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Microelectromechanical System (MEMS) Simulation

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A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Master of Applied Science

Ottawa-Carleton Institute for Electrical Engineering,
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June 2001

Abstract

Microelectromechanical systems (MEMS) depend on the interaction between electrical, mechanical and fluidic forces. Since most MEMS devices are innately three-dimensional and geometrically complicated, simulating these coupled domain problems involves many challenges. This thesis presents a methodology to simulate MEMS. Reduced order dynamic macromodels are automatically generated from three-dimensional, domain-specific, physical simulations. Specifically, the Arnoldi method for model order reduction, commonly used for high-speed circuit interconnect problems, is extended to microelectromechanical system (MEMS) problems. Linear models are used to simulate single and coupled domain MEMS problems. The automatic extraction of reduced order models is demonstrated. The model order reduction algorithm is applied to a representative set of MEMS operating in a linear region. The results show that very-low order reduced models can capture the dynamics of the large linear systems. The nonlinear behavior of MEMS is also addressed. Waveform relaxation techniques for self-consistent analysis

of three-dimensional MEMS are presented. Finally, techniques for nonlinear model order reduction of MEMS are discussed.

In Memory of Bear...

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List of Symbols

$\Delta \mathbf{P}$ components of a force vector acting on an infinitesimal area ΔA

ΔA area

τ shear stress

σ normal stress

ϵ extensional or normal strain

$\mathbf{S}$ symmetric stress matrix

ρ density

$\mathbf{K}$ volume (body) forces

γ shear strain

ν Poisson's ratio

E Young's modulus

G shear modulus

u, v, w	displacement/deformation components
t	time
p	pressure
p_a	ambient pressure
θ	angle
I	moment of inertia
$\bar{\rho}$	radius of curvature
M	internal bending moment
$\bar{V}$	shear force
$\bar{q}$	force perunit length
P_o	uniform pressure load
σ_o, S	residual stress
σ_a	axial stress
q_0	per unit length load
W	width
H	height
F_{elec}, F_e, F	electrostatic force
q	surface charge density at point x

ϕ	electrostatic potential at point x
ρ	electric charge density
$\mathbf{E}$	electric field intensity
$\mathbf{D}$	electric flux density
$\mathbf{B}$	magnetic flux density
$\mathbf{H}$	magnetic field intensity
$\mathbf{J}$	current density
ϵ_e	permittivity
μ_e	permeability
σ_e	conductivity
$G(x, x')$	Green's function
dS'	incremental surface area
ϵ_o	permittivity of free space
F	finger driving force
h	finger height
V	voltage
g_f	finger spacing
U_e^*	electrostatic co-energy

C	capacitance
A	area
l	length
w	width
d	distance
ρ_m	density of fluid
U	velocity
P	pressure
g	acceleration due to gravity
η, μ	viscosity
K	Knudsen number
u	displacement
h_0	initial gap height
$k(A, R, n)$	Kyrlov subspace

List of Abbreviations

MEMS	Microelectromechanical Systems
FEM	Finite Element Method
FD	Finite Differences
AWE	Asymptotic Waveform Evaluation
PRIMA	Passive Reduced-Order Interconnect Macromodeling Algorithm
PDE	Partial Differential Equation
ODE	Ordinary Differential Equation
CAD	Computer Aided Design
IC	Integrated Circuit
CPU	Central Processing Unit
VLSI	Very Large Scale Integration

Chapter 1

Introduction

1.1 Introduction to MEMS

Microelectromechanical system (MEMS) technology offers unique opportunities for scientific and technological progress. The explosive growth of new process technologies, new device configurations and new markets for microsensors, microactuators and microsystems makes MEMS a very exciting field with great promise for the future. The advantages that follow miniaturization extend far beyond the dramatic decreases in volume and weight. Microelectromechanical devices and systems are inherently faster and usually more precise than their macroscopic counterparts. MEMS provide new

functionality, increased performance, improved reliability and decreased cost. MEMS uniquely enable complete system control on the micro or nano scale. MEMS sense, compute and actuate functionality enables contingency tolerance and performance enhancement. MEMS have supported large families of highly successful products. They form the front-ends of global information networks and bridge microelectronics with biotechnology.

Due to the considerable overlap between physical domains in terms of applications, no rigid domain based taxonomy for MEMS has been developed. Microelectromechanical systems are known as microsystems in Europe and micromachines in Japan. A microelectromechanical system contains inseparably combined, integrated mechanical and electronic components. Complete microelectromechanical systems are sensors and actuators integrated with signal conditioning and feedback-control circuitry.

Individual MEMS components include transducers, a term encompassing both sensors and actuators, which convert one form of energy to another. These miniature devices often convert physical parameters to and from electrical signals. MEMS structures include gears, motors and deformable thin membranes. MEMS based actuators perform useful work in the form of mechanical motion on the environment in response to a command or a control

Figure 1.1: Microelectromechanical System Components

signal. An important class of MEMS devices use electrostatic forces as a means of actuation. The electrical actuation method relies on the coulombic attraction/repulsion between charged bodies. Electrostatic actuation uses attraction between dissimilar charges to exert a force on a deformable or mobile body. A typical MEMS electrostatic actuation device, the comb-drive shown in Figure 1.2, can be envisioned as a collection of thin beams and plates with cross-sections on the order of microns (μm) and lengths on the order of tens of microns. The comb-drive consists of two sets of capacitor banks, each containing a number of capacitors in parallel [1]. The two capacitor banks can be excited with a voltage sequentially to produce a push-pull

Figure 1.2: Comb Drive

effect. If the capacitance as a function of time of one of the capacitor banks is monitored, it can be used as a sense arm to sense the magnitude and frequency of the actuation. Electrostatic comb drives are used in a wide variety of applications including optical switches [2] and micromechanical resonators [3] to avoid the drawbacks of parallel-plate capacitor actuators [4].

Since MEMS have a small physical size, significant forces and, thus, deformations are obtained from the application of voltages around ten volts. When many elements are arranged in series, deformations are magnified. If the elements are arranged in parallel, the forces are cumulative. Aside from

sensing, controlling and actuating on the micro scale, MEMS can function in arrays to generate effects on a macro scale.

The application area of MEMS has grown past microsensors and microactuators: the range of applications appears limitless. Current MEMS devices form a new class of mechanical, optical, fluid, thermal and chemical machines. Examples include picosatellites, microturbomachinery and micro-rocket engines [5]. In essence, most MEMS structures are integrated movable structures for precision motion and actuation.

The materials and techniques commonly applied to MEMS are closely related to very large scale integration (VLSI) technology. Early microstructures were fabricated with polycrystalline silicon with sacrificial layers to release movable elements [6]. However, since MEMS actuators require three-dimensional geometries, MEMS structures are now machined on semiconductor wafers using a combination of well-known integrated circuit (IC) processes and advanced micromachining techniques [7],[8]. Suitable materials must have desirable electromechanical material properties. In particular, small structures made of these materials must maintain their shapes when released; therefore, residual stresses in the released structures must be low [9]. In general, it is difficult to obtain materials to fit both the stress and

compatibility criteria.

MEMS have a long history dating back to the earliest days of micro-electronics. After more than twenty-five years, the field has reached critical mass. Full production of MEMS based inkjet print heads, pressure sensors, flowmeters, accelerometers, gyros, uncooled infrared imagers, and optical projection displays occurred in the past decade. The most advanced devices include DNA analyzers, integrated gas chromatography systems and miniature mass spectrometers [9]. MEMS design cuts across most disciplines in Engineering and Science. To solve problems at the micro level, the MEMS designer employs principles of materials, mechanics, fluidics and electronics. This thesis addresses the complex range of problems involved in simulating entire systems at the micro domain.

1.2 Motivation

When compared to the IC industry, MEMS design, production, implementation and performance quality control is more complicated for any single application. However, MEMS have a vastly broader and more diverse range of utility [7]. While many MEMS devices contain only a few electrical and

mechanical elements, Figure 1.3 illustrates the trend toward more advanced devices with hundreds or thousands of components.

Figure 1.3: Bright Future for MEMS

MEMS designers require a platform to create and optimize MEMS designs. If such a design package existed, it would lead to improved device performance and higher yields with fewer fabrication trials, reduced development time and lower manufacturing costs. The state of the art in MEMS CAD lags significantly behind the state of the art in device technologies: only a small fraction of the total potential MEMS has been explored. At this time, accurate and efficient MEMS simulation requires new mathematical formulations, analysis techniques and simulation strategies. Designers must be free to be creative, thus, to perform many simulations of advanced

structures and systems.

MEMS included coupled subsystems based on distinct physical principles such as electronics, mechanics, optics or fluids. To simulate most structures, it is necessary to capture the complex multi-energy domain physics of many components: this task falls far beyond the capacity of traditional finite or boundary element analysis. Additional problems include nonlinearities and instabilities due to the electrostatic forces and the small physical size of MEMS. Many commercially available finite element method (FEM) solvers use high-resolution volume discretizations to simulate mechanical structures with complex geometries. However, an accurate solution in the presence of large aspect ratios, typically found in MEMS, requires prohibitive amounts of memory and computation time.

Presently, MEMS designers often resort to ad-hoc approaches that include capacitance modeling for the electrostatics and simple lumped, mass-spring-damper, models for dynamics. This approach does not work for even moderately complicated structures or when nonlinear behavior, including the coupling of physical domains, is important.

MEMS require quasistatic, frequency domain and general dynamical behavior analyses. Most of the current approaches for developing CAD tools for

MEMS [10] use full, self-consistent, three-dimensional simulation of coupled electrostatic-elastostatic devices. Accuracy is preserved with these implementations, but the computational demands associated with dynamic analysis quickly overwhelm typical workstations.

To solve this problem, the dynamical behavior must be solved using low-order models that agree with full three-dimensional analysis. Thus, capturing the complex behavior that is only discovered through experiment or full three-dimensional simulation is the great challenge for MEMS designers. Low-order models are used in lieu of computationally expensive full three-dimensional analysis. When reduced-order macromodels are available, it is possible to solve system level dynamics without a significant computational effort. At the same time, since the macromodels agree with the full three-dimensional simulations, they can provide accuracy sufficient for design purposes. At the moment, macromodel construction is tedious and problem specific [11]. It is the main focus of this research to further develop macromodeling techniques for MEMS simulation. Specifically, congruent transformation techniques for macromodeling of MEMS are investigated.

Also, current CAD tools share an architecture in which fast numerical algorithms are used to optimize simulation in each energy domain separately

[10]. Then, iteration is used to obtain a self-consistent solution for the behavior of the system. Waveform relaxation techniques are proposed to provide convergent and correct solutions given the nonlinearities and characteristics of MEMS structures.

1.3 Objectives

The thesis addresses the simulation of microelectromechanical systems. The main objectives are summarized below.

1. The heterogeneous structure of microelectromechanical systems mandates the intensive use of mixed-mode/multi-level simulators. This research provides the framework and develops supporting algorithms to extend successful methodologies found in state-of-the-art integrated circuit design to MEMS design.
2. Strategies are derived to manage the computational complexity of MEMS subjected to linearity assumptions. The Arnoldi method for model order reduction, successfully used for high-speed interconnect problems, is extended

to MEMS problems. This emerging computational technology provides an improvement in speed of several orders of magnitude.

3. Mathematically, the coupling of physical domains in MEMS results in a system of coupled partial differential equations. The majority of MEMS are described by a nonlinear system: due to the dependency of the applied force on vibrational amplitude, even if the decoupled problems are linear, the resulting MEMS equation is nonlinear. A new technique, employing waveform relaxation, is demonstrated for self-consistent analysis of MEMS.

4. Identifying the sources of nonlinearities found in MEMS leads to understanding of the impact these nonlinearities have on the resulting system of equations. Designers of MEMS operating in nonlinear regions need dynamically accurate nonlinear macromodels. The construction of such macromodels is an open research area. For this reason, the extension of concurrent research in nonlinear model order reduction for circuit simulation to MEMS is discussed.

1.4 Thesis Outline

Chapter 2 introduces the set of partial differential equations in continuum coordinates that govern MEMS devices. The examination is presented one energy domain at a time to prepare the reader for MEMS modeling and simulation. Chapter 3 presents the interaction between electrical, mechanical and fluidic forces in linear MEMS. The resulting systems of equations are discretized using finite element and finite difference schemes. Chapter 4 describes the Arnoldi method for model order reduction, the basis for a fast, accurate and general MEMS simulator. Chapter 5 presents examples of the Arnoldi method applied to a representative set of complex devices. Chapter 6 describes techniques for nonlinear MEMS simulation including waveform relaxation and nonlinear model order reduction. Conclusions and a summary of proposed future research is given in Chapter 7.

Chapter 2

Fundamental Physics of MEMS

2.1 Introduction

A framework for the computation of MEMS phenomena is presented. The physical level of device modeling and analysis is presented one energy domain at a time. This chapter describes structural mechanics, fluid mechanics and electrostatics. The governing equations in Section 2.2 and Section 2.4 are well known among Mechanical Engineers. Similarly, the equations in Section 2.3 are familiar to the Electrical Engineer. However, as often the case with MEMS, the reader may not have sufficient background in both Mechanical and Electrical Engineering. Thus, a brief review of all the governing equa-

tions is in place. The symbols used for the variables in different domains overlap, which has the potential for confusion. This is a very serious problem because each discipline has its own standard notation. Some attempt has been made to differentiate similar symbols. However, it is impossible to create a completely unique non-overlapping set of symbols that can be easily understood and easily related to reference material in the various disciplines.

2.2 Structural Mechanics

When a force is applied to a microelectromechanical structure, it deforms and, in most cases, recovers its original shape when the force is removed. Indeed, all MEMS can, to some extent, deform elastically without breaking or developing a permanent shape change, but all will break if the applied forces is too large. The elastic behavior of materials is a very large field, and only the key ideas required for MEMS analysis are presented in the following subsections. First, the description begins with basic definitions of stress and strain. The linear relation between stress and strain that applies in all solid materials for sufficiently small deformations is presented in Section 2.2.1. Next, the fundamental equations for describing the behavior of a solid are

described. The equations are used to determine the elastic deformation of an arbitrarily shaped three-dimensional structure in response to applied forces. Additionally, a brief overview of material and geometric nonlinearities is included. Microsystems contain many different kinds of structural elements. The devices and the packages that hold them are solid objects governed by the laws of continuum solid mechanics. Section 2.2.3 examines a basic structural element, the cantilever beam, and develops the Euler differential equation for beam bending. This equation can be used if the device under consideration can be approximated by a one-dimensional beam. Some elementary solutions to simple forms of this beam equation can be found in [9]. However, the form of the beam equation presented does not have an analytical solution. Both the Navier equation of Section 2.2.1 and the Euler beam equation of Section 2.2.3 describe mechanical devices. The application of these equations to MEMS devices and their solution in a MEMS context is the subject of chapters 3 through 6.

