Low Complexity Normal Bases

by

David Thomson, B.Math

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To the number π , may you always
be curiously identically equal to three.

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Abstract

We develop an efficient algorithm to enumerate and examine all normal bases over finite fields of characteristic two and degrees up to $n = 39$. This improves on previous tables given in [22] which search to $n = 30$. We create a table of minimum complexities for $n \leq 607$ using our new data and theorems from the literature. We conjecture on the average and variance of the distribution of complexities in a finite field. We further conjecture that there is no constant k such that the minimum complexity of a normal basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ is bounded below by kn .

We also give a new construction of low complexity normal bases for finite fields of any characteristic. We do this by projecting to subfields using the trace function from extensions where an optimal normal basis is known to exist. This is performed for both Type I and Type II optimal normal bases. We also examine the dual bases of our construction and give their resulting complexities. We compare the five fields recommended by NIST [33] for elliptic curve cryptography to fields with similar characteristics where we apply our construction and note that if our fields are cryptographically strong they admit a much lower complexity than the given fields.

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Chapter 1

Introduction

The choice of representation of elements in a finite field is critical for performing efficient calculations in the field. There are two standard bases used to represent a finite field: *polynomial bases* and *normal bases*.

A polynomial basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is given by $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$, where α is a root of an irreducible polynomial f of degree n over $\mathbb{F}_q$. In a polynomial basis, multiplication of elements becomes more efficient as the number of nonzero terms in f decreases. For a discussion of polynomial bases and constructions of low-weight polynomials see, for example, [13, 18, 21].

An alternate basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is given when an element $\alpha \in \mathbb{F}_{q^n}$ and its *conjugates* $\alpha^q, \alpha^{q^2}, \dots, \alpha^{q^{n-1}}$ are linearly independent. A basis formed by the conjugates of α is called a *normal basis*. Normal bases are efficient for applications where large powers of elements are needed, such as in public key cryptography, but have the drawback that ordinary multiplication is computationally expensive. A measure of

this expense is called the *complexity* of the normal basis. We examine methods and constructions for optimizing the complexity of a normal basis in a finite field.

We make every effort to make this work self-contained. In Chapter 2 we begin by giving some key results on finite fields. When feasible, theorems and definitions are given in generality and then narrowed to the scope of finite fields. Critical notions such as the *trace* of a field element are introduced. We give known results on normal bases that are necessary for all results we derive later. We give a discussion of normal basis arithmetic and a proof of the normal basis theorem. The complexity of a normal basis is introduced and we show the constructions of an *optimal* normal basis. We give Gao's original proof [10] that all optimal normal bases are equivalent to these constructions. We derive the number of normal bases in a finite field generally and give some computationally feasible bounds. We then give some previously known constructions of normal bases with given complexities. Finally, we discuss the dual of a normal basis, prove that this is again a normal basis, give necessary conditions for the existence of the dual of a normal basis and cite some results on the complexities of the dual of a normal basis.

We begin giving new results in Chapter 3. Some of the contents of this chapter are given in [27]. We develop an efficient algorithm to enumerate and examine all normal bases over finite fields of characteristic two and degrees up to $n = 39$. This improves on previous tables given in [22] which search until $n = 30$. We analyze the cost complexity of the algorithm and compare the run time of our algorithm with

a non-optimized version. We use the data obtained by our search to explore the distribution of normal bases. We then create a table of minimum complexities for degrees $n \leq 607$ using our new data and theorems from the literature. We end by giving conjectures on distribution of normal elements in a finite field with respect to their complexities.

The results from Chapter 3 indicate experimentally that there is a very low probability that any randomized algorithm will provide normal bases with low complexity, and so in Chapter 4 we give a direct construction of normal bases in finite fields with low complexity. The contents of this chapter are outlined in [4]. We give our construction by projecting to subfields from fields which contain an optimal normal basis using the field trace function. This projection again yields normal bases with a computable (low) upper bound on their complexities. We also construct the dual bases of these projections and give upper bounds on their complexities. We discuss some deep questions on how large our field must reach to contain the desired extension as a subfield and also contain an optimal normal basis.

We give a summary of all results obtained in Chapter 5. Section 5.2 concludes by posing some open problems which arose in conducting the research outlined above. The appendices present tables of data referred to in Chapters 3 and 4.

Chapter 2

Normal Bases in Finite Fields

The following assumes some basic knowledge of rings and fields. Some key results and definitions of field extensions and Galois theory are given for completeness.

2.1 Finite Fields

All definitions and theorems can be found in the text and exercises of Dummit and Foote's *Abstract Algebra* [6, Chapter 13-14], unless otherwise indicated. Some proofs are given for clarity.

Theorem 2.1.1. *Let $f \in F[x]$ be an irreducible polynomial of degree n over the field F and let $K = F[x]/(f)$. Let $\theta \equiv x \pmod{f}$. Then, the powers*

$$1, \theta, \theta^2, \dots, \theta^{n-1}$$

are a basis for K viewed as a vector space over F and therefore $[K : F] = n$. We have

$$K = \{a_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1}; \quad \text{with } a_0, a_1, \dots, a_{n-1} \in \mathbb{F}\}$$

and so K consists of all polynomials of degree up to $n - 1$ in $F[\theta]$. Addition in K is defined by polynomial addition and multiplication in K is defined by

$$a(\theta)b(\theta) = r(\theta)$$

where $a(x)b(x) = q(x)f(x) + r(x)$ for some $q, r \in F[x]$ where $\deg(r) < n$.

Theorem 2.1.2. *Let F be a field and let $f \in F[x]$ be an irreducible polynomial of degree n . Suppose K is an extension field of F containing a root α of f . Let $F(\alpha)$ denote the subfield of K generated over F by the element α . Then $F(\alpha) \cong F[x]/(f)$ and*

$$F(\alpha) = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}; \quad \text{with } a_0, a_1, \dots, a_{n-1} \in F\} \subseteq K.$$

The following theorem is the keystone upon which Galois theory is built.

We now begin to characterize all finite fields. For the remainder of this discussion, we always assume that p is a prime integer, and q denotes the prime power p^n for some integer $n \geq 1$.

Proposition 2.1.3. *Let $\mathbb{F}$ be a finite field of characteristic p . Then $\mathbb{F}$ has p^n elements*

for some integer $n \geq 1$.

PROOF. Let $\mathbb{F}$ be a finite field of characteristic p . Then $\mathbb{F}$ has prime subfield isomorphic to $\mathbb{Z}_p$. In particular, $\mathbb{F}$ is a vector space over $\mathbb{Z}_p$ and so it has a basis $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ over $\mathbb{Z}_p$ for some integer $n \geq 1$. Any element in $\mathbb{F}$ is of the form

$$z_0\alpha_0 + z_1\alpha_1 + \dots + z_{n-1}\alpha_{n-1}; \quad \text{with } z_0, z_1, \dots, z_{n-1} \in \mathbb{Z}_p$$

and so there are p^n such elements. ■

Definition 2.1.4. *The extension field K of F is called a splitting field for the polynomial $f \in F[x]$ if f factors completely into linear factors in K and f does not factor into linear factors in any subfield of K containing $\mathbb{F}$.*

Theorem 2.1.5. (Existence and Uniqueness of Splitting Fields) *Let $f \in F[x]$. Then there exists a splitting field E of f over F and any two splitting fields of f over F are isomorphic under an isomorphism which fixes F and maps roots of f into each other.*

Lemma 2.1.6. *Let $f \in F[x]$, then f has a multiple root α if and only if α is also a root of f' . In particular, f is separable if and only if $\gcd(f, f') = 1$.*

Example 2.1.7. *Consider $f(x) = x^{p^n} - x$ over $\mathbb{F}_p$, then $f'(x) = p^n x^{p^n-1} - 1 = -1$ since $\mathbb{F}_p$ has characteristic p . Therefore f' has no roots and f is separable.*

Theorem 2.1.8. (Existence and Uniqueness of Finite Fields) *For every prime p and*

positive integer n , there is a finite field with p^n elements. Any finite field with p^n elements is isomorphic to the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$.

PROOF. Let n be a positive integer and let $f(x) = x^{p^n} - x$ over $\mathbb{F}_p$. By Example 2.1.7, f is separable and so f has exactly p^n distinct roots. If α and β are roots of f then $f(\alpha) = 0 = f(\beta)$ so $\alpha^{p^n} = \alpha$ and $\beta^{p^n} = \beta$.

Denote $\mathbb{F}$ as the set of roots of $x^{p^n} - x$ over $\mathbb{F}_p$. Then $\mathbb{F}$ is closed under addition, multiplication, and inversion, so $\mathbb{F}$ is a subfield of the splitting field of f over $\mathbb{F}_p$. By the minimality of the splitting field, $\mathbb{F}$ is the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$. This shows that, for any positive integer n , there exists a finite field of degree n over $\mathbb{F}_p$.

To prove uniqueness, let $\mathbb{F}$ be a finite field of characteristic p . If $\mathbb{F}$ has dimension n over $\mathbb{F}_p$ then $\mathbb{F}$ has p^n elements, and the multiplicative group of $\mathbb{F}$, $\mathbb{F}^*$ has order $p^n - 1$. Therefore for any $\alpha \in \mathbb{F}^*$, $\alpha^{p^n - 1} = 1$, and so $\alpha^{p^n} = \alpha$ and α is a root of f . Since the splitting field of f has p^n elements, $\mathbb{F}$ is a splitting field of f over $\mathbb{F}_p$. Uniqueness of finite fields then follows directly from uniqueness of splitting fields. ■

We now explore the structure of all subfields of a given finite field $\mathbb{F}_q$.

Lemma 2.1.9. *Let n, d be integers, then d divides n if and only if $x^d - 1$ divides $x^n - 1$.*

PROOF. By the Division Algorithm, $n = qd + r$ for integers q, r , $0 \leq r < d$. So $x^n - 1 = (x^{qd+r} - x^r) + (x^r - 1)$. Then $x^d - 1$ divides $x^n - 1$ if and only if $x^d - 1$ divides $x^r - 1$ and $x^d - 1$ divides $x^{qd+r} - x^r = x^r(x^{qd} - 1)$. The first condition holds if and only

if $r = 0$. The second condition holds since $x^{qd} - 1 = (x^d - 1)(x^q - x^{q-1} + \cdots - x + 1)$, as a telescoping sum. ■

Lemma 2.1.10. *Let n, d be integers, then d divides n if and only if $a^d - 1$ divides $a^n - 1$, for any integer a .*

PROOF. The previous lemma holds over any polynomial ring so, in particular, it holds over the ring of integers $\mathbb{Z}$. By the previous lemma, d divides n if and only if $x^n - 1 = (x^d - 1)q(x)$ for some $q(x) \in \mathbb{Z}[x]$. Since the evaluation map is a homomorphism it follows that for any integer a , d divides n if and only if $a^n - 1 = (a^d - 1)q(a)$ where $q(a) \in \mathbb{Z}$. ■

Theorem 2.1.11. (Subfield Criterion) *Let $\mathbb{F}_{p^n}$ be the finite field with p^n elements. Every subfield of $\mathbb{F}_{p^n}$ has order p^d for d is a positive divisor of n . Conversely, if d is a positive divisor of n , there is a unique subfield of $\mathbb{F}_{p^n}$ with p^d elements*

PROOF. If $\mathbb{F}$ is a subfield of $\mathbb{F}_{p^n}$, then it is again a finite field of characteristic p so therefore has p^d elements for some integer $d \leq n$. We can view $\mathbb{F}_{p^n}$ as a finite dimensional vector space (say, of dimension m) over $\mathbb{F}$ and so $\mathbb{F}_{p^n}$ has a basis of m elements $\{b_1, b_2, \dots, b_m\} \subseteq \mathbb{F}_{p^n}$. Then every element in $\mathbb{F}_{p^n}$ is of the form $a_1b_1 + a_2b_2 + \cdots + a_mb_m$ for $a_1, a_2, \dots, a_m \in \mathbb{F}$. Therefore, $p^n = (p^d)^m$.

Conversely, if d is a positive divisor of n then, by the above lemma, $p^d - 1$ divides $p^n - 1$, for any prime p . Consequently $x^{p^d} - x$ divides $x^{p^n} - x$. Then every root of $x^{p^d} - x$ is a root of $x^{p^n} - x$. Therefore $\mathbb{F}_{p^n}$ contains a splitting field of $x^{p^d} - x$ as a

subfield. If $\mathbb{F}_{p^n}$ contained two distinct subfields which split $x^{p^d} - x$ then there would be more than p^d roots, which is a contradiction.

This gives a unique lattice of subfields for any finite field $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$, for any positive integer n with respect to its positive divisors.

Corollary 2.1.12. *Let $\mathbb{F}_p$ be the finite field of characteristic p , then $\mathbb{F}_{p^n}$ is Galois over $\mathbb{F}_p$ and $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \phi_p \rangle$, the Frobenius p -automorphism.*

PROOF. The extension $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ is Galois since it is the splitting field of $x^{p^n} - x \in \mathbb{F}_p^*$. The Frobenius map was shown above to be an automorphism, and is of order n , so must generate the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$. ■

Proposition 2.1.13. *Let $q = p^m$ for a prime p and let $\mathbb{F}_q = \mathbb{F}_{p^m}$ be the finite field with q elements. Let $\phi_q = \phi_p^m$ where ϕ_p is the Frobenius automorphism. Then any extension of $\mathbb{F}_q$ of degree n is the splitting field of $x^{q^n} - x$ over $\mathbb{F}_q$, and therefore unique and Galois. Furthermore, ϕ_q generates the Galois group of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Finally, the subfields of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ are in bijective correspondence with the divisors of n .*

PROOF. We first show that ϕ_q fixes $\mathbb{F}_q$. Let $\alpha \in \mathbb{F}_q$, then $\phi_q(\alpha) = \phi_p^m(\alpha) = \phi_p(\alpha)^m = \alpha^{p^m} = \alpha$ since $\mathbb{F}_q$ is the splitting field of $x^{p^m} - x$ over $\mathbb{F}_p$.

Let $\mathbb{F}_{q^n}$ be a finite extension of $\mathbb{F}_q$ of degree n . Then $\mathbb{F}_{q^n}$ is the finite extension of $\mathbb{F}_p$ of degree mn , and is the splitting field of $x^{p^{mn}} - x$ over $\mathbb{F}_p$ and therefore unique and Galois. It is an easy exercise, therefore, to show that ϕ_q is an automorphism of degree n in $\mathbb{F}_{q^n}$ and we assert that ϕ_q generates the Galois group of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

The subfield criterion is a direct result of the Fundamental Theorem of Galois Theory and Lagrange's theorem on the (cyclic) Galois group of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ of order n . ■

Definition 2.1.14. Let K be a Galois extension of F . If $\alpha \in K$ the elements $\sigma(\alpha)$ for $\sigma \in \text{Gal}(K/F)$ are called the Galois conjugates (or simply conjugates) of α over F .

Fact 2.1.15. The Galois conjugates of $\alpha \in \mathbb{F}_q$ are $\alpha, \alpha^q, \alpha^{q^2}, \dots, \alpha^{q^{n-1}}$.

2.1.1 Traces and Bases of Finite Fields

We have derived the existence and uniqueness of finite fields using some elementary field extension theory. This provides motivation for the definitions of some key functions we need for the later chapters.

Definition 2.1.16. Let K be a Galois extension of F , and let $\alpha \in K$. Define the trace of $\alpha \in K$ over F by

$$\text{Tr}_{K/F}(\alpha) = \sum_{\sigma \in \text{Gal}(K/F)} \sigma(\alpha).$$

Similarly the norm of $\alpha \in K$ over F is defined by

$$N_{K/F}(\alpha) = \prod_{\sigma \in \text{Gal}(K/F)} \sigma(\alpha).$$

When the context is clear, we drop the subscripts denoting the extension and base fields.

Fact 2.1.17. *Let $\alpha \in \mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Then the trace of α is given by*

$$\text{Tr}(\alpha) = \alpha + \alpha^q + \cdots + \alpha^{q^{n-1}}.$$

Similarly, the norm of α is given by

$$N(\alpha) = \alpha \alpha^q \cdots \alpha^{q^{n-1}}.$$

Now we give some properties of the trace function, and note that analogous results can be obtained for the norm, for example see [6, 25].

Proposition 2.1.18. [25] *Let K be a Galois extension of a field F and let $\alpha, \beta \in K$. Then $\text{Tr}_{K/F}(\alpha)$ is a linear projection from K to F . If $K = \mathbb{F}_{q^n}$ and $F = \mathbb{F}_q$ then also*

1. $\text{Tr}_{K/F}(a) = na$ for all $a \in F$;
2. $\text{Tr}_{K/F}(\alpha^q) = \text{Tr}_{K/F}(\alpha)$ for all $\alpha \in K$.

PROOF. Let $\alpha \in K$, $\sigma \in \text{Gal}(K/F)$. Then

$$\sigma(\text{Tr}(\alpha)) = \sigma \left(\sum_{\gamma \in \text{Gal}(K/F)} \gamma(\alpha) \right),$$

and σ is a homomorphism of the additive group of K , so

$$\sigma(\text{Tr}(\alpha)) = \sum_{\gamma \in \text{Gal}(K/F)} \sigma\gamma(\alpha).$$

Left action by σ is transitive on $\text{Gal}(K/F)$, so $\sigma(\text{Tr}(\alpha)) = \text{Tr}(\alpha)$ and therefore $\text{Tr}(\alpha) \in F$.

Let $\beta \in K$ as well, then

$$\text{Tr}(\alpha + \beta) = \sum_{\gamma \in \text{Gal}(K/F)} \gamma(\alpha + \beta) = \sum_{\gamma \in \text{Gal}(K/F)} \gamma\alpha + \sum_{\gamma \in \text{Gal}(K/F)} \gamma\beta$$

since γ is a homomorphism of the additive group of K . Also, for any $c \in F$

$$\text{Tr}(c\alpha) = \sum_{\gamma \in \text{Gal}(K/F)} \gamma(c\alpha)$$

with γ a homomorphism of the multiplicative group of K which fixes F . Composing these two facts shows the linearity of the trace function.

Now let $K = \mathbb{F}_{q^n}$ and $F = \mathbb{F}_q$, let $a \in \mathbb{F}_q$ and compute

$$\text{Tr}_{q^n/q}(a) = \sum_{i=0}^{n-1} a^{q^i} = na$$

since $a^q = a$. Similarly let $\alpha \in \mathbb{F}_{q^n}$, then

$$\text{Tr}_{q^n/q}(\alpha^q) = \sum_{i=0}^{n-1} \alpha^{q^{i+1}} = \text{Tr}_{q^n/q}(\alpha)$$

since $\alpha^{q^n} = \alpha$. ■

The trace function is especially important as it serves as a representation of all linear transformations from K onto F independently of the chosen basis.

Theorem 2.1.19. [25] *Consider $\mathbb{F}_{q^n}$ as a vector space over $\mathbb{F}_q$. Then the linear transformations from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$ are precisely the mappings L_β where $\beta \in \mathbb{F}_{q^n}$ such that $L_\beta(\alpha) = \text{Tr}(\beta\alpha)$ for all $\alpha \in \mathbb{F}_{q^n}$. Furthermore $L_\beta \neq L_\gamma$ whenever $\beta \neq \gamma$.*

PROOF. We have shown that each L_β is a linear transformation from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$. For $\beta \neq \gamma$ we have $L_\beta(\alpha) - L_\gamma(\alpha) = \text{Tr}(\beta\alpha) - \text{Tr}(\gamma\alpha) = \text{Tr}((\beta - \gamma)\alpha)$ which is nonzero for some $\alpha \in \mathbb{F}_{q^n}$ as the trace function is an onto mapping. This proves the final assertion and therefore there are q^n different mappings L_β from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$. But each mapping onto $\mathbb{F}_q$ can be obtained by assigning arbitrary elements of $\mathbb{F}_q$ to the n basis elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ for any basis. There are q^n ways of doing this, exhausting all possible linear transformations. ■

The following is a simple yet important rule when dealing with compositions of field extensions.

Theorem 2.1.20. (Transitivity of Trace) [25] *Let $F = \mathbb{F}_q$, let K be an extension of F of degree m and let E be an extension of K of degree n . Then for all $\alpha \in E$,*

$$\text{Tr}_{E/F}(\alpha) = \text{Tr}_{K/F}(\text{Tr}_{E/K}(\alpha)).$$

PROOF. Define F, K, E as in the hypothesis, then

$$\begin{aligned} \text{Tr}_{K/F}(\text{Tr}_{E/K}(\alpha)) &= \sum_{i=0}^{m-1} \text{Tr}_{E/K}(\alpha)^{q^i} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \alpha^{q^{jm+i}} \\ &= \sum_{k=0}^{mn-1} \alpha^{q^k} \\ &= \text{Tr}_{E/F}(\alpha). \end{aligned}$$

■

Finally, we use the notion of the trace function to define a property which allows us to easily check linear independence of a group of elements.

Definition 2.1.21. [10] Let $F = \mathbb{F}_q$ and $K = \mathbb{F}_{q^n}$, let $\alpha_1, \alpha_2, \dots, \alpha_n \in K$. Then the discriminant of $\alpha_1, \alpha_2, \dots, \alpha_n$, denoted $\Delta_{K/F}(\alpha_1, \dots, \alpha_n)$ is given by

$$\Delta_{K/F}(\alpha_1, \dots, \alpha_n) = \begin{vmatrix} \text{Tr}_{K/F}(\alpha_1\alpha_1) & \text{Tr}_{K/F}(\alpha_1\alpha_2) & \cdots & \text{Tr}_{K/F}(\alpha_1\alpha_n) \\ \text{Tr}_{K/F}(\alpha_2\alpha_1) & \text{Tr}_{K/F}(\alpha_2\alpha_2) & \cdots & \text{Tr}_{K/F}(\alpha_2\alpha_n) \\ \vdots & \vdots & & \vdots \\ \text{Tr}_{K/F}(\alpha_n\alpha_1) & \text{Tr}_{K/F}(\alpha_n\alpha_2) & \cdots & \text{Tr}_{K/F}(\alpha_n\alpha_n) \end{vmatrix}. \quad (2.1)$$

The discriminant is an F -linear combination of elements in F , and therefore $\Delta_{K/F}(\alpha_1, \alpha_2, \dots, \alpha_n) \in F$.

Theorem 2.1.22. [10] Let $N = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be any n elements in $\mathbb{F}_{q^n}$, then N is a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if and only if $\Delta_{\mathbb{F}_{q^n}/\mathbb{F}_q}(N) \neq 0$.

PROOF. Suppose $N = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ forms a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Let $\beta = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$ for $c_1, c_2, \dots, c_n \in \mathbb{F}_q$, and suppose that $\text{Tr}(\beta\alpha_j) = 0$ for $1 \leq j \leq n$. For any $\alpha \in \mathbb{F}_{q^n}$, $\alpha = d_1\alpha_1 + d_2\alpha_2 + \dots + d_n\alpha_n$, so

$$\text{Tr}(\beta\alpha) = d_1\text{Tr}(\beta\alpha_1) + d_2\text{Tr}(\beta\alpha_2) + \dots + d_n\text{Tr}(\beta\alpha_n) = 0.$$

This can only occur for all $\alpha \in \mathbb{F}_{q^n}$ if $\beta = 0$ and since $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ form a basis for $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ we have $c_1 = c_2 = \dots = c_n = 0$. Therefore the rows of the matrix in Equation (2.1) are linearly independent and so $\Delta(N) \neq 0$.

