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THE ROLE OF INTERFACE ELEMENTS IN FINITE
ELEMENT ANALYSIS OF GEOTECHNICAL
ENGINEERING PROBLEMS

by

Md. Omar Faruque, B.Sc. Engg.

A thesis submitted to the Faculty of Graduate Studies
and Research in partial fulfillment of the requirements
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ABSTRACT

The axially symmetric problems in Geotechnical engineering which include interfaces between dissimilar materials are examined by the finite element method. The stress-strain relationships for the interface material are assumed to be nonlinear while the adjacent continuum material is assumed to be isotropic, homogeneous and linearly elastic. There are several models for the nonlinear stress-strain relationships for the interface material. For simplicity, it is assumed that the shear stress-shear strain relationship is elastic-perfectly plastic. Under compressive normal stresses, the interface material compresses purely elastically while it is incapable of sustaining any tensile stresses.

A four noded axisymmetric interface elements are used to model the joints and material interfaces. A finite joint thickness is assumed for formulating the element stiffness matrix. Two generalized displacements (e.g. radial and vertical) are considered at each node. To model the isotropic elastic solid medium adjacent to the joints or interfaces, axisymmetric solid elements of triangular cross-section are used.

On the basis of these models, a finite element computer program is developed in FORTRAN IV language. In order to simulate the nonlinear interface behavior, the load transfer method is employed as an iterative scheme in the finite element program.

The work presented in this thesis involves in investigation of the effects of interface elements in the finite element analysis of Geotechnical engineering problems. Few selected problems of this class are examined in order to investigate the effects of interface conditions on the displacements, stresses etc. The convergency of the iterative scheme employed in the program is checked. Effects of aspect ratio (defined by the ratio of the joint length to the joint thickness), and stiffnesses of the joint element on the rate of convergence is also studied.

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LIST OF SYMBOLS

SYMBOL	DEFINITION
[A]	Matrix which relates the nodal displacement vector $\{\delta\}$ with the generalized coordinate vector $\{a\}$.
a	Radius of circular plate, deformable testing block and rigid cylindrical anchor.
$\{a\}$	Generalized coordinate vector.
$a_i (i=1,6)$	Generalized coordinates.
[B]	Matrix which relates the strain vector $\{\epsilon\}$ with the displacement vector $\{\delta\}$, for an element.
b	Inner radius of a thick cylinder.
[c]	Matrix which relates the strain vector with the displacement vector.
D	Rigidity of circular plate.
[D]	Constitutive relation matrix (i.e. it relates the stress vector $\{\sigma\}$ with the strain vector $\{\epsilon\}$, for an element.
dA	Differential area.
dV	Differential volume.
E	Young's modulus for elastic solid medium.
E_b	Young's modulus for circular plate.
E_{nn}	Joint modulus in the normal direction.
E_{ss}	Joint modulus in the shearing direction.
E_{ns}, E_{sn}	Joint modulus values which account for volume changes (dilatancy) during joint shearing.

SYMBOL	DEFINITION
F_i, F_j, F_m	Nodal forces at nodes i, j, m respectively.
G	Shearing modulus for the elastic solid.
H	Height of the elastic solid medium.
h_1, h_2	Interpolation functions (see eqns. 2.89 and 2.90).
i, j, m	Node numbers. When used as subscript, indicates that the quantity is referred to a particular node (e.g. w_i denotes the vertical displacement at node i).
K_r	Relative rigidity of the plate.
K_{sp}, K_e, K_i	Stiffnesses for spring.
$[K^b]_e$	Plate element stiffness matrix.
$[K]_e$	Stiffness matrix for axisymmetric triangular element.
$[K_J]_e$	Joint element stiffness matrix.
$[K]$	Global stiffness matrix.
$\ln$	Natural logarithm (i.e. to the base e).
M_i	Equivalent externally applied radial moment at node i.
M_r	Radial component of bending moment.
M_θ	Circumferential component of bending moment.
P	Externally applied concentrated load.

SYMBOL	DEFINITION
p	Intensity of externally applied uniform pressure (normal).
p_0	Internal pressure for thick cylinder.
Q_r	Shearing force (normal) at a point having radial coordinate r .
r	Radial coordinate of a point.
$\bar{r}$	Distance of centroid of an element from the R-axis.
r_i, r_j, r_m	Radial coordinates of nodes i, j, m respectively.
t	Thickness of circular plate.
t_j	Thickness of joint element.
U	Component of the displacement along the radial direction.
U_i, U_j, U_m	Radial displacement of nodes i, j, m respectively.
$\{U\}$	Global displacement vector.
U_e, U_i	Energy stored in the springs.
w	Component of the displacement along z direction.
w_i, w_j, w_m	Vertical displacement at nodes i, j, m respectively.
w_a^d	Work done by the actual system of forces (uniformly distributed load).
w_e^d	Work done by the equivalent system of forces (uniformly distributed load).

SYMBOL	DEFINITION
W_a^G	Work done by the actual system of forces (gravity load).
W_e^G	Work done by the equivalent system of forces (gravity load).
x, y	Radial and axial coordinate measured from the centroid of a triangular element.
z	Axial coordinate of a point.
z_i, z_j, z_m	Axial coordinate of nodes i, j, m respectively.
$\bar{z}$	Distance of the centroid of a triangular element from the global z -axis.
α_a	Embedment length for the rigid cylindrical anchor.
α_s	Modular ratio for the joint element (E_{ss}/E_{nn}).
L_j	Length of the joint element.
L	Length of the rigid cylindrical anchor.
β_a	Length of the cylindrical anchor.
η, n	Local coordinate axis for the joint element (normal to the joint axis).
$\{\delta\}_e$	Nodal displacement vector for an element.
ν_b	Poisson's ratio for plate.
ν	Poisson's ratio for the elastic solid medium.
$\{\sigma\}$	Stress vector for an element.

SYMBOL	DEFINITION
σ_r	Radial component of stress.
σ_z	Axial component of stress.
σ_θ	Circumferential component of stress.
σ_{rz}	Shear stress.
σ_{ss}	Joint shearing stress.
σ_{nn}	Joint normal stress.
$\sigma_{\theta\theta}$	Joint circumferential stress.
ΣV_s	Sum of the excess shear forces in all the joint elements.
ΣV_{ps}	Sum of the shear forces in all the joint elements when peak shear strength is mobilized.
γ	Unit weight for the elastic solid medium.
Δu	Relative displacement in the radial direction.
Δu_s	Relative displacement along joint axis (for joint element).
Δw_n	Relative displacement normal to the joint axis (for joint element).
$\{\epsilon\}$	Strain vector for an element.
ϵ_r	Radial component of strain.
ϵ_z	Normal component of strain.
ϵ_θ	Circumferential component of strain.

SYMBOL	DEFINITION
γ_{rz}	Shear strain.
ϵ_{ss}	Shear strain for the joint element.
ϵ_{nn}	Normal strain for the joint element.
$\epsilon_{\theta\theta}$	Circumferential strain for the joint element.
χ_r	Curvature of the circular plate in radial direction.
χ_θ	Curvature of the circular plate in circumferential direction.
ψ	Orientation of the joint element.
ξ, s	Local coordinate axis for the joint element (shearing direction).
Δ	Sum of the difference of the nodal displacements between i th and $(i+1)$ th iteration.
ϵ_v, ϵ_c	Small quantities.
θ, z, R	Components of circular cylindrical coordinate system.
τ_y	Peak shear strength of a joint.

1. INTRODUCTION

1.1 General

The finite element method has been used quite successfully for solution of a wide range of problems in Geomechanics and structural mechanics. The configurations for many problems of these classes include interfaces between dissimilar materials. If, throughout the loading history, perfect bonding is maintained, then the presence of an interface offers no difficulties; the interface merely serves as a boundary between neighboring rows or columns of elements with dissimilar properties. However, if the bond breaks down and there is relative movement of the two mating surfaces, then special analysis procedures are necessary. Examples of problems in which material interfaces play a prominent role include the following. There are numerous problems that involve soil-structure interaction, e.g., soil-retaining wall, soil-footing, soil-culvert systems, underground structures, dams. The modelling of the bond between the reinforcing members and the surrounding matrix material is quite important for the understanding of the behavior of reinforced systems, such as reinforced concrete and reinforced earth. The results of laboratory studies of material behavior are often highly influenced

by the interface conditions between the material samples and the testing apparatus. The inclusion of active joints in the modelling of rock systems is necessary if meaningful analysis are to be obtained. There are, of course, many classical contact problems in engineering mechanics, such as bearing-race contact, etc.

Application of the finite element method for solving this class of problems needs special elements to model the joints and the interfaces (Goodman et al., 1968; Clough and Duncan, 1969; Mahtab and Goodman, 1970; Zienkiewicz et al., 1970; Chang et al., 1972; Desai and Holloway, 1972; Ghaboussi et al., 1973; Tsui and Clough, 1979; Desai, 1975; Herrmann, 1978). In general the interface behavior is quite different from the adjacent continuum. An interface may exhibit properties such as nonlinearity, dilatancy or strain softening. Hence the special elements which are used to model the joints and the interfaces should reflect these interface behavior.

While dilatancy or strain-softening is difficult to include in the analysis, joint nonlinearity can be easily incorporated by defining stress-strain relationships for the special interface elements. Such stress-strain relationships can be obtained by conducting suitable laboratory tests on intact specimens of joints (e.g. Goodman, 1976; Desai, 1977).

The stress-strain curves as obtained from the laboratory tests cannot be utilized directly in the finite element calculations owing to their irregular nature. For this reason, the stress-strain relationships are idealized on the basis of the available test data.

The interface element, that has been used in this thesis, is illustrated in Fig. 2.19. The idealized stress-strain relationships for the interface element both in the normal and the shearing direction is shown in Fig. 2.19. It should be noted that in the present analysis the shearing stress-shearing strain relationship is assumed to be elastic-perfectly plastic (Fig. 2.1). The joint element has a finite thickness t_j and its properties in the elastic range are represented by joint normal modulus E_{nn} (in compression) and joint shearing modulus E_{ss} . Joint normal modulus in tension is assumed to be zero. In order to simulate the nonlinear stress-strain behavior (as shown in Fig. 2.1) the Load transfer method is used in the analysis. For simplicity it is assumed that the joint is non-dilatant and non-strain softening.

1.2 A Summary of Literature on Interface Elements

Many investigators have developed interface elements to model joints and material interfaces in the finite element analysis. A summary of these interface elements are given in this section.

Ngo and Scordelis (1967) developed a one-dimensional linkage element in connection with the problem of simulating the bond condition between the reinforcement and the surrounding concrete (Fig. 1.1). The element can be oriented at any arbitrary angle with the global system of axes. This linkage element has no physical dimension at all, and only its mechanical properties are of importance. Since this element has two linear springs, two spring constants (stiffnesses) completely define the mechanical behavior of such an element. Nonlinearity was considered by specifying a limiting elongation for the springs.

Goodman, Taylor, and Brekke (1968) developed a one dimensional interface element to represent discontinuities in rocks. This element is a four-nodal point element with a finite length but zero width (Fig. 1.2). At each node there are two degrees of freedom - horizontal and vertical displacements. In evaluating the stiffness matrix, it is assumed that both normal and shear displacements vary linearly along the length of the element, which is compatible

with the external boundary displacements of the two-dimensional continuum elements. The properties of this interface element are related to the unit joint stiffness in the normal and the tangential directions. These unit joint stiffnesses are evaluated by carrying out direct shear tests on intact specimens of joints (e.g. Goodman, 1976).

. Modes of behavior for the interface element are such that under compressive load the adjacent two-dimensional elements penetrate each other. This is due to the fact that compressive stresses require compressive relative displacements across the interface. In order to prevent significant overlapping of the adjacent continuum elements, the unit joint stiffness in the normal direction is made large compared to the adjacent continuum. When an element fails in shear, with the interface still in compression, the value of unit joint stiffness in the tangential direction is reduced to a negligible value but the value of unit joint stiffness in the normal direction is kept large. However, for tension or joint opening both joint stiffnesses are assigned very small values.

Zienkiewicz, Best, Dullage, and Stagg (1970) used joint elements for solving jointed rock problems. They used both four noded rectangular joint elements and curved

and variable thickness joint elements (Fig. 1.3). The shape functions used for rectangular joint elements allow the strain components to vary linearly along and perpendicular to the longitudinal axis of the joint. However, for the curved isoparametric elements higher order shape functions may be used in order to allow the strain components to vary parabolically or cubically. For example, when eight nodes are used to define a curved joint element, the strain components vary cubically along joint axis, this element is termed as a cubic joint element. When the elastic stiffness characteristics of such joint elements differ from the adjacent continuum elements by large amounts, the assembled elastic equations may suffer numerical ill-conditioning. In order to overcome this difficulty Zienkiewicz et al. (1970) used stiffness ratios of 1:1000 or less as a safe rule. However, exact limits of possible stiffness ratios depend on the precision of computer word length and on the total number of elements used. Again to deal with the nonlinearity and no-tension feature of the interface element, they made use of initial stress method (Zienkiewicz, 1970).

Mahtab and Goodman (1970) extended the one dimensional joint element developed by Goodman, Taylor and Brekke (1968) to triangular and quadrilateral plane, joint elements in

order to use these elements in three-dimensional finite element analysis of jointed rock slopes (Fig. 1.4). The strains in such joint elements are given by the relative displacement between top and bottom faces while stresses are given by stress-strain relationships derived from direct shear tests (see Goodman, Taylor, and Brekke; 1968). The joint element stiffness matrix is obtained by minimizing the potential energy stored in the joint.

Ghaboussi, Wilson, and Isenberg (1973) developed four-nodal point plane strain and axisymmetric joint elements to model discontinuities and material interfaces (Fig. 1.5). In contrast to previous studies, one significant departure in their formulation is the use of relative nodal displacements as the independent degrees-of-freedom for the joint element. The strains are assumed to vary linearly along the axis of the joint element. It should be noted that the thickness of the joint element is used to formulate the joint element stiffness matrix. The stresses and the strains are related through a material property matrix $[C]$ in the form

$$[C] = \begin{bmatrix} C_{nn} & C_{ns} \\ C_{sn} & C_{ss} \end{bmatrix}$$

C_{nn} and C_{ss} are the unit joint stiffnesses in the normal

and tangential directions respectively. These are obtained by conducting laboratory tests on intact specimens of joints (see Goodman, 1976; Desai, 1977). The material constants C_{ns} and C_{sn} arise when there is volume expansion during shearing (dilatancy) of the joint which couples the unit normal and the unit tangential stiffnesses. It may be noted that for non-dilatant joints, the shear and the normal components of deformation are uncoupled. Hence $C_{ns} = C_{sn} = 0$. They also outline special techniques to incorporate joint dilatancy in the analysis.

Goodman (1976) introduced joint elements in connection with the use of initial stress method as an iterative technique to simulate the nonlinear joint behavior. The geometry and the mechanical properties of these joint elements are directly calculated from the actual joint elements used (Goodman, 1976). He also explained the possible techniques to incorporate strain-softening characteristics of the joint element in the analysis.

Wilson (1977) extended the joint element developed by Ghaboussi, et al. (1973) to two-dimensional and three-dimensional interface elements (Fig. 1.6). The relative nodal displacements are used as the independent degrees-of-freedom. He advocated that numerical ill-conditioning of the assembled elastic equations may be avoided by using relative displacements as an independent parameter for the joint elements.

Desai (1977) used an eight noded quadrilateral joint element to model the interface between piles and soil (Fig. 1.7). This element was originally used by Mahtab and Goodman (1970) for three-dimensional analysis of jointed rock slopes. However, for two dimensional problems he used the interface element developed by Goodman et al. (1968). Nonlinearity in both the normal and the shearing stress-strain curves were approximated by simple bilinear relationships. He also suggested possible techniques to incorporate joint dilatancy and stress-softening in the analysis.