2.2.1 Navier Equation for Linear Geometry

General structural mechanical modeling means finding the relationship between the stresses and strains in a material. Stress is defined microscopically

as the force per unit area acting on the surface of a differential volume element of a solid body [9]. Since the components of stress, the intensity of force per unit area, hold true only at a point, the definition of stress is [12]

$$\tau_{xx} = \lim_{\Delta A \rightarrow 0} \frac{\Delta P_x}{\Delta A} \quad \tau_{xy} = \lim_{\Delta A \rightarrow 0} \frac{\Delta P_y}{\Delta A} \quad \tau_{xz} = \lim_{\Delta A \rightarrow 0} \frac{\Delta P_z}{\Delta A} \quad (2.1)$$

where ΔP represents the components of a force vector acting on an infinitesimal area ΔA . In all three cases, the first subscript of τ indicates that the plane perpendicular to the x axis is considered, and the second designates the direction of the stress component. The intensity of the force perpendicular to or normal to the section is called the normal stress at a point. Normal stresses that cause traction or tension on the surface of a section are called tensile stresses. Also, those that are pushing against it are compressive stresses. Normal stresses are denoted by the letter σ instead of a double subscript on τ . The other components of the intensity of force act parallel to the plane of the section. These components are called shear stresses and are designated by τ .

When forces are applied to a solid body, it can deform. The differential deformation is called the strain, expressed as change in length per unit length

[9]. The definition of strain also follows from an examination of a section of a structure. If the extensional strain taking place in one direction as shown in Figure 2.1 is considered, some points like A and B move to A' and B' ,

Figure 2.1: Strain

respectively. During straining, point A experiences a displacement u . The displacement of point B is $u + \Delta u$. In addition to the rigid body displacement u , common to the whole element Δx , a stretch Δu takes place within the element. On this basis, the definition of the extensional or normal strain [12] is

$$\epsilon = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \quad (2.2)$$

A more complete description of strain is given in [13].

For a total description of the stress conditions in a point, the symmetric stress matrix,

$$\mathbf{S} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} \quad (2.3)$$

can be used, where σ_i are the stress components, and τ_i are the shear stresses.

To study the variation of the stress as the position of the point changes, the conditions of equilibrium of a small rectangular parallelepiped with sides $\delta x, \delta y, \delta z$ are considered [14]. The components of stresses acting on the sides

Figure 2.2: Small Rectangular Parallelepiped

of this small element and their positive directions are indicated in Figure 2.2. The mid-points of the sides of this element are designated by 1, 2, 3, 4, 5 and 6. The small changes of the components of stress due to the small increases δx , δy and δz of the coordinates. In Figure 2.2, the value of σ_x at point 1 is distinguished from its value at point 2 by writing $(\sigma)_1$ and $(\sigma)_2$ respectively. The symbol $(\sigma)_1$ denotes the value of this stress component at the point x , y , z . In calculating the forces acting on the element, the sides are considered very small, and the force is obtained by multiplying the stress, a force per unit area, at the centroid of a side by the area of this side [14]. Also, the body force acting on the element, such as the weight of the element, is of the same order of magnitude as the terms due to the variations of the stress components. The equations of static equilibrium are obtained by summing all the forces acting on the element in each direction. For example, in the x direction, the equation is

$$\begin{aligned}
 & [(\sigma_x)_1 - (\sigma_x)_2](\delta y \delta z) + [(\tau_{xy})_3 - (\tau_{xy})_4](\delta x \delta z) \\
 & + [(\tau_{xz})_5 - (\tau_{xz})_6](\delta x \delta y) + K_x(\delta x \delta y \delta z) = 0
 \end{aligned} \tag{2.4}$$

The two other equations of equilibrium are obtained in the same manner. After dividing by δx , δy and δz and proceeding to the limit by shrinking the

element down to the point x, y, z , the relation between the different stress components can be found: the stress equilibrium conditions, a set of force balance equations, for moving material with small deformations is

$$\begin{aligned}\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_{xz}}{\partial z} &= K_x \\ \rho \frac{\partial^2 v}{\partial t^2} - \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau_{yz}}{\partial z} &= K_y \\ \rho \frac{\partial^2 w}{\partial t^2} - \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \sigma_z}{\partial z} &= K_z\end{aligned}\tag{2.5}$$

where u, v and w are the deformation components, $\mathbf{K}$ denotes the volume (body) forces, ρ is the density of the material and t is time.

The strain conditions at a point x, y, z can be completely described from the deformation components and their derivatives. The elongation strain (ϵ) components and the shearing strain (γ) components are given from the equations

$$\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x} & \epsilon_{xy} &= \frac{\gamma_{xy}}{2} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \epsilon_y &= \frac{\partial v}{\partial y} & \epsilon_{yz} &= \frac{\gamma_{yz}}{2} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \epsilon_z &= \frac{\partial w}{\partial z} & \epsilon_{zx} &= \frac{\gamma_{zx}}{2} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)\end{aligned}\tag{2.6}$$

where u, v , and w are the deformation components. For elastic materials,

there is an unambiguous relation between the stress (σ , τ) and the strain (ϵ , γ) in the material. This relation, known as Hooke's law [12], is under the assumptions of isotropic and isothermal conditions, described by the equations

$$\begin{aligned}\sigma_x &= \frac{E}{1+\nu}(\epsilon_x + \frac{\nu}{1-2\nu}(\epsilon_x + \epsilon_y + \epsilon_z)) & \tau_{xy} &= G\gamma_{xy} \\ \sigma_y &= \frac{E}{1+\nu}(\epsilon_y + \frac{\nu}{1-2\nu}(\epsilon_x + \epsilon_y + \epsilon_z)) & \tau_{yz} &= G\gamma_{yz} \\ \sigma_z &= \frac{E}{1+\nu}(\epsilon_z + \frac{\nu}{1-2\nu}(\epsilon_x + \epsilon_y + \epsilon_z)) & \tau_{zx} &= G\gamma_{zx}\end{aligned}\quad (2.7)$$

where ν is Poisson's ratio, E is the elasticity modulus or Young's modulus, and G is the shear modulus defined by

$$G = \frac{E}{2(1+\nu)} \quad (2.8)$$

In isothermal systems, any temperature rise or any temperature drop is quickly compensated by flow to or from the walls of the system. Isotropic materials have the same properties in all directions. Hooke's law becomes more complex for anisotropic materials: fully anisotropic crystalline materials can have 21 material constants [12]. If a material has different properties in the three orthogonal directions, it is known as orthotropic and has nine

independent material constants. With proper selection of the directions of axes, the following discussion can be extended to orthotropic materials.

Using Hooke's law, all stress and strain components can be derived once the derivatives of the deformations have been obtained. To do this, the balance of force can be formulated in terms of the deformation components, formulating a system of three equations known as Navier's equations. In other words, substituting information from (2.7) into (2.5) yields

$$\rho \frac{\partial^2 u}{\partial t^2} - (\mu + G) \frac{\partial e}{\partial x} + G \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + K_x = 0 \quad (2.9)$$

where $e = \epsilon_x + \epsilon_y + \epsilon_z$ and μ is defined as

$$\mu = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}. \quad (2.10)$$

The other two equations can be derived in the same manner. Thus, when modeling arbitrarily shaped, three-dimensional MEMS devices with Navier's

equations, the system of three equations

$$\begin{aligned}
\rho \frac{\partial^2 u}{\partial t^2} - (\mu + G) \frac{\partial e}{\partial x} + G \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + K_x &= 0 \\
\rho \frac{\partial^2 v}{\partial t^2} - (\mu + G) \frac{\partial e}{\partial y} + G \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + K_y &= 0 \\
\rho \frac{\partial^2 w}{\partial t^2} - (\mu + G) \frac{\partial e}{\partial z} + G \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + K_z &= 0
\end{aligned} \tag{2.11}$$

is solved.

2.2.2 Nonlinear Geometry and Stress-Strain Relations

When displacements are large, the assumption that the strains vary linearly with the displacements does not hold [15]. In that case, the dependence can be divided into a linear and a nonlinear part.

The nonlinear strain-displacement relations, for the two-dimensional case, require Green's strains where the strains depend on the displacements

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right) \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right)
\end{aligned} \tag{2.12}$$

When the stress-strain relation does not follow the usual linear form, the problem is treated as a nonlinear one. Types of nonlinearities include

polynomial stress-strain relations, plasticity with isotropic hardening and plasticity with kinematic hardening.

2.2.3 Euler Beam Equation

The one-dimensional Euler beam equation

$$EI \frac{\partial^4 u}{\partial x^4} - S \frac{\partial^2 u}{\partial x^2} = F_{electrostatic} + F_{air} - \rho \frac{\partial^2 u}{\partial t^2} \quad (2.13)$$

can be used if the device can be approximated by a one-dimensional beam.

In this equation, E is Young's modulus, I is the moment of inertia, S is the residual stress, ρ is the density of the material, u is the height of the one-dimensional beam above a reference point and t is time. The Euler beam equation is an expressive form of Newton's second law. It has four force terms: the elastic bending force of the beam, the residual stress, the electrostatics actuation force and the damping force from the air pressure.

The damping force has the form

$$F_{air} = \int_0^w (p - p_a) dy \quad (2.14)$$

where p is pressure, p_a is the ambient pressure and w is the width of the beam. This equation is an integro-differential equation and is linear in the dependent variables $u(x, t)$ and $p(x, y, t)$. Some of the force terms are nonlinear in the dependent variables $u(x, t)$ and $p(x, y, t)$. In this section, the Euler beam equation will be examined in detail. A complete definition of the force terms is delayed until Section 3.2.

Figure 2.3 shows a cantilever beam bending under some applied force.

Figure 2.3: Cantilever Beam Bending

This one-dimensional representation depicts the position of the neutral axis $w(x)$ as a function of position along the unloaded beam x . A length ds along

the neutral axis is related to dx by

$$ds = \frac{dx}{\cos\theta} \quad (2.15)$$

and the slope of the beam at any point is

$$\frac{dw}{dx} = \tan\theta \quad (2.16)$$

Also, for a given radius of curvature $\bar{\rho}$ at position x , ds is related to the incremental subtended angle $d\theta$ by

$$ds = \bar{\rho}d\theta \quad (2.17)$$

In most MEMS, a small-angle approximation is valid; therefore, it is appropriate to assume $ds \approx dx$. Thus,

$$\frac{d\theta}{dx} \approx \frac{1}{\bar{\rho}} \quad (2.18)$$

and

$$\theta \approx \frac{dw}{dx} \quad (2.19)$$

If (2.19) is differentiated with respect to x and (2.18) is substituted, the radius of curvature of the beam segment,

$$\frac{1}{\bar{\rho}} = \frac{d^2w}{dx^2} \quad (2.20)$$

can be written as

$$\frac{1}{\bar{\rho}} = -\frac{M}{EI} \quad (2.21)$$

where M is the internal bending moment and EI is the flexural rigidity.

Then, the basic differential equation for small-angle bending of slender beams is

$$\frac{d^2w}{dx^2} = -\frac{M}{EI} \quad (2.22)$$

[9]. In equilibrium, externally applied loads match the internal bending mo-

ment M . Since point loads $\tilde{V}$ and distributed loads $\tilde{q}$ are often found in MEMS problems, it is useful to work with the first and second derivatives of (2.22)

$$\frac{d^3 w}{dx^3} = -\frac{\tilde{V}}{EI} \quad (2.23)$$

or

$$\frac{d^4 w}{dx^4} = \frac{\tilde{q}}{EI}. \quad (2.24)$$

Also, as discussed in [9], if the angles grow such that the small-angle approximation is invalidated, a more exact expression for the radius of curvature can be obtained.

Figure 2.4 illustrates a thin beam that has an axial stress σ_a and has been bent with a radius of curvature $\tilde{\rho}$ [9]. The axial stresses do not produce a net horizontal force. However, when the beam is bent, a net downward force acts on the beam. Equilibrium is maintained if an upward load on the beam is postulated. In other words, a uniformly distributed effective pressure load P_0 acts on the lower surface of the beam to counteract the downward force due to axial tension. Figure 2.5 shows a thin beam with cross-sectional area

Figure 2.4: Beam Under Axial Stress

Figure 2.5: Effect of Residual Stress on Bending

$W \times H$ and in pure bending through an angle π . The horizontal component of both the axial stress and the uniform pressure load sum to zero. Summing the forces in the vertical direction yields

$$2\bar{\rho}WP_0 = 2\sigma_aWH \quad (2.25)$$

Thus,

$$P_0 = \frac{\sigma_a H}{\bar{\rho}} \quad (2.26)$$

The uniform pressure load, P_0 , corresponds to a per unit length beam load of $q_0 = P_0W$. Also, if $\bar{\rho}$ is replaced with the beam curvature, an equivalent beam load, valid for small displacements and angles, is

$$q_0 = \sigma_a WH \frac{d^2w}{dx^2} \quad (2.27)$$

where w is the vertical displacement of the beam. This accounts for the details of internal stresses associated with the bending depicted in Figure 2.3. With this distributed load compensating for the bending when stress is

present, the differential equation governing beam bending is

$$EI \frac{d^4 w}{dx^4} = \bar{q} + q_0 \quad (2.28)$$

where $\bar{q}$ is some external load. Again, q_0 is an equivalent load which accounts for the axial stress contribution to beam bending stiffness. Substituting (2.27) for q_0 , the equation becomes the Euler beam equation:

$$EI \frac{d^4 w}{dx^4} - (\sigma_a W H) \frac{d^2 w}{dx^2} = \bar{q} \quad (2.29)$$

where the quantity $\sigma_a W H$ is the tension in the beam and has the dimensions of force. In summary, the new tension term works against the $\frac{d^4 w}{dx^4}$ making the beam appear stiffer in response to externally applied loads [9]. This equation, in the form shown in (2.13), is revisited in later chapters.

2.3 Electrostatics

The subject of this section is the calculation of the electrostatic fields and forces in MEMS. The induced deformations and motions enable precision motion and control. MEMS structures can include conductors and coatings.

Charges accumulate on the conductor surfaces under an applied voltage. Imposed electric fields produce tractions on the surfaces of the structures and cause deformations. Although there are MEMS designs that use piezoelectric, thermal, pneumatic, and magnetic actuation, the use of electrostatic forces to move micromachined parts is the most popular approach in present day microsensor and microactuator designs.

2.3.1 Preliminaries of Electrostatic Simulation for MEMS

The theory and formulations for computing the charge density on the surfaces of conductors is fundamental to many electrostatic applications. Potential theory is often applied to electrostatics simulation in exterior regions. A familiar application is the calculation of transmission line parameters in electrical power systems. MEMS is another important application area.

In electrostatics for MEMS, of particular interest is the surface charge density distribution on the surface of the conductors, which is required to calculate the electrostatic forces on the conductors. Specifically, the electrostatic force $\mathbf{F}$ can be computed from

$$\mathbf{F} = -\frac{1}{2} \frac{q^2}{\epsilon_e} \mathbf{n} \quad (2.30)$$

where $q(x)$ is the surface charge density at point x on the surface of conductor, $\mathbf{n}$ is the inward normal at x and ϵ_e is the dielectric constant of the medium in which the conductor is placed [6].

