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COZERO-SETS AND FUNCTION SPACE TOPOLOGIES

by

Denis Riordan

A thesis submitted to the Faculty of
Graduate Studies in partial fulfilment
of the requirements of the degree of

Doctor of Philosophy

Department of Mathematics

Carleton University

Ottawa, Ontario

September 5, 1972

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Cozero-sets and function space topologies

by

Denis Riordan

ABSTRACT

In the ring $C(X)$ of all continuous real-valued functions on the completely regular space X the set $C_K(X)$ of functions vanishing on the complements of compact sets forms an ideal about which a considerable literature has accumulated. In this dissertation a special study is made of the larger ideal $C_S(X)$ formed by the functions vanishing on the complements of pseudocompact subsets, together with several other ideals closely related to it.

In the first chapters the results relate algebraic properties of the ring $C(X)$ with topological properties of X . A typical example is the following purely algebraic characterization obtained for $C_S(X)$: it is precisely the intersection of all hyper-real maximal ideals of $C(X)$. This result supplements in a satisfactory way the existing information on intersection of various classes of maximal ideals of $C(X)$. It is shown inter alia that $C_S(X) = C_K(\cup X)$ and analogues of the above facts are proved for the subring $C^*(X)$ of bounded functions in $C(X)$ where the role of $C_S(X)$ is taken over by the larger ideal $C_S^\infty(X)$ consisting of all f in $C(X)$ for which the sets $\{x : |f(x)| \geq \frac{1}{n}\}$ are pseudocompact for every natural number n .

With X specialized to be locally pseudocompact it is shown inter alia that $C_S(X)$ is a proper free ideal if and only if X is locally pseudocompact but not pseudocompact.

λ -compact spaces are introduced and studied; they are the spaces for which the intersection of all free real maximal ideals in $C(X)$ agrees with the intersection of all free real O^P -ideals. It is shown that these spaces fall naturally between realcompact spaces and the μ -compact spaces introduced recently by Mandelker.

In the latter part of the dissertation $C(X)$ is topologized by Hewitt's m -topology and also by Naimpally's graph topology. Some connections between these topologies and previous results are established. The ideal $C_S(X)$ turns out to be precisely the connected component of 0 in the m -topology of $C(X)$. An analogous fact for the graph topology is also proved. Moreover the spaces X for which the m -topology and graph topology on $C(X)$ coincide turn out to be precisely the cb -spaces studied by Mack from a different point of view. The graph topology is shown to be compatible with the ring structure, yielding a ring with continuous inverse.

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BACKGROUND INTRODUCTION TERMINOLOGY

0.1. BACKGROUND AND INTRODUCTION. The material included in this dissertation concerns the ring $C(X)$ of all continuous real-valued functions on a topological space X . It falls naturally into two parts. The first, consisting of Chapters 1, 2, 3 and the beginning of Chapter 6, constitutes a study of several ideals of $C(X)$ which generalize the ideal $C_K(X)$ of all functions with compact support. The ideals $C_S(X)$, of all functions with pseudo-compact support, and $C_C(X)$, of all functions with relatively countably compact cozero-sets, receive particular attention.

The second part, consisting of Chapters 4, 5 and 6, concerns the ring $C(X)$ in certain well known function space topologies. Those given most attention are the m -topology and the graph topology. These will be denoted by $\langle C(X), m \rangle$ and $\langle C(X), g \rangle$ respectively.

The underlying space X will nearly always be assumed to be completely regular and Hausdorff. This point will be clarified in the paragraphs concerned.

In order to motivate the discussion we give some problems concerning $C_K(X)$ which are typical of those in the literature. We begin by mentioning the early and stimulating work done by

Kaplansky in $[K_1]$. He showed that, for discrete spaces X , $C_K(X)$ is the intersection of the free maximal ideals in $C(X)$. He left open the question of whether or not this equality held in general. This has since been settled negatively.

Other questions concerning $C_K(X)$ are now given.

- (a) Can $C_K(X)$ be characterized algebraically?
- (b) For which spaces X is $C_K(X)$ a proper free ideal?
- (c) Can the spaces X for which the above mentioned Kaplansky equality holds be characterized nicely?

The existing states of solution of these questions are as follows.

- (a) Kohls has given a ring characterization of $C_K(X)$, where X is a locally compact Hausdorff space, $[K_2]$.
- (b) It is known that $C_K(X)$ is a proper free ideal if and only if X is locally compact but not compact, $[GJ, 4D]$.
- (c) Several authors have characterized the spaces for which the Kaplansky equality holds. They have been called μ -compact spaces and Kaplansky spaces, see Mandelker $[M_3]$ and Lavigne $[4]$. Properties of these spaces have been studied.

The subring $C_\infty(X)$ of all functions f in $C(X)$ for which the set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for every $n \in \mathbb{N}$ has also been studied.

In chapter 1 we consider the ideal $C_S(X)$ defined above, and also the ideal $C_S^\infty(X)$ of all functions f in $C(X)$ for which the set

$$cl_X\{x : |f(x)| > \frac{1}{n}\}$$

is pseudocompact for every $n \in \mathbb{N}$. We ask and settle some questions of the above types for these subrings. For example we characterize $C_S(X)$ algebraically within $C(X)$ as follows.

[1.9] For completely regular spaces X , $C_S(X)$ is the intersection of the hyper-real maximal ideals in $C(X)$.

In chapter 2 we specialize the discussion to the case of X locally pseudocompact. Some typical results are as follows.

[2.2] X is locally pseudocompact if and only if X is open in $X \cup (\beta X \setminus \nu X)$.

[2.4] $C_S(X)$ is a proper free ideal if and only if X is locally pseudocompact but not pseudocompact.

In chapter 3 we consider problems which resemble type (c) above. We define and discuss a class of spaces which we call λ -compact. Some of the results which are obtained are now given.

[3.6] If a space X is realcompact then it is λ -compact and if X is λ -compact it is μ -compact.

[3.13] The class of λ -compact spaces is closed under direct union, subject to a non-measurable cardinality restriction.

In Chapters 4, 5 and 6 we characterize the topological components of $C(X)$ in the m -topology and the graph topology in terms of the ideals $C_S(X)$ and $C_C(X)$. We obtain as a corollary the following result.

[4.7] $C(X)$ in the m -topology is totally disconnected if and only if $\beta X \setminus \alpha X$ is dense in βX .

As a side result we characterize the spaces X for which the m -topology and the graph topology coincide in $C(X)$.

[5.5] An arbitrary topological space is a cb -space if and only if the m -topology and the graph topology coincide in $C(X)$.

This seems to be of independent interest in view of the recent work of Poppe, see [P₁, 9].

Chapter 7 concerns a related subalgebra of $C(X)$. The main results concern the compactification corresponding to this subalgebra.

0.2. TERMINOLOGY. Let X denote any topological space and let f be any function in $C(X)$. The set $Z(f)$ of all zeros of f is called the zero-set of f . $\text{coz}(f)$ will denote the set $X \setminus Z(f)$. By the support of f we shall mean the set $S(f) = \text{cl}(X \setminus Z(f))$. The subring of all bounded functions of the ring $C(X)$ will be denoted by $C^*(X)$.

0.3. IDEALS AND z-FILTERS. Let I denote any proper ideal of C . Then the set $Z[I] = \{Z(f) : f \in I\}$ is a filter in the lattice of zero-sets. Conversely, to every filter $\mathcal{F}$ in the lattice of zero-sets there corresponds an ideal $Z(\mathcal{F}) = \{f \in C(X) : Z(f) \in \mathcal{F}\}$ in $C(X)$. Such ideals and filters will be called z-ideals and z-filters respectively. Maximal ideals correspond precisely to maximal z-filters. Maximal z-filters are called z-ultrafilters.

0.4. STONE-CECH COMPACTIFICATION. Every completely regular space X can be densely imbedded as a subspace of a compact space βX (the Stone-Cech compactification of X) such that every $f \in C^*(X)$ has a unique continuous extension to $C(\beta X)$. This extension will be denoted by f^β .

Since much of our work involves imbedded subspaces it will sometimes be necessary to specify to which subspace we may be referring. This will be done by means of subscripts. Thus we will sometimes write $Z_X(f)$, $\text{coz}_X(f)$ and $S_X(f)$ in place of the above notation for $Z(f)$, $\text{coz}(f)$ and $S(f)$. The same applies to the closure and interior operators.

0.5. GELFAND-KOLMOGOROFF. For every point p of βX the set

$$M^p = \{f \in C(X) ; p \in \text{cl}_{\beta X} Z(f)\}$$

is a maximal ideal of $C(X)$. Conversely, for every maximal ideal M of $C(X)$ there exists a unique point $p \in \beta X$ such

that $M = M^p$. If $p \in X$, then M^p is fixed. Otherwise M^p is free. The corresponding ultrafilter is denoted by A^p . Similarly the maximal ideals in $C^*(X)$ are precisely the sets $M^{*p} = \{f \in C^*(X) : f^{\beta}(p) = 0\}$ and they are distinct for distinct p .

For any point $p \in \beta X$, O^p denotes the set of all functions $f \in C(X)$ such that $cl_{\beta X} Z(f)$ is a neighbourhood of p in βX . It can be shown that

$$(a) \quad C_K(X) = \bigcap_{p \in \beta X \setminus X} O^p, \text{ and that}$$

$$(b) \quad C_{\infty}(X) = \bigcap_{p \in \beta X \setminus X} M^{*p}.$$

0.6. HEWITT REALCOMPACTIFICATION. Every completely regular space X can be densely imbedded as a subspace of a realcompact space υX (the Hewitt realcompactification of X) such that every $f \in C(X)$ has a unique continuous extension to $C(\upsilon X)$. This extension will be denoted by f^{υ} .

It is clear that $X \subset \upsilon X \subset \beta X$. An ideal M^p is said to be real if $p \in \upsilon X$ and hyper-real otherwise. It can be shown that

$$Z_{\upsilon X}(f^{\upsilon}) = cl_{\upsilon X} Z(f), \text{ [GJ, 8.8(b)]}.$$

0.7. STONE EXTENSION f^* . Let R denote the space of real numbers.

Any continuous function f in $C(X)$ may be regarded as a continuous mapping of X into the one point compactification

$R^* = R \cup \{\infty\}$ of R . If X is completely regular this function has an extension $f^* : \beta X \rightarrow R^*$. With this notation a maximal ideal M^p is real if and only if $f^*(p) \neq \infty$ for all $f \in C(X)$, [GJ, 8.4].

Finally, for any point p , the ultrafilter A^p and the ideals O^p and M^{*p} will be called fixed or free, real or hyper-real according to the corresponding character of M^p .

0.8. MISCELLANEOUS. Let Q denote the space of rational numbers and N the natural numbers.

Let X and Y denote arbitrary sets. By Y^X we shall mean the collection of all mappings $f : X \rightarrow Y$.

A subset A of a topological space X is said to be C -(C^* -)embedded in X if every continuous (bounded) function from A to R can be extended continuously to a function from X to R .

Chapter 1

RELATIVE PSEUDOCOMPACTNESS AND IDEALS IN $C(X)$

1.1. The concept of relative pseudocompactness, as defined by Noble $[N_2]$, is essential for the development below.

DEFINITION. A subset A of a space X is called relatively pseudocompact if each function f in $C(X)$ maps A to a bounded set.

We commence by obtaining characterizations of relatively pseudocompact subsets which will be used repeatedly. We also obtain a characterization of a relatively pseudocompact zero-set which is analogous to the following well known result of Hewitt, $[H_3]$, Theorem 37] or $[GJ]$, Lemma 4.10].

- (a) A zero-set Z is compact if and only if it belongs to no free z -filter.

Hewitt's result is of considerable importance in the z -ultrafilter construction of βX .

In this chapter we assume that our domain space X is completely regular and Hausdorff.

1.2. THEOREM. Let X denote any topological space and A any

subspace of X . The following are equivalent.

- (1) A is relatively pseudocompact.
- (2) $cl_{\cup X} A$ is compact.
- (3) $\beta X \setminus A$ is a neighbourhood of $\beta X \setminus \cup X$.
- (4) A contains no copy of N , C -embedded in X .

PROOF. The equivalence of (1) and (2) has been noted by Mandelker [M₃].

(2) implies (3). Suppose that $cl_{\cup X} A$ is compact. Then clearly $cl_{\beta X} A = cl_{\cup X} A$ and $cl_{\beta X} A \subset \cup X$. This implies that $\beta X \setminus cl_{\cup X} A$ is a neighbourhood of $\beta X \setminus \cup X$, and the result follows.

(3) implies (4). Suppose that A contains a copy of N , which is C -embedded in X . Denote this copy of N by $T = \{x_n, n \in N\}$. The function f mapping T to R defined by $f(x_n) = n$ has a continuous extension to a function, also denoted by f , from X to R . Let p be any accumulation point of T in βX . Clearly $f^*(p) = \infty$. Hence $p \in \beta X \setminus \cup X$, [0.7], and no neighbourhood of p is contained in $\beta X \setminus A$. Hence $\beta X \setminus A$ is not a neighbourhood of $\beta X \setminus \cup X$.

(4) implies (1). This is given in [GJ, Corollary 1.20].

1.3. We record the following result of Mandelker for future reference. It is true in arbitrary (not necessarily completely regular or Hausdorff) spaces.

- (a) In any topological space X , any relatively pseudocompact support is pseudocompact.

The converse is obviously true. He also notes that for a normal space the two concepts are equivalent for any closed set. He gives an example to show that this is not necessarily true for every completely regular Hausdorff space, see $[M_3]$.

1.4. In the case of a realcompact space X , relatively pseudocompact subsets coincide with relatively compact subsets. A relatively compact subset is a subset A such that $cl_X A$ is compact. This result appears in $[GJ, 8E]$ and is an obvious consequence of $[Theorem\ 1.2(2)]$.

1.5. We now record some "permanence" properties of relatively pseudocompact subsets. The proofs are obvious.

THEOREM. Let X and Y be topological spaces.

- (a) Subsets of relatively pseudocompact subsets of any space X are relatively pseudocompact.
- (b) A subset A of X is relatively pseudocompact if and only if $cl_X A$ is relatively pseudocompact.
- (c) If $f : X \rightarrow Y$ is continuous and A is relatively pseudocompact in X then $f(A)$ is relatively pseudocompact in Y .
- (d) Finite unions of relatively pseudocompact sets are relatively pseudocompact.

1.6. RELATIVELY PSEUDOCOMPACT ZERO-SETS. Mandelker has shown, again for arbitrary topological spaces, that a zero-set that belongs to no free z -filter is relatively pseudocompact, $[M_3]$. He has noted that the converse is false even for completely regular Hausdorff spaces.

We clarify this situation in the setting of completely regular Hausdorff spaces.

Given a z -filter $\mathcal{F}$ let $\theta(\mathcal{F})$ denote the set

$$\bigcap_{Z \in \mathcal{F}} \text{cl}_{\beta X} Z(f) \text{ of cluster points of } \mathcal{F} \text{ in } \beta X .$$

DEFINITION. A z -filter $\mathcal{F}$ is hyper-real if $\theta(\mathcal{F})$ meets $\beta X \setminus \upsilon X$.

THEOREM. A zero-set is relatively pseudocompact if and only if it belongs to no hyper-real z -filter.

PROOF. Suppose that the zero-set Z is relatively pseudocompact. then, by [Theorem 1.2(1)] $\text{cl}_{\beta X} Z = \text{cl}_{\upsilon X} Z$ and so $\text{cl}_{\beta X} Z \subset \upsilon X$. Hence if Z is a member of some z -filter $\mathcal{F}$ then $\theta(\mathcal{F}) \subset \upsilon X$.