Pande and Sharma (1978) developed an eight noded isoparametric parabolic joint element to represent discontinuities (Fig. 1.8). In order to take into account sharp variations in strains in the thickness direction they introduced mid-side nodes so that the strains can vary parabolically across the interface. The formulation of the joint element stiffness matrix was attempted in two separate ways - (1) assuming the absolute nodal displacements as independent parameters and (2) assuming relative displacements on independent parameters. However, they have shown that the difference in results when these two separate formulations are used is quite insignificant even for extremely large aspect ratio (they define the aspect ratio as the ratio of the length to the thickness of the joint element) of the interface element.

Herrmann (1978) used a 'bond link element' (see Ngo and Scordelis, 1967) to model the interfaces between the reinforcement and the surrounding concrete. Geometrically this element consists of two nonlinear springs normal and tangential to the steel. The mechanical behavior of this linkage element is completely defined by the two nonlinear spring stiffnesses. He attempted to select the nonlinear stiffness characteristics of the tangential bond springs so as to model the force slip relationship observed in a "pull-out" test. However, when the bond breaks down and slippage or separation, or both, occurs, this procedure tends to exhibit very undesirable numerical characteristics. The difficulty is that the bond links are a compatibility model, and thus require stiffness information. However, when slippage or separation, or both, is occurring, what is known is equilibrium information, i.e., a limiting value for bond stress. To avoid this objectionable feature he defined three behavioural regimes, i.e., the "non-stop", the "slip", and the "separation" modes (Fig. 1.9). The limiting bond stress is determined by the Coulomb law, i.e., $\tau_{\max} = c + f\sigma_n$, where, c is the cohesion and f is the coefficient of friction. He also assumed that, even though slippage has taken place, cohesion is maintained.

In addition to the basic development, many researchers

used such interface elements for solving many practical problems in Geo-mechanics and Geo-technical engineering. Clough and Duncan (1971) used the four noded rectangular joint element developed by Goodman et al. (1968) for solving retaining wall structures. Incremental solution technique was adopted to simulate the nonlinear interface behavior. Withiam and Kulhawy (1979) used an axisymmetric interface element to model the drilled shaft foundation-soil interface. Ke Hsu-Jun (1979) carried out a non-linear analysis of the mechanical properties of joints and discontinuities in rock. On the basis of the joint element suggested by Goodman et al. (1968), he suggested several models of linear and non-linear types to generalize the mechanical non-linear properties of weak planes. It may be noted that for a sheared off joint element he used non-associated flow rule in order to define the elasto-plastic stress-deformation relationship in the incremental forms.

1.3 Scope and Objectives of the Work

Joints and interfaces are encountered in most Geotechnical Engineering problems. For example, discontinuous rock masses. The properties of these joints and discontinuities are inherently nonlinear. Hence the presence of these

discontinuities greatly affect the overall response of such systems when subjected to loads. Therefore, in the finite element analysis these joint properties should be considered in order to arrive at an acceptable solution. In the past special joint elements were developed to model the joints and discontinuities. The properties of such joints are obtained by conducting laboratory tests on intact specimens of joints (see Goodman, 1976; Desai, 1977).

The works presented in this thesis involve investigation of the influence of joint and interface conditions on the overall system. The primary objectives of this work are as mentioned below:

- i) To develop a finite element computer program which is capable of solving axisymmetric problems with joints and interfaces.
- ii) To examine the behavior of a system where joints are encountered.
- iii) To examine the influence of joint elements on the performance of certain selected Geotechnical engineering problems.

1.4 Content of the Thesis

A brief summary of literature on interface elements is presented in Chapter 1.

Chapter 2 is concerned with the development and calibration of a finite element computer-program which is capable of handling axisymmetric problems with joints and interfaces. Detail derivation of the stiffness matrix for a typical joint element is also presented in this chapter. The convergency of the iterative scheme employed in this work is also examined.

Chapters 3, 4 and 5 is devoted to the analysis of certain selected Geotechnical engineering problems with joints and interfaces.

Conclusions and recommendations are made in Chapter 6.

Figure 1.1 Linkage element

Figure 1.2 "Joint" element

Figure 1.3(a) A simple joint element of rectangular shape. Points 1-4 and 2-3 assumed to have same co-ordinates.

Figure 1.3(b) Cubic isoparametric elements in analysis of a cavern. A curved, variable thickness, cubic joint element.

Figure 1.4(a) Triangular joint element.

Figure 1.4(b) Quadrilateral joint element

Figure 1.5 Coordinate System for Plane Joint Element

Figure 1.6 Three-dimensional interface element

Figure 1.7 Three-dimensional interface element

Figure 1.8 Isoparametric parabolic interface element

Figure 1.9 Idealized Model of Bond between Contacting Surfaces

2. FINITE ELEMENT ANALYSIS OF AXISYMMETRIC PROBLEMS WITH JOINTS AND INTERFACES

Finite element analysis is a digital computer method for stress analysis and other field problems (Goodman, 1976). In the past few decades, it has been extensively used to solve many problems of Geomechanics and structural mechanics.

Joints and interfaces are common features of Geotechnical Engineering problems. For example, jointed rock masses. The bulk behavior of such rock masses are greatly influenced by the presence of joints and pre-existing weak planes of potential discontinuity. In the past, the finite element method has been successfully used to solve many problems of this class (Goodman et al., 1968 ; Ghaboussi et al., 1973; Zienkiewicz, Stagg, et al., 1970; Mahtab and Goodman, 1970; Desai, 1975, 1977). The solution of a continuum problem by the finite element method follows as orderly step-by-step procedure. This procedure is given in most standard texts on finite element method (e.g. Zienkiewicz, 1967; Bowes and Russell, 1975).

The stress-strain relationships for joints and interfaces are quite often nonlinear. This is due to the fact that slip and debonding are always associated with the joints. The presence of such interfaces in a system may

result in weak nonlinearity or large scale nonlinearity. The term weak nonlinearity is used to describe systems where nonlinearity is local and constitutes a small portion of an otherwise linear system (Ghaboussi, et al., 1973). Owing to the nonlinearity in the system an iterative or incremental-iterative technique has to be incorporated in the solution procedure. In subsequent sections of this chapter, the development of axisymmetric finite element technique for solving Geotechnical problems with joints and interfaces will be described. The numerical modelling of this class of problems are based on certain assumptions and it is necessary to outline these assumptions before presenting a detail development of the technique.

2.1 Assumptions

(1) The physical behavior of joints and discontinuities is nonlinear. This can be realized by performing laboratory tests on intact specimens of discontinuities (e.g. Goodman, 1976; Desai, 1977; Huck, et al., 1974). The nonlinear joint behavior is characterized by the following assumptions:

- (a) Joint material is assumed to be elastic-perfectly plastic (Fig. 2.1)

- (b) Peak shear strength of the joint is not a function of the normal stress developed in the joint. However, many investigators expressed the shear strength as a function of cohesion, normal stress and roughness angle. (e.g. Goodman, 1976; Pande and Sharma, 1979).
- (c) Joints behave purely elastically, when the shear stress developed in the joint is less than the peak shear strength.
- (d) Joints are incapable of withstanding any tension in the normal direction.
- (e) Dilatancy and strain softening effect has been neglected (see e.g. Ghaboussi, et al., 1973; Desai, 1977).

(2) The material properties of the continuum on either side of the joint is assumed to be homogeneous, isotropic and linearly elastic.

2.2 Development of Axisymmetric Solid Element

Axisymmetric solids under axisymmetric loading is

a three dimensional problem. However, by symmetry, two components of displacements in any plane section of the body along its axis of symmetry define completely the state of strain and, therefore, the state of stress. Hence the problem is a two-dimensional one. The first step in the finite element analysis consists of subdividing the continuum region into elements by a set of imaginary lines. The selection of element for a given problem is an exercise of engineering judgement and greatly influenced by the nature of the problem to be solved (Desai and Abel, 1972). In general, the number, shape, size and configuration of the elements should be such that the original body is simulated as closely as possible. The factors to be considered in this process of discretization are the geometry, loading, and material properties of the structure, bandwidth, element aspect ratio, finite representation of infinite bodies, etc.

In the past, various elements (e.g. three noded triangular element, six noded triangular element, four noded rectangular element, eight noded isoparametric element) have been employed in the finite element analysis of axisymmetric solids under axisymmetric loading (e.g. Clough and Rashid, 1965; Smith, 1967; Boswell and Scott, 1975; Hooper, 1974). In this work, an axisymmetric solid element of triangular cross-section is used.

2.2.1 Choice of Element

In general, the choice of an element (Desai and Abel, 1972) greatly depends on the nature of the problem to be solved and is influenced by the geometry of the structure and also by the number of independent spatial coordinates (e.g. R , θ , Z referring to the cylindrical polar coordinate system) necessary to describe the problem. Since the present problem possesses an axis of symmetry, and the solid elements are assumed to have no rotational stiffness, only the two generalized displacements per node can be specified. In this situation, an improved higher order polynomial approximation could be used by increasing the number of nodes per element. But, incorporation of secondary nodes will obviously increase the band width and require more core space and computing cost. On the other hand, the selection of lower order polynomial requires, in general, the use of relatively refined mesh for adequate accuracy. In all practical problems, there is a compromise between the two methods. However, the reasons for the author's option for three noded triangular elements are stated below:

1. A three noded triangular element is the simplest of all two-dimensional elements, but still very efficient in subdividing a continuum region, especially one which has an irregular boundary.

2. For the same number of elements,
the band width is considerably
less.

2.2.2 Derivation of the Element Stiffness Matrix

A typical three noded axisymmetric triangular element is shown in Fig. 2.2 with nodes 1, j, m numbered in the counter-clockwise order. At each nodal point there are two degrees of freedom, one in the radial direction (r- direction) and the other in the axial direction (Z- direction). The nonzero components of displacement in the r and z direction are denoted by u and w respectively. The unknown nodal displacements for a three noded axisymmetric triangular element can be expressed in the form

$$\{\delta\} = \begin{Bmatrix} u_1 \\ w_1 \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix} \quad (2.1)$$

The general displacement functions (u, w) required to describe the behavior of the elements can be simply represented by two first order polynomials in r and Z.

$$u = a_1 + a_2 r + a_3 z \quad (2.2)$$

$$w = a_4 + a_5 r + a_6 z \quad (2.3)$$

where a_i ($i=1, \dots, 6$) are the undetermined coefficients. The values of a_i ($i=1, \dots, 6$) are obtained by evaluating the functions u and w at the nodal points i, j and m . This procedure introduces six linear equations:

$$\begin{aligned} u_i &= a_1 + a_2 r_i + a_3 z_i \\ u_j &= a_1 + a_2 r_j + a_3 z_j \\ u_m &= a_1 + a_2 r_m + a_3 z_m \\ w_i &= a_4 + a_5 r_i + a_6 z_i \\ w_j &= a_4 + a_5 r_j + a_6 z_j \\ w_m &= a_4 + a_5 r_m + a_6 z_m \end{aligned} \quad (2.4)$$

Operating on the system of linear equations (2.4), the coefficients a_i 's ($i=1, \dots, 6$) can be expressed in the form:

$$\{a_i\} = [A^{-1}] \{\delta\} = \begin{bmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix} \{\delta\} \quad (2.5)$$

where

$$[A^*] = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix} \quad (2.6)$$

The strain components at any point within the element can be written as:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_z \\ \epsilon_r \\ \epsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \end{Bmatrix} \quad (2.7)$$

where

ϵ_z = strain in the z-direction

ϵ_r = strain in the r-direction

ϵ_θ = strain in the θ -direction

γ_{rz} = shearing strain in r-z plane

Substituting the polynomial expressions for u, w in (2.7),

the strain components can be expressed in the matrix form:

$$\{\epsilon\} = [B][A^{-1}]\{\delta\} \quad (2.8)$$

where

$$[B] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{r} & 1 & \frac{z}{r} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad (2.9)$$

It may be noted that for two-dimensional problems the elements of the matrix [B] for three noded triangular elements are constant. However, in axisymmetric analysis the matrix [B] contain terms in r and z (see eqn. 2.9).

This is due to the presence of the strain component

$\epsilon_{\theta} = \frac{u}{r}$ in the strain vector $\{\epsilon\}$.

For homogeneous, isotropic and elastic material the stress-strain relationships can be expressed in the form:

$$\{\sigma\} = [D]\{\epsilon\} \quad (2.10)$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_z \\ \sigma_r \\ \sigma_\theta \\ \sigma_{rz} \end{Bmatrix} \quad (2.11)$$

$$[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \quad (2.12)$$

and E is Young's modulus and ν is poisson's ratio of the elastic material.

Matrix [D] in (2.12) is sometimes referred to as the constitutive relation matrix or the elasticity matrix.

The stiffness matrix of the element can now be computed following the standard steps of the equilibrium method (Zienkiewicz, 1967; Bowes and Russell, 1975). The required element stiffness matrix $[K]_e$ can be expressed in the matrix form

$$[K]_e = [A^{-1}]^T \left(\int_{Vol} [B]^T [D] [B] dV \right) [A^{-1}] \quad (2.13)$$

where

$$[A^{-1}] = \begin{bmatrix} A^{*-1} & 0 \\ 0 & A^{*-1} \end{bmatrix} \quad (2.14)$$

and

$$dV = r dr d\theta dz \quad (2.15)$$

The integrals in (2.13) involves functions of r and z only. Closed form solutions can be obtained for these integrals. However, numerical integrations, using the Gaussian quadrature formulae (Cowper, 1972) with area coordinate system, give satisfactory results for all practical purposes.

Once the element stiffness matrices are obtained, the next step in the formulation is to assemble these matrices to form the global stiffness matrix $[K]$. The code number method (Beaufait, F.W. et al., 1970) is used to assemble the element stiffnesses to arrive at $[K]$.

It should be mentioned here that the present finite element analysis cannot be employed for situations where ν approaches 0.5 (i.e. incompressible). This is due to the presence of a term $1/(1-2\nu)$ in the matrix $[D]$, (see eqn. 2.12). However, techniques are available which can be employed to

analyze nearly incompressible (ν is very close to 0.5) to completely incompressible ($\nu = 0.5$) continuum problems (Herrmann, 1965).

2.2.3 Consistent Load Vector for Uniformly Distributed Loads

Fig. 2.3 shows an element subjected to a distributed vertical load of intensity p on the surface $i-m$. The radii r_1 and r_m represents the radial distance of the nodes i and m respectively. Since in the finite element analysis only concentrated nodal forces can be employed, the distributed load has to be decomposed into two concentrated forces F_1 and F_m which act at the nodal points i and m . As a simple method static equivalents at the nodes may be utilized. However, a more accurate method would be to find a set of equivalent nodal forces which constitute the consistent load vector. These nodal forces are obtained by comparing the work done in the both loading systems.