A system of n ideal conductors embedded in a uniform lossless dielectric medium is considered. The electrostatic potential on each of the conductors is a constant. The charges in each conductor are distributed on its surface and satisfy [6]

$$q_i(x) = -\epsilon_e \frac{\partial \phi_i(x)}{\partial \mathbf{n}} \quad (2.31)$$

where $q_i(x)$ is the surface charge density at point x on the surface of conductor i , $\phi_i(x)$ is the electrostatic potential of conductor i , $\mathbf{n}$ is the outward normal at x and ϵ_e is the dielectric constant of the medium in which the conductor is placed. The surface charge density $q(x)$ is needed in (2.30) to compute the electrostatic force on a conductor. The charge density $q(x)$ can be obtained by the theories developed in [6]. For review purposes, the remaining paragraphs survey some background information related to finding $\phi(x)$ and, thus, $q(x)$.

2.3.2 Review of Electromagnetics

The fundamental physical quantities in the field of electromagnetics are the electric field intensity $\mathbf{E}$, the electric flux density or electric displacement $\mathbf{D}$, the magnetic flux density $\mathbf{B}$, the magnetic field intensity $\mathbf{H}$, the current density $\mathbf{J}$ and the electric charge density ρ_e .

Maxwell's Equations

The governing partial differential equations for all macroscopic electromagnetic problems are the well-known Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.32)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (2.33)$$

$$\nabla \cdot \mathbf{D} = \rho_e \quad (2.34)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.35)$$

The first three equations are also known as Faraday's law, Ampere's law and Gauss' law, respectively. The last equation denotes the assumption that isolated magnetic charges, monopoles, do not exist in nature. The equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_e}{\partial t} \quad (2.36)$$

is also of interest: it specifies the conservation of charge. In total, nine equations are written in compact vector notation. Each field quantity has three components in the x, y and z directions, respectively. For example, the electric field can be written as $\mathbf{E} = (E_x, E_y, E_z)$. Only three of the above equations are independent. Equations 2.32, 2.33, 2.34 or 2.32, 2.33 together with the equation of continuity, 2.36, form a set of independent equations. The other equations can be derived from these independent equations.

Constitutive Relations

To solve Maxwell's equations for the unknowns $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{B}$, more information is needed. This information will be provided from the material dependent relations between the field quantities, the constitutive relations,

$$\mathbf{D} = \epsilon_e \mathbf{E} \quad (2.37)$$

$$\mathbf{B} = \mu_e \mathbf{H} \quad (2.38)$$

and

$$\mathbf{J} = \sigma_e \mathbf{E}, \quad (2.39)$$

where ϵ_e is the permittivity, μ_e is the permeability, and σ_e is the conductivity. For anisotropic media, ϵ_e , μ_e and σ_e are tensors. For isotropic media, these quantities are scalars. For inhomogeneous media, ϵ_e , μ_e and σ_e vary with position. If the medium is linear, then ϵ_e , μ_e and σ_e are constants.

Boundary Conditions

To get a full description of an electromagnetic problem, the boundary conditions must be specified at all the geometrical interfaces. The boundary conditions can be derived from the integral form of Maxwell's equations [16]. The first condition states that the tangential component of $\mathbf{E}$ is always continuous at the interface of two media. The second condition states that the tangential component of $\mathbf{H}$ is continuous unless there is a surface current density. The third condition states that the normal component of $\mathbf{D}$ is continuous unless there is a surface charge. The fourth condition states that the normal component of $\mathbf{B}$ is always continuous [17]. The boundary conditions for electrostatic problems can be of Dirichlet or Neumann type. For Dirichlet conditions, the electrostatic potential is specified on the boundary. For Neumann conditions, the surface charge is specified on the boundary.

Electrostatic Approximation

In practical applications of the theory presented above, the most general form of Maxwell's equations are rarely used. In MEMS problems, Maxwell's equations can often be simplified by assuming time-invariance. The static field solution is evaluated because it gives a good approximation to the dynamic

field solution and it is easier to calculate than the dynamic field. By definition, static fields are independent of time or invariant with time. In practice, the statics implies that the time rate of change is slow and the wavelengths are very large compared to the size of the domain of interest.

From Maxwell's equations, the expressions that characterize an electrostatic problem in an isotropic homogeneous material are

$$\nabla \times \mathbf{E} = 0 \quad (2.40)$$

$$\nabla \times \mathbf{E} = \frac{1}{\epsilon_e} \rho_e \quad (2.41)$$

The vector identity

$$\nabla \times \nabla \psi = 0 \quad (2.42)$$

serves as the basis for the definition of the static electric scalar potential ϕ .

Thus, ϕ is defined such that

$$\mathbf{E} = -\nabla \phi \quad (2.43)$$

where the units of ϕ are volts. Substituting the above into (2.41) yields Poisson's equation

$$\nabla^2 \phi = -\frac{\rho_e}{\epsilon_e} \quad (2.44)$$

In materials where $\rho_e = 0$, the above simplifies to Laplace's equation [18]:

$$\nabla^2 \phi = 0 \quad (2.45)$$

Thus, four of Maxwell's equations are reduced to a single scalar equation, a second-order scalar partial differential equation (PDE).

The basic problem in electrostatics is to solve Poisson's or Laplace's equation for the potential function ϕ that satisfies the relevant boundary conditions. The treatment here is of necessity quite abbreviated. References to more detailed treatments include [16] and [18].

Most importantly, the heart of the MEMS electrostatics problem is usually expressed as the computation of the surface charges on the conductors of the system. This entails numerically solving the integral form of Laplace's

equation [4]

$$\phi(x) = \int_{Surfaces} G(x, x')q(x')dS' \quad (2.46)$$

by a variety of numerical techniques, where q is the surface charge density, dS' is the incremental surface area, ϕ is the applied surface potential, and

$$G(x, x') = \frac{1}{4\pi\epsilon_e||x - x'||} \quad (2.47)$$

is the Green's function. For accurate electrostatic analysis, numerically methods must be employed. For illustration purposes, in later chapters, some popular but inaccurate simplifying assumptions to the electrostatics problem are introduced.

2.4 Fluid Mechanics

The subject of this section is the calculation of fluidic forces in MEMS. In general, fluids are difficult to analyse: the governing partial differential equations are intrinsically nonlinear. The goal of this chapter is to introduce some basic fluidic concepts and focus on the phenomena that are important in the

pressure sensor device of Section 3.2. However, all fluid mechanics concepts are relevant to MEMS: fluid mechanics texts, like [19], should be viewed for a complete treatment. Under certain assumptions, the continuity equation [20], the Navier-Stokes equation and the ideal gas law can be combined to yield the Reynolds equation. In this work, the Reynolds equation is used in the pressure sensor analysis. A more general treatment of fluids will be required when the proposed future work listed in Chapter 7 is undertaken.

2.4.1 Navier Stokes Equation

Newton's second law of motion, in differential form, is the Navier-Stokes equation

$$\rho_m \frac{D\mathbf{U}}{Dt} = -\nabla P + \rho_m \mathbf{g} + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} (\nabla \cdot \mathbf{U}) \quad (2.48)$$

where ρ_m is the density of the fluid, $\mathbf{U}$ is velocity, P is pressure, $\mathbf{g}$ is acceleration due to gravity and η is viscosity. It describes the motion of a fluid in the presence of external and internal viscous forces. When studying liquid flows, it is often safe to assume that the density of the material is almost constant [17]. This is referred to as incompressible fluid flow. Simplified

expressions can be derived from this equation. They are discussed in the following section and used to analyse an example later in this work.

2.4.2 Squeezed-Film Damping

In pressure sensors like the one presented in Section 3.2 and many other MEMS devices, the situation illustrated in Figure 2.6 occurs. A gas, air for

Figure 2.6: Squeezed-Film Damping

example, fills the space between a fixed plate and a moveable plate. The gap between the plates h is a function of time. If the upper plate moves toward the lower plate as a result of a force, the pressure inside the gas increases and the gas is squeezed out from the edges of the plate. If the upper plate

moves away from the lower plate, the pressure drops and the gas is sucked back. Most importantly, the viscous drag of the air during the flow creates a dissipative mechanical force on the plate. The damping force, known as squeezed-film damping, opposes the motion of the plate. To analyse this problem, the following simplifying assumptions are made [9]:

- The height of the gap is small when compared to the horizontal extent of the plates.
- The motion of the gas is slow so Stokes flow is applied.
- There is no pressure gradient from plate to plate.
- The lateral flow has zero transverse velocity at the upper and lower plates.
- The gas obeys the ideal gas law.
- The system is isothermal.

Thus, under these assumptions, together with the continuity equation [19] based on conservation of mass, and the ideal gas law [20], the Navier Stokes equation for fluids is used to derive the two-dimensional, nonlinear Reynolds

equation

$$\nabla \cdot (u^3 p \nabla p) = \frac{12\mu}{1 + 6K} \frac{\partial(pu)}{\partial t} \quad (2.49)$$

where the pressure p does not depend on z , the deflection of the gap u is time and position dependent, μ is the viscosity of air and K is a new factor called the Knudsen number. The Knudsen number is the ratio of the mean free path of the gas molecules to the gap.

In summary, the two-dimensional, nonlinear Reynolds equation is derived from the Navier-Stokes equation for fluids. The derivation of the nonlinear Reynolds equation reveals the underlying physical assumptions that make the further linear derivation presented in [21] possible.

Chapter 3

MEMS Mathematical Models

MEMS designers are primarily interested in the natural modes of vibration of a structure and the deformation of a device due to electrostatic, mechanical or thermal excitation force, whether individual or coupled. The values of capacitance, stress, and displacement are of particular interest. Chapter 2 surveys the physical domains relevant to MEMS including an examination of the physical level of device modeling and analysis, one energy domain at a time. This information forms the basis for the proposed methodology for the modeling and simulation of MEMS. The central theme of this chapter, the overall solution flow, follows an introduction to self-consistent MEMS simulation. Next, the Euler and Reynolds equation are related directly to

an electrostatically actuated plate structure shown in Figure 3.3. Finding solutions to the systems of equations that MEMS designers are interested in requires a discretization scheme. The finite element method (FEM) and finite differences (FD) are introduced. Lastly, the problems in MEMS simulation that this chapter highlights are summarized. This is the first of two chapters to explain the proposed solution of the linear MEMS systems.

3.1 Principles of Coupled Simulation

Microelectromechanical systems typically involve a mechanical structure which undergoes deformation when subjected to external forces including electrostatic actuation and fluidic forces.

3.1.1 MEMS-Based Projection Display

In a MEMS-based projection display, a large two-dimensional array of aluminum micro-mirrors is fabricated over a silicon integrated circuit that contains drive electronics along with electrodes for electrostatically tipping the mirrors to various positions [9]. Each mirror controls one pixel of the image. The electromechanical coupling is illustrated with Figure 3.1, the simplified

schematic of a single element in a micromirror array. The tipping motion is

Figure 3.1: Mirror Schematic

based on the change of capacitance between two initially parallel plates when one plate is free to rotate about a support. If the mirror support is grounded and a voltage is applied between the actuation electrode and the mirror-support, the resulting electric field produces a force on the mirror-support and causes it to deform. The structure's new shape leads to a change in

capacitance and, hence, a change in the applied force. In other words, when a voltage is applied between the mirror-support and a ground plane, a charge distribution is induced on the surface of the mirror-support. This charge distribution causes the mirror-support to deflect. Also, the charge distribution on the mirror-support surface changes because of the mirror-support deflection. When the forces due to the mirror-support deflection and the surface charges balance each other, a state of equilibrium is obtained.

3.1.2 Electrostatic Comb Drive

The electrostatic comb drive shown in Figure 1.2 is one of the most important MEMS devices. In [6], the full characterization of a silicon comb drive has been reported. The evaluation of the comb drive serves to illustrate the principles of electrostatics applied to MEMS calculations. The driving forces and the bending forces on the comb drive are of interest.

An inaccurate but popular formula for evaluating the driving force of a comb drive is

$$F = \frac{\epsilon_0 h V^2}{g_f} \quad (3.1)$$

where F is the driving force on each moveable finger, V is the voltage applied between the electrodes and ground, h is the height of a single finger, g_f is the spacing between the fingers and ϵ_0 is the permittivity of air. This assumes the comb-drive capacitance is linear with the displacement of the combs, and, thus, the electrostatic driving force is independent of vibrational amplitude. In fact, [6] examines the performance of (3.1) and demonstrates its inaccuracy: the driving force has greater dependence on the geometry of the comb. Thus, only full numerical electrostatic analysis can obtain many of the important details. Full three-dimensional, electrostatic analysis is essential for better understanding of the behavior of MEMS devices and optimal design of MEMS. Also, if the overlap between the movable fingers and the fixed fingers is small, the driving force on the movable finger becomes nonlinear [6]. This is seen in comb drives with short moveable fingers or with small overlap between movable and fixed fingers. These comb drives do not have a significant region of constant force as a function of axial displacement. This nonlinearity is more pronounced with higher applied voltage and requires coupled nonlinear mechanical-electrical analysis.

3.1.3 Parallel Plate Capacitor Approximation

Some MEMS structures may be approximated as parallel-plate capacitors for a rough calculation of the electrostatic forces. The force F_e that puts energy into the electrostatic domain of the MEMS system is given by the gradient of the electrostatic co-energy U_e^*

$$F_e = -\nabla U_e^* \quad (3.2)$$

where $U_e^* = \frac{1}{2}CV^2$, V is the applied voltage and C is the capacitance between the conductors. Since the applied voltage is independent of motion, the gradient need only be applied to the capacitance, thus

$$F_e = -\left(\frac{1}{2}V^2\right) \nabla C \quad (3.3)$$

The capacitance of a large parallel-plate capacitor neglecting fringe field-effects is given by

$$C = \frac{\epsilon_0 A}{d} \quad (3.4)$$

where A is the area of the plate, and d is the distance between the plates. This introduces some inaccuracy, but it replaces the computationally costly, full electrostatics analysis described in Chapter 2. Thus, the electrostatic force F_e is given by

$$F_e = -\frac{1}{2} \frac{\epsilon_0 l w V^2}{d^2} \quad (3.5)$$

where l is the length of the plate, w is the width of the plate, ϵ_0 is the permittivity of air, V is the voltage applied between the plates and d is the gap between the plates.

3.2 Conventional MEMS Simulation

3.2.1 Conventional Solution Steps

In the following sections, the information presented in Chapter 2 is related directly to MEMS simulation. The present, overall method for analyzing the dynamics of a microelectromechanical system is depicted in Figure 3.2. The partial differential governing equations of Chapter 2, which have no analytical solution, require full numerical treatment. In summary, after the equations

are discretized, the resulting large set of algebraic differential equations are solved using conventional techniques.

3.2.2 Mixed Electrostatics/Fluids/Mechanics Problem

The structure of Figure 3.3 is an electrostatically actuated plate structure with squeezed-film damping. The top plate is flexible and clamped on both sides, while the bottom plate is fixed. The applied voltage, V , causes an electrostatic force to develop thus actuating the flexible top plate. As the top plate vibrates, the air in the gap is squeezed out and sucked in. The force varying with the pressure of the air in the gap dampens the motion of the plate. This is described in Section 2.6 and is known as squeezed-film damping.