Suppose conversely $\Delta(N) \neq 0$ and let $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0$ for some $c_1, c_2, \dots, c_n \in \mathbb{F}_q$. Then

$$c_1\alpha_1\alpha_j + c_2\alpha_2\alpha_j + \dots + c_n\alpha_n\alpha_j = 0$$

for $1 \leq j \leq n$ and taking the trace we find

$$c_1\text{Tr}(\alpha_1\alpha_j) + c_2\text{Tr}(\alpha_2\alpha_j) + \dots + c_n\text{Tr}(\alpha_n\alpha_j) = 0.$$

Since $\Delta(N) \neq 0$, the rows of the matrix in Equation (2.1) are linearly independent, giving $c_1 = c_2 = \dots = c_n = 0$. ■

Corollary 2.1.23. *Let $N = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \mathbb{F}_{q^n}$ and let*

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^q & \alpha_2^q & \cdots & \alpha_n^q \\ \vdots & \vdots & & \vdots \\ \alpha_1^{q^{n-1}} & \alpha_2^{q^{n-1}} & \cdots & \alpha_n^{q^{n-1}} \end{pmatrix}, \quad (2.2)$$

then N forms a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if and only if $\det(A) \neq 0$.

PROOF. This follows directly from $\det(A^t \cdot A) = (\det(A))^2 = \Delta(N)$. ■

2.2 Introduction to Normal Bases

This remainder of the chapter provides an overview of some key definitions and results regarding normal bases. Unless otherwise indicated, all definitions and theorems are from the Ph.D thesis of S. Gao [10].

For the remainder of this work we make the following assumptions: let p be a prime, q a power of p . Let $\mathbb{F}_q$ be the finite field with q elements, and $\mathbb{F}_{q^n}$ the extension of dimension n over $\mathbb{F}_q$. We note that $\mathbb{F}_{q^n}$ can be viewed as a vector space over $\mathbb{F}_q$, and therefore any basis has precisely n elements.

Definition 2.2.1. *A normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is a basis of the form $B = \{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\}$ for some $\alpha \in \mathbb{F}_{q^n}$. We say that the element α generates the normal basis B and that α is a normal element.*

We observe that the basis B is the set α together with its (Galois) conjugates. For the sake of simplicity we represent the i th conjugate $\alpha^{q^i} = \alpha_i$ unless the context requires that we use the original form.

We first begin by giving the classical result that normal bases exist for any finite field. This theorem was first proven by Hensel [19] in 1888, and was later generalized for any finite Galois extension by Noether in 1934. Ore independently proved this result in 1934 using “linearized polynomials”; for a discussion of this method see [25]. First, we introduce a notion from linear algebra necessary to prove the result.

Definition 2.2.2. *Let V be an n -dimensional vector space over a field F and let T be a linear transformation $V \rightarrow V$. The minimal polynomial m of T is the (monic) polynomial of least degree such that $m(T) = 0$. The characteristic polynomial f of T is the product of $(x - \lambda)$ over all eigenvalues λ of T .*

The Cayley-Hamilton theorem states that the minimal polynomial of T always divides the characteristic polynomial of T .

Definition 2.2.3. *A cyclic vector $\alpha \in V$ of a linear transformation T is an element such that $\alpha, T(\alpha), T^2(\alpha), \dots, T^{n-1}(\alpha)$ forms a basis for V over F .*

We give as a lemma a standard result from linear algebra.

Lemma 2.2.4. [20] *Let V be an n -dimensional vector space over a field F and let T be a linear transformation $V \rightarrow V$. Then V contains a cyclic vector of T if and only if the minimal polynomial of T equals the characteristic polynomial of T .*

PROOF. Let m be the minimal polynomial of a linear transformation T from $V \rightarrow V$ over F , and let f be the characteristic polynomial of T . Suppose that T admits a cyclic vector $\alpha \in V$. Then $\alpha, T(\alpha), \dots, T^{n-1}(\alpha)$ form a basis of V over F . That is, $\alpha, T(\alpha), \dots, T^{n-1}(\alpha)$ are linearly independent over F . Suppose the degree of m is less than n , say $m(x) = m_0 + m_1x + \dots + m_kx^k$ for some $k < n$. Then $m(T) = m_01 + m_1T + \dots + m_kT^k = 0$. In particular,

$$m_0\alpha + m_1T(\alpha) + \dots + m_kT^k(\alpha) = 0,$$

but linear independence of $\alpha, T(\alpha), \dots, T^{n-1}(\alpha)$ guarantees that $m_0 = m_1 = \dots = m_k = 0$, contradicting the choice of m . Thus, the degree of m is at least n , but since m divides f and f is monic of degree n , we have $m = f$.

Suppose the converse, that the minimal polynomial of a linear transformation $T : V \rightarrow V$ over F is equal to the characteristic polynomial of T . Then the degree of the minimal polynomial of T is n . Suppose there is no cyclic vector $\alpha \in V$; that is $\alpha, T(\alpha), \dots, T^{n-1}(\alpha)$ is linearly dependent for all $\alpha \in V$. Then there is a non-trivial second block in the rational decomposition of T . Let

$$A = \begin{pmatrix} M_1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & M_k \end{pmatrix}$$

be the rational decomposition of T , where M_2 is the companion matrix to a polynomial p , and p divides m . By the assumption, M_2 is non-trivial so p is non-constant. The matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & I \end{pmatrix}$$

commutes with A . We show an equivalent form of the theorem: that B is not a polynomial of degree less than n . If $B = f(A)$ for some polynomial f , then $f(M_1) = 0$ and therefore m divides f . We have that p divides m and so $f(M_2) = 0$, which contradicts construction of B . ■

Theorem 2.2.5. (Normal basis theorem for finite fields) *Every extension $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ has a normal basis.*

PROOF. We have shown that the Galois group of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is generated by the Frobenius q -automorphism ϕ_q . Since ϕ_q is a field automorphism that fixes $\mathbb{F}_q$, we have that $\phi_q(a + b) = \phi_q(a) + \phi_q(b)$ and $\phi_q(ca) = \phi_q(c)\phi_q(a) = c\phi_q(a)$ for all $a, b \in \mathbb{F}_{q^n}, c \in \mathbb{F}_q$. In other words, we can consider ϕ_q as a linear operator on the n -dimensional vector space $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

Since $\phi_q^n = \text{id}$, the polynomial $x^n - 1 \in \mathbb{F}_q[x]$ annihilates ϕ_q . Since the first $n - 1$ powers of ϕ_q are linearly independent, no polynomial of degree less than n can annihilate ϕ_q . It follows that both the characteristic polynomial and minimal polynomial of

ϕ_q are $x^n - 1$. By Lemma 2.2.4 there exists an $\alpha \in \mathbb{F}_{q^n}$ such that $\{\alpha, \phi_q(\alpha), \phi_q^2(\alpha), \dots\}$ spans $\mathbb{F}_{q^n}$. Since $x^n - 1$ annihilates ϕ_q we have $\{\alpha, \phi_q(\alpha), \dots, \phi_q^{n-1}(\alpha)\}$ spans $\mathbb{F}_{q^n}$. Thus, the element α and its conjugates form a basis for $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. ■

Viewing $\mathbb{F}_{q^n}$ as an n -dimensional vector space over $\mathbb{F}_q$, we can identify any element of $\mathbb{F}_{q^n}$ as an n -tuple with respect to a basis $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ over $\mathbb{F}_q$. For any $\alpha, \beta \in \mathbb{F}_{q^n}$ we have $\alpha = a_0\alpha_0 + a_1\alpha_1 + \dots + a_{n-1}\alpha_{n-1}$ and $\beta = b_0\alpha_0 + b_1\alpha_1 + \dots + b_{n-1}\alpha_{n-1}$ for $a_i, b_i \in \mathbb{F}_q$; $0 \leq i \leq n-1$.

2.2.1 Arithmetic Using Normal Bases

We identify an arbitrary $\alpha \in \mathbb{F}_{q^n}$ with the vector $A = (a_0, a_1, \dots, a_{n-1})$ and $\beta \in \mathbb{F}_{q^n}$ with $B = (b_0, b_1, \dots, b_{n-1})$. Then addition of α and β is component-wise, i.e:

$$A + B = (a_0 + b_0, a_1 + b_1, \dots, a_{n-1} + b_{n-1}).$$

If our basis is a normal basis consider the vector $A^q = (a_{n-1}, a_0, a_1, \dots, a_{n-2})$; taking q th powers is simply a cyclic bit-shift of the coefficients, and is cost-negligible. For this reason normal bases are highly used in all applications that require taking large powers, particularly in public-key cryptography.

Multiplication requires more computation. If $AB = C = (c_0, c_1, \dots, c_{n-1})$ then the goal is to represent the c_i coefficients in terms of the a_i and b_i coefficients. We

begin by defining the $n \times n$ matrix $T_k = (t_{ij}^{(k)})$ by

$$\alpha_i \alpha_j = \sum_{k=0}^{n-1} t_{ij}^{(k)} \alpha_k \quad t_{ij}^{(k)} \in \mathbb{F}_q. \quad (2.3)$$

Then,

$$\begin{aligned} \alpha\beta &= (a_0\alpha_0 + a_1\alpha_1 + \cdots + a_{n-1}\alpha_{n-1})(b_0\alpha_0 + b_1\alpha_1 + \cdots + b_{n-1}\alpha_{n-1}) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_i \alpha_i b_j \alpha_j \\ &= \sum_{k=0}^{n-1} \sum_{i,j=0}^{n-1} a_i b_j t_{ij}^{(k)} \alpha_k \end{aligned}$$

and the k th coefficient of C is given by

$$c_k = \sum_{i,j=0}^{n-1} a_i b_j t_{ij}^{(k)} = AT_k B^t, \quad 0 \leq k \leq n-1.$$

The problem of multiplication reduces to finding the collection of matrices T_k .

Definition 2.2.6. *The multiplication table of a basis $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is the collection of n matrices T_k as outlined above.*

As the dimension n of the extension increases we notice that the computations required to find the T_k matrices become prohibitive. For this reason we must find bases where we can take advantage of symmetry of the T_k matrices.

Consider our basis to be a normal basis. Then we can take advantage of the exponentiation property in the following way. We take the q^{-l} th power of Equation (2.3)

to find the identity

$$\alpha_{i-l}\alpha_{j-l} = \sum_{k=0}^{n-1} t_{ij}^{(k)} \alpha_{k-l} \quad (2.4)$$

Matching up the coefficient of α_0 in Equations (2.3) and (2.4) gives the identity

$$t_{ij}^{(l)} = t_{i-l, j-l}^{(0)}.$$

Therefore we find that once a computation scheme is built to calculate the first coefficient c_0 , we can simply run the system using $A^{q^{-l}}$ and $B^{q^{-l}t}$ with negligible computation cost as inputs to find c_l , $1 \leq l \leq n-1$. The number of operations required in any such computational scheme is the number of non-zero entries in the matrix T_0 .

Again, let $N = \{\alpha, \alpha_1, \dots, \alpha_{n-1}\}$ be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. We begin by defining the $n \times n$ matrix $T = (t_{ij})$ by the following system

$$\alpha\alpha_i = \sum_{j=0}^{n-1} t_{ij}\alpha_j \quad 0 \leq i \leq n-1 \quad (2.5)$$

Recall, for any $i, j, k \leq n-1$ we have

$$\alpha\alpha_{i-j} = \sum_{l=0}^{n-1} t_{i-j, l}\alpha_l.$$

Taking both sides to the q^j th power we find,

$$\alpha_j \alpha_i = \sum_{l=0}^{n-1} t_{i-j,l} \alpha_{l+j}.$$

Reindexing with $k = l + j$ gives that for any $i, j, k \leq n - 1$

$$t_{ij}^{(k)} = t_{i-j,k-j}. \quad (2.6)$$

In particular, this shows that the number of non-zero entries in the matrix T_0 is equal to the number of non-zero entries in T . Moreover, the collection of matrices $\{T_k\}$ are completely determined by the matrix T .

Definition 2.2.7. *If $N = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ is a normal basis, the matrix $T = (t_{ij})$ defined as in Equation (2.5) is called the multiplication table of N .*

Definition 2.2.8. *Let T be the multiplication table of a normal basis N . Then the complexity of T is the number of non-zero entries of T . The complexity of the normal basis N is denoted c_N . Also, if α generates a normal basis, i.e.: α is a normal element, then the complexity of α is the number of non-zero entries in the multiplication table of the basis generated by α .*

2.3 Optimal Normal Bases

The following theorem is due to Mullin et al., and gives a very important lower bound on the complexity of normal bases.

Theorem 2.3.1. [31] *Let N be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, then $c_N \geq 2n - 1$.*

PROOF. Let N be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ generated by α with multiplication table T . Let

$$\begin{aligned} a = \text{Tr}(\alpha) &= \sum_{i=0}^{n-1} \alpha_i \quad \text{then} \\ a \cdot \alpha &= \sum_{i=0}^{n-1} \alpha \alpha_i \end{aligned}$$

so that the sum of the rows of T is an n -tuple with an a in the first position and 0 elsewhere. Since the rows of T are linearly independent, there must be at least two non-zero terms in each column of T except for the first column. Thus, the total number of non-zero entries of T is at least $2n - 1$. ■

Definition 2.3.2. *Consider $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. The complexity of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, denoted $C_q(n)$, is the minimum complexity of a normal basis in $\mathbb{F}_{q^n}$.*

Definition 2.3.3. *Let N be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ then N is an Optimal Normal Basis (ONB) if $c_N = 2n - 1$.*

Two constructions for optimal normal bases were given by Mullin et al., and it was conjectured that all optimal normal bases were given by these constructions. This

was later proven by Gao [10].

2.3.1 Characterization of Optimal Normal Bases

We begin by giving the constructions shown by Mullin et al.

Theorem 2.3.4. [31] *Let $n + 1$ be a prime in $\mathbb{Z}$ and suppose q is a prime or a prime power such that q is primitive in $\mathbb{Z}_{n+1}$. Then the n non-unit $(n + 1)$ th primitive roots of unity form an optimal normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.*

PROOF. We have $(n + 1)$ is a prime so $\mathbb{Z}_{n+1}$ is a field. If q is primitive in $\mathbb{Z}_{n+1}$ then q generates the multiplicative group of $\mathbb{Z}_{n+1}$ and so $q^n \equiv 1 \pmod{n + 1}$. Therefore $n + 1$ divides $q^n - 1$. Let η be primitive element of $\mathbb{F}_{q^n}$, then let $\beta = \eta^{q^n - 1/(n+1)}$ so that β is a primitive $(n + 1)$ th root of unity and all the conjugates $\beta, \beta^q, \dots, \beta^{q^{n-1}}$ are primitive $(n + 1)$ th roots of unity.

Since q is primitive in $\mathbb{Z}_{n+1}$ the minimal polynomial of β is $(x^{n+1} - 1)/(x - 1)$, with the conjugates as the (linearly independent) roots. This shows that the primitive $(n + 1)$ th roots of unity form a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

Let $\beta_i = \beta^{q^i}$ in the usual convention. If $\beta_i \neq \beta^{-1}$ then $\beta\beta_i = \beta_j$ for some $1 \leq j \neq i \leq n - 1$. If $\beta_i = \beta^{-1}$ then $\beta\beta_i = 1 = -\text{Tr}(\beta)$ and so row i of the multiplication table is $(-1, -1, \dots, -1)$. In total, the multiplication table of this basis has $(n - 1) + n = 2n - 1$ non-zero terms, and therefore is optimal. ■

Theorem 2.3.5. [31] *Let $2n + 1$ be prime and suppose that*

(1) 2 is primitive in $\mathbb{Z}_{2n+1}$, or

(2) $2n + 1 \equiv 3 \pmod{4}$ and 2 generates the quadratic residues in $\mathbb{Z}_{2n+1}$.

Then $\beta = \gamma + \gamma^{-1}$ generates an optimal normal basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$, where γ is a primitive $2n + 1$ root of unity.

PROOF. We begin by restating condition (2): $(2n + 1) \equiv 3 \pmod{4}$ implies that $(2n + 1)$ cannot be written as the sum of two squares. If $a^2 \equiv -1 \pmod{2n + 1}$ for some $a \in \mathbb{Z}_{2n+1}$. Then $a^{(2n+1)-1} = (-1)^{(2n)/2} \equiv -1 \pmod{2n + 1}$. But, by Fermat's little theorem $a^{2n} \equiv 1 \pmod{2n + 1}$. Thus, (-1) is not a quadratic residue $\pmod{2n + 1}$. Also, 2 generates the quadratic residues in $\mathbb{Z}_{2n+1}$ is equivalent to saying that the multiplicative order of 2 $\pmod{2n + 1}$ is n . Thus, $\mathbb{Z}_{2n+1}$ is generated by 2 and -1 .

If the first condition holds, then $2^{2n} \equiv 1 \pmod{2n+1}$ so $2^{2n} - 1 \equiv 0 \pmod{2n+1}$, or $2n+1$ divides $2^{2n} - 1$. If the second condition holds then $2^n = 2^{2n} \equiv 1 \pmod{2n+1}$ and so also $2n + 1$ divides $2^{2n} - 1$.

So, as in the proof of Theorem 2.3.4, in $\mathbb{F}_{2^{2n}}$ there is an element γ that is a primitive $(2n + 1)$ th root of unity in $\mathbb{F}_{2^{2n}}$. We note that $(2^n)^2 = 2^{2n} \equiv 1 \pmod{2n+1}$ so $2^n \equiv \pm 1 \pmod{2n + 1}$. Letting $\beta = \gamma + \gamma^{-1}$ we note that $\gamma^{2^n} = \gamma$ or $\gamma^{2^n} = \gamma^{-1}$. In either case, we have that $\beta^{2^n} = \beta$ so $\beta \in \mathbb{F}_{2^n} \subseteq \mathbb{F}_{2^{2n}}$.

To prove that $N = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ is linearly independent, suppose $\sum_{i=0}^{n-1} c_i \beta_i = 0$.

Then,

$$\sum_{i=0}^{n-1} c_i (\gamma^{2^i} + \gamma^{-2^i}) = \sum_{i=0}^{n-1} c_i \gamma^{2^i} + \sum_{i=0}^{n-1} c_i \gamma^{-2^i} = 0.$$

We recall that $\gamma \in \mathbb{F}_{2^{2n}}$, and 2 generates the multiplicative group of $\mathbb{Z}_{2n+1}$ or 2 generates the quadratic residues in $\mathbb{Z}_{2n+1}$ with $2n+1 \equiv 3 \pmod{4}$. In either case the exponents 2^i and -2^i run through all non-zero residues $j \pmod{2n+1}$ so we have,

$$\sum_{j=1}^{2n} \delta_j \gamma^j = 0$$

and each $c_i \in \{\delta_1, \delta_2, \dots, \delta_{2n}\}$.

Let $f(x) = \sum_{i=0}^{2n-1} \delta_{i+1} x^i$, then γ is a root of f and the minimal polynomial of γ (the $(2n+1)$ th cyclotomic polynomial over $\mathbb{Z}_{2n+1}$), divides f . If condition (1) holds, then $m_\gamma(x) = 1 + x + x^2 + \dots + x^{2n}$. Since m_γ divides f we have $f = 0$ and thus all $c_i = 0$. If condition (2) holds, then the degree of both $m_\gamma(x)$ and $m_{\gamma^{-1}}(x)$ is n . This gives that

$$x^{2n} - 1 = (x - 1)m_\gamma(x)m_{\gamma^{-1}}(x)$$

but γ and γ^{-1} are roots of f , so $m_\gamma(x)m_{\gamma^{-1}}(x) = 1 + x + x^2 + \dots + x^{2n}$ divides f , once again giving $f = 0$ and all $c_i = 0$. This shows linear independence of $N = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$.

Let $N = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ be the normal basis as above with $\beta = \gamma + \gamma^{-1}$. Then $\beta_i = \gamma^{2^i} + \gamma^{-2^i} = \gamma^j + \gamma^{-j}$ for some $1 \leq j \leq n-1$. First, we note that $\beta\beta = \beta^2 \in N$.

Now

$$\begin{aligned}
 \beta\beta_i &= (\gamma + \gamma^{-1})(\gamma^j + \gamma^{-j}) \\
 &= \gamma^{j+1} + \gamma^{-(j+1)} + \gamma^{j-1} + \gamma^{-(j-1)} \\
 &= \beta_k + \beta_l \quad \text{for } 0 \leq k, l \leq n-1.
 \end{aligned} \tag{2.7}$$

To conclude, we note that the first row of the multiplication table has a 1 in the first row, and two 1s in every subsequent row, for a total complexity $c_N = 2n - 1$.

■

Definition 2.3.6. *Let N be the optimal normal basis as described in Theorem 2.3.4, then N is called a Type I Optimal Normal Basis.*

Let N be the optimal normal basis as described in Theorem 2.3.5, then N is called a Type II Optimal Normal Basis.

By analyzing the construction of some optimal normal bases, we have a quick method for determining some other extension fields with optimal normal bases.

Proposition 2.3.7. *Let $\mathbb{F}_{2^n}$ be an extension of $\mathbb{F}_2$ containing a Type I optimal normal basis with n even. Then, $\mathbb{F}_{2^{n/2}}$ contains a Type II optimal normal basis.*

PROOF. Let n be an even positive integer, and let $m = n/2$. If $\mathbb{F}_{2^n}$ contains a Type I Optimal normal basis then $n + 1$ is prime and 2 is primitive modulo $(n + 1)$. This implies that $2m + 1$ is prime, and 2 is primitive modulo $(2m + 1)$, or that $\mathbb{F}_{2^m}$ contains a Type II optimal normal basis. ■

We end this part by giving an equivalence class on normal bases.

Definition 2.3.8. *Suppose $N = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ is a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Then, for any nonzero $a \in \mathbb{F}_q$, $aN = \{a\beta_i : 0 \leq i \leq n-1\}$ is an equivalent basis of N .*

We are concerned with the property that equivalent bases preserve complexity. We show this in the case of optimal normal bases but note that the proof holds for any normal basis.

Proposition 2.3.9. *Suppose $N = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ is an optimal normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Then the equivalent basis $aN = \{a\beta_i : 0 \leq i \leq n-1\}$ is an optimal normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ for any non-zero $a \in \mathbb{F}_q$.*

PROOF. It is clear that the linear independence of the elements of aN follows from the linear independence of N . Also, for β generating N , $a\beta(a\beta)_i = aa^{q^i}\beta\beta_i = a^2\beta\beta_i$ which indicates that the number of nonzero terms of the multiplication table remains the same. ■

2.3.2 Determination of all Optimal Normal Bases

The natural question to ask is: do any other optimal normal bases exist over finite fields? In 1988, Mullin et al. [31] performed exhaustive computer searches over $\mathbb{F}_{2^n}$ for $n \leq 30$, and no new optimal normal bases were found. We present the original proof of Gao: that Theorems 2.3.4, 2.3.5 and their equivalent bases describe all optimal

normal bases in finite field extensions. Later, Gao and Lenstra [12] generalized the notion of an optimal normal basis to any finite Galois extension of fields.

Theorem 2.3.10. [10] *Let $N = \{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\}$ be an optimal normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ and let $b = \text{Tr}_{q^n/q}(\alpha)$. Then either*

1. *$n + 1$ is prime, q primitive in $\mathbb{Z}_{n+1}$ and $-\alpha/b$ is a primitive $(n + 1)$ th root of unity; or*
2. (a) *$q = 2^\nu$ for some integer ν such that $\gcd(\nu, n) = 1$,*
 (b) *$2n + 1$ is a prime, 2 and -1 generate the multiplicative group $\mathbb{Z}_{2n+1}^*$, and*
 (c) *$\alpha/b = \gamma + \gamma^{-1}$ for some primitive $(2n + 1)$ th root of unity γ .*

Remark. *We note that this is the sufficiency condition that any optimal normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is equivalent to a Type I or Type II optimal normal basis as introduced above. We also note, for the case $q = 2$, that this ensures uniqueness of optimal normal bases of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$.*

PROOF. Let $N = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be an optimal normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ with $n \times n$ multiplication table $T = (t_{ij})$ and let $M = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ be the dual (normal) basis of N (see Section 2.6) with multiplication table $D = (d_{ij})$. By Lemma 2.6.5 we have $D = T^t$. Also by Equation (2.6) we have $d_{ij} = d_{i-j, -j}$.