Referring to Fig. 2.3, the work done by the equivalent nodal forces F_1 and F_m can be written as

$$W_e^d = F_1 w_1 + F_m w_m \quad (2.16)$$

where w_1 and w_m are the axial displacements at the nodes i

and m respectively. Since z is constant along i - m , the general displacement function w can be expressed in the form

$$w = a_1 + a_2 r \quad (2.17)$$

where a_1 and a_2 are the undetermined constants. These are obtained by evaluating w at the nodes i and m i.e.,

$$\begin{aligned} w_i &= a_1 + a_2 r_i \\ w_m &= a_1 + a_2 r_m \end{aligned} \quad (2.18)$$

The linear equations in (2.18) can be solved to obtain the coefficients a_i ($i=1, \dots, 2$). In matrix form

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} \frac{r_m}{(r_m - r_i)} & -\frac{r_i}{(r_m - r_i)} \\ -\frac{1}{(r_m - r_i)} & \frac{1}{(r_m - r_i)} \end{bmatrix} \begin{Bmatrix} w_i \\ w_m \end{Bmatrix} \quad (2.19)$$

The work done by the distributed load is given by

$$W_a^d = 2\pi p \int_{r_i}^{r_m} w \cdot r \, dr \quad (2.20)$$

Substituting the polynomial expression for w in (2.20)

$$w_a^d = 2\lambda p \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}^T \int_{r_1}^{r_m} \begin{Bmatrix} r \\ r^2 \end{Bmatrix} dr \quad (2.21)$$

The integral in (2.21) is evaluated in terms of r_1 and r_m .

$$w_a^d = \frac{\lambda p}{3} \{\delta\}^T \begin{bmatrix} (r_m^2 + r_1 r_m - 2r_1^2) \\ (2r_m^2 - r_1 r_m - r_1^2) \end{bmatrix} \quad (2.22)$$

where

$$\{\delta\}^T = [w_1 \quad w_m] \quad (2.23)$$

Equation (2.16) can be written as

$$w_e^d = \{\delta\}^T \begin{Bmatrix} F_1 \\ F_m \end{Bmatrix} \quad (2.24)$$

By equating (2.22) and (2.24), the consistent load vector

$\{F\}_d^T = [F_1 \quad F_m]$ can be expressed in the form

$$\{F\}_d = \frac{\gamma p}{3} \begin{bmatrix} (r_m^2 + r_i r_m - 2r_i^2) \\ (2r_m^2 - r_i r_m - r_i^2) \end{bmatrix} \quad (2.25)$$

2.2.4 Consistent Load Vector for Gravity Load

Fig. 2.4 shows an element subjected to gravity load of intensity γ per unit volume. The consistent load vector for gravity forces consists of three equivalent nodal forces F_i , F_j and F_m at the nodes i , j and m respectively. The work done by this equivalent system can be expressed as

$$W_e^a = \{\delta\}^T \begin{Bmatrix} F_i \\ F_j \\ F_m \end{Bmatrix} \quad (2.26)$$

where

$$\{\delta\} = \begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix} \quad (2.27)$$

Also, the work done by gravity forces is given by the integral

$$w_a^G = 2\lambda\gamma \int_{\text{Area}} w.r \, dA \quad (2.28)$$

where

$$dA = r \, dr \, dZ \quad (2.29)$$

w is a linear displacement function in r and z. This can be expressed as

$$w = a_1 + a_2 r + a_3 z \quad (2.30)$$

a_i 's ($i=1, \dots, 3$) are the undetermined constants and they are obtained by evaluating w at the nodes i, j, m (see Section 2.2.2).

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A^*]^{-1} \begin{Bmatrix} w_1 \\ w_j \\ w_m \end{Bmatrix} \quad (2.31)$$

where $[A^*]$ is given by eqn. (2.6).

Substituting w and a_i 's in (2.28)

$$W_a^G = 2\lambda_Y \begin{Bmatrix} w_1 \\ w_j \\ w_m \end{Bmatrix}^T [A^{*-1}] \int_{\text{Area}} \begin{Bmatrix} r \\ r^2 \\ rz \end{Bmatrix} dA \quad (2.32)$$

The integral in (2.32) can be evaluated exactly over the triangle 1, j, m if the origin of the coordinate system is located at the centroid of the triangle. Referring to Fig. 2.4, r and z is substituted by x and y in the form

$$\begin{aligned} r &= x + \bar{r} \\ z &= y + \bar{z} \end{aligned} \quad (2.33)$$

where $\bar{r}$ and $\bar{z}$ represents the centroidal distances of the triangle from the global r and z axis. Substituting the expressions for r and z in (2.32)

$$W_a^G = 2\lambda_Y \begin{Bmatrix} w_1 \\ w_j \\ w_m \end{Bmatrix}^T [A^{*-1}] \int_{\text{Area}} \begin{Bmatrix} x + \bar{r} \\ x^2 + x\bar{r} + \bar{r}^2 \\ xy + x\bar{z} + y\bar{r} + \bar{r}\bar{z} \end{Bmatrix} dA \quad (2.34)$$

Noting that $\int x \, dA = \int y \, dA = 0$, eqn. (2.34) can be expressed in the form

$$W_a^G = 2\bar{\Lambda}\gamma \begin{Bmatrix} w_1 \\ w_j \\ w_m \end{Bmatrix}^T [\bar{A}^{-1}] \begin{Bmatrix} \bar{r}A \\ \frac{A}{12} (x_1^2 + x_j^2 + x_m^2) + \bar{r}A \\ \frac{A}{12} (x_1 y_1 + x_j y_j + x_m y_m) + \bar{r}\bar{z}A \end{Bmatrix} \quad (2.35)$$

where A is the area of the triangle i, j, m and is given by

$$A = \frac{1}{2} (r_1 z_j + r_j z_m + r_m z_1 - z_1 r_j - z_j r_m - z_m r_1) \quad (2.36)$$

By equating (2.26) and (2.35), the consistent load vector can be expressed in the form

$$\{F\}_G = \begin{Bmatrix} F_1 \\ F_j \\ F_m \end{Bmatrix} = 2\bar{\Lambda}\gamma [\bar{A}^{-1}]^T \begin{Bmatrix} \bar{r}A \\ \frac{A}{12} (x_1^2 + x_j^2 + x_m^2) + \bar{r}A \\ \frac{A}{12} (x_1 y_1 + x_j y_j + x_m y_m) + \bar{r}\bar{z}A \end{Bmatrix} \quad (2.37)$$

2.2.5 Numerical Accuracy

A computer program has been developed using the three noded axisymmetric triangular element described in the previous sections of this chapter. To check the accuracy of the program, two problems were solved and the computed results were compared with the known exact solutions.

1) Thick Cylinder Under Internal Pressure: The problem investigated by Bowes and Russell (1975) is illustrated in Fig. 2.5. The finite element subdivisions are also shown in the same figure. The exact solutions for the elastic state of stress is given by (Timoshenko and Goodier, 1970)

$$\sigma_r = \frac{b^2 p_0}{(a^2 - b^2)} \left(1 - \frac{a^2}{r^2} \right) \quad (2.38)$$

$$\sigma_\theta = \frac{b^2 p_0}{(a^2 - b^2)} \left(1 + \frac{a^2}{r^2} \right) \quad (2.39)$$

where

a = outer radius of the cylinder

b = inner radius of the cylinder

p_0 = internal pressure intensity

Fig. 2.6 shows the comparison between the finite

element solutions and the elasticity solutions. It is seen that both solutions compare very closely to within an accuracy of 0.2 to 0.3 percent.

ii) Uniformly Distributed Surface load on an Isotropic, Homogeneous and Elastic Half-Space: Finite element representation of an elastic half-space subjected to an uniformly distributed load of intensity 100. psi is shown in Fig. 2.7. The half-space is idealized as a region in which the outer boundaries extend to eighteen times the radius of the loaded area in both r and z directions. The exact solutions for the surface displacements are given by (see Selvadurai, 1979):

For $0 \leq r \leq a$

$$w(r,0) = \frac{2(1-\nu)pa}{\pi G} \int_0^{\pi/2} \left(1 - \frac{r^2}{a^2} \sin^2 \theta\right)^{\frac{1}{2}} d\theta \quad (2.40)$$

and for $a \leq r \leq \infty$

$$w(r,0) = \frac{2(1-\nu)pa}{\pi G} \int_0^{\pi/2} \left(1 - \frac{a^2}{r^2} \sin^2 \theta\right)^{\frac{1}{2}} d\theta - \left(1 - \frac{a^2}{r^2}\right) \int_0^{\pi/2} \left(1 - \frac{a^2}{r^2} \sin^2 \theta\right)^{-\frac{1}{2}} d\theta \quad (2.41)$$

In (2.40) and (2.41)

a = radius of the loaded area

G = shearing modulus for the elastic material

ν = poisson's ratio

The values of the complete elliptic integrals in (2.41) can be obtained from tables (see Janke and Emde, 1960) for various values of a/r . For comparison the surface displacements $w(r,0)$ is normalized as follows:

$$w(r,0) = \frac{E}{pa} \bar{w}(r,0) \quad (2.42)$$

where

$\bar{w}(r,0)$ = displacement influence coefficient.

For $\nu = 0.0$ finite element results for $\bar{w}(r,0)$ are accurate within 8 percent. When $\nu = 0.49$, the corresponding difference is about 10 percent. Figs. 2.8 and 2.9 shows the comparison between the finite element results and the analytical results.

2.3 Development of Axisymmetric Plate Bending Element

Annular plate bending elements are used to model

the axisymmetrically loaded circular plates. The derivation of the element stiffness matrix is based on the Poisson-Kirchhoff thin plate assumptions:

- i) The thickness of the plate is small compared to its other dimensions.
- ii) The deflection of the plate is small compared to its thickness.
- iii) The strain energy of the plate is composed of flexural energy only.

It may be noted that various plate bending elements are available for analyzing plates under general loading conditions (e.g. Cheung and Zienkiewicz, 1965; Hooper, 1974). Since the present study is restricted only to the analysis of circular plates subjected to axisymmetric loads, an annular plate element would be advantageous. This type of element was used by Smith (1970) for the analysis of circular plates resting on Winkler medium and on elastic half-space.

2.3.1 Derivation of the Plate Element Stiffness Matrix

Fig. 2.10(a) shows an annular plate bending element

with the associated degrees-of-freedom. The assumed displacement functions are given by

$$u = a_1 + a_2 r \quad (2.43)$$

$$w = a_3 + a_4 r + a_5 r^2 + a_6 r^3 \quad (2.44)$$

The rotation of the plate is obtained by differentiating w with respect to the radial coordinate r .

$$\theta = \frac{dw}{dr} = a_4 + 2a_5 r + 3a_6 r^2 \quad (2.45)$$

where a_i , s ($i=1, \dots, 6$) are the undetermined coefficients.

These are obtained by evaluating the functions u , w , θ at the nodes i and j (Fig. 2.10(a)). Resulting linear equations can be written as

$$\begin{aligned} u_i &= a_1 + a_2 r_i \\ w_i &= a_3 + a_4 r_i + a_5 r_i^2 + a_6 r_i^3 \\ \theta_i &= a_4 + 2a_5 r_i + 3a_6 r_i^2 \\ u_j &= a_1 + a_2 r_j \\ w_j &= a_3 + a_4 r_j + a_5 r_j^2 + a_6 r_j^3 \\ \theta_j &= a_4 + 2a_5 r_j + 3a_6 r_j^2 \end{aligned} \quad (2.46)$$

Eqns. (2.46) can be valued to determine the coefficients a_i ($i=1, \dots, 6$). In matrix form

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} = [c^{-1}] \begin{Bmatrix} u_i \\ w_i \\ \theta_i \\ u_j \\ w_j \\ \theta_j \end{Bmatrix} \quad (2.47)$$

where

$$[c] = \begin{bmatrix} 1 & r_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & r_i & r_i^2 & r_i^3 \\ 0 & 0 & 0 & 1 & 2r_i & 3r_i^2 \\ 1 & r_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & r_j & r_j^2 & r_j^3 \\ 0 & 0 & 0 & 1 & 2r_j & 3r_j^2 \end{bmatrix} \quad (2.48)$$

The strains and curvatures at any point within the element can be expressed in the form

$$\{\epsilon^b\} = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \chi_r \\ \chi_\theta \end{Bmatrix} = \begin{Bmatrix} \frac{du}{dr} \\ \frac{u}{r} \\ \frac{d^2 w}{dr^2} \\ \frac{1}{r} \frac{dw}{dr} \end{Bmatrix} \quad (2.49)$$

The moment-curvature relationships and the stress-strain relationships for the plate element can be expressed in the matrix form

$$\{\sigma^b\} = [D] \{\epsilon^b\} \quad (2.50)$$

where

$$\{\sigma^b\} = \begin{Bmatrix} N_r \\ N_\theta \\ M_r \\ M_\theta \end{Bmatrix} \quad (2.51)$$

$$[D] = \begin{bmatrix} D_1 & D_2 & 0 & 0 \\ & D_1 & 0 & 0 \\ & & D_3 & D_4 \\ \text{Symmetric} & & & D_3 \end{bmatrix} \quad (2.52)$$

$$D_1 = \frac{E_b t}{(1-\nu_b^2)} \quad (2.53)$$

$$D_2 = \frac{\nu_b E_b t}{(1-\nu_b^2)} \quad (2.54)$$

$$D_3 = \frac{E_b t^3}{12(1-\nu_b^2)} \quad (2.55)$$

$$D_4 = \frac{\nu_b E_b t^3}{12(1-\nu_b^2)} \quad (2.56)$$

and

E_b = Young's modulus

ν_b = poisson's ratio

t = thickness of the plate

Operating on (2.49) and substituting the polynomial expressions for u and w defined by (2.43) and (2.44), it can be shown that

$$\{\epsilon^b\} = [B] [C^{-1}] \{\delta\}_e \quad (2.57)$$

where

$$\{\delta\}_e = \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_j \\ w_j \\ \theta_j \end{Bmatrix} \quad (2.58)$$

and

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 6r \\ 0 & 0 & 0 & \frac{1}{r} & 2 & 3r \end{bmatrix} \quad (2.59)$$

Now following the standard steps of equilibrium method (e.g. Zienkiewicz, 1967; Bowes and Russell, 1975), the element stiffness matrix $[K^b]_e$ can be expressed in the integral form

$$[K^b]_e = 2\pi [\bar{C}^{-1}]^T \left(\int_{r_1}^{r_j} [B]^T [D] [B] r dr \right) [C^{-1}] \quad (2.60)$$

The integral in (2.60) can be evaluated exactly for nonzero r_1 and the element stiffness matrix can be expressed in the matrix form

$$[K^b]_e = 2\pi [C^{-1}]^T [K_D] [C^{-1}] \quad (2.61)$$

where

$$[K_D] = \begin{bmatrix} c_1 & c_2 & 0 & 0 & 0 & 0 \\ & c_3 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & c_4 & c_5 & c_6 \\ & & & & c_7 & c_8 \\ & & & & & c_9 \end{bmatrix}$$

Symmetric

(2.62)

and

$$\begin{aligned} c_1 &= D_1 \ln \left(\frac{r_j}{r_1} \right) \\ c_2 &= (D_1 + D_2) (r_j - r_1) \\ c_3 &= (D_1 + D_2) (r_j^2 - r_1^2) \\ c_4 &= D_3 \ln \left(\frac{r_j}{r_1} \right) \end{aligned}$$

$$\begin{aligned}
 C_5 &= 2 (D_3 + D_4) (r_j - r_1) \\
 C_6 &= 1.5 (D_3 + 2D_4) (r_j^2 - r_1^2) \\
 C_7 &= 4 (D_3 + D_4) (r_j^2 - r_1^2) \\
 C_8 &= 6 (D_3 + D_4) (r_j^3 - r_1^3) \\
 C_9 &= 2.25 (5D_3 + 4D_4) (r_j^4 - r_1^4)
 \end{aligned} \tag{2.63}$$

The terms C_1 and C_4 in the matrix $[K_D]$ cannot be evaluated when $r_1 = 0$. This can be overcome by introducing a special solid plate element for the central part of the plate. The derivation of the stiffness matrix for such an element is given in the following section.

