, \dots, n$, K is the constant in (1.1.13) and $v_p(f) := v_p(f; [a, b])$. The right hand side of (1.1.15) tends to 0 as $|\kappa| \rightarrow 0$ since all f_l are continuous functions on $[a, b]$. Therefore the right hand side of (1.1.14) tends to the right hand side of (1.1.12) as $|\kappa| \rightarrow 0$. $\square$

Next is an extension of an *integration by parts* formula for the Riemann-Stieltjes integral when functions may have unbounded variation.

Corollary 3A. *Let $f, h \in \mathcal{CW}_p([a, b])$, $1 \leq p < 2$. Then the following two integrals*

exist and satisfy the relation:

$$(RS) \int_a^b f dh + (RS) \int_a^b h df = f(b)h(b) - f(a)h(a). \quad (1.1.16)$$

Proof. The claim follows when Theorem 3 is applied to the function $g(u, v) = uv$ and $f \equiv 1$. $\square$

Let $E_a(f)(x) := \exp \{f(x) - f(a)\}$, $x \in [a, b]$.

Theorem 4. *Let f be a function in $CW_p([a, b])$ with $1 \leq p < 2$. Then for any $r \geq p$ such that $1/p + 1/r > 1$, $E_a(f)$ is the unique solution in $\mathcal{W}_r([a, b])$ of the integral equation*

$$F(y) = 1 + (RS) \int_a^y F df, \quad y \in [a, b],$$

where $(RS) \int$ refers to Riemann-Stieltjes integral (cf. Definition A.1 in T.1).

Proof. Let $g(u) := \exp\{u - f(a)\}$ for $u \in R$. Then by Theorem 3 (with $d = 1$), we have

$$E_a(f)(y) - 1 = g \circ f(y) - g \circ f(a) = (RS) \int_a^y g' \circ f df = (RS) \int_a^y E_a(f) df$$

for all $y \in [a, b]$. Uniqueness is proved by Dudley and Norvaiša(1997, Theorem 5.21) (stated in T.5). $\square$

Proposition 1. *(substitution rule) Let f , g , and h be functions in $CW_p([a, b])$, $1 \leq p < 2$. Then*

$$(RS) \int_a^b f(x) d((RS) \int_a^x g(y) dh(y)) = (RS) \int_a^b f(x)g(x)dh(x). \quad (1.1.17)$$

On occasion, we will denote $(RS) \int$ as $\int$.

Proof. We start by showing that the following integrals exist:

$$\Phi(x) := \int_a^x g(y) dh(y), \quad x \in [a, b], \quad (1.1.18)$$

$$A := \int_a^b f(x) d\Phi(x), \quad (1.1.19)$$

and

$$B := \int_a^b f(x)g(x)dh(x) \quad (1.1.20)$$

The indefinite integral (1.1.18) exists by Young's Stieltjes integrability theorem (Theorem 2) because g and h have finite p -variation with $0 < p < 2$ and do not have discontinuities at all.

To prove the existence of (1.1.19), again by the Young's Stieltjes integrability theorem, we only need to show that $\Phi \in W_p$ since by assumption $f \in W_p$ and both Φ and f are continuous. For any partition $\kappa := \{a = x_0 < x_1 < \dots < x_n = b\}$ of $[a, b]$ we have to prove that $\Phi \in \mathcal{W}_p$. That is,

$$\begin{aligned} \sum_{i=1}^n |\Phi(x_i) - \Phi(x_{i-1})|^p &= \sum_{i=1}^n \left| \int_a^{x_i} g(y)dh(y) - \int_a^{x_{i-1}} g(y)dh(y) \right|^p \\ &= \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} g(y)dh(y) - g(\xi_i)[h(x_i) - h(x_{i-1})] + g(\xi_i)[h(x_i) - h(x_{i-1})] \right|^p \\ &\leq 2^{p-1} \left(\sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} g(y)dh(y) - g(\xi_i)[h(x_i) - h(x_{i-1})] \right|^p \right. \\ &\quad \left. + \sum_{i=1}^n |g(\xi_i)[h(x_i) - h(x_{i-1})]|^p \right). \end{aligned} \quad (1.1.21)$$

The first equality is by the additivity of Riemann-Stieltjes integrals and the second one by adding and subtracting $g(\xi_i)[h(x_i) - h(x_{i-1})]$ with $\xi_i \in [x_{i-1}, x_i]$, $i = 1, \dots, n$

and the last inequality by Lemma 5. By Theorem 2 with $p = q$ and Lemma 1, we then have

$$\begin{aligned} & \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} g(y) dh(y) - g(\xi_i)[h(x_i) - h(x_{i-1})] \right|^p \\ & \leq (1 + \zeta(2/p))^p \sum_{i=1}^n v_p(h; [x_{i-1}, x_i]) v_p(g; [x_{i-1}, x_i]) \\ & \leq (1 + \zeta(2/p))^p v_p(h; [a, b]) v_p(g; [a, b]). \end{aligned}$$

Now consider the second sum of (1.1.21):

$$\sum |g(\xi_i)[h(x_i) - h(x_{i-1})]|^p \leq \|g\|_\infty v_p(h; [a, b]).$$

Since $h \in \mathcal{W}_p$, $\|g\|_\infty < \infty$ and partition κ is arbitrary, it follows that $\Phi \in \mathcal{W}_p$. So the integral (1.1.19) exists.

To prove the existence of integral (1.1.20), by Young's Stieltjes integrability theorem again we only have to check $f g \in \mathcal{W}_p$. Let $\kappa := \{a = x_0 < x_1 < \dots < x_n = b\}$ be a partition of $[a, b]$. Then for each $i = 1, \dots, n$ we have

$$\begin{aligned} & |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})|^p \\ & = |f(x_i)g(x_i) - f(x_i)g(x_{i-1}) + f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1})|^p \\ & = |f(x_i)[g(x_i) - g(x_{i-1})] + g(x_{i-1})[f(x_i) - f(x_{i-1})]|^p \\ & \leq 2^{p-1} (|f(x_i)[g(x_i) - g(x_{i-1})]|^p + |g(x_{i-1})[f(x_i) - f(x_{i-1})]|^p) \\ & \leq 2^{p-1} (\|f\|_\infty^p |g(x_i) - g(x_{i-1})|^p + \|g\|_\infty^p |f(x_i) - f(x_{i-1})|^p). \end{aligned}$$

Here we used Lemma 5 on the third inequality. Summing over i , we get the following estimate

$$\sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})|^p \leq 2^{p-1} (\|f\|_\infty^p v_p(g; [a, b]) + \|g\|_\infty^p v_p(f; [a, b])).$$

Since κ is arbitrary, $v_p(fg; [a, b]) < \infty$. Therefore the integral (1.1.20) exists.

We are left to prove equality (1.1.17). Let $\kappa = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ and $y_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$. Define

$$A(\kappa) := \sum_{i=1}^n f(y_i) [\Phi(x_i) - \Phi(x_{i-1})]$$

and

$$B(\kappa) := \sum_{i=1}^n f(y_i) g(y_i) [h(x_i) - h(x_{i-1})].$$

We have already proved that $A = \lim_{|\kappa| \rightarrow 0} A(\kappa)$ and $B = \lim_{|\kappa| \rightarrow 0} B(\kappa)$. Therefore it is enough to show that

$$\lim_{|\kappa| \rightarrow 0} |A(\kappa) - B(\kappa)| = 0.$$

Let $\epsilon > 0$ and $p < r < 2$. Since g is continuous on a bounded interval $[a, b]$ it is uniformly continuous. Therefore we can choose a partition κ such that $|g(u) - g(v)| \leq \epsilon$ whenever $u, v \in [x_{i-1}, x_i]$ for some $i = 1, \dots, n$. Then by Theorem 2, we have

$$\begin{aligned} |A(\kappa) - B(\kappa)| &= \left| \sum_{i=1}^n f(y_i) [\Phi(x_i) - \Phi(x_{i-1})] - \sum_{i=1}^n f(y_i) g(y_i) [h(x_i) - h(x_{i-1})] \right| \\ &= \left| \sum_{i=1}^n f(y_i) \left[\int_{x_{i-1}}^{x_i} g(z) dh(z) - g(y_i) [h(x_i) - h(x_{i-1})] \right] \right| \\ &\leq \sum_{i=1}^n |f(y_i)| \left| \int_{x_{i-1}}^{x_i} g(z) dh(z) - g(y_i) [h(x_i) - h(x_{i-1})] \right| \\ &\leq \|f\|_{\infty} \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} g(z) dh(z) - g(y_i) [h(x_i) - h(x_{i-1})] \right| \\ &\leq \|f\|_{\infty} (1 + \zeta(2/r)) \sum V_r(g; [x_{i-1}, x_i]) V_r(h; [x_{i-1}, x_i]). \end{aligned} \tag{1.1.22}$$

Let $(1 + \zeta(2/r)) = C(r)$, and take $\delta := r - p > 0$. Then for any partition $\kappa_i :=$

$\{x_{i-1} = y_0^i < y_1^i < \dots < y_{m(i)}^i = x_i\}$ and for each $i = 1, \dots, n$, we have

$$\begin{aligned} \sum_{j=1}^{m(i)} |g(y_j^i) - g(y_{j-1}^i)|^r &\leq \max_j |g(y_j^i) - g(y_{j-1}^i)|^\delta \sum_{j=1}^{m(i)} |g(y_j^i) - g(y_{j-1}^i)|^p \\ &\leq \epsilon^\delta v_p(g; [x_{i-1}, x_i]). \end{aligned}$$

Since κ_i are arbitrary partitions it follows that, for each $i = 1, \dots, n$,

$$v_r(g; [x_{i-1}, x_i]) \leq \epsilon^\delta v_p(g; [x_{i-1}, x_i]).$$

Continuing with (1.1.22) and using Hölder's inequality, we then have

$$\begin{aligned} |A(\kappa) - B(\kappa)| &\leq \|f\|_\infty C(r) \epsilon^{\frac{\delta}{r}} \left(\sum_{i=1}^n v_p(g; [x_{i-1}, x_i]) \right)^{1/r} \left(\sum_{i=1}^n v_r(h; [x_{i-1}, x_i]) \right)^{1/r} \\ &\leq \|f\|_\infty C(r) \epsilon^{\frac{\delta}{r}} (v_p(g; [a, b]))^{1/r} (v_r(h; [a, b]))^{1/r} \end{aligned}$$

where the last inequality follows from Lemma 1. Since ϵ is arbitrary, it follows that $\lim_{|\kappa| \rightarrow 0} |A(\kappa) - B(\kappa)| = 0$. The proof of the proposition is now complete. $\square$

Proposition 2. *Let $f, g \in CW_p([a, b])$ with $1 \leq p < 2$ and let ϕ be a differentiable function on $\mathbb{R}$ with derivative ϕ' satisfying a local Lipschitz condition (cf. definition on p.11). Then*

$$(RS) \int_a^b f d(\phi \circ g) = (RS) \int_a^b f(\phi' \circ g) dg.$$

Proof. By Theorem 3 and Proposition 1, we have

$$\begin{aligned} \int_a^b f(x) d\phi(g(x)) &= \int_a^b f(x) d[\phi \circ g(a) + \int_a^x \phi' \circ g dg] \\ &= \int_a^b f(x) d[\phi \circ g(a)] + \int_a^b f(x) d\left(\int_a^x \phi' \circ g dg\right) \\ &= \int_a^b f(x)(\phi' \circ g)(x) dg(x). \end{aligned}$$

The second equality follows from the linearity of Riemann-Stieltjes integrals and the first integral of the second equality is obviously zero. The proof is complete. $\square$

Proposition 3. *Let h be a Lipschitz function and let $g \in CW_p([a, b])$, $1 \leq p < 2$.*

Then

$$(RS) \int_a^b h[g(x)] dg(x) = (RS) \int_{g(a)}^{g(b)} h(v) dv.$$

Proof. Let $H(y) := \int_{g(a)}^y h(v) dv$ for $y \in [a, b]$. Then

$$\int_a^b h[g(x)] dg(x) = \int_a^b H'[g(x)] dg(x) = (H \circ g)(b) - (H \circ g)(a) = \int_{g(a)}^{g(b)} h(v) dv.$$

The first equality is by the fundamental theorem of calculus for Riemann integral and the second equality follows from the chain rule formula (1.1.12). $\square$

1.2 Fractional Brownian Motion

Fractional Brownian motion (fBm) has recently become a “hot” choice for modeling in mathematical finance and other sciences. On purely empirical data, some believe that fBm is an *ideal* candidate since it is a long-term dependent and self-similar process. Even with its popularity, our understanding of the properties and behaviour of fBm is limited.

Kolmogorov (1940, 1941) was the first to introduce the Gaussian process which is now known as fractional Brownian motion in the theory of probability. This class of processes was studied by Kolmogorov in detail and it played an essential role in the series of problems of the statistical theory of turbulence. Yaglom (1955, 1958) discussed the spectral density and correlation function of fBm. A quadratic variation formula for fractional Brownian motion follows from a general result of Baxter (1956). Gladyshev (1961) extended Baxter’s (1956) result and provided a theoretical result to determine the value of the Hurst effect denoted by H (cf. definition below). However, most of the encomium to fractional Brownian motion has been given to Mandelbrot and Van Ness (1968) who used fBm to model natural phenomena such as the speculative market fluctuations.

There are several equivalent definitions of the fractional Brownian motion (cf. Lin, 1995). We will begin our discussion with these definitions. Since Gaussian

processes G are determined by their covariance structure, $\{Cov(G(t), G(s)) : t, s \geq 0\}$, the proof of the equivalency is not difficult (however too messy to be presented here).

We denote by $B_H = \{B_H(t) : t \geq 0\}$ fractional Brownian motion with index $0 < H < 1$. On occasion, we will use the notation $X_t := X(t)$ also to represent fractional Brownian motion. The symbol $\stackrel{D}{=}$ will denote the equality in distribution.

Definition 2a. A centered Gaussian process $\{B_H(t) : 0 \leq t < \infty\}$ with $B_H(0) = 0$ is a fBm, if

$$(1) (B_H(t_2) - B_H(t_1), B_H(s_2) - B_H(s_1)) \stackrel{D}{=} (B_H(t_2 + h) - B_H(t_1 + h), B_H(s_2 + h) - B_H(s_1 + h)) \text{ for all } t_1, t_2, s_1, s_2 \text{ and } h \geq 0.$$

$$(2) \text{ There is an } H \in (0, 1) \text{ such that}$$

$$B_H(t + \tau) - B_H(t) \stackrel{D}{=} h^{-H} (B_H(t + h\tau) - B_H(t))$$

$$\text{for all } t, \tau, h \geq 0.$$

Definition 2b. A centered Gaussian process $\{B_H(t) : 0 \leq t < \infty\}$ with $B_H(0) = 0$ is a fBm, if

$$Var(B_H(t) - B_H(s)) = Var(B_H(1))|t - s|^{2H}$$

$$\text{for all } t, s \geq 0.$$

Definition 2c. A centered Gaussian process $\{B_H(t) : 0 \leq t < \infty\}$ with $B_H(0) = 0$ is a fBm, if

$$Cov(B_H(t), B_H(s)) = \frac{1}{2} Var(B_H(1))(|t|^{2H} + |s|^{2H} - |t - s|^{2H})$$

$$\text{for all } t, s \geq 0.$$

Fractional Brownian motion is a centered Gaussian process with stationary increments that is stochastically self-similar. The H is called the *Hurst constant* and it determines the sign of the covariance of the future and past increments. As an aside, if $0 \leq t_1 \leq t_2 \leq s_1 \leq s_2$, then

$$Cov(B_H(t_2) - B_H(t_1), B_H(s_2) - B_H(s_1)) = \frac{1}{2} Var(B_H(1))\{(s_2 - t_2)^{2H}$$

$$+(s_1 - t_1)^{2H} - (s_1 - t_2)^{2H} - (s_2 - t_1)^{2H}\}. \quad (*)$$

If $H = 1/2$ and $Var(B_H(1)) = 1$, then $Cov(B_H(t), B_H(s)) = 1/2(|t| + |s| - |t - s|) = t \wedge s$. Therefore when $H = 1/2$, $B_{1/2} := B$ is ordinary Brownian motion. Since a Gaussian random function can be completely described by its covariance structure, we have

$$B_H(t) \implies B(t) \quad \text{for every } t \text{ as } H \rightarrow 1/2, \quad (**)$$

where $\implies$ refers to convergence in distribution. In addition, when $H > 1/2$, $(*)$ is positive and when $H < 1/2$, $(*)$ is negative. This is derived from the convexity/concavity of the power function. Hereafter, we will consider the case when $1/2 < H < 1$ which reflects the long-range dependence property.

It is a well-known fact that B_H has almost all sample paths continuous (cf. Mandelbrot and van Ness (1968)), but the next theorem provides an alternative proof of this fact.

Theorem 5. *Let B_H be fractional Brownian motion with index $0 < H < 1$. Then almost all sample paths of B_H are continuous.*

Proof. It is sufficient only to check the metric entropy condition in Dudley's (1973) Theorem. That is, recall for fractional Brownian motion $B_H := X$, $Var(X(t) - X(s)) = VarX(1)|t - s|^{2H}$ for all $t, s \geq 0$. Set $d(s, t) := [E(X(t) - X(s))^2]^{1/2}$. Then if X is fBm then $d(s, t) = c|t - s|^H$ where $c = \sqrt{VarX(1)}$. Let $A_j = [j(\frac{2\epsilon}{c})^{1/H}, (j+1)(\frac{2\epsilon}{c})^{1/H}]$ for $j = 1, \dots, n$. Then for $s, t \in A_j$, we have $|t - s| \leq (j+1)(\frac{2\epsilon}{c})^{1/H} - j(\frac{2\epsilon}{c})^{1/H} = (\frac{2\epsilon}{c})^{1/H}$. This gives us $d(s, t) \leq 2\epsilon$. Now we choose n such that $[0, T] \subset \cup_{j=1}^n A_j$. Note that $(j+1)(\frac{2\epsilon}{c})^{1/H} \geq T$. Hence

$$N(\epsilon) := \inf\{n : \exists A_j, j = 1, \dots, n : [0, T] \subset \cup_{j=1}^n A_j, \sup_{s, t \in A_j} d(s, t) \leq 2\epsilon\}$$

$$\leq T(\frac{2\epsilon}{c})^{1/H} = C\epsilon^{-1/H}.$$

That is, $N(\epsilon) \leq C\epsilon^{-1/H}$. (The value of C may change from line to line.) Taking

logarithms on both sides gives

$$\begin{aligned}\log N(\epsilon) &\leq \log C + \log \epsilon^{-1/H} \\ &\leq C \log \frac{1}{\epsilon} \quad \text{as } \epsilon \downarrow 0 \\ &\leq C \frac{1}{\epsilon}.\end{aligned}$$

Therefore

$$\int_0^1 [\log N(x)]^{1/2} dx \leq \int_0^1 [C \frac{1}{x}]^{1/2} dx = 2\sqrt{C} < \infty.$$

Hence X is continuous by Dudley's Theorem (1973, cf. Theorem A.12 in T.5). $\square$

1.3 Stochastic Calculus of Fractional Brownian Motion.

In this section, we propose a stochastic calculus of fractional Brownian motion without probability. The pathwise integral of fractional Brownian motion is based on the p -variation property. That is, Proposition 4, we show that if $H \in (1/2, 1)$ then almost all sample paths of the stochastic process $P_H = \{P_H(t): 0 \leq t \leq T\}$ with

$$P_H(t) := \exp \{ \mu t + \sigma B_H(t) \} \quad (1.3.1)$$

are unique solutions (in a certain class) of the (pathwise) integral equation:

$$F(t) = 1 + (RS) \int_0^t F(s) d(\mu s + \sigma B_H(s)), \quad t \in [0, T], \quad (1.3.2)$$

where $(RS) \int$ denotes the Riemann-Stieltjes integral (and not Itô's). Hereafter, we will refer to P_H as geometric fractional Brownian motion.