Since the response of the top plate is very sensitive to the air pressure in the air film under it, this structure can be used as a pressure sensor. When this structure is subjected to forces that cause large deformations, this coupled domain problem is strongly nonlinear. This structure has a large length to width aspect ratio. Since the top plate is clamped on both edges in the y direction, the displacement, $u(x, y, t)$, and its first derivative along y are much smaller than the displacement and its first derivative along

Figure 3.2: Conventional MEMS Simulation

Figure 3.3: Pressure Sensor

the x axis. This leads to the assumption that $\frac{\partial u}{\partial y} = 0$. In essence, the two dimensional plate is modeled as a fixed-fixed beam and the displacement of the plate is uniformly distributed along the y axis. Thus, $u(x, y, t) = u(x, t)$.

This configuration represents a coupled domain problem and requires a model with equations selected from several physical domains. Specifically, the model, first proposed by [22] and later used in [21], consists of two equations:

$$EI \frac{\partial^4 u}{\partial x^4} - S \frac{\partial^2 u}{\partial x^2} = F_{elec} + \int_0^w (p - p_a) dy - \rho \frac{\partial^2 u}{\partial t^2} \quad (3.6)$$

$$\nabla \cdot (u^3 p \nabla p) = \frac{12\mu}{1+6K} \frac{\partial(pu)}{\partial t} \quad (3.7)$$

where $\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j}$, u is deflection, p is air pressure, p_a is ambient air pressure, EI is flexural rigidity, S is stress, w is the dimension of the beam in the y direction, ρ is density, μ is air viscosity, K is Knudsen's number and F_{elec} is the electrostatic force approximated by

$$F_{elec} \cong -\frac{\epsilon_0 w V^2}{2u^2} \quad (3.8)$$

as in Section 3.1.3. The electrostatic force depends on the permittivity of air ϵ_0 , the width of the beam w , the voltage applied between the upper and lower plates V and the deflection of the beam u . Recall, Equation 3.6 is the one-dimensional Euler beam equation, and Equation 3.7 is the two-dimensional Reynolds equation. The system composed of 3.6 and 3.7 has three independent variables, namely x, y and t . The beam equation, an integro-partial differential equation, is linear in the dependent variables $u(x, t)$ and $p(x, y, t)$. The Reynolds equation, a parabolic partial differential equation, is nonlinear in the dependent variables.

Many MEMS devices with structures similar to the one shown in Fig-

ure 3.3 operate in a linear region. The actuation signal produces a small input force; thus, the beam deflection is small when compared to the gap distance. To this end, the system of coupled nonlinear governing equations, (3.6) and (3.7), are linearized around an initial equilibrium point [21]. Figure 3.3 depicts the initial position of the top plate. Thus,

$$\begin{aligned} u &= h_0 + \delta u \\ p &= p_a + \delta p \end{aligned} \tag{3.9}$$

where δu and δp are perturbations. Equation 3.9 is substituted in (3.7) and the δu and δp terms of order two and higher are dropped. Equation 3.9 is substituted into (3.6), thus, linearizing F_{elec} and K . Further, (3.6) and (3.7) are normalized using the initial gap h_0 and p_a as

$$\begin{aligned} u &= \frac{\delta u}{h_0} \\ p &= \frac{\delta p}{p_a} \end{aligned} \tag{3.10}$$

The normalized linear Reynolds equation is

$$h_0^2 p_a \nabla^2 p = \frac{12\mu}{1 + 6K} \left(\frac{\partial p}{\partial t} + \frac{\partial u}{\partial t} \right) \tag{3.11}$$

and the normalized beam equation is

$$EI \frac{\partial^4 u}{\partial x^4} - S \frac{\partial^2 u}{\partial x^2} = F_{elec}^* + \frac{p_a}{h_0} \int_0^w p dy - \rho \frac{\partial^2 u}{\partial t^2} \quad (3.12)$$

where $F_{elec}^* = -\frac{\epsilon_0 w V^2}{2h_0^3}$. With a constant supply voltage, the charges are uniformly distributed over the top plate. Thus, apart from the small deflection approximation which allows the linearization of the actuation force, a further assumption that the load is uniformly distributed over the one-dimensional beam is made.

To recap, physically, (3.12) and (3.11) are coupled because, as the beam deflects downwards, it squeezes out the air between the gap. However, the back pressure force of the air acting on the beam is a function of the deflection.

Compact macromodels have been constructed using valid assumptions for this pressure sensor [22]. However, these approaches require that the designer have knowledge about the characteristics of the device a priori. When frequent design changes are required, the cost of human insight may limit the design possibilities.

3.2.3 Discretization Schemes

First, the relevant partial differential equations are combined to form MEMS models. Next, these partial differential equations are discretized. This results in a system of ordinary differential equations. The next chapter deals with the application of model order reduction to these ordinary differential equations. Model order reduction presents the key to fast simulation of the large systems of equations that result from high resolution discretizations.

For a dynamical system, a system with n states can be represented by n first-order ordinary differential equations. This system is then combined into a first-order vector-matrix differential equation. Similarly, n algebraic equations can describe a static system. Increasing the accuracy and, thus, the resolution of the mesh, means an increase in the number of state variables, but it does not increase the complexity of the ODEs. The equations 3.12, 3.11 and 2.11 have no analytical solution; therefore, full numerical treatment is necessary to obtain a solution.

In the system described by (3.12) and (3.11), position, velocity and pressure are the state variables and are defined as

$$x_1 = u \quad x_2 = \frac{\partial u}{\partial t} \quad x_3 = p \quad (3.13)$$

Substituting (3.13) into (3.12) and (3.11), the state-space representation of the linear dynamical system is:

$$\dot{x}_1 = x_2 \quad (3.14)$$

$$\dot{x}_2 = \left[-\left(\frac{EI}{\rho}\right) \frac{\partial^4}{\partial x^4} + \left(\frac{S}{\rho}\right) \frac{\partial^2}{\partial x^2} \right] x_1 + \left[\frac{p_a}{\rho h_0} \int_0^w dy \right] x_3 + \frac{\epsilon_0 w V^2}{2h_0^3 \rho} \quad (3.15)$$

$$\dot{x}_3 = -x_2 + \left[\frac{(1 + 6K)h_0^2 p_a}{12\mu} \nabla^2 \right] x_3 \quad (3.16)$$

In this case, the discrete forms for the linear operators are then substituted.

For the simple geometry of the pressure sensor problem, finite differences are used for the discretization. However, the non-trivial geometries found in the other examples require the finite element method, a more robust discretization technique. The two discretization schemes are described in the following sections.

Finite Difference Analysis

Finite differences is a spatial discretization scheme used to generate a system of linear dynamic ordinary differential equation in the state space representation. For an accurate solution, high resolution discretization is needed. The few thousand state variables that result make this type of analysis impractical for higher level simulation.

In this section, a uniform discretization of the pressure sensor of Figure 3.3, a simple geometry, will illustrate the use of finite differences. The top

Figure 3.4: Finite Differences Discretization

plate in Figure 3.4 illustrates an $(N + 1) \times (M + 1)$ meshing. For each of the

three states, one discrete variable per inner grid point is specified. Thus, the pressure state $\mathbf{x}_3$ becomes,

$$\mathbf{x}_3 = \mathbf{p}_{M \times N} = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & p_{22} & \dots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \dots & p_{MN} \end{bmatrix} \quad (3.17)$$

From the one dimensional beam equation, the position and velocity states discretize into vectors of $\mathfrak{R}^N$

$$\mathbf{x}_1 = \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_{N \times 1} \quad \mathbf{x}_2 = \dot{\mathbf{u}} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_N \end{bmatrix}_{N \times 1} \quad (3.18)$$

At the boundary grid points in Figure 3.4, the boundary conditions specify the value of the variables. Thus,

$$u(0, t) = u(l, t) = 0 \quad \frac{\partial(x, t)}{\partial x} \Big|_{x=0} = \frac{\partial(x, t)}{\partial x} \Big|_{x=l} = 0 \quad (3.19)$$

and

$$p(x, 0, t) = p(x, w, t) = p_a \quad \frac{\partial(x, y, t)}{\partial x} \Big|_{x=0} = \frac{\partial(x, y, t)}{\partial x} \Big|_{x=l} = 0. \quad (3.20)$$

These boundary conditions are derived from the set-up of the fixed-fixed beam. The displacements at both ends are zero. Also, the pressure is equal to the ambient pressure along the two edges that are parallel to the x axis.

Now that the problem domain has been discretized and the boundary conditions applied, discrete forms of the linear operators must be substituted into equations 3.15 and 3.16. In this work, central difference schemes are used for the spacial differential operators, and the Trapezoidal rule is used for the integral operator. Applying the fourth order central difference scheme for $\frac{\partial^4 u}{\partial x^4}$

$$u_i^{(4)} = \frac{1}{(\Delta x)^4} (u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}) \quad (3.21)$$

and the Neumann boundary condition at both ends, the term $\frac{\partial^4 u}{\partial x^4}$ can be

transformed into $[M]u$ where

$$M = \frac{1}{(\Delta x)^4} = \begin{bmatrix} 7 & -4 & 1 & \dots & 0 \\ -4 & 6 & -4 & \ddots & \vdots \\ 1 & -4 & 6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -4 \\ 0 & \dots & \dots & -4 & 7 \end{bmatrix} \quad (3.22)$$

The term $\frac{\partial^2 u}{\partial x^2}$ can be written as $[S]u$ where matrix S is derived in a similar manner. For the integral operator, the Trapezoidal rule is applied so

$$\int_0^w p dy \approx \Delta y \left(\frac{1}{2} p_0 + p_1 + p_2 + \dots + p_M + \frac{1}{2} p_{M+1} \right). \quad (3.23)$$

Thus, the matrix operator of the integral takes the form $T^T p_i$ where

$$T = \Delta y \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{M \times 1} \quad p_i = \begin{bmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{Mi} \end{bmatrix}_{M \times 1} \quad (3.24)$$

Lastly, the discrete form for the Laplace operator in Equation 3.16 is found

from a five-point centroid scheme [21]

$$\nabla^2 p_{ij} \approx \frac{1}{\Delta x^2} (p_{(i-1)j} - 2p_{ij} + p_{(i+1)j}) + \frac{1}{\Delta y^2} (p_{i(j-1)} - 2p_{ij} + p_{i(j+1)}) \quad (3.25)$$

where the single vector of pressure states is

$$\mathbf{p} = \begin{bmatrix} p_{11} \\ \vdots \\ p_{M1} \\ p_{12} \\ \vdots \\ p_{M2} \\ \vdots \\ \vdots \\ p_{1N} \\ \vdots \\ p_{MN} \end{bmatrix}_{MN \times 1} \quad (3.26)$$

Together with the boundary conditions, the discrete form of the Laplace operator is $\nabla^2 p \approx \mathbf{Lp}$. Finally, with all the matrix operators substituted,

the state-space equations in discrete form are

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}v(t) \quad (3.27)$$

$$r(t) = \mathbf{C}^T \mathbf{x}(t) \quad (3.28)$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \\ \mathbf{p} \end{bmatrix} \quad (3.29)$$

The output can be any linear combination of state variables. The displacement of the center point of the beam is of particular interest in the pressure sensor example.

Finite Element Analysis

To carry out the discretization required for frequency domain and transient analyses, the finite element method (FEM) is a standard method for realistic industrial problems. A complete discussion of the finite element method is

beyond the scope of this thesis. A comprehensive description may be found in [23]. The brief introduction to the theory of FEM will be presented here in the discussion of discretization schemes. However, the relation between the general variational methods presented in [9] and finite element methods using meshes is not covered here.

Although finite elements accomplish the same task as finite differences, they give a more robust, versatile method of approximation to continuum problems. The finite element method is a powerful numerical technique used to solve MEMS problems, problems which are described by partial differential equations. A MEMS model in continuum coordinates is transformed into a discretized finite element problem with unknown nodal values. The nodal values of the desired physical field are found from the approximating functions in finite elements. For a linear problem, a system of linear algebraic equations is solved. Values inside finite elements can be recovered using nodal values. A piecewise approximation of the physical field on finite elements provides good precision even with simple approximating functions: when the number of elements is increased, any degree of precision can be achieved [24]. Also, the locality of the approximation leads to sparse matrices.

Several approaches can be used to transform the physical formulation of the problem to its finite element discrete analogue [23]. If the physical formulation of the problem is known as a differential equation, the most popular method of its finite element formulation is the Galerkin method.

A simple, one-dimensional example will illustrate the finite element formulation using the Galerkin method. The differential equation

$$\begin{aligned} a \frac{d^2 u}{dx^2} + b &= 0, & 0 \leq x \leq 2L \\ u|_{x=0} &= 0 \\ d \frac{du}{dx} \Big|_{x=2L} &= R, \end{aligned} \tag{3.30}$$

where u is an unknown, can be solved numerically. The problem can be solved using two linear one-dimensional finite elements as shown in Figure 3.5. The

Figure 3.5: One-Dimensional Domain Divided into Two Finite Elements

finite element presented in Figure 3.6 has two nodes. The approximation of the function $u(x)$ can be performed as follows:

Figure 3.6: Function Approximation in a One-Dimensional Element

$$\begin{aligned}
 u &= N_1 u_1 + N_2 u_2 = [N]\{u\} \\
 [N] &= [N_1 \quad N_2], \quad \{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\
 N_1 &= 1 - \frac{x - x_1}{x_2 - x_1}, \quad N_2 = \frac{x - x_1}{x_2 - x_1}
 \end{aligned} \tag{3.31}$$

where the N_i are the shape functions which are used to interpolate $u(x)$ using its nodal values. The nodal values u_1 and u_2 are the unknowns which are

found from the discrete global system of equations

$$a \frac{d^2}{dx^2} [N] \{u\} + b = \psi \quad (3.32)$$

where ψ is a nonzero residual. The Galerkin method multiplies the terms of equation 3.32 by shape functions, integrates over the element and equates it to zero. Thus, it provides residual minimization. The final form for a finite element is [24]

$$[k] \{u\} = \{f\}$$

$$[k] = \int_{x_1}^{x_2} \left[\frac{dN}{dx} \right]^T a \left[\frac{dN}{dx} \right] dx, \quad \{f\} = \int_{x_1}^{x_2} [N]^T b dx + a \frac{du}{dx} \Big|_{x=x_i}.$$

(3.33)

In structural mechanics, $[k]$ is the stiffness matrix and $\{f\}$ is the load vector. In the end, an assembly of element equations forms a global system of equations for the entire domain.

The variational formulation of the finite element equations are used if the physical problem can be formulated in terms of minimizing the potential energy functional. The equations obtained from the variational formulation coincide with the equations obtained from the Galerkin method. In this

work, MEMS structural mechanics problems described by the partial differential equations of Chapter 2 are discretized using the commercial finite element tool FEMLAB[®]. Also, FEMLAB[®] converts the linear, time-dependent MEMS models into quasi-state-space models which contain sparse matrices. The equations are of the form

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{M}_A\mathbf{x} + \mathbf{M}_B\mathbf{u} \quad (3.34)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \quad (3.35)$$

where $\mathbf{u}$ is a vector with inputs to the FEMLAB[®] model, $\mathbf{y}$ is the output vector, $\mathbf{x}$ is a vector of state variables and $\mathbf{M}$, $\mathbf{M}_A$, $\mathbf{M}_B$ and $\mathbf{C}$ are constant matrices.