2.3.2 Derivation of the Stiffness Matrix for the Special Plate Bending Element

Fig. 2.10(b) shows a typical solid circular plate bending element with its associated degrees of freedom. It should be noted that due to the radial symmetry u_1 and θ_1 are zero at the central node 1. Hence the displacement functions should be such that the symmetry conditions at $r=r_1=0$ are satisfied. The following displacement functions are compatible with the force boundary conditions:

$$u = a_1 r \quad (2.64)$$

$$w = a_2 + a_3 r^2 + a_4 r^3 \quad (2.65)$$

and the rotation at any point within the element is given by

$$\theta = 2a_3 r + 3a_4 r^2 \quad (2.66)$$

where a_i 's ($i=1, \dots, 4$) are the undetermined coefficients. Evaluating the functions u , w , θ at the nodes i and j , the coefficients a_i can be expressed in the form

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = [C^{-1}] \begin{Bmatrix} w_1 \\ u_j \\ w_j \\ a_j \end{Bmatrix} \quad (2.67)$$

where

$$[C] = \begin{bmatrix} 0 & 1 & r_1^2 & r_1^3 \\ r_j & 0 & 0 & 0 \\ 0 & 1 & r_j^2 & r_j^3 \\ 0 & 0 & 2r_j & 3r_j^2 \end{bmatrix} \quad (2.68)$$

The element stiffness matrix can be expressed as

$$[K^b]_e = 2\pi [C^{-1}]^T \left(\int_{r_1=0}^{r_j} [B]^T [D] [B] r dr \right) [C^{-1}] \quad (2.69)$$

where

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6r \\ 0 & 0 & 2 & 3r \end{bmatrix} \quad (2.70)$$

and matrix $[D]$ is given by (2:52). The integral in (2.69) can be evaluated in exact form and $[K^b]_e$ can be expressed as

$$[K^b]_e = 2\pi [C^{-1}]^T [K_D] [C^{-1}] \quad (2.71)$$

where

$$[K_D] = \begin{bmatrix} (D_1+D_2)r_j^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & 4(D_3+D_4)r_j^2 & 6(D_3+D_4)r_j^3 & \\ \text{Symmetric} & & & 2.25(5D_3+4D_4)r_j^4 \end{bmatrix} \quad (2.72)$$

2.3.3 Consistent Load Vector for Uniformly Distributed Load

Fig. 2.10(a) shows an annular plate element. The consistent load vector $\{F^b\}_e$ for an uniformly distributed load of intensity p acting on the annular element is obtained following a similar method outlined in section 2.2.3. This can be expressed in the matrix form

$$\{F^b\}_e = \begin{Bmatrix} F_1 \\ M_1 \\ F_j \\ M_j \end{Bmatrix} = 2\pi p [C^{-1}]^T \begin{Bmatrix} \frac{r_j^2 - r_1^2}{2} \\ \frac{r_j^3 - r_1^3}{3} \\ \frac{r_j^4 - r_1^4}{4} \\ \frac{r_j^5 - r_1^5}{5} \end{Bmatrix} \quad (2.73)$$

where

$$[C] = \begin{bmatrix} 1 & r_1 & r_1^2 & r_1^3 \\ 0 & 1 & 2r_1 & 3r_1^2 \\ 1 & r_j & r_j^2 & r_j^3 \\ 0 & 1 & 2r_j & 3r_j^2 \end{bmatrix} \quad (2.74)$$

The corresponding expressions of $\{F^b\}_e$ for the special plate element are

$$\{F\}_e = \begin{Bmatrix} F_1 \\ F_j \\ M_j \end{Bmatrix} = 2\pi p [C^{-1}]^T \begin{Bmatrix} \frac{r_j^2}{2} \\ \frac{r_j^4}{4} \\ \frac{r_j^5}{5} \end{Bmatrix} \quad (2.75)$$

where

$$[C] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & r_j^2 & r_j^3 \\ 0 & 2r_j & 3r_j^2 \end{bmatrix} \quad (2.76)$$

2.3.4 Flexural Moments and Shearing Forces

The flexural moments at any point within the element is calculated from the expressions (Timoshenko and Woinowsky-Krieger, 1959).

$$M_r = -D \frac{\partial^2 w}{\partial r^2} + \frac{\nu_b}{r} \frac{\partial w}{\partial r} \quad (2.77)$$

$$M_\theta = -D \frac{1}{r} \frac{\partial w}{\partial r} + \nu_b \frac{\partial^2 w}{\partial r^2} \quad (2.78)$$

where M_r and M_θ are the moments in radial and circumferential

directions respectively.

$$D = \frac{E_b t^3}{12(1-\nu_b^2)} \quad (2.79)$$

Substituting the polynomial expression for $w(r)$ from (2.44), the equations (2.77) and (2.78) can be expressed in the form

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = -D \begin{bmatrix} 0 & \frac{\nu_b}{r} & 2(1+\nu_b) & 3(2+\nu_b)r \\ 0 & \frac{1}{r} & 2(1+\nu_b) & 3(2\nu_b+1)r \end{bmatrix} [C^{-1}] \{\delta\}_e \quad (2.80)$$

where

$$\{\delta\}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_j \\ \theta_j \end{Bmatrix} \quad (2.81)$$

and $[C]$ is defined in (2.74).

The corresponding expressions for the special solid element is given by

$$\begin{Bmatrix} M_{\theta} \\ M_{\theta} \end{Bmatrix} = -D \begin{bmatrix} 0 & 2(1+\nu_b) & 3(2+\nu_b)r \\ 0 & 2(1+\nu_b) & 3(2\nu_b+1)r \end{bmatrix} [C^{-1}] \{\delta\}_e \quad (2.82)$$

where

$$\{\delta\}_e = \begin{Bmatrix} w_1 \\ w_j \\ \theta_j \end{Bmatrix} \quad (2.83)$$

and $[C]$ is given by (2.76).

Similarly the shearing force per unit length Q_r can be calculated from the expression

$$Q_r = D \left[\frac{\partial^3 w}{\partial r^3} + \frac{1}{r} \frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2} \frac{\partial w}{\partial r} \right] \quad (2.84)$$

The final expressions for the shearing force at any point within the element can be expressed as

$$Q_r = D \begin{bmatrix} 0 & -\frac{1}{r^2} & 0 & 9 \end{bmatrix} [C^{-1}] \{\delta\}_e \quad (2.85)$$

where $[C]$ and $\{\delta\}_e$ are defined by (2.74) and (2.81).

Equation (2.83) cannot be evaluated when $r=0$. The shearing force at the centre of the plate is calculated from the expression

$$Q_r = D \begin{bmatrix} 0 & 0 & 9 \end{bmatrix} [c^{-1}] \{\delta\}_e \quad (2.86)$$

where $[C]$ and $\{\delta\}_e$ are defined by (2.76) and (2.83). It should be noted that due to the symmetry of the problem, the shearing force at the centre of the plate is identically equal to zero.

2.3.5 Numerical Accuracy

In order to check the accuracy of the plate bending elements developed in the previous sections, the following problems have been solved and compared with the available analytical solutions.

(a) A simply supported circular plate:

- i) For uniformly distributed load
- ii) For concentrated load.

(b) A clamped circular plate:

- i) For uniformly distributed load
- ii) For concentrated load.

Graphical results for the finite element solutions and analytical solutions (Timoshenko and Woinowsky-Krieger, 1959) are shown in Figs. (2.11) through (2.18). It can be seen that a close agreement is obtained between the two solutions. Finite element results for deflection compare almost exactly with the corresponding exact solutions. For concentrated loads, the bending moments also agree very closely. However, for distributed load, the maximum error in bending moments is of the order of 1 percent or less.

2.4 Interface Elements for Joints and Material Interfaces

The finite element analysis of Geotechnical engineering problems such as footings, piles, anchors, buildings and dams etc. are complicated by the fact that interfaces are encountered in these problems. Normal practice is to assume that the interfaces are either smooth or bonded. In reality they are neither perfectly smooth nor perfectly bonded. In the past few years, many authors have developed special elements both with zero and nonzero thicknesses to model such interface behavior (e.g. Goodman et al., 1968; Zienkiewicz et al., 1970; Ghaboussi et al., 1973; Pandey and Sharma, 1979). These were used successfully for solving many problems of practical interest.

2.4.1 Choice of Interface Element and Few Salient Features

In this thesis, a four noded ~~axisymmetric~~ interface element with nonzero thickness is used to model joints and material interfaces. This type of element was developed by Ghaboussi et al. (1973). Goodman (1968, 1976) also used a four noded joint element for solving plane strain problems of discontinuous rocks. Isoparametric elements were also developed by Zienkiewicz et al, (1970), Desai (1977), Pande and Sharma (1979) and others. Few salient features of the four noded interface element can be summarized as follows:

- i) The thickness of the interface element is small compared to its length.
- ii) The strains are assumed to be constant along the thickness of the element. This assumption is reasonable when the thickness of the interface element is small compared to its length. When the ratio of the thickness to the length is large, a constant strain assumption across the joint element may not be reasonable. In these situations, a linear variation in strain can be obtained by inserting mid-side nodes.

iii) Nodal displacements are used as the independent degrees-of-freedom for the interface element. Wilson (1973, 1977) advocated the use of relative nodal displacements as an independent parameter for avoiding possible numerical ill-conditioning. However, Pande and Sharma (1979) have developed an eight noded isoparametric parabolic element both with relative nodal displacements and nodal displacements as independent degrees of freedom. They have shown that the difference between the results obtained by using these two different formulations is quite insignificant up to an extremely large aspect ratio of the interface element. They have also shown that in non-linear analysis involving many resolutions, the difference is insignificant. Also it may be noted here that when relative nodal

displacements are used to formulate the joint element stiffness matrices, the assembly process becomes complicated. Hence this additional effort involved in adopting relative nodal displacements as an independent parameter may not be worth while (Pande and Sharma, 1979).

- iv) Aspect ratio of the interface element is defined by the ratio of its thickness to the length. In this thesis, aspect ratios up to 10^4 have been used.

2.4.2 Derivation of Joint Element Stiffness Matrix

A typical four noded axisymmetric interface element with its associated degrees-of-freedom is shown in Fig. 2.19.

The nodes i, j, k, & are numbered in counter-clockwise order. Referring to Fig. 2.19, the relative displacements in radial (Δu) and vertical directions (Δw) can be expressed as

$$\Delta u = h_1(u_k - u_1) + h_2(u_k - u_j) \quad (2.87)$$

$$\Delta w = h_1(w_k - w_1) + h_2(w_k - w_j) \quad (2.88)$$

where h_1 and h_2 are linear interpolation functions

$$h_1 = \frac{1}{2}(1-\xi) \quad (2.89)$$

$$h_2 = \frac{1}{2}(1+\xi) \quad (2.90)$$

and ξ = nondimensional coordinate. The quantities $u_1, \dots$, and $w_1, \dots$ etc. are the nodal displacements in the radial and axial directions respectively. The same interpolation functions h_1 and h_2 can be used to represent the variation of the radial displacement (u^B) along the bottom line ij of the joint element

$$u^B = h_1 u_1 + h_2 u_j \quad (2.91)$$

The orientation of the joint is represented by the angle ψ (see Fig. 2.16). The components of the relative displacements along and normal to the ξ axis can be

expressed as

$$\Delta u_s = \Delta u \cos \psi + \Delta w \sin \psi \quad (2.92)$$

$$\Delta w_n = -\Delta u \sin \psi + \Delta w \cos \psi \quad (2.93)$$

The strain components ϵ_{ss} , ϵ_{nn} and $\epsilon_{\theta\theta}$ for the interface are represented by

$$\epsilon_{ss} = \frac{\Delta u_s}{t_J} \quad (2.94)$$

$$\epsilon_{nn} = \frac{\Delta w_n}{t_J} \quad (2.95)$$

$$\epsilon_{\theta\theta} = \frac{1}{r} \left(u^B + \frac{\Delta u}{2} \right) \quad (2.96)$$

where

ϵ_{ss} = strain in s-direction

ϵ_{nn} = strain normal to s-axis

$\epsilon_{\theta\theta}$ = circumferential strain

t_J = thickness of the interface element

Substituting the expressions for Δu_s , Δw_n and u^B into eqns. (2.94) through (2.96) and substituting the nodal displacements for Δu and Δw , the strain-displacement relationships can be written explicitly as

$$\epsilon_{ss} = \frac{1}{t_j} [h_1(u_l - u_1)\cos \psi + h_2(u_k - u_j)\cos \psi + h_1(w_l - w_1)\sin \psi + h_2(w_k - w_j)\sin \psi] \quad (2.97)$$

$$\epsilon_{nn} = \frac{1}{t_j} [-h_1(u_l - u_1)\sin \psi - h_2(u_k - u_j)\sin \psi + h_1(w_l - w_1)\cos \psi + h_2(w_k - w_j)\cos \psi] \quad (2.98)$$

$$\epsilon_{\theta\theta} = \frac{1}{2r} [h_1 u_1 + h_2 u_j + h_2 u_k + h_1 u_l] \quad (2.99)$$

In matrix form

$$\{\epsilon\} = [B]\{\delta\}_e \quad (2.100)$$

where

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_{ss} \\ \epsilon_{nn} \\ \epsilon_{\theta\theta} \end{Bmatrix} \quad (2.101)$$

$$\{\delta\}_e = \langle u_i \ w_i \ u_j \ w_j \ u_k \ w_k \ u_l \ w_l \rangle^T \quad (2.102)$$

and

$$[B] = \begin{bmatrix} -B_1 & -B_2 & -B_3 & -B_4 & B_3 & B_4 & B_1 & B_2 \\ B_2 & -B_1 & B_4 & -B_3 & -B_4 & B_3 & -B_2 & B_1 \\ B_5 & 0 & B_6 & 0 & B_6 & 0 & B_5 & 0 \end{bmatrix} \quad (2.103)$$

where

$$B_1 = \frac{h_1 \cos \psi}{t_J}$$

$$B_2 = \frac{h_1 \sin \psi}{t_J}$$

$$B_3 = \frac{h_2 \cos \psi}{t_J} \quad (2.104)$$

$$B_4 = \frac{h_2 \sin \psi}{t_J}$$

$$B_5 = \frac{h_1}{2r}$$

$$B_6 = \frac{h_2}{2r}$$

The constitutive relationships for the linearly elastic interface material is expressed in the form

$$\{\sigma\} = [D] \{\epsilon\} \quad (2.105)$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_{ss} \\ \sigma_{nn} \\ \sigma_{\theta\theta} \end{Bmatrix} \quad (2.106)$$

and

$$[D] = \begin{bmatrix} E_{ss} & E_{sn} & 0 \\ E_{ns} & E_{nn} & 0 \\ 0 & 0 & E_{\theta\theta} \end{bmatrix} \quad (2.107)$$

where

σ_{ss} = joint shearing stress

σ_{nn} = joint normal stress

$\sigma_{\theta\theta}$ = circumferential stress

E_{nn} = joint modulus in the normal direction.

E_{ss} = joint modulus in the shearing direction

The material constants E_{ns} and E_{sn} account for coupling between shearing and normal displacements in the joint. Physically this can be interpreted as joint dilatancy. Since in this thesis, only nondilatant (no volume change due to shearing strain) joints are considered, the shear and the normal components of deformation are uncoupled. Hence the constants E_{ns} and E_{sn} are zero. It is also assumed that $\sigma_{\theta\theta}$ is zero for the joints. Hence the material constant $E_{\theta\theta}$ is made zero. It may be noted here that Ghaboussi et al. (1973) used this assumption on the basis of the uncertainty involved in getting a realistic value for $E_{\theta\theta}$. The final form of the constitutive relation matrix for the linearly elastic joint material can be expressed as

$$[D] = \begin{bmatrix} E_{ss} & 0 & 0 \\ 0 & E_{nn} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.108)$$

Using standard steps of the equilibrium method, the joint

element stiffness matrix $[K_J]_e$ can be expressed in the matrix form

$$[K_J]_e = \int [B]^T [D] [B] dV \quad (2.109)$$

It is possible to evaluate the integrals in (2.109) in explicit form. However, numerical integration gives satisfactory results for most practical purposes.