Recall that $B_{1/2} := B$ is a (ordinary) Brownian motion and geometric Brownian motion $P = \{P(t): 0 \leq t \leq T\}$ which is defined by

$$P(t) := \exp \{ (\mu - \sigma^2/2)t + \sigma B(t) \}. \quad (1.3.3)$$

are unique solutions of the integral equation:

$$F(t) = 1 + (I) \int_0^t F(s) d(\mu s + \sigma B(s)), \quad t \in [0, T]$$

where $(I) \int$ denotes the Itô's integral (cf. Karatzas & Shreve (1991)).

To solve equation (1.3.2), we need to extend the “chain rule” formula to include functions with bounded p -variation, $1 \leq p < 2$. This replaces the Itô's formula used in the case of the Brownian motion. The usual chain rule known in analysis becomes Itô's formula if the quadratic variation exists. Since the quadratic variation for fBm is zero (cf. Lin (1995) and Proposition 6 below), Itô's formula is not valid. So at most, we can prove the usual chain rule for fBm using a different stochastic integral (cf. Theorem 3, for example, and Lin (1995)). The next lemma illustrates this difference.

Lemma. *Geometric fBm does not weakly converge to geometric BM as $H \downarrow 1/2$.*

Proof. Let $B_{1/2} = B$ be ordinary Brownian motion. As $H \rightarrow 1/2$,

$$\begin{aligned} P_H(t) &:= \exp \{ \mu t + \sigma B_H(t) \} \implies \exp \{ \mu t + \sigma B(t) \} \quad \text{by } (**) \\ &\neq P(t) := \exp \{ (\mu - \sigma^2/2)t + \sigma B(t) \}. \end{aligned}$$

□

Proposition 4. *Let $1/2 < H < 1$. Then almost all sample paths of the stochastic process $P_H = \{P_H(t) : t \geq 0\}$ given by (1.3.1) are unique solutions of the integral equation (1.3.2).*

Proof. Let $B_H = \{B_H(t) : t \geq 0\}$ be a fractional Brownian motion with index H , $0 < H < 1$. By Theorem 5, B_H has almost all sample paths continuous. Therefore Theorem 3 of Kawada and Kôno (1973, cf. Theorem A.13 in T.5) implies that

$$\limsup_{\delta \downarrow 0} \left\{ \sum_{i=1}^n |B_H(t_i) - B_H(t_{i-1})|^p : \kappa = \{0 = t_0 < \dots t_n = T\}, |\kappa| \leq \delta \right\} < \infty$$

almost surely for any $p > 1/H$. Then one can show (cf. Lemma 2.3, Dudley and Norvaiša (1997) in T.5) that $v_p(B_H; [0, T]) < \infty$ with $p > 1/H$ for almost all sample paths of B_H . Therefore the claim of Proposition follows from Theorem 4. $\square$

The proof of Proposition 4 tells us that almost all sample paths of B_H with $1/2 < H < 1$ have bounded p -variation with $1/H < p < 2$. So then, almost all sample paths of geometric fractional Brownian motion are included in the class of bounded p -variation functions, $1 \leq p < 2$, and this provides us with the tools to adapt fractional Brownian motion to the behaviour of stock price fluctuations in the coming sections.

Remark. The p -variation of fBm with index H is infinite whenever $p = 1/H$ (cf. Norvaiša, 1997b). The actual proof of this is short. However, it involves some non-trivial steps. For example, when $H = 1/2$. Norvaiša gives the following argument:

Given a function $f : [0, T] \rightarrow \mathbb{R}$ and a partition $\kappa := \{0 = t_0 < \dots < t_n = T\}$ of $[0, T]$, let

$$v_p(f; \kappa) := \sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p \quad \text{and} \quad v_p(f; [0, T]) := \sup\{v_p(f; \kappa) : \kappa \in Q([0, T])\}$$

for $0 < p < \infty$, where $Q([0, T])$ denotes the class of all partitions κ of $[0, T]$. The mesh of $\kappa \in Q([0, T])$ will be denoted by $|\kappa|$. By a result of Taylor (1972) on the exact asymptotic estimate of Brownian path variation, it follows that with probability 1,

$$\limsup_{\epsilon \downarrow 0} \{v_2(B_{1/2}; \kappa) : \kappa \in Q([0, T]), |\kappa| \leq \epsilon\} = \infty.$$

Thus, $v_2(B_{1/2}; [0, T]) = \infty$ with probability 1 and the integral (1.3.2) does not exist in the Riemann-Stieltjes sense.

The next proposition shows us that the sample paths of fBm can not have bounded variation.

Proposition 5. *For almost all sample paths B_H of fBm with $0 < H < 1$, $v_p(B_H) = +\infty$ whenever $p < 1/H$.*

Proof. Theorem 2 of Kawada and Kôno (1973, cf Theorem A.13b in T.5) states that if X is a continuous centered Gaussian process then either $v_p(B_H) = +\infty$ almost surely (a.s.) or $v_p(B_H) < +\infty$ a.s. Suppose that $v_p(B_H) < +\infty$. Let $\kappa := \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. Then by Jain and Monrad (1983), $\sup_{\kappa} \sum_{i=1}^n |t_i - t_{i-1}|^{pH} < \infty$. This means that the function $f(t) = t$ belongs to $\mathcal{W}_{pH}$. However if $p < 1/H$ then this contradicts a result of Dudley and Norvaiša (1997, Theorem 2.11) (stated in T.5) that the only continuous functions in $\mathcal{W}_{pH}$ are constants. Hence $v_p(B_H) = +\infty$ whenever $p < 1/H$. $\square$

Next we will provide an alternative proof of Theorem 2.1 of Lin (1995) who proved that the quadratic variation of fractional Brownian motion is zero. Let $|\kappa| = \max\{|t_j - t_{j-1}|, 1 \leq j \leq n\}$ denote the mesh. Then the quadratic variation is defined as the limit

$$\lim_{|\kappa^m| \rightarrow 0} \sum_{i=1}^{n(m)} |X(t_i^m) - X(t_{i-1}^m)|^2 \rightarrow 0 \quad \text{in probability as } m \rightarrow \infty$$

where $\kappa^m := \{a = t_0^m < t_1^m < \cdots < t_{n(m)}^m = b\}$. Recall that quadratic variation (cf. $v_2^r(f; [a, b])$ defined on page 3) of ordinary Brownian motion $\{B_{1/2}(t) : 0 \leq t < \infty\}$ is t , and $B_{1/2}^2(t) - t$ is a martingale (cf. Karatzas and Shreve (1991)).

In fact, the proof of the following statement implies that this limit is zero almost surely.

Proposition 6. *Let $1/2 < H < 1$. Then fractional Brownian motion B_H has zero quadratic variation.*

Proof. By Proposition 5, almost all sample paths of B_H are continuous and have bounded p -variation with $1/H < p < 2$. Let $\kappa^m := \{a = t_0^m < t_1^m < \cdots < t_{n(m)}^m = b\}$ be a sequence of partitions of $[a, b]$. Let B_H be a continuous sample path with $v_p(B_H) < \infty$. Then given $\epsilon > 0$ there exists a $\delta > 0$ such that $|B_H(t) - B_H(s)| < \epsilon$ whenever $|t - s| < \delta$. Let m be such that $|\kappa^m| < \delta$. Then

$$\sum_{i=1}^{n(m)} |B_H(t_i^m) - B_H(t_{i-1}^m)|^2 \leq \max_{1 \leq i \leq n(m)} |B_H(t_i^m) - B_H(t_{i-1}^m)|^{2-p} v_p(B_H)$$

$$\leq \epsilon^{2-p} v_p(B_H).$$

Since ϵ is arbitrary and $v_p(B_H) < \infty$,

$$\lim_{m \rightarrow \infty} \sum_{i=1}^{n(m)} |B_H(t_i^m) - B_H(t_{i-1}^m)|^2 = 0$$

for almost every sample path of B_H . So B_H has zero quadratic variation. $\square$

The next result is well known (cf. Liptser & Shirayev (1986,1989,p 300), Cutland, Kopp & Willinger (1995, Proposition 4.1), Lin (1995, Corollary 2.2), Rogers (1995, Section 2). The only difference is that we are not using the fact that B_H has Hausdorff dimension $2 - H > 1$.

Proposition 7. *Fractional Brownian motion B_H is not a semimartingale.*

Proof. Assume that B_H is a semimartingale. Then we can represent B_H by the Doob-Meyer decomposition $B_H(t) = M(t) + V(t)$ where M is a continuous local martingale and V is a finite variation process with $M_0 = V_0 = 0$. Denote by $[Z, Z]$ the quadratic variation of a semimartingale Z . By Proposition 6, B_H has zero quadratic variation, so we have

$$0 = [B_H, B_H]_t = [M + V, M + V]_t = [M, M]_t + 2[M, V]_t + [V, V]_t = [M, M]_t.$$

So M must be itself a zero process by the Burkholder-Gundy-Davis inequality, and hence $B_H = V$ has finite variation. However, this contradicts Proposition 5. Therefore, B_H is not a semimartingale. $\square$

Since fBm is not a semimartingale, the usefulness of semimartingale techniques are limited for this process. Hence in this section, we proposed a viable alternative to the analysis of fBm. In the next section we will define trading strategies for geometric fBm based on the p -variation property. It should be stressed that a trading strategy defined via the Ito integral has no meaning for a given trajectory of price process (cf. Föllmer (1981) and Bick and Willinger (1994)). The reason for this is that Ito's integral is not defined pathwise. In addition, it is desirable to have

either a more general model or an alternative model to existing semimartingale models. The purpose of financial mathematical models is not restricted to option pricing theory where the martingale property is desirable. It has been suggested by Maheswaran and Sims (1993) that price processes which follow a semimartingale is a substantive restriction, and not a mere regularity condition.

In the coming section, we include fBm as a possible candidate to model the behaviour of stock price fluctuations.

2 Arbitrage Opportunities

The first use of a Wiener process to model the behaviour of security prices appeared in Louis Bachelier's (1900) thesis "Theory of Speculation". However, because of limited liability (prices can never be negative), Samuelson (1964) introduced to us geometric Brownian motion. It has become the accepted model for stock price fluctuations because Black and Scholes (1973) and Merton (1973) were able to derive an explicit formula for the fair price of an European call option using geometric Brownian motion (BM) as one of their assumptions. Continuous-time trading, no arbitrage opportunities, constant interest rate and volatility, to name a few, were the other assumptions. Later, Harrison & Kreps (1979) introduced a more general version of the Black-Scholes (BS) model by replacing geometric BM with a martingale. They were able to do this using the representation theorem of Kunita & Watanabe (1967). Harrison & Pliska (1981) then extended the martingale model to semimartingales.

In the celebrated work of Harrison and Pliska (1981), it was shown that (i) the semimartingale model is complete (i.e. every contingent claim is attainable) if and only if the discounted price process has the martingale representation property under a given reference measure (cf. Theorem 3.35 and Corollary 3.36 of Harrison & Pliska (1981)), and (ii) the semimartingale model contains no arbitrage opportunities if and only if the set of all probability measures Q which are equivalent to P and are such that the discounted price process is a (vector) martingale under Q , is nonempty (cf. Theorem 2.7 of Harrison & Pliska (1981)). Furthermore, Cutland et al. (1993) state that Stricker's theorem (1990, cf. Theorem A.17 in T.6) about the equivalence between the existence of equivalent martingale measure for continuous stochastic processes and the absence of arbitrage opportunities applies directly to any geometric fBm P_H with $1/2 < H < 1$. Using Kunitomo's Theorem 3 (1993), they claim that their fBm model based on nonstandard analysis contains arbitrage opportunities. However, Stricker's result does not provide any information about identifying or actually constructing some of these arbitrage opportunities (cf. Re-

mark A.18 in T.6).

Huang (1985) showed that if the information structure is generated by a Wiener process, then the equilibrium prices of securities are Itô processes. In addition, Harrison, Pitbladdo, & Schaefer (1984) proved that if there are no arbitrage strategies and prices have continuous sample paths, then the price process must have infinite variation. They also gave examples of arbitrage opportunities when price process are of bounded variation.

The discussion of arbitrage is of importance to finance since we believe that the stock market should be a *fair play ground* or *fair game*. That is, we should not be able to make large profits without any risk. This leads financial economists to assume that if arbitrage opportunities exist, then we will not be in economic equilibrium for a frictionless market (cf. for example, p.24 of Dothan (1990)). However, there are many empirical studies on the various stock markets which indicate the existence of arbitrage in the futures markets (cf., for example, Dwyer, Locke. Yu (1996), Bühler & Kempf (1995), Holden (1995), Moosa & Al-Loughani (1995), Grünbichler & Callahan (1994)).

The present analysis is an extension of Harrison, Pitbladdo and Schaefer (1984). The aim of this section is to construct trading strategies which will lead to arbitrage opportunities when the price process is a continuous, strictly positive function of bounded p -variation with $1 \leq p < 2$. Therefore, Harrison, Pitbladdo, and Schaefer's conclusion (1984) is extended to: if there is no arbitrage and price process is continuous, then p -variation with $p < 2$ must be infinite. In the earlier section, we showed that almost all sample paths of fractional Brownian motion B_H with $1/2 < H < 1$ have bounded p -variation with $1/H < p < 2$ (cf. Proposition 4 in §1.3). Recall that fBm is not a semimartingale (cf. Liptser & Shirvyayev (1986,1989, p.130) and Proposition 7 in §1.3).

Possibly the first mention of fractional Brownian motion (fBm) as a price process of stock fluctuations was in the paper by Mandelbrot and van Ness (1968) who briefly alluded to using fBm to describe economic time series. Since then there have been several papers which discuss the possibility of long range dependence

or long term memory in the stock price fluctuations (cf. Greene & Fielitz (1977), Lo (1991), Kunitomo (1993)). Moreover, in the recent works of Kopp (1996) and Cutland, Kopp and Willinger (1995), the discussion of using fBm as possible model for stock price fluctuations was renewed on *empirical* grounds. Using techniques of nonstandard analysis (NSA), they were able to establish that fBm cannot be turned into a martingale by any equivalent change of measure, and their NSA model is *complete* and allows for arbitrage opportunities. However, Kopp (1996) and Cutland et al (1995) are unable to make sense of equation (1.3.1) as an Itô integral equation but it was shown in Section 1.3 that equation (1.3.1) does have meaning if we use Riemann-Stieltjes integrals.

As the interest towards fractional Brownian motion grows, there are others who wish to discard fBm since fBm admits arbitrage. For example, Rogers (1995) argues that since fractional Brownian motion is not a semimartingale (except in the Brownian case), there cannot be equivalent martingale measure, and therefore by Stricker's (1990) result (cf. Theorem A.17 in T.6), arbitrage opportunities must exist. Rogers was able to construct an arbitrage opportunity by manipulating the σ -algebras to illustrate that when the prices move continuously in an upward trend, then one can make riskless profits (ie. construct arbitrage opportunities). Rogers concludes from this that fractional Brownian motion is an *absurd candidate* for a (log) price process, and proposes an alternative process which has the same long-range dependence as fractional Brownian motion but is also a semimartingale. However, no one has clearly define trading strategies and arbitrage for fBm. Trading strategies in a semimartingale model are defined as processes that are integrable with respect a semimartingale price process. Since fBm is not a semimartingale, trading strategies have to be redefined. In this work, we appropriately define trading strategies for fBm, and provide a formal construction of arbitrage using this definition of trading strategy.

In the coming subsections, we will follow Harrison et al (1984) and extend their argument of portfolio policies that lead to arbitrage when prices are functions of *bounded p -variation*, $1 \leq p < 2$ (cf. Definition 1 in §1.1). This includes the sample

paths of geometric fractional Brownian motion with $1/2 < H < 1$. Since fBm has the property of long range dependence, it may enable us to spot the direction of an upward trend of the price process and exploit this in trading.

2.1 Notation and Definitions.

Consider a market in which K securities are traded during the time interval $[0, T]$. Let X_k , $k = 1, 2, \dots, K$, be continuous and strictly positive functions with bounded p -variation, $1 \leq p < 2$ (cf. Definition 1 in §1.1), where for each security k , $X_k(t)$ represents the price of one share of security k at time t . Here X_k denotes the price process which is assumed to be non-random. We call $X = (X_1, \dots, X_K)$ a *price system* of the K security assets.

Before continuing, we would like to state what we mean by trading strategies, self-financing and attainability of contingent claims. Let $RS(h; [a, b]) := \{f : [a, b] \rightarrow \mathbb{R} : f \text{ is Riemann-Stieltjes integrable with respect to } h\}$ (cf. Definition A.1 in T.1). Informally, a trading strategy is a set of instructions specifying what to buy or sell at each time upon each possible development.

Definition 3. Let X be a price system as above. Then any collection of $\phi = (\phi_1, \dots, \phi_K)$ where each $\phi_k \in RS(X_k; [0, T])$, $k = 1, \dots, K$, will be called a *trading strategy with respect to X* .

The functions $\phi_k(t)$ represents the number of shares of security k held at time t .

Denote by $P = P^{\phi, X}$ the total value of the portfolio held at time $t \in [0, T]$ so that

$$P^{\phi, X}(t) := \sum_{k=1}^K X_k(t) \phi_k(t), \quad (2.1.1)$$

and the total gains at $t \in [0, T]$

$$G^{\phi, X}(t) := \sum_{i=1}^K \int_0^t \phi_i(u) dX_i(u) \quad (2.1.2)$$

where $\int$ denotes the Riemann-Stieltjes integral. From hereafter, unless otherwise stated, the integral sign $\int$ will refer to the Riemann-Stieltjes integral.

Remark. From (2.1.1), it is clear that we can acquire larger profits if we increase the number of the stocks of which were performing better since time zero and decrease the number of those stocks which were doing poorly since time zero.

In the same spirit as Harrison and Pliska (1981), we give the following:

Definition 4. Let X be a price system. A trading strategy ϕ with respect to X is *self-financing* if

$$P^{\phi, X}(t) = P^{\phi, X}(0) + G^{\phi, X}(t) \quad (2.1.1a)$$

for each $t \in [0, T]$. Furthermore, ϕ is said to be admissible if $P^{\phi, X}(t) \geq 0$ for all $t \in [0, T]$.

Alternatively, we can check if a trading strategy ϕ is self financing by the following lemma.