By the proof of Theorem 2.3.1 we have that each column of D has exactly two non-zero entries which are additive inverses except for the first column, which has

only one non-zero entry with value $b = \text{Tr}(\alpha)$. Therefore

$$\alpha\beta_i = a\beta_j - a\beta_k; \quad a \in \mathbb{F}_q, 0 \leq i \neq j \neq k \leq n-1, \text{ and} \quad (2.8)$$

$$\alpha\beta_0 = b\beta_m; \quad \text{for } 0 \leq m \leq n-1. \quad (2.9)$$

Replacing α with α/b and β with $b\beta$ without loss of generality we may assume that

$\text{Tr}(\alpha) = -1$ and $\alpha\beta_0 = -\beta_m$. By the proof of Theorem 2.3.1 we have

$$\text{Tr}(\alpha)\text{Tr}(\beta) = \sum_{i,j} \alpha_i\beta_j = \sum_{k=0}^{n-1} \alpha\beta_k = 1$$

and so $\text{Tr}(\beta) = -1$.

If $m = 0$ then $\alpha\beta_0 = -\beta_m$ and $\alpha = -1$ which gives $n = 1$. This covers the trivial case so for the remainder of the proof assume $m > 0$.

Suppose $2m \equiv 0 \pmod{n}$. Then

$$(\alpha\beta_0)^{q^m} = \alpha_m\beta_m = -\beta_{2m} = -\beta_0 = \beta_m/\alpha.$$

Rearranging gives

$$\alpha\alpha_m = 1 = -\text{Tr}(\alpha) = \sum_{i=0}^{n-1} -\alpha_i$$

so for each $i \neq 0$, there is a unique $s \neq m$ such that $\alpha\beta_i = \beta_s - \beta_m$. Define

$$\phi : \{1, 2, \dots, n-1\} \rightarrow \{0, 1, \dots, n-1\} - \{m\}$$

by $\phi(i) = s$ as above. If $i \neq j$ then $\alpha\beta_i \neq \alpha\beta_j$; so if $t \neq m$ is the unique integer such that $\alpha\beta_j = \beta_t - \beta_m$ we have $s \neq t$ or $\phi(i) \neq \phi(j)$. This shows that ϕ is a bijection. This, along with $\alpha\alpha_m = 1$ shows that $S = \{1\} \cup \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ is closed under multiplication by α . The set S is also closed under the Frobenius map and we find S is a multiplicative group of order $n+1$. Then $\alpha^{n+1} = 1$, $\alpha \neq 1$ and so α is a root of $x^{n+1} - 1/(x-1) = x^n + x^{n-1} + \dots + x + 1$. Since α is of degree n over $\mathbb{F}_q$ we have $x^n + x^{n-1} + \dots + x + 1$ is irreducible, which occurs if and only if $n+1$ is prime, see [6, Section 13.6]. This means that N is equivalent to a Type I Optimal normal basis.

Now suppose $m \neq 0$ and $2m \not\equiv 0 \pmod{n}$. What remains is to show that N is equivalent to a Type II Optimal normal basis. Recall if $i = 0$ then $\alpha\beta_i = \alpha\beta_0 = -\alpha_m$ and so $d_{0,m} = -1$ and $d_{0,i} = 0$ for all other i . Using Equation (2.6) gives us that

$$d_{-i,-i} = \begin{cases} -1 & \text{if } i = m \\ 0 & \text{if } i \neq m \end{cases}$$

So $\alpha\beta_{-m} = \beta_l - \beta_{-m}$ for some $0 \leq l \leq n-1$.

To show that $\mathbb{F}_q$ is of characteristic two, we consider

$$\alpha_m(\alpha\beta_0) = -\alpha_m\beta_m = -(\alpha\beta)^{q^m} = \beta_{2m}.$$

However, viewing this product another way,

$$\alpha(\alpha_m\beta_0) = \alpha(\alpha\beta_{-m})^{q^m} = \alpha(\beta_l - \beta_{-m})^{q^m} = \alpha\beta_{l+m} - \alpha\beta_0 = \alpha\beta_{l+m} + \beta_m.$$

Combining these two expressions gives

$$\alpha\beta_{l+m} = \beta_{2m} - \beta_m. \quad (2.10)$$

We calculate $\alpha\alpha_l\beta_{-m}$ also in two ways. We have that $d_{-m,l} = 1 = d_{-(l+m),-l}$ and $l+m \neq 0$ so that

$$\alpha\beta_{-(l+m),-l} = \beta_{-l} - \beta_j$$

for some $j \neq -l, -(l+m)$. We proceed as above by calculating

$$\begin{aligned} \alpha_l(\alpha\beta_{-m}) &= \alpha_l(\beta_l - \beta_{-m}) = (\alpha\beta_0 - \alpha\beta_{-(l+m)})^{q^l} \\ &= (-\beta_m - \beta_{-l} + \beta_j)^{q^l} = -\beta_{l+m} - \beta_0 + \beta_{l+j}. \end{aligned} \quad (2.11)$$

We also calculate the product as

$$\begin{aligned} \alpha(\alpha_l\beta_{-m}) &= \alpha(\alpha\beta_{-(l+m)})^{q^l} = \alpha(\beta_{-l} - \beta_j)^{q^l} \\ &= \alpha\beta_0 - \alpha\beta_{l+j} = -\beta_m - \alpha\beta_{l+j}. \end{aligned} \quad (2.12)$$

Comparing Equations 2.11 and 2.12 gives the identity

$$\alpha\beta_{l+j} = \beta_0 + \beta_{l+m} - \beta_m - \beta_{l+j}. \quad (2.13)$$

We note that β generates an optimal normal basis, so $\alpha\beta_{l+j}$ has at most two nonzero terms. Since $l+j \neq 0$, β_{l+j} must equal one of β_{l+m} and $-\beta_m$.

Suppose then, that $\beta_{l+j} = \beta_{l+m}$ i.e: that $l+j = l+m$. Then $\alpha\beta_{l+j} = \beta_0 - \beta_m$ and also by Equation (2.10) $\alpha\beta_{l+j} = \beta_{2m} - \beta_m$. Combining these give $\beta_{2m} = \beta_0$, and therefore $2m \equiv 0 \pmod{n}$, contradicting the assumption.

In the second case, $\beta_{l+j} = -\beta_m$ and since β_{l+j} and β_m are basis elements, $l+j = m$ and so $2 = 0$. This shows we are in characteristic two, i.e: $q = 2^\nu$.

By Equation (2.13) we have $\alpha\beta_m = \beta_{l+m} + \beta_0$. We note that Equations (2.8) and (2.9) reduce to $\alpha\beta_0 = \beta_m$ and $\alpha\beta_{-m} = \beta_l + \beta_{-m}$ in the case of characteristic two. Raising Equation (2.9) to the q^m th power we find $\alpha_m\beta_0 = \beta_{l+m} + \beta_0 = \alpha\beta_m$. Combining this expression with Equation (2.8) we find $\alpha^2 = \alpha_m$. Inductively, for every non-negative integer k , we have $\alpha^{2^k} = \alpha^{q^{mk}}$.

We pick $k = n/\gcd(m, n)$ to find $\alpha^{2^k} = \alpha$, or $\alpha \in \mathbb{F}_{2^k}$ and so of degree at most $k \leq n$ over $\mathbb{F}_2 \subseteq \mathbb{F}_q$. Since α is of degree n over $\mathbb{F}_q$ it must have degree at least n over $\mathbb{F}_2$ and thus $k = n$. Since α has the same degree over $\mathbb{F}_2$ as $\mathbb{F}_q$, with $q = 2^\nu$, we have ν and n are relatively prime and the conjugates of α over $\mathbb{F}_q$ are $\{\alpha, \alpha^2, \dots, \alpha^{2^{n-1}}\}$. This proves the first condition of the second hypothesis in the statement.

Let m_1 be a positive integer such that $mm_1 \equiv 1 \pmod{n}$. Since $\alpha_m\beta = \alpha\beta_m$ we

have $\alpha\beta^{-1} = \alpha_m\beta_m^{-1}$, and raising the expression to the q^m th power we find $\alpha_m\beta_m^{-1} = \alpha_{2m}\beta_{2m}^{-1}$. Inductively we have

$$\alpha_{mm_1}\beta_{mm_1}^{-1} = \alpha\beta^{-1} = \alpha^q\beta^{-q},$$

and so $\alpha\beta^{-1} \in \mathbb{F}_q$. We have

$$\text{Tr}(\alpha) = 1 = \text{Tr}(\alpha\beta^{-1}\beta) = \alpha\beta^{-1}\text{Tr}(\beta)$$

and so $\alpha = \beta$ since $\text{Tr}(\beta) = 1$ (as a corollary, upon completion of the proof this shows that all bases equivalent to Type II Optimal normal bases are self-dual, see Section 2.6).

Let γ be a root of the polynomial $x^2 - \alpha x + 1$ considered in $\mathbb{F}_{q^{2n}}$ over $\mathbb{F}_{q^n}$ such that $\gamma + \gamma^{-1} = \alpha$. Let the order of γ be $2t+1$ for some t . For any integer i let $\eta_i = \gamma^i + \gamma^{-i}$. Then $\eta_i = \eta_j$ if and only if $i \equiv \pm j \pmod{2t+1}$ and thus there are exactly t non-zero distinct values of η . Any conjugate of α is of the form $\alpha^{2^j} = \gamma^{2^j} + \gamma^{-2^j} = \eta_{2^j}$ where η_{2^j} is one of $\eta_1, \eta_2, \dots, \eta_t$. Thus, the degree of α , n , is at most t .

To show that $n = t$, observe that $\eta_0 = 0$, $\eta_1 = \alpha$ and $\eta_2 = \alpha^2$ so consider i such that $3 \leq i \leq t$. We show by induction that η_i is a conjugate of α . Consider

$$\alpha\eta_{i-2} = (\gamma + \gamma^{-1})(\gamma^{i-2} + \gamma^{-(i-2)}) = \eta_{i-1} + \eta_{i-3},$$

with η_{i-1} conjugate to α and either η_{i-3} conjugate to α (for $i \neq 3$) or $\eta_{i-3} = 0$. The

latter case proves the assertion, and the former case gives that η_{i-1} occurs in the normal basis N with a coefficient of 1. Since $\beta = \alpha$ we have that the multiplication tables T and D are equal and symmetric. Therefore under a reordering of bases we have that $\alpha\eta_{i-1}$ is the sum of η_{i-2} and another conjugate of α . Since $\alpha\eta_{i-1} = \eta_{i-2} + \eta_i$, this gives that η_i is a conjugate of α and thus every $\eta_1, \eta_2, \dots, \eta_t$ are conjugates of α .

For each integer i not divisible by $2n + 1 = 2t + 1$ there is an integer j such that $i \equiv \pm 2^j \pmod{2n + 1}$. Since $2n + 1$ is odd, each i is relatively prime to $2n + 1$ and so $2n + 1$ is prime, and $\mathbb{Z}_{2n+1}$ is generated by 2 and -1 . This shows that the Optimal normal basis N satisfies the second assertion in the statement; i.e: that N is equivalent to a Type II Optimal normal basis. ■

2.4 Distribution of Normal Bases

The following theorem, due to Hensel [19], gives a criterion for determining if a field element is normal. This method is used extensively in Chapter 3. To prove this result we introduce a certain class of matrices whose structure is related to powering normal elements.

Definition 2.4.1. *Let $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}_{q^n}$, the circulant matrix on $\alpha_1, \alpha_2, \dots, \alpha_n$ is*

given by

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_n & \alpha_1 & \cdots & \alpha_{n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_2 & \alpha_3 & \cdots & \alpha_1 \end{pmatrix}. \quad (2.14)$$

That is, each row in Equation (2.14) is a cyclic shift of the previous row.

Lemma 2.4.2. *Let A be the circulant matrix,*

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_n & \alpha_1 & \cdots & \alpha_{n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_2 & \alpha_3 & \cdots & \alpha_1 \end{pmatrix},$$

then A is invertible if and only if $\sum_{i=1}^n \alpha_i x^i$ and $x^n - 1$ are relatively prime.

PROOF. Define the $n \times n$ matrix P by

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and notice that

$$A = \sum_{i=1}^n \alpha_i P^i.$$

Define $f(x) = \sum_{i=1}^n \alpha_i x^i$ then $A = f(P)$. By cofactor expansion we observe that the characteristic polynomial of P is $x^n - 1$. The matrix P is a permutation matrix of order n and the minimal polynomial of P is $x^n - 1$. Suppose f is relatively prime to $x^n - 1$. Then there are polynomials $a, b \in \mathbb{F}_q[x]$ such that

$$f(x)a(x) + (x^n - 1)b(x) = 1.$$

Evaluating at P and applying the Cayley-Hamilton theorem gives $f(P)a(P) = 1$.

That is, $A \cdot a(P) = 1$ and A is invertible.

Suppose to the contrary that $f(x) = \sum_{i=1}^n \alpha_i x^i$ and $x^n - 1$ have a common divisor $d \in \mathbb{F}_q[x]$. Then $f(x) = g(x)d(x)$ and $x^n - 1 = h(x)d(x)$. Evaluating at P we have $h(P)d(P) = 0$, but since the degrees of h, d are less than n , we have that $h(P), d(P) \neq 0$ and so $d(P)$ is not invertible. This implies that $A = g(P)d(P)$ is not invertible. ■

Theorem 2.4.3. *For any $\alpha \in \mathbb{F}_{q^n}$, α is a normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if and only if $\alpha^{q^{n-1}}x^{n-1} + \alpha^{q^{n-2}}x^{n-2} + \cdots + \alpha^q x + \alpha$ and $x^n - 1$ are relatively prime.*

PROOF. By definition, $\alpha \in \mathbb{F}_{q^n}$ is a normal element if and only if $\alpha, \alpha^q, \alpha^{q^2}, \dots, \alpha^{q^{n-1}}$

are linearly independent. By Corollary 2.1.23, this occurs if and only if the matrix

$$\begin{pmatrix} \alpha & \alpha^q & \cdots & \alpha^{q^{n-1}} \\ \alpha^q & \alpha^{q^2} & \cdots & \alpha \\ \vdots & \vdots & & \vdots \\ \alpha^{q^{n-1}} & \alpha^{q^{n-2}} & \cdots & \alpha \end{pmatrix} \quad (2.15)$$

is non-singular.

It is clear that Equation (2.15) forms a circulant matrix. By Lemma 2.4.2, this is non-singular if and only if $\alpha^{q^{n-1}}x^{n-1} + \alpha^{q^{n-2}}x^{n-2} + \cdots + \alpha^q x + \alpha$ and $x^n - 1$ are relatively prime. ■

We state without proof an important corollary we require to prove the structure of a dual element. The proof is quite short but requires some results about circulant matrices beyond the scope of this work; see [10, 22] for a rigorous discussion of these results.

Corollary 2.4.4. *Let $\alpha \in \mathbb{F}_{q^n}$, $\alpha_i = \alpha^{q^i}$ and let $t_i = \text{Tr}(\alpha_0 \alpha_i)$. Then α generates a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if and only if $h(x) = t_{n-1}x^{n-1} + \cdots + t_1x + t_0$ and $x^n - 1$ are relatively prime.*

To establish the number of normal bases, we define an analog of Euler's ϕ function for polynomials.

Definition 2.4.5. [6] *Let $f \in \mathbb{F}_q[x]$ be a polynomial of degree m . Then $\Phi(f)$ denotes the number of polynomials of degree less than m that are relatively prime to f .*

From Theorem 2.4.3, it is clear that the number of normal elements in a finite field $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is $\Phi(x^n - 1) \in \mathbb{F}_q[x]$. Since each normal element and its conjugates form the same basis, the number of normal bases of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is $\frac{1}{n}\Phi(x^n - 1)$. An explicit expression for the number of normal elements was established by Ore.

Theorem 2.4.6. [22] *Let q be a power of the prime p , let n be a positive integer and write $n = p^a m$, where p does not divide m . Then the number of normal bases of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ equals*

$$\frac{1}{n}\Phi_q(x^n - 1) = \frac{q^n}{n} \prod_{d|m} (1 - q^{-o_d(q)})^{\phi(d)/o_d(q)},$$

where $\Phi_q(f)$ is the number of polynomials of degree smaller than the degree of f which are relatively prime to f , $o_n(a)$ is the order of a modulo n , and $\phi(d)$ is the number of positive integers smaller than d that are relatively prime to d .

Several authors have provided computationally feasible lower and upper bounds for the number of normal bases of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Here, we present Gao and Panario's bounds [11]:

$$\frac{q^n}{e^{0.83}n(1 + \log_q n)} \leq \frac{1}{n}\Phi_q(x^n - 1) \leq \frac{q^n}{e^{\gamma - c_q}n\sqrt{1 + \log_q n}}, \quad (2.16)$$

where $\gamma = 0.577216\dots$ is Euler's constant and $c_q = q/(q-1)(\sqrt{q}-1)$; see also [9]. Von zur Gathen and Giesbrecht [16] showed that the probability that an arbitrary element in $\mathbb{F}_{q^n}$ forms a normal basis is larger than $1/(16 \log_q n)$.

2.5 Constructions of Normal Bases

Though normal bases are relatively dense in finite fields (see Chapter 3), it is non-trivial to find normal bases with prescribed properties, such as low-complexity. By constructing new normal bases from other known normal bases, we can inherit some of the desired properties from the known basis.

As the proof of Theorem 2.3.5 uses a projection from $\mathbb{F}_{2^{2n}}$ to $\mathbb{F}_{2^n}$, the following theorem works in reverse and allows us to construct new normal bases with known complexity over particular extensions using previously known normal bases.

Theorem 2.5.1. [10, 35] *Let $n = mk$, all positive integers with m, k relatively prime. Let $\alpha \in \mathbb{F}_{q^m}$ and $\beta \in \mathbb{F}_{q^k}$ be normal elements over $\mathbb{F}_q$. Then $\gamma = \alpha\beta$ is a normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, with complexity equal to the product of the complexities of α and β .*

PROOF. Define α, β, γ as in the hypothesis. We must first show that the conjugates of γ are linearly independent. We have $\gamma^{q^i} = \alpha^{q^i} \beta^{q^i}$. By the Chinese Remainder theorem, since $\gcd(m, k) = 1$ there exist unique positive integers λ, μ such that $i \equiv \lambda \pmod{m}$ and $i \equiv \mu \pmod{k}$, so $\gamma^{q^i} = \alpha^{q^\lambda} \beta^{q^\mu}$. Therefore, the conjugates of γ are $\alpha^{q^s} \beta^{q^t}$ with $1 \leq s \leq m$, and $1 \leq t \leq k$. Suppose there are constants $a_{st} \in \mathbb{F}_q$ such that

$$\sum_{s,t} a_{st} \alpha^{q^s} \beta^{q^t} = 0$$

and define $b_t = \sum_{s=0}^{m-1} a_{st} \alpha^{q^s} \in \mathbb{F}_{q^m}$. Then

$$\sum_{t=0}^{k-1} b_t \beta^{q^t} = 0$$

and we require the conjugates of β to be linearly independent over $\mathbb{F}_{q^m}$.

For any integer j ,

$$\left(\sum_{t=0}^{k-1} b_t \beta_t \right)^{q^{jm}} = \sum_{t=0}^{k-1} b_t^{q^{jm}} \beta_t^{q^{jm}} = \sum_{t=0}^{k-1} b_t \beta_t^{q^{jm}}.$$

Since m and k are relatively prime, as j runs from $0, 1, \dots, k-1$, then jm runs from $0, 1, \dots, k-1 \pmod{k}$. But this means that $\sum_{t=0}^{k-1} b_t \beta_t^{q^j} = 0$ for $0 \leq j \leq k-1$. We construct the $k \times k$ matrix in expression Equation (2.15) using $(\beta_0, \beta_1, \dots, \beta_{k-1})$ as the first row. The matrix is invertible since $\beta_0, \beta_1, \dots, \beta_{k-1}$ are linearly independent over $\mathbb{F}_q$. So $b_0 = b_1 = \dots = b_{k-1} = 0$ and $\beta_0, \beta_1, \dots, \beta_{k-1}$ are linearly independent over $\mathbb{F}_{q^m}$.

Since $b_t = 0$ for $0 \leq t \leq k-1$ we find

$$\sum_{s=0}^{m-1} a_{st} \alpha^{q^s} = 0.$$

The conjugates of α are linearly independent over $\mathbb{F}_q$ and therefore $a_{st} = 0$ for all $0 \leq s \leq m-1, 0 \leq t \leq k-1$, proving independence of the conjugates of γ .

Now, to show that the complexity of γ is the product of the complexities of α and

β , let $T_\alpha = (t_{ij}), T_\beta = (s_{uv})$ be the multiplication table of α, β respectively. Then $\gamma\gamma^{q^i} = (\alpha\beta)\alpha^{q^i}\beta^{q^i}$ by the Chinese Remainder theorem, as above. Using the common notation we find

$$\begin{aligned}\gamma\gamma_i &= \alpha\alpha_\lambda \cdot \beta\beta_\mu = \left(\sum_{j=0}^{m-1} t_{\lambda j} \alpha_j \right) \left(\sum_{v=0}^{k-1} s_{\mu v} \beta_v \right) \\ &= \sum_{j=0}^{m-1} \sum_{v=0}^{k-1} t_{\lambda j} s_{\mu v} \alpha_j \beta_v.\end{aligned}\tag{2.17}$$

Reapplying the Chinese Remainder theorem as above, there exists $\gamma_l = \alpha_j \beta_v$ for some $0 \leq l \leq n-1$ and for all $0 \leq j \leq m-1$ and $0 \leq v \leq k-1$ such that Equation (2.17) becomes

$$\gamma\gamma_i = \sum_{j=0}^{m-1} \sum_{v=0}^{k-1} t_{\lambda j} s_{\mu v} \gamma_l.$$

Consequently, the number of non-zero terms in the multiplication table of the basis generated by γ is the product of the number of non-zero terms in T_α and T_β . ■

Corollary 2.5.2. *Let $\mathbb{F}_{q^n}$ and $\mathbb{F}_{q^m}$ be finite extensions of $\mathbb{F}_q$ where $\gcd(m, n) = 1$.*

Then

$$C_q(mn) \leq C_q(m)C_q(n).$$

If $\mathbb{F}_{q^n}$ and $\mathbb{F}_{q^m}$ contain optimal normal bases then

$$C_q(mn) \leq 4mn - 2(m + n) + 1.$$

We present a more general theorem due to Ash, Blake and Vanstone for $q = 2$,

and independently proved by Beth, Geiselmann and Meyer for general q . The authors give a way of constructing a normal element with a low complexity, under certain conditions. We state the result for general q and follow with specific results for $q = 2$.

Theorem 2.5.3. [1, 29] *Let k and n be positive integers. Assume $kn + 1$ is a prime, and let γ be a primitive k th root of unity in $\mathbb{F}_{kn+1}$. If q and γ generate the multiplicative group $\mathbb{F}_{kn+1}^*$, and if ξ is a primitive $(kn + 1)$ th root of unity in $\mathbb{F}_{q^{kn}}$, then*

$$\alpha = \xi + \xi^\gamma + \cdots + \xi^{\gamma^{k-1}}$$

generates a normal basis N of $\mathbb{F}_{q^n}/\mathbb{F}_2$ with complexity $c_N \leq (k + 1)n - k$. If n is a multiple of k , this bound can be improved to $c_N \leq kn - 1$.

Specific results can be obtained from Theorem 2.5.3 for the case $q = 2$. We present some bounds, the lower bounds due to Ash, Blake and Vanstone and the upper bounds due to Beth, Geiselmann and Meyer.

Theorem 2.5.4. [22] *Suppose $q = 2$ in Theorem 2.5.3. Then*

$$\begin{aligned} (k + 1)n - (k^2 + k + 1) &\leq c_N \leq (k + 1)n - 2k + 1, & \text{if } k \text{ is odd;} \\ kn - (k^2 - 3k + 3) &\leq c_N \leq kn - k + 1, & \text{if } k \text{ is even.} \end{aligned}$$

Ash, Blake and Vanstone also showed that the lower bounds in the above theorem are quite often sharp. More precisely, it holds for all sufficiently large n as long as k is a power of 2, an odd prime, twice an odd prime, or four times an odd prime.