On the other hand, if Z is not relatively pseudocompact then by [Theorem 1.2(1)] there exists some point $p \in (\beta X \setminus \upsilon X) \cap \text{cl}_{\beta X} Z$. Clearly then, Z is a member of the hyper-real z -ultrafilter A^p and the theorem is proved.

1.7. COROLLARY. A zero-set is relatively pseudocompact if and

only if it belongs to no hyper-real z -ultrafilter.

PROOF. A zero-set belongs to a hyper-real z -filter if and only if it belongs to a hyper-real z -ultrafilter.

1.8. Hewitt has obtained the following result, [H₃, Theorem 44]

Let X be a non-pseudocompact space. A function f in $C(X)$ has the property that $M(f)$ is non-real for every free maximal ideal M in $C(X)$ if and only if the set $Z(f - a)$ is compact for every $a \in \mathbb{R}$.

For a point p of βX we know that $M^p(f)$ is real for every function f in $C(X)$ if and only if $p \in \upsilon X$. Hence, any space X , for which there is a function f in $C(X)$ satisfying the condition of this theorem, is realcompact.

We are now able, using the concept of relative pseudocompactness, to obtain a result of this type with no such implicit restriction on the class of spaces considered.

COROLLARY. For any space X , a function f in $C(X)$ has the property that $M(f)$ is non-real for every hyper-real maximal ideal M in $C(X)$ if and only if the set $Z(f - a)$ is relatively pseudocompact for every $a \in \mathbb{R}$.

PROOF. For a point p of βX we have that $M^p(f - a)$ is hyper-real for every $a \in \mathbb{R}$. The result then follows from [Corollary 1.7].

1.9. THE HYPER-REAL MAXIMAL IDEALS. Mandelker has obtained the following result, [M₃, Theorem 2.2]. Again, his result holds without the assumption of complete regularity.

(a) Let X be any topological space. Every function in $C(X)$ that belongs to all the free maximal ideals has pseudocompact support.

The converse is not true, for if X is pseudocompact but not compact then X is a support of the constant function 1 which is pseudocompact. 1 does not belong to any proper ideal.

We now give a result which clarifies this situation in the completely regular case.

We will need the following result which is not new. Mandelker has in fact obtained it as a corollary of a more general theorem [M₂, Corollary 5.4]. We include a direct proof.

LEMMA. For any space X and zero-set $Z(f)$ in $Z[X]$ the following holds. $cl_{\beta X} Z(f) \supset \beta X \setminus \nu X$ if and only if $cl_{\beta X} Z(f)$ is a neighbourhood of $\beta X \setminus \nu X$.

PROOF. Suppose that $cl_{\beta X} Z(f) \supset \beta X \setminus \nu X$. Because $Z_{\nu X}(f^\nu) = cl_{\nu X} Z(f)$, [0.6] and $cl_{\beta X} Z(f) = cl_{\beta X}(cl_{\nu X} Z(f))$, we see that f^ν is in the intersection of the free maximal ideals in $C(\nu X)$. Hence f^ν is in $C_K(\nu X)$, [GJ, 8.10]. This in turn

implies that $cl_{\beta X \cup X} Z(f^u)$ is a neighbourhood of $\beta X \setminus uX$, [0.5(a)].
Hence $cl_{\beta X} Z(f)$ is a neighbourhood of $\beta X \setminus uX$.

1.10. THEOREM. A function f in $C(X)$ belongs to the intersection of the hyper-real maximal ideals in $C(X)$ if and only if it has pseudocompact support.

PROOF. Suppose that f belongs to the intersection of the hyper-real maximal ideals in $C(X)$. Then $cl_{\beta X} Z(f) \supset \beta X \setminus uX$, [0.5]. By the lemma, equivalently, $cl_{\beta X} Z(f)$ is a neighbourhood of $\beta X \setminus uX$. It follows that $cl_{\beta X} S(f) \subset uX$ and that $cl_{uX} S(f)$ is compact, and the result follows by [Theorem 1.2(2)].

The converse follows from the reverse argument.

1.11. Henceforth we shall denote the ideal of all functions in $C(X)$ with pseudocompact support by $C_S(X)$.

COROLLARY. For any space X , $C_S(X) = C_K(uX)$.

PROOF. The hyper-real maximal ideals in $C(X)$ are the free maximal ideals in $C(uX)$.

1.12. It is also clear that, since $C(X)$ and $C(uX)$ are isomorphic rings for any space X , $C_S(X)$ is an invariant of the ring structure of $C(X)$. More precisely, we have the following result.

COROLLARY. For any spaces X and Y such that $C(X)$ is

isomorphic to $C(Y)$ we have that $C_S(X)$ is isomorphic to $C_S(Y)$.
Note that C_K does not have this invariance.

1.13. THE RING $C_S^\infty(X)$. Recall the ring $C_\infty(X)$ consisting of all functions f in $C(X)$ for which the set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for every $n \in \mathbb{N}$. $C_\infty(X)$ is an ideal in $C^*(X)$ and its properties are well known, [GJ, 7F] and Kohls [K₂].

Let us denote the set $\text{cl}_X\{x : |f(x)| > \frac{1}{n}\}$ by $S_n(f)$. Since $S_n(f) \subset \{x \in X : |f(x)| \geq \frac{1}{n}\} \subset S_{n+1}(f)$ for every $n \in \mathbb{N}$ we see that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for every $n \in \mathbb{N}$ if and only if $S_n(f)$ is compact for every $n \in \mathbb{N}$.

Thus we could as well have defined $C_\infty(X)$ to be the set $\{f \in C(X) : S_n(f) \text{ is compact for every } n \in \mathbb{N}\}$.

DEFINITION. For any space X , let $C_S^\infty(X)$ denote the set of all functions f in $C(X)$ for which the set $S_n(f)$ is pseudocompact for every $n \in \mathbb{N}$.

From the definition and the above remark we obtain the following inclusion relationships between the ideals considered above.

THEOREM. For any space X the following inclusion relationships hold. Each inclusion can be proper.

$$(1) \quad C_K(X) \subset C_S(X) \subset C_S^\infty(X) \subset C^*(X).$$

$$(2) \ C_S^\infty(X) \subset C_S(X) .$$

PROOF. We show that the given inclusions can be proper. To see this let W denote the space of all countable ordinals in the order topology. W is well known to be pseudocompact and not compact. A closed subspace is compact if and only if it is cofinal. Let

$$X = W \times N , \text{ the cartesian product.}$$

Consider the following functions.

$$(a) \ f(x, n) = 1 , \text{ when } n = 1 \text{ and zero when } n \neq 1 .$$

$$(b) \ g(x, n) = 1/n \text{ for each } n \in N .$$

$$(c) \ h(x, n) = 1 \text{ for all } n \in N \text{ and all } x \in W .$$

Then $f \in C_S \setminus C_K$, $g \in C_S^\infty \setminus C_S$ and $h \in C^* \setminus C_S^\infty$. Also $f \in C_S^\infty \setminus C_\infty$.

1.14. As with the ideal C_S , we now characterize $C_S^\infty(X)$ in terms of known subrings of $C(X)$.

THEOREM. For any space X , $C_S^\infty(X)$ is an ideal in $C^*(X)$ and $C_S^\infty(X) = C_\infty(\cup X)$.

PROOF. The proof that $C_S^\infty(X) = C_\infty(\cup X)$ would constitute a needless repetition of the reasoning of the proof of [Theorem 1.9]. That $C_S^\infty(X)$ is an ideal in $C^*(X)$ then follows from the known result that for any space Y , $C_\infty(Y)$ is an ideal in $C^*(Y)$.

1.15. Corresponding to [Theorem 1.10] we obtain the following result.

COROLLARY. For any space X , $C_S^\infty(X)$ is the intersection of all the hyper-real maximal ideals in $C^*(X)$.

1.16. We now characterize the spaces X for which $C_S(X)$ and $C_S^\infty(X)$ are not proper ideals of $C(X)$.

THEOREM. For any space X , $\beta X \setminus \nu X$ is dense in βX if and only if $C_S(X) = (0)$ or $C_S^\infty(X) = (0)$.

PROOF. Using [Corollary 1.15] and the known fact, [GJ, 7F4], $\beta X \setminus X$ is dense in βX if and only if $C_\infty(X) = (0)$ we deduce that $\beta X \setminus \nu X$ is dense in βX if and only if $C_S^\infty(X) = (0)$.

From the inclusions of theorem 1.13 (1) we see that $C_S^\infty(X) = (0)$ implies that $C_S(X) = (0)$. It remains to be shown that $C_S^\infty(X) \neq (0)$ implies that $C_S(X) \neq (0)$. Suppose that $f \neq 0$ and let $f \in C_S^\infty(X)$. Since $f(p) \neq 0$ for some $p \in X$, there exists an $\alpha \in \mathbb{R}$ such that $|f(x)| > \alpha$ for some $x \in X$. Then $(|f| \vee \alpha) - \alpha$ is in $C_S(X)$ and is not the zero function. This completes the proof.

On the other hand, we can easily show that a space X is pseudocompact if and only if either $C_S(X) = C(X)$ or $C_S^\infty(X) = C(X)$.

1.17. In view of the inclusion $C_S \subset C_S^\infty$, it is natural to characterize the spaces X for which $C_S(X) = C_S^\infty(X)$.

DEFINITION. A subset A of a space X is called σ -relatively pseudocompact if it is a countable union of relatively pseudocompact spaces.

THEOREM. For any space X the subrings $C_S(X)$ and $C_S^\infty(X)$ coincide if and only if every σ -relatively pseudocompact cozero-set is relatively pseudocompact.

PROOF. Necessity. Let $A = \bigcup_{n=1}^{\infty} \text{coz}(f_n)$, where $\text{coz}(f_n)$ is relatively pseudocompact for every $n \in \mathbb{N}$ and each f_n is such that $\sup_{x \in X} |f_n(x)| \leq 1$. Then clearly $A = \text{coz}(f)$ where

$$f = \sum_{n=1}^{\infty} |f_n| \frac{1}{2^n}.$$

Now

$$\begin{aligned} S_n(f) &= \text{cl}\{x : \sum_{n=1}^{\infty} 2^{-n} |f_n(x)| > \frac{1}{n}\} \\ &\subseteq \text{cl}\{x : |f_1(x)| > \frac{1}{2n}\} = S_{2n}(f_1). \end{aligned}$$

But $S_{2n}(f_1)$ is pseudocompact for every $n \in \mathbb{N}$. Hence $S_n(f)$ is pseudocompact for every $n \in \mathbb{N}$, and so $f \in C_S^\infty(X)$. The result then follows from the hypothesis.

Sufficiency. This is clear because, if $f \in C_S^\infty(X)$, then

$\text{coz } f = \bigcup_{n=1}^{\infty} \{x : |f(x)| > \frac{1}{n}\}$ and each member of the union is relatively pseudocompact.

1.18. EXAMPLE. A space X such that,

- (1) $C_S(X) = C_S^{\infty}(X)$,
- (2) $C_K(X) \neq C_{\infty}(X)$, and
- (3) each of these is a proper non-trivial subring of $C(X)$.

Let $X = T \oplus Q$ where T denotes the Tychonoff plank. Since T is locally compact and not countable compact we have that $C_K(T) \neq C_{\infty}(T)$, [GJ, 7G(2)]. Clearly $(0) \subset C_K(T)$. Also since T is pseudocompact $C_S(T) = C_S^{\infty}(T) = C(T)$.

Also $C_K(Q) = C_{\infty}(Q) = (0) = C_S(Q) = C_S^{\infty}(Q)$.

Then using arguments like the one below we can show that X has the required properties.

To see that $C_S(X) = C_S^{\infty}(X)$ note that

$$\begin{aligned} C_S(X) &= C_S(T) \oplus C_S(Q) \\ &= C_S^{\infty}(T) \oplus (0) \\ &= C_S^{\infty}(T) . \end{aligned}$$

Chapter 2

LOCAL PSEUDOCOMPACTNESS

2.1. The concept of local pseudocompactness was first studied by Comfort $[C_1]$ or $[C_2]$. A space is locally pseudocompact if each point admits a pseudocompact neighbourhood. Equivalently, since the closure of an open subset of a pseudocompact space is pseudocompact, a space is locally pseudocompact if at each point p of X there is a local neighbourhood base consisting of pseudocompact sets.

We wish to show how the results of chapter 1 can be sharpened under the assumption that X or $\cup X$ is locally pseudocompact. Again we shall make the assumption that all spaces considered are completely regular and Hausdorff.

2.2. We now characterize local pseudocompactness in terms of βX .

THEOREM. A space X is locally pseudocompact if and only if X is open in $X \cup (\beta X \setminus \cup X) = Y$.

PROOF. Necessity. Let X be locally pseudocompact and let U be any open relatively pseudocompact neighbourhood of a point $x \in X$. Then there exists an f in $C^*(X)$ such that

$$x \in \text{coz } f \subset S(f) \subset U.$$

By [1.2(2)], $x \in \text{coz } f^B \subset \text{cl}_{\beta X} S(f) \subset \upsilon X$. This shows that the Y -neighbourhood $Y \cap \text{coz } f^B \subset X$.

Sufficiency. Assume that X is open in $X \cup (\beta X \setminus \upsilon X)$ and let $U(x)$ be a neighbourhood of a point $x \in X$ open in $X \cup (\beta X \setminus \upsilon X)$ such that $U(x) \subset X$. Then

$$U(x) = V(x) \cap (X \cup (\beta X \setminus \upsilon X)) \text{ with } V(x)$$

open in βX and $V(x) \subset \upsilon X$.

Since βX is completely regular, there exists an $f^B \in C(\beta X)$ such that $x \in \text{coz}(f^B)$ and

$$\text{cl}_{\beta X} \text{coz}(f^B) \subset V(x) \subset \upsilon X.$$

By [1.2(2)] we have that $\text{coz } f = \text{coz}(f^B) \cap X$ is relatively pseudocompact and by [1.3] that

$$\text{cl}_X(\text{coz } f)$$

is a pseudocompact neighbourhood of x .

2.3. At this point we must mention some recent results of Mack, Rayburn and Woods, [MRW]. In particular, the following theorem is proved, the conclusion of which, he remarks, is true for locally pseudocompact spaces, see [MRW, Theorem 4.1].

(R) For any $\mathcal{P}$ -regular space X , the following are equivalent:

- (a) X is locally $\mathcal{P}$.
- (b) X is open in $\gamma_K X$.
- (c) X has a one point $\mathcal{P}$ -extension.
- (d) X is a dense subspace of a $\mathcal{P}$ -space T for which $T \setminus X$ is compact.
- (e) X is an open subspace of some $\mathcal{P}$ -space.

Moreover if γX belongs to $\mathcal{P}$, then each of the above statements is equivalent to X being open in γX . In the case of local pseudocompactness,

$$\gamma X = \bigcap \{T : X \subset T \subset \beta X, T \text{ pseudocompact}\},$$

$$\gamma_K X = \bigcap \{T : X \subset T \subset \beta X, T \text{ pseudocompact, } cl_T(T \setminus X) \text{ is compact}\}$$

and $\mathcal{P}$ is the class of pseudocompact spaces.

We show that the previous result [2.2] is at least, not an obvious consequence of the [MRW] theorem by noting locally compact spaces for which

$$\gamma X \neq X \cup (\beta X \setminus \alpha X)$$

and
$$\gamma_K(X) \neq X \cup (\beta X \setminus \alpha X).$$

Firstly, if $X = R$, then for any point p in $\beta R \setminus R$ we have that $T = \beta R \setminus \{p\}$ is pseudocompact. Thus, in this case $\gamma R = R$.