Lemma 15. Let $X = (X_1, \dots, X_K)$ be a price system where each $X_k \in CW_p([0, T])$ with $1 \leq p < 2$ and let $\phi = (\phi_1, \dots, \phi_K)$ be a trading strategy with respect to X where each $\phi_k \in \mathcal{W}_p([0, T])$. Then ϕ is self-financing if and only if

$$\sum_{k=1}^K \int_0^t X_k(u) d\phi_k(u) = 0 \quad \forall t \in [0, T].$$

Proof. Apply the Corollary 3A of Theorem 3 in §1.1 and the result will follow.

$$\begin{aligned} \sum_{k=1}^K \int_0^t X_k(u) d\phi_k(u) &= \sum_{k=1}^K \{X_k(t)\phi_k(t) - X_k(0)\phi_k(0) - \int_0^t \phi_k(u) dX_k(u)\} \\ &= P^{\phi, X}(t) - P^{\phi, X}(0) - G^{\phi, X}(t) = 0. \end{aligned}$$

□

In finance, a contingent claim is a contract or agreement which pays a certain amount, say M , at time T if a certain event occurs. In our context, we will define a

contingent claim as a non-negative real number. In this section, we will concentrate on a pathwise approach, so this definition of contingent claim will be sufficient and in a later work we will introduce the probabilistic version of these notions.

Definition 5. Let X be a price system. A contingent claim $\xi \in (0, \infty)$ is *attainable* at T if there exists a self-financing strategy $\phi = (\phi_1, \dots, \phi_K)$ with respect to X such that each $\phi_k \in \mathcal{W}_p([0, T])$ with $1 \leq p < 2$ and

$$\xi = P^{\phi, X}(T).$$

We will take later $\xi = (\frac{1}{K} \sum_{k=1}^K [X_k(T)]^\alpha)^{1/\alpha}$. Recall that a model is complete if every contingent claim is attainable (cf. Harrison and Pliska (1981)). Using nonstandard analysis, Cutland, Kopp, and Willinger (1995) proved that the model of fBm is complete.

Before proceeding, it would be nice that we state what we mean by an arbitrage opportunity.

Definition 6. A self-financing trading strategy ϕ is an arbitrage opportunity for X if $P^{\phi, X}(0) \leq 0$ and $P^{\phi, X}(T) > 0$.

In other words, an arbitrage opportunity is a riskless plan to make profits without investment.

Alternatively, under the semimartingale model, they require that P must be tame (i.e. the discounted gain process must be bounded below by some real constant), and an arbitrage opportunity exists if there is a tame self-financing trading strategy ϕ that satisfies $\mathbb{P}[P^{\phi, X}(T) \geq 0] = 1$ and $\mathbb{P}[P^{\phi, X}(T) > 0] > 0$. That is, starting at time zero with no initial investment, and employing your strategy ϕ , then at time T , you end up with a portfolio $P^{\phi, X}(T)$ with no exposure to risk ($X(T) \geq 0$, a.s.), and positive probability of gain ($\mathbb{P}[P^{\phi, X}(T) > 0] > 0$) (cf. Karatzas (1997)).

2.2. Simple Trading Strategies.

In this section, we will justify the definition of trading strategies via Riemann-Stieltjes integrability, and show that there exist discrete-time trading strategies

which are also self-financing and each attainable contingent claim can be approximated by them. We interchangeably use discrete-time trading strategies and simple trading strategies. A *simple trading strategy* is defined as a step function where trading takes places at finitely many times.

Proposition 8. *Let $X = (X_1, \dots, X_K)$ be a price system such that, for each k , $X_k \in \mathcal{CW}_p([0, T])$, $1 \leq p < 2$. If ξ is an attainable contingent claim then there exists a sequence $\{\phi^N : N \geq 1\}$ of simple self-financing trading strategies such that*

$$\lim_{N \rightarrow \infty} P^{\phi^N, X}(T) = \xi.$$

Proof. Assume that $X = (X_1, \dots, X_K)$ is a price system and assume that ξ is an attainable contingent claim (cf. Definition 5) with respect to X . Then there exists a self-financing trading strategy $\phi = (\phi_1, \dots, \phi_K)$ such that each $\phi_k \in \mathcal{W}_p([0, T])$ with $1 \leq p < 2$ and

$$\xi = P^{\phi, X}(T).$$

Let N be an integer and let $\{0 = t_0 < t_1 < \dots < t_N = T\}$ be any partition of $[0, T]$. Define $\phi^N = (\phi_1^N, \dots, \phi_K^N)$ recursively. That is, let

$$\phi_k^N(t) := \phi_k(0) \quad \text{for } t \in [0, t_1) \quad \text{and } k = 1, \dots, K.$$

Assume that $\phi_k^N(t)$ are defined for $t \in [0, t_n)$. Let

$$P^N(t_n) := \sum_{k=1}^K \phi_k^N(t_{n-1}) X_k(t_n). \tag{1}$$

Define, for each $k = 1, \dots, K$,

$$\phi_k^N(t) := \frac{P^N(t_n)}{P^{\phi, X}(t_n)} \phi_k(t_n) \tag{2}$$

either for $t \in [t_n, t_{n+1})$ if $n \leq N - 1$ or $t = \{T\}$ if $n = N$.

It follows by this construction that

$$\phi_k^N(t_n-) := \lim_{x \uparrow t_n} \phi_k^N(x) = \phi_k^N(t_{n-1})$$

for each $n = 1, \dots, N$. Recall that $f(x-) = \lim_{y \uparrow x} f(y)$. Then, denoting the integer part of t by $[t]$, we have for each $t \in [0, T]$,

$$\begin{aligned} \sum_{k=1}^K \int_0^t X_k(u) d\phi_k^N(u) &= \sum_{k=1}^K \sum_{n=1}^{[t]} X_k(t_n) [\phi_k^N(t_n) - \phi_k^N(t_{n-1})] \\ &= \sum_{n=1}^{[t]} \left[\sum_{k=1}^K X_k(t_n) \phi_k^N(t_n) - \sum_{k=1}^K X_k(t_n) \phi_k^N(t_{n-1}) \right] \\ \{\text{by (1) and (2)}\} &= \sum_{n=1}^{[t]} \left[\frac{P^N(t_n)}{P^{\phi, X}(t_n)} \sum_{k=1}^K X_k(t_n) \phi_k(t_n) - P^N(t_n) \right] = 0. \end{aligned}$$

So by Lemma 15, ϕ^N is self-financing.

Notice that $\delta := \inf_{0 \leq t \leq T} P^{\phi, X}(t) > 0$ (This can also be shown by following the same argument presented in the proof of Lemma 7 and Theorem A.16). By Lemma 4, we can find a partition $\{0 = t_0 < t_1 < \dots < t_N = T\}$ of $[0, T]$ such that $v_p(\phi_k, [t_n, t_{n+1}]) < \epsilon$ and $v_p(X_k, [t_n, t_{n+1}]) < \delta/K$, for $k = 1, \dots, K$ and $n = 0, \dots, N$. By (2), it follows that

$$\frac{\phi_k(t_n)}{P^{\phi, X}(t_n)} = \frac{\phi_k^N(t_n)}{P^N(t_n)}$$

for $n = 0, \dots, N$, so that

$$\begin{aligned} \sum_{k=1}^K \phi_k(t_n) X_k(t_{n+1}) &= \sum_{k=1}^K \frac{\phi_k^N(t_n) P^{\phi, X}(t_n)}{P^N(t_n)} X_k(t_{n+1}) \\ &= \frac{P^{\phi, X}(t_n)}{P^N(t_n)} \sum_{k=1}^K \phi_k^N(t_n) X_k(t_{n+1}) = \frac{P^{\phi, X}(t_n)}{P^N(t_n)} P^N(t_{n+1}). \end{aligned}$$

So we have,

$$|P^{\phi,X}(t_{n+1}) - P^{\phi,X}(t_n) \frac{P^N(t_{n+1})}{P^N(t_n)}| = |P^{\phi,X}(t_{n+1}) - \sum_k \phi_k(t_n) X_k(t_{n+1})| \quad (3)$$

Add and subtract $\sum_k \phi_k(t_n) X_k(t_n)$ on right hand side of (3) to give

$$\begin{aligned} &= |P^{\phi,X}(t_{n+1}) - \sum_k \phi_k(t_n) [X_k(t_{n+1}) - X_k(t_n)] - \sum_k \phi_k(t_n) X_k(t_n)| \\ &= |P^{\phi,X}(t_{n+1}) - P^{\phi,X}(t_n) - \sum_k \phi_k(t_n) [X_k(t_{n+1}) - X_k(t_n)]| \end{aligned}$$

$$\{\text{by (2.1.1a) and (2.1.2)}\} \leq \sum_k \left| \int_{t_n}^{t_{n+1}} \phi_k(u) dX_k(u) - \phi_k(t_n) [X_k(t_{n+1}) - X_k(t_n)] \right|$$

$$\{\text{by Theorem 2}\} \leq C \sum_k V_p(\phi_k, [t_n, t_{n+1}]) V_p(X_k; [t_n, t_{n+1}])$$

$$\{\text{by Lemma 4}\} \leq C \epsilon^{1/p} \sum_k V_p(X_k; [t_n, t_{n+1}]),$$

where $C = 1 + \zeta(1/p + 1/q)$. Set $\epsilon' = C \epsilon^{1/p}$. Now, if we divide the left hand side by $P^{\phi,X}(t_{n+1})$ and right hand side by $\delta (\leq P^{\phi,X}(t_{n+1}))$, we have

$$\left| 1 - \frac{P^{\phi,X}(t_n)}{P^{\phi,X}(t_{n+1})} \frac{P^N(t_{n+1})}{P^N(t_n)} \right| \leq \epsilon' / \delta \sum_k V_p(X_k; [t_n, t_{n+1}]),$$

or

$$\begin{aligned} 1 - \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]) &\leq \frac{P^{\phi,X}(t_n)}{P^{\phi,X}(t_{n+1})} \frac{P^N(t_{n+1})}{P^N(t_n)} \\ &\leq 1 + \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]). \end{aligned}$$

Since $V_p(X_k; [t_n, t_{n+1}]) < \delta/K < 1$, we may take the logarithm of each term to give,

$$\begin{aligned} \log \left[1 - \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]) \right] &\leq \log \left[\frac{P^{\phi, X}(t_n)}{P^{\phi, X}(t_{n+1})} \frac{P^N(t_{n+1})}{P^N(t_n)} \right] \\ &\leq \log \left[1 + \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]) \right]. \end{aligned}$$

Summing up,

$$\begin{aligned} \sum_{n=0}^{N-1} \log \left[1 - \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]) \right] &\leq \log \left[\frac{P^N(T)}{P^{\phi, X}(T)} \right] \\ &\leq \sum_{n=0}^{N-1} \log \left[1 + \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]) \right]. \end{aligned}$$

Now by the mean value theorem, there exists $\nu \in \{-\epsilon'/\delta \sum_k V_p(X_k; [t_n, t_{n+1}]), 0\}$, such that

$$\log \left(1 - \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]) \right) = \frac{-1}{1+\nu} \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]).$$

Similarly, there exists $\eta \in \{0, -\epsilon'/\delta \sum_k V_p(X_k; [t_n, t_{n+1}])\}$ such that

$$\log \left(1 + \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]) \right) = \frac{1}{1+\eta} \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [t_n, t_{n+1}]).$$

Now $\epsilon'/\delta \sum_k V_p(X_k; [t_n, t_{n+1}]) \leq \epsilon'$, so that $\nu \geq -\epsilon'$, and

$$\begin{aligned} \frac{-1}{1-\epsilon'} \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [0, T]) &= \frac{-1}{1-\epsilon'} \frac{\epsilon'}{\delta} \sum_{n=0}^{N-1} \sum_k V_p(X_k; [t_n, t_{n+1}]) \\ &\leq \log P^N(T) - \log P^{\phi, X}(T) \\ &\leq \frac{\epsilon'}{\delta} \sum_{n=0}^{N-1} \sum_k V_p(X_k; [t_n, t_{n+1}]) \\ &\leq \frac{\epsilon'}{\delta} \sum_k V_p(X_k; [0, T]). \end{aligned}$$

Since the first and last terms are arbitrarily small by taking ϵ' small, the same holds true for the middle term. $\square$

2.3 Heuristic explanation.

Let α be a real number and $t \in [0, T]$. Let $X = (X_1, \dots, X_K)$ be a price system such that each $X_k(0) = 1$. Here we will construct trading strategies $S_k(t)$, $k = 1, \dots, K$ with respect to X in such a way that it will produce arbitrage.

We follow Harrison et al (1984) to explain this construction of arbitrage opportunity. Consider a portfolio with an initial investment of one dollar, where no further investment is required. Now, if we continuously adjust this investment so that the dollar value of holdings of security k at time t is proportional to $[X_k(t)]^\alpha$, then we have

$$\frac{X_k(t)S_k(t)}{P(t)} = \frac{[X_k(t)]^\alpha}{\sum_{j=1}^K [X_j(t)]^\alpha}, \quad (2.3.1)$$

where

$$P(t) := \sum_{k=1}^K S_k(t)X_k(t). \quad (2.1.1b)$$

For large values of α , (2.3.1) says that relatively larger amounts of the higher priced securities are held, and vice versa for low values α . If $\alpha = 1$ and $S_k(t) = 1/K$ for all k and t in (2.3.1), then we have a “buy and hold” strategy.

How do we adjust the portfolio over time to achieve (2.3.1)? If we re-arrange (2.3.1) in terms of $S_k(t)$, we get the following trading strategy

$$S_k(t) = \frac{P(t)[X_k(t)]^{\alpha-1}}{\sum [X_k(t)]^\alpha}. \quad (2.3.2)$$

However, P is still expressed in terms of $S_1, \dots, S_K$. Since we want to construct self-financing strategies, we are not allowed to put in additional money to the portfolio fund. This requirement can be written as

$$dP(t) = \sum S_k(t)dX_k(t). \quad (2.3.3)$$

This is equivalent to saying that all changes in the portfolio fund at time t are due to capital gains (2.1.2).

For $\alpha \neq 0$ and putting together (2.3.1) and (2.3.3), we have

$$\begin{aligned} dP(t) &= \frac{P(t) \sum [X_k(t)]^{\alpha-1} dX_k(t)}{\sum [X_k(t)]^\alpha} \\ &= \frac{P(t) \sum \frac{1}{\alpha} d[X_k(t)]^\alpha}{\sum [X_k(t)]^\alpha} \end{aligned}$$

so that

$$\frac{dP(t)}{P(t)} = \frac{1}{\alpha} \frac{d(\sum [X_k(t)]^\alpha)}{\sum [X_k(t)]^\alpha} \quad (2.3.4)$$

or

$$d \log P(t) = \frac{1}{\alpha} d \log \sum [X_k(t)]^\alpha. \quad (2.3.5)$$

Using the fact that $P(0) = X_k(0) = 1$, we get from (2.3.4) or (2.3.5) that

$$P(t) = \left\{ \frac{1}{K} \sum [X_k(t)]^\alpha \right\}^{1/\alpha}. \quad (2.3.6)$$

$P(t)$ is the α th order power mean of the prices $X_1(t), \dots, X_K(t)$. Denote this α th power mean by $P^\alpha(t)$ rather than $P(t)$ indicating the dependence on α , and trading strategy (2.3.2) as the $S := S^\alpha$ -strategy. The geometric mean is the case when $\alpha = 0$:

$$P^0(t) = \left\{ \prod_{k=1}^K [X_k(t)] \right\}^{1/K}.$$

Equations (2.3.2) through (2.3.6) explicitly give a trading strategy that yields a portfolio fund of $P^\alpha(t)$ at time t . This trading strategy will give arbitrage opportunities because of a result proven by Hardy, Littlewood, and Pólya (1959) for power means. That is,

Proposition 9. *If $-\infty < \alpha < \beta < +\infty$ then $P^\alpha(t) \leq P^\beta(t)$, with equality only when $X_1(t) = \dots = X_K(t)$.*

The proof of this is an easy application of Jensen's inequality, and proven in Hardy, Littlewood, and Pólya's (1959) *Inequalities*, pages 26-27.

Let $\alpha < \beta$. If we wish to make arbitrage opportunities, then we would buy $1/K$ shares of each security at time zero and continue to buy using the β -strategy, and sell $1/K$ shares of each security at time zero and continue to sell by the α -strategy. This will give us a profit of $P^\beta(T) - P^\alpha(T)$ at time T with no investment. This is arbitrage!

What is the economic interpretation to all of this? It may not be clear if the S^α -strategy requires zero net new investment after time zero. That is, is S^α -strategy self-financing (cf. Definition 4)? The usual condition given to justify manipulations like (2.3.3) - (2.3.6) is differentiability of the price process. In the next subsection, we extend this justification to a possibly non-differentiable class of functions and hence show that S^α -strategy is indeed self-financing.

It should be noted that a clever strategy to follow would be to take β very large and α very negative since the limiting values are

$$\lim_{\beta \rightarrow \infty} \left(\frac{1}{K} \sum_{k=1}^K X_k(t)^\beta \right)^{1/\beta} = \max_{1 \leq k \leq K} \{X_k(t)\}$$

and

$$\lim_{\alpha \rightarrow -\infty} \left(\frac{1}{K} \sum_{k=1}^K X_k(t)^\alpha \right)^{1/\alpha} = \min_{1 \leq k \leq K} \{X_k(t)\},$$

Furthermore, as $\beta \rightarrow \infty$, the associated strategy comes closer to holding only the highest price security (equal division when there is more than one security at the maximum). The fact that $P^\beta(t) \rightarrow \max_k X_k(t)$ as $\beta \rightarrow \infty$ lends credibility to such a strategy of “going with the leader”, provided the price process is smooth enough to justify this argument. However, when the price process are diffusions, the above strategy is subject to discontinuities that are not easily approximated by simple strategies. In trying to go with the leader in a clever way, one cannot keep up when the lead changes hands according to Harrison et al (1984).

2.4 Formal Construction.

Now we will formally verify that P^α satisfies (2.3.3). That is, we will give a rigorous justification of formulae (2.3.4) to (2.3.6). Assume that $\alpha > 0$ and define

$$S_k^\alpha(t) := P^\alpha(t) \frac{[X_k(t)]^{\alpha-1}}{\sum_j [X_j(t)]^\alpha} \quad (2.4.1)$$

where $P^\alpha(t) := \{(1/K) \sum_k [X_k(t)]^\alpha\}^{1/\alpha}$ and $X_k \in \mathcal{CW}_p([0, T])$ with $1 \leq p < 2$ for each k .

Proposition 10. *Let $X = (X_1, \dots, X_K)$ be a price system, where each $X_k \in \mathcal{CW}_p$ with $1 \leq p < 2$ and let $\alpha \in (0, \infty)$. Then $S^\alpha = (S_1^\alpha, \dots, S_K^\alpha)$ as defined by (2.4.1) are trading strategies with respect to X . Furthermore, S^α are self-financing. That is,*

$$P^\alpha(t) = P^\alpha(0) + \sum_{k=1}^K \int_0^t S_k(u) dX_k(u) \quad \text{for every } t \in [0, T].$$

Proof. First, we will prove that S^α are trading strategies. Since $X_k \in \mathcal{CW}_p([0, T])$, where $1 \leq p < 2$, then, by Young's Stieltjes integrability theorem (cf. Theorem 2 in §1.1), it is enough to check if S^α has bounded p -variation, $1 \leq p < 2$. This will be done in several steps.

By continuity of trading strategies, $\delta := \min_k \inf_t X_k(t) > 0$ (cf. Theorem A.16 in T.5). Let $u = X_k(t)$. By Lemma 12, $\phi(u) = u^{\alpha-1}$ has bounded p -variation, and it follows from Lemma 9 that $\sum u^{\alpha-1}$ has bounded p -variation. So then, by Lemma 14, $u^{\alpha-1} / \sum u^{\alpha-1}$ also has bounded p -variation. Now putting everything together, we have S^α has bounded p -variation since the product of two bounded p -variation also has p -variation (cf. Lemma 13).