Meyer showed that for $k \leq 7$ the lower bounds hold with just four exceptions.

For $q = 2$ and specific n, k the bases constructed using this theorem have prescribed complexity. Some results are presented below.

Theorem 2.5.5. [22] *Suppose $q = 2$ in Theorem 2.5.3. Then for specific values of n, k we find*

$$c_N = \begin{cases} 4n - 7, & \text{for } k = 3 \text{ and } k = 4; \\ 6n - 21, & \text{for } k = 5 \text{ and } n > 2, \text{ and for } k = 6 \text{ and } n > 6; \\ 8n - 43, & \text{for } k = 7 \text{ and } n > 4. \end{cases}$$

We present direct constructions of low complexity normal bases of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ by Blake, Gao and Mullin using the theory of linear fractional forms. These results are mainly interesting for large q .

Theorem 2.5.6. [3] *For every $a, \beta \in \mathbb{F}_q^*$ with $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\beta) = 1$,*

$$x^p - \frac{1}{\beta}ax^{p-1} - \frac{1}{\beta}a^p,$$

is irreducible over $\mathbb{F}_q$ and its roots form a normal basis of $\mathbb{F}_{q^p}$ over $\mathbb{F}_q$ with complexity at most $3p - 2$.

Theorem 2.5.7. [3] *Let n be any factor of $q - 1$. Let $\beta \in \mathbb{F}_q$ with multiplicative order*

t such that n is relatively prime to $\frac{q-1}{t}$ and let $a = \beta^{(q-1)/n}$. Then

$$x^n - \beta(x - a + 1)^n$$

is irreducible over $\mathbb{F}_q$ and its roots form a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ with complexity at most $3n - 2$.

We present some new constructions of low complexity normal bases using projections from optimal extension fields in Chapter 4.

2.6 The Dual of a Normal Basis

We begin by giving the classical definition of a dual vector space.

Definition 2.6.1. *Let V be a vector space over a field F . The dual space of V is the space of all linear transformations from V into F , denoted $\text{Hom}_F(V, F)$*

We present the dual basis for the case of finite fields. Many of the results hold more generally over any finite dimensional vector space; for a discussion of these results, see [20].

Definition 2.6.2. *Let $N = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ and let $M = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ be another basis. Then M is the dual basis of N if $\text{Tr}(\alpha_i \beta_j) = \delta_{ij}$ for $0 \leq i, j \leq n - 1$, where δ is the Kronecker delta function. A basis is called self-dual if it is a dual of itself, i.e. N is self-dual if $\text{Tr}(\alpha_i \alpha_j) = \delta_{ij}$ for all i, j .*

Theorem 2.6.3. *Let $N = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ and let $M = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ be a dual basis of N . Then M is the unique dual basis of N .*

PROOF. Let $N = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ and $M = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ be a dual basis of N for the dual vector space $\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$. Then M consists of the n linear mappings from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$ defined by $\beta_i(\alpha_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta function. By Theorem 2.1.19 there is an isomorphism from $\mathbb{F}_{q^n}$ to the homomorphisms of $\mathbb{F}_{q^n}$ into $\mathbb{F}_q$. Define γ_i to be the pre-image of β_i under this isomorphism, $0 \leq i \leq n-1$, and let $C = \{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$. Since isomorphisms preserve linear independence, C is a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, and the construction of these isomorphisms guarantees $\text{Tr}(\beta_i \gamma_j) = \delta_{ij}$ for $0 \leq i, j \leq n-1$. This uniquely determines C , and therefore uniquely determines M . ■

Theorem 2.6.4. *Let $N = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, and $M = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ be the dual basis of N . Then M is a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.*

PROOF. We have that M is a basis, so it remains to show that M is generated by the conjugates of β_0 . We show, by a similar argument to the above, that the i th conjugate of β_0 induce the same linear mapping from $\mathbb{F}_{q^n}$ onto $\mathbb{F}_q$ as β_i . By Theorem 2.1.19 this guarantees that they are the same element.

Since M is a dual basis we have $\text{Tr}(\alpha_i \beta_j) = \delta_{ij}$ for $0 \leq i, j \leq n-1$. Consider the

special case $j = 0$, and apply the Frobenius automorphism ϕ_q to obtain

$$\mathrm{Tr}(\beta_{i+1}\gamma_0^q) = \mathrm{Tr}((\beta_i\gamma_0)^q) = \mathrm{Tr}(\beta_i\gamma_0) = \delta_{i0}$$

for $0 \leq i \leq n-1$. This shows that γ_1 and γ_0^q induce the same mapping. By reapplying this argument for the first $n-1$ powers of the Frobenius automorphism this proves $\gamma_j = \gamma_0^{q^j}$ for $0 \leq j \leq n-1$. ■

Lemma 2.6.5. *Let $N = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ with multiplication table $T = (t_{ij})$ and let $M = \{\beta_0, \beta_1, \dots, \beta_{n-1}\}$ be the dual basis of N with multiplication table $D = (d_{ij})$. Then $(d_{ij}) = (t_{ji})$.*

PROOF. By definition of the dual basis, we have

$$\mathrm{Tr}(\alpha_i\beta_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

. We consider the quantity

$$\mathrm{Tr}(\alpha\beta_i\alpha_k) = \mathrm{Tr}\left(\sum_{j=0}^{n-1} d_{ij}\beta_j\alpha_k\right) = \sum_{j=0}^{n-1} d_{ij}\mathrm{Tr}(\beta_j\alpha_k) = d_{ik}$$

but we also find

$$\mathrm{Tr}(\alpha\beta_i\alpha_k) = \mathrm{Tr}\left(\sum_{j=0}^{n-1} t_{kj}\alpha_j\beta_i\right) = \sum_{j=0}^{n-1} t_{kj}\mathrm{Tr}(\alpha_j\beta_i) = t_{ki}.$$

This shows that the multiplication table for the dual basis is the transpose of the multiplication table of the normal basis. ■

Gao showed that the dual basis of a normal basis can be computed explicitly. This result is used in Chapter 4 to determine the complexity of a given basis.

Theorem 2.6.6. [10] *Let $N = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ and let $t_i = \text{Tr}(\alpha_0 \alpha_i)$. Let $h(x) = t_{n-1}x^{n-1} + \dots + t_1x + t_0$ and let $g(x) = d_0 + d_1x + \dots + d_{n-1}x^{n-1}$ be the unique monic polynomial of degree less than n such that*

$$g(x)h(x) \equiv 1 \pmod{x^n - 1}.$$

The dual basis of N is generated by the element

$$\gamma = d_0\alpha_0 + d_1\alpha_1 + \dots + d_{n-1}\alpha_{n-1}.$$

PROOF. Define $N, h(x), g(x), \gamma$ as in the hypothesis. Corollary 2.4.4 gives that h is invertible $\pmod{x^n - 1}$, so g is well-defined. The condition that $g(x)h(x) \equiv 1 \pmod{x^n - 1}$ gives

$$\sum_{k=0}^{n-1} d_k t_{i-k} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

and we find for any $0 \leq j \leq n-1$ with $\gamma_j = \gamma^{q^j}$

$$\begin{aligned} \text{Tr}(\alpha_i \gamma_j) &= \text{Tr} \left(\alpha_i \sum_{k=0}^{n-1} d_k \alpha_{k+j} \right) = \sum_{k=0}^{n-1} d_k \text{Tr}(\alpha_i \alpha_{k+j}) \\ &= \sum_{k=0}^{n-1} d_k \text{Tr}(\alpha_{i-j-k} \alpha_0) = \sum_{k=0}^{n-1} d_k t_{i-j-k} = \delta_{ij}. \end{aligned}$$

Thus, the conjugates of γ form the dual basis of N . ■

We end this section by presenting some results about self-dual normal bases which are useful when considering the results in Chapter 3.

Proposition 2.6.7. *The multiplication table of a self-dual normal basis is symmetric.*

Furthermore, any Type II Optimal normal basis of F_{2^n} over $\mathbb{F}_2$ is self-dual.

This is a consequence of the proof of Theorem 2.3.10.

Theorem 2.6.8. [22] *A self-dual normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ exists if and only if either q is even and n is not a multiple of 4 or both q and n are odd.*

As a consequence, there is no self dual normal basis over $\mathbb{F}_{2^n}$ if 4 divides n . If $n \equiv 2 \pmod{4}$, the next two results show how to obtain a self-dual normal basis over $\mathbb{F}_{2^n}$, provided that a basis of the same type is given, and how their complexities are related.

Proposition 2.6.9. [22] *Let $\alpha \in \mathbb{F}_{2^n}$, where n is even, and put $\gamma := 1 + \alpha$. Then α generates a self-dual normal basis if and only if γ does.*

Proposition 2.6.10. [22] *Let α generate a self-dual normal basis B for $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$, and assume that n is even. Put $\gamma := 1 + \alpha$, and let B^C be the self-dual normal basis generated by γ . Then the complexities of B and B^C are related as follows*

$$C_{B^C} = n^2 - 3n + 8 - C_B.$$

Corollary 2.6.11. [22] *For $n \equiv 2 \pmod{4}$, the average complexity of a self-dual normal basis for $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ is $\frac{1}{2}(n^2 - 3n + 8)$. Also, if c_B is the complexity of a self-dual normal basis B then*

$$2n - 1 \leq c_B \leq n^2 - 5n + 9.$$

Equality holds in one of these bounds if and only if either B or B^C is optimal.

Chapter 3

Distribution of Complexities

We know that optimal normal bases do not exist in every finite field $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, see Section 2.3.2. In these fields, the problem of finding the lowest complexity normal bases still remains. Ideally, we would like to find an upper-bound with the degree n of the extension on the minimum complexity of an element in a finite field.

For a better understanding of the behavior of the complexities of normal elements in $\mathbb{F}_{2^n}$, tables summarizing the complexity distribution are important tools. In [22], Jungnickel provides a table with minimum and maximum complexities of normal elements in $\mathbb{F}_{2^n}$, for each n smaller than or equal to 30. He also provides a table due to Geiselmann [17] with the lowest complexities found via free polynomials.

In this chapter, we present the distribution of the complexity of normal elements for $n \leq 39$. We gather, for each n in that range, the frequency of each possible complexity value. Summary tables with minimum, maximum, average and variance values for the complexities of normal elements and self-dual normal elements in $\mathbb{F}_{2^n}$

are found in Appendix B. Our data is obtained via an exhaustive search in $\mathbb{F}_{2^n}$ which is substantially sped up by visiting the finite field elements in a Gray code order. An extension of the table of the smallest known complexities is given in Appendix C for other practical values of interest ($40 \leq n \leq 607$), using a combination of known results and a random search.

3.1 Exhaustive Search of Normal Bases

In this section, we describe and analyze two variants of an exhaustive algorithm for computing the complexity of each normal element in $\mathbb{F}_{q^n}$. Algorithm `StandardNCD` (for “Standard Normal Complexity Distribution”) is the simplest variant. Algorithm `GrayCodeNCD` (for “Gray Code Normal Complexity Distribution”), the second variant, uses Gray codes in order to efficiently update the current finite field element and its conjugates. We use basic operations in $\mathbb{F}_q$ as our time complexity measure. We show that asymptotically the Gray code variant reduces the time complexity of the standard algorithm by 28.6%. We present the algorithms and their analyses for $q = 2$, but their extensions to general q are straightforward.

3.1.1 Description of the Algorithms

We now describe Algorithm `StandardNCD`; its pseudocode appears below. At each step, we compute $\alpha \in \mathbb{F}_{2^n}$, represented as a polynomial over $\mathbb{F}_2$ with coefficients stored in the tuple $\alpha_{\text{coeff}} = [a_{n-1}, \dots, a_0]$. Variables $\alpha_j = \alpha^{2^j}, j = 0, 1, \dots, n-1$,

store α and its conjugates. The main loop runs through $\alpha \in \mathbb{F}_{2^n}^*$ by updating the tuple α_{coeff} with the next binary tuple using an arbitrary order computed via function **NextTuple()** in line 12.

Each field element α is processed in lines 6-11. Since conjugates form an equivalence class for the computation of the complexity of normal elements, we only compute the complexity for one canonical representative of the normal basis. We stipulate that α is canonical if and only if α is the unique minimum element among $\{\alpha^{2^j}\}_{j=0}^{n-1}$, when using the lexicographical ordering of the coefficients of their polynomial representations. For a canonical element α , we check if it is normal using procedure **IsNormal** which employs a gcd test based on Theorem 2.4.3. If α is normal, then we calculate its complexity via procedure **CalculateComplexity**, which is an implementation of the complexity definition. This amounts to solving an $n \times n$ system of equations with coefficients in $\mathbb{F}_2$. Since the only difference between these systems of equations are their right-hand sides, we calculate the inverse of the system's matrix (matrix P) and then apply n multiplications of P by vectors in $\mathbb{F}_{2^n}$.

The values of α and its conjugates are updated in lines 12-15. Algorithm **StandardNCD** does a straight-forward update, while Algorithm **GrayCodeNCD** employs a more efficient update that uses a Gray code.

3.1.2 Gray Codes

A *Gray code* is a sequential ordering of the 2^n binary vectors of length n such that any two consecutive vectors have Hamming distance equal to one, i.e. any two consecutive vectors differ in a unique position k . Gray codes are well studied (see [34]) and they exist for every n . In particular, we can build a Gray code in which the first vector is the zero vector, and each subsequent vector can be computed in time linear with n . In `GrayCodeNCD` algorithm, this computation is done by calling function `NextGrayTuple()` in line 12. There are multiple possible Gray Codes for a given n , one possible algorithm is given below.

```

Algorithm NextGrayTuple( $[a_0, a_1, \dots, a_{n-1}]$ )
1.  if ( $a_{n-1} == 0$ ) then
2.       $a_{n-1} = 1$  /* flip the last digit */
3.  else
4.      for  $i = 0$  to  $n - 1$  do
5.          if  $a_{n-(i+1)} == 1$  then
6.               $a_{n-i} \leftarrow a_{n-i} + 1 \pmod{2}$  /* flip the next digit */
7.          endif
8.      endfor
9.  endif
end NextGrayTuple

```

Now we describe how Gray codes are used to update the field element $\alpha \in \mathbb{F}_{2^n}$ in the `GrayCodeNCD` algorithm. Let α' be the field element in the previous iteration of the Gray code and α be the field element in the current iteration, represented as polynomials in $\mathbb{F}_2[x]$. We represent the polynomial coefficients as the successive tuples in a Gray code, so we have $\alpha = \alpha' + x^k$, for some $k \in \{0, 1, \dots, n-1\}$. Therefore, $\alpha^{2^j} = (\alpha' + x^k)^{2^j} = (\alpha')^{2^j} + (x^k)^{2^j}$. In lines 1-2 of algorithm `GrayCodeNCD`, the values

$e_i^j = (x^i)^{2^j}$ are precomputed, and used in line 15 for updating α . This reduces the computation in lines 14-15 from $n - 1$ squarings, as in the standard algorithm, to n additions in the Gray code variant.

3.1.3 StandardNCD and GrayCodeNCD Algorithms

Now we present the StandardNCD and GrayCodeNCD algorithms.

Algorithm StandardNCD(n)

```

1.   $\alpha_{\text{coeff}} = \text{NextTuple}([0, 0, \dots, 0]);$ 
2.   $\alpha_0 = \text{polynomial}(\alpha_{\text{coeff}});$  /* initialize  $\alpha$  */
3.  for  $j = 1$  to  $n - 1$  do
4.       $\alpha_j = (\alpha_{j-1})^2;$  /* calculate conjugates  $\alpha_j = (\alpha)^{2^j}$  */
5.  for  $i = 1$  to  $2^n - 1$  do /* run through  $\mathbb{F}_{2^n}^*$  */
6.      if  $((\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \text{ is canonical})$  then
7.          if  $(\text{IsNormal}(\alpha_0, \alpha_1, \dots, \alpha_{n-1}))$  then
8.               $\text{Compl} = \text{CalculateComplexity}(\alpha_0, \alpha_1, \dots, \alpha_{n-1});$ 
9.               $\text{UpdateStats}(\alpha_0, \text{Compl})$ 
10.         endif
11.     endif
12.      $\alpha_{\text{coeff}} = \text{NextTuple}(\alpha_{\text{coeff}});$  /* go to next */
13.      $\alpha_0 = \text{polynomial}(\alpha_{\text{coeff}});$ 
14.     for  $j = 1$  to  $n - 1$  do
15.          $\alpha_j = (\alpha_{j-1})^2;$  /* calculate  $\alpha_j = \alpha^{2^j}$  */
16.     endfor
17. PrintFinalStats();
end StandardNCD
```

Algorithm GrayCodeNCD(n)

```

1.  for  $i = 0$  to  $n - 1$  do  $e_i^0 = x^i;$  /* precomputation */
2.  for  $j = 1$  to  $n - 1$  do  $e_i^j = (e_i^{j-1})^2;$  /* precomputation */
3.   $\alpha_{\text{coeff}} = \text{NextGrayTuple}([0, 0, \dots, 0]) = [0, \dots, 0, 1];$ 
4.  for  $j = 0$  to  $n - 1$  do  $\alpha_j = e_0^j;$  /* initial.  $\alpha$  & conjug. */
5.  for  $i = 1$  to  $2^n - 1$  do /* run through  $\mathbb{F}_{2^n}^*$  */
6.      if  $((\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \text{ is canonical})$  then
7.          if  $(\text{IsNormal}(\alpha_0, \alpha_1, \dots, \alpha_{n-1}))$  then
8.               $\text{Compl} = \text{CalculateComplexity}(\alpha_0, \alpha_1, \dots, \alpha_{n-1});$ 
9.               $\text{UpdateStats}(\alpha_0, \text{Compl})$ 
10.         endif
```

```

11.      endif
12.       $\alpha'_{\text{coeff}} = \alpha_{\text{coeff}}$ ;  $\alpha_{\text{coeff}} = \text{NextGrayTuple}(\alpha'_{\text{coeff}})$ ; /* next */
13.      Let  $k$  be the index  $\ell$  such that  $\alpha_{\text{coeff}}[\ell] \neq \alpha'_{\text{coeff}}[\ell]$ ;
14.      for  $j = 0$  to  $n - 1$  do
15.           $\alpha_j = \alpha_j + e_k^j$ ; /* efficient  $\alpha_j$  computation */
16.      endfor
17.      PrintFinalStats();
end GrayCodeNCD

```

```

Procedure CalculateComplexity( $\alpha^0, \alpha^1, \dots, \alpha^{n-1}$ )
1.  for  $i = 0$  to  $n - 1$  do
2.       $\beta^i = \alpha^0 * \alpha^i$ ;
3.      for  $i = 0$  to  $n - 1$  do
4.          for  $j = 0$  to  $n - 1$  do
5.               $P_{ij} = j$ -th coefficient of  $\alpha^i$ ;
6.               $A_{ij} = j$ -th coefficient of  $\beta^i$ ;
7.          endfor
8.      Calculate  $P_{\text{inv}} = P^{-1}$ ;
9.      Calculate  $T = A \times P_{\text{inv}}$ ; /*  $T = (t_{ij})$  */
10.  Compl =  $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} t_{ij}$ ;
11.  return Compl;
end CalculateComplexity

```

```

Procedure IsNormal( $\alpha^0, \alpha^1, \dots, \alpha^{n-1}$ )
1.  if  $(\gcd(x^n - 1, \alpha^0 + \alpha^1 x + \dots + \alpha^{n-1} x^{n-1}) = 1)$ 
2.      then return true;
3.      else return false;
end IsNormal

```

We recall Proposition 2.6.7: that the multiplication table of a self-dual normal bases is symmetric. Therefore, in order to adapt the given algorithms to search exclusively for self-dual normal elements, it is enough to alter the `CalculateComplexity` procedure to solve for t_{ij} row-by-row (i.e. solving the system up to n times), and check the non-diagonal elements (up to the index of the calculated row) for symmetry. If this follows to completion, the normal element is self-dual and the complexity

is stored. We call this algorithm `SelfDualNCD`. A further refinement of this algorithm, considered after implementation, is due to Proposition 2.6.9: when n is even, α generates a self-dual normal basis if and only if $\alpha + 1$ generates a self-dual normal basis. If implemented, for $n \equiv 2 \pmod{4}$, the running time should be reduced approximately by half.

3.1.4 Karatsuba Multiplication Scheme

Elements in a finite field are classically represented as polynomials. Let $A, B \in \mathbb{F}_{q^n}$ be represented as $a, b \in \mathbb{F}_q[x]$, where $a(x) = a_{n-1}x^{n-1} + \dots + a_1x + a_0$ and $b(x) = b_{n-1}x^{n-1} + \dots + b_1x + b_0$ where $a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1} \in \mathbb{F}_q$. Consider multiplication $A \cdot B = a(x) \cdot b(x) = c(x) = c_{n-1}x^{n-1} + \dots + c_1x + c_0$ for some $c_{n-1}, \dots, c_1, c_0 \in \mathbb{F}_q$.

The resultant c is given as the residue modulo the reduction polynomial $f \in \mathbb{F}_q[x]$ where $\mathbb{F}_{q^n} = \mathbb{F}_q/(f)$. Classically, this is computed as term-by-term multiplication of a and b followed by the reduction step. This is classically performed in time $O(n^2)$.

Karatsuba and Ofman (1963) presented the first multiplication scheme for integers which runs in time less than $O(n^2)$ [23]. More precisely, Karatsuba presented a fast squaring scheme and noted the identity

$$ab = 1/4 \left((a+b)^2 + (a-b)^2 \right)$$

We present Karatsuba's multiplication scheme for finite fields of characteristic two [8].

We express the polynomial $A \in \mathbb{F}_{2^n}$ in two parts $A = A_1x^{n/2} + A_0$. Consider

$$\begin{aligned} C &= A \cdot B = (A_1x^{n/2} + A_0)(B_1x^{n/2} + B_0) \\ &= A_1B_1x^n + (A_1B_0 + A_0B_1)x^{n/2} + A_0B_0. \end{aligned}$$

To simplify this expression we define

$$\begin{aligned} T_1 &= A_1B_1 \\ T_2 &= (A_1 + A_0)(B_1 + B_0) \\ &= A_1B_0 + A_0B_1 + A_1B_1 + A_0B_0 \\ T_3 &= A_0B_0. \end{aligned}$$

which reduces the expression to

$$C = AB = T_1x^n + (T_2 - T_1 - T_3)x^{n/2} + T_3.$$

This gives that an n bit multiplication can be expressed by 3 $n/2$ bit multiplications and some additional additions.

We conclude this discussion by giving the time complexity of the Karatsuba multiplication scheme.

Theorem 3.1.1. [14] *Karatatsuba's algorithm for multiplying polynomials of degree less than a power n of 2 over a ring can be done with at most $9n^{\log_2 3}$ or $O(n^{1.59})$ ring*

operations.

3.1.5 Analyses of the Algorithms

In this section, we analyze the amortized worst-case running time per finite field element, which we call $TS(n)$ and $TG(n)$, for standard and Gray code variants, respectively. Therefore, the total running time of each algorithm will be at most $2^n TS(n)$ and $2^n TG(n)$, respectively. We do our analysis in terms of $M(n)$, the number of basic steps in $\mathbb{F}_2$ used in a multiplication of polynomials of degree smaller than n over $\mathbb{F}_2$. We then compute the savings in running time given by the Gray code variant with respect to the standard one.

We analyze the steps that are common to both algorithms. The canonicity test in line 6 of both algorithms can be done in n^2 comparisons in $\mathbb{F}_2$. Procedure `IsNormal` called in line 7 is only run for $1/n$ of the field elements. Each time it is run, it involves the computation of a gcd between polynomials of degrees $n - 1$ and n over $\mathbb{F}_{2^n}$, see Theorem 2.4.3. This gcd can be computed via the classical method in time $(5/2)n^2 M(n) + O(n)$ operations in $\mathbb{F}_2$, see [14]. Therefore the amortized time for line 7 is $(5/2)nM(n) + O(1)$.