3.3 Computational Complexity

With the methodology depicted in Figure 3.2 and described in detail above, discrete simulation time steps may be on the order of hours for current desktop workstations. However, as Chapter 4 describes in detail, an emerging

computational technology, model order reduction, is the key to a fast, accurate solution to the MEMS equation set. Thus, the proposed method for MEMS simulation is illustrated in Figure 3.7.

The type of model order reduction depends critically on whether or not the system is linear. Many MEMS devices including accelerometers, resonators, gyroscopes and pressure sensors have linear working regions [21]. A linear system is characterized by equations that depend only on the first powers of the various dependent variables. Model order reduction for linear MEMS is the subject of Chapter 4. Nonlinear MEMS are discussed in Chapter 6.

Figure 3.7: Improved MEMS Simulation Methodology

Chapter 4

Simulation and Analysis of MEMS Structures: The Linear Case

The ultimate goal of this research, to develop MEMS simulators that enable in-depth understanding of MEMS behavior, will help in the design and fabrication of optimal MEMS. The focus here is on fast, accurate simulation of MEMS operating in a linear region where the fabricated shapes, material properties and applied voltages are used as inputs to predict their behavior. As Chapter 3 reviews, simulation of MEMS involves computing the defor-

mations and stresses in MEM structures. For linear MEMS, the problem translates to a very large system of ordinary differential equations (ODEs). The equations are solved using any numerical integration technique. However, for large systems of equations this becomes impractical. To reduce the computational complexity, the original system of equations are projected into a subspace of lower dimension. The matrices associated with this reduced set of equations are much smaller when compared to that of the original system. Hence, solution of the reduced set of equations is computationally less expensive when compared to the original system. The solution obtained from the reduced system is then mapped back to the original system space. Essentially, model reduction techniques reduce the number of ODEs to ultimately solve by several orders of magnitude. This solution flow is first introduced in Chapter 3.

This chapter explains how to create dynamically accurate macromodels for micromachined devices. Although it is possible to create these macromodels by hand, the technique shown here is vastly more efficient. It can derive such macromodels directly from full three-dimensional physical simulations. This means the designer does not have to attempt simplifying assumptions based on knowledge of the physics of the problem. Also, since many micro-

machined devices are at least mildly nonlinear, nonlinear reduction methods are discussed in Chapter 6.

4.1 Simulation of Large Linear Systems

Structural mechanics problems described by partial differential equations are translated into a set of ordinary differential equations, known as the macro-model, through some kind of discretization. Then, the solution to this large linear system must be found. Time domain simulation of such systems conventionally involves solving

$$\mathbf{M}\dot{\mathbf{x}}(t) = \mathbf{M}_A\mathbf{x}(t) + \mathbf{M}_B u(t) \quad (4.1)$$

$$w = \mathbf{L}^T \mathbf{x}(t) \quad (4.2)$$

where $\mathbf{x} \in \Re^n$ is the vector of unknowns, $\mathbf{M}, \mathbf{M}_A, \mathbf{M}_B \in \Re^{n \times n}$ are constant matrices describing the material properties of the system and $u(t)$ represents the excitation of the system. Integration techniques are used to convert a set of time-domain differential equations into a set of difference equations

[25]. For example, when the trapezoidal rule is applied to (4.1), a set of difference equations is generated. The CPU cost of a time-domain analysis is very expensive for large matrices found in MEMS. Another computationally demanding task, frequency-domain simulation, includes taking the Laplace transformation of (4.1) and solving for the unknown vector

$$\mathbf{X}(s) = (\mathbf{U}_n - s\mathbf{M}\mathbf{A}^{-1}\mathbf{M})\mathbf{M}\mathbf{A}^{-1}\mathbf{M}\mathbf{B} \quad (4.3)$$

where $\mathbf{U}_n \in \mathbb{R}^{n \times n}$ is the identity matrix. The objective of MEMS simulation algorithm presented here is handle large linear systems with minimal CPU expense.

4.2 MEMS Model Order Reduction Based Simulation Algorithms

The following presents an approach that automatically extracts macromodels for MEMS devices that operate in a linear region. When the input is limited to small signals, MEMS devices such as accelerometers, resonators, gyroscopes and pressure sensors have linear working regions [21].

In general, MEMS devices have a large number of poles, spread over a wide frequency range. Normally, the majority of these poles have little effect on simulation results. However, with a large number of poles, the simulation is CPU intensive since the simulator must use smaller step sizes.

4.2.1 Dominant Poles

Poles close to the imaginary axis are dominant poles and significantly influence the time and frequency characteristics of the system. Even if a system contains a large number of poles, only the dominant poles are sufficient to accurately characterize the system. To illustrate this concept, a simple example is shown in Figure 4.1. Pole P_2 will have a negligible effect on the final transient result.

Figure 4.1: Illustration of Dominant Poles

4.2.2 Moment-Matching Techniques Concepts

If a single-input/single-output system is considered, $H(s)$ is the transfer function. $H(s)$ can be represented in rational form as

$$H(s) = \frac{P(s)}{Q(s)} \quad (4.4)$$

where $P(s)$ and $Q(s)$ are polynomials in s . Equivalently, (4.4) can be written as

$$H(s) = c + \sum_{i=1}^{N_p} \frac{k_i}{s - p_i} \quad (4.5)$$

where p_i and K_i are the i_{th} pole-residue pair, N_p is the total number of system poles and c is the direct coupling constant. The closed form, time-domain impulse response can be computed using the inverse Laplace transform,

$$h(t) = c\delta t + \sum_{i=1}^{N_p} K_i e^{p_i t} \quad (4.6)$$

With large microelectromechanical systems, the total number of poles,

N_p , can be on the order of thousands. Thus, finding a solution if all the poles are considered will have high a CPU cost. Model-reduction techniques find reduced-order approximations in terms of the dominant system poles. Thus, computing only a small number of dominant poles, leads to significant savings in terms of CPU cost. Assuming L dominant poles are extracted, (4.4) can be re-written to obtain approximate frequency and time responses, as

$$H(s) \approx \hat{H}(s) = \frac{\hat{P}(s)}{\hat{Q}(s)} = \hat{c} + \sum_{i=1}^L \frac{\hat{k}_i}{s - \hat{p}_i} \quad (4.7)$$

$$h(t) \approx \hat{h}(t) = \hat{c}\delta t + \sum_{i=1}^L \hat{K}_i e^{\hat{p}_i t} \quad (4.8)$$

The Taylor series expansion of a transfer function $H(s)$ at point $s = 0$ is

$$H(s) \approx \hat{H}(s) = H(0) + s \frac{(H(0))^{(1)}}{1!} + s^2 \frac{(H(0))^{(2)}}{2!} + \dots + s^n \frac{(H(0))^{(n)}}{n!} \quad (4.9)$$

where the superscript n denotes the n_{th} derivative. Equation (4.9) can also

be written as

$$H(s) \approx \hat{H}(s) = m_0 + m_1(s) + M_2s^2 + \dots + m_ns^n = \sum_{i=0}^n s^i m_i; \quad m_i = \frac{H(0)^{(i)}}{i!}. \quad (4.10)$$

The inverse Laplace transform of $h(t)$,

$$\begin{aligned} H(s) &= \int_0^\infty h(t)e^{-st}dt = \int_0^\infty h(t) \left[1 - st + \frac{s^2t^2}{2!} - \dots \right] dt \\ &= \int_0^\infty h(t)dt + s \int_0^\infty (-1)th(t)dt + s^2 \int_0^\infty \frac{t^2h(t)}{2!}dt + \dots \\ &= \sum_{i=0}^\infty s^i \left(\frac{(-1)^i}{i!} \int_0^\infty t^i h(t)dt \right), \end{aligned} \quad (4.11)$$

reveals the coefficients of the Taylor series expansion, m_i , to be identical to the time-domain moments of the impulse response $h(t)$. Hence, the coefficients of Taylor series expansion, m_i , are usually referred to as moments.

In order to get accurate prediction of MEMS behavior, it is essential that the reduced-order model preserve as many moments as possible. The literature contains several algorithms for the reduction of large linear circuits. These techniques, which can be extended to MEMS, can be broadly classified into two categories. The first category is based on explicitly matching the moments to a reduced-order model, and the second category is based on

implicitly matching the moments. Techniques like AWE [26] belong to the first category, while techniques such as PRIMA [27], which are based on Krylov subspace formulation, belong to the second category.

4.2.3 MEMS Model Reduction Based on Krylov-Subspace Techniques

Model reduction based on explicit moment-matching employ Padé approximation to extract the dominant poles and residues of a given system. These techniques discussed in [25] have some disadvantages. First, the moment-matrix is ill-conditioned. Thus, the number of good poles that can be extracted from any expansion is generally fewer than ten. At the same time, these techniques do not guarantee the passivity of the reduced-order models. In order to address these difficulties, a parallel class of algorithms, indirect moment-matching techniques, were developed. These algorithms, which are based on Krylov-subspace formulation and Congruent transformation, have been used successfully for efficient simulation of high-speed VLSI interconnects [27]. This research extends the applicability of these techniques to general, three-dimensional MEMS. Specifically, congruent transformation techniques create a practical and accurate representation of realistic devices

in the three dimensional continuum. In Chapter 5, to validate the concept, the Arnoldi algorithm [27] was applied to several examples of electrostatically actuated microstructures. A significant reduction in the size of the systems was achieved. It is expected these results will continue to improve when more complex problems are considered.

These algorithms, based on the Krylov-subspace formulation and Congruent transformation, construct reduced models based on the extraction of leading eigenvalues, eigenvalues with the largest magnitude, of a given system. The reduced models from the direct moment matching techniques discussed in [25] are based on extracting the dominant poles of a given system. The following sections describe the concept of model reduction together with the important features of Krylov-subspace techniques.

Preliminaries

Recall the time domain equations describing linear MEMS problems can be represented in the quasi-state-space form [15]:

$$\begin{aligned} \mathbf{M}\dot{\mathbf{x}}(t) &= \mathbf{M}_A\mathbf{x}(t) + \mathbf{M}_B u(t); & \mathbf{M}, \mathbf{M}_A &\in \mathfrak{R}^{n \times n} & \mathbf{M}_B &\in \mathfrak{R}^{n \times 1} & \mathbf{x} &\in \mathfrak{R}^{n \times 1} \\ w &= \mathbf{L}^T \mathbf{x}(t); & \mathbf{L} &\in \mathfrak{R}^{n \times 1} \end{aligned} \tag{4.12}$$

where n represents the total number of state variables. Pre-multiplying both sides of (4.12) by $-\mathbf{M}_A^{-1}$ yields

$$\begin{aligned} \mathbf{A}\dot{\mathbf{x}}(t) &= \mathbf{x}(t) - \mathbf{R}u(t); & \mathbf{A} &= \mathbf{M}_A^{-1}\mathbf{M}; & \mathbf{R} &= -\mathbf{M}_A^{-1}\mathbf{M}_B \\ w &= \mathbf{L}^T \mathbf{x}(t) \end{aligned} \tag{4.13}$$

Taking the Laplace transform of (4.13) yields

$$\begin{aligned} s\mathbf{A}\mathbf{X}(s) &= \mathbf{X}(s) - \mathbf{R}U(s) \\ W(s) &= \mathbf{L}^T \mathbf{X}(s) \end{aligned} \tag{4.14}$$

and rearranging (4.14), the transfer function, $Y(s)$, of the given system is

$$Y(s) = \frac{W(s)}{U(s)} = \mathbf{L}^T (\mathbf{I} - s\mathbf{A})^{-1} \mathbf{R} \quad (4.15)$$

where $\mathbf{I}$ is an identity matrix.

A: Ill-Conditioning Associated With Direct Padé Based Approximation

Expanding the transfer function of (4.15) in terms of a Taylor series yields

$$\begin{aligned} Y(s) &= \mathbf{L}^T (\mathbf{I} + s\mathbf{A} + s^2\mathbf{A}^2 + s^3\mathbf{A}^3 + \dots + s^q\mathbf{A}^q) \mathbf{R} \\ &= \sum_{k=0}^q s^k (\mathbf{L}^T \mathbf{A}^k \mathbf{R}) = \sum_{k=0}^q s^k m_k \end{aligned} \quad (4.16)$$

where $m_k = \mathbf{L}^T \mathbf{A}^k \mathbf{R}$. According to (4.16), increasing the order of the Padé approximation, which is equivalent to matching more moments, should give a better approximation. However, in practice, beyond a very limited order, Padé approximation will not yield any better results [25]. The higher order moments are given by $m_k = \mathbf{L}^T \mathbf{A}^k \mathbf{R}$. Thus, when successive moment are explicitly calculated, they are obtained as powers of $\mathbf{A}$. As k increases, this process quickly converges to an eigenvector corresponding to an eigenvalue

of $\mathbf{A}$ with the largest magnitude. As a result, as k grows, the explicitly calculated moments, $m_k, m_{k+1}, m_{k+2}, \dots$, do not add any extra information to the moment matrix: all the moments contain information only about the largest eigenvalue. In other words, the rows beyond k of the moment matrix are almost identical, or parallel to each other, making the matrix very ill-conditioned.

B: Relationship Between Eigenvalues and Poles of The System

Krylov subspace based techniques extract the leading eigenvalues of a given system to obtain the reduced models. However, the leading eigenvalues correspond to the poles of a system. If the matrix $\mathbf{A}$ from (4.13) can be diagonalized in the form

$$\mathbf{A} = \mathbf{F} \boldsymbol{\lambda} \mathbf{F}^{-1} \quad (4.17)$$

where $\boldsymbol{\lambda} = \text{diag}[\lambda_1 \lambda_2 \dots \lambda_n]$ is a diagonal matrix whose diagonal elements represent the eigenvalues of matrix $\mathbf{A}$, the matrix $\mathbf{F}$ contains the eigenvectors of matrix $\mathbf{A}$. Using (4.17), the transfer function represented by (4.15) can

be re-written as

$$\begin{aligned}
Y(s) &= \mathbf{L}^T (\mathbf{I} - s\mathbf{F}\mathbf{A}\mathbf{F}^{-1})^{-1} \mathbf{R} \\
&= \mathbf{L}^T \mathbf{F} (\mathbf{I} - s\mathbf{A})^{-1} \mathbf{F}^{-1} \mathbf{R} \\
&= \mathbf{L}^T \mathbf{F} \begin{bmatrix} \frac{1}{1-s\lambda_1} & & \\ & \dots & \\ & & \frac{1}{1-s\lambda_n} \end{bmatrix} \mathbf{F}^{-1} \mathbf{R}
\end{aligned} \tag{4.18}$$

which can be simplified to

$$Y(s) = \sum_i \frac{\eta_i}{1 - s\lambda_i} = \sum_i \frac{-(\eta_i/\lambda_i)}{s - (1/\lambda_i)} = \sum_i \frac{k_i}{s - p_i} \tag{4.19}$$

where η_i is a function of the eigenvectors of matrix $\mathbf{A}$, and k_i represent the residues. This form reveals that the poles p_i are the reciprocals of the eigenvalues of matrix $\mathbf{A}$, and the leading eigenvalues, those with the largest magnitudes, correspond to the poles closer to the origin. Also, once the eigenvalues and eigenvectors of $\mathbf{A}$ are available, the transfer function of $Y(s)$ can be obtained in terms of poles and residues.