We can apply Proposition 3 in §1.1 to the function $h(u) = (1/K \sum_k u_k^\alpha)^{1/\alpha}$ with $u = (u_1, \dots, u_K) \in [\delta, \infty)^K$, and it is clear that this h is differentiable with

Lipschitz derivative. So by (2.4.1), we have

$$\begin{aligned}
\sum_{k=1}^K \int_0^t S_k^\alpha(u) dX_k(u) &= \sum_{k=1}^K \int_0^t \left(\frac{1}{K}\right)^{1/\alpha} \left\{ \sum_j [X_j(u)]^\alpha \right\}^{1/\alpha-1} [X_k(u)]^{\alpha-1} dX_k(u) \\
\{\text{by Proposition 2}\} &= \sum_{k=1}^K \int_0^t \left(\frac{1}{K}\right)^{1/\alpha} \left\{ \sum_j [X_j(u)]^\alpha \right\}^{1/\alpha-1} d\left\{ \left(\frac{1}{\alpha}\right) [X_k(u)]^\alpha \right\} \\
\{\text{by bilinearity}\} &= \left(\frac{1}{K}\right)^{1/\alpha} \int_0^t \left(\frac{1}{\alpha}\right) \left\{ \sum_{j=1}^K [X_j(u)]^\alpha \right\}^{1/\alpha-1} d\left\{ \sum_{k=1}^K [X_k(u)]^\alpha \right\} \\
\{\text{by Proposition 3}\} &= \left(\frac{1}{K}\right)^{1/\alpha} \int_{\sum_k [X_k(0)]^\alpha}^{\sum_k [X_k(t)]^\alpha} \left(\frac{1}{\alpha}\right) v^{1/\alpha-1} dv \\
&= \left(\frac{1}{K}\right)^{1/\alpha} \left(\left\{ \sum_k [X_k(t)]^\alpha \right\}^{1/\alpha} - \left\{ \sum_k [X_k(0)]^\alpha \right\}^{1/\alpha} \right) \\
&= P^\alpha(t) - P^\alpha(0).
\end{aligned}$$

□

This proposition tells us that the trading strategies $S_1^\alpha, \dots, S_K^\alpha$ are self-financing. The second term on the right hand side of the proposition represents the investor's total earnings (or capital gains) up to time t . Then we have that all changes in the portfolio fund at time t are due to capital gains or losses, as opposed to cash withdrawals or additional cash deposits. In other words, the change of money is generated by the sale of securities over any period is exactly balanced by the cost of security purchases over that same period.

We need the next proposition to conclude the construction of arbitrage.

Proposition 11. *Let X_t' and X_t'' be two independent identically distributed Gaussian random variables. Then $\mathbb{P}(X_t' \neq X_t'') = 1$.*

Proof. Let $A_n = \{|X_t' - X_t''| > 1/n\}$.

FACT: Let X_t be a Gaussian random variable. Then

$$X_t \stackrel{D}{=} \frac{X_t' - X_t''}{\sqrt{2}}$$

where X'_t and X''_t are two independent identically distributed Gaussian random variables (cf. Lifshitz (1995)).

Therefore,

$$\mathbb{P}(X'_t \neq X''_t) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{2}|X_t| > 1/n) = 1$$

where the above fact is used at the second equality. $\square$

Consider a market with two stocks. One stock follows B'_H and the other follows B''_H . Assume that B'_H and B''_H are independent. Therefore, by Proposition 11, for each t , $B'_H(t) \neq B''_H(t)$ almost surely. So then, for $\alpha < \beta$, $P^\alpha(t) < P^\beta(t)$ (cf. Proposition 9 in §2.3), and this confirms the possibility of arbitrage opportunities since $P^\beta(t) - P^\alpha(t) > 0$.

2.5 When is the stop-loss start-gain strategy self-financing?

In this section, we define the stop-loss start-gain strategy and review the cases in which this strategy produces arbitrage. The answer to this depends on the underlying assumptions on the price process and the definition of trading strategy. We direct the reader to Carr and Jarrow (1990) for the history of the problem. Hereafter, we will refer to the Carr and Jarrow's paper as C-J.

In C-J, the Black and Scholes model is adapted to illustrate that the stop-loss start-gain strategy is not self-financing, and hence the stop-loss start-gain strategy will not yield arbitrage opportunities. Recall that in the Black and Scholes model, there exist perfect bond and stock markets, constant interest rate, no payouts, and the stock price process follows a geometric BM. The notation for the stop-loss start-gain strategy is given below.

Denote by $m(t)$ the number of bonds and by $n(t)$ the number of shares in one stock, held in the portfolio. The value of this portfolio at time t is

$$P(t) = m(t)B(t) + n(t)S(t), \quad \text{for all } t \in [0, T], \quad (2.5.1)$$

where $B(t) = e^{r(T-t)}$ and $S(t)$ denotes the stock price process (T is the time period). Let K be an arbitrary floor. The stop-loss start-gain trading strategy is defined by

$$m(t) = -1_{S(t) > KB(t)}K$$

and

$$n(t) = 1_{S(t) > KB(t)} \quad (2.5.2)$$

where 1_A is the indicator function of the set A . In other words, one maintains a short (selling) position in K whenever the stock price is greater than the present value of an arbitrarily chosen floor K , and holding one share of stock. At time T , one liquidates all one has.

An insightful observation of the Black-Scholes formula (given in T.4) is that it is not necessary to buy the option to obtain the final payoff of an European call

option. The Black-Scholes formula specifically tells us (given continuous trading) the amount we should invest in a bond and the underlying stock. It so happens that the stop-loss start-gain strategy can replicate the payoff function of the Black and Scholes formula. If we substitute (2.5.2) into (2.5.1), we obtain a lower bound for this call option at time t with exercise K and maturity date T ,

$$\begin{aligned} P(t) &= -1_{S(t) > KB(t)}KB(t) + 1_{S(t) > KB(t)}S(t) \\ &= \max[0, S(t) - KB(t)] \quad \text{for all } t \in [0, T] \end{aligned} \quad (2.5.3)$$

and, because $B(T) = 1$, the terminal portfolio replicates this call's payoff:

$$P(T) = \max[0, S(T) - K]. \quad (2.5.4)$$

It should be noted that, when the replicated option starts out of the money (ie. the stock price is below the present value of the exercise price), the SLSG strategy is initially costless, so that the SLSG strategy appears to be self-financing (ie. an arbitrage opportunity). Hence, it leads to the question of whether the Black-Scholes model allows for arbitrage opportunities, even though it assumes that there is an absence of arbitrage to price options? As stated earlier, whether SLSG strategy is self-financing or not depends on the assumptions on the price process and its corresponding trading strategies. So far, it is known that

- (1) SLSG strategy is not self-financing if the price process follows geometric BM and trading strategies are defined via Itô's integrals. (cf. Carr and Jarrow's main result, Proposition 1).
- (2) SLSG strategy is self-financing if the price process is a continuous, bounded variation function and its corresponding trading strategies are defined via Lebesgue-Stieltjes integrals (cf. Proposition 13 below and stated as a footnote in Carr and Jarrow, p 475).
- (3) SLSG is self-financing if the price process is continuous, bounded 1-variation function and its corresponding trading strategies are defined via Riemann-Stieltjes integrals (cf. Proposition 14 below).

However, when price processes are of bounded p -variation with $1 < p < 2$, we have been unable to conclude that the SLSG strategy is self-financing. We will discuss this case at the end of this section because of its difficulty. Now, we will review cases (2) and (3). As for (1), we refer to C-J.

The definition of trading strategy is specific for each price process. So before proceeding, we will recall the definitions of trading strategies for the Itô, Lebesgue-Stieltjes, and Riemann-Stieltjes integrals.

- (1) *Itô integrals*: The trading strategy is a stochastic process $\{m(t), n(t) : t \in [0, T]\}$ which satisfies the following conditions (i) $m(t), n(t)$ are adapted to $\{\mathcal{F}_t : t \in [0, T]\}$ and (ii)

$$\tilde{E}_0\left[\int_0^T m^2(t)dt\right] < \infty \quad \text{and} \quad \tilde{E}_0\left[\int_0^T n^2(t)S^2(t)dt\right] < \infty,$$

where $\tilde{E}(\cdot) = \tilde{E}[\cdot|\mathcal{F}_0] = \int \cdot d\tilde{Q}$ and S is a geometric BM. That is, the expectation is taken under the equivalent martingale measure.

- (2) *Lebesgue-Stieltjes integrals*: The trading strategy $\{m(t), n(t) : t \in [0, T]\}$ is defined to be Lebesgue-Stieltjes integrable $[0, T]$ with respect to the price process.
- (3) *Riemann-Stieltjes integrals*: The trading strategy is defined to be Riemann-Stieltjes integrable with respect to the price process (cf. Definition 3 in §2.1).

The function (2.5.2) may or may not be a trading strategy depending on whether it is integrable with respect to a price process.

Let $CBV([a, b])$ denote that class of functions that have bounded variation and are continuous. Recall that $CW_p([a, b])$ denotes the class of functions that have bounded p -variation and are continuous.

We will show later that the RS integral does exist if $S \in CW_p$ with $p = 1$. For the Lebesgue-Stieltjes integrability, it is sufficient to show that m is Borel-measurable and bounded. Recall that every function of bounded variation is Borel-measurable (cf. Lemma A.26). Hence $1_{S(t) > KB(t)}$ is Borel-measurable, and obviously, $|m| \leq 1$.

In the case of Itô integrability, cf. C-J. By definition, the resulting portfolio values are self-financing

- (1) if S is geometric BM, and for all $t \in [0, T]$,

$$P(t) - P(0) = (L) \int_0^t m(v) dB(v) + (I) \int_0^t n(v) dS(v), \quad (2.5.5)$$

where $(I) \int$ and $(L) \int$ denote Ito and Lebesgue-Stieltjes integrals, respectively (cf. Carr and Jarrow (1990)).

- (2) if $S \in \mathcal{CBV}$, and for all $t \in [0, T]$,

$$P(t) - P(0) = (L) \int_0^t m(v) dB(v) + (L) \int_0^t n(v) dS(v), \quad (2.5.6)$$

where $(L) \int$ denotes Lebesgue-Stieltjes integral.

- (3) if $S \in \mathcal{CW}_p$ with $1 \leq p < 2$

$$P(t) - P(0) = (RS) \int_0^t m(v) dB(v) + (RS) \int_0^t n(v) dS(v), \quad (2.5.7)$$

where $(RS) \int$ denotes Riemann-Stieltjes integral.

Dividing through equations (2.5.5) to (2.5.7) by $B(\cdot)$, we obtain

- (1) when S is geometric BM and

$$\frac{P(t)}{B(t)} = \frac{P(0)}{B(0)} + (I) \int_0^t n(v) d\left(\frac{S(v)}{B(v)}\right). \quad (2.5.5a)$$

- (2) when $S \in \mathcal{CBV}$ and

$$\frac{P(t)}{B(t)} = \frac{P(0)}{B(0)} + (L) \int_0^t n(v) d\left(\frac{S(v)}{B(v)}\right). \quad (2.5.6a)$$

- (3) when $S \in \mathcal{CW}_p$ with $1 \leq p < 2$, and

$$\frac{P(t)}{B(t)} = \frac{P(0)}{B(0)} + (RS) \int_0^t n(v) d\left(\frac{S(v)}{B(v)}\right). \quad (2.5.7a)$$

Define

$$F(t) = \frac{S(t)}{B(t)} = S(t)e^{r(T-t)}, \quad \text{for all } t \in [0, T]. \quad (2.5.8)$$

Here $F(t)$ is known as the forward price of the stock. Therefore, the stop-loss start-gain strategy (2.5.3) is self-financing if and only if, for all $t \in [0, T]$,

(1) when S is geometric BM and

$$\max[0, F(t) - K] = \max[0, F(0) - K] + (I) \int_0^t 1_{F(v) > K} d(F(v)). \quad (2.5.5b)$$

(2) when $S \in \mathcal{CBV}$ and

$$\max[0, F(t) - K] = \max[0, F(0) - K] + (L) \int_0^t 1_{F(v) > K} d(F(v)). \quad (2.5.6b)$$

(3) when $S \in \mathcal{CW}_p$ with $1 \leq p < 2$ and

$$\max[0, F(t) - K] = \max[0, F(0) - K] + (RS) \int_0^t 1_{F(v) > K} d(F(v)). \quad (2.5.7b)$$

Let $D^-g(y) \equiv \lim_{\epsilon \downarrow 0} \frac{g(y) - g(y-\epsilon)}{\epsilon}$, be the left derivative of $g(\cdot)$.

Proposition 12. (*Chain Rule using Lebesgue-Stieltjes integrals for the composite function $g \circ X$ when g is convex and X is a continuous bounded variation process*)

Let X be a continuous, bounded variation process. For every convex function $g : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$g(X_t) = g(X_0) + (L) \int_0^t D^-g(X_s) dX_s, \quad 0 \leq t < \infty \quad \text{a.s. } P. \quad (2.5.9)$$

Proof. The proof follows from Theorem 7.1 of Karatzas and Shreve (1991) (stated as Theorem A.24 in T.7).

Proposition 13. *The forward price process F as defined by (2.5.8) is self-financing when $S(t)$ is continuous and of bounded variation and trading strategy is defined via Lebesgue-Stieltjes integral.*

Proof. Let $g(y) = \max[0, y - K]$. Then $D^-g(y) = 1_{y>K}$. Now applying (2.5.9) to the process $g \circ F$, we have

$$\max[0, F_t - K] = \max[0, F_0 - K] + (L) \int_0^t 1_{F(v)>K} dF(v) \quad 0 \leq t < \infty, \quad \text{a.s. } P,$$

where $(L) \int$ refers to Lebesgue-Stieltjes integral. Therefore the SLSG strategy is self-financing in this case. $\square$

In the next proposition, we define the trading strategies via Riemann-Stieltjes integrals in the case of $S \in CW_1$. This proposition agrees with the above proposition but we need the existence of Young's integral (cf. Definition A.3 in T.1) to obtain the result for functions of bounded 1-variation.

Proposition 14. *The forward price process F as defined by (2.5.9) is self-financing when S is continuous and of bounded 1-variation and trading strategy is defined via Riemann-Stieltjes integral.*

Proof. Let $g(y) = \max[0, y - K]$. Then $D^-g(y) = 1_{y>K}$ is regulated so that $g \in \mathcal{DR}$ (cf. definition in T.7). By assumption, F is continuous, hence condition (4.7.1) $\Lambda(g, D^-g, F) = 0$ (cf T.7). Now we can apply Theorem A.19 which is a chain rule for such a composite function $g \circ F$ to get

$$\max[0, F(t) - K] = \max[0, F(0) - K] + (Y) \int_a^b 1_{F(v)>K} dF(v), \quad \text{for all } t \in [0, \infty).$$

Since the integral $(Y) \int$ exists (cf. Definition A.3 in T.1) and F is continuous, then by Theorem A.22 the Riemann-Stieltjes integral also exists and has the same value. That is, F is self-financing when $S \in CW_1$. $\square$

However, in the case when $S \in CW_p$ with $1 < p < 2$, the SLSG strategy may not be self-financing as stated earlier. By Theorem A.20 in T.7, condition (4.7.3)

is not satisfied. That is, when g is given to be $g(y) = \max[0, y - K]$, we have that $D^-g(y) = 1_{y>K}$. So then

$$\begin{aligned} & \sum |D^-g(f(x_i + y)) - D^-g(f(x_{i-1} + y))|^r \\ (\text{set } y = 0) \\ &= \sum |1_{f(x_i)>K} - 1_{f(x_{i-1})>K}|^r \leq n. \end{aligned}$$

If the process has infinite crossings then n can be larger than arbitrary A^r . (It is shown in T.7 (cf. Example 1) that functions of bounded p -variation with $p \geq 1$ have the property of infinite crossing of zero.) Therefore, we can not conclude that the SLSG is self-financing. It should be noted that Theorem A.20 only gives us a sufficient condition, but not the necessary condition. Furthermore, the SLSG may not necessarily be attainable (cf. Definition 5 in §2.1) either. Here we have an example where the Riemann-Stieltjes integral may not exist, so this motivates us to introduce a new concept of self-financing for functions of bounded p -variation with $1 < p < 2$. That is, we need a generalization of the notion of the integral to include the property of infinite crossings. We propose this problem as a future direction in research. We direct the reader to the section of future research for a brief discussion.

An interesting point of the SLSG strategy is its relation to the infinite crossing property and the connection to local time. As pointed out by C-J, $2\Lambda_t(X)$ (defined in T.7, Theorem A.24) can viewed as the density for the amount of quadratic variation (cf. p 26) the process experiences in a particular interval A :

$$\int_A 2\Lambda_t(x)dx = \int_0^t 1_{F(v) \in A} d\langle F \rangle_v, \quad \text{for any } t \in [0, T].$$

Recall that the quadratic variation of F as defined by (2.5.9) when S is geometric Brownian motion is given by

$$\langle F \rangle_t = \sigma^2 \int_0^t [F(v)]^2 dv.$$

To illustrate the connection between local times and the property of infinite crossing, C-J propose an altered SLSG. The heuristic explanation is as follows: the player holds no shares when $F(t) = K$ and buys one share if $F(t) = K + \epsilon$. That is, the player buys if the price process F rises from K to $K + \epsilon$, where $\epsilon > 0$ is a small number. When the player buys a share, K bonds are sold and the player requires an additional ϵ bonds in funds to finance the transaction. When $F(t)$ returns back to K , the player sells the share. Now, let $U_t(\epsilon)$ be the number of times that the price process F rises from K to $K + \epsilon$. The total amount of external funding needed for this altered SLSG strategy is given by the product of ϵ and $U_t(\epsilon)$. As ϵ becomes smaller, this altered strategy behaves like (2.5.2). It was shown by El Karoui (1978, p 67) that the external funding converges a.s. to the semimartingale local time:

$$\lim_{\epsilon \downarrow 0} \epsilon U_t(\epsilon) = \Lambda_t(X).$$

That is, as $\epsilon \rightarrow 0$, $U_t(\epsilon) \rightarrow \infty$ so that the local time does not vanish. Thus, the infinite crossing property holds for any continuous semimartingale with nonvanishing local time (cf. C-J (1990, p 477)). From the above argument and Tanaka-Meyer formula (cf. Theorem A.24 in T.7), we have that

$$\max[0, F(t) - K] = \max[0, F(0) - K] + \int_0^t 1_{F(v) > K} dF(v) + \Lambda_t(X)$$

for all $t \in [0, T]$. Since the last term is positive with positive probability for any positive t , Carr and Jarrow were able to prove that the SLSG strategy could not be self-financing and resolve the conflict with the Black and Scholes model.

The relationship between the continuous local times and the property of infinite crossing for functions of bounded p -variation with $1 < p < 2$ is not known as of yet. (cf. Taylor (1986)). Even though fBm has zero quadratic variation (cf. Proposition 6 in §1.2), it is not obvious how this relates to its continuous version of local times of fBm. From Davies (1977), we know that a continuous version of local time exists for fBm. We observe that a process of bounded variation has vanishing local time

(cf. Lemma A.25). Are we able to infer anything about the local time of functions of bounded p -variation with $1 < p < 2$ from this?