The number of normal elements is $\Phi_2(x^n - 1)$ by Theorem 2.4.6 therefore lines 8-9 are only executed $\Phi_2(x^n - 1)/n$ times in total. Each execution of these lines runs the `CalculateComplexity` algorithm. For this algorithm, steps 1-2 take $O(nM(n))$, steps 3-7 and 9-10 take $O(n^2)$ and step 8 takes $O(n^3)$. In the classical multiplication scheme

$M(n) = n^2$, and for Karatsuba's multiplication scheme $M(n) = n^{1+\log_2 3} \leq n^{1.59}$. There are other asymptotically faster multiplication schemes but for the sizes of n we are considering it suffices to consider these two schemes, see [15]. So, the running time of `CalculateComplexity` is $O(n^3)$. Thus, the amortized cost for steps 8-9 is in $O(n^3 \Phi_2(x^n - 1)/(n2^n))$, which, by the upper bound in Equation (2.16), is contained in $O(n^2/\sqrt{\log n})$.

The iterations of the `StandardNCD` and `GrayCodeNCD` algorithms differ in the updates in lines 12-16. For the standard algorithm, the cost of these lines is dominated by $n - 1$ squarings of polynomials of degree smaller than n over $\mathbb{F}_2$, which are done in $(n - 1)M(n)$ operations in $\mathbb{F}_2$. For the Gray code algorithm, the cost of lines 12-16 is dominated by n additions of polynomials of degree smaller than n over $\mathbb{F}_2$, which is done in time $O(n^2)$, see [14].

Combining the above analyses for the steps of each algorithm, we get

$$\begin{aligned} TS(n) &= \frac{5}{2}nM(n) + nM(n) + O(n^2) = \frac{7}{2}nM(n) + O(n^2), \\ TG(n) &= \frac{5}{2}nM(n) + O(n^2). \end{aligned}$$

Using the above equations, we conclude that the Gray code variant gives (asymptotically) savings of $(\frac{7}{2} - \frac{5}{2})nM(n)/(\frac{7}{2}nM(n)) \approx 28.6\%$.

We conclude with Table 3.1 where we give asymptotic estimates of $TS(n)$ and $TG(n)$ using classical and Karatsuba's multiplication methods.

Method	Karatsuba's	Classical
$TS(n)$	$31.5n^{1+\log_2 3} + O(n^2)$	$3.5n^3 + O(n^2)$
$TG(n)$	$22.5n^{1+\log_2 3} + O(n^2)$	$2.5n^3 + O(n^2)$

Table 3.1: Running time of StandardNCD and GrayCodeNCD using Karatsuba and classical multiplication methods.

3.2 Results

The experiments were performed on individual Pentium IV 3.0 GHz systems with 1.5 GB of DDR RAM. The operating system was Red Hat Linux Enterprise Edition kernel 2.6.9-34.EL. C++, compiled using g++ 3.4.3, was used for all programming. For arithmetic performed over $\mathbb{F}_2$, Shoup's NTL package version 5.3.2 was used [36], specifically taking advantage of the optimized binary arithmetic "GF2" libraries. For $n = 37, 38, 39$ the Gray code was split into multiple sections and run simultaneously across many machines which reduces the run-time linearly with the number of machines used.

3.2.1 Running Time Comparison

Table 3.2 gives the CPU User-time for the StandardNCD, GrayCodeNCD and the SelfDual-

NCD respectively, measured in seconds, followed by a figure for graphical comparison.

We observe from Table 3.2 that the running time of the SelfDualNCD algorithm is approximately ten percent faster than the GrayCodeNCD algorithm running time, and so it was not included in Figure 3.1. Moreover, a comparison of the running time per

n	Standard	GrayCode	SelfDual	n	Standard	GrayCode	SelfDual
2	0.00	0.00	0.00	17	3.20	2.42	2.21
3	0.00	0.00	0.00	18	6.37	4.96	4.75
4	0.00	0.00	0.00	19	14.11	10.52	9.67
5	0.00	0.00	0.00	20	29.18	21.50	20.49
6	0.00	0.00	0.00	21	57.34	41.74	39.67
7	0.00	0.00	0.00	22	127.11	92.23	83.80
8	0.00	0.00	0.00	23	281.38	194.05	176.11
9	0.00	0.01	0.00	24	546.18	375.23	355.41
10	0.01	0.01	0.01	25	1157.33	833.03	741.50
11	0.03	0.03	0.02	26	2416.47	1672.35	1513.56
12	0.07	0.06	0.05	27	4983.14	3252.12	3011.80
13	0.16	0.13	0.11	28	9116.89	6776.99	7276.91
14	0.43	0.27	0.23	29	21179.50	14994.80	13115.90
15	0.65	0.52	0.48	30	40738.50	27515.20	25103.80
16	1.48	1.13	1.02				

Table 3.2: Running time in seconds for several algorithms.

element shows an improvement in the GrayCodeNCD algorithm over the StandardNCD ranging from 26 and 35 percent for $20 \leq n \leq 30$. This mirrors the expected 28.6% decrease in run-time from the theoretical analysis.

Figure 3.1: Average runtime per element for two exhaustive algorithms.

3.3 Conjectures

To examine the distribution of normal bases of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$, in Figure 3.2 we plot the number of normal bases found against their complexity. In each case we observe a

Figure 3.2: Sample distributions of complexity of (a) normal elements and (b) self-dual normal elements.

Gaussian shaped curve. This leads to the following conjecture.

Conjecture 3.3.1. *The number of normal bases over $\mathbb{F}_{2^n}$ is normally distributed with respect to their complexities.*

This conjecture is verified for the cases $n \leq 39$ by running the Shapiro-Wilk normality test [28] on the results of the GrayCodeNCD experiments, with over 99.9 percent certainty for $15 \leq n \leq 39$. Finding the general form of the distribution curve requires calculating the average complexity and the variance. As a consequence of this conjecture, although probabilistic algorithms to find normal elements exist [16], most normal elements exist close to the peak of the curve and these will not give

low complexity normal elements. For efficient computation, new searching methods for normal elements must be developed. Alternatively, direct constructions of normal elements with low complexity can be explored, as in Chapter 4.

Alternatively, an enveloping curve requires bounds (as functions of n) to find the averages and variances of the complexities. We observe that the probability of finding a normal element of complexity $c_N \leq kn$ is analogous to finding the density under the normalized curve as follows, $P(c_N \leq kn) = P[Z \leq (kn - \mu)/\sigma]$. A low complexity is analogous to a low Z-score on the normalized curve.

The data lends itself to the following conjecture on a bound for the average complexity of normal elements.

Conjecture 3.3.2. *The average complexity of normal elements in a finite field $\mathbb{F}_{q^n}$ with $n \geq 8$ has an upper bound of $(n^2 - n + 3)/2$.*

This conjecture is based upon the data found using the GrayCodeNCD algorithm, and observing the symmetry of the Gaussian curve conjectured previously. This is very close to half of the largest possible complexity $n^2 - n$, which is always even, so the extra factor of adding 3 in the numerator accounts for any decimal places calculated in the average. This conjectured bound is quite tight for many values of $n \leq 39$, and only for $n = 14, 23, 25$ the numerator requires a constant term of 3 rather than 2.

Conjecture 3.3.3. *For sufficiently large n the variance of complexities of normal bases is bounded above by $n^2/2$.*

This conjecture comes from the side-by-side comparison in Appendix B of the

average and variances of the complexities. For this data $n \geq 19$ suffices. The variance and the average complexity are quite similar, which implies that there could probably be a stronger conjecture here, since this conjecture is loose in comparison to the one on the average complexity.

We recall that the average complexity of self-dual normal elements is given in Theorem 2.6.10, so the next step would naturally be to conjecture on the variance of the complexity of self-dual normal elements. A theoretical average for $n \equiv 2 \pmod{4}$ is given in Corollary 2.6.11, and our data agrees with this theory. However, since there are very few self-dual normal bases for $n \leq 36$ more experiments are required to give sufficient data to support any such conjecture.

The following argument leads to our final conjecture. Let us assume that the above conjectures on the average and variance of complexities of all normal elements in a finite field hold. Moreover, let us consider the following problem: is there some constant k such that the minimum complexity element in a finite field $\mathbb{F}_{2^n}$ is bounded above by kn ? Recall that an upper bound on the complexity of a normal element corresponds to a Z-score on the standardized normal curve. Given that by our conjectures $\mu \approx (n^2 - n + 3)/2$ and $\sigma^2 \approx n^2/2$, calculating the Z-score of kn gives

$$z_p = \frac{kn - \frac{n^2 - n + 3}{2}}{n/\sqrt{2}}.$$

If k is a constant, then, asymptotically, the Z-score becomes infinitely small. Relating this to the complexity distribution, this implies that the upper bound on the minimum

complexity vanishes. Since the minimum possible complexity is $2n - 1$, this analysis leads to the following conclusion.

Conjecture 3.3.4. *There is no constant k such that the complexity C_n of $\mathbb{F}_{2^n}$ is bounded above by kn for all n . Furthermore, if the average and variance of the complexities of all normal elements in $\mathbb{F}_{2^n}$ are both of order n^2 , then an upperbound for C_n , for all n , is also of order n^2 .*

Our final observation concerns the distribution of the normal elements themselves. To achieve results for $36 \leq n \leq 39$ we distributed the computation across many processors by letting each processor deal with a range of field elements that were consecutive in the Gray code order. This divided our time linearly with the number of processors used. For example, the estimated run time of $n = 39$ was 8 months using a single processor, but we were able to complete the simulation in just under 3 weeks using 13 processors. Furthermore, we observed that there were large blocks along the run of the Gray code in which there were no normal elements found. This is an interesting topic for future research, which could lead to conclusions on the existence of normal elements with prescribed coefficients.

Chapter 4

Traces of Optimal Normal Bases

As seen in Chapter 3 it is likely that the complexity of a random normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ will be quite high: approximately $n^2/2$ in the $q = 2$ case. For this reason, constructions such as those given in Section 2.5 are required to find normal bases with low complexity. Ideally, these constructions should be easy to analyze and implement, and we would like the complexity of the basis to be close to kn for some small value of k .

In this chapter we give constructions of low-complexity normal bases by projecting to subfields from fields containing both Type I and Type II optimal normal bases for all q . We do this by taking the trace function of the optimal normal element in the larger field. We also find the complexity of the dual basis of the bases obtained by these projection maps. A discussion of how large an extension must be to project down to a desired subfield is given, followed by a summary and comparison of these constructions to commonly used extensions.

4.1 The Trace of a Normal Element

This chapter makes extensive use of the observation that the trace of a normal element is again normal. This is formally stated as the following proposition.

Proposition 4.1.1. *Let $n = mk$ where n, m, k are all positive integers, and let α be a normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Then $\text{Tr}_{q^n/q^m}(\alpha)$ is a normal element of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.*

PROOF. Define $n, \alpha \in \mathbb{F}_{q^n}$ as above, and let $\beta = \text{Tr}_{q^n/q^m}(\alpha)$. Then

$$\beta_j = \beta^{q^j} = \sum_{i=0}^{k-1} \alpha^{q^{mi+j}}, \quad 0 \leq j \leq m-1,$$

and so for a fixed j we have

$$\sum_{i=0}^{k-1} c_j \alpha^{q^{mi+j}} = \left(\sum_{i=0}^{k-1} c_j \alpha^{q^{mi}} \right)^{q^j} \quad (4.1)$$

for some $c_j \in \mathbb{F}_q$. The conjugates of α are linearly independent so setting Equation (4.1) equal to zero guarantees $c_j = 0$ for all $0 \leq j \leq m-1$. Therefore the conjugates of β are linearly independent and form a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. ■

4.2 Main results

We have seen that the trace of a normal element is again normal, so we use the trace function as a projection map from fields which contain an optimal normal basis to

their subfields. We calculate upper bounds on the complexity of the basis generated by these projections for Type I and Type II Optimal normal bases, and for all q .

4.2.1 Type I Optimal Normal Bases: q odd

Theorem 4.2.1. *Let $\alpha \in \mathbb{F}_{q^n}$ generate an optimal normal basis of Type I of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, q odd, and let $\beta = \text{Tr}_{q^n/q^m}(\alpha) \in \mathbb{F}_{q^m}$ with $m = n/k$ and $k \leq m$. Then, the complexity of the normal basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ generated by β is bounded by $(k + 2)m - 3k + 1$, if m is even and k is odd and by $(k + 1)m - k$ in all other cases.*

Furthermore, for $1 \leq j \leq m - 1$ row j of the multiplication table of β is a cyclic permutation of j positions of row $(m - j)$.

PROOF. Let $n + 1$ be a prime, q a primitive root modulo $n + 1$ where q is prime or prime power, $q \neq 2$ and $\alpha \in \mathbb{F}_{q^n}$ an optimal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Let $N = \{\alpha, \alpha^q, \alpha^{q^2}, \dots, \alpha^{q^{n-1}}\}$ be the optimal normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ which is generated by α .

The multiplication table $C_{[n \times n]}$ of the linear map

$$C_\alpha: \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}, \quad C_\alpha(x) = \alpha \cdot x$$

has exactly $2n - 1$ non-zero terms with the following properties:

$$\alpha \cdot \alpha^{q^i} = \alpha^{q^j}, \quad i = 0, 1, \dots, \frac{n}{2} - 1, \frac{n}{2} + 1, \dots, n - 1, \quad j = 0, 1, \dots, n - 1, \quad (4.2)$$

$$\alpha \cdot \alpha^{q^{n/2}} = \sum_{s=0}^{n-1} -\alpha^{q^s}. \quad (4.3)$$

The above equations imply that there is exactly one 1 in each row except for the row $n/2$, when n is even, where all the n entries are -1.

Suppose that $\beta = \text{Tr}_{q^n/q^m}(\alpha) \in \mathbb{F}_{q^m}$ with $m = n/k$. Then,

$$\beta = \text{Tr}_{q^n/q^m}(\alpha) = \sum_{i=0}^{k-1} \alpha^{q^{mi}} = \alpha + \alpha^{q^m} + \alpha^{q^{2m}} + \cdots + \alpha^{q^{(k-1)m}},$$

generates a normal basis M of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ of the form

$$M = \{\beta, \beta^q, \beta^{q^2}, \dots, \beta^{q^{m-1}}\}.$$

We observe that, for $j = 0, \dots, m-1$, we have

$$\beta^{q^j} = \sum_{i=0}^{k-1} \alpha^{q^{mi+j}} = \alpha^{q^j} + \alpha^{q^{j+m}} + \cdots + \alpha^{q^{j+(k-1)m}}.$$

Let $D = D_{[m \times m]}$ be the multiplication table of the linear map

$$D_\beta: \mathbb{F}_{q^m} \rightarrow \mathbb{F}_{q^m}, \quad D_\beta(x) = \beta \cdot x.$$

The first row of the table D is given by

$$\beta \cdot \beta = \left(\sum_{i=0}^{k-1} \alpha^{q^{mi}} \right) \cdot \left(\sum_{i=0}^{k-1} \alpha^{q^{mi}} \right)$$

$$\begin{aligned}
&= \sum_{i=0}^{k-1} (\alpha \cdot \alpha)^{q^{mi}} + \sum_{i=0}^{k-1} (\alpha \cdot \alpha^{q^m})^{q^{mi}} + \cdots + \sum_{i=0}^{k-1} (\alpha \cdot \alpha^{q^{(km)/2}})^{q^{mi}} \\
&\quad + \cdots + \sum_{i=0}^{k-1} (\alpha \cdot \alpha^{q^{(k-1)m}})^{q^{mi}}.
\end{aligned}$$

Using Equations (4.2) and (4.3), there are $\mu_0, \mu_1, \dots, \mu_{k-2} \in \mathbb{Z}_n$ such that

$$\alpha \cdot \alpha = \alpha^{q^{\mu_0}}, \alpha \cdot \alpha^{q^m} = \alpha^{q^{\mu_1}}, \dots, \alpha \cdot \alpha^{q^{(k-1)m}} = \alpha^{q^{\mu_{k-2}}}$$

and

$$\sum_{i=0}^{k-1} (\alpha \cdot \alpha^{q^{n/2}})^{q^{mi}} = \sum_{i=0}^{k-1} \left(- \sum_{s=0}^{n-1} \alpha^{q^s} \right)^{q^{mi}} = \sum_{i=0}^{k-1} \left(- \sum_{s=0}^{m-1} \beta^{q^s} \right)^{q^{mi}} = -k \sum_{s=0}^{m-1} \beta^{q^s}.$$

Thus, we get

$$\begin{aligned}
\beta \cdot \beta &= \sum_{i=0}^{k-1} (\alpha^{q^{\mu_0}})^{q^{mi}} + \sum_{i=0}^{k-1} (\alpha^{q^{\mu_1}})^{q^{mi}} + \cdots + \sum_{i=0}^{k-1} (\alpha^{q^{\mu_{k-2}}})^{q^{mi}} - k \sum_{s=0}^{m-1} \beta^{q^s} \\
&= \beta^{q^{\mu_0}} + \beta^{q^{\mu_1}} + \cdots + \beta^{q^{\mu_{k-2}}} - k(\beta + \beta^q + \cdots + \beta^{q^{m-1}}) \\
&= -k\beta - k\beta^q + \cdots + (1-k)\beta^{q^{\mu_0}} + (1-k)\beta^{q^{\mu_1}} + \cdots + (1-k)\beta^{q^{\mu_{k-2}}} \\
&\quad + \cdots + (-k)\beta^{q^{m-1}}.
\end{aligned}$$

The coefficients of β^{q^j} are computed modulo q , so the first row of the table has at most m non-zero terms.

Then, we calculate the remaining rows $j = 1, \dots, m-1$ of the table by computing

$$\begin{aligned}
 \beta \cdot \beta^{q^j} &= \left(\sum_{i=0}^{k-1} \alpha^{q^{mi}} \right) \cdot \left(\sum_{u=0}^{k-1} \alpha^{q^{mu+j}} \right) \\
 &= \sum_{0 \leq u, i \leq k-1} \left(\alpha^{q^{mi}} \right) \left(\alpha^{q^{mu+j}} \right) \\
 &= \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^j} \right)^{q^{im}} + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{j+m}} \right)^{q^{im}} + \dots + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{j+(k-1)m}} \right)^{q^{im}}.
 \end{aligned}$$

By Equation (4.2) there are $\lambda_0, \lambda_1, \dots, \lambda_{k-1} \in \mathbb{Z}_n$ such that

$$\alpha \cdot \alpha^{q^j} = \alpha^{q^{\lambda_0}}, \quad \alpha \cdot \alpha^{q^{j+m}} = \alpha^{q^{\lambda_1}}, \quad \dots, \quad \alpha \cdot \alpha^{q^{j+(k-1)m}} = \alpha^{q^{\lambda_{k-1}}}, \quad (4.4)$$

which implies,

$$\begin{aligned}
 \beta \cdot \beta^{q^j} &= \sum_{i=0}^{k-1} \left(\alpha^{q^{\lambda_0}} \right)^{q^{im}} + \sum_{i=0}^{k-1} \left(\alpha^{q^{\lambda_1}} \right)^{q^{im}} + \dots + \sum_{i=0}^{k-1} \left(\alpha^{q^{\lambda_{k-1}}} \right)^{q^{im}} \\
 &= \beta^{q^{\lambda_0}} + \beta^{q^{\lambda_1}} + \dots + \beta^{q^{\lambda_{k-1}}}.
 \end{aligned}$$

Finally, for the row $(m-j)$ of the table D , we have

$$\beta \cdot \beta^{q^{m-j}} = \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{m-j}} \right)^{q^{im}} + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{2m-j}} \right)^{q^{im}} + \dots + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{-j}} \right)^{q^{im}}.$$

Using the identities (4.4) it follows that

$$\beta \cdot \beta^{q^{m-j}} = \beta^{q^{\lambda_{k-1}+m-j}} + \beta^{q^{\lambda_{k-2}+2m-j}} + \dots + \beta^{q^{\lambda_0-j}},$$

and since $\beta^{q^m} = \beta$ we get

$$\beta \cdot \beta^{q^{m-j}} = \beta^{q^{\lambda_{k-1}-j}} + \beta^{q^{\lambda_{k-2}-j}} + \dots + \beta^{q^{\lambda_0-j}}.$$

Thus, the row j of the multiplication table of β is a cyclic permutation of j positions of row $(m-j)$.

If $m = n/k$ is an even number, to calculate the row $m/2$ of the table D we must consider both the cases where k is even and where k is odd. For the odd case we have

$$\begin{aligned} \beta \cdot \beta^{q^{m/2}} &= \left(\sum_{i=0}^{k-1} \alpha^{q^{mi}} \right) \cdot \left(\sum_{u=0}^{k-1} \alpha^{q^{mu+m/2}} \right) \\ &= \sum_{0 \leq u, i \leq k-1} \left(\alpha^{q^{mi}} \right) \left(\alpha^{q^{mu+m/2}} \right) \\ &= \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{m/2}} \right)^{q^{im}} + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{m/2+m}} \right)^{q^{im}} + \dots \\ &\quad + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{m/2+m(k-1)/2}} \right)^{q^{im}} + \dots + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{m/2+(k-1)m}} \right)^{q^{im}} \\ &= \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{m/2}} \right)^{q^{im}} + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{m/2+m}} \right)^{q^{im}} + \dots \\ &\quad + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{n/2}} \right)^{q^{im}} + \dots + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{q^{m/2+(k-1)m}} \right)^{q^{im}}. \end{aligned} \quad (4.5)$$

By Equations (4.2) and (4.3), there are $\delta_0, \delta_1, \dots, \delta_{k-2} \in \mathbb{Z}_n$ such that

$$\beta \cdot \beta^{q^{m/2}} = \beta^{q^{\delta_0}} + \beta^{q^{\delta_1}} + \dots + (-\beta - \beta^q - \beta^{q^2} - \dots - \beta^{q^{m-1}}) + \dots + \beta^{q^{\delta_{k-2}}}.$$

Thus, the row $m/2$ in this case has at most $m - k + 1$ non-zero terms. For the case

where k is even, the computations of (4.5) are similar to the calculations involving the identities (4.4) above, and yield at most k non-zero terms.

In conclusion, we observe that an upper bound for the complexity of the normal basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ generated by β , when m is even and k is odd is $k(m-2) + 2m - k + 1 = (k+2)m - 3k + 1$ since the first row of the table gives at most m non-zero entries, the $m/2$ row gives at most $m - k + 1$ entries and all other rows give at most k entries. In all other cases, the upper bound for the complexity of the normal basis is $k(m-1) + m = (k+1)m - k$, since the first row gives at most m non-zero terms and all other rows give at most k non-zero terms. ■

4.2.2 Type I Optimal Normal Bases: q even

Theorem 4.2.2. *Let $\alpha \in \mathbb{F}_{2^n}$ generate an optimal normal basis of Type I of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$, $n > 2$, and let $\beta = \text{Tr}_{2^n/2^m}(\alpha) \in \mathbb{F}_{2^m}$ with $m = n/k$ and $k \leq m$. Then, an upper bound for the complexity of the normal basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$ generated by β is $(k+1)m - 3k + 2$ if m is even and k is odd, or $km - k + 1$ otherwise.*

Furthermore, for $1 \leq j \leq m-1$ row j of the multiplication table of β is a cyclic permutation of j positions of row $(m-j)$.

PROOF. We observe that the proof of this claim is nearly identical to the case where q is odd except for the following changes.

Let $n+1$ be a prime, 2 a primitive root modulo $n+1$, and let $\alpha \in \mathbb{F}_{2^n}$ be an optimal element of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$. Let $N = \{\alpha, \alpha^2, \alpha^{2^2}, \dots, \alpha^{2^{n-1}}\}$ be the optimal normal basis

of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ which is generated by α .