However, for the large systems of equations typically found in MEMS, it is impractical to compute all of the eigenvalues and eigenvectors. Thus,

efficient techniques to extract leading eigenvalues are required.

C: Computation of Eigenvalues of Matrix A

In general, the numerical computation of all the eigenvalues and eigenvectors of a given matrix A becomes too expensive as its size gets above a few hundred by a few hundred. To reduce the computational cost, A is approximated with a smaller matrix $\hat{A}$ such that the eigenvalues of $\hat{A}$ are a reasonable approximation of the leading eigenvalues of A . Since $\hat{A}$ is small, finding its eigenvalues will be much easier than finding the eigenvalues of A .

Specifically, a similarity transformation

$$AK = KH_n \quad (4.20)$$

may be performed matrix A where H_n has an upper-Hessenberg companion form [28], and the transformation matrix K is defined as

$$K = [\mathbf{R} \quad \mathbf{AR} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{R}]. \quad (4.21)$$

$\mathbf{R}$ is an upper-triangular matrix with positive diagonals ($r_{ii} > 0$) such that

$\mathbf{K} = \mathbf{QR}$. The modified Gram-Schmidt orthogonalization process is one of several techniques for performing $\mathbf{QR}$ decompositions [28]. Since $\mathbf{H}_n$ is related to the matrix $\mathbf{A}$ through a similarity transformation, its eigenvalues are the same as that of $\mathbf{A}$. This approach has some important limitations. For example, the computation of $\mathbf{H}_n$ using the relation ($\mathbf{H}_n = \mathbf{K}^{-1}\mathbf{AK}$) requires the inverse of the dense matrix $\mathbf{K}$, an expensive operation. As depicted in Figure 4.2 , $\mathbf{K}$ is usually ill-conditioned since the columns of $\mathbf{K}$ are formed

Figure 4.2: Convergence To Dominant Eigenvector

based on the sequence $\mathbf{A}^i\mathbf{R}$ which quickly converges to the eigenvector corresponding to the largest eigenvalue. Krylov subspace techniques overcome these problems.

Krylov-Subspace Methods For Iterative Computation of Eigenvalues

First, the matrix $\mathbf{K}$ in (4.20) is replaced with an orthogonal matrix $\mathbf{Q}$ such that, for all n , the leading columns of $\mathbf{K}$ and $\mathbf{Q}$ span the same space. This space is called the Krylov-subspace and is denoted $\kappa(\mathbf{A}, \mathbf{R}, n)$. This means any vector which is a linear combination of the leading n columns of $\mathbf{K}$ can also be expressed as a linear combination of the leading n columns of $\mathbf{Q}$. Thus,

$$\begin{aligned}\kappa(\mathbf{A}, \mathbf{R}, n) &= \text{ColumnSpace}([\mathbf{R} \quad \mathbf{AR} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{R}]) \\ &= \text{ColumnSpace}[\mathbf{Q}]\end{aligned}\tag{4.22}$$

In contrast to matrix $\mathbf{K}$, the matrix $\mathbf{Q}$ is well conditioned and is easy to invert since $\mathbf{Q}^{-1} = \mathbf{Q}^T$. Most importantly, only as many leading columns of $\mathbf{Q}$ as necessary are computed to get an accurate solution. Using $\mathbf{QR}$ decomposition, the matrix $\mathbf{K}$ is expressed as $\mathbf{K} = \mathbf{QR}_u$. $\mathbf{Q}$ is an orthonormal matrix, and $\mathbf{R}_u$ is an upper-triangular matrix. Thus, (4.20) can be modified

to

$$\mathbf{H}_n = \mathbf{K}^{-1} \mathbf{A} \mathbf{K} = (\mathbf{Q} \mathbf{R}_u)^{-1} \mathbf{A} (\mathbf{Q} \mathbf{R}_u) = (\mathbf{R}_u^{-1} \mathbf{Q}^T) \mathbf{A} (\mathbf{Q} \mathbf{R}_u) \quad (4.23)$$

or

$$\mathbf{Q}^T \mathbf{A} \mathbf{Q} = \mathbf{R}_u \mathbf{H}_n \mathbf{R}_u^{-1} = \mathbf{H} \quad (4.24)$$

Since $\mathbf{R}_u$ and $\mathbf{R}_u^{-1}$ are both upper triangular and $\mathbf{H}_n$ is upper Hessenberg, the new matrix, $\mathbf{H} = \mathbf{R}_u \mathbf{H}_n \mathbf{R}_u^{-1}$, is also upper Hessenberg [25].

If the first leading q columns ($q < n$) of $\mathbf{Q}$ are used, the matrices under consideration will have the following dimensions: $\mathbf{Q} \in \mathbb{R}^{n \times q}$ and $\mathbf{H} \in \mathbb{R}^{q \times q}$. Equation (4.24) implies that the matrix $\mathbf{A}$ of dimension $n \times n$ can be reduced to a smaller upper-Hessenberg matrix $\mathbf{H}$ of dimension $q \times q$ using orthogonal transformation. In addition, the eigenvalues of the smaller system $\mathbf{H}$ are approximations of the first q leading eigenvalues of the larger system represented by $\mathbf{A}$.

Additionally, the columns of $\mathbf{Q}$ can be computed one at a time giving the advantage of computing only as many leading columns of $\mathbf{Q}$ as needed. One of the popular approaches for partial reduction of a large matrix to a smaller

upper Hessenberg matrix using $\mathbf{Q}$ is known as Arnoldi's Algorithm [29]. The Arnoldi approach was applied to MEMS problems in this thesis.

As described above, the Arnoldi method computes a specified order of a macromodel by matching the moments of a truncated Taylor series of the transfer function. The Arnoldi method also guarantees the stability of the reduced order model.

Assume $\mathbf{Q} = [q_1 \ q_2 \ \dots \ q_k]$, where q_i represents the i^{th} column of matrix $\mathbf{Q}$. From (4.24)

$$\mathbf{A}\mathbf{Q} = \mathbf{Q}\mathbf{H} \quad (4.25)$$

or

$$\mathbf{A} \begin{bmatrix} \begin{bmatrix} q_1 \end{bmatrix} & \dots & \begin{bmatrix} q_k \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} q_1 \end{bmatrix} & \dots & \begin{bmatrix} q_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} & \dots & \dots & \dots \\ h_{21} & h_{22} & h_{23} & \dots & \dots & \dots \\ 0 & h_{32} & h_{33} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \end{bmatrix} \quad (4.26)$$

Recall $q_i^T q_i = 1$ and $q_i^T q_j = 0$. Using these properties, the first few steps in obtaining the $\mathbf{Q}$ and $\mathbf{H}$ matrices are outlined below.

Since $\|q_i\|_2 = 1$, a convenient way to calculate the vector q_i is to divide the vector $\mathbf{R}$ from (4.13) by its magnitude $\|\mathbf{R}\|_2$. This results in a unit vector in the direction of $\mathbf{R}$ where $q_1 = \frac{\mathbf{R}}{\|\mathbf{R}\|_2}$. The computation of q_1 , q_2 , h_{11} and h_{21} are illustrated in Figure 4.3. Next, to determine q_2 and the first column

Figure 4.3: Steps in Arnoldi Algorithm

of $\mathbf{H}$, $\mathbf{A}$ is multiplied by the first column of $\mathbf{Q}$ giving $\mathbf{A}q_1$, which is the first

column on the LHS of (4.26). Equating it with the first column of the RHS yields

$$\mathbf{A}q_1 = h_{11}q_1 + h_{21}q_2 \quad (4.27)$$

If both sides are premultiplied by q_1^T

$$\begin{aligned} q_1^T \mathbf{A}q_1 &= h_{11}q_1^T q_1 + h_{21}q_1^T q_2 \\ q_1^T \mathbf{A}q_1 &= h_{11} \end{aligned} \quad (4.28)$$

results. The value of h_{11} is known and $\|q_2\| = 1$; therefore, h_{21} can be computed from (4.27) as

$$h_{21} = \|\mathbf{A}q_1 - h_{11}q_1\|. \quad (4.29)$$

As illustrated in Figure 4.3, the direction for q_2 can be obtained using (4.27) as

$$q_2 = \frac{\mathbf{A}q_1 - h_{11}q_1}{h_{21}} \quad (4.30)$$

Similarly, the rest of the columns of the $\mathbf{Q}$ and $\mathbf{H}$ matrices can be obtained

from generalization of the above steps as shown in Figure 4.4 [29]. With

```

begin function_Arnoldi
     $q_1 = R / \|R\|_2$            \ \ get direction for  $q_1$ 
    begin for j = 1 to k       \ \ k number of columns for Q and H matrices
         $z = A * q_j$ 
        begin for i = 1 to j   \ \ begin nested for loop
             $h_{ij} = q_i^T * z$ 
             $z = z - h_{ij} * q_i$ 
        end                   \ \ for i
         $h_{j+1,j} = \|z\|$ ;
        if  $h_{j+1,j} = 0$       exit   \ \ quit function
        else
             $q_{j+1} = z / h_{j+1,j}$ ;
        end                   \ \ for j
    end function_Arnoldi

```

Figure 4.4: Arnoldi Algorithm

the Arnoldi algorithm, the product $A^k R$ is not computed; therefore, the ill-conditioning problem arising due to the quick convergence of the sequence $[R \quad AR \quad A^2R \quad A^3R \quad \dots]$ to the eigenvector of the largest eigenvalue is avoided.

The columns q_j computed by the Arnoldi algorithm are known as Arnoldi

vectors. The loop over i updating z corresponds to the modified Gram-Schmidt algorithm [29]: the components in the directions q_1 to q_j are subtracted from z , leaving them orthogonal to z . Computing k Arnoldi vectors costs k matrix-vector multiplications involving $\mathbf{A}$ plus $O(k^2n)$ related cost.

Several other methods may be used to find the basis for Krylov-subspace. For example, multiple passes of orthogonalization may be used to increase the robustness of the modified Gram-Schmidt orthogonalization process.

In summary, the equations are $\mathbf{M}\dot{\mathbf{x}}(t) = \mathbf{M}_\mathbf{A}\mathbf{x}(t) + \mathbf{M}_\mathbf{B}u(t)$ and $w = \mathbf{L}^T\mathbf{x}(t)$. From these, the product $\mathbf{A} = \mathbf{M}_\mathbf{A}^{-1}\mathbf{M}$ may be formed. Then, using orthogonal transformation, the leading eigenvalues of $\mathbf{A}$, which correspond to the dominant poles of the transfer function, are extracted.

MEMS reduction could be achieved by applying Arnoldi on the set of equations (4.14); however, this approach does not preserve the passivity of the reduced system. Instead, Arnoldi is applied on the original set of circuit equations represented by (4.12) [30] [31]. If $\mathbf{M}$ and $\mathbf{M}_\mathbf{A}$ are reduced directly, passivity can be preserved [25]. The remaining sections discuss the application of this procedure to perform MEMS model order reduction.

MEMS Reduction Using the Arnoldi Algorithm

Finding the reduced order MEMS equations shown in Figure 4.5 can be

Figure 4.5: MEMS Reduction

explained by a change of variables in (4.12). The vector $\mathbf{x}$ of dimension n is mapped into a smaller vector $\hat{\mathbf{x}}$ of dimension q ($q \ll n$) using the orthogonal matrix $\mathbf{Q}$:

$$\mathbf{x} = \mathbf{Q}_{n \times q} \hat{\mathbf{x}} \quad (4.31)$$

This is known as a congruence transformation. The system represented by (4.12) then becomes

$$\widehat{\mathbf{M}}\dot{\hat{\mathbf{x}}} = \widehat{\mathbf{M}}_A\hat{\mathbf{x}} + \widehat{\mathbf{M}}_B u \quad (4.32)$$

where

$$\widehat{\mathbf{M}} = \mathbf{Q}^T \mathbf{M} \mathbf{Q}; \quad \widehat{\mathbf{M}}_A = \mathbf{Q}^T \mathbf{M}_A \mathbf{Q}; \quad \widehat{\mathbf{M}}_B = \mathbf{Q}^T \mathbf{M}_B. \quad (4.33)$$

Using (4.31) and (4.32), the response of the original system can be calculated from the reduced model as

$$\mathbf{X}(s) = \mathbf{Q}(\mathbf{U}_q - s\widehat{\mathbf{M}}_A^{-1}\widehat{\mathbf{M}})\widehat{\mathbf{M}}_A^{-1}\widehat{\mathbf{M}}_B \quad (4.34)$$

The accuracy of the response of the reduced system given by (4.34) is an important criterion in the reduction schemes. The frequency response of the reduced system must be a good approximation of the frequency response of the original transfer function. An indicator for the accuracy of the response of the reduced system is the total number of moments it can preserve for a given order of reduction. It can be shown that the reduced system of order

q preserves the first q moments of the original system [29]. The Arnoldi based reduction of order q matches the first q moments, where a Padé approximation, a direct moment matching technique, of order q matches the first $2q$ moments. Thus, for comparable accuracy, the reduced model from Arnoldi will be twice as large as the reduced model from direct Padé based approximation. However, due to the ill-conditioning, direct Padé based approximation can not achieve the higher-order approximation available from Arnoldi based approximation.

Passivity Preservation

Passivity implies that a system of components cannot generate more energy than it absorbs. Also, it implies that no passive termination of the network will make the system unstable. Passivity is an important property because stable, but not passive macromodels can lead to unstable systems when connected to other passive systems. For example, without passivity preservation, transient simulations of nonpassive systems may encounter artificial oscillations. Again, the Arnoldi method used in this work guarantees the stability of the reduced order model.

4.3 Numerical Issues

4.3.1 Scaling

The natural frequencies of MEMS are usually on the order of KHz, MHz or even GHz; therefore, nondimensionalizing or rescaling quantities is necessary for simulation [6]. Such properties do not mean that a particular linear system will be inaccurately solved; however, it does raise suspicion. The ill-conditioning comes from a combination of sources. First, as the mesh density is increased to meet accuracy requirements, the ill-conditioning grows. Also, the different state variables have vastly different scales. A simple scaling technique involving scaling $\mathbf{M}$ and t of equation 4.12 is used in this work. The issues of ill-scaling/ill-conditioning are still under investigation.

Chapter 5

Computational Results

In this chapter, the reduced order models generated using the technique presented in Chapter 4 are compared to the full finite element and finite difference solutions. As the following examples illustrate, reduced order models of very low order accurately describe the original large linear systems. Even the most advanced finite element schemes result in a very large set of ordinary differential equations (ODEs). The model reduction technique reduces the number of ODEs to ultimately solve by several orders of magnitude; thus, a significant reduction in computation time is achieved.