Another claim of Carr and Jarrow which needs further justification: If the stock price process $F(t)$ as defined by (2.5.8) is absolutely continuous, then the self-financing condition (2.5.6) is claimed to hold. In particular, the relation

$$(g \circ f)(b) - (g \circ f)(a) = (L) \int_a^b (D^-g) \circ f df \quad (*)$$

follows from the Fundamental Theorem of Calculus (FTC) (Royden (1968), p 107 and also see the end of §4.7). As pointed out by Stromberg (1981, p 323), the validity of $(*)$ has a long history. Instead, we can achieve $(*)$ from a theorem due to Serrin and Varberg (1969, Theorem 3) (stated below). By this theorem, $(*)$ holds if $g \circ f$ is absolutely continuous. Thus to apply their result the composite function should also be absolutely continuous which is not realistic in applications. The next theorem can be found in Stromberg (1981, p 325).

Change of Variable Theorem. *Let $a < b$ and $c < d$ in $\mathbb{R}$. Suppose that the function $f : [a, b] \rightarrow [c, d]$ is differentiable almost everywhere on $[a, b]$, and g is a real-valued function that is differentiable almost everywhere on $[c, d]$. Also assume that g is absolutely continuous on $[c, d]$. Then the following two statements are equivalent.*

- (1) $g \circ f$ is absolutely continuous on $[a, b]$.
- (2) $(g' \circ f) \cdot f' \in L_1([a, b])$ and

$$\int_{f(\alpha)}^{f(\beta)} g'(x) dx = \int_{\alpha}^{\beta} g'(f(t)) f'(t) dt$$

for all $\alpha, \beta \in [a, b]$.

Proof. See proof of (6.95) of Stromberg (1981, p.325).

In this theorem, we also require that the composite $(g \circ f)$ be absolutely continuous on $[0, T]$. However the following example shows that we are not guaranteed such a result.

Counterexample. Let $f(t) = t^2 \sin^2(1/t)$ for $t \neq 0$, $\phi(0) = 0$, and $g(x) = \sqrt{x}$ for $x \geq 0$. Then total variation of $g \circ f$ on $[0, 1]$ is infinite so $g \circ f$ is not absolutely continuous on $[0, 1]$. But f is absolutely continuous on $[0, 1]$ since f' is bounded there. Also, g is also absolutely continuous on $[0, 1]$. (cf. Stromberg (1981, pp 325-6)).

Hence, this example shows that the above change of variable formula may fail to hold unless the composite function is absolutely continuous.

2.6 Further comments on SLSG strategy.

This section is a possible direction for the SLSG strategy with respect to price processes of bounded p -variation with $1 < p < 2$.

Assume that the price process, S follows

$$dS(t) = [\alpha dt + \sigma dW_H(t)]S(t)$$

and define trading strategies $(m(t), n(t))$ as in §2.5. Assume the bond price $B(t) = e^{r(t-T)}$. We define the portfolio value to be

$$P(t) := m(t)Ke^{r(t-T)} + n(t)S(t)$$

and we say tha the portfolio value is self-financing if

$$P(t) = P(0) + \int_0^t m(v)dB(v) + \int_0^t n(v)dS(v) \quad \text{for every } t \in [0, t]$$

Let g_ϵ be a function such that:

$$\begin{aligned} g''_\epsilon(x) &= \frac{1}{\epsilon} \quad 0 < x < \epsilon \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and

$$g_\epsilon(0) = g'_\epsilon(0) = 0.$$

We define a g_ϵ -trading strategy by

$$\begin{aligned} n_\epsilon(t) &= g'_\epsilon(S(t) - Ke^{r(t-T)}) \\ m_\epsilon(t) &= -g'_\epsilon(S(t) - Ke^{r(t-T)}). \end{aligned}$$

An example of such a function is

$$\begin{aligned} g_\epsilon(x) &= 0 \quad \text{for } x \leq 0 \\ &= x^2/(2\epsilon) \quad \text{for } 0 \leq x \leq \epsilon \\ &= x \quad \text{for } x \geq \epsilon. \end{aligned}$$

This g_ϵ has a continuous derivative, g''_ϵ exists except for the points 0 and ϵ by Darboux Theorem. Then by the chain rules, we have

$$\begin{aligned} g_\epsilon(S(t) - Ke^{r(t-T)}) &= \int m_\epsilon(v)dB(v) + \int n_\epsilon(v)dS(v), \quad \text{for fBm} \\ &= \int m_\epsilon(v)dB(v) + \int n_\epsilon(v)dS(v) \\ &\quad + \frac{1}{2\epsilon} \int 1_{(0,\epsilon)}(S(u) - Ke^{r(u-T)})S^2(u)du \\ &\xrightarrow{\epsilon \rightarrow 0} \int m(v)dB(v) + \int n(v)dS(v) \\ &\quad + \text{term involving local time on the curve } Ke^{r(t-T)} \text{ for BM} \end{aligned}$$

Define P_ϵ as

$$P_\epsilon(t) := m_\epsilon(t)Ke^{r(t-T)} + n_\epsilon(t)S(t).$$

Assume that $P_\epsilon(0) = 0$. P_ϵ is not self-financing. However, it is “almost” self financing in the sense that

$$\begin{aligned} &|P_\epsilon(t) - \int_0^t m_\epsilon(v)dB(v) + \int_0^t n_\epsilon(v)dS(v)| \\ &= |g_\epsilon(t)(S(t) - Ke^{r(t-T)}) - g'_\epsilon(S(t) - Ke^{r(t-T)})[S(t) - Ke^{r(t-T)}]| \\ &= 0 \quad \text{if } S(t) \leq Ke^{r(t-T)} \text{ or } S(t) - Ke^{r(t-T)} > \epsilon \\ &\leq \frac{\epsilon}{2} + \epsilon = \frac{3}{2}\epsilon. \end{aligned}$$

We say that g_ϵ -trading is ϵ -self-financing. Note that

$$\sup_{0 \leq t \leq T} |P_\epsilon(t) - \int_0^t m_\epsilon(v) dB(v) + \int_0^t n_\epsilon(v) dS(v)| \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

Therefore, $g_\epsilon(t)$ is an ϵ -self-financing, that is an ϵ -self financing arbitrage opportunity. So, we propose the following:

Definition 7. $(m(t), n(t))$ (as defined by (2.5.2)) is a *weak arbitrage opportunity* for the price process $\{S(\cdot)\}$ if

$$\lim_{\epsilon \rightarrow 0} |(m(t), n(t)) - (m_\epsilon(t), n_\epsilon(t))| = 0 \quad \text{for every } 0 \leq t \leq T,$$

and $(m_\epsilon(t), n_\epsilon(t))$ is ϵ -self-financing for every $\epsilon > 0$.

Combining the above with the arguments of Carr and Jarrow, we conclude with the next theorem.

Theorem 6. *The SLSG trading strategy is a weak arbitrage opportunity for the fractional Brownian motion price process with $H > 1/2$ but is not a weak arbitrage opportunity for the Brownian motion price process.*

Remarks: In the case of BM one can take the limit as $\epsilon \rightarrow 0$ and get the generalized Itô formula as in Carr and Jarrow. It is an open problem as to whether the SLSG strategy is an “strong” arbitrage opportunity (as defined in §2.5) for the fBm case because the limiting integral is not defined.

3 Simulations and Statistical Analyses

In this section, we adapt a modified quadratic variation to test the index H of fractional Brownian motion. We shall use a result of Gladyshev (1961) for this purpose and we will refer to Gladyshev's modified quadratic variation as *Gladyshev's estimator* of the index H .

Most empirical analysis in economic time series is based on the R/S statistic, also called the *rescaled range* to detect long-range dependence (cf, for example, Mandelbrot (1971), Lo (1991), Peters (1994)). The R/S statistic is the range of partial sums of deviations over a time series from the mean, rescaled by its standard deviation. For the history of the R/S statistics, see Campbell et al (1997).

Using *Mathematica*, we simulated fractional Brownian motion to test how fast Gladyshev's estimator converges. (The program is given in the appendix.) For typical sample paths of geometric fractional Brownian motion, see Figure 1 and Figure 2. In addition, we use Gladyshev's estimator on exchange rate data under the *assumption* that returns have the distribution of fractional Brownian motion. Surprisingly, we found strong evidence that exchange rate data does not have independent increments. In particular, there is a strong positive autocorrelation among consecutive trading days. This analysis is consistent with Peters (1994, Ch. 12). We discuss the exchange rate analysis in §3.1. Before proceeding, we should provide a little background to Gladyshev's estimator.

The next theorem is applicable to processes of stationary Gaussian increments with a spectral decomposition of the form

$$\xi(t) = \xi(0) + \int_{-\infty}^{\infty} (e^{it\lambda} - 1) dz(\lambda),$$

where $z(\lambda)$ is a stochastic process with independent increments. If moreover,

$$E[|dz(\lambda)|^2] = f(\lambda)d\lambda,$$

it is said that the process $\xi(t)$ has the spectral density $f(\lambda)$.

Gladyshev's Theorem 3. *Let $\xi(t)$ be a real stochastic process with stationary Gaussian increments and mean zero, have the spectral density $f(\lambda)$ such that*

$$\frac{f(\lambda)}{|\lambda|^{-\alpha}} = c + O(\lambda^{-2}) \quad \text{for large } \lambda, \quad c > 0, \quad 1 < \alpha < 3.$$

Then with probability 1

$$\lim_{n \rightarrow \infty} \left\{ 2 - \frac{1}{n} \log_2 \sum_{k=1}^{2^n} \left(\xi\left(\frac{k}{2^n}\right) - \xi\left(\frac{k-1}{2^n}\right) \right)^2 \right\} = \alpha$$

and with probability 1

$$\lim_{n \rightarrow \infty} \left\{ -\cos \frac{\pi\alpha}{2} \Gamma(\alpha) 2^{n(\alpha-2)-1} \sum_{k=1}^{2^n} \left(\xi\left(\frac{k}{2^n}\right) - \xi\left(\frac{k-1}{2^n}\right) \right)^2 \right\} = c,$$

where $\Gamma(\lambda)$ is the Euler gamma function.

Now, consider a real-valued Gaussian stochastic process $\xi_t^{(c,\delta)}$, $-\infty < t < \infty$, with stationary increments for which

$$E(\xi_t - \xi_s) = 0$$

and

$$E(\xi_{t+\tau} - \xi_t)^2 = c|\tau|^\delta,$$

where $c > 0$, $0 < \delta < 2$.

This stochastic process is known as fractional Brownian motion, and this process also satisfies the conditions of the above theorem. Recall that the spectral density $f(\lambda)$ of this process $\xi_t^{(c,\delta)}$ is given

$$f(\lambda) = 2^{-1} c \Gamma(\delta + 1) \sin \frac{\pi\delta}{2} |\lambda|^{-\delta-1}$$

(cf. Yaglom (1955)). Now applying the above theorem to this process, Gladyshev (1961) proved that with probability 1,

$$\lim_{n \rightarrow \infty} \left\{ 1 - \frac{1}{n} \log_2 \sum_{k=1}^{2^n} \left[\xi_{k/2^n}^{(c,\delta)} - \xi_{(k-1)/2^n}^{(c,\delta)} \right]^2 \right\} = \delta = 2H. \quad (3.1.1)$$

and with probability 1

$$\lim_{n \rightarrow \infty} 2^{n(\delta-1)} \sum_{k=1}^{2^n} [\xi_{k/2^n}^{(c,\delta)} - \xi_{(k-1)/2^n}^{(c,\delta)}]^2 = c. \quad (3.1.2)$$

This estimator tells us that, as n becomes very large (and taking finer partitions), we will obtain the *true* index $2H$. Thus, intuitively, we know that

$$E \left[\sum_{k=1}^{2^n} [\xi_{\frac{k}{2^n}} - \xi_{\frac{k-1}{2^n}}]^2 \right] \approx \frac{c2^n}{2^{n\delta}}.$$

Hence, by the law of large numbers behaviour,

$$1 - \frac{1}{n} \log_2 \left[\sum_{k=1}^{2^n} [\xi_{k/2^n} - \xi_{(k-1)/2^n}]^2 \right] \approx 1 - \left\{ \frac{\log_2 c}{n} + (1 - \delta) \right\} \rightarrow \delta \text{ as } n \rightarrow \infty.$$

In order to obtain an empirical evaluation of how fast (3.11) is converging, we generated $2^n + 1$ points of fBm with $H = 0.75$ and $n = \{18, 17, 16, 15, 14, 13, 12, 11\}$. A summary of these results can be found in Tables 13-45 (cf. §3.2). It appears that this estimator may be useful to test the index $2H$. However, we cannot draw rigorous conclusions at this stage of research. For the following values of $H = 0.45, 0.5, 0.6, 0.75$, we generated $K = 100$ independent samples, each of 8193 and 2049, of fractional Brownian motion, say $\{H_i, i = 1, \dots, 100\}$. We calculated their mean, (unbiased) standard deviation, $\hat{\sigma}$ and mean squared error, MSE

$$MSE = \frac{1}{K} \sum_{i=1}^K (2H_i - 2(\text{True } H))^2,$$

where the true H is the value of H we used to generate the fractional Brownian motion. Recall the mean squared error has the following decomposition

$$\frac{1}{K} \sum_{i=1}^K (2H_i - 2(\text{True } H))^2 = \frac{1}{K} \sum_{i=1}^K (2H_i - \overline{2H})^2 + (\overline{2H} - 2 \text{ True } H)^2.$$

The quantity $(\overline{2H} - 2 \text{ True } H)^2$ is called the bias of the estimator. These calculations are presented in Table 1 and Table 2 (below).

Let $T_{2H} = 2 \text{ True } H$.

Table 1. Estimation of $2H$ using 100 independent samples of 8193.

	True 2H			
	0.90	1.00	1.20	1.50
$\overline{2H}$	0.965699	1.06459	1.2659536	1.55892
$\hat{\sigma}$	0.00202232	0.00208516	0.036421	0.00326492
$\sqrt{MSE}$	0.06572298	0.0646282	0.0624413	0.0590076
$(\overline{2H} - T_{2H})^2$	0.00431636	0.00417187	0.00434988	0.00347157

Table 2. Estimation of $2H$ using 100 independent samples of 2049.

	True 2H			
	0.90	1.00	1.20	1.50
$\overline{2H}$	0.977475	1.07566	1.27415	1.56939
$\hat{\sigma}$	0.00459959	0.00471056	0.00485768	0.0087942
$\sqrt{MSE}$	0.0776097	0.0758097	0.0743103	0.0699322
$(\overline{2H} - T_{2H})^2$	0.00600231	0.00572515	0.00549866	0.00481472

As one can observe, there is an obvious bias. However, as you increase the sample size, the bias becomes smaller and we also see this reduction in the individual simulations (cf. Tables 13 - 45). However, further study is needed to test the accuracy and bias of this estimator. As for further research, we require theoretical statistical properties for this estimator. In particular, to establish a Type 1 and Type 2 error analysis for this estimator. In addition, it would be nice to obtain

a central limit theorem and to clarify if this estimator is an asymptotically biased estimator.

Figure 1. Typical Sample Paths of Geometric Fractional Brownian Motion with $H = 0.9, 0.7, 0.5, 0.3, 0.1$.

Figure 2. Typical Sample Paths of Geometric Fractional Brownian Motion with $H = 0.6, 0.55, 0.5$.

3.1 Exchange Rate Data Analysis.

We assume that the returns (defined below) have a distribution of fractional Brownian motion with index H . We test real data taken from the Monetary Yearbook of the Chicago Mercantile Exchange using Gladyshev's estimator. These are the daily closing bids for five currencies (German Mark, Japanese Yen, Swiss franc, Pound Sterling and Canadian Dollar) with respect to the US Dollar. Originally, Weigend, Huberman, and Rumelhart (1992) do not consider the problem of testing of a particular distribution of the data. Their aim is to predict *returns* defined by

$$R_t := \log(P_t/P_{t-1})$$

using *connectionist networks*, where P_t denotes the price at time t . That is, given exchange rate information up to and including a Monday (starting from the previous Tuesday), predict the German Mark-US dollar rate for the following Tuesday. This data set has been studied by others (cf., for example, Hsieh, D (1988, 1989a, 1989b), Bhar (1994), Peters (1994)).

Peters (1994) argues that currencies are true Hurst processes. In his analysis using the R/S statistics, he found that the estimated index for the Yen/Dollar to be $H = 0.642$, for the Mark/Dollar, $H = 0.624$, and for the Pound/Dollar, $H = 0.626$. Peters further discussed the economic implications of currencies. In particular, currencies are not securities, however, they are heavily traded. The largest participants, the central banks, are not return maximizers and their objectives may not be rational. At the same time, there is little evidence of cycles in the currency markets, although they do have strong trends (cf. Peters (1994)). Peters concludes that currencies are characterized by infinite-memory processes so long-term investors should be cautious in their trading of currencies. In particular, risk increases through time and does not decline with time. This is the motivation for our analysis.

Remark. These data were given to ASW by Blake LeBaron.

We apply Gladyshev's estimator on various times of this data. We consider two definitions of returns:

$$R_t := \log(P_t/P_{t-1}) \quad (1)$$

and

$$R'_t := (P_t - P_{t-1})/P_{t-1}. \quad (2)$$

From a practical viewpoint, these returns are approximately the same (cf. Granger and Morgenstern (1970, p.73) and Taylor (1986, p.12). However, if returns defined by (1) have a stable distribution, then returns defined by (2) can never be stable (cf. Elton, Gruber, Kleindorfer (1975)).

The results are found in Table 3.1 to Table 12.2. The interpretation of these tables is as follows: when $n = 11$, we use all the trading days (ie. $2^{11} + 1$), when $n = 10$, we use every second trading day, when $n = 9$, every three trading days apart, and so further. It should be noted that the lower values of n do not have any meaning.

From these tables (cf. Tables 3.1 - 12.2), there is strong evidence that $H \neq 0.5$ (ie. do not have independent increments among consecutive trading days). In particular, there is a positive autocorrelation among consecutive trading days which is inconsistent with the assumption that returns follow geometric BM (ie. increments are not correlated). This analysis complements Peters (1994). However, we propose further investigation since we have very few theoretical statistical results for the Gladyshev's estimator. In particular, it is desirable to have statistical properties such as Type 1 and Type 2 error analysis for this estimator.

It is also be noted that for the Canadian Dollar the returns calculated by (1) and (2) are different. As commented on the previous page, this could indicate that returns defined by (1) come from a stable distribution. However, further analysis is required to establish if the Canada/Dollar has a stable distribution.