The multiplication table $C_{[n \times n]}$ of the linear map

$$C_\alpha: \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}, \quad C_\alpha(x) = \alpha \cdot x$$

has exactly $2n - 1$ non-zero terms with the following properties:

$$\alpha \cdot \alpha^{2^i} = \alpha^{2^j}, \quad i = 0, 1, \dots, \frac{n}{2} - 1, \frac{n}{2} + 1, \dots, n - 1, \quad j = 0, 1, \dots, n - 1, \quad (4.6)$$

$$\alpha \cdot \alpha^{2^{n/2}} = \sum_{s=0}^{n-1} -\alpha^{2^s} = \sum_{s=0}^{n-1} \alpha^{2^s}. \quad (4.7)$$

The above equations imply that there is exactly one 1 in each row except for the row $n/2$ where all the n entries are 1.

Suppose that $\beta = \text{Tr}_{2^n/2^m}(\alpha) \in \mathbb{F}_{2^m}$ with $m = n/k$. Then,

$$\beta = \text{Tr}_{2^n/2^m}(\alpha) = \sum_{i=0}^{k-1} \alpha^{2^{mi}} = \alpha + \alpha^{2^m} + \alpha^{2^{2m}} + \dots + \alpha^{2^{(k-1)m}},$$

generates a normal basis M of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$ of the form

$$M = \{\beta, \beta^2, \beta^{2^2}, \dots, \beta^{2^{m-1}}\}.$$

We have that, for $j = 0, \dots, m-1$,

$$\beta^{2^j} = \sum_{i=0}^{k-1} \alpha^{2^{mi+j}} = \alpha^{2^j} + \alpha^{2^{j+m}} + \dots + \alpha^{2^{j+(k-1)m}}.$$

Let $D = D_{[m \times m]}$ be the multiplication table of the linear map

$$D_\beta: \mathbb{F}_{2^m} \rightarrow \mathbb{F}_{2^m}, \quad D_\beta(x) = \beta \cdot x.$$

The first row of the table D is given by $\beta \cdot \beta = \beta^2$. Thus, the first row of the table D has a 1 in the second position.

If $m = n/k$ is an even number then for the row $m/2$ of the table, by Equation (4.5), we have that this row contributes at most $m - k + 1$ ones to D if k is odd, and at most k ones to D if k is even.

For the remaining rows the proof is identical to the q odd case. We recall that each of the remaining rows contribute to the complexity with at most k non-zero entries. The proof that row j of the multiplication of β is a cyclic permutation of j positions of row $(m - j)$ is also identical to the q odd case.

In conclusion, we observe that an upper bound for the complexity of the normal basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$ generated by β , when m is odd or if both m and k are even, is $k(m-1) + 1 = km - k + 1$. Otherwise, if m is even and k is odd, the complexity is at most $k(m-2) + m - k + 2 = (k+1)m - 3k + 2$. ■

4.2.3 Type II Optimal Normal Bases

Theorem 4.2.3. *Let α generate a Type II optimal normal basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ and let $\beta = \text{Tr}_{2^n/2^m}(\alpha) \in \mathbb{F}_{2^m}$ with $m = n/k$ and $k \leq m$. Then the complexity of the normal basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$ generated by β is $2km - 2k + 1$.*

PROOF. Let $2n + 1$ be prime, and suppose that either

1. 2 is a primitive element of $\mathbb{Z}_{2n+1}$, or
2. $2n + 1 \equiv 3 \pmod{4}$ and 2 generates the quadratic residues in $\mathbb{Z}_{2n+1}$.

Let $\alpha = \gamma + \gamma^{-1}$, where γ is a primitive $(2n + 1)$ th root of unity. Let $N = \{\alpha, \alpha^2, \alpha^{2^2}, \dots,$

$\alpha^{2^{n-1}}\}$ be the optimal normal basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ which is generated by α .

The multiplication table $C_{[n \times n]}$ of the linear map

$$C_\alpha: \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}, \quad C_\alpha(x) = \alpha \cdot x$$

has exactly $2n - 1$ non-zero terms with the following property:

$$\alpha \cdot \alpha^{2^i} = \alpha^{2^j} + \alpha^{2^k}, \quad i = 1, \dots, n-1, \quad j, k = 0, 1, \dots, n-1, j \neq k. \quad (4.8)$$

Therefore, there are exactly two ones in each row of C except for the first row, where there is one 1 in the second position.

Suppose that $\beta = \text{Tr}_{2^n/2^m}(\alpha) \in \mathbb{F}_{2^m}$ with $m = n/k$. Then,

$$\beta = \text{Tr}_{2^n/2^m}(\alpha) = \sum_{i=0}^{k-1} \alpha^{2^{mi}} = \alpha + \alpha^{2^m} + \alpha^{2^{2m}} + \cdots + \alpha^{2^{(k-1)m}}$$

generates a normal basis M of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$ of the form

$$M = \{\beta, \beta^2, \beta^{2^2}, \dots, \beta^{2^{m-1}}\}.$$

We observe that, for $j = 0, \dots, m-1$, we have

$$\beta^{2^j} = \sum_{i=0}^{k-1} \alpha^{2^{j+mi}} = \alpha^{2^j} + \alpha^{2^{j+m}} + \cdots + \alpha^{2^{j+(k-1)m}}.$$

Let $D = D_{[m \times m]}$ be the multiplication table of the linear map

$$D_\beta: \mathbb{F}_{2^m} \rightarrow \mathbb{F}_{2^m}, \quad D_\beta(x) = \beta \cdot x.$$

The first row of the table D is given by $\beta \cdot \beta = \beta^2$. Thus, it has 1 non-zero term in the second position.

As in Theorem 4.2.2, computing the j th row of D gives

$$\beta \cdot \beta^{2^j} = \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{2^j} \right)^{2^{mi}} + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{2^{j+m}} \right)^{2^{mi}} + \cdots + \sum_{i=0}^{k-1} \left(\alpha \cdot \alpha^{2^{j+(k-1)m}} \right)^{2^{mi}}.$$

By Equation (4.8), there exist $\lambda_i, \mu_i \in \mathbb{Z}_n$ such that $\alpha \cdot \alpha^{2^{j+mi}} = \alpha^{2^{\lambda_i}} + \alpha^{2^{\mu_i}}, i =$

$0, \dots, k-1$. So,

$$\begin{aligned} \beta \cdot \beta^{2^j} &= \sum_{i=0}^{k-1} \left(\alpha^{2^{\lambda_0}} + \alpha^{2^{\mu_0}} \right)^{2^{mi}} + \dots + \sum_{i=0}^{k-1} \left(\alpha^{2^{\lambda_{k-1}}} + \alpha^{2^{\mu_{k-1}}} \right)^{2^{mi}} \\ &= \beta^{2^{\lambda_0}} + \beta^{2^{\mu_0}} + \dots + \beta^{2^{\lambda_{k-1}}} + \beta^{2^{\mu_{k-1}}}. \end{aligned}$$

Thus, there are $2k$ ones appearing in each remaining row of D .

In conclusion, we observe that the complexity of the basis generated by $\beta \in \mathbb{F}_{2^m}$ over $\mathbb{F}_2$ is $2k(m-1) + 1$. ■

The following corollary shows that, in contrast with the case Type I normal basis and q even, we do not obtain optimal normal elements for Type II normal basis. Hence, this corollary gives new low complexity normal elements for q even.

Corollary 4.2.4. *Let α generate a Type II optimal normal basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ and let $\beta = \text{Tr}_{2^n/2^m}(\alpha) \in \mathbb{F}_{2^m}$ with $m = n/2$. Then the complexity of the normal basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$ generated by β is $4m - 3$.*

4.3 The Dual of the Trace of an Optimal Element

Let $N = \{\alpha, \alpha_1, \dots, \alpha_{n-1}\}$ be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ and $M = \{\gamma, \gamma_1, \dots, \gamma_{n-1}\}$ be another basis. Then M is the dual basis of N if $\text{Tr}(\alpha_i \gamma_j) = \delta_{ij}$ for $1 \leq i, j \leq n-1$ where δ_{ij} is the Kronecker delta function. Proposition 4.1.1 states that the dual basis of a normal basis is again normal, so let $\gamma_i = \gamma^{q^i}$. Theorem 4.3.1 gives the statement and proof of an upper bound of the complexity of the dual basis of the trace of a Type

I optimal normal basis when q is odd. Following the proof we present a statement of the theorem using Type II optimal normal bases and a summary table outlining upper bounds for any q and for both Type I and Type II optimal normal bases.

Theorem 4.3.1. *Let $\alpha \in \mathbb{F}_{q^n}$ generate an optimal normal basis of Type I of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ and let $\beta = \text{Tr}_{q^n/q^m}(\alpha) \in \mathbb{F}_{q^m}$ with $m = n/k$, $k \leq m$. Then the complexity of the normal basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ generated by γ , which is the dual element of β , is $(k+2)m - 2$ when m is odd. Further, for $1 \leq j \leq m-1$ row j of the multiplication table of γ is a cyclic permutation of j positions of row $(m-j)$.*

PROOF. As in Theorem 4.2.1, let $\alpha \in \mathbb{F}_{q^n}$ generate an optimal normal basis of Type I and $\beta = \text{Tr}_{q^n/q^m}(\alpha) \in \mathbb{F}_{q^m}$ with $m = n/k$. Let $\gamma \in \mathbb{F}_{q^m}$ be a dual element of β . According to Theorem 2.6.6, γ is of the form

$$\gamma = d_0\beta + d_1\beta^q + \cdots + d_{m-1}\beta^{q^{m-1}},$$

where $d_0, d_1, \dots, d_{m-1} \in \mathbb{F}_q$ are the coefficients of the unique polynomial $g(x)$ of degree $\leq m-1$ satisfying

$$g(x)h(x) \equiv 1 \pmod{x^m - 1}$$

and $h(x)$ is of degree $\leq m-1$ with coefficients $t_0, t_1, \dots, t_{m-1}$ where,

$$t_i = \text{Tr}_{q^m/q}(\beta \cdot \beta^{q^i}), \quad i = 0, 1, \dots, m-1.$$

Since, $\text{Tr}_{q^m/q}(\beta) \cdot \text{Tr}_{q^m/q}(\gamma) = 1$ and $\text{Tr}_{q^m/q}(\beta) = -1$ we get,

$$d_0 + d_1 + \cdots + d_{m-1} = 1.$$

We compute $t_0 = \text{Tr}_{q^m/q}(\beta \cdot \beta)$ separate from $t_i = \text{Tr}_{q^m/q}(\beta \cdot \beta^{q^i}), i = 1, \dots, m-1$.

$$\begin{aligned} t_0 &= \text{Tr}_{q^n/k/q}(\beta \cdot \beta) = \frac{1}{k} \text{Tr}_{q^n/q}(\beta \cdot \beta) = \frac{1}{k} \text{Tr}_{q^n/q} \left(\left(\sum_{i=0}^{k-1} \alpha^{q^{mi}} \right) \left(\sum_{i=0}^{k-1} \alpha^{q^{mi}} \right) \right) \\ &= \frac{1}{k} \text{Tr}_{q^n/q} \left(\sum_{i=0}^{k-1} (\alpha \cdot \alpha)^{q^{mi}} + \sum_{i=0}^{k-1} (\alpha \cdot \alpha^{q^m})^{q^{mi}} + \cdots + \sum_{i=0}^{k-1} (\alpha \cdot \alpha^{q^{(km)/2}})^{q^{mi}} + \right. \\ &\quad \left. + \cdots + \sum_{i=0}^{k-1} (\alpha \cdot \alpha^{q^{(k-1)m}})^{q^{mi}} \right) \end{aligned}$$

using Equations (4.2),(4.3) there are $\mu_0, \mu_1, \dots, \mu_{k-1}, \in \mathbb{Z}_m$ such that

$$\alpha \cdot \alpha = \alpha^{q^{\mu_0}}, \alpha \cdot \alpha^{q^m} = \alpha^{q^{\mu_1}}, \dots, \alpha \cdot \alpha^{q^{(k-1)m}} = \alpha^{q^{\mu_{k-2}}}.$$

Thus

$$\begin{aligned} t_0 &= \frac{1}{k} \text{Tr}_{q^n/q} \left(\sum_{i=0}^{k-1} (\alpha^{q^{\mu_0}})^{q^{mi}} + \sum_{i=0}^{k-1} (\alpha^{q^{\mu_1}})^{q^{mi}} + \cdots + k + \cdots + \sum_{i=0}^{k-1} (\alpha^{q^{\mu_{k-2}}})^{q^{mi}} \right) \\ &= \frac{1}{k} \left(\sum_{i=0}^{k-1} \text{Tr}_{q^n/q} (\alpha^{q^{\mu_0}})^{q^{mi}} + \sum_{i=0}^{k-1} \text{Tr}_{q^n/q} (\alpha^{q^{\mu_1}})^{q^{mi}} + \cdots + \text{Tr}_{q^n/q}(k) + \cdots + \right. \\ &\quad \left. + \cdots + \sum_{i=0}^{k-1} \text{Tr}_{q^n/q} (\alpha^{q^{\mu_{k-2}}})^{q^{mi}} \right) \\ &= \frac{1}{k} ((-k) + (-k) + \cdots + kn + \cdots + (-k)) \\ &= \frac{1}{k} ((-k)(k-1) + kn) = n - k + 1. \end{aligned}$$

Now, we calculate $t_i = \text{Tr}_{q^m/q}(\beta \cdot \beta^{q^i})$, $i = 1, \dots, m-1$. By Theorem 4.2.1, there are $\lambda_0, \lambda_1, \dots, \lambda_{k-1} \in \mathbb{Z}_m$ such that

$$\beta \cdot \beta^{q^i} = \beta^{q^{\lambda_0}} + \beta^{q^{\lambda_1}} + \dots + \beta^{q^{\lambda_{k-1}}}.$$

$$\begin{aligned} t_i &= \text{Tr}_{q^m/q}(\beta \cdot \beta^{q^i}) = \text{Tr}_{q^m/q}(\beta^{q^{\lambda_0}} + \beta^{q^{\lambda_1}} + \dots + \beta^{q^{\lambda_{k-1}}}) \\ &= \text{Tr}_{q^m/q}(\beta^{q^{\lambda_0}}) + \text{Tr}_{q^m/q}(\beta^{q^{\lambda_1}}) + \dots + \text{Tr}_{q^m/q}(\beta^{q^{\lambda_{k-1}}}) \\ &= (-1) + (-1) + \dots + (-1) = -k. \end{aligned}$$

This implies,

$$h(x) = -k(x^{m-1} + x^{m-2} + \dots + x) + n - k + 1.$$

As in Theorem 2.6.6, we compute the d_i coefficients by the system

$$\begin{aligned} d_0 t_0 &+ d_1 t_{-1} + \dots + d_{m-1} t_{-(m-1)} &= 1 \\ d_0 t_1 &+ d_1 t_0 + \dots + d_{m-1} t_{-(m-2)} &= 0 \\ &\vdots &\vdots &\vdots &\vdots \\ d_0 t_{m-2} &+ d_1 t_{m-1} + \dots + d_{m-1} t_{-1} &= 0 \\ d_0 t_{m-1} &+ d_1 t_{m-2} + \dots + d_{m-1} t_0 &= 0. \end{aligned}$$

The indices of t_i are computed (mod m) and the $d_i \in \mathbb{F}_q$ are found by solving the

system

$$\begin{pmatrix} n-k+1 & -k & \dots & -k & -k \\ -k & n-k+1 & \dots & -k & -k \\ \vdots & \dots & \dots & \vdots & \vdots \\ -k & -k & \dots & n-k+1 & -k \\ -k & -k & \dots & -k & n-k+1 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{m-2} \\ d_{m-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

which implies that

$$d_0 = \frac{k+1}{n+1}, \quad d_i = \frac{k}{n+1}, \quad i = 1, \dots, m-1.$$

Note that $n+1$ is a prime different from zero in $\mathbb{F}_q$, and therefore has an inverse.

Then using that $\text{Tr}_{q^m/q}(\beta) = -1$ the dual element $\gamma \in \mathbb{F}_{q^m}$ is :

$$\begin{aligned} \gamma &= \frac{k+1}{n+1}\beta + \frac{k}{n+1}(\beta^q + \beta^{q^2} + \dots + \beta^{q^{m-1}}) \\ &= \frac{1}{n+1}\beta + \frac{k}{n+1}(\beta + \beta^q + \beta^{q^2} + \dots + \beta^{q^{m-1}}) \\ &= \frac{1}{n+1}\beta + \frac{k}{n+1}\text{Tr}_{q^m/q}(\beta) \\ &= \frac{1}{n+1}\beta - \frac{k}{n+1}, \end{aligned}$$

$$\text{and } \gamma^{q^i} = \frac{1}{n+1}\beta^{q^i} - \frac{k}{n+1}, \quad i = 0, \dots, m-1.$$

Let $C = C_{[m \times m]}$ be the multiplication table of the linear map

$$C_\gamma : \mathbb{F}_{q^m} \rightarrow \mathbb{F}_{q^m}, \quad C_\gamma(x) = \gamma \cdot x.$$

By Theorem 4.2.1, there exist $\mu_0, \mu_1, \dots, \mu_{k-2} \in \mathbb{Z}_m$ such that

$$\beta \cdot \beta = \beta^{q^{\mu_0}} + \beta^{q^{\mu_1}} + \dots + \beta^{q^{\mu_{k-2}}} - k \left(\beta + \beta^q + \dots + \beta^{q^{m-1}} \right).$$

The first row of the table C is given by:

$$\begin{aligned} \gamma \cdot \gamma &= \left(\frac{1}{n+1} \beta - \frac{k}{n+1} \right) \cdot \left(\frac{1}{n+1} \beta - \frac{k}{n+1} \right) \\ &= \frac{1}{(n+1)^2} \beta \cdot \beta - \frac{2k}{(n+1)^2} \beta + \frac{k^2}{(n+1)^2} \\ &= \frac{1}{(n+1)} \left(\left(\frac{1}{n+1} \beta^{q^{\mu_0}} - \frac{k}{n+1} \right) + \dots + \left(\frac{1}{n+1} \beta^{q^{\mu_{k-2}}} - \frac{k}{n+1} \right) \right. \\ &\quad \left. - k \operatorname{Tr}_{q^m/q}(\beta) \right) + \frac{k(k-1)}{(n+1)^2} - \frac{2k}{(n+1)^2} \beta + \frac{k^2}{(n+1)^2} \\ &= \frac{1}{(n+1)} \left(\gamma^{q^{\mu_0}} + \gamma^{q^{\mu_1}} + \dots + \gamma^{q^{\mu_{k-2}}} \right) - \frac{2k}{n+1} \gamma \end{aligned}$$

Thus, the first row of the table C has at most k non-zero. Next we prove that each one of the remaining rows has at most $k+2$ non-zero terms. For every $i = 1, \dots, m-1$ we compute

$$\gamma \cdot \gamma^{q^i} = \left(\frac{1}{n+1} \beta - \frac{k}{n+1} \right) \cdot \left(\frac{1}{n+1} \beta^{q^i} - \frac{k}{n+1} \right)$$

$$\begin{aligned}
&= \frac{1}{(n+1)^2} \beta \cdot \beta^{q^i} - \frac{k}{(n+1)^2} \beta - \frac{k}{(n+1)^2} \beta^{q^i} + \frac{k^2}{(n+1)^2} \\
&= \frac{1}{(n+1)^2} \left(\beta^{q^{\lambda_0}} + \beta^{q^{\lambda_1}} + \dots + \beta^{q^{\lambda_{k-1}}} \right) - \frac{k}{(n+1)^2} \beta - \frac{k}{(n+1)^2} \beta^{q^i} \\
&\quad + \frac{k^2}{(n+1)^2} \\
&= \frac{1}{(n+1)} \left(\left(\frac{1}{n+1} \beta^{q^{\lambda_0}} - \frac{k}{n+1} \right) + \dots + \left(\frac{1}{n+1} \beta^{q^{\lambda_{k-1}}} - \frac{k}{n+1} \right) \right) + \\
&\quad + \frac{2k^2}{(n+1)^2} - \frac{k}{(n+1)^2} \beta - \frac{k}{(n+1)^2} \beta^{q^i} \\
&= \frac{1}{(n+1)} \left(\gamma^{q^{\lambda_0}} + \gamma^{q^{\lambda_1}} + \dots + \gamma^{q^{\lambda_{k-1}}} \right) - \frac{k}{n+1} \gamma - \frac{k}{n+1} \gamma^{q^i}.
\end{aligned}$$

Hence, the multiplication table has at most $(k+2) \cdot (m-1) + k = (k+2)m - 2$ non-zero terms, so the complexity is at most $(k+2)m - 2$.

Finally, for the row $(m-i)$ of the table C , using Theorem 4.2.1 we get,

$$\begin{aligned}
\gamma \cdot \gamma^{q^{m-i}} &= \left(\frac{1}{n+1} \beta - \frac{k}{n+1} \right) \cdot \left(\frac{1}{n+1} \beta^{q^{m-i}} - \frac{k}{n+1} \right) \\
&= \frac{1}{(n+1)^2} \beta \cdot \beta^{q^{m-i}} - \frac{k}{(n+1)^2} \beta - \frac{k}{(n+1)^2} \beta^{q^{m-i}} + \frac{k^2}{(n+1)^2} \\
&= \frac{1}{(n+1)^2} \left(\beta^{q^{\lambda_{k-1}-i}} + \dots + \beta^{q^{\lambda_0-i}} \right) - \frac{k}{(n+1)^2} \beta - \frac{k}{(n+1)^2} \beta^{q^{m-i}} \\
&\quad + \frac{k^2}{(n+1)^2} \\
&= \frac{1}{(n+1)} \left(\left(\frac{1}{n+1} \beta^{q^{\lambda_{k-1}-i}} - \frac{k}{n+1} \right) + \dots + \left(\frac{1}{n+1} \beta^{q^{\lambda_0-i}} - \frac{k}{n+1} \right) \right) \\
&\quad + \frac{2k^2}{(n+1)^2} - \frac{k}{(n+1)^2} \beta - \frac{k}{(n+1)^2} \beta^{q^{m-i}} \\
&= \frac{1}{(n+1)} \left(\gamma^{q^{\lambda_{k-1}-i}} + \dots + \gamma^{q^{\lambda_0-i}} \right) - \frac{k}{n+1} \gamma - \frac{k}{n+1} \gamma^{q^{m-i}}.
\end{aligned}$$

Thus, the row j of the multiplication table of γ is a cyclic permutation of j positions of row $(m-j)$. ■

We note that the above proof is analogous in the case where q is even with the exception that the first row contributes only 1 to the complexity as $\gamma \cdot \gamma = \gamma^2$ is an element of the normal basis generated by γ . Also, when m is even, we must use the bound that the $m/2$ row of the multiplication table C has at most m non-zero entries. The resulting complexity for q odd is bounded above by $(m-2) \cdot (k+2) + m + k = m(k+3) - k - 4$, and for q even is bounded above by $(k+3)m - 2k - 3$.

Recall that the coefficients t_i of the polynomial $h(x)$ defined in Theorem 4.3.1 are given by

$$t_i = \text{Tr}_{q^m/q} \left(\beta \cdot \beta^{q^i} \right), \quad i = 0, 1, \dots, m-1.$$

If β generates a Type II optimal normal bases, by Theorem 4.2.3 there exist $\lambda_i, \mu_i \in \mathbb{Z}_n$ such that

$$\beta \cdot \beta^{2^j} = \beta^{2^{\lambda_0}} + \beta^{2^{\mu_0}} + \dots + \beta^{\lambda_{k-1}} + \beta^{\mu_{k-1}}.$$

Thus, $t_0 = 1$ and $t_i = 0$ for $1 \leq i \leq m-1$. This provides the analogous result for Type II optimal normal bases.