2.4.3 Nonlinearity of the Interface Element

Joints and interfaces quite often show nonlinear properties when subjected to shearing loads (e.g. Goodman, 1976). These nonlinear joint properties are obtained by carrying out direct shear tests or triaxial tests on intact specimens of joints: Fig. 2.20 shows a typical shear stress-vs.-shear deformation curve (Desai, 1977) for a joint. The irregular stress-strain curve obtained from laboratory tests is, however, very difficult to utilize in the finite element calculations. The behavior of the joint material is, therefore, idealized as an elastic-perfectly plastic material (Fig. 24). The peak shear strength (τ_y) cannot be assigned arbitrarily. It has to be determined from laboratory tests conducted on intact joint materials.

Alternatively, plausible limits of joint shear strength can be assumed by examining the micro-structural behavior of the joint material. The idealized normal stress- normal strain behavior of the joint element is shown in Fig. 2.1. The modulus of the joint in the normal direction, in compression, is set equal to a high value. When tensile stresses are developed in the joint, the joint material is assumed to have zero peak shear strength and allowed to separate both in the normal and shearing directions.

2.5 An Axisymmetric Finite Element Program to Solve Problems With Joints and Interfaces

The interface element developed in section 2.4 is used to model the joints and interfaces in the analysis of axisymmetric problems. As a first step, the elastic stiffness matrix of the joint element is considered. The nonlinear properties of the joint is simulated by using an iterative technique.

2.5.1 Nonlinear Solution Technique Used in the Program

The linear equations developed for the displacement of nodal points provide a first solution. These displacements may cause excessive shear in some of the joint elements which exceed their peak shear strength. Therefore, the

displacements obtained from the first solution may not be acceptable. In order to arrive at a completely acceptable solution of this problem an iterative solution technique is used in the program. The load transfer method developed by Zienkiewicz (1970) and others has been adopted as the basis for the iterative technique. The steps involved in this method can be summarized as follows:

- 1) The given problem is discretized into consistent finite elements. For example, solid continuum is modelled by axisymmetric triangular elements and joints, and material interfaces are modelled as joint elements. Stiffness matrices for each of these elements are derived and assembled to form the global stiffness matrix $[K]$. The force vector $\{F\}$ for the total system is determined by making use of consistent load vectors as outlined in sections 2.2.3, 2.2.4 and 2.3.3. The system of linear equations $[K]\{U\} = \{F\}$ for the unknown nodal displacements $\{U\}$ is solved by Gaussian elimination along with the technique outlined by Payne and Iron (see Bowes and Russell, 1975).

11) The nodal displacements $\{U\}$ as obtained from the first solution may presume tension in some joints or excessive shear in others. Therefore, the first solution must be examined at each nodal point along the joints and compared with real behavior. This comparison yields information as to how the problem can be restarted or continued in order to produce an entirely acceptable answer. As a first step, the nodal displacement vector $\{\delta\}_e$ is formed for each of the joint element by using the global $\{U\}$. Using the strain-displacement relationships from (2.105), the stresses at any point in a joint element can be determined from the expression

$$\{\sigma\} = [D][B]\{\delta\}_e \quad (2.110)$$

where $\{\sigma\}$ is given by (2.106), $[D]$, $[B]$ $\{\delta\}_e$ are given by the eqns. (2.107), (2.103) and (2.102) respectively.

However, since the matrix $[B]$ is a linear

2

9

function of the nondimensional variable ξ , stresses will be

different at different points along the length of joint element. In this thesis, the stresses $\{\sigma\}$ are evaluated at the midpoint of the element (e.g., $\xi=0$) and is used as average stresses.

A more accurate method would be to evaluate $\{\sigma\}$ at a number of point along the length of the element.

- iii) The average shear stress, σ_{ss} calculated in step (ii) is compared with the peak shearing strength (τ_y) of that element. If σ_{ss} does not exceed (τ_y), step (ii) is repeated for the next joint element.
- iv) When $|\sigma_{ss}| > \tau_y$, the global force vector $\{F\}$ has to be modified for the excess shear stress ($|\sigma_{ss}| - \tau_y$). If σ_{nn} is a tensile stress, the peak shear strength of the joint (τ_y) is set equal to zero. It should be noted that in the load transfer method (see Goodman, 1976), the stiffness of the system $[K]$ is kept

the same while the load vector $\{F\}$ is modified to restart the problem.

Referring to the Fig. 2.22 excess

shear in the joint element is

expressed as

$$V_s = \Lambda L_J (r_j + r_i) (|\sigma_{ss}| - \tau_y) \quad (2.111)$$

where,

L_J = length of the joint element

r_i and r_j are the radial coordinates of the node i and j .

This excess shear force can be replaced by the nodal forces F_{is} , F_{js} , F_{in} , F_{jn} , F_{ks} , F_{ls} , F_{kn} , F_{ln} in the local system of coordinates as follows:

$$F_{is} = -\frac{\Lambda L_J}{2} \left(\frac{3r_i + r_j}{2} \right) (|\sigma_{ss}| - \tau_y) \left(\frac{\sigma_{ss}}{|\sigma_{ss}|} \right)$$

$$F_{js} = -\frac{\Lambda L_J}{2} \left(\frac{r_i + 3r_j}{2} \right) (|\sigma_{ss}| - \tau_y) \left(\frac{\sigma_{ss}}{|\sigma_{ss}|} \right) \quad (2.112)$$

$$F_{ks} = -F_{js}$$

$$F_{ls} = -F_{is}$$

and $F_{in} = F_{jn} = F_{kn} = F_{ln} = 0$.

These forces can be replaced by their components along the global axes of r and z , i.e.

$$F_{ir} = F_{is} \cos \psi$$

$$F_{iz} = F_{is} \sin \psi$$

$$F_{jr} = F_{js} \cos \psi$$

$$F_{jz} = F_{js} \sin \psi$$

(2.113)

$$F_{kr} = -F_{jr}$$

$$F_{kz} = -F_{jz}$$

$$F_{lr} = -F_{ir}$$

$$F_{lz} = -F_{iz}$$

Using (2.113) the global force vector

$\{F\}$ is modified to perform the next iteration. It should be noted here

that when $\sum V_s$ (sum of the excess shear in all the joint elements) is such that

the ratio $\frac{\sum V_s}{\sum V_{ps}} \leq \epsilon_v$, the program prints

out the final nodal displacements and stresses. ΣV_{ps} is the sum of the shear forces in all the joint elements when peak shear strength (τ_y) is mobilized. ϵ_v is a small number assigned arbitrarily. When $\frac{\Sigma V_s}{\Sigma V_{ps}} > \epsilon_v$, the program performs

the next iteration with the modified global force vector $\{F\}$.

- v) This step checks the convergency of the displacements between successive iterations. The difference of the displacements between two successive iterations is calculated as follows:

$$\Delta = \{U\}_{i+1} - \{U\}_i \quad (2.114)$$

where (i) and (i+1) indicates the iteration numbers. Δ is compared with a small number ϵ_c which is set arbitrarily. The displacements obtained from the (i+1)th iteration is considered to have attained the required convergency when $\Delta \leq \epsilon_c$.

If $\Delta > \epsilon_c$, the program goes back to step

- (ii) to perform the next iteration with the already modified force vector $\{F\}$ from step (iv).
- vi) Convergent displacements are printed out. Stresses are calculated by using the convergent displacements and printed out.

2.5.2 Effect of Some Parameters on Convergency

In order to check the convergency of the iterative scheme utilized in the program, a small problem has been examined. Fig. 2.23 shows two rigid blocks separated by an interface. One of the blocks is supported by a spring of stiffness k_{sp} and the other block is resting on a immovable support. The length and the thickness of the interface is represented by L_J and t_J . E_{nn} and E_{ss} are the joint normal modulus and shearing modulus respectively. The system is loaded by a concentrated force P as shown in the figure.

As a first step in the analysis, k_{sp} is kept constant while the aspect ratio (L_J/t_J) is varied. Table 2.2 shows the number of iterations required for the convergency of displacements for different (L_J/t_J) ratio.

It can be seen from the table that as the aspect ratio increases, number of iterations required for a given convergence tolerance (in this case tolerance = .00001) also increases and even may not converge at all. The reason for this increasing number of iterations required for convergency can be explained by evaluating the ratio of K_{sp} to (E_{ss}/t_J) . It should be mentioned here that the stiffness of the joint element in shearing direction is directly proportional to E_{ss}/t_J ratio. The last column of Table 2.2 shows the values of $K_{sp}/(E_{ss}/t_J)$ for each L_J/t_J ratio. It is seen that as the ratio of $K_{sp}/(E_{ss}/t_J)$ decreases, number of iterations required for convergence increases. This is due to the fact that the ratio $K_{sp}/(E_{ss}/t_J)$ represents influence of joint stiffness in the overall structural stiffness of the system. As this ratio decreases, the joint stiffness becomes more prominent and governs the whole problem. The iterative method employed in this analysis is not suitable for such cases and depending on the value of $K_{sp}/(E_{ss}/t_J)$ it may not converge at all. Table 2.2 shows that as $K_{sp}/(E_{ss}/t_J) = .02$ the number of iterations required is more than 1400. In order to establish that

the ratio $K_{sp}/(E_{ss}/t_J)$ is the most important factor in the process of convergence, the same problem has been investigated keeping the aspect ratio (L_J/t_J) constant and varying the spring stiffness K_{sp} . Table 2.3 shows the number of iterations required for the convergence for different values of K_{sp} and $K_{sp}/(E_{ss}/t_J)$. It is seen that on the ratio $K_{sp}/(E_{ss}/t_J)$ decreases the process of convergence becomes slower and at certain points may not converge at all. Table 2.4 shows the results on convergence process for different aspect ratio (L_J/t_J) while $K_{sp}/(E_{ss}/t_J)$ is kept constant. It can be seen that as long as the value of $K_{sp}/(E_{ss}/t_J)$ remains unchanged, the convergence process is not affected very much by the increasing aspect ratios. It can be seen from Table 2.4 that for aspect ratio of 100000, the number of iterations required for the displacements to converge is 28. However, if the aspect ratio (L_J/t_J) of the interface element is made very large the resulting equations may be numerically ill-conditioned.

Table 2.5 shows how the displacement converges to the actual value. A graphical representation is given in Fig. 2.25. It can be seen that the displacement converges monotonically.

Fig. 2.26 shows the load-displacement curve for the problem shown in Fig. 2.23. It is seen that for different τ_y/E_{ss} ratio, the overall load-displacement curve is bilinear in nature. This is due to the fact that a sheared off joint has zero stiffness and does not contribute to the overall stiffness of the system.

Fig. 2.24 shows two rigid blocks separated by an interface and supported by two springs of stiffnesses K_e and K_1 . The interface is modelled by a four noded rectangular interface element as developed in section 2.4. The mechanical properties of the joint element is completely defined by the joint normal modulus E_{nn} and joint shearing modulus E_{ss} . The nonlinear stress-strain relationships employed in the analysis are shown in Fig. 2.1. It should be noted that the peak shear strength of the joint is assumed to be independent of the joint normal stress.

The system is loaded by a concentrated force P applied as in Fig. 2.24. The results of the analysis are shown in Figs. 2.27 and 2.28. Fig. 2.27 shows the influence of the stiffness ratio, K_e/K_1 on the number of iterations required for convergence. A tolerance of .00001 is used as a convergent criteria in this analysis. It can be seen that the number of iterations required for convergence is

maximum when the stiffness ratio, K_e/K_1 is about 0.1 and as the stiffness ratio increases the curve monotonically converges. The variation of the energy ratios (defined by the ratio of the energy stored in the exterior spring to the energy stored in the interior spring) against stiffness ratios (K_e/K_1) is plotted in Fig. 2.28. It is seen from the Table 2.8 that as K_e/K_1 exceeds a value of 5.0, no. of iterations required for convergency becomes constant. The corresponding energy ratio (U_e/U_1) becomes an extremum (Fig. 2.28).

2.5.3 Numerical Accuracy

Few axisymmetric problems with interfaces have been solved and compared with the corresponding known solutions.

1) Embedded Rigid Circular Foundation

Fig. 2.29 shows the dimensions and the location of interface elements used. This problem was solved by Ghaboussi et al. (1973). It should be noted here that Ghaboussi et al. (1973) used relative nodal displacement as the independent degrees of freedom to formulate the joint element stiffness matrix. Table 2.9 shows a comparison between the results obtained from the present

analysis and the corresponding results obtained by Ghaboussi et al. (1973). It is seen that a close agreement is reached between these results.

ii) Circular Plates on Elastic Half-space

Fig. 2.30 shows a circular plate resting on an isotropic, homogeneous elastic half-space wherein interface elements are incorporated at the contact surface. Analytical solutions are available for the limiting conditions of perfectly smooth and perfectly bonded interfaces. The comparison is shown in Table 2.10.

2.5.4 Limitations

The present finite element program for solving axisymmetric problems which incorporate joints and interfaces has the following limitations:

- 1) The material of the continuum on either side of the joint is isotropic, homogeneous and linearly elastic. Therefore, the program can not be utilized to solve anisotropic or nonlinear continuum problems. It should be noted that there is a provision for considering inhomogeneity

in Young's modulus (E) in the form

$$E_z = E_0 + \lambda z \quad (1.115)$$

where

E_0 = Young's modulus at $z=0$

E_z = Young's modulus at a depth z
below the surface

λ = a constant

- ii) The program cannot be employed for poisson's ratio $\nu=0.5$ (incompressible). A hybrid finite element technique can be used for solving problems where the continuum material is nearly incompressible (ν is close to 0.5) or incompressible ($\nu=0.5$) (Herrmann, 1965).
- iii) For the joint material, only nonlinearity in the shearing stress-shearing strain relationship has been considered. In addition to the nonlinearity, dilatancy or strain softening effects may also be associated with the joints. The present

finite element program cannot be employed for solving problems where interfaces exhibit strain softening effect, dilatancy or nonlinearity in the normal stress-normal strain relationship.

However, the constitutive relation matrix $[D]$ in (2.107) contains terms like E_{sn} and E_{ns} . Physically these material constants describe the contribution to normal stress because of dilatancy (Goodman and Dubois, 1972). Therefore, joint dilatancy can be incorporated in the analysis by assigning appropriate values to E_{sn} and E_{ns} .

- iv) The present finite element program is suitable for solving problems where interface nonlinearity is very much localized. This class of problems may be termed as weakly nonlinear problems (e.g. Ghaboussi et al., 1973). In other words, in the weakly nonlinear problems the variation of some norm of the

stiffness matrix of the system,

$||K||$, is small compared to the
considerable variation of such norm
in strongly nonlinear problems.

The iterative method employed in the
program may not converge when the
problem is strongly nonlinear (Goodman,
1976). In such situations, normal
practice is to employ an incremental or
incremental-iterative solution technique.

Figure 2.1 Idealized stress-strain curves of interfaces.

Figure 2.2 A Typical Axisymmetric Triangular Ring Element.

Figure 2.3 Assumed Deflection Model Due to Uniform Pressure.

Figure 2.4 Consistent Load Vector Due to Gravity Load.

Figure 2.5 Thick Cylinder Under Internal Pressure:
Finite element subdivisions.

Figure 2.6 Thick Cylinder Under Internal Pressure: Comparison of Finite Element Results.

Figure 2.7 Idealized half-space Subjected to an Uniformly Distributed Load.

Figure 2.8 Elastic Half-Space Subjected to an Uniformly Distributed Surface Load. Comparison of $w(r)$ for $\nu=0.0$

Figure 2.9 Elastic Half-space Subjected to an Uniformly Distributed Surface Load. Comparison of $\bar{w}(r)$ for $\nu=0.5$.