5.1 Cantilever Beam

The first simulation results are for a cantilever beam shown in Figure 5.1. This figure depicts the meshed solid model of the beam in its initial, undeformed position. The three-dimension, thin beam has dimensions of $100 \times$

Figure 5.1: Mesh Structure for the Cantilever Beam

$20 \times 10 \mu\text{m}$. The beam has one clamped edge and one free end. For the entire length of the beam, an electrode is placed underneath the beam, separated from the beam by a $10 \mu\text{m}$ gap. The beam is made out of polysilicon with a Young's modulus of 165 GPa and a Poisson ratio of 0.23. This structure deforms upon actuation with a voltage.

First, a solid model of the beam is constructed using the commercial software tool FEMLAB[®]. Modeling of this structural mechanics problem

in FEMLAB[®] entails solving Navier's equations (2.11) in three-dimensions. An actuation voltage of 5.3 Volts is applied between the substrate and the cantilever beam. The resulting electrostatic force will be a function of beam deflection. To characterize this force accurately, coupled electro-mechanical simulation must be used where a numerical electrostatic solver must be used to find the force. The focus of this example is to prove the feasibility of model order reduction applied to MEMS. Thus, as a practical matter, an approximate calculation of the electrostatic force is used in lieu of full three-dimensional electrostatic analysis. The cantilever beam and the substrate are approximated as parallel-plate capacitors, and the electrostatic force F_e is calculated using the formula

$$F_e = -\frac{1}{2} \frac{\epsilon_0 l w V^2}{z^2} \quad (5.1)$$

which is presented in more detail in Section 3.1.3. A further simplification is made: F_e is assumed to be constant with deflection; thus, z is replaced with z_0 , a constant equal to $10\mu\text{m}$. Values for the above parameters, the structure's dimensions and the relevant material properties are summarized in Table 5.1.

While this model for the applied force may not be highly physically accurate,

Table 5.1: Parameters for Cantilever Beam Problem

Height=Gap, z_0	$10 \times 10^{-6} \text{ (m)}$	Young's Modulus, E	165 (GPa)
Length, l	$100 \times 10^{-6} \text{ (m)}$	Density, ρ	2330 (kg/m ³)
Width, w	$20 \times 10^{-6} \text{ (m)}$	Poisson Ratio, ν	0.23
Applied Voltage, V	5.3 (volts)	Permittivity, ϵ_0	$8.85 \times 10^{-12} \text{ (F/m)}$

it serves to validate the concept. Since equations governing the mechanical and electrical domains are, in reality, coupled through the elastic deformation of this structure that simultaneously serves as a plate of a capacitor, this difficulty is addressed in Chapter 6.

In this example, a constant force of $-1.25 \times 10^{-10} \text{ N}$ is used as input to FEMLAB[®]. It is uniformly distributed over the top surface of the beam and will cause the beam to deflect downward as shown in Figure 5.2.

After the force is applied, the structure is meshed to create the finite-element model. The linear, time dependent model available from FEMLAB[®] is converted to a quasi-state-space model [17]. The quasi-state -space equations are of the form

$$\mathbf{M}\dot{\mathbf{x}}(t) = \mathbf{M}_A\mathbf{x}(t) + \mathbf{M}_B\mathbf{u}(t) \quad (5.2)$$

Figure 5.2: Beam Deflection from The Equilibrium Position

$$w = \mathbf{L}^T \mathbf{x}(t) \quad (5.3)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector of unknowns, $\mathbf{M}, \mathbf{M}_A, \mathbf{M}_B \in \mathbb{R}^{n \times n}$ are constant matrices describing the material properties of the system and $\mathbf{u}(t)$ represents the excitation of the system. In this example, the size of the matrices $\mathbf{M}$ and $\mathbf{M}_A$ is 1326×1326 . This is called the original system.

Finally, the dynamics of the beam are simulated. Model order reduction was performed using the technique described in Chapter 4. An important issue is the accuracy of the reduced order model. Hence, the dynamical tran-

sients calculated using the reduced order model and the original system are compared. The full-finite element solution of the original system is obtained using the MATLAB® ODE suite.

The order of the original system is 1326×1326 . For a time domain comparison, the time history of the downward or z displacement of the endpoint of the beam, node 24, is included. Figure 5.3 shows a significant reduction in the size of the system without loss of accuracy. The responses at node

Figure 5.3: Time History of the Cantilever Beam Endpoint

24 of the original system of size 1326×1326 and the reduced system of size 20×20 match.

The results in this section demonstrate that the macromodel generated automatically by the Arnoldi method is efficient and accurate in solving the large linear MEMS model. This is a very useful step toward improving overall MEMS simulation.

5.2 Suspended Clamped Plate

A suspended clamped plate, a structure with nontrivial geometry, is analysed in this section. The $125 \times 155 \times 15\mu\text{m}$ plate is suspended by four $85 \times 15\mu\text{m}$ beams of unequal thickness. Two of the beams are $15\mu\text{m}$ thick, one is $10\mu\text{m}$ thick and one is $5\mu\text{m}$ thick. As in the previous example, an electrode is placed under the plate separated from the plate by a $15\mu\text{m}$ gap. This device is made out of polysilicon with a Young's modulus of 165 GPa and a Poisson ratio of 0.23. Due to the unequal beam thickness, this structure displaces, bends and tilts upon actuation with a voltage.

Again, Navier's equations (2.11) in three-dimensions must be solved. Thus, a solid model of the plate is constructed using the commercial software tool FEMLAB[®]. An actuation voltage of 19.5 Volts is applied between the substrate and the plate portion of the structure. To characterize the

Table 5.2: Parameters for Suspended Plate Problem

	Plate	Beam 1, Beam 4	Beam 2	Beam3
Length	$125 \times 10^{-6} \text{ (m)}$	$85 \times 10^{-6} \text{ (m)}$	$85 \times 10^{-6} \text{ (m)}$	$85 \times 10^{-6} \text{ (m)}$
Width	$155 \times 10^{-6} \text{ (m)}$	$15 \times 10^{-6} \text{ (m)}$	$15 \times 10^{-6} \text{ (m)}$	$15 \times 10^{-6} \text{ (m)}$
Height=Gap	$15 \times 10^{-6} \text{ (m)}$	$15 \times 10^{-6} \text{ (m)}$	$10 \times 10^{-6} \text{ (m)}$	$5 \times 10^{-6} \text{ (m)}$

resultant electrostatic force F_e accurately, highly meshed coupled numerical electro-mechanical simulation must be used.

The purpose of this example is to further demonstrate the benefits of model order reduction in the area of MEMS. Thus, as in the previous example, an approximate calculation of the electrostatic force substitutes for full three-dimensional electrostatic analysis. The plate and the electrode are approximated as parallel-plate capacitors, so the electrostatic force F_e is calculated using the formula

$$F_e = -\frac{1}{2} \frac{\epsilon_0 l w V^2}{z^2}. \quad (5.4)$$

A further simplification is made: F_e is assumed to be constant with deflection; thus, z is replace with z_0 , a constant equal to $15\mu\text{m}$. Values for the above parameters, the structure's dimensions and the relevant material properties are summarized in Table 5.2.

A constant force of $-1.7 \times 10^{-9} \text{ N}$ is used as input to FEMLAB[®]. It is uniformly distributed over the top surface of the plate and will cause the plate to deflect and tilt downward.

After the force is applied, the structure is meshed to create the finite-element model. The linear, time dependent model available from FEMLAB[®] is converted to a quasi-state-space model [17]. In this mechanically nontrivial example, the order of the original system as determined by the size of the matrices $\mathbf{M}$ and $\mathbf{MA}$ is $10,344 \times 10,344$.

Finally, the dynamics of the suspended plate are simulated. A reduced order model is constructed using the technique described in Chapter 4, and, using the MATLAB[®] ODE suite, the full-finite element solution of the original system is obtained.

FEMLAB[®] creates a model of size $10,344 \times 10,344$ for this suspended plate. However, a typical MEMS device is much thinner than the plate in this example; therefore, the size of the system would be much larger than $10,344 \times 10,344$.

To compare the original and reduced systems, the response of all the nodes in the structure at a point in time is included. Figure 5.4 shows a significant reduction in the size of the system without loss of accuracy. The

leftmost indicator bars are the z or downward deflection in meters, and the rightmost indicator bars measure the stress [17]. The response of the original system of size $10,344 \times 10,344$ and the reduced system of size 20×20 match. A reduction of 517 : 1 is achieved.

As in the previous section, the results demonstrate that the macromodel generated automatically by the Arnoldi method is efficient and accurate in solving the large linear MEMS model. Most importantly, it is expected that the reduction ratio and, hence, speed-up will continue to improve when yet more complex problems are considered.

5.3 Pull-in Time Pressure Sensor

The dynamics of a squeezed-film damping, MEMS problem involving mechanical, electrostatic and fluidic components are examined. The device analysed here and shown in Figure 3.3 is described in detail in Section 3.2.2. The x and z axes of Figure 3.3 are in the direction of the length and width of the beam respectively. The structure extends into the page in the y direction. In this configuration, the flexible top plate is clamped on both edges in the y direction and will deform upon actuation with a voltage. The response

Figure 5.4: Original and Reduced Systems for the Suspended Plate Problem

of the top-plate is sensitive to the air pressure in the air film under it [22]. The beam has fixed supports, and the fluid system is open. Thus, ambient pressure is assumed along the sides of the beam, and no flow is assumed at the ends of the beam.

Following the assumptions described in Section 3.3, the device is modeled using the linearized, normalized, one-dimensional Euler beam equation and the linearized, normalized, two-dimensional Reynolds equation:

$$h_0^2 p_a \nabla^2 p = \frac{12\mu}{1+6K} \left(\frac{\partial p}{\partial t} + \frac{\partial u}{\partial t} \right) \quad (5.5)$$

and

$$EI \frac{\partial^4 u}{\partial x^4} - S \frac{\partial^2 u}{\partial x^2} = F_{elec}^* + \frac{p_a}{h_0} \int_0^w p dy - \rho \frac{\partial^2 u}{\partial t^2} \quad (5.6)$$

where $F_{elec}^* = -\frac{\epsilon_0 w V^2}{2h_0^3}$.

Under these assumptions, the displacement of the plate u is uniformly distributed along the y axis, and, since the air film between the gap is very thin, the pressure p is uniform across the film in the z direction.

The beam has a length l of $610\mu\text{m}$, a width w of $40\mu\text{m}$ and a height h of $2.2\mu\text{m}$. The undeflected gap between the plate and substrate h_0 is $2.3\mu\text{m}$.

Table 5.3: Parameters for Pressure Sensor Problem

Height, h	$2.2 \times 10^{-6} \text{ (m)}$
Young's Modulus, E	149 (GPa)
Length, l	$610 \times 10^{-6} \text{ (m)}$
Density, ρ	$2330 \text{ (kg/m}^3\text{)}$
Width, w	$40 \times 10^{-6} \text{ (m)}$
Air Viscosity, μ	$1.82 \times 10^{-5} \text{ (kg/m} \cdot \text{s)}$
Undelected Gap, h_0	$2.3 \times 10^{-6} \text{ (m)}$
Mean Free Path of Air, λ	$0.064 \times 10^{-6} \text{ (m)}$
Moment of Inertia, I	$(w \cdot h_0^3)/12$
Stress, S	-3.7 (MPa)
Permittivity of air, ϵ_0	$8.85 \times 10^{-12} \text{ (F/m)}$
Applied Voltage, V	$0.1 \text{ and } 2 \text{ (Volts)}$
Knudsen's Number, K	λ/h_0
Ambient Air Pressure, p_a	$1.013 \times 10^5 \text{ (Pa)}$

As in the previous examples, the aspect ratio of length to width is very large. This is a feature typical of MEMS structures. The values of E , I , S , ρ , ϵ_0 , μ and V are summarized in Table 5.3. This is the same set of parameters used in [22].

Finite differences, which are described in Section 3.2.3, are used to transform the continuous state equations into discrete form. The top plate is discretized using the same scheme as in [22], and a system of order 880×880 results. Next, a reduced order model is constructed using the reduction technique presented in Chapter 4.

A comparison of the reduced system and the full finite difference time

domain simulation is shown in Figures 5.5 and 5.6. In Figure 5.5, the input voltage is 0.1 volts which ensures that the device is operating in the linear region. In Figure 5.6, the device is operated at an input voltage of 2 volts. At this voltage, the pressure sensor device is still operating in the linear region. In both cases, the response from the reduced order model, a 10×10 system, matches the response from the original system, a 880×880 system.

Figure 5.5: Pressure Sensor Response at $V=0.1$ Volts

This mechanical system is usually operated at low frequencies. The frequency response generated using the original system and the reduced system are compared in Figure 5.7. The response of the macromodel of order 10

Figure 5.6: Pressure Sensor Response at $V=2$ Volts

follows the behavior of the original system up until 10^7 Hz. Thus, this macro-model is accurate enough for most purposes.

These results are in good agreement with the work reported in [21]. The results show that very low order models can accurately describe the time and frequency responses of the original linear coupled domain MEMS problem.

In Figure 5.6, the reduced model matches the original linear model; however, the nonlinear simulation results presented in [22] show, at an input voltage of 9 volts, the pull-in voltage, the top plate collapses onto the substrate at the bottom. The original large linear model does not capture this

Figure 5.7: Frequency Response Comparison for Pressure Sensor

nonlinear behavior. This voltage is used as a reference to locate the linear operating region of the structure. Linear models are accurate to approximately fifteen percent of the total gap distance [21]. Then, nonlinear simulation techniques such as those discussed in Chapter 6 become important.

Chapter 6

Nonlinear MEMS

6.1 The Nonlinear MEMS Simulation Problem Formulation

In MEMS simulation, many of the difficulties arise from the implicit dependence of the forces on the displacements, the high frequencies of the structures and the nonlinear behaviors of the devices. In this chapter, dynamic simulation and analysis methods for nonlinear MEMS are studied. In Section 6.2, waveform relaxation, is demonstrated to simulate the nonlinear dynamic behavior of MEMS. In MEMS analysis, this is a new approach for coupled field simulation. This numerical approach has been applied on a clamped-clamped

beam structure. As in the linear MEMS simulation problem of Chapter 3, dynamically accurate nonlinear macromodels are needed to make simulation time feasible. Chapter 4 describes a technique to derive such macromodels directly from physical coupled-domain simulation. This technique works when the problem is linear. However, since most micromachined devices are at a minimum mildly nonlinear, new techniques must be developed. At this time, there have been few successes in nonlinear and time-varying model order reduction. Only ad-hoc approaches for well understood applications have been reported. In Section 6.3, the problems associated with extending nonlinear circuit model order reduction to MEMS are presented. Further research in nonlinear numerical macromodeling of MEMS forms the basis for automatic compact model generation and new device or technology models. This chapter gives an overview of the problems associated with modeling and simulation of nonlinear MEMS. It briefly introduces new solutions, one of which is supported by a simple example. Most importantly, it provides a useful starting point for future MEMS research. In essence, this chapter provokes many new and exciting questions.