Theorem 4.3.2. *Let $\alpha \in \mathbb{F}_{q^n}$ generate an optimal normal basis of Type II of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ and let $\beta = \text{Tr}_{q^n/q^m}(\alpha) \in \mathbb{F}_{q^m}$ with $m = n/k$, $k \leq m$. Then β is self-dual and consequently the complexity of the dual basis of β is $2k(m-1) + 1$.*

We summarize the results in this section in the following table.

	Type I (q odd)	Type I (q even)	Type II (q even)
m odd	$(k+2)m - 2$	$(k+2)(m-1) + 1$	$2k(m-1) + 1$
m even	$(k+3)m - k - 4$	$(k+3)m - 2k - 3$	$2k(m-1) + 1$

Table 4.1: Upper bounds on complexities for $\mathbb{F}_{q^m}$ obtained by the dual of the trace of ONBs, where $m = n/k$.

4.4 Existence of Optimal Extensions

A question that naturally arises is whether, given a prime power q and a natural number m , there exists an extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_{q^m}$, such that $\mathbb{F}_{q^n}$ has an optimal normal basis over $\mathbb{F}_q$. This is a hard question, and certainly it is not the subject of this work. We give a brief discussion of known results that provide partial answers to this and related questions.

Dirichlet's prime number theorem states that for any two integers a, d which are relatively prime, there exist infinitely many primes congruent to $a \pmod{d}$. Under the Generalized Riemann Hypothesis, for every prime p there is a primitive root $\pmod{p}$ which is less than $70(\ln p)^2$.

The extension $\mathbb{F}_{q^n}$ contains a Type I optimal normal basis over $\mathbb{F}_q$ implies that $n = \ell - 1$ for a prime ℓ and q is primitive modulo ℓ . The requirement that $\mathbb{F}_{q^n}$ is an extension of $\mathbb{F}_{q^m}$ implies that $\ell \equiv 1 \pmod{m}$. One would be interested to know if such a prime always exists and what is its order of magnitude in terms of m . This is already a refinement of Artin's conjecture on primitive roots which states that an integer $b \neq 1, -1$ which is not a perfect square is a primitive root modulo infinitely many primes [32]. We note further, that if q is a square it cannot be primitive modulo any odd prime. Suppose that q is an odd prime power. Then the work of Moree [30]

and Lenstra [24], imply that under the Generalized Riemann Hypothesis (GRH) there exist infinitely many primes ℓ such that $\ell \equiv 1 \pmod{m}$ and q is primitive modulo ℓ . Thus, under the GRH, one is assured that Type I Optimal extensions such as those used in this work exist.

The extension $\mathbb{F}_{2^n}$ contains a Type II optimal normal basis of $\mathbb{F}_2$ implies that $n = 1/2(\ell - 1)$ for a prime ℓ and that either 2 is primitive modulo ℓ or $n \equiv 1 \pmod{4}$ (equivalently $\ell \equiv 3 \pmod{4}$) and that 2 has multiplicative order n modulo ℓ . In the first case we can apply the same argument as in the Type I case above. In the second case the works of Lenstra and Moree do not apply: we still require that $\ell \equiv 1 \pmod{m}$, but now we require that 2 generates the quadratic residues modulo ℓ .

The ratio $k = n/m$ is clearly of importance for the bounds that we have given. In the terminology of this section one would like to know the smallest prime ℓ in the arithmetic progression of 1 modulo m such that q is primitive modulo ℓ . This however is a much harder question, and a good bound seems to be out of reach even under the GRH.

4.5 Summary and Comparisons

In this chapter we give low complexity normal elements for $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, when $m = n/k$ and there is an optimal normal element in $\mathbb{F}_{q^n}$. Table 4.2 gives a summary of the complexities obtained in this paper.

In practice, we are mainly interested in fields with q even where we have low

	Type I (q odd):	Type I (q even):	Type II (q even):
m odd	$(k+1)m - k$	$km - k + 1$	$2km - 2k + 1$
m even, k odd	$(k+2)m - 3k + 1$	$(k+1)m - 3k + 2$	(for all m)
m even, k even	$(k+1)m - k$	$km - k + 1$	

Table 4.2: Summary of best-case low complexities for $\mathbb{F}_{q^m}$ obtained by traces, where $m = n/k$.

complexity normal bases. As a result of our constructions, we are able to find low complexity normal elements in intermediate fields using tables from [10]. The optimal normal bases in characteristic two were completely characterized in [12], but there is still a need to find low complexity normal bases in extensions for which there is no optimal normal basis. Appendix D gives n -degree extensions of $\mathbb{F}_2$, $278 \leq n \leq 1026$, in which there exists a Type II optimal normal basis in $\mathbb{F}_{2^n}$ but no such basis exists in $\mathbb{F}_{2^{n/2}}$. We also provide the resulting complexities of the found bases.

The National Institute of Standards and Technology (NIST) recommends a series of five elliptic curves over binary fields for United States federal government use in cryptography [33]. The complexities of the normal basis representatives were found by [26], and Table 4.3 compares our best-found constructions with the NIST standard curves.

m	NIST- c_N	Our c_N
163	645	649
233	465*	465*
283	1677	-
307	-	1225
409	1629	1633
571	5637	-
577	-	2305

Table 4.3: Comparison of NIST-standard normal basis representatives of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$ with our construction.

The complexities of the extensions where $m = 163$ and $m = 409$ given by NIST satisfy the relation $c_N = 4m - 7$, which is due to the construction in Theorem 2.5.3. This construction requires finding primitive roots of unity in large composite extensions of $\mathbb{F}_{q^n}$, which is certainly computationally more difficult than finding the trace of a known optimal normal element. The basis used for $m = 233$ is a Type II ONB, and for the $m = 283$ and $m = 571$ existing tables in [10] only give extensions for which ONBs exist up to $m = 2000$, and so we could not apply our construction. We have provided complexities using our construction for extensions $m = 307$ and $m = 577$ which have the properties that their degrees are prime, close to an extension given by NIST, and the order of the multiplicative group $2^m - 1$ has no small prime factors. This could be indicative that elliptic curve cryptography is computationally desirable over these and similar fields. However, further research into the cryptographic strength of these fields is necessary.

Chapter 5

Conclusions

5.1 Summary of Results

We examine the distribution of normal bases in finite fields of characteristic two by running an exhaustive search for normal elements for all extension degrees $n \leq 39$. The search is significantly improved by the use of a Gray code ordering. We perform exhaustive searches of all self-dual normal bases for $n \leq 37$. We count the number of normal bases, tabulate the complexities of all bases and calculate their average and variance. We give plots of the number of normal bases versus the complexity of a normal basis. We use the above data to form the following conjectures.

- The number of normal bases of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ are normally distributed with respect to their complexities.
- The average complexity of a normal basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$, $n \geq 8$ is bounded

above by $(n^2 - n + 3)/2$.

- The variance of the complexities of normal bases of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ has a strong positive correlation with the average complexity of a normal basis. Loosely, the variance of the complexities of normal bases of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ is bounded above by $n^2/2$ for sufficiently large n .
- There is no constant k such that the minimum complexity of a normal basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ is bounded below by $k \cdot n$ for all n .

We give new low complexity normal bases in subfields $\mathbb{F}_{q^m}$ of $\mathbb{F}_{q^n}$, where $n = mk$ for some k and $\mathbb{F}_{q^n}$ contains an Optimal normal basis. This projection is performed by means of the field trace function. We do this for both even and odd characteristics and for both Type I and Type II Optimal normal bases. We also give the complexity of the dual bases of all bases constructed. Fixing n , these complexities depend on k and are summarized in the following tables.

	Type I (q odd):	Type I (q even):	Type II (q even):
m odd	$(k+1)m - k$	$km - k + 1$	$2km - 2k + 1$
m even, k odd	$(k+2)m - 3k + 1$	$(k+1)m - 3k + 2$	(for all m)
m even, k even	$(k+1)m - k$	$km - k + 1$	

Table 5.1: Summary of best-case low complexities for $\mathbb{F}_{q^m}$ obtained by traces, where $m = n/k$.

	Type I (q odd)	Type I (q even)	Type II (q even)
m odd	$(k+2)m - 2$	$(k+2)(m-1) + 1$	$2k(m-1) + 1$
m even	$(k+3)m - k - 4$	$(k+3)m - 2k - 3$	$2k(m-1) + 1$

Table 5.2: Upper bounds on complexities for $\mathbb{F}_{q^m}$ obtained by the dual of the trace of ONBs, where $m = n/k$.

5.1.1 Explanation of Tables in Appendices B and C

Appendix B shows findings on the complexity of normal elements for every finite field $\mathbb{F}_{2^n}$ with $n \leq 39$, computed with the implementation of the `GrayCodeNCD` algorithm. Due to the field size and the computational complexity, no search was performed for self-dual normal elements for $n \geq 37$. The optimal normal bases found are in agreement with the theorem of Gao and Lenstra [12] that characterizes for which fields optimal normal bases exist.

Appendix C is devoted to the cases when $40 \leq n \leq 607$. For these values of n , no self-dual testing has been performed. When an optimal normal element is known not to exist, we show the lowest complexity found by using three methods. In the first one we apply Theorem 2.5.1 which allows us to combine two complexities previously computed. Secondly, we apply the results obtained in Chapter 4. The final method is based on a random search, which starts with a random finite field element, and randomly flips one of its polynomial coefficients a prescribed number of times, keeping the normal element found with lowest complexity. If the first method yields the lowest complexity then we indicate in the Property column which co-prime factorization of n led us to obtain the low complexity. If the second method yields the smaller complexity we note this in the Property column by “TraceONB”, and indicate both the type of the optimal normal basis and the value of k required. Theorem 2.5.1 does not apply for n prime or prime-power, and for these values the random search is required. However, random search yields quite large minimum complexities, so it

appears only when the other methods fail. We observe that the smallest complexity found is not necessarily the minimum complexity possible in the field, and for this reason we denote best found complexity as $\min c_N$.

5.2 Further Work and Questions

This discussion of normal bases, their distribution and some constructions of low-complexity of normal bases lends itself to even more questions on these topics.

Question 1. *Can the conjectures in Chapter 3 be refined or proven? In particular, can it be shown that normal bases of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ are normally distributed with respect to their complexities? If so, can we envelope this bound and find a probabilistic (at worst) or deterministic (at best) upper-bound on the lowest-expected complexity of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$?*

Question 2. *There is an increasing interest in $\mathbb{F}_3$ for use in pairing-based cryptosystems (see [2, 7]), and in the use of elliptic curves over prime fields (see [37]). The natural extension is to ask if the above conjectures hold for general q . It should be noted that experiments over higher q , even $q = 3$ and moderate n , are computationally out of reach using the methods presented here.*

Question 3. *It was noted in the cases $n = 37, 38, 39$, when the Gray code ordering was split across multiple machines, that there were large blocks of the field (under the Gray code ordering) where no normal elements were obtained. This indicates*

some regular structure on normal elements. Similar to work performed by Cohen [5] on finding primitive normal polynomials with prescribed coefficients, it appears that normal elements could be found with prescribed structure. This seems very difficult, however, even in the binary case.

Question 4. *What can be said about the orders of certain normal elements? It seems, from tables obtained in the research pertaining to Chapter 3, that often low-complexity normal elements are primitive, but when is this true, in general? The primitive normal element theorem states that in all finite fields there exists a normal element that generates the multiplicative group of the field. Is it desirable to find these elements? Cohen [5] attempts to answer this question, though from an existence point of view.*

The conjectures given in Chapter 3 and the random search in Appendix C indicate that, though normal elements are relatively dense in finite fields, specific constructions are required to be able to find normal elements of low-complexity.

Question 5. *It is known that all linear transformations from $\mathbb{F}_{q^n}$ into $\mathbb{F}_q$ are precisely the maps $L_\beta, \beta \in \mathbb{F}_{q^n}$ where $L_\beta(\alpha) = \text{Tr}_{q^n/q}(\beta \cdot \alpha)$ for all $\alpha \in \mathbb{F}_{q^n}$. Using the methods of Chapter 4, are there any such $\beta \in \mathbb{F}_{q^n}$ such that the projection $L_\beta(\alpha)$ gives a low-complexity normal basis?*

Question 6. *In a similar vein, are there other projection maps $\mathbb{F}_{q^n}$ to $\mathbb{F}_{q^m}$ that give similar results? The natural next candidate is the field norm function (the multiplicative version of the trace). It is simple to see due to the multiplicative nature of*

the norm and the cyclic structure imposed by the Frobenius q -automorphism that the conjugates of the norm of a normal element fail to be linearly independent. Other more complex projection maps may still exist.

Question 7. *This is restating Section 4.4. The question is, given any desired q and extension degree m , what is the smallest k such that $\mathbb{F}_{q^n} = \mathbb{F}_{q^{mk}}$ admits an optimal normal basis? Does such a k exist for all q and m ? This is a deep question related to the Generalized Riemann Hypothesis, but a study for certain desired values of q and m could be of interest.*

Question 8. *In Section 4.5 we give a table of field extensions recommended by NIST for elliptic curve cryptography and compare them to fields which have some similar properties (such as prime degree, etc.) but to which the traces obtained in Chapter 4 yield low complexity normal bases. A further study of these fields is necessary to determine their cryptographic strength. If not, are there other fields with similar properties to which our construction applies?*

Appendix A

Optimal Normal Bases over $\mathbb{F}_2$

2†	113	293	473	676*	873	1110	1310	1533	1790
3	119	299	483	683	876*	1116*	1323	1539	1791
4*	130*	303	490*	686	879	1118	1329	1541	1806
5	131	306	491	690	882*	1119	1331	1548*	1811
6	134	309	495	700*	891	1121	1338	1559	1818
9	135	316*	508*	708*	893	1122*	1341	1570*	1821
10*	138*	323	509	713	906*	1133	1346	1583	1829
11	146	326	515	719	911	1134	1349	1593	1835
12*	148*	329	519	723	923	1146	1353	1601	1838
14	155	330	522*	725	930	1154	1355	1618*	1845
18†	158	338	530	726	933	1155	1359	1620*	1850
23	162*	346*	531	741	935	1166	1370	1626	1854
26	172*	348*	540*	743	938	1169	1372*	1636*	1859
28*	173	350	543	746	939	1170*	1380*	1649	1860*
29	174	354	545	749	940*	1178	1394	1653	1863
30	178*	359	546*	755	946*	1185	1398	1659	1866†
33	179	371	554	756*	950	1186*	1401	1661	1876*
35	180*	372*	556*	761	953	1194	1409	1666*	1883
36*	183	375	558	765	965	1199	1418	1668*	1889
39	186	378†	561	771	974	1211	1421	1673	1898
41	189	378*	562*	772*	975	1212*	1425	1679	1900*
50	191	386	575	774	986	1218	1426*	1685	1901
51	194	388*	585	779	989	1223	1430	1692*	1906*
52*	196*	393	586*	783	993	1228*	1439	1703	1923
53	209	398	593	785	998	1229	1443	1706	1925
58*	210†	410	606	786*	1013	1233	1450*	1730	1926
60*	221	411	611	791	1014	1236*	1451	1732*	1930*
65	226*	413	612*	796*	1018*	1238	1452*	1733	1931
66*	230	414	614	803	1019	1251	1454	1734	1938
69	231	418*	615	809	1026	1258*	1463	1740*	1948*
74	233	419	618†	810	1031	1265	1469	1745	1953

81	239	420*	629	818	1034	1269	1478	1746*	1955
82*	243	426	638	820*	1041	1271	1481	1749	1958
83	245	429	639	826*	1043	1274	1482*	1755	1959
86	251	431	641	828*	1049	1275	1492*	1758	1961
89	254	438	645	831	1055	1276*	1498*	1763	1965
90	261	441	650	833	1060*	1278	1499	1766	1972*
95	268*	442*	651	834	1065	1282*	1505	1769	1973
98	270	443	652*	846	1070	1289	1509	1773	1978*
99	273	453	653	852*	1090*	1290*	1511	1778	1983
100*	278	460*	658*	858*	1103	1295	1518	1779	1986*
105	281	466*	659	866	1106	1300*	1522*	1785	1994
106*	292*	470	660*	870	1108*	1306*	1530*	1786*	1996*

* Denotes existence of a Type I optimal normal basis

† Denotes existence of both Type I and Type II optimal normal bases

Appendix B

Exhaustive Search of $\mathbb{F}_{2^n}, n < 40$

This table displays the results found by the exhaustive search described in Chapter 3. The table gives the number of normal bases found $((\Phi_2(x^n - 1))/n)$, smallest and largest complexities (m_{c_N}, M_{c_N}) , average complexity, variance and standard deviation $(Avg_{c_N}, Var_{c_N}, \sigma_{c_N})$ of all complexities, then the smallest and largest complexities for self-dual normal elements.

n	$\frac{\Phi_2(x^n - 1)}{n}$						self-dual		Notes
		m_{c_N}	M_{c_N}	Avg_{c_N}	Var_{c_N}	σ_{c_N}	m_{c_N}	M_{c_N}	
2	1	3	3	3.00	0	0	3	3	Optimal, sd
3	1	5	5	5.00	0	0	5	5	Optimal, sd
4	2	7	9	8.00	1.00	1.00	-	-	
5	3	9	15	11.67	6.20	2.49	9	9	Optimal, sd
6	4	11	17	15.00	6.00	2.45	11	15	Optimal, sd
7	7	19	27	23.00	9.12	3.02	21	21	3n-2
8	16	21	35	29.00	11.02	3.32	-	-	3n-3
9	21	17	45	35.57	41.6	6.45	17	29	Optimal, sd
10	48	19	61	44.83	61.31	7.83	27	51	
11	93	21	71	55.82	57.61	7.59	21	57	Optimal, sd
12	128	23	83	64.13	139.48	11.81	-	-	
13	315	45	101	78.38	71.06	8.43	45	81	Best, sd
14	448	27	135	91.07	108.37	10.41	27	135	Optimal, sd
15	675	45	137	105.89	127.46	11.29	45	105	Best, sd
16	2048	85	157	115.82	731.7	27.05	-	-	
17	3825	81	177	132.77	671.84	25.92	81	171	Best, sd
18	5376	35	243	153.51	189.5	13.77	35	243	Optimal, sd
19	13797	117	229	172.00	174.05	13.19	117	201	Best, sd
20	24576	63	257	190.80	207.28	14.40	-	-	
21	27783	95	277	210.97	216.43	14.71	105	237	
22	95232	63	363	231.93	238.56	15.45	63	363	3n-3
23	182183	45	325	254.02	254.6	15.96	45	309	Optimal, sd
24	262144	105	375	276.82	281.01	16.76	-	-	
25	629145	93	383	301.01	300.37	17.33	93	357	Best, sd
26	1290240	51	555	325.96	328.59	18.13	51	555	Optimal, sd
27	1835001	141	443	351.99	351.38	18.75	141	413	
28	3670016	55	517	378.98	379.12	19.47	-	-	Optimal
29	9256395	57	521	407.00	406.22	20.15	57	465	Optimal, sd
30	11059200	59	759	435.95	438.52	20.94	59	759	Optimal, sd
31	28629151	237	587	466.00	465.2	21.57	237	537	Best, sd
32	67108864	361	621	497.00	495.95	22.27	-	-	
33	97327197	65	693	529.00	528.48	22.99	65	693	Optimal, sd

							self-dual		
n	$\frac{\Phi_2(x^n-1)}{n}$	m_{c_N}	M_{c_N}	Avg_{c_N}	Var_{c_N}	σ_{c_N}	m_{c_N}	M_{c_N}	Notes
34	250675200	243	819	562.00	561.52	23.70	243	819	Best, sd
35	352149515	69	779	596.00	595.03	24.39	69	693	Optimal, sd
36	704643060	71	1017	630.99	630.51	25.11	-	-	Optimal
37	1857283155	171	823	667.00	666.04	25.81			
38	3616800703	207	1131	704.00	703.18	26.52			
39	5282242828	77	933	742.00	741.09	27.22			Optimal

Appendix C

Complexities Found $40 \leq n \leq 607$

n	$\min c_N$	Property	Method	n	$\min c_N$	Property	Method
40	189	5,8	Thm 2.5.1	84	275	3,28	Thm 2.5.1
41	81	Optimal	[31]	85	729	5,17	Thm 2.5.1
42	135	3,14	Thm 2.5.1	86	171	Optimal	[31]
43	169	TraceONB	Type 2, $k = 2$	87	285	3,29	Thm 2.5.1
44	147	4,11	Thm 2.5.1	88	441	8,11	Thm 2.5.1
45	153	5,9	Thm 2.5.1	89	177	Optimal	[31]
46	135	2,23	Thm 2.5.1	90	179	Optimal	[31]
47	645	TraceONB	Type 2, $k = 7$	91	541	TraceONB	Type 2, $k = 3$
48	425	3,16	Thm 2.5.1	92	315	4,23	Thm 2.5.1
49	193	TraceONB	Type 2, $k = 2$	93	369	TraceONB	Type 2, $k = 2$
50	99	Optimal	[31]	94	931	TraceONB	Type 2, $k = 5$
51	101	Optimal	[31]	95	189	Optimal	[31]
52	103	Optimal	[31]	96	1805	3,32	Thm 2.5.1
53	105	Optimal	[31]	97	385	TraceONB	Type 2, $k = 2$
54	209	TraceONB	Type 1, $k = 3$	98	195	Optimal	[31]
55	189	5,11	Thm 2.5.1	99	197	Optimal	[31]
56	399	7,8	Thm 2.5.1	100	199	Optimal	[31]
57	585	3,19	Thm 2.5.1	101	601	TraceONB	Type 2, $k = 3$
58	115	Optimal	[31]	102	303	2,51	Thm 2.5.1
59	697	TraceONB	Type 2, $k = 6$	103	613	TraceONB	Type 2, $k = 3$
60	119	Optimal	[31]	104	945	8,13	Thm 2.5.1
61	361	TraceONB	Type 2, $k = 3$	105	209	Optimal	[31]
62	367	TraceONB	Type 2, $k = 3$	106	211	Optimal	[31]
63	323	7,9	Thm 2.5.1	107	1485	TraceONB	Type 2, $k = 7$
64	1829	Prime Power	Random	108	635	TraceONB	Type 1, $k = 5$
65	129	Optimal	[31]	109	1081	TraceONB	Type 2, $k = 5$
66	131	Optimal	[31]	110	399	10,11	Thm 2.5.1
67	265	TraceONB	Type 2, $k = 2$	111	2201	TraceONB	Type 2, $k = 10$
68	567	4,17	Thm 2.5.1	112	1615	7,16	Thm 2.5.1
69	137	Optimal	[31]	113	225	Optimal	[31]
70	207	2,35	Thm 2.5.1	114	1287	6,19	Thm 2.5.1
71	841	TraceONB	Type 2, $k = 6$	115	405	5,23	Thm 2.5.1
72	357	8,9	Thm 2.5.1	116	399	4,29	Thm 2.5.1
73	289	TraceONB	Type 2, $k = 2$	117	765	9,13	Thm 2.5.1
74	147	Optimal	[31]	118	703	TraceONB	Type 2, $k = 3$
75	465	3,25	Thm 2.5.1	119	237	Optimal	[31]
76	819	4,19	Thm 2.5.1	120	945	3,40	Thm 2.5.1
77	399	7,11	Thm 2.5.1	121	1441	TraceONB	Type 2, $k = 6$
78	231	2,39	Thm 2.5.1	122	4599	TraceONB	Type 2, $k = 19$
79	313	TraceONB	Type 2, $k = 2$	123	405	3,41	Thm 2.5.1
80	765	5,16	Thm 2.5.1	124	489	TraceONB	Type 1, $k = 3$
81	161	Optimal	[31]	125	745	TraceONB	Type 2, $k = 3$
82	163	Optimal	[31]	126	459	9,14	Thm 2.5.1
83	165	Optimal	[31]	127	505	TraceONB	Type 2, $k = 2$