Secondly, let X denote the "spiral" space of Comfort $[C_1]$. This locally compact space has the properties that $\iota X \setminus X$ consists of exactly one point, say, p , and that any neighbourhood of this point p in βX meets $\beta X \setminus \iota X$. Hence any compact set containing $\beta X \setminus \iota X$ contains p . But, by [MRW, Theorem 2.7] $\gamma_K X = X \cup K$ where K is compact. This implies that $p \in \gamma_K X$ and $\gamma_K X \neq X \cup \beta X \setminus \iota X$.

2.4. We have mentioned, [0.1(b)] that $C_K(X)$ is a proper free ideal if and only if X is locally compact but not compact. Since $Z(f^\cup) \supset Z(f)$ it is clear in view of [1.11] that $C_S(X)$ is a proper free ideal if $C_K(\iota X)$ is proper and free. The following result then follows from the fact that a pseudocompact space is compact if and only if it is realcompact.

(a) If ιX is locally pseudocompact but not pseudocompact then $C_S(X)$ is a proper free ideal.

That the converse is not true will follow from the following theorem by an example of Comfort.

THEOREM. For any space X , $C_S(X)$ is a proper free ideal if and only if X is locally pseudocompact but not pseudocompact.

PROOF. If X is locally pseudocompact then for each point $x \in X$ there exists an $f \in C_S(X)$ such that $x \in \text{coz } f$. Hence $\bigcap_{f \in C_S(X)} Z(f) = \emptyset$ and so $C_S(X)$ is free.

Conversely. Suppose that $C_S(X)$ is free. Then for every point x in X there exists a function $f \in C_S(X)$ such that $x \notin Z(f)$. Then $x \in \text{coz } f$, which is relatively pseudocompact.

The theorem follows if we note that the ideal $C_S(X)$ is proper if and only if X is not pseudocompact [1.16].

The "spiral" space of Comfort is locally compact but its Hewitt realcompactification is not locally pseudocompact. So we see that $C_S(X)$ may be free and proper without $C_K(\cup X)$ being free and proper.

2.5. Using [Theorem 2.4] and remark [2.4(a)] above we can obtain as a corollary the result of Comfort $[C_1]$ or $[C_2]$.

COROLLARY. A space X is locally pseudocompact if $\cup X$ is locally compact.

2.6. A ONE POINT PSEUDOCOMPACT EXTENSION. We now describe the construction of a one point pseudocompact extension of a locally pseudocompact space which proves useful in the discussion of the rings $C_S^\infty(X)$ and $C_S(X)$. Although the method discussed is not

as elegant as that of [MRW] it is much simpler and more directly suited to our purposes.

THEOREM. Let X be locally pseudocompact but not pseudocompact. Then there exists a (completely regular) space $\hat{X}$ with the following properties,

- (1) X is densely embedded in $\hat{X}$ and $\hat{X} \setminus X$ consists of exactly one point.
- (2) $\hat{X}$ is pseudocompact.

PROOF. Consider an arbitrary point ∞ which is not in X . Put $\hat{X} = X \cup \{\infty\}$. Put $\hat{X} \cup \{\infty\}$. Extend $f \in C_S(X)$ to a map f' from $\hat{X}$ to $\mathbb{R}$ by defining

$$f'(\infty) = 0$$

and $f'(x) = f(x)$, for every $x \in X$.

Let $\hat{X}$ have the weak topology determined by this set of functions. Since X is locally pseudocompact $\hat{X}$ is Hausdorff. The complete regularity of $\hat{X}$ then follows because the topology of $\hat{X}$ is determined by the subfamily

$$\{f' : f \in C_S(X)\} \text{ of } C(\hat{X}).$$

To show that X is dense in $\hat{X}$ we have only to show that ∞ is not an isolated point. But, if ∞ were isolated

then the pseudocompactness of $\hat{X}$ would imply the pseudocompactness of X . The result will follow when we establish the pseudocompactness of $\hat{X}$.

Suppose that f is a positive function in $C(\hat{X})$, unbounded on $\hat{X}$. Let $V(\infty)$ be any basic neighbourhood of ∞ which is of the form

$$V(\infty) = \{x : |g'(x)| < \epsilon\}$$

where $g \in C_S(X)$ and $\epsilon > 0$, and where $|f(x)| \leq m$ for every $x \in V(\infty)$, and some $m \in \mathbb{R}$. Then

$$\hat{X} \setminus V(\infty) = \{x : |g'(x)| \geq \epsilon\} \subset \text{cl}\{x : |g'(x)| > \frac{\epsilon}{2}\}.$$

Since $g \in C_S(X)$ we see that the set

$$\text{cl}\{x : |g'(x)| > \frac{\epsilon}{2}\}$$

is pseudocompact [1.3]. Hence f must be bounded.

2.7. Using this extension we obtain a characterization of the ring $C_S^\infty(X)$ in the case of X locally pseudocompact and not pseudocompact.

THEOREM. Let X be locally pseudocompact and not pseudocompact. Then $C_S^\infty(X)$ is isomorphic to the ideal M^∞ in $C(\hat{X})$.

PROOF. Let each function g in $\bigwedge C(X)$ be identified with its restriction to X . If f is in $\bigwedge C_S(X)$ then if f is extendable, clearly $f(\infty) = 0$ and $f \in M^\infty$. We have only to show that f is extendable. It will suffice to show that for any $\epsilon > 0$, there is a neighbourhood of ∞ upon which $|f|$ is less than ϵ . Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. Let $g \in C(X)$ be the function defined by $g = (|f| \vee \frac{1}{n_0}) - \frac{1}{n_0}$.

Then $S(g) = \text{cl}_X \{x : |f(x)| > \frac{1}{n_0}\}$ and so $S(g)$ is pseudocompact, (by definition). Now, there exists a real number ρ such that $\epsilon > \rho > \frac{1}{n_0}$. Clearly $\{x : |g(x)| < \rho\}$ is a neighbourhood of ∞ contained in $\{x : |f(x)| < \epsilon\}$.

Conversely. Suppose that f is in M^∞ . Then, for every $n \in \mathbb{N}$ there exists a basic neighbourhood $V_n(\infty)$ of ∞ upon which $|f|$ is less than $\frac{1}{n+1}$. As before we may assume that $V_n(\infty)$ is of the form

$$\{x : |g'(x)| < \epsilon\}, \text{ where } g' \in C_S(X)$$

and ϵ is a positive real number. As before

$$\bigwedge X \setminus V_n(\infty) = \{x : |g'(x)| \geq \epsilon\} \subset \text{cl}\{x : |g'(x)| > \frac{\epsilon}{2}\}.$$

Since the set $\text{cl}\{x : |g'(x)| > \frac{\epsilon}{2}\}$ is pseudocompact [1.3], we can deduce that $\text{cl}\{x : |f(x)| > \frac{1}{n}\}$ is pseudocompact if we

note that the set $\{x : |f(x)| > \frac{1}{n}\} \subset X \setminus V_n(\infty)$.

Hence $f \in C_S^\infty(X)$.

2.8. Concerning the ideal $C_S(X)$ we have the following result.

THEOREM. Let X be locally pseudocompact and not pseudocompact. Then 0^∞ is an ideal in $C_S(X)$.

PROOF. If we identify functions in $C(X)^\wedge$ with their restrictions to X the result will follow from set theoretic inclusion.

Suppose therefore, that f is a function in 0^∞ . Then $Z_X(f)$ is a neighbourhood of ∞ . There will exist a basic neighbourhood $V(\infty)$ of ∞ such that $V(\infty) \subset Z_X(f)$. As before $X \setminus V(\infty) \subset \text{cl}\{x : g'(x) > \frac{\epsilon}{2}\}$ where $g' \in C_S(X)$ and $\text{cl}\{x : |g'(x)| > \frac{\epsilon}{2}\}$ is pseudocompact. Since

$$S(f) \subset \text{cl}\{x : |g'(x)| > \frac{\epsilon}{2}\}$$

we see that $S(f)$ is pseudocompact and hence that f is in $C_S(X)$.

2.9. At this point questions naturally arise concerning the relationship between the extension $X^\wedge$ and the Alexandroff one point extension X^* in the case of X locally compact.

We find that, in general, $X^\wedge$ need not be homeomorphic to X^* by the map which leaves X pointwise fixed and maps the

point ∞ in $\hat{X}$ onto the point $w \in X^* \setminus X$. However, if X is a ψ -compact space, as defined by Mandelker in [M₃], then $\hat{X}$ and X^* are indeed homeomorphic by this map, see [2.10]. As the ψ -compact spaces include the realcompact spaces we see that this homeomorphism holds for a wide class of spaces.

EXAMPLE. A locally compact space X for which $\hat{X}$ is not homeomorphic to X^* .

CONSTRUCTION. Let X denote the product $T \times N$, where T denotes the Tychonoff plank. Then the subset $T \times (N \setminus \{1\})$ of X is open in $\hat{X}$ but not in X^* .

2.10. We now recall the concept of ψ -compactness, see Mandelker [M].

DEFINITION. A space X is ψ -compact if $C_S(X) = C_K(X)$. We show that if X is locally compact and ψ -compact then $\hat{X}$ is homeomorphic to X^* .

THEOREM. Let X denote any locally compact space, X^* its Alexandroff compactification and $\hat{X}$ the one point extension described in [2.6]. If $C_S(X) = C_K(X)$, then $\hat{X}$ is homeomorphic to X^* by the bijection which leaves X pointwise fixed.

PROOF. We need only show that an open subbasic neighbourhood of w in the topology of X^* is open in $\hat{X}$ and that an open subbasic neighbourhood of ∞ in the topology of $\hat{X}$ is open in X^* .

Consider a basic open neighbourhood of ∞ in $\hat{X}$. We may suppose that this is of the form $\{x : |f(x)| < \epsilon\}$ for some $f \in C_S(X)$ and some $\epsilon > 0$. Note that, as before, we are identifying functions in $\overset{\wedge}{C(X)}$ with their restrictions to X . Because $C_S(X) = C_K(X)$ it follows that $\{x : |f(x)| \geq \epsilon\}$ is compact and that $\{x : |f(x)| < \epsilon\}$ is open in the topology of X^* .

Consider a basic open neighbourhood of w in X^* . This is of the form $X^* \setminus K$ where K is compact in X . Let $x \in X^* \setminus K$. We must show the existence of some set open in the topology of $\hat{X}$, contained in $X^* \setminus K$ and containing x . If $x \neq \infty$ this is obvious. Suppose however that $x = \infty$. Since X is locally compact there exist compact sets K_1 and K_2 such that $K \subset K_1 \subset K_2$, K_1 being a neighbourhood of K and K_2 a neighbourhood of K_1 . Using this we can conclude the existence of a function f in $C(X)$ with the following properties.

$$f(x) = 1 \text{ for every } x \in K_1$$

$$\text{and} \quad f(x) = 0 \text{ for every } x \in X \setminus K_2.$$

Clearly $S(f)$ is compact, hence pseudocompact. It follows that the set $\{x : |f(x)| < \frac{1}{2}\}$ is a neighbourhood of ∞ in $\hat{X}$ which is contained in $X^* \setminus K$. Note that, again, we have identified $f \in C_S(X)$ with its extension to $\hat{X}$.

2.11. In this paragraph we establish another condition on the space X under which $\hat{X}$ is very simply described. The condition which we impose on X is that $\cup X$ be locally compact. This property may seem somewhat unnatural, and perhaps it is, but it is not without significance in the theory of $C(X)$, see Comfort $[C_2]$.

THEOREM. If $\cup X$ is locally compact, then $\hat{X}$ is homeomorphic, by the natural embedding, to the subspace $X \cup \{\infty\}$ of the Alexandroff compactification $(\cup X)^*$ of $\cup X$.

PROOF. For simplicity let us identify $\hat{X}$ with the subspace $X \cup \{\infty\}$ of $(\cup X)^*$ set theoretically. Our task then reduces to showing that the topology of $\hat{X}$ is the same as that which it inherits as a subspace of $(\cup X)^*$. The proof which we give is similar to that of [Theorem 2.10].

Clearly, any subset of $\hat{X}$ which does not contain ∞ , is open in $\hat{X}$ if and only if it is open as a subset of $(\cup X)^*$. Hence we need only consider open subsets which contain ∞ .

Consider a basic neighbourhood $V(\infty)$ of ∞ in the topology of $(\cup X)^*$. We may assume that $V(\infty) = \cup X \setminus K$ where K is a compact subset of $\cup X$. Using the local compactness of $\cup X$, we can conclude the existence of compact subsets K_1 and K_2 of $\cup X$ with $K \subset K_1 \subset K_2$, and of a function $f \in C_K(\cup X)$

such that

$$f(x) = 1 \text{ for every } x \in K$$

$$\text{and } f(x) = 0 \text{ for every } x \in \cup X \setminus K_2.$$

The restriction of f to X , denoted also by f , is in $C_S(X)$, [1.11].

If we identify f with its extension to $\hat{X}$ we can conclude that the set $\{x : |f(x)| < \frac{1}{2}\}$ is a basic neighbourhood of ∞ in $\hat{X}$ which is contained in $V(\infty)$. Hence the topology of $\hat{X}$ is at least as strong as that which it inherits as a subspace of $(\cup X)^*$.

We now show that these topologies are in fact equal.

Consider a basic neighbourhood of ∞ in the topology of $\hat{X}$. We may assume that this is of the form

$$V(\infty) = \{x : |g(x)| < \epsilon\}$$

where $g \in C_S(X)$ and $\epsilon > 0$. Note that, once more, we have merely identified each g in $C_S(X)$ with its extension $\hat{g} \in C(\hat{X})$ as defined in [2.6].

Let $W = \text{cl}\{x : |g(x)| > \frac{\epsilon}{2}\}$. As in Theorem [2.8], $\text{cl}\{x : |g(x)| > \frac{\epsilon}{2}\}$ is pseudocompact. Hence W is pseudocompact.

It follows that the set $cl_{\cup X} W$ is compact, [1.2(2)]. This implies that $\cup X \setminus cl_{\cup X} W$ is a neighbourhood of ∞ in $(\cup X)^*$. But $(\cup X \setminus cl_{\cup X} W) \cap X \cup \{\infty\} \subset V(\infty)$, and we have found a neighbourhood of ∞ open in its relative topology which is contained in $V(\infty)$. This shows that the two topologies for X coincide.

2.12. In the case of $\cup X$ locally compact we obtain the following simple description of the ideal $C_S(X)$ in terms of X .

THEOREM. Let X be a topological space with the property that $\cup X$ is locally compact. Then $C_S(X)$ is isomorphic to the ideal O^∞ of $C(X)$.

PROOF. By [1.11] we have that $C_S(X)$ is isomorphic to $C_K(\cup X)$. By [GJ, 7G1] and the hypothesis, $C_K(\cup X)$ is isomorphic to the ideal O^W in $C((\cup X)^*)$. The result will follow from [Theorem 2.11] if we can show that the ideal O^W in $C((\cup X)^*)$ is isomorphic to the ideal O^W in the ring $C(X \cup \{w\})$ where $X \cup \{w\}$ has the relative topology which it inherits as a subspace of $(\cup X)^*$.

To see this proceed as follows. Suppose that f is a function in the ideal O^W of $C((\cup X)^*)$. Clearly the restriction of f to $X \cup \{w\}$ is in the ideal O^W of $C(X \cup \{w\})$. On the other hand, suppose that the function f is in the ideal O^W of $C(X \cup \{w\})$. Then $Z_{X \cup \{w\}}(f)$ is a neighbourhood of w . This

implies that there exists a set θ such that

$$Z_{X \cup \{w\}}(f) \supset \theta$$

and
$$\theta = (X \cup \{w\}) \cap (\cup X \setminus K) ,$$

where K is a compact subset of $\cup X$.