6.1.1 Self-Consistent Analysis

Figure 6.1 resumes the coupled-field requirements for many MEMS. Arrows represent interaction between different fields. Chapter 3 introduced the need for self-consistent analysis when it described the dependence of the electrostatic force on vibrational amplitude. Both discrete and analytical methods are often applied to perform analysis of MEMS. The coupling between different fields renders analytical methods complex and often intractable except for simple device geometries and boundary conditions. In this work, waveform relaxation is adapted from high-speed interconnect simulation to solve MEMS problems. This is the subject of Section 6.2.

6.1.2 Nonlinearities In Individual Physical Domains

Chapter 2 presents the equations that govern MEMS. Apart from the nonlinearity due to the coupling of different domains, in some cases, the equations of each individual physical domain are nonlinear.

In electrostatically actuated MEMS, as the input voltage grows, although, from a structural mechanics analysis perspective, the structure is still undergoing small deformation and in a linear elastic state, the behavior of the electrostatic force is nonlinear. Also, if the loads are sufficient to create a

Figure 6.1: Coupled Fields in MEMS

non-negligible change in the in-plane stress or tension, the structure becomes a nonlinear elastic element with an incremental stiffness that increases with load. When displacements are large, the problem has nonlinear geometry; thus, the strains no longer vary linearly with the displacements. Material nonlinearities occur when a stress level sufficient to cause plastic flowing occurs. The plastic region is characterized by the fact that strains will remain in the body even after it is completely unloaded [15]. Many electrostatically actuated MEMS configurations exhibit an important behavior called pull-in. The equilibrium that must exist between the electrostatic force pulling the

structure down and the spring force opposing the motion of the structure is examined. There is a specific voltage at which the stability of the equilibrium is lost, and the structure will collapse onto the substrate. This is known as the pull-in voltage. Spring softening is another important MEMS phenomenon. As pull-in is approached, the effect of the nonlinear electrostatic force in reducing the net restoring force is evident and shows up in the spring softening behavior. Nonlinear behaviors are almost always important in MEMS fluids analysis. When subjected to shear forces, fluids deform without limit. Also, the viscosity makes this deformation inherently dissipative and irreversible.

The nonlinearities described above intersect many different issues involved in adapting model order reduction for MEMS. The problem is much more difficult than in the linear case. This problem is analogous to model order reduction of an electrical network containing a nonlinear element, a diode for example, at every node. However, considering a rich array of nonlinear model order reduction tools are being developed in parallel with this work, many new contributions are in the near vicinity; thus, nonlinear model order reduction for MEMS, the subject of Section 6.3, is one of the main motivations for the proposed future work.

6.2 Waveform Relaxation Techniques for Coupled MEMS Simulation

The principle of waveform relaxation analysis is presented in [32]. In this section, the fundamental concept of waveform relaxation is introduced with some basic examples. Then, Cheng's adaptation of waveform relaxation for high-speed interconnect circuit simulation [33] is further developed for MEMS simulation.

The following two equations

$$f_1(x_1, x_2) = 0 \quad (6.1)$$

and

$$g_1(x_1, x_2) = 0 \quad (6.2)$$

with unknowns x_1 and x_2 are considered. To solve this problem using relaxation techniques, a solution for x_1 is assumed. For example, $x_1 = 5$. Of course, $x_1 = 5$ is not the correct solution. However, substituting $x_1 = 5$ into (6.1) yields an equation with only one unknown. Thus, x_2 can be calculated.

This value for x_2 is then substituted into (6.2) to get a new value for x_1 . Once again, this new value for x_1 is substituted into (6.1) to get x_2 . In most cases, this process eventually converges to the solution of x_1 and x_2 . This basic idea can be extended to solve, for example, a set of linear algebraic equations where x_1, x_2 and x_3 are the unknowns. First a solution for x_2 and x_3 are assumed so that x_1 may be calculated. This value for x_1 along with the guess for x_3 are used to find x_2 . This idea can be extended further to solve a set of nonlinear differential equations [32] where the unknowns are waveforms. The following two, nonlinear, differential equations

$$f_1(\dot{x}_1, \dot{x}_2, x_1, x_2) = 0 \quad (6.3)$$

and

$$g_1(\dot{x}_1, \dot{x}_2, x_1, x_2) = 0 \quad (6.4)$$

with unknowns $x_1(t)$ and $x_2(t)$ are considered. Again, to solve this problem, $x_1(t) = r_1(t)$ is assumed. Substituting $r_1(t)$ into (6.3) yields an equation in one unknown. Next, $x_2(t)$ is calculated. As in the algebraic equation case presented above, this value of $x_2(t)$ can be substituted into (6.4) to find the

new value for $x_1(t)$. This process is repeated until the solution of $x_1(t)$ and $x_2(t)$ are obtained.

Since the equations are decoupled, at each step, there is a single equation and a single unknown. In the algorithm described below, the same basic ideas are used to solve MEMS nonlinear differential equations. This was proposed by Cheng in [33] for mixed frequency-time domain simulation.

6.2.1 MEMS Waveform Relaxation Algorithm

In this work, waveform relaxation as presented in [33] is applied for the first time to MEMS. The method takes the form of the algorithm described below.

1. A force $f^{(k-1)}(t)$, some initial guess for the force at all time points, is applied to the structure, and the simulation is carried out for the entire time interval. In terms of the *new* displacement waveform $u^{(k)}(t)$, the *new* waveform $f^{(k)}(t)$ is generated by recalculating the applied force. The new waveforms $u^{(k)}(t)$ and $f^{(k)}(t)$ for $0 < t < T$ are stored.

2. The *new* waveform $f^{(k)}(t)$ obtained in Step 1 is used for the applied force, and the simulation is carried out for the entire time interval. In terms of the *new* displacement waveform $u^{(k+1)}(t)$, the applied force waveform is updated by recalculating the applied force. The new waveforms $u^{(k+1)}(t)$ and

$f^{(k+1)}(t)$ for $0 \leq t \leq T$ are stored for the next iteration.

3. This iteration process continues until the difference between the *new* and the *old* displacement waveforms $u^{(k+1)} - u^{(k)}$ is sufficiently small. Otherwise, $k = k + 1$ and the iteration process continues.

6.2.2 Example

The top view of an electrostatically actuated, clamped-clamped beam problem is shown in Figure 6.2. This example is chosen to illustrate waveform

Figure 6.2: Clamped-Clamped Beam

relaxation applied to MEMS. This figure depicts a two-dimensional repre-

sentation of the problem. The geometry is specified with a width of $40\mu\text{m}$, a length of $600\mu\text{m}$ and a height of $2.2\mu\text{m}$. The beam is fixed at both ends. An electrode is place at a distance of $2\mu\text{m}$ under the entire length of the beam. The beam is made out of polysilicon with a Young's of 149 GPa and a Poisson ratio of 0.23. This structure vibrates upon actuation with a voltage.

A two-dimensional model is constructed using FEMLAB[®]'s structural mechanics module [15]. The dynamic problem has the following form:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}(t) \quad (6.5)$$

where $\mathbf{K}$ is called the stiffness matrix from linear static problems and $\mathbf{F}$ is the externally applied forces. $\mathbf{M}$ is the mass matrix for the object and $\mathbf{C}$ is the damping matrix. The damping matrix is assumed to be some linear combination of the stiffness and the mass matrix. This is known as Rayleigh damping. Discrete Kirchhoff triangles are selected as the element type. With this approach, applied forces in the z direction and moments about the x and y axis are allowed. This model is converted to a quasi-state-space model [17]. The structure's dimensions and material properties are summarized in Table

Table 6.1: Parameters for the Beam Problem

Height	$2.2 \times 10^{-6} \text{ (m)}$	Young's Modulus, E	149 (GPa)
Length	$610 \times 10^{-6} \text{ (m)}$	Density, ρ	2330 (kg/m ³)
Width	$40 \times 10^{-6} \text{ (m)}$	Poisson Ratio, ν	0.23

6.1.

The electrostatic force due to the actuation voltage V of 5 Volts will cause the beam to deform. The electrostatic force and the elastic deformation of the structure are coupled. Thus, the dynamics of this structure are evaluated using waveform relaxation. FEMLAB[®] was used to solve for the deflection at each iteration, and the simple, approximate formula presented in Section 3.1.3,

$$F_e = -\frac{1}{2} \frac{\epsilon_0 l w V^2}{(gap - z)^2}, \quad (6.6)$$

was used to solve for the electrostatic force at each iteration. Under this approximation, the electrostatic force is distributed uniformly over the top surface of the beam. However, this force is still dependent on the deflection of the beam center point; therefore, a coupled simulation is performed. While this model may not be highly physically accurate, it serves to illustrate the technique. This equation can easily be replaced with a commercial, elec-

trostatic solver. In general, waveform relaxation preserves the “black-box” nature of coupled MEMS simulation.

In this example, fifteen iterations were required to obtain the final solution. It is expected that this algorithm may take significantly longer to converge when the structures are too flexible or the electric fields are too large. However, to eliminate this problem, Newton waveform relaxation, a technique with better convergence properties, is proposed.

For comparison purposes, an uncoupled simulation is performed in FEMLAB[®] assuming a constant, force which is independent of vibrational amplitude. This constant force is calculated using the initial value for the gap distance $g_0 = 2\mu\text{m}$. Results in Figure 6.3 show that the waveform relaxation approach which accounts for the electro-mechanical coupling is much more accurate than using an uncoupled approximation. Therefore, waveform relaxation is a promising new technique for coupled, three-dimensional MEMS analysis. The observations derived from this simple example validate the call for further study discussed in Chapter 7.

Figure 6.3: Coupled Versus Uncoupled Simulation

6.3 Nonlinear Model Order Reduction

Integrated circuit designers have sophisticated tools to handle the problems associated with high-speed interconnect analysis. Accurate macromodels can be automatically generated for use in a circuit-level simulation of the entire system. MEMS designers have the same requirement. To this end, throughout this thesis, the need for nonlinear reduced-order models is highlighted. However, unlike high-speed interconnects, many MEMS are nonlinear. In general, MEMS properties make model order reduction of MEMS very different from high-speed interconnect reduction: the problem of nonlinear model order reduction for MEMS is an open research area.

The first success in true nonlinear model order reduction is reported in [34]. An algorithm is described for projecting full *nonlinear* circuit equations into a Krylov subspace of lower dimension. The reduced system can be solved using a conventional numerical integration technique. A significant reduction in circuit simulation time is achieved because, with the reduced system, at each time point, a much smaller set of nonlinear algebraic equations is solved. Also, in [35], nonlinear model order reduction techniques have been used to find the steady-state solution of nonlinear circuits. There are many approximate approaches to nonlinear model order reduction based on adapting the

linear modes to describe the nonlinear problem. The idea of nonlinear dynamics using linear modes is presented in [36] and [37]. Using the linear modes is considered a justified approximation because present restrictions make the procedure of projecting the full nonlinear MEMS equations into a Krylov subspace of lower dimension too costly.

Chapter 7

Conclusions and Future Work

MEMS science and technology are growing rapidly in response to an ever-increasing demand for faster, cheaper, smaller and more sensitive means to monitor and control the chemical, biological and physical world. MEMS sensors will have a global impact in environmental cleanup, industrial process control, emissions monitoring, aeronautical and space systems, screening for explosives, home and workplace safety, and medical diagnosis and care. MEMS sensors provide some of the functionality of macroscale instrumentation, but with vastly reduced cost, size and power consumption, and an ability for real-time, in situ measurement. A fast growing demand for MEMS computer aided engineering and design systems follows naturally from the

explosive growth of the MEMS industry.

Most MEMS technologies are based on batch processing and dedicated to mass production. The wide variety found in MEMS simulation makes it impossible to rely on time consuming, cost-ineffective trial and error design methodologies. The magnitude of the computational expense associated with MEMS system simulation places enormous premiums on fast, accurate simulation methodologies. Finite element based, commercially available CAD tools handle simulations of complex geometry and coupled domains. However, to achieve an accurate solution, the resulting large, coupled systems require prohibitive amounts of memory and computation time, often days or weeks, to solve. The development of fast, accurate MEMS design tools will reduce the entry cost of MEMS.

To support the continued success of MEMS, new computer aided design techniques are reported. First, a macromodeling technique, based on the Arnoldi algorithm, allows the generation of manageable, compact representations of the dynamics of MEMS devices operating in the linear region. The theory and numerical tools developed are supported by examples in view of validating the approach. The macromodels are fast to simulate, requiring only the initial investment of computation time to construct them. Using

these reduced-order models, further simulations involving the same device can be performed with no significant computational cost. This modeling strategy makes system level design possible: numerous simulations of MEMS containing many devices may be carried out on a typical workstation in just seconds or minutes. Also, a new technique for MEMS simulation, waveform relaxation, is introduced to handle the coupling between the various domains. The approach may be implemented using existing, highly optimized, domain-specific solvers. Lastly, to accent the need for further research, a discussion of nonlinear model order reduction for MEMS is presented. Together, the methods presented here enable macromodel construction for a wide variety of complex MEMS that would otherwise be too difficult for designers to macromodel. In summary, the resultant methodology shows progress towards the flexibility, speed and accuracy required for solutions of interest to the MEMS community.

There are some clear directions for continued evolution of this research. First and foremost, nonlinear model order reduction must be applied to a representative set of MEMS. This might include exploring techniques to accelerate the computational expense associated with the formation of the reduced order models. For other work, the macromodeling implementation

should be extended to other conservative and nonconservative domains, such as magnetics. If the macromodel has multiple input and output ports, the number of poles in the reduced order model is typically the number of ports multiplied by the number of poles required to match the response at a single port. Eigenvalue decomposition may be used to remove redundant poles, thus, reduce the size of the macromodel further [38]. The efficiency and the convergence properties of waveform relaxation, a new technique for coupled MEMS simulation, require further study. Using sensitivity analysis and optimization in future MEMS research is an open area that holds a key to the invention of new MEMS devices. In this case, multidimensional model order reduction [39] for MEMS can be applied. For example, optimizing the shape and configuration of a MEM structure allows the designer to maximize the linear operating range of a MEM structure, to modify the structure's natural frequencies and mode shapes, or to reduce air damping on the structure. In general, highly meshed simulations with state-of-the-art finite element modeling programs are sufficiently accurate to serve as the exact solution. However, the correctness of meshed models depends on the accuracy with which such effects as real geometric variations, boundary conditions and constitutive properties, especially residual stress variations, are

captured [9]. Thus, fabrication followed by careful experiment, while slow and expensive to obtain, provides the only true “exact” solution. In this work, the responses generated using the reduced order models match the responses obtained from the original finite element simulations. In the end, as expected, the correctness of the reduced order models depends on the accuracy of the original model. An increase in the accuracy of the original model usually leads to an increase in the size of the resulting system of equations. Model order reduction makes it possible to reduce the size of the systems while retaining the important model properties. This leads to dynamically accurate macromodels that are practical in a high level simulation setting. To this end, research must begin to construct macromodels which account for more effects such as stress-stiffening in MEMS structures. True reality simulation is far beyond the reach of MEMS designers today. However, before expensive physical prototypes are constructed, the completely mathematical approaches suggested here reveal important design issues. The prospect for a design tool that performs accurate, numerical macromodeling and couples the effects of different types of physical phenomena will have tremendous impact on the field of MEMS design.

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