Table 3.1 Returns defined by (1) for the
British Pound from June 01, 1973 to August 10, 1981

n	$2H$
11	1.2731211
10	1.4052653
9	1.1551862
8	1.6909113
7	2.0688340
6	2.3726697
5	2.9438500
4	3.4425650
3	4.0757234
2	6.5901485
1	13.455184

Table 4.1 Returns defined by (1) for the
British Pound from April 09, 1979 to May 21, 1987

n	$2H$
11	1.1924323
10	1.3185940
9	1.4466235
8	1.6449293
7	1.8070007
6	2.1022587
5	2.7992275
4	3.2883685
3	3.8805340
2	6.1254793
1	16.6964044

Table 3.2 Returns defined by (2) for the
British Pound from June 01, 1973 to August 10, 1981

n	$2H$
11	1.2735127
10	1.4057283
9	1.5528466
8	1.6931447
7	2.0711723
6	2.3759103
5	2.9451946
4	3.4448356
3	4.0783377
2	6.5975926
1	13.4701521

Table 4.2 Returns defined by (2) for the
British Pound from April 09, 1979 to May 21, 1987

n	$2H$
11	1.1921490
10	1.3187012
9	1.4471768
8	1.6452750
7	1.8072920
6	2.1018023
5	2.7971707
4	3.2849041
3	3.8768820
2	6.1118502
1	16.6990140

Table 5.1 Returns defined by (1) for the
Canadian Dollar from June 01, 1973 to August 10, 1981

n	$2H$
11	1.5319753
10	1.6830155
9	1.8738312
8	2.1214019
7	2.4809689
6	2.9056461
5	3.5238827
4	4.6733424
3	6.7586852
2	9.7098524
1	17.9790291

Table 6.1 Returns defined by (1) for the
Canadian Dollar from April 09, 1979 to May 21, 1987

n	$2H$
11	1.4677374
10	1.6251946
9	1.8266756
8	2.0805632
7	2.3453136
6	2.7261560
5	3.4553159
4	4.3849764
3	5.7069291
2	8.6670887
1	19.1541411

Table 5.2 Returns defined by (2) for the
Canadian Dollar from June 01, 1973 to August 10, 1981

n	$2H$
11	1.7407383
10	1.9116912
9	2.1230261
8	2.4172195
7	2.8014237
6	3.2854603
5	3.9953249
4	5.3244327
3	7.4905822
2	10.7967345
1	19.9867246

Table 6.2 Returns defined by (2) for the
Canadian Dollar from April 09, 1979 to May 21, 1987

n	$2H$
11	1.6570780
10	1.8331608
9	2.0577400
8	2.3478246
7	2.6387084
6	3.0659407
5	3.8551999
4	4.9213381
3	6.3249132
2	9.5966520
1	21.1113863

Table 7.1 Returns defined by (1) for the
German Mark from June 01, 1973 to August 10, 1981

n	$2H$
11	1.2216650
10	1.3435881
9	1.4829410
8	1.6164847
7	1.9893540
6	2.2181639
5	2.6853207
4	3.3032644
3	4.0663762
2	6.1369346
1	11.301010

Table 8.1 Returns defined by (1) for the
German Mark from April 09, 1979 to May 21, 1987

n	$2H$
11	1.1821881
10	1.3238775
9	1.4610417
8	1.6566776
7	1.8802623
6	2.1272573
5	2.5962694
4	3.0552665
3	3.9280626
2	6.3830406
1	16.7286000

Table 7.2 Returns defined by (2) for the
German Mark from June 01, 1973 to August 10, 1981

n	$2H$
11	1.2220602
10	1.3445836
9	1.4849537
8	1.6202956
7	1.9903340
6	2.2187083
5	2.6850692
4	3.3009681
3	4.0638423
2	6.1255596
1	11.2840467

Table 8.2 Returns defined by (2) for the
German Mark from April 09, 1979 to May 21, 1987

n	$2H$
11	1.1817970
10	1.3237759
9	1.4610256
8	1.6564653
7	1.8797086
6	2.1265092
5	2.5987767
4	3.0578849
3	3.9251777
2	6.3741537
1	16.7260136

Table 9.1 Returns defined by (1) for the
Japanese Yen from June 01, 1973 to August 10, 1981

n	$2H$
11	1.2479994
10	1.3798255
9	1.5165034
8	1.6849196
7	1.9758474
6	2.3142918
5	2.3848077
4	3.7182994
3	4.3913374
2	7.1504682
1	13.8628091

Table 10.1 Returns defined by (1) for the
Japanese Yen from April 09, 1979 to May 21, 1987

n	$2H$
11	1.2040215
10	1.3408083
9	1.4870474
8	1.6786671
7	1.3979342
6	2.2840798
5	2.8043404
4	3.3820202
3	4.6205924
2	7.3421836
1	14.3614281

Table 9.2 Returns defined by (2) for the
Japanese Yen from June 01, 1973 to August 10, 1981

n	$2H$
11	1.2482721
10	1.3799336
9	1.5172125
8	1.6873063
7	1.9760037
6	2.3126062
5	2.3838702
4	3.7202036
3	4.3944181
2	7.1589211
1	13.8617488

Table 10.2 Returns defined by (2) for the
Japanese Yen from April 09, 1979 to May 21, 1987

n	$2H$
11	1.2033683
10	1.3402892
9	1.4865722
8	1.6780074
7	1.8973608
6	2.2831361
5	2.8035144
4	3.3796316
3	4.6176635
2	7.3381150
1	14.8606181

Table 11.1 Returns defined by (1) for the
Swiss Franc from June 01, 1973 to August 10, 1981

n	$2H$
11	1.1688141
10	1.3055722
9	1.4300314
8	1.5854633
7	1.9003981
6	2.1356354
5	2.5137387
4	3.2165919
3	3.9518607
2	6.7850015
1	12.7327279

Table 12.1 Returns defined by (1) for the
Swiss Franc from April 09, 1979 to May 21, 1987

n	$2H$
11	1.1633444
10	1.2977007
9	1.4355663
8	1.6425575
7	1.8533975
6	2.1808514
5	2.6783613
4	3.1891343
3	3.9408213
2	6.5632683
1	16.6681993

Table 11.2 Returns defined by (2) for the
Swiss Franc from June 01, 1973 to August 10, 1981

n	$2H$
11	1.1690928
10	1.3061765
9	1.4317771
8	1.5888861
7	1.9013630
6	2.1374172
5	2.5145504
4	3.2145460
3	3.9492921
2	6.7837745
1	12.7193585

Table 12.2 Returns defined by (2) for the
Swiss Franc from April 09, 1979 to May 21, 1987

n	$2H$
11	1.1630120
10	1.2974990
9	1.4354689
8	1.6424030
7	1.8529315
6	2.1791388
5	2.6765728
4	3.1848242
3	3.9355553
2	6.5535906
1	16.663081

3.2 Simulated Data Tables.

Table 13. $2^{18} + 1$ and $H = 0.75$

n	$2H$
18	1.54298
17	1.53636
16	1.53569
15	1.53723
14	1.54046
13	1.54554
12	1.54979
11	1.55575
10	1.56526
9	1.57583
8	1.59753
7	1.61901
6	1.62900
5	1.74957
4	1.82789
3	2.45083
2	3.21861
1	7.18523

Table 14. $2^{17} + 1$ and $H = 0.75$

n	$2H$
17	1.54424
16	1.53676
15	1.53547
14	1.53708
13	1.53903
12	1.54410
11	1.54817
10	1.55217
9	1.55916
8	1.56696
7	1.58545
6	1.61477
5	1.63144
4	1.74934
3	1.78966
2	2.33706
1	5.79276

Table 15. $2^{16} + 1$ and $H = 0.75$

n	$2H$
16	1.54771
15	1.54004
14	1.53889
13	1.54018
12	1.54341
11	1.54823
10	1.55726
9	1.56396
8	1.57840
7	1.59255
6	1.64947
5	1.72349
4	1.75674
3	2.05468
2	2.31415
1	2.86793

Table 16. $2^{16} + 1$ and $H = 0.75$

n	$2H$
16	1.54674
15	1.53917
14	1.53743
13	1.53869
12	1.53991
11	1.54137
10	1.54416
9	1.54255
8	1.54301
7	1.52871
6	1.51089
5	1.53731
4	1.59585
3	1.50525
2	1.73732
1	2.59517

Table 17. $2^{16} + 1$ and $H = 0.75$

n	$2H$
16	1.54773
15	1.54004
14	1.53907
13	1.54187
12	1.54554
11	1.55058
10	1.55601
9	1.56805
8	1.58669
7	1.61570
6	1.62544
5	1.66264
4	1.70846
3	1.78645
2	2.59424
1	4.77634

Table 19. $2^{16} + 1$ and $H = 0.75$

n	$2H$
16	1.54876
15	1.54171
14	1.54191
13	1.54488
12	1.55005
11	1.55751
10	1.56560
9	1.57300
8	1.59084
7	1.60316
6	1.71694
5	1.82744
4	1.97612
3	2.28893
2	2.50719
1	3.76384

Table 18. $2^{16} + 1$ and $H = 0.75$

n	$2H$
16	1.54834
15	1.54089
14	1.53998
13	1.54061
12	1.54293
11	1.54469
10	1.54934
9	1.56233
8	1.56924
7	1.57521
6	1.56053
5	1.59168
4	1.62347
3	1.62827
2	1.80922
1	2.17588

Table 20. $2^{15} + 1$ and $H = 0.75$

n	$2H$
15	1.55251
14	1.54472
13	1.54604
12	1.54907
11	1.55441
10	1.56452
9	1.57167
8	1.58812
7	1.62098
6	1.67226
5	1.72415
4	1.78762
3	2.06316
2	2.91541
1	5.16591

Table 21. $2^{15} + 1$ and $H = 0.75$

n	$2H$
15	1.55129
14	1.54386
13	1.54295
12	1.54645
11	1.55126
10	1.55842
9	1.56534
8	1.57771
7	1.59756
6	1.60713
5	1.57260
4	1.56174
3	1.56811
2	1.97858
1	3.13065

Table 23. $2^{15} + 1$ and $H = 0.75$

n	$2H$
15	1.55124
14	1.54360
13	1.54338
12	1.54623
11	1.55008
10	1.55550
9	1.55988
8	1.56680
7	1.56633
6	1.56981
5	1.60655
4	1.74983
3	1.75886
2	1.82798
1	2.30837

Table 22. $2^{15} + 1$ and $H = 0.75$

n	$2H$
15	1.55069
14	1.54291
13	1.54169
12	1.54288
11	1.54455
10	1.54849
9	1.54638
8	1.55906
7	1.56705
6	1.58154
5	1.60447
4	1.64955
3	1.77538
2	2.09616
1	2.78194

Table 24. $2^{15} + 1$ and $H = 0.75$

n	$2H$
15	1.55064
14	1.54262
13	1.54181
12	1.54306
11	1.54707
10	1.55264
9	1.56211
8	1.57165
7	1.59207
6	1.61605
5	1.64398
4	1.65617
3	1.81691
2	2.08381
1	2.99598

Table 25. $2^{14} + 1$ and $H = 0.75$

n	$2H$
14	1.55170
13	1.54290
12	1.54158
11	1.54104
10	1.54378
9	1.54769
8	1.55450
7	1.55501
6	1.56458
5	1.57268
4	1.59095
3	1.57996
2	1.56393
1	1.08355

Table 27. $2^{14} + 1$ and $H = 0.75$

n	$2H$
14	1.5560192770
13	1.5486274145
12	1.5472758440
11	1.5492201776
10	1.5522956449
9	1.5581781511
8	1.5680694535
7	1.5722660880
6	1.5499438370
5	1.5439293783
4	1.5322029072
3	1.7225926411
2	1.8738115521
1	2.0739870202

Table 26. $2^{14} + 1$ and $H = 0.75$

n	$2H$
14	1.55246
13	1.54420
12	1.54200
11	1.54183
10	1.54533
9	1.55273
8	1.55049
7	1.55608
6	1.53440
5	1.56100
4	1.59324
3	1.76313
2	1.99878
1	2.20334

Table 28. $2^{14} + 1$ and $H = 0.75$

n	$2H$
14	1.5597680936
13	1.5542798118
12	1.5584233341
11	1.5628911111
10	1.5714484761
9	1.5820360823
8	1.5985279051
7	1.6079292706
6	1.6105377361
5	1.6059847362
4	1.5675584737
3	1.5474620000
2	1.5529650485
1	1.1252282095

Table 29. $2^{14} + 1$ and $H = 0.75$

n	$2H$
14	1.55123
13	1.54219
12	1.5399
11	1.54094
10	1.54213
9	1.5367
8	1.53851
7	1.55101
6	1.5353
5	1.50347
4	1.48228
3	1.5714
2	1.50238
1	1.79345

Table 31. $2^{13} + 1$ and $H = 0.75$

n	$2H$
13	1.55533
12	1.54515
11	1.53957
10	1.53795
9	1.53367
8	1.53642
7	1.5302
6	1.5286
5	1.56876
4	1.51712
3	1.45039
2	1.23581
1	1.38766

Table 30. $2^{13} + 1$ and $H = 0.75$

n	$2H$
13	1.56098
12	1.55359
11	1.55439
10	1.5606
9	1.5685
8	1.57553
7	1.58256
6	1.66023
5	1.73245
4	1.80152
3	2.14739
2	2.56395
1	3.94277

Table 32. $2^{13} + 1$ and $H = 0.75$

n	$2H$
13	1.55894
12	1.55064
11	1.55037
10	1.54924
9	1.55199
8	1.55701
7	1.55026
6	1.56665
5	1.58221
4	1.56978
3	1.52381
2	1.54343
1	1.31682

Table 33. $2^{13} + 1$ and $H = 0.75$

n	$2H$
13	1.55989
12	1.55196
11	1.55221
10	1.5566
9	1.56372
8	1.57971
7	1.58488
6	1.57442
5	1.55795
4	1.65642
3	1.85858
2	2.41273
1	2.98648

Table 35. $2^{12} + 1$, $H = 0.75$

n	$2H$
12	1.56817
11	1.56229
10	1.56662
9	1.57126
8	1.57631
7	1.58492
6	1.60694
5	1.6236
4	1.61358
3	1.62754
2	1.63765
1	1.96241

Table 34. $2^{13} + 1$ and $H = 0.75$

n	$2H$
13	1.56265
12	1.5545
11	1.55548
10	1.555589
9	1.5592
8	1.56597
7	1.57747
6	1.58727
5	1.61023
4	1.61877
3	1.61056
2	1.44453
1	1.13799

Table 36. $2^{11} + 1$, $H = 0.75$

n	$2H$
11	1.57265
10	1.56246
9	1.56849
8	1.57799
7	1.5958
6	1.62578
5	1.60841
4	1.60645
3	1.92781
2	2.37181
1	5.72645

Table 37. $2^{11} + 1$, $H = 0.75$

n	$2H$
11	1.57614
10	1.57088
9	1.57458
8	1.59705
7	1.60902
6	1.62921
5	1.71573
4	1.99135
3	2.64067
2	3.47482
1	5.04912

Table 38. $2^{11} + 1$ and $H = 0.55$

n	$2H$
11	1.17505
10	1.14182
9	1.13955
8	1.13594
7	1.1323
6	1.16105
5	1.09967
4	1.06416
3	1.12269
2	1.52832
1	1.22513

Table 39. $2^{11} + 1$ and $H = 0.55$

n	$2H$
11	1.17807
10	1.15587
9	1.15258
8	1.15023
7	1.16115
6	1.17311
5	1.14655
4	1.08135
3	1.14228
2	1.2382
1	0.736843

Table 40. $2^{10} + 1$, $H = 0.55$

n	$2H$
10	1.17577
9	1.14249
8	1.13374
7	1.12542
6	1.14401
5	1.13729
4	1.13187
3	1.06019
2	1.21694
1	1.53992

Table 41. $2^{10} + 1$ and $H = 0.55$

n	$2H$
10	1.18548
9	1.15616
8	1.15636
7	1.17451
6	1.19414
5	1.25479
4	1.23244
3	1.17746
2	1.20341
1	1.55356

Table 42. $2^{10} + 1$ and $H = 0.55$

n	$2H$
10	1.17493
9	1.16014
8	1.1555
7	1.14599
6	1.09361
5	1.08496
4	0.966103
3	0.70253
2	0.468162
1	0.592839

Table 43. $2^{10} + 1$ and $H = 0.55$

n	$2H$
10	1.18529
9	1.1607
8	1.14343
7	1.14657
6	1.17217
5	1.1928
4	1.27504
3	1.19398
2	1.02932
1	2.35847

Table 44. $2^{10} + 1$ and $H = 0.55$

n	$2H$
10	1.18488
9	1.15626
8	1.16136
7	1.16385
6	1.15139
5	1.15028
4	1.21712
3	1.09661
2	1.52168
1	2.05981

Table 45. $2^{10} + 1$ and $H = 0.55$

n	$2H$
10	1.18231
9	1.16229
8	1.16045
7	1.1671
6	1.14717
5	1.17212
4	1.33343
3	1.50868
2	1.46054
1	8.03489

4 Conclusions and Future Research

In the previous section, we constructed trading strategies which will lead to arbitrage opportunities when the price process is a continuous, strictly positive function of bounded p -variation with $1 \leq p < 2$. That is, we extended the result of Harrison et al (1984) as follows: if there is no arbitrage and the price process is continuous then p -variation of the price process with $p < 2$ must be infinite.

Furthermore, we set the foundation of a calculus for fractional Brownian motion without probability. The pathwise integral of fractional Brownian motion is based on the property of p -variation. Recall that almost all sample paths of fractional Brownian motion are included in the class of bounded p -variation with $1 \leq p < 2$ (cf. Proposition 4 in §1.3). In addition, stochastic calculus using pathwise integrals based on the class of functions of bounded p -variation can be easily adapted to handle other stochastic processes which have been proposed as a viable alternative to the study of stock price fluctuations (such as α -stable Lévy motions with index $\alpha \in (0, 2)$ (Mandelbrot (1963), Norvaiša (1997b)), hyperbolic Lévy motions, the normal inverse Levy process of Barndorff-Nielsen (cf. Eberlein and Keller (1995)), and subordinate processes (cf. Clark (1973), Madan and Milne (1991), Madan and Senata (1990)). In particular, the first two processes have bounded p -variation with $p > 1$ and the last process has bounded p -variation with $p > 0$ (cf. Norvaiša (1997b). For a nice expository survey on modeling stock price changes and p -variation, we refer the reader to Norvaiša (1997b).

We proposed the Gladyshev's estimator to estimate the index H under the assumption of stationary Gaussian increments. Our analysis has opened many theoretical research problems for the use of this estimator. In particular, we require statistical results such a Type 1 and Type 2 error analysis for the Gladyshev's estimator.

From our empirical analysis, there is a strong indication that the exchange rate data is consistent with function of bounded p -variation with $p < 2$. So in the case of exchange rates, it is appropriate to use the fractional Brownian motion

calculus rather than Itô's calculus. This is also consistent with Peters (1994, Ch. 12) analysis.

4.1 Models and Empirical Tools.

There are several possibilities to pursue in this area. One possible direction of work is to extend my earlier results to stochastic processes with possibly discontinuous sample paths. It is known that almost all paths of the α -stable processes with independent increments have bounded p -variation with $0 < \alpha < p < 2$. However, the sample paths of this process are discontinuous. For this case, there exists a chain rule that can be applied to this problem as well (cf. Norvaiša (1997a)). It should be noted that the chain rule plays an essential role in these problems, and is the analogue of Itô's Lemma in semimartingale theory.

The next logical step after applying p -variation theory to fBm and α -stable processes is to apply it to random Fourier series. It is well-known that Brownian motion has a random Fourier representation (cf. Lévy, 1951). Therefore, it is reasonable to consider a more general class of random Fourier series which may or may not be Gaussian processes; in particular, to investigate this alternative model to stock price fluctuations using random Fourier series.

Furthermore, both fractional Brownian motion and the α -stables have a *harmonizable representation*. The harmonizable representation of fractional Brownian motion (cf. Samorodnitsky and Taqqu (1994)) is given by the following proposition.