Let us extend the function f to a mapping $f' : (\cup X)^* \rightarrow R$ by defining $f'(x) = f^{\cup}(x)$ for each $x \in \cup X \setminus X$.

Since $f'(x) = 0$ for every $x \in \cup X \setminus K$ we see that f' is continuous and that f' is in the ideal 0^w of $C((\cup X)^*)$. This proves the theorem.

2.13. Theorem 2.2 gave us a characterization of local pseudo-compactness in terms of βX . Using this characterization we now characterize $\hat{X}$ as a quotient of the space $X \cup (\beta X \setminus \cup X)$.

Firstly we establish some notation. Let X and Y be topological spaces and f a continuous function from X to Y . We shall denote the equivalence relation determined by the decomposition classes

$$\{f^{-1}(y)\}_{y \in Y} \quad \text{by } R(f) .$$

THEOREM. Let $\hat{X}$ denote any locally pseudocompact space which is not pseudocompact. Then $\hat{X}$ is homeomorphic by the natural

embedding to the topological space obtained from the space $X \cup (\beta X \setminus \nu X)$ by identifying the points in $\beta X \setminus \nu X$.

PROOF. Let a mapping $f : X \cup (\beta X \setminus \nu X) \rightarrow X^\wedge$ be defined by the following conditions:

$$f(x) = x \text{ for every point } x \in X .$$

$$f(x) = \infty \text{ for every point } x \in \beta X \setminus \nu X .$$

We show that f is continuous. By [Theorem 2.2], X is open in $X \cup (\beta X \setminus \nu X)$ and so sets open in X are open in $X \cup (\beta X \setminus \nu X)$ and it follows that f is continuous at every point x in X .

Now suppose that $x \in \beta X \setminus \nu X$. Let

$$V = \{x : |g(x)| < \epsilon\} \cup \{\infty\} ,$$

where $g \in C_S(X)$ and $\epsilon > 0$, be a subbasic neighbourhood of $\infty (= f(x))$ in $X^\wedge$. Since g is in $C_S(X)$ the Stone-Cech extension g^β of g is zero on $\beta X \setminus \nu X$. Hence the set

$$\{x : |g^\beta(x)| < \epsilon\} \cap (X \cup (\beta X \setminus \nu X))$$

is open in $X \cup \beta X \setminus \nu X$ and is mapped into V by f . Hence f is continuous.

Given the function f as defined above, consider the

equivalence relation $R(f)$. Let A be a set open in $X \cup (\beta X \setminus \upsilon X)$. We will show that the union of the equivalence classes of the relation $R(f)$ which are contained in A is open in $X \cup (\beta X \setminus \upsilon X)$. It will then follow from the de Morgan formulas, that R is a closed equivalence relation and that

$$X \cup (\beta X \setminus \upsilon X)/R(f)$$

is homeomorphic to $\hat{X}$, see [E, Theorem 2.4.4].

We proceed as follows. Suppose that $A \supset \beta X \setminus \upsilon X$. Then

$$\bigcup_{f^{-1}(y) \subset A} \{f^{-1}(y)\} = A, \text{ and } A \text{ is open.}$$

On the other hand, suppose that $A \not\supset \beta X \setminus \upsilon X$. Then

$$\bigcup_{f^{-1}(y) \subset A} \{f^{-1}(y)\} = A \cap X.$$

This is open in $X \cup (\beta X \setminus \upsilon X)$ because X is open in $X \cup (\beta X \setminus \upsilon X)$. This completes the proof of the theorem.

2.14. Since the whole purpose of introducing the extension $\hat{X}$ of a locally pseudocompact space X was to obtain specialized characterizations of the rings $C_S(X)$ and $C_S^\infty(X)$ in this case, and since such pleasant specializations are known in the case of X locally compact, it seems appropriate at this point to motivate

this discussion by giving an example illustrating the non-triviality of the situation.

EXAMPLE. A locally pseudocompact non-locally compact space X which is not pseudocompact.

If, for the moment, we assume the existence of a pseudocompact non-locally compact space L , then the product

$$X = L \otimes N$$

will obviously have the required properties.

We now show that a well known pseudocompact space (L) is not locally compact. Consider the Tychonoff cube $\bigotimes_{r \in \mathbb{R}} I_r$ where for each $r \in \mathbb{R}$, I_r denotes a copy of the unit interval. Let L denote the subspace consisting of all points which have at most countably many coordinates different from zero. It is well known that L is countably compact and not compact.

Suppose that L is in fact locally compact, and let K denote a compact neighbourhood of the element p of L which is zero at every coordinate. Clearly each element of K is such that only countably many coordinates are different from zero. But, since L is dense in $\bigotimes_{r \in \mathbb{R}} I_r$ K is a compact neighbourhood of p in $\bigotimes_{r \in \mathbb{R}} I_r$, see for instance [GJ, 3.15]. As such it clearly contains elements of $\bigotimes_{r \in \mathbb{R}} I_r$ which differ from zero at more than finitely many coordinates. The assertion is thus established by contradiction.

Chapter 3

IDEALS AND RELATED COMPACTNESS PROPERTIES

3.1. Mandelker $[M_3]$ has introduced the notions of μ -compactness and ψ -compactness. A space X is μ -compact if every function f in $C(X)$ that belongs to all the free maximal ideals has compact support. As we have already mentioned, a space X is ψ -compact if every pseudocompact support set is compact.

We recall some of the results concerning these concepts as background against which the results of this chapter are developed.

$[M_3, \text{Theorem 3.1}]$ shows that for arbitrary topological spaces realcompactness implies ψ -compactness and ψ -compactness implies μ -compactness. Counterexamples to the converses are given.

$[M_3, \text{Theorem 3.2}]$ shows that P -spaces and spaces which admit complete uniform structures are ψ -compact. It follows that a metrizable space is ψ -compact whether or not it is of non-measurable cardinal.

$[M_3, \text{Theorem 3.3}]$ notes that, since a space X is itself always a support, every pseudocompact ψ -compact space is compact.

Several characterizations of μ -compactness are given in the completely regular case.

[M₃, Theorem 4.2]: For any completely regular Hausdorff space X , the following are equivalent.

- (1) X is μ -compact.
- (2) $\beta X \setminus X$ is a round subset of βX .
- (3) For any $Z \in Z(X)$, $\text{int}_{\beta X \setminus X}(\text{cl}_{\beta X} Z \setminus X) = \text{int}_{\beta X} \text{cl}_{\beta X} Z \setminus X$.
- (4) Every cozero-set with noncompact closure contains a noncompact zero-set.
- (5) Every function in $C(X)$ that belongs to all the free maximal ideals has realcompact support.

We recall that a subset A of the Stone-Cech compactification βX is said to be round if whenever $\text{cl}_{\beta X} Z$ contains A , where $Z \in Z(X)$, then $\text{cl}_{\beta X} Z$ is a neighbourhood of A .

It is clear, that given a completely regular space X , if we let $C_M(X)$ be the set of all functions which are in every free maximal ideal we have the following:

- (a) X is μ -compact if and only if $C_M(X) = C_K(X)$, [1.15].
- (b) X is ψ -compact if and only if $C_S(X) = C_K(X)$.

Many other topological properties can be defined in this way. For example:

- (c) X is pseudocompact if and only if $C_S(X) = C(X)$,
 (d) X is compact if and only if $C_K(X) = C(X)$.

It is our intention in this chapter, to study several classes of spaces which are defined by specifying relationships between certain of the subrings which we have considered above and others, which we have yet to introduce.

In this chapter we restrict the discussion entirely to completely regular Hausdorff spaces.

3.2. We begin by listing the subrings to be used in the following paragraphs.

Let X denote any topological space. The subrings (I1) to (I5) have been discussed.

$$(I1) \quad C_K(X) = \bigcap_{p \in \beta X \setminus X} O^p = \{f \in C(X) : S(f) \text{ is compact}\}$$

[0.1] and [0.5(a)]

$$(I2) \quad C_\infty(X) = \bigcap_{p \in \beta X \setminus X} M^{*p} = \{f \in C(X) : S_n(f) \text{ is compact}$$

for every $n \in \mathbb{N}\}$, [0.5(b)] and [1.13]

$$(I3) \quad C_S(X) = \bigcap_{p \in \beta X \setminus \cup X} O^p = \bigcap_{p \in \beta X \setminus \cup X} M^p , \quad [\text{Lemma 1.9}]$$

$$= \{f \in C(X) : S(f) \text{ is pseudocompact}\} \quad [1.10]$$

$$(I4) \quad C_S^\infty(X) = \bigcap_{p \in \beta X \setminus \cup X} M^{*p} = \{f \in C(X) : S_n(f) \text{ is pseudocompact} \\ \text{for every } n \in \mathbb{N}\} , \quad [1.15]$$

$$(I5) \quad C_M(X) = \bigcap_{p \in \beta X \setminus X} M^p , \quad [3.1]$$

$$(I6) \quad C_L(X) = \bigcap_{p \in \cup X \setminus X} O^p$$

$$(I7) \quad C_L^\infty(X) = \bigcap_{p \in \cup X \setminus X} M^{*p}$$

$$(I8) \quad C_N(X) = \bigcap_{p \in \cup X \setminus X} M^p$$

REMARK. It follows from the definitions that $C_\infty(X)$, $C_S^\infty(X)$ and $C_L^\infty(X)$ are ideals in $C^*(X)$. However they need not necessarily be ideals in $C(X)$. For example, consider the function j in $C(\mathbb{N})$ defined by

$$j(n) = \frac{1}{n} .$$

It is clear that $j \in C_\infty(\mathbb{N})$. j is, however, a unit in $C(\mathbb{N})$. All the other subrings are ideals in $C(X)$.

3.3. REMARK. A cursory glance at the definition might lead one to suspect that for any space X , $C_\infty(X) = C_M(X) \cap C^*(X)$. However the previous remark provides a counterexample. For $C_N(X)$ and $C_L^\infty(X)$ the situation is different. It, however, follows

immediately from [GJ, 7.9(c)] that for any space X ,

$$C_L^\infty(X) = C_N(X) \cap C^*(X).$$

3.4. We now give the inclusion relationships holding between the subrings considered above. The known results are included.

THEOREM. For any space X the following inclusion relationships hold.

All inclusions can be proper.

$$(1) \quad (0) \subset C_K \subset C_S \subset C_S^\infty \subset C^* \subset C$$

$$(2) \quad C_K \subset C_\infty \subset C_S^\infty$$

$$(3) \quad C_K \subset C_\infty \subset C_L^\infty \subset C_N \subset C$$

$$(4) \quad C_K \subset C_M \subset C_N$$

$$(5) \quad C_K \subset C_L \subset C_N$$

PROOF. The inclusion $C_L^\infty \subset C_N$ follows from [Theorem 3.3]. The remaining inclusions follow from the respective definitions and the following known facts

(a) For every point $p \in \beta X$, $0^p \subset M^p \subset M^{*p}$, and

(b) $X \subset \cup X \subset \beta X$.

That each of these inclusions can be proper is shown by the following examples.

- (1) This has already been shown, [Theorem 1.13].
- (2) It is well known that $C_K(N) \neq C_\infty(N)$. Since $C_\infty(W) = C_K(W) \neq C(W)$, [GJ, 7G2], and since $C_S^\infty(W) = C(W)$ we see that $C_S^\infty(W) \neq C_\infty(W)$.
- (3) Consider the space $X = W \oplus N$. The function $f \in C(X)$ defined by

$$f(x) = 0 \quad \text{for all } x \in W$$

$$f(x) = 1 \quad \text{for all } x \notin W$$

is such that $f \in C_L^\infty \setminus C_\infty$.

The function defined by

$$f(x) = 0 \quad \text{for all } x \in W$$

$$\text{and} \quad f(x) = x \quad \text{for all } x \notin W$$

is such that $f \in C_N \setminus C_L^\infty$.

Clearly $C_N(Y) = C(Y)$ if and only if Y is realcompact, hence if Y is not realcompact, $C(Y) \neq C_N(Y)$.

- (4) Let T denote the Tychonoff Plank and let p denote the unique point in $\beta T \setminus T$. In this case $M^P = C_M$ and $C_K = 0^P$. It is well known that $M^P \neq 0^P$. Also $C_N(N) = C(N)$ but $C_M(N) \neq C(N)$. Hence $C_N(N) \neq C_M(N)$.
- (5) Let T and p be as in (4). Then $C_N(T) = M^P$ and $C_L(T) = 0^P$ and as before they are not equal.

3.5. It is immediate that X is realcompact if and only if either of C_L , $C_N = C$ or $C_L^\infty = C^*$.

At the other extreme we have the following result.

THEOREM. For any space X , $\cup X \setminus X$ is dense in βX if and only if either $C_L(X) = (0)$, $C_N(X) = (0)$ or $C_L^\infty(X) = (0)$.

PROOF. Necessity. Suppose that $\cup X \setminus X$ is dense in βX and let f be a function in $C_N(X)$. Then clearly $\text{cl}_{\beta X} Z(f) \supset \cup X \setminus X$ and so $\text{cl}_{\beta X} Z(f) = \beta X$ and $f = 0$. Hence $C_N(X) = (0)$.

Sufficiency. Suppose that $\cup X \setminus X$ is not dense in βX . Then there exists a point p in X and a function denoted by f^β in $C(\beta X)$ such that

$$f^\beta(p) = 1, f^\beta(x) = 0$$

for all x in some neighbourhood of $\cup X \setminus X$ and $0 \leq f^\beta(x) \leq 1$ for every $x \in \beta X$.

Let f denote the restriction of f^β to X . We wish to show that $\text{cl}_{\beta X} Z(f)$ is a neighbourhood of $\cup X \setminus X$. From this we can conclude that $f \in C_L(X) \cap C_N(X) \cap C_L^\infty(X)$ and the result will follow.

To the contrary, suppose that $\text{cl}_{\beta X} Z(f)$ is not a neighbourhood some point p in $\cup X \setminus X$. Since $X = Z(f) \cup \text{coz } f$

it follows that $p \in \text{cl}_{\beta X} \text{coz } f$ and this contradicts the fact that $f^\beta(x) = 0$ for all x in some neighbourhood of p .

3.6. λ -COMPACTNESS. It has been noted that for a topological space X , $C_N(X) \supseteq C_L(X)$ and that this inclusion can be proper. Thus we have the following definition.

DEFINITION. A space X is λ -compact if $C_N(X) = C_L(X)$.

We begin by obtaining characterizations of λ -compactness, some of which will be used below.

THEOREM. For any space X , the following are equivalent.

- (1) X is λ -compact
- (2) $\cup X \setminus X$ is a round subset of βX
- (3) If f is in the intersection of the free real maximal ideals in $C(X)$ then $\text{cl}_{\cup X} S(f) = S(f)$.

PROOF. (1) implies (2). Suppose that the zero-set $Z(f)$ is such that $\text{cl}_{\beta X} Z(f) \supset \cup X \setminus X$. This implies that f is in $\bigcap_{p \in \cup X \setminus X} M^p$ and so by the hypothesis that f is in $\bigcap_{p \in \cup X \setminus X} O^p$. This in turn implies that $\text{cl}_{\beta X} Z(f)$ is a neighbourhood of $\cup X \setminus X$. Hence $\cup X \setminus X$ is a round subset of βX .