PLAN

SECTION

Figure 2.10(a) Annular Plate Element with Associated Degrees of Freedom.

PLAN

Figure 2.10(b) Solid Circular Plate Element with Associated Degrees of Freedom.

Figure 2.11 Simply Supported Circular Plate Under Uniform Pressure: Comparison of Finite Element Results with the Corresponding Exact Solutions.

Figure 2.12 Simply Supported Circular Plate Under Uniform Pressure: Comparison of Finite Element Results With the Corresponding Exact Solutions.

Figure 2.13 Simply Supported Circular Plate Under Concentrated Loads: Comparison of Finite Element Results with the Corresponding Exact Solutions.

Figure 2.14 Simply Supported Circular Plate Under Concentrated Load: Comparison of Finite Element Results with the Corresponding Exact Solutions.

Figure 2.15 Clamped Circular Plate Under Uniform Pressure: Comparison of Finite Element Results with the Corresponding Exact Solutions.

Figure 2.16 Clamped Circular Plate Under Uniform Pressure: Comparison of Finite Element Results with the Corresponding Exact Solutions.

Figure 2.17

Clamped Circular Plate Under Concentrated Load: Comparison of Finite Element Results with the Corresponding Exact Solutions.

Figure 2.18 Clamped Circular Plate Under Concentrated Load: Comparison of Finite Element Results with the Corresponding Exact Solutions.

Coordinates of node i : (r_i, z_i)

Coordinates of node j : (r_j, z_j)

Coordinates of node k : (r_k, z_k)

Coordinates of node l : (r_l, z_l)

Figure 2.19 A typical four noded axisymmetric interface element.

Figure 2.20 A typical shear stress-shear strain relationship for the joint.

Figure 2.21 A typical normal stress- normal strain relationship for the joint.

Coordinates of node i : (r_i, z_i)

Coordinates of node j : (r_j, z_j)

Coordinates of node k : (r_k, z_k)

Coordinates of node l : (r_l, z_l)

Figure 2.22 Interface element with the associated nodal forces.

Figure 2.23 Two Rigid Blocks Separated by an Interface.

Figure 2.24 Two Rigid Blocks Separated by an Interface and Supported by two Springs.

Figure 2.25 Convergence of Displacements. For Aspect Ratio (L_J/t_J) = 100.0.

Figure 2.26 Load-Displacement Curve Showing the Effects of Joint Nonlinearity.

Figure 2.27 Effect of Stiffness Ratio, (K_e/K_1) on the no. of Iterations Required for Convergence.

U_e = Energy stored in exterior spring.
 U_1 = Energy stored in interior spring.
 K_e = Stiffness of the exterior spring.
 K_1 = Stiffness of the interior spring.

Figure 2.28 Variation of Energy Ratio (U_e/U_1) Against Stiffness Ratio (K_e/K_1).

Foundation

$E = 1000000$ psi

$\nu = 0.25$

Soil

$E = 10000$ psi

$\nu = 0.45$

Figure 2.29 Embedded Circular Foundation in Layer of Soft Soil.

Figure 2.30 Circular Plate Resting on an Isotropic, Homogeneous Elastic Half-Space.

Table 2.1 Elastic half-space subjected to an uniformly distributed surface load. Influence coefficients for surface displacements.

r/a	v=0.0		v=0.5	
	F.E.M. $\bar{w}(r)$	Theoretical $\bar{w}(r)$	F.E.M. $\bar{w}(r)$	Theoretical $\bar{w}(r)$
0.0	1.82481	2.00000	1.30361	1.5
0.05	1.82395	1.99721	1.30221	1.49791
0.1	1.82187	1.99500	1.30111	1.49625
0.2	1.80825	1.97450	1.29865	1.48088
0.3	1.78458	1.9350	1.28554	1.45125
0.45	1.72307	1.89500	1.24445	1.42125
0.6	1.62651	1.80000	1.17570	1.35000
0.8	1.42066	1.6125	1.02546	1.20938
1.0	1.11237	1.27300	.79159	.95475
1.3	.74262	.83750	.53871	.62813
1.8	.49067	.57500	.36308	.48125
2.5	.32711	.41500	.24640	.31125
3.5	.21410	.29100	.16488	.21825
5.0	.13302	.21120	10639	.15825

$$* w(r) = \frac{pa}{E} \bar{w}(r)$$

Table 2.2 Effect of aspect ratio of the interface element on the number of iterations required for convergence. $K_{sp} = 100000$.

τ_y/E_{ss} Aspect Ratio L_J/t_J	0	1.0	3.0	$\frac{K_{sp}}{(E_{ss}/t_J)}$
	No. of Iterations Req'd.	No. of Iterations Req'd.	No. of Iterations Req'd.	
10	3	0	0	200
100	7	6	4	20
1000	29	28	27	2
10000	182	179	172	.2
100000	Does not converge	-	-	.02

* Convergence tolerance of .00001 has been used.

where K_{sp} = Spring constant

E_{ss} = Joint shearing modulus

t_J = Joint thickness

τ_y = Limiting shear strength

L_J = Length of the joint element

Table 2.3 Effect of spring stiffness (K_{sp}) on the number of iterations required for convergence. For aspect ratio = 100.0

K_{sp}	τ_y/E_{ss}	$\frac{K_{sp}}{(E_{ss}/t_J)}$
	No. of iterations required.	
100000	7	2.0
50000	10	10
10000	36	2
5000	69	1
1000	182	.2

Table 2.4 Effect of aspect ratio of the interface element (L_J/t_J) on the number of iterations required for "convergency" displacements. $K_{sp}/(E_{ss}/t_J)$ is kept constant.

Aspect ratio τ_y/E_{ss}	0.0	$K_{sp}/(E_{ss}/t_J)$
	No. of iteration Reqd.	
10	6	20
100	7	20
1000	7	20
10000	10	20
100000	28	20

Table 2.5 Effect of τ_y/E_{ss} on the convergence of the iterative process. For aspect ratio = 100.0.

τ_y/E_{ss} No. of iterations required	0.0	1.0	3.0
	$w_0 \times 10^{-1}$ (INCH)	$w_0 \times 10^{-1}$ (INCH)	$w_0 \times 10^{-1}$ (INCH)
0	.304025	.304025	.304025
1	.376314	.352650	.305322
2	.394150	.364647	.305641
3	.39855	.367606	.305720
4	.399636	.368337	.305740
5	.399904	.368517	
6	.399970	.368562	
7	.399986		

* Convergence tolerance used is .00001

Table 2.6 Effect of τ_y/E_{ss} on the convergence of the iterative process. For aspect ratio = 10.0.

τ_y/E_{ss} No. of iterations required	0.0	1.0	3.0
	$w_0 \times 10^{-1}$ (INCH)	$w_0 \times 10^{-1}$ (INCH)	$w_0 \times 10^{-1}$ (INCH)
0	.387643	.387643	.387643
1	.399587		
2	.399966		
3	.399978		

Table 2.7 Effect of (τ_y/E_{ss}) on the load-displacement curves. For $k_{sp}/(E_{ss}/t_J) = 20.0$.

Load, p(lbs.)	τ_y/E_{ss}	0	1	4
		Displacement w_0 (inch)	w_0 (Inch)	w_0 (Inch)
500.0		$.499878 \times 10^{-1}$	$.380030 \times 10^{-2}$	$.380030 \times 10^{-2}$
1000.0		$.999923 \times 10^{-2}$	$.760061 \times 10^{-2}$	$.760061 \times 10^{-2}$
1500.0		$.149989 \times 10^{-1}$	$.118577 \times 10^{-1}$	$.114009 \times 10^{-1}$
2000.0		$.199985 \times 10^{-1}$	$.168565 \times 10^{-1}$	$.152012 \times 10^{-1}$
4000.0		$.399986 \times 10^{-1}$	$.368562 \times 10^{-1}$	$.304025 \times 10^{-1}$
6000.0		$.599978 \times 10^{-1}$	$.568566 \times 10^{-1}$	$.474322 \times 10^{-1}$
8000.0		$.799969 \times 10^{-1}$	$.768557 \times 10^{-1}$	$.674303 \times 10^{-1}$

K_{sp} = Spring constant

E_{ss} = Joint shearing modulus

t_J = Joint thickness

τ_y = Limiting shear

Table 2.8 Effect of stiffness ratio (K_e/K_i) and the energy ratio (U_e/U_i) on the convergency of displacements.

K_e/K_i	Vertical displacement at node a $\times 10^{-1}$ (Inch)	Vertical displacement at node b $\times 10^{-2}$ (Inch)	No. of iterations required.	Energy Ratio U_e/U_i
0.0	.499813	4.99813	0	0
.05	.476388	3.88603	0	.043406
.075	.472505	3.25517	22	.054586
.1	.472488	2.57498	22	.057574
.2	.472453	1.41904	16	.063457
.3	.472447	.979154	14	.065679
.5	.472435	.604578	11	.067589
.6	.472428	.507550	10	.068091
.7	.472433	.437229	10	.068433
.8	.472423	.384189	9	.068721
.9	.472428	.342494	9	.068921
1.0	.472431	.308973	9	.069084
2.0	.472430	.156005	8	.069763
5.0	.472426	.0628116	7	.070221
10.0	.472429	.0315442	7	.070330
20.0	.472430	.0157540	7	.070449
100.0	.472430	.0031534	7	.070508
10000.0	.472430	.000031541	7	.070521

K_e = Stiffness of the exterior spring.

K_i = Stiffness of the interior spring.

U_e = Energy stored in the exterior spring.

U_i = Energy stored in the interior spring.

Table 2.9 Variation of vertical stress σ_z (psi) with radius. Depth = 5 ft. Comparison of the results obtained in the present study with that of Ghaboussi et al. (1973).

Radial Distance in Ft.	With Slip		Without Slip	
	Ghaboussi et al. (1973) (psi)	Present Analysis (psi)	Ghaboussi et al. (1973) (psi)	Present Analysis (psi)
1	45.50	44.45	48.50	47.33
2.5	45.75	44.96	48.75	47.72
3.5	47.25	46.77	51.00	50.55
4.5	53.00	52.83	57.25	56.96
5.25	55.00	55.29	59.00	60.03
5.75	51.00	51.35	54.75	54.91
6.25	45.00	45.41	47.00	47.17
7.0	36.50	36.60	37.00	36.87
8.0	22.0	21.66	22.00	21.73
10.5	4.0	3.54	7.00	1.34

Figure 2.10 Effect of interface conditions on the central displacement of a circular plate resting on an isotropic, homogeneous and elastic half-space

Poisson's Ratio	R I G I D P L A T E					
	F.E.M.			T H E O R E T I C A L		
	Smooth Interface $\bar{w}_S(0)$	Bonded Interface $\bar{w}_B(0)$	$\frac{\bar{w}_B(0)}{\bar{w}_S(0)}$	Smooth Interface $\bar{w}_S(0)$	Bonded Interface $\bar{w}_B(0)$	$\frac{\bar{w}_B(0)}{\bar{w}_S(0)}$
0.0	1.29059	1.17149	0.908	1.57080	1.42980	.910
0.40	1.06970	1.05148	0.983	1.31947	1.30716	.991
F L E X I B L E P L A T E						
0.0	1.82481	1.52225	0.834	2.0	1.68169	0.841
0.40	1.49402	1.47886	0.990	1.68	1.65029	0.982

* Note: Subscripts S and B stands for smooth interface and bonded interface respectively.

3. ANALYSIS OF CIRCULAR PLATES RESTING ON HOMOGENEOUS, ISOTROPIC AND ELASTIC HALF-SPACE.

3.1 General

Circular plates are normally used as raft foundations for structures such as nuclear reactors, storage tanks, silos etc. In the past few years, many investigators solved this interaction problem by using different methods. A complete exposition of these methods is given by Selvadurai (1978). Borowicka (1936) used a power series expansion technique to solve circular plates on isotropic elastic half-space. This same technique was adopted by Benito (1948) to solve circular plates stiffened at the centre by an infinitely rigid concentric region. Subsequently Ishkova (1951), Brown (1969), Zémochkin (1947), Holmberg (1946) and Borden (1965) also utilize the same technique for solving circular plates resting on elastic half-space. Chattopadhyay and Ghosh (1969) employed finite difference method for solving circular plates resting on elastic-half space. The technique was extended subsequently by Chakravorty and Ghosh (1975). Cheung and Zienkiewicz (1965) employed finite element method to investigate slabs and tanks on both elastic half-space and on so-called

Winkler media. Smith (1967) investigated axisymmetric flexure of circular plates resting on idealized Winkler medium or on homogeneous elastic half-space by using finite element method. Subsequently Christian (1968), Gladwell (1973), Carrier and Christian (1973), Bowswell and Scott (1975) and Hooper (1974, 1975) employed finite element method for solving this interaction problem. Selvadurai (1979) employed an energy method for the analysis of axially symmetric flexural interaction of a uniformly loaded circular plate resting in smooth contact with an isotropic elastic half-space and subsequently extended this analysis for thick plates and plates with a restrained edge. With the exception of Hooper (1974, 1975) most of other investigators assumed the interface between plate and elastic half-space to be smooth. Hooper examined the problem by considering an adhesive contact between plate and half-space. In reality the interface is neither perfectly smooth nor perfectly bonded. An axisymmetric circular plate resting on elastic half-space and subjected to a uniform pressure has been solved to show the effect of interface condition on the displacements, contact stresses and bending moments. The interface element developed in section 2.4 is used to simulate the contact

between circular plate and elastic half-space.

Axisymmetric plate bending element as developed in section 2.3 has been used to model the circular plate.

It should be mentioned here that the plate has been considered sufficiently thin compared to its radial dimension to make use of poisson - Kirchhoff's thin plate theory.

The relative rigidity of the circular plate is defined by

$$K_r = \frac{\pi(1-\nu^2) E_b}{6(1-\nu_b^2) E} (h/a)^3 \quad (3.1)$$

where

E = Young's modulus of elastic half-space

E_b = Young's modulus of plate

ν = poisson's ratio of elastic half-space

ν_b = poisson's ratio of plate

h = thickness of the plate

a = radius of the plate.

3.2 Rigid Plates Resting on Homogeneous Elastic Half-Space

Physically a rigid plate has infinite relative rigidity. However, numerical analysis is carried out with a finite plate rigidity because of the difficulty involved in handling infinite numbers with computers. When K_r is made sufficiently large, the circular plate can be considered rigid for all practical purposes. In this analysis, a relative rigidity of 10000.0 has been used to represent a rigid circular plate.

The results relating to the case of rigid plate on elastic half-space are shown in Figs. 3.1 through 3.6. For $\nu=0.0$, the rigid plate displacement for perfectly smooth interface is approximately 10.17% higher than the corresponding displacement for perfectly bonded interface. However, for $\nu=0.40$ the corresponding difference is only 1.73%. It should be mentioned here that as the poisson's ratio approaches a value of 0.5, the difference between these two displacements becomes negligible. This problem has also been solved for $\nu=0.49$ and the results are shown in Fig. 3.6. Figs. 3.2 and 3.3 shows the contact stress distribution under the rigid circular plate for $\nu=0.0$ and $\nu=0.40$. It is seen that when $\nu=0.0$, the contact stress for smooth interface is 12.39% less than the

corresponding value for bonded interface. Again difference becomes quite small when $\nu=0.40$. Figs. 3.3 and 3.4 represents the variation of normal stress along axis of symmetry for both $\nu=0.0$ and $\nu=0.40$. The variations of vertical displacements along the axis of symmetry are shown in Figs. 3.5 and 3.6 for both bonded and smooth interface conditions.