n	$\min c_N$	Property	Method	n	$\min c_N$	Property	Method
128	7821	Prime Power	Random	188	1115	TraceONB	Type 1, $k = 5$
129	1281	TraceONB	Type 2, $k = 5$	189	377	Optimal	[31]
130	259	Optimal	[31]	190	567	2,95	Thm 2.5.1
131	261	Optimal	[31]	191	381	Optimal	[31]
132	455	4,33	Thm 2.5.1	192	9145	3,64	Thm 2.5.1
133	2223	7,19	Thm 2.5.1	193	769	TraceONB	Type 2, $k = 2$
134	267	Optimal	[31]	194	387	Optimal	[31]
135	269	Optimal	[31]	195	645	3,65	Thm 2.5.1
136	1701	8,17	Thm 2.5.1	196	391	Optimal	[31]
137	817	TraceONB	Type 2, $k = 3$	197	3529	TraceONB	Type 2, $k = 9$
138	275	Optimal	[31]	198	591	2,99	Thm 2.5.1
139	553	TraceONB	Type 2, $k = 2$	199	793	TraceONB	Type 2, $k = 2$
140	483	4,35	Thm 2.5.1	200	1953	8,25	Thm 2.5.1
141	1681	TraceONB	Type 2, $k = 6$	201	4001	TraceONB	Type 2, $k = 10$
142	847	TraceONB	Type 2, $k = 3$	202	1207	TraceONB	Type 2, $k = 3$
143	853	TraceONB	Type 2, $k = 3$	203	1083	7,29	Thm 2.5.1
144	1445	9,16	Thm 2.5.1	204	707	4,51	Thm 2.5.1
145	513	5,29	Thm 2.5.1	205	729	5,41	Thm 2.5.1
146	291	Optimal	[31]	206	817	TraceONB	Type 1, $k = 3$
147	877	TraceONB	Type 2, $k = 3$	207	765	9,23	Thm 2.5.1
148	295	Optimal	[31]	208	3825	13,16	Thm 2.5.1
149	2073	TraceONB	Type 2, $k = 7$	209	417	Optimal	[31]
150	495	3,50	Thm 2.5.1	210	419	Optimal	[31]
151	901	TraceONB	Type 2, $k = 3$	211	2101	TraceONB	Type 2, $k = 5$
152	2457	8,19	Thm 2.5.1	212	735	4,53	Thm 2.5.1
153	609	TraceONB	Type 2, $k = 2$	213	849	TraceONB	Type 2, $k = 2$
154	567	11,14	Thm 2.5.1	214	2131	TraceONB	Type 2, $k = 5$
155	309	Optimal	[31]	215	1285	TraceONB	Type 2, $k = 3$
156	515	3,52	Thm 2.5.1	216	2961	8,27	Thm 2.5.1
157	1561	TraceONB	Type 2, $k = 5$	217	1297	TraceONB	Type 2, $k = 3$
158	315	Optimal	[31]	218	1295	TraceONB	Type 1, $k = 5$
159	525	3,53	Thm 2.5.1	219	873	TraceONB	Type 2, $k = 2$
160	3249	5,32	Thm 2.5.1	220	873	TraceONB	Type 1, $k = 3$
161	855	7,23	Thm 2.5.1	221	441	Optimal	[31]
162	323	Optimal	[31]	222	735	3,74	Thm 2.5.1
163	649	TraceONB	Type 2, $k = 2$	223	2665	TraceONB	Type 2, $k = 6$
164	567	4,41	Thm 2.5.1	224	6859	7,32	Thm 2.5.1
165	585	5,33	Thm 2.5.1	225	1581	9,25	Thm 2.5.1
166	495	2,83	Thm 2.5.1	226	451	Optimal	[31]
167	2325	TraceONB	Type 2, $k = 7$	227	5877	TraceONB	Type 2, $k = 13$
168	1995	3,56	Thm 2.5.1	228	2255	TraceONB	Type 1, $k = 9$
169	673	TraceONB	Type 2, $k = 2$	229	5017	TraceONB	Type 2, $k = 11$
170	1539	10,17	Thm 2.5.1	230	459	Optimal	[31]
171	1989	9,19	Thm 2.5.1	231	461	Optimal	[31]
172	343	Optimal	[31]	232	1197	8,29	Thm 2.5.1
173	345	Optimal	[31]	233	465	Optimal	[31]
174	347	Optimal	[31]	234	867	9,26	Thm 2.5.1
175	697	TraceONB	Type 2, $k = 2$	235	937	TraceONB	Type 2, $k = 2$
176	1785	11,16	Thm 2.5.1	236	937	TraceONB	Type 1, $k = 3$
177	705	TraceONB	Type 2, $k = 2$	237	2361	TraceONB	Type 2, $k = 5$
178	355	Optimal	[31]	238	711	2,119	Thm 2.5.1
179	357	Optimal	[31]	239	477	Optimal	[31]
180	359	Optimal	[31]	240	3825	3,80	Thm 2.5.1
181	1081	TraceONB	Type 2, $k = 3$	241	1441	TraceONB	Type 2, $k = 3$
182	721	TraceONB	Type 1, $k = 3$	242	1447	TraceONB	Type 2, $k = 3$
183	365	Optimal	[31]	243	485	Optimal	[31]
184	945	8,23	Thm 2.5.1	244	4825	TraceONB	Type 1, $k = 19$
185	2209	TraceONB	Type 2, $k = 6$	245	489	Optimal	[31]
186	371	Optimal	[31]	246	815	3,82	Thm 2.5.1
187	1117	TraceONB	Type 2, $k = 3$	247	1477	TraceONB	Type 2, $k = 3$

n	$\min c_N$	Property	Method	n	$\min c_N$	Property	Method
248	4977	8,31	Thm 2.5.1	308	1155	11,28	Thm 2.5.1
249	825	3,83	Thm 2.5.1	309	617	Optimal	[31]
250	5479	TraceONB	Type 2, $k = 11$	310	927	2,155	Thm 2.5.1
251	501	Optimal	[31]	311	1861	TraceONB	Type 2, $k = 3$
252	935	9,28	Thm 2.5.1	312	1617	8,39	Thm 2.5.1
253	945	11,23	Thm 2.5.1	313	1873	TraceONB	Type 2, $k = 3$
254	507	Optimal	[31]	314	1871	TraceONB	Type 1, $k = 5$
255	909	5,51	Thm 2.5.1	315	1173	9,35	Thm 2.5.1
256	32033	Prime Power	Random	316	631	Optimal	[31]
257	1537	TraceONB	Type 2, $k = 3$	317	8217	TraceONB	Type 2, $k = 13$
258	855	3,86	Thm 2.5.1	318	1055	3,106	Thm 2.5.1
259	2581	TraceONB	Type 2, $k = 5$	319	1197	11,29	Thm 2.5.1
260	903	4,65	Thm 2.5.1	320	16461	5,64	Thm 2.5.1
261	521	Optimal	[31]	321	3841	TraceONB	Type 2, $k = 6$
262	783	2,131	Thm 2.5.1	322	1215	14,23	Thm 2.5.1
263	5241	TraceONB	Type 2, $k = 10$	323	645	Optimal	[31]
264	1365	8,33	Thm 2.5.1	324	1127	4,81	Thm 2.5.1
265	945	5,53	Thm 2.5.1	325	1297	TraceONB	Type 2, $k = 2$
266	3159	14,19	Thm 2.5.1	326	651	Optimal	[31]
267	885	3,89	Thm 2.5.1	327	14345	TraceONB	Type 2, $k = 22$
268	535	Optimal	[31]	328	1701	8,41	Thm 2.5.1
269	3753	TraceONB	Type 2, $k = 7$	329	657	Optimal	[31]
270	539	Optimal	[31]	330	659	Optimal	[31]
271	2701	TraceONB	Type 2, $k = 5$	331	1981	TraceONB	Type 2, $k = 3$
272	6885	16,17	Thm 2.5.1	332	1155	4,83	Thm 2.5.1
273	545	Optimal	[31]	333	8721	9,37	Thm 2.5.1
274	2731	TraceONB	Type 2, $k = 5$	334	2653	TraceONB	Type 1, $k = 7$
275	1953	11,25	Thm 2.5.1	335	4009	TraceONB	Type 2, $k = 6$
276	959	4,69	Thm 2.5.1	336	8075	3,112	Thm 2.5.1
277	1105	TraceONB	Type 2, $k = 2$	337	3361	TraceONB	Type 2, $k = 5$
278	555	Optimal	[31]	338	675	Optimal	[31]
279	1113	TraceONB	Type 2, $k = 2$	339	1125	3,113	Thm 2.5.1
280	1449	8,35	Thm 2.5.1	340	5103	4,85	Thm 2.5.1
281	561	Optimal	[31]	341	4081	TraceONB	Type 2, $k = 6$
282	1687	TraceONB	Type 2, $k = 3$	342	2047	TraceONB	Type 2, $k = 3$
283	11845	TraceONB	Type 2, $k = 21$	343	1369	TraceONB	Type 2, $k = 2$
284	1129	TraceONB	Type 1, $k = 3$	344	15771	8,43	Thm 2.5.1
285	945	3,95	Thm 2.5.1	345	1233	5,69	Thm 2.5.1
286	1071	11,26	Thm 2.5.1	346	691	Optimal	[31]
287	1539	7,41	Thm 2.5.1	347	2077	TraceONB	Type 2, $k = 3$
288	6137	9,32	Thm 2.5.1	348	695	Optimal	[31]
289	3457	TraceONB	Type 2, $k = 6$	349	3481	TraceONB	Type 2, $k = 5$
290	1035	5,58	Thm 2.5.1	350	699	Optimal	[31]
291	1741	TraceONB	Type 2, $k = 3$	351	3501	TraceONB	Type 2, $k = 5$
292	583	Optimal	[31]	352	7581	11,32	Thm 2.5.1
293	585	Optimal	[31]	353	4929	TraceONB	Type 2, $k = 7$
294	975	3,98	Thm 2.5.1	354	707	Optimal	[31]
295	6469	TraceONB	Type 2, $k = 11$	355	2125	TraceONB	Type 2, $k = 3$
296	10773	8,37	Thm 2.5.1	356	1239	4,89	Thm 2.5.1
297	1777	TraceONB	Type 2, $k = 3$	357	1185	3,119	Thm 2.5.1
298	2955	TraceONB	Type 1, $k = 9$	358	1071	2,179	Thm 2.5.1
299	597	Optimal	[31]	359	717	Optimal	[31]
300	995	3,100	Thm 2.5.1	360	3213	5,72	Thm 2.5.1
301	3001	TraceONB	Type 2, $k = 5$	361	10801	TraceONB	Type 2, $k = 15$
302	1201	TraceONB	Type 1, $k = 3$	362	10831	TraceONB	Type 2, $k = 15$
303	605	Optimal	[31]	363	1449	TraceONB	Type 2, $k = 2$
304	9945	16,19	Thm 2.5.1	364	1957	7,52	Thm 2.5.1
305	6081	TraceONB	Type 2, $k = 10$	365	9465	TraceONB	Type 2, $k = 13$
306	611	Optimal	[31]	366	1095	2,183	Thm 2.5.1
307	1225	TraceONB	Type 2, $k = 2$	367	3661	TraceONB	Type 2, $k = 5$

n	$\min c_N$	Property	Method	n	$\min c_N$	Property	Method
368	3825	16,23	Thm 2.5.1	428	2555	TraceONB	Type 1, $k = 5$
369	1377	9,41	Thm 2.5.1	429	857	Optimal	[31]
370	1323	5,74	Thm 2.5.1	430	1539	5,86	Thm 2.5.1
371	741	Optimal	[31]	431	861	Optimal	[31]
372	743	Optimal	[31]	432	11985	16,27	Thm 2.5.1
373	1489	TraceONB	Type 2, $k = 2$	433	1729	TraceONB	Type 2, $k = 2$
374	1489	TraceONB	Type 1, $k = 3$	434	4315	TraceONB	Type 1, $k = 9$
375	749	Optimal	[31]	435	1737	TraceONB	Type 2, $k = 2$
376	19173	8,47	Thm 2.5.1	436	14727	TraceONB	Type 1, $k = 33$
377	2565	13,29	Thm 2.5.1	437	5265	19,23	Thm 2.5.1
378	755	Optimal	[31]	438	875	Optimal	[31]
379	6805	TraceONB	Type 2, $k = 9$	439	4381	TraceONB	Type 2, $k = 5$
380	1323	4,95	Thm 2.5.1	440	3969	5,88	Thm 2.5.1
381	7601	TraceONB	Type 2, $k = 10$	441	881	Optimal	[31]
382	1143	2,191	Thm 2.5.1	442	883	Optimal	[31]
383	16045	TraceONB	Type 2, $k = 21$	443	885	Optimal	[31]
384	39105	3,128	Thm 2.5.1	444	1475	3,148	Thm 2.5.1
385	1449	11,35	Thm 2.5.1	445	1593	5,89	Thm 2.5.1
386	771	Optimal	[31]	446	2671	TraceONB	Type 2, $k = 3$
387	1545	TraceONB	Type 2, $k = 2$	447	2677	TraceONB	Type 2, $k = 3$
388	775	Optimal	[31]	448	34751	7,64	Thm 2.5.1
389	14745	TraceONB	Type 2, $k = 19$	449	6273	TraceONB	Type 2, $k = 7$
390	1295	3,130	Thm 2.5.1	450	1683	9,50	Thm 2.5.1
391	3645	17,23	Thm 2.5.1	451	1701	11,41	Thm 2.5.1
392	21021	8,49	Thm 2.5.1	452	1575	4,113	Thm 2.5.1
393	785	Optimal	[31]	453	905	Optimal	[31]
394	3915	TraceONB	Type 1, $k = 9$	454	9025	TraceONB	Type 1, $k = 19$
395	2365	TraceONB	Type 2, $k = 3$	455	2451	7,65	Thm 2.5.1
396	1379	4,99	Thm 2.5.1	456	12285	3,152	Thm 2.5.1
397	22969	TraceONB	Type 2, $k = 29$	457	13681	TraceONB	Type 2, $k = 15$
398	795	Optimal	[31]	458	11883	TraceONB	Type 2, $k = 13$
399	4777	TraceONB	Type 2, $k = 6$	459	4581	TraceONB	Type 2, $k = 5$
400	7905	16,25	Thm 2.5.1	460	919	Optimal	[31]
401	4801	TraceONB	Type 2, $k = 6$	461	9201	TraceONB	Type 2, $k = 10$
402	1335	3,134	Thm 2.5.1	462	1383	2,231	Thm 2.5.1
403	8845	TraceONB	Type 2, $k = 11$	463	5545	TraceONB	Type 2, $k = 6$
404	1609	TraceONB	Type 1, $k = 3$	464	4845	16,29	Thm 2.5.1
405	1449	5,81	Thm 2.5.1	465	1545	3,155	Thm 2.5.1
406	1539	14,29	Thm 2.5.1	466	931	Optimal	[31]
407	10773	11,37	Thm 2.5.1	467	2797	TraceONB	Type 2, $k = 3$
408	2121	8,51	Thm 2.5.1	468	1751	9,52	Thm 2.5.1
409	1633	TraceONB	Type 2, $k = 2$	469	1873	TraceONB	Type 2, $k = 2$
410	819	Optimal	[31]	470	939	Optimal	[31]
411	821	Optimal	[31]	471	9401	TraceONB	Type 2, $k = 10$
412	1641	TraceONB	Type 1, $k = 3$	472	32067	8,59	Thm 2.5.1
413	825	Optimal	[31]	473	945	Optimal	[31]
414	827	Optimal	[31]	474	1575	3,158	Thm 2.5.1
415	1485	5,83	Thm 2.5.1	475	1897	TraceONB	Type 2, $k = 2$
416	16245	13,32	Thm 2.5.1	476	1659	4,119	Thm 2.5.1
417	1665	TraceONB	Type 2, $k = 2$	477	1785	9,53	Thm 2.5.1
418	835	Optimal	[31]	478	1431	2,239	Thm 2.5.1
419	837	Optimal	[31]	479	5737	TraceONB	Type 2, $k = 6$
420	839	Optimal	[31]	480	16245	3,160	Thm 2.5.1
421	11761	TraceONB	Type 2, $k = 14$	481	2881	TraceONB	Type 2, $k = 3$
422	10947	TraceONB	Type 2, $k = 13$	482	8659	TraceONB	Type 2, $k = 9$
423	1689	TraceONB	Type 2, $k = 2$	483	965	Optimal	[31]
424	2205	8,53	Thm 2.5.1	484	1929	TraceONB	Type 1, $k = 3$
425	2545	TraceONB	Type 2, $k = 3$	485	8713	TraceONB	Type 2, $k = 9$
426	851	Optimal	[31]	486	1455	2,243	Thm 2.5.1
427	7669	TraceONB	Type 2, $k = 9$	487	1945	TraceONB	Type 2, $k = 2$

n	$\min c_N$	Property	Method	n	$\min c_N$	Property	Method
488	34041	8,61	Thm 2.5.1	548	3275	TraceONB	Type 1, $k = 5$
489	5857	TraceONB	Type 2, $k = 6$	549	7673	TraceONB	Type 2, $k = 7$
490	979	Optimal	[31]	550	2079	11,50	Thm 2.5.1
491	981	Optimal	[31]	551	3301	TraceONB	Type 2, $k = 3$
492	1863	12,41	Thm 2.5.1	552	2877	8,69	Thm 2.5.1
493	1969	TraceONB	Type 2, $k = 2$	553	2209	TraceONB	Type 2, $k = 2$
494	1969	TraceONB	Type 1, $k = 3$	554	1107	Optimal	[31]
495	989	Optimal	[31]	555	2217	TraceONB	Type 2, $k = 2$
496	20145	16,31	Thm 2.5.1	556	1111	Optimal	[31]
497	9921	TraceONB	Type 2, $k = 10$	557	24465	TraceONB	Type 2, $k = 22$
498	1815	6,83	Thm 2.5.1	558	1115	Optimal	[31]
499	1993	TraceONB	Type 2, $k = 2$	559	2233	TraceONB	Type 2, $k = 2$
500	5969	TraceONB	Type 1, $k = 11$	560	5865	16,35	Thm 2.5.1
501	5001	TraceONB	Type 2, $k = 5$	561	1121	Optimal	[31]
502	1503	2,251	Thm 2.5.1	562	1123	Optimal	[31]
503	3013	TraceONB	Type 2, $k = 3$	563	7869	TraceONB	Type 2, $k = 7$
504	6783	7,72	Thm 2.5.1	564	2249	TraceONB	Type 1, $k = 3$
505	5041	TraceONB	Type 2, $k = 5$	565	2025	5,113	Thm 2.5.1
506	2835	2,253	Thm 2.5.1	566	14691	TraceONB	Type 2, $k = 13$
507	2025	TraceONB	Type 2, $k = 2$	567	2265	TraceONB	Type 2, $k = 2$
508	1015	Optimal	[31]	568	48615	8,71	Thm 2.5.1
509	1017	Optimal	[31]	569	6817	TraceONB	Type 2, $k = 6$
510	1919	10,51	Thm 2.5.1	570	2079	6,95	Thm 2.5.1
511	3061	TraceONB	Type 2, $k = 3$	571	26221	TraceONB	Type 2, $k = 23$
512	131072	Prime Power	Random	572	2163	11,52	Thm 2.5.1
513	2049	TraceONB	Type 2, $k = 2$	573	1905	3,191	Thm 2.5.1
514	17443	TraceONB	Type 2, $k = 17$	574	2187	14,41	Thm 2.5.1
515	1029	Optimal	[31]	575	1149	Optimal	[31]
516	1715	3,172	Thm 2.5.1	576	31093	9,64	Thm 2.5.1
517	2065	TraceONB	Type 2, $k = 2$	577	2305	TraceONB	Type 2, $k = 2$
518	2793	7,74	Thm 2.5.1	578	3463	TraceONB	Type 2, $k = 3$
519	1037	Optimal	[31]	579	5781	TraceONB	Type 2, $k = 5$
520	2709	8,65	Thm 2.5.1	580	2313	TraceONB	Type 1, $k = 3$
521	18721	TraceONB	Type 2, $k = 18$	581	3135	7,83	Thm 2.5.1
522	1043	Optimal	[31]	582	1935	3,194	Thm 2.5.1
523	5221	TraceONB	Type 2, $k = 5$	583	2205	11,53	Thm 2.5.1
524	1827	4,131	Thm 2.5.1	584	50211	8,73	Thm 2.5.1
525	8835	3,175	Thm 2.5.1	585	1169	Optimal	[31]
526	5251	TraceONB	Type 2, $k = 5$	586	1171	Optimal	[31]
527	18937	TraceONB	Type 2, $k = 18$	587	8205	TraceONB	Type 2, $k = 7$
528	5525	16,33	Thm 2.5.1	588	1955	3,196	Thm 2.5.1
529	14785	TraceONB	Type 2, $k = 14$	589	2353	TraceONB	Type 2, $k = 2$
530	1059	Optimal	[31]	590	7049	TraceONB	Type 1, $k = 11$
531	1061	Optimal	[31]	591	3541	TraceONB	Type 2, $k = 3$
532	5295	TraceONB	Type 1, $k = 9$	592	43605	16,37	Thm 2.5.1
533	3645	13,41	Thm 2.5.1	593	1185	Optimal	[31]
534	1775	3,178	Thm 2.5.1	594	8883	2,297	Thm 2.5.1
535	2137	TraceONB	Type 2, $k = 2$	595	2133	5,119	Thm 2.5.1
536	42063	8,67	Thm 2.5.1	596	10679	TraceONB	Type 1, $k = 17$
537	1785	3,179	Thm 2.5.1	597	2385	TraceONB	Type 2, $k = 2$
538	24703	TraceONB	Type 2, $k = 23$	598	1791	2,299	Thm 2.5.1
539	6457	TraceONB	Type 2, $k = 6$	599	8373	TraceONB	Type 2, $k = 7$
540	1079	Optimal	[31]	600	9765	3,200	Thm 2.5.1
541	11881	TraceONB	Type 2, $k = 11$	601	6001	TraceONB	Type 2, $k = 5$
542	3247	TraceONB	Type 2, $k = 3$	602	3249	7,86	Thm 2.5.1
543	1085	Optimal	[31]	603	7225	TraceONB	Type 2, $k = 6$
544	29241	17,32	Thm 2.5.1	604	4813	TraceONB	Type 1, $k = 7$
545	1089	Optimal	[31]	605	10873	TraceONB	Type 2, $k = 9$
546	1091	Optimal	[31]	606	1211	Optimal	[31]
547	25117	TraceONB	Type 2, $k = 23$	607	3637	TraceONB	Type 2, $k = 3$

Appendix D

Low-Complexity Subfields

Type I q even, $m = n/4$

n	$m = n/4$	$C_m = 4m - 3$	n	$m = n/4$	$C_m = 4m - 3$
52	13	49	612	153	609
60	15	57	652	163	649
100	25	97	660	165	657
148	37	145	676	169	673
172	43	169	700	175	697
180	45	177	708	177	705
196	49	193	756	189	753
268	67	265	772	193	769
292	73	289	796	199	793
316	79	313	820	205	817
348	87	345	828	207	825
372	93	369	852	213	849
388	97	385	876	219	873
420	105	417	940	235	937
460	115	457	1060	265	1057
508	127	505	1108	277	1105
540	135	537	1116	279	1113
556	139	553			

Type II $m = n/2$

n	$m = n/2$	$C_m = 4m - 3$	n	$m = n/2$	$C_m = 4m - 3$
278	139	553	650	325	1297
306	153	609	686	343	1369
326	163	649	690	345	1377
330	165	657	726	363	1449
338	169	673	746	373	1489
350	175	697	774	387	1545
354	177	705	810	405	1617
386	193	769	818	409	1633
398	199	793	834	417	1665
410	205	817	846	423	1689
414	207	825	866	433	1729
426	213	849	870	435	1737
438	219	873	930	465	1857
470	235	937	938	469	1873
530	265	1057	950	475	1897
554	277	1105	974	487	1945
558	279	1113	986	493	1969
614	307	1225	998	499	1993
638	319	1273	1026	513	2049

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