(2) implies (3). Suppose that $f \in \bigcap_{p \in \cup X \setminus X} M^p$. Then $\text{cl}_{\beta X} Z(f) \supset \cup X \setminus X$ and so $\text{cl}_{\beta X} Z(f)$ is a neighbourhood of $\cup X \setminus X$. Since $X = Z(f) \cup S(f)$ this implies that $\text{cl}_{\beta X} S(f) \subset X \cup (\beta X \setminus \cup X)$.

This in turn ensures that $cl_{\cup X} S(f) \subset X$ and hence that $cl_{\cup X} S(f) = S(f)$.

(3) implies (1). Suppose that $f \in \bigcap_{p \in \cup X \setminus X} M^p$. We wish to show that $cl_{\beta X} Z(f)$ is a neighbourhood of $\cup X \setminus X$. Suppose to the contrary that there exists a point p of $\cup X \setminus X$ which is in the boundary of $cl_{\beta X} Z(f)$. Then since $X = Z(f) \cup S(f)$ we see that $p \in cl_{\beta X} S(f)$. This implies that $p \in cl_{\cup X} S(f)$ and therefore that $S(f) \neq cl_{\cup X} S(f)$. This contradicts (3) and the theorem follows.

3.7. We now establish the position of λ -compactness with respect to realcompactness and μ -compactness.

THEOREM. For a topological space X , each of the following conditions implies the next.

- (1) X is realcompact.
- (2) X is λ -compact.
- (3) X is μ -compact.

PROOF. (1) implies (2). We have noted that if X is realcompact $C_N(X) = C_L(X) = C(X)$.

(2) implies (3). Let $f \in \bigcap_{p \in \beta X \setminus X} M^p$. Then by (2) $f \in \bigcap_{p \in \beta X \setminus X} O^p$.

By [3.2, I3] we have that $f \in \bigcap_{p \in \beta X \setminus \cup X} M^p$ implies that

$f \in \bigcap_{p \in \beta X \setminus \cup X} O^p$. Hence $f \in \bigcap_{p \in \beta X \setminus X} O^p$. This completes the proof.

3.8. Mandelker remarks [M₃, page 78] that pseudocompactness together with μ -compactness does not imply compactness. We are now able to clarify this situation slightly.

THEOREM. Every pseudocompact μ -compact space is λ -compact.

PROOF. Let $f \in \bigcap_{p \in \cup X \setminus X} M^p$. Since X is pseudocompact, $\beta X = \cup X$, [GJ, 8A], and so $f \in \bigcap_{p \in \beta X \setminus X} M^p$. It follows that $f \in \bigcap_{p \in \cup X \setminus X} 0^p$ and the theorem is proved.

3.9. We now show that the λ -compact spaces include a large class of ψ -compact spaces which need not be realcompact.

THEOREM. Every p -space is λ -compact.

PROOF. Suppose that X is a p -space. Then $\cup X$ is a p -space, [GJ, 8A5]. Let $f \in \bigcap_{p \in \cup X \setminus X} M^p$. Then $f^u(p) = 0$ for every $p \in \cup X \setminus X$. We must show that $cl_{\beta X} Z(f)$ is a neighbourhood of $\cup X \setminus X$. Let us suppose, to the contrary, that there exists a point $p \in \cup X \setminus X$ such that p is in the boundary of $cl_{\beta X} Z(f)$. Then as before $p \in cl_{\beta X} S(f)$ and we see that in every neighbourhood of p there exists a point x in X at which $f(x) \neq 0$. This is a contradiction to the fact that $\cup X$ is a p -space and the result follows.

3.10. COROLLARY. Any discrete space is λ -compact.

3.11. Concerning the permanence properties of the class of λ -compact spaces, not much can be said at the moment. The Tychonoff plank is not λ -compact. Considering T as an open subset of βT we see that open subsets of λ -compact spaces need not be λ -compact.

It is not known to the author, whether or not the class of λ -compact spaces is closed under products or quotients or whether closed subspaces of λ -compact spaces are λ -compact.

We do however know that in the absence of measurable cardinals, this class is closed under the formation of direct unions. To show this we need the following lemma.

3.12. LEMMA. Let $(X_i)_{i \in I}$ denote a family of topological spaces indexed by I . Then, for every point i in I

$$\beta X_i \text{ is an open-closed subset of } \bigoplus_{i \in I} X_i.$$

PROOF. Denote $\bigoplus_{i \in I} X_i$ by X . For every $i \in I$ X_i is C^* -embedded in X and so $\beta X_i = cl_{\beta X} X_i$, [GJ, 6.9(a)]. Since each such X_i is open and closed in X , it follows as a basic property of the Stone-Cech compactification that $cl_{\beta X} X_i$ is open and closed in βX . The result follows.

3.13. THEOREM. Let $(X_i)_{i \in I}$ denote a family of λ -compact spaces indexed by I . If I is of nonmeasurable cardinal,

$\bigoplus_{i \in I} X_i$ is λ -compact.

PROOF. Let us denote $\bigoplus_{i \in I} X_i$ by X . Since I is nonmeasurable we have that $\cup X = \bigoplus_{i \in I} \cup X_i$, [H₂, Satz 7.22]. Hence

$$\bigcap_{p \in \cup X \setminus X} M^p = \bigcap_{i \in I} \left(\bigcap_{p \in \cup X_i \setminus X_i} M^p \right).$$

Now $f \in \bigcap_{p \in \cup X_i \setminus X_i} M^p$ if and only if $cl_{\beta X \setminus X}^Z(f) \supset \cup X_i \setminus X_i$ [0.5].

By [lemma 3.12] $cl_{\beta X \setminus X}^Z(f) \cap \beta X_i = cl_{\beta X_i \setminus X_i}^Z(f)$. Then using the

fact that each X_i is λ -compact we see that for each $i \in I$

$cl_{\beta X \setminus X}^Z(f)$ is a neighbourhood of $\cup X_i \setminus X_i$. Hence

$f \in \bigcap_{p \in \cup X_i \setminus X_i} O^p$. Thus we have that

$$\bigcap_{i \in I} \left(\bigcap_{p \in \cup X_i \setminus X_i} M^p \right) \subset \bigcap_{i \in I} \left(\bigcap_{p \in \cup X_i \setminus X_i} O^p \right)$$

and so $\bigcap_{p \in \cup X \setminus X} M^p = \bigcap_{p \in \cup X \setminus X} O^p$. This concludes the proof of the theorem.

3.13a. REMARKS. We have shown that any realcompact space is λ -compact. We now show that the class of λ -compact spaces strictly includes the class of realcompact spaces.

EXAMPLE. A λ -compact space which is not realcompact. Mandelker

[M₃] remarks that the space of countable ordinals is μ -compact and pseudocompact. It follows by [Theorem 3.8] that it is λ -compact. It is also well known not to be realcompact.

At this point we are unable to give an example of a μ -compact space which is not λ -compact. We are also unable to show the equivalence of the respective classes.

The following diagram summarizes much of the work of paragraphs [3.6-3.12], if we bear in mind that proper inclusion between the outer classes is still uncertain.

3.14. δ -COMPACTNESS. Pursuing the theme of the chapter we consider another class of spaces defined by means of ideals in $C(X)$.

DEFINITION. A space X will be called δ -compact if $C_N(X) = C_K(X)$.

THEOREM. Every δ -compact space is λ -compact. The converse is false.

PROOF. By [Theorem 3.4(5)] we have that for any space X ,

$C_K(X) \subset C_L(X) \subset C_N(X)$. Hence if $C_N(X) = C_K(X)$ then
 $C_L(X) = C_N(X)$. N is a λ -compact space which is not δ -compact.

3.15. THEOREM. A space X is compact if and only if it is both realcompact and δ -compact.

PROOF. If X is realcompact then $\cup X \setminus X$ is empty and
 $C_N(X) = C(X)$. Hence if X is δ -compact $C(X) = C_K(X)$ and X
 is compact. Clearly any compact space is both realcompact and
 δ -compact.

3.16. We conclude with examples and a diagram.

EXAMPLE (1). W is a δ -compact which is not realcompact.

(2) $N \oplus W$ is a λ -compact space which is neither realcompact
 nor δ -compact.

These two examples enable us to give the following
 diagrammetric representation of the situation with all inclusions
 proper.

Chapter 4

THE COMPONENTS OF $C(X)$ IN THE m -TOPOLOGY

4.1. In this chapter we establish a link between the ideal $C_S(X)$ and the components of $C(X)$ in the m -topology. This allows us to characterize some of the connectedness properties of $C(X)$ in the m -topology in terms of topological properties of the underlying space X .

The m -topology for $C(X)$ was first studied by Hewitt in the classic paper [H₃]. For each function f in $C(X)$ the sets of the form

$$V_p(f) = \{g \in C(X) : |f - g| < p\}$$

where p ranges over the never vanishing positive functions in $C(X)$, form a local neighbourhood base at f for this topology. We denote $C(X)$ in the m -topology by $\langle C(X), m \rangle$.

Throughout this chapter we confine our attention to completely regular Hausdorff spaces.

4.2. Before we study the components of $C(X)$ in the m -topology we shall survey the situation for $C^*(X)$ and $C(X)$ in another well known topology, namely, the topology of uniform convergence,

(denoted by u -topology).

For each function f in $C(X)$ the sets of the form

$$V_{\epsilon}(f) = \{g \in C(X) : |g - f| < \epsilon\}$$

where ϵ ranges over the positive reals, form a local neighbourhood base at f for this topology.

We denote $C(X)$ in the u -topology by $\langle C(X), u \rangle$.

The u -topology in $C^*(X)$ can be similarly defined and is just the relative topology of $C^*(X)$ as a subspace of $\langle C(X), u \rangle$.

It is well known that $\langle C^*(X), u \rangle$ is a Banach algebra, because the topology generated by the norm

$$\|f\| = \sup_{x \in X} |f(x)|$$

is the same as the u -topology. As such it is connected.

The same is not true for $\langle C(X), m \rangle$ as we shall see in a moment.

DEFINITION. The principal component of $\langle C(X), u \rangle$ resp. $\langle C(X), m \rangle$ is the component containing the zero function. As a preliminary result we have the following.

THEOREM. The principal component of $\langle C(X), u \rangle$ is the subring $C^*(X)$.

PROOF. Let $f \in C^*(X)$. As remarked above $\langle C^*(X), u \rangle$ is connected as a topological space in its own right. In particular it must be connected as a subspace of $C(X)$. This implies that f is in the principal component of $C(X)$.

Conversely. We have only to note that if $f \notin C^*(X)$ then the sets $C^*(X)$ and $C(X) \setminus C^*(X)$ form a disconnection of $C(X)$ separating f and the zero function.

4.3. COROLLARY. The components of $\langle C(X), u \rangle$ are the equivalence classes of $C(X)$ modulo $C^*(X)$.

PROOF. This follows from the facts that $C(X)$ is a topological group in the u -topology and that continuous maps preserve connectedness.

4.4. We now turn our attention to $\langle C(X), m \rangle$ and prove a deeper result.

THEOREM. The principal component of $\langle C(X), m \rangle$ is the ideal $C_S(X)$.

PROOF. Let f be a function in $C_S(X)$. Let

$$A(f) = \{g \in C(X) : S(g) \subset S(f)\}.$$

Since $S(f)$ is pseudocompact $A(f)$ is a normed algebra, the relative m -topology being generated by the supremum norm. As

such it is connected. Since it is a connected subset containing both f and the zero function, f is in the principal component.

Conversely. Suppose that f is a function in $C(X) \setminus C_S(X)$. Then $\text{coz } f$ is not relatively pseudocompact and contains a copy of N , denoted by $T = \{t_n, n \in N\}$, which is C -embedded in X , [Theorem 1.2(4)]. Since T is C -embedded in X , there exists a function h in $C(X)$ such that

$$h(t_n) = \frac{1}{|f(t_n)|} \quad \forall n.$$

The function K in $C(X)$, defined pointwisely by

$$K(x) = \frac{1}{|fh(x)|},$$

has the following properties.

- (1) K is strictly positive.
- (2) $K(t_n) \leq |f(t_n)|$ for every $n \in N$.
- (3) $K(t_n) \rightarrow 0$ as $n \rightarrow \infty$.

We construct a disconnection of $\langle C(X), m \rangle$ separating f and the zero function. Put

$$A = \{g \in C(X) : gK^{-2} \text{ is bounded on } T\}.$$

Clearly the zero function is in A . Also, since for every $n \in N$ $K(t_n) \leq |f(t_n)|$, we have that

$$1 \leq \frac{|f(t_n)|}{K(t_n)} .$$

This implies that

$$0 \leq \frac{|f(t_n)|}{K(t_n)} - 1$$

and

$$0 \leq \left(\frac{|f(t_n)|}{K(t_n)} - 1 \right) \frac{1}{K(t_n)}$$

and hence that

$$\frac{1}{K(t_n)} \leq \frac{f(t_n)}{K^2(t_n)} , \text{ for every } n \in \mathbb{N} .$$

Since $K(t_n) \rightarrow 0$ as $n \rightarrow \infty$ we see that fK^{-2} is not bounded on T and $f \in C(X) \setminus A$.

The result will now follow if we can show that A is open and closed in $\langle C(X), m \rangle$.

To see that A is open: suppose that $g \in A$ and consider $g' \in V_p(g)$ where $p = K^2$. Clearly

$$|g'| (t_n) \leq (|g' - g| + |g|)(t_n) .$$

But $|g|(t_n) < CK^2(t_n)$ for some real $C > 0$. Also

$|g' - g|(t_n) < K^2(t_n)$ for all $n \in \mathbb{N}$. It follows that

$$|g'|(\tau_n) \leq (C + 1)K^2(\tau_n) \quad \text{for all } n \in \mathbb{N}$$

and hence that $g' \in A$. This implies that A is open.

To see that $C(X) \setminus A$ is open: suppose that $g \in C(X) \setminus A$ and consider $g' \in V_\pi(g)$ where $\pi = K^4$. Clearly

$$|g'|(\tau_n) \geq ||g| - |g' - g||(\tau_n).$$

Since gK^{-2} is unbounded on T , the set

$$I_C = \{n : |g(\tau_n)| \geq CK^2(\tau_n)\}$$

is infinite for every positive real $C \geq 1$.

So for $n \in I_C$, $|g'|(\tau_n) \geq |CK^2 - |g' - g||(\tau_n)$ because for any $n \in \mathbb{N}$, $|g' - g|(\tau_n) \leq K^4(\tau_n) \leq K^2(\tau_n)$.

Thus we see that, for $n \in I_C$

$$\begin{aligned} |g'|(\tau_n) &\geq |CK^2 - K^4|(\tau_n) \\ &\geq |K^2(C - K^2)|(\tau_n). \end{aligned}$$

Hence if we choose an $n_0 \in \mathbb{N}$ beyond which $K^2(\tau_n) < \frac{1}{4}$ we can conclude that for all such that $n \in I_C$ and $n \geq n_0$, we have

$$|g'|(\tau_n) \geq |K^2(C - \frac{1}{4})|(\tau_n).$$

This shows that $g'K^{-2}$ is unbounded and that $g' \in C(X) \setminus A$.

This completes the proof.

4.5. COROLLARY. The components of $\langle C(X), m \rangle$ are the equivalence classes of $C(X)$ modulo $C_S(X)$.

PROOF. The proof is similar to that of [Corollary 4.3].

4.6. Let the space X be ψ -compact. Then the components of $\langle C(X), m \rangle$ are the equivalence classes of $C(X)$ modulo $C_K(X)$. In particular this is true for realcompact X .

PROOF. This follows from the definition of ψ -compactness.

4.7. We are now able to characterize some connectedness properties of $\langle C(X), m \rangle$ in terms of topological properties of X .