3.3 Flexible Plates Resting on Homogeneous Elastic Half-Space

In this section flexible plates with relative rigidities 10, 1, and .01 has been solved and the corresponding results are shown in Figs. 3.8 through 3.30. Again analysis is carried out for poisson's ratio $\nu=0.0$ and $\nu=0.40$. For $\nu=0.0$ and relative rigidity of 10 the central plate displacement for smooth case is 10.31% higher than for bonded contact. However, on the relative rigidity of the plate is decreased this value increases. Fig. 3.9 shows the surface displacements for both smooth and bonded contact for a relative rigidity of 1 ($K_p=1.0$). It is seen that for $\nu=0.0$ the central plate deflection for smooth contact is 11.59% higher than for bonded contact. When the relative rigidity of the plate is reduced to 0.01 ($K_p=.01$), the circulate plate behaves as a completely

flexible plate. In this case, the central displacement for smooth case is 19.88% higher than the corresponding value for bonded interface condition. However, as poisson's ratio of the elastic solid media is increased, the difference of central displacements for smooth and bonded cases is reduced. For $\nu=0.40$, the variations of central displacements for smooth and bonded interface conditions are 1.66% for $K_r=10.0$, 1.33% for $K_r=1.0$ and 1.03% for $K_r=0.01$. As poisson's ratio approaches a value of 0.5, these differences become insignificant. Figs. 3.10 through 3.13 shows the variations of contact stress distribution for relative rigidities of 10.0 and 1.0. It is seen from the Figures that contact stresses at the central location increases as the relative rigidity decreases. Also the difference between the contact stresses for smooth and bonded interface conditions increases as the relative rigidity of the plate increases. For example, when $\nu=0.0$, the difference is 9.9% for $K_r=1.0$ whereas 12.16% for $K_r=10.0$. However, for $\nu=0.0$ these differences are quite small and become negligible as poisson's ratio approaches 0.5.

For a plate of finite flexibility, bending moments for both radial and circumferential directions are affected

by the interface conditions. Figs. 3.22 through 3.29 show how the interface conditions affects the bending moments M_r and M_θ . It can be seen from the Figures that the influence of interface conditions on moments is maximum when $\nu=0.0$. For $\nu=0.40$, the influence is quite small and becomes negligible as ν approaches 0.5. Also as the relative rigidity of the plate increases the difference of the bending moments between smooth and bonded interface conditions decreases. For example, when $\nu=0.0$ the difference in radial moments M_r (between smooth and bonded cases) is 21.79% for $K_r=1.0$ and 16.33% for $K_r=10.0$. This is also true for circumferential bending moment M_θ .

Table 3.6 shows a numerical comparison of the bending moments M_r and M_θ for smooth and bonded contact between plate and the elastic half-space.

Fig. 3.30 shows the variations of central plate displacements for both smooth and bonded contacts as the relative rigidity is changed. A similar variation in central moment is shown in Fig. 3.31.

Figure 3.1(a) Axisymmetric Circular plate resting on an isotropic, homogeneous and elastic half-space (finite element idealization of the half-space).

Figure 3.1(b) Rigid plate on homogeneous elastic half-space: Influence coefficients for surface displacements.

Figure 3.2 Rigid circular plate on homogeneous elastic half-space: contact stress distribution for $\nu=0.0$.

Figure 3.3(a) Contact stress distribution.
For $\nu=0.40$.

Figure 3.3(b) Variation of normal stress with depth.
For $\nu=0.40$.

Figure 3.4 Variation of normal stress with depth for $\nu=0.0$

Figure 3.5 Variation of vertical displacements with depth for $\nu=0.40$

Figure 3.6 Rigid plate on homogeneous elastic half-space
 Influence coefficients for surface displacement

Figure 3.7 Variation of vertical displacements with
 depth for $\nu=0.0$.

Figure 3.8 Flexible plates on homogeneous elastic half-space: Influence coefficients for surface displacements for $K_r=10.0$.

Figure 3.9 Flexible plates on homogeneous elastic half-space: Influence coefficients for surface displacements for $K_r=1.0$.

Figure 3.10 Contact stress distribution for $\nu=0.0$; $K_r=1.0$

Figure 3.11 Contact stress distribution for $\nu=0.40$; $K_r=1.0$

Figure 3.14 Variation of vertical displacement with depth for $v=0.0$; $K_r=1.0$.

Figure 3.15 Variation of vertical displacement with depth for $v=0.40$; $K_r=1.0$.

Figure 3.16 Variation of vertical displacement with depth for $v=0.0$; $K_r=10.0$.

Figure 3.17 Variation of vertical displacements with depth for $v=0.40$; $K_r=10.0$.

Figure 3.18 Variation of normal stress with depth for $\nu=0.0$; $K_r=1.0$.

Figure 3.19 Variation of normal stress with depth for $\nu=0.40$; $K_r=1.0$.

Figure 3.20 Variation of normal stress with depth for $\nu=0.0$; $K_r=10.0$

Figure 3.21 Variation of normal stress with depth for $\nu=0.40$; $K_r=10.0$

Figure 3.22 Variation of flexural moment $\bar{M}_r$ for $v=0.0$; $K_r=1.0$.

Figure 3.23 Variation of flexural moment $\bar{M}_r$ for $v=0.40$; $K_r=1.0$.

Figure 3.24 Variation of flexural moment $\bar{M}_r$ for $v=0.0$; $K_r=10.0$.

Figure 3.25 Variation of flexural moment $\bar{M}_r$ for $v=0.40$; $K_r=10.0$.

Figure 3.26 Variation of flexural moment $\bar{M}_\theta$ for $\nu=0.0; K_r=1.0$

Figure 3.27 Variation of flexural moment $\bar{M}_\theta$ for $\nu=0.40; K_r=1.0$

Figure 3.28 Variation of flexural moment $\bar{M}_\theta$ for $\nu=0.0$; $K_r=10.0$

Figure 3.29 Variation of flexural moment $\bar{M}_\theta$ for $\nu=0.40$; $K_r=10.0$.

Figure 3.30 Effect of relative rigidity K_r on the central plate displacement.

(a)

(b)

Figure 3.31(a) & (b) Effect of relative rigidity K_r on the central plate moment.

Table 3.1 Rigid Circular Plate on Homogeneous Elastic Half-space: Influence Coefficients for Surface Displacement.

$K_r = 10000.0$

$$w(r) = \frac{pa}{E} \bar{w}(r)$$

	$\nu=0.0$		$\nu=.40$	
r/a	ROUGH $\bar{w}(r)$	SMOOTH $\bar{w}(r)$	ROUGH $\bar{w}(r)$	SMOOTH $\bar{w}(r)$
0.0	1.17149	1.29059	1.05148	1.06970
0.05	1.17149	1.29059	1.05148	1.06970
.1	1.17149	1.29059	1.05148	1.06970
.2	1.17149	1.29059	1.05148	1.06970
.3	1.17149	1.29059	1.05148	1.06970
.45	1.17149	1.29059	1.05148	1.06970
.6	1.17149	1.29059	1.05147	1.06969
.8	1.17149	1.29058	1.05147	1.06968
1.0	1.17149	1.29057	1.05146	1.06967
1.3	.77250	.79085	.64578	.64784
1.8	.50153	.50498	.41843	.41847
2.5	.33100	.33176	.27832	.27815
3.5	.21554	.21567	.18338	.18320
5.0	.13360	.13354	.11609	.11592

Table 3.2 Rigid Circular Plate on Homogeneous Elastic Half-space: Variation of Vertical Displacement with Depth Z

$K_r = 10000.0$

	$\nu = 0.0$		$\nu = .40$	
r/a	ROUGH σ_z/p	SMOOTH σ_z/p	ROUGH σ_z/p	SMOOTH σ_z/p
.025	.4503	.3945	.4065	.4188
.075	.4511	.3945	.4015	.4109
.15	.4543	.3945	.3995	.4044
.25	.4614	.4024	.4016	.4046
.375	.4775	.4168	.4146	.4146
.525	.5088	.4446	.4398	.4344
.70	.5787	.5117	.4990	.4831
.90	.7675	.8276	.7653	.9466

Table 3.3 Rigid Circular Plate on Homogeneous Elastic Half-space. Variation of Vertical Displacements with Depth :

$K_r = 10000.0$

$$w(z) = \frac{pa}{E} \bar{w}(z)$$

	$\nu=0.0$		$\nu=.40$	
z/a	ROUGH $\bar{w}(z)$	SMOOTH $\bar{w}(z)$	ROUGH $\bar{w}(z)$	SMOOTH $\bar{w}(z)$
0.0	1.17149	1.29059	1.05148	1.06970
.1	1.12649	1.25109	1.03249	1.05892
.3	1.03959	1.17032	.99361	1.03427
.6	.31686	1.04031	.92165	.97057
1.0	.76924	.86667	.80489	.84848
1.7	.57073	.62928	.61427	.64165
2.7	.39499	.42515	.42777	.44148
4.0	.26782	.28301	.29149	.29823
6.0	.16802	.17509	.18505	.18820
3.0	.0493	.09803	.10512	.10699

Table 3.4 Rigid Circular Plate on Homogeneous Elastic Half-space: Variation of Normal Stress with Depth z .

$K_r = 10000.0$

z/a	$\nu = 0.0$		$\nu = .40$	
	ROUGH σ_z/p	SMOOTH σ_z/p	ROUGH σ_z/p	SMOOTH σ_z/p
.05	.4503	.3945	.4065	.4188
.2	.4346	.4043	.4025	.4181
.45	.4093	.4334	.4051	.4351
.8	.3692	.4342	.3854	.4252
1.35	.2835	.3390	.2998	.3296
2.2	.1757	.2040	.1769	.1886
3.35	.0978	.1093	.0970	.1017
5.0	.0499	.0540	.0493	.0510
7.5	.0244	.0257	.0246	.0252

Table 3.5 Flexible Circular Plate on Homogeneous Elastic Half-space: Influence Coefficients for Surface Displacements.

$$K_r = 1.0$$

r/a	$\nu=0.0$		$\nu=0.40$	
	ROUGH $\bar{w}(r)$	SMOOTH $\bar{w}(r)$	ROUGH $\bar{w}(r)$	SMOOTH $\bar{w}(r)$
0.0	1.28035	1.42872	1.18250	1.19789
.05	1.27991	1.42819	1.18200	1.19739
.1	1.27860	1.42659	1.18049	1.19591
.2	1.27337	1.42022	1.17448	1.18997
.3	1.26476	1.40976	1.16458	1.18021
.45	1.24595	1.38691	1.14295	1.15886
.6	1.22100	1.35663	1.11422	1.13053
.8	1.18069	1.30778	1.06776	1.08477
1.0	1.13666	1.25450	1.01699	1.03481
1.3	.76249	.78088	.63686	.63914
1.8	.49852	.50198	.41571	.41585
2.5	.33003	.33078	.27739	.27726
3.5	.21521	.21534	.18305	.18290
5.0	.13349	.13343	.11599	.11582

Table 3.6 Flexible Circular Plates on Homogeneous Elastic Half-space: Variation of Radial Moments M_r .

$$M_r = pa^2 \bar{M}_r$$

$\nu=0.0$			$\nu=0.40$		
r/a	ROUGH $\bar{M}_r$	SMOOTH $\bar{M}_r$	SMOOTH $\bar{M}_r$	ROUGH $\bar{M}_r$	
0.0	.07259	.08841	.08343	.08226	$K_r=1.0$
.05	.07247	.08825	.08331	.08216	
.1	.07229	.08793	.08308	.08196	
.2	.07043	.08559	.08102	.07997	
.3	.06737	.08170	.07751	.07648	
.45	.05957	.07216	.06874	.06773	
.6	.04804	.05875	.05622	.05521	
.8	.03341	.03413	.03300	.03217	
1.0	0.0	0.0	0.0	0.0	
0.0	.9565	.11127	.11007	.10795	$K_r=10.0$
.05	.09542	.11099	.10981	.10771	
.1	.09491	.11039	.10924	.10715	
.2	.09199	.10704	.10597	.10396	
.4	.08720	.10152	.10054	.09859	
.45	.07581	.08853	.08774	.08591	
.6	.06039	.07074	.07016	.06847	
.8	.03367	.03970	.03941	.03819	
1.0	0.0	0.0	0.0	0.0	

Figure 3.7 Flexible Circular Plate on Homogeneous Elastic Half-space: Influence Coefficients for Surface Displacements.

$$K_r = 0.01$$

	$\nu = 0.0$		$\nu = .40$	
	ROUGH	SMOOTH	ROUGH	SMOOTH
r/a	$\bar{w}(r)$	$\bar{w}(r)$	$\bar{w}(r)$	$\bar{w}(r)$
0.0	1.52225	1.82481	1.47886	1.49402
.05	1.52182	1.82395	1.47833	1.49357
.1	1.52043	1.82127	1.47665	1.49208
.2	1.51272	1.80825	1.46748	1.48347
.3	1.49830	1.78458	1.45010	1.46658
.45	1.45891	1.72307	1.40307	1.42006
.6	1.38577	1.62651	1.32791	1.34491
.8	1.25203	1.42066	1.16258	1.17809
1.0	1.01864	1.11237	.90394	.91633
1.3	.72995	.74262	.60861	.61067
1.8	.48894	.49067	.40726	.40748
2.5	.32696	.32711	.27452	.27444
3.5	.21420	.21410	.18206	.18193
5.0	.13318	.13302	.11565	.11550

4. CABLE JACKING TEST

4.1 General

The cable⁰ method of in-situ testing provides an efficient technique for the determination of the in-situ deformability characteristics of soil and rock masses (Selvadurai, 1979). It is essentially a plate loading test in which the load is provided by reaction against a steel cable which is anchored at some depth in a small diameter borehole. This method was originally suggested by Stagg and Zienkiewicz (1966). The only available theoretical solution for this problem was developed by Selvadurai (1979) considering the rock mass as isotropic and homogeneous. He also extended this solution for rock masses idealized as a transversely isotropic elastic half-space (Selvadurai, 1980). However, both solutions were developed assuming the interface between the test plate and the rock mass (isotropic or transversely isotropic) to be smooth but capable of sustaining tensile tractions. This smooth interface condition is not always physically admissible. For example, when concrete loading pad is used for cable jacking test, the interface between the concrete block and the rock mass may be far from being smooth and shear traction is likely to be developed at the interface. This chapter shows the influence of the interface conditions on the displacements and stresses

for this plate-anchor interaction problem. The axisymmetric interface element developed in section 2.4 has been used to model the interface between the test plate/block and the rock mass. It should be mentioned here that in the present analysis the rock mass is assumed to be homogeneous and linearly elastic.

4.2 Rigid Test Plates Anchored to an Isotropic Rock Mass

Fig. 4.1 shows a rigid circular plate anchored to an isotropic rockmass. The test plate diameter is represented by $2a$ while depth of anchor and anchor length is presented by αa and βa respectively. The anchor load is assumed to be uniformly distributed along the anchor length. The analysis has been performed for $\nu=0.0$ and $\nu=0.49$ and the results are shown in Figs. 4.2 through 4.15. It should be noted here that only two limiting interface conditions (perfectly bonded and perfectly smooth) are checked in this analysis.

Figs. 4.2 through 4.4 shows variation of rigid plate displacement for different α and β values. The poisson's ratio of the rock mass is taken as $\nu=0.0$. It is seen that as the value of α increases, the difference between the displacements for bonded and smooth interface

also increases. However, beyond $\alpha=8$, this difference in the displacements becomes almost constant. It should be noted that the percentage of this difference is almost the same for different α values. It is seen from Fig. 4.2 that, for $\alpha=8$ the rigid plate displacement for bonded interface condition is 9.63% lower than the displacement for smooth interface condition. For $\alpha=2$, the corresponding difference is about 10%. Fig. 4.4 shows the influence of the parameter β on the rigid plate displacements. It can be seen that for lower values of α , the difference in the plate displacements for $\beta=.5$ and $\beta=2.0$ is quite appreciable. However, as α is made large, the influence of β on the rigid plate displacements is reduced. Figs. 4.7 and 4.8 shows the variations of normal displacement along the line of symmetry. It can be seen that the interface conditions only affect the portion compressed by the plate load. Figs. 4.13 through Figs. 4.16 shows some results when poisson's ratio $\nu=0.49$. However, for lower values of α rigid plate displacements for bonded and smooth contact deviates substantially. For example, when $\alpha=.5$, $\beta=2.0$, the displacement for bonded contact is 10.16% higher than the corresponding displacement for smooth contact.