Proposition. *Let $0 < H < 1$. Then standard fractional Brownian motion $\{B_H(t), t \in \mathbb{R}\}$ has the integral representation*

$$\frac{1}{C_2(H)} \int_{-\infty}^{\infty} \frac{e^{ixt} - 1}{ix} |x|^{-(H-1/2)} \widetilde{M}(dx), \quad t \in \mathbb{R}, \quad (1)$$

where

$$C_2(H) = \left(\frac{\pi}{H\Gamma(2H)\sin H\pi} \right)$$

and $\widetilde{M}$ is a complex isotropic random measure.

This representation has a physical interpretation (cf. Samorodnitsky and Taqqu (1994, p. 332)). The factor $(e^{ixt} - 1)/ix$ is the Fourier transform of the indicator function on the interval $[0, t]$ and ensures the stationarity of the increments. The term $|x|^{-(H-1/2)}$ colours the underlying Gaussian noise. This gives rise to a spectral density $||x|^{-(H-1/2)}|^2$ which diverges at low frequencies (ie. $x \rightarrow 0$) when $1/2 < H < 1$, is colourless (equal to 1) when $H = 1/2$ and tends to 0 at low frequencies when $0 < H < 1/2$.

We require a more systematic attempt to identify the true nature of stock price fluctuations in the context of this very large class of processes. In particular, there is a real possibility of a multifractal nature in these processes which will have impact at different time scales. Observation of multifractal behaviour has been discussed in physics, (cf. for example, Pesin and Weiss (1997), Jaffard (1996, 1997) and Cheng and Agterberg (1996)) but has not been investigated in economic time series.

4.2 Derivative Pricing.

With the assumption that the stock price process follows fractional Brownian motion, it is natural to discuss the pricing of options and derivatives. Investors generally require liquidity from a market. Liquidity will ensure that the investor will get close to what the market considers as a fair price, efficient trading practices among different investors, and no market crashes from supply and demand imbalances (cf. Peters (1994, p. 42)). If the market is liquid, then the price can be considered close to be *fair* and hence we have a stable market (cf. Peters (1994)). Hence, the aim is to have a liquid market and so alternative stock price processes other than semimartingales can be used in derivative pricing. In particular, since fBm has the long-range dependence property, how would it influence the derivation of the fair price of an option? So far, no one has succeeded to price options under the fBm assumption. There have been attempts using nonstandard analysis and semimartingale theory, but the problem is to solve a linear integral equation driven by fBm. However, fBm is not a semimartingale. The purpose of pricing under alternative processes is to give a model of investor behaviour and market

price movements that fits our observation.

APPENDIX: Technical Supplement

T.1 The Riemann-Stieltjes Integral.

Note that the Riemann-Stieltjes integral $\int f dg$ is used in this paper even though f and g may not have bounded variation. In most textbooks, the Riemann-Stieltjes integral is considered when one of the functions has bounded variation. Therefore, this is an attempt to collect and give references to necessary facts that we used without the assumption of bounded variation.

This was taken from Natanson (1955).

Definition A.1. Let f and g be real valued functions defined on an interval $[a, b]$. Let us subdivide the interval $[a, b]$ by the points

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Let ξ_k be any point in $[x_k, x_{k+1}]$. We will set up the sum

$$s(\kappa; f, g) = \sum_{k=0}^{n-1} f(\xi_k)[g(x_{k+1}) - g(x_k)].$$

If $s(\kappa; f, g)$ tends to a limit as $|\kappa| := \max(x_{k+1} - x_k) \rightarrow 0$ and this limit does not depend on either the method of dividing the interval or on the choice of the points ξ_k , then this limit is denoted by

$$\int_a^b f dg = \int_a^b f(x) dg(x)$$

and is called the *Riemann-Stieltjes* integral of f with respect to g .

It is clear that the Riemann integral is a special case of the Riemann-Stieltjes integral when $g(x)$ replaced by x .

A more general Riemann-Stieltjes integral is the Moore-Pollard-Stieltjes. These two integrals are equivalent under the assumptions of this thesis. This is taken from Dudley and Norvaiša (1997).

Let $Q([a, b])$ denote the set of all partitions κ of the interval $[a, b]$ and let λ be a refinement of κ .

The *Moore-Pollard-Stieltjes* integral

$$(MPS) \int_a^b f dg$$

is defined as the real number T , whenever such exists, if for every $\epsilon > 0$ there is a subdivision λ such that

$$|s(\kappa; f, g) - T| < \epsilon$$

for all refinements $\kappa \in Q([a, b])$ of λ and all $\xi_k \in [x_k, x_{k+1}]$. In this case, we say that f is *(MPS)-integrable* with respect to g .

The next definition is due to Dudley (1992). The Left Young and Right Young integrals are defined by Dudley and Norvaiša (1997).

A real-valued function f on the closed interval $[a, b]$ is called *regulated* if it has a left limit at each point of $(a, b]$ and a right limit at each point of $[a, b)$, and we denote such a function $f \in \mathcal{R}([a, b])$. Note that every function of bounded p -variation is regulated (cf. Lemma 2.4 of Dudley and Norvaiša (1997) in T.5). Given a regulated function f , defined on $[a, b]$, a left continuous function $f_-^{(a)}$ and a right continuous function $f_+^{(b)}$ by

$$f_a^-(x) := f^-(x) := f(x-) := \lim_{y \uparrow x} f(y), \quad a < x \leq b, \quad f_a^-(a) = f(a)$$

and

$$f_b^+(x) := f^+(x) := f(x+) := \lim_{y \downarrow x} f(y), \quad a \leq x < b, \quad f_b^+(b) = f(b),$$

respectively. Let $\Delta^- f := f - f_a^-$ and $\Delta^+ f := f_b^+ - f$, so that $\Delta^- f(x)$ is the left jump of f at $x \in (a, b]$ and $\Delta^+ f(y)$ be the right jump of f at $y \in [a, b)$

Defintion A.2. Let $f, g \in \mathcal{R}([a, b])$. If the corresponding *(MPS)-integral* exists and the sum converges absolutely, then define the Young (Y_1) -integral as

$$(Y_1) \int_a^b f dg := (MPS) \int_a^b f_a^+ dg_a^- - [\Delta^+ f \Delta^+ g](a) - \sum_{(a, b)} \Delta^+ f (g^+ - g^-) + [f \Delta^- g](b),$$

the Young (Y_2)-integral

$$(Y_2) \int_a^b f dg := (MPS) \int_a^b f^- dg_b^+ + [f \Delta^+ g](a) + \sum_{(a,b)} \Delta^- f (g^+ - g^-) + [\Delta^- f \Delta^- g](b),$$

and the Left Young (LY)-integral

$$(LY) \int_a^b f dg := (MPS) \int_a^b f_a^- dg_b^+ + [h \Delta^+ g](a) + \sum_{(a,b)} \Delta^- f \Delta^+ g.$$

The definition of the (Y)-integral can be found in Hildebrandt (1963). The motivation of such an integral was initiated by WH Young (1914). This integral is defined if jumps of $g(x-)$ and $g(x+)$ exist finitely for every points.

Definition A.3. If f and g are real-valued functions.

$$(Y) \int_a^b f dg := \lim_{(\kappa)} \sum_{i=1}^n \psi([x_{i-1}, x_i])$$

if it exists, where the limit is taken under successive refinements of partitions of $[a, b]$ and we assume for any interval $I = [c, d]$, ψ is defined by

$$\psi(I) := \Delta^+ g(c) f(c) + [g(x-) - g(x+)] f(y) + \Delta_g(d) f(d)$$

for $y \in (c, d)$.

T.2 Some Properties of the Riemann-Stieltjes Integral.

1. Bilinearity of the Riemann-Stieltjes Integral:

Theorem. (a) If f_1, f_2 are integrable with respect to g on $[a, b]$ and α, β are real numbers, then $\alpha f_1 + \beta f_2$ is integrable with respect to g on $[a, b]$ and

$$\int_a^b (\alpha f_1 + \beta f_2) dg = \alpha \int_a^b f_1 dg + \beta \int_a^b f_2 dg. \quad (4)$$

(b) If f is integrable with respect to g_1 and g_2 on $[a, b]$ and α, β are real numbers, then f is integrable to $g = \alpha g_1 + \beta g_2$ on $[a, b]$ and

$$\int_a^b f dg = \alpha \int_a^b f dg_1 + \beta \int_a^b f dg_2. \quad (5)$$

Proof. See Bartle's (1976) proof of Theorem 29.5

2. Additivity of the Riemann-Stieltjes Integral

Theorem. (a) Suppose that $a \leq c \leq b$ and that f is integrable with respect to g over both the subintervals $[a, c]$ and $[c, b]$. Then f is integrable with respect to g on the interval $[a, b]$ and

$$\int_a^b f dg = \int_a^c f dg + \int_c^b f dg. \quad (6)$$

(b) Let f be integrable with respect to g on the interval $[a, b]$ and let c satisfy $a \leq c \leq b$. Then f is integrable with respect to g on the subinterval $[a, c]$ and $[c, b]$ and formula (6) holds.

Proof. See Bartle's (1976) proof of Theorem 29.6.

3. Integration by parts

Theorem. A function f is integrable with respect to g over $[a, b]$ if and only if g is integrable with respect to f over $[a, b]$. In this case,

$$\int_a^b f dg + \int_a^b g df = f(b)g(b) - f(a)g(a). \quad (7)$$

Proof. See Bartle's (1976) proof of Theorem 29.7

T.3 Properties of Convexity.

Definition A.6. A real valued function ϕ defined on the interval (a, b) , $-\infty \leq a < b \leq \infty$, is called convex if the inequality

$$\phi(\alpha x + (1 - \alpha)y) \leq \alpha\phi(x) + (1 - \alpha)\phi(y)$$

holds whenever $a < x < b$, $a < y < b$ and $\alpha \in [0, 1]$.

Proposition A.7. *If ϕ is a real-valued function, then the following statements are equivalent*

- (i) $\phi(\alpha x + (1 - \alpha)y) \leq \alpha\phi(x) + (1 - \alpha)\phi(y)$.
- (ii) For every $x_1 < x_2 < x_3$, $(x_3 - x_1)\phi(x_2) \leq (x_3 - x_2)\phi(x_1) + (x_2 - x_1)\phi(x_3)$.
- (iii) The derivative ϕ' is a monotonically increasing function, that is, if $a < s < t < b$, $\phi'(s) \leq \phi'(t)$.

Proof. (i) $\Rightarrow$ (ii) : Write $x_2 = \alpha x_1 + (1 - \alpha)x_3$ for $\alpha \in [0, 1]$ or $\alpha = \frac{x_2 - x_3}{x_1 - x_3}$. Therefore,

$$\phi(x_2) = \phi(\alpha x_1 + (1 - \alpha)x_3) \leq \alpha\phi(x_1) + (1 - \alpha)\phi(x_3)$$

by (i). Multiply both sides by $(x_3 - x_1)$ and substitute the value of α to give

$$(x_3 - x_1)\phi(x_2) = (x_3 - x_2)\phi(x_1) + (x_2 - x_1)\phi(x_3)$$

as desired.

(ii) $\Rightarrow$ (i) : Set $x_2 = \alpha x + (1 - \alpha)y$ so $x < x_2 < y$. Then $\phi(\alpha x + (1 - \alpha)y) = \phi(x_2) \leq \alpha\phi(x) + (1 - \alpha)\phi(y)$ by (ii).

(i) $\Leftrightarrow$ (iii): If $x < t < y$, then the point $(t, \phi(t))$ should lie below or on the line connecting the points $(x, \phi(x))$ and $(y, \phi(y))$ in the plane. So (i) is equivalent to the requirement that

$$\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(t)}{u - t} \quad (*)$$

whenever $a < s < t < u < b$. By the mean value theorem combined with (*) shows us immediately that a real differentiable function ϕ is convex if and only if $a < s < t < u < b$ implies $\phi'(s) \leq \phi'(t)$, that is, if and only if the derivative ϕ' is an monotonically increasing function. $\square$

Proposition A.8. *If ϕ is convex and ψ is convex and nondecreasing on the range of ϕ , then the composition $\psi \circ \phi$ is convex in (a, b) .*

Proof. If ϕ is convex, it means that $\phi((1-\alpha)x + \alpha y) \leq (1-\alpha)\phi(x) + \alpha\phi(y)$ whenever $\alpha \in [0, 1]$, $-\infty \leq a < b \leq \infty$, $a < x < b$ and $a < y < b$. Then

$$\begin{aligned} (\psi \circ \phi)[(1-\alpha)x + \alpha y] &= \psi[\phi[(1-\alpha)x + \alpha y]] \\ &\leq \psi[(1-\alpha)\phi(x) + \alpha\phi(y)] \\ &\leq (1-\alpha)\psi(\phi(x)) + \alpha\psi(\phi(y)) \\ &= (1-\alpha)(\psi \circ \phi)(x) + \alpha(\psi \circ \phi)(y). \end{aligned}$$

The second line follows since ϕ is convex and third line since ψ is convex and non-decreasing. Therefore, $\psi \circ \phi$ is convex.

Proposition A.9. *For $\phi > 0$, if the $\log \phi$ is convex on (a, b) then ϕ is convex on (a, b) .*

Proof. Assume that $\log \phi$ convex. Using the notation of Proposition A.8, take ψ to be the exponential function on $(-\infty, \infty)$ (since the exponential function is nondecreasing and convex). Then by Proposition A.8, it follows that $\exp(\log \phi) = \phi$ is convex on (a, b) . $\square$

Theorem A.10. *If ϕ is convex on (a, b) , then ϕ is continuous on (a, b) .*

Proof. See the proof of Theorem 3.2, page 61 of Rudin (1987).

Theorem A.11. Jensen's inequality *If ϕ is continuous and convex function, f and $p \geq 0$ are defined on $[a, b]$, f is measurable and finite almost everywhere, and p, pf are summable then*

$$\phi\left[\frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx}\right] \leq \frac{\int_a^b \phi[f(x)]p(x)dx}{\int_a^b p(x)dx}.$$

Proof. See the proof of Theorem 6 on p.46 Natanson (1955) Volume 2.

T.4 Economic and Related Material.

This is a collection of financial terms which appear in the thesis.

Arbitrage is a riskless plan to make money without risk.

A financial model is *complete* if every contingent claim is attainable.

A *contingent claim* is a contract or agreement which pays a certain amount, say M , at time T if a certain event occurs.

Limited liability means that the investor cannot lose more than the amount he has paid for his shares.

A call option is *out of the money* if at time t the stock price $S(t)$ starts below the discounted exercise price, $e^{-r(T-t)}K$, and it is *in the money* if $S(t)$ starts above $e^{-r(T-t)}K$.

A trading strategy is *self-financing* if the changes of your initial portfolio value is due only to capital gains. You are not allowed to add or subtract funds from this initial portfolio fund.

A *trading strategy* is a set of instructions specifying what to buy or sell at each time upon each possible development.

Black and Scholes formula for the European call option is

$$c(S, \tau; K) = S\Phi(d_1) - K\Phi(d_2),$$

where Φ is the cumulative standard normal distribution function, S is the stock price, σ^2 is the instantaneous variance of the return on the stock price,

$$d_1 \equiv \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\tau^{1/2}},$$

r is risk-free interest rate, and $d_2 \equiv d_1 - \sigma\tau^{1/2}$ (cf. Merton (1973)).

T.5 Miscellaneous.

Theorem A.12 (Dudley (1973)). *Let $X = \{X(t) : t \geq 0\}$ be a mean zero Gaussian process. Let $d(s, t) := [E(X(t) - X(s))^2]^{1/2}$. Then X has almost all sample paths continuous if*

$$\int_0^1 \{\log N(x)\}^{1/2} dx < \infty$$

where for any $\epsilon > 0$,

$$N(\epsilon) := N([0, T], d, \epsilon) := \inf\{n : \exists A_1, \dots, A_n : [0, T] \subset \cup_{j=1}^n A_j, \\ \sup_{s, t \in A_j} d(s, t) \leq 2\epsilon, j = 1, \dots, n\}.$$

That is, a useful sufficient condition for sample continuity can be given by $N(\epsilon)$, the metric entropy condition.

Theorem A.13 (Thm 3 of Kawada and Kôno (1973)). *Let $\{X(t, \omega); 0 \leq t \leq 1\}$ be a centered Gaussian process with continuous sample functions. Assume that there exists a continuous function $\sigma(t)$ with $\sigma(0) = 0$ such that (i) $E[(X(s) - X(t))^2] \leq \sigma^2(|t - s|)$, (ii) $\sigma(t)$ is a non-decreasing regular varying function (iii) $\sigma(t)\sqrt{2 \log \log 1/t}$ is strictly increasing near the origin.*

Denote by Π the family of all finite partitions of $[0, 1]$ and $Q(\delta)$ the family of all finite partitions $P = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ such that $\max |t_i - t_{i-1}| \leq \delta$.

Set $\Phi(t) =$ the inverse function of $\sigma(t)\sqrt{2 \log \log 1/t}$ near the origin. Then we have, under assumptions (i)-(iii),

$$\lim_{\delta \rightarrow 0} \sup_{P \in Q(\delta)} V_\Phi(X; P) \leq 1$$

holds with probability 1.

Theorem A.13b (Thm 2 of Kawada and Kôno (1973)). *Let $\{X(t; \omega); 0 \leq t \leq 1\}$ be a centered Gaussian process with continuous sample functions. If a continuous function $\Phi(t)$ satisfies the inequality*

$$\Phi(at + bs) \leq C_1(a)\Phi(t) + C_2(b)\Phi(s),$$

for any constants a and b , then either

$$\sup_{P \in \Pi} V_{\Phi}(X; P) = +\infty \quad \text{with probability 1}$$

or

$$\sup_{P \in \Pi} V_{\Phi}(X; P) < +\infty \quad \text{with probability 1 holds.}$$

Theorem A.14 (Heine-Borel). *If the interval $[a, b]$ is covered by an infinite system $\mathcal{M}$ of open intervals, then it is possible to select a finite subsystem $\mathcal{M}^*$ of $\mathcal{M}$ which also covers $[a, b]$.*

Proof. See Proof of Theorem 7, §2.2, of Natanson (1955).

Let T be either $\mathbb{R}$, $\mathbb{R}^+ = \{t : t \geq 0\}$ or $\{t : t > 0\}$.

Definition A.15. (cf. Samorodnitsky and Taqqu, 1994) The real valued process $\{X(t), t \in T\}$ is *self similar* with index $H > 0$ (H - ss) if for all $a > 0$, the finite dimensional distributions of $\{X(at), t \in T\}$ are identical. That is, for any $d \leq 1$, $t_1, \dots, t_d \in T$, and any $a > 0$

$$(X(at_1), X(at_2), \dots, X(at_d)) \stackrel{D}{=} (a^H X(t_1), \dots, a^H X(t_d)) \quad (*)$$

Assume $H > 0$, we can express * as

$$\{X(at) : t \in T\} \stackrel{D}{=} \{a^H X(t), t \in T\}.$$

Theorem A.16. *If f is a positive and continuous functions on $[a, b]$ then $\delta := \inf_{x \in [a, b]} f(x) > 0$.*

Proof. FACT: under these condition, there exists an $\alpha \in [a, b]$ such that $\inf_{x \in [a, b]} f(x) = f(\alpha)$.

Suppose $\inf f(x) = 0$. So by the fact, there exists a x_0 such that $f(x_0) = 0$. This is a contradiction.

We want to prove that there exists an $\alpha \in [a, b]$ such that $c = f(\alpha)$ where $c = \inf_{x \in [a, b]} f(x)$.

This means that for every $\epsilon > 0$, there exists a $x \in [a, b]$ such that $f(x) = c + \epsilon$. Set $\epsilon = 1/n$.