COROLLARY. For any topological space X , $\langle C(X), m \rangle$ is totally disconnected if and only if $\beta X \setminus \cup X$ is dense in βX .

PROOF. $\langle C(X), m \rangle$ is totally disconnected if and only if its components are its points. This is clearly the case if and only if $C_S(X) = (0)$. The result then follows by [Theorem 1.16].

4.8. COROLLARY. $\langle C(Q), m \rangle$ and $\langle C(R \setminus Q), m \rangle$ are totally disconnected.

4.9. We also obtain the following result which is essentially well known.

COROLLARY. For any space X , $\langle C(X), m \rangle$ is connected if and only if X is pseudocompact.

PROOF. $\langle C(X), m \rangle$ is connected if and only if $C(X) = C_S(X)$. This holds if and only if X is pseudocompact.

4.10. Remark. Although $\langle C^*(X), u \rangle$ is connected it is clear that $\langle C^*(X), m \rangle$ has no such property. In fact we can see, at this point, that $\langle C^*(X), m \rangle$ is very similar to $\langle C(X), m \rangle$ in this respect, namely, Corollary 4.5 holds verbatim with $C^*(X)$ replacing $C(X)$.

THEOREM. The components of $\langle C^*(X), m \rangle$ are the equivalence classes of $C^*(X)$ modulo $C_S(X)$.

4.11. GROUP OF POSITIVE UNITS IN $C(X)$. Let $G^+(X) = \{f \in C(X) : f(x) > 0 \text{ for every } x \in X\}$. $G(X)$ is a group under the operation of pointwise multiplication. It is also a topological group in the relative m -topology.

We show that the above study is in many ways compatible with a study of subgroups of $G^+(X)$.

Consider the map

$$\ell_n : G^+(X) \rightarrow C(X)$$

defined pointwisely by

$$(\ln(g))x = \ln(g(x)) .$$

The additive group structure carried over from $G^+(X)$ by the logarithm function is the same as the ordinary pointwise addition in $C(X)$. The map is also onto having the antilog function as an obvious inverse.

It is also a topological group homomorphism.. Hence we have the following theorem.

THEOREM. $\langle C(X), m \rangle$ is homomorphic as a topological group to its set of positive of units considered as a topological group under multiplication with the relative m -topology. The homomorphism is the logarithm function described above.

It follows that the results contained in corollaries [4.5, 6, 7, 8, 9] remain true when $C(X)$ is replaced by $G^+(X)$ and $C_S(X)$ is replaced by the appropriate subgroup of $G^+(X)$.

Chapter 5

$C(X)$ IN THE GRAPH TOPOLOGY

5.1. The uniform topology is well known to be compatible with the ring structure of $C^*(X)$ but not with $C(X)$. In contrast, the m -topology, as remarked by Hewitt [H₃, pg 49], has a very natural character for the ring $C(X)$. For instance, $C(X)$ is a topological ring in the m -topology.

In the remaining chapters we consider $C(X)$ in yet another topology, this time the graph topology (denoted by g -topology).

The g -topology for function spaces was first studied by Naimpally, [N₁]. It has since been applied to the study of $C(X)$ by Poppe [P, p 2], Hansard, [H₁] et al., and it is this aspect of the study which we pursue.

We begin by comparing some properties of $C(X)$ in the g -topology with those of $C(X)$ in the m -topology. We find, [5.2], that $C(X)$ in the g -topology shares the pleasant property of being a topological ring with $C(X)$ in the m -topology.

Another problem which we attempt is the following. When do the g -topology and the m -topology coincide in $C(X)$? It is

known that, in general, the m -topology is stronger than the g -topology, $[P_1, (9)]$. Hansard has shown that the g -topology and the u -topology coincide in $C(X)$ if and only if X is countably compact, $[H_1]$. We prove the following.

[5.5]: A space X is a cb -space if and only if $g = m$ on $C(X)$: cb -spaces have been studied by Mack $[M_1]$ from a different point of view.

Using this, we are able to obtain the following theorem.

[5.6]: An arbitrary topological space is countably compact if and only if it is a pseudocompact cb -space.

This sharpens a result of Mack, which we now quote.

$[M_1, \text{Theorem 9}]$: Let X be a completely regular pseudocompact space. Then the following statements are equivalent.

- (1) X is countably compact.
- (2) X is a cb -space.
- (3) X is countably paracompact.

In this chapter we shall assume that our domain space X is T_1 .

5.2. THE GRAPH TOPOLOGY AND $C(X)$. If f denotes a function from X to Y then the graph of f is the set $G(f) = \{(x, f(x)) : x \in X\}$.

Let X denote any topological space. The graph topology

in $C(X)$ has a basis consisting of sets of the form

$$F_U = \{f \in C(X) : G(f) \subset U\}$$

where U ranges over the sets open in $X \times \mathbb{R}$, $[H_1]$.

For any function f in $C(X)$ and any subset A of X , by the oscillation of f on A we shall mean the number

$$\text{osc}_{x \in A} f(x) = \sup_{x \in A} |f(x)| - \inf_{x \in A} |f(x)|$$

which includes the infinite case.

THEOREM. Let X denote any topological space. Then $\langle C(X), g \rangle$ is a topological ring in which inversion is a continuous operation.

PROOF. Firstly we show that addition is a continuous operation.

Let f and g be functions in $C(X)$, and let U be an open set in $X \times \mathbb{R}$ such that $G(f + g) \subset U$. We shall exhibit open subsets V and W of $X \times \mathbb{R}$ with the properties that $G(f) \subset V$ and $G(g) \subset W$, and if f' and g' are functions such that $G(f') \subset V$ and $G(g') \subset W$ then $G(f' + g') \subset U$.

Corresponding to each point $x \in X$ there exists a neighbourhood $A(x)$ of x and an open interval $B((f + g)(x), r(x))$, with center $(f + g)(x)$ and radius $r(x)$ such that

$$(1) \quad A(x) \times B((f + g)(x), r(x)) \subset U, \text{ and}$$

$$(2) \quad \operatorname{osc}_{y \in A(x)} (f + g)(y) < \frac{1}{2} r(x) .$$

Furthermore, corresponding to each point $x \in X$ there exist an open neighbourhood $A_1(x)$ of x and a positive real number $r_1(x)$ subject to the following conditions.

$$(3) \quad A_1(x) \subset A(x) ,$$

$$(4) \quad r_1(x) < \frac{1}{2} r(x) ,$$

$$(5) \quad \operatorname{osc}_{y \in A_1(x)} f(y) < \frac{1}{8} r_1(x) ,$$

$$(6) \quad \operatorname{osc}_{y \in A_1(x)} g(y) < \frac{1}{8} r_1(x) .$$

We now define V and W and then show that they have the required properties.

Let

$$(7) \quad V = \bigcup_{x \in X} (A_1(x) \times B(f(x), \frac{1}{8} r_1(x))) \quad \text{and}$$

$$(8) \quad W = \bigcup_{x \in X} (A_1(x) \times B(g(x), \frac{1}{8} r_1(x))) .$$

Suppose now that the functions f' and g' are such that $G(f') \subset V$ and $G(g') \subset W$. Then, for every point $x \in X$, there exist points $y, z \in X$ such that

$$(9) \quad (x, f'(x)) \in A_1(y) \times B(f(y), \frac{1}{8} r_1(y)) \quad \text{and}$$

$$(10) \quad (x, g'(x)) \in A_1(z) \times B(f(z), \frac{1}{8} r_1(z)) .$$

We may suppose, without any loss that

$$(11) \quad r_1(y) \leq r_1(z) .$$

In this case we have, using (9), (5) and (10), (6) and (11), that

$$\begin{aligned} |(f'+g')(x) - (f+g)(x)| &\leq |f'(x) - f(x)| + |g'(x) - g(x)| \\ (12) \quad &\leq \frac{1}{4} r_1(y) + \frac{1}{4} r_1(z) \\ &\leq \frac{1}{2} r_1(z) \leq \frac{1}{4} r(z) . \end{aligned}$$

Furthermore, using (12) and (2) we have that

$$\begin{aligned} |(f'+g')(x) - (f+g)(z)| &\leq |(f'+g')(x) - (f+g)(x)| + |(f+g)(x) - (f+g)(z)| \\ &\leq \frac{1}{4} r(z) + \frac{1}{2} r(z) \\ &< r(z) . \end{aligned}$$

Hence, by (1)

$$(x, (f'+g')(x)) \in A(z) \times B(f+g)(z), r(z)) \subset U ,$$

• and therefore

$$G(f'+g') \subset U .$$

This establishes that $\langle C(X), g \rangle$ is a topological group.

To show that multiplication is continuous requires even more manipulation. Let f, g and U be given as before. We show the existence of sets V and W having the required properties.

Corresponding to each $x \in X$ there exists a neighbourhood $A(x)$ of x and an open interval $B((fg)(x), r(x))$ of center $(fg)(x)$ and radius $r(x)$ with the following properties.

$$(1') \quad A(x) \times B((fg)(x), r(x)) \subset U, \text{ and}$$

$$(2') \quad \operatorname{osc}_{y \in A(x)} (fg)(y) < r(x).$$

Furthermore, corresponding to each point $x \in X$ there exists an open neighbourhood $A_1(x)$ of x and a positive real number $r_1(x)$ with the following properties.

$$(3') \quad A_1(x) \subset A(x)$$

$$(4') \quad r_1(x) < [\{ (|f| + |g|)^2(x) + 2r(x) \}^{1/2} - (|f| + |g|)(x)/2]$$

$$(5') \quad \operatorname{osc}_{y \in A_1(x)} f(y) < \frac{1}{8} r_1(x)$$

$$(6') \quad \operatorname{osc}_{y \in A_1(x)} g(y) < \frac{1}{8} r_1(x).$$

As before, let

$$(7') \quad V = \bigcup_{x \in X} (A_1(x) \times B(f(x), \frac{1}{8} r_1(x)))$$

$$(8') \quad W = \bigcup_{x \in X} (A_1(x) \times B(g(x), \frac{1}{8} r_1(x)))$$

Clearly $G(f) \subset V$ and $G(g) \subset W$.

Suppose, as before, that the functions f' and g' are such that $G(f') \subset V$ and $G(g') \subset W$. Then, for every point $x \in X$ there exist points $y, z \in X$ such that

$$(9') \quad (x, f'(x)) \in A_1(y) \times B(f(y), \frac{1}{8} r_1(y)) \text{ , and}$$

$$(10') \quad (x, g'(x)) \in A_1(z) \times B(g(z), \frac{1}{8} r_1(z)) \text{ .}$$

Suppose, without loss of generality, that $r_1(y) < r_1(z)$.

We wish to show that $(x, (f'g')(x)) \in A(z) \times B((fg)(z), r(z))$. The result will then follow.

To proceed, by (10'), we have that

$$|g'(x)| \leq |g(z)| + \frac{1}{8} r_1(z)$$

and so, using (6') we have that

$$(11') \quad |g'(x)| \leq |g(x)| + \frac{1}{4} r_1(z) \text{ .}$$

Also, using (6') and (10') we have that

$$\begin{aligned}
 |g - g'| (x) &\leq |g(x) - g(z)| + |g(z) - g'(x)| \\
 (12') \quad &\leq \frac{1}{8} r_1(z) + \frac{1}{8} r_1(z) \\
 &\leq \frac{1}{4} r_1(z) .
 \end{aligned}$$

Using (12'), (11'), (9'), (5') we have that

$$\begin{aligned}
 |fg - f'g'| (x) &\leq |f| |g - g'| (x) |g'| |f - f'| (x) \\
 &\leq |f| (x) \frac{1}{4} r_1(z) + (|g| (x) + \frac{1}{4} r_1(z)) \frac{1}{4} r_1(y)
 \end{aligned}$$

$$\text{Using (11') : L.H.S.} \leq |f| (x) \frac{1}{4} r_1(z) + (|g| (x) + \frac{1}{4} r_1(z)) \frac{1}{4} r_1(z) .$$

$$\begin{aligned}
 \text{Then by (5') and (6') : L.H.S.} &\leq [|f(z) + \frac{1}{4} r_1(z) + |g(z) + \frac{1}{4} r_1(z) + \frac{1}{4} r_1(z)| \\
 &\quad \frac{1}{4} r_1(z)] \\
 &\leq [|f(z)| + |g(z)| + r_1(z)] r_1(z) .
 \end{aligned}$$

Then using (4')

$$\begin{aligned}
 \text{L.H.S.} &\leq [|f(z)| + |g(z)| + \{(|f| + |g|)^2(z) + 2r(z)\}^{1/2}/2 - \\
 (13') \quad &\quad (|f| + |g|)(z)/2] r_1(z) \\
 &\leq ((|f| + |g|)^2(z) + 2r(z))/4 - (|f| + |g|)^2(z)/4 \\
 &\leq \frac{1}{2} r(z) .
 \end{aligned}$$

But

$$\begin{aligned} |(f'g')(x) - (fg)(z)| &\leq |(f'g')(x) - (fg)(x)| + |(fg)(x) - (fg)(z)| \\ &\leq \frac{1}{2} r(z) + \frac{1}{2} r(z) , \text{ by (13') and (2') .} \end{aligned}$$

Hence

$$(x, (f'g')(x)) \in A(z) \times B((fg)(z), r(z)).$$

It is a simple matter to see that inversion is a continuous operation. Suppose that f is an invertible function in $C(X)$ and that $G(f) \subset U$ where U is open in $X \times \mathbb{R}$.

Corresponding to each point $x \in X$ there exist a neighbourhood $A(x)$ of x and a positive real number $r(x)$ with the following properties.

- (1) $r(x) < |f(x)|$.
- (2) $A(x) \times B(f(x), r(x)) \subset U$.
- (3) $\operatorname{osc}_{y \in A(x)} f(y) < r(x)$.

Let

$$U' = \bigcup_{x \in X} A(x) \times B\left(f^{-1}(x), \frac{1}{f(x) + (-1)^n r(x)}\right) ,$$

where $n = 1$ if $f(x) < 0$, and $n = 2$ if $f(x) > 0$.

It is clear that $G(f^{-1}) \subset U'$ and that if f' is any function in $C(X)$ such that $G(f') \subset U'$, then $G(f) \subset U$.

5.3. COROLLARY. The g -closure of a proper ideal in $C(X)$ is a proper ideal.

PROOF. Let I be a proper ideal in $C(X)$. Since the g -topology is stronger than the m -topology, [5.1] we see that

$$g - cl I \subset m - cl I .$$

But we know that $m - cl I$ is a proper ideal, [GJ, 2N(5)], and so the result follows from [Theorem 5.2], and the well known fact that in a topological ring the closure of an ideal is an ideal.

5.4. COROLLARY. The group of units of $C(X)$ is a topological group in the relative g -topology. It is g -open in $C(X)$ and the map $f \rightarrow f^{-1}$ is a topological group automorphism.

PROOF. Since $g \supseteq m$, and since the group of units is open in the m -topology we see that it is g -open. The rest of the assertion follows from the theorem.

5.5. THE GRAPH TOPOLOGY AND THE m -TOPOLOGY. We now recall the concept of a cb -space as characterized by Mack in $[M_1]$.

DEFINITION. An arbitrary topological space X is a cb -space if, given a positive (nonvanishing) lower-semicontinuous function

g on X , there exists a function f in $C(X)$ such that

$$0 < f(x) \leq g(x) \text{ for every } x \in X .$$

Although Mack studied these spaces from a different point of view, we find that these are precisely the spaces X for which $\Gamma = m$ in $C(X)$.

THEOREM. An arbitrary topological space X is a cb-space if and only if $m = g$ in $C(X)$.

PROOF. Since both $\langle C(X), g \rangle$ and $\langle C(X), m \rangle$ are topological groups under addition, [Theorem 5.1] and [H_3 , Theorem 3] or [GJ, 2N(1)], we need only consider a local neighbourhood base at 0 , where 0 denotes the zero function.

Necessity. Suppose that U is open in $X \times R$ and that $G(0) \subset U$. We may assume that U is of the form

$$\bigcup_{s \in S} A_s \times B(0, r_s)$$

where for every $s \in S$, A_s is open in X and $B(0, r_s)$ is an open interval in R with center 0 and radius r_s .

For every $s \in S$ let the function $f_s \in R^X$ be defined by

$$g_s(x) = \frac{1}{2} r_s \text{ for every } x \in A_s$$

and

$$g_s(x) = 0 \text{ for every } x \in X \setminus A_s.$$

Let g be the function in R^X defined by $\bigvee_{s \in S} g_s$. Since $(A_s)_{s \in S}$ is a cover of X it follows that g is positive and nonvanishing. Also, since each f_s is obviously lower semi-continuous it follows by a standard result [B, Theorem 4.6.4] that g is lower semi-continuous.

We now show that $G(g) \subset U$. For any $x \in X$ let there exists $s \in S$ such that

$$\frac{1}{2} r_s = f_s(x) > \frac{3}{4} f(x)$$

This implies that $r_s > f(x)$ and hence that $(x, f(x)) \in A_s \times B(0, r_s) \subset U$.

Now, by the definition there exists a function $f \in C(X)$ such that $0 < f(x) \leq g(x)$ for every $x \in X$. It is then clear that if h is any function in $C(X)$ such that

$$|h(x)| < f(x) \text{ for all } x \in X$$

then $G(h) \subset U$, and the result follows.

Sufficiency. Suppose that g is a nonvanishing, positive and lower semi-continuous function in R^X . Corresponding

to each point $x \in X$ there exists an open neighbourhood $A(x)$ of x in X and a positive real number $r(x)$ with the property that $g(y) > r(x)$ for every $y \in A(x)$.

Put

$$U = \bigcup_{x \in X} A(x) \times B(0, r(x)) .$$

Clearly $G(0) \subset U$. By hypothesis there exists a positive function $p \in C(X)$ such that for any function $f \in C(X)$

$$f \in V_p(0) \text{ implies that } G(f) \subset U .$$

The function p has the required properties.

5.6. COROLLARY. An arbitrary topological space X is countably compact if and only if it is a pseudocompact cb-space.

PROOF. Hansard has shown that a space X is countably compact if and only if $g = u$ on $C(X)$, where u denotes the topology of uniform convergence, $[H_1]$. It is essentially well known that $u = m$ on $C(X)$ if and only if X is pseudocompact. We have shown that $g = m$ on $C(X)$ if and only if X is a cb-space. The result follows.

5.7. THE GRAPH TOPOLOGY AND THE 0^P IDEALS IN $C(X)$. The problem of characterizing the closure of an ideal I in $C(X)$ in the

m-topology was solved by Gillman, Henriksen & Jerison, see [GJ, 7Q]. The corresponding problem for $C(X)$ in the uniform topology has recently been solved by Nanzetta and Plank [NP]. This paragraph constitutes an attempt to work the corresponding problem for the g-topology. However we have had to settle for less.

We naturally make the assumption that the domain space X is completely regular and Hausdorff.

Firstly, we establish some notation.

DEFINITION. (1). Let f denote a function in $C(X)$. We say that two subsets A and B of X are f -separated if $f(x) = 0$ for $x \in A$ for some positive real number ϵ .

either $f(x) > \epsilon > 0$ for all $x \in B$
or $f(x) < -\epsilon < 0$ for all $x \in B$.

DEFINITION. (2). We call an open cover $(O_s)_{s \in S}$ of X an f -cover if every point $x \in \text{coz } f$ is contained in some set O_s which is f -separated from $Z(f)$.

THEOREM. Let X denote a topological space and let p denote any point in βX . For any function f in $C(X)$ the following are equivalent.

$$(1) \quad f \in g - cl \, O^p.$$

$$(2) \quad Z(f) \neq \emptyset \text{ and, for every } f\text{-cover } (O_s)_{s \in S} \text{ of } X,$$

there exists a neighbourhood $V(p)$ of p in βX such that

$$X \cap V(p) \subset U\{0_s : 0_s \cap Z(f) \neq \emptyset\} .$$

Necessity. This we establish by contraposition.

Suppose that $(0_s)_{s \in S}$ is an f -cover of X such that for any neighbourhood $V(p)$ of p in βX we have that

$$X \cap V(p) \not\subset U\{0_s : 0_s \cap Z(f) \neq \emptyset\} .$$

For each point $x \in \text{coz } f$ there exist $s_x \in S$ and an open interval (t_x, t'_x) such that $(x, f(x)) \in 0_{s_x} \times (t_x, t'_x)$ and $0 \notin (t_x, t'_x)$.

For each $x \in Z(f)$ there exists $s_x \in S$ such that $x \in 0_{s_x}$. Let

$$U = \left(\bigcup_{x \in \text{coz } f} (0_{s_x} \times (t_x, t'_x)) \right) \cup \left(\bigcup_{x \in Z(f)} 0_{s_x} \times (-1, 1) \right) .$$

Clearly $G(f) \subset U$ and U is open in $X \times \mathbb{R}$. We now show that any function $g \in C(X)$ with $G(g) \subset U$ is such that $g \notin 0^P$.

This implies that $f \notin g - \text{cl } 0^P$. Suppose therefore that $g \in C(X)$ has $G(g) \subset U$, and let $V(p)$ be any neighbourhood of p in βX . By the assumption it is clear that, for some $z \in (X \cap V(p)) \setminus U\{0_s : 0_s \cap Z(f) \neq \emptyset\}$ we have that $g(z) \neq 0$.

Hence $g \notin 0^P$ and the result follows.

Sufficiency. Let f be a function in $C(X)$ with $Z(f) \neq \emptyset$ and let U be an open set in $X \times R$ with $G(f) \subset U$. We may assume that U is of the form

$$\left(\bigcup_{x \in \text{coz } f} 0_{s_x} \times (t_x, t'_x) \right) \cup \left(\bigcup_{x \in Z(f)} 0_{s_x} \times (-t_x, t_x) \right)$$

where when $x \in \text{coz } f$, $0 \notin (t_x, t'_x)$ and for every corresponding $s_x \in S$, $x \in 0_{s_x}$ implies $f(x) \in (t_x, t'_x)$.

Let $V(p)$ be a neighbourhood of p in βX with the hypothesized property. We know that there exists a function h in $C(\beta X)$ such that

$$h(x) = 0 \text{ on some neighbourhood of } p,$$

$$h(x) = 1 \text{ for all } x \in \beta X \setminus V(p),$$

$$\text{and } 0 \leq h(x) \leq 1 \text{ for all } x \in \beta X.$$

Since $h \in 0^P$ it is clear that $hf \in 0^P$. We show that $G(hf) \subset U$.

Suppose that $x \in V(p) \cap X$. Then by hypothesis there exists $x' \in Z(f)$ such that

$$(x', f(x')) \in 0_{s_{x'}} \times (-t_{x'}, t_{x'}) .$$

Also $hf(x) = f(x) \cdot h(x)$ where $0 \leq h(x) \leq 1$. Hence

$hf(x) \in (-t_{x'}, t_{x'})$ and $(x, hf(x)) \in 0_{s_{x'}} \times (t_{x'}, t_{x'})$.

On the other hand, if $x \notin V(p)$ then $hf(x) = f(x)$ and it is clear that $(x, hf(x)) \in U$.

Hence $G(hf) \subset U$ and so $f \in g - cl\ 0^P$.

5.8. COROLLARY. If the ideal 0^P in $C(X)$ is fixed, then $g - cl\ 0^P = M^P$.

PROOF. Since $g \supset m$ on $C(X)$ it follows that $g - cl\ 0^P \subset M^P$. That equality actually holds follows trivially from the theorem.

5.9. We now give an example of a g -closed ideal in $C(X)$ which is not m -closed.

EXAMPLE. Let T denote the Tychonoff plank and let p denote the unique point in $\beta T \setminus T$. We have mentioned before that $M^P \neq 0^P$. This means that 0^P is not m -closed, [GJ, 7Q].

To see that 0^P is g -closed, let $f \in M^P \setminus 0^P$. Then $f(n_k, w_1) \neq 0$ for some infinite sequence $(n_k)_{k \in \mathbb{N}}$. For each $k \in \mathbb{N}$ let $0_k = \{(n_k, \gamma), \gamma > \gamma_k \in W^*\}$ where γ_k is chosen such that f is constant on 0_k . Clearly each 0_k is an open subset of T . Also $\{(0_k)_{k \in \mathbb{N}}\}$ may be enlarged to form an f -cover of T in such a way that no new members contain any of the points (n_k, γ) .

It is then clear that no neighbourhood $V(p)$ of p in βX will have the property of the theorem.

Chapter 6

THE COMPONENTS OF $C(X)$ IN THE GRAPH TOPOLOGY

6.1. In this chapter we introduce a slight generalization of the concept of relative pseudocompactness, that is relative countable compactness. The set of all functions in $C(X)$ with relatively countably compact zero-sets turns out to be an ideal. Using this ideal and [Theorem 5.2], we characterize the components of $\langle C(X), g \rangle$.

In this chapter we shall assume that our domain space X is completely regular and Hausdorff.

6.2. DEFINITION. A subset A of a space X will be called relatively countably compact if every countably infinite subset of A has an accumulation point in $cl_X A$.

Clearly any subset of a countably compact space is relatively countably compact. Also, a closed subset of a space X is relatively countably compact if and only if it is countably compact.

We now give the relationship between relative countable compactness and relative pseudocompactness.

THEOREM. Let X denote any topological space. Every relatively

countably compact subset A of X is relatively pseudocompact.
The converse is false.

PROOF. Suppose that the subset A of X is not relatively pseudocompact. Then A contains a copy of N , C -embedded in X [1.2(4)]. This copy of N has no accumulation point in X let alone $cl_X A$. The result follows. The following example shows that the converse is false.

6.3. In view of the importance of cozero-sets in this study we show that the converse of [Theorem 6.2] is false even for cozero-subsets.

EXAMPLE. A relatively pseudocompact cozero-set which is not relatively countably compact. Since the whole space is a cozero-set we see that the Tychonoff plank or any pseudocompact non-countably compact space suffices as an example.

6.4. In the case of X normal the concepts coincide, in fact we have the following theorem.

THEOREM. For a normal space X the following are equivalent for a subset A of X .

- (1) A is relatively pseudocompact.
- (2) $cl_X A$ is countably compact.
- (3) A is relatively countably compact.

PROOF. In this case if A is relatively pseudocompact then $cl_X A$ is pseudocompact, hence countably compact. Clearly then A is relatively countably compact, and the result follows from [Theorem 6.2].

6.5. We now note that, in contrast to the situation of [1.3], a relatively countably compact cozero-set need not have a countably compact closure, if X is not normal. To provide an example we consider the space ψ of [GJ, 5I].

EXAMPLE. The space ψ contains a cozero-set N such that

- (1) N is relatively countably compact, and
- (2) $cl_{\psi} N$ is not countably compact.

PROOF. ψ has the following properties, see [GJ, 5I].

(a) ψ can be written as a disjoint union $N \cup D$ where N is a countable set of isolated points dense in ψ , and also a cozero-set.

(b) ψ is pseudocompact but not countably compact.

We show that the cozero-set N is relatively countably compact. Suppose, to the contrary that there exists a countably infinite subset $T = \{t_n, n \in \mathbb{N}\}$ of N with no accumulation point in ψ . Define the mapping $f \in R^{\psi}$ as follows.

$$f(t_n) = n \text{ for all } n \in \mathbb{N}$$

and $f(x) = 0$ for all $x \in \psi \setminus T$.

Since the point t_n is isolated for every $n \in \mathbb{N}$, f is continuous on T . Also if $x \in \psi \setminus T$ then x is not an accumulation point of T and so there exists a neighbourhood of x which contains no member of T . Hence f is continuous at each point of ψ . This contradicts (b).

Since $cl_{\psi} N = \psi$ we see that this is not countably compact.

6.6. We now wish to show that the set

$$C_c(X) = \{f \in C(X) : \text{coz } f \text{ is relatively countably compact}\}$$

is an ideal in $C(X)$.

We first observe something slightly more general.

THEOREM. Let X denote any topological space and let $\mathcal{P}$ denote a class of subsets of X . If $\mathcal{P}$ is closed under finite unions and subsets then

$$I(\mathcal{P}) = \{f \in C(X) : \text{coz } f \in \mathcal{P}\}$$

is an ideal in $C(X)$.

PROOF. Let $f, g \in C(X)$. The result then follows from the two

inclusions,

$$\text{coz}(f + g) \subset \text{coz } f \cup \text{coz } g ,$$

$$\text{and} \quad \text{coz}(f \cdot g) \subset \text{coz } f \cap \text{coz } g .$$

6.7. THEOREM. Let X denote any topological space. The class of relatively countably compact subsets is closed under finite unions and subsets.

PROOF. This follows from the definition.

6.8. COROLLARY. Let X denote any topological space. Then $C_C(X)$ is an ideal in both $C(X)$ and $C^*(X)$.

PROOF. That $C_C(X)$ is an ideal in $C(X)$ follows from [Theorems 6.6, 7]. It follows from [Theorem 6.2] that $C_C(X) \subset C^*(X)$ and hence that $C_C(X)$ is an ideal in $C^*(X)$.

6.9. For completeness we introduce the following notion.

Let $C_C^\infty(X)$ denote the class of all functions f in $C(X)$ for which the set $\{x : |f(x)| > \frac{1}{n}\}$ is relatively countably compact for every $n \in \mathbb{N}$.

THEOREM. Let X denote any topological space. Then $C_C^\infty(X)$ is an ideal in $C^*(X)$.

PROOF. Let $f, g \in C_C^\infty(X)$. Then

$$\begin{aligned} \{x : |f + g|(x) > \frac{1}{n}\} &\subset \{x : |f(x)| + |g(x)| > \frac{1}{n}\} \\ &\subset \{x : |f(x)| > \frac{1}{2n}\} \cup \{x : |g(x)| > \frac{1}{2n}\} \end{aligned}$$

and by [Theorem 6.7] it follows that $f + g \in C_S^\infty(X)$.

Let $f \in C_C^\infty(X)$ and let $h \in C^*(X)$. Then there exists an integer m in N such that $\sup_{x \in X} |h(x)| < m$. Clearly,

$$\{x : |hf|(x) > \frac{1}{n}\} \subset \{x : |f(x)| > \frac{1}{nm}\}$$

and again the result follows from [Theorem 6.7].

6.10. The ideals $C_C(X)$ and $C_C^\infty(X)$ may now be included in the network of [Theorem 3.4].

THEOREM. For any topological space X the following inclusion relationships hold. All inclusions can be proper.

$$(1) \quad C_C \subset C_S.$$

$$(2) \quad C_C \subset C_C^\infty \subset C_S^\infty.$$

PROOF. (1) follows from [Theorem 6.2]. That the inclusion can be proper follows from [Example 6.3].

(2) Clearly $C_C \subset C_C^\infty$. When $X = R$, $C_C(X) = C_K(X)$ and $C_C^\infty(X) = C_\infty(X)$. The fact that the inclusion can be proper follows. From [Theorem 6.2] it follows that $C_C^\infty \subset C_S^\infty$. To see that this

inclusion can be proper let $X = T \times N$ where T denotes the Tychonoff plank.

Consider the function $f \in C(X)$ defined by

$$f(x, n) = \frac{1}{n} \text{ for all } x \in T.$$

Then clearly $f \in C_S^\infty(X)$. Since T is not countably compact $f \notin C_C^\infty(X)$.

6.11. We now characterize the components of $\langle C(X), g \rangle$ by obtaining a theorem similar in many respects to [Theorem 4.4]. In this case it is more difficult to show that $C_C(X)$ is connected. Fortunately, a technique developed by Hansard [H_1 , Lemma 2 and converse] is of great utility.

THEOREM. The principal component of $\langle C(X), g \rangle$ is the ideal $C_C(X)$.

PROOF. Suppose that the function f is in $C_C(X)$. Consider the subalgebra

$$A(f) = \{g \in C(X) : \text{coz } g \subset \text{coz } f\}$$

of $C(X)$. We will show that $A(f)$ in the relative graph topology is isomorphic to $A(f)$ in the relative uniform topology. From this we will conclude as before that $A(f)$ is connected and that f is in the principal component of $\langle C(X), g \rangle$.

To this end, suppose that $g \in A(f)$ and let $G(g) \subset U$ where U is open in $X \times R$. We will show that there exists an $\epsilon > 0$ such that $\{x\} \times N_\epsilon(h) \subset U$ for every $x \in \text{coz } f$. Assume, to the contrary, that for each $n \in N$ there exists an x_n in X such that

$$\{x_n\} \times N_{\frac{1}{n}}(g(x_n)) \not\subset U.$$

Let (x_n) accumulate at $p \in \text{cl}_X \text{coz } f$. But $(p, g(p)) \in U$, and every neighbourhood of $(p, g(p))$ in $X \times R$ meets at least one member of the sequence $(x_n, g(x_n))$. This contradicts the fact that U is open in $X \times R$.

Hence, we can conclude that any function $h \in V_\epsilon(g) \cap A(f)$ has $(x, h(x)) \in U$ for all $x \in \text{coz } f$. But when $x \in X \setminus \text{coz } f$ we have that $h(x) = 0$. Hence $G(h) \subset U$.

Conversely. Suppose that $f \notin C_c(X)$. Then there exists a sequence $(x_n) \subset \text{coz } f$ such that (x_n) has no accumulation point in $\text{cl}_X \text{coz } f$, and therefore none in X .

Also, there exists a sequence of positive real numbers (ϵ_n) such that, for every $n \in N$, $\epsilon_n < \frac{1}{2} |f(x_n)|$ and $(\epsilon_n) \rightarrow 0$.

Using these facts we construct a disconnection of $\langle C(X), g \rangle$ separating f and the zero function.

Put

$$A = \{g \in C(X) : (g(x_n)\epsilon_n^{-2}) \text{ is bounded}\} .$$

As before, the zero function is in A , and since for every $n \in \mathbb{N}$, $\epsilon_n < |f(x_n)|$ we have that

$$1 < \frac{f(x_n)}{\epsilon_n} .$$

This implies that

$$0 < \frac{|f(x_n)|}{\epsilon_n} - 1$$

and

$$0 < \left(\frac{|f(x_n)|}{\epsilon_n} - 1 \right) \frac{1}{\epsilon_n}$$

and hence that

$$\frac{1}{\epsilon_n} < \frac{|f(x_n)|}{\epsilon_n^2} , \text{ for every } n \in \mathbb{N} .$$

Since $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ we see that $(f(x_n)\epsilon_n^{-2})$ is not bounded.

Hence $f \notin A$.

We must show that A is open and closed. Suppose that $g \in A$.

For every $n \in \mathbb{N}$ the set

$$U_n = \{x_n\} \times \mathbb{R} / [g(x_n) - \epsilon_n^2, g(x_n) + \epsilon_n^2]$$

is closed in $X \times \mathbb{R}$.

Let $U = \bigcup_{n \in \mathbb{N}} U_n$. Then U is closed in $X \times \mathbb{R}$, for suppose to the contrary that there exists a point $(x, y) \in X \times \mathbb{R}$ such that $(x, y) \in \text{cl}_{X \times \mathbb{R}} U \setminus U$. Then clearly x is a point of accumulation of the sequence (x_n) in X , which contradicts the fact that $\text{coz } f$ is relatively countably compact.

It follows that $U^C = X \times \mathbb{R} \setminus U$ is open in $X \times \mathbb{R}$. Furthermore $G(g) \subset U^C$.

Suppose now that the function $g' \in C(X)$ has $G(g') \subset U^C$. Clearly

$$|g'| (x_n) \leq (|g' - g| + |g|)(x_n).$$

But

$$|g|(x_n) < C \epsilon_n^2 \text{ for some } C > 0 \text{ and all } n \in \mathbb{N}.$$

Also

$$|g' - g|(x_n) < \epsilon_n^2 \text{ for all } n \in \mathbb{N}.$$

Hence

$$|g'| (x_n) \leq (C + 1) \epsilon_n^2 \text{ for all } n \in \mathbb{N} \text{ and } g' \in A .$$

This implies that A is open.

To see that $C(X) \setminus A$ is open. Suppose again that $g \in C(X) \setminus A$.

For every $n \in \mathbb{N}$ the set

$$U_n = \{x_n\} \times \mathbb{R} \setminus [g(x_n) - \epsilon_n^4, g(x_n) + \epsilon_n^4]$$

is closed in $X \times \mathbb{R}$. As before the set

$$U^C = X \times \mathbb{R} \setminus \left(\bigcup_{n \in \mathbb{N}} U_n \right)$$

is open in $X \times \mathbb{R}$ and $G(g) \subset U^C$. Consider now any function $g' \in C(X)$ such that $G(g') \subset U^C$. Clearly

$$|g'| (x_n) \geq ||g| - |g' - g| | (t_n) .$$

Since the sequence $(g(x_n) \epsilon_n^2)$ is unbounded, the set

$$I_C = \{n : g(x_n) \geq C \epsilon_n^2\}$$

is infinite for every positive real $C \geq 1$.

As in [Theorem 4.4] we can conclude for any such C

there exists integers $n \in \mathbb{N}$ such that

$$|g'| (x_n) \geq \epsilon_n^2 (C - \frac{1}{4})$$

and hence that the sequence

$$(g'(x_n) \epsilon_n^{-2})$$

is unbounded. This concludes the proof of the theorem.

6.12. COROLLARY. The components of $\langle C(X), g \rangle$ are the equivalence classes of $C(X)$ modulo $C_c(X)$.

PROOF. This follows from [Theorem 5.2] and the fact that continuous maps preserve connectedness.

6.13. COROLLARY. For any space X , $\langle C(X), g \rangle$ is connected if and only if X is countably compact.

PROOF. This follows from the obvious fact that X is countably compact if and only if it is relatively countably compact.

6.14. REMARK. It follows from [Theorems 5.5, 6.4] that [Theorem 6.11] is equivalent to [Theorem 4.4] when X is either normal or a cb-space. In general this equivalence does not hold. We know for instance [M_1 , corollary 3] that every cb-space is countably paracompact. It follows that both theorems are necessary for any spaces which are neither countably paracompact nor normal.

ADDENDUM

NOTE ON A SUBALGEBRA OF $C(X)$

The material presented in this addendum is to appear in the Canadian Mathematical Bulletin in a paper of the same title authored by L. D. Nel and D. Riordan.

7.1. Corresponding to every ideal I in $C(X)$ there is a subalgebra of the form $I \oplus \mathbb{R}$. Since the class of subalgebras of an algebra is closed under intersection we have immediately at our disposal a number of new subalgebras of $C(X)$ which are obtained in a manner similar to that in which we obtained the ideals of Chapter 3.

We now study one such algebra but, from a different point of view to that adopted in Chapter 3.

7.2. DEFINITION. Let $C^\#(X)$ consist of all $f \in C(X)$ whose image $M(f)$ in the residue class ring $C(X)/M$ is real for every maximal ideal M in $C(X)$.

$C^\#$ shares with C^* the property of being an intrinsically determined subalgebra of C . The compactification corresponding to $C^\#$ (as uniformity determining subalgebra of C^*) is thus also an intrinsically determined one. We show that

this compactification is well-known and "natural" in the cases of several elementary spaces X . Some characterizations of $C^\#(X)$ are first obtained. All spaces are assumed completely regular and Hausdorff.

7.3. The straight forward proof of the following theorem is omitted. Equivalence (4) shows that $C^\#(X)$ is a subalgebra of the above mentioned type.

THEOREM. For a function $f \in C(X)$ the following are equivalent.

- (1) $f \in C^\#(X)$.
- (2) Every z -ultrafilter on X has a member on which f is constant.
- (3) For every z -ultrafilter $\mathcal{Q}$ on X the family $f^\# \mathcal{Q}$ of all closed sets in R whose pre-images under f belong to $\mathcal{Q}$, is again a z -ultrafilter.
- (4) $f \in \bigcap_{p \in \beta X \setminus X} M^p \oplus R$.

7.4. COROLLARY. $C^\#(X)$ is an m -closed sublattice of $C(X)$.

PROOF. Let $f, g \in C(X)$ and $r, r' \in R$. From the equality $(f + r) \vee (g + r') = 2^{-1}(f + g + r + r' + |f - g + r - r'|)$ we can conclude that $M + R$ is a sublattice of $C(X)$ for every ideal M of $C(X)$. It then follows from [Theorem 7.3(4)] that $C^\#(X)$ is a sublattice of $C(X)$.

We now show that for any point p in βX , $M^P \oplus R$ is m -closed. Suppose that the function f is in $C(X) \setminus (M^P \oplus R)$. Then clearly $M(f - f^\beta(p)) \neq 0$, hence infinitely small. Then, by [GJ, 7B], there exists a neighbourhood V of p such that $|f - f^\beta(p)|(x) > 0$ for all $x \in V \cap X$. Also there exists a function $g \in C(\beta X)$ such that g is zero on some neighbourhood of p in βX , $g(x) = 1$ for all $x \in \beta X \setminus V$ and $0 \leq g \leq 1$.

Let h be the function defined by

$$h(x) = |f - f^\beta(p)|(x) + g(x) \text{ for all } x \in X.$$

Clearly h is positive. Also $V_h(f) \subset C(X) \setminus (M^P \oplus R)$. Hence $M^P \oplus R$ is closed.

The result follows from [Theorem 7.3(4)].

$C^\#(X)$ need not be closed in the uniform topology.

7.5. We now proceed to obtain another characterization of $C^\#(X)$ which is useful in special cases.

LEMMA. Let $D = \{d_n : n \in \mathbb{N}\}$ be a C -embedded copy of $\mathbb{N}$ in X . There exists a neighbourhood W_n of d_n for each n such that for every zero-set $Z_n \subset W_n$ and for every $M \subset \mathbb{N}$, $\bigcup_{m \in M} Z_m$ is a zero-set. (Hence in a G_δ -space every C -embedded copy of $\mathbb{N}$ is a zero-set).

PROOF. There exists $u \in C(X)$ such that $u(d_n) = n$. Put

$W_n = \{x : |u(x) - n| \leq \frac{1}{3}\}$ and let Z_n be any zero-set contained in W_n . Put $Y_n = \{x : |u(x) - n| \geq \frac{2}{3}\}$. Note that Z_n is disjoint from Y_n and $W_m \subset Y_n$ for all $n \in \mathbb{N}$ and $m \neq n$.

We can choose a non-negative $h_n \in C(X)$ which has the value 0 precisely on Z_n and the value 1 precisely on Y_n . Since each point x has a neighbourhood (e.g. $\{y : |u(y) - u(x)| < 1\}$) on which all but finitely many h_n have the same value, it follows that the function $\inf_{m \in \mathbb{M}} h_m$ belongs to $C(X)$ and we have $Z(\inf_{m \in \mathbb{M}} h_m) = \bigcup_{m \in \mathbb{M}} Z_m$ as required.

7.6. THEOREM. $f \in C^\#(X)$ if and only if f is bounded and $f[D]$ is closed for every C -embedded copy of $\mathbb{N}$.

PROOF. Suppose D is a C -embedded copy of $\mathbb{N}$ such that $\text{cl } f[D] - f[D]$ contains a point r . Choose $y_n \in D$ ($n \in \mathbb{N}$) such that $\lim_n f(y_n) = r$ and put $V_n = \{x : |f(x) - f(y_n)| \leq \frac{1}{n}\}$. Choose W_n to be a nbhd of y_n as described in the above lemma. Then $Z_n = V_n \cap W_n$ is a zero-set nbhd of y_n such that $A_m = \bigcup_{n \geq m} Z_n$ is a zero-set for each m . The family $\{A_m : m \in \mathbb{N}\}$ has the finite intersection property, so there exists a z -ultrafilter $Z[M]$ to which each A_m belongs. For any $\epsilon > 0$ we can take m so large that $0 < |f(x) - r| \leq |f(x) - f(y_m)| + |f(y_m) - r| < \epsilon$ holds for all $x \in A_m$. It follows that $M(f - \underline{r}) = M(f) - M(\underline{r})$ is infinitely small [GJ, ch 5] so $M(f)$ cannot be real.

To prove the converse, take $f \in C^*(X)$ and suppose that $M^P(f)$ fails to be real for some maximal ideal M^P corresponding to $p \in \beta X$. Since M^P is hyper-real, there exists $g \in C(X)$ with $|M^P(g)|$ infinitely large. At the same time $|M^P(f) - f^\beta(p)|$ is infinitely small but positive. Hence for each $n \in \mathbb{N}$ there is a neighbourhood U_n of p such that

$$0 < |f(x) - f^\beta(p)| < \frac{1}{n} < n < |g(x)|$$

for all $x \in U_n \cap X$. It follows that there exists a sequence (x_n) in X such that $g(x_n)$ is strictly increasing to ∞ , $f(x_n) \rightarrow f^\beta(p)$ while $f(x_n) \neq f^\beta(p)$ for all n . We conclude that $D = \{x_n : n \in \mathbb{N}\}$ is a C -embedded copy of $\mathbb{N}$ [GJ, 1.20] and that $f[D]$ is not closed. This completes the proof.

7.7. We now turn to some special cases. The C -embedded copies of $\mathbb{N}$ in any space X are formed by sequences (x_n) satisfying the condition $h(x_n) \rightarrow \infty$ for some $h \in C(X)$. This condition reduces in the case $X = \mathbb{R}$ to the requirement that (x_n) tends to $\pm\infty$; in $\mathbb{R}^2$ it is equivalent to saying that the distance from x_n to 0 tends to ∞ ; in $\mathbb{Q}$ it becomes (x_n) tends to $\pm\infty$ or to an irrational limit; in $\mathbb{N}$ it is automatically satisfied. Using [Theorem 7.6] we conclude easily that $C^\#(\mathbb{R})$ consists of all f which have a constant value on $\{x : x \leq a\}$ and on $\{x : x \geq b\}$ for some $a, b \in \mathbb{R}$. It is not difficult to verify that the compactification determined by $C^\#(\mathbb{R})$ is the

extended real line. $C^\#(\mathbb{R}^2)$ consists of all f which are constant on the complement of some compact set; the one point compactification is determined in this case. Both $C^\#(Q)$ and $C^\#(N)$ consist of functions which attain only finitely many values. Any two disjoint closed sets in Q (resp. N) have disjoint open-closed neighbourhoods and so they can be separated by a function in $C^\#$. The corresponding compactification can be verified to be βQ (resp. βN).

We note in conclusion that the compactification $[0, 1]$ of the space of rational numbers in this interval appears to be a difficult one to describe intrinsically.

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