4.3 Deformable Testing Block Anchored to an Isotropic Elastic Rock Mass

Fig. 4.16 shows a cylindrical block of diameter '2a' anchored to an isotropic elastic rock mass. The anchor region is located at a depth of 'aa' below the surface of the half-space and the anchor length is represented by '8a'. It is assumed that the anchor load is uniformly distributed along the length of the anchor. Analysis is carried out for two limiting interface conditions i.e. perfectly bonded interface condition and perfectly smooth interface condition. To take into account the effect of the deformability of the test block, the analysis has been carried out for modular ratios (ratio of the block modulus to the rock modulus) of 1.0 and 4.0. Figs. 4.17 through 4.21 shows the results of the analysis for poisson's ratio $\nu=0.0$.

Fig. 4.17 shows the variations of central displacement for different values of α . It is seen that, when $E_B/E_R = 1.0$, the difference of the central displacements for smooth and bonded contact is about 17.7%, for $\alpha=8$, while the corresponding difference is about 15.8% for $E_B/E_R = 4.0$. Figs. 4.20 Shows the variations of surface displacement with radial distance. It is seen that interface

conditions only influence the displacement below the block. For $r/a > 1.0$ the surface displacement is not affected by the interface condition at all. Variations of the vertical displacement along the line of symmetry are shown in Figs. 4.20 and 4.21. Figs. 4.22 through 4.26 shows some of the results for poisson's ratio of 0.45. It can be seen that for higher poisson's ratios the effects of the interface conditions on displacements is reduced. For $\nu=0.5$, the effect of interface conditions does not affect the results. Fig. 4.22 shows the variations of central displacement with α . It is seen that, the difference between the displacements for smooth and bonded interface conditions is substantially less than the corresponding difference for $\nu=0.0$. Fig. 4.22 shows that, for $E_B/E_R = 1.0$ and $\alpha=8$, the central displacement for bonded case is about 5% lower than the displacement for smooth case. It should be mentioned here that the corresponding difference was about 17.7% for $\nu=0.0$. Variation of vertical displacement along the line of symmetry is shown in Fig. 4.26. It can be seen that the difference between the results obtained for smooth and bonded case is quite small. It may be noted that as the poisson's ratio $\nu \rightarrow 0.5$, this difference vanishes.

Figure 4.1 Geometry of the plate and anchor region.

Figure 4,2 Variations of rigid plate displacements with α for $\nu=0.0$; $\beta=0.5$.

Figure 4.3 Variation of rigid plate displacements with α for $\nu=0.0$; $\beta=2.0$.

Figure 4.4 Variations of rigid plate displacements with α for $\nu=0.0$. Effect of β .

Figure 4.5 Influence coefficients for surface displacement for $\nu=0.0$

Figure 4.6. Influence coefficients for surface displacements for $\nu=0.0; \beta=2.0$

Figure 4.7 Variations of vertical displacements with depth for $\nu=0.0$. Effect of α .

Figure 4.8 Variations of vertical displacements with depth for $v=0.0$. Effect of β .

Figure 4.9 Contact stress distribution for $\nu = .0; \alpha = 1.0$. Effect of β .

Figure 4.10 Contact stress distribution for $\nu=0.0$; $\beta=0.5$. Effect of α

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Figure 4.11 Contact stress distribution for $\nu=0.0$; $\beta=0.5$. Effect of α .

Figure 4.12 Contact stress distribution for $\nu=0.0$; $\alpha=8.0$. Effect of the parameter β .

Figure 4.13 Variation of rigid plate displacement with α for $\nu=0.49$. Effect of β .

Figure 4.14 Influence coefficient of surface displacement for $\nu=0.49$; $\beta=0.5$. Effect of α .

$$W(r) = \frac{p(1-\nu)}{4aG} \bar{W}(r).$$

Figure 4.15 Influence coefficients of surface displacements for $\nu=0.49; \beta=2.0$. Effect of α .

Figure 4.16 Geometry of the testing block and anchor region.

Figure 4.17 Variations of central block displacements (at block base) with α for $\nu=0.0$. Effect of E_B/E_R .

Figure 4.18 Variations of central block displacements (at block top with α for $\nu=0.0$. Effect of E_B/E_R .

Figure 4.19 Influence coefficients of surface displacement for $\nu=0.0$. Effect of E_B/E_R ratio.

Figure 4.20 Variations of vertical displacements with depth for $\nu = .0$; $\alpha = 8.0$. Effect of E_B/E_R ratio.

Figure 4.21 Variations of vertical displacements with depth for $\nu=0.0$; $\alpha=4.0$. Effect of E_B/E_R ratio.

Figure 4.22 Variations of central block displacements (at block base) with α for $\nu=0.45$.

Figure 4.23 Variations of central block displacements (at block top) with α for $\nu=0.45$.

Figure 4.24 Influence coefficients of surface displacements for $\nu=0.45$; $\alpha=4.0$. Effect of E_B/E_R ratio.

Figure 4.25 Influence coefficients of surface displacements for $\nu=0.45$; $\alpha=8$.-- Effect of E_B/E_R ratio.

Figure 4.26 Variations of vertical displacements with depth for $\nu=0.45$; $\alpha=8.0$. Effect of E_B/E_R ratio.

Table 4.1 Rigid test plate anchored to an isotropic, homogeneous and elastic solid medium. Influence coefficients for surface displacements.

r/a	$\nu=0.0$		$\nu=0.49$	
	SMOOTH $\bar{W}(r)$	ROUGH $\bar{W}(r)$	SMOOTH $\bar{W}(r)$	ROUGH $\bar{W}(r)$
0.0	.739421	.668658	.537895	.534787
0.15	.739417	.668653	.537890	.534783
0.40	.739403	.668640	.537870	.534762
0.70	.739379	.668617	.537838	.534729
1.00	.739354	.668592	.537802	.534693
1.30	.435728	.422677	.344330	.322130
1.80	.258390	.256164	.185290	.184813
2.50	.152641	.152203	.095784	.095755
3.50	.084530	.024542	.038588	.038629
5.00	.040501	.040714	.007395	.007421

Table 4.2 Deformable testing block anchored to an isotropic, homogeneous and elastic solid medium. Influence coefficients $\bar{W}(r)$ for surface displacements. For $E_B/E_R=4.0$; $\alpha=4.0$.

	$\nu=0.0$		$\nu=0.45$	
	SMOOTH	ROUGH	SMOOTH	ROUGH
r/a	$\bar{W}(r)$	$\bar{W}(r)$	$\bar{W}(r)$	$\bar{W}(r)$
0.0	3.07806	2.56609	2.17026	2.08704
0.15	3.05949	2.56013	2.16373	2.08271
0.40	2.99164	2.53233	2.11191	2.03160
0.70	2.83166	2.46315	1.37201	1.31484
1.0	2.47470	2.31253	1.65407	1.64880
1.3	1.39399	1.37036	0.87214	0.86790
1.8	0.74981	0.74811	0.42116	0.42199
2.5	0.38083	0.38134	0.16551	0.16637
3.5	0.16971	0.17014	0.03527	0.03570
5.0	0.06081	0.06195	-0.01078	-0.01065

$$W(r) = \frac{2pa}{(E_B + E_R)} \bar{W}(r)$$

Table 4.3 Deformable testing block anchored to an isotropic, homogeneous and elastic solid medium. Variation of vertical displacement with depth z . For $E_B/E_R=1.0$; $\alpha=8.0$

	$\nu=0.0$		$\nu=0.45$	
z/a	SMOOTH $\bar{W}(z)$	ROUGH $\bar{W}(z)$	SMOOTH $\bar{W}(z)$	ROUGH $\bar{W}(z)$
1.0	1.53609	1.26405	1.07971	1.02774
1.2	1.38111	1.13679	1.07403	.97950
1.5	1.14960	.96129	.98895	.88383
2.0	.82746	.71715	.77709	.69996
3.0	.45603	.41355	.44514	.41319
4.0	.17064	.15744	.13104	.12213
7.0	.00066	-.00504	-.11031	-.11367
8.5	-.25563	-.25908	-.56712	-.56916
9.0	-1.07382	-1.07676	-1.42767	-1.42944

$$W(z) = \frac{2pa}{(E_B + E_R)} \bar{W}(z)$$

5. EMBEDDED RIGID CYLINDRICAL ANCHOR

Fig. 5.1 shows a rigid cylindrical anchor embedded to an isotropic, homogeneous and elastic solid medium. The anchor has radius a and length L . The depth of embedment is represented by h . The four noded axisymmetric interface elements (as developed in section 2.4) are used to model the interface between the rigid cylindrical anchor and the surrounding elastic solid medium. It may be noted that in this work, it is assumed that the bottom face of the anchor is in smooth contact with the adjacent elastic continuum. The cylindrical anchor is subjected to a concentrated load P along the axis of symmetry as shown in Fig. 5.1. In order to investigate the influence of the interface nonlinearity on the overall load displacement response of the rigid anchor, the following interface properties are used:

peak shear strength of interface $\tau_y = 25.0$ psi

joint modulus in the normal direction, $E_{nn} = 10^{10}$ psi

joint modulus in the shearing direction, $E_{ss} = 1.0$ psi

The displacement is measured at the centre of the top face of the anchor. The results of the analysis are

shown in Figs. 5.2 through 5.11. Figs. 5.2 and 5.3 show the load-displacement curve for $v=0.0$ and $v=0.49$ respectively. It can be seen that for smooth and bonded interface conditions load-displacement responses are linear while for partial slip the response is nonlinear. Since the peak shear strength is independent of the joint normal stress, it is likely that for partial slip the load-displacement response would be bilinear (see Fig. 2.26 in Chapter 2). However, if the peak shear stress cannot be mobilized to all the joint elements at one time, the load-displacement response would have a curved transition zone (Figs. 5.2 through 5.9). When all the joint elements are sheared off, the load-displacement response follows a parallel line obtained for smooth interface conditions. This is due to the fact that a sheared off joint has zero shearing stiffness.

The nonlinear load-displacement response of a rigid cylindrical anchor is influenced by many factors. Few salient factors are investigated here.

- a) Effect of poisson's ratio: Figs. 5.4 and 5.5 shows the load-displacement curves for $v=0.0$ and $v=0.49$. It is seen that as

poisson's ratio approaches a value of 0.5 (incompressible), the effects of interface nonlinearity on the load-displacement curve diminishes. The maximum effect is observed when the poisson's ratio of the surrounding elastic medium is zero.

- b) Effect of L/a ratio: Fig. 5.6 and 5.7 shows the influence of L/a ratio on the load-displacement curve for a rigid cylindrical anchor.
- c) Effect of h/a (embedment length): Fig. 5.8 shows the load-displacement curves for $h/a = 3.0$ and $h/a = 5.0$. It can be seen that the nonlinearity in the load-displacement response increases as the embedment length h/a decreases.
- d) Effect of the stiffness ratio E_{nn}/E_{ss} : Fig. 5.9 shows the variation of rigid anchor displacement with $-\log (E_{nn}/E_{ss})$ for $\nu=0.0$ and $\nu=0.49$.

Figure 5.1 Geometry of a typical rigid cylindrical anchor embedded to an isotropic homogeneous and elastic solid medium.

Figure 5.2 Load-displacement curve for Rigid Cylindrical anchor for $v=0.0$. Effect of interface nonlinearity.

Figure 5.3 Load-Displacement Curve for Rigid Cylindrical Anchor. For $\nu = 0.49$.

Figure 5.4 Load-displacement curve for rigid cylindrical anchor for $h/a=6.0$; $l/a=3.0$. Effect of poisson's ratio.

Figure 5.5 Load-Displacement curve for a rigid cylindrical anchor.
For $h/a=3.0$; $L/a=5.0$. Effect of Poisson's ratio.

Figure 5.6 Load-displacement curve for Rigid Cylindrical anchor.
For $\nu=0.0$; $h/a=6.0$. Effect of L/a ratio.

Figure 5.7 Load-displacement curve for Rigid Cylindrical anchor.
For $\nu=0.49$; $L/a=6.0$.

Figure 5.8 Load-Displacement curve for Rigid Cylindrical anchor.
For $L/a=5.0$; $\nu=0.49$. Effect of h/a ratio.

Figure 5.9 Effect of (E_{ss}/E_{nn}) on the rigid anchor displacement.
For $h/a=6.0$; $L/a=5.0$.

Figure 5.10 Vertical Displacement of soil beneath the Cylindrical Anchor. For $\nu = 0.49$.

Figure 5.11 Distribution of vertical stress with radial distance for $\nu=0.49$. Depth $z/a=8.0$.

6. CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

1) In the context of the present study, it is evident that the existence of joints and interfaces in a system substantially affects the displacements and the stress distributions within the body.

2) The load displacement response of a system where joints are encountered, is also affected by the nonlinear joint properties. When the peak shear strength is mobilized in all the joint elements, the load-displacement response is bilinear. This is due to the fact that a sheared joint is assigned zero post yield shearing stiffness (see Fig. 2.1). When the peak shear stress is not mobilized in all the joint elements then the load-displacement response would have a curved transition zone (Fig. 5.2).

3) In the analysis of joints with nonlinear properties, Load Transfer method is used as the iterative technique. Under such an iterative scheme, the displacements converge monotonically.

4) If the joints and interfaces constitute a very small portion of linearly elastic system or if the joint stiffness is very much small compared to the overall stiffness of the system, the aspect ratio (defined by the ratio of the length of a joint element to its thickness) of the joint elements do not affect the convergence process substantially.

5) In the context of the present study, the influence of the interface condition is most prominent when $\nu=0.0$. As the poisson's ratio approaches 0.5 (incompressible material), the effect due to the presence of interfaces tends to diminish.

6.2 Recommendations for Future Work

This study has attempted to investigate the influence of joints and interfaces on some selected Geotechnical engineering problems. The study has indicated the need for further research in the following areas:

- 1) In this work, peak shear strength of the joint is assumed to be independent of the joint normal stress. To express the peak shear strength as a function of the normal stress, Mohr-Coulomb criterion

can be utilized. The finite element computer program developed herein could be extended to handle such cases.

- 2) A simple elastic-perfectly plastic shear stress-shear strain relationship is employed in this study. Possible alternatives are bilinear or parabolic relationships. It would be desirable to utilize such stress-strain relationships in the analysis.
- 3) Joint dilatancy and strain softening effects are neglected in the present study. It may be noted that many natural joints and weak planes in a rock mass shows dilatancy and strain softening characteristics. It is therefore recommended that dilatancy and strain-softening effect be investigated.
- 4) The present study assumes that the continuum material on either side of the joint or interface is linearly elastic. The assumption may not be feasible

for materials such as soil or rock masses for all range of loadings. Therefore, it would be desirable to extend the work which assumes the continuum material adjacent to the interface to be nonlinear.

- 5) Displacement formulation of the finite element method is not applicable to a situation where the poisson's ratio of the supporting elastic medium equals 0.5 (incompressible). Special formulation, known as 'hybrid formulation', is required for such a case. The finite element computer program, developed in this work, could be extended to handle such cases.

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APPENDIX A

OUTLINE FLOW DIAGRAM FOR THE
FINITE ELEMENT PROGRAM

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