FACT: If the sequence $\{x_n : n \geq 1\}$, $\lim_{n \rightarrow \infty} x_n = \alpha$, $\{x_{n_k}\}$ is a subsequence $\{x_n : n \geq 1\}$ then $\lim_{k \rightarrow \infty} x_{n_k} = \alpha$. (cf Stromberg (1980), page 52).

Then $f(\alpha) = c$. Suppose not. Therefore $f(\alpha) - c = \epsilon_0 > 0$. So there exists a δ_0 such that $|f(\alpha) - f(x)| < \epsilon_0/2$ whenever $|x - \alpha| < \delta_0$. That is, for all x ,

$$f(\alpha) - \epsilon_0/2 \leq f(x) \leq f(\alpha) + \epsilon_0/2.$$

Choose $x_n \in [a, b]$ so that (*) is true. Then we have

$$f(\alpha) - \epsilon_0/2 \leq f(x_n) \leq f(\alpha) + \epsilon_0/2.$$

Then $f(\alpha) - \epsilon_0/2 \leq c + 1/n$ or $c + \epsilon_0/2 \leq c + 1/n$. Now letting $n \rightarrow \infty$ we have that $\epsilon_0/2 \leq 0$ which contradicts our assumption. Therefore $f(\alpha) = c$ and $c = \inf_x f(x)$. $\square$

Let $Q([a, b])$ denote the set of all partitions κ of the interval $[a, b]$ and let λ be a refinement of κ . Let $v_p(f; \kappa) := \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p$ and let $v * _p(f; \kappa) := \sup\{v_p(f; \lambda) : \lambda \in Q([a, b]), \lambda \supset \kappa\}$.

Define

$$V_p^*(f; [a, b]) := \inf_{\kappa \in Q([a, b])} v_p^*(f; \kappa)^{1/p}.$$

Lemma 2.3 of Dudley and Norvaiša (1997). *If $V_p^*(f, [a, b]) < \infty$, then $f \in \mathcal{W}_p([a, b])$. Moreover,*

$$V_p^*(f; [a, b]) := \inf \left\{ \sum_{i=1}^n v_p(f; [x_{i-1}, x_i]); \{x_i : i = 0, \dots, n\} \in Q([a, b]) \right\}.$$

Lemma 2.4 of Dudley and Norvaiša (1997). *Every function with finite p -variation, $0 < p < \infty$ is regulated.*

Theorem 2.11 of Dudley and Norvaiša (1997). *Let $B = (B, \|\cdot\|)$ be a Banach space, $a < b$, $0 < p < 1$ and $f \in \mathcal{W}_p([a, b])$. Then f is a pure jump function: for $a < x \leq b$,*

$$f(x) = f(a) + \Delta^+ f(a) + \sum_{a < y < x} \Delta^\pm f(y) + \Delta^- f(x),$$

where the series converges absolutely.

Theorem 5.21 of Dudley and Norvaiša (1997). *Let B be a Banach algebra with unit and let $f \in \mathcal{W}_p([a, b]; B)$ for $0 < p < 2$. Then the indefinite integral $E_a(f)$ exists, is in $\mathcal{W}_p([a, b]; B)$ and is the unique solution in $\mathcal{W}_r([a, b]; B)$ for any $r \geq p$ such that $1/p + 1/r > 1$, of the forward integral equation*

$$F(y) = 1 + (LY) \int_a^y dfF,$$

where $(LY) \int$ refers to Left-Young integral (cf. Definition A.2 in T.1).

T.6 Stricker's (1990) Result.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with the filtration $\mathcal{F} = \{F_t : 0 \leq t \leq 1\}$ satisfying the usual conditions and let $X = \{X_t : 0 \leq t \leq 1\}$ be $\mathcal{F}$ -adapted stochastic process with values in $\mathbb{R}^d$ and almost all (a.a.) right continuous with left limits sample paths. Given $1 \leq p < \infty$, we say that $(X, P) \in \mathcal{M}^p$ if

- (1) there exists a probability measure Q equivalent to P such that the density of $dQ/dP \in L^q$ with $1/p + 1/q = 1$
- (2) X is a martingale with respect to Q , that is, $E_Q(X_t | F_s) = X_s$ a.s for $0 \leq s \leq t \leq 1$.

Let $K = \{\int_0^1 H dX : H \text{ is predictable, bounded and simple}\}$ and the closure of K is denoted by $\bar{K}$.

That is, $H = \sum_{i=1}^n \lambda_i 1_{[t_i, t_{i+1}]}$ where $\lambda_i = (\lambda_i^1, \dots, \lambda_i^d)$ is F_{t_i} -measurable such that

$$\int_0^1 H dX = \sum_{i=0}^{n-1} (\lambda_i | X_{t_{i+1}} - X_{t_i}) = \sum_{j=1}^d \left(\sum_{i=0}^{n-1} \lambda_i^j (X_{t_{i+1}} - X_{t_i}) \right) = \sum_{j=1}^d \int_0^1 H^j dX^j$$

Theorem A.17 (Stricker (1990)). *Let $1 \leq p < \infty$. Let $X_t = (X_t)$ be a continuous stochastic process with values in $\mathbb{R}^d$, adapted to $\mathcal{F}$ and $X_t \in L^p$ for each $t \in [0, 1]$. Then $(X, P) \in \mathcal{M}^p$ if and only if $\tilde{K} \cap L_+^p = \{0\}$.*

Remark A.18 If $Y \in \tilde{K}$, then there exist a sequence of bounded simple, predictable processes (H^n) such that

$$(H^n \cdot X)_1 = \int_0^1 H^n dX \rightarrow Y.$$

In general, Y may not be represented as $\int_0^1 H dX$. This is so in the following case. Let Q be an equivalent measure to P with density $Z = dQ/dP \in L^q$ such that X is a martingale under Q . Then $\int_0^1 H^n dX$ converges under $L^1(Q)$ to $\int_0^1 H dX$, so one can apply the above theorem or Corollary 2.5.2 of Yor (1978) so that there exist a predictable process $H \in \mathbb{R}^d$ such that $\int_0^1 H dX = Y$. However,

Proposition A.18 (Proposition 2 of Stricker (1990)). *Let $G \subset F$ and let X be a random vector with $X \in G$ under $\mathbb{R}^d$. If (H^n) are also random vector with $H^n \in G$ under $\mathbb{R}^d$ such that $(H^n|X)$ converges under L^0 to Y , then there exist a random vector $H \in G$ under $\mathbb{R}^d$ such that $Y = (H|X)$.*

T.7 Results used for the SLSG strategy.

Before we proceed, we show in Example 1 that functions of bounded p -variation with $p \geq 1$ can have the property of infinite crossing of zero. Recall the definition of Banach indicatrix (cf. Natanson (1955, Vol 1, p 225)).

Definition. Let f be continuous and defined on $[a, b]$, and let $m = \min[f(x)]$ and $M = \max[f(x)]$ for all $x \in [a, b]$. Define the function, N_f , on $[m, M]$ such that $N_f(y)$ is the number of roots of the equation $f(x) = y$ for all $y \in [m, M]$. That is,

$$N_f(y) = \#\{x : f(x) = y\}.$$

The function N_f is called the *Banach indicatrix*. If the set of these roots is infinite, then $N_f(y) = +\infty$.

Example 1. Define the function f as

$$\begin{aligned} f(x) &= x^2 \sin[\pi/x] & 0 < x \leq 1 \\ &= 0 & x = 0. \end{aligned}$$

This function has bounded variation but $N_f(0) = +\infty$.

The following notation is used for the next two theorems. Let $a < b$ be real numbers. A real-valued function f on a closed interval $[a, b]$ is called *regulated* if it has a left limit at each point of $(a, b]$ and a right limit at each point of $[a, b)$, and we will denote such a function by $f \in \mathcal{R}([a, b])$.

Consider a function ϕ on $[c, d]$ having a left derivative $D^-\phi$ on $(c, d]$, where $D^-\phi(y) \equiv \lim_{\epsilon \downarrow 0} \frac{\phi(y) - \phi(y-\epsilon)}{\epsilon}$. Let f be a continuous function on $[a, b]$ which is not constant on the whole interval, and the range of f is contained in the open interval (c, d) . Write $\phi \in \mathcal{DR}(c, d)$ if $D^-\phi$ is a regulated function. If f is also a regulated function on $[a, b]$, then we set

$$\Lambda(\phi, D^-\phi, f) := \sum_{(a, b]} [\Delta^-(\phi \circ f) - (D^-\phi \circ f) \Delta^- f] + \sum_{[a, b)} [\Delta^+(\phi \circ f) - (D^-\phi \circ f) \Delta^+ f] \quad (4.7.1)$$

whenever the two sums converge absolutely. Here $\Delta^- f$ and $\Delta^+ f$ denote a left- and right-sided jumps of a regulated function f , respectively, and are zero everywhere except at countably many points. The next two chain rules for the composite $D^-\phi \circ f$, where ϕ is a convex function and $f \in \mathcal{W}_p$ are proved by Norvaiša (1997a).

Theorem A.19. *Let $\phi \in \mathcal{DR}([c, d])$ and let f be a function of bounded variation on $[a, b]$. Then the composition $D\phi \circ f$ is Y -integrable, the sum Λ as defined by converges absolutely and the chain rule holds:*

$$\phi(f(b)) - \phi(f(a)) = (Y) \int_a^b D^-\phi \circ f df + \Lambda(\phi, D^-\phi, f), \quad (4.7.2)$$

where $(Y) \int$ denotes Young's integral (cf. Definition A.3 in T.1).

Proof. See the proof of Theorem 1.2 of Norvaiša (1997).

Let f be a continuous function on $[a, b]$ which is not constant on the whole interval. A point $x \in [a, b]$ is called a *point of constancy* of the function f if a neighbourhood $[x - \delta, x + \delta]$, $\delta > 0$, can be found such that f has the same value at all points in this neighbourhood. In the interval $[a, b]$, there may be finite or at most countably many nonoverlapping intervals of constancy (cf. Hobson (1957, p 348)). The next theorem is the best known result at this time.

Theorem A.20. *Let $1 < p < 2$ and let f be a continuous function of bounded p -variation on the interval $[a, b]$ which has at most finitely many points of constancy. Let ϕ be a convex function on $[c, d]$ such that for a constant $A (< \infty)$ and a number $r \leq 1$*

$$\sum_{i=1}^n |D^- \phi(f(x_i) + y) - D^- \phi(f(x_{i-1}) + y)|^r \leq A^r, \quad (4.7.3)$$

for all y and for all $a = x_0 < x_1 < x_2 < \dots < x_n = b$.

Then the composite function $(D^- \phi) \circ f$ is R - S integrable with respect to f and the relation

$$\phi(f(b)) - \phi(f(a)) = (RS) \int_a^b (D^- \phi) \circ f df.$$

holds.

Proof. See the proof of Theorem 1.4 Norvaiša (1997). $\square$

Remark. If f is a monotone function, then (4.7.3) holds with $r = 1$ because $D^- g$ is nondecreasing.

The definition of Young's integral (ie. $(Y) \int$) is given in T.1.

Theorem A.21. *If $\int_a^b f dg$ exists, then $(Y) \int_a^b f dg$ exists and the two integrals are the same.*

Proof. See the proof of Theorem 19.3.3, Hildebrandt (1963, p 89). $\square$

Theorem A.22. *If $(Y) \int_a^b fdg$ exists, and if g is continuous, then $\int_a^b fdg$ exists.*

Proof. See the proof of Theorem 19.3.4, Hildebrandt (1963, p 89). $\square$

Here, we have heavily used Karatzas and Shreve (1991) as our reference to semimartingale local times.

Theorem A.24. *Let X be a continuous semimartingale of the form $X_t = X_0 + M_t + V_t$ for $0 \leq t < \infty$ on some probability space $(\Omega, \mathcal{F}, P)$. There exists then a semimartingale local time for X , ie., a nonnegative random field $\Lambda = \{\Lambda_t(a, \omega); (t, a) \in [0, \infty) \times \mathbb{R}, \omega \in \Omega\}$ such that the following hold:*

- (1) *The mapping $(t, a, \omega) \mapsto \Lambda_t(a, \omega)$ is measurable and, for each (t, a) , the random variable $\Lambda_t(a)$ is $\mathcal{F}_t$ -measurable*
- (2) *For every fixed $a \in \mathbb{R}$, the mapping $t \mapsto \Lambda_t(a, \omega)$ is continuous and non-decreasing with $\Lambda_0(a, \omega) = 0$, and*

$$\int_0^\infty 1_{\mathbb{R} \setminus \{a\}}(X_t(\omega)) d\Lambda_t(a, \omega) = 0, \quad \text{for } P\text{-a.e. } \omega \in \Omega. \quad (4.7.4)$$

- (3) *For every Borel-measurable $k : \mathbb{R} \rightarrow [0, \infty)$, the identity*

$$\int_0^t k(X_s(\omega)) d\langle M \rangle_s(\omega) = 2 \int_{-\infty}^\infty k(a) \Lambda_t(a, \omega) da; \quad 0 \leq t < \infty$$

holds for P -a.e. $\omega \in \Omega$.

- (4) *For P -a.e. $\omega \in \Omega$, the limits*

$$\lim_{\substack{r \rightarrow t \\ b \downarrow a}} \Lambda_r(b, \omega) = \Lambda_t(a, \omega) \quad \text{and} \quad \Lambda_t(a-, \omega) \equiv \lim_{\substack{r \rightarrow t \\ b \uparrow a}} \Lambda_r(b, \omega)$$

exists for all $(t, a) \in [0, \infty) \times \mathbb{R}$.

- (5) *For every convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, we have the generalized change of variable formula*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t D^- f(X_s) dM_s + \int_0^t D^- f(X_s) dV_s \\ &\quad + \int_{-\infty}^\infty \Lambda_t(a) \mu(da); \quad 0 \leq t < \infty, \quad \text{a.s. } P, \end{aligned} \quad (4.7.5)$$

where D^-f is the left derivative and μ is the second derivative measure

$$\mu([a, b]) \equiv D^-f(b) - D^-f(a); \quad -\infty < a < b < \infty.$$

Proof. See Karatzas and Shreve (1991), p 223.

Lemma A.25. *The semimartingale local time of a continuous process of bounded variation is identically zero.*

Proof.

If X, Y be continuous semimartingales of the form $X_t = X_0 + M_t + V_t$ and $Y_t = Y_0 + N_t + U_t$ for $t \in [0, \infty)$, then

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle M, N \rangle_t. \quad (4.7.6)$$

Set $X_0 = Y_0 = 0$ and $X_t = Y_t$ for all t . So (4.7.6) becomes

$$\int_0^t X_s dX_s = X_t^2 - \int_0^t X_s dX_s - \langle M, M \rangle_t.$$

(Recall that $\langle M, M \rangle_t = \langle M \rangle_t$.) This implies that

$$\langle M \rangle_t = X_t^2 - 2 \int_0^t X_s dX_s. \quad (4.7.7)$$

Now, if $f(x) = x^2$, then (4.7.5) becomes

$$\begin{aligned} X_t^2 - X_0^2 &= 2 \int_0^t X_s dM_s + 2 \int_0^t X_s dV_s + 2 \int_{-\infty}^{\infty} \Lambda_t(x) dx \\ &= 2 \int_0^t X_s dX_s + 2 \int_{-\infty}^{\infty} \Lambda_t(x) dx. \end{aligned}$$

With $X_0 = 0$, we have

$$X_t^2 - 2 \int_0^t X_s dX_s = 2 \int_{-\infty}^{\infty} \Lambda_t(x) dx. \quad (4.7.8)$$

By (4.7.7), left hand side of (4.7.8) is $\langle M \rangle_t$. Hence,

$$\langle M \rangle_t = 2 \int_{-\infty}^{\infty} \Lambda_t(x) dx.$$

If your semimartingale is only a continuous process of bounded variation (ie., $M_t = 0$ for all t), then $0 = \langle M \rangle_t$. So

$$0 = 2 \int_{-\infty}^{\infty} \Lambda_t(x) dx. \quad (4.7.9)$$

Hence, the semimartingale local time is identically zero. $\square$

Lemma A.26. *Every function of bounded variation is Borel-measurable.*

Proof. Borel sets are generated by intervals $I = [a, \infty]$, so it is enough to check the inverse image of such intervals under the function f . If f is monotone, then the inverse image of I under f , is again an interval, so Borel. So every monotone function is Borel-measurable. Differences of monotone functions are Borel-measurable. So every function of bounded variation is Borel-measurable. $\square$

T.8 Mathematica simulation programs.

Source: Maeder, RE (1995) Fractional Brownian Motion. *Mathematica Journal*, 6 (1), 38-48.

```

randDir := 2 Random[Integer] -1

randWalk[n_] := NestList[# + randDir&, 0, n-1]

Needs["Statistics`NormalDistribution`"]

With[{nd = NormalDistribution[], pi2 = N[2Pi]},
  gauss := Random[nd]
]

inter[x_,m_] :=
  With[{interpol = Interpolation[x, InterpolationOrder -> 1],
    lx = Length[x]},
    interpol[Range[1, lx, (lx -1)/(m-1) ]]
  ]

fbmRA[e_Integer, args_] :=
fbmRA[ Array[gauss/2&, Table[2, {e}]] args ]

interpolate[x_, ndims_List] :=
  With[ {e = TensorRank[x]},
    With[{trans = RotateRight[Range[e]]},
      Fold[ Map[Function[slice, inter[slice, #2]],
        Transpose[#1, trans], {e-1} ]&,
        x, ndims]
    ]
  ]

fbmRAStep[mat_List, h_, r_:1/2] :=
  Module[{t,T, oddims,ndims},
    oddims = Dimensions[mat];
    ndims = Ceiling[oddims/r];
    t = 1.0 /(Max[ndims] -1); T = 1.0 /(Max[oddims]-1);
    With[{delta = Sqrt[0.5] Sqrt[1 - (t/T)^(2 - 2h)] t^h},
      Map[delta gauss + # &, interpolate[mat, ndims], {-1}]
    ]
  ]

fbmRA[mat_List, n_Integer, h_, r_:1/2] :=
  Module[{x=mat},
    While[ Length[x] < n,
      x= fbmRAStep[x,h, Max[r, Length[x]/n]]
    ];
    x
  ]

fbmRA [ (g:(SurfaceGraphics|ContourGraphics|DensityGraphics)) [
  mat_, opts___], args_] :=
  g[ fbmRA[mat,args], opts ]

plots =
  Table[ ListPlot[Exp[fbmRa[1,1024, h]],
```

```

PlotJoined -> True, DisplayFunction -> Identity,
Axes -> None, PlotRange -> All, AspectRatio -> 0.3,
PlotLabel -> StringForm["h= '1'", h]],
{h, 0.6, 0.45, -.05} ];

Show[ GraphicsArray[Partition[plots, 1]],
GraphicsSpacing -> 0.5];

-----

Gladyshev's statistics:
-----

data1 = fbmRA[1, 32769, 0.75];

Length[data1]

delta1[data1_, m_Integer] := Sum[(data1[[i]] - data1[[i+2^m]])^2,
{i, 1, (Length[data1] -1), 2^m}]

deltas[m_] := Table[{m, delta1[data1,m]}, {m,0,15}]

deltas1[m_] := Table[{m, (1 - (1/15-m) Log[2, delta1[data1,m]])}, {m, 0, 14}]

Save["results",deltas,deltas1]

```

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