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**DYNAMIC RESPONSE OF HIGHWAY BRIDGES
TO MOVING VEHICLES**

by

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B. Eng.(Hons) - Cairo University, Egypt

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**A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY**

**DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING
CARLETON UNIVERSITY
OTTAWA, CANADA
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TO MOVING VEHICLES**

submitted by
Ahmed Hassan Kashif
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Engineering

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ABSTRACT

The dynamic response to moving vehicle load is an important consideration in the design of bridges. Interaction between the vehicle and the bridge affects design in different ways. First, the dynamic nature of the response amplifies the deflections and stresses induced in the bridge structure which should therefore be designed to sustain such increased stresses and to keep the deflection within limits. Second, the vibrations induced by dynamic interaction may have adverse psychological effect on the pedestrians using the bridge.

A number of theoretical investigations, most of them based on an idealized one dimensional beam model of the bridge, have been carried out during the past 30 years to study the dynamic response of different kinds of bridges. Codes have traditionally allowed for the dynamic effect of moving loads on stresses and deflections by specifying empirical impact factors. These impact factors have been progressively refined on the basis of results obtained from dynamic response analysis of one dimensional models. Further, most of the codes have prescribed limits on live load deflections and span-to-depth ratios to keep the vibrations within comfort level.

In spite of a number of analytical and experimental investigations on the dynamic response of bridges to moving vehicle loads, the controlling parameters that govern the response have not been clearly identified. This has, in turn, inhibited the development of rational design procedures. Based on an analytical investigation of the response of a simplified beam model traversed by a moving mass, the first part of the present study identifies the governing parameters. The results clearly show why attempts to correlate the response to a single parameter, either the span length or the fundamental frequency, are unsuccessful. The bridge response is controlled by three major parameters, namely the speed parameter α , the frequency ratio ϕ , and the mass ratio κ . In addition the initial conditions of the vehicle on entry to the bridge also significantly influence the response.

The investigation is then extended to include two-axle vehicle model. The dynamic response of a bridge traversed by a two-axle vehicle model is governed by six parameters, namely, κ, ϕ, α , the location of the center of mass, a_1 , the spacing between the axles, S/L , and the mass moment of inertia of the vehicle body, $\dot{I}$. Based on the studies related to

single and two-axle vehicle, simple design procedures are developed taking into account the relationship between the speed ratio, the weight ratio and the dynamic amplification factors and a set of design curves are developed for both a single-axle vehicle model and a two-axle model.

The second part of this work is related to the analysis of two dimensional plate models so as to account for the transverse flexural and the torsional modes. The bridge is idealized by a rectangular isotropic or orthotropic plate and the vehicle is represented by a single sprung mass moving along the deck. For the purpose of analysis, the plate is discretized by quadrilateral thin plate elements. The resulting nonlinear differential equations are solved in their incremental form using a step by step numerical integration technique. The results show that the governing parameters for a bridge modelled by an isotropic plate are: the aspect ratio, the speed parameter, the frequency ratio, and the mass ratio. For the orthotropic idealization, the characterizing parameters that govern the dynamic response of a bridge structure are: parameters χ and θ related to the geometry and stiffness of the bridge, the speed parameter, the frequency ratio, and the mass ratio. The response of a bridge under the passage of multiple vehicles is also investigated. Response is calculated for two vehicles traveling parallel to each other and side by side, parallel but staggered, or in series. The response is then compared to that produced by a single vehicle of the same mass traveling along the center line. It is found that the response produced by two vehicles each of mass m_v , traveling abreast of each other along paths symmetrically located with respect to the center line, is similar to that produced by a single vehicle of mass $2m_v$ traveling along the center line. For the other two cases – two vehicles in series or staggered – the response is smaller than that produced by a single vehicle of mass $2m_v$ moving along the center line.

In the third part of the study a 3D model of the bridge is used to analyze the dynamic response of simple span box girder bridges. The finite element procedure is used in the analyses. The bridge is modelled by using quadrilateral thin shell elements. The vehicle is idealized as a two-axle, four wheel system with sprung and unsprung masses. The sprung mass represents the vehicle body while the unsprung masses represent the mass of the wheels and the axles. The unsprung masses of the vehicle are assumed to be always in contact with the bridge surface during the motion of the vehicle. Damping in both the vehicle and the bridge structure is assumed to be viscous. The natural frequencies and mode shapes

of the bridge model strongly influence its response and permit a better understanding of its behaviour under moving loads. The first phase of the 3D study is therefore devoted to the determination of the natural frequencies and mode shapes of bridge models and to an examination of the effect of several parameters on these characteristics of the structure.

The second phase of the 3D analysis is focused on the study of forced vibration response of a box girder bridge traversed by a vehicle. The effect of several parameters on the response of this type of bridges is investigated. The analyses reveal that a beam or an isotropic plate model is incapable of representing the behaviour of box girder bridges. The study also shows that the governing parameters of box girder bridges are not simply ϕ , κ , α , and the aspect ratio. In this case, the relative value of the flexural and torsional rigidities and their distribution also affect the response.

To my parents, with affection and gratitude

To my sisters, with respect and love

and

To my wife, with all my love

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List of Symbols

CHAPTER 1:

a	amplitude of the dynamic component of acceleration in the fundamental mode
e_1	first mode eigenvalue
DI	dynamic impact factor, the ratio of the maximum dynamic deflection to the maximum static deflection under the moving force
DI_{max}	maximum dynamic impact factor
DIN	the dynamic increment
E	Young's modulus
$E_b I_b$	the stiffness of the stringer
f_{b1}	first natural frequency of bridge in Hz.
f_b°	the observed fundamental frequency of bridge structure
f_{b1}	the loaded fundamental frequency of the bridge
$[F]$	the flexibility matrix
g	gravitational acceleration

h	thickness of the slab of the bridge
$1/H$	stringer flexibility
I_{peak}	a measure of dynamic amplification
I_{range}	a measure of dynamic amplification
J	a parameter, determined experimentally, reflects the maximum possible contribution to dynamic impact from the alternating component of the truck load
l_2	length of the second span of a continuous bridge
L	span length of the bridge
m_b	mass of the bridge
m_1	mass of the load
p	percentage of critical damping
S	axle spacing of two-axle vehicle model
t	the time after the entry of the load on the span
T_b	fundamental period of the bridge structure
v	the speed of the vehicle
W_{DL}	self-weight of the beam per unit length

$Y_{st(+)}$	positive static response of a component of the bridge
$Y_{st(-)}$	negative static response of a component of the bridge
$Y_{(+)}$	positive dynamic response of a component of the bridge
$Y_{(-)}$	negative dynamic response of a component of the bridge
α	speed parameter
δ_s	static component of the deflection
δ_D	dynamic component of the deflection
δ	logarithmic decrement
η_c	coefficient of vibration
ν	Poisson's ratio

CHAPTER 2:

A, B	constants
A_m	dimensionless amplitude
a_1, a_2	centeroidal distance parameters as shown in the vehicle model
c	damping constant of the damper of the vehicle
c_f, c_r	the viscous damping constants for the front and the rear axle

suspension system of the vehicle

$\mathbf{c}_v, \mathbf{c}_h$	time dependence matrices
$\mathbf{c}^*$	the damping matrix superimposed to the bridge equations of motion due to the vehicle inertia effect
$\mathbf{C}$	damping matrix of the bridge
$\mathbf{D}$	vector of the nodal d.o.f. of the bridge
EI	flexural rigidity of the bridge
k	stiffness of the spring of the vehicle
k_f, k_r	the spring stiffnesses constants for the front and the rear axle suspension system of the vehicle
$\mathbf{k}_h, \mathbf{k}_v$	time dependence matrices
k^*	effective stiffness
$\mathbf{k}^*$	the stiffness matrix added to the bridge equations of motion due to the vehicle inertia effect
$\mathbf{K}$	stiffness matrix of the bridge
m_t	unsprung mass of the vehicle
m_v	sprung mass of the vehicle
m^*	effective mass

$\mathbf{m}^*$	the mass matrix augmented to the bridge equations of motion due to the vehicle inertia effect
m	mass of the bridge per unit length
$\mathbf{M}$	mass matrix of the bridge
n	integer value
$\mathbf{N}$	row vector of the shape functions
h	maximum depth (height) of a bump
l	length of a bump
$\mathbf{H}_1, \mathbf{H}_2$	matrices each of size 1×1
$\mathbf{f}_1, \mathbf{f}_2$	matrices each of size 1×2
F_1, F_2	the spring forces in the front and rear axles suspension systems
$\mathbf{g}_1, \mathbf{g}_2$	matrices each of size 1×2
$\mathbf{G}_1, \mathbf{G}_2$	matrices each of size 1×1
p^*	effective external load
P_1, P_2	the contact forces between the vehicle and the bridge
$s_1, s_2,$ s_3	amplitudes of the static deflection for the lowest three modes of the bridge

q_1	bounce D.O.F of the vehicle
q_2	pitch D.O.F of the vehicle
Q	a moving oscillating force
$\bar{t}$	time variable
T	the traversing time of a vehicle crossing a bridge
u_g	a harmonic function represents the shape of a bump
u_v	vertical displacement of the sprung mass relative to its position of equilibrium under self weight
W	weight of the vehicle
$z_1, z_2,$ z_3	amplitudes of dynamic deflection the lowest three modes of the bridge
$\bar{z}$	impact factor
Z_1, Z_2	matrices each of size 1×1
β	dimensionless parameter
η	a function represents the displacement of the point of contact between the vehicle and the bridge deck
ϕ	frequency ratio

κ	mass ratio
ρ	amplitude of vibration of the sprung mass as it enters the bridge
$\psi(x)$	a function represents an assumed deflected shape of the bridge
ω_v	bounce frequency of the sprung mass
ω_b	fundamental frequency of the bridge structure in rad/sec.

CHAPTER 3:

b	half width of the plate model
c_v	damping constant in the vehicle model
C^*	the damping matrix superimposed to the bridge equations of motion due to the vehicle inertia effect
D_x	longitudinal flexural rigidity per unit width
D_y	transverse flexural rigidity per unit length
D_{xy}	torsional rigidity per unit width
D_t	coupling rigidity per unit width
$E_x, E_y,$ E', G	constants needed to define the elastic properties of an orthotropic material
f	time-dependent function representing the distribution of the load over the surface of the bridge

h	plate thickness
H	$D_1 + 2D_{xy}$
k_v	stiffness of vehicle spring
$\mathbf{k}^*$	the stiffness matrix added to the bridge equations of motion due to the vehicle inertia effect
M_x	the longitudinal bending moments per unit width of the deck
$\mathbf{M}^*$	the mass matrix augmented to the bridge equations of motion due to the vehicle inertia effect
$\mathbf{Q}$	the modal matrix of size $N \times M$
$T(t)$	harmonic function
u_v^t	the total displacement of the sprung mass relative to the undeformed position of the spring
$\tilde{u}_v$	the transformed deflection of the vehicle spring
$w(x, y, t)$	vertical displacement function of the plate model
W	width of the plate model
y	represents the M normal coordinates included in the modal transformation

$\bar{y}$	represents the M normal transformed coordinates
δ	the static deflection of the vehicle spring
ρ	mass of the plate per unit area
γ	vertical displacement of the contact point
ζ, η	dimensionless coordinates
$\varphi(x, y)$	a mode shape function
θ	$= \left(\frac{b}{L}\right) \left(\frac{D_x}{D_y}\right)^{0.25}$
χ	$= \frac{2H}{\sqrt{D_x D_y}}$
λ	$= \frac{\omega L b \sqrt{\rho}}{(D_x D_y)^{0.25}}$
ω_i	the i^{th} frequency of a plate

CHAPTER 4:

b	is the width of the vehicle
$d_{13}, d_{14},$ d_{15}	the sprung mass degrees of freedom
$d_{23}, d_{24},$ d_{33}, d_{34}	the unsprung mass degrees of freedom
$\mathbf{D}$	the vector of the nodal d.o.f. of the bridge

$\mathbf{f}_1, \mathbf{f}_2,$ $\mathbf{f}_3, \mathbf{f}_4$	matrices each of size 1×3
$\mathbf{g}_1, \mathbf{g}_2,$ $\mathbf{g}_3, \mathbf{g}_4$	matrices each of size 1×3
$H_1, H_2,$ H_3, H_4	matrices each of size $1 \times N$
$G_1, G_2,$ G_3, G_4	matrices each of size $1 \times N$
I_{11}, I_{12}	the mass moments of inertia of m_1 about p_1 and p_2
I_{21}, I_{31}	the mass moments of inertia of m_2 and m_3 about p_1
$\mathbf{K}_b$	matrix represents the plate bending stiffness
$\mathbf{K}_m$	matrix represents the plane stress stiffness
L	is the wheel base of the vehicle
m_1	the sprung mass of the vehicle body
m_2, m_3	the unsprung masses of the vehicle
p_1	the longitudinal axis along the direction of motion for the vehicle local reference system
p_2	the transverse axis for the vehicle local reference system
p_3	the vertical axis for the vehicle local reference system

$Z_1, Z_2,$
 Z_3, Z_4 matrices each of size $1 \times N$

CHAPTER 5:

Θ phase angle

Φ the modal matrix of the structure

CHAPTER 6:

J is the number of the lowest mode shapes of the bridge included
in the analysis

Λ_b *spectral matrix* of the bridge

Λ_v *spectral matrix* of the vehicle

Ψ_b transformed damping matrix of the bridge

Ψ_v transformed damping matrix of the vehicle

1.1 General

Vibration of bridge structures under the passage of vehicles is an important consideration in the design of bridges. Dynamic response of bridges has assumed added significance with the advent of faster and heavier vehicles and the modern tendency to design bridges with large span-to-depth ratios. Interaction between the vehicle and the bridge affects design in two ways. First, the dynamic nature of the response amplifies both the deflections and the stresses induced in the bridge structure, which should therefore be designed to sustain such increased stresses and to keep the deflection within limits. Second, the vibrations induced by dynamic interaction may have adverse psychological effect on pedestrians using the bridge, and the level of vibrations should therefore be kept within the comfort zone.

1.2 Nature of the Problem

The analysis of a complex dynamic system comprising a moving vehicle and the bridge is a mathematically intractable problem. The complexity of the problem arises because of the interaction between the bridge and the vehicle. A number of variables control the bridge-vehicle dynamic interaction. An insight into the mechanics of this interaction is essential in developing an efficient numerical method for the analyses of bridge response.

The nature and magnitude of the displacements and stresses caused in a bridge on the arrival of a moving vehicle at some point on the bridge differ from those caused by a stationary vehicle at the same point. The time-dependent force imposed by the moving vehicle excites vibration of the bridge and leads to an additional loading on the bridge. The deflections and vibrations of the bridge caused by the force pulses of a moving vehicle axles are considered as a profile variation causing excitation for the moving vehicle itself. Thus, the vehicle and bridge are considered as inseparably coupled.

The magnitude of the additional dynamic load exerted on the bridge by a moving vehicle is governed by a large number of different parameters. The following is a list of the important parameters.

- (1) Characteristics of the vehicle which include: the number of axles, the axle spacing, axle loads, the natural frequencies of the vehicle along its degrees of freedom, the damping provided by the shock absorbers and interleaf friction in the springs.
- (2) Characteristics of the bridge structure, that is, its dimensions, support conditions and mass and stiffness distribution. Often, these characteristics can be represented by the mode shapes and frequencies of the structure.
- (3) Vehicle speed.
- (4) Profile of the bridge deck, and the vibrations introduced by surface irregularities and surface misalignments at the abutments and expansion joints.
- (5) The travel paths of vehicles on the bridge deck.
- (6) The number of vehicles and their relative positions on the deck.
- (7) Forces caused by braking and acceleration.

As a result of the large volume of research, analytical models and methods have now been developed that will permit rational study of the dynamic response of a bridge structure of given geometry and other physical characteristics under the passage of a vehicle modelled in an adequate manner. In most cases the sophisticated analytical techniques still fail to identify the major parameters that govern bridge response. In fact, the use of simplified models may be more effective in identifying the characteristics that govern response and in developing rational procedures for design.

The experimental investigations have provided important information on the dynamic behaviour of bridges, yet it has not been possible to develop clear correlations between the nature of response and the parameters that are known to affect such response. For example, attempts have been made to obtain relationships between dynamic amplification of deflections and the bridge frequency, bridge weight, irregularity of the riding surface and the approaches, vehicle weight and somewhat indirectly, the vehicle frequency by plotting the amplification data against one of these parameters. In each case, the data shows a large amount of scatter and the interpretations are often speculative and subjective.

In spite of the limitations of various analytical and experimental studies, certain broad conclusions have been drawn. Among the conclusions most often cited, the following are of particular interest.

1. Since the response of a bridge structure under a moving load is a complex dynamic phenomenon, simple geometrical and statical parameters such as depth-span ratio and static deflection under live load do not provide an adequate basis for predicting or estimating the response. The use of such parameters by certain codes to define design requirements is not therefore entirely rational.
2. The first flexural mode of the bridge usually dominates the response.
3. A phenomenon of quasi-resonance appears to exist when the vehicle bounce or pitch frequency is close to the fundamental natural frequency of the bridge.
4. The dynamic amplification of response predicted by a two-dimensional plate model is generally lower than that predicted by a beam model even when both models are traversed by an identical vehicle.
5. The passage of multiple vehicles in parallel or in sequence in a random manner will, in general, lead to lower dynamic amplification than the passage of a single vehicle.
6. Irregularity in the riding surface or in the approach affects the response very significantly and may be more critical than the normal interaction between the moving vehicle and the bridge deck.

Many of the conclusions cited above are well substantiated; others, although widely accepted, may need further verification. In any case, there still remains a need for clearly identifying the important parameters that govern response. The lack of understanding of such parameters has inhibited the development of more rational code requirements, and even where code revisions have been made the revised provisions are not necessarily more rational.

1.3 Review of Previous Analytical and Experimental Studies

A number of analytical and experimental investigations have been carried out in the past to study the dynamic behaviour of bridge decks traversed by moving vehicles. Lists of relevant references on the subject have been provided by the Committee on Deformation

Limitations of Bridges (1958), Jacobsen and Ayre (1958) and Veletsos and Huang (1970). The reported analytical studies involve various degrees of idealization of the bridge-vehicle system. The behaviour of highway bridges under the passage of heavy vehicles has been analyzed using several different approaches. The first and the most commonly used approach is to model the bridge deck as a uniform prismatic beam. In another approach the deck is analyzed as a plate continuous over a series of flexible beams, yet another approach is to carry out a full 3 dimensional analysis. Previous analytical studies based on the three approaches are reviewed in the following paragraphs. Also included is a brief review of some of the field tests carried out in the past.

1.3.1 Beam Model of the bridge

The earliest studies by Willis (1849 & 1851), which were applicable to railway bridges, treated the beam model of the bridge structure as massless and the vehicle as a moving load possessing mass, Fig. 1.1a. Stokes (1867) solved the case of massless load traversing a beam of uniform mass, Fig. 1.1b. Jeffcott (1929) was the first to include both the mass of the beam and the moving load. Inglis (1934) solved the problem of moving mass by making use of the assumption that the deflection shape of the beam at any time was that of the first natural mode of the unloaded beam, that is, a half sinusoidal wave. Hillerborg (1948) studied the problem of a straight prismatic beam under moving loads with regard to vibrations of railway bridges. He made the approximation that the dynamic deflection curve of the beam at any time was proportional to its instantaneous static deflection curve due to the moving load (the deflection curve if the load on the beam at that instant were applied statically). Wen (1960) improved Hillerborg's assumption by including the weight of the beam itself in the determination of the basic shape of the structure in motion. He considered the case of a two-axle load composed of a sprung mass connected to two unsprung masses through linearly elastic springs. Wen also included in his study the rotary inertia of the vehicle as well as the unevenness of the surface of the approaches and the beam. He obtained time histories of mid-span deflection and moment for a 45 ft span bridge traversed by a 20 ton two-axle vehicle with an axle spacing, S , of 30 ft traveling at a speed of 51 miles per hour. Wen observed that for both moment and deflection, the dynamic curve oscillated about the static curve; the maximum dynamic moment occurred when the rear axle was positioned at $0.55L$, where L was the span of the bridge, and was

about 15 % larger than the maximum static moment. The maximum dynamic deflection was 25 % larger than the maximum static deflection. Wen also obtained spectrum curves relating the maximum dynamic response to various parameters of the beam-vehicle system. He concluded that augmented response of a bridge occurred at values of $S/2L$ that were multiples of $\alpha = vT_b/2L$, where T_b was the fundamental period of the bridge structure and v the speed of the vehicle. He also observed that the peak values of the response spectrum plotted as function of α were appreciably higher and more sensitive to α for a section under the rear axle in comparison to a section under the front axle.

Csagoly et al. (1972) studied bridge vibrations using the general purpose finite element program ICES STRUDL II. The bridge was idealized as a straight continuous elastic beam subdivided into five elements with lumped masses. They obtained the natural frequencies of continuous beam bridges and also measured the dynamic response of such bridges. On the basis of such measurements they observed that the impact factor increased up to spans of approximately 100 ft length and then started to decrease with an increase in the span length. This behaviour was not exhibited by simple span bridges. Csagoly et al. explained that the larger response of continuous bridges up to 100 ft span was mainly because of the nature of the unloaded frequencies of continuous bridges. Generally, the lower natural frequencies of these bridges were closely grouped together in a frequency band of 2 to 5 cps which matched bounce and pitching natural frequencies of most modern vehicles. This could lead to a resonance, so that for a short period, the bridge and the vehicle would vibrate as one system, thereby intensifying the vibrations. The results of the field tests clearly indicated that values of impact factor higher than those specified by AASHTO and of the order of 25 % were common for bridges having dominant frequencies between 2 to 5 cps. The impact factor was measured to be as high as 87 % in case of a voided post-tensioned continuous concrete deck bridge with five spans (48, 85, 80, 85, 48 ft). It should be noted that the impact factor is defined as the difference between dynamic and static deflection expressed as a percentage of the static deflection. Csagoly and coworkers concluded that if resonance with heavy vehicles was to be avoided, a continuous bridge must be proportioned such that its fundamental frequency was greater than 5 cps.

In their report, Csagoly et al., examined the effect of several parameters which might have influenced the dynamic behaviour of continuous bridges. These parametric studies were focussed on two and three span bridges. The three span bridges considered were

symmetrical with their end spans being equal in length. They reported that two major parameters had the greatest effect: the ratio of the adjacent spans, and the ratio of the dead load to the flexural rigidity of the structure. The results of their analyses showed that as the number of spans increased, the grouping of natural frequencies became denser allowing for a higher possibility of resonance. On the other hand, the difference between the first and second natural frequencies increased as the span ratio decreased. Thus, for ratio of the adjacent spans of 0.63 for a two-span bridge the frequency band of 2 to 5 cps was contained entirely within the first and second natural frequencies. For a three-span bridge the corresponding span ratio was 0.52. Although these ratios represent the optimum from the point of view of dynamic behaviour to avoid dynamic resonance, they are not the most economical when designing for static loads and may be impossible to achieve in view of geometric constraints imposed by the construction site.

Comparisons were made between the theoretical results using the STRUDL program and field tests on three existing bridges all having solid-voided continuous prestressed concrete decks. The Ministry's test vehicle with a gross weight of approximately 90,000 lb was used in the field tests. The test vehicle was driven over the bridge in two different fashions with a view to exciting the different modes of vibrations of the bridges. In general, the measured frequencies were slightly higher than their theoretical values. Impact factors were reported to range from 40 to 100 percent. The 100 percent value was observed where the test vehicle entered the bridge from a traffic lane which showed a distinct depression (about 2.5 in. deep) in the pavement just prior to the bridge and also while the deck was still vibrating from the passage of a preceding vehicle.

Csagoly et al also commented upon the inadequacy of the AASHTO provisions which specified a deflection limitation related to span. As most of the observations made during field tests on bridges revealed, the impact factor was not mainly dependent on the value of the live load but rather on frequency match between the bridge and the traversing vehicle as well as on the state of excitation of the vehicle as it entered the bridge. Thus, a better deflection control criteria could be achieved by imposing limitations on the frequency range of the bridge. For the types of two and three span bridges they studied, the investigation led to the following recommendations related to the value of η , the so called coefficient of

vibration,

$$\eta = \left(\frac{100\pi^2}{g e_1^4} \right) \cdot \left(\frac{W_{DL} l_2^4}{EI} \right) \leq 1.0$$

where e_1 was the normalized first mode eigenvalue, g the gravitational acceleration, l_2 the length of the larger span of a continuous bridge, and W_{DL} the self-weight of the beam per unit length. The normalized eigenvalue e_1 was a function of the span ratio and can be obtained from charts provided in the report for two and three span bridges. The charts related the eigenvalues for several modes to the span ratio. The above equation was derived from the stipulation that the first natural frequency of bridge be higher than 5.0 cps:

$$f_{b1} = \frac{e_1^2}{2\pi l_2^2} \sqrt{\frac{EIg}{W_{DL}}} \geq 5$$

As seen from the equation, the coefficient of vibration depends on two terms, the first of which contains the span ratio, while the second is a function of the dead load deflection. Thus for a given span ratio the second term should be kept as small as possible to ensure that resonance in the bridge structure is avoided. It was concluded that the thin-walled box-type of cross section was best for satisfying such a requirement as it possessed a high stiffness to weight ratio. Csagoly et al. also studied the effect of the structural depth of a bridge on its natural frequency by monitoring the variation of inertia to dead weight ratio with the depth, for three different types of cross section, voided slab, solid slab, and box girders with web contribution being ignored. The theoretical analysis showed that the square root of the inertia to weight ratio increased linearly with an increase in the depth and that the rate of increase for the box girder sections was 1.73 times that of the solid section, while for the voided slab it was 1.12 that of the solid slab.

Csagoly et al. pointed out that in some cases a two dimensional analysis was indispensable and if one dimensional analysis was carried out significant errors could be involved in the results. Such situations arise, for example, in the cases of skewed bridges, curved spans, and wide bridges. They proposed a grid system to idealize the bridge. The idealization comprised linear members with total weight of the bridge being distributed to certain selected nodes. Each node would have only one degree of freedom, namely the vertical deflection, and the flexibility matrix, $[F]$, would be formulated by placing a unit load at each node in turn. The idealization was applied in the analyses of Conestogo River Bridge and good results were obtained. The analyzed bridge consisted of steel trusses supporting

cross steel beams which in turn supported the concrete deck by means of longitudinal steel stringers.

Csagoly et al. also reported that the impact factor did change as a function of the traversing vehicle speed. Fleming and Romualdi (1960) have proved that maximum impact occurs when the vehicle traverses a span in a time equal to the fundamental period of vibration of the structure, that is for $\alpha = 0.50$. However, as observed by Csagoly et al, in most practical situations such a value of α can be produced only by unrealistically high vehicle speed.

Shepherd and Aves (1973) analyzed bridges with simply supported spans for their response to a moving truck. In their analysis the bridge was idealized as a single simply supported beam and the vehicle as a single sprung load. The impact factors predicted by the mathematical model were compared with those obtained from field tests on ten different concrete bridges supporting a moving truck. The study led to the observation that the dynamic impact allowance specified by the code was insufficient for most of the tested bridges. A new and simple method for the prediction of impact factor was proposed. The tested bridges were designed in accordance with the New Zealand Ministry of Works standard for highway bridges and were of flat slab or T beam type of cross section. Seven of these bridges were simply supported, while the other three were continuous. The test truck was the standard Bedford truck which was used either unladen, gross weight 4.1 tons with 2.15 tons on the rear axle, or laden, its weight being either 8.0 tons, with 6.5 tons being on the rear axle, or 9.45 tons, with 7.9 tons being on the rear axle. A simple forced vibration resonance test gave a value of 3.6 Hz for the natural frequency of vertical vibration of the fully laden truck. The stiffness of the rear axle leaf spring was evaluated experimentally to be 10,000 lb/in. and that of the rear axle combined spring and tire suspension to be 4500 lb/in.

The field tests revealed that the observed values of impact factors for fully loaded truck were smaller than those for corresponding unladen runs. Road irregularities were thought to be the cause for such observations since they are more able to excite a lighter truck. Also the code impact factors ($I = \frac{50}{L+160}$, L span in feet) were mostly exceeded for simply supported bridges of relatively short span, especially for relatively high frequency ratios.

The theoretical analyses carried out by Shepherd and Aves were based on the following

assumptions; (1) the bridge had a smooth riding surface, (2) only the first mode of vibration of both the bridge and truck was considered in the analysis, (3) the damping in the bridge, as well as in the vehicle suspension, was negligible, (4) only one vehicle at a time was considered to traverse the bridge, (5) the vehicle speed was constant, and (6) the force exerted by the vehicle on the bridge was assumed to have specific idealized characteristics. Thus the interaction between the bridge and the vehicle was not considered in this analysis. Based on the correlation between the theoretical analyses and the field tests, Shepherd and Aves suggested a simplified expression to determine the impact factor $I_{max} = \left(J + \frac{\alpha}{1-\alpha} \right) 100\%$ in which α was the speed parameter. The expression implied that the impact factor was composed of two different components. The second term, $\left(\frac{\alpha}{1-\alpha} \right)$, in fact, represented the dynamic response due to a moving constant force, while, J the value of which was to be determined experimentally, reflected the maximum possible contribution to impact from the alternating component of the truck load.

Billing (1979) idealized a continuous bridge by multi-span uniform beams and obtained tabulated natural frequencies for such bridges. Toshiro and Noboru (1981) idealized the continuous beam as a Bernoulli-Euler beam and developed analytical methods for determining the eigenvalues and eigenfunctions of the beam with arbitrary boundary conditions. They observed that the dynamic amplification factors decreased as the number of spans increased.

A good model of a vehicle must comprise a moving mass supported by springs and dash-pots, known as a sprung mass, and a mass in contact with the deck known as an unsprung mass. All of the studies referred to in the previous paragraphs represented the vehicle by a single sprung or unsprung load or mass. The case of a moving vehicle with both sprung and unsprung masses was analyzed by Tung et. al (1956), and Biggs et al. (1959), Fig. 1.1c.

In the sixties and early seventies several studies were published in which the finite element method was used to represent the beam model of the bridge. These include studies by Fleming and Romualdi (1961), Veletsos and Huang (1970), and Yoshida and Weaver (1971). Yoshida and Weaver (1971) found a great deal of similarity between the linear dynamic responses of the beam and plate models subjected to moving forces. The largest *dynamic magnification factors* for the beam and plate models were 1.68 and 1.58 respectively, corresponding to a travel time equal to the fundamental period. On including in the analysis

the mass of the vehicle as a moving unsprung mass, Yoshida obtained very large values of magnification factors relative to the moving-force problem. The effect of the inclusion of the mass increased dramatically as the mass ratio, that is the ratio of the vehicle mass to the structure mass, increased and higher modes of vibration were excited. Tan and Shore (1968) studied the response of a horizontally curved bridge to moving load. They idealized the bridge as a single prismatic simply supported beam with curvature in a horizontal plane and the vehicle as a set of moving sprung and unsprung masses connected to each other by a spring and dash-pot. They found a great deal of similarity between the response curves for a moving mass and a moving force, although the magnitudes of the *dynamic increments* were greater for the moving mass. Also, they drew the conclusion that for a mass ratio equal to 0.30 or less, the response of the bridge could, for all practical purposes, be assumed to be equal to that caused by a moving force. Gupta (1980) used three mathematical models to idealize the single span multi-girder highway bridges: (1) beam representation; (2) beam with torsional freedom; and (3) uniform orthotropic plate. The vehicle was idealized as a planar, two-axle, sprung mass system. It was found that the unidimensional bridge models led to higher estimates of impact factors than the two-dimensional plate models.

Hugh and Ghali (1981) proposed an analytical procedure called **IDS**M, iterative dynamic substructuring method, to assess the response of a beam-slab bridge structure under the passage of multi-axle trucks. In this method an iterative technique was used to ensure the compatibility of forces and displacements at the points of contacts between the bridge and the vehicle. The bridge and the vehicle were both assumed to have elastic properties. The method was employed to obtain the dynamic response of the Conestogo River Bridge and to study the effect of different load parameters such as vehicle speed, number of vehicles, relative positions, and the travel direction. The method was also capable of including the bridge irregularities. The Conestogo River Bridge, which is a three-span continuous steel plate-girder with a longitudinally prestressed concrete deck, was dynamically tested by Dorton and Csagoly (1977). In their comparative theoretical study, the bridge was idealized using finite element discretization and a lumped mass matrix. The concrete strength was assumed to be 25 % higher than the specified 28 day strength (28 MPa) and the dynamic modulus of elasticity was taken as 35 GPa. Also the damping of the bridge was set in the analysis to be 1 % of the critical. The stiffness and the weight of the barrier walls were included in the analysis and the bridge was modelled as having an equivalent

steel cross section. The Ontario MTC load-test vehicle having the first natural frequency of 3.5 Hz was idealized as two rigid beam frames (the tractor and trailer) connected by hinges. A system of elastic springs with 1 % damping was used in the model to support the beams. The total mass was assumed to consist of an as unsprung mass (12 %) lumped at the bottom ends of the springs, and a sprung mass attached to the springs at the top ends and having a vertical degree of freedom. The vehicle model was so designed as to retain the first natural frequency, the total mass, and rigid body mode shapes. Hugh and Ghali concluded that the relative vehicle position had a significant influence on dynamic response of the bridge. According to them, the dynamic increment for internal forces for multi-vehicle load case was higher than that for the vertical displacement. Multiple-vehicle load case also produced higher impact factors than a single vehicle load case, and positioning the vehicles in two different lanes strongly excited bridge vibrations.

Walker and Veletsos (1966) derived expression for the dynamic increment *DIN*, defined as the difference between the instantaneous value of a dynamic effect and the corresponding value of the static effect normalized by the maximum static value of the effect ($DIN = \frac{\delta_D - \delta_s}{(\delta_s)_{max}}$), based on solutions for a moving constant magnitude force. In this approach the main parameter which controls the dynamic response is the speed parameter ($\alpha = v/2f_b L$ where v is the speed of the vehicle and f_b the bridge fundamental frequency). Their approach yielded the following approximate expression for the dynamic increment for deflection and moment for a single smoothly moving force

$$DIN \approx \alpha$$

They also suggested an extra constant term to take account of the surface roughness and the initial vehicle conditions in case of simple span bridges, giving

$$DIN \approx 0.15 + \alpha$$

Wright and Walker adopted the same expression for continuous span bridges, since *DIN* values will be smaller for such bridges in comparison to a single span bridge and the use of the aforementioned expression will be conservative.

Wright and Walker also examined the effects of the flexibility of the stringers on live load stresses and deflection distribution, dynamic response, deck deterioration, and on human

reaction to bridge vibrations. They defined flexibility by the parameter $1/H$, obtained from

$$1/H = [ELh^3/12(1 - \nu^2)]/E_b I_b$$

where E , h , and ν are respectively the Young's modulus, thickness, and Poisson's ratio for the slab, and $E_b I_b$ is the stiffness of the stringer. Obviously, an increase in the flexibility will mean a decrease in I_b and a consequently decrease in f_b . The change in f_b is about one-half the percent change in I_b since the frequency f_b is given by $f_b = \frac{\pi}{2L^2} \sqrt{\frac{E_b I_b g}{w}}$. This in turn means a proportionate increase in the speed parameter. Thus, a reduced I_b will tend to increase DIN and at the same time will increase the static component of the deflection δ_s ; consequently, the dynamic component of the deflection δ_D will be increased by an amount equal to the percent change in $(DIN + \delta_s)$. The above argument can be used to show that an increase of 10 % in the flexibility will cause an increase of roughly 12.5 % in δ_D . The amplitude of the dynamic component of acceleration in the fundamental mode can be expressed as

$$a = \delta_D (2\pi f_b)^2$$

and therefore it can be proved that the increase in a is proportional only to the increase in DIN . Finally, an increase in the flexibility term will increase δ_s linearly, increase δ_D slightly more than linearly, and increase a only slightly. Also, conclusions have been reached that changes in the stringer flexibility have a small effect on the dynamic component of moment; in fact, the total moment might be reduced as a result of the decrease of the static component.

An agreement between the field measurements reported by Wright and Green (1964) and the analytical results discussed above has been observed and evidently shows that bridge flexibility and slenderness have limited effect on the dynamic components of the deflections and stresses.

1.3.2 Plate Model of the bridge

The idealization of a bridge deck by a single beam implies that the deflections do not vary in the transverse direction at any cross section of the bridge. This idealization will fail to account for the transverse flexural and torsional modes of the bridge. These modes of vibrations may dominate the response of a bridge of low aspect ratio, whose width is of the

same order as the span. Transverse flexural and torsional modes may also be expected if the vehicle follows a path other than the center-line of the bridge. A single beam idealization would not thus be appropriate for such bridges. Studies have therefore been carried out in which the deck is treated as an isotropic plate (Yoshida and Weaver, 1971) or as an orthotropic plate (Sundara and Jagadish 1968, 1970). The orthotropic plate simulation of the bridge can be applied either to slab bridges or to beam and slab bridges. Sundara and Jagadish (1970) conducted parametric studies on two typical highway bridges using a sprung mass to approximate the vehicle and considering only the first five modes of vibration to contribute to the bridge response. They concluded that the *amplification factors* given by the beam theory were slightly on the conservative side as compared to those obtained by the orthotropic plate theory. Also, they obtained the maximum *amplification factors* at points in the bridge away from the line of loading especially for the case of eccentrically loaded bridge. In this type of loading, the unloaded edge experiences quite large amplifications.

In other studies, a combination of beam elements to represent the girders and plate elements to represent the slab has been used (Yamada and Veletsos, 1958 and Oran and Veletsos, 1961). Chaudhuri (1975) studied the dynamic response of curved bridges using a simulated two-axle, four wheel vehicle and including the centrifugal forces. He modelled the bridge deck by thin plate annular elements, the girders by one-dimensional thin-walled circular beam elements, and the diaphragms as straight prismatic beams. Damping in the bridge was ignored. The computation times were relatively high, which imposed a restriction on the size of problems analyzed. Therefore, a parametric study or the development of an impact formula could not be pursued. Instead, Chaudhuri obtained the static and dynamic responses for a number of bridges and gave some tentative values of the *impact factors*. The *impact factor* for the deflections was reported to be 30 %. The study found out that torsional modes of vibration had a significant influence on the dynamic responses. Also, it was found that the inclusion of the vehicle mass gave rise to a response that contained higher frequencies.

1.3.3 Three Dimensional Model of the bridge

Like slab and beam bridges the response of box girder and slab bridges of low aspect ratio may be dominated by the transverse flexural and torsional modes. A two or three dimensional analysis is necessary for predicting the behaviour of such bridges. Cheung et al.

(1982) conducted a free vibration analysis on curved and straight beam-slab or box-girder bridges using the finite strip method. Raizadeh and Shore (1975) studied the dynamic response of curved box-girder bridges using the finite element method. Annular plate and cylindrical shell elements were used to idealize the slabs, bottom flanges, and webs. The diaphragms were modelled by rectangular plate elements and pin-jointed bar elements. The vehicle was simulated by two sets of concentrated forces, having components in the radial and transverse directions and moving with constant angular velocities on circumferential paths of the bridge. Mirza et al. (1985) carried out an analytical and experimental study of the dynamic behaviour of a composite concrete deck-steel box girder bridge. They used the finite element program **SAP IV** to analyze the bridge. In their study they were concerned mainly with the effect of the bracing systems on the composite bridge response which was dominated by the flexural mode. Recently Megnounif (1988) has carried out an investigation on the frequencies, mode shapes and forced response of box girder bridges using the finite element method. He used the computer program **ADINA** to analyze the bridges. The vehicle was idealized by a set of moving forces; the vehicle mass was not included in the analysis.

1.3.4 Field Tests

In order to understand the dynamic behaviour of bridges under moving load and to establish correlations between the analytical and the experimental studies, a number of field tests have been carried out over the years, some of these tests are reviewed in this section.

Biggs et al. (1959) carried out field tests on two bridges, one a stringer type and the other a typical through girder bridge. The aim of these tests was to verify an analytical method developed by the researchers to obtain the magnitude and characteristics of highway bridge vibrations caused by the passage of heavy vehicles. The bridges were medium-span structures. The test vehicle was a 1952 two-axle (six-wheel) 10-ton dump truck. The vehicle had a wheel base of 13 ft, a tread of 6 ft 6 in., 11:00-24 front wheel tires, and 12:00-24 rear-wheel tires. The suspension system consisted of leaf springs with a pair of auxiliary springs on the rear axle that were engaged under heavy load. The truck was loaded with steel snow-plough blades to produce a total weight of 41,625 lb, of which 33,875 was on the rear axle (81.38 %). The tests were carried out at different speeds and with the vehicle running in both directions. The speed of the vehicle was kept constant during each run

and the vehicle was running along the center-line. In each case, the midspan deflection was measured during the passage of the truck. The fundamental frequency of the vehicle was 2.3 Hz. and the average viscous damping coefficient was 70.0 lb-sec/in. The natural frequencies of the test bridges were: 3.77 Hz. for stringer bridge, and 4.17 Hz. for through bridge.

In the theoretical analysis, Biggs et al., modeled the vehicle as one degree of freedom system which had sprung and unsprung masses connected to each other by elastic spring and dash-pot. The bridge was modelled as a simple beam single degree of freedom system. Viscous damping was assumed in both bridge and vehicle. The two tested bridges were idealized theoretically and solved using two different approaches. In the first approach, the dynamic deflection was computed using the measured load variation (a real vehicle behaviour) so as to investigate the validity of the proposed analytical method in application to actual bridges. The second approach was entirely analytical and the vehicle behaviour was idealized. Only the initial conditions and the values of bridge and vehicle stiffnesses were taken from the field measurements. The second approach was used to study the practicality of the vehicle model. In the theoretical solution, damping was ignored. A satisfactory agreement between the theoretical results and field tests was observed when the first approach was used. The second approach did not produce as good a comparison as the first approach.

Biggs et al. concluded that there was a critical speed of the running vehicle that caused maximum load variation. This critical speed was a function of the roughness of the roadway surface and the natural frequency of the vehicle. Also, the major part of the load variation was caused by surface roughness on the approaches rather than on the bridge itself. Expansion joints at the entrance to the span of the tested bridges strongly influenced the subsequent amplitude of vehicle oscillation. They concluded also that for a truck moving along the center-line, the movement of the bridge in a transverse direction was closer to that of a rigid body than was the case for static loading. They found that the maximum dynamic deflection of a bridge occurred for a combination of small mass ratio (the ratio of mass of the vehicle to mass of the bridge), low speed, and unit frequency ratio (the ratio of the fundamental frequencies of the vehicle and the bridge). They reported also that for design purposes the effect of surface roughness on the bridge response can be neglected. The theoretical investigation showed that the most important parameter

affecting a bridge response was the amplitude of the initial condition and that the only other significant parameter was the mass ratio which in fact became important only when the frequency ratio was near unity.

Wright and Green (1964) carried out a number of experimental measurements of the dynamic response of numerous bridges of different types. A report by these authors presented the results of an Ontario program on field studies, which included field observation of the motion of 52 representative highway bridges under the action of ordinary vehicle traffic. Comparison between the observed and the calculated bridge frequencies showed that the former were always larger than the latter. This was attributed to several reasons; first the calculated frequencies were obtained using beam formulae which were based on one dimensional analysis of the structure, and further, the contributions of deck stiffnesses and joint fixities were absent from the formulae used. A least square analysis of the observed frequencies gave the following general expression

$$f_o = 0.72 + 0.95 f_b$$

where f_o and f_b are the observed and the calculated frequencies respectively. They also reported that the value of the observed stiffnesses of the bridges were always greater than those of the calculated stiffnesses — in some instances three times as great.

The report indicated that the frequency of vibration of a bridge over which a loaded vehicle is passing may produce a forced vibration that is either in phase with the truck motion or at a frequency somewhat smaller than the unloaded natural frequency of the bridge. In general, the loaded bridge will exhibit a lower natural frequency than the unloaded bridge because of the presence of the vehicle mass. The report presented the following expression for calculating the fundamental frequency of the loaded bridge

$$f_{b1} = f_b \sqrt{\frac{m_b}{m_b + 2m_1 \sin^2\left(\frac{\pi vt}{L}\right)}}$$

where f_{b1} and f_b are the loaded and unloaded frequencies of the bridge respectively, m_b and m_1 the masses of the bridge and the load respectively, v , the velocity of the load, L , the span length and t the time after the entry of the load on the span, (t varies between 0 and L/v). The report also indicated that the higher natural frequencies of the bridge were activated by the passing of automobiles of rather small weights and soft springs.

It was observed in the report that in a substantial number of cases the measured dynamic increment was 100 percent of the static deflection. In general, the dynamic increments of deflection and moment (or stress) were closely related, but the deflection increments were higher. The report tried to correlate the observed dynamic increments with several parameters which are believed to influence the dynamic response and the impact factor. The investigation showed that the relationship between the impact factor and the span did not follow the usually assumed relationship, namely inversely proportional to L . The investigation also concluded that there was no significant correlation between the dynamic increments and most of the parameters studied, such as the ratios of span to depth, span/calculated live load deflection, and the natural frequencies of vibration for the bridges investigated.

The measured values of the damping factors were observed to be between 1 and 2 percent for most bridges. The damping factor was thought to depend primarily on the details of the structural system of a given bridge and on the characteristics of the energy absorption by bearings, connections, etc.

The study also described the intensity of vibration in relation to human perception for each test bridge using both the Reiher-Meister scale (Reiher and Meister, 1931), and Isosensor scale (Wright and Green, 1963). The results revealed that perceptible vibrations were always detected and ranged between unpleasant and discomforting vibrations. It appeared from the investigation that the (span/depth) ratio had no influence on the susceptibility of bridges to vibrate. Also the (span/live load deflection) ratio had a minimal effect on the tendency of a bridge to vibrate.

The report observed that the bridge surface condition had a significant influence on the vibrations of a bridge. Tests were conducted on bridges with two different types of surface conditions with a view to comparing their response. The first series of tests were carried out before the completion of final asphalt pavement with the running surface being just the finished concrete deck. In the second series of tests the bridge surface was covered with a freshly finished asphalt pavement. Eventhough the concrete deck could hardly be classified as being irregular or rough, since the profile variations were evidently quite small, the dynamic increment for the second set of tests was almost one third that for the first set. The two sets of tests produced identical values of stiffness and natural frequencies.

The motion induced in the vehicle springing system as a result of the passage of the vehicle over an approach with undulating surface was shown to greatly affect the response of the bridge. This effect was more evident in the case of a match between the time of travel over the undulation and the period of the vehicle springing system. Bumps caused by expansion joints were also seen as an important contributing factor in increasing the dynamic response of a bridge.

The investigation included tests conducted with two or more vehicles running across a bridge. The vehicles crossed the bridge either in succession or in opposite directions at close time intervals. The results of this study revealed that in general the cumulative effects were always less than the single vehicle effects.

Page (1976) tested thirty motorway and over-motorway bridges by running a test lorry, AEC Mandator two-axle, with axle loads of 60 kN (front) and 100 kN (rear) instrumented to measure dynamic variations of the load on its rear offside wheel. The investigation studied the effect of the irregularities in the bridge surfacing including discrete bumps, vehicle speed, and bridge span on the dynamic variations of the wheel load. The bridges ranged in span from between 13 to 84 m and comprised a wide range of types of construction and ages. Two of the bridges were of steel and concrete composite construction, the rest were of concrete. The speed of the vehicle varied between 48 and 60 km/hr.

This study differentiated between the “dynamic response factor” — the ratio of the maximum dynamic bridge deflection when the vehicle crosses the bridge at the test speed to the maximum static deflection when the vehicle crosses at a crawl speed, and “vehicle impact factor” — the ratio of the dynamic wheel load to static wheel load. The latter was defined in the investigation as being the impact factor.

The report prepared by Page concluded that the peak impact factors for the motorway bridges (1.09 to 1.43) were on the average smaller than for the overbridges (1.16 to 1.75). The peak values of the impact factors occurred predominantly within the first or last 10 % of the length of the bridges. Also, it was found that a bridge with a step of 25 to 31 mm height between the bridge surface and the approach produced peak impact factor of 2.77 which was considerably larger than the design value (1.25). The effect of the speed of the vehicle on response was studied for three motorway bridges with different surface conditions ranging from good to poor. For a bridge with good surface quality, the dynamic variation

of the wheel load was recognized to be very little. The peak impact factors for the average and poor surface conditions were found to increase noticeably for the speeds between 18 to 48 km/hr, while for the speeds between 48 to 60 km/hr the change in the impact factors was almost negligible.

The Swiss Federal Laboratories for Materials Testing and Research (EMPA) have conducted standardized dynamic load tests on 226 beam and slab-type highway bridges, most of them built in concrete, between 1958 and 1981 (Cantieni, 1984). The bridges were tested by passing a single, fully loaded, two-axle truck. The normal axle spacing was 4.5 m and the gross weight of the vehicle usually lay within the legal range of 160 kN. The two important frequencies of a normal vehicle, the body bounce and wheel hop respectively, were in range of 2-5 Hz. and 10-15 Hz. respectively. Of the bridges tested, 109 were straight and without skew, 97 were curved or skewed, and 20 were both curved and skewed. Half of the bridges had a box-shaped cross section, 26 % a multibeam deck, and 24 % a solid or hollow-core slab cross section. The width of the cross section varied from 4.3 m to 30.4 m. The spring constant of the test vehicle was between 7 kN/mm and 800 kN/mm.

The study attempted to establish a relationship between the fundamental frequency of a bridge and its maximum span for the various bridges tested. A regression analysis was performed on the gathered data although it was recognized that there were some limitations on the applicability of the process.

The variation of the dynamic increment with the mean span of the bridge was also studied but it was found that the two did not follow a systematic pattern. The fundamental frequency of the bridge was found to have a great influence on the dynamic increment. Therefore, the maximum dynamic increments were plotted as a function of the fundamental bridge frequency for two different cases. The first curve enveloped the measured values of the test series on undisturbed pavement. The curve showed a peak in the region of 2.5 to 4 Hz. (the region of the bounce frequency of the vehicle body) The second envelope curve was for the test series conducted with a plank lying on the roadway (to simulate road roughness). This curve had two peaks, the first in the range 1.5 and 3 Hz. and the second one at frequencies above 7 Hz. It was observed that the maximum dynamic increments observed for a bridge with a plank were much higher than those for a bridge with a smooth pavement.

The study concluded that a highway bridge exhibited amplified response only if (1) its fundamental frequency was in the same region as one of the vehicle frequencies and (2) the vehicle speed and pavement roughness were such that the corresponding vibrations of the vehicle were excited.

The structural damping values were determined for 198 test bridges by plotting the distribution of logarithmic decrement, δ , determined from a free vibration decay analysis. In this study, material damping was neglected. Long bridges with total length of more than 125 m (where the response is dominated by the longitudinal flexure) exhibited a mean $\delta = 0.056$ (corresponding to $p =$ the percentage of critical damping $= 0.89 \%$); while short bridges with total length less than 75 m (where the response is a superposition of the longitudinal flexure, torsion, and transverse flexure) showed a mean $\delta = 0.098$ ($p = 1.56 \%$). Straight bridges with narrow, closed cross sections showed a mean $\delta = 0.063$ ($p = 1 \%$), whereas curved or skewed bridges with wide cross sections exhibited a mean $\delta = 0.087$ ($p = 1.38 \%$).

In 1982, Billing conducted dynamic tests on 27 selected bridges of various configurations and different construction materials. The spans of the bridges ranged from 5 to 122 m. The distribution of dynamic load and its relationships to span length and vehicle weight were observed to generate the data required to examine the validity of the novel design provisions in the first edition of the Ontario Highway Bridge Design Code (1979). The approaches, expansion joints and decks of all these bridges were reported to have been in good to excellent conditions. Damping ratios of first flexural modes of steel bridges were measured to be in the range of 0.4 – 0.7 % of critical. Concrete bridges had damping ratios in a wider range, 0.8 – 3.8 % of critical. The investigation resulted in several conclusions, the following are some of them

1. The modes of vibration of the bridge that were excited depended on both the characteristics of the truck and of the travel lane. For simple spans, the first flexural and torsional modes were always excited and dominated the response. The transverse flexural modes were excited in some multi-girder simple span bridges. For continuous bridges, especially those of longer spans, the excited modes included the first several flexural and torsional modes.
2. The dynamic amplification of the bridge under two trucks running side by side was

generally less than that caused by a single truck.

3. When test runs were made with test vehicles following each other separated by one or two span lengths of a simple or continuous span, the dynamic increments under the passage of second vehicle were generally between 10 – 30 % higher than those obtained from an initially quiescent bridge traversed by a single vehicle.
4. The frequency match of vehicle and bridge was observed only for one bridge whose fundamental flexural frequency, 1.31 Hz., was excited by body heave of passing inter-city busses.
5. A fully effective section (one which comprises barrier walls, curbs, deck, and secondary members) is considerably stiffer under a single truck than the idealized section used in design.
6. The data generally confirmed that the dynamic load and serviceability provisions in the first edition of the Ontario Bridge Design Code and the proposed minor revisions for the second edition were valid. A further refinement of dynamic load provisions was proposed to account for the relationship between dynamic load distribution and span length and vehicle weight.

Cheung et al. (1982) carried out a test program on a number of steel-concrete box-girder bridges with different span lengths and number of spans. The program was to determine the dynamic characteristics and the dynamic responses of the tested bridges under normal traffic and special test vehicles. The dynamic characteristics included natural frequencies, and mode shapes, the response quantity included impact factors, and damping ratios.

A field test was carried out by Agarwal and Billing (1990) to assess the dynamic behaviour of St. Vincent Street Bridge on Hwy. 400 in Ontario following complaints of excessive vibration and movement of the bridge deck experienced by pedestrians using the bridge. The bridge consists of four continuous spans of voided prestressed concrete slab supported over a pair of columns at each intermediate pier locations. The bridge was built in 1967. The test was conducted using the Ministry of Transportation's bridge testing vehicles. The bridge responses, namely accelerations, deflections and strains histories, corresponding to four cases of loading were recorded. In the first load case, one vehicle at a time ran in both of the two directions. In the other load cases two trucks were used, moving simultaneously in various patterns. These patterns included: two vehicles in the same lane

at a clear distance of 10 m; side by side in the same direction; and in different lanes in the same direction maintaining a clear longitudinal distance of 10 m. The test trucks were run at load levels ranging from empty to legal load limit and at different speeds.

To determine the natural frequencies of the bridge, the bridge structure was idealized as a grid supported at discrete locations at the pier columns and on bearings at the abutments and was analyzed using the ICES STRUDL program. The bridge displayed high dynamic response consisting of complex vibration caused by coupling of the first flexural (3.60 Hz) and torsional modes (4.18 Hz) into a dominant flexural frequency of approximately 3.10 Hz. The close values of the first flexural and torsional frequencies gave rise to an interesting phenomena called “beating effect” representing back and forth transfer of energy between the first flexural and torsional modes. This back and forth action took place every five cycles and was the main reason for the feeling of discomfort experienced by the pedestrians. The researchers defined the dynamic amplifications for continuous bridges in two ways to account for the phenomena of reversal of responses in these type of bridges. The two measures of dynamic amplifications were

$$I_{peak} = \text{dynamic amplification for peak response} = \frac{Y_{(+)}}{Y_{st(+)}} - 1$$

$$I_{range} = \text{dynamic amplification for response range} = \frac{Y_{(+)} + Y_{(-)}}{Y_{st(+)} + Y_{st(-)}} - 1$$

where $Y_{st(+)}$ and $Y_{st(-)}$ were respectively the positive and negative magnitudes of static deflection responses of a component of the bridge and $Y_{(+)}$ and $Y_{(-)}$ were the magnitudes of dynamic positive and negative responses respectively.

In the findings of the study by Agarwal and Billing, dynamic effects were, surprisingly, recognized to be larger at slower vehicle speeds. This is contrary to the common belief and was explained by the fact that the presence of the truck on the bridge for longer time period in case of low speeds caused a greater build up of the dynamic response. Maximum accelerations were larger under simultaneous movement of two truck, especially at high speeds of about 60 km/hr, in comparison to those resulting from the motion of a single truck. Despite the bridge vibrations being highly perceptible to pedestrians, they did not have any detrimental effect on the structural performance of the bridge. Dynamic amplifications were detected to be larger in case of single truck than for multi-lane loading with trucks side by side. For the other cases of multiple loading, the two vehicles running

in the same lane or staggered, dynamic amplifications were much larger than those for a single truck. Also, in continuous bridges dynamic amplification of response range in a component were generally larger than that of the peak response ($I_{range} > I_{peak}$). For runs by a single test truck the ratio I_{range}/I_{peak} was 2.35 in positive moment region and 1.25 in negative moment region. This could present an important consideration in the fatigue of bridge structure. Normally, the code considers the peak response to determine the dynamic load allowance but this underestimates the fatigue effects in such structures. The study also found that the dynamic amplification diminished as the level of load of the vehicle increased.

1.4 Ontario Code Provisions Regarding Dynamic Response of Bridges

Based on existing data on dynamic amplification and certain assumptions regarding the variability of dynamic loading (Campbell et al., 1971, and Wright and Green, 1964), the first edition of the Ontario Highway Bridge Design Code (1979) included significant new provisions regarding dynamic loading and dynamic response of bridges. These provisions were in conformity with the limit states philosophy of the new Code. The provisions remain essentially unchanged in the current version (1983) of the Code. The significant among them, are

- (1) For deck slabs whose design would be governed by a single axle unit and where dynamic loading would be due to axle hop, the dynamic load allowance is 0.40.
- (2) A dynamic load allowance of 0.35 should be used for other slabs, and beams having spans up to 12 m, whose design would be governed by a dual axle unit or part of a truck and where dynamic loading would be due to axle unit dynamics.
- (3) A dynamic load allowance dependent upon the unloaded first flexural frequency of the structure, Fig. 1.2, should be used for other main longitudinal members, whose design would be governed by a truck or lane loading, and where dynamic loading would be due to response of the structure in its modes of vibration.
- (4) A criterion in terms of superstructure deflection is provided to limit vibration which might be perceived by pedestrians on the bridge, dependent also upon anticipated pedestrian usage of the bridge and the first flexural frequency of the bridge, Fig. 1.3.

In the revised edition of the Code (1983), some modifications were made in the above provisions, such as

- (1) For loading by any two or more axle loads of spans greater than 22 m in length, and spans continuous therewith, the dynamic load allowance shall be obtained from Fig. 1.2.
- (2) The dynamic load allowance shall be 0.30 for loading by other than a single wheel or single axle of all transverse members and spans, all simple spans not greater than 22 m in length, and all superstructures composed of continuous spans with no span length greater than 22 m.
- (3) Pedestrian bridges shall be proportioned so that a single pedestrian weighing 700 N does not produce an acceleration exceeding the limit given in Fig. 1.4 as a function of the first flexural frequency of the superstructure. Also, pedestrian bridges shall be designed such that the longitudinal and the lateral frequency of vibration shall not be less than 1.5 times the first flexural frequency of the superstructure, nor less than 4 Hz., whichever is the smaller.

The Code has not provided limitations such as deflection-to-span or span-to-depth ratios except for timber bridges. Such limitations were specified in the older Ontario Codes and are still used by other codes to limit vibrations, control stresses in secondary members, and reduce the susceptibility to fatigue failure. The new Ontario Code covers these issues explicitly in other sections.

With the above-mentioned provisions the Code recognized an amplified response for bridges having vibration modes with frequencies in the 2 – 4 Hz. range. This range of frequency lies in the frequency range of truck body bounce on tires and suspension. The amplification of response caused by resonance was observed in earlier dynamic tests on bridges conducted by MTC (Campbell et al., 1971).

1.5 Proposed Scope of Present Study

The previous sections have indicated the importance of including the vehicle inertia in the analysis of the dynamic response of a bridge rather than treating the vehicle as a single or a set of moving forces. The effect of vehicle mass on the calculated *impact factors* has become even more important with the advent of faster and heavier trucks. The complexity of the bridge-vehicle dynamic system and the larger number of parameters that

affect the response of such a system have frustrated attempts at identifying the essential characteristics of the bridge behaviour and the important parameters that govern such behaviour. The initial parts of this work use simplified beam and plate models to study the effect of governing parameters. The second part is devoted to a more in depth study using a 3D model for the bridge as well as a sophisticated model for the vehicle consisting of sprung and unsprung masses connected together by a system of springs and dash-pots. Most of the difficulty in such an analysis arises in solving the system of nonlinear differential equations that are generated by treating the bridge-vehicle as an integrated system. On account of the large number of such equations involved, attempts to carry out a complete study have been in the past restricted to relatively small size problems which have not represented the actual bridges dimensions. Methods of solution that rely on "*Ritz Vector*" reduction of the governing equations are developed in the present study.

Both the first and the second part of the study use the finite element method to analyze the dynamic response of bridge structures. The prismatic beam model is also modeled by one or more sinusoidal shapes. The vehicle idealization ranges from a single-axle single-wheel model having sprung and unsprung masses, a two-axle two-wheel model with a sprung mass and two unsprung masses, or as two-axle four-wheel model having one sprung mass and four unsprung masses. All the bridge models studied in this work are single span, simply supported bridges. However, the methodology and the analyses procedures developed here are equally applicable to multiple span continuous bridges. The type of bridges that can be modelled includes: slab bridges, voided slab bridges, deck and girder bridges, and box girder bridges.

The objectives of this study can be summarized under two main categories:

I. Simplified Models — 1D and 2D models

1. Analytical investigation of the dynamic response of simplified models of a bridge. These include beam as well as plate models.
2. Determination of control parameters governing the response of bridges for various types of idealizations.
3. Study of the effect of different parameters on bridge response. The parameters include: damping of the vehicle, initial conditions of the vehicle, interaction between vehicle and bridge, and number of axles of the vehicle.

4. Development of simplified but rational procedures for the design of bridges for dynamic effects.
5. Comparison between the responses for the two type of idealizations of the bridge. This comparison will allow the determination of situations in which the bridge can not be represented by a 1D model.
6. Study of the effect of the path of traveling vehicle and combinations of vehicles on the response of a bridge idealized by a 2D model.
7. Comparison between the responses of isotropic and orthotropic plate models.

In order to achieve the above-mentioned goals, the following steps have been taken:

- a. Various dynamic models for the bridge-vehicle system have been developed and the equations of motion have been derived for each model. These models include, beam models and plate models. The analyses of beam models are carried out by using either sinusoidal mode shapes or the finite element procedure. Several support programs are developed for the analysis of beam models using sinusoidal mode shapes. The output of these programs is used to develop design charts for bridge responses due to single-axle and two-axle vehicle idealizations. The plate models cover isotropic as well as orthotropic plate representations.
- b. The vehicle is idealized using models ranging from a moving force to a two-axle two-wheel vehicle.

II. 3D Analyses — Simple Span Box-girder bridges

1. Analytical studies of 3 dimensional models of simple span box girder bridges to determine the effect of several parameters on the natural frequencies and mode shapes. These parameters include: span-to-width ratio, the presence of diaphragms, and the existence of bracings.
2. Study of the effect on the forced dynamic response of the bridge of several parameters such as span-to-width ratio, the presence of diaphragms, vehicle speed and weight, the ratio between the flexural and torsional rigidities, and the transverse position of the vehicle on the bridge. The parametric studies will be focussed on bridge types which have not been adequately investigated in the past, yet are some of the most widely used, because of their great adaptability to the complex geometry involved in the highway

systems. These are the box girder and deck type bridges.

To meet the above objectives, the following steps have been taken:

- a. A mesh-generation routine for box girder bridges of any number of cells has been developed to facilitate the analytical studies.
- b. A shell element has been developed by combining existing membrane and bending elements and tested for both dynamic and static responses.

A general computer program has been developed for the analysis of the different bridge models for their dynamic response to various moving vehicle idealizations. The program is capable of obtaining the static response due to vehicle load as well as the free vibration response. The computer program is written in **FORTRAN** and algorithms are designed so as to minimize both the needed storage size and the computation time. The program accepts any homogeneous set of units. In solving the eigenvalue problem, the program uses either the *determinant search* or the *subspace iteration* techniques, depending on the size of the matrices involved. Also a mass condensation procedure is carried out on the mass matrix before the final solution to avoid the zero diagonal elements encountered in the lumped mass matrix (Cook, 1970 and Bathe and Wilson, 1976). Double precision has been used throughout the program. An efficient equation solver has been derived and utilized in solving the nonlinear equations associated with the analysis of the complex bridge-vehicle system. Portions of the program have been based on the available source code of the program **SAP IV** (Bathe et al., 1974)

1.6 Arrangement of the Thesis

Chapter 1 presents the nature and mechanics of the complex dynamic system of a moving vehicle and bridge and discusses the different parameters that govern the vehicle-bridge interaction and consequently the bridge dynamic response. This chapter also includes a literature survey that includes references that deal with the various models and idealizations proposed for the vehicle-bridge dynamic system as well as the different methods used in analyzing these models. Chapter 1 also highlights the recent provisions introduced in the Ontario Highway Bridge Design Code regarding the dynamic loading and dynamic response of bridges. Finally, the scope of the study is presented followed by the arrangement of the thesis.

In Chapter 2, an analytical investigation is presented in which the bridge is idealized by a one dimensional beam model that vibrates in a superposition of the lowest few mode shapes which can either be assumed to be harmonic or obtained from a finite element representation of the beam. The response of the beam model for a moving force is first presented. Next a vehicle model comprising a sprung mass and an unsprung mass connected together by an elastic spring and dash-pot is used to obtain the dynamic response of the beam model. The response is calculated using one sinusoidal mode shape and the governing parameters are determined. The practical range of values in which the parameters lie are discussed, and the effect of the parameters on the response of the beam model is analyzed. The response is then calculated using three mode shapes. A brief description of the derivation of the equations of motion using the finite element technique is also presented in this chapter followed by a comparison between the response produced using the finite element method and that obtained using three sinusoidal mode shapes.

The effects of initial conditions and vehicle damping on the response of the beam model are studied. The results of this study are presented in Chapter 2. The development of a design procedure based on a single axle vehicle is described. The response under two-axle vehicle is then discussed and design charts are generated. Finally, a comparison between the responses under two single sprung vehicles in sequence and a two-axle vehicle is presented. A summary and conclusions are also included in Chapter 2.

Chapter 3 is devoted to dynamic response of 2D bridge models. The bridge is modeled either as an orthotropic plates or as an isotropic plates. For each type of idealization, the governing differential equations are discussed and the controlling parameters for both free and forced vibrations are identified. The effect of vehicle path on the response is studied. The response of the bridge under the passage of two vehicles traversing the bridge along different paths is determined and the results presented. The transverse distributions of deflections and moments at the middle cross-section arising from the passage of a single vehicle are obtained and displayed in the form of plots. Finally, a comparison is made between the response of orthotropic plate model and that of isotropic plate model. The chapter concludes with a summary and conclusions.

In Chapter 4, a general formulation of the equations of motion for a 3D model of the bridge is presented. The bridge is modelled by using quadrilateral thin shell elements. The vehicle is idealized as a two-axle, four wheel system with sprung and unsprung masses.

A comparative study is carried out to help in the selection of the finite element that can adequately simulate the behaviour of the bridge superstructure.

Chapter 5 presents a parametric study of 3 dimensional models of simple span box girder bridges. The natural frequencies and mode shapes of such a bridge are studied and the effect of different parameters such as span-to-width ratio, the presence of diaphragms, and the existence of bracings on the frequencies and mode shapes is investigated.

Chapter 6 presents the methods proposed for the solution of the nonlinear differential equations of motion for the vehicle-bridge system. First the average acceleration method is discussed briefly. This is followed by a presentation of the incremental form of the equations of equilibrium. Two equations solvers are described in this chapter, the performance of each of them is investigated and a selection is made of the most efficient solver to be used in the present study. The chapter provides several numerical examples to support the validity of the developed computer program developed for the present study.

In Chapter 7, a parametric study is conducted to investigate the effect on the forced dynamic response of box girder bridges of several parameters such as span-to-width ratio, the presence of diaphragms, vehicle speed and weight, the ratio between the flexural and torsional rigidities, and the transverse position of the vehicle on the bridge.

A summary of the present study and the conclusions drawn are presented in Chapter 8. Also, recommendations are presented for future research.

FIGURE 1.1 MODELLING OF BRIDGE-VEHICLE SYSTEM

FIGURE 1.2 Dynamic Load Allowance
[OHBD, 1983, clause 2-4.3.2.4]

FIGURE 1.3 Deflection Limits for Serviceability
[OHBD, 1983, clause 2-6.2.2.2]

FIGURE 1.4 Acceleration Limit for Pedestrian Bridge Serviceability
[OHBDC, 1983, clause 2-6.2.2.3]

2.1 Introduction

In spite of a number of analytical and experimental investigations on the dynamic response of bridges to moving vehicle loads, the controlling parameters that govern the response have not been clearly identified. This has, in turn, inhibited the development of rational design procedures. An analytical investigation of the response of a simplified beam model traversed by a moving mass that identifies the governing parameters is presented in this Chapter. The results clearly show why attempts to correlate the response to a single parameter, either the span length or the fundamental frequency, are unsuccessful.

Simple design procedures are developed based on relationships between the speed ratio, the weight ratio and the dynamic amplification factors and a set of design curves are provided.

2.2 Simplified Models for Analysis

In order to clearly identify the important parameters, it is necessary to study the response of a simplified model. A simplified bridge-vehicle model could comprise

- (1) A one dimensional beam model of the bridge
- (2) A vehicle model consisting of an unsprung mass in contact with the deck and a sprung mass supported by a spring and a damper.

Further simplification can be achieved by assuming that the beam vibrations can be represented by the superposition of the lowest few mode shapes. These mode shapes can be taken as harmonic functions or can be obtained from a finite element representation. In addition, the damping in the system can be neglected. Also, the unsprung mass can be assumed to be negligible.

2.3 Beam with a Moving Force Model

The simplest model of a bridge-vehicle system consists of a beam with a moving force, the latter representing the vehicle (Fig. 2.1). The response of the system can be represented by the impact factor or dynamic amplification factor, DI , defined as the ratio of the maximum dynamic deflection to the maximum static deflection under the moving force. Considering that the beam vibrates in its first sinusoidal mode, the impact factor is given by (Humar, 1990)

$$DI\left(\frac{L}{2}, t\right) = \left[\frac{1}{(1-\alpha^2)} \sin \alpha \omega_b t - \frac{\alpha}{(1-\alpha^2)} \sin \omega_b t \right] \quad (2.1)$$

where

ω_b = is the fundamental frequency of the bridge structure

$$= \pi^2 \sqrt{\frac{EI}{mL^4}}$$

m = the mass of the bridge per unit length

$$\alpha = \text{speed parameter} = \frac{\pi v}{\omega_b L} = \frac{v}{2f_b L}$$

L = length of bridge

v = speed of the vehicle

Equation 2.1 can be rewritten in the form

$$DI\left(\frac{L}{2}, t\right) = \left[\frac{1}{(1-\alpha^2)} \sin 2\pi \alpha \frac{t}{T_b} - \frac{\alpha}{(1-\alpha^2)} \sin 2\pi \frac{t}{T_b} \right] \quad (2.2)$$

where T_b is the fundamental period of the bridge structure.

By equating the differential of Eq. 2.2 to zero it can be shown that the maximum deflection occurs when

$$\sin \frac{2\pi t}{T_b} = -\sin \frac{2\pi \alpha t}{T_b} \quad (2.3)$$

or

$$\frac{2\pi t}{T_b} = 2n\pi - \frac{2\pi \alpha t}{T_b}$$

so that

$$\frac{t}{T_b} = \frac{n}{1+\alpha} \quad (2.4)$$

where n takes the values $(1, 2, \dots, n)$. The value of t/T_b can not exceed the limiting value T/T_b where T is the traversing time (L/v). Since

$$\frac{T}{T_b} = \frac{L}{vT_b} = \frac{1}{2\alpha} ,$$

for each value of α the maximum impact factor occurs when

$$\frac{t}{T_b} = \frac{1}{1+\alpha}, \frac{2}{1+\alpha}, \dots, \frac{n}{1+\alpha} \neq \frac{1}{2\alpha} \quad (2.5)$$

Substituting for t/T_b from Eq. 2.4 in Eq. 2.2 we get

$$DI\left(\frac{L}{2}, \alpha\right) = \frac{1}{(1-\alpha)} \sin \frac{2n\pi\alpha}{1+\alpha} \quad (2.6)$$

The maximum value of DI is obtained from Eq. 2.6 by substituting the appropriate value of n . The calculations of DI_{max} for different values of α is as shown in Table 2.1. The results lead to a relationship between DI_{max} and α as shown in Fig. 2.2.

2.4 Beam with a Moving Sprung Mass Model

The model shown in Fig. 2.1 does not take into account the interaction of the vehicle and the bridge. In order to account for such interaction, the model is refined so that the vehicle is represented by a sprung mass m_v and an unsprung mass m_t . The mass m_v is supported by a spring of stiffness k and a damper of damping constant c as shown in Fig. 2.3. The unsprung mass remains in contact with the deck. The bridge is assumed to vibrate in its first mode, $\psi(x) = \sin(\pi x/L)$, and the initial conditions of the vehicle on entry to the bridge are assumed to be zero. Damping of the bridge system is ignored.

In order to find the equations of motion, we note that the displacement of the point of contact between the vehicle and the bridge deck is given by

$$\eta = z \sin \frac{\pi vt}{L} \quad (2.7)$$

The time derivatives of η are given by

$$\dot{\eta} = \frac{\pi v}{L} z \cos \frac{\pi vt}{L} + \dot{z} \sin \frac{\pi vt}{L} \quad (2.8a)$$

$$\ddot{\eta} = -\left(\frac{\pi v}{L}\right)^2 z \sin \frac{\pi vt}{L} + 2\frac{\pi v}{L} \dot{z} \cos \frac{\pi vt}{L} + \ddot{z} \sin \frac{\pi vt}{L} \quad (2.8b)$$

Using the principle of virtual work (Humar, 1990), expressions for the effective mass m^* , effective stiffness k^* , and effective external load p^* are given by;

$$m^* = \int_0^L \bar{m} \psi^2(x) dx = \frac{mL}{2} \quad (2.9)$$

$$k^* = \int_0^L EI \psi''^2(x) dx = \frac{\pi^4 EI}{2L^3} \quad (2.10)$$

$$p^* = \int_0^L p \psi(x) dx \quad (2.11)$$

Where EI is the flexural rigidity of the bridge, and $\bar{m}$ is the mass per unit length. Also, as shown in Fig. 2.3

$$p = k(u_v - \eta) + c(\dot{u}_v - \dot{\eta}) - m_t g - m_t \ddot{\eta} - m_v g \quad (2.12)$$

Substituting for $\eta, \dot{\eta}, \ddot{\eta}$ from Eqs. 2.7 and 2.8 into 2.12 and using the resulting value of p in Eq. 2.11

$$p^* = \left[k \left(u_v - \sin \frac{\pi vt}{L} z \right) + c \left(\dot{u}_v - \frac{\pi v}{L} \cos \frac{\pi vt}{L} z - \sin \frac{\pi vt}{L} \dot{z} \right) - m_t \left(- \left(\frac{\pi v}{L} \right)^2 \sin \frac{\pi vt}{L} z + 2 \frac{\pi v}{L} \cos \frac{\pi vt}{L} \dot{z} + \sin \frac{\pi vt}{L} \ddot{z} \right) - (m_t + m_v) g \right] \sin \frac{\pi vt}{L} \quad (2.13)$$

The equation of motion of the bridge deck are now obtained from

$$m^* \ddot{z} + k^* z = p^* \quad (2.14)$$

Referring to the free body diagram of the sprung mass of the vehicle system (Fig. 2.3), the equation of motion of the vehicle can be expressed in the form

$$m_v \ddot{u}_v + c \left(\dot{u}_v - \frac{\pi v}{L} z \cos \frac{\pi vt}{L} - \dot{z} \sin \frac{\pi vt}{L} \right) + k \left(u_v - z \sin \frac{\pi vt}{L} \right) = 0 \quad (2.15)$$

where u_v is the vertical displacement of the sprung mass relative to its position of equilibrium under self weight.

On Substituting for m^*, k^* and p^* in Eq. 2.14 and combining the resulting equation with Eq. 2.15, the equations of motion of the bridge-vehicle system can be written in the form

$$\begin{aligned} & \begin{bmatrix} m_v & 0 \\ 0 & \frac{\bar{m}L}{2} + m_t \sin^2 \left(\frac{\pi vt}{L} \right) \end{bmatrix} \begin{bmatrix} \ddot{u}_v \\ \ddot{z} \end{bmatrix} \\ & + \begin{bmatrix} c & -c \sin \left(\frac{\pi vt}{L} \right) \\ -c \sin \left(\frac{\pi vt}{L} \right) & c \sin^2 \left(\frac{\pi vt}{L} \right) + 2m_t \left(\frac{\pi v}{L} \right) \sin \left(\frac{\pi vt}{L} \right) \cos \left(\frac{\pi vt}{L} \right) \end{bmatrix} \begin{bmatrix} \dot{u}_v \\ \dot{z} \end{bmatrix} \\ & + \begin{bmatrix} k & -k \sin \left(\frac{\pi vt}{L} \right) - c \left(\frac{\pi v}{L} \right) \cos \left(\frac{\pi vt}{L} \right) \\ -k \sin \left(\frac{\pi vt}{L} \right) & k \sin^2 \left(\frac{\pi vt}{L} \right) + c \left(\frac{\pi v}{L} \right) \sin \left(\frac{\pi vt}{L} \right) \cos \left(\frac{\pi vt}{L} \right) - m_t \left(\frac{\pi v}{L} \right)^2 \sin^2 \left(\frac{\pi vt}{L} \right) + \frac{\pi^4 EI}{2L^3} \end{bmatrix} \begin{bmatrix} u_v \\ z \end{bmatrix} \\ & = -g \begin{bmatrix} 0 \\ (m_t + m_v) \sin \left(\frac{\pi vt}{L} \right) \end{bmatrix} \end{aligned} \quad (2.16)$$

In order to simplify the problem, m_t is taken to be negligible and damping c is ignored. This gives

$$\begin{bmatrix} m_v & 0 \\ 0 & \frac{mL}{2} \end{bmatrix} \begin{bmatrix} \ddot{u}_v \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} k & -k \sin\left(\frac{\pi vt}{L}\right) \\ -k \sin\left(\frac{\pi vt}{L}\right) & k \sin^2\left(\frac{\pi vt}{L}\right) + \frac{\pi^4 EI}{2L^3} \end{bmatrix} \begin{bmatrix} u_v \\ z \end{bmatrix} = -g \begin{bmatrix} 0 \\ m_v \sin\left(\frac{\pi vt}{L}\right) \end{bmatrix} \quad (2.17)$$

Now we define

$$\tilde{z} = \frac{zk^*}{m_v g} = \frac{z\omega_b^2}{2\kappa g} \quad (2.18a)$$

and

$$\tilde{u}_v = \frac{u_v k}{m_v g} = \frac{u_v \omega_v^2}{g} \quad (2.18b)$$

where

$$\begin{aligned} \kappa &= \text{mass ratio} = \frac{m_v}{m_b} \\ m_b &= mL = \text{mass of the bridge} \\ \omega_b &= \sqrt{\frac{k^*}{m^*}} \\ \omega_v &= \sqrt{\frac{k}{m_v}} = \text{bounce frequency of the sprung mass} \\ \frac{m_v g}{k^*} &= \text{maximum static deflection of the center point of the bridge} \\ \frac{m_v g}{k} &= \text{maximum static deflection of the vehicle spring} \end{aligned}$$

Note that $\tilde{z}$ is equivalent to the impact factor. Transformation to coordinates $\tilde{z}$ and $\tilde{u}_v$ gives

$$\begin{bmatrix} \frac{1}{\omega_v^2} & 0 \\ 0 & \frac{2\phi^4 \kappa}{\omega_v^2} \end{bmatrix} \begin{bmatrix} \ddot{\tilde{u}}_v \\ \ddot{\tilde{z}} \end{bmatrix} + \begin{bmatrix} 1 & -2\kappa\phi^2 \sin \frac{\omega_v \alpha t}{\phi} \\ -2\kappa\phi^2 \sin \frac{\omega_v \alpha t}{\phi} & 4\kappa^2\phi^4 \sin^2 \frac{\omega_v \alpha t}{\phi} + 2\kappa\phi^2 \end{bmatrix} \begin{bmatrix} \tilde{u}_v \\ \tilde{z} \end{bmatrix} = - \begin{bmatrix} 0 \\ 2\kappa\phi^2 \sin \frac{\omega_v \alpha t}{\phi} \end{bmatrix} \quad (2.19)$$

where ϕ is frequency ratio (ω_v/ω_b).

Equation 2.19 indicates that the bridge response is controlled by four parameters ω_v , κ , ϕ , and α . In fact, it will be seen that the response is independent of ω_v . To prove this, a single span solid slab concrete bridge is analyzed for its response to a moving vehicle. The only parameter varied is ω_v . In order that the frequency ratio ϕ remain constant, ω_b

should be varied accordingly. Parameter α is kept constant by varying v appropriately. The following data is invariant in this study

$$\text{Mass density } m = 2.4465 \text{ t/m}^3$$

$$\text{Bridge width} = 10.0 \text{ m}$$

$$\text{Bridge thickness} = 0.525 \text{ m}$$

$$\text{Bridge span } L = 20 \text{ m}$$

The relationships between the impact factor and $f_v = \omega_v/2\pi$ are plotted in Figs. 2.5 through 2.7 for different values of κ, ϕ , and α . It is clear that the bridge response does not depend on the absolute value of f_v . This result can be explained theoretically as follows. The parameter ω_v appears in the equations of motion, Eq. 2.19, in two ways, in the mass matrix as $\left(\frac{1}{\omega_v^2}\right)$ and in the stiffness matrix as $(\omega_v t)$. A careful examination of the output results shows that upon varying ω_v the time at which the maximum impact factor occurs changes such that the term $(\omega_v t)$ has a constant value regardless of the values of ω_v . Also, for constant values of κ, ϕ , and α the vehicle and bridge accelerations at the time when the maximum impact factor occurs are $\ddot{z} = -Aw_v^2 z$ and $\ddot{u}_v = Bw_v^2 u_v$, where A and B are constants which depend only on the parameters κ, ϕ , and α . Thus the product of mass matrix and acceleration vector can be expressed as the product of a matrix which is independent of ω_v and the displacement vector. Another way to explain the above conclusion is by introducing a new time variable $t = \omega_v t$. Noting that $\partial^2 \ddot{u}_v / \partial t^2 = \omega_v^2 \partial^2 \ddot{u}_v / \partial \bar{t}^2$ and $\partial^2 \ddot{z} / \partial t^2 = \omega_v^2 \partial^2 \ddot{z} / \partial \bar{t}^2$, the response is found to be independent of ω_v .

The foregoing discussion leads to the important conclusion that, with the idealizations made, the dynamic amplification factor is dependent only on the value of κ, ϕ and α . In other words, two different bridges vibrating under the passage of a moving vehicle but having identical values of κ, ϕ and α will have the same dynamic amplification.

2.4.1 Practical Ranges of the Control Parameters

It is seen from the previous paragraph that only three parameters control the dynamic response of a bridge, namely κ, ϕ , and α . It is useful to know the practical range of values that these parameters can take. This is possible by examining the data collected from a number of different references.

A list of 45 bridges together with the description of their characteristics is presented in Table 2.2. These data were obtained in the context of field investigations and tests

carried out in 1964 as a part of the Ontario Joint Highway Research program. The first four columns of Table 2.2, which contain bridge classifications, dead weight, span, and fundamental frequency, are taken directly from this study. The rest of the table displays the values of the three controlling parameters for these bridges. It should be mentioned that these values are calculated on the basis of the following information;

v : vehicle speed = 100 km/hr

f_v : vehicle frequency = 1.5 - 4.0 Hz. average value 2.75 Hz

W_v : vehicle weight = 140 kips

The bridges referred to in Table 2.2 were generally designed for *H20 - S16* loading, representing a truck weighting 72 kips, although larger loads up to 120 kips were permitted on the bridges with special license. The value of 140 kips is taken from a technical report on a recent study of weights and dimensions of heavy trucks in Canada (RTAC Report, 1986). It corresponds to the heaviest weight in this report, 8 axle b-train doubles (3-axle 1st semi, or 2-axle semi's + belly axle). In fact the weights given in the report range from 87 to 140 kips.

Another study was carried out in 1973 on twenty-two spans of ten different bridges designed according to the New Zealand Ministry of Works standard for highway bridges (Shepherd and Aves, 1973). A list of some of these bridges is presented in Table 2.3 along with their relevant characteristics and types. Again the parameters ϕ , and α are calculated on the basis of information provided earlier. The weights of the bridges were not provided in the referred study.

As a part of the present study four concrete slab bridges have been designed in accordance with the Ontario Highway Bridges Design Code, the details of which are presented in Fig. 2.8. Table 2.4 shows the values of the three governing parameters κ , ϕ , and α for these bridges. The values of the lowest fifteen natural frequencies are displayed in Tables 2.5 through 2.8.

Based on the data provided in Tables 2.2 through 2.4, the practical limits for the parameters ϕ , κ and α can be established as follows;

mass ratio $\kappa \leq 1.50$

frequency ratio $\phi \leq 1.20$

speed parameter $\alpha \leq 0.30$

Examining the calculated values of the three parameters presented in Tables 2.2 through 2.4, one can see that approximately 95% of the values falls under the above-mentioned practical limits.

In the following paragraphs a study will be made of the effect of parameters κ , ϕ , and α on the response of the bridge. To study the effect of a certain parameter, only that parameter will be varied while the other two are kept fixed.

2.4.2 The Effect of Weight Ratio κ

The dependence of DI_{max} on parameter κ for various values of α , and ϕ is shown in Figs. 2.9a through 2.9c. The bridge used in the study is a single span slab type bridge with span length 20m. The bridge was designed according to the Ontario Highway Bridge Design Code. The relation $DI_{max} - \kappa$ is fairly complicated such that no systematic trend can be observed. The effect of κ is quite different for different values of ϕ indicated in Figs. 2.9a through 2.9c. While for $\phi = 0.10$, κ does not seem to have a significant effect on DI_{max} , the effect of κ is quite strong for the other values of ϕ . The common feature among the response curves for different values of ϕ is that for $\alpha = 0.05$, the response is independent of κ .

For values of ϕ greater than 0.40 the response curves exhibit a more systematic behaviour in which the response will increase with an increase in values of κ and α with a number of local troughs.

2.4.3 The Effect of Frequency Ratio ϕ

Figures 2.10a through 2.10c present the relationship between DI_{max} and ϕ , again for different values of α , and κ . The effect of ϕ becomes more significant with an increase in κ . Thus, when $\kappa = 0.0$ (moving force case), ϕ does not have any effect on the bridge response. Comparing the response curves for $\phi = 0.10$ (Fig. 2.9a), and for $\kappa = 0.0$ (Fig. 2.10a), one can observe that for small values of κ , ϕ , or both, the response depends mainly on α . Also, for all practical purposes, DI_{max} is independent of ϕ in case of $\alpha = 0.05$ regardless of the value of κ .

For high values of α , that is 0.25 and 0.30, the response curves exhibit a series of local troughs. The location and depth of the first trough depends on the parameter κ . As κ increases, this trough becomes deeper and is located at lower values of ϕ . For $\alpha = 0.15$

and 0.20, the relation $DI_{max} - \phi$ tends to have the first distinct crest located around $\phi = 0.60$ and its height increases with an increase of κ . What is interesting is that, contrary to the generally held belief, maximum response does not occur for $\phi = 1.0$. For $\alpha = 0.1$ the peaks are around $\phi = 0.4$; for $\alpha = 0.15$ they occur when ϕ is approximately equal to 0.6; for $\alpha = 0.20$, the peaks are spread between $\phi = 0.6$ to 0.9; and for $\alpha = 0.3$ the spread is from $\phi = 0.6$ to 1.2.

2.4.4 The Effect of Speed Ratio α

The relationships between DI_{max} and α for different values of ϕ and κ are illustrated in Figs. 2.11a and 2.11b. The curves display a generally ascending tendency with a number of local peaks.

2.5 Beam with a Moving Sprung Mass Model - Effect of Higher Modes

In order to assess the effect of higher modes on the response of a simple beam model, the vibration shape of the beam is approximated by

$$u(x, t) = z_1(t)\psi_1(x) + z_2(t)\psi_2(x) + z_3(t)\psi_3(x) \quad (2.20)$$

where

$$\psi_1(x) = \sin \frac{\pi x}{L}$$

$$\psi_2(x) = \sin \frac{2\pi x}{L}$$

$$\psi_3(x) = \sin \frac{3\pi x}{L}$$

On using the vibration shape defined in Eq. 2.20, the following equations of motion are obtained for the response of the bridge under a moving sprung mass

$$\begin{bmatrix} m_v & 0 & 0 & 0 \\ 0 & \frac{mL}{2} & 0 & 0 \\ 0 & 0 & \frac{mL}{2} & 0 \\ 0 & 0 & 0 & \frac{mL}{2} \end{bmatrix} \begin{bmatrix} u_v \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} +$$

$$= -m_s g \begin{bmatrix} 0 \\ \sin \frac{\pi v t}{L} \\ \sin \frac{2\pi v t}{L} \\ \sin \frac{3\pi v t}{L} \end{bmatrix} \quad (2.21)$$

$$\begin{bmatrix} k & -k \sin \left(\frac{\pi v t}{L} \right) & -k \sin \left(\frac{2\pi v t}{L} \right) & -k \sin \left(\frac{3\pi v t}{L} \right) \\ -k \sin \left(\frac{\pi v t}{L} \right) & k \sin^2 \left(\frac{\pi v t}{L} \right) + \frac{\pi^4 E I}{2L^3} & k \sin \left(\frac{\pi v t}{L} \right) \sin \left(\frac{2\pi v t}{L} \right) & k \sin \left(\frac{\pi v t}{L} \right) \sin \left(\frac{3\pi v t}{L} \right) \\ -k \sin \left(\frac{2\pi v t}{L} \right) & k \sin \left(\frac{\pi v t}{L} \right) \sin \left(\frac{2\pi v t}{L} \right) & k \sin^2 \left(\frac{2\pi v t}{L} \right) + \frac{16\pi^4 E I}{2L^3} & k \sin \left(\frac{2\pi v t}{L} \right) \sin \left(\frac{3\pi v t}{L} \right) \\ -k \sin \left(\frac{3\pi v t}{L} \right) & k \sin \left(\frac{\pi v t}{L} \right) \sin \left(\frac{3\pi v t}{L} \right) & k \sin \left(\frac{2\pi v t}{L} \right) \sin \left(\frac{3\pi v t}{L} \right) & k \sin^2 \left(\frac{3\pi v t}{L} \right) + \frac{81\pi^4 E I}{2L^3} \end{bmatrix} \begin{bmatrix} u_v \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

In deriving Eq. 2.21a, the damping has been neglected. On defining

$$z_1 = \frac{2\kappa g}{\omega_{1b}^2} \tilde{z}_1$$

$$z_2 = \frac{2\kappa g}{\omega_{2b}^2} \tilde{z}_2$$

$$z_3 = \frac{2\kappa g}{\omega_{3b}^2} \tilde{z}_3$$

and transforming to coordinates $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3$ and $\tilde{u}_v$ the equations of motion reduce to

$$\begin{bmatrix} \frac{1}{\omega_v^2} & 0 & 0 & 0 \\ 0 & \frac{2\phi^4 \kappa}{\omega_v^2} & 0 & 0 \\ 0 & 0 & \frac{2\phi^4 \kappa}{\omega_v^2} \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^4 & 0 \\ 0 & 0 & 0 & \frac{2\phi^4 \kappa}{\omega_v^2} \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^4 \end{bmatrix} \begin{bmatrix} \tilde{u}_v \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{bmatrix} +$$

$$\begin{bmatrix} 1 & -2\kappa\phi^2\psi_1 & -2\kappa\phi^2\psi_2 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^2 & -2\kappa\phi^2\psi_3 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^2 \\ -2\kappa\phi^2\psi_1 & 4\kappa^2\phi^4\psi_1^2 + 2\kappa\phi^2 & 4\kappa^2\phi^4 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^2 \psi_1\psi_2 & 4\kappa^2\phi^4 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^2 \psi_1\psi_3 \\ -2\kappa\phi^2\psi_2 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^2 & 4\kappa^2\phi^4 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^2 \psi_1\psi_2 & 4\kappa^2\phi^4\psi_2^2 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^4 + 2\kappa\phi^2 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^2 & 4\kappa^2\phi^4 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^2 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^2 \psi_2\psi_3 \\ -2\kappa\phi^2\psi_3 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^2 & 4\kappa^2\phi^4 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^2 \psi_1\psi_3 & 4\kappa^2\phi^4 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^2 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^2 \psi_2\psi_3 & 4\kappa^2\phi^4\psi_3^2 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^4 + 2\kappa\phi^2 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{u}_v \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{bmatrix} = - \begin{bmatrix} 0 \\ 2\kappa\phi^2\psi_1 \\ 2\kappa\phi^2 \left(\frac{\omega_{1b}}{\omega_{2b}} \right)^2 \psi_2 \\ 2\kappa\phi^2 \left(\frac{\omega_{1b}}{\omega_{3b}} \right)^2 \psi_3 \end{bmatrix} \quad (2.22)$$

where $\psi_i = \sin \frac{i\pi v t}{\phi}$

Note that the impact factor in this case is given by

$$DI = \frac{(z_1 + z_3)}{2\kappa g \left(\frac{1}{\omega_{1b}^2} + \frac{1}{\omega_{3b}^2} \right)}$$

Figure 2.12 displays the response behaviour of a simple beam obtained by including three sinusoidal shape functions in the analysis. Generally, the inclusion of higher modes reduces the response of a bridge. It is clear from the derived equations of motion, Eq. 2.22, that the contribution of the higher mode shapes to the displacement response is very small as the terms related to these modes contain the term $\left(\frac{\omega_b}{\omega_{ib}}\right)^2$ or its higher powers, all of which are very small. This observation can be confirmed by referring to Fig. 2.12 in which curves obtained using three mode shapes practically coincide with those obtained using one sinusoidal shape function for values of $\kappa = 0.0$ and 0.50 . For κ equal 1.5 , the inclusion of higher modes has a more significant effect on the response especially for higher values of α .

2.6 The Effect of the Inclusion of the Interaction

It is well recognized that interaction of the vehicle with the bridge significantly affects the response of the bridge. Thus a moving force model is quite inadequate in predicting the response. This will be evident from Fig. 2.13 which shows the relationship between the dynamic amplification factor DI , and the speed parameter α for one value of the weight ratio κ and three values of frequency ratio ϕ . It will be observed that the envelope of response including interaction is higher than that for a moving force, the difference between the two increasing with α . It is also of interest to note that the highest dynamic amplification does not always occur at $\phi = 1.0$.

2.7 Beam with a Moving Sprung Mass Model - Analysis by Finite Element Method

In this section a comparison is made between response results obtained by using sinusoidal shape functions and those obtained by using a finite element analysis. Before carrying out the comparison, a brief description of the derivation of the equations of motion using the finite element technique will be presented. The complete derivation will be given in detail in a Chapter 4.

2.7.1 Formulation of the Equations of Motion

In finite element analysis, the structure is first subdivided into a number of regions (elements). The displacement field within each region is described by the superposition of a set of a shape functions of spatial coordinates, each function being weighted by a generalized coordinate. For a beam model, the generalized coordinates are the deflection and its first derivative at the interconnecting nodes between the regions. The trial displacement fields should meet the criteria of completeness and admissibility. To satisfy the criteria of completeness, the element displacement field should be able to represent any possible rigid body motion of the element as well as any possible states of constant strain. For the assumed displacement field to be admissible, the compatibility condition along the boundaries between adjacent elements must be satisfied and the continuity of displacements within each element should be maintained. In fact, the finite element procedure as described above is viewed as a piecewise Rayleigh-Ritz procedure in which the solution is made more systematic by the use of matrices. Therefore, the requirements for element displacement fields bear a distinct relationship to the requirements for overall displacement fields in the conventional Rayleigh-Ritz procedure.

For the specific case of flexural vibrations of a beam, since the virtual work contains a derivative of order 2, the nodal generalized coordinates should contain the displacement as well as its first derivative, the slope. Thus, each element will have four nodal degrees of freedom. Within each element, the lateral displacement u_b is expressed as the superposition of four shape functions. The four shape functions are as follows;

$$\begin{aligned}
 \psi_1(x) &= x \left(1 - \frac{x}{L}\right)^2 \\
 \psi_2(x) &= 1 - 3 \left(\frac{x}{L}\right)^2 + 2 \left(\frac{x}{L}\right)^3 \\
 \psi_3(x) &= \left(\frac{x^2}{L}\right)^2 \left(\frac{x}{L} - 1\right) \\
 \psi_4(x) &= 3 \left(\frac{x}{L}\right)^2 - 2 \left(\frac{x}{L}\right)^3
 \end{aligned} \tag{2.23}$$

In the present study, the mass matrix is taken as a lumped mass matrix which is a diagonal matrix with the rotational degrees of freedom being eliminated. The lumped mass matrix is obtained by using shape functions of lower order than those used for obtaining the stiffness

matrix. The shape functions in this case will be :

$$\begin{aligned} \psi_1 &= 0 \\ \psi_2 &= \begin{cases} 1 & r \leq \frac{L}{2} \\ 0 & r > \frac{L}{2} \end{cases} \\ \psi_3 &= 0 \\ \psi_4 &= \begin{cases} 0 & r < \frac{L}{2} \\ 1 & r \geq \frac{L}{2} \end{cases} \end{aligned} \quad (2.24)$$

Following the procedure used earlier for the sinusoidal shape function approach, it can be easily proved that the final equations of motion are;

$$\begin{bmatrix} \mathbf{m} & 0 \\ 0 & \mathbf{M} + \mathbf{m}^* \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_v \\ \ddot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} \mathbf{c} & \mathbf{c}_h \\ \mathbf{c}_v & \mathbf{C} + \mathbf{c}^* \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_v \\ \dot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} \mathbf{k} & \mathbf{k}_h \\ \mathbf{k}_v & \mathbf{K} + \mathbf{k}^* \end{bmatrix} \begin{bmatrix} \mathbf{u}_v \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{p}^* \end{bmatrix} \quad (2.25)$$

where

$$\begin{aligned} \mathbf{m}^*_{(N \times N)} &= m_t \mathbf{N}^T \mathbf{N} \\ \mathbf{c}^*_{(N \times N)} &= 2m_t(v + at) \mathbf{N}^T \mathbf{N}_{,x} + c \mathbf{N}^T \mathbf{N} \\ \mathbf{k}^*_{(N \times N)} &= m_t(v + at)^2 \mathbf{N}^T \mathbf{N}_{,xx} + m_t a \mathbf{N}^T \mathbf{N}_{,x} + k \mathbf{N}^T \mathbf{N} + c(v + at) \mathbf{N}^T \mathbf{N}_{,x} \\ \mathbf{c}_{h(1 \times N)} &= -c \mathbf{N}^T \\ \mathbf{c}_{v(N \times 1)} &= -c \mathbf{N} \\ \mathbf{k}_{v(N \times 1)} &= -k \mathbf{N}^T \\ \mathbf{k}_{h(1 \times N)} &= -k \mathbf{N} - c(v + at) \mathbf{N}_{,x} \\ \mathbf{p}^*_{(N \times 1)} &= -(m_v + m_t) g \mathbf{N}^T \end{aligned}$$

v is the speed of vehicle and a is its acceleration, $\mathbf{N}$ is the row vector of the shape functions. $\mathbf{D}$ is the vector of the nodal d.o.f. of the bridge, $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ the damping matrix, and $\mathbf{K}$ the stiffness matrix of the bridge structure. Equation 2.25 is a set of second-order nonlinear ordinary differential equations. The nonlinearity (time dependence) of the equations is embodied in matrices $\mathbf{m}^*$, $\mathbf{c}^*$, $\mathbf{k}^*$, $\mathbf{c}_v$, $\mathbf{k}_v$, $\mathbf{k}_h$ and $\mathbf{c}_h$. This system of nonlinear equations is generally solved by step-by-step integration methods. However, an approach which combines two concepts, the *modal method* and the *step-integration* will be used to solve these nonlinear equations. In this method, the free vibrations analysis is first performed for both the bridge structure and the vehicle system in order to calculate their characteristics, namely, the natural frequencies and the mode shapes. Both the bridge and vehicle nodal displacements are then expressed in terms of modal coordinates by means of

a transformation. The details of this method as well as its validity are discussed in depth in a later chapter.

2.7.2 Comparison between F.E.M. and Sinusoidal Approach Results

A beam is discretized into twenty finite elements, each of one meter length. Different number of mode shapes are used to assess the bridge response in order to examine the convergence of the solution. In the present case, the results revealed that the response obtained with three modes is as good as that obtained with ten modes. Therefore three modes are considered sufficient for obtaining accurate results. Table 2.9 presents the frequencies and periods of the lowest ten mode shape of the analyzed bridge. Figure 2.14 displays the convergence in the finite element solution by showing the bridge response obtained with one and three mode shapes. As seen from the figure, the response is almost matching for all values of α . Figures 2.15a and 2.15b present comparison between the response obtained using three finite element mode shapes and that obtained using three sinusoidal shape functions for $\kappa = 0.30$. As seen from the figures the trend of the curves corresponding to different values of α is the same for both approaches but the values obtained by using the finite element approach are generally less than those obtained by the other method, especially for high values of α . Also, the troughs and crests in the finite element response curves are shallower than those in the curves obtained by the sinusoidal shape functions approach.

2.7.3 Comparison between The Deflection Impact Factor and Moment Impact Factor

The maximum deflection of the beam occurs at mid-span. If the deflected shape under a static load P acting at the mid-span is also assumed to be proportionate to $\sin \pi x/L$, the maximum static mid-span deflection becomes $2PL^3/\pi^4 EI$. This value is very close to the exact value of $PL^3/48EI$ signifying that the function $\sin \pi x/L$ closely approximates the static deflected shape.

It should be pointed out that for a bridge modelled by a beam, if beam is considered to vibrate only in its first mode of vibration, the dynamic amplification of moment or shear is exactly equal to that of the deflection. The validity of this statement follows from the following argument. If the dynamic displacement of the bridge is represented by $u_b(x, t) = z \sin \pi x/L$, the same spatial function should be used to calculate the static

displacement. The maximum mid-span static deflection under the moving mass will then work out to $\Delta_{st} = 2m_v g L^3 / (\pi^4 EI)$. Since the moment is given by $M = EI \partial^2 u / \partial x^2$, the dynamic value of the mid-span moment is $\pi^2 EI z / L^2$. In a similar manner, when the static displacement shape is assumed to be a sine function, the maximum midspan moment is given by $M_{st} = \pi^2 EI \Delta_{st} / L^2 = 2m_v g L / \pi^2$ compared to the exact value of $m_v g L / 4$. Obviously $M / M_{st} = z / \Delta_{st}$, signifying that the moment amplification is identical to the deflection amplification.

If more than one mode shape is included in the modeling of the deflected shape of the beam, deflection and moment impact factors may be different. In the following paragraph, a simple analysis is carried out to investigate this relationship.

If the shear deformation and rotary inertia are neglected, then for a uniform beam the frequencies and the normalized mode shapes are (Humar, 1990)

$$\omega_n = n^2 \pi^2 \sqrt{\frac{EI}{mL^4}} \quad (2.26a)$$

$$\phi_n(x) = \sqrt{\frac{2}{\bar{m}L}} \sin \frac{n\pi x}{L} \quad n = 1, 2, 3, \dots \quad (2.26b)$$

where $\bar{m}$ is the mass per unit length of the beam. When the lowest three mode shapes are included the static and dynamic deflected shapes, denoted by u_s and u_b respectively, are given by

$$u_s(x) = s_1 \sqrt{\frac{2}{\bar{m}L}} \sin \frac{\pi x}{L} + s_2 \sqrt{\frac{2}{\bar{m}L}} \sin \frac{2\pi x}{L} + s_3 \sqrt{\frac{2}{\bar{m}L}} \sin \frac{3\pi x}{L} \quad (2.27a)$$

and

$$u_b(x, t) = z_1(t) \sqrt{\frac{2}{\bar{m}L}} \sin \frac{\pi x}{L} + z_2(t) \sqrt{\frac{2}{\bar{m}L}} \sin \frac{2\pi x}{L} + z_3(t) \sqrt{\frac{2}{\bar{m}L}} \sin \frac{3\pi x}{L} \quad (2.27b)$$

where s_1, s_2 , and s_3 are the amplitudes of the static deflection for the lowest three modes and z_1, z_2 , and z_3 are the amplitudes of dynamic deflection in the same three modes. The generalized load vector, p^* , is given by

$$p_i^* = m_v g \int \sqrt{\frac{2}{\bar{m}L}} \sin \frac{n\pi x}{L} \delta \left(x - \frac{L}{2} \right) dx \quad (2.28a)$$

Evaluation of Eq. 2.28a gives

$$\begin{aligned} p_1^* &= \sqrt{\frac{2}{\bar{m}L}} m_v g \\ p_2^* &= 0 \\ p_3^* &= -\sqrt{\frac{2}{\bar{m}L}} m_v g \end{aligned} \quad (2.28b)$$

The i^{th} modal equation of the static deflection of the beam model is given by

$$\omega_i^2 s_i = p_i^* \quad (2.29)$$

Therefore, the static amplitudes of deflection at mid-point can be calculated by substituting from Eq. 2.28b and Eq. 2.26a into Eq. 2.29 giving

$$\begin{aligned} u_{s1}\left(\frac{L}{2}\right) &= \frac{2m_v g L^3}{\pi^4 EI} \\ u_{s2}\left(\frac{L}{2}\right) &= 0 \\ u_{s3}\left(\frac{L}{2}\right) &= \frac{-2m_v g L^3}{81\pi^4 EI} \end{aligned} \quad (2.30)$$

Note that

$$s_1 = \frac{1}{\omega_1^2} \sqrt{\frac{2}{\bar{m}L}} m_v g = \frac{\bar{m}L^4}{\pi^4 EI} \sqrt{\frac{2}{\bar{m}L}} m_v g$$

Using Eq. 2.27a

$$u_{s1}\left(\frac{L}{2}\right) = \frac{\bar{m}L^4}{\pi^4 EI} \sqrt{\frac{2}{\bar{m}L}} m_v g \sqrt{\frac{2}{\bar{m}L}} \sin \frac{\pi}{2} = \frac{2m_v g L^3}{\pi^4 EI}$$

Thus, when the lowest three mode shapes are included in the analysis, the deflection impact factor at mid-point becomes

$$DI_{max} = \frac{z_1 + z_3}{s_1 + s_3} \quad (2.31a)$$

Since the moment is given by $EI\partial^2 u/\partial x^2$, the moment impact factors as obtained by differentiating Eqs. 2.27a and b twice becomes

$$MI_{max} = \frac{z_1 + 9z_3}{s_1 + 9s_3} \quad (2.31b)$$

Examining Eqs. 2.31a and 2.31b, one can observe that the impact factors for deflection and moment are exactly similar if only one mode shape is considered. When more than one mode shape is included in the analysis, the relationship between the two impact factors depend on the relative contributions of the higher modes. These contributions depend on the values of the parameters, α , ϕ , and κ . Figures 2.16a and 2.16b show the effect of various values of the parameters α , ϕ , and κ on the deflection and moment impact factors. In some cases the values of deflection impact factors are higher than those of moment; in other cases, the moment impact factors are higher and no systematic trend can be observed. Generally, the values of deflection and moment impact factors are very close.

2.8 Effect of Initial Conditions

Initial conditions of the vehicle at the time of its entry into a bridge can have significant influence on response. Initial displacement and velocity may arise due to a bump in the approach ramp or due to uneven joints over the abutments. To get an appreciation for the magnitude of such initial conditions, consider the simple case of a sprung mass riding over a bump as shown in Fig. 2.17.

The equation of motion is given by

$$m\ddot{u}_v + k(u_v - u_g) = 0 \quad (2.32)$$

or

$$m\ddot{u}_v + ku_v = kh \sin \frac{\pi vt}{H} \quad (2.33)$$

where u_v is the displacement of the sprung mass relative to its displacement due to gravity. The solution to this equation is

$$u_v = \frac{h}{1 - \left(\frac{\pi v}{\omega_v H}\right)^2} \left[\sin \frac{\pi vt}{H} - \frac{\pi v}{\omega_v H} \sin \omega_v t \right] \quad (2.34)$$

At the end of the bump $vt = H$, and

$$u_v = \frac{h}{1 - \left(\frac{\pi v}{\omega_v H}\right)^2} \left(-\frac{\pi v}{\omega_v H} \sin \frac{\omega_v H}{v} \right) \quad (2.35a)$$

$$\dot{u}_v = \frac{h}{1 - \left(\frac{\pi v}{\omega_v H}\right)^2} \left[-\frac{\pi v}{H} - \frac{\pi v}{H} \cos \frac{\omega_v H}{v} \right] \quad (2.35b)$$

The sprung mass now undergoes free vibration with the above initial conditions u_v and $\dot{u}_v$. The amplitude of vibration is given by

$$\begin{aligned} \rho &= h \frac{\frac{\pi v}{\omega_v H}}{\left| 1 - \left(\frac{\pi v}{\omega_v H}\right)^2 \right|} \sqrt{2 \left(1 + \cos \frac{\omega_v H}{v} \right)} \\ &= h \frac{\beta}{|1 - \beta^2|} \sqrt{2 \left(1 + \cos \frac{\pi}{\beta} \right)} \end{aligned} \quad (2.36)$$

where $\beta = \frac{\pi v}{\omega_v H}$

If the interaction between the vehicle and bridge is neglected, the residual vibrations caused by the bump can be accounted for by assuming that the vehicle is represented by a moving oscillating force Q given by

$$\begin{aligned} Q &= -m_v g - \rho k \cos \omega_v t \\ &= -m_v g \left[1 + \frac{\rho k}{m_v g} \cos \omega_v t \right] \\ &= -W [1 + A_m \cos \omega_v t] \end{aligned}$$

where

$$\begin{aligned} W &= m_v g \quad \text{is the weight of the vehicle} \\ A_m &= \frac{\rho k}{m_v g} = \frac{\rho w_v^2}{g} \quad \text{is the dimensionless amplitude} \end{aligned}$$

Table 2.10 provides values for the ratio A_m/h for different values of β and f_v , where f_v is the vehicle frequency in Hz. Figure 2.37 represents this relationship graphically. The figure shows that the dimensionless amplitude A_m is quite small for larger values of β which correspond to small values of H . Evidently irregularities of short width in the riding surface or uneven expansion joints do not affect the response as much as bumps and depressions in the approaches.

Neglecting the interaction and assuming that the bridge vibrates in its first sinusoidal mode, the impact factor due to the oscillating force is given by (Humar, 1990)

$$\begin{aligned} DI \left(\frac{L}{2}, t \right) &= A_m \left[D_1 \sin 2\pi \alpha \frac{t}{T_b} \cos 2\pi \frac{t}{T_v} + D_2 \cos 2\pi \alpha \frac{t}{T} \sin 2\pi \frac{t}{T_v} - \alpha D_3 \sin 2\pi \frac{t}{T_b} \right] \\ &\quad + \left[\frac{1}{(1 - \alpha^2)} \sin 2\pi \alpha \frac{t}{T_b} - \frac{\alpha}{(1 - \alpha^2)} \sin 2\pi \frac{t}{T_b} \right] \end{aligned}$$

or

$$\begin{aligned} DI \left(\frac{L}{2}, t \right) &= A_m \left[D_1 \sin 2\pi \alpha \frac{t}{T_b} \cos 2\pi \phi \frac{t}{T_b} + D_2 \cos 2\pi \alpha \frac{t}{T_b} \sin 2\pi \phi \frac{t}{T_b} - \alpha D_3 \sin 2\pi \frac{t}{T_b} \right] \\ &\quad + \left[\frac{1}{(1 - \alpha^2)} \sin 2\pi \alpha \frac{t}{T_b} - \frac{\alpha}{(1 - \alpha^2)} \sin 2\pi \frac{t}{T_b} \right] \end{aligned} \quad (2.37)$$

where

$$D_1 = [1 - (\phi^2 + \alpha^2)]/\Delta$$

$$D_2 = 2\alpha\phi/\Delta$$

$$D_3 = [1 + (\phi^2 - \alpha^2)]/\Delta$$

$$T_v = 2\pi/\omega_v$$

$$\phi = T_b/T_v$$

$$\Delta = [1 - (\phi + \alpha)^2][1 - (\phi - \alpha)^2]$$

As seen from Eq. 2.37, the impact factor depends on the two parameters ϕ and α as opposed to the case of moving force, Eq. 2.2, where the response depends only on α . In fact, Eq. 2.31 can be looked at as being composed of two parts, the first part which includes the term A_m reflects the effect of initial conditions and the second part which is common in the two Eqs. 2.2 and 2.37 predicts the bridge response to a moving force. Figure 2.19 shows the effect of ϕ on DI_{max} for different values of α . The localized infinite peaks shown in Fig. 2.19 represents the situations in which $\Delta = 0$, that is either $(\phi + \alpha) = 1$ or $(\phi - \alpha) = 1$. It is clear that the DI_{max} versus ϕ relationships will have a peak near $\phi = 1.0$ for very small value of α . For larger values of α the peak will be significantly removed from $\phi = 1$.

Figure 2.20 shows the relationship between DI_{max} and ϕ for $A_m = 0.30$. The relationships are similar to those for $A_m = 0.15$. Figure 2.21 shows a comparison between the response curves for the two values of A_m , 0.15, and 0.30. For α equal to 0.05 and 0.15 the response corresponding to $A_m = 0.30$ is generally greater than that corresponding to $A_m = 0.15$. The difference between the two responses is larger for small values of $\alpha = 0.05$. As the value of α increases, the difference decreases. These results imply that the effect of initial conditions, represented by the first part of Eq. 2.18, is quite significant for a combination of small values of α , values of ϕ around 1.0, and very small values of the weight of the traversing vehicle. Large values of A_m can result under critical combinations of v , f_v and H . For example with $v = 100$ km/hr, $f_v = 5$ Hz., and $H = 3.8$ m, β is 0.73, $\rho/h = 1.72$ and a 25 mm depression will give $A_m = 4.33$. This, in turn, will lead to very high values of dynamic amplification. Fortunately, such critical combinations are expected to be rare. Nevertheless experimental measurements have highlighted excessive amplification factors resulting from rough uneven approaches (Wright and Green, 1964).

In assessing the effect of initial conditions, interaction can be taken into account for the sprung mass model by a simple modification in which the initial displacement of the vehicle is taken as $(-\rho)$. The initial velocity is taken as zero and the initial acceleration works out to $\rho\omega_v^2$ where

$$\rho = \frac{A_m m_v g}{k} = \frac{A_m g}{\omega_v^2} \quad (2.38)$$

A negative sign for the initial displacement produces the most critical response because the static deflection of the bridge under gravity load is also negative.

As stated earlier, initial oscillations, measured by ρ or A_m , strongly influence the vibrations of the bridge. The magnitude of A_m is subject to considerable uncertainty and can be determined only on the basis of extensive statistical studies. However, some field investigations have indicated a range of 0.1 to 0.4 (Biggs, 1959, Shepherd and Aves, 1973).

It is of interest to note that for $\kappa = 0$ the amplification factor obtained from Eq. 2.37 is identical to that obtained by solving Eq. 2.17 with initial conditions described above (Fig. 2.22). This serves as one verification of the formulations. Some researchers have used the oscillating force model to obtain estimates of dynamic amplification. This is not a correct approach because interaction between the bridge and the vehicle alters the response. This will be evident from Fig. 2.23 which show the relationship between DI_{max} and ϕ for two different values of α and three different values of κ . The results obtained from Eq. 2.31 are also shown. It will be noted that in some cases interaction reduces the response, while in others it increases the response, and the changes can be very significant.

It was proved earlier that the response does not depend on the absolute value of f_v . This fact is not affected by the inclusion of the initial conditions, thus two bridges with identical values of α , κ , ϕ and A_m will have identical responses. To prove this, two different values of f_v , namely 1.0 and 3.0 Hz, are chosen and the corresponding values of ρ are calculated using Eq. 2.38 for $A_m = 0.30$. The values of ρ are 0.07455 and 0.00828 m respectively. The bridge response corresponding to the two cases are presented in Figs. 2.24a and 2.24b respectively. It is clear that the two sets of results are exactly the same. Therefore f_v can be chosen to have any value without affecting the generality of the analysis.

Figures 2.25a and 2.25b present the effect of initial conditions on bridge response for different weight ratios. A distinct peak is observed at $\phi = 1.0$ when $\kappa = 0.0$. This peak disappears for other values of κ . Further, as seen by comparing Figs. 2.25a and 2.25b with

Figs. 2.26a and 2.26b, although doubling the value of ρ or A_m results in almost two fold increase in the response of the bridge at $\phi = 1.0$ for $\kappa = 0.0$; the same increase in A_m amplifies the response by a maximum of only 20% for values of $\kappa = 0.1$, and 0.30. As κ increases the difference between the responses corresponding to different values of A_m becomes quite small. In interpreting Figs. 2.25a through 2.26b, it should be noted that for $f_v = 1$, $A_m = 0.15$ corresponds to $\rho = 0.03727$ and $A_m = 0.30$ corresponds to $\rho = 0.07455$.

In a previous paragraph we compared the dynamic amplifications caused by initial conditions with and without interaction between the bridge and the vehicle. We found that interaction significantly affects the response, except in the case of vehicles with very low weight ratio. It would be of interest to study the effect of initial conditions on the dynamic amplifications caused by interaction between the bridge and the vehicle. This can be done by referring to Figs. 2.27a and 2.27b which display the response curves for the two cases $\rho = 0.0$ and 0.03727 for values of $\kappa = 0.0, 1.5$. It is seen from the figures that the effect of the initial conditions is quite significant on the behaviour of the bridge especially for small values of $\kappa = 0.0$. For $\kappa = 0.0$, the inclusion of initial conditions amplifies the response at $\phi = 1.0$ by almost 3 and 1.5 for $\alpha = 0.05$, and 0.15 respectively. For $\kappa = 1.5$, the initial conditions do increase the peak response but the effect becomes less pronounced as α increases. From the previous observations one can conclude that the effect of initial conditions can not be ignored in determining the bridge response, especially for small values of κ .

Interaction and initial conditions can be incorporated in a finite element analysis by considering isolated irregularities (localized bumps, expansion joints, localized potholes, etc). In the finite element program developed as a part of the present study, the initial conditions can be accounted for in two different ways. The first method is to consider an approach ramp of certain length and infinite rigidity on which the vehicle passes before entering the bridge. The location of the bump, Fig. 2.17, on the ramp should be specified as a part of the input data together with the bump length H and height h . The program produces the values of the vehicle vertical displacement, velocity and accelerations just before entering the bridge which become the initial conditions of the vehicle. In order to be able to compare the results obtained by using the finite element analysis results with those obtained by using sinusoidal shape functions, the same initial conditions should be

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used. Therefore, at the end of the ramp the vehicle initial conditions should be;

$$\text{vertical displacement} = \rho,$$

$$\text{vertical velocity} = 0.0, \text{ and}$$

$$\text{vertical acceleration} = -\rho\omega_v^2$$

To maintain such a status, the value of h is determined from the expression

$$h = \rho \frac{|1 - \beta^2|}{\beta \sqrt{2 \left(1 + \cos \frac{\pi}{\beta}\right)}} \quad (2.39)$$

and then the length of the ramp is adjusted so that at the end of it the vehicle vertical velocity is zero and therefore the displacement is equal ρ .

The second approach to account for the vehicle initial conditions is to input the values of initial displacement, initial velocity, and acceleration directly. The second approach is used to produce Figs. 2.28a and 2.28b. In these figures a comparison is made between the results of the finite element analysis and those of the sinusoidal shape functions approach for different values of α and κ . It should be mentioned that in Figs. 2.28a and 2.28b, the vehicle mass is taken to be equal to 1.0 ton, which means that $\kappa = 0.00391$. Figures 2.28a and 2.28b show good agreement between the two methods of analysis.

2.9 Effect of the Vehicle Spring Damping

2.9.1 Introduction

In the present section we consider the effect of a parameter which was neglected earlier, that is the vehicle damping c . Consideration of the characteristics of vehicle suspension system is of considerable importance in the analysis of vehicle-bridge interaction, especially when vehicle initial conditions are involved. Attempts at the evaluation and simulation of the suspension properties of the vehicle have been carried out by many researchers (Fenves et al, 1962; Veletsos and Huang, 1970; Whittemore et al, 1970; Winkler, 1973; Mulcahy et al, 1983). As a result of these attempts, several models for vehicles suspensions have been developed. A reasonable mathematical model of the suspension system is one in which the suspension is idealized as the parallel combination of a suspension spring, a viscous damper, and a friction device in series with a spring called the "friction spring" (Mulcahy

et al, 1983). The suspension spring is, in general, nonlinear whereas the friction spring is linear.

2.9.2 Formulation of Equations of Motions

In the following analysis, the suspension system is modelled by a linear suspension spring and a viscous damper (Fig. 2.3). When the vibration shape of the bridge is represented by a sine function, the equation of motion is given by Eq. 2.16. When the vibration shape is represented by a superposition of three sinusoidal shape functions, the equations of motion become

$$\begin{aligned}
 & \begin{bmatrix} m_v & 0 & 0 & 0 \\ 0 & \frac{mL}{2} & 0 & 0 \\ 0 & 0 & \frac{mL}{2} & 0 \\ 0 & 0 & 0 & \frac{mL}{2} \end{bmatrix} \begin{bmatrix} \ddot{u}_v \\ \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_3 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} \begin{bmatrix} \dot{u}_v \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} \\
 & + \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_v \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = -gm_v \begin{bmatrix} 0 \\ \sin \frac{\pi vt}{L} \\ \sin \frac{2\pi vt}{L} \\ \sin \frac{3\pi vt}{L} \end{bmatrix} \quad (2.40)
 \end{aligned}$$

where

$$c_{11} = c$$

$$c_{12} = c_{21} = -c \sin \left(\frac{\pi vt}{L} \right)$$

$$c_{13} = c_{31} = -c \sin \left(\frac{2\pi vt}{L} \right)$$

$$c_{14} = c_{41} = -c \sin \left(\frac{3\pi vt}{L} \right)$$

$$c_{22} = c \sin^2 \left(\frac{\pi vt}{L} \right)$$

$$c_{23} = c_{32} = c \sin \left(\frac{\pi vt}{L} \right) \sin \left(\frac{2\pi vt}{L} \right)$$

$$c_{24} = c_{42} = c \sin \left(\frac{\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right)$$

$$c_{33} = c \sin^2 \left(\frac{2\pi vt}{L} \right)$$

$$c_{34} = c_{43} = c \sin \left(\frac{2\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right)$$

$$c_{44} = c \sin^2 \left(\frac{3\pi vt}{L} \right)$$

$$k_{11} = k$$

$$k_{12} = -k \sin \left(\frac{\pi vt}{L} \right) - c \frac{\pi v}{L} \cos \left(\frac{\pi vt}{L} \right)$$

$$k_{13} = -k \sin \left(\frac{2\pi vt}{L} \right) - c \frac{2\pi v}{L} \cos \left(\frac{2\pi vt}{L} \right)$$

$$k_{14} = -k \sin \left(\frac{3\pi vt}{L} \right) - c \frac{3\pi v}{L} \cos \left(\frac{3\pi vt}{L} \right)$$

$$k_{21} = -k \sin \left(\frac{\pi vt}{L} \right)$$

$$k_{22} = k \sin^2 \left(\frac{\pi vt}{L} \right) + c \frac{\pi v}{L} \cos \left(\frac{\pi vt}{L} \right) \sin \left(\frac{\pi vt}{L} \right) + \frac{\pi^4 EI}{2L^3}$$

$$k_{23} = k \sin \left(\frac{\pi vt}{L} \right) \sin \left(\frac{2\pi vt}{L} \right) + c \frac{2\pi v}{L} \cos \left(\frac{2\pi vt}{L} \right) \sin \left(\frac{\pi vt}{L} \right)$$

$$k_{24} = k \sin \left(\frac{\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right) + c \frac{3\pi v}{L} \cos \left(\frac{3\pi vt}{L} \right) \sin \left(\frac{\pi vt}{L} \right)$$

$$k_{31} = -k \sin \left(\frac{2\pi vt}{L} \right)$$

$$k_{32} = k \sin \left(\frac{\pi vt}{L} \right) \sin \left(\frac{2\pi vt}{L} \right) + c \frac{\pi v}{L} \cos \left(\frac{\pi vt}{L} \right) \sin \left(\frac{2\pi vt}{L} \right)$$

$$k_{33} = k \sin^2 \left(\frac{2\pi vt}{L} \right) + c \frac{2\pi v}{L} \cos \left(\frac{2\pi vt}{L} \right) \sin \left(\frac{2\pi vt}{L} \right) + \frac{16\pi^4 EI}{2L^3}$$

$$k_{34} = k \sin \left(\frac{2\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right) + c \frac{3\pi v}{L} \cos \left(\frac{3\pi vt}{L} \right) \sin \left(\frac{2\pi vt}{L} \right)$$

$$k_{41} = -k \sin \left(\frac{3\pi vt}{L} \right)$$

$$k_{42} = k \sin \left(\frac{\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right) + c \frac{\pi v}{L} \cos \left(\frac{\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right)$$

$$k_{43} = k \sin \left(\frac{2\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right) + c \frac{2\pi v}{L} \cos \left(\frac{2\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right)$$

$$k_{44} = k \sin^2 \left(\frac{3\pi vt}{L} \right) + c \frac{3\pi v}{L} \cos \left(\frac{3\pi vt}{L} \right) \sin \left(\frac{3\pi vt}{L} \right) + \frac{81\pi^4 EI}{2L^3}$$

2.9.3 Discussion of Results

Figures 2.29 through 2.30 display the effect of vehicle damping on bridge response. The figures are drawn for different values of α , κ , and $\zeta = c/2\sqrt{km_v}$. The figures include two set of results; the first set is drawn without considering the initial conditions of the vehicle,

that is $\rho = 0.0$ (Fig. 2.29); and the second set accounts for the initial conditions by setting a value for $\rho = 0.03727$ m (Fig. 2.30). Generally, the inclusion of vehicle damping reduces the response of the bridge. The amount of reduction depends on the values of ϕ , α , κ , and ζ .

The first set of results, Figs. 2.29a through 2.29c, correspond to the values of $\kappa = 0.0, 0.5$, and 1.5 . It is clear that for $\kappa = 0.0$, which represents the moving force case, vehicle damping does not have any effect on the bridge response. In other cases the reduction in response is greater for larger values of κ . It is interesting to observe that the inclusion of vehicle damping in the analysis does not change the nature of the variation of response with variation of ϕ .

The second set of results presented in Figs. 2.30a through 2.30c are for the same values of κ as the first set. The figures reveal that the inclusion of vehicle damping is quite significant when initial conditions exist, especially for combinations of small vehicle weights and low values α .

2.10 Response History

At this stage it would be of interest to present the curves of response history for midspan deflection of the bridge corresponding to different values of ϕ , α , κ , ρ , and ζ . Figures 2.31a and 2.31b show the response history curves for values of κ ranging from 0.0 to 1.5 and values of α ranging from 0.05 to 0.30 . Results are presented for two different cases; with and without initial conditions.

When the initial conditions are neglected (Fig. 2.31a), the response history curves show many local peaks for values of $\alpha = 0.05$ and 0.15 . For $\alpha = 0.30$ the response curves have one well-defined peak. The location of the peak depends on the value of κ . For $\kappa = 0.0$, which corresponds to the moving force case, the peak occurs at $t/T = 0.45$. For larger values of κ the peak shifts towards the right, that is just before the vehicle leaves the bridge.

With the initial conditions taken into account (Fig. 2.31b), the response history curve corresponding to $\kappa = 0.0$ and $\alpha = 0.05$ is markedly different from that obtained with no initial conditions. For other combination of κ and α the nature of the two sets of curves is similar.

2.11 Response under a Two Axle Vehicle

2.11.1 Formulation of Equations of Motion

In this section we will investigate the effect of modeling the vehicle as two-axle vehicle on the response of the bridge structure traversed by such a vehicle model. The two-axle vehicle idealization has two degrees of freedom, D.O.F., q_1 representing the bounce of the vehicle, and q_2 representing the pitch (Fig. 2.32). Referring to Fig. 2.32, the following geometric relationships can be written;

$$y_1 = q_1 - a_1 S q_2 = d_1 - a_1 d_2 \quad (2.41a)$$

$$y_2 = q_1 + a_2 S q_2 = d_1 + a_2 d_2 \quad (2.41b)$$

where

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ S q_2 \end{bmatrix}$$

The different forces acting on the vehicle body are now identified. Forces F_1 and F_2 at the rear and front of the body respectively, caused by the springs and dampers are given by

$$F_1 = k_r(y_1 - zN_1) + c_r(\dot{y}_1 - \frac{dN_1}{dt}z - N_1\dot{z}) \quad (2.42a)$$

$$F_2 = k_f(y_2 - zN_2) + c_f(\dot{y}_2 - \frac{dN_2}{dt}z - N_2\dot{z}) \quad (2.42b)$$

where

$$N_1 = \sin \frac{\pi}{L}(vt - S) = \sin \left(\frac{\alpha}{\phi} \omega_v t - \frac{\pi S}{L} \right),$$

$$N_2 = \sin \frac{\alpha}{\phi} \omega_v t$$

$$\begin{array}{ll} \text{Also} & \frac{\partial y_1}{\partial d_1} = 1 & \frac{\partial y_1}{\partial d_2} = -a_1 \\ & \frac{\partial y_2}{\partial d_1} = 1 & \frac{\partial y_2}{\partial d_2} = a_2 \end{array}$$

The equation of virtual work for the vehicle body is given by

$$F_1 \delta y_1 + F_2 \delta y_2 + m_v \ddot{q}_1 \delta q_1 + I_v \ddot{q}_2 \delta q_2 = 0 \quad (2.43)$$

or

$$F_1(\delta d_1 - a_1 \delta d_2) + F_2(\delta d_1 + a_2 \delta d_2) + m_v \ddot{d}_1 \delta d_1 + \frac{I_v}{S^2} \ddot{d}_2 \delta d_2 = 0 \quad (2.44)$$

Substituting for the forces F_1 , and F_2 from Eqs. 2.42 into Eqs. 2.44 gives

$$\begin{aligned}
 & k_r(d_1 - a_1 d_2 - N_1 z)(\delta d_1 - a_1 \delta d_2) + k_f(d_1 + a_2 d_2 - N_2 z)(\delta d_1 + a_2 \delta d_2) \\
 & + c_r(\dot{d}_1 - a_1 \dot{d}_2 - N_1 \dot{z} - \frac{dN_1}{dt}z)(\delta d_1 - a_1 \delta d_2) + c_f(\dot{d}_1 + a_2 \dot{d}_2 - N_2 \dot{z} - \frac{dN_2}{dt}z)(\delta d_1 + a_2 \delta d_2) \\
 & + m_v \ddot{d}_1 \delta d_1 + \frac{I_v}{S^2} \ddot{d}_2 \delta d_2 = 0
 \end{aligned} \tag{2.45}$$

Rearranging of Eq. 2.45 yields

$$\begin{aligned}
 & [(k_r + k_f)d_1 + (-a_1 k_r + a_2 k_f)d_2] \delta d_1 + [(-a_1 k_r + a_2 k_f)d_1 + (a_1^2 k_r + a_2^2 k_f)d_2] \delta d_2 \\
 & + [(c_r + c_f)\dot{d}_1 + (-a_1 c_r + a_2 c_f)\dot{d}_2] \delta d_1 + [(-a_1 c_r + a_2 c_f)\dot{d}_1 + (a_1^2 c_r + a_2^2 c_f)\dot{d}_2] \delta d_2 \\
 & + m_v \ddot{d}_1 \delta d_1 + \frac{I_v}{H^2} \ddot{d}_2 \delta d_2 - \left[(k_r N_1 + k_f N_2 + c_r \frac{dN_1}{dt} + c_f \frac{dN_2}{dt}) z \right] \delta d_1 \\
 & - [(c_r N_1 + c_f N_2) \dot{z}] \delta d_1 + \left[(a_1 k_r N_1 - a_2 k_f N_2 + a_1 c_r \frac{dN_1}{dt} - a_2 c_f \frac{dN_2}{dt}) z \right] \delta d_2 \\
 & + [(a_1 c_r N_1 - a_2 c_f N_2) \dot{z}] \delta d_2 = 0
 \end{aligned} \tag{2.46}$$

Since δd_1 and δd_2 are both arbitrary virtual quantities, Eq. 2.46 leads to two independent equations, which can be expressed in matrix form as

$$m\ddot{d} + c\dot{d} + kd + k_h z + c_h \dot{z} = 0 \tag{2.47}$$

where

$$m_{(2 \times 2)} = \begin{bmatrix} m_v & 0 \\ 0 & \frac{I_v}{S^2} \end{bmatrix} \quad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$k_{(2 \times 2)} = \begin{bmatrix} k_r + k_f & -a_1 k_r + a_2 k_f \\ -a_1 k_r + a_2 k_f & a_1^2 k_r + a_2^2 k_f \end{bmatrix}$$

$$c_{(2 \times 2)} = \begin{bmatrix} c_r + c_f & -a_1 c_r + a_2 c_f \\ -a_1 c_r + a_2 c_f & a_1^2 c_r + a_2^2 c_f \end{bmatrix}$$

$$\begin{aligned}
 k_{h(2 \times 1)} &= \begin{bmatrix} -(k_r N_1 + k_f N_2 + c_r \frac{dN_1}{dt} + c_f \frac{dN_2}{dt}) \\ a_1 k_r N_1 - a_2 k_f N_2 + a_1 c_r \frac{dN_1}{dt} - a_2 c_f \frac{dN_2}{dt} \end{bmatrix} \quad c_{h(2 \times 1)} = \begin{bmatrix} -c_r N_1 - c_f N_2 \\ a_1 c_r N_1 - a_2 c_f N_2 \end{bmatrix}
 \end{aligned}$$

Contact Forces

Figure 2.33 shows the contact forces, P_1 , and P_2 , between the vehicle and bridge surface. It is easily shown that they can be expressed as follows

$$P_1 = \mathbf{f}_1 \mathbf{d} + \mathbf{g}_1 \dot{\mathbf{d}} + Z_1 z + G_1 \dot{z} + H_1 \ddot{z} + Q_1 \quad (2.48)$$

where,

$$\begin{aligned} \mathbf{f}_1 &= [k_r \quad -a_1 k_r], \quad \mathbf{g}_1 = [c_r \quad -a_1 c_r] \\ Z_1 &= -\left(k_r N_1 + \frac{dN_1}{dt} c_r + \frac{d^2 N_1}{dt^2} m_{tr}\right) \\ G_1 &= -\left(c_r N_1 + 2 \frac{dN_1}{dt} m_{tr}\right) \\ H_1 &= -m_{tr} N_1 \\ Q_1 &= -(a_2 m_v + m_{tr}) g \end{aligned}$$

Note that $\mathbf{f}_1$ and $\mathbf{g}_1$ are each of size 1×2 and Z_1 , G_1 , and H_1 are each of size 1×1 . Similarly

$$P_2 = \mathbf{f}_2 \mathbf{d} + \mathbf{g}_2 \dot{\mathbf{d}} + Z_2 z + G_2 \dot{z} + H_2 \ddot{z} + Q_2 \quad (2.49)$$

where,

$$\begin{aligned} \mathbf{f}_2 &= [k_f \quad a_2 k_f], \quad \mathbf{g}_2 = [c_f \quad a_2 c_f] \\ Z_2 &= -\left(k_f N_2 + \frac{dN_2}{dt} c_f + \frac{d^2 N_2}{dt^2} m_{tf}\right) \\ G_2 &= -\left(c_f N_2 + 2 \frac{dN_2}{dt} m_{tf}\right) \\ H_2 &= -m_{tf} N_2 \\ Q_2 &= -(a_1 m_v + m_{tf}) g \end{aligned}$$

Equations of motion of the Bridge-Vehicle System

Since the objective here is to find out the controlling parameters for the case of the two-axle vehicle model, the following simplifications are made;

$$c_f = c_r = 0, \quad k_f = k_r = \frac{k}{2}, \quad m_{tr} = m_{tf} = 0$$

With the above simplifications, the equations of motion for bridge are given by

$$m^* \ddot{z} + k^* z = P_1 N_1 + P_2 N_2 \quad (2.50)$$

On substituting for P_1 and P_2 from Eqs. 2.48 and 2.49 in Eq. 2.50 and combining the result with Eq. 2.47, the equations of motion for bridge-vehicle system can be expressed as

$$\begin{bmatrix} m_v & 0 & 0 \\ 0 & \frac{I_v}{S^2} & 0 \\ 0 & 0 & m_s \end{bmatrix} \begin{bmatrix} \ddot{d}_1 \\ \ddot{d}_2 \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} k & \frac{k}{2}(a_2 - a_1) & -\frac{k}{2}(N_1 + N_2) \\ \frac{k}{2}(a_2 - a_1) & \frac{k}{2}(a_1^2 + a_2^2) & \frac{k}{2}(a_1 N_1 - a_2 N_2) \\ -\frac{k}{2}(N_1 + N_2) & \frac{k}{2}(a_1 N_1 - a_2 N_2) & k_s + \frac{k}{2}(N_1^2 + N_2^2) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ z \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ m_v g(a_2 N_1 + a_1 N_2) \end{bmatrix} \quad (2.51)$$

2.11.2 Discussion of Response under Two Axle Vehicle

On examining Eq. 2.51, one can identify the controlling parameters associated with the two-axle idealization of the vehicle. In addition to the three parameters obtained earlier for the case of single sprung mass, namely α , ϕ , and κ , three more parameters appear to affect the response of the bridge structure. These parameters are; S/L ; a_1 ; and $\hat{I} = I_v/(a_1 a_2 S^2 m_v)$ where $\hat{I}$ is the dynamic index of the sprung mass m_v . The dynamic index, $\hat{I}$, is assumed to fall in the range 0.0 to 1.0. It should be noted that when $a_1 = a_2 = 1/2$, the dynamic index $\hat{I} = \omega_v^2/\omega_\theta^2$, where ω_θ is the pitch frequency of the vehicle.

To verify that the controlling parameters are as stated above, the response of three bridges of different spans, 10, 20 and 30 m, having the same values of $\kappa = 0.50$, $\alpha = 0.15$, $\phi = 1.0$, $\hat{I} = 1.0$, $a_1 = 0.50$, and $S/L = 0.30$ are analyzed. The dynamic response under the two axle vehicle of mass m_v is determined and normalized by the maximum static mid-point deflection for a single sprung mass vehicle model also of mass m_v . The normalized dynamic responses is found to be 1.083 in each case.

In order to study the response of bridges under the passage of a two-axle vehicle, the dynamic amplification factors obtained for several different cases are shown in Figs. 2.34. These figures are drawn for $\alpha = 0.15$ and 0.30, $\kappa = 0.50$, $a_1 = 0.50$, $\hat{I} = 0.0$, two values of S/L ; 0.15 and 0.30, and different values of ϕ . The deflections are normalized with respect to the maximum static mid-point deflection of the model of the bridge under consideration for a single sprung mass vehicle model. For the purpose of comparison, the response curves for a single vehicle of the same mass are also shown. The figures show the strong effect of the axle spacings on the dynamic response of the bridge especially for higher values of α .

In general, the two axle vehicle produces significantly smaller response than a single axle vehicle of the same mass.

The effect of the dynamic index $\dot{I}$ on the response of the bridge is shown in Figs. 2.35a through 2.35b. Again, the deflections are normalized by the maximum static mid-point deflection for a single sprung mass vehicle model. It is clear from the figures that the effect of this parameter is quite negligible.

2.11.3 Development of a Design Procedure for single-axle vehicle model

The present codes generally relate the dynamic amplification factor either to the bridge span or to the first natural frequency of the bridge. Neither approach is completely rational. In fact, relating the amplification factor to the bridge frequency is not an improvement over the older method of relating it to the bridge span. The correlation between the measured data on amplification and the bridge frequency is not strong (Wright and Green, 1964, Cantieni 1989), and there is considerable scatter in the data. This is to be expected on the basis of analytical results presented in this chapter. Two bridges with equal spans or equal fundamental natural frequencies can have widely different values of α , ϕ and κ and hence large differences in the calculated amplification factors.

In developing a design methodology for dynamic loading, we note that as soon as a preliminary design for the bridge has been completed, the weight and the frequency of the bridge are determined. Knowing the weight of the design vehicle, the weight ratio κ can thus be calculated. The speed parameter can vary over a range depending on the speed of vehicle. However, because the response, in general, increases with α , it is only the maximum value of α that is of interest. Like the speed parameter, the frequency ratio ϕ can also vary over a range of values depending on the bounce frequency of the vehicle; however, unlike the case of α , the dynamic amplification does not vary systematically with ϕ . This will be clear from Figs. 2.36a through 2.36c. Dynamic amplifications should therefore be determined for a range of possible values of ϕ and the maximum amplification used in design.

Consider now the results presented in Fig. 2.37 which shows the variation of DI_{max} with α for a number of different values of ϕ varying from 0 to 1.35. The curves have been drawn for a single value of κ . The envelope to these curves gives the largest value of amplification factor for any given value of α . The envelope value also increase with α . Obviously curves

such as the one in Fig. 2.37 can be used to select the design value of DI_{max} for a given κ . With this in view, envelope curves have been drawn in Fig. 2.38 for several different values of κ . In order to obtain the design amplification factor for a given bridge, a preliminary design must be performed first in order to obtain the bridge weight and the fundamental frequency. Knowing the weight of the heaviest design vehicle and the maximum speed, the design weight ratio and speed parameter can be determined. The curves in Fig. 2.38 can now be entered with known values of κ and α to determine the value of DI_{max} which can then be used in the final design.

2.11.4 Development of a Design Procedure for two-axle vehicle

Following the same technique as used in the previous section, design charts can be drawn for the case of two-axle vehicle. Consider the results presented in Fig. 2.39 which shows the variation of DI_{max} with α for a number of different values of ϕ varying from 0 to 1.35. The curves have been drawn for single values of κ and S/L . As usual, dynamic amplification is defined as the ratio of the maximum dynamic mid-span deflection produced by the two axle vehicle to the maximum mid-span static deflection produced by the same vehicle. The envelope to these curves gives the largest value of amplification factor for any given value of α . The envelope value also increase with α . Obviously curves such as the one in Fig. 2.39 can be used to select the design value of DI_{max} for given κ and S/L .

Following the methodology outlined above and assuming that the design vehicle can be represented by a two axle model, envelope curves for dynamic amplifications have been drawn for several different values of κ and S/L . The effect of initial conditions has not been included in these curves. A similar procedure will apply if initial conditions were to be included, but a design value for A_m must be established first, as well a reasonable value of vehicle damping must be selected.

For $S/L \leq 0.05$ (Fig. 2.40), the two axle vehicle produces approximately the same response as an equivalent single axle vehicle. For other cases the two sets of curves are different. A few of the design envelope curves are shown in Fig. 2.41. In order to obtain the design amplification factor for a given bridge, a preliminary design must be performed first so as to obtain the bridge weight and the fundamental frequency. Knowing the weight of the two axle design vehicle and the maximum speed, the design weight ratio, and the speed parameter can be determined. The curves in Fig. 2.41 can now be entered with known

values of κ , α , and S/L to determine the value of the amplification factor DI_{max} to be used in the final design.

2.12 Response under Two Single Sprung Mass Vehicles in Sequence

For bridges of larger spans, it is possible for more than one vehicle to be traveling in sequence. Figures 2.42 through 2.43 show the normalized mid-point deflection produced by two vehicles each of mass m_v and spring stiffness k traveling in sequence along the center line of a bridge of span 20 m. The distance between the vehicles is taken to be 3 and 6 m. For the purpose of comparison, the normalized deflections produced by a single vehicle represented by a one-axle model of spring stiffness $2k$ and that produced by a single vehicle represented by a two-axle model, with $k_f = k_r = k$, each of mass $2m_v$, are also shown. All values of deflections have been normalized by the maximum static mid-span deflection caused by a single vehicle idealized by a single-axle model of mass $2m_v$.

In general, the response due to two vehicles in series is less than that due to single vehicle of equivalent characteristics. The reduction in response is greater for higher values of S/L . For lower values of S/L , the dynamic response of a bridge due to a two-axle vehicle is quite similar to that due to two single vehicles in sequence, especially for higher values of α (Fig. 2.42). The difference between the two is more pronounced for higher values of S/L (Figs. 2.43).

2.13 Summary and Conclusions

The dynamic response of a bridge under a moving vehicle is a highly complex phenomenon. In order to identify the important parameters governing the response, it is useful to represent the bridge vehicle system by a simplified model that will still capture the essential dynamics characteristics of the problem. In this chapter analytical results for the dynamic response of a bridge are obtained on the basis of a simple sprung mass model for the vehicle. It is shown that for such a model, a Rayleigh-Ritz procedure, in which the first sinusoidal shape of the bridge is used as a Ritz shape, is quite effective for obtaining the response. As an alternative, a finite element procedure is developed in which the bridge

is divided into a series of beam elements and a modal superposition method of analysis is used. Again the first mode is adequate for representing the displacement response.

Whether the Rayleigh-Ritz method with sinusoidal Ritz shape or the finite element method is used, the resulting equations are nonlinear and are solved by an appropriate numerical integration scheme.

The analytical results are used to identify the controlling parameters that govern the response. Also, a design methodology for selecting an impact factor or dynamic amplification is proposed.

The following conclusions can be drawn from the analytical results presented in this chapter

1. Bridge-vehicle interaction is a very important consideration in determining the response of a bridge under the action of a moving vehicle.
2. The dynamic response of the bridge is independent of the absolute value of the vehicle sprung mass frequency ω_v .
3. Besides the initial conditions of the vehicle on entry to the bridge, only three other parameters control the dynamic response of a bridge, namely κ , ϕ , and α .
4. The practical ranges for the values of parameters of κ , ϕ , and α are as follows;

$$\kappa \leq 1.50, \quad \phi \leq 1.20, \quad \alpha \leq 0.30$$

5. When the initial conditions are not accounted for and ϕ and κ are small, the response of the bridge depends mainly on α .
6. Generally, for values of ϕ greater than 0.40 the response increases with an increase in the values of both κ and α except that there may be a number of local troughs.
7. The relationship $DI_{max}-\phi$ does not always exhibit a peak and even if there is a peak it does not occur at $\phi = 1.0$. The maximum values of response may correspond to ϕ values ranging from 0.40 to more than 1. Also, the response increases with both κ and α .
8. The response of the bridge to a moving oscillating force of amplitude A_m is identical

to that of a moving sprung mass of zero weight and initial conditions as follows

$$\text{vertical displacement} = -A_m g / \omega_v^2,$$

$$\text{vertical velocity} = 0.0, \text{ and}$$

$$\text{vertical acceleration} = A_m g$$

9. When initial conditions of the vehicle are taken into account, the effect of the parameters κ , ϕ , and α is different from that with no initial conditions. For $\kappa = 0.0$, the response exhibits a marked peak at $\phi = 1.0$ when initial conditions are included in the analysis. For other values of κ there are no distinct peaks and the effect of initial conditions is less pronounced, especially for higher values of α .
10. In general, the inclusion of vehicle damping reduces the response of the bridge.
11. The dynamic response of a bridge traversed by a two-axle vehicle model is governed by six parameters, namely, κ , ϕ , α , S/L , a_1 , and $\dot{I}$. In fact, the effect of $\dot{I}$ on the response of the bridge is very small and can be neglected.
12. The response of a bridge to a moving two-axle vehicle model is less than that to a moving single axle vehicle model of equivalent mass and spring stiffnesses.
13. In general, higher values of S/L parameter produce smaller responses, especially for higher values of α .
14. For $S/L \leq 0.05$, the two axle vehicle produces approximately the same response as an equivalent single axle vehicle.
15. Based on the results of the analytical investigations, a series of design envelope curves are derived. These curves relate the dynamic amplification factors to parameters κ and α . As in the case of bridges designed according to the current Ontario Highway Bridge Design code, a preliminary design must be carried out first to obtain values of κ and α . Using these values the design curves can be entered to obtain the dynamic amplification factor. The design charts are drawn for both a single-axle vehicle model and a two-axle model.
16. In general, the response due to two vehicles in series is less than that due to a single vehicle of equivalent characterist

**TABLE 2.1 MAXIMUM IMPACT FACTORS FOR BEAM WITH
A MOVING FORCE MODEL**

α	$\frac{1}{2\alpha}$	n = 1		n = 2		n = 3		n = 4		n = 5	
		$\frac{1}{1+\alpha}$	DI _{max}	$\frac{2}{1+\alpha}$	DI _{max}	$\frac{3}{1+\alpha}$	DI _{max}	$\frac{4}{1+\alpha}$	DI _{max}	$\frac{5}{1+\alpha}$	DI _{max}
0.10	5.0	0.909	0.601	1.818	1.011	2.727	0.300	3.636	0.840	4.545	0.313
0.15	3.3	0.870	0.860	1.739	1.124	2.609	0.743				
0.20	2.5	0.833	1.000	1.667	1.083	2.500	0.000				
0.25	2.0	0.800	1.200	1.600	0.784						
0.30	1.7	0.769	1.412	1.538	0.341						
0.35	1.4	0.741	1.536								

TABLE 2.2 DESCRIPTION OF BRIDGES

Type	Weight (kips)	Span (ft)	F_b (Hz.)	ϕ	κ	α
TSD	322.0	96.0	4.42	0.62	0.434	0.107
	2500.0	136.0	3.06	0.90	0.056	0.109
	236.0	96.0	3.36	0.82	0.593	0.141
	150.0	105.0	4.20	0.65	0.933	0.103
TST	540.0	180.0	3.30	0.83	0.259	0.077
	1296.0	198.0	2.42	1.14	0.108	0.095
	1160.0	174.0	2.58	1.06	0.121	0.102
	2480.0	320.0	1.70	1.61	0.056	0.084
	1450.0	200.0	2.27	1.21	0.097	0.100
	258.0	129.0	3.41	0.81	0.543	0.104
	359.0	146.0	3.28	0.83	0.390	0.095
	695.0	136.0	2.80	0.98	0.201	0.120
	234.0	101.5	4.50	0.61	0.598	0.100
	620.0	129.0	3.23	0.85	0.226	0.109
	105.0	133.5	3.66	0.75	1.333	0.093
	63.0	105.0	6.13	0.45	2.222	0.071
	105.0	133.5	3.66	0.75	1.333	0.093
	131.0	157.0	4.46	0.61	1.068	0.065
	148.0	138.0	5.00	0.55	0.946	0.066
	111.0	144.0	4.91	0.56	1.261	0.064
	226.0	101.0	3.82	0.70	0.619	0.118
	403.0	138.0	3.56	0.77	0.347	0.093
	242.0	152.0	3.45	0.80	0.579	0.087
	91.0	114.0	4.30	0.74	1.538	0.093
	490.0	117.0	4.15	0.66	0.286	0.094
	86.0	134.0	5.50	0.50	1.628	0.062
	74.0	94.0	6.90	0.39	1.892	0.070
	53.0	95.0	5.52	0.50	2.642	0.087

TABLE 2.2 DESCRIPTION OF BRIDGES (contd.)

Type	Weight (kips)	Span (ft)	F_b (Hz.)	ϕ	κ	α
TSP	97.0	70.5	3.88	0.71	1.443	0.167
	184.0	52.5	6.08	0.45	0.761	0.143
	518.0	89.0	3.18	0.86	0.270	0.167
	129.0	59.0	5.28	0.52	1.085	0.146
	232.0	80.0	3.95	0.70	0.603	0.144
	345.0	79.0	4.95	0.55	0.406	0.117
	450.0	98.0	3.40	0.81	0.311	0.137
TC'D	800.0	190.0	3.01	0.92	0.175	0.080
BC'D	175.0	100.0	7.88	0.35	0.800	0.058
	1075.0	107.0	2.28	1.21	0.130	0.187
	256.0	80.0	2.34	1.17	0.547	0.243
	830.0	100.0	2.86	0.96	0.169	0.159
BSD	687.0	78.0	5.23	0.52	0.204	0.112
	1265.0	63.3	4.87	0.57	0.111	0.148
	66.0	66.0	8.70	0.31	2.121	0.079
	179.0	50.0	10.0	0.28	0.782	0.091
TC'P	533.0	90.0	2.70	1.02	0.263	0.188
			Average	0.744	0.722	0.11039

The three letters used in the classifications stand for the following;

B : beam

or plate girder

T : truss

C : continuous

or cantilever span

S : simple span

D : deck structure

P : pony,

or half-through truss

T : through truss

TABLE 2.3 DESCRIPTION OF BRIDGES

Type	Span (ft)	F_b (Hz.)	ϕ	α
17 span, simply supported, prestressed beam-concrete deck bridge	80.0	6.0	0.46	0.10
13 span, simply supported, prestressed beam-concrete deck bridge	61.0	8.0	0.34	0.10
4 span, simply supported, prestressed precast deck unit bridge	36.5 and 37.25	8.0	0.34	0.15
3 span, simply supported, prestressed precast deck unit bridge	35.75	10.2	0.27	0.12
3 span, simply supported, skew pier, prestressed precast deck unit bridge	31.0	10.1	0.27	0.15
2 span, continuous, cast in situ reinforced concrete slab bridge	48.0	9.0	0.31	0.10
3 span, continuous, cast in situ reinforced concrete slab bridge	26.0	11.0	0.25	0.15
9 span, continuous, cast in situ reinforced concrete slab bridge	22.5 and 30.0	12.0	0.23	0.12
		Average	0.31	0.12

The weights of the bridges are not available

TABLE 2.4 DESCRIPTION OF BRIDGES

Weight (KN)	Span (m)	F_b (Hz.)	ϕ	κ	α
992.0	10.0	5.114	0.54	0.63	0.27
1757.0	15.0	2.792	0.96	0.35	0.33
2943.0	20.0	2.058	1.33	0.211	0.34
4579.0	25.0	1.692	1.63	0.14	0.33
		Average	1.12	0.33	0.32

**TABLE 2.5 THE LOWEST FIFTEEN NATURAL FREQUENCIES OF
10X10m SIMPLE SLAB-BRIDGE
PRESENTED BY FINITE ELEMENTS**

Mode Number	Circular Frequency (rad/sec)	Frequency (cycles/sec)	Period (sec)	Type
1	0.3213E+02	0.5114E+01	0.1955E+00	LF1
2	0.5526E+02	0.8795E+01	0.1137E+00	TO1
3	0.1240E+03	0.1974E+02	0.5065E-01	TR1
4	0.1291E+03	0.2055E+02	0.4866E-01	LF2
5	0.1563E+03	0.2488E+02	0.4020E-01	TO2
6	0.2414E+03	0.3841E+02	0.2603E-01	TR2
7	0.2425E+03	0.3860E+02	0.2591E-01	TR3
8	0.2918E+03	0.4645E+02	0.2153E-01	LF3
9	0.3193E+03	0.5083E+02	0.1968E-01	TO3
10	0.3666E+03	0.5835E+02	0.1714E-01	TR4
11	0.4111E+03	0.6591E+02	0.1517E-01	TR5
12	0.4152E+03	0.6607E+02	0.1513E-01	TR6
13	0.5215E+03	0.8299E+02	0.1205E-01	LF4
14	0.5318E+03	0.8464E+02	0.1181E-01	TR7
15	0.5485E+03	0.8729E+02	0.1146E-01	TO4

**TABLE 2.6 THE LOWEST FIFTEEN NATURAL FREQUENCIES OF
10X15m SIMPLE SLAB-BRIDGE
PRESENTED BY FINITE ELEMENTS**

Mode Number	Circular Frequency (rad/sec)	Frequency (cycles/sec)	Period (sec)	Type
1	0.1754E+02	0.2792E+01	0.3582E+00	LF1
2	0.4107E+02	0.6537E+01	0.1530E+00	TO1
3	0.7043E+02	0.1121E+02	0.8921E-01	LF2
4	0.1016E+03	0.1617E+02	0.6183E-01	TO2
5	0.1197E+03	0.1906E+02	0.5217E-01	TR1
6	0.1589E+03	0.2529E+02	0.3954E-01	LF3
7	0.1924E+03	0.3062E+02	0.3266E-01	TO3
8	0.1935E+03	0.3079E+02	0.3248E-01	TR2
9	0.2662E+03	0.4237E+02	0.2360E-01	TR3
10	0.2834E+03	0.4510E+02	0.2217E-01	LF4
11	0.2971E+03	0.4728E+02	0.2115E-01	TR4
12	0.3173E+03	0.5050E+02	0.1980E-01	TO4
13	0.3409E+03	0.5425E+02	0.1843E-01	TR5
14	0.4314E+03	0.6867E+02	0.1456E-01	TR6
15	0.4442E+03	0.7069E+02	0.1415E-01	LF5

**TABLE 2.7 THE LOWEST FIFTEEN NATURAL FREQUENCIES OF
10X20m SIMPLE SLAB-BRIDGE
PRESENTED BY FINITE ELEMENTS**

Mode Number	Circular Frequency (rad/sec)	Frequency (cycles/sec)	Period (sec)	Type
1	0.1293E+02	0.2058E+01	0.4858E+00	LF1
2	0.3888E+02	0.6187E+01	0.1616E+00	TO1
3	0.5191E+02	0.8261E+01	0.1210E+00	LF2
4	0.8927E+02	0.1421E+02	0.7038E-01	TO2
5	0.1171E+03	0.1863E+02	0.5367E-01	LF3
6	0.1402E+03	0.2232E+02	0.4480E-01	TR1
7	0.1591E+03	0.2533E+02	0.3948E-01	TO3
8	0.2004E+03	0.3189E+02	0.3136E-01	TR2
9	0.2086E+03	0.3320E+02	0.3012E-01	LF4
10	0.2525E+03	0.4019E+02	0.2488E-01	TO4
11	0.2843E+03	0.4525E+02	0.2210E-01	TR3
12	0.3266E+03	0.5198E+02	0.1924E-01	LF5
13	0.3338E+03	0.5313E+02	0.1882E-01	TR4
14	0.3711E+03	0.5906E+02	0.1693E-01	TO5
15	0.3899E+03	0.6205E+02	0.1612E-01	TR5

**TABLE 2.8 THE LOWEST FIFTEEN NATURAL FREQUENCIES OF
10X25m SIMPLE SLAB-BRIDGE
PRESENTED BY FINITE ELEMENTS**

Mode Number	Circular Frequency (rad/sec)	Frequency (cycles/sec)	Period (sec)	Type
1	0.1063E+02	0.1692E+01	0.5909E+00	LF1
2	0.3923E+02	0.6244E+01	0.1602E+00	TO1
3	0.4266E+02	0.6790E+01	0.1473E+00	LF2
4	0.8629E+02	0.1373E+02	0.7281E-01	TO2
5	0.9621E+02	0.1531E+02	0.6531E-01	LF3
6	0.1473E+03	0.2345E+02	0.4265E-01	TO3
7	0.1696E+03	0.2699E+02	0.3705E-01	TR1
8	0.1713E+03	0.2727E+02	0.3667E-01	LF4
9	0.2225E+03	0.3542E+02	0.2823E-01	TR2
10	0.2261E+03	0.3599E+02	0.2779E-01	TO4
11	0.2682E+03	0.4268E+02	0.2343E-01	LF5
12	0.2975E+03	0.4734E+02	0.2112E-01	TR3
13	0.3246E+03	0.5167E+02	0.1935E-01	TO5
14	0.3869E+03	0.6157E+02	0.1624E-01	LF6
15	0.3904E+03	0.6214E+02	0.1609E-01	TR4

**TABLE 2.9 THE LOWEST TEN MODE SHAPES OF A SIMPLE BEAM
REPRESENTED BY FINITE ELEMENTS**

MODE NUMBER	CIRCULAR FREQUENCY (RAD/SEC)	FREQUENCY (CYCLES/SEC)	PERIOD (SEC)
1	0.1276E+02	0.2058E+01	0.4923E+00
2	0.5105E+02	0.8261E+01	0.1231E+00
3	0.1149E+03	0.1863E+02	0.5470E-01
4	0.2042E+03	0.3320E+02	0.3077E-01
5	0.3190E+03	0.5200E+02	0.1970E-01
6	0.4592E+03	0.7308E+02	0.1368E-01
7	0.6246E+03	0.9940E+02	0.1006E-01
8	0.8148E+03	0.1297E+03	0.7711E-02
9	0.1029E+04	0.1638E+03	0.6104E-02
10	0.1267E+04	0.2017E+03	0.4959E-02

**TABLE 2.10 RELATIONSHIP BETWEEN THE DIMENSIONLESS
AMPLITUDE AND THE PARAMETER β**

$F_v = 1. \text{ Hz.}$		$F_v = 2. \text{ Hz.}$		$F_v = 3. \text{ Hz.}$		$F_v = 4. \text{ Hz.}$		$F_v = 5. \text{ Hz.}$	
β	$\frac{\Delta m}{h}$	β	$\frac{\Delta m}{h}$	β	$\frac{\Delta m}{h}$	β	$\frac{\Delta m}{h}$	β	$\frac{\Delta m}{h}$
2.77	2.81	1.38	20.48	0.92	59.29	0.69	110.31	0.55	153.95
2.92	2.67	1.46	19.70	0.97	58.56	0.73	110.69	0.58	160.80
3.08	2.54	1.54	18.88	1.02	55.32	0.77	110.37	0.61	166.21
3.26	2.41	1.63	18.03	1.08	54.10	0.81	109.36	0.65	170.09
3.47	2.27	1.73	17.15	1.15	52.25	0.86	107.66	0.69	172.35
3.70	2.13	1.85	16.23	1.23	50.12	0.92	105.10	0.74	172.94
3.96	1.99	1.98	15.29	1.32	47.77	0.99	106.77	0.79	171.81
4.27	1.86	2.13	14.31	1.42	45.23	1.06	97.13	0.85	168.98
4.62	1.72	2.31	13.31	1.54	42.50	1.15	92.89	0.92	164.69
5.05	1.58	2.52	12.29	1.68	39.59	1.26	87.76	1.01	151.09
5.55	1.43	2.77	11.25	1.85	36.53	1.38	81.95	1.11	148.68
5.84	1.43	2.92	11.25	1.94	36.53	1.46	81.95	1.16	148.68
6.17	1.29	3.08	10.18	2.05	33.31	1.54	75.55	1.23	139.23
6.53	1.22	3.26	9.64	2.17	31.65	1.63	72.14	1.30	133.84
6.94	1.15	3.47	9.10	2.31	29.96	1.73	68.60	1.38	128.06
7.4	1.08	3.70	8.55	2.46	28.24	1.85	64.94	1.48	121.91
7.9	1.01	3.96	7.99	2.64	26.49	1.98	61.16	1.58	115.42
8.3	0.96	4.16	7.62	2.77	25.31	2.08	58.58	1.66	110.91
8.5	0.93	4.27	7.44	2.84	24.71	2.13	57.27	1.70	108.60
8.7	0.91	4.38	7.25	2.92	24.11	2.19	55.95	1.75	106.26
9.2	0.86	4.62	6.88	3.08	22.91	2.31	53.27	1.85	101.47
9.8	0.81	4.90	6.50	3.26	21.69	2.45	50.55	1.96	96.56

**TABLE 2.10 RELATIONSHIP BETWEEN THE DIMENSIONLESS
AMPLITUDE AND THE PARAMETER β (Contd.)**

$F_v = 1. \text{ Hz.}$		$F_v = 2. \text{ Hz.}$		$F_v = 3. \text{ Hz.}$		$F_v = 4. \text{ Hz.}$		$F_v = 5. \text{ Hz.}$	
β	$\frac{A_m}{h}$	β	$\frac{A_m}{h}$	β	$\frac{A_m}{h}$	β	$\frac{A_m}{h}$	β	$\frac{A_m}{h}$
10.10	0.79	5.05	6.32	3.36	21.08	2.52	49.18	2.02	94.06
10.41	0.77	5.20	6.13	3.47	20.47	2.60	47.79	2.08	91.53
11.11	0.72	5.55	5.75	3.70	19.24	2.77	45.00	2.22	86.38
11.69	0.68	5.84	5.47	3.89	18.30	2.92	42.87	2.33	82.45
11.90	0.67	5.95	5.37	3.96	17.99	2.97	42.16	2.38	81.12
12.34	0.65	6.17	5.18	4.11	17.37	3.08	40.73	2.46	78.46
12.82	0.62	6.41	4.99	4.27	16.74	3.20	39.29	2.56	75.76
13.07	0.61	6.53	4.90	4.35	16.43	3.26	38.57	2.61	74.41
13.88	0.57	6.94	4.61	4.62	15.48	3.47	36.40	2.77	70.31
14.61	0.55	6.94	4.61	4.87	14.72	3.65	34.64	2.92	66.99
14.81	0.54	7.30	4.38	4.93	14.53	3.70	34.20	2.96	66.16
15.15	0.53	7.40	4.33	5.05	14.22	3.78	33.47	3.03	64.77
15.43	0.52	7.57	4.23	5.14	13.96	3.85	32.88	3.08	63.65
15.87	0.50	7.71	4.15	5.29	13.58	3.96	31.99	3.17	61.96
16.33	0.49	7.93	4.04	5.44	13.20	4.08	31.10	3.26	60.27
16.66	0.48	8.16	3.92	5.55	12.95	4.16	30.51	3.33	59.14
17.09	0.47	8.33	3.85	5.69	12.63	4.27	29.77	3.41	57.72
17.36	0.46	8.54	3.75	5.78	12.43	4.34	29.32	3.47	56.87
18.51	0.43	8.68	3.70	6.17	11.67	4.62	27.53	3.70	53.44
19.84	0.40	9.25	3.47	6.61	10.90	4.96	25.73	3.96	49.99
20.20	0.39	9.92	3.24	6.73	10.71	5.05	25.28	4.04	49.12
20.83	0.38	10.10	3.18	6.94	10.38	5.20	24.53	4.16	47.68

FIGURE 2.1 MOVING FORCE ON SIMPLE BEAM MODEL

FIGURE 2.2 RELATIONSHIP BETWEEN THE SPEED PARAMETER AND THE MAXIMUM IMPACT FACTOR

FIGURE 2.3 MOVING SPRUNG AND UNSPRUNG MASSES ON SIMPLY SUPPORTED BEAM VIBRATES IN HARMONIC FASHION

FIGURE 2.4 THE LOWEST THREE HARMONIC MODE SHAPES OF A SIMPLY SUPPORTED BEAM

FIGURE 2.5a EFFECT OF VEHICLE FREQUENCY ON BRIDGE RESPONSE
 ($\phi = 0.1$)

FIGURE 2.5b EFFECT OF VEHICLE FREQUENCY ON BRIDGE RESPONSE
($\phi = 0.1$)

FIGURE 2.6a EFFECT OF VEHICLE FREQUENCY ON BRIDGE RESPONSE
 ($\Phi = 0.4$)

FIGURE 2.6b EFFECT OF VEHICLE FREQUENCY ON BRIDGE RESPONSE
 ($\Phi = 0.4$)

FIGURE 2.7a EFFECT OF VEHICLE FREQUENCY ON BRIDGE RESPONSE
 ($\Phi = 0.70$)

FIGURE 2.7b EFFECT OF VEHICLE FREQUENCY ON BRIDGE RESPONSE
($\Phi = 0.70$)

Material Properties :

Concrete : unit weight = 24.00 kN/m³
 f_c = 30.00 MPa
 E_c = 2.85×10^7 kN/m²

Steel : E_s = 2×10^5 kN/m²
 f_y = 400 MPa

Bridge Span (m)	t (m)	Fundamental Frequency (Hz.)		D.L.A †††
		(1)†	(2)††	
10	0.325	4.463	5.114	1.30
15	0.400	2.492	2.792	1.30
20	0.525	1.880	2.058	1.30
25	0.675	1.573	1.692	1.28

† values calculated according to ONTARIO HIGHWAY BRIDGE DESIGN CODE- 1983

†† values obtained by the finite element analysis

††† dynamic load allowance obtained from ONTARIO HIGHWAY BRIDGE DESIGN CODE- 1983

(b)

FIGURE 2.8 DETAILS OF FOUR BRIDGES DESIGNED IN ACCORDANCE WITH ONTARIO HIGHWAY BRIDGE DESIGN CODE- 1983

(a) TYPICAL CROSS SECTION

(b) STATIC AND DYNAMIC CHARACTERISTICS

FIGURE 2.9a EFFECT OF WEIGHT RATIO ON BRIDGE RESPONSE
(Using one sinusoidal shape function)

FIGURE 2.9b EFFECT OF WEIGHT RATIO ON BRIDGE RESPONSE
(Using one sinusoidal shape function)

FIGURE 2.9c EFFECT OF WEIGHT RATIO ON BRIDGE RESPONSE
(Using one sinusoidal shape function)

FIGURE 2.10a EFFECT OF FREQUENCY PARAMETER ON BRIDGE RESPONSE
 (Using one sinusoidal shape function)

FIGURE 2.10b EFFECT OF FREQUENCY PARAMETER ON BRIDGE RESPONSE
 (Using one sinusoidal shape function)

FIGURE 2.10c EFFECT OF FREQUENCY PARAMETER ON BRIDGE RESPONSE
(Using one sinusoidal shape function)

FIGURE 2.11a THE EFFECT OF SPEED PARAMETER ON BRIDGE RESPONSE
(Using one sinusoidal shape function)

FIGURE 2.11b THE EFFECT OF SPEED PARAMETER ON BRIDGE RESPONSE
(Using one sinusoidal shape function)

FIGURE 2.12 EFFECT OF FREQUENCY PARAMETER ON BRIDGE RESPONSE

— results obtained by using three sinusoidal shape functions

FIGURE 2.13 DYNAMIC RESPONSE OF A BEAM UNDER MOVING SPRUNG MASS

**FIGURE 2.14 EFFECT OF FREQUENCY PARAMETER ON BRIDGE RESPONSE - USING FINITE ELEMENT ANALYSIS
20 ELEMENTS - BRIDGE SPAN 20.0 m - $h = 0.0$ m**

(Note : Solid lines indicate results obtained by using
3 finite element mode shapes)

FIGURE 2.15a COMPARISON BETWEEN THE FINITE ELEMENT ANALYSIS AND SINUSOIDAL SHAPE FUNCTION ANALYSIS - $\kappa = 0.30$

- Three sinusoidal shape functions
- Finite element analysis - 20 elements & three mode shapes

FIGURE 2.15b COMPARISON BETWEEN THE FINITE ELEMENT ANALYSIS AND SINUSOIDAL SHAPE FUNCTION ANALYSIS - $\kappa = 0.30$

— Three sinusoidal shape functions
 ■ Finite element analysis - 20 elements & three mode shapes

FIGURE 2.16a COMPARISON BETWEEN DEFLECTION AND MOMENT IMPACT FACTORS

FIGURE 2.16b COMPARISON BETWEEN DEFLECTION AND MOMENT IMPACT FACTORS

FIGURE 2.17 A SINE BUMP AT BRIDGE ENTRY

FIGURE 2.18 RELATIONSHIP BETWEEN THE DIMENSIONLESS AMPLITUDE AND THE PARAMETER β

FIGURE 2.19 EFFECT OF FREQUENCY PARAMETER ON BRIDGE RESPONSE TO A MOVING PULSATING FORCE - $A_m = 0.15$

FIGURE 2.20 EFFECT OF FREQUENCY PARAMETER ON BRIDGE RESPONSE TO A MOVING PULSATING FORCE - $A_m = 0.30$

FIGURE 2.21 COMPARISON BETWEEN BRIDGE RESPONSE TO A MOVING PULSATING FORCE OF DIFFERENT AMPLITUDES

FIGURE 2.22 COMPARISON OF RESULTS OBTAINED USING ONE SINUSOIDAL SHAPE FUNCTION AND MOVING PULSATING FORCE- $\kappa = 0.0$, $A_m = 0.15$ ($-0.03727m$)

— moving pulsating force solution

□ One sinusoidal shape function

FIGURE 2.23 EFFECT OF INITIAL DISPLACEMENT OF SPRUNG MASS ON THE BRIDGE RESPONSE

FIGURE 2. 24a BRIDGE RESPONSE CORRESPONDING TO DIFFERENT VALUES OF VEHICLE FREQUENCY - $\kappa = 0.0$

$\square$ $\rho = 0.07455 \text{ m}$ — $\rho = 0.00828 \text{ m}$

FIGURE 2. 24b BRIDGE RESPONSE CORRESPONDING TO DIFFERENT VALUES OF VEHICLE FREQUENCY - $\kappa = 0.30$

$\square$ $\rho = 0.07455 \text{ m}$ — $\rho = 0.00828 \text{ m}$

FIGURE 2.25a EFFECT OF INITIAL CONDITIONS ON BRIDGE RESPONSE
 $\rho = 0.03727$ m, $f_v = 1.0$ Hz. (Using one sinusoidal shape function)

FIGURE 2.25b EFFECT OF INITIAL CONDITIONS ON BRIDGE RESPONSE
 $\rho = 0.03727 \text{ m}$, $f_v = 1.0 \text{ Hz}$. (Using one sinusoidal shape function)

FIGURE 2.26a EFFECT OF INITIAL CONDITIONS ON BRIDGE RESPONSE
 $\rho = 0.0747$ m, $f_v = 1.0$ Hz. (Using one sinusoidal shape function)

FIGURE 2.26b EFFECT OF INITIAL CONDITIONS ON BRIDGE RESPONSE
 $\rho = 0.0747$ m, $f_v = 1.0$ Hz. (Using one sinusoidal shape function)

FIGURE 2.27a EFFECT OF INITIAL CONDITIONS ON BRIDGE RESPONSE - $\kappa = 0.0$
 (using one sinusoidal shape function)

FIGURE 2.27b EFFECT OF INITIAL CONDITIONS ON BRIDGE RESPONSE - $\kappa = 1.5$
 (using one sinusoidal shape function)

FIGURE 2.28a COMPARISON BETWEEN THE FINITE ELEMENT ANALYSIS AND SINUSOIDAL SHAPE FUNCTION ANALYSIS - $\kappa \approx 0.0$ - $A_m = 0.15$

- Three sinusoidal shape functions
- Finite element analysis - 20 elements & three mode shapes

FIGURE 2.28b COMPARISON BETWEEN THE FINITE ELEMENT ANALYSIS AND SINUSOIDAL SHAPE FUNCTION ANALYSIS - $\kappa = 0.30$, $A_m = 0.15$

— Three sinusoidal shape functions
 □ Finite element analysis - 20 elements & three mode shapes

FIGURE 2. 29a EFFECT OF VEHICLE DAMPING ON BRIDGE
RESPONSE - $\rho = 0.0$ m

FIGURE 2.29b EFFECT OF VEHICLE DAMPING ON BRIDGE RESPONSE - $\rho = 0.0$ m

FIGURE 2.29c EFFECT OF VEHICLE DAMPING ON BRIDGE RESPONSE - $\rho = 0.0$ m

FIGURE 2. 30a EFFECT OF VEHICLE DAMPING ON BRIDGE RESPONSE - $\rho = 0.03727 \text{ m}$

FIGURE 2. 30b EFFECT OF VEHICLE DAMPING ON BRIDGE RESPONSE - $\rho = 0.03727$ m

FIGURE 2. 30c EFFECT OF VEHICLE DAMPING ON BRIDGE RESPONSE- $\rho = 0.03727$ m

FIGURE 2.31a CURVES OF RESPONSE HISTORY OF THE BRIDGE
 $\phi = 1.0, F_v = 1.0 \text{ Hz}, \rho = 0.0 \text{ m}, \zeta = 0.0$

FIGURE 2.31b CURVES OF RESPONSE HISTORY OF THE BRIDGE
 $\phi = 1.0$, $F_v = 1.0$ Hz, $\rho = 0.03727$ m, $\zeta = 0.0$

FIGURE 2.32 TWO-AXLE VEHICLE MODEL

$$\begin{aligned}
 & k_r (q_1 - a_1 S q_2 - z N_1) + \\
 & c_r (\dot{q}_1 - a_1 S \dot{q}_2 - \dot{z} N_1 - \frac{\pi v}{L} z N_1) \\
 & \quad \quad \quad \uparrow \\
 & \quad \quad \quad (a_2 m_v + m_{tr}) g + \\
 & m_{tr} \left(\left(\frac{\pi v}{L} \right)^2 z + 2 \frac{\pi v}{L} \dot{z} + \ddot{z} \right) N_1
 \end{aligned}
 \qquad
 \begin{aligned}
 & k_r (q_1 + a_2 S q_2 - z N_2) + \\
 & c_r (\dot{q}_1 + a_2 S \dot{q}_2 - \dot{z} N_2 - \frac{\pi v}{L} z N_2) \\
 & \quad \quad \quad \uparrow \\
 & \quad \quad \quad (a_1 m_v + m_{tr}) g + \\
 & m_{tr} \left(\left(\frac{\pi v}{L} \right)^2 z + 2 \frac{\pi v}{L} \dot{z} + \ddot{z} \right) N_2
 \end{aligned}$$

FIGURE 2.33 CONTACT FORCES

FIGURE 2.34 EFFECT OF S/L PARAMETER ON THE BRIDGE RESPONSE

FIGURE 2.35a EFFECT OF DYNAMIC INDEX PARAMETER ON THE BRIDGE RESPONSE

FIGURE 2.35b EFFECT OF DYNAMIC INDEX PARAMETER ON THE BRIDGE RESPONSE

FIGURE 2.36a EFFECT OF DIFFERENT PARAMETERS ON BRIDGE RESPONSE

FIGURE 2.36b EFFECT OF DIFFERENT PARAMETERS ON BRIDGE RESPONSE

FIGURE 2.36C EFFECT OF DIFFERENT PARAMETERS ON BRIDGE RESPONSE

FIGURE 2.37 GENERATION OF ENVELOPE CURVES RELATING D , α , AND κ

FIGURE 2.38 DESIGN ENVELOPE CURVES RELATING D , α , AND κ

FIGURE 2.39 GENERATION OF ENVELOPE CURVES RELATING D , α , AND κ FOR TWO-AXLE VEHICLE

FIGURE 2.40 COMPARISON BETWEEN DESIGN ENVELOPE CURVES OF SINGLE-AXLE VEHICLE AND TWO-AXLE VEHICLE

**FIGURE 2.41a DESIGN ENVELOPE CURVES RELATING
D, α , AND κ**

Note: Range of ϕ , 0 to 2.5

**FIGURE 2.41b DESIGN ENVELOPE CURVES RELATING
D, α , AND κ**

Note: Range of ϕ , 0 to 2.5

**FIGURE 2.41c DESIGN ENVELOPE CURVES RELATING
D, α , AND κ**

Note: Range of ϕ , 0 to 2.5

FIGURE 2.42 COMPARISON BETWEEN BRIDGE RESPONSES DUE TO TWO AXLE VEHICLE AND TWO VEHICLES IN SERIES

FIGURE 2.43 COMPARISON BETWEEN BRIDGE RESPONSES DUE TO TWO AXLE VEHICLE AND TWO VEHICLES IN SERIES

3 DYNAMIC RESPONSE ANALYSIS OF SHALLOW SUPERSTRUCTURE BRIDGES

3.1 Introduction

As stated earlier, the dynamic response of a bridge structure under a moving vehicle is affected by the interaction between the bridge and the vehicle. The complexity introduced by such interaction has made it difficult to identify the significant governing parameters. In Chapter 2 we showed that the controlling parameters in case of simple beam models are; the speed parameter, the frequency ratio, and the mass ratio. A beam model can not truly represent a two or three dimensional behaviour as it neglects the effect of transverse flexibility on the bridge response. This chapter therefore extends the previous study to a two dimensional plate model.

The bridge is idealized by a rectangular plate, of isotropic or orthotropic characteristics, and the vehicle is represented by a single sprung mass moving along the deck (Fig. 3.1). For the purpose of analysis, the plate is discretized by quadrilateral thin plate elements. The resulting nonlinear differential equations are solved in their incremental form using step by step numerical integration.

3.2 Modeling of the Bridge and the Vehicle

The objective of this chapter is to investigate the response of a shallow bridge superstructure which can be idealized either as an isotropic or an orthotropic plate. In the attempt to identify the important parameters that govern the response, the bridge and the vehicle are both represented by highly idealized models. In many recent studies on the static response of bridges, a shallow bridge deck has been modelled as an orthotropic plate (Bakht and Jaeger, 1985). Such a model is also expected to provide a good representation of the deck for the purpose of a dynamic analysis. In the present study two different plate

models are used. In the first model, the slab deck has similar stiffnesses in both longitudinal and transverse directions so that the deck is 'isotropic'. In the second model different stiffnesses are considered in the two directions, and the slab is 'orthotropic'. It is of interest to record the governing differential equations for both types of idealizations.

Consider an orthotropic plate, subjected to a moving sprung load of mass m_v and spring constant k_v . The equation of motion of the orthotropic-plate, assuming that the plate is initially at rest and has no damping, may be written as follows (Sundara and Jagadish, 1970):

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \rho \frac{\partial^2 w}{\partial t^2} = -m_v(g + \ddot{u}_v^t)f(x - vt, y) \quad (3.1)$$

where

w = deflection of the bridge, positive upwards

u_v^t = absolute deflection of the sprung mass, positive upwards

D_x = longitudinal flexural rigidity per unit width
 $= \frac{h^3 E_x}{12}$

D_y = transverse flexural rigidity per unit length
 $= \frac{h^3 E_y}{12}$

D_{xy} = torsional rigidity per unit width
 $= \frac{h^3 G}{12}$

h = plate thickness

D_1 = coupling rigidity per unit width
 $= \frac{h^3 E'}{12}$

H = $D_1 + 2D_{xy}$

E_x, E_y, E', G = constants needed to define the elastic properties of
 an orthotropic material

ρ = mass of the plate per unit area

v = vehicle speed

f = time-dependent function representing the
 distribution of the load over the surface of the bridge

The isotropic representation of a bridge is a special case of the orthotropic idealization where the material can be identified completely by knowing two constants E , the Young's modulus and ν , the Poisson's ratio. In this case we have

$$\begin{aligned}
 E_x &= E_y = \frac{E}{1-\nu^2} \\
 E' &= \frac{\nu E}{(1-\nu^2)} \\
 G &= \frac{E}{2(1+\nu)}
 \end{aligned}$$

and therefore

$$\begin{aligned}
 D_x &= D_y = D = \frac{h^3 E}{12(1-\nu^2)} \\
 H &= D
 \end{aligned}$$

The equation of motion reduces to

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\rho}{D} \frac{\partial^2 w}{\partial t^2} = -\frac{m_v}{D} (g + \ddot{u}_v^t) f(x - vt, y) \quad (3.2)$$

A moving vehicle represents a highly complex dynamic system. In the present chapter a very simple model is used; it consists of a single unsprung mass and a sprung mass supported by a spring and a dash-pot, the latter representing a viscous damping mechanism (Fig. 3.1b). The vehicle has a point contact with the bridge deck and maintains that contact as it rolls along the deck. During the course of analysis, further simplifications are introduced in the basic model described above. Such simplifications will be described as they are used.

3.3 Free Vibrations Analysis – Recognition of the Controlling Parameters

Before analyzing the response of plate models to moving vehicles, it is important to study the parameters that govern the free vibrations of such models. In fact, free vibration analysis is an essential first step in obtaining the forced vibration response of a structure. Moreover, the parameters that affect the free vibration response also affect the response to moving loads.

3.3.1 Orthotropic Plate – Free Vibrations Analysis

The differential equation governing the free vibration of orthotropic plate models can be obtained by setting the right-hand side in Eq. 3.1 to be equal to zero. The governing differential equation in this case becomes

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \rho \frac{\partial^2 w}{\partial t^2} = 0 \quad (3.3)$$

The displacement function $w(x, y, t)$ can be expressed as the product of two functions, one involving only the space coordinates x and y , called a mode shape function, and other involving the variable time (Gorman, 1982)

$$w(x, y, t) = \varphi(x, y)T(t) \quad (3.4)$$

Substituting from Eq. 3.4 into Eq. 3.3, it can be shown that the function $T(t)$ varies sinusoidally with time and that the frequency of these sinusoidal oscillation is the quantity ω . Equation 3.4 can be then expressed as (Gorman 1982);

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - \rho \omega^2 w = 0 \quad (3.5)$$

Now we introduce the dimensionless coordinates;

$$\zeta = \frac{x}{L} \quad \& \quad \eta = \frac{y}{b} \quad (3.6)$$

where $b = W/2$, then

$$\frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial \zeta} \quad \& \quad \frac{\partial}{\partial y} = \frac{1}{b} \frac{\partial}{\partial \eta} \quad (3.7)$$

Expressing Eq. 3.5 in terms of the dimensionless coordinates, ζ and η , we get

$$\frac{D_x}{L^4} \frac{\partial^4 w}{\partial \zeta^4} + \frac{2H}{L^2 b^2} \frac{\partial^4 w}{\partial \zeta^2 \partial \eta^2} + \frac{D_y}{b^4} \frac{\partial^4 w}{\partial \eta^4} - \rho \omega^2 w = 0 \quad (3.8)$$

Multiplying Eq. 3.8 by $(L^2 b^2 / \sqrt{D_x D_y})$ yields

$$\theta^2 \frac{\partial^4 w}{\partial \zeta^4} + \chi \frac{\partial^4 w}{\partial \zeta^2 \partial \eta^2} + \frac{1}{\theta^2} \frac{\partial^4 w}{\partial \eta^4} - \lambda^2 w = 0 \quad (3.9)$$

where

$$\theta = \left(\frac{b}{L} \right) \left(\frac{D_x}{D_y} \right)^{0.25}$$

$$\chi = \frac{2H}{\sqrt{D_x D_y}}$$

$$\lambda = \frac{\omega L b \sqrt{\rho}}{(D_x D_y)^{0.25}}$$

Bakht and Jaeger (1985) have concluded that two different orthotropic plates will have the same patterns of static deflection if the values of χ and θ are the same for the two plates. Examination of Eq. 3.9 leads to the conclusion that two different orthotropic plates will have the same mode shapes when expressed in terms of normalized coordinates provided

they have the same values of χ and θ . Such plates will also have the same frequencies to within a scale factor.

3.3.2 Isotropic Plate – Free Vibrations Analysis

For isotropic plates parameters χ and θ take the following values ;

$$\frac{1}{\theta} = \left(\frac{L}{b}\right) = \text{twice the aspect ratio}$$

$$\chi = 2.0$$

Substituting these values in Eq. 3.9, the governing equation becomes

$$\left(\frac{b}{L}\right)^2 \frac{\partial^4 w}{\partial \zeta^4} + 2 \frac{\partial^4 w}{\partial \zeta^2 \partial \eta^2} + \left(\frac{L}{b}\right)^2 \frac{\partial^4 w}{\partial \eta^4} - \frac{L^2 b^2 \omega^2 \rho}{D} w = 0 \quad (3.10)$$

In this case the the mode shape and static deflection are controlled only by the aspect ratio, so that two isotropic plates with the same aspect ratios will have the same mode shapes as well as similar frequencies to within a scale factor.

3.4 Isotropic Plate – Forced Vibrations Analysis

The slab bridge is a common type of highway bridge which can be conveniently analyzed by idealizing it as an isotropic plate. Figure 3.1a shows an isotropic plate model of a single span simply supported bridge. The standard finite element method is used in the analysis of the bridge model. The plate is discretized by rectangular thin plate elements (Fig 3.1c). The plate bending element used for the present study is based on the triangular element derived by Hsieh, Clough and Tocher (Clough and Tocher, 1966). This element ensures full compatibility of slopes and therefore provides monotonic convergence to the exact solution.

3.4.1 Formulation of the Equations of Motion

Referring to Figure 3.1b, the equation of motion of the sprung mass is given by

$$m_v \ddot{u}_v^t + c_v (\dot{u}_v^t - \dot{\gamma}) + k_v (u_v^t - \gamma) = -m_v g \quad (3.11)$$

where

u_v^t = the total displacement of the sprung mass relative to the undeformed position of the spring

k_v = stiffness of vehicle spring

c_v = damping constant in the vehicle model

m_v = sprung mass of the vehicle

γ = vertical displacement of the contact point

The equations of motion for the finite element representation of the bridge deck are given by

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{C}\dot{\mathbf{D}} + \mathbf{K}\mathbf{D} = \mathbf{P}^* \quad (3.12)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are respectively the mass matrix, the damping matrix, and the stiffness matrix of the finite element model; $\mathbf{D}$ is the vector of displacements and $\mathbf{P}^*$ is the effective force. Matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are of size $N \times N$ and vectors $\mathbf{D}$ and $\mathbf{P}^*$ are of size N , where N is the number of the nodal degrees of freedom in the finite element model.

The vertical displacement of the point of contact, γ , can be represented as

$$\gamma = \mathbf{N}\mathbf{D} \quad (3.13)$$

where $\mathbf{N}$ is a shape function row vector which relates γ to $\mathbf{D}$. The non-zero elements of $\mathbf{N}$ depend on the position of the element in which the contact point falls. These elements of $\mathbf{N}$ are functions of the coordinates x and y of the contact point. As the vehicle moves across the deck along a straight line parallel to the span, y remains constant but x varies. Coordinate x and hence $\mathbf{N}$ are thus functions of time. Differentiation of Eq. 3.13 yields

$$\dot{\gamma} = \frac{\partial \mathbf{N}}{\partial x} \mathbf{D}\dot{x} + \mathbf{N}\dot{\mathbf{D}} \quad (3.14)$$

$$\ddot{\gamma} = \frac{\partial^2 \mathbf{N}}{\partial x^2} \mathbf{D}\dot{x}^2 + 2 \frac{\partial \mathbf{N}}{\partial x} \dot{\mathbf{D}}\dot{x} + \frac{\partial \mathbf{N}}{\partial x} \mathbf{D}\ddot{x} + \mathbf{N}\ddot{\mathbf{D}} \quad (3.15)$$

Also, the effective force vector $\mathbf{P}^*$ is given by

$$\mathbf{P}^* = \mathbf{N}^T [-m_t \ddot{\gamma} + c_v (\dot{u}_v^t - \dot{\gamma}) + k_v (u_v^t - \gamma) - m_t g] \quad (3.16)$$

where m_t = unsprung mass of the vehicle

When it is assumed that the vehicle moves with a constant speed v , substitution of Eqs. 3.14 through 3.16 in Eqs. 3.11 and 3.12 and assembly of the two sets of equations yield

$$\begin{bmatrix} m_v & 0 \\ 0 & \mathbf{M} + m_t \mathbf{N}^T \mathbf{N} \end{bmatrix} \begin{bmatrix} \ddot{u}_v^t \\ \ddot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} c_v & -c_v \mathbf{N} \\ -c_v \mathbf{N}^T & \mathbf{C} + 2m_t v \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} + c_v \mathbf{N}^T \mathbf{N} \end{bmatrix} \begin{bmatrix} \dot{u}_v^t \\ \dot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} k_v & -k_v \mathbf{N} - c_v v \frac{\partial \mathbf{N}}{\partial x} \\ -k_v \mathbf{N}^T & \mathbf{K} + m_t v^2 \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} + c_v v \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} + k_v \mathbf{N}^T \mathbf{N} \end{bmatrix} \begin{bmatrix} u_v^t \\ \mathbf{D} \end{bmatrix} = - \begin{bmatrix} m_v g \\ m_t g \mathbf{N}^T \end{bmatrix} \quad (3.17)$$

In order to simplify the problem, m_t is taken to be negligible and damping in both the vehicle and the bridge is ignored. With these assumptions Eq. 3.17 reduces to

$$\begin{bmatrix} m_v & 0 \\ 0 & \mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{u}_v^t \\ \dot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} k_v & -k_v \mathbf{N} \\ -k_v \mathbf{N}^T & \mathbf{K} + k_v \mathbf{N}^T \mathbf{N} \end{bmatrix} \begin{bmatrix} u_v^t \\ \mathbf{D} \end{bmatrix} = -m_v g \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (3.18)$$

Since $\mathbf{N}$ is a function of time, the equations of motion represented by Eq. 3.18 are nonlinear. They can be solved in an incremental form by using step-by-step numerical integration. Because the size of Eq. 3.18 is quite large, the step integration requires considerable computation. In order to reduce the amount of computations, the size of the equations is reduced by applying a normal coordinate transformation represented by

$$\mathbf{D} = \mathbf{Q}\mathbf{y} \quad (3.19)$$

where $\mathbf{Q}$ is the modal matrix of size $N \times M$, and $\mathbf{y}$ represents the M normal coordinates included in the transformation. Good accuracy can be obtained by using only a few mode shapes so that $M \ll N$. With the transformation represented by Eq. 3.19, the equations of motion reduce to

$$\begin{bmatrix} m_v & 0 \\ 0 & \mathbf{M}^* \end{bmatrix} \begin{bmatrix} \ddot{u}_v^t \\ \ddot{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} k_v & -k_v \mathbf{N}\mathbf{Q} \\ -k_v \mathbf{Q}^T \mathbf{N}^T & \mathbf{K}^* + k_v \mathbf{Q}^T \mathbf{N}^T \mathbf{N} \mathbf{Q} \end{bmatrix} \begin{bmatrix} u_v^t \\ \mathbf{y} \end{bmatrix} = -m_v g \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (3.20)$$

where

$$\mathbf{M}^* = \mathbf{Q}^T \mathbf{M} \mathbf{Q}$$

$$\mathbf{K}^* = \mathbf{Q}^T \mathbf{K} \mathbf{Q}$$

As stated earlier, Eqs. 3.20 are solved in an incremental form using a technique of step-by-step numerical integration, for example, the average acceleration method. The coefficient matrices are assumed to remain constant over the integration step; but the errors involved in this assumption are small, provided the selected time step is short enough.

3.4.2 Identification of the Control Parameters

In the previous chapter it has been shown that the response of beam models is governed by three characterizing parameters α , ϕ , and κ .

In order to define the characterizing parameters for a bridge modelled by a plate, it is useful to introduce the concept of corresponding plane structures (Bakht and Jaeger, 1985). Consider, for example, the two plate models shown in Fig. 3.2. If appropriate scales

are applied to the plate in Fig. 3.2a (the scales in the two directions may be different) it will map identically to the plate in Fig. 3.2b. The mapping includes the finite element discretization as well as the path of the moving vehicle. The two plate models are said to correspond to each other. Points such as P_A and P_B , which coincide when the models are mapped to each other, are called corresponding points. For the special case of isotropic plates, the scales in the two directions are same and the plates have the same aspect ratio.

As stated earlier, when expressed in terms of normalized coordinates, the mode shapes of plate A are identical to those of plate B, provided they are normalized so that any two corresponding elements have the same magnitude. In addition

$$\frac{\omega_{iA}}{\omega_{iB}} = \frac{\omega_{jA}}{\omega_{jB}} \quad \begin{matrix} i = 1, 2, \dots, N \\ j = 1, 2, \dots, N \end{matrix} \quad (3.21)$$

where ω_{iA} is the i^{th} frequency of plate A. Further, it can be shown that if the transformed mass matrix of plate A is expressed as

$$\mathbf{M}_A^* = m_A \mathbf{M} \quad (3.22a)$$

where m_A is the mass of plate A and $\mathbf{M}$ is a normalized mass matrix, then

$$\mathbf{M}_B^* = m_B \mathbf{M} \quad (3.22b)$$

In a similar manner if

$$\mathbf{K}_A^* = m_A \omega_{iA}^2 \mathbf{K} \quad (3.23a)$$

then

$$\mathbf{K}_B^* = m_B \omega_{iB}^2 \mathbf{K} \quad (3.23b)$$

We now define the following set of coordinates

$$\tilde{u}_v = \frac{u_v}{\delta} = \frac{u_v \omega_1^2}{g} \quad ; \quad \tilde{\mathbf{y}} = \frac{\omega_1^2}{\kappa g} \mathbf{y} \quad (3.24)$$

where δ is the static deflection of the vehicle spring and ω_1 is the first natural frequency of the bridge. Transformation to coordinates $\tilde{u}_v$ and $\tilde{\mathbf{y}}$ yields

$$\begin{bmatrix} \frac{1}{\omega_v^2} & 0 \\ 0 & \frac{\phi^4 \kappa^2 \mathbf{M}^*}{\omega_v^2 m_1} \end{bmatrix} \begin{bmatrix} \ddot{\tilde{u}}_v \\ \ddot{\tilde{\mathbf{y}}} \end{bmatrix} + \begin{bmatrix} 1 & -\kappa \phi^2 \mathbf{N} \mathbf{Q} \\ -\kappa \phi^2 \mathbf{Q}^T \mathbf{N}^T & \kappa^2 \phi^4 \mathbf{Q}^T \mathbf{N}^T \mathbf{N} \mathbf{Q} + \frac{\kappa^4 \omega_1^4}{m_1 \omega_1^4} \mathbf{K}^* \end{bmatrix} \begin{bmatrix} \tilde{u}_v \\ \tilde{\mathbf{y}} \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (3.25)$$

Using Eqs. 3.21 through 3.23, Eqs. 3.25 can be expressed as

$$\begin{bmatrix} \frac{1}{2L} & 0 \\ 0 & \frac{\phi^4 \kappa M}{2L^3} \end{bmatrix} \begin{bmatrix} \ddot{u}_v \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 1 & -\kappa \phi^2 \mathbf{N} \mathbf{Q} \\ -\kappa \phi^2 \mathbf{Q}^T \mathbf{N}^T & \kappa^2 \phi^4 \mathbf{Q}^T \mathbf{N}^T \mathbf{N} \mathbf{Q} + \kappa \phi^2 \mathbf{K} \end{bmatrix} \begin{bmatrix} \tilde{u}_v \\ \tilde{y} \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (3.26)$$

where $\phi = \omega_v / \omega_1$ is the frequency ratio and $\kappa = m_v / m$ is the mass ratio.

It should be noted that vector $\mathbf{N}$ is a function of vt/L and $\tilde{y}/W$, and since $vt/L = \alpha \omega_v t / \pi \phi$, here the speed parameter α is calculated with reference to frequency ω_1 , $\mathbf{N}$ becomes a function of α, ϕ, ω_v and y/W , $\tilde{y}$ being the coordinate of vehicle path and W the width of the plate.

Equation 3.26 thus shows that if two corresponding plates subjected to a moving sprung mass have the same aspect ratios and the same values of α, ϕ, ω_v and κ their deflections will be identical to within a scale factor. The same scale factor will also apply to the static deflections. Hence if the dynamic deflections of such plates are normalized with respect to any representative static deflection, the normalized values will be exactly equal. In fact, using the same argument as in Chapter 2, it can be observed that the response is also independent of ω_v .

The conclusion we have just arrived at is, in fact, expected based on what we have concluded earlier regarding the forced vibration analysis of beam models and the free vibration analysis of plate models. The governing parameters for the response of beam models are α, ϕ , and κ , whereas the governing parameter for the free vibration analysis of isotropic plate models is the aspect ratio. Combining the two sets of results leads to the four controlling parameters we have just obtained from Eq. 3.26. For the forced vibration of orthotropic plate the governing parameters are $\alpha, \phi, \kappa, \theta$, and χ .

3.4.3 Results of Analytical Investigation of Isotropic Plates

3.4.3.1 Response of Bridge Models with Identical Values of Characterizing Parameters

As shown in the previous sections of this chapter, isotropic bridge models with the same aspect ratios and the same values of the parameters α, ϕ , and κ will have identical normalized response. To verify this the responses of the two plate models shown in Fig. 3.2 are calculated for a vehicle moving at the center line. The following dimensions and properties are used in the analysis

$$\rho = 2.4465 \text{ ton/m}^3$$

Plate A : -

$$W_A = 10.0 \text{ m}$$

$$L_A = 10.0 \text{ m}$$

$$E = 2.85 \times 10^7 \text{ kN/m}^2$$

$$\nu = 0.15$$

$$t_A = 0.325 \text{ m}$$

Plate B : -

$$W_B = 20.0 \text{ m}$$

$$L_B = 20.0 \text{ m}$$

$$t_B = 0.525 \text{ m}$$

Analyses are carried out for $\kappa = 0.50$ and 0.90 , two different values of α , namely 0.15 and 0.30 and several values of ϕ . In each case the response is normalized by the maximum static deflection at the middle point of the plates. The Figures 3.3a and 3.3b show the normalized dynamic deflections of the middle point of the plates. It is evident from the results presented in these figures, the response of the two plates are identical.

Similar calculation are performed for two plates with different aspect ratios but with identical values of α , ϕ , and κ . One plate (plate A) is 10 m by 10 m while the other (plate B) is 10 m wide by 20 m span. The static and dynamic characteristics of plate B are presented in Fig. 2.8 of Chapter 2. It should be noted that the frequency ratios are matched corresponding to the fundamental frequencies of the plates; for higher frequencies the ratios are different for the two plates.

Figures 3.4a to 3.4d show the normalized displacements at the middle point of the plates for a vehicle moving also along the center line. The normalization is with respect to the maximum mid-point displacement for the same model for which the response is being plotted. It is seen that even though the frequency ratios are matched only for the lowest frequency, the responses are very similar. Since the response is dominated by the lowest mode which is flexural in nature, this is to be expected. For the purpose of comparison, the normalized response of a beam model with identical values α , ϕ , and κ is also shown. It is found to be close to the responses of the plates. As it will be seen later (Figs. 3.11 and 3.12, section 3.4.4), the observation, regarding the similarity between the responses of two bridges having different aspect ratio, holds for the moment responses as well. However,

for the normalized moment responses, the difference is a little larger for two bridges having different aspect ratios.

As stated earlier, for plates of different aspect ratios having the same value ϕ , only the first frequency is matched. It is therefore possible that when modes other than the first make significant contribution to the response, the normalized response of the two plates may be different. To verify the extent of the difference, two plates of spans 10 and 20 m respectively but the same width, 10 m, are analyzed for a vehicle moving at the center line of a lane which is off-set from the center line of the bridge as shown in Fig. 3.5. Maximum response values are calculated on a cross-section at the mid-span of the bridge. Maximum normalized deflections at several locations on a transverse section at the mid span are plotted as function of ϕ for $\kappa = 0.50$ and two different values of α : 0.15 and 0.30. The deflections are normalized with respect to the maximum static mid-point deflection of the model of the bridge under consideration for a vehicle moving along the center line. At locations 2 and 5 differences do exist between the deflections of the two bridges, but they are small. At locations 8, which is farther away from the vehicle, the differences are much larger. This is not of much significance because the magnitudes of these deflections are small and they are therefore not critical for the purpose of design.

Figure 3.6 shows the variation of normalized dynamic deflection with ϕ at several points on the cross-section of the bridge of span 20 m and width 10 m caused by a vehicle moving along the center line of a lane as shown in Fig. 3.5a. A value of $\kappa = 0.50$ and two different values of α namely 0.15, 0.30 are used. For the purpose of comparison the normalized dynamic mid-point deflection of the model of the bridge for a vehicle moving along the center is also shown. All values have been normalized by the maximum static mid-span deflection at the center line for a vehicle moving also along the center line. The largest deflections caused by an off-center vehicle are at the bridge edge closer to the vehicle. The deflections at the center line caused by off center vehicle are similar to those caused by a vehicle moving along the center line of the plate.

3.4.3.2 Response under two parallel vehicles

When the bridge has more than one lane, it is possible for all lanes to be occupied by the design vehicles traveling in parallel. In recognition of the reduced probability of having more than one maximum vehicle occupying parallel positions, the code specifies a reduction

factor to be applied to the design vehicle load.

It would be of interest to calculate the dynamic response of bridge under the action of two or more vehicles traveling in parallel. For brevity, this is done here for only the simple case of two vehicles traveling side by side symmetrically with respect to the center line. If the mass of each vehicle is m_v , the mass ratio is $2m_v/m_k$ where m_k is the mass of the bridge. The normalized dynamic deflections at mid point of the bridge are plotted in Fig. 3.7 for $\kappa = 0.50$, two different values of α , namely 0.15 and 0.30 and several values of ϕ . For the purpose of comparison, deflections produced at the the mid point due to a vehicle of mass $2m_v$ traveling along the center line are also shown. All deflections have been normalized by the maximum static mid-point deflection due to the vehicle of mass $2m_v$ traveling along the center line. The two sets of deflections are very close to each other. Evidently, for vehicles traveling in parallel along paths that are symmetrical with respect to the center line, the response can be estimated by using a single vehicle of appropriate mass traveling along the center line. As shown earlier, for this latter case the deflection responses are closely predicted by a beam model.

3.4.3.3 Response under multiple vehicles in a lane

For bridges of longer spans, it is possible for more than one vehicle to be traveling in sequence. Figure 3.8, shows the normalized mid-point deflection produced by two vehicles each of mass m_v traveling in sequence along the center line of a bridge of span 20 m. The distance between the vehicles is 5 m. For the purpose of comparison, the normalized deflections produced by single vehicles of mass $2m_v$ and m_v are also shown.

The normalized mid-point deflections produced by two vehicles each of mass m_v traveling in sequence along the center line of the off center lane of bridge of span 20 are shown in Figs. 3.9a and 3.9b. Again the distance between the two vehicles is 5 m. All deflections have been normalized by the maximum static mid point deflection due to the vehicle of mass $2m_v$ traveling along the center line. The figure displays also, for comparison, the normalized middle point deflections produced by single vehicle of mass $2m_v$ traveling along the center line.

It is evident from Figs. 3.8 and 3.9 that the multiple vehicles in a lane passing over a bridge generally produce smaller response in comparison to that produced by a single vehicle of an equivalent weight traversing the bridge on a similar path. The amount of

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PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
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reduction in response depends on the distance between the traveling vehicles, the larger this distance the greater is the reduction in bridge response.

Although Fig. 3.9 shows the normalized middle point deflections produced by single vehicle of mass $2m_v$ traveling along the center line, yet as mentioned earlier, there is a great deal similarity of deflections at center line between those produced by a vehicle moving along the center line and the deflections caused by an off center vehicle of the same weight ($2m_v$).

3.4.3.4 Response under Multiple Staggered Vehicles

It would also be of interest to examine the response produced by two vehicles running along parallel lines but not abreast of each other. It is done here for the case when the travel paths are symmetrical with respect to the center line.

Figures 3.10a, and 3.10b show the normalized dynamic deflections at several locations along a transverse section at the mid-span caused by two staggered vehicles each of mass m_v . All deflections have been normalized by the maximum static mid-point deflection due to the vehicle of mass $2m_v$ traveling along the center line. The bridge span is 20 m and the longitudinal distance between vehicles is 5 m. For comparison, the mid-point deflections produced by a single vehicle of mass $2m_v$ traveling along the center line are also shown. It is clear that the staggered vehicles produce much smaller response.

3.4.4 Comparison between the Deflection Impact Factor and Moment Impact Factor

Figures 3.11 and 3.12 display the normalized mid-point deflections and mid-element moments for two aspect ratios, namely 1 and 2, as a function of ϕ . The results produced by a moving vehicle along the center line of the bridge are shown in Fig. 3.11, whereas those produced by a moving vehicle along the center line of a lane are shown in Fig. 3.12. For the central load the normalization is done by dividing the maximum dynamic response, deflection or moment, by the static response at the mid-point of the bridge due a load placed at this point. In case of the lane load, the maximum dynamic response at location 5, deflection or moment, is divided by the static response at the mid-point of the bridge due a load placed at that point. For all values of ϕ and for both the two aspect ratios, namely 1 and 2, the normalized deflections are, in general, greater than the normalized moments although the difference is fairly small. It is also clear from the figures that the effect of

aspect ratio on both the normalized deflection and the normalized moment is very small.

3.4.5 Transverse Distributions of Deflections and Moments

It is of interest to monitor the distribution of the deflections and longitudinal moments, M_x , across a transverse section of the bridges.

For an orthotropic plate model, The longitudinal bending moments per unit width may be expressed, in the conventional coordinates, as follows (Timoshenko 1970):

$$M_x = - \left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2} \right) \quad (3.27a)$$

or in terms of dimensionless coordinates as

$$M_x = - \frac{D_x}{L^2} \left(\frac{\partial^2 w}{\partial \zeta^2} + \frac{D_1 L^2}{D_x b^2} \frac{\partial^2 w}{\partial \eta^2} \right) \quad (3.27b)$$

Since w is a function of the parameters χ , θ , α , ϕ , and κ , M_x is also a function of these parameters. In addition, M_x is function of the nondimensional parameter $(D_1 L^2 / D_x b^2)$. For most bridges, the coupling rigidity D_1 is small (Bakht and Jaeger 1985), so that the second term in Eq. 3.28b is negligible in comparison to the first, and for all practical purposes, parameters χ , θ , α , ϕ , and κ can be considered to be the governing parameters in calculating longitudinal moments for orthotropic plate models.

For isotropic plate models the expression for the dimensionless moments M_x becomes

$$M_x = - \frac{D}{L^2} \left(\frac{\partial^2 \bar{w}}{\partial \zeta^2} + \frac{\nu L^2}{b^2} \frac{\partial^2 \bar{w}}{\partial \eta^2} \right) \quad (3.28)$$

Thus moment M_x is governed by four parameters, namely α , ϕ , κ , and the aspect ratio. It will be seen from the results, that the distribution is indeed only dependent on these parameters.

Figures 3.13a and 3.13b present the distribution of the maximum static and dynamic deflections and moments across the middle cross section of a simply supported bridge of span 20 m and width 20 m subjected to a moving single vehicle at the center line of the bridge. The response is drawn for values of $\kappa = 0.50$, $\alpha = 0.15$, and $\phi = 0.50$. The maximum static distribution is obtained by placing the vehicle at the mid point of the bridge. The dynamic values are obtained by moving the vehicle across the center line of the bridge and recording the deflections and moments along the middle cross section at all times.

The values of moments and deflections are scanned for the entire period during which the vehicle is traveling across the bridge and the maximum values are selected for all locations along the cross-section (method 1). As expected, these maximum values across the mid cross section occur at different times. The times of occurrence of the maxima are found to be very close to each other. Figure 3.15a shows this fact, where the dynamic values of moments are obtained at the time of occurrence of the maximum value at middle element (method 2). Almost complete similarity exists between the moments obtained by the two methods.

Comparison between the distributions of the deflections and of the moments across the mid cross section, shown in Figs. 3.13a and 3.13b respectively, reflects two important characteristics. The dynamic distribution is similar to the static distribution for both deflections (Fig. 3.13a) and moments (Fig. 3.13b). Also the transverse distribution of moments, dynamic and static, is more peaked than that of deflections. The same observations can be made in reference to Figs. 3.13c and 3.13d, drawn for values of $\phi = 1.0$, $\kappa = 0.50$, and $\alpha = 0.15$.

Figures 3.14a through 3.14d display the distributions of deflections and moments across the middle cross section for a single vehicle moving along the center line of a lane for values of $\alpha = 0.15$, $\kappa = 0.50$, and two values of $\phi = 0.5$ and 1.0 . The dynamic values are the maximum observed and the maxima at different locations along the cross-section may not occur at the same time. However, as seen for the case of a central vehicle the times at which the maximum occur are quite close. This can be verified by reference to Fig. 3.15b where the dynamic values of moments are obtained at the time of occurrence of the maximum at middle elements. It is interesting to note that when the vehicle is moving along the center line of an off-center lane, the torsional and transverse modes of vibration are readily excited and the plate responses, deflections and moments, are dominated by such modes. Again the moment distributions are more localized than those for deflections.

To see the effect of the aspect ratio on the distributions, a bridge of span 20 m and width 10 m is analyzed. The deflections and moments produced at the middle cross section are shown in Figs. 3.16a through 3.17d. The results correspond to values of $\kappa = 0.50$, $\alpha = 0.15$ and two values of ϕ , namely 0.50 and 1.0. The observations made for plate of aspect ratio 1.0 regarding the static and dynamic transverse distributions of deflections and moments are valid in this case too except that the static and dynamic distributions of

deflections and moments are less localized than those for aspect ratio = 1.0. As the aspect ratio increases, the effect of transverse modes on plate responses decreases and the plate cross section tends to behave as one unit.

3.5 Orthotropic Plate – Forced Vibrations Analysis

The orthotropic representation of a plate allows a more general study of the response than an isotropic representation. The analysis can be applied either to slab bridges or to beam and slab bridges. The difference between isotropic and orthotropic analyses of a plate is in the introduction of a different transverse rigidity in the latter case. It is expected that this difference will also be reflected in the characterizing parameters for a bridge modelled by an orthotropic plate. As in the case of an isotropic plate model, the controlling parameters for the forced vibration analysis of orthotropic plate models can be obtained by combining the parameters which govern the forced vibration of beam models and those which control the free vibration of plate models. Thus, the controlling parameters in this case will be; χ , θ , α , ϕ and κ .

3.5.1 Results of Analytical Investigation

Three plate models, A, B and C, are analyzed to verify the prediction of control parameters. The analyses are carried out for a vehicle moving along the center line as well as for a vehicle moving along an off-center line. The following dimensions and properties of the three plates are used in the analysis

$$\rho = 0.090 \text{ ton/m}^3$$

Plate A : –

$$W_A = 10.0 \text{ m}$$

$$L_A = 20.0 \text{ m}$$

$$D_{xA} = 6.13125 \times 10^6 \text{ kN.m}^2/\text{m}$$

$$D_{yA} = \frac{1}{100} D_{xA} = 6.13125 \times 10^4 \text{ kN.m}^2/\text{m}$$

$$H_A = 2.4525 \times 10^5 \text{ kN.m}^2/\text{m}$$

$$\theta = 0.7906$$

$$\lambda = 0.80$$

$$t_A = 1.0 \text{ m}$$

Plate B : –

$$\begin{aligned}
 W_B &= 5.0 \text{ m} \\
 L_B &= 10.0 \text{ m} \\
 D_{xB} &= 9.5801 \times 10^4 \text{ kN.m}^2/\text{m} \\
 D_{yB} &= \frac{1}{100} D_{xB} = 9.5801 \times 10^2 \text{ kN.m}^2/\text{m} \\
 H_B &= 3.8320 \times 10^3 \text{ kN.m}^2/\text{m} \\
 \theta &= 0.7906 \\
 \chi &= 0.80 \\
 t_B &= 0.25 \text{ m}
 \end{aligned}$$

Plate C : -

$$\begin{aligned}
 W_C &= 20.0 \text{ m} \\
 L_C &= 20.0 \text{ m} \\
 D_{xC} &= 6.13125 \times 10^6 \text{ kN.m}^2/\text{m} \\
 D_{yC} &= \frac{1}{8.25} D_{xC} = 9.81020 \times 10^5 \text{ kN.m}^2/\text{m} \\
 H_C &= 9.810 \times 10^5 \text{ kN.m}^2/\text{m} \\
 \theta &= 0.7906 \\
 \chi &= 0.80 \\
 t_C &= 1.0 \text{ m}
 \end{aligned}$$

Analyses are carried out for $\kappa = 0.50$, two values of α namely 0.15 and 0.30, and $\phi = 0.60$. The three plates are analyzed for a single vehicle moving along the center line of the bridge and along the center line of a lane which is off-center. Figure 3.18a displays comparisons between the maximum static and dynamic deflections along the central cross-section of the three plate models. The responses due to moving lane load are shown in Fig. 3.18b. Both Figs. 3.18a and 3.18b show that bridge models with the same χ , θ and the same values of the parameters α , ϕ , and κ will have identical static and dynamic responses.

Figure 3.19a demonstrates the maximum static and dynamic moment distributions along the central cross section for the two plate models, A and C, analyzed for a vehicle traversing along the center line, $\kappa = 0.50$, $\phi = 0.60$, and two values of α , 0.15 and 0.30. It is evident from Fig. 3.19a that the moment responses of the two plate models are identical. Figure 3.19b shows the moment response of plate B for the same values of parameters κ , ϕ , and α . The two figures, Fig. 3.19a and 3.19b, are identical within a scale factor of 16. The

value of the scale factor is obtained from the following simple calculations;

$$\begin{aligned}
 M_{xA} \text{ per unit width} &= \frac{P_A L_A}{4W_A} \\
 &= \frac{(16P_B)(2L_B)}{4(2W_B)} \\
 &= 16M_{xB} \text{ per unit width}
 \end{aligned}$$

Note that since κ is the same for plates A and B, the static load due to vehicle should be proportional to the mass of the plates. Since $m_A = 16m_B$, $P_A = 16P_B$. By a similar argument it can be shown that response of plates A and C have identical scales.

Figures 3.19c and 3.19d display the moment responses of the three plate models for the lane load case. Again, The responses of the two plate models, A and C are identical with each other and with plate model B within the same scale factor.

3.5.2 Effect of the Parameters χ and θ

Figures 3.20a through 3.21b show the effect of the parameters χ and θ on the response of an orthotropic plate. The analysis is carried out for two values of α , namely 0.15 and 0.30, $\kappa = 0.5$, and several values of ϕ . The figures (3.20a through 3.21b) show the normalized mid-point deflections caused by a vehicle moving along the center line of the bridge. As before, the normalization is done by dividing the maximum dynamic deflection by the static one at the mid-point of the bridge due a load placed at this point.

Figure 3.20a displays the effect of θ for a fixed value of $\chi = 0.80$ and values of $\kappa = 0.50$ and $\alpha = 0.15$. For values of ϕ higher than 0.60, an increase in the value of θ increases the normalized response of the bridge. The effect of θ is once more investigated and plotted in Fig. 3.21a for the same values of $\chi = 0.80$ and values of $\kappa = 0.50$ and $\alpha = 0.30$. Figure 3.21a shows that smaller values of θ produce higher response for values of ϕ less than 0.60. For higher values of ϕ the response does not follow a systematic behaviour.

Figure 3.20b shows the effect of the parameter χ for a fixed value of $\theta = 0.7906$ and values of $\kappa = 0.50$ and $\alpha = 0.15$. Generally, higher values of χ produces lower values of response. The effect of χ is again drawn in Fig. 3.21b for value of $\alpha = 0.30$, while the values of κ and θ are maintained to be the same as in Fig. 3.20b. For values of ϕ less than 0.60 greater values of χ produce greater response. For higher values of ϕ the response does not follow a systematic behaviour.

3.6 Comparison between the Isotropic and Orthotropic Models

To appreciate the difference between the isotropic and orthotropic models, the same orthotropic plate model A, span 20 m and width 10 m, used in previous section, is analyzed first as an orthotropic plate with $\chi = 0.80$ and $\theta = 0.7906$ and second as an isotropic plate with $\chi = 1.0$ and $\theta = 0.25$. Both plate models have the same value of $D_x = 6.13125 \times 10^6 \text{ k.N.m}^2/\text{m}$. The analyses are carried out for both a moving central load and a lane load and for values of $\kappa = 0.50$, two values of α , namely 0.15 and 0.30, and several values of ϕ .

The normalized mid-point deflection response for central load for value of $\alpha = 0.15$ is shown in Fig. 3.22a. The normalization is done by dividing the maximum dynamic deflection by the static response at the mid-point of the bridge due a load placed at this point. The response of the isotropic plate model is higher than that of the orthotropic one up to a value of $\phi = 0.60$; for higher values of ϕ , the orthotropic model displays a higher response. Figure 3.22b shows the response for value of $\alpha = 0.30$, in which the response of the orthotropic plate is generally less than that for the isotropic idealization.

Figures 3.23a and 3.23b display the response of the two plate models for a lane load. Maximum normalized deflections at several locations on a transverse section at the mid span are plotted as function of ϕ for $\kappa = 0.50$ and two different values of α : 0.15 and 0.30. The deflections are normalized with respect to the maximum static mid-point deflection of the model of the bridge under consideration for a vehicle moving along the center line of the lane. At locations 2 and 5 differences do exist between the deflections of the two bridges, and generally the response of the orthotropic model is higher than that of isotropic one. At locations 8, which is farther away from the vehicle, the differences are much larger and the response of the orthotropic model is much less than that of isotropic one.

3.7 SUMMARY AND CONCLUSION

The dynamic response of a shallow superstructure bridge produced by moving vehicles is studied. The bridge is idealized as an isotropic or orthotropic plate and the vehicle as a single sprung mass traveling along the deck in a direction parallel to span. Damping in

both the bridge and the vehicle is ignored. The vehicle is assumed to remain in contact with the deck. The response is measured in terms of the normalized dynamic deflection. Several important conclusions can be drawn.

1. The characterizing parameters that govern the dynamic response of a bridge structure modelled by an isotropic plate are: aspect ratio, the speed parameter α , the frequency ratio ϕ , and the mass ratio κ . Two different bridges that have identical values of these parameters will have identical responses.
2. The maximum responses of two different bridges with identical values of α , ϕ , κ but different aspect ratio are not exactly the same but are close enough.
3. The response produced by two vehicles each of mass m_v , traveling abreast of each other along paths symmetrically located with respect to the center line, is similar to that produced by a single vehicle of mass $2m_v$ traveling along the center line.
4. The response produced by two vehicles each of mass m_v traveling in sequence is less than that produced by a single vehicle of mass $2m_v$.
5. The response produced by two vehicles each of mass m_v traveling along parallel path but one lagging the other is significantly less than that produced by the same vehicles traveling abreast of each other.
6. For isotropic plate idealization, the transverse distribution of longitudinal moments are flatter than the static one. Also, the normalized moments are always less than the normalized deflections.
7. The effect of aspect ratio on the normalized deflection is very small.
8. As the aspect ratio increases, the normalized moment increases and the variation with ϕ becomes very similar to that of the normalized deflection.
9. The characterizing parameters that govern the dynamic response of a bridge structure modelled by an orthotropic plate are: χ , θ , the speed parameter α , the frequency ratio ϕ , and the mass ratio κ . Two different bridges that have identical values of these parameters will have identical responses.

FIGURE 3.1 MODELLING OF THE BRIDGE AND VEHICLE

FIGURE 3.2 CORRESPONDING PLATE MODELS

FIGURE 3.3 RESPONSE OF CORRESPONDING PLATE MODELS, NORMALIZED DEFLECTIONS AT MIDDLE POINT

Fig. 3.4b

Fig. 3.4c

Fig. 3.4d

FIGURE 3.4 COMPARISON OF NORMALIZED MIDDLE POINT DEFLECTIONS FOR PLATES WITH DIFFERENT ASPECT RATIOS BUT IDENTICAL α, c, κ UNDER A CENTRAL LOAD

Fig. 3.5a Cross-section of bridge

Fig. 3.5b

Fig. 3.5c

FIGURE 3.5 COMPARISON OF NORMALIZED CENTER-LINE DEFLECTIONS FOR PLATES WITH DIFFERENT ASPECT RATIOS BUT IDENTICAL α , ϕ , κ UNDER AN OFF-CENTER VEHICLE

FIGURE 3.6 NORMALIZED CENTER-LINE DEFLECTIONS OF A BRIDGE OF ASPECT RATIO 2 UNDER LANE LOAD

Fig. 3.7a

Fig. 3.7b

FIGURE 3.7 NORMALIZED MIDDLE POINT DEFLECTIONS OF A BRIDGE UNDER TWO SYMMETRICAL PARALLEL VEHICLES

FIGURE 3.8 NORMALIZED DEFLECTION OF A BRIDGE UNDER TWO VEHICLES IN SERIES

FIGURE 3.9 NORMALIZED CENTER-LINE DEFLECTION OF A BRIDGE UNDER TWO VEHICLES IN SERIES MOVING ALONG THE CENTER-LINE OF A LANE

FIGURE 3.10 NORMALIZED CENTER LINE DEFLECTION OF A BRIDGE UNDER TWO STAGGERED VEHICLES

FIGURE 3.11 EFFECT OF ASPECT RATIO ON NORMALIZED MID POINT DEFLECTIONS AND MOMENTS FOR CENTRAL VEHICLE
 $\alpha = 0.15 - \kappa = 0.50$

FIGURE 3.12 EFFECT OF ASPECT RATIO ON NORMALIZED MID POINT DEFLECTIONS AND MOMENTS FOR OFF CENTER VEHICLE
 $\alpha = 0.15 - \kappa = 0.50$

Fig. 3.13a

Fig. 3.13b

FIGURE 3.13 RESPONSE OF ISOTROPIC PLATE MODELS TO MOVING CENTRAL LOAD

Fig. 3.13c

Fig. 3.13d

FIGURE 3.13 RESPONSE OF ISOTROPIC PLATE MODELS TO MOVING CENTRAL LOAD

Fig. 3.14a

Fig. 3.14b

FIGURE 3.14 RESPONSE OF ISOTROPIC PLATE MODELS TO MOVING LANE LOAD, DISTRIBUTION OF DEFLECTIONS AND MOMENTS AT MIDDLE CROSS-SECTION

Fig. 3.14c

Fig. 3.14d

FIGURE 3.14 RESPONSE OF ISOTROPIC PLATE MODELS TO MOVING LANE LOAD, DISTRIBUTION OF DEFLECTIONS AND MOMENTS AT MIDDLE CROSS-SECTION

Fig. 3.15a Response Due to Central Load

Fig. 3.15b Response Due to Lane Load

**FIGURE 3.15 COMPARISON OF MAXIMUM DYNAMIC RESPONSES
OBTAINED BY TWO ALTERNATIVE MEANS.**

$\phi = 0.50 - \alpha = 0.15 - \kappa = 0.50 - \text{aspect ratio} = 1.0$

Fig. 3.16a Response Due to Central Load

Fig. 3.16b Response Due to Central Load

Fig. 3.16c Response Due to Lane Load

Fig. 3.16d Response Due to Lane Load

FIGURE 3.16 RESPONSE OF ISOTROPIC PLATE MODELS UNDER VEHICLES
MOVING ALONG DIFFERENT PATHS, MAXIMUM DEFLECTIONS AND MOMENTS
AT MIDDLE CROSS SECTION

Fig. 3.17a Response Due to Central Load

Fig. 3.17b Response Due to Central Load

Fig. 3.17c Response Due to Lane Load

Fig. 3.17d Response Due to Lane Load

FIGURE 3.17 RESPONSE OF ISOTROPIC PLATE MODELS UNDER VEHICLES
MOVING ALONG DIFFERENT PATHS, MAXIMUM DEFLECTIONS AND MOMENTS
AT MIDDLE CROSS SECTION

Fig. 3.18a Response Due to Central Load

- Static deflection - Br. 10x20
- ◆— Dynamic deflection - $\alpha = 0.15$ - Br. 10x20
- Dynamic deflection - $\alpha = 0.30$ - Br. 10x20
- ◆— Static deflection - Br. 5x10
- Dynamic deflection - $\alpha = 0.15$ - Br. 5x10
- Dynamic deflection - $\alpha = 0.15$ - Br. 5x10
- ▲— Static deflection - Br. 20x20
- ▲— Dynamic deflection - $\alpha = 0.15$ - Br. 20x20
- Dynamic deflection - $\alpha = 0.15$ - Br. 20x20

Fig. 3.18b Response Due to Lane Load

FIGURE 3.18 RESPONSE OF ORTHOTROPIC PLATE MODELS
DUE TO DIFFERENT PATHS OF LOAD

$$\phi = 0.60 - \kappa = 0.50$$

Fig. 3.19c Br. 10x20 - Response Due to Lane Load

Fig. 3.19d Br. 5x10 - Response Due to Lane Load

Fig. 3.19a Br. 10x20 - Response Due to Central Load

Fig. 3.19b Br. 5x10 - Response Due to Lane Load

FIGURE 3.19 RESPONSE OF ORTHOTROPIC PLATE MODELS UNDER VEHICLES MOVING ALONG DIFFERENT PATHS OF LOADS

Fig. 3.20a $\alpha = 0.15 - \kappa = 0.50 - \chi = 0.80$ Fig. 3.20b $\alpha = 0.15 - \kappa = 0.50 - \theta = 0.7906$ FIGURE 3.20 RESPONSE OF ORTHOTROPIC PLATE
DUE TO A MOVING CENTRAL LOAD

Fig. 3.21a $\alpha = 0.30 - \kappa = 0.50 - \chi = 0.80$ Fig. 3.21b $\alpha = 0.30 - \kappa = 0.50 - \theta = 0.7906$ **FIGURE 3.21 RESPONSE OF ORTHOTROPIC PLATE
DUE TO A MOVING CENTRAL LOAD**

Fig. 3.22a

Fig. 3.22b

FIGURE 3.22 COMPARISON BETWEEN ISOTROPIC AND ORTHOTROPIC PLATE MODELS RESPONSE

Fig. 3.23a

Fig. 3.23b

FIGURE 3.23 COMPARISON BETWEEN ISOTROPIC AND ORTHOTROPIC PLATE MODELS RESPONSES DUE TO LANE LOAD

4 DYNAMIC RESPONSE OF BOX GIRDER HIGHWAY BRIDGES EQUATIONS OF MOTION

4.1 General

The present chapter is devoted to a study of the dynamic response of the box girder and deck type bridges. These bridges have not been adequately investigated in the past, yet they are some of the most widely used because of their great adaptability to the complex geometry involved in the highway systems. A thorough investigation of this type of bridges has not been successfully conducted due to the complexity involved in solving a system of large number of nonlinear differential equations. This system of equations is encountered when the bridge and vehicle are treated as an integrated system in which the bridge is idealized using a three-dimensional model and the vehicle is modeled by sprung and unsprung masses connected together by a system of springs and dash-pots.

4.2 Idealization of Bridge and vehicle

As mentioned earlier, the finite element method has been employed in this study to obtain an analytical model for the bridge system. Both the natural frequencies and the dynamic response of the box-girder bridge will be assessed using this analytical model. The accuracy of a mathematical model in predicting response depends greatly on the closeness of the proposed idealization to the realistic behaviour of different components comprising the bridge system. Therefore a comparison study has been carried out to determine the finite elements that are suitable for representing the different components of the bridge. Many different models have been used in the past (Jeffcott, 1929, Oran and Veletsos, 1961, and Walker and Veletsos, 1963) to simulate the vehicle system. A reasonable model of the vehicle dynamical system is one which represents the vehicle as a sprung mass supported

by springs and dash-pots on two-axle, four wheels. The axles and wheels represent the unsprung masses in the system. This is the model adopted in the present study.

4.2.1 Model of the Bridge Structure

The three distinct parts of the bridge structure, the deck, the box girders, and the diaphragms, usually have a small thickness compared with the other dimensions of the bridge. Examination of the structural behaviour of these three components under external loads suggests that a flat thin quadrilateral shell elements will best model them all. The shell element shown in Fig. 4.1 has 5 degrees of freedom per node: two in-plane or membrane displacements along x and y axes, the transverse displacement and the rotations. The last three, which can be viewed as bending deformations, are uncoupled from the membrane displacements. As a result, the stiffness matrix for flat thin shell element can be derived by combining the stiffness matrices of a plate bending element and a plane stress element. Thus, if matrix K_b represents the plate bending stiffness and matrix K_m represents the plane stress stiffness, the stiffness matrix of the shell element corresponding to the degrees of freedom shown in Fig. 4.1 is given by

$$K = \begin{bmatrix} K_b & 0 \\ 0 & K_m \end{bmatrix} \quad (4.1)$$

Matrix K_b is of size 12×12 and matrix K_m is of size 8×8 . Hence matrix K is of size 20×20 .

The plate bending element used for the present study is based on the triangular element HCT derived by Hsieh, (Clough and Tocher, 1966 and Clough and Fleippa, 1968). This element ensures full compatibility of slopes and should therefore provide monotonic convergence to the exact solution. A series of rectangular plates modelled by the above-said finite element were analyzed in a previous research (Gorman, 1982) to obtain their mode shapes and frequencies. The results of the finite element free vibration analysis were then compared with the available closed form solutions. The two sets of results showed very close agreement, Table 4.1. In the computer programs developed for the study, the algorithms and source code corresponding to the HCT plate bending element were taken from program SAP IV (Bathe et al., 1974)

In choosing a membrane element, special care should be exercised to avoid the main sources of inaccuracy. Lower-order finite elements— such as the constant strain elements—

are known to perform poorly in situations where bending deformation are dominant, which is the case for box girders in the bridge. In this case the element experiences what is called *locking phenomenon* which gives rise to *parasitic* transverse shear stresses especially in case of thin shell elements. As a result *spurious shear energy* term is present in the final strain energy formulation. A higher order element such as a linear strain element provides much improved accuracy. However, for dominant bending modes, even the use of linear strain elements, where *Gauss-Quadrature* is performed in the usual manner for deriving the stiffness matrix, leads to a model that overestimates the stiffness. As a result the calculated natural frequencies are higher than the true frequencies.

Several attempts have been made to overcome the problem of spurious shear energy. A method of avoiding the inclusion of such energies by using a uniform *reduced integration scheme*, wherein a lower order of integration for all stress components is performed, was first introduced by Zienkiewicz et al (1971). A *selective integration scheme* was suggested by Doherty et al (1969) and by Pawsey (1971). In this scheme, the shear strain energy is evaluated by a lower-order integration than that used for other types of energies. This technique significantly mitigates the *locking phenomena* and also avoids engendering the dangerous *spurious modes* associated with the uniform reduced integration. However, the application of modified integration schemes has been limited to idealized geometries and isotropic materials. Also, convergence may not be assured with their use (Wilson 1971). Furthermore, mechanisms or spurious zero energy modes may form which may either lead to no solution or pollute the solution. Wilson (1971) proposed another technique based on the introduction of *incompatible displacement modes* in the basic quadrilateral element. When this technique is used, the resulting element stiffness matrix is 12×12 rather than 8×8 . However, by minimizing the strain energy within the element, the magnitude of the unknown modes can be determined. Before the assembly of the global stiffness element, the additional displacements fields are eliminated and a reduced 8×8 stiffness matrix is obtained. Although, these extra displacement modes violate the inter-element compatibility, excellent results have been obtained using this technique.

To determine the suitability of any one of the above cited techniques for the purpose of this study, two cantilever beams were analyzed for their response to static loads. The details of the beams and the loading are shown in Fig. 4.2. A number of different finite elements were used to model the beam. The results of the analysis are presented in Ta-

ble 4.2 where for comparison the theoretical solution obtained by using the standard beam theory is also shown. Of the finite elements referred to in the table, **QUAD4** is the standard isoparametric quadrilateral element, **QUAD4SR** is a quadrilateral with a restricted integration on the shear strain, **QUAD6** is the quadrilateral associated with incompatible displacement modes, **SHELLN** is composed of four linear strain triangles with reduced shear integrations, **SHELLO** is composed of four constant strain triangles, **Quad83** is an eight noded isoparametric element in which the shear strains are integrated with (3×3) *Gauss points*, and **Quad82** is an eight noded isoparametric element in which the shear strains are integrated with (2×2) *Gauss points*. As seen from Table 4.2, **QUAD6** finite element gives results that are closest to the beam theory. Free vibrations analysis were carried out on the same two beams to establish the validity of the different elements in predicting dynamic responses. The computer programs required for this study were developed by the author except for element **QUAD4** the source code for which was taken from program **SAP IV** (Bathe et al., 1974) and element **SHELLN** for which a program developed by Algor (1989) was used. Tables 4.3 and 4.4 show the results of these studies. Again **QUAD6** element, in general, gives the best results. Hence, **QUAD6** finite element is chosen for the present study.

4.2.2 Dynamic Model of the Vehicle

For an analytical study of the bridge-vehicle system, the vehicle is represented by a two axle, four wheel system as shown in Fig. 4.3. The mass of the vehicle body is assumed to be concentrated at its centroid. The body itself is considered to be rigid and is supported by a system of springs and dash-pots attached to the front and rear axles. The dash-pots provide viscous damping in the system. The mass of the axles, wheels, driveshafts, brakes, and the suspension system is supported by the wheels which always remain in contact with the bridge surface.

The sprung mass m_1 of the vehicle body has three independent degrees of freedom (d.o.f.) defined with respect to a set of orthogonal axes with origin at the centroid of the body. The set of axes consists of a longitudinal axis p_1 along the direction of motion, the transverse axis p_2 , and the vertical axis p_3 . The three independent degrees of freedom comprise: translation d_{13} along axis p_3 , referred to as the bounce; rotation d_{14} about p_1 , referred to as the rolling motion; and rotation d_{15} about p_2 , referred to as the pitch. Translations d_{11} and d_{12} are assumed to be rigid body motions and do not affect the

equations of motion of the bridge-vehicle system. The effect of rotation about p_3 , or the yawing motion, is assumed to be negligible. All displacements are measured relative to the deformed position under gravity loads.

The unsprung masses m_2 and m_3 are assumed to be concentrated at the centers of the front and rear axles respectively. Mass m_2 can undergo vertical translation d_{23} and a rotation d_{24} about the longitudinal axis. In a similar manner, mass m_3 has a translation d_{33} and a rotation d_{34} . Because masses m_2 and m_3 are not supported by springs and the wheels are assumed to remain in contact with the bridge surface, degrees of freedom d_{23} , d_{24} , d_{33} , and d_{34} are not independent but are related to the vertical motions of the contact points between the wheels and the bridge surface. The vehicle thus contributes three independent degrees of freedom and the total number of d.o.f. of the bridge-vehicle system is $N + 3$, N being the number of d.o.f of the bridge structure.

4.2.3 Assumptions Made in the Idealization of Bridge and Vehicle

In the process of modeling of the bridge and the vehicle, certain assumptions are made. These assumptions can be summarized as follows:

Regarding the bridge structure

- The materials used in the bridge are assumed to be linearly elastic and the deformations of the structure are assumed to remain within the limits of small displacements theory.
- The damping in the bridge structure is assumed to be viscous.

Regarding the vehicle dynamic system

- The suspension springs are assumed to be linearly elastic and the vehicle damping is assumed to be of viscous nature

Regarding the bridge-vehicle system

- The surface roughness of the bridge is not considered in this study.
- During the motion, the vehicle is always assumed to remain in contact with the bridge.

4.3 Formulation of the Equations of Motion

In considering the