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T_A has non-zero preprojective and preinjective direct summands.

This thesis contributes to that question by describing a class of tilting modules T_A over a wild hereditary algebra A which have the property that $B = \text{End } T_A$ has a proper hereditary factor algebra of wild representation type. The description is given by a construction process of a tilting module. This process is explained for an arbitrary algebra in chapter IV and studied for hereditary algebras in chapter V.

Chapter I - III provides the background of representation theory. Chapter I the theory of Auslander-Reiten, chapter II tilting theory and chapter III the theory of hereditary algebras. Most of the results are only stated and the reader is referred to the given literature for the proofs. Although there are many links between these theories, influencing the development of each other quite a bit, each stands on its own.

Bongartz showed in [Bo] that every module S_A satisfying T 1 and T 2 is a direct summand of a tilting module. In chapter IV this extension is studied under the additional assumption that $S = \bigoplus_{k=1}^n S_k$ with

- iii) Each $\text{End } S_k$ is a division ring,
- iv) $\text{Hom}_A(S_k, S_l) = 0$ for all $k \neq l$;
- v) $\text{Hom}_A(S, A) = 0$.

These conditions were studied in an unpublished paper by W. Geigle [Ge]. They are easily satisfied in the hereditary case. The resulting tilting module $T(S) = V \oplus S$ is given by a short exact sequence

$$0 \rightarrow A_A \rightarrow V' \rightarrow S' \rightarrow 0$$

with $\text{add } S' = \text{add } S$ and $\text{add } V' = \text{add } V$, where V is the minimal Ext-projective generator of $A(S)$. $C(S) = \text{End } V$ has a simple modules less than A , satisfies $\text{gl d } C(S) \leq \text{gl d } A$ and its module category is naturally equivalent to $A(S)$ via the tilting functors

TILTING MODULES OVER WILD HEREDITARY ALGEBRAS

by

Hubertus Strauß

A thesis submitted to

the Faculty of Graduate Studies and Research

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

November, 1986

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Tilting Modules over Wild Hereditary Algebras

submitted by Hubertus Strauß, Dipl. Math., in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

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November, 1986

ABSTRACT

Let A be a finite dimensional algebra over a field k with precisely n non-isomorphic simple modules. A module T_A is called a tilting module if

- i) $\text{p.d. } T_A \leq 1$;
- ii) $\text{Ext}_A^1(T, T) = 0$,
- iii) There exists a short exact sequence $0 \rightarrow A_A \rightarrow T' \rightarrow T'' \rightarrow 0$
with $T', T'' \in \text{add } T$.

If a module $S = \bigoplus_{k=1}^s S_k$ satisfies

- i) $\text{p.d. } S_A \leq 1$,
- ii) $\text{Ext}_A^1(S, S) = 0$,
- iii) $\text{End } S_k$ is a division algebra for $1 \leq k \leq s$,
- iv) $\text{Hom}_A(S_k, S_l) = 0$ for $1 \leq k \neq l \leq s$,
- v) $\text{Hom}_A(S, A) = 0$,

then there exists a tilting module $T(S) = V \oplus S$. The algebra $C(S) = \text{End } V_A$ is the main object of this thesis. It enjoys the following properties

- i) $\text{mod } C(S)$ is naturally equivalent to the full subcategory $\mathcal{A}(S)$ of $\text{mod } A$ consisting of all $X \in \text{mod } A$ satisfying $\text{Hom}_A(S, X) = 0$ and $\text{Ext}_A^1(S, X) = 0$.
- ii) $\text{gl.d. } C(S) \leq \text{gl.d. } A$.
- iii) $B(S) := \text{End } T(S)_A$ is an s -point extension of $C(S)$.
- iv) If $T = \bigoplus_{k=1}^{n-s} T_k^{t_k} \oplus \bigoplus_{k=1}^s S_k^{s_k}$ is a tilting module having S as a sink summand, then $\text{Hom}_A(T, T) = W \oplus \bigoplus_{k=1}^s (e_k B(S))^{s_k}$ is a tilting module over $B(S)$ with W a tilting module over $C(S)$.

If A is hereditary and S is a regular A -module without self-extensions, then $C(r^z S)$ is well-defined for each integer z and the underlying valued graphs of $C(S)$ and $C(r^z S)$ coincide.

i.e. the algebras are "almost identical" from a representation theory point of view. If A is hereditary and S_A is an indecomposable regular A -module without self-extensions, then

$$C(S) = C^{fin}(S) \times \underline{C^{inf}(S)},$$

where $C^{fin}(S)$ is the path algebra of the quiver $\bullet \leftarrow \bullet \leftarrow \dots \leftarrow \bullet \leftarrow \bullet$ on $(ql(S) - 1)$ points over the division algebra $End S$, while $C^{inf}(S)$ inherits the representation type and indecomposability from A .

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TABLE OF CONTENTS

Acknowledgements	
Table of contents	
Introduction	1
Preliminaries	5
Chapter I: Elements of the Auslander-Reiten theory	10
Chapter II: Elements of tilting theory	18
Chapter III: Hereditary algebras	24
Chapter IV: The general construction	31
Chapter V: The hereditary case	53
References	68

INTRODUCTION

Bringing to an end (at least for the time being) a development highlighted by the names of Gelfand-Ponomarev (1970), Bernstein-Gelfand-Ponomarev (1973), Dlab-Ringel [DR 2, 1976], Auslander-Platzek-Reiten [APR, 1979] and Brenner-Butler [BB 1, 1980], Happel-Ringel introduced in [HR 1, 1982] the concept of a tilting module. A tilting module T_A is an A -module satisfying

$$T\ 1: \text{p.d. } T \leq 1;$$

$$T\ 2: \text{Ext}_A^1(T, T) = 0;$$

T 3: There exists a short exact sequence of A -modules

$$0 \rightarrow A_A \rightarrow T' \rightarrow T'' \rightarrow 0 \quad \text{with } T', T'' \in \text{add } T$$

Tilting theory relates the representation theory of A and that of $B = \text{End } T_A$. The main result is that $\mathcal{G}(T)$ and $\mathcal{Y}(T)$ are equivalent as well as $\mathcal{F}(T)$ and $\mathcal{X}(T)$ via $\text{Hom}_A(T, -)$, $- \otimes_B T$ and their derived functors. When A is hereditary, B is called a tilted algebra. In that case every indecomposable B -module is obtained either as an image of $\text{Hom}_A(T, -)$ or $\text{Ext}_A^1(T, -)$.

In contrast to its predecessors, the reflection at a sink for a tensor algebra [DR 2] and the process studied in [APR], the general tilting process might change the representation type. There are examples of wild hereditary algebras, which admit tilted algebras of finite representation type as well as of tame type. It is also known that some wild hereditary algebras do not have tilted algebras of finite representation type. Happel-Ringel showed in [HR 2] that if $B = \text{End } T_A$ is a tilted algebra of finite representation type, then T_A has non-zero preprojective and preinjective direct summands. Moreover, they showed that this condition is sufficient in case A is of tame representation type. It is natural to ask whether any change of the representation type of $B = \text{End } T_A$ implies necessarily that

T_A has non-zero preprojective and preinjective direct summands.

This thesis contributes to that question by describing a class of tilting modules T_A over a wild hereditary algebra A which have the property that $B = \text{End } T_A$ has a proper hereditary factor algebra of wild representation type. The description is given by a construction process of a tilting module. This process is explained for an arbitrary algebra in chapter IV and studied for hereditary algebras in chapter V.

Chapter I - III provides the background of representation theory: Chapter I the theory of Auslander-Reiten, chapter II tilting theory and chapter III the theory of hereditary algebras. Most of the results are only stated and the reader is referred to the given literature for the proofs. Although there are many links between these theories, influencing the development of each other quite a bit, each stands on its own.

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$F = \text{Hom}_A(T(S), -)$ and $- \otimes_B T$. The assumptions on S_A imply that $\tau_A S$ satisfies dual properties. This enables one to describe the minimal Ext-injective cogenerator W of $\mathcal{A}(S)$ in form of a short exact sequence

$$0 \rightarrow S' \rightarrow W' \rightarrow DA \rightarrow 0$$

with $\text{add } S' = \text{add } \tau_A S$ and $\text{add } W' = \text{add } W$. The indecomposable injective $C(S)$ -modules correspond to the indecomposable direct summands of W under the functor F .

The algebra $B(S) = \text{End}(T(S))$ has the property that every tilting module T having S as sink summand decomposes in $\text{mod}(B(S))$ as $FT = U \oplus FS$ with U a tilting module over $C(S)$. The last property, as obvious as it is, is a main point of chapter IV and the author is convinced that it will be useful in the future for induction arguments.

In chapter V two main results are derived

- 1) If for a hereditary algebra A the modules S_A and $\tau_A S$ satisfy the assumptions of chapter IV, then $C(S)$ and $C(\tau_A S)$ have the same valued graphs. In particular if S_A is a regular module, then the valued graph of $C(S)$ is an invariant of the τ_A -orbit of S .
- 2) For an indecomposable hereditary algebra A and an indecomposable regular module S_A without self-extensions, the "regular submodules" are recognized as direct summands of V . The algebra $C(S)$ decomposes into the direct product of the path algebra of the quiver

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \dots \leftarrow (ql(S) - 2) \leftarrow (ql(S) - 1)$$

and an algebra of infinite representation type, which inherits the representation type of A .

Hoshino's result, that the regular length of a module without self-extension is bounded by $n - 2$ for $n =$ number of simple A -modules [Ho], is now an easy consequence. 2) might be regarded as a generalization of Hoshino's result which extended Ringel's result that the regular representations of a bimodule have non-trivial self-extensions [Ri 1]

PRELIMINARIES

0.1 Notation. The symbol $:=$ defines the term of the left-hand side by the term of the right hand side. $\cong$ will denote isomorphism between groups, algebras, modules or vector spaces, the context will specify the structure. Algebras under consideration will be finite dimensional algebras over some field k and will have a unity. Modules will be unitary modules. To stress right or left action of the algebra A , we will write X_A or ${}_A X$, respectively. For an algebra A denote by

$Mod A$ the category of right A -modules;

$mod A$ the full subcategory of $Mod A$ consisting of all modules having finite composition length - these are precisely the modules which are finite dimensional over k ;

$A - Mod$ the category of left A -modules;

$A - mod$ the full subcategory of $A - Mod$ consisting of all modules having finite composition length.

$D := Hom_k(-, k)$ denotes the usual duality between $Mod A$ and $A - Mod$

$ind A$ denotes the set of isomorphism classes of indecomposable modules in A , where A is a subcategory of $mod A$ or $A - mod$.

For $S \in mod A$ full subcategories of $mod A$ are denoted as follows

$add S$ consisting of all modules which are isomorphic to a finite direct sum of direct summands of S ;

$\mathcal{F}(S)$ consisting of all $X \in mod A$ satisfying $Hom_A(S, X) = 0$;

$\mathcal{G}(S)$ consisting of all modules $X \in mod A$ satisfying $Ext_A^1(S, X) = 0$.

For a module X denote by $End X$ its endomorphism ring; by $rad X$, $Soc X$ and $top X := X/rad X$ the radical, socle and radical factor of X , respectively; by $p.d X$ and

$\text{p.d. } X$ the projective dimension and the injective dimension of X , respectively, by $|X|$ the composition length of X ; by $[X]$ the isomorphism class of X .

For a homomorphism $f: X \rightarrow Y$ the image, kernel and cokernel of f are denoted by $\text{im } f$, $\text{ker } f$ and $\text{coker } f$, respectively.

The global dimension of the algebra A is denoted by $\text{g.l.d. } A$.

Since A is finite dimensional, there is always a decomposition

$$(0.1.1) \quad A \cong \bigoplus_{k=1}^n P_k^{n_k}$$

with $P_1, P_2, \dots, P_n$ indecomposable and pairwise non-isomorphic. A is called indecomposable (as algebra), if there is no non-trivial partition $I \cup J = \{k \mid 1 \leq k \leq n\}$ with $I \cap J = \emptyset$ satisfying $\text{Hom}_A(P_i, P_j) = 0 = \text{Hom}_A(P_j, P_i)$ for all $i \in I$ and $j \in J$. This condition is satisfied if and only if A does not have non-trivial central idempotents.

An algebra A is said to be of finite representation type, if the set $\text{ind(mod } A)$ is finite; otherwise A is said to be of infinite representation type. A is called of strictly wild representation type, if there exists an exact embedding

$$\text{mod}(k \langle X, Y \rangle) \rightarrow \text{mod } A,$$

where $k \langle X, Y \rangle$ denotes the free associative (infinite dimensional) k -algebra in the non-commuting variables X and Y .

0.2 The Grothendieck group $G_0(A)$. $G_0(A)$, the Grothendieck group of the algebra A , is the abelian group generated by $\{|X| : X \in \text{mod } A\}$ subject to the relations

$$|Y| - |X| - |Z| = 0$$

for each short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$. If $A_A \cong \bigoplus_{k=1}^n P_k^{n_k}$, where P_k are indecomposable and pairwise non-isomorphic and $n_k \geq 1$ ($1 \leq k \leq n$), then $G_0(A)$ can be

identified with the free abelian group $\mathbb{Z}^n$ and $[X]$ with $\dim X \in \mathbb{Z}^n$ with k -th coordinate

$$(\dim X)_k := | \operatorname{Hom}_A(P_k, X)_{\operatorname{End} P_k} |$$

In order to have arguments from linear algebra at hand, we think of $\mathbb{Z}^n$ as embedded in F^n for a subfield $F \subseteq \mathbb{C}$; formally define $G_0(A, F) := G_0(A) \otimes_{\mathbb{Z}} F$

If for $X, Y \in \operatorname{mod} A$ $\operatorname{Ext}_A^i(X, Y) \neq 0$ for only finitely many indices i , then denote

$$\langle \dim X, \dim Y \rangle_A := \sum_{i \geq 0} (-1)^i \operatorname{Ext}_A^i(X, Y)$$

In particular if $\operatorname{gl} d A$ is finite, then

$$\langle -, - \rangle_A : G_0(A) \times G_0(A) \rightarrow \mathbb{Z}$$

is a bilinear form on $G_0(A)$. The associated quadratic form will be denoted by

$$q_A(\dim X) := \langle \dim X, \dim X \rangle_A$$

Set $Q_k = D\operatorname{Hom}_A(P_k, A)$, the indecomposable injective satisfying $\operatorname{Soc} Q_k = \operatorname{top} P_k$. If both $\{\dim P_k \mid 1 \leq k \leq n\}$ and $\{\dim Q_k \mid 1 \leq k \leq n\}$ are linearly independent, then there is an isomorphism of abelian groups or F -spaces

$$\Phi_A : G_0(A) \rightarrow G_0(A) \quad \text{with} \quad \Phi_A(\dim P_k) = \dim Q_k$$

Φ_A and Φ_A^{-1} are called the Coxeter transformations of A

0.3 Quiver; species; path algebra; tensor algebra. A quiver is defined to be a set of points I together with a set of arrows $i \leftarrow j$ for $i, j \in I$. Note that a quiver might have loops or multiple arrows. A path of length k from $j \in I$ to $i \in I$ is a sequence of arrows

$$i \xrightarrow{\alpha_1} i_1 \xrightarrow{\alpha_2} i_2 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{k-1}} i_{k-1} \xrightarrow{\alpha_k} j$$

Denote by $e_i := i \rightarrow i$ the empty path in $i \in I$, which is by definition a path. The path algebra $P(Q)$ of a quiver Q is defined in the following way: A k -basis is the set of all paths. The multiplication is induced by composing paths whenever this is possible. If I is finite, then $\{e_i : i \in I\}$ is a complete set of orthogonal primitive idempotents of $P(Q)$.

A k -species $S = (k_i, {}_iM_j)_{1 \leq i, j \leq n}$ consists of division rings k_i , each containing k in its center and finite dimensional over k ; and $k_i - k_j$ -bimodules ${}_iM_j$, each finite dimensional over k .

The quiver $Q(S)$ of a k -species S consists of points $\{i : 1 \leq i \leq n\}$ and arrows $i \rightarrow j$ whenever ${}_iM_j \neq 0$.

A representation $(V_i, {}_i\varphi_j)_{1 \leq i, j \leq n}$ of the k -species S consists of right k_i -spaces V_i and k_j -linear maps ${}_i\varphi_j : V_i \otimes_{k_i} {}_iM_j \rightarrow V_j$. A morphism

$$f = (f_i)_{1 \leq i \leq n} : (V_i, {}_i\varphi_j)_{1 \leq i, j \leq n} \rightarrow (V'_i, {}'_i\varphi_j)_{1 \leq i, j \leq n}$$

consists of k_i -linear maps $f_i : V_i \rightarrow V'_i$ such that ${}_i\varphi'_j (f_i \otimes 1_{{}_iM_j}) = f_j {}_i\varphi_j$. The representations of S form an abelian category, denoted by $\mathcal{R}(S)$.

The tensor algebra $\Theta(S) = \bigoplus_{k \geq 0} M^{(k)}$ of a k -species is defined in the following way: Consider the semi-simple algebra $\Lambda := \prod_{i=1}^n k_i$ and the $\Lambda - \Lambda$ -bimodules $M^{(0)} = \Lambda$, $M^{(1)} := \bigoplus_{1 \leq i, j \leq n} {}_iM_j$, and define inductively $M^{(k)} := M^{(k-1)} \otimes_{\Lambda} M^{(1)}$ for $k \geq 2$. $\Theta(S)$ becomes a k -algebra via the canonical maps $M^{(k)} \otimes_{\Lambda} M^{(l)} \rightarrow M^{(k+l)}$.

$\{e_i := 1_{k_i} \in M^{(0)} : 1 \leq i \leq n\}$ is a complete set of primitive orthogonal idempotents of $\Theta(S)$. The categories $\text{Mod } \Theta(S)$ and $\mathcal{R}(S)$ are equivalent for any k -species S , as has been proven in [DR 1]. In a way a representation gives a better view into the internal structure of a $\Theta(S)$ -module.

If all $k_i = k$ then the structure of $\Theta(S)$ is completely determined by the integers $d_{ij} := \dim_k {}_iM_j$. Put d_{ij} arrows $\alpha_{ij}^1, \alpha_{ij}^2, \dots, \alpha_{ij}^{d_{ij}} : i \rightarrow j$ in $Q(S)$. It is easy to see that

the algebras $\Theta(S)$ and $P(Q(S))$ are isomorphic.

Note that $\Theta(S)$ is in general not finite dimensional over k . In fact it is finite dimensional if and only if $Q(S)$ does not have oriented cycles. In that case in particular, $M_i \neq 0$ implies $M_i = 0$. Thus we can define a valuation on the arrows $i \xrightarrow{(d_{ij}, d_{ji})} j$ by setting

$$d_{ij} := \dim M_{ij,k}; \quad d_{ji} := \dim_{k_i} M_j; \quad f_i := [k_i : k]$$

These integers satisfy $d_{ii} = 0$ and $d_{ij} f_j = d_{ji} f_i$ for all $1 \leq i, j \leq n$. Thus,

$$(Q(S), (d_{ij})_{1 \leq i, j \leq n})$$

is a valued graph with an admissible orientation in the sense of Dlab-Ringel [DR 2].

0.4 Species and Gabriel quiver of an algebra. Let A be of a finite dimensional algebra and $A_A = \bigoplus_{k=1}^n P_k^{n_k}$ with $P_k, 1 \leq k \leq n$ indecomposable and pairwise non-isomorphic. Set $S_i = \text{top } P_i$. Then

$$S(A) := (End S_i, End S, Ext_A^1(S_j, S_i)_{End S_i})_{1 \leq i, j \leq n}$$

is called the k -species of A . $Q(A) := Q(S(A))$ is called the Gabriel quiver of A . Note that there is an arrow $i \leftarrow j$ in $Q(A)$ if and only if $\text{Hom}_A(P_i, \text{rad } P_j / \text{rad}^2 P_j) \neq 0$. Note also that $Q(A)$ might have loops or other oriented cycles, for example if A is a self-injective algebra.

Chapter I

ELEMENTS OF THE AUSLANDER-REITEN THEORY

This chapter provides the basic facts of the theory. The exposition follows [AR 1], where most of the proofs can be found. [Ga] and [Ri 3; 2.4] present most of the material as well.

1.1 AR sequences. A short exact sequence

$$\mathcal{E}: 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

is called an **Auslander-Reiten sequence**, in short **AR sequence**, or almost split sequence, if the following three conditions are satisfied:

AR 1: $\mathcal{E}$ does not split;

AR 2: X and Z are indecomposable;

AR 3: For each homomorphism $h: Z' \rightarrow Z$ which is not a splitting epimorphism, there exists $h': Z' \rightarrow Y$ such that $h = gh'$.

Theorem on existence of AR sequences. a) If $Z \in \text{mod } A$ is indecomposable non-projective, then there exists an AR sequence

$$\mathcal{E}: 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0.$$

$\mathcal{E}$ is uniquely determined up to isomorphism of short exact sequences, i.e. if

$$\mathcal{E}': 0 \rightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \rightarrow 0$$

is an AR sequence, then there exist isomorphisms $\alpha: X \rightarrow X'$ and $\beta: Y \rightarrow Y'$ such that $f'\alpha = \beta f$ and $g = g'\beta$.

b) If $X \in \text{mod } A$ is indecomposable and non-injective, then there exists an AR sequence

$$\mathcal{E}: 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

Again $\mathcal{E}$ is uniquely determined up to an isomorphism of short exact sequences

Characterization of AR sequences. For a short exact sequence

$$\mathcal{E}: 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

satisfying AR 1 and AR 2, the following statements are equivalent

- a) $\mathcal{E}$ is an AR sequence.
- b) If $h: Z' \rightarrow Z$ is a homomorphism but not a splitting epimorphism, then the short exact sequence $\text{Ext}_A^1(h, X)(\mathcal{E})$ splits.
- c) For any homomorphism $k: X \rightarrow X'$ which is not a splitting monomorphism, there exists $k': Y \rightarrow X'$ such that $k = k'f$.
- d) If $k: X \rightarrow X'$ is a homomorphism but not a splitting monomorphism, then the short exact sequence $\text{Ext}_A^1(Z, k)(\mathcal{E})$ splits.

Remark. c) and d) imply that the images of the connecting homomorphisms

$$\delta: \text{Hom}_A(-, Z) \rightarrow \text{Ext}_A^1(-, X) \text{ and } \delta': \text{Hom}_A(X, -) \rightarrow \text{Ext}_A^1(Z, -)$$

are simple subfunctors.

1.2 AR translations. a) Let

$$(1.2.1) \quad P_1 \xrightarrow{\alpha} P_0 \xrightarrow{\epsilon} X \rightarrow 0$$

be the first terms of a minimal projective resolution of $X \in \text{mod } A$. The AR translation $\tau_A X$ is given by the exact sequence

$$(1.2.2) \quad 0 \rightarrow \tau_A X \rightarrow D\text{Hom}_A(P_1, A) \xrightarrow{D\text{Hom}_A(\alpha, A)} D\text{Hom}_A(P_0, A)$$

Note that this sequence gives the first terms of a minimal injective coresolution of $\tau_A X$, if X has no non-zero projective direct summands.

b) Let

$$(1.2.3) \quad 0 \rightarrow X \xrightarrow{\mu} E_0 \xrightarrow{\beta} E_1$$

be the first terms of a minimal injective coresolution of $X \in \text{mod } A$. The AR translation $\tau_A^{-1} X$ is given by the exact sequence

$$(1.2.4) \quad \text{Hom}_A(DE_0, A) \xrightarrow{\text{Hom}_A(D\beta, A)} \text{Hom}_A(DE_1, A) \rightarrow \tau_A^{-1} X \rightarrow 0.$$

Note that this sequence gives the first terms of a minimal projective resolution of $\tau_A^{-1} X$, if X has no non-zero injective direct summands.

Theorem. Let $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ be an AR sequence. Then $Z \cong \tau_A^{-1} X$ and $X \cong \tau_A Z$.

1.3 The categories $\text{mod } A$ and $\overline{\text{mod } A}$.

a) A homomorphism $f: X \rightarrow Y$ is said to factor over the module Z , if there are homomorphisms $g: X \rightarrow Z$ and $h: Z \rightarrow Y$ such that $f = hg$.

b) For $X, Y \in \text{mod } A$ define

$$P(X, Y) := \{f \in \text{Hom}_A(X, Y) : f \text{ factors over some projective module}\};$$

$$I(X, Y) := \{f \in \text{Hom}_A(X, Y) : f \text{ factors over some injective module}\}.$$

Certainly $P(X, Y)$ and $I(X, Y)$ are subspaces of $\text{Hom}_A(X, Y)$. Thus, the following is well-defined:

$$\underline{\text{Hom}}_A(X, Y) := \text{Hom}_A(X, Y) / P(X, Y) \text{ and}$$

$$\overline{\text{Hom}}_A(X, Y) := \text{Hom}_A(X, Y) / I(X, Y).$$

For $f \in \text{Hom}_A(X, Y)$ denote $\underline{f} := f + P(X, Y)$ and $\overline{f} := f + I(X, Y)$

c) The category $\underline{\text{mod}} A$ is defined by the following data :

- The objects are $X \in \text{mod } A$ which do not have projective direct summands
- Take $\underline{\text{Hom}}_A(X, Y)$ as the set of morphisms.
- The composition of morphisms is given by $\underline{f} \underline{g} := \underline{f} \underline{g}$.

d) The category $\overline{\text{mod}} A$ is defined by the following data :

- The objects are $X \in \text{mod } A$ which do not have injective direct summands
- Put $\overline{\text{Hom}}_A(X, Y)$ as the set of morphisms.
- The composition of morphisms is given by $\overline{f} \overline{g} := \overline{f} \overline{g}$

Note that the composition in $\underline{\text{mod}} A$ and $\overline{\text{mod}} A$ is well defined, since if either f or g factors over some projective (injective) module, so does fg . These categories are k -categories, but in general not abelian.

Proposition. a) τ_A induces an equivalence $\underline{\text{mod}} A \rightarrow \overline{\text{mod}} A$. Its inverse is induced by τ_A^{-1} . In particular

$$(1.3.1) \quad \underline{\text{End}} X_A \cong \overline{\text{End}}(\tau_A X)_A \text{ for all } X \in \underline{\text{mod}} A \text{ and}$$

$$(1.3.2) \quad \overline{\text{End}} X_A \cong \underline{\text{End}}(\tau_A^{-1} X)_A \text{ for all } X \in \overline{\text{mod}} A$$

b) Auslander-Reiten formulas : For all $X, Y \in \text{mod } A$

$$(1.3.3) \quad \text{Ext}_A^1(X, Y) \cong D\overline{\text{Hom}}_A(Y, \tau_A X) \cong D\underline{\text{Hom}}_A(\tau_A^{-1} Y, X)$$

1.4 The homological dimension ≤ 1 case. Let $X, Y \in \text{mod } A$. Then

a) $p.d. X \leq 1$ if and only if $\text{Hom}_A(DA, \tau_A X) = 0$.

b) If $p.d. X \leq 1$, then

$$\tau_A X \cong D\text{Ext}_A^1(X, A) \text{ and}$$

$$\text{Ext}_A^1(X, Y) \cong D\text{Hom}_A(Y, \tau_A X) \text{ for all } Y \in \text{mod } A.$$

c) $\text{i.d. } Y \leq 1$ if and only if $\text{Hom}_A(\tau_A^{-1}Y, A) = 0$.

d) If $\text{i.d. } Y \leq 1$, then

$$\tau_A^{-1}Y \cong \text{Ext}_A^1(DY, A) \text{ and}$$

$$\text{Ext}_A^1(X, Y) \cong D\text{Hom}_A(\tau_A^{-1}Y, X) \text{ for all } X \in \text{mod } A.$$

e) $X \in \text{mod } A$ satisfies $\text{p.d. } X \leq 1$ and $\text{Hom}_A(X, A) = 0$, if and only if $\text{i.d. } \tau_A X \leq 1$ and $\text{Hom}_A(DA, \tau_A X) = 0$. In that case

i) $\dim \tau_A X = \Phi_A(\dim X)$;

ii) $\text{End } X_A \cong \text{End}(\tau_A X)_A$;

iii) $\text{Ext}_A^1(X, X) \cong \text{Ext}_A^1(\tau_A X, \tau_A X)$.

f) If $X \in \text{mod } A$ satisfies $\text{i.d. } X \leq 1$ and $\text{Hom}_A(DA, X) = 0$, then

i) $\dim \tau_A^{-1}X = \Phi_A^{-1}(\dim X)$;

ii) $\text{End } X_A \cong \text{End}(\tau_A^{-1}X)_A$;

iii) $\text{Ext}_A^1(X, X) \cong \text{Ext}_A^1(\tau_A^{-1}X, \tau_A^{-1}X)$.

Proof. See [Ri 3; 2.4], [Bo; 2.2] and [HR 1; 2.5]. Denote $[X, Y] = \text{Hom}_A(X, Y)$ for X, Y either right or left A -modules.

a) Consider the diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{\alpha} & P_0 & \xrightarrow{\epsilon} & X & \rightarrow & 0 \\ \alpha_{P_1} \downarrow & & \downarrow \alpha_{P_0} & & & & \\ 0 \rightarrow [D\tau_A X, A] \rightarrow & [[P_1, A], A] & \xrightarrow{[[\alpha, A], A]} & [[P_0, A], A] & & & \end{array}$$

where the upper row is the beginning of a minimal projective resolution of X and the lower row is obtained by applying $\text{Hom}_A(D-, A)$ to the sequence (1.2.2). For any $Z \in \text{mod } A$ let $\alpha_Z: Z \rightarrow \text{Hom}_A(\text{Hom}_A(Z, A), A)$ be the double-dual map which assigns to each $z \in Z$ the "functional" $\text{Hom}_A(Z, A) \rightarrow A; \varphi \mapsto \varphi(z)$. It is well known that $\alpha: 1_{\text{mod } A} \rightarrow \text{Hom}_A(\text{Hom}_A(-, A), A)$ is a natural transformation, i.e.

the above diagram is commutative and that α_Z is an isomorphism for any projective $Z \in \text{mod } A$. Thus, $\text{Hom}_A(D\tau_A X, A) \cong \ker \alpha$. From this follows that $\text{pd } X < 1$ if and only if $0 = \text{Hom}_A(D\tau_A X, A) \cong \text{Hom}_A(DA, \tau_A X)$

b) In this case a minimal projective resolution is of the form $0 \rightarrow P_1 \xrightarrow{\alpha} P_0 \xrightarrow{\beta} X \rightarrow 0$

Hence,

$$0 \rightarrow \text{DExt}_A^1(X, A) \rightarrow D[P_1, A] \xrightarrow{D[\alpha, A]} D[P_0, A]$$

is exact, i.e. $\text{DExt}_A^1(X, A) \cong \tau_A X$.

$I(Y, \tau_A X) = 0$ for all $Y \in \text{mod } A$ by a). Thus, using (1.3.3)

$$\text{Ext}_A^1(X, Y) \cong D\overline{\text{Hom}}_A(Y, \tau_A X) = D\text{Hom}_A(Y, \tau_A X)$$

c) and d) are proven in a "dual" fashion.

e) The equivalence is just given by a) and c) Let

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

be a minimal projective resolution of X . Then $\dim X = \dim P_0 - \dim P_1$ and since $\text{Hom}_A(X, A) = 0$, the minimal injective coresolution of $\tau_A X$ is of the form

$$0 \rightarrow \tau_A X \rightarrow D\text{Hom}_A(P_1, A) \rightarrow D\text{Hom}_A(P_0, A) \rightarrow 0$$

Note that $\dim(D\text{Hom}_A(P_i, A)) = -\Phi_A(\dim P_i)$. Thus,

$$\dim \tau_A X = -\Phi_A(\dim P_1) + \Phi_A(\dim P_0) = \Phi_A(\dim P_0 - \dim P_1) = \Phi_A(\dim X)$$

ii) $\text{End } X_A = \underline{\text{End}} X_A$, since $\text{Hom}_A(X, A) = 0$

From a) follows $\text{Hom}_A(DA, \tau_A X) = 0$. Hence, $\text{End}(\tau_A X)_A \cong \underline{\text{End}}(\tau_A X)_A$

Thus, $\text{End } X_A \cong \text{End}(\tau_A X)_A$ using (1.3.1).

iii) $\text{Ext}_A^1(X, X) \cong D\text{Hom}_A(X, \tau_A X) \cong \text{Ext}_A^1(\tau_A X, \tau_A X)$ using b) and d)

f) "Dualize" the proof of e).

1.5 Irreducible maps. A homomorphism $f: X \rightarrow Y$ is called **irreducible**, if it satisfies

IR 1: f is neither a splitting monomorphism nor a splitting epimorphism

IR 2: If $f = hg$ for some homomorphisms $g: X \rightarrow Z$ and $h: Z \rightarrow Y$, then either g is a splitting monomorphism or h is a splitting epimorphism.

Proposition 1. Let $f: X \rightarrow Y$ be an irreducible homomorphism

- a) f is either a monomorphism or an epimorphism.
- b) If X is indecomposable and $g: Y \rightarrow Z$ is a splitting epimorphism, then $gf: X \rightarrow Z$ is irreducible.
- c) If Y is indecomposable and $h: Z \rightarrow X$ is a splitting monomorphism, then $fh: Z \rightarrow Y$ is irreducible.

Irreducible homomorphisms are the "building blocks" of AR sequences according to the following

Proposition 2. a) Let $X \in \text{mod } A$ be indecomposable non-injective, then $f: X \rightarrow Y$ is irreducible if and only if $Y \neq 0$ and there exists $g: X \rightarrow Y'$ such that

$$0 \rightarrow X \xrightarrow{\begin{bmatrix} f \\ g \end{bmatrix}} Y \oplus Y' \rightarrow \text{coker} \begin{bmatrix} f \\ g \end{bmatrix} \rightarrow 0$$

is an AR sequence.

- b) Let $Z \in \text{mod } A$ be indecomposable non-projective. Then $g: Y \rightarrow Z$ is irreducible if and only if $Y \neq 0$ and there exists $f: Y' \rightarrow Z$ such that

$$0 \rightarrow \ker[g, f] \rightarrow Y \oplus Y' \xrightarrow{[g, f]} Z \rightarrow 0$$

is an AR sequence.

- c) If $E \in \text{mod } A$ is indecomposable injective, then $f: E \rightarrow X$ is irreducible if and only if there exists a splitting epimorphism $g: E/\text{Soc } E \rightarrow X$ such that $f = g\pi$ for $\pi: E \rightarrow E/\text{Soc } E$ the canonical epimorphism
- d) If $P \in \text{mod } A$ is indecomposable projective, then $f: X \rightarrow P$ is irreducible if and only if there exists a splitting monomorphism $g: X \rightarrow \text{rad } P$ such that $f = \mu g$ for $\mu: \text{rad } P \rightarrow P$ the embedding

1.6 The Auslander-Reiten quiver of an algebra. The Auslander-Reiten quiver short AR quiver $\Gamma(A)$ of an algebra A has as points the isomorphism classes $[X]$ of indecomposable $X \in \text{mod } A$ and an arrow $[X] \rightarrow [Y]$, if there exists an irreducible map $f: X \rightarrow Y$. Call a point $[X]$ in $\Gamma(A)$ projective (injective), if X is projective (injective). τ induces a map from non-projective points in $\Gamma(A)$ to non-injective points with the property: For a non-projective point $[X]$ there is an arrow $[Y] \rightarrow [\tilde{X}]$ if and only if there is an arrow $[\tau X] \rightarrow [Y]$. Similar τ_A^{-1} induces a map from non-injective points to non-projective points with the property: For a non-injective $[X]$ there is an arrow $[X] \rightarrow [Y]$ if and only if there is an arrow $[Y] \rightarrow [\tau_A^{-1} X]$.

Intuitively the Auslander-Reiten quiver mimicks the Gabriel quiver on the level of the module category. For example if A is of finite representation type, then $\text{End } \bigoplus_{k=1}^m X_k$, where $X_1, X_2, \dots, X_m$ represents up to isomorphism all indecomposable A -modules, is again a finite dimensional k -algebra and its Gabriel quiver coincides with the Auslander-Reiten quiver $\Gamma(A)$; see [Ga, 6.2].

Chapter II

ELEMENTS OF TILTING THEORY

This chapter provides the basic facts of tilting theory as used in the sequel. The results are taken from [HR 1], [Bo] and [R1 3; Chapter 4], where the missing proofs are given.

2.1 Torsion Theory. Recall that a pair $(\mathcal{G}, \mathcal{F})$ is called a torsion theory in $\text{mod } A$, if $\mathcal{G}, \mathcal{F}$ are full subcategories of $\text{mod } A$ satisfying

TT 1: If $X \in \text{mod } A$, then $X \in \mathcal{G}$ if and only if $\text{Hom}_A(X, F) = 0$ for all $F \in \mathcal{F}$

TT 2: If $Y \in \text{mod } A$ satisfies $\text{Hom}_A(G, Y) = 0$ for all $G \in \mathcal{G}$, then $Y \in \mathcal{F}$

$\mathcal{G}$ is called the torsion class and $\mathcal{F}$ is called the torsion-free class of the torsion theory. It is easy to see that $\mathcal{G}$ is closed under factor modules and extensions, while $\mathcal{F}$ is closed under submodules and extensions. Associated with any torsion theory $(\mathcal{G}, \mathcal{F})$ is a torsion functor $t: \text{mod } A \rightarrow \mathcal{G}$ which assigns to each $X \in \text{mod } A$ the uniquely determined, maximal submodule of X contained in $\mathcal{G}$. Thus, for any $X \in \text{mod } A$ there exists a short exact sequence

$$0 \rightarrow t(X) \rightarrow X \rightarrow X/t(X) \rightarrow 0$$

with $X/t(X) \in \mathcal{F}$. For more details see [St, chapter VI].

2.2 Definition and notation: A module $T \in \text{mod } A$ (or $A - \text{mod}$) is called a tilting module, if it satisfies the following three conditions

T 1: $\text{p.d. } T \leq 1$;

T 2: $\text{Ext}_A^1(T, T) = 0$;

T 3: There exists a short exact sequence of right (left) A -modules

$$0 \rightarrow A \rightarrow T' \rightarrow T'' \rightarrow 0 \quad \text{with } T', T'' \in \text{add } T.$$

Associated with a tilting module T_A are

- $B := \text{End } T_A$. T is regarded as a left B -module

- The tilting functors

$$F = \text{Hom}_A(T, -); \quad F' := \text{Ext}_A^1(T, -) \text{ mod } A \rightarrow \text{mod } B \text{ and}$$

$$G = - \otimes_B T, \quad G' := \text{Tor}_1^B(-, T) \text{ mod } B \rightarrow \text{mod } A$$

- The full subcategories $\mathcal{G}(T)$ and $\mathcal{F}(T)$ of $\text{mod } A$

- The full subcategories $\mathcal{X}(T)$ and $\mathcal{Y}(T)$ of $\text{mod } B$, where

$\mathcal{X}(T)$ contains all $X \in \text{mod } B$ satisfying $G(X) = X \otimes_B T = 0$ and

$\mathcal{Y}(T)$ contains all $X \in \text{mod } B$ satisfying $G'(X) = \text{Tor}_1^B(X, T) = 0$

The following theorem shows the relations between these objects

2.3 Theorem of Brenner-Butler. Let T_A be a tilting module. With the notation of 2.2 the following statements hold.

a) ${}_B T$ is a tilting module. The canonical map $A \rightarrow \text{End } {}_B T$, $a \mapsto \varphi_a$ with $\varphi_a(t) = ta$ for all $t \in T$ and $a \in A$ is a ring isomorphism, i.e. ${}_B T_A$ is a balanced bimodule

b) The tilting functors satisfy $G'F = GF' = 0$ and $F'G = FG' = 0$, i.e.

$$F: \text{mod } A \rightarrow \mathcal{Y}(T); \quad F': \text{mod } A \rightarrow \mathcal{X}(T); \quad G: \text{mod } B \rightarrow \mathcal{G}(T) \text{ and } G': \text{mod } B \rightarrow \mathcal{F}(T)$$

c) $(\mathcal{G}(T), \mathcal{F}(T))$ is a torsion theory in $\text{mod } A$. For any $X \in \text{mod } A$ the exact sequence in 2.1 is of the form

$$0 \hookrightarrow GF(X) \xrightarrow{\epsilon_X} X \rightarrow G'F'(X) \rightarrow 0$$

with $\epsilon_X(\varphi \otimes t) = \varphi(t)$ for $\varphi \in F(X)$ and $t \in T$. Moreover, $\mathcal{G}(T) \cap \mathcal{F}(T) = 0$. T

is a generator of $\mathcal{G}(T)$, i.e. for any $X \in \mathcal{G}(T)$ there is an epimorphism $\varphi: T' \rightarrow X$

with $T' \in \text{add } T$. ${}_A T$ is a cogenerator of $\mathcal{F}(T)$, i.e. for any $X \in \mathcal{F}(T)$ there is a

monomorphism $\psi: X \rightarrow W$ with $W \in \text{add } {}_A T$.

d) $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion theory in $\text{mod } B$. The short exact sequence in 2.1 is of the form

$$0 \rightarrow F'G'(Y) \rightarrow Y \xrightarrow{\eta_Y} FG(Y) \rightarrow 0$$

with $\eta_Y(y)$ maps $t \mapsto y \otimes t$ for $y \in Y$ and $t \in T$.

e) $\mathcal{G}(T)$ and $\mathcal{Y}(T)$ are equivalent via the restriction of the tilting functors F and G .

Moreover $\text{Ext}_A^k(X, Y) \cong \text{Ext}_B^k(F(X), F(Y))$ for all $X, Y \in \mathcal{G}(T)$ and $k \geq 0$. Note that the restrictions are exact, i.e. short exact sequences are mapped to short exact sequences.

f) $\mathcal{F}(T)$ and $\mathcal{X}(T)$ are equivalent via the restriction of the tilting functors F' and G' , which again are exact.

g) There is an isomorphism $\sigma_T: G_0(A) \rightarrow G_0(B)$ satisfying

$$\sigma_T(\dim X) = \dim (FX)_B - \dim (F'X)_B$$

In particular if n is the number of isomorphism classes of simple A -modules, then there are $T_1, T_2, \dots, T_n$ non-zero, indecomposable and pairwise non-isomorphic A -modules such that $T \cong \bigoplus_{k=1}^n T_k^{t_k}$ for some $t_k \geq 1$. If the bilinear forms q_A and q_B are well-defined on $G_0(A)$ and $G_0(B)$, respectively, then σ_T is an isometry.

2.4 Connecting lemma. Let T_A be a tilting module. Let $P_k \in \text{mod } A$ be indecomposable projective and $Q_k \in \text{mod } A$ be indecomposable injective such that $\text{Soc } Q_k \cong \text{top } P_k$. Then

a) $r_B^{-1}(FQ_k) \cong F'P_k$

b) $(FQ_k)_B$ is injective if and only if $P_k \in \text{add } T$. Any injective B -module in $\mathcal{Y}(T)$ is of this type.

c) $(FQ_k)_B$ is projective-injective if and only if $P_k, Q_k \in \text{add } T$. Any projective-injective B -module is of that type.

Other injective B-modules. If T_k is an indecomposable direct summand of T satisfying $\text{Hom}_A(T_k, A) = 0$, then $\text{Ext}_A^1(T, r_A T_k) \cong D\text{Hom}_A(T_k, T)$ is an indecomposable injective B-module contained in $\mathcal{I}(T)$

2.5 Homological behaviour. Let T_A be a tilting module

a) If $X \in \mathcal{G}(T)$, then

$$\text{i) } p.d. FX_B \leq p.d. X_A,$$

$$\text{ii) } i.d. FX_B \leq i.d. X_A + 1,$$

$$\text{iii) } \text{If } p.d. X_A \leq 1, \text{ then } F(r_A X) \cong r_B FX$$

$$\text{iv) } \text{If } p.d. X \leq m, \text{ then } \text{Ext}_B^m(FX, Z) = 0 \text{ for all } Z \in \mathcal{I}(T)$$

b) If $X \in \mathcal{F}(T)$, then

$$\text{i) } p.d. F'X_B \leq 1 + \max\{1, p.d. X_A\},$$

$$\text{ii) } i.d. F'X \leq i.d. X_A,$$

$$\text{iii) } \text{If } i.d. X_A = m, \text{ then } \text{Ext}_B^m(Z, F'X) = 0 \text{ for all } Z \in \mathcal{Y}(T)$$

$$\text{c) } gl.d. A - 1 \leq gl.d. B \leq gl.d. A + 1$$

2.6 Lemma. Let $S \in \text{mod } A$ satisfying $p.d. S \leq 1$ and $\text{Ext}_A^1(S, S) = 0$. Then there exists $V \in \text{mod } A$ such that $S \oplus V$ is a tilting module

Corollary 1. If $T \in \text{mod } A$ satisfies T1 and T2, then T is a tilting module if and only if there are non-zero indecomposable, pairwise non-isomorphic modules $T_1, T_2, \dots, T_n$ such that $T \cong \bigoplus_{k=1}^n T_k^{t_k}$ for some $t_k \geq 1$

Corollary 2. If T_A is a tilting module, then any indecomposable projective-injective A-module is isomorphic to a direct summand of T_A

2.7 APR and BB tilting modules. Let $A_A \cong \bigoplus_{k=1}^n P_k^{n_k}$ with P_k indecomposable and pairwise non-isomorphic. Set $S_k := \text{top } P_k$. Assume that S_k is not injective for a fixed k

Define

$$T^{(k)} := \bigoplus_{l=1}^n P_l \oplus \tau_A^{-1} S_k.$$

$T^{(k)}$ is a tilting module if and only if $\text{Hom}_A(DA, S_k) = 0$ and $\text{Ext}_A^1(S_k, S_k) = 0$.

In that case $T^{(k)}$ is called the **Brenner-Butler tilting module**, in short the **BB tilting module** at the point k , see [BB 1]

Proof. See [TW 1, §2]. $\text{p.d. } \tau_A^{-1} S_k \leq 1$ if and only if $\text{Hom}_A(DA, S_k) = 0$ by 1.4 a. In that case,

- i) $\text{Ext}_A^1(S_k, S_k) \cong D\text{Hom}_A(\tau_A^{-1} S_k, S_k)$ by (1.3.3),
- ii) $\text{Ext}_A^1(\tau_A^{-1} S_k, \tau_A^{-1} S_k) \cong D\text{Hom}_A(\tau_A^{-1} S_k, S_k)$ by 1.4 b

Thus $\text{Ext}_A^1(\tau_A^{-1} S_k, \tau_A^{-1} S_k) = 0$ implies $\text{Ext}_A^1(S_k, S_k) = 0$ showing the necessary part. Conversely if $\text{Ext}_A^1(S_k, S_k) = 0$, then every homomorphism $\psi: \tau_A^{-1} S_k \rightarrow S_k$ factors over a projective module. From this follows that ψ factors over the projective hull $\epsilon: P_k \rightarrow S_k$, i.e. $\psi = \epsilon\varphi$. But since $\tau_A^{-1} S_k$ is indecomposable non-projective $\text{im } \varphi \subseteq \text{rad } P_k = \ker \epsilon$. Hence $\psi = 0$. This shows $\text{Ext}_A^1(\tau_A^{-1} S_k, \tau_A^{-1} S_k) = 0$. Certainly $\text{Ext}_A^1(P_l, \tau_A^{-1} S_k) = 0$. $\text{Ext}_A^1(\tau_A^{-1} S_k, P_l) \cong D\text{Hom}_A(P_l, S_k) = 0$, since $\text{p.d. } \tau_A^{-1} S_k \leq 1$. This shows TT2. Thus, $T^{(k)}$ is a tilting module by corollary 1 of 2.6. ■

Certainly the conditions for a tilting module are satisfied if S_k is simple projective non-injective. In that case $T^{(k)}$ is called the **Auslander-Platzek-Reiten**, short **APR**, tilting module at the point k , see [APR]. The associated tilting functors were first discovered in the representation theory of tensor algebras as reflection functors at a sink k [DR 2] and played a driving force for the development of the tilting theory, see [APR] and [BB 1]

The following statements hold for an APR tilting module $T^{(k)}$ with $B = \text{End } T_A^{(k)}$

- i) If $X \in \text{mod } A$ is indecomposable, then $X \in \mathcal{G}(T^{(k)})$ if and only if $X \not\cong S_k$
- $X \in \mathcal{F}(T^{(k)})$ if and only if $X \cong S_k$.

ii) Assume in addition $\text{id. } S_k = 1$. Take the following enumeration of the indecomposable projective B -modules: $e_l B = \begin{cases} FP_l, & \text{if } l \neq k, \\ F(\tau_A^{-1} S_k), & \text{if } l = k. \end{cases}$

Then

$$(\sigma_{T^{(k)}}(\underline{x}))_i = \begin{cases} -x_k + \sum_{i=1}^n d_{ik} x_i, & \text{if } k = i, \\ x_i, & \text{otherwise,} \end{cases}$$

where $d_{ik} = \dim \text{Ext}_A^1(S_i, S_k)_{\text{End}(S_k)}$

Proof. i) Certainly $\text{Ext}_A^1(P_l, X) = 0$ $\text{Ext}_A^1(\tau_A^{-1} S_k, X) \simeq D\text{Hom}_A(X, S_k) = 0$ if and only if $X \not\cong S_k$ $S_k \in \mathcal{F}(T^{(k)})$, since $\text{Hom}_A(P_l, S_k) = 0$ for $l \neq k$ and $\text{Hom}_A(\tau_A^{-1} S_k, S_k) = 0$, see ii) of the previous proof

ii) 2.3.e and 1.4 f imply

$$(1) \quad \text{End}(FP_l)_B \cong \text{End}(P_l)_A \quad \text{for } l \neq k,$$

$$(2) \quad \text{End}(F(\tau_A^{-1} S_k))_B \cong \text{End}(\tau_A^{-1} S_k)_A \simeq \text{End}(S_k)_A$$

In particular the isomorphic algebras have the same k dimension. We check the equation on the standard basis $\{\dim S_1, \dim S_2, \dots, \dim S_n\}$ of $G_0(A)$: i) implies

$$\sigma_{T^{(k)}}(\dim S_i) = \dim (FS_i)_B - \dim (F'S_i)_B = \begin{cases} \dim (FS_i)_B, & \text{if } i \neq k \\ -\dim (F'S_k)_B \end{cases}$$

$S_i \in \mathcal{G}(T^{(k)})$ for $i \neq k$ implies $\text{Hom}_B(FP_l, FS_i) \cong \text{Hom}_A(P_l, S_i)$. Thus,

$$(\dim (FS_i)_B)_l = (\dim (S_i)_A)_l - \delta_{il}$$

using (1). Since $\text{Ext}_A^1(P_l, S_k) = 0$, also $(\dim (F'S_k)_B)_l = 0$

$\text{Hom}_B(F(\tau_A^{-1} S_k), FS_i) \cong \text{Hom}_A(\tau_A^{-1} S_k, S_i) \cong D\text{Ext}_A^1(S_i, S_k)$ implies with (2)

$$(\dim (FS_i)_B)_k = d_{ik}$$

$(F'S_k)_B$ is simple, since it is the only indecomposable module in the torsion class $\chi(T^{(k)})$

Hence, $F'S_k \cong \text{top } F(\tau_A^{-1} S_k)$ and $(\dim (F'S_k)_B)_k = 1$. ■

HEREDITARY ALGEBRAS

3.1 Definition and basic properties. For a finite dimensional algebra A , the following statements are equivalent as H. Cartan and S. Eilenberg showed in [CE; 15.4]

- i) Every right ideal is projective.
- ii) Every submodule of a projective right module is projective.
- iii) Every factor module of an injective right module is injective
- iv) $\text{p.d. } X_A \leq 1$ for all $X \in \text{mod } A$.
- v) $\text{i.d. } X_A \leq 1$ for all $X \in \text{mod } A$.

The same statements with "right" replaced by "left".

An algebra which satisfies one of these conditions is called **hereditary**. Examples of hereditary algebras are the tensor algebras of species having no oriented cycle in their quiver, Dlab-Ringel's algebra $\tilde{A}(\epsilon, \delta)$ providing examples which are not tensor algebras [DR 3]. From now on A will always denote a hereditary algebra.

From ii) follows immediately that if there exists a non-zero homomorphism $f: X \rightarrow P$ for some projective module P , then X has a non-zero projective direct summand (isomorphic to $\text{im } f$). Thus, for all $X \in \text{mod } A$ and $Y \in \text{mod } A$

$$(3.1.1) \quad P(X, Y) = 0 \quad \text{and} \quad \underline{\text{Hom}}_A(X, Y) = \text{Hom}_A(X, Y)$$

By duality iii) implies for all $X \in \text{mod } A$ and $Y \in \overline{\text{mod } A}$

$$(3.1.2) \quad I(X, Y) = 0 \quad \text{and} \quad \overline{\text{Hom}}_A(X, Y) = \text{Hom}_A(X, Y)$$

In other words $\text{mod } A$ and $\overline{\text{mod } A}$ are full subcategories consisting of all modules having no projective or no injective direct summand, respectively. With 1.3 it is easily seen that

$\tau_A^- \cong D\text{Ext}_A^1(-, A)$ and $\tau_A^{-1} \cong \text{Ext}_A^1(D-, A)$. In particular τ_A preserves monomorphisms while τ_A^{-1} preserves epimorphisms. 1.3 also implies

$$(3.1.3) \quad \text{Hom}_A(X, Y) \cong \text{Hom}_A(\tau_A X, \tau_A Y) \quad \text{for all } X, Y \in \text{mod } A$$

$$(3.1.4) \quad \text{Hom}_A(X, Y) \cong \text{Hom}_A(\tau_A^{-1} X, \tau_A^{-1} Y) \quad \text{for all } X, Y \in \text{mod } A$$

These formulas imply a particular shape of the AR quiver $\Gamma(A)$, see [Ri 2, Lemma 2.1]

- i) If $[X]$ is a point in $\Gamma(A)$ with the property that $[\tau_A^m X]$ is projective for some m , then for each $[Y]$ in the connected component of $[X]$ there exists an integer $m(Y) \geq 0$ such that $[\tau_A^{m(Y)} Y]$ is projective. Such a component is called a **preprojective component** of $\Gamma(A)$.
- ii) If $[X]$ is a point in $\Gamma(A)$ such that $[\tau_A^m X]$ is injective for some integer $m \leq 0$, then for each point $[Y]$ in the connected component of $[X]$ there exists an integer $m(Y) \leq 0$ such that $[\tau_A^{m(Y)} Y]$ is injective. A connected component with that property is called a **preinjective component**.
- iii) Components which are neither preprojective nor preinjective are called **regular**. A point $[X]$ in a regular component is characterized by the property that $[\tau_A^m X]$ is neither projective nor injective for every integer m .

A point $[X]$ in $\Gamma(A)$ contained in a preprojective, preinjective or regular component is called preprojective, preinjective or regular, respectively. A module $X \in \text{mod } A$ is called preprojective, preinjective or regular, if each indecomposable direct summand of X is preprojective, preinjective or regular, respectively. Note that A has regular modules if and only if A is of infinite representation type.

3.2 Ringel's result on regular components. C. M. Ringel determined the structure

of regular components. The main result is summarized here [Ri 2; §2-3 and 3.1] Let S be an indecomposable regular A -module.

- a) Either the middle term of an Auslander-Reiten sequence starting (or ending) in S is indecomposable — in that case S is called **quasi-simple** — or there is a chain of irreducible monomorphisms

$$(3.2.1) \quad S_1 \xrightarrow{\mu_1} S_2 \xrightarrow{\mu_2} \dots \xrightarrow{\mu_{r-2}} S_{r-1} \xrightarrow{\mu_{r-1}} S_r = S \xrightarrow{\mu_r} S_{r+1}$$

such that $r \geq 2$, S_k and $\tau_A^{-1} S_{k-1} = S_k/S_1$ are indecomposable for $2 \leq k \leq r+1$, S_1 is quasi-simple and the short exact sequence

$$(3.2.2) \quad 0 \rightarrow S = S_r \xrightarrow{\begin{bmatrix} \mu_r \\ \pi_r \end{bmatrix}} S_{r+1} \oplus S_r/S_1 \xrightarrow{[-\pi_{r+1}, \mu_{r,-1}]} S_{r+1}/S_1 \rightarrow 0$$

is an Auslander-Reiten sequence, where $\pi_r: S_r \rightarrow S_r/S_1$ and $\pi_{r+1}: S_{r+1} \rightarrow S_{r+1}/S_1$ are the natural epimorphisms and $\mu_{r,-1}: S_r/S_1 \rightarrow S_{r+1}/S_1$ the monomorphism induced by μ_r .

$r = ql(S)$ is called the **quasi-length** of S and is defined to be 1 for S quasi-simple

- b) The irreducible monomorphism $\mu_k: S_k \rightarrow S_{k+1}$, ($1 \leq k \leq r$) in (3.2.1) enjoys the following lifting property of homomorphisms: If X is an indecomposable A -module not isomorphic to $\tau_A^{-i} S_{k-1} \cong S_k/S_1$ for all $0 \leq i \leq k-1$, where $S_0 := 0$, and $\varphi: S_k \rightarrow X$ is a homomorphism, then there exists $\varphi': S_{k+1} \rightarrow X$ such that $\varphi = \varphi' \mu_k$, in short the map

$$\text{Hom}_A(\mu_k, X): \text{Hom}_A(S_{k+1}, X) \rightarrow \text{Hom}_A(S_k, X)$$

is an epimorphism.

- c) The natural epimorphism $\pi_k: S_k \rightarrow \tau_A^{-1} S_{k-1} = S_k/S_1$, ($2 \leq k \leq r+1$) is an irreducible epimorphism and enjoys a similar property: If Y is an indecomposable

A -module not isomorphic to $\tau_A^{-1}S_i = S_{i+1}/S_1$ for $1 \leq i \leq k-1$, then

$$\text{Hom}_A(Y, \pi_k): \text{Hom}_A(Y, S_k) \rightarrow \text{Hom}_A(Y, \tau_A^{-1}S_{k-1})$$

is an epimorphism.

3.3 The Coxeter transformation. If $\psi: P \rightarrow P'$ is a non-zero homomorphism between projective A -modules, then $\ker \psi$ is a direct summand of P by definition ii) of hereditary. Consequently non-zero homomorphisms between indecomposable projective modules are monomorphisms. This implies in particular that the Gabriel quiver $Q(A)$ has no oriented cycles. Thus, the valuation $(d_{ij})_{1 \leq i, j \leq n}$ is well-defined. Furthermore the indecomposable projective modules $P_1, P_2, \dots, P_n$ can be listed in such a way that $\text{Hom}_A(P_i, P_j) \neq 0$ implies $i \leq j$. Such a choice of the indices is called **admissible**. Assume from now on admissibility. The sets

$$\{\dim P_k : 1 \leq k \leq n\} \quad \text{and} \quad \{\dim Q_k : 1 < k \leq n\}$$

are linearly independent in $G_0(A)$, since they correspond to triangular matrices with diagonal entries all = 1. Thus the Coxeter transformations Φ_A and Φ_A^{-1} are well-defined. They can be computed as Coxeter element of the Weyl group associated with the valued graph $(d_{ij})_{1 \leq i, j \leq n}$, see [DR 2], as done in [APR] and outlined in the following. Since P_1 is simple projective, there is an APR tilting module $T^{(1)}$ at 1. Set $A^{(0)} = A$, $A^{(1)} = \text{End } T_A^{(1)}$ and

$$F_1 := \text{Hom}_{A^{(0)}}(T^{(1)}, -) \cdot \text{mod } A^{(0)} \rightarrow \text{mod } A^{(1)}$$

It is easy to see that $A^{(1)}$ is again hereditary and that $F_1(P_2)$ is a simple projective $A^{(1)}$ -module. Thus the APR tilting module $T_{A^{(1)}}^{(2)}$ is defined. Continuing in that fashion yields hereditary algebras $A^{(k)} = \text{End } T_{A^{(k-1)}}^{(k)}$ with tilting functor

$$F_k = \text{Hom}_A(T^{(k)}, -) \cdot \text{mod } A^{(k-1)} \rightarrow \text{mod } A^{(k)}$$

such that $F_k F_{k-1} \dots F_2 F_1(P_{k+1})$ is a simple $A^{(k)}$ -module. Since for $i, j \geq k+1$

$$\text{Ext}_A^1(S_i, S_j) \cong \text{Ext}_{A^{(k)}}^1(F_k \dots F_1(S_i), F_k \dots F_1(S_j))$$

by 2.3.e, the "dimension shift" $\Phi_k := \sigma_{T^{(k-1)}} : G_0(A^{(k-1)}) \rightarrow G_0(A^{(k)})$ has k -th coordinate function

$$(\Phi_k(\underline{x}))_k = -x_k + \sum_{i=1}^n d_{ik} x_i,$$

while the other coordinates are unchanged. The n -th iteration yields an algebra $A^{(n)} \cong A$. It has been shown in [APR] that $F_n F_{n-1} \dots F_2 F_1 \cong \tau_A$. Thus,

$$(3.3.1) \quad \Phi_A = \Phi_n^* \Phi_{n-1} \dots \Phi_2 \Phi_1,$$

as known for tensor algebras by combining the results of [DR 2] and [BB 2]

Recall the bilinear form on $G_0(A)$

$$\langle \dim X, \dim Y \rangle_A = \dim_{\mathbf{k}} \text{Hom}_A(X, Y) - \dim_{\mathbf{k}} \text{Ext}_A^1(X, Y)$$

It follows easily from 1.4 that

$$(3.3.2) \quad \langle \underline{x}, \underline{y} \rangle_A = -\langle \underline{y}, \Phi_A \underline{x} \rangle_A = -\langle \Phi_A^{-1} \underline{y}, \underline{x} \rangle_A \quad \text{for all } \underline{x}, \underline{y} \in \mathbb{C}^n$$

In particular the bilinear form is invariant under Φ_A and Φ_A^{-1} , i.e.

$$(3.3.3) \quad \langle \underline{x}, \underline{y} \rangle_A = \langle \Phi_A \underline{x}, \Phi_A \underline{y} \rangle_A = \langle \Phi_A^{-1} \underline{x}, \Phi_A^{-1} \underline{y} \rangle_A \quad \text{for all } \underline{x}, \underline{y} \in \mathbb{C}^n$$

A useful generalization of the behaviour of homomorphisms between projective modules is obtained in [HR 1; 4.1 and 4.2] as

3.4 Lemma (Happel, Ringel). Let X, Y be indecomposable A -modules satisfying $\text{Ext}_A^1(X, Y) = 0$. Then any non-zero homomorphism $\varphi: Y \rightarrow X$ is either a monomorphism

• or an epimorphism. In particular, if X is an indecomposable A -module with

$\text{Ext}_A^1(X, X) = 0$, then $\text{End } X$ is a division algebra

3.5 Corollary . If X_A satisfies $\text{Ext}_A^1(X_A, X_A) = 0$, then the Gabriel quiver of $\text{End } T_A$ has no oriented cycles.

3.6 Representation type. Any hereditary algebra A is either of finite, tame or wild representation type. If A is indecomposable, then the following characterizations hold

- a) A is of finite representation type if and only if the quadratic form q_A is positive definite if and only if the valued graph $(d_{i,j})_{1 \leq i,j \leq n}$ is up to a multiple a Dynkin diagram. It has been shown in [DR 1] that the hereditary algebras of finite representation type are all tensor algebras of some species
- b) A is of tame representation type if and only if the quadratic form q_A is positive semidefinite (but not definite) if and only if the valued graph $(d_{i,j})_{1 \leq i,j \leq n}$ is a Euclidean diagram. In that case A is either a tensor algebra or an algebra of the type $\tilde{A}(\epsilon, \delta)$ as shown in [DR 3].
- c) A is of wild representation type if and only if the quadratic form q_A is indefinite. Ringel showed in [Ri 1] that a k -species with indefinite quadratic form allows an exact embedding of the category of representations of a wild bimodule. The proof is based on the behaviour of preinjective dimension types of a tame factor algebra, thus as outlined in 3.3 accessible for the general hereditary case. His results on wild bimodules show furthermore that there exist indecomposable modules X with $q_A(\dim X) \leq -1$.

3.7 Baer's result on regular dimension types. D. Baer proved recently the following result [Ba, 2.1] for a wild hereditary algebra A : Given regular modules $X, S \in \text{mod } A$

with X indecomposable, then there exist integers $N < M$ such that

$$\langle \Phi_A^j(\dim S), \dim X \rangle_A > 0 \quad \text{for all } j \leq N,$$

$$\langle \Phi_A^j(\dim S), \dim X \rangle_A < 0 \quad \text{for all } j \geq M$$

Chapter IV

The general construction

The following is based on an idea of W. Geigle in [Ge]

4.1 The category $\mathcal{A}(S)$. For $S \in \text{mod } A$ denote by $\mathcal{A}(S)$ the full subcategory of $\text{mod } A$ consisting of all modules $X \in \text{mod } A$ which satisfy

$$(4.1.1) \quad \text{Hom}_A(S, X) = 0 \quad \text{and} \quad \text{Ext}_A^t(S, X) = 0 \quad \text{for all } t \geq 1$$

$\mathcal{A}(S)$ enjoys the following properties

a) If $\text{p.d. } S \leq m$, then $X \in \mathcal{A}(S)$ if and only if

$$\text{Hom}_A(S, X) = 0 \quad \text{and} \quad \text{Ext}_A^t(S, X) = 0 \quad \text{for all } 1 \leq t < m$$

b) If $S = \bigoplus_{k=1}^r S_k$, then $\mathcal{A}(S) = \bigcap_{k=1}^r \mathcal{A}(S_k)$

c) $\mathcal{A}(S)$ is closed under direct summands and finite direct sums — finite direct products

in $\text{mod } A$, i.e. $\bigoplus_{k=1}^m X_k \in \mathcal{A}(S)$ if and only if $X_k \in \mathcal{A}(S)$ for all $1 \leq k \leq m$

d) $\mathcal{A}(S)$ is closed under extensions in $\text{mod } A$; i.e. for any short exact sequence

$$(4.1.2) \quad 0 \rightarrow X_1 \rightarrow Y \rightarrow X_2 \rightarrow 0$$

in $\text{mod } A$ with $X_1, X_2 \in \mathcal{A}(S)$, also $Y \in \mathcal{A}(S)$.

e) If $\text{p.d. } S \leq 1$, then $\mathcal{A}(S)$ is closed under images, kernels and cokernels in $\text{mod } A$, i.e.

for any homomorphism $f: X \rightarrow Y$ in $\mathcal{A}(S)$ also $\text{im } f$, $\ker f$ and $\text{coker } f \in \mathcal{A}(S)$

Remark. The condition $\text{p.d. } S \leq 1$ in e) cannot be weakened in general as the following example shows. Let A be the path algebra of the quiver with relation

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3, \quad \alpha\beta = 0$$

lower part of the commutative diagram

$$\begin{array}{ccccccc}
 \varepsilon_k & 0 \rightarrow & X & \xrightarrow{\mu_k} & Y_k & \xrightarrow{\sigma_k} & S_k \rightarrow 0 \\
 & & \downarrow \varrho_k & & \downarrow \sigma_k & & \downarrow \tau_k \\
 (4.2.3) & & S & 0 \rightarrow & X^\vee & \xrightarrow{\oplus \mu_k} & \bigoplus_{k=1}^n Y_k \xrightarrow{\oplus \sigma_k} S(X) \rightarrow 0 \\
 & & \downarrow \nabla & & \downarrow \alpha & & \parallel \\
 & & \varepsilon(X) & 0 \rightarrow & X & \xrightarrow{\mu_X} & V(X) \xrightarrow{\alpha} \bigoplus_{k=1}^n S_k^{\vee k} \rightarrow 0
 \end{array}$$

Since $\text{Hom}_A(S_k, X) = 0 = \text{Ext}_A^1(S_k, S(X))$, $\text{Hom}_A(S_k, \quad)$ applied to $\varepsilon(X)$ yields a long exact sequence

(*)

$$0 \rightarrow \text{Hom}_A(S_k, V(X)) \rightarrow \text{Hom}_A(S_k, S(X)) \xrightarrow{\delta_k} \text{Ext}_A^1(S_k, X) \rightarrow \text{Ext}_A^1(S_k, V(X)) \rightarrow 0$$

Let $(\varrho_k, \sigma_k, \tau_k) : \varepsilon_k \rightarrow S$, the upper part of (4.2.3), be the embedding into the (k, i) -th summand. Then

$$(1_X, \alpha \sigma_k, \tau_k) = (\nabla, \alpha, 1_{S(X)}) \circ (\varrho_k, \sigma_k, \tau_k) : \varepsilon_k \rightarrow \varepsilon(X)$$

is the pull-back of $\varepsilon(X)$ along τ_k , i.e. $\varepsilon_k \equiv \delta_k(\tau_k)$. Hence, δ_k is an epimorphism. Since it is E_k -linear and $\dim \text{Hom}_A(S_k, S(X))_{E_k} = v_k = \dim \text{Ext}_A^1(S_k, X)_{E_k}$, δ_k is actually an isomorphism. Thus, the exactness of (*) implies $V(X) \in \mathcal{A}(S_k)$. This, for each k , yields $V(X) \in \mathcal{A}(S)$.

In order to show the uniqueness of $V(X)$ and $S(X)$ let

$$\varepsilon'_i : 0 \rightarrow X \xrightarrow{\mu'} Y = S' \rightarrow \bigoplus_{k=1}^n S_k^{\vee k} \rightarrow 0$$

be a short exact sequence with $Y \in \mathcal{A}(S)$. Let $S' \in \text{add } S$. Applying $\text{Hom}_A(S', \quad)$ to ε' yields an exact sequence

$$\text{Hom}_A(S', Y) = 0 \rightarrow \text{Hom}_A(S', \bigoplus_{k=1}^n S_k^{\vee k}) \xrightarrow{\delta_{S'}} \text{Ext}_A^1(S', X) \rightarrow 0 \rightarrow \text{Ext}_A^1(S', Y)$$

Since $X, Y \in \mathcal{A}(S)$, each of the Hom - and Ext -spaces is zero. Thus $\text{im } f$, $\ker f$ and $\text{coker } f \in \mathcal{A}(S)$. ■

4.2 Lemma. Let A be a finite dimensional k -algebra and $S = \bigoplus_{k=1}^s S_k \in \text{mod } A$ satisfying the following conditions

- i) $\text{p.d. } S \leq 1$,
 - ii) $\text{Ext}_A^1(S, S) = 0$;
 - iii) $E_k = \text{End}(S_k)_A$ is a division algebra for $1 \leq k \leq s$ and
 - iv) $\text{Hom}_A(S_k, S_l) = 0$ for $1 \leq k \neq l \leq s$
- a) (Geigle) For each module $X \in \mathcal{F}(S)$ there exist a up to isomorphism (of exact sequences) uniquely determined short exact sequence

$$(4.2.1) \quad \ell(X) \quad 0 \rightarrow X \xrightarrow{\mu_X} V(X) \xrightarrow{\nu_X} S(X) \rightarrow 0$$

with $V(X) \in \mathcal{A}(S)$ and $S(X) \in \text{odd } S$. Moreover $S(X) \sim \bigoplus_{k=1}^s S_k^{c_k}$ with $c_k = \dim \text{Ext}_A^1(S_k, X)_{E_k}$.

- b) If $X \in \mathcal{F}(S)$ and $Y \in \text{mod } A$, then

$$\text{Ext}_A^k(\mu_X, Y) : \text{Ext}_A^k(V(X), Y) \rightarrow \text{Ext}_A^k(X, Y)$$

is an epimorphism for $k = 1$ and an isomorphism for $k \geq 2$. If $Y \in \mathcal{G}(S)$, then

$\text{Ext}_A^1(\mu_X, Y) : \text{Ext}_A^1(V(X), Y) \rightarrow \text{Ext}_A^1(X, Y)$ is an isomorphism and

$\text{Hom}_A(\mu_X, Y) : \text{Hom}_A(V(X), Y) \rightarrow \text{Hom}_A(X, Y)$ is an epimorphism.

If $Y \in \mathcal{A}(S)$, then $\text{Hom}_A(\mu_X, Y)$ is an isomorphism too.

- c) If $X \in \mathcal{F}(S)$ and $Z \in \text{mod } A$ satisfies $\text{Hom}_A(Z, S) = 0$, then

$$\text{Hom}_A(Z, \mu_X) : \text{Hom}_A(Z, X) \rightarrow \text{Hom}_A(Z, V(X))$$

is an isomorphism. If, in addition, there exists an integer l such that $\text{Ext}_A^k(Z, S) = 0$ for all $1 \leq k \leq l$, then

$$\text{Ext}_A^k(Z, \mu_X) : \text{Ext}_A^k(Z, X) \rightarrow \text{Ext}_A^k(Z, V(X))$$

is an isomorphism for $1 \leq k \leq l$

d) For any homomorphism $f: X \rightarrow Y \in \mathcal{F}(S)$ there exists a unique homomorphism $V(f): V(X) \rightarrow V(Y)$ satisfying $V(f)\mu_X = \mu_Y f$. Thus in particular

$$V: \mathcal{F}(S) \rightarrow A(S)$$

is a k -additive, covariant functor and $\mu: 1_{\mathcal{F}(S)} \rightarrow V$ is a natural transformation

Moreover $V(\bigoplus_{k=1}^r X_k) \cong \bigoplus_{k=1}^r V(X_k)$ for $\bigoplus_{k=1}^r X_k \in \mathcal{F}(S)$

The map $V: \text{Hom}_A(X, Y) \rightarrow \text{Hom}_A(V(X), V(Y))$

(4.2.2)

$$f \mapsto V(f)$$

is injective for all $X, Y \in \mathcal{F}(S)$ and bijective if $\text{Hom}_A(X, S) = 0$

Remark. a) shows that $X \cong V(X)$ if and only if $X \in A(S)$. Certainly $\mathcal{E}(X)$ does not split for $\text{Ext}_A^1(S, X) \neq 0$

Proof. a) This is the construction Bongartz used for the proof of 2.6 [Bo, §2]. Clearly each E_k is finite dimensional over k and $v_k := \dim \text{Ext}_A^1(S_k, X)_{E_k} < \infty$. Set $v = \sum_{k=1}^r v_k$ and $S(X) := \bigoplus_{k=1}^r S_k^{v_k}$. Let

$$\mathcal{E}_k: 0 \rightarrow X \xrightarrow{\mu_k} Y_k \xrightarrow{\eta_k} S_k \rightarrow 0, \quad i = 1, \dots, v_k$$

be an E_k -basis of $\text{Ext}_A^1(S_k, X)$. Let $\mathcal{E}(X)$ be the short exact sequence given by the push-out of $S := \bigoplus_{k=1}^r \bigoplus_{i=1}^{v_k} \mathcal{E}_k$ along the codiagonal map $\nabla: X^v \rightarrow X$, i.e. $\mathcal{E}(X)$ is given by the

lower part of the commutative diagram

$$(4.2.3) \quad \begin{array}{ccccccc} \varepsilon_{k1} & 0 \rightarrow & X & \xrightarrow{\mu_{k1}} & Y_{k1} & \xrightarrow{\sigma_{k1}} & S_k \rightarrow 0 \\ & & \downarrow \rho_{k1} & & \downarrow \sigma_{k1} & & \downarrow r_{k1} \\ S & 0 \rightarrow & X^\vee & \xrightarrow{\oplus \mu_{k1}} & \bigoplus_{k=1}^n \bigoplus_{i=1}^{v_k} Y_{ki} & \xrightarrow{\oplus \sigma_{k1}} & S(X) \rightarrow 0 \\ & & \downarrow \nabla & & \downarrow \alpha & & \downarrow \parallel \\ \mathcal{E}(X) & 0 \rightarrow & X & \xrightarrow{\mu_X} & V(X) & \xrightarrow{\quad} & \bigoplus_{k=1}^n S_k^{\vee v_k} \rightarrow 0 \end{array}$$

Since $\text{Hom}_A(S_k, X) = 0 = \text{Ext}_A^1(S_k, S(X))$, $\text{Hom}_A(S_k, \quad)$ applied to $\mathcal{E}(X)$ yields a long exact sequence

$$(*) \quad 0 \rightarrow \text{Hom}_A(S_k, V(X)) \rightarrow \text{Hom}_A(S_k, S(X)) \xrightarrow{\delta_k} \text{Ext}_A^1(S_k, X) \rightarrow \text{Ext}_A^1(S_k, V(X)) \rightarrow 0$$

Let $(\rho_{k1}, \sigma_{k1}, r_{k1}) : \varepsilon_{k1} \rightarrow S$, the upper part of (4.2.3), be the embedding into the $(k, 1)$ -th summand. Then

$$(1_X, \alpha \sigma_{k1}, r_{k1}) = (\nabla, \alpha, 1_{S(X)}) \circ (\rho_{k1}, \sigma_{k1}, r_{k1}) : \varepsilon_{k1} \rightarrow \mathcal{E}(X)$$

is the pull-back of $\mathcal{E}(X)$ along r_{k1} , i.e. $\varepsilon_{k1} \equiv \delta_k(r_{k1})$. Hence, δ_k is an epimorphism. Since it is E_k -linear and $\dim \text{Hom}_A(S_k, S(X))_{E_k} = v_k = \dim \text{Ext}_A^1(S_k, X)_{E_k}$, δ_k is actually an isomorphism. Thus, the exactness of $(*)$ implies $V(X) \in \mathcal{A}(S_k)$. This, for each k , yields $V(X) \in \mathcal{A}(S)$.

In order to show the uniqueness of $V(X)$ and $S(X)$ let

$$\xi'_1 : 0 \rightarrow X \xrightarrow{\mu'} Y \xrightarrow{\sigma'} \bigoplus_{k=1}^n S_k^{\vee v_k} \rightarrow 0$$

be a short exact sequence with $Y \in \mathcal{A}(S)$. Let $S' \in \text{add } S$. Applying $\text{Hom}_A(S', \quad)$ to ξ'_1 yields an exact sequence

$$\text{Hom}_A(S', Y) = 0 \rightarrow \text{Hom}_A(S', \bigoplus_{k=1}^n S_k^{\vee v_k}) \xrightarrow{\delta_{S'}} \text{Ext}_A^1(S', X) \rightarrow 0 \rightarrow \text{Ext}_A^1(S', Y)$$

Hence $\delta_{S'}$ is an isomorphism. Taking $S' := S_k$ implies that

$$\delta_{S_k}: \text{Hom}_A(S_k, \bigoplus_{i=1}^n S_i^{v'_i}) \rightarrow \text{Ext}_A^1(S_k, X)$$

is an isomorphism. In particular

$$v'_k = \dim \text{Hom}_A(S_k, \bigoplus_{i=1}^n S_i^{v'_i})_{E_k} = \dim \text{Ext}_A^1(S_k, X)_{E_k} = v_k.$$

Thus, $S(X) \cong \bigoplus_{k=1}^n S_k^{v'_k}$.

Taking $S' := S(X)$ yields that

$$\delta_{S(X)}: \text{Hom}_A(S(X), \bigoplus_{k=1}^n S_k^{v'_k}) \rightarrow \text{Ext}_A^1(S(X), X)$$

is an isomorphism. In particular there exists $\tau \in \text{Hom}_A(S(X), \bigoplus_{k=1}^n S_k^{v'_k})$ allowing a commutative diagram

$$\begin{array}{ccccccc} \delta_{S(X)}^*(\tau) \equiv \mathcal{E}(X): & 0 \rightarrow & X & \xrightarrow{\mu_X} & V(X) & \xrightarrow{\epsilon_X} & S(X) \rightarrow 0 \\ & & \parallel & & \downarrow \beta & & \downarrow \tau \\ \mathcal{E}': & 0 \rightarrow & X & \xrightarrow{\mu'} & Y & \xrightarrow{\epsilon'} & \bigoplus_{k=1}^n S_k^{v'_k} \rightarrow 0. \end{array}$$

It follows from the snake lemma that $\text{coker } \beta \cong \text{coker } \tau$. $\text{coker } \beta \in \mathcal{A}(S)$ by 4.1.e) This implies that $\text{coker } \tau = 0$, i.e. τ is an epimorphism. Since τ is $\mathbf{k}$ -linear and $\dim_{\mathbf{k}} \bigoplus_{k=1}^n S_k^{v'_k} = \dim_{\mathbf{k}} S(X) < \infty$, τ is actually an isomorphism. The 3-lemma implies that β is an isomorphism too.

- b) Apply $\text{Hom}_A(-, X)$ to the short exact sequence (4.2.1) and use $p.d. S(X) \leq 1$
- c) Apply $\text{Hom}_A(Z, -)$ to the short exact sequence (4.2.1)
- d) Let $f: X \rightarrow Y$ be an homomorphism in $\mathcal{F}(S)$. Consider the diagram

$$\begin{array}{ccccccc} \mathcal{E}(X): & 0 \rightarrow & X & \xrightarrow{\mu_X} & V(X) & \xrightarrow{\epsilon_X} & S(X) \rightarrow 0 \\ & & \downarrow f & & & & \\ \mathcal{E}(Y): & 0 \rightarrow & Y & \xrightarrow{\mu_Y} & V(Y) & \xrightarrow{\epsilon_Y} & S(Y) \rightarrow 0 \end{array}$$

Since

$$(4.2.4) \quad \text{Hom}_A(\mu_X, V(Y)): \text{Hom}_A(V(X), V(Y)) \xrightarrow{\sim} \text{Hom}_A(X, V(Y))$$

is an isomorphism by b), there exists a unique $V(f) \in \text{Hom}_A(V(X), V(Y))$ such that $V(f)\mu_X = \mu_Y f$. The uniqueness of $V(f)$ forces V to be a $\mathbf{k}$ -additive functor

$$- 1_{V(X)}\mu_X = \mu_X 1_X \text{ implies } V(1_X) = 1_{V(X)}$$

$$- \text{If } g: Y \rightarrow Z \in \mathcal{F}(S), \text{ then } V(g)V(f)\mu_X = V(g)\mu_Y f = \mu_Z gf, \text{ thus}$$

$$V(gf) = V(g)V(f).$$

$$- \text{If } f, g \in \text{Hom}_A(X, Y), \text{ then } (V(f) + V(g))\mu_X = \mu_Y(f + g), \text{ hence}$$

$$V(f + g) = V(f) + V(g).$$

$$- \text{If } \alpha \in \mathbf{k}, \text{ then } (\alpha V(f))\mu_X = \alpha(\mu_Y f) = \mu_Y(\alpha f), \text{ thus } V(\alpha f) = \alpha V(f)$$

Certainly μ is a natural transformation

If $V(f) = 0$, then $\mu_Y f = 0$. Hence, since μ_Y is a monomorphism, $f = 0$. This shows that $V: \text{Hom}_A(X, Y) \rightarrow \text{Hom}_A(V(X), V(Y))$ is an injective $\mathbf{k}$ -linear map

Now assume in addition $\text{Hom}_A(X, S) = 0$. Then c) implies

$\text{Hom}_A(X, Y) \cong \text{Hom}_A(X, V(Y))$. Thus, $\text{Hom}_A(X, Y) \cong \text{Hom}_A(V(X), V(Y))$ as $\mathbf{k}$ -space by (4.2.4). This forces (4.2.2) to be an isomorphism. ■

4.3 Theorem. Let A be a finite dimensional $\mathbf{k}$ -algebra with precisely n non-isomorphic simple modules; let $S = \bigoplus_{k=1}^n S_k \in \text{mod } A$ satisfying the following five conditions

$$\text{i) } p.d.S \leq 1,$$

$$\text{ii) } \text{Ext}_A^1(S, S) = 0,$$

$$\text{iii) } E_k := \text{End}(S_k)_A \text{ is a division algebra for all } 1 \leq k \leq n,$$

$$\text{iv) } \text{Hom}_A(S_k, S_l) = 0 \text{ for all } 1 \leq k \neq l \leq n \text{ and}$$

$$\text{v) } \text{Hom}_A(S, A) = 0.$$

Then there exists pairwise non-isomorphic, non-zero indecomposable modules

$$(4.3.1) \quad V_1, V_2, \dots, V_{n-s} \in \mathcal{A}(S)$$

with the following properties:

- a) There exist integers $a_k \geq 1$, $(1 \leq k \leq n-s)$ and a non-split short exact sequence

$$(4.3.2) \quad \mathcal{E}: 0 \rightarrow A \xrightarrow{\mu_A} \bigoplus_{k=1}^{n-s} V_k^{a_k} \xrightarrow{\epsilon_A} \bigoplus_{k=1}^s S_k^{v_k} \rightarrow 0$$

with $v_k = \dim \operatorname{Ext}_A^1(S_k, A)_{E_k} \geq 1$, $(1 \leq k \leq s)$

- b) $T(S) := \bigoplus_{k=1}^{n-s} V_k \oplus S$ is a tilting module with $\mathcal{G}(T(S)) = \mathcal{G}(S)$.
- c) $V := \bigoplus_{k=1}^{n-s} V_k$ is a generator of $\mathcal{A}(S)$, i.e. for each $X \in \mathcal{A}(S)$ there exist $V' \in \operatorname{add} V$ and an epimorphism $\varphi: V' \rightarrow X$
- d) For an indecomposable non-zero module $U \in \operatorname{mod} A$ the following statements are equivalent
- i) $U \cong V_k$ for some $1 \leq k \leq n-s$.
 - ii) There is an indecomposable projective module P such that U is isomorphic to a direct summand of $V(P)$.
 - iii) $U \in \mathcal{A}(S)$, $\operatorname{p.d.} U \leq 1$ and $\operatorname{Ext}_A^1(U, \bigoplus_{i=1}^{n-s} V_i) = 0$.
 - iv) $U \in \mathcal{A}(S)$ and $\operatorname{Ext}_A^1(U, X) = 0$ for all $X \in \mathcal{A}(S)$, i.e. U is Ext-projective in $\mathcal{A}(S)$ in the sense of Auslander-Smalø [see AS1 and AS2]

- e) If for $1 \leq k \leq s$

$$(4.3.3) \quad 0 \rightarrow \tau S_k \rightarrow Y_k \rightarrow S_k \rightarrow 0$$

is a non-splitting short exact sequence, then $Y_k \in \mathcal{A}(S)$.

- f) Let $Y \rightarrow S_k$ be an irreducible monomorphism with $Y \neq 0$ indecomposable and $\operatorname{p.d.} S_k/Y \leq 1$, then $Y \cong V_t$ for some $1 \leq t \leq n-s$.

g) $\text{Hom}_A(V_t, S) = 0$ implies V_t is projective

Remark 1. Certainly an indecomposable projective module $P \in \mathcal{A}(S)$ if and only if $P \cong V_t$ for some $1 \leq t \leq n$. Thus in particular a BB tilting module at some point k is of the form in the theorem if and only if $\text{Hom}_A(\tau_A^{-1} S_k, A) = 0$ if and only if $\tau d S_k = 1$ by 1.4. This is not always the case as the following example shows. Consider the bounded path algebra of the quiver with relations

$$\begin{array}{ccccc} & & 2 & & \\ & \swarrow \alpha_1 & & \nwarrow \beta_1 & \\ 1 & \xleftarrow{\alpha_2} & 3 & \xleftarrow{\beta_2} & 5 \\ & \searrow \alpha_3 & & \swarrow \beta_3 & \\ & & 4 & & \end{array} \quad \text{with} \quad \alpha_1 \beta_1 = \alpha_2 \beta_2 = \alpha_3 \beta_3$$

There is an APR tilting module at 1 but $\tau d S_1 = 2$, while the BB tilting modules at 2, 3 and 4 are well-defined and the corresponding simple modules have $\tau d = 1$.

Remark 2. For a given tilting module T_A Tachikawa-Wakamatsu introduced in [TW 2] the concept of a torsion resolution of $X \in \text{mod } A$. This is a short exact sequence

$$0 \rightarrow X \rightarrow V' \rightarrow T' \rightarrow 0$$

such that $V' \in \mathcal{G}(T)$ and $T' \in \text{add } T$. They proved that for any $X \in \text{mod } A$ there exists a torsion resolution

$$(*) \quad 0 \rightarrow X \xrightarrow{\alpha_X} V(X) \xrightarrow{\beta_X} T(X) \rightarrow 0$$

uniquely determined up to isomorphism of short exact sequences such that any torsion resolution of X is of the form

$$0 \rightarrow X \xrightarrow{\begin{bmatrix} \alpha_X \\ 0 \end{bmatrix}} V(X) \oplus T_0 \xrightarrow{\begin{bmatrix} \beta_X & \alpha \\ 0 & 1 \end{bmatrix}} T(X) \oplus T_0 \rightarrow 0$$

(*) is called the **minimal torsion resolution** of X . Certainly the short exact sequence $\mathcal{E}(X)$ for $X \in \mathcal{F}(S)$ in 4.2 is a minimal torsion resolution for the tilting module $T(S)$.

from the previous theorem. Note that $T(X) \in \text{add } S$. From Tachikawa-Wakamatsu's proof, it follows immediately that $T(X) \in \text{add } S$ for all minimal torsion resolutions and $\text{add } X \in \text{mod } A$.

Remark 3. If $E_k \cong k$ for all $1 \leq k \leq s$, for example if k is algebraically closed, then $T(S)$ is exactly the tilting module from 2.6, constructed by K. Bongartz in [Bo].

Proof. The assumptions on S allow us to apply 4.2 to $X_A = A_A$. Hence, there is a short exact sequence

$$(4.3.4) \quad \mathcal{E}(A) : 0 \rightarrow A \xrightarrow{\mu_A} V(A) \xrightarrow{\epsilon_A} S(A) \rightarrow 0$$

with $V(A) \in \mathcal{A}(S)$ and $S(A) \cong \bigoplus_{k=1}^s S_k^{v_k}$. Condition v) implies that each S_k is non-projective. Thus, $\text{Ext}_A^1(S_k, A) \cong D\text{Hom}_A(A, \tau_A S_k) \neq 0$ and consequently $v_k \geq 1$ for each k . In particular $\mathcal{E}(A)$ does not split by 4.2.a)

(1) $T = V(A) \oplus S$ is a tilting module.

Proof. T 1. $p.d. S(A) \leq 1$, since $p.d. S \leq 1$. Hence, (4.3.4) yields $p.d. V(A) \leq 1$.

Thus $p.d. T \leq 1$.

T 2. Since $V(A) \oplus S \in \mathcal{G}(S)$, 4.2.b) implies that

$$\text{Ext}_A^1(V(A), V(A) \oplus S) \cong \text{Ext}_A^1(A, V(A) \oplus S) = 0.$$

$$\text{Ext}_A^1(S, V(A)) = 0, \text{ since } V(A) \in \mathcal{A}(S). \text{ Altogether } \text{Ext}_A^1(T, T) = 0$$

T 3: (4.3.4) has the desired form.

Decompose

$$(4.3.5) \quad V(A) \cong \bigoplus_{k=1}^r V_k^{a_k}$$

with $V_1, V_2, \dots, V_r$ pairwise non-isomorphic indecomposable and set $V = \bigoplus_{k=1}^r V_k$. The assumption iv) and $V(A) \in \mathcal{A}(S)$ assures that

$$S_1, S_2, \dots, S_s, V_1, V_2, \dots, V_r$$

are pairwise non-isomorphic and indecomposable, hence $r = n - s$ by 2.3 g or 2.6. This shows the existence of the modules in (3.4.1) and a).

b) As a multiplicative free version of T , $T(S)$ is also a tilting module using 2.6

$\text{Ext}_A^1(V(A), X) \cong \text{Ext}_A^1(A, X) = 0$ for all $X \in \mathcal{G}(S)$ by 4.2 b implies that

$$\mathcal{G}(T(S)) = \mathcal{G}(S).$$

c) Let $X \in \mathcal{A}(S)$. There is an epimorphism $\psi: A^p \rightarrow X$ 4.2 d) assures the existence of a homomorphism $V(\psi): V(A^p) \rightarrow V(X)$ such that $\mu_X \psi = V(\psi) \mu_{A^p}$. Since μ_X is an isomorphism by 4.2 a), $\mu_X^{-1} V(\psi) \cdot V(A^p) \rightarrow X$ is an epimorphism $V(A^p) \simeq (V(A))^p \in \text{add } V$, by 4.2 d).

d) "i) $\Leftrightarrow$ ii)": Let $A_A \cong \bigoplus_{i=1}^n P_i^{n_i}$ with P_i indecomposable, pairwise non-isomorphic projective A -modules. Then $V(A) \cong \bigoplus_{i=1}^n V(P_i)^{n_i}$ by 4.2 d). Since U is indecomposable, the Krull-Schmidt theorem implies the equivalence of i) and ii)

- "i) $\Rightarrow$ iii)": See b)

- "iii) $\Rightarrow$ iv)": Let $X \in \mathcal{A}(S)$. By c) there is $V' \in \text{add } V$ and a short exact sequence

$$0 \rightarrow K \rightarrow V' \xrightarrow{\varphi} X \rightarrow 0$$

Applying $\text{Hom}_A(U, -)$ yields an exact sequence

$$\text{Ext}_A^1(U, V') \rightarrow \text{Ext}_A^1(U, X) \rightarrow \text{Ext}_A^2(U, K)$$

Hence iii) yields $\text{Ext}_A^1(U, X) = 0$.

- "iv) $\Rightarrow$ i)": Since $U \in \mathcal{A}(S)$, there is $V' \in \text{add } V$ and a short exact sequence

$$(4.3.6) \quad 0 \rightarrow K \rightarrow V' \xrightarrow{\varphi} U \rightarrow 0$$

applying c). Since $K \in \mathcal{A}(S)$ by 4.1.e), iv) implies that (4.3.6) splits. Thus, since U is indecomposable, the theorem of Krull-Schmidt implies that U is isomorphic to some V_k .

e) $\text{Hom}_A(S_l, \tau S_k) \cong D \text{Ext}_A^1(S_k, S_l) = 0$ since $p \cdot d \cdot S_k \leq 1$. Thus, $\text{Hom}_A(S_l, -)$ applied to (4.3.3) yields an exact sequence

$$(*) \quad 0 \rightarrow \text{Hom}_A(S_l, Y_k) \rightarrow \text{Hom}_A(S_l, S_k) \xrightarrow{\delta_l} \text{Ext}_A^1(S_l, \tau S_k) \rightarrow \text{Ext}_A^1(S_l, Y_k) \rightarrow 0.$$

If $l \neq k$, then $\text{Hom}_A(S_l, S_k) = 0$ and $\text{Ext}_A^1(S_l, \tau S_k) \cong D \text{Hom}_A(S_k, S_l) = 0$ since $i \cdot d \cdot \tau S_k \leq 1$. Thus, the exactness of $(*)$ implies that $Y_k \in \mathcal{A}(S_l)$.

If $l = k$, then $\delta_k \neq 0$, since (4.3.3) does not split and is congruent to $\delta_k(1_{S_k})$. Since

$$\dim \text{Hom}_A(E_k S_k, S_k)_{E_k} = \dim E_k E_k = 1 \quad \text{and}$$

$$\dim \text{Ext}_A^1(E_k S_k, \tau S_k)_{E_k} = \dim(D \text{Hom}_A(S_k, E_k S_k))_{E_k} = \dim(D_{E_k} E_k)_{E_k} = 1,$$

δ_k is an isomorphism. Thus, the exactness of $(*)$ implies that $Y_k \in \mathcal{A}(S_k)$.

Altogether $Y_k \in \mathcal{A}(S)$.

f) Check condition d iv): $Y \in \mathcal{A}(S)$ by e). Let $X \in \mathcal{A}(S)$. Then $\text{Hom}_A(-, X)$ applied to $0 \rightarrow Y \rightarrow S_k \rightarrow S_k/Y \rightarrow 0$ yields an exact sequence

$$\text{Ext}_A^1(S_k, X) \rightarrow \text{Ext}_A^1(Y, X) \rightarrow \text{Ext}_A^2(S_k/Y, X)$$

Since $\text{Ext}_A^1(S_k, X) = 0$ and $\text{Ext}_A^2(S_k/Y, X) = 0$ by assumption, $\text{Ext}_A^1(Y, X) = 0$ for all $X \in \mathcal{A}(S)$. Hence, $Y \cong V_t$ for some $1 \leq t \leq n - s$, using d).

g) Suppose $\text{Hom}_A(V_t, S) = 0$ and V_t is not projective. Then

$$\text{Ext}_A^1(S, \tau_A V_t) \cong D \text{Hom}_A(V_t, S) = 0.$$

Thus, $\tau_A V_t \in \mathcal{G}(S)$ and b) implies that $\text{Ext}_A^1(V_t, \tau_A V_t) = 0$ - a contradiction. ■

4.4 The endomorphism algebra. Notation and assumptions as in 4.3. Define in addition

$$(4.4.1) \quad T(S) := T := V \oplus S; \quad C(S) := C := \text{End } V_A; \quad B(S) := B := \text{End } T_A$$

where the "S-free notation" will be used throughout this section. $C(S)$ and $B(S)$ are basic rings by construction. The reader is reminded of the bimodule structures of ${}_C V_A$ and ${}_B T_A$. Denote by

$$\begin{aligned} \pi_X: T \rightarrow X & \text{ the natural projection} \\ \mu_X: X \rightarrow T & \text{ the natural injection} \end{aligned} \quad \text{for } X \text{ a direct summand of } T_A$$

Then a complete list of orthogonal primitive idempotents of B is given by

$$\begin{aligned} (4.4.2) \quad e_k &= \mu_{S_k} \pi_{S_k}, \quad (1 \leq k \leq s), \\ f_k &= \mu_{V_k} \pi_{V_k}, \quad (1 \leq k \leq n-s) \end{aligned}$$

A complete list of indecomposable projective right B -modules is given by

$$\begin{aligned} (4.4.3) \quad e_k B_B &\cong \text{Hom}_A({}_B T, S_k)_B, \quad (1 \leq k \leq s), \\ f_k B_B &\cong \text{Hom}_A({}_B T, V_k)_B, \quad (1 \leq k \leq n-s) \end{aligned}$$

A complete list of indecomposable projective left B -modules is given by

$$\begin{aligned} (4.4.4) \quad {}_B B e_k &\cong {}_B \text{Hom}_A(S_k, {}_B T), \quad (1 \leq k \leq s), \\ {}_B B f_k &\cong {}_B \text{Hom}_A(V_k, {}_B T), \quad (1 \leq k \leq n-s) \end{aligned}$$

Also

$$(4.4.5) \quad e = \sum_{k=1}^s e_k = \mu_S \pi_S \quad \text{and} \quad f = \sum_{k=1}^{n-s} f_k = \mu_V \pi_V$$

are orthogonal idempotents of B satisfying $1_B = e + f$

The following properties are easily verified :

$e_k B$, $(1 \leq k \leq s)$ and $eB = eBe$ are 2-sided ideals of B due to 4.3.iv and $V \in \mathcal{A}(S)$

$eBe \cong \text{End } S_A$ is the direct product of the division algebras $E_1, \dots, E_s$ due to condition 4.3.iv).

$fBf \cong C$ as algebras.

$fBe_k \cong \text{Hom}_A(S_k, V) = 0$.

$E_k e_k Bf_C \cong_{E_k} \text{Hom}_A({}_C V, E_k S_k)_C$ as bimodules

Thus there is a ring isomorphism

$$(4.4.6) \quad B \cong \begin{pmatrix} E_1 & 0 & \dots & 0 & e_1 Bf \\ 0 & E_2 & 0 & 0 & e_2 Bf \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & E_{s-1} & 0 & e_{s-1} Bf \\ 0 & 0 & \dots & 0 & E_s & e_s Bf \\ 0 & 0 & \dots & 0 & 0 & C \end{pmatrix}.$$

The category of right modules over that triangular matrix ring is equivalent to the category

$\mathcal{R}$, which is given by the following data.

- The objects of $\mathcal{R}$ are systems

$$\underline{X} = (X_1, \dots, X_s, \hat{X}, \varphi_1, \dots, \varphi_s)$$

where X_k is an E_k -space, $\hat{X}$ is a C -module and $\varphi_k: X_k \otimes_{E_k} e_k Bf \rightarrow \hat{X}$ is a homomorphism of C -modules

- A morphism

$$h: \underline{X} = (X_1, \dots, X_s, \hat{X}, \varphi_1, \dots, \varphi_s) \rightarrow \underline{Y} = (Y_1, \dots, Y_s, \hat{Y}, \psi_1, \dots, \psi_s)$$

is a system $h = (h_1, \dots, h_s, \hat{h})$, where $h_k: X_k \rightarrow Y_k$ is an E_k -linear map and $\hat{h}: \hat{X} \rightarrow \hat{Y}$

a C -homomorphism, such that for $1 \leq k \leq s$ the following diagram commutes

$$\begin{array}{ccc} X_k \otimes_{E_k} e_k Bf & \xrightarrow{\varphi_k} & \hat{X} \\ \downarrow h_k \otimes e_k Bf & & \downarrow \hat{h} \\ Y_k \otimes_{E_k} e_k Bf & \xrightarrow{\psi_k} & \hat{Y} \end{array}$$

Modules in $\text{mod } B$ correspond to systems $X \in \mathcal{R}$ with each X_k and $\hat{X}$ finite dimensional k -spaces.

There is a full exact embedding $\text{mod } C \rightarrow \text{mod } B$. Its image will be identified with $\text{mod } C$ from now on. Then $X \in \text{mod } B$ is a C -module if and only if $Xe = 0$.

For $X \in \text{mod } A$ we have

$$\text{Hom}_A(T, X)_B = (\text{Hom}_A(S_1, X), \quad , \text{Hom}_A(S_s, X), \text{Hom}_A(V, X), \varphi_1, \quad , \varphi_s)$$

with

$$\varphi_k: \text{Hom}_A(S_k, X) \otimes_{E_k} \text{Hom}_A(V, S_k) \rightarrow \text{Hom}_A(V, X), \quad \alpha \otimes \beta \mapsto \alpha\beta$$

In particular

$$(4.4.3') \quad e_k B = (0, \quad , 0, E_k, 0, \quad , 0, \text{Hom}_A(V, S_k), \varphi_1, \quad 0, \quad , 1_{\text{Hom}_A(V, S_k)}, \quad , 0)$$

$$(4.4.3'') \quad f_k B = (0, \quad , 0, \text{Hom}_A(V, V_k), 0, \quad , 0)$$

Note that (4.4.3'') provides a complete list of indecomposable projective right C -modules.

Similar for $X \in \text{mod } A$

$$\text{Ext}_A^1(T, X) = (\text{Ext}_A^1(S_1, X), \quad , \text{Ext}_A^1(S_s, X), \text{Ext}_A^1(V, X), \varphi_1, \quad , \varphi_s)$$

with

$$\varphi_k: \text{Ext}_A^1(S_k, X) \otimes_{E_k} \text{Hom}_A(V, S_k) \rightarrow \text{Ext}_A^1(V, X), \quad \xi \otimes \beta \mapsto \text{Ext}_A^1(\beta, X)(\xi)$$

$$(4.4.7) \quad \text{Ext}_A^1(T, \tau_A S_k) \cong \text{top}(e_k B_B) \text{ is simple injective for all } 1 \leq k \leq s$$

The injectivity is given by 2.4. For $1 \leq l \leq s$

$$\text{Ext}_A^1(S_l, \tau_A S_k) \cong D\text{Hom}_A(S_k, S_l) \cong \delta_{k,l} D E_k \quad \text{and}$$

$$\text{Ext}_A^1(V, \tau_A S_k) \cong D\text{Hom}_A(S_k, V) = 0$$

shows that $\text{Ext}_A^1(T, \tau_A S_k)_B \cong \text{top}(e_k B)$.

With the notation of this section, we have the following properties of the tilting process $\text{mod } A \leftrightarrow \text{mod } B$.

4.5 Proposition. a) $A(S) \subset \mathcal{G}(T)$ and $\text{mod } C \subset \mathcal{Y}(T)$

$A(S)$ and $\text{mod } C$ are equivalent via the restriction of the tilting functors

$$F = \text{Hom}_A({}_B T, -): A(S) \rightarrow \text{mod } C \text{ and } G = - \otimes_B T_A: \text{mod } C \rightarrow A(S)$$

b) $\text{gl } d C \leq \text{gl } d A$

c) If $f: X \rightarrow Y$ is an irreducible map in $\text{mod } A$ and $X, Y \in A(S)$, then

$\text{Hom}_A(T, f): \text{Hom}_A(T, X) \rightarrow \text{Hom}_A(T, Y)$ is an irreducible map in $\text{mod } C$

Proof. $A(S) \subseteq \mathcal{G}(S) = \mathcal{G}(T)$ by 4.3 e)

Applying $\text{Hom}_A(-, T)$ to (4.3.2) yields an exact sequence

$$0 \rightarrow \text{Hom}_A\left(\bigoplus_{k=1}^s S_k^{v_k}, T\right) \rightarrow \text{Hom}_A\left(\bigoplus_{k=1}^{n-s} V_k^{a_k}, T\right) \rightarrow \text{Hom}_A(A, T) \rightarrow 0$$

Using ${}_B T \cong {}_B \text{Hom}_A(A, {}_B T)$ and (4.4.4), this sequence is a projective resolution of ${}_B T$ (the canonical resolution) of the form

$$0 \rightarrow \bigoplus_{k=1}^s (Be_k)^{v_k} \rightarrow \bigoplus_{k=1}^{n-s} (Bf_k)^{a_k} \rightarrow {}_B T \rightarrow 0$$

Thus, for any $Y \in \text{mod } B$, there is an exact sequence

$$(4.5.1) \quad 0 \rightarrow \text{Tor}_1^B(Y, T) \rightarrow Y \otimes_B \bigoplus_{k=1}^s (Be_k)^{v_k} \rightarrow Y \otimes_B \bigoplus_{k=1}^{n-s} (Bf_k)^{a_k} \rightarrow Y \otimes_B T \rightarrow 0.$$

Since $Y \otimes_B Be_k \cong Ye_k$, $Y \in \text{mod } C$ implies $Y \otimes_B Be_k \cong Ye_k = 0$ for $1 \leq k \leq s$

Hence, exactness of (4.5.1) yields $\text{Tor}_1^B(Y, T) = 0$ for all $Y \in \text{mod } C$

Thus, $\text{mod } C \subseteq \mathcal{Y}(T)$

The claimed equivalence is proven if we show that these subcategories correspond to each other under the equivalence between $\mathcal{G}(T)$ and $\mathcal{Y}(T)$, see 2.3 c. Thus it remains only to show:

i) $X \in \mathcal{A}(S)$ implies $F(X) = \text{Hom}_A(T, X) \in \text{mod } C$ and

ii) $Y \in \text{mod } C$ implies $G(Y) \in \mathcal{A}(S)$

- "i)" $F(X)e = \text{Hom}_A(eT, X) \cong \text{Hom}_A(S, X) = 0$ for $X \in \mathcal{A}(S)$

- "ii)" Let $Y \in \text{mod } C$. Then $Y \in \mathcal{Y}(T_A)$, implying that $G(Y) \in \mathcal{G}(T_A) = \mathcal{G}(S)$

Since $S, G(Y) \in \mathcal{G}(T_A)$, applying twice 2.3 c yields isomorphisms

$$\text{Hom}_A(S, G(Y)) \simeq \text{Hom}_B(F(S), FG(Y)) \simeq \text{Hom}_B(eB, Y) \simeq Ye = 0$$

Thus, $G(Y) \in \mathcal{A}(S)$

b) If $\text{gl } d A = \infty$ there is nothing to show. So assume $\text{gl } d A = g < \infty$. Let

$Y \in \text{mod } C$. Then $Y \in \mathcal{Y}(T_A)$ by a) $p d Y_B = g$ by 2.5 a). Thus a minimal projective B -resolution of Y is of the form

$$(4.5.2) \quad 0 \rightarrow Q_t \xrightarrow{f_t} Q_{t-1} \rightarrow \dots \rightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} Y_B \rightarrow 0$$

with $t = p d Y_B \leq g$. Since $Ye = 0$ and δ_0 is a projective cover of Y_B ,

$Q_0 \in \text{add } \bigoplus_{k=1}^{n-s} f_k B$. Since $f_k B e = 0$ for $1 \leq k \leq n-s$ and (4.5.2) is a minimal

projective B -resolution, all $Q_i \in \text{add } \bigoplus_{k=1}^{n-s} f_k B$. Hence, all Q_i are actually projective C -modules and (4.5.2) is also a minimal projective C -resolution of Y_C .

c) If $\text{Hom}_A(T, f) = h'g'$ for some $g' \in \text{Hom}_A(T, X) \rightarrow Z'$ and $h' \in Z' \rightarrow \text{Hom}_A(T, Y)$ then a) implies $Z' = \text{Hom}_A(T, Z)$ for some $Z \in \mathcal{A}(S)$, $g' = \text{Hom}_A(T, g)$ for some $g: X \rightarrow Z$; and $h' = \text{Hom}_A(T, h)$ for some $h: Z \rightarrow Y$. Thus, $f = hg$. Consequently either g is a splitting monomorphism or h is a splitting epimorphism, forcing, since the

equivalence is exact, either g' to be a splitting monomorphism or h' to be a splitting epimorphism. ■

4.6 Sink summands. Call a direct summand S of a module X a **sink summand** of X , if for each indecomposable direct summand S' of S and each indecomposable direct summand X' of X satisfying $S' \not\cong X'$, $\text{Hom}_A(S', X') = 0$. Certainly each S_k in 4.2 is a sink summand of S ; actually S is a sink summand of itself. In 4.3 the module S is a sink summand of the tilting module $T(S)$. This tilting module is distinguished from all tilting modules having S as sink summand in the following way. Let

$$T = \bigoplus_{k=1}^{n-s} T_k^{t_k} \oplus \bigoplus_{k=1}^s S_k^{s_k}$$

be a tilting module having $S = \bigoplus_{k=1}^s S_k$ as sink summand, where S satisfies the assumptions of theorem 4.3. Let $F := \text{Hom}_A(T(S), -) \text{ mod } A \rightarrow \text{mod } B(S)$. Then $T \in \mathcal{G}(T(S))$ by 4.3 e). Thus, $\text{p.d. } FT_{B(S)} \leq 1$ by 2.5.a), $\text{Ext}_{B(S)}^1(FT, FT) = 0$ by 2.3 e) and FT has precisely n non-isomorphic direct summands. Thus,

$$FT_{B(S)} = \bigoplus_{k=1}^{n-s} (FT_k)^{t_k} \oplus \bigoplus_{k=1}^s (e_k B(S))^{s_k}$$

is a tilting module over $B(S)$ with $\text{End } FT_{B(S)} \cong \text{End } T_A$ and $\bigoplus_{k=1}^{n-s} FT_k$ is a tilting module over $C(S)$.

4.7 Dual lemma of 4.2. Let $S = \bigoplus_{k=1}^s S_k \in \text{mod } A$ satisfy the assumptions of theorem 4.3. Then

- a) For each $X \in \mathcal{G}(S)$ there exists a up to isomorphism uniquely determined short exact sequence

$$(4.7.1) \quad \mathcal{E}^*(X): \quad 0 \rightarrow S^*(X) \xrightarrow{\mu_X} W(X) \xrightarrow{\epsilon_X} X \rightarrow 0$$

with $W(X) \in \mathcal{A}(S)$ and $S'(X) \in \text{add}(r_A S)$. Moreover, $S'(X) \sim \bigoplus_{k=1}^r (r_A S_k)^{w_k}$ with

$$w_k = \dim_{\text{End } r_A S_k} \text{Ext}_A^1(X, r_A S_k) \cdot \dim_{E_k} \text{Hom}_A(S_k, X)$$

b) If $Y \in \text{mod } A$ and $X \in \mathcal{G}(S)$, then

$$\text{Ext}_A^k(Y, \epsilon_X) = \text{Ext}_A^k(Y, W(X)) \rightarrow \text{Ext}_A^k(Y, X)$$

is an epimorphism for $k = 1$ and an isomorphism for $k > 2$. If $Y \in \mathcal{F}(S)$, then

$$\text{Ext}_A^1(Y, \epsilon_X) = \text{Ext}_A^1(Y, W(X)) \rightarrow \text{Ext}_A^1(Y, X) \quad \text{is an isomorphism and}$$

$\text{Hom}_A(Y, \epsilon_X) = \text{Hom}_A(Y, W(X)) \rightarrow \text{Hom}_A(Y, X)$ is an epimorphism. If $Y \in \mathcal{A}(S)$, then also $\text{Hom}_A(Y, \epsilon_X)$ is an isomorphism.

c) Suppose $\text{Hom}_A(r_A S, Z) = 0$ and there exists $l > 0$ such that $\text{Ext}_A^i(r_A S, Z) = 0$ for all $1 \leq i \leq l$. Then

$$\text{Hom}_A(\epsilon_X, Z) = \text{Hom}_A(X, Z) \rightarrow \text{Hom}_A(W(X), Z) \quad \text{and}$$

$$\text{Ext}_A^i(\epsilon_X, Z) = \text{Ext}_A^i(X, Z) \rightarrow \text{Ext}_A^i(W(X), Z)$$

are isomorphisms for $1 \leq i \leq l$.

d) For each $f: X \rightarrow Y \in \mathcal{G}(S)$ there exists a unique $W(f): W(X) \rightarrow W(Y)$ such that $f \epsilon_X = \epsilon_Y W(f)$. $W: \mathcal{G}(S) \rightarrow \mathcal{A}(S)$ is a k -additive functor and $\epsilon: W \rightarrow 1_{\mathcal{G}(S)}$ is a natural transformation.

In particular $W(\bigoplus_{k=1}^r X_k) \cong \bigoplus_{k=1}^r W(X_k)$ for all $\bigoplus_{k=1}^r X_k \in \mathcal{G}(S)$.

The map $W: \text{Hom}_A(X, Y) \rightarrow \text{Hom}_A(W(X), W(Y))$
(4.7.2)

$$f \mapsto W(f)$$

is injective for all $X, Y \in \mathcal{G}(S)$ and an isomorphism, if $\text{Hom}_A(r_A S, Y) = 0$.

Proof. The assumptions on S imply using 1.4 that

1) $\text{i.d. } r_A S \leq 1$;

$$\text{ii) } \text{Ext}_A^1(\tau_A S, \tau_A S) \cong \text{Ext}_A^1(S, S) = 0;$$

$$\text{iii) } \text{End}(\tau_A S_k)_A \cong \text{End}(S_k)_A = E_k;$$

$$\text{iv) } \text{Hom}_A(\tau_A S_k, \tau_A S_l) \cong \text{Hom}_A(S_k, S_l) = 0 \text{ for } k \neq l,$$

$$\text{v) } X \in \mathcal{G}(S) \text{ if and only if } \text{Hom}_A(X, \tau_A S) = 0.$$

$\text{Ext}_A^1(X, \tau_A S_k) \cong \text{DHom}_A(S_k, X)$ as k -spaces implies the dimension equation in a)

Dualizing the proof of 4.2 where S is replaced by $\tau_A S$ is now a matter of routine ■

4.8 Proposition. Let $S = \bigoplus_{k=1}^n S_k \in \text{mod } A$ satisfy the assumptions of theorem 4.3. Adopt the notation of 4.3 and 4.4. Then there exist non-zero indecomposable, pairwise non-isomorphic modules

$$(4.8.1) \quad W_1, W_2, \dots, W_{n-s} \in \mathcal{A}(S)$$

with the following properties :

a) $W := \bigoplus_{k=1}^{n-s} W_k$ is a cogenerator of $\mathcal{A}(S)$, i.e. for each $X \in \mathcal{A}(S)$ there exists $W' \in \text{add } W$ and a monomorphism $\psi: X \rightarrow W'$.

b) For an indecomposable non-zero module $U \in \text{mod } A$ the following statements are equivalent:

i) $U \cong W_k$ for some $1 \leq k \leq n-s$.

ii) $\text{Hom}_A(T(S), U)$ is an indecomposable injective $C(S)$ -module.

iii) $U \in \mathcal{A}(S)$ and $\text{Ext}_A^1(X, U) = 0$ for all $X \in \mathcal{A}(S)$, i.e. U is Ext-injective in $\mathcal{A}(S)$ in the sense of Auslander-Smalø.

iv) $U \in \mathcal{A}(S)$, $\text{id.} U \leq 1$ and $\text{Ext}_A^1(W, U) = 0$.

c) $\text{Ext}_A^1(X, W) = 0$ for all $X \in \mathcal{F}(S)$.

d) There exist $b_k \geq 1$ and a non-splitting short exact sequence

$$(4.8.2) \quad 0 \rightarrow \bigoplus_{k=1}^s (\tau_A S_k)^{w_k} \rightarrow \bigoplus_{k=1}^{n-s} W_k^{b_k} \rightarrow DA \rightarrow 0$$

with $w_k = \dim \text{Hom}_A(S_k, DA)_{E_k} \geq 1$

e) $W^* := W \oplus \tau_A S$ satisfies $\text{i.d.} W^* = 1$ and $\text{Ext}_A^1(W^*, W^*) = 0$, i.e. ${}_A DW^*$ is a tilting module.

f) $\text{Hom}_A(\tau_A S, W_k) = 0$ implies $(W_k)_A$ is injective.

Remark. Certainly an indecomposable injective module Q_A is contained in $\mathcal{A}(S)$ if and only if $Q_A \cong W_k$ for some $1 \leq k \leq n-s$.

Proof. This is the dual version of 4.3. Since $(DA)_A \in \mathcal{G}(S)$, there is a short exact sequence

$$0 \rightarrow \bigoplus_{k=1}^{n-s} (\tau_A S_k)^{w_k} \rightarrow W(DA) \rightarrow DA \rightarrow 0$$

with $w_k = \dim \text{Hom}_A(S_k, DA)_{E_k} \geq 1$ and $W(DA) \in \mathcal{A}(S)$

$\text{Ext}_A^1(X, W(DA)) \cong \text{Ext}_A^1(X, DA) = 0$ for all $X \in \mathcal{F}(S)$ by 4.7 b

In particular $\text{Ext}_A^1(\tau_A S \oplus W(DA), W(DA)) = 0$.

$\text{Ext}_A^1(W(DA), \tau_A S) \cong D\text{Hom}_A(S, W(DA)) = 0$, since $\text{i.d.} \tau_A S < 1$. Thus, $\tau_A S \oplus W(DA)$ has no self-extensions. $\text{i.d.} \tau_A S \leq 1$ implies $\text{i.d.} W(DA) \leq 1$. Together with the above sequence we see that $D(\tau_A S \oplus W(DA))$ is a tilting module. Consequently 2.3 g implies $W(DA) \cong \bigoplus_{k=1}^{n-s} W_k^{b_k}$ with $b_k \geq 1$ and W_k non-zero, indecomposable and pairwise non-isomorphic. This proves c), d) and e).

a) A monomorphism $\phi : X \rightarrow (DA)^t$ factors over $W(DA)^t$ by 4.7 e for all $X \in \mathcal{A}(S)$

Certainly the lift is a monomorphism

b) - "ii) $\Leftrightarrow$ iii)": Follows immediately from 4.5 a, using 2.3 g

- "i) $\Rightarrow$ iv)": See e).

- "iv) $\Rightarrow$ iii)": Let $X \in \mathcal{A}(S)$. a) assures the existence of a short exact sequence

$$0 \rightarrow X \rightarrow W' \rightarrow C \rightarrow 0$$

with $W' \in \text{add } W$. Hence, there is an exact sequence

$$0 = \text{Ext}_A^1(W', U) \rightarrow \text{Ext}_A^1(X, U) \rightarrow \text{Ext}_A^2(C, U) = 0$$

"iii) $\Rightarrow$ i)" : a) asserts the existence of a short exact sequence

$$0 \rightarrow U \rightarrow W' \rightarrow C \rightarrow 0$$

with $W' \in \text{add } W \subset \mathcal{A}(S)$. Thus $C \in \mathcal{A}(S)$ by 4.1.e) and the sequence splits by iii).

f) Suppose $\text{Hom}_A(\tau_A S, W_k) = 0$ and W_k is not injective. Then

$$\text{Hom}_A(S, \tau_A^{-1} W_k) = \underline{\text{Hom}}_A(S, \tau_A^{-1} W_k) \cong \overline{\text{Hom}}_A(\tau_A S, W_k) = 0$$

Thus by c) $\text{Ext}_A^1(\tau_A^{-1} W_k, W_k) = 0$, an obvious contradiction. ■

THE HEREDITARY CASE

In this chapter A denotes always a hereditary finite dimensional k -algebra with precisely n non-isomorphic simple modules. The results from chapter I and chapter III will be freely used, in particular 1.4 and 3.1

5.1. Let $S = \bigoplus_{k=1}^s S_k \in \text{mod } A$ satisfying

- i) $\text{Ext}_A^1(S, S) = 0$;
- ii) S_k is indecomposable and non-projective for $1 \leq k \leq s$ and
- iii) $\text{Hom}_A(S_k, S_l) = 0$ for all $1 \leq k \neq l \leq s$

Then S satisfies the hypothesis of theorem 4.3 by 3.4 and (3.1.1). Thus, there is the tilting module $T(S)_A = \bigoplus_{t=1}^{n-s} V_t \oplus S$ with the properties stated in 4.3. $C(S) = \text{End}(\bigoplus_{t=1}^{n-s} V_t)_A$ is a hereditary algebra by 4.5. Set $B(S) = \text{End } T(S)_A$. The cotilting module W^* from 4.8 is now also a tilting module

Conversely any tilting module T_A has a non-trivial sink summand by 3.5. If T_A has non-projective direct summands, then it has also non-projective sink summand by (3.1.1). Let $S = \bigoplus_{k=1}^s S_k$ a multiplicative free sink summand of T with each S_k non-projective. Certainly S satisfies the above three conditions. Hence, there exists also the tilting module $T(S)_A$. Then $\text{Hom}_A(T(S), T) = W \oplus \bigoplus_{k=1}^s e_k B(S)$ is a tilting module over $B(S)$ with W a tilting module over $C(S)$ and $\text{End}(W \oplus \bigoplus_{k=1}^s e_k B)_B \simeq \text{End } T_A$

5.2 Lemma. Let $S = \bigoplus_{k=1}^s S_k \in \text{mod } A$ with each S_k indecomposable non-projective

Then the map

$$\Psi: \text{ind } A(S) \longrightarrow \text{ind } A(r_A S)$$

$$\Psi(X) := \begin{cases} r_A X, & \text{if } X \text{ is not projective;} \\ DAe, & \text{if } X = eA \text{ is projective,} \end{cases}$$

is a bijection.

Proof. If X is not projective, then $\text{Hom}_A(S, X) \cong \text{Hom}_A(\tau_A S, \tau_A X)$ and

$\text{Ext}_A^1(S, X) \cong \text{Ext}_A^1(\tau_A S, \tau_A X)$ by 3.1. Thus $X \in \mathcal{A}(S)$ if and only if $\tau_A X \in \mathcal{A}(\tau_A S)$

If $X = eA$ is indecomposable projective, then certainly $\text{Hom}_A(S, eA) = 0$ and

$\text{Ext}_A^1(\tau_A S, DAe) = 0$. Now

$$\text{Hom}_A(\tau_A S_k, DAe) \cong \text{Hom}_A(D \text{Ext}_A^1(S_k, A), DAe) \cong D \text{Hom}_A(Ae, \text{Ext}_A^1(S_k, A))$$

$$\cong D[e \text{Ext}_A^1(S_k, A)] \cong D \text{Ext}_A^1(S_k, eA)$$

shows that $eA \in \mathcal{A}(S)$ if and only if $DAe \in \mathcal{A}(\tau_A S)$. ■

5.3 Theorem. Let A be a hereditary, finite dimensional k -algebra with precisely n non-isomorphic simple modules and $S = \bigoplus_{k=1}^s S_k \in \text{mod } A$ satisfying

i) $\text{Ext}_A^1(S, S) = 0$;

ii) $\text{Hom}_A(S_k, S_l) = 0$ for all $1 \leq k \neq l \leq s$ and

iii) S_k and $\tau_A S_k$ are indecomposable and non-projective for $1 \leq k \leq s$

Let $T := T(S) = \bigoplus_{t=1}^{n-s} V_t \oplus S$. Assume the enumeration satisfies $\text{Hom}_A(V_u, V_t) = 0$

for all $1 \leq t < u \leq n-s$ and

$V_1, V_2, \dots, V_r$ are projective, whereas

$V_{r+1}, V_{r+2}, \dots, V_{n-s}$ are non projective.

Let $U: \mathcal{F}(\tau_A S) \rightarrow \mathcal{A}(\tau_A S)$ be the functor described in 4.2 for $\tau_A S$. Then

a) $T(\tau_A S) = \bigoplus_{t=r+1}^{n-s} \tau_A V_t \oplus \bigoplus_{k=1}^r U(V_k) \oplus \tau_A S$

b) Let $C := C(S) = \text{End}(\bigoplus_{t=1}^{n-s} V_t)_A$. Let f_t be the idempotent in $B := B(S) = \text{End } T_A$

with $f_t B_B \cong \text{Hom}_A(T, V_t)_B$. If each V_t ($1 \leq t \leq r$) is non-injective, then

$$C(\tau_A S) \cong \text{End} \left(\bigoplus_{t=r+1}^{n-s} f_t C \oplus \bigoplus_{t=1}^r \tau_C^{-1} f_t C \right)_C.$$

Thus $C(\tau_A S)$ is an r -iterated APR-tilt of $C(S)$. In particular the valued graphs of $C(S)$ and $C(\tau_A S)$ coincide and the algebras have the same representation type

Remark. Note that only a uniserial hereditary algebra i.e. the path algebra of the quiver $\bullet \leftarrow \bullet \leftarrow \dots \leftarrow \bullet \leftarrow \bullet$ has projective-injective modules

Corollary. If $S_A = \bigoplus_{k=1}^r S_k$ is regular with $\text{Ext}_A^1(S, S) = 0$ and $\text{Hom}_A(S_k, S_l) = 0$ for $l \neq k$, then the valued graphs, connectivity and representation type of the algebras $C(\tau_A^z S)$ coincide for all integer z , i.e. are invariant of the τ_A -orbit of S_A

Proof. Note first that S and $\tau_A S$ satisfy the three conditions in 5.1, i.e. the tilting modules $T(S)$ and $T(\tau_A S)$ are well-defined. 3.5 and (3.1.1) assure that the labeling of the V_t can be chosen as assumed.

a) The indecomposable direct summands of $U_t := U(V_t)$, $(1 \leq t \leq r)$ are direct summands of $T(\tau_A S)$ by 4.3 d. $\text{Hom}_A(V_t, V_u) \cong \text{Hom}_A(U_t, U_u)$ by 4.2 e, since

$$\text{Hom}_A(V_t, \tau_A S) \cong D \text{Ext}_A^1(S, V_t) = 0 \text{ Hence}$$

i) $\text{End } V_t \cong \text{End } U_t$, in particular U_t is indecomposable and

ii) $\text{Hom}_A(U_t, U_u) = 0$ for $1 \leq u < t \leq r$

Altogether $U_1, U_2, \dots, U_r$ are indecomposable and pairwise non-isomorphic direct summands of $T(\tau_A S)$

Now let $r+1 \leq t \leq n-s$. Then $\tau_A V_t \in \mathcal{A}(\tau_A S)$ by 5.2. Let $Y \in \mathcal{A}(\tau_A S)$ be indecomposable. Certainly if Y is injective, then $\text{Ext}_A^1(\tau_A V_t, Y) = 0$. If Y is not injective, then there is $X \in \mathcal{A}(S)$ such that $Y \cong \tau_A X$ by 5.2. Hence

$$\text{Ext}_A^1(\tau_A V_t, Y) \cong \text{Ext}_A^1(V_t, X) = 0 \text{ by 4.3.b applied to } S. \text{ Thus, 4.3 d applied to } \tau_A S$$

yields $\tau_A V_t$ is a direct summand of $T(\tau_A S)$. Certainly $\tau_A V_{r+1}, \tau_A V_{r+2}, \dots, \tau_A V_n$

are indecomposable and pairwise non-isomorphic. For $1 \leq t \leq r < u \leq n-s$

$$\begin{aligned} \text{Hom}_A(U_t, \tau_A V_u) &\cong \text{Hom}_A(V_t, \tau_A V_u) \quad \text{by 4.2.b, since } \tau_A V_u \in \mathcal{A}(\tau_A S), \\ &\cong D \text{Ext}_A^1(V_u, V_t) = 0 \quad \text{by 4.3.b.} \end{aligned}$$

Thus,

$$U_1, U_2, \dots, U_r, \tau_A V_{r+1}, \tau_A V_{r+2}, \dots, \tau_A V_{n-s}$$

are indecomposable and pairwise non-isomorphic direct summands of $T(\tau_A S)$ and that are all of them except those of $\tau_A S$.

b) Note that the defining sequence (4.3.2) of U_t is of the form

$$(5.3.1) \quad 0 \rightarrow V_t \rightarrow U_t \rightarrow \bigoplus_{k=1}^s (\tau_A S_k)^{u_k} \rightarrow 0$$

with $u_k = \dim \text{Ext}_A^1(\tau_A S_k, V_t)_{\text{End } \tau_A S_k}$. If U_t is injective, then since no $\tau_A S_k$ is injective, $\text{Hom}_A(U_t, \tau_A S_k) = 0$; thus $U_t \cong V_t$ is projective-injective, contradicting the assumptions. Thus U_t is not injective; hence by 5.2 and 4.3 b)

$$(5.3.2) \quad \tau_A^{-1} U_t \in \mathcal{A}(S) \subseteq \mathcal{G}(T), \quad (1 \leq t \leq r)$$

Let $F := \text{Hom}_A(T, -): \text{mod } A \rightarrow \text{mod } B$. Applying F to (5.3.1) yields $F(U_t) \cong F(V_t)$, since $\tau_A S \in \mathcal{F}(T)$. Thus, using 2.5.a.iii)

$$\tau_B F(\tau_A^{-1} U_t) \cong F(\tau_A \tau_A^{-1} U_t) \cong F(U_t) \cong F(V_t) = f_t B$$

$F(\tau_A^{-1} U_t)$ is a C -module by (5.3.2) and $f_t B = f_t C$ is a C -module, see (4.4.3) and (4.4.3"). Thus, an Auslander-Reiten sequence in $\text{mod } B$

$$0 \rightarrow f_t B \rightarrow Y \rightarrow F(\tau_A^{-1} U_t) = \tau_B^{-1}(f_t B) \rightarrow 0$$

is a short exact sequence of C -modules and therefore $\tau_C^{-1} f_t C = F(\tau_A^{-1} U_t)$. Hence there are ring isomorphisms using 2.3 e and (3.1.3)

$$\begin{aligned} \text{End} \left(\bigoplus_{t=r+1}^{n-s} f_t C \oplus \bigoplus_{t=1}^r \tau_C^{-1} f_t C \right)_C &= \text{End} \left(\bigoplus_{t=r+1}^{n-s} F(V_t) \oplus \bigoplus_{t=1}^r F(\tau_A^{-1} U_t) \right)_B \\ &\cong \text{End} \left(\bigoplus_{t=r+1}^{n-s} V_t \oplus \bigoplus_{t=1}^r \tau_A^{-1} U_t \right)_A \\ &\cong \text{End} \left(\bigoplus_{t=r+1}^{n-s} \tau_A V_t \oplus \bigoplus_{t=1}^r U_t \right)_A = C(\tau_A S) \quad \square \end{aligned}$$

5.4 Lemma. Let S be an indecomposable regular A -module with $ql(S) = r$ and $\text{Ext}_A^1(S, S) = 0$. Let

$$(5.4.1) \quad S_1 \xrightarrow{\mu_1} S_2 \xrightarrow{\mu_2} \dots \xrightarrow{\mu_{r-2}} S_{r-1} \xrightarrow{\mu_{r-1}} S_r = S$$

be a chain of irreducible monomorphisms and $T = T(S) = \bigoplus_{t=1}^{n-1} V_t \oplus S$

- a) i) S_k is isomorphic to a direct summand of $\bigoplus_{t=1}^{n-1} V_t$ for each $1 \leq k \leq r-1$
 ii) If $X \in \mathcal{A}(S)$ is indecomposable and satisfies $X \not\cong \tau_A^{-1} S_k$, for each $0 \leq i < k \leq r-1$, then $X \in \mathcal{A}(\bigoplus_{t=1}^r S_t)$ and $\bigoplus_{t=1}^{r-1} S_t \in \mathcal{A}(X)$
 iii) If $V_t \not\cong S_k$ for $1 \leq t \leq n-1$ and $1 \leq k \leq r-1$ then $V_t \in \mathcal{A}(\bigoplus_{k=1}^r S_k)$

b) Let

$$(5.4.2) \quad S_{m+1} \xrightarrow{\mu_{m+1}} S_m \xrightarrow{\mu_m} S_{m-1} \xrightarrow{\mu_{m-1}} \dots \xrightarrow{\mu_{r+2}} S_{r+1} \xrightarrow{\mu_{r+1}} S_r = S$$

be a chain of irreducible epimorphisms with indecomposable S_k , $(r+1 \leq k \leq m+1)$ and $\text{Ext}_A^1(S_m, S_m) = 0$. Then

- i) $S_k \in \mathcal{A}(\bigoplus_{t=1}^r S_t) \subseteq \mathcal{G}(T)$ for $r+1 \leq k \leq m+1$
 ii) If a pair of integers (i, k) satisfies

$$\begin{aligned} &\text{either} \quad 1 \leq k \leq r \quad \text{and} \quad k+1 \leq i \leq m-r+1, \\ &\text{or} \quad r+1 \leq k \leq m-r \quad \text{and} \quad r+1 \leq i \leq m-k+1, \end{aligned}$$

then $\tau_A^i S_k \in \mathcal{A}(\bigoplus_{t=1}^r S_t)$

Remark 1. From a.i) follows immediately for $0 \leq i \leq k \leq r-1$, that

$\tau_A^i S_{k-i} = S_k/S_i \in \mathcal{A}(S) \subseteq \mathcal{G}(T)$ and $S/S_k = \tau_A^{-k} S_{r-k} \in \mathcal{G}(T)$, since $\mathcal{A}(S)$ and $\mathcal{G}(T)$ are closed under factor modules. Certainly also $\tau_A S_k \in \mathcal{F}(T)$ for $1 \leq k \leq r$

Remark 2. Since $\bigoplus_{t=1}^r S_t$ is a direct summand of the tilting module T , $ql(S) = r \leq n$ by 2.3 g or 2.6, corollary 1.

Proof. Set $V = \bigoplus_{t=1}^{n-1} V_t$

a) i) Use induction on k (from above)

$\Rightarrow r-1$: S_{r-1} is a direct summand of V by 4.3 f)

Induction: Assume S_{k+1} is a direct summand of V for $2 \leq k+1 \leq r-1$. Consider the non-splitting short exact sequence

$$0 \rightarrow S_k \xrightarrow{\mu_k} S_{k+1} \rightarrow \tau_A^{-k} S_1 \rightarrow 0$$

$\text{Hom}_A(S, S_k) = 0$, since $\text{Hom}_A(S, -)$ is left exact

Since $ql(S_k) = k < r = ql(\tau_A S)$, $\tau_A S \not\cong \tau_A^{-i} S_{k-i}$, for $0 \leq i < k-1$. Hence

$$\text{Hom}_A(\mu_k, \tau_A S) : \text{Hom}_A(S_{k+1}, \tau_A S) \rightarrow \text{Hom}_A(S_k, \tau_A S)$$

is an epimorphism by 3.2.b). $\text{Hom}_A(S_{k+1}, \tau_A S) \cong \text{DExt}_A^1(S, S_{k+1}) = 0$ by induction hypothesis. Thus, $\text{Hom}_A(S_k, \tau_A S) = 0$ and $\text{Ext}_A^1(S, S_k) = 0$ as well

This shows $S_k \in \mathcal{A}(S)$.

Let $X \in \mathcal{A}(S)$. Then $\text{Ext}_A^1(\mu_k, X) : \text{Ext}_A^1(S_{k+1}, X) \rightarrow \text{Ext}_A^1(S_k, X)$ is an epimorphism and $\text{Ext}_A^1(S_{k+1}, X) = 0$ by induction hypothesis and 4.3 b). Hence $\text{Ext}_A^1(S_k, X) = 0$ as well. Thus, 4.3.d implies that S_k is a direct summand of V .

ii) Let $X \in \mathcal{A}(S)$ be indecomposable and satisfy the condition. Then

$\text{Ext}_A^1(\bigoplus_{t=1}^r S_t, X) = 0$ by combining i) and 4.3.d. To show $\text{Hom}_A(\bigoplus_{t=1}^r S_t, X) = 0$,

use again induction from above on t

$t = r$ is just the assumption $X \in \mathcal{A}(S)$.

So assume $\text{Hom}_{\mathcal{A}}(S_{t+1}, X) = 0$ for some $2 \leq t+1 \leq r$. Since $X \not\cong \tau_{\mathcal{A}}^{-1} S_t$, for $0 \leq i < t$, $\text{Hom}_{\mathcal{A}}(\mu_i, X) \cdot \text{Hom}_{\mathcal{A}}(S_{t+1}, X) \rightarrow \text{Hom}_{\mathcal{A}}(S_t, X)$ is an epimorphism by 3.2 b). Hence, $\text{Hom}_{\mathcal{A}}(S_t, X) = 0$ as well. This shows $X \in \bigoplus_{i=1}^r S_i$.

For $1 \leq k < r-1$ there is an AR sequence of the form

$$0 \rightarrow \tau_{\mathcal{A}} S_k \rightarrow \tau_{\mathcal{A}} S_{k+1} \oplus \tau_{\mathcal{A}} S_{k-1} \rightarrow S_k \rightarrow 0$$

with $\tau_{\mathcal{A}} S_{k+1} \oplus \tau_{\mathcal{A}} S_{k-1} \in \mathcal{F}(T)$, see the remark 1. Since any non-zero homomorphism $\varphi: X \rightarrow S_k$ factors over the middle term of the AR sequence,

$$\text{Hom}_{\mathcal{A}}(X, S_k) = 0.$$

Let $\pi: \tau_{\mathcal{A}} S_{k+1} \rightarrow S_k$ be an irreducible epimorphism. Certainly

$\text{Ext}_{\mathcal{A}}^1(X, \pi): \text{Ext}_{\mathcal{A}}^1(X, \tau_{\mathcal{A}} S_{k+1}) \rightarrow \text{Ext}_{\mathcal{A}}^1(X, S_k)$ is an epimorphism and

$$\text{Ext}_{\mathcal{A}}^1(X, \tau_{\mathcal{A}} S_{k+1}) \cong D\text{Hom}_{\mathcal{A}}(S_{k+1}, X) = 0. \text{ Consequently } \text{Ext}_{\mathcal{A}}^1(X, S_k) = 0.$$

Altogether $\bigoplus_{k=1}^r S_k \in \mathcal{A}(X)$

iii) By ii) it is enough to show $V_t \not\cong S_k/S_t \cong \tau_{\mathcal{A}}^{-1} S_k$, for $1 \leq t \leq n-1$ and

$$0 \leq i < k \leq r-1.$$

Suppose $V_t \cong S_k/S_t$. Then there is a short exact sequence of the form

$$0 \rightarrow S_t \rightarrow S_k \rightarrow V_t \rightarrow 0$$

which splits by 4.3 b) and i). Thus, $S_k \cong V_t$ contradicting the assumption

b) i) Let

$$X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_{m-1} \rightarrow X_m = 0$$

be a maximal chain of irreducible monomorphisms. Then by a i) $\bigoplus_{k=1}^m X_k$ is a direct summand of a tilting module (constructed from S_m according theorem

4.3), thus in particular $\text{Ext}_A^1(X_k, X_k) = 0$ for $1 \leq k \leq m$. Since $S_k = r_A^{-m+k} X_k$ for $r+1 \leq k \leq m$, also $\text{Ext}_A^1(S_k, S_k) = 0$.

Show first by induction on k , ($r+1 \leq k \leq m+1$) that $S_k \in \mathcal{A}(S)$.

$S_{r+1} \in \mathcal{A}(S)$ by 4.3.f).

Assume now inductively that $S_k \in \mathcal{A}(S)$ for $r+1 \leq k \leq m$. Let

$$V_1 \rightarrow V_2 \rightarrow \dots \rightarrow V_{k-1} \rightarrow V_k = S_k$$

a chain of irreducible monomorphisms with V_1 quasi-simple. Since $V_k = S_k$ is without self-extensions (as shown above. This is also the reason why it stops by $m+1$), 4.3.f) implies that $S_{k+1} \in \mathcal{A}(S_k)$ and a.ii) that $S_{k+1} \in \mathcal{A}(\bigoplus_{i=1}^k V_i)$.

Applying $\text{Hom}_A(-, S_{k+1})$ to the short exact sequence

$$0 \rightarrow V_1 \rightarrow V_k \rightarrow S \rightarrow 0$$

induces exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_A(S, S_{k+1}) &\rightarrow \text{Hom}_A(V_k, S_{k+1}) \rightarrow 0 \quad \text{and} \\ 0 = \text{Hom}_A(V_1, S_{k+1}) &\rightarrow \text{Ext}_A^1(S, S_{k+1}) \rightarrow \text{Ext}_A^1(V_k, S_{k+1}) = 0 \end{aligned}$$

Thus $S_{k+1} \in \mathcal{A}(S)$ and the induction is done.

Applying a.ii) to the last result yields $S_k \in \mathcal{A}(\bigoplus_{i=1}^r S_i)$ for $r+1 \leq k \leq m+1$.

ii) Show $r_A^i S_k$ is kernel of some homomorphism $S_q \rightarrow S_p \rightarrow 0$ for

$$r+1 \leq p < q \leq m+1.$$

Case $1 \leq k \leq r$ and $k+1 \leq i \leq m-r+1$: Then

$r+1 \leq r+i-k < r+i \leq m+1$ and $0 \rightarrow r_A^i S_k \rightarrow S_{r+i} \rightarrow S_{r+i-k} \rightarrow 0$ is a short exact sequence of the desired form.

Case $r+1 \leq k \leq m-r$ and $r+1 \leq i \leq m-k+1$: Then

$r+1 \leq i \leq i+k \leq m+1$ and $0 \rightarrow r_A^i S_k \rightarrow S_{i+k} \rightarrow S_i \rightarrow 0$ is a short exact

sequence of the desired form

Now ii) follows from 4.1.e) ■

5.5 Corollary. Let S_A be an indecomposable non-projective A -module without self extensions, let $T_A = T(S)$. Let U be an indecomposable regular A -module with $q(l)$ and

$$(5.5.1) \quad U_1 \xrightarrow{\nu_1} U_2 \xrightarrow{\nu_2} \dots \xrightarrow{\nu_{q-1}} U_q \xrightarrow{\nu_q} U$$

be a chain of irreducible monomorphisms. Set $U_0 = 0$.

a) If $U \in \mathcal{A}(S)$ and $S \not\cong U/U$, for $1 \leq i < q-1$, then $U_{q-1} \in \mathcal{A}(S)$.

b) If U is a direct summand of T , then

i) $S \not\cong U/U$, for all $0 \leq i \leq q-1$ and

ii) U_k is a direct summand of T for all $1 \leq k \leq q$.

Proof. a) $\text{Hom}_A(S, U_{q-1}) = 0$, since $\text{Hom}_A(S, _)$ is left exact.

From the assumption follows $U_{q-1}/U \not\cong r_A S$ for all $0 \leq i \leq q-2$. Thus

$$\text{Hom}_A(\nu_{q-1}, r_A S) = \text{Hom}_A(U, r_A S) = \text{Hom}_A(U_{q-1}, r_A S)$$

is an epimorphism by 3.2 b) $\text{Hom}_A(U, r_A S) \cong \text{DExt}_A^1(S, U) = 0$ by 4.3. Thus

$\text{Hom}_A(U_{q-1}, r_A S) = 0$ and $\text{Ext}_A^1(S, U_{q-1}) = 0$ as well. Altogether $U_{q-1} \in \mathcal{A}(S)$.

b) By 5.4 a) it remains only to consider the case $U \not\cong S$. Show first $S \not\cong U/U$, for

$$0 \leq i \leq q-1$$

Suppose $S \cong U/U$, for some $1 \leq i < q-1$. Then in particular S is regular and there is

a chain of irreducible epimorphisms from U to S . Let $\pi: U_{q+1} \rightarrow U$ be an irreducible

epimorphism with U_{q+1} indecomposable. Then $U_{q+1} \in \mathcal{A}(S)$ by 5.4 b i). Thus

$$0 \rightarrow r_A U_1 \rightarrow U_{q+1} \rightarrow U \rightarrow 0$$

is a short exact sequence with $r_A U_1 \in \mathcal{A}(S)$ by 4.1.e), thus splitting by 4.3.d) an obvious contradiction.

Hence $U_{q-1} \in \mathcal{A}(S)$ by a)

$\text{Ext}_A^1(U_{q-1}, T): \text{Ext}_A^1(U, T) \rightarrow \text{Ext}_A^1(U_{q-1}, T)$ is an epimorphism with

$\text{Ext}_A^1(U, T) = 0$ by 4.3.d) Hence $\text{Ext}_A^1(U_{q-1}, T) = 0$ as well. Thus, U_{q-1} is a direct summand of T_A by 4.3.d. The case $1 \leq k \leq q-2$ follows now inductively. ■

5.6 Theorem. Let A be a hereditary finite dimensional k -algebra with precisely n pairwise non-isomorphic simple modules. Let S be an indecomposable regular A -module without self-extensions and $ql(S) = r$. Let

$$S_1 \xrightarrow{\mu_1} S_2 \xrightarrow{\mu_2} \dots \xrightarrow{\mu_{r-2}} S_{r-1} \xrightarrow{\mu_{r-1}} S_r = S$$

a chain of irreducible monomorphisms and $S_0 = 0$. Let $T = T(S) = \bigoplus_{i=1}^{n-r} V_i \oplus \bigoplus_{k=1}^r S_k$

Set $C^{fin}(S) = \text{End}(\bigoplus_{k=1}^{r-1} S_k)_A$ and $C^{inf}(S) = \text{End}(\bigoplus_{i=1}^{n-r} V_i)_A$

a) $C(S) \cong C^{fin}(S) \times C^{inf}(S)$ as algebras.

b) $C^{fin}(S)$ is isomorphic to the path algebra of the quiver $\bullet \leftarrow \bullet \leftarrow \dots \leftarrow \bullet$ on $(r-1)$ points over the division algebra $E = \text{End } S_A$. The indecomposable $C^{fin}(S)$ -modules correspond exactly to the modules S_k/S_i for $0 \leq i < k \leq r-1$ via the functor $\text{Hom}_A(T, -)$

c) $C^{inf}(S)$ is a hereditary algebra of infinite representation type. $\text{mod } C^{inf}(S)$ is equivalent to $A(\bigoplus_{k=1}^r S_k)$.

If $\pi_{r+1}: S_{r+1} \rightarrow S_r$ is an irreducible epimorphism with S_{r+1} indecomposable, then $\text{Hom}_A(T, S_{r+1})$ is an indecomposable regular, quasi-simple $C^{inf}(S)$ -module

d) If A is connected, so is $C^{inf}(S)$.

c) If q_A , the quadratic form associated with A , is indefinite, so is $q_{C^{fin}(S)}$, the quadratic form of $C^{fin}(S)$

Proof. Set $C = C(S)$ and $B = B(S)$

a) $\text{Hom}_A(\bigoplus_{k=1}^{r-1} S_k, \bigoplus_{i=1}^{n-r} V_i) = 0$ and

$\text{Hom}_A(\bigoplus_{i=1}^{n-r} V_i, \bigoplus_{k=1}^{r-1} S_k) = 0$ by 5.4.a iii) combined with 5.4.a ii)

b) The only indecomposable modules in $A(S)$ having $\text{Hom}_A(V_k, X) \neq 0$ for some $1 \leq k \leq r-1$ are just the given ones by 5.4.a ii). Thus, $C^{fin}(S)$ is of finite representation type. Since the modules $F(S_k)$ are projective and F preserves irreducible maps, the AR quiver of $C^{fin}(S)$ is embedded in the triangle determined by S_1, S_{r-1} and S_{r-1}/S_{r-2} . But it is well-known that the only hereditary algebra with this number of indecomposables and projectives distributed in that way is the indicated path algebra. Of course one also can compute the dimensions of the $\text{Hom}_A(\cdot, \cdot)$ -sets using just the properties of irreducible maps between the V_k 's.

Set $E_k = \text{End}(S_k)_A$, ($1 \leq k \leq r$), which is a division algebra by 3.4

(1) $\text{Hom}_A(\mu_k, S_{k+1}) \cdot \text{Hom}_A(S_{k+1}, S_{k+1}) \rightarrow \text{Hom}_A(S_k, S_{k+1})$ is an isomorphism for $1 \leq k \leq r-1$

Proof. Since $ql(S_k) < ql(S_{k+1})$, $\text{Hom}_A(\mu_k, S_{k+1})$ is an epimorphism by 3.2 b)

Now $\text{Hom}_A(S_k, S_{k+1}) \neq 0$ implies that the E_{k+1} -dimension of domain and range are both = 1, which proves (1).

(2) $\text{Hom}_A(S_k, \mu_l) \cdot \text{Hom}_A(S_k, S_l) \rightarrow \text{Hom}_A(S_k, S_{l+1})$ is an isomorphism for $1 \leq k \leq l \leq r-1$

Proof: By the same argument as in the previous proof, it is enough to show that $\text{Hom}_A(S_k, \mu_l)$ is an epimorphism. Note that the middle term of an Auslander-Reiten sequence ending in S_{l+1} is either $r_A S_{l+2} \oplus S_l$, if $1 \leq l \leq r-2$ or

$S_{r+1} \oplus S_{r-1}$, if $k = l - 1$. Since any $\psi \in \text{Hom}_A(S_k, S_{l+1})$ is not a splitting epimorphism, it factors over these modules. But $\text{Hom}_A(S_k, r_A S_{l+2}) = 0$ by remark 1 to 5.4 and $\text{Hom}_A(S_{r-1}, S_{r+1}) = 0$ by 5.4. Thus, $\psi = \mu_k \varphi = \text{Hom}_A(S_k, \mu_l)(\varphi)$ for some $\varphi \in \text{Hom}_A(S_k, S_l)$.

(3) $E_k \cong \text{End } S_A$ for $1 \leq k \leq r$.

Proof. It is enough to show $E_k \cong E_{k+1}$ for $1 \leq k \leq r - 1$. Let

$1 \leq k \leq r - 1$. Then for any $\varphi \in \text{Hom}_A(S_k, S_k)$ there exists by (1) a unique $V(\varphi) \in \text{Hom}_A(S_{k+1}, S_{k+1})$ such that $\mu_k \varphi = V(\varphi) \mu_k$. The map $V: E_k \rightarrow E_{k+1}$ is an injective algebra homomorphism by the same reasoning as in the proof of 4.2.d). Since $\dim_{\mathbb{K}} E_k = \dim_{\mathbb{K}} E_{k+1}$ by combining (1) and (2), V is actually an isomorphism.

If $1 \leq l < k \leq r$, then $\text{Hom}_A(S_k, S_l) = 0$ by 5.4.a.1). Thus (3) and (2) shows b)

c) As direct factor of the hereditary algebra $C(S)$, $C^{nf}(S)$ is also a hereditary algebra

By remark 2 of 5.4 there exists a chain of irreducible epimorphisms

$$(5.6.1) \quad S_{m+1} \xrightarrow{\pi_{m+1}} S_m \xrightarrow{\pi_m} S_{m-1} \xrightarrow{\pi_{m-1}} \dots \xrightarrow{\pi_{r+2}} S_{r+1} \xrightarrow{\pi_{r+1}} S_r = S$$

with S_k indecomposable, $\text{Ext}_A^1(S_m, S_m) = 0$, but $\text{Ext}_A^1(S_{m+1}, S_{m+1}) \neq 0$. Then by 5.4.b.i) $S_k \in A(\bigoplus_{i=1}^r S_i)$ for $r+1 \leq k \leq m+1$. Thus $\text{Hom}_A(T, S_k)$ is an indecomposable $C^{nf}(S)$ -module. In particular $\text{Hom}_A(T, S_{m+1})$ is an indecomposable $C^{nf}(S)$ -module having non-trivial self-extensions by 2.3 c). This shows that $C^{nf}(S)$ is of infinite representation type and $\text{Hom}_A(T, S_{m+1})$ is a regular $C^{nf}(S)$ -module. $\text{Hom}_A(T, \pi_k)$, ($r+1 \leq k \leq m+1$), is an irreducible map in $\text{mod } C^{nf}(S)$ by 4.5 c), thus $\text{Hom}_A(T, S_{r+1})$ is also an indecomposable regular $C^{nf}(S)$ -module.

There is a non-zero map $V_l \rightarrow S_{r+1}$ by 4.3.c), which factors over S_{r+2} since the other middle-term in the AR sequence is $r_A S \in \mathcal{F}(T)$. Thus S_{r+2} has a non-trivial torsion

part and being indecomposable, $F(S_{r+1})$ is quasi-simple

d) Recall from 4.3

(4) If $\text{Hom}_A(V_t, S) = 0$ for some $1 \leq t \leq n - r$, then V_t is projective

(5) $\text{Hom}_A(X, \pi_{r+1}): \text{Hom}_A(X, S_{n+1}) \xrightarrow{\sim} \text{Hom}_A(X, S)$ is an isomorphism for

$$1 \leq t \leq n - r \text{ and } X \in \mathcal{A}(\bigoplus_{k=1}^r S_k)$$

Proof: There is a short exact sequence $0 \rightarrow \tau_A S_1 \rightarrow S_{r+1} \xrightarrow{\pi_{r+1}} S_r \rightarrow 0$. Applying

$\text{Hom}_A(X, -)$ yields an exact sequence

$$\text{Hom}_A(X, \tau_A S_1) \rightarrow \text{Hom}_A(X, S_{r+1}) \xrightarrow{\pi_{r+1}} \text{Hom}_A(X, S_r) \rightarrow \text{Ext}_A^1(X, \tau_A S_1)$$

$$\text{Now } \text{Hom}_A(X, \tau_A S_1) = 0 \text{ and } \text{Ext}_A^1(X, \tau_A S_1) = 0.$$

Now we are ready to prove d). It is enough to exhibit a sincere indecomposable $C^{nJ}(S)$ -module. We will distinguish the cases q_A positive semidefinite and q_A indefinite.

q_A positive semidefinite. Consider the sequence (5.6.1). Suppose $\text{Hom}_A(T, S_{m+1})$ is not a sincere $C^{nJ}(S)$ -module. Then there exists $1 \leq t \leq n - r$ such that

$\text{Hom}_A(V_t, S_{m+1}) = 0$. If $m \geq r + 1$ consider the short exact sequence

$$0 \rightarrow K \rightarrow S_{m+1} \xrightarrow{\pi} S_{r+1} \rightarrow 0$$

with $\pi = \pi_{r+2}\pi_{r+3} \dots \pi_m\pi_{m+1}$. With $S_{m+1}, S_{r+1} \in \mathcal{A}(\bigoplus_{t=1}^r S_t)$ also

$K \in \mathcal{A}(\bigoplus_{t=1}^r S_t) \subseteq \mathcal{G}(T)$ by 4.1.e) and 4.3.b). Thus applying $\text{Hom}_A(V_t, -)$ yields

$\text{Hom}_A(V_t, S_{r+1}) = 0$. This is plain in the case $m = r$. So $\text{Hom}_A(V_t, S) = 0$ by (5)

hence V_t is projective by (4). Thus S_{m+1} is not a sincere A -module having non-trivial self-extensions. This is a contradiction, since the support algebra of S_{m+1} is of finite representation type, thus S_{m+1} has no nontrivial self-extensions. So $\text{Hom}_A(T, S_{m+1})$ is a sincere indecomposable $C^{nJ}(S)$ -module.

- q_A indefinite : Assume first S is a sincere A -module. Then $\text{Hom}_A(T, S_{r+1})$ is a sincere $C^{nf}(S)$ -module. For assume not, then there exists t with $\text{Hom}_A(V_t, S_{r+1}) = 0$. So $\text{Hom}_A(V_t, S) = 0$ by (5), hence V_t projective by (4), contradicting that S is sincere. If S is not sincere, then there exists an integer m such that $r_A^m S$ is sincere. (Here the assumptions q_A is indefinite and A is indecomposable are used both.) Consider

$$T(r_A^m S) = \bigoplus_{t=1}^{n-r} W_t \oplus \bigoplus_{k=1}^r r_A^m S_k.$$

Then, depending whether S is quasi-simple or not,

$$C(r_A^m S) = \text{End}\left(\bigoplus_{t=1}^{n-r} W_t \oplus \bigoplus_{k=1}^{r-1} r_A^m S_k\right)_A$$

has either one or two connected components as just proven. Thus the same holds for $C(S)$ by theorem 5.3, i.e. $C^{nf}(S)$ is connected.

c) Let

$$f := \sigma_T: G_0(A) \rightarrow G_0(B)$$

be the isometry of the Grothendieck groups of A and B , see 2.3.g. $A(S)$ generates in $G_0(A)$ a free subgroup of rank $n-1$

$$H^S = \{\underline{x} \in Z^n : \langle \dim S, \underline{x} \rangle_A = 0\}$$

By identifying $G_0(C)$ with its canonical image in $G_0(B)$ and using 4.5 a, f induces by restriction an isometry

$$g: H^S \rightarrow G_0(C).$$

Since q_A is indefinite, there is an indecomposable regular A -module Y' with $q_A(\dim Y') \leq -1$, see 3.6. By 3.7 there are integers $N < M$ such that

$$\langle \dim r_A^j S, \dim Y' \rangle_A = \langle \dim S, \dim r_A^{-j} Y' \rangle_A > 0, \quad \text{for all } j \leq N;$$

$$\langle \dim S, \dim r_A^{-j} Y' \rangle_A < 0, \quad \text{for all } j \geq M.$$

Let N be maximal with that property. Set $Y = r_A^{-N} Y'$, $y = \dim Y$ and $s = \dim S$. Then

$$1) \quad \langle \underline{y}, \underline{y} \rangle_A \leq -1,$$

$$2) \quad \langle \underline{s}, \Phi_A^t \underline{y} \rangle_A > 0 \text{ for all } t > 0,$$

$$3) \quad \langle \underline{y}, \underline{s} \rangle_A \geq 0$$

$\langle \underline{s}, \underline{y} \rangle_A = \dim_{\mathbf{k}} \operatorname{Hom}_A(S, Y) - \dim_{\mathbf{k}} \operatorname{Ext}_A^1(S, Y) > 0$ is divisible by $\langle \underline{s}, \underline{s} \rangle_A = \dim_{\mathbf{k}} E$

thus there is an integer $\alpha > 0$ with $\langle \underline{s}, \underline{y} \rangle_A = \alpha \langle \underline{s}, \underline{s} \rangle_A$. Set $\underline{x} = \underline{y} - \alpha \underline{s}$. Then

$$\langle \underline{s}, \underline{x} \rangle_A = \langle \underline{s}, \underline{y} - \alpha \underline{s} \rangle_A = \langle \underline{s}, \underline{y} \rangle_A - \alpha \langle \underline{s}, \underline{s} \rangle_A = 0$$

So $\underline{x} \in H^S$ and

$$q_A(\underline{x}) = \langle \underline{y} - \alpha \underline{s}, \underline{x} \rangle_A = \langle \underline{y}, \underline{x} \rangle_A = \langle \underline{y}, \underline{y} - \alpha \underline{s} \rangle_A = \langle \underline{y}, \underline{y} \rangle_A - \alpha \langle \underline{y}, \underline{s} \rangle_A = \langle \underline{y}, \underline{y} \rangle_A - 1$$

Write $g(\underline{x}) = \underline{z}_{f,n} + \underline{z}_{i,nf}$, where the support of $\underline{z}_{f,n}$ and $\underline{z}_{i,nf}$ is the support of $C^{f,n}(S)$ and $C^{i,nf}(S)$ in $G_0(C)$, respectively. Let $\hat{\underline{z}}_{f,n} \in G_0(C^{f,n}(S))$ the projection of $\underline{z}_{f,n}$ and $\hat{\underline{z}}_{i,nf} \in G_0(C^{i,nf}(S))$ the projection of $\underline{z}_{i,nf}$ a) implies

$$q_{C^{f,n}(S)}(\hat{\underline{z}}_{i,nf}) + q_{C^{i,nf}(S)}(\hat{\underline{z}}_{f,n}) = q_C(\underline{z}_{i,nf}) + q_C(\underline{z}_{f,n}) = q_C(g(\underline{x})) = q_A(\underline{x}) = -1$$

Thus, since $q_{C^{f,n}(S)}$ is positive definite by b), $q_{C^{i,nf}(S)}$ is indefinite

5.7 Corollary (Hoshino). Let A be a hereditary algebra with precisely n pairwise non-isomorphic simple modules. Let S_A be an indecomposable regular module without self-extensions. Then $ql(S) \leq n - 2$.

Proof. By remark 2 of 5.4 we only have to exclude the cases $r = n$ and $r = n - 1$. In the first case $C^{i,nf} = 0$ and in the second case $C^{i,nf} = \operatorname{End} V_1$ a division ring by 3.4 both of course of finite representation type. ■

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