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STRESS AND DISPLACEMENT WAVES
IN SOLID ELASTIC BARS

by

Leslie Tremaine Russell, M.Sc

A thesis submitted to the Faculty of
Graduate Studies in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Engineering

Carleton University

Ottawa, Ontario

May 27, 1970

c. Leslie T. Russell, 1971

ABSTRACT

This study is concerned with the propagation of stress and displacement waves in solid bars of various shapes. The nature of these waves is of interest in the design of ultrasonic concentrators and acoustic transducers. Two methods, the Finite Element Method and the Moiré Fringe Method, have been used in this study. In the Finite Element Method, rectangular and quadrilateral elements, both plane and ring, have been used. The load was applied to one end of the bars and the wavelength of this applied load was equal to the width of the bars of rectangular cross section and equal to the diameter of the axisymmetric bars. The Moiré Fringe Method using a grille of 1000 lines per inch and single exposure technique was used to record photographically the Moiré Fringe pattern at various times during the propagation of the displacement wave. The numerical and experimental results have been interpreted and are presented in graphical form. The results confirm the validity of the one dimensional theory for $\lambda \gg d$ but indicate that for a square load pulse with $\lambda \approx d$ the effect of the change in the apex angle from 0° to 20° is not appreciable.

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1. INTRODUCTION

1.1 BACKGROUND

The theory of elasticity used for determining stresses within a body under a system of loads is based on the principle of equilibrium. Both static and dynamic problems can be solved with this theory since many dynamic problems may be considered as quasi steady-state problems. For example, the stresses in a vibrating body may be found at any instant by considering the bar to be at rest for that particular instant. Further, stress waves will propagate continuously in a body under the action of a time-varying load system. These problems must be considered as dynamic problems since the stress at any position within the stressed body is a result of the propagation of stress waves during the time immediately after the application of a load to the body, until the steady-state condition is reached. It is this latter stress wave condition which is of interest to this work.

A most interesting case of an elastic solid is a rod, either axisymmetric or not, with a load applied at one end. In this rod, three types of stress waves may propagate: namely, (i) longitudinal waves, (ii) torsional waves, and (iii) flexural waves. The motion of the elements of a rod is dependent on the type of stress wave being propagated.

Longitudinal waves cause extension and contraction of the elements with no transverse motion of the axis of the rod. The torsional wave causes each section of the rod to rotate in its own plane with no disturbance of the axis. The flexural wave causes transverse motion of the elements and, hence, the axis of the rod will also be disturbed. The differential equations describing the motion of an elastic solid under the action of a loading system have been derived and are readily available to the reader (1). Theoretically, these equations of motion can be solved for any particular shape under any loading and with any support condition. However, it is a fact that, except for a very limited number of simple problems, the application of the boundary conditions complicates the equations, rendering them unsolvable. As a result, only a limited number of problems have been solved theoretically. Even the simple case of a cylindrical rod of finite length has been solved only with the limitation that the length of the rod must be long compared to the radius of the rod (1).

A large amount of the theoretical work has been based on the so-called one-dimensional wave equation. In particular, one well-known equation of this type is the "Webster's Horn Equation" (1) which is based on the assumption of no lateral motion in the bar, i.e. $\nu = 0$, (ν = Poisson's Ratio)

$$\frac{\partial}{\partial x} \left[EA \frac{\partial u}{\partial x} \right] = \rho A \frac{\partial^2 u}{\partial t^2} \quad (1)$$

In Equation 1,

x = axial length along the bar

t = time

$u(x,t)$ = displacement

E = Young's Modulus

ρ = density

$A(x)$ = cross-sectional area

The displacement, u , is a function of axial length and time and is not dependent on the other co-ordinates even though the equation allows for a variation of Young's Modulus and cross-sectional area with axial length.

The "exact" theory of the stress or displacement propagation in an elastic solid was developed by Pochhammer (1876) and independently by Chree (1889). The resulting equations describing the actual behaviour in bars are commonly known as the Pochhammer-Chree equations and a complete formulation of these equations will be found in Love (55). In theory, it is possible to solve any problem of vibration or stress propagation in an elastic medium by the insertion of the appropriate boundary conditions in these equations. In practice, exact solutions, because of the difficulties imposed by the boundary conditions, have only been obtained for a limited number of bars. Davies (40) has investigated the propagation of waves in a bar of infinite length in order to shed more light on the usefulness of the Hopkinson pressure bar. Davies has limited his investigation

to a cylindrical bar of infinite length and shows that as a/λ becomes small, the variation of stress and displacement across the cylindrical bar becomes very small.

Raleigh (54) developed, because of the difficulties in solving the exact theory, a correction to the one dimensional theory. This frequency dependent correction was applied to the velocity of sound and was the result of addition of the effects of the lateral inertia to the one dimensional theory. Davies (40) has shown using the one dimensional theory, including the effect of the lateral energy, that the errors are small provided a/λ is small.

Eisner (2) has given an excellent review of the theoretical work, a large amount of which is based on Webster's Horn equations. He states that this equation is a good approximation to the actual situation provided the bar is thin relative to the wave length of the wave being propagated, is straight and does not taper too fast and provided the waves are of small amplitude. These restrictions are inherent in much of the theoretical work based on the one dimensional equation but recently two papers (3), (4) have appeared which take into account the effect of Poisson contraction. It is interesting to note that in the work of Chehil and Heaps (3) that even though the Poisson contraction is considered, a basic assumption is that the longitudinal displacement is dependent only on the axial distance and is independent of the radial and angular coordinates. A further assumption is

that each cross-section remains plane and that the longitudinal component of stress is uniformly distributed over the cross-section of the bar. Martin (4) develops a new wave equation which also includes the lateral inertia effects and concludes that the lateral effects are small but significant if the horn is long and thin.

The work of Mason (5) and Rozenberg and Lozinskii (6) around 1945 provided a stimulus to the research in wave propagation in elastic bodies. The discovery by Mason (5) and Rozenberg and Lozinskii (6) that a tapered solid rod would magnify the velocity of a particle at the small end of the rod as compared to the particle velocity at the large end of the rod has formed a basis for much of the subsequent work. This work has been devoted to a large extent to the determination of the shape of the rod to give the best amplification factor. Many designs have been produced and Eisner (7) has shown a number of shapes with very large amplifications. Care must, however, be exercised else the rod becomes very slender and its lack of lateral stiffness renders the rod useless for practical purposes. Another factor which must be considered is the maximum stress induced in any vibrator since obviously this should be kept within the elastic region so as to prevent permanent deformation in the rod.

Hopkinson (8) in 1914 devised an experimental apparatus capable of measuring pressures within a rod. It consisted

of a long rod to which an impulse was applied at one end. The magnitude of the pressure was measured with the aid of another shorter piece of rod which was wrung to the opposite end of the long rod. By recording the momentum trapped in this detachable piece or rod, as it left the main bar (due to the passage of the pressure pulse), the pressure could be deduced. Improvements have been made to the above technique by incorporating electronic measuring devices. The Hopkinson pressure bar is still used today in situations where the rod is long and the radius of the bar is small compared to the wave length of the pressure pulse being propagated. Usually the measurements are now made with either a plate type condenser or a ring type condenser mounted on the rod. With such a system the pressure-time history can easily be measured but the variation of the pulse shape over the area of the bar cannot be determined.

The work of Mason and others (5), (6) has supplied stimulation to the interest in the wave shape over any one particular cross-section of a bar. Baker and Dove (9) have used commercially available strain gauges and by embedding these at various positions in a circular bar were able to determine that the changes in the strain profile caused by a central impact on one end of the bar ceased in the region of two diameters from the impact end.

Mindlin and Herrmann (45) have derived an improved one dimensional theory of wave propagation in an elastic rod.

They have included both lateral inertia and shear but they did not require that that stress resultant vanish on the lateral free surface of the bar. In their solution, Mindlin and Herrmann (45) assumed displacement functions of the same form as used by Love (55) and have shown improved accuracy in the high frequency region.

Another improved theory by Mindlin and McNiven (46) used a system of one dimensional equations which incorporated coupling between the longitudinal, axial shear and radial modes. The authors point out that with common values of Poisson's ratio, this coupling is possible and has an important influence on wave velocities and on high-frequency mode shapes. Their analysis shows improved agreement between theory and experimental results.

Photoelasticity is a natural experimental tool to study the propagation of stress waves and as is evidenced by the review of Goldsmith (10) much work has been done in this area. The main requirement to satisfactorily record the photoelastic fringe pattern produced by the passage of a stress wave in a body is to be able to instantaneously record the pattern at any time after loading. This can be done by using a high speed framing camera or a single shot short duration light source together with a still camera. Hemann (11) has applied the principle of Scattered Light Photoelasticity to dynamic stress analysis and has shown, with the aid of a ruby laser light source, that this technique

will produce a fringe pattern provided the light source is sufficiently intense and of sufficiently short duration. Another technique which has provided very clear results is the embedded polariscope method. The clarity of results obtainable by this method is shown in the work of Brillhart and Dally (12) and Okada, Cunningham and Goldsmith (13). Both of these papers show clearly that the stress wave is markedly altered in either a cone or a pyramid when the apex angle becomes large. Brillhart and Dally (12) have stated that the one dimensional or elementary theory will give a good approximation provided the ratio of wave length to maximum cone diameter is greater than 3.

The Moiré method has as its basis the fringe pattern produced when two grid systems of closely spaced fine parallel lines are displaced relative to one another. Each fringe produced by keeping one grid fixed when the second, which is integrally attached to the model, is distorted by the strain in the model is the locus of points in the model having the same component of displacement. The geometry of Moiré fringes is well explained in a paper by Morse, Durelli and Sciammarella (14). This technique has been used by Riley and Durelli (15) to study transient stress and strain. They used a framing camera operating at 6400 frames/sec to study the stress waves while Kobayashi, Harris and Engstrom (16) used a laser light source coupled with a Kerr cell to obtain the transient displacement in the vicinity of a moving crack.

Theocaris, Davids, Gillich and Calvit (17) have used the Moiré method to study shock waves produced by explosives. They were able to determine the velocity of the wave as well as the displacement field with this technique. In order to obtain the displacement field at an internal plane in a body Sciammarella and Chiang (18) have used the embedded grid principle where a grid is placed on the plane of interest inside the model being studied. Kerber and Whittier (19) have studied embedded grids in relation to the effect of optical refraction. They conclude that internal refraction effects can be expected in dynamic experiments.

Kenner, Goldsmith and Sackman (51) have studied the effect of longitudinal impact on a hollow cone. They compare the results of a one dimensional equation, in which it was assumed that plane waves remain plane and in which the effects of lateral inertia and radial shear are neglected, to an experimental investigation. The experiments were performed on a hollow aluminum cone with a 20-degree apex angle to which semiconductor strain gauges were bonded. They found that although the one dimensional theory was reasonably accurate for the low frequencies, it became more inaccurate as the frequency became higher. Bending in the cone wall was also observed and this feature increased as the frequency of the applied pulse increased.

Most theoretical analyses of stress wave problems can be reduced to a numerical procedure solvable with the aid of a digital computer. Bertholf (39) has presented a finite

difference solution to the elastic wave propagation problem in a cylindrical bar of semi-infinite length. Two solutions, one for a plane, harmonic, axial displacement and a second for a step change in pressure at the loaded end of the bar, are presented and compared with theoretical and experimental solutions. The analysis for the plane, harmonic, displacement with $a/\lambda = .3$ is shown to compare well with the Pochhammer-Chree solution and the author concludes that the values of the displacement at four diameters from the end of the bar agrees more closely with the Pochhammer-Chree solution as time increases. For the second case of a step change in pressure, reasonably close agreement between the numerical solution and the experimental data of Micklowitz and Nisewanger (49) was found.

It is interesting to note that the finite element technique has recently been applied to the stress wave problem (20). Costantino (20) concludes that the finite element method will predict satisfactorily the displacement-time histories but will be less accurate for the stress-time histories. This method was first developed by Turner et al (21) to permit the analysis of complicated structures such as are found in aircraft and has been extensively used to study a great variety of problems ranging from earth dams to shell structures to fluid flow problems. A very clear exposition of the physics of this finite element method as applied to dynamic elasticity and thermo-elasticity is given by Jones, R.E. (52) in his discussion

of a paper by Nickell and Sackman. The method consists of dividing the structure into a series of small elements connected to each other at the nodal points. A displacement function is then assumed, valid over each element, which defines the displacement within the element in terms of the nodal point displacements. From the assumed displacement function an element stiffness, which relates the nodal point forces to the nodal point displacements, can be calculated. The stiffness of the total assembly of elements is then determined by matrix addition and hence for any loading condition the displacements can be found and from the displacements using the stress-displacement relations, the stress at each node may be determined.

1.2 GENERAL PROBLEM

The problem is to investigate the nature of the displacement and stress in bodies of various shapes under the action of a given load system and support condition during the time before the steady state stress condition is reached. The displacement and stress will vary with time during this period and hence the study involves the nature of the stress and displacement waves as they propagate through the body under consideration.

1.3 SCOPE

Because it is impossible to study each and every shape of body, a limitation of the number and shapes of bodies to

be studied is necessary. Two general shapes will hence be studied: (a) flat plate type bodies, and (b) axisymmetric bodies. The nature of the waves is expected to be dependent upon the shape of the body and hence the flat plate and axisymmetric bars will be varied in shape in the direction transverse to the direction of travel of the wave.

Again, one could choose numerous methods to attempt an analysis of these problems, but based on an assessment of the various methods, it has been decided to use the numerical technique of finite elements together with the integration* scheme presented in Chapter 2 and the Moiré fringe experimental technique.

*At the time of submission of this thesis, a paper by C.C. Fu (53) describing a method for the numerical integration of equations of the type generated by the dynamic analysis of bars using the Finite Element Method appeared in the literature. This method described as "an extended de Vogelaere method" has, according to the author, advantages over other methods of increased speed of computation and requires a minimum amount of computer storage space while maintaining good accuracy.

2. FINITE ELEMENT METHOD

2.1 GENERAL PRINCIPLES

A very large number of problems in stress analysis, either plane stress, plane strain or in the area of shell analysis are not amenable to closed form solutions. Even though the differential equations for many problems may be written, the nature of the boundary conditions makes the attainment of a closed form solution extremely difficult, if not impossible. Two important approximate methods have been used to analyze such problems; the finite difference method and the finite element method.

In the finite difference method, the governing differential equations together with the boundary conditions are replaced by finite difference approximations in terms of a finite number of unknown values of dependent variables at discrete points in the region of interest. Although this method has the obvious advantage of replacing the awkward differential equations by a set of algebraic equations any irregularity in the shape of the boundary will make the application of this method extremely cumbersome to use. A good example of a finite difference solution to an elastic wave propagation problem is found in the work of Bertholf (39).

The fundamental concept of the finite element method of analysis is that every structure or continuum may be considered as consisting of a finite number of individual elements attached to one another at a finite number of

points called nodal points. A frame structure is a natural for this type of analysis as it consists of structural elements such as beams, columns and girders. Any elastic continuum may be considered as a finite number of elements of various shapes and sizes connected to one another at their nodal points. The number, size and shape of these elements depends on the shape of the body and the nature of the loading and it is this part of the analysis that requires a large amount of experience and good judgement. A typical problem of a plate with a hole in it is shown subdivided into elements in Figure 2.1. The finite element method may be visualized in three distinct steps, (i) the idealization or subdivision of the body into finite elements, (ii) the determination of the element properties, and (iii) the analysis of the completely assembled elements which now approximates the original body. Since the analysis is done on the idealized body, that is, the assembly of elements, it is extremely important to use good judgement when the body is subdivided into these finite elements. A very important part of the analysis is the determination of the element properties. The stiffness of the element relates the nodal point deflections to the nodal point forces. If F_e represents a column vector of nodal point forces acting on the assumed element, k the element stiffness matrix and δ_e the column vector of the element nodal

Fig. 2.1 ELEMENT SUBDIVISION OF A PLATE

Fig. 2.2 ELEMENTS FRAMING INTO A COMMON NODE

point displacements then

$$F_e = k \delta_e \quad (1)$$

The displacement of any nodal point depends on the stiffness of all the elements framing into that particular node. Hence the stiffness of the complete assembly of elements is the combination of the individual element stiffnesses. In Figure 2.2 three elements are shown all framing into node i . The stiffness for one degree of freedom, j , at node i will be assembled from the components of stiffness, for the degree of freedom, j , at node i , of each of the elements framing into node i . Hence using a subscript notation, the term in the stiffness matrix relating the nodal force at node i in the direction j to the displacement at node i in the direction of j is K_{ij} and is given by

$$K_{ij} = k_{ij}^{(1)} + k_{ij}^{(2)} + k_{ij}^{(3)} + \dots \quad (2)$$

where the superscripts refer to the elements framing into node i . The total stiffness matrix, K , can hence be assembled by considering each node and each degree of freedom in the above manner. If K is the combined stiffness matrix, U is the column vector of the displacements for the total assembly of elements and P is the column vector of externally applied loads then for the total structure

the relationship is

$$P = K U \quad (3)$$

The obvious advantages of this method are evident from Equation (3). The boundary conditions may be easily substituted directly in the appropriate location in Equation (3). Further the applied loads may also be easily applied by substitution into the external force vector, P . Since any geometry may easily be represented by suitably chosen shapes and numbers of elements, this method may be used to approximate a body of any shape no matter how irregular. In the present analysis, a uniformly distributed load was applied over the large end of each bar through equivalent nodal point forces. Under the action of a uniformly distributed load, the axis of the symmetric bar would not move radially, hence only one side of the bar was investigated. The base of the bar, small end, was restrained from axial motion by setting the axial displacements, at the nodes on the base, equal to zero. In like manner, the axis of the bar was prevented from moving laterally by setting the lateral nodal displacements along the axis equal to zero. These boundary conditions which permitted axial motion throughout the bar and lateral motion throughout the bar, except on the axis of symmetry, matched closely the actual condition of the bar. The boundary conditions for the axisymmetric bars were similar to those of

the flat bar. Because ring elements were used for this analysis, a small hole remained in the center of the axisymmetric bars. This hole was .000002 inches in diameter and no lateral motion was permitted at the circumference of this hole. These boundary conditions, no axial motion on the base of the bar and no lateral motion at the circumference of the small hole, located at the axis of the bar, approximated closely the actual conditions.

The body under stress is approximated by an assemblage of elements and it is necessary to consider both equilibrium and compatibility within the individual elements and throughout the body as a whole. The displacement within the element is assumed to vary in a specified way. The element stiffness depends on this assumed displacement function and compatibility is automatically satisfied within the element provided the assumed function is continuous. Equilibrium within each element requires that the assumed displacement function satisfy the equilibrium equations although these more severe restrictions have not always been met by workers in the field. Throughout the body as a whole, compatibility is achieved at nodal points but not necessarily along the boundary of the elements. This boundary compatibility depends on the assumed element displacement function. Equilibrium for the entire structure is approximated by ensuring equilibrium of element forces at common nodes.

In the analysis used here, the assumed displacement function ensures that both equilibrium and compatibility are satisfied within each element and that the element boundaries remain in line-to-line contact in the deflected state.

2.2 ELEMENT STIFFNESS

The procedure for the element stiffness derivation is well established (21,22,23,24) and only an outline is presented here. The stiffness details for the elements used in this project, each element having eight degrees of freedom, are found in Appendix A1.

The displacements within any element will be related to the displacements in other elements through the compatibility of the nodal point displacements and, hence, the displacements within any element are functions of the eight permitted degrees of freedom. If u is the displacement in the x co-ordinate direction and v the displacement in the y co-ordinate direction, the following displacement functions may be assumed for u and v :

$$u = a_1 + a_2x + a_3y + a_4xy \quad (4)$$

$$v = a_5 + a_6x + a_7y + a_8xy \quad (5)$$

In expressions (4) and (5) x and y are the coordinates of any point in question and $a_1 \dots a_8$ are generalized coordinates to be determined in terms of the eight nodal point displacements. If δ_e represents a column vector of the nodal point displacements, A , a matrix of the x and y coordinates of the nodes, and, a , a column vector of the eight generalized coordinates, the matrix equation relating all of the element nodal point displacements to the generalized coordinates is:

$$\delta_e = A a \quad (6)$$

The strains within the element, namely ϵ_{xx} , ϵ_{yy} , and ϵ_{xy} can be found from the strain displacement relations giving

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} = a_2 + a_4 y \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} = a_7 + a_8 x \\ \epsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_3 + a_4 x + a_6 + a_8 y \end{aligned} \quad (7)$$

or in matrix form

$$\epsilon = B a \quad (8)$$

At this point the type of stress condition of interest must be specified (i.e., plane stress, plane strain or axi-symmetric). For example if the plane stress condition is used the relation between stress and strain is

$$\sigma = D \epsilon \quad (9)$$

where

$$\sigma = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}$$

and

$$D = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (9a)$$

Substitution of Equation (8) into Equation (9) yields

$$\sigma = D B a \quad (10)$$

The total analysis is accomplished in terms of nodal point forces and nodal point displacements and it is necessary for the nodal point forces to be statically equivalent to the boundary stresses and any distributed loads on the element. If arbitrary nodal displacements are imposed on the element then the above requirement will be satisfied provided the work done by the equivalent nodal point forces during an arbitrary or virtual displacement is equated to the work done by the internal stresses and strains of the element during that displacement. If δ_e^* represents the virtual nodal point displacements virtual strains, ϵ^* , will be developed within the elements. The work done by the nodal forces, F_e , during the imposed virtual displacements is

$$dW_E = \delta_e^{*T} F_e \quad (11)$$

and the internal work done by the internal stresses σ_e during the imposed virtual displacements, per unit volume is

$$dW_I = \epsilon^* T \sigma_e \quad (12)$$

Equating Equations (11) and (12)

$$\delta_e^* T F_e = \epsilon^* T \sigma_e$$

and further substitution yields

$$\delta_e^* T F_e = \delta_e^* T A^{-1} T B^T D B A^{-1} U_e \quad (13)$$

Equation (13) must be valid for all δ_e^* therefore

$$F_e = A^{-1} T B^T D B A^{-1} U_e \quad (14)$$

Equation (14) gives the nodal point forces in terms of the nodal point displacements per unit volume and hence the stiffness required becomes

$$k = A^{-1} T \int_{vol} B^T D B dv A^{-1} \quad (15)$$

All of the matrix expressions in the above relation are known for any element and hence, provided the integral can be evaluated, the element stiffness can be found.

2.3 CHOICE OF ELEMENTS

The nature of the elements chosen for any particular stress problem depends on the physical dimensions, geometric

symmetry, and the type of loading, together with the nature of the support of the body under consideration. The elements must be chosen such that the assembly of elements can represent as closely as possible the geometry of the body itself, while the type of loading and boundary conditions dictate the variation of stress that might be expected and hence the type of displacement function which should be assumed for the elements.

The present study was subdivided into two main areas, (i) the plane stress case and (ii) the axisymmetric case. For the plane stress case the boundaries were not, for all cases, parallel to the axis system although they varied in a uniform manner. Since this analysis was done to detect any possible variation in the stress and displacement waves, the most refined, well known elements were chosen. These elements allowed a linear varying stress field within the element. Two elements were chosen for case (i) namely, the rectangular element and the general quadrilateral element.

For the second case a ring type element was the most appropriate and again two types of element were chosen, namely, the rectangular ring element and the quadrilateral ring element. As in the previous case, this allowed for a linearly varying stress within the element and permitted the assembled elements to reasonably approximate the actual body.

The four elements used in this project are shown in Figure 2.3.

Fig. 2.3 ELEMENTS

The stiffness expressions for the plane stress rectangular element and the rectangular ring element were found by the evaluation of Equation (15) in the following manner. The analytic expression for the integral portion of Equation (15) was found and is detailed in Appendix A.1. These expressions were programmed along with the expressions for A^{-1} and hence with the matrix multiplication performed by the machine, the stiffness of these elements was evaluated.

The irregularity of the limits of integration for the plane stress quadrilateral element and the quadrilateral ring element dictated the use of a numerical integration procedure for evaluation of the volume integral. From among the available numerical integration procedures the Gaussian Quadrature formula for double integration was chosen. Provided the limits are -1 to +1 this formula is

$$\iint f(\eta, \xi) \, d\eta \, d\xi = \sum_{i=1}^n \sum_{j=1}^m H_i H_j f(a_i, a_j) \quad (15)$$

where a_i , a_j are specified internal points and H_i , H_j are weighting coefficients. Reference (25) and (26) give a good account of this Gaussian Quadrature formula together with the values of a_i , a_j , H_i and H_j .

2.4 DYNAMIC CONSIDERATIONS

The finite element analysis of a static problem after the stiffness is assembled reduces to solving Equation (3).

The problem posed for this thesis was to determine the nature of the stress wave as it moved through bodies of various shapes. Hence the problem studied was dynamic.

In a normal spring mass system with damping, the differential equation as found in any standard text on vibration is:

$$m \frac{d^2 u}{dt^2} + c \frac{du}{dt} = f \quad (17)$$

where m is the mass, c is the damping coefficient, u is the displacement, t is time and f is the applied force. The thesis problem was solved by considering the bar to consist of a number of lumped masses at the nodal points. In matrix form the equation describing the motion in the bar was

$$m \frac{d^2 U}{dt^2} + C \frac{dU}{dt} = \Sigma F \quad (18)$$

where m is the mass matrix, C is the damping matrix, U is the displacement vector and ΣF is the unbalanced force vector. For any particular node the unbalanced force was determined by combining the applied nodal force, F_a , with the force due to any displacement, hence ΣF becomes

$$\Sigma F = F_a - KU \quad (19)$$

To obtain the solution to the above equation it was

necessary to determine the mass matrix, m , as well as the damping matrix, C . Although a real system has a certain amount of damping due to internal friction, C was reduced to damping relative to a fixed support and considered as a one degree of freedom system for each nodal mass. The damping was introduced into the analysis to permit the solution to converge to the static case, providing a means of checking the validity of the program. Damping was introduced in this analysis for use solely in the static cases. The value of C was kept at zero for all the dynamic cases because of the poorly understood mechanism of damping within a bar and because the results of this investigation are only for the time period necessary for the complete passage of one wave during which time the effect of C would be insignificant. For the static test cases convergence to the steady state condition was attained by using as the value for the damping, C , a fraction of the critical value of damping for each mass vibrating alone. Thus

$$C = \frac{1}{p} \sqrt{2KM}$$

where p is some integer value, K is the nodal stiffness and M is the lumped nodal mass. A value of damping was calculated for each node resulting in the C matrix mentioned above. Throughout the investigation involving the test cases, a number of different values of p were used resulting only in a variation in the time required for convergence to the steady state condition. The mass matrix, m , was determined on a lumped

mass assumption with a portion of each element being assigned to each node. For the rectangular elements, one quarter of the element mass was assigned to each node. The quadrilateral element was divided into four masses by joining the mid points of the opposite sides and assigning the portion adjacent to each corner to that corner node.

A technique to solve Equation (18) was necessary. Finite difference approximations using the central difference formula for both the first and second derivatives was used. Using the approximations

$$\frac{d^2U}{dt^2} = \frac{U_B - 2U_C + U_F}{\Delta t^2} \quad (20)$$

and

$$\frac{dU}{dt} = \frac{U_B - U_F}{2\Delta t} \quad (21)$$

where U_F is the column matrix of nodal displacements one time unit in the future, U_C is the column vector of the current displacements and U_B is the column vector of displacements one time unit in the past and Δt the chosen time interval. On substitution into Equation (18), the final form of the Equation was

$$m \frac{U_B - 2U_C + U_F}{\Delta t^2} + c \frac{U_B - U_F}{2\Delta t} = F_a - KU_C \quad (22)$$

The terms of the K , m , C matrices were first determined and then the values of the displacement vector U_F , with $U_B = 0$ and $U_C = 0$ initially, were determined in accordance with the flow chart of section 2.6. Hence, for each time interval a complete description of the nodal point displacements was found.

2.5 STRESS DETERMINATION

The solution of Equation (22) yielded the displacements at each time interval but not the nodal point stresses. Since the stress strain relation is already known in Equation (9), namely

$$\sigma = D \epsilon \quad (9)$$

and since a relation between strain and the element nodal point displacements is found from Equations (6) and (8) to be

$$\epsilon = B A^{-1} \delta_e \quad (23)$$

the element nodal point stress can be determined from

$$\sigma = D B A^{-1} \delta_e \quad (24)$$

2.6 COMPUTER ANALYSIS

The solution of the matrix Equation (22) yields the displacement vector U while the solution of the matrix

Equation (24) yields the element stress vector σ . As is detailed in Appendix A.1, the solution of these equations for the axisymmetric case is different than the solution for the flat bar case. There are of course numerous similarities and where possible the same subroutines have been used for both problems. These similarities and differences can be seen in the flow charts and in more detail in the program listings to be found in Appendix A.3. A number of points may not be at once obvious to the reader and are worthy of further comment.

One limitation in a computer program of this type is the size of the computer and in order to save space only the diagonal elements and those essential elements below the diagonal of the symmetric stiffness matrix were stored. This is evident in the subroutines ESA1R and ESA2R. Another necessary quantity is the value of Δt since if this is too large the solution of Equation (22) will not converge and if it is too small convergence will be slow and wasteful of computer time. If the system is considered as a series of lumped masses connected in the x-y coordinate directions by springs of stiffnesses calculated by the analysis of Appendix A.1 then an approximation of the highest mode of vibration will serve to determine Δt . The approximation is found by considering at some instant one lumped mass to be moving in one direction with the remainder moving in the opposite direction. The subroutines TIMER, TIM2R con-

sider each lumped mass in turn as the one moving opposite to the others and determines the highest frequency of vibration using the relation

$$W = \sqrt{\frac{K_T}{M}} \quad (25)$$

For each node the natural frequency, W , can be found from Equation 25 where K_T is equal to twice the nodal stiffness and M is the lumped nodal mass. A survey of all the values of W as calculated from Equation 25 determines the highest natural frequency and hence, the largest allowable value for Δt can then be found as some fraction of $1/W$. Although this approximation in no way solves the eigenvalue problem for the complete system of lumped masses, the ease with which a suitable value of Δt can be found is perhaps sufficient justification for its use.

The procedure for the computer program can best be seen from the flow charts on pages 32 and 33 which are presented with a view to the understanding of the overall philosophy of the program and not to present the fine detail. This fine detail can best be found in the listings presented in Appendix A.3.

FLOW CHART — FLRT P.A.P.SUBROUTINE

READ: NODE POINT COORD.
 RXY4R
 CALC: NO. OF NODES & NO. OF DEGREES OF FREEDOM

 READ: ELEMENT & NODE NOS., DENSITY, YOUNG'S MOD.,
 POISSON'S RATIO
 RECONR
 CALC: NO. OF ELEMENTS & BAND WIDTH

 READ: BOUNDARY CONDITIONS
 RBCTR

 SET CONSTANTS AND D MATRIX
 INOIR
 DAH2R
 DAHLR

 DO FOR EACH ELEMENT
 CALC: ELEMENT STIFFNESS, NODAL MASSES.
 ASSEMBLE STIFFNESS & MASS MATRICES.
 AIN2R
 GSRTX
 MTMLR
 ESBIR
 VFJDR
 PMATR
 KMATR
 ESA2R
 NMASR

 CALC: TIME INTERVAL
 TIMER

 READ: INITIAL NODE POINT FORCES
 RINFR

FLOW CHART — AXI-SYMMETRIC BARSUBROUTINE

READ: NODE POINT COORD.
 RXY1R CALC: NO. OF NODES & NO. OF DEGREES OF FREEDOM

 READ: ELEMENT & NODE NOS., DENSITY, YOUNG'S MOD.,
 REGR POISSON'S RATIO
 CALC: NO. OF ELEMENTS & BAND WIDTH

 R3CTR READ: BOUNDARY CONDITIONS

 DAN3R SET CONSTANTS AND D MATRIX

 AMTIR
 INOBB
 GSRRR
 MTMLR
 ESB1R
 VFJDR
 PMT2R
 KMT2R
 ESA2R
 NMS2R

 TIMBR CALC: TIME INTERVAL

 R1NFR READ: INITIAL NODE POINT FORCES

3. THE MOIRÉ FRINGE METHOD

3.1 GENERAL PRINCIPLES

The Moiré Fringe Method is an experimental method using grids or grilles of closely spaced fine lines. When two such grilles are superimposed and one is displaced relative to the other, the light passing through the grilles is altered, resulting in a series of light and dark fringes. Obviously since the fringes are the result of the relative displacement of the two grilles, the Moiré Method is a tool by which displacement and hence strain measurements may be made. The grilles are applied to the area of interest and hence a full field representation of the displacement can be easily recorded.

Consider a sufficiently small part of a non-homogeneous displacement field to which the Moiré method has been applied. In general the fringe pattern in this area is caused by the combined effect of rotation and differences in pitch of the straight parallel lines of the model grille as compared to the master or reference grille. The derivation of the equations describing the Moiré Method can be found in a number of references (27),(28),(29) however the method detailed in reference (28) is general and is repeated here for completeness.

Consider the master grille to consist of straight parallel lines spaced at a pitch P , and the model grille

with a pitch P' at the same point, as in Figure 3.1. This model grille pitch P' may be the result of the distortion within the model as well as any intended difference in pitch between the master and undistorted model grille. If the acute angle between the master grille and the distorted model grille at any point is θ , and the angle between the master grille and the fringe at a point is $\emptyset$ then the normal distance between fringes is derived as follows. The distance AB along any fringe between two adjacent lines on the master grille may be determined as

$$AB = \frac{P}{\cos(\emptyset - \frac{\pi}{2})} = \frac{P}{\sin \emptyset}$$

$$\text{and } AB = \frac{P'}{\cos(\emptyset - \frac{\pi}{2} - \theta)} = \frac{P'}{\sin(\emptyset - \theta)}$$

$$P' = P \frac{\sin(\emptyset - \theta)}{\sin \emptyset} \quad (26)$$

Combining these relations gives

$$\tan \emptyset = \frac{P \sin \theta}{P' \cos \theta - P} \quad (27)$$

$$\text{Now } a = \frac{P}{\sin \theta}$$

$$\text{and } \delta = a \cos(\emptyset - \frac{\pi}{2} - \theta) = P \frac{\sin(\emptyset - \theta)}{\sin \theta}$$

$$= P' \frac{\sin \emptyset}{\sin \theta} \quad (28)$$

Fig. 3.1 MOIRÉ FRINGES

Using Equation (27) to determine $\sin \theta$ as

$$\sin \theta = \frac{P \sin \theta}{\sqrt{P^2 \sin^2 \theta + (P \cos \theta - P')^2}}$$

the normal spacing between fringes δ is

$$\delta = \frac{P P'}{\sqrt{P^2 \sin^2 \theta + (P \cos \theta - P')^2}} \quad (29)$$

In this expression for δ if $\theta = 0$, that is, if there is no rotation of the model grille relative to the master, then

$$\delta = \frac{P P'}{P - P'} \quad (30)$$

and if the number of fringes per inch is f

$$\text{then } f = 1/\delta = \frac{P - P'}{P P'} \quad (31)$$

Designating the number of lines per inch in the master and model grilles by N and N' respectively, f becomes

$$f = N' - N \quad (32)$$

Considering now the special case for pure rotation and no difference in the pitch of the two grilles, then provided θ is small

$$\delta = \frac{P}{\theta} \quad (33)$$

since $\sin \theta \doteq \theta$, $\cos \theta \doteq 1$ and $p = p'$ in Equation (29).

A fringe will be produced whenever the model displacement perpendicular to the master grille equals the pitch of the master grille and hence a Moiré fringe is the locus of all points having the same displacement perpendicular to the master grille. Thus the accumulated displacement at any point is the fringe order multiplied by the pitch of the master grille or

$$U = n P \quad (34)$$

where n is the fringe order referred to some point of zero displacement.

The displacement curves throughout the body under examination are determined by the fringe pattern and hence the accuracy of this method depends on the line density of the grille. Although the line density may be increased, the cost of suitable grilles of high line density is very high. Further, the difficulty of recording the fringes with a very high line density dictates highly sophisticated equipment. It has been reported in the literature (30) that a line density of 1000 lines per inch is an upper limit for use without elaborate equipment.

A method of increasing the accuracy without increasing the line density has been detailed in a paper by Chiang (30). This method is based on a difference in the line density between the model and master grilles, that is a mismatch.

Two types of mismatch may be employed to obtain greater accuracy. A linear mismatch occurs when the pitch of the model grille is different from the pitch of the master grille, while a rotational mismatch occurs when the model grille is rotated on angle θ relative to the master grille. If the pitch of the model grille is smaller than the pitch of the master grille the result is a linear compressive mismatch since this corresponds to a fictitious compression in the model. Similarly a linear tensile mismatch is created when the pitch of the model grille is greater than the pitch of the master grille. In both the linear mismatch and rotational mismatch situation, a series of fringes appear and these fringes represent a zero state of strain in the model. On the application of a load, causing strain within the model, the fringe pattern will change and the fringe density will either increase or decrease depending on the nature of the load compared to the type of mismatch. Since the actual displacement at a point in the model is found by algebraically combining the initial and final fringe patterns any errors resulting from the application of the grilles will automatically be eliminated. Equation (33) describes the fringe spacing as a function of the pitch P , and the angle of rotation θ provided the angle of rotation is small. The relation between the fringe spacing and the pitch of the master and model grilles for the linear mismatch is given by Equation (30). Both of these expressions, as shown in the previous derivations, are

special cases of the more general expression, Equation (29). In order to eliminate the initial errors caused by distortion of the grilles at the time of application and to create a reasonable fringe density it was decided to use a line density of 1000 lines per inch coupled with a linear mismatch. Complete detail of the way of obtaining the mismatch together with the remaining experimental technique is given in the section entitled EXPERIMENTAL PROCEDURE.

3.2 EXPERIMENTAL APPARATUS

The main requirement in the design of the experimental apparatus was the necessity of recording the Moire fringe pattern of a moving wave front. This main requirement dictated the consideration of the light sources, the loading device and the means of recording the fringe pattern. In an experiment where the stress distribution in the model is static, a number of alternatives are available for the placement of the master grille relative to the model grid. Among these are, (1) the superposition of the master grille over a photograph of the distorted model grille, (2) a double exposure of the distorted model grille over the undistorted model grille and (3) the placement of the master grille directly in front of the model grille hence recording directly the fringe pattern. Because of the motion of the

displacement waves in a dynamic experiment, it was decided to attempt to record the complete fringe pattern at various times after the initiation of the wave. Again, two alternatives were available for recording the fringes: (1) a fast framing camera, and (2) a single exposure technique. Mainly, because of the expense associated with a fast-framing camera, it was decided to use a single exposure technique. The design of the experimental apparatus was based on this decision to record the complete fringe pattern with a single exposure technique and is detailed in the following sections.

3.3 PHOTOGRAPHIC RECORDING

If a master grille is located at the film plane of one camera, then a fringe pattern can easily be produced by focusing this camera on a model grille. It is then necessary to use a second camera to record this fringe pattern. A block diagram of the two cameras relative to the model grille is given in Figure 3.2.

This two-camera technique was adopted for this experiment. A further advantage of this technique is that a mismatch may easily be obtained by proper placement of Camera 2 relative to the model grille. Uniform magnification at the film plane of Camera 2 was obtained by ensuring that the film plane and the plane of the lens of Camera 1 were parallel to the plane of the model grille. This was

Fig. 3.2 CAMERA ARRANGEMENT

accomplished by proper adjustment of the film and lens planes of Camera 2. Since only a linear mismatch was desired, elimination of all unwanted rotational mismatch was obtained by proper alignment of the master grille relative to the model grille. A 4" x 5" Orbit Monorail plate camera with adjustable lens and film planes and a rotatable back provided the necessary flexibility to obtain the desired alignment. A Hasselblad Model 500C camera was used to record the fringe pattern created on the back of Camera 2 and is shown schematically in Figure 3.2 as Camera 1.

3.4 LIGHT SOURCE

The decision to use a single exposure for permanently recording the image meant that either an extremely fast shutter or a high intensity light of very short duration was necessary. It was decided to attempt to use a short-duration light source manufactured by E. G. & G., Inc. of Boston. This light source was a spark gap light of high intensity with a maximum aperture duration time less than 1 microsecond. This duration time was sufficiently short to adequately stop the motion of the fringe and could be triggered through a time delay circuit. A dependable time delay was easily obtainable with a Model 564 Tektronix Oscilloscope and a 3B3 time base plug-in unit.

Fig. 3.3 LOADING FRAME

Fig. 3.4 LOADING MECHANISM

3.5 LOADING FRAME

The loading frame, Figure 3.3, was constructed of 3" x 2" x 5/16" aluminum channel welded together. Surfaces A and B were machined parallel to each other after welding, preparatory to mounting the model support apparatus and loading mechanism. The model was supported perpendicular to the base of the loading frame with a support mechanism located on Surface A. The loading mechanism, Figure 3.4, consisted of an aluminum tube mounted perpendicular to Surface B through which a steel ball of 1 3/8" diameter weighing .381 pound was dropped. The ball was held in the tube by an electromagnet which was adjustable in height, thus enabling the ball to be dropped from a preset height.

It was necessary to trigger the microflash unit, previously mentioned, when the displacement wave had moved to the desired position within the model. A reference from which all timing was made was obtained by the ball's interrupting a light beam which was directed at a photodiode, hence, obtaining a voltage pulse usable to activate the oscilloscope.

3.6 MODEL

The major considerations in the construction of the models were the selection of the model material and the application of a suitably prepared grille to the model

Fig. 3.5 RECTANGULAR AND 20° TAPERED MODELS

Fig. 3.6 ASSEMBLY JIG FOR MODELS

surface. Although the speed of propagation of a displacement wave in "Perspex" Acrylic material was reasonably fast, (being approximately one-third the speed of propagation of a wave in steel), the short duration light from the micro-flash unit reduced any blurr to a negligible amount; and it was decided to proceed with this material. Another advantage to the use of this material was its very high degree of transparency.

The grilles were produced on Kodak Ortholith stripping film from a master glass grille of 1000 lines/inch kindly presented to the author by Dr. F. P. Chiang of State University of New York at Stony Brook. The grilles were then cemented to the accurately machined model with the aid of a jig such that the grille lines were parallel to the model base. The model dimensions are given in Figure 3.5. The jig used for ensuring the proper orientation of the grille lines in relation to the model base was constructed as shown in Figure 3.6. The vertical surface of Part 1 was ground to a 12 micro-inch finish; and it was found that, if a piece of stripping film placed between the base and Part 1 were viewed with a microscope along the axis shown, both a direct and a reflected image could be viewed as in Figure 3.7. Lines both parallel and non-parallel to the vertical polished surface of Part 1 are shown in Figure 3.7. It was then easy to adjust the film such that the lines in the reflected image

LINES PARALLEL TO JIG FACE

LINES NON-PARALLEL TO JIG FACE

Fig. 3.7 MICROSCOPE IMAGE OF GRILLE LINES IN ASSEMBLY
JIG

and the direct image were parallel, hence, ensuring that the grille lines were parallel to the polished surface of Part 1. The face of the model to receive the film was roughened with Number 360 grit surface paper and cleaned. After application of Photoelastic VPAC cement with UPH-1 hardener, the model was placed on top of the film as shown in Figure 3.6. Sufficient pressure was then applied, thus keeping the base of the model in contact with the polished surface, hence, parallel to the grille lines. Pressure was also applied so that the cement thickness was kept to a minimum. After a drying time of 24 hours, the model was removed from the jig, and the backing of the stripping film was removed, leaving the emulsion with the grille lines firmly attached to the model. In all cases the grilles were placed on the outer surface of the models.

3.7 EXPERIMENTAL PROCEDURE

The equipment described in Sections 3.3 to 3.7 is shown assembled in Figure 3.8. A rigid table was used to firmly support the loading frame and the Zeiss profile optical bench. The Zeiss profile bench was used to support the lens, the plate camera, and the recording camera. The other equipment--the light source, the oscilloscopes, the Ellis bridge amplifier, and the various power supplies--are also shown in this figure. The use of each piece of

FIG. 3.8 EXPERIMENTAL APPARATUS

equipment can best be explained by detailing the experimental procedure.

The model, together with the attached grille, was placed in the model support clamp making sure that the model center line and the loading mechanism center line were properly aligned. The plate camera was then adjusted so that the lens plane and the film plane were parallel to the plane of the model. This was accomplished by viewing the fringe pattern on the film plane of the plate camera with the aid of an intense light source, A in Figure 3.8. To enable the fringe pattern to be formed, a sandwich arrangement of a piece of frosted glass and the master grille with the frosted surface and the line surface placed together was formed and mounted on the film plane of the plate camera. In this way, an image of the interference of the Moire grilles was formed on the frosted glass immediately next to the master grille. When the film plane and lens plane were adjusted parallel to the model grille, i.e. uniform magnification, the fringe pattern was a series of equally spaced parallel straight lines. After elimination of any rotational mismatch, the chosen linear mismatch was set and the camera then firmly clamped to the Zeiss profile optical bench. A compressive mismatch was used here in order that the number of fringes might be increased during the loading of the model. A 6-volt D.C. power supply was used to provide power for the

light located at point B in Figure 3.8. This light was focused on a photodiode diametrically opposite to the light as shown in Figure 3.4. The photodiode was powered by power supply C, in Figure 3.8. A 1 3/8" steel ball bearing was supported inside the aluminum tube D, by the electromagnet. This magnet was adjustable in height, and a clamp was provided so that the ball could be dropped repeatedly from the same height. The output from the photodiode circuit was used to trigger the Tekronix Oscilloscope; and an output pulse from this unit, delivered after a preset time delay, was used as a trigger for the microflash. The sequence of operations was to open the shutter of the recording camera and then to de-energize the electromagnet. This caused the ball to fall, interrupting the light beam, hence, triggering the oscilloscope time-delay circuit. The output pulse from the time-delay unit then caused the microflash to operate, resulting in an exposed frame on the recording camera. The shutter of the recording camera was then closed; and after adjusting the time delay on the time-base unit and advancing the film, the experiment was repeated. In all experimental runs, a reference or zero fringe pattern was obtained at the beginning and at the end of the run.

A transition piece of 2 1/4-inch diameter "Perspex" 1 inch long was used between the falling ball and the model to prevent damage to the model at impact. In order to ensure good transfer of energy, a light film of grease was

placed between the transition piece and the model. A recording was made of the strain pulse in the model at a point $1/8$ " below the loading end and on the center line with an $1/8$ " strain gauge. This enabled the determination of the pulse length and, hence, an indication of the wave-length to lateral dimension ratio. Two factors affected the choice of a transition piece, (1) the flatness of the wave produced, and (2) the length of the pulse generated within the bar. These two factors were investigated experimentally by choosing transition pieces of varying lengths and examining the pulse length and wave shape within the bar. The resulting choice of a transition piece was, therefore, an experimental compromise between obtaining a plane wave and a short pulse. The one chosen gave a plane wave within the accuracy of the experiment and a short pulse length.

4. RESULTS AND DISCUSSION

4.1 FINITE ELEMENT ANALYSIS

The scope of the work for this thesis was stated in Section 1.3 as an investigation of the nature of the stress and the displacement waves in flat and axisymmetric bodies. From a practical standpoint axisymmetric bars are more important than flat bars and, for this reason, the axisymmetrical bar was studied in slightly more detail than the flat bar. The change in the shape of the stress and displacement wave front, propagating in any bar, will depend on the lateral motion as well as the shape of the bar itself. It was, therefore, decided to attempt to separate these two effects by using two values of Poisson's Ratio, 0.0 and 0.35. For each value of Poisson's Ratio, three flat bars were studied. These bars had apex angles of 0° , 10° , and 20° . The apex angles for the axisymmetric bars were 0° , 5° , 10° , 15° , and 20° and these were also studied for both values of Poisson's Ratio.

Flat plate elements were used for the flat bars and ring elements were used for the axisymmetric bars. The organization of these elements, together with node numbers and the physical size of the bars, is shown in Figures 4.1 to 4.6. The number of elements were chosen so that a

FIG. 4.1 ELEMENTS FOR FLAT AND AXISYMMETRIC BARS
OF 0° APEX ANGLE

FIG. 4.2 ELEMENTS FOR AN AXISYMMETRIC BAR OF 5° APEX ANGLE

FIG. 4.3 ELEMENTS FOR FLAT AND AXISYMMETRIC BARS
OF 10° APEX ANGLE

FIG. 4.4 ELEMENTS FOR AN AXISYMMETRIC BAR OF 15° APEX ANGLE

FIG. 4.5 ELEMENTS FOR AN AXISYMMETRIC BAR OF 20° APEX ANGLE

FIG. 4.6 ELEMENTS FOR A FLAT BAR OF 20° APEX ANGLE

reasonable approximation to the actual bars was maintained but not so great in number to prevent analysis on the available computers. The elements were graded smaller toward the free edge of the bars because it appeared logical to expect that, because of the lesser restraint in this area, there would be greater variation of deflection and stress there than at the bar axis and a large number of small elements would make better allowance for this variation. In order to keep the value of Δt within a reasonable value, it was decided to permit several very small steps in the edges of the grid work at the edge of the tapered bars. Because the shape of the bars were well maintained, a reasonably large number of elements were used and because the elements used were the most refined, well-known elements presently developed, only the assembly of elements presented in Figures 4.1 - 4.6 were used in this investigation. All of the results presented have been for sections which do not include a step at the boundary.

Two computers were used extensively, an IBM 1130 and and IBM 360 Model 50. The IBM 1130 was used mainly in the assembly, testing and verification of the program, while it was found necessary to use the larger, faster machine, IBM 360, for the actual test cases mentioned in the foregoing. A damping factor, as explained in Chapter 2, was included in the formulation of the problem and because of this feature, static, as well as dynamic, stress conditions can be studied

by the program since, by using the appropriate value of the damping factor, the vibrations are damped out, resulting in the static stress condition. This feature permitted testing of the program by analyzing static stress cases. Because the loading system in this investigation was kept as a uniformly distributed load, the loading for the test cases was also uniformly distributed. The test cases were simply supported and nodal forces statically equivalent to a uniformly distributed load of 1000 psi were applied. A number of combinations of elements were used in these tests--all rectangular elements, all quadrilateral elements, and a combination of both rectangular and quadrilateral elements--in order to determine if any change in accuracy occurred from one system of elements to another. The incorporation of damping in the program permitted the analysis to reach equilibrium and hence a comparison with the calculated uniform stress in the rectangular bar of 1000 psi. A selection of six test cases is shown in Figures 4.7 and 4.8 and a copy of the computer output of the stress at each node in the direction of the applied load is given in Table 1. The computer values of stress given in Table 1 show an error of 0.1% when compared to the theoretical uniform stress of 1000 psi. This accuracy of 0.1% is only valid for rectangular bars under the action of uniformly distributed load and it is not intended to depict the accuracy for other types of problems but was useful during the verification of this program. For all the dynamic cases studied, no damping was included.

FIG. 4.7 STATIC TEST CASE CONFIGURATIONS

FIG. 4,8 STATIC TEST CASE CONFIGURATIONS

[illegible]

TABLE 1. AVERAGE NODAL STRESSES FOR TEST CASES SHOWN IN FIGURE 4.9

In section 2.6, a technique for the determination of the value of the integration time was explained. The appropriate value, $\Delta t'$, being some multiple of the smallest period of vibration, $\Delta t = \frac{1}{W}$, where W is the natural frequency as defined by Equation 25. If $\Delta t'$ is chosen too large, the results of the forward integration of the mass acceleration equation would diverge while, if $\Delta t'$ is very small, the results would be very accurate but the analysis would require an extremely large amount of computer time. A preliminary experiment yielded a good compromise for $\Delta t'$. In order to determine this value, a flat bar consisting of eight elements with loads and supports as shown in Figure 4.9 was programmed to obtain the displacements as a function of time, after application of the continuously applied load, for various values of integration time. Figure 4.9 shows that divergence occurs when $\Delta t' = \frac{\Delta t}{.5}$, no variation with values of $\Delta t' = \frac{\Delta t}{5}$ and $\Delta t' = \frac{\Delta t}{10}$ and little variation between these curves and that of $\Delta t' = \frac{\Delta t}{1}$. Hence, the decision to use $\Delta t = \frac{\Delta t}{2}$ was a reasonable compromise. This method of determining the appropriate value of $\Delta t'$ to be used was an engineering approach to the eigenvalue problem and enabled a useful value of $\Delta t'$ to be easily determined. A more accurate value of $\Delta t'$ could have been determined by the method described by Leech (41), however, this analysis was made to determine the variation of the shape of the stress and displacement wavefronts and not to solve the eigenvalue

FIG. 4.9 AXIAL DISPLACEMENT OF NODE NO. 1

problem. The determination of $\Delta t'$ by this method was justified by the results shown in Figure 4.9 and it was, therefore, decided not to solve the more involved eigenvalue problem and hence the alternative method of determining $\Delta t'$ (41) was not used.

In the present analysis, a lumped mass system was used resulting in a diagonal matrix for the equivalent nodal masses. This lumped mass system was used primarily to conserve computer space and to reduce computer time. Another approach would have been to use a consistent mass matrix as detailed by Archer (31). This consistent mass matrix would, however, not have been a diagonal matrix as was the mass matrix found by lumping the masses. Because the inverse of the consistent mass matrix would, in general, tend to be filled, it would, therefore, give rise immediately to displacements throughout the bar rather than propagating throughout the bar as in the case for the lumped nodal masses. Arguments may be presented for both approaches but papers by Costantino (20) and Baker and Daly (32) support the approach taken in this thesis; that is, a lumped mass system.

An infinite number of different loads could be applied to the bars under examination. Unlike most previous work in this field, it was decided to apply the load to the larger ends of the bars since, in most real cases, this is the situation. The nature of the application of this load could vary from an impulse load of short duration to a sinusoidally

varying load at any frequency. It was decided to apply an impulse load rising to a maximum value in zero time, remaining constant for a specified period of time, and returning to zero in zero time, hence, a square pulse. The duration time of the load was chosen such that the pulse wave length was approximately equal to the width of the flat bar and approximately equal to the diameter of the axisymmetric bar. For the program, the uniformly distributed loads had to be applied as equivalent nodal point loads. These nodal point equivalent loads were determined so as to maintain equilibrium conditions and were read into the computer as an input data. For the computer analysis, a uniformly distributed load of 1000 psi was applied to both the flat bar and the axisymmetric bar, thus maintaining initially the same stress in all of the bars.

Most studies of stress wave propagation have either assumed the stress and displacement wave to remain constant across the bar as the wave propagates along the bar or have concentrated on the determination of an accurate value for the natural frequency under various loading conditions. The present analysis was set up primarily to investigate the shape of these waves and, hence, the axial displacement and axial stress are the most important results, and curves presenting this data for all cases studied are found in Appendix A2. Because the stresses in each element at a common node will have a different value, the nodal values of

stress were found as the average of the stresses in each element at the nodal point. Only this average value of stress at the nodal points was obtained from the computer program and no individual nodal point stresses for each element were obtained. The curves in Appendix A2 were plotted using these average values of nodal point stresses. Although the axial displacement and axial stress are of prime interest, they are not independent of the displacement and stress in the other directions. This is evident from consideration of the stress-displacement relations to be found in the theory of elasticity and quoted here for completeness. For the flat bars in rectangular co-ordinates

$$\sigma_{xx} = \frac{E}{(1 - 2\nu)(1 + \nu)} \left\{ (1 - \nu) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\nu \partial w}{\partial z} \right\}$$

with similar expressions for σ_{yy} , σ_{zz} . For the axisymmetric bars, the stress displacement relations take the form

$$\sigma_{xx} = \frac{E}{(1 - 2\nu)(1 + \nu)} \left\{ (1 - \nu) \frac{\partial u}{\partial x} + \frac{\nu \partial v}{\partial y} + \frac{\nu u}{x} \right\}$$

with similar expressions for the σ_{yy} , σ_{zz} stresses. These expressions show the interdependence of the stress in one direction on the displacements or the change in displacements in the other directions. The condition of $\nu = 0.0$ removes any lateral motion in the bar, and hence, a distinct difference would be expected between $\nu = 0.0$ and $\nu = .35$. This

expected difference was detected and is easily seen in the curves in Appendix A2. Although the condition of $\nu = 0.0$ does not exist in practice, a theoretical analysis for this case shows the effect of the shape of the bar not complicated by the effects of lateral motion.

Three flat bars of dimensions shown in Figures 4.1, 4.3, and 4.6 were analyzed first with $\nu = 0.0$ and second with $\nu = 0.35$. The numerical results of axial stress and displacement at the sections indicated are interpreted in graphical form in Figures A2-1 to A2-12 located in Appendix A2. These curves show that, for 0° apex angle, that is, a parallel sided bar, and for $\nu = 0.0$ both the stress and displacement are constant across any section of the bar. With $\nu = 0.0$, two trends are noted as the apex angle increases: first, the average value of both stress and displacement increase with increase in time after application of the load; and second, the difference between the edge displacements and edge stresses compared to axial values in general also increase with this increase in time. This increase in both stress and displacement at the bar edge as compared to the axial value is attributed to the redistribution of the stress pattern caused by the change in shape. Another way of viewing this change is to consider that, because some material, compared to the straight bar, has been removed, a more flexible portion of the bar exists near the bar edge, hence, permitting larger displacements in that area.

With $\nu = 0.35$ lateral expansion and , hence lateral vibration of the material in the bar is permitted. This results in a more complex situation which is evident from the increased complexity of the resulting curves of displacement and axial stress. In this case, even with a 0° apex angle, a considerable increase is noted in both stress and displacement from the axial position to the bar edge. A distinct difference from the $\nu = 0.0$ case is evident. This is shown in Figure 4.10 where the variation of the stress, from the axial value, σ_c , is plotted, as a non-dimensional ratio $\Delta\sigma/\sigma_c$, against a non-dimensional base of w_b/w_0 . w_b is the distance from the axis to the point being considered and w_0 is the total width of the bar at the section of interest. A similar non-dimensional plot of axial displacement is also shown in Figure 4.10. The values of stress and displacement used for these curves were taken from the curves in Appendix A2. The values used were taken from the curves which show the maximum difference, with respect to time and position along the bar, between the edge values and the axial values for the various bars. Here $\Delta\delta$ is the variation of the displacement from the axial value δ_c . For $\nu = 0.0$ an increase in $\Delta\sigma/\sigma_c$ is noted as the apex angle increases from 0° to 20° while for $\nu = 0.35$ little change is noted in the curves as the apex angle is increased. A similar trend is noted for displacement. The lack of change in the non-dimensional stress and displacement values as the apex angle is changed, for the

Fig. 4.10 AXIAL STRESS AND DISPLACEMENT IN FLAT BARS
OF STATED TAPER ANGLE

case where $\nu = 0.35$, is a result of the greater effect of the lateral motion within the bar.

Another interesting aspect of the data is shown in Figure 4.11. Here two curves depict the axial stress in a 20° flat bar at the axis and the edge of a section, $\frac{1}{2}$ " from the large end of the bar, which was initially parallel to the loaded end of the bar. These curves show that the maximum value of the stress occurs earlier at the bar axis than at the bar edge. This difference in the arrival times of the peak value of stress can be accounted for by considering the velocity of propagation of a stress wave in the bar. The elapsed time, from the time load application to the occurrence of the maximum values of stress, is 5.9% greater at the bar edge than at the bar axis. This indicates the velocity of wave propagation must be greater at the bar axis than at the bar edge by 5.9%. The material along the bar axis, being contained by the material surrounding it, is essentially in a condition of plain strain and the material at the bar edge is in a condition of plain stress. The velocity of wave propagation varies with Young's Modulus, E , and the mass density of the material being $\sqrt{\frac{E}{\rho}}$. The value of E used in this analysis was 30×10^6 psi and Poisson's ratio was set at 0.35. The effective value of E varies from E for a plain stress condition to $\frac{E}{1 - \nu^2}$ for a plain strain condition. A change from a plain stress condition to a plain strain condition suggests that a variation in velocity of 6% could be expected. This variation in velocity would only be evident until the

FIG. 4.11 AXIAL STRESS FOR 20° FLAT BAR

effects of shape and lateral motion become predominant. The curves show, by the 5.9% difference in the arrival times of the peak stress condition at the bar axis and edge, that the change in velocity is as expected.

Axisymmetric bars or solid horns of apex angles 0° , 5° , 10° , 15° , and 20° with dimensions shown in Figures 4.1 to 4.5 were analyzed for the two values of Poisson's Ratio, $\nu = 0.0$ and $\nu = 0.35$. The graphical presentation of both axial stress and displacement for a section of $1\frac{1}{4}$ inches from the loaded end are found in Appendix A2. For the case of $\nu = 0.0$, a very similar trend to the case of $\nu = 0.0$ and the flat bar is shown in Figure 4.12. Here the non-dimensional stress, $\Delta\sigma/\sigma_c$, and the non-dimensional displacement, $\Delta\delta/\delta_c$, taken from the curves in Appendix A2 which show a maximum difference between the edge values and center line values, with $\nu = 0.0$ show an increase similar to that of the flat bar as the apex angle increases from 0° to 20° . Although the non-dimensional stress increases toward the bar edge, similar to the flat bar, the maximum value obtained with $\theta = 20^\circ$ is not as great in the axisymmetric bar as in the flat bar.

The change of ν from 0.0 to 0.35 results in a more complicated wave shape as was evident for the flat bar case. Detailed plots for axial stress and displacement are found in Appendix A2. As with the flat bar, the effect of the change in apex angle is very small when the non-dimensional displacement is considered. The difference between the edge value and the axial value of displacement is essentially con-

Fig. 4.12 AXIAL STRESS AND DISPLACEMENT IN AXISYMMETRIC BARS
OF STATED TAPER ANGLES

stant for all bars from 0° to 20° apex angle although the non-dimensional displacement at the edge decreases with increase in the axial value of displacement as the apex angle is increased. The difference between the edge stress and the axial stress decreases slightly with increase in the apex angle and since the axial stress increases with increase in the apex angle, the non-dimensional stress decreases as the apex angle is increased.

Figures 4.10 and 4.12 show that for some bars the differences between the edge values and axial values of stress and displacement, relative to the axial values, are significant. In Figure 4.13 the variations of the values of the stress and displacement on the center line of the bar, as the apex angle is increased, are shown in non-dimensional form. The static values of stress and displacement, σ_{st} and δ_{st} , were based on equilibrium at the section of interest in the bar and were used for non-dimensionalizing in both the $\nu = 0.0$ and $\nu = 0.35$ cases. The value of the center line stress was taken from the curve in Appendix A2 which has the maximum edge to center line difference. The value of the center line displacement was found in a similar manner. These curves show that the center line stress and displacement tends to decrease as the apex angle is increased. That is, for the particular instant in time in the propagation of the waves when the difference between the edge values and axial values is greatest, the centre line values of stress and displacement are smaller

θ = APEX ANGLE

Fig. 4.13 VARIATION WITH APEX ANGLE OF CENTER LINE STRESS AND DISPLACEMENT

for large θ than for small θ . Another way of looking at this is that this maximum difference between the edge and axial values occurs in the bars of greater taper before the axial values of stress and displacement have built up to large values.

Figures 4.14 and 4.15 are curves of the lateral stress and the axial stress on a base of time after application of the load. These curves show that the times at which these stresses reach their maximum do not coincide. Since the periods of vibration for the axial and lateral stress waves do not coincide nor do these periods appear to be any integral multiple of each other, the lateral stress will affect the axial stress differently, depending on how close they are to being in phase at any point in time. This complicated interaction of these stress waves would also appear to be the cause of the differences in the shape of the various stress waves.

N. P. Suh (33) has investigated the reflected stress pulse from the lateral surface of a truncated cone in an experimental technique using strain gauges. He concludes that, in an elastic medium, part of the wave of a long wavelength elastic pulse is reflected from the lateral surface. The results of the present work would substantiate this since, in fact, it may be inferred by a comparison of the displacement for two bars of different apex angle. The displacement for any bar such as given in Figure A2.21 for

given time is greater toward the edge of the bar than at the center. Thus, the wavefront defined as a line of equal displacement across the bar at a particular time must be advanced at the edges relative to the center. By comparing Figure A2.19 and A2.21 for 15° and 20° axisymmetric bars, it is noted that the ratio of the displacement at the edge to that at the center at a fixed time (.898 sec) is greater for the 20° bar (1.096) than for the 15° bar (1.058). Of course, for 0° the ratio is 1.00. It is considered that this change in the shape of the wavefront is due to the reflection from the lateral sides of the bar.

Others, Brillhart and Dally (12) and Lewis, Goldsmith and Cunningham (34) have also studied this problem experimentally. Brillhart and Dally conclude that, if the ratio of wavelength to maximum cone diameter is greater than 3, then the axial stress can be closely predicted from the elementary theory, while Lewis et al state that the non-uniformity of strain increases as d/λ increases. This would imply a change in the waveshape with axial distance in a tapering bar. In contrast to this, the present work shown that for $\frac{d}{\lambda} = 1$, although the waveshape changes with increase in apex angle, the variation of stress and displacement from an axial position to an edge position does not vary substantially with an increase in apex angle.

Fig. 4.14 RADIAL (X) AND AXIAL (Y) STRESSES AT NODE 55 OF 20° AXISYMMETRIC BAR

Fig. 4.15 RADIAL (X) AND AXIAL (Y) STRESSES AT NODE 45 OF 20° FLAT BAR

4.2 MOIRÉ FRINGE ANALYSIS

The Moiré Fringe experimental apparatus was set up as described in Chapter 3. The 1 3/8" diameter ball bearing used in creating the displacement wave weighed .381 pound and was dropped from a height of 8.32 inches above the transition piece. With this arrangement, a strain pulse as shown in Figure 4.16 was obtained. The transition piece was selected from a number of different transition pieces after consideration of the resulting strain pulse generated in the bar. From among the various trials, the transition piece selected gave the smallest pulse wavelength. By observing successive photographs of fringe patterns, beginning immediately after the initiation of the displacement wave and ending when the wave had propagated well into the bar, it was ascertained that the displacement wave began and remained flat to within the accuracy of the method. The accuracy of the Moiré Fringe method is discussed in section 4.3 of this thesis. It was found necessary to apply a thin film of grease between the transition piece and the model in order to obtain a uniform transfer of energy into the model. The strain pulse was measured with 1/8" strain gauge applied on the center line of the model immediately below the loading surface. If the speed of propagation of a displacement wave in the "Perspex" model is calculated based on a value of $E = 4.2 \times 10^5$ psi and a density of .043 lb/in.³ and measuring

Fig. 4.16 MODEL STRAIN PULSE

the effective pulse duration as 230 microseconds, the pulse length is 14.1 inches. This compares with pulse lengths of 13.3 inches (34), 7 inches (35), 5 inches (13), and 33 inches (33) which have been obtained by other investigators. Although the pulse length was not as short as desired, it was decided to proceed with the experiment using the two models shown in Figure 3.5.

Many photographs of fringe patterns at various times during the propagation of the wave were taken and examples are shown in Figure 4.17. In order to interpret these Moiré fringe patterns, full size drawings of the models were made and a grid of parallel lines spaced at $1/4"$ was superimposed on them as seen in Figure 4.18. The film was then placed in a Durst L 1000 photographic enlarger and the model image, together with the fringes, was superimposed on the full size drawings. The fringe number and position was then easily recorded on the drawing.

It was not practical to show the complete data from this experiment but a carpet plot of the fringe order at the various grid points for the times shown are recorded in Figures 4.19 and 4.20. It is evident from these patterns by comparing the patterns with the static or no-load pattern that a definite change in the fringe pattern occurs as the time after the initial loading increases. In fact, the progression of the displacement wave can be seen more clearly

20° TAPERED BAR

RECTANGULAR BAR

Fig. 4.17 MOIRÉ FRINGE PATTERNS 50 μ SEC. AFTER
APPLICATION OF THE LOAD

20° TAPERED BAR

RECTANGULAR BAR

Fig.4.18 MODEL INTERPRETATION GRID

Fig 4.19 MOIRÉ FRINGE FOR 20° TAPERED BAR

Fig 4.20 MOIRÉ FRINGE FOR RECTANGULAR BAR

in Figures 4.21 and 4.22 which is a plot of the center-line fringe order at various times after the beginning of the displacement wave.

The relation between the speed of propagation of the displacement wave and the duration time of the microflash unit indicates that a small amount of movement of the fringe could take place during the exposure of the film. Because of this and the fact that the position of the fringe was found using the human eye, the position of the center of the fringe could be ascertained with $\pm 1/8$ of a fringe spacing. In this experiment, a master grille of 1000 lines/inch was used and a mismatch resulting in 10 fringes per inch shows that effectively a line density of 1010 lines/inch was being used for the model.

The displacement at any point can be calculated from the initial or zero fringe pattern and the fringe pattern for the time of interest. The actual displacement is N at time t minus N at time $t = 0$, multiplied by the pitch of the master grille. In this manner, a plot of the actual displacement across the rectangular bar for a section 2 inches from the fixed end of the bar at three different times is shown in Figure 4.23. A similar plot for the tapered bar for a section $1 \frac{3}{4}$ " from the fixed end is shown in Figure 4.24. Although both of these figures show some variation in the shape of the wave across the bar, the amount of variation is not large and it could be argued that within the accuracy

Fig. 4.21 MOIRÉ FRINGE ORDER ON CENTER LINE OF
RECTANGULAR BAR

Fig. 4.22 MOIRÉ FRINGE ORDER ON CENTER LINE OF
20° TAPERED BAR

Fig. 4.23 AXIAL DISPLACEMENT AT 2" FROM FIXED
END OF RECTANGULAR BAR

Fig. 4.24 AXIAL DISPLACEMENT AT $1 \frac{3}{4}$ " FROM FIXED
END OF 20° TAPERED BAR

mentioned above the wave remains essentially straight with the passage of time. This is in agreement with other authors who have indicated that, for a $\lambda/d > 3$, the distortion across any section of the bar should be negligible. In all cases, an attempt was made to analyze the wave shape sufficiently early so that no reflection from the fixed boundary would effect the wave shape. One notable exception is in Figure 4.24 curve labeled $t = 80$ microseconds. There the large distortion is attributed to a non-uniform reflection from the support of the model, there being sufficient elapsed time for the initial portion of the wave to have returned to this point on reflection.

4.3 DISCUSSION

A number of interesting results have appeared in the literature (34),(33),(39),(12) and a comparison with the results presented in Sections 4.1 and 4.2 would be useful. The finite element results of Section 4.1 and the experimental results of Section 4.2 are for different values of a/λ ratios where a is the lateral distance from the bar axis to the bar edge and λ is the wavelength of the applied load, and are, therefore, not directly comparable. Interesting finite element results have, however, been obtained for specific values of a/λ which permit comparisons with the experimental work of this thesis and with the results presented in a paper by Lewis, Goldsmith and Cunningham (34). The input pulse for the finite element results remained as in the main investigation, a square pulse, and this may account for some of the discrepancy in the comparisons which follow.

The numerical results of a finite element analysis for a flat bar with $a/\lambda = .142$ and a 20° apex angle are displayed in a graphical form in Figure 4.25. The displacement at the bar edge, for this case in which the maximum difference between the edge and axial displacement exists, is greater than at the bar axis by 10% or the displacement curve is flat to within $\pm 5\%$. The load for this analysis was applied at the large end of the bar and the larger displacement at the bar

Fig. 4.25 AXIAL DISPLACEMENT FOR 20° FLAT BAR AT
 NODES 43-51, 13 μ SEC AFTER APPLICATION
 OF THE LOAD

edge, although small, is in the part of the bar which could be considered to be more flexible as compared to the portion near the bar axis. This curve compares favourably with the results of the Moiré fringe experimental investigation as given in Section 4.2 where it was concluded that the wave was essentially flat within the experimental accuracy.

A 20° tapered axisymmetric solid bar, loaded at the small end of the bar in a manner to yield a/λ ratios of .019 at $\frac{1}{2}$ " and .175 at $1\frac{1}{2}$ " from the small end of the bar, was analyzed using the finite element method. The results of this analysis are presented graphically in Figure 4.26 where the curves represent the axial stress at a time when the difference between the edge and axial values is greatest. These curves show that the percentage variation of the stress across the bar section, as compared to the axial value, is greater for $a/\lambda = .175$ than for $a/\lambda = .091$. Other investigators (33),(12),(34) (13) have shown that, as λ increased or a/λ decreased, the variation of the stress across the bar section increases. This increase in the variation of the stress across the bar section as λ is increased is also shown for the present work in Figure 4.26.

The accuracy of any finite element analysis is difficult to ascertain because it involves many possible errors such as that due to the geometric approximation, that due to the form of the assumed displacement function and its relation to the true displacement within the element, that due to the round-

Fig. 4.26 AXIAL STRESS FOR 20° AXISYMMETRIC BAR
LOADED AT THE SMALL END
(FINITE ELEMENT METHOD)

off error in the computing machine and finally human errors. The elements used in this analysis together with the assumed displacement functions permit rigid body motion, permit a linear variation of strain together with maintaining compatibility along the element boundaries, hence as the element size reduces to zero, the error due to the assumed displacement function also reduces to zero. The geometric approximation to the actual solid bars of various tapers in this investigation is very good and, because the matrices involved in this analysis have been kept to a reasonable size, the machine round-off error will not be excessive.

Among the various papers (34),(33),(39),(12) which could be used for comparison, perhaps the most appropriate is a paper by Lewis, Goldsmith and Cunningham (34). The conclusions found in this paper were based on an experimental investigation using embedded strain gauges. Strain gauges were embedded in a cone with an apex angle of 20° and the cone was loaded with an impulse load at its small end. Three gauges were used on one half of the bar at the small section and four on one half of the bar at the larger section and, from the output of these gauges, Lewis was able to discern the strain and hence the stress distribution across the bar. Because the bar diameter varied in these two sections, the data was presented at a/λ ratios of .091 and .175. The value of Poisson's ratio for the bar in Lewis' investigation was identical to the value used in the finite

element investigation, namely .35. Lewis compared the variation of stress and strain across the bar section with the results of the Pochhammer-Chree theory for a cylindrical bar. The stress variations were calculated from the Pochhammer-Chree relations given by Davies (40) and were plotted together with Lewis' experimental results in his Figures 5 and 6. The results of the finite element analysis for $a/\lambda = .091$ and $.175$, shown in Figure 4.26, were normalized to yield the same axial values, at $r/r_0 = 0$, as those presented by Lewis et al (34) and are plotted together with the reproduced results of Lewis' work in Figure 4.27. The stress variation across the section of the bar is seen to be greater, for all three results, for the larger a/λ as compared to the smaller a/λ . The results of Lewis et al (34) and the present finite element results are in qualitative agreement since they both show a greater variation in stress across the section of the bar, for both values of a/λ , than that predicted by the Pochhammer-Chree theory for a circular bar of the same a/λ . Lewis' curves show a variation of 8% in the stress from the bar axis to the bar edge for $a/\lambda = .091$ while the present finite element investigation predicts the variation to be 15%. Similarly for $a/\lambda = .175$ the variation across the bar is shown to be 32.5% by Lewis and 27.5% by the present finite element results. Because no theoretical solution for this problem was available, a comparison with the work of Lewis et al (34) was used to obtain an estimate

POCHHAMMER-CHREE - - - - -
 LEWIS et al —————
 RUSSELL (FINITE ELEMENT) - · - · - ·

Fig. 4.27 NORMALIZED AXIAL STRESS

of the error in the finite element investigation. Lewis' work yielded results for both $a/\lambda = .091$ and $a/\lambda = .175$ and the comparison was made for $a/\lambda = .175$ since, in this case, larger strain values were obtained resulting in better experimental accuracy. Comparing the per cent variation of stress across the bar, from axis to edge, it is seen that the present work shows a 5% lesser variation than the work of Lewis. This represents an accuracy, if Lewis' work is assumed correct, of 5% in 32.5% or 15%. That is, the percentage variation of stress across the bar is estimated to be accurate within 15%. A similar comparison of the actual edge value predicted by Lewis and that of this work shows a variation of stress of 2 psi in 27 psi, or an estimated error of 7.3%. The accuracy of the predictions for the displacements will be better than for the stress predictions because the finite element method used here assumes a linear variation of stress across the element which is more restrictive than the permitted variation of displacement across the element. The displacement will, therefore, be predicted more accurately than the stress in a region in which the spatial variation of stress and displacement is large.

It was stated earlier that the fringe order, in the Moiré fringe analysis, could be ascertained to within $\pm 1/8$ of a fringe order. Because the displacement is found by numerically combining the initial or zero reference pattern with the fringe pattern resulting from a travelling wave, the error involved in the displacement measurement is $\pm 1/4$

of a fringe order or ± 0.00025 inches. This represents an error of $\pm 6\%$ at the maximum displacement measured. The results shown in Figure 4.25 indicate that the displacement wave was flat to within $\pm 5\%$ based on the finite element analysis. The accuracy of the finite element predictions for stress was estimated at 15% and was stated to be even better for displacement predictions, hence the conclusion that, for an $a/\lambda = .142$ in a flat bar, the displacement wave was essentially flat is justified.

5. CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The following conclusions are based on the finite element analysis and the Moiré Fringe experimental results. The loading in the finite element analysis was a square pulse of wavelength approximately equal to the width and diameter of the bars analysed. The load pulse in the experimental work was large relative to the physical size of the bars with a resulting $\lambda/d = 7.05$

1. The stress and displacement, away from the bar axis, is greater than the corresponding axial values with $\nu = 0.0$ for all apex angles except 0° and the difference between these off axis values and axial values increases with increase in the apex angle.
2. The variation of the axial stress and displacement across the bars is significant for 0° apex angle with $\nu = 0.35$.
3. The variation of the axial stress and displacement across the bars does not change significantly as the apex angle increases from 0° to 20° for the flat bar with $\nu = 0.35$.
4. The difference between the off axis and axial values of stress and displacement decreases with increase in the apex angle for the axisymmetric bar with $\nu = 0.35$ and remains significant in all cases studied.

5. The finite element analysis predicts a greater wave velocity on axis than on the bar edge which is in agreement with the theory.
6. The finite element analysis shows that the lateral motion is not in phase with the axial motion in the bar.
7. The finite element method, with a lumped mass approximation for the nodal masses, can be used to predict the stress and displacement in flat and axisymmetric bars of various shapes as stress and displacement waves propagate through the bar.
8. The Moiré Fringe method can be used to photographically record the displacement field in plane bodies under dynamic loading.
9. The experimental results with $\lambda/d = 7.05$ are in qualitative agreement with other authors (12), (35) and with the one dimensional theory which states that the wave will remain plane provided the wave length of the applied load is large compared to the lateral dimensions of the bar.

5.2 Recommendations

There are, obviously, many questions concerning the propagation of stress and displacement waves in bars of various shapes yet to be answered. Based on the experience of the author in the work described in this thesis, the author recommends the following for future work.

1. Investigate the use of the consistent mass matrix as compared to the lumped mass assumption for stress wave investigations and determine if any improvement in the prediction of stress and displacement occurs when other elements are used in a finite element analysis.
2. Investigate the dependence of the wave shape on rod taper and physical shape of the bar and on the pulse length and shape of the applied load using the finite element method.
3. Investigate experimentally a displacement wave as it propagates in a larger model than used in this thesis and using a shorter pulse length. A fast framing camera together with a Moiré Fringe experimental apparatus could be used to record the growth of the wave in the model.
4. Investigate methods of obtaining load pulses of shorter duration and faster rise time.
5. Develop a plate camera for both viewing the Moiré Fringes and also photographically recording them, hence, eliminating the need of a second camera in the mismatch technique used in this experiment.
6. Investigate the effects that the use of different types of elements, together with different grid spacings and consistent mass matrices may have on convergence and accuracy of solutions.

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APPENDIX A1

Element Stiffness

The expression for the element stiffness has been derived in the main body of this thesis and is denoted by Equation 15, namely,

$$K = A^{-1}{}^T \int_{vol} B^T DB \, dv \, A^{-1} \quad A1-1$$

where the terms have the following meaning:

A relates the element nodal point displacements, δ_e , to the generalized coordinates, a .

B relates the strains to the generalized coordinates.

D is the matrix relating stress to strain

and hence $B A^{-1}$ relates the strains to the nodal point displacements. The manner in which Equation 15 is fulfilled depends on the shape of the element and the choice of the local coordinate axis system. Since this equation is executed in slightly different ways for each of the four elements used a more detailed derivation of the stiffness for each element follows.

A. Rectangular Element - Plane Stress

The A matrix in the expression for the element stiffness, A1-1, since it relates nodal point displacements to the generalized coordinates, depends on the assumed displacement functions. If the displacement functions are assumed to have the form given by Equations 4 and 5 a linear variation of strain and hence stress is permitted. Further if the local coordinate axis is assumed to be located parallel to the sides of the element as shown in Figure A1-1 the displacements vary linearly along the element sides and hence the element boundary, after distortion, remains a straight line. This ensures compatibility of the element edges in the distorted state.

Assuming the displacement functions in the form given by Equations 4 and 5 namely,

$$u = a_1 + a_2x + a_3y + a_4xy$$

$$v = a_5 + a_6x + a_7y + a_8xy$$

the B matrix given by Equation 8,

$$\epsilon = B a \quad \text{becomes}$$

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & y & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \quad \text{A1-2}$$

Fig. A1.1 PLANE RECTANGULAR ELEMENT

Using B as detailed in A1-2 and for this plane stress case the D matrix given by Equation 9a,

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

the matrix expression $B^T D B$ is detailed by Equation A1-3.

Assuming thickness to be unity and the symbol for the element length is l and for the element height, h , the integral portion of A1-1 reduces to

$$\int_{vol} B^T D B = \int_0^h \int_0^l B^T D B \, dx \, dy$$

Performing this integration results in quation A1-4

$$B^T D B = \frac{E}{1-\nu^2}$$

A1-3

0	0	0	0	0	0	0	0	0	0
1	0	0	y	$\frac{1-\nu}{2} x$	0	$\frac{1-\nu}{2} y$	νx	0	$\frac{1-\nu}{2} y$
			y^2	$\frac{1-\nu}{2} x^2$	0	$\frac{1-\nu}{2} xy$	νy		
					0	0	0		0
						$\frac{1-\nu}{2} y$	0		$\frac{1-\nu}{2} y$
							1		x
									x^2
									$+ \frac{1-\nu}{2} y^2$

SYM

The A matrix relates the element nodal point displacements, δ_e , to the generalized coordinates, a, and its form will depend on the order of the terms in the nodal point displacement vector. For this element it was convenient to use the nodal point displacement vector in the form

$$[\delta_e] = [u_i, v_i, u_j, v_j, u_k, v_k, u_l, v_l] \quad A1-5$$

With the displacement vector given above and the local coordinate axis as shown in Figure A1-1 the A matrix was of simple form and its inverse was easily found to be

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{l} & 0 & \frac{1}{l} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & \frac{1}{h} & 0 \\ \frac{1}{lh} & 0 & -\frac{1}{lh} & 0 & \frac{1}{lh} & 0 & -\frac{1}{lh} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{l} & 0 & \frac{1}{l} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & \frac{1}{h} \\ 0 & \frac{1}{lh} & 0 & -\frac{1}{lh} & 0 & \frac{1}{lh} & 0 & -\frac{1}{lh} \end{bmatrix} \quad A1-6$$

As is shown in the section describing the computer program the stiffness was found using the results given in Equations A1-4 and A1-6.

B. Quadrilateral Element - Plane Stress

The quadrilateral element together with the coordinate axis system is shown in Figure A1-2. The continuity of the conditions within all the elements and the compatibility condition along the element boundaries are the conditions governing the choice of the displacement functions for the quadrilateral element. It may be observed in Figure A1-2 that for the quadrilateral element neither x or y remains constant along the element boundaries and hence if the displacement functions specified by equations 4 and 5 were used, the compatibility condition along the element boundaries would be violated. A method suggested by Zienkiewicz (22) is used here and insures the compatibility of edge displacements as well as maintaining the linear stress and strain condition. The method consists of establishing a new coordinate system as shown in Figure A1-2 and assuming the displacement functions

$$u = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

$$v = a_5 + a_6\xi + a_7\eta + a_8\xi\eta$$

A1-7

Fig. A1.2 PLANE QUADRILATERAL ELEMENT

Provided ξ and η have the limits of ± 1 on the boundary as shown the displacements will vary linearly along the edges ensuring compatibility of edge displacements.

Substitution of the various values of ξ and η verifies that the relationship between the x-y and ξ - η coordinate system is

$$\begin{aligned} x &= \frac{1}{4} \{ (1-\xi)(1-\eta)x_i + (1+\xi)(1-\eta)x_j + (1+\xi)(1+\eta)x_k + (1-\xi)(1+\eta)x_l \} \\ y &= \frac{1}{4} \{ (1-\xi)(1-\eta)y_i + (1+\xi)(1-\eta)y_j + (1+\xi)(1+\eta)y_k + (1-\xi)(1+\eta)y_l \} \end{aligned}$$

A1-8

If $u_i, v_i \dots u_l, v_l$ represent the nodal point displacements then u and v may be written in terms of these obtaining

$$u = \frac{1}{4} \{ (1-\xi)(1-\eta)u_i + (1+\xi)(1-\eta)u_j + (1+\xi)(1+\eta)u_k + (1-\xi)(1+\eta)u_l \}$$

A1-9

$$v = \frac{1}{4} \{ (1-\xi)(1-\eta)v_i + (1+\xi)(1-\eta)v_j + (1+\xi)(1+\eta)v_k + (1-\xi)(1+\eta)v_l \}$$

The strain displacement relations are

$$\begin{bmatrix} \epsilon \\ \epsilon \\ \epsilon \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

A1-10

and since u and v are expressed as functions of ξ and η , and ξ and η are not expressed explicitly in terms of x and y , the following relations are needed,

$$\frac{\partial u}{\partial \xi} = \frac{\partial x}{\partial \xi} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial \xi} \cdot \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial x}{\partial \eta} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial u}{\partial y}$$

or in general

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi} \cdot \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial}{\partial y}$$

A1-11

resulting in the general matrix relation

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

$$= [J] \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

A1-12

Hence

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} J^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \quad \text{A1-13}$$

where J^{-1} is a 2×2 matrix and for convenience the terms of this matrix are designated as

$$\begin{bmatrix} J^{-1} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \quad \text{A1-14}$$

Using the above relation, the partial derivatives of the displacements become

$$\frac{\partial u}{\partial x} = \left\{ J_{11} \frac{\partial u}{\partial \xi} + J_{12} \frac{\partial u}{\partial \eta} \right\}$$

$$\frac{\partial v}{\partial y} = \left\{ J_{21} \frac{\partial v}{\partial \xi} + J_{22} \frac{\partial v}{\partial \eta} \right\}$$

$$\frac{\partial u}{\partial y} = \left\{ J_{21} \frac{\partial u}{\partial \xi} + J_{22} \frac{\partial u}{\partial \eta} \right\} \quad \text{A1-15}$$

$$\frac{\partial v}{\partial x} = \left\{ J_{11} \frac{\partial v}{\partial \xi} + J_{12} \frac{\partial v}{\partial \eta} \right\}$$

The above partial derivatives when combined in accordance with Equation A1-10 yield the strain displacement relations and since u and v are, by Equation A1-9,

functions of the nodal point displacements, the strain is related to the nodal point displacements or

$$[\epsilon] = [P] \begin{bmatrix} \delta_e \end{bmatrix} \quad A1-16$$

where

$$\begin{bmatrix} \delta_e \end{bmatrix} = \begin{bmatrix} u_i & u_j & u_k & u_l & v_i & v_j & v_k & v_l \end{bmatrix} \quad A1-17$$

The order of the nodal point displacements is arbitrary and was chosen as in A1-17 for convenience in programming. Using Equations A1-15 and A1-7 the expression for the matrix, P, was calculated as found in Equation A1-18. This P matrix relates the strain to the nodal point displacements and hence replaces $B A^{-1}$ in Equation 15. Hence for this quadrilateral element the stiffness expression is

$$\bar{K} = \int_{vol} P^T D P dv \quad A1-19$$

where the bar over the K indicates that the nodal point displacements are in the order specified in Equation A1-17.

$$P = \frac{1}{4} \begin{bmatrix} -J_{11}(1-n)-J_{12}(1-\xi) & J_{11}(1-n)-J_{12}(1+\xi) & J_{11}(1+n)+J_{12}(1+\xi) & -J_{11}(1+n)+J_{12}(1-\xi) \\ 0 & 0 & 0 & 0 \\ -J_{21}(1-n)-J_{22}(1-\xi) & J_{21}(1-n)-J_{22}(1+\xi) & J_{21}(1+n)+J_{22}(1+\xi) & -J_{21}(1+n)+J_{22}(1-\xi) \end{bmatrix}$$

A1-18

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -J_{21}(1-n)-J_{22}(1-\xi) & J_{21}(1-n)-J_{22}(1+\xi) & J_{21}(1+n)+J_{22}(1+\xi) & -J_{21}(1+n)+J_{22}(1-\xi) \\ -J_{11}(1-n)-J_{12}(1-\xi) & J_{11}(1-n)-J_{12}(1+\xi) & J_{11}(1+n)+J_{12}(1+\xi) & -J_{11}(1+n)+J_{12}(1-\xi) \end{bmatrix}$$

The integration in A1-19 must be performed with respect to the variables x and y with P expressed in terms of ξ , η . As is shown in various texts (36) this is accomplished with the aid of the determinant of the J matrix hence the relation

$$dx dy = |J| d\xi d\eta \quad A1-20$$

This determinant is commonly called the Jacobian. Equation A1-19 with the aid of A1-10 is

$$K = \int_{-1}^1 \int_{-1}^1 P^T D P |J| d\xi d\eta \quad A1-21$$

The form of the expressions for P and D indicate that a closed form of solution for the integral will be extremely difficult and a numerical procedure for solving A1-21 is indicated. This is accomplished in the present thesis using the Gaussian Quadrature resulting in

$$K = \sum_{i=1}^n \sum_{j=1}^m P^T D P H_i H_j |J| \quad A1-22$$

In this expression P is evaluated at a number of specified points i , j where the weighting functions are H_i and H_j . Although almost any number can be chosen for n and m , six was found to be adequate for this problem and the weighting

functions are to be found in a number of books (22,25) together with the coordinates for the positions for evaluation of P.

Ring Elements

In an axi-symmetric problem there exists four components of strain as compared to three for the plane stress problem. These components are the radial component, ϵ_{xx} , the axial component ϵ_{yy} , the shear component ϵ_{xy} , and the circumferential component $\epsilon_{\theta\theta}$. Because of this additional strain component the strain displacement relations take the form

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$

A1-23

$$\epsilon_{\theta\theta} = \frac{u}{x}$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Although there is an additional strain component two displacement components completely describe the distortion and hence two degrees of freedom at each nodal point are sufficient to describe the element displacements.

C. Rectangular Ring Element

The axes of the coordinate system were chosen parallel to the sides of the element as shown in Figure A1-3. As with the plane stress rectangular element the displacement functions 4, and 5 namely

$$u = a_1 + a_2x + a_3y + a_4xy$$

A1-24

$$v = a_5 + a_6x + a_7y + a_8xy$$

insure compatibility of edge displacements as well as a linear varying strain and stress within each element. The assumed displacement functions together with the element displacement vector in the form

$$[\delta_e] = [u_i \ v_i \ u_j \ v_j \ u_k \ v_k \ u_l \ v_l] \quad \text{A1-25}$$

are sufficient to determine the A matrix which relates the element displacement vector to the generalized coordinates, a. The A matrix defined by Equation 6, or

$$\delta_e = A \ a$$

is

Fig. A1.3 RECTANGULAR RING ELEMENT

$$A = \begin{bmatrix} 1 & x_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_i & 0 & 0 \\ 1 & x_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_j & 0 & 0 \\ 1 & x_k & y_k & x_k y_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_k & y_k & x_k y_k \\ 1 & x_l & y_l & x_l y_l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_l & y_l & x_l y_l \end{bmatrix} \quad A1-26$$

The B matrix relating the strain vector to the generalized coordinates is found by using Equations A1-23 and A1-24 to be

$$B = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 1/x & 1 & y/x & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \quad A1-27$$

The remaining matrix expression needed for the evaluation of the element stiffness, K, from the equation

$$K = A^{-1T} \int_{vol} B^T D B \, dv \, A^{-1} \quad A1-28$$

is the D matrix which relates the stress to the strain. This matrix is now a 4 x 4 matrix and is shown in reference (22) to be

$$D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \quad A1-29$$

The elemental volume dv in Equation A1-28 is expressed as

$$dv = 2\pi x \, dx \, dy$$

and Equation A1-28 becomes

$$K = A^{-1T} 2\pi \int \int x B^T D B \, dx \, dy A^{-1} \quad A1-30$$

where the integration is performed over the length, l , and height, h , of the element and hence the limits of integration in the x direction are x_i to $x_i + l$ and in the y direction 0 to h . Using for convenience the following expression for the D matrix

$$D = E a \begin{bmatrix} 1 & b & b & 0 \\ b & 1 & b & 0 \\ b & b & 1 & 0 \\ 0 & 0 & 0 & c \end{bmatrix} \quad A1-31$$

the result of matrix multiplication of the terms within the integral is

$$xB^T_{DB} = Ea \begin{bmatrix} \frac{1}{x} & b+1 & y/x & y(b+1) & 0 & 0 & b & bx \\ 2x(b+1) & y(b+1) & 2xy(b+1) & 0 & 0 & 2bx & 2bx^2 \\ \frac{y^2}{x} + cx & y^2(b+1)+cx^2 & 0 & cx & by & xy(c+b) \\ 2y^2x(b+1)+cx^3 & 0 & cx^2 & 2bxy & x^2y(2b+c) \\ SYM & 0 & 0 & 0 & 0 \\ & cx & 0 & cxy \\ & & x & x^2 \\ & & & x^3+cx^2y^2 \end{bmatrix}$$

A1-32

Integration of the terms in the matrix of Equation A1-32 over the length and height of the element yield the following expressions where all zero terms have been omitted. These

terms when arranged in a symmetric matrix and post multiplied by A^{-1} and pre multiplied by $2 \pi E a A^{-1T}$ gives the required stiffness matrix for the element.

$$K_{11} = h \ln(1 + \frac{1}{x_i})$$

$$K_{21} = (b+1)lh$$

$$K_{22} = (b+1)(hl^2 + 2h x_i l)$$

$$K_{31} = \frac{h^2}{2} \ln(1 + \frac{1}{x_i})$$

$$K_{32} = \frac{h^2 l}{2} (b+1)$$

$$K_{33} = \frac{h^3}{3} \ln(1 + \frac{1}{x_i}) + ch(\frac{l^2}{2} + x_i l)$$

$$K_{41} = K_{32}$$

$$K_{42} = \frac{h^2}{2} (b+1)(l^2 + 2x_i l)$$

$$K_{43} = \frac{h^3 l}{3} (b+1) + \frac{ch}{3}(3x_i^2 l + 3x_i l^2 + l^3)$$

$$K_{44} = \frac{h^3}{3} (b+1)(l^2 + 2x_i l) + \frac{ch}{4}(4x_i^3 l + 6x_i^2 l^2 + 4x_i l^3 + l^4)$$

$$K_{63} = \frac{ch}{2} (l^2 + 2x_i l)$$

$$K_{64} = \frac{ch}{3} (3x_i^2 l + 3x_i l^2 + l^3)$$

$$K_{66} = K_{63}$$

$$K_{71} = b l h$$

$$K_{72} = b h (l^2 + 2 x_i l)$$

$$K_{73} = \frac{b h^2 l}{2}$$

$$K_{74} = \frac{b h^2}{2} (l^2 + 2 x_i l)$$

$$K_{77} = \frac{h}{2} (l^2 + 2 x_i l)$$

$$K_{81} = \frac{b h}{2} (l^2 + 2 x_i l)$$

$$K_{82} = \frac{2 b h}{3} (3 x_i^2 l + 3 x_i l^2 + l^3)$$

$$K_{83} = \frac{h^2 (b+c)}{4} (l^2 + 2 x_i l)$$

$$K_{84} = \frac{h^2}{6} (2b+c) (3 x_i^2 l + 3 x_i l^2 + l^3)$$

$$K_{86} = \frac{c h^2}{4} (l^2 + 2 x_i l)$$

$$K_{87} = \frac{h}{3} (3 x_i^2 l + 3 x_i l^2 + l^3)$$

$$K_{88} = \frac{h}{4} (4 x_i^3 l + 6 x_i^2 l^2 + 4 x_i l^3 + l^4) + \frac{c h^3}{6} (l^2 + 2 x_i l)$$

D. Quadrilateral Ring Element

Because it is desirable to maintain the edge compatibility and to have the same linear strain variation occurring in the previous three elements displacement functions identical with those of the plane stress quadrilateral element are assumed for this element. This element together with the x-y axis system and the ξ - η coordinates is shown in Figure A1-4. The assumed displacement functions in terms of ξ, η are

$$u = a_1 + a_2x + a_3y + a_4xy$$

$$v = a_5 + a_6x + a_7y + a_8xy$$

A1-33

and the relationship between the x-y and ξ - η coordinate systems is

$$x = \frac{1}{4} \left\{ (1-\xi)(1-\eta)x_i + (1+\xi)(1-\eta)x_j + (1+\xi)(1+\eta)x_k + (1-\xi)(1+\eta)x_l \right\}$$

$$y = \frac{1}{4} \left\{ (1-\xi)(1-\eta)y_i + (1+\xi)(1-\eta)y_j + (1+\xi)(1+\eta)y_k + (1-\xi)(1+\eta)y_l \right\}$$

A1-34

Denoting the nodal point displacements by $u_i, v_i \dots u_l, v_l$ the displacement functions may be written in terms of these obtaining

Fig. A1.4 QUADRILATERAL RING ELEMENT

$$u = \frac{1}{4} \left\{ (1-\xi)(1-\eta)u_i + (1+\xi)(1-\eta)u_j + (1+\xi)(1+\eta)u_k + (1-\xi)(1+\eta)u_l \right\}$$

$$v = \frac{1}{4} \left\{ (1-\xi)(1-\eta)v_i + (1+\xi)(1-\eta)v_j + (1+\xi)(1+\eta)v_k + (1-\xi)(1+\eta)v_l \right\}$$

A1-35

The strain displacement relations for this ring element are identical to A1-23 and the expressions for the partial derivatives given in A1-15 together with the expression for u/x are sufficient to determine the P matrix which relates the element strain to the nodal point displacements, hence

$$\epsilon = P \delta_e \quad \text{A1-36}$$

The nodal point displacement vector for this element is in the form

$$\left[\delta_e \right] = \left[u_i \ u_j \ u_k \ u_l \ v_i \ v_j \ v_k \ v_l \right] \quad \text{A1-37}$$

Combining Equations A1-23, A1-15 and noting that J_{11} , J_{12} , J_{21} , J_{22} are the terms of the inverted J matrix results in the P matrix, Equation A1-38.

The stiffness for this axi-symmetric ring element has the form

$$\overline{K} = 2 \pi \iint x P^T D P dx dy \quad \text{A1-39}$$

$$\begin{array}{ccccccc}
 \left[\begin{array}{ccccccc}
 -J_{11}(1-n) - J_{12}(1-\varepsilon) & J_{11}(1-n) - J_{12}(1+\varepsilon) & J_{11}(1+n) + J_{12}(1+\varepsilon) & -J_{11}(1+n) + J_{12}(1-\varepsilon) \\
 0 & 0 & 0 & 0 \\
 \frac{(1-\varepsilon)(1-n)}{x} & \frac{(1+\varepsilon)(1-n)}{x} & \frac{(1+\varepsilon)(1+n)}{x} & \frac{(1-\varepsilon)(1+n)}{x} \\
 -J_{21}(1-n) - J_{22}(1-\varepsilon) & J_{21}(1-n) - J_{22}(1-\varepsilon) & J_{21}(1+n) + J_{22}(1+\varepsilon) & -J_{21}(1+n) + J_{22}(1-\varepsilon)
 \end{array} \right] \\
 \\
 \left[\begin{array}{ccccccc}
 -J_{21}(1-n) - J_{22}(1-\varepsilon) & J_{21}(1-n) - J_{22}(1-\varepsilon) & J_{21}(1+n) + J_{22}(1+\varepsilon) & -J_{21}(1+n) + J_{22}(1-\varepsilon) \\
 0 & 0 & 0 & 0 \\
 -J_{21}(1-n) - J_{22}(1-\varepsilon) & J_{21}(1-n) - J_{22}(1-\varepsilon) & J_{21}(1+n) + J_{22}(1+\varepsilon) & -J_{21}(1+n) + J_{22}(1-\varepsilon) \\
 -J_{11}(1-n) - J_{12}(1-\varepsilon) & J_{11}(1-n) - J_{12}(1+\varepsilon) & J_{11}(1+n) + J_{12}(1+\varepsilon) & -J_{11}(1+n) + J_{12}(1-\varepsilon)
 \end{array} \right]
 \end{array}$$

A1-38

where the integration must be performed over the area of the element. It is extremely difficult if not impossible to obtain an analytic expression for the integral and hence a numerical integration using Gaussian Quadrature is utilized. The equation of K then reduces to

$$\bar{K} = 2\pi \sum_{i=1}^n \sum_{j=1}^m x P^T D P H_i H_j |J|$$

where H_i , H_j are the weighting functions and $|J|$ is the Jacobian.

APPENDIX A2

The following are the graphical interpretations of the numerical results of the finite element analysis for flat and axisymmetric bars.

FIG.A2.1 AXIAL DISPLACEMENT FOR 0° FLAT BAR AT NODES 56-66

FIG.A2.2 AXIAL STRESS FOR 0° FLAT BAR AT NODES 56-66

FIG.A2.3 AXIAL DISPLACEMENT FOR 10° FLAT BAR AT NODES 55-64

FIG.A2.4 AXIAL STRESS FOR 10° FLAT BAR AT NODES 55-64

FIG.A2.5 AXIAL DISPLACEMENT FOR 20° FLAT BAR AT NODES 43-51

FIG.A2.6 AXIAL STRESS FOR 20° FLAT BAR AT NODES 43-51

FIG.A2.7 AXIAL DISPLACEMENT FOR 0° FLAT BAR AT NODES 56-66

FIG.A2.8 AXIAL STRESS FOR 0° FLAT BAR AT NODES 56-66

FIG. A2.9 AXIAL DISPLACEMENT FOR 10° FLAT BAR AT NODES 55-64

Fig. A2.10 AXIAL STRESS FOR 10° FLAT BAR AT NODES 55-64

Fig. A2.11 AXIAL DISPLACEMENT FOR 20° FLAT BAR AT NODES 43-51

Fig. A2.12 AXIAL STRESS FOR 20° FLAT BAR AT NODES 43-51

Fig. A2.13 AXIAL DISPLACEMENT FOR 0° AXISYMMETRIC BAR
AT NODES 56-66

Fig. A2.14 AXIAL STRESS FOR 0° AXISYMMETRIC BAR

AT NODES 56-66

Fig. A2.15 AXIAL DISPLACEMENT FOR 5° AXISYMMETRIC BAR
AT NODES 56-65

Fig. A2.16 AXIAL STRESS FOR 5° AXISYMMETRIC BAR
AT NODES 56-65

Fig. A2.17 AXIAL DISPLACEMENT FOR 10° AXISYMMETRIC BAR
AT NODES 55-64

Fig. A2.18 AXIAL STRESS FOR 10° AXISYMMETRIC BAR
AT NODES 55-64

Fig. A2.19 AXIAL DISPLACEMENT FOR 15° AXISYMMETRIC BAR
AT NODES 51-59

Fig. A2.20 AXIAL STRESS FOR 15° AXISYMMETRIC BAR
AT NODES 51-59

Fig. A2.21 AXIAL DISPLACEMENT FOR 20° AXISYMMETRIC BAR
AT NODES 52-60

Fig. A2.22 AXIAL STRESS FOR 20° AXISYMMETRIC BAR
AT NODES 52-60

Fig. A2.23 AXIAL DISPLACEMENT FOR 0° AXISYMMETRIC BAR
AT NODES 56-66

Fig. A2.24 AXIAL STRESS FOR 0° AXISYMMETRIC BAR
AT NODES 56-66

Fig. A2.25 AXIAL DISPLACEMENT FOR 5° AXISYMMETRIC BAR
AT NODES 56-65

Fig. A2.26 AXIAL STRESS FOR 5° AXISYMMETRIC BAR
AT NODES 56-65

Fig. A2.27 AXIAL DISPLACEMENT FOR 10° AXISYMMETRIC BAR
AT NODES 55-64

Fig. A2.28 AXIAL STRESS FOR 10° AXISYMMETRIC BAR
AT NODES 55-64

Fig. A2.29 AXIAL DISPLACEMENT FOR 15° AXISYMMETRIC BAR
AT NODES 51-59

Fig. A2.30 AXIAL STRESS FOR 15° AXISYMMETRIC BAR
AT NODES 51-59

Fig. A2.31 AXIAL DISPLACEMENT FOR 20° AXISYMMETRIC BAR
AT NODES 52-60

Fig. A2.32 AXIAL STRESS FOR 20° AXISYMMETRIC BAR
AT NODES 52-60

APPENDIX A3

The computer programs for the Finite Element Method used in this study are presented in three sections as follows:

- A. Program and suboutlines for the analysis of rectangular bars.
- B. Program and subroutines for the analysis of axisymmetric bars.
- C. Subroutines which were used for both the rectangular and axisymmetric bars.

AXISYMMETRIC BAR

```

      DIMENSION A(4),H(4),D(4,4),V(2,4),CA(4,8),P(4,8),DUM1(4),DUM2(4)
      DIMENSION CSR(8),AR(8),FJ(2,2)
      DIMENSION X(140),Y(140),PH(4,160),IND(160),ASR(26,380),FA(380)
      DIMENSION ANM(380),UP(380),UC(380),NAF(380),UR(380),UF(380)
      DIMENSION NN(380),DF(380),DAMP(380),FNC(380),STR(4,380)
1  WRITE (3,99)
      PFAD (2,97) THETA
      WRITE (3,98) THETA
      CALL RXY4R(X,Y,INDF,NNT)
      DO 9 I=1,NNT
          IF(X(I)-.00001) 8,8,9
          X(I)=.000001
9  CONTINUE
      CALL RCONR(INN,IND,NFT,NRAND,F,GNU,RHO,11)
      CALL DPCTR (UP,NRC,C,NDF,UC,NB)
      CALL DAH3P(F,GNU,A,H,D,UP,UF,FA,ANM,ASR,NRAND,NDF,DUM1,DUM2)
      DO 101 IF=1,NFT
          IP=NN(1,IF)
          JP=NN(2,IF)
          KP=NN(3,IF)
          LP=NN(4,IF)
          FL=X(JP)-X(IP)
          FH=Y(LP)-Y(IP)
          IF(IND(1E)) 20,11,20
11  CALL AMT1R(X,IP,JP,KP,LP,FH,CA)
          CALL INDRR(CA)
          CALL GSRRR(F,GNU,FH,FL,X,IP,JP,GS)
          CALL VTMLD(CA,GS,AK)
          CALL FSR1R(INN,IF,ASR,AK)
          GO TO 100
20  DO 25 I=1,P
          DO 25 J=1,P
25  AK(I,J)=0.0
          DO 30 I=1,11
          DO 30 J=1,11
          ETA=A(I)
          XI=A(J)
          HJ=H(J)
          HI=H(1)
          CALL VFJDR(ETA,XI,X,Y,IP,JP,KP,LP,V,FJ,DJ)
          CALL PMT2R(IP,JP,KP,LP,XI,ETA,X,FJ,V,P,XX)
          CALL KMT2P(HI,HJ,DJ,D,P,XX,AK)
30  CALL FSA2R(INN,AK,ASR,IF)
100 CALL NMS2R(NNT,FL,FH,IP,JP,KP,LP,X,Y,RHO,IND,1E,NDF,C,ASR,ANM,DAMP)
101 CONTINUE
      CALL TIM2P(NNT,ANM,ASR,X,DELT,NRC,NB,UP)
      T=0.0
      CALL RINFR(FA,NAF,NANF,NTI,INDF,INP)
      IMP=INP
      IT=0
      NP=0
      DO 401 IN=1,NTI
          IF(INDF) 35,42,35
35  IF(IT) 200,36,200
36  CALL DELFR(DF,NP,IFIN,NDF)
          NP=NP+IN-1
          IT=1
          IF(IFIN) 200,500,200
200 DO 41 I=1,NANF
          M=NAF(I)
41  FA(M)=FA(M)+DF(M)
42  CALL DSPLR(NDF,DELT,ASR,UR,UC,UF,NRAND,FA,DAMP,T,ANM,UP,NRC,NB)
          IF(IN=IMP) 400,43,400
43  IMP=IMP+1NP
          CALL WDSPLR(IT,UC,NDF)
          DO 51 J=1,NNT
          DO 50 I=1,4
          STR(I,J)=0.0
50  FNC(J)=0.0
51  FNC(J)=0.0

```



```

DO 300 IF=1,NET
  IP=NN(1,IF)
  JP=NN(2,IF)
  KP=NN(3,IF)
  LP=NN(4,IF)
  DO 300 J=1,4
    IO=NN(J,IF)
    IF(IND(1,IF)) 45,46,45
  45 ETA=DUM1(J)
    XI=DUM2(J)
    CALL VFJDP(ETA,XI,X,Y,IP,JP,KP,LP,V,FJ,DJ)
    CALL PMT2P(IP,JP,KP,LP,XI,ETA,X,FJ,V,P,XX)
    CALL STORR(D,P)
    CALL SXY3R(NN,IO,P,IF,UC,STR,FNC1)
    GO TO 300
  46 FH=Y(LP)-Y(IP)
    CALL AMT1P(X,IP,JP,KP,LP,FH,CA)
    CALL INORP(CA)
    CALL STRRP(X,Y,IP,IO,D,CA,P)
    CALL SXY2R(IO,IF,P,UC,NN,STR,FNC1)
  300 CONTINUE
    CALL WAS2R(FNC,STR,NN,X)
  400 IF(IN=NP) 401,37,401
  37 IN=IN+1
    GO TO 36
  401 CONTINUE
  500 READ (2,RCO) NEXT
    IF(NEXT) 501,501,1
  501 CALL EXIT
  800 FORMAT (I5)
  97 FORMAT (F10.5)
  98 FORMAT ( ' SIDES OF BAR - SLOPE', F10.5, 'DEGREES' '/')
  99 FORMAT (// '1**** STRESS AND DISPLACEMENT WAVES IN AXI-SYMMETRIC BAR
    15 ****'//)
  END

```


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AMTR

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```

SUBROUTINE AMTR(K,IP,JP,KP,LP,HP,CA)
DIMENSION CA(100),X(1)
DO 10 I=1,9
DO 10 J=1,9
----- 10 CA(I,J)=0.0
CA(I,1)=1.
CA(1,J)=X(I)
CA(2,1)=1.
CA(2,2)=X(I)
CA(3,1)=1.
CA(3,2)=X(JP)
CA(4,1)=1.
CA(4,2)=X(JP)
CA(5,1)=1.
CA(5,2)=X(JP)
CA(6,1)=1.
CA(6,2)=X(JP)
CA(7,1)=1.
CA(7,2)=X(JP)
CA(8,1)=1.
CA(8,2)=X(JP)
CA(9,1)=1.
CA(9,2)=X(JP)
CA(10,1)=1.
CA(10,2)=X(JP)
CA(11,1)=1.
CA(11,2)=X(JP)
CA(12,1)=1.
CA(12,2)=X(JP)
CA(13,1)=1.
CA(13,2)=X(JP)
CA(14,1)=1.
CA(14,2)=X(JP)
CA(15,1)=1.
CA(15,2)=X(JP)
CA(16,1)=1.
CA(16,2)=X(JP)
CA(17,1)=1.
CA(17,2)=X(JP)
CA(18,1)=1.
CA(18,2)=X(JP)
CA(19,1)=1.
CA(19,2)=X(JP)
CA(20,1)=1.
CA(20,2)=X(JP)
CA(21,1)=1.
CA(21,2)=X(JP)
CA(22,1)=1.
CA(22,2)=X(JP)
CA(23,1)=1.
CA(23,2)=X(JP)
CA(24,1)=1.
CA(24,2)=X(JP)
CA(25,1)=1.
CA(25,2)=X(JP)
CA(26,1)=1.
CA(26,2)=X(JP)
CA(27,1)=1.
CA(27,2)=X(JP)
CA(28,1)=1.
CA(28,2)=X(JP)
CA(29,1)=1.
CA(29,2)=X(JP)
CA(30,1)=1.
CA(30,2)=X(JP)
CA(31,1)=1.
CA(31,2)=X(JP)
CA(32,1)=1.
CA(32,2)=X(JP)
CA(33,1)=1.
CA(33,2)=X(JP)
CA(34,1)=1.
CA(34,2)=X(JP)
CA(35,1)=1.
CA(35,2)=X(JP)
CA(36,1)=1.
CA(36,2)=X(JP)
CA(37,1)=1.
CA(37,2)=X(JP)
CA(38,1)=1.
CA(38,2)=X(JP)
CA(39,1)=1.
CA(39,2)=X(JP)
CA(40,1)=1.
CA(40,2)=X(JP)
CA(41,1)=1.
CA(41,2)=X(JP)
CA(42,1)=1.
CA(42,2)=X(JP)
CA(43,1)=1.
CA(43,2)=X(JP)
CA(44,1)=1.
CA(44,2)=X(JP)
CA(45,1)=1.
CA(45,2)=X(JP)
CA(46,1)=1.
CA(46,2)=X(JP)
CA(47,1)=1.
CA(47,2)=X(JP)
CA(48,1)=1.
CA(48,2)=X(JP)
CA(49,1)=1.
CA(49,2)=X(JP)
CA(50,1)=1.
CA(50,2)=X(JP)
CA(51,1)=1.
CA(51,2)=X(JP)
CA(52,1)=1.
CA(52,2)=X(JP)
CA(53,1)=1.
CA(53,2)=X(JP)
CA(54,1)=1.
CA(54,2)=X(JP)
CA(55,1)=1.
CA(55,2)=X(JP)
CA(56,1)=1.
CA(56,2)=X(JP)
CA(57,1)=1.
CA(57,2)=X(JP)
CA(58,1)=1.
CA(58,2)=X(JP)
CA(59,1)=1.
CA(59,2)=X(JP)
CA(60,1)=1.
CA(60,2)=X(JP)
CA(61,1)=1.
CA(61,2)=X(JP)
CA(62,1)=1.
CA(62,2)=X(JP)
CA(63,1)=1.
CA(63,2)=X(JP)
CA(64,1)=1.
CA(64,2)=X(JP)
CA(65,1)=1.
CA(65,2)=X(JP)
CA(66,1)=1.
CA(66,2)=X(JP)
CA(67,1)=1.
CA(67,2)=X(JP)
CA(68,1)=1.
CA(68,2)=X(JP)
CA(69,1)=1.
CA(69,2)=X(JP)
CA(70,1)=1.
CA(70,2)=X(JP)
CA(71,1)=1.
CA(71,2)=X(JP)
CA(72,1)=1.
CA(72,2)=X(JP)
CA(73,1)=1.
CA(73,2)=X(JP)
CA(74,1)=1.
CA(74,2)=X(JP)
CA(75,1)=1.
CA(75,2)=X(JP)
CA(76,1)=1.
CA(76,2)=X(JP)
CA(77,1)=1.
CA(77,2)=X(JP)
CA(78,1)=1.
CA(78,2)=X(JP)
CA(79,1)=1.
CA(79,2)=X(JP)
CA(80,1)=1.
CA(80,2)=X(JP)
CA(81,1)=1.
CA(81,2)=X(JP)
CA(82,1)=1.
CA(82,2)=X(JP)
CA(83,1)=1.
CA(83,2)=X(JP)
CA(84,1)=1.
CA(84,2)=X(JP)
CA(85,1)=1.
CA(85,2)=X(JP)
CA(86,1)=1.
CA(86,2)=X(JP)
CA(87,1)=1.
CA(87,2)=X(JP)
CA(88,1)=1.
CA(88,2)=X(JP)
CA(89,1)=1.
CA(89,2)=X(JP)
CA(90,1)=1.
CA(90,2)=X(JP)
CA(91,1)=1.
CA(91,2)=X(JP)
CA(92,1)=1.
CA(92,2)=X(JP)
CA(93,1)=1.
CA(93,2)=X(JP)
CA(94,1)=1.
CA(94,2)=X(JP)
CA(95,1)=1.
CA(95,2)=X(JP)
CA(96,1)=1.
CA(96,2)=X(JP)
CA(97,1)=1.
CA(97,2)=X(JP)
CA(98,1)=1.
CA(98,2)=X(JP)
CA(99,1)=1.
CA(99,2)=X(JP)
CA(100,1)=1.
CA(100,2)=X(JP)
RETURN
END

```

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INORR

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SUBROUTINE INORR(A)
C INVERTS A 8x8 MATRIX A
DIMENSION A(8,8),B(8,8)
N=P
NR=N
SUM=0.
DO 4 I=1,N
DO 4 J=1,N
2 B(I,J)=0.
SUM=SUM+A(I,I)
4 B(I,I)=1.
A(I,I)=SUM/N
NR=NR-1
DO 40 I=1,N
IF (ABS(A(I,I))-NR)6,6,20
6 IF (I-N) 10,7,10
7 WRITE (3,100)
CALL EXIT
10 DO 12 II=1,N
IF (ABS(A(II,I))-NR)12,12,14
12 CONTINUE
CC 10 7
DO 16 JJ=1,N
A(I,JJ)=A(I,JJ)+A(II,JJ)
16 B(I,JJ)=B(I,JJ)+B(II,JJ)
20 R=1./A(I,I)
DO 22 J=1,N
A(I,J)=C+A(I,J)
22 B(I,J)=R+A(I,J)
DO 37 II=1,N
IF (II-I)22,37,32
32 C=-A(II,I)
DO 35 J=1,N
A(II,J)=A(II,J)+C*A(I,J)
B(II,J)=B(II,J)+C*B(I,J)
35 CONTINUE
37 CONTINUE
40 CONTINUE
DO 42 I=1,N
DO 42 J=1,N
42 A(I,J)=B(I,J)
RETURN
100 FORMAT(20HOMATRIX IS SINGULAR )
END

```

IN IV G LEVEL 1. PCD 4

KMT2R

DATE = 69233

08/57/18

```

SUBROUTINE KMT2R(HI,HJ,DJ,D,XX,AK)
C CALC. OF K PRIME MATRIX FOR QUAD. RING ELEMENT
DIMENSION P(4,8),J(4,4),GS(8,8),AKT(8,8),AK(8,8)
CON=.0025*(HI+HJ+DJ+6.24318527XX)
DO 10 I=1,4
DO 10 J=1,4
GS(I,J)=0.0
DO 10 K=1,4
10 GS(I,J)=GS(I,J)+D(I,K)*P(K,J)
DO 11 I=1,8
DO 11 J=1,8
AKT(I,J)=0.0
DO 11 K=1,4
11 AKT(I,J)=AKT(I,J)+P(K,I)*GS(K,J)
DO 12 I=1,8
DO 12 J=1,8
12 AK(I,J)=(AK(I,J)+CON)*AKT(I,J)
RETURN
END

```


AN IV G LEVEL 1. MOD 4

TIM2R

DATE = 69233

08/57/18

```

SUBROUTINE TIM2R(NNT,ANN,ANM,X,DELTA,NHICAT0,UP)
DIMENSION ANN(1),ASB(20,1),X(1),NO(1),UP(1)
HLAD (2,101) FM,FR
DT1=100.
DO 14 I=1,NNT
  J=2*1
  UT=ANN(J)*.5/ASB(1,J)
  IF (DT1-DT) 12,13,13
13 DT=DT
12 JJ=2*1-1
  IF (X(1)-.0001) 16,16,11
11 DT=ANN(JJ)*.5/ASB(1,JJ)
  IF (DTT-DT) 14,15,15
15 DT1=DT
  GO TO 14
16 NHIC=NHIC+1
  NH(N)=JJ
  UP(JJ)=0.0
14 CONTINUE
  DT=SCRT(TIMT)
  AN=DT*FM
  DO 5 I=1,10
    J=1
    KEN/(10**1)
    IF (K) 5,8,5
5 CONTINUE
8 K=J-2
  K=(K/(10**K))*(10**K)
  FN=K
  DELT = FN/(FM*FR)
  WRITE(3,100) DELT
100 FORMAT (/ ' TIME INCREMENT = ',E12.5, ' SECONDS'/)
101 FORMAT (E12.5,F10.5)
  RETURN
END

```

IV G LEVEL 1. MOD 4

STRRR

DATE = 69233

08/57/18

```

C
SUBROUTINE STRRR(X,Y,IP,IO,D,CA,P)
CALC. OF STRESS MATRIX FOR RECT. RING ELEMENT
DIMENSION X(1),Y(1),D(4,4),CA(3,8),B(4,8),DB(4,8),P(4,8)
DO 10 I=1,4
  DO 10 J=1,8
10 U(1,J)=0.0
  B(1,2)=1.0
  B(1,4)=Y(IO)-Y(IP)
  B(2,7)=1.0
  B(2,8)=X(IO)
  B(3,1)=1./X(IO)
  U(3,2)=1.0
  B(3,7)=B(1,4)/X(IO)
  B(3,4)=B(1,4)
  B(4,3)=1.0
  B(4,4)=B(2,8)
  D(4,6)=1.0
  B(4,8)=B(1,4)
  DO 20 I=1,4
    DO 20 K=1,8
      DO 20 J=1,4
        DO 20 J=1,4
          UJ(1,K)=DB(1,K)+D(1,J)*B(J,K)
20 DO 25 I=1,4
  DO 25 J=1,8
  P(1,J)=0.0
  DO 25 K=1,8
25 P(1,J)=P(1,J)+DB(1,K)*CA(K,J)
  RETURN
END

```

IV G LEVEL 1. MOD 4

WAS2R

DATE = 69233

08/57/18

```

C
SUBROUTINE WAS2R(FNC,STR,NNT,X)
PRINT AVERAGE STRESS AT EACH NODE
DIMENSION FNC(1),STR(4,1),X(1)
DO 10 I=1,4
  DO 10 J=1,NNT
10 STR(1,J)=STR(1,J)/FNC(J)
  WRITE (3,100)
  DO 12 J=1,NNT
    IF (X(J)-.00001) 12,12,11
11 WRITE (3,101) J,(STR(1,J),I=1,4)
12 CONTINUE
100 FORMAT(// ' NODE NO.      AVG. SIGX      AVG. SIGY      AVG. SIGMETHA
101 FORMAT(3A,14,5X,E11.4,5X,E11.4,5X,E11.4,5X,E11.4)
  RETURN
END

```

G LEVEL 1. MOD 4 SKY1R DATE = 69233 06/57/18

```

C      SUBROUTINE SKY1R(NN,IO,P,IF,UC,STR,FNC)
      CALC. OF STRESS FROM DISPLACEMENT (QUAD. RING ELEMENT)
      DIMENSION TS(4),NN(4,1),P(4,8),UC(1),STR(4,1),FNC(1)
      DO 11 I=1,4
      TS(I)=0.0
      DO 10 J=1,4
      JJ=2*NN(J,1I)-1
10      TS(I)=TS(I)+P(1,J)*UC(JJ)+P(1,J+4)*UC(JJ+1)
      TS(I)=TS(I)*.25
11      STR(I,IO)=STR(I,IO)+TS(I)
      FNC(1)=FNC(1)+1
      RETURN
      END

```

G LEVEL 1. MOD 4 SKY2R DATE = 69233 06/57/18

```

C      SUBROUTINE SKY2R(NN,IE,P,UC,NN,STR,FNC)
      CALC. OF STRESS FROM DISPLACEMENT (RECT. RING ELEMENT)
      DIMENSION TS(4),NN(4,1),P(4,8),UC(1),STR(4,1),FNC(1)
      DO 11 I=1,4
      TS(I)=0.0
      DO 10 J=1,4
      II=2*J-1
      JJ=2*NN(J,IE)-1
10      TS(I)=TS(I)+P(1,II)*UC(JJ)+P(1,II+1)*UC(JJ+1)
11      STR(I,IO)=STR(I,IO)+TS(I)
      FNC(1)=FNC(1)+1
      RETURN
      END

```

FLAT BAR

```

LEVEL 1. MOD 4          MAIN          DATE = 6/21/85          11/54/10

DIMENSION A(16),H(16),D(1,3),V(2,4),DUM1(4),DUM2(4),CJ(2,2),P(3,3)
DIMENSION AIN(2,4),GS(4,4),AK(4,4),AKT(4,4)
DIMENSION X(100),Y(100),M(4,100),IND(100),ASB(25,200),FA(100)
DIMENSION ANP(100),DMP(100),UC(100),UF(100),NAND(100),UP(100),UPC(100)
DIMENSION DFI(100),DFI2(100),DAMP(100),TNC(100),STO(3,300)

1  WRITE(3,99)
  READ(2,97) THETA
  WRITE(3,98) THETA
  CALL SXFOR(X,Y,NOI,NNT)
  CALL DCONR(NS,IND,NF1,NRAND,F,GNU,RHO,IF)
  CALL DCTD (UP,DMP,C,AK,UC,NB)
  D(1,1-1) 3,4,4
4  CALL DAMP(1,F,GNU,A,H2D)
  GO TO 2
3  CALL PAHUB(1,GNU,A,H,D)
2  CALL IND1(NP,NRAND,UP,UF,FA,ANN,ASB,DUM1,DUM2)
  DO 101 IF=1,NET
  IP=NN(1,1F)
  JP=NN(2,1F)
  KP=NN(3,1F)
  LP=NN(4,1F)
  FX=X(JP)-X(IP)
  FY=Y(LP)-Y(IP)
  IF(IND(IF)) 20,11,20
11  CALL AIN2R(AIN,IL,IM)
  CALL GS2R(GS,GNU,1,FH,FL)
  CALL MMLR(AIN,GS,AK)
  CALL FSHIP(NN,IL,ASB,AK)
  GO TO 100
20  DO 25 J=1,8
  DO 25 J=1,8
25  AKT(1,J)=0.0
  DO 30 J=1,11
  DO 30 J=1,11
  ETA=A(1)
  XI=A(J)
  HJ=R(J)
  HI=H(1)
  CALL VFJDR(ETA,XI,X,Y,IP,JP,KP,LP,V,FJ,DJ)
  CALL PMATR(FJ,V,P)
30  CALL KMATD(P,D,HI,HJ,DJ,AKT)
  CALL FSAPR(NN,AKT,ASB,IF)
100  CALL NMASR(IND,IND,IF,FL,FH,IP,JP,KP,LP,X,Y,RHO,ASB,C,ANN,DAMP)
101  CONTINUE
  CALL TMR(IND,ANN,ASB,DEL1)
  T=0.0
  CALL MINFR(FA,NAF,NANF,NTI,INDF,INP)
  IMP=INP
  IT=0
  NP=0
  DO 401 IN=1,NTI
  IF(INDF) 35,42,35
35  IF(IT) 200,36,200
36  CALL DELIR (DF,NP,IFIN,NDF)
  NP=NP+IN-1
  IT=1
  IF(IFIN) 200,500,200
200  DO 41 I=1,NANF
  W=NAF(I)
41  FA(M)=FA(M)+DF(W)
42  CALL DSDIR(NDF,DEL1,ASB,UP,UC,UF,NRAND,FA,DAMP,T,ANN,UP,NBC,NB)
  IF(IN=IMP) 400,43,400
43  IMP=IMP+IND
  CALL WSPR(T,UC,NDF)
  DO 51 J=1,NNT
  DO 50 J=1,3
50  STR(1,J)=0.0
51  FNC(J)=0.0
  DO 300 IF=1,NET
  IP=NN(1,1F)
  JP=NN(2,1F)
  KP=NN(3,1F)
  LP=NN(4,1F)
  DO 300 J=1,4
  IO=NN(J,1F)
  IF(IND(IE)) 45,46,45
45  ETA=DUM1(J)
  XI=DUM2(J)
  CALL VFJDR(ETA,XI,X,Y,IP,JP,KP,LP,V,FJ,DJ)
  CALL PMATR(FJ,V,P)
  CALL STCLR(D,P)
  CALL SXFOR (IO,P,UC,IF,NN,STR,FNC)
  GO TO 300
46  XX=X(IO)-X(IP)
  YY=Y(IO)-Y(IP)
  CALL STRFR(X,Y,IP,JP,KP,F,GNU,XX,YY,P)
  CALL SXVR(STR,IO,P,UC,FNC,IF,NN)
300  CONTINUE
  CALL WASTR(FNC,STR,NNT)
400  IF(IN=NP) 401,37,401
37  IN=IN+1
  GO TO 36
401  CONTINUE
500  READ(2,900) ATXT
  IF(ATXT) 501,501,1
501  CALL FKIT
  97  FORMAT (F10.5)
  98  FORMAT (1 * SIDES OF PLATE SLOPE*, F10.5, *DEGREES **/)
  99  FORMAT (// *10000 STRESS AND DISPLACEMENT WAVES IN FLAT PLATES ****
  //)
800  FORMAT (15)
  END

```

LEVEL 1, MOD 4 GSOR DATE = 69185 11/54/10

```

C      SUBROUTINE GSOR(GN,GNU,F,FL,FL1)
C      CALC. OF GENERALIZED STIFFNESS FOR RECT. ELEMENT
C      DIMENSION GS(8,8)
C      DO 10 I=1,8
C      DO 10 J=1,8
10     GS(I,J)=0.0
C      GNAT=(1.-GNU)/2.
C      FL1=FL*(1.-GNU*GNU)
C      FL2=FL*FL
C      FL3=FL*FL1
C      FL4=FL1*FL1
C      GS(2,2)=FL4*FH*FF
C      GS(2,4)=FL1*FH*FL/2.
C      GS(2,7)=GNU*GS(2,2)
C      GS(2,8)=GNU*FL1*FH*FL/2.
C      GS(3,3)=GNAT*GS(2,2)
C      GS(3,4)=GNAT*FL2*FH*FL/2.
C      GS(3,6)=GNAT*GS(2,2)
C      GS(3,8)=GNAT*GS(2,4)
C      GS(4,4)=FL4*FH3*GNAT*FL3*FH*FF/3.
C      GS(4,6)=GS(3,4)
C      GS(4,7)=GNU*GS(2,4)
C      GS(4,8)=(1.+GNU)*FL2*FH2*FF/8.
C      GS(6,6)=GNAT*GS(2,2)
C      GS(6,8)=GNAT*GS(2,4)
C      GS(7,7)=GS(2,2)
C      GS(7,8)=FL2*FH*FF*.5
C      GS(8,8)=GNAT*FL*FH3*FL*FL*FH*FF/3.
C      NU=2
C      MD=2
C      DO 12 I=NO,8
C      DO 13 J=MD,8
13     GS(J,I)=GS(I,J)
C      NO=NO+1
12     MD=MD+1
C      RETURN
C      END

```

LEVEL 1, MOD 4 AIN2R DATE = 69185 11/54/10

```

C      SUBROUTINE AIN2R(AIN,FL,FH)
C      CALC. OF A INVERSE
C      DIMENSION AIN(8,8)
C      DO 10 I=1,8
C      DO 10 J=1,8
10     AIN(I,J)=0.0
C      AIN(1,1)=1.
C      AIN(2,1)=-1./FL
C      AIN(2,3)=-AIN(2,1)
C      AIN(3,1)=-1./FH
C      AIN(3,7)=-AIN(3,1)
C      AIN(4,1)=1./FL*FH
C      AIN(4,5)=-AIN(4,1)
C      AIN(4,7)=AIN(4,3)
C      DO 11 I=1,4
C      II=I+4
C      DO 11 J=1,7
C      JJ=J+1
11     AIN(II,JJ)=AIN(I,J)
C      RETURN
C      END

```

G LEVEL 1, MOD 4 PMATR DATE = 69185 11/54/10

```

C      SUBROUTINE PMATR(FJ,V,P)
C      CALC. OF P MATRIX
C      DIMENSION P(3,8),FJ(2,2),V(2,4)
C      DO 10 I=1,2
C      DO 10 K=1,4
10     IF(I-2) 5,6,6
5      P(I,K)=0.0
C      M=1
C      GO TO 9
6      DO 7 M=2,3
7      P(M,K)=0.0
C      M=3
9      DO 10 J=1,2
10     P(M,K)=P(M,K)+FJ(I,J)*V(J,K)
C      DO 11 I=1,4
C      J=I+4
C      P(3,J)=P(1,1)
C      P(2,J)=P(3,1)
11     P(1,J)=P(2,1)
C      RETURN
C      END

```

G LEVEL 1, MOD 4 STOR DATE = 69185 11/54/10

```

C      SUBROUTINE STOR(D,P)
C      CALC. OF STRESS MATRIX FOR QUAD.ELEMENT
C      DIMENSION D(3,3),P(3,8),ST(3,8)
C      DO 10 I=1,3
C      DO 10 J=1,8
10     ST(I,J)=0.0
C      DO 10 K=1,3
C      ST(I,J)=ST(I,J)+D(I,K)*P(K,J)
C      DO 11 I=1,3
C      DO 11 J=1,8
11     P(I,J)=ST(I,J)
C      RETURN
C      END

```


LEVEL 1, MOD 4 DAH2R DATE = 69185 11/54/10

SUBROUTINE DAH2R(E,GNU,A,H,D)
C DIMENSION A(16),H(16),D(3,3)
DAH2R CALC. D MATRIX AND SETS A AND H
EF=1/(1.-GNU*GNU)

D(1,1)=EF
D(1,2)=GNU*EF
D(1,3)=0.0
D(2,1)=0(1,2)
D(2,2)=D(1,1)
D(2,3)=0.0
D(3,1)=0.0
D(3,2)=0.0
D(3,3)=(1.-GNU)*EF/2.

A(1)=.0004000149
A(2)=.0045750230
A(3)=.0006312023
A(4)=.7554044003
A(5)=.6170762444
A(6)=.4580167776
A(7)=.2016035507
A(8)=.0050125529
A(9)=.0050125529
A(10)=.2016035507
A(11)=.4580167776
A(12)=.6170762444
A(13)=.7554044003
A(14)=.6006312023
A(15)=.0045750230
A(16)=.0004000149

H(1)=.0271524594
H(2)=.0022535239
H(3)=.0051565116
H(4)=.1246209712
H(5)=.1495950088
H(6)=.1691565193
H(7)=.1626034150
H(8)=.1634506104
H(9)=.1694506104
H(10)=.1626034150
H(11)=.1691565193
H(12)=.1495950088
H(13)=.1246209712
H(14)=.0051565116
H(15)=.0022535239
H(16)=.0271524594

RETURN
END

LEVEL 1, MOD 4 DAHLR DATE = 69185 11/54/10

SUBROUTINE DAHLR(E,GNU,A,H,D)
C DIMENSION A(6),H(6),D(3,3)
DAHLR CALC. D MATRIX AND SETS A AND H
EF=1/(1.-GNU*GNU)

A(1)=.66120939
A(2)=.23461919
A(3)=.23461919
A(4)=.23461919
A(5)=.66120939
A(6)=.93246951
H(1)=.17132449
H(2)=.36076157
H(3)=.46791393
H(4)=.46791393
H(5)=.36076157
H(6)=.17132449

D(1,1)=EF
D(1,2)=GNU*EF
D(1,3)=0.0
D(2,1)=D(1,2)
D(2,2)=D(1,1)
D(2,3)=0.0
D(3,1)=0.0
D(3,2)=0.0
D(3,3)=(1.-GNU)*EF/2.
RETURN
END

LEVEL 1, MOD 4 IN01R DATE = 69185 11/54/10

SUBROUTINE IN01R(NDF,NRAND,UN,UF,FA,ANM,ASR,DUM1,DUM2)
C INITIALIZATION SUBROUTINE
DIMENSION UN(1),UF(1),FA(1),ANM(1),ASR(26,1),DUM1(4),DUM2(4)
DO 10 I=1,NDF

UN(I)=0.0
UF(I)=0.0
FA(I)=0.0
ANM(I)=0.0
DO 10 J=1,NRAND
10 ASR(J,I)=0.0
DUM1(I)=1.
DUM1(2)=1.
DUM1(3)=1.
DUM1(4)=1.
DUM2(I)=1.
DUM2(2)=1.
DUM2(3)=1.
DUM2(4)=1.

RETURN
END

LEVEL 1, MOD 4 NNASR DATE = 69185 11/54/10

```

C      SUBROUTINE NNASR(NDF,IND,IF,FL,FH,IP,JP,KD,LP,X,Y,RHO,ASB,C,ANM,
      1DAMP)
      DIMENSION ANM(180),IND(1),ASB(26,1),ANM(1),DAMP(1),X(1),Y(1)
      NNT=NDF/7
      DO 10 I=1,NNT
      10 FM(I)=0.0
      IF(IND(I))12,11,12
      11 FMT=FL*FH*RHO
      FM(IP)=FMT*.25
      FM(JP)=FM(IP)
      FM(KD)=FM(IP)
      FM(LP)=FM(IP)
      GO TO 40
      12 FM(IP)=.015625*((3.*(Y(LP)-Y(IP))*Y(KD)-Y(JP))*(3.*(X(JP)-X(IP))
      1+X(KD)-X(LP))+(3.*(X(LP)-X(IP))*X(KD)-X(JP))*(3.*(Y(IP)-Y(JP))
      1-Y(KD)+Y(LP)))*RHO
      FM(JP)=.015625*((3.*(Y(KD)-Y(JP))*Y(LP)-Y(IP))*(3.*(X(JP)-X(IP))+
      1X(KD)-X(LP))*(3.*(X(KD)-X(JP))*X(LP)-X(IP))*(3.*(Y(IP)-Y(JP))
      1-Y(KD)+Y(LP)))*RHO
      FM(KD)=.015625*((3.*(Y(KD)-Y(JP))*Y(LP)-Y(IP))*(3.*(X(KD)-X(LP))-
      1X(IP)+X(JP))*(3.*(X(KD)-X(JP))*X(LP)-X(IP))*(3.*(Y(LP)-Y(KD))+
      1Y(IP)-Y(JP)))*RHO
      40 DO 25 I=1,NNT
      25 ANM(I)=ANM(I)+FM(I)
      25 ANM(I-1)=ANM(I-1)+FM(I)
      DO 35 J=1,NDF
      35 DAMP(J)=C*.2*SORT(ASB(1,J)*ANM(J))
      RETURN
      END

```

LEVEL 1, MOD 4 KMATR DATE = 69185 11/54/10

```

C      SUBROUTINE KMATR(P,D,H1,HJ,DJ,AKT)
      CALC. OF K PRIME FOR QUAD. ELEMENT
      DIMENSION P(3,8),D(3,3),GS(8,8),AKT(8,8),AK(8,8)
      CON = H1*HJ*DJ/16.
      DO 10 I=1,3
      DO 10 J=1,8
      GS(I,J)=0.0
      DO 10 K=1,3
      10 GS(I,J)=GS(I,J)+D(I,K)*P(K,J)
      DO 11 I=1,8
      DO 11 J=1,8
      AK(I,J)=0.0
      DO 11 K=1,3
      11 AK(I,J)=AK(I,J)+P(K,1)*GS(K,J)
      DO 12 I=1,8
      DO 12 J=1,8
      12 AKT(I,J)=AKT(I,J)+CON*AK(I,J)
      RETURN
      END

```

G LEVEL 1, MOD 4 SKYQR DATE = 69185 11/54/10

```

C      SUBROUTINE SKYQR(ID,P,UC,IF,NN,STR,FNC)
      CALC. OF STRESS FROM DISPLACEMENT (QUAD. EL.)
      DIMENSION TS(3),STR(3,1),P(3,8),UC(1),FNC(1),NN(4,1)
      DO 11 I=1,3
      TS(I)=0.0
      DO 10 J=1,4
      JJ=2*NN(J,IF)-1
      10 TS(I)=TS(I)+P(1,J)*UC(JJ)+P(1,J+4)*UC(JJ+1)
      TS(I)=TS(I)*.25
      11 STR(I,1)=STR(I,1)+TS(I)
      FNC(1C)=FNC(1C)+1
      RETURN
      END

```

G LEVEL 1, MOD 4 STRR DATE = 69185 11/54/10

```

C      SUBROUTINE STRR(X,Y,IP,JP,KP,F,GNU,XX,YY,P)
      CALC. OF STRESS MATRIX FOR RECT. ELEMENT
      DIMENSION P(4,9),X(1),Y(1)
      A=1./2*(X(JP)-X(IP))
      B=1./2*(Y(KP)-Y(IP))
      CC=1./2*(X(JP)-X(IP))*(Y(KP)-Y(IP))
      E=1./2*(1.-GNU*GNU)
      CD=1./2*(1.-GNU)/2.0
      P(1,1)=C-A*(CC*YY)
      P(2,1)=GNU*P(1,1)
      P(2,2)=(C+XX-P(1,1)
      P(1,2)=GNU*P(2,2)
      P(1,3)=1-C*(CC*YY)
      P(2,3)=GNU*P(1,3)
      P(2,4)=C-CC*XX
      P(1,4)=GNU*P(2,4)
      P(1,5)=C*YY
      P(2,5)=GNU*P(1,5)
      P(2,6)=P(2,4)
      P(1,6)=GNU*P(2,6)
      P(1,7)=P(1,5)
      P(2,7)=GNU*P(1,7)
      P(2,8)=(B-CC*XX)
      P(1,8)=GNU*P(2,8)
      P(3,1)=E*(C+XX-0)
      P(3,2)=E*(C+YY-A)
      P(3,3)=E*(C+CC*XX)
      P(3,4)=E*(C+CC*YY)
      P(3,5)=E*(C+CC*XX)
      P(3,6)=E*(C+CC*YY)
      P(3,7)=E*(B-CC*XX)
      P(3,8)=E*(B-CC*YY)
      RETURN
      END

```

G LEVEL 1, MOD 4 SKYLR DATE = 69185 11/54/10

```

C      SUBROUTINE SKYLR(STR,IO,P,UC,FNC,IF,NN)
      CALC. OF STRESS FROM DISPLACEMENT(RECT. EL.)
      DIMENSION TS(3),STR(3,1),P(3,8),UC(1),FNC(1),NN(4,1)
      DO 11 I=1,3
      TS(I)=0.0
      DO 10 J=1,4
      II=2*J-1
      JJ=2*NN(J,IF)-1
10  TS(I)=TS(I)+P(I,II)*UC(JJ)+P(I,II+1)*UC(JJ+1)
11  STR(I,IO)=STR(I,IO)+TS(I)
      FNC(IO)=FNC(IO)+1
      RETURN
      END

```

G LEVEL 1, MOD 4 WASTR DATE = 69185 11/54/10

```

C      SUBROUTINE WASTR(FNC,STR,NN)
      PRINT AVERAGE STRESS IN EACH ELEMENT
      DIMENSION FNC(1),STR(3,1)
      DO 10 I=1,3
      DO 10 J=1,NN
10  STR(I,J)=STR(I,J)/FNC(J)
      WRITE (3,100)
      DO 12 J=1,NN
12  WRITE (3,101) J,(STR(I,J),I=1,3)
100 FORMAT(/' NODE NO.      AVG. SIGX      AVG. SIGY      AVG. SIGXY'/)
101 FORMAT(3X,14.5X,E11.4,5X,E11.4,5X,E11.4)
      RETURN
      END

```

G LEVEL 1, MOD 4 TIMER DATE = 69185 11/54/10

```

C      SUBROUTINE TIMER(NOF,ANM,ASB,DELT)
      CALCULATION OF DELTA T
      DIMENSION ANM(1),ASB(26,1)
      READ (2,101) FM,FR
      DTT=100.
      DO 11 J=1,NOF
      DT=ANM(J)*.5/ASB(1,J)
      IF(DTT-DT) 11,10,10
10  DTT=DT
11  CONTINUE
      DT=SQRT(DTT)
      N=DT*FM
      DO 5 I=1,10
      J=I
      M=N/(10**I)
      IF(M) 5,8,5
      5  CONTINUE
      8  K=J-2
      N=(N/(10**K))*(10**K)
      FR=FR
      DELT=FM/(FR*FR)
      WRITE (3,100) DELT
100 FORMAT(/' TIME INCREMENT = '.E12,5.' SECONDS'/)
101 FORMAT(12.5,F10.5)
      RETURN
      END

```

COMMON SUBROUTINES

G LEVEL 1, MOD 4 RDCR DATE = 69185 11/54/10

```

C      SUBROUTINE RDCR(UP,NDC,C,NDC,UC,NDC)
      READS U, C, AND DAMPING CONSTANT
      DIMENSION UP(1),UC(1),NDC(1)
      DO 1 I=1,NDC
1      UC(I)=0.0
      NDC=0
      I=1
4      READ(2,100) N
      IF(N) 5,6,5
5      NDC(I)=N
      READ(2,101) UP(N)
      I=I+1
      NDC=NDC+1
      GO TO 4
6      READ(2,101) C
      WRITE(3,103)
      DO 7 I=1,NDC
      N=NDC(I)
      WRITE(3,102) N,UP(N)
7      UC(I)=UP(N)
      WRITE(3,104) C
100  FORMAT (15)
101  FORMAT (2F12.5)
102  FORMAT (1, U('13.0') = 'F12.5)
103  FORMAT (2, F12.5 DISPLACEMENTS)
104  FORMAT (2, DAMPING = 'F5.2,' X CRIT. DAMP. %)
      RETURN
      END

```

G LEVEL 1, MOD 4 RCONR DATE = 69185 11/54/10

```

C      SUBROUTINE RCONR(NN,IND,NNT,NBAND,E,GNU,RHO,IT)
      READS (L, NO, AND CONN. NODE NUMBERS
      DIMENSION NN(4,1),IND(1)
      READ(2,92) L,GNU,RHO,IT
      IF(IT) 30,31,30
30  IT=16
      GO TO 32
31  IT=6
32  ARITH(3,103) E,GNU,RHO
      NBAND=0
      NNT=0
      WRITE(3,100)
4      READ(2,101) IE,I,J,K,L,IDENT
      IF(IE) 5,6,5
5      WRITE(3,102) IE,I,J,K,L,IDENT
      NN(1,IE)=I
      NN(2,IE)=J
      NN(3,IE)=K
      NN(4,IE)=L
      IND(IE)=IDENT
      NNT=NNT+1
      NS=I
      NL=I
      DO 10 M=2,4
      IF(NL-NN(M,IE)) 7,7,8
7      NL=NN(M,IE)
      GO TO 10
8      IF(NS-NN(M,IE)) 10,9,9
9      NS=NN(M,IE)
10  CONTINUE
      NT=(N-NS+1)*2
      IF(NBAND-NT) 11,12,12
11  NBAND=NT
12  IF(NBAND-26) 4,4,21
21  WRITE(3,104)
92  FORMAT (7F12.5,15)
103  FORMAT (2, E = 'F12.5,' NO = 'F6.4,' MASS DENSITY = '
      IE12.5)
100  FORMAT (2, EL, ASSOCIATED NODES IDENT)
101  FORMAT (1015)
102  FORMAT (14,2X,415,4X,15)
104  FORMAT (2, NBAND EXCEEDS 26)
6      RETURN
      END

```

G LEVEL 1, MOD 4 DELFR DATE = 69185 11/54/10

```

C      SUBROUTINE DELFR(DF,NP,IFIN,NDF)
      READS LINEARLY VARYING FORCE, NO. OF STEPS, END OF PROBLEM INDICATOR
      DIMENSION DF(10)
      DO 10 I=1,NDF
10  DF(I)=0.0
      READ(2,97) NP,IC,IFIN
      IF(IFIN) 5,7,5
5      IF(IC) 12,13,12
12  WRITE(3,99) NP
      WRITE(3,98)
      FND=NP
2      READ(2,100) M,FORCE
      IF(M) 6,7,6
6      DF(M)=FORCE/FND
      WRITE(3,102) M,DF(M)
      GO TO 2
13  WRITE(3,103) NP
7      RETURN
97  FORMAT (315)
98  FORMAT (2, FORCE CHANGE PER TIME INTERVAL)
99  FORMAT (2, FORCE CHANGES LINEARLY IN '15,' TIME INTERVALS)
100  FORMAT (15,12.5)
102  FORMAT (2, DF('14.0') = 'F12.5)
103  FORMAT (2, FORCE REMAINS CONSTANT FOR '15,' TIME INTERVALS)
      END

```

```

: LEVEL 1, MOD 4          VFJDR          DATE = 69185          11/54/10

C      SUBROUTINE VFJDR(I1A,X1,Y1,IP,JP,KP,LP,V,FJ,DJ)
      CALC. OF V, FJ INVERSE, AND DJ
      DIMENSION V(2,4),X(1),Y(1),XY(4,2),FJ(2,2)
      V(1,1)=-1.-F1A
      V(1,2)=V(1,1)
      V(1,3)=1.-F1A
      V(1,4)=-V(1,3)
      V(2,1)=-(-1.-Y1)
      V(2,2)=-(-1.-X1)
      V(2,3)=-V(2,2)
      V(2,4)=-V(2,3)
      XY(1,1)=X(IP)
      XY(1,2)=Y(IP)
      XY(2,1)=X(JP)
      XY(2,2)=Y(JP)
      XY(3,1)=X(KP)
      XY(3,2)=Y(KP)
      XY(4,1)=X(LP)
      XY(4,2)=Y(LP)
      DO 10 I=1,2
      DO 10 K=1,2
      FJ(1,K)=0.0
      DO 10 J=1,4
      10 FJ(1,K)=FJ(1,K)+V(1,J)*XY(J,K)
      DO 12 I=1,2
      DO 12 J=1,2
      12 FJ(1,J)=.25*FJ(1,J)
      DJ=FJ(1,1)*FJ(2,2)-FJ(1,2)*FJ(2,1)
      TR=FJ(1,1)
      FJ(1,1)=FJ(2,2)/DJ
      FJ(1,2)=-FJ(1,2)/DJ
      FJ(2,1)=-FJ(2,1)/DJ
      FJ(2,2)=FJ(2,2)/DJ
      DJ=ABS(DJ)
      RETURN
      END

```

```

: LEVEL 1, MOD 4          DSPLR          DATE = 69185          11/54/10

C      SUBROUTINE DSPLR(NDF,DELT,ASR,UB,UC,UF,NBAND,FA,DAMP,T,ANM,UP,
      CALC. OF NODAL POINT DISPLACEMENTS
      INHC,NRI
      DIMENSION ASR(26,1),UC(1),UB(1),UF(1),FA(1),DAMP(1),ANM(1)
      1 INR(1)=UP(1)
      DELT2=DELT*DELT
      DO 20 J=1,NDF
      FORCE=ASR(1,J)+UC(J)
      DO 25 K=1,NBAND
      JL=J+1-K
      IF(JL) 20,28,29
      29 FORCE=FORCE+ASR(K,JL)*UC(JL)
      28 JR=J-1-K
      IF(NDF-JR) 25,24,24
      24 FORCE=FORCE+ASR(K,JR)*UC(JR)
      25 CONTINUE
      20 UF(J)=(2.*UC(J)-UB(J)+DELT2*(FA(J)-FORCE)/ANM(J)+UB(J)*DAMP(J)*
      1 DELT+.5/ANM(J))/(1.+DAMP(J)*DELT*.5/ANM(J))
      DO 30 I=1,NRC
      IP=NH(I)
      30 UF(IP)=UP(IP)
      T=T+DELT
      DO 40 N=1,NDF
      UB(N)=UC(N)
      UC(N)=UF(N)
      40 UF(N)=0.0
      RETURN
      END

```

```

: LEVEL 1, MOD 4          ESRIR          DATE = 69185          11/54/10

C      SUBROUTINE ESRIR(NN,IC,ASB,AK)
      ASSEMBLES STIFFNESS IN RECT. FORM
      DIMENSION JJ(8),NN(4,1),ASB(26,1),AK(8,8)
      DO 10 I=1,4
      10 I=7+I
      IO=II-1
      JJ(11)=2*NN(1,IE)
      10 JJ(10)=JJ(11)-1
      DO 12 J=1,8
      JP=JJ(J)
      DO 12 I=1,8
      IP=JJ(11)+1-JP
      12 IF(IP) 12,12,13
      13 ASB(IP,JP)=ASB(IP,JP)+AK(1,J)
      12 CONTINUE
      RETURN
      END

```

```

: LEVEL 1, MOD 4          WDSPR          DATE = 69185          11/54/10

C      SUBROUTINE WDSPR(T,UC,NDF)
      PRINTS DISPLACEMENTS
      DIMENSION UC(1)
      WRITE(3,99) T
      99 WRITE(17,100) T,UC(1),T+1,NDF
      100 FORMAT ('(1 UC',12,' ) = ',(12,5))
      99 FORMAT ('// DISPLACEMENTS AT T = ',E14.7,' SEC.//')
      RETURN
      END

```

```

LEVEL 1. MOD 4          RXY4R          DATE = 69185          11/54/10
SUBROUTINE RXY4R(X,Y,NDF,K)
  DIMENSION X(1),Y(1)
  READS COORDINATES X, Y
  K=0
  WRITE(3,103)
  4 READ(2,102) I,XX,YY
  IF(1) 5,6,5
  5 X(1)=XX
  Y(1)=YY
  WRITE(3,103) I,X(1),Y(1)
  K=K+1
  GO TO 4
  6 NDF=2+K
  102 FORMAT(15,2F10.5)
  103 FORMAT(14,1X,10,4,2X,F8.4)
  104 FORMAT(// ' NODE' X Y '/')
  RETURN
END

```

```

LEVEL 1. MOD 4          MTMLR          DATE = 69185          11/54/10
SUBROUTINE MTMLR(A,R,AK)
  MATRIX MULTIPLICATION A(TRANSPOSE) * B * A
  DIMENSION A(R,R),B(R,B),AK(R,B),STIF(R,B)
  DO 10 I=1,R
  DO 10 J=1,B
  STIF(I,J)=0.0
  DO 10 K=1,B
  10 STIF(I,J)=STIF(I,J)+B(I,K)*A(K,J)
  DO 12 I=1,R
  DO 12 J=1,B
  AK(I,J)=0.0
  DO 12 K=1,B
  12 AK(I,J)=AK(I,J)+A(K,I)*STIF(K,J)
  RETURN
END

```

```

LEVEL 1. MOD 4          RINFR          DATE = 69185          11/54/10
SUBROUTINE RINFR(FA,NAF,NANF,NTI,INDF,INP)
  READS INITIAL NODAL FORCES AND PRINT INFORMATION
  DIMENSION FA(1),NAF(1)
  NANF=0
  4 READ(2,100) M
  IF(M) 5,6,5
  5 NANF=NANF+1
  NAF(NANF)=M
  READ(2,101) FA(M)
  GO TO 4
  6 READ(2,100) NTI,INDF,INP
  WRITE(3,102)
  DO 10 I=1,NANF
  M=NAF(I)
  10 WRITE(3,103) M,FA(M)
  100 FORMAT(4I5)
  101 FORMAT(F10.5)
  102 FORMAT(// ' DEGREE OF FREEDOM APPLIED FORCE'//)
  103 FORMAT(5X,14,15X,F10.3)
  RETURN
END

```

```

LEVEL 1. MOD 4          ESA2R          DATE = 69185          11/54/10
SUBROUTINE ESA2R(NN,AKT,ASR,IE)
  ASSEMBLES STIFFNESS FROM AKT IN PRIME FORM
  DIMENSION NN(4,1),AKT(R,8),ASR(26,1)
  DO 10 K=1,2
  DO 10 L=1,2
  DO 10 IH=1,4
  DO 10 JH=1,4
  JJ=2*NN(JH,IF)-2+L
  II=2*NN(IH,IF)+K-JJ-1
  IF(1) 10,10,14
  14 I=4*(K-1)+IH
  J=4*(L-1)+JH
  ASR(I,J)=ASR(I,J)+AKT(I,J)
  10 CONTINUE
  RETURN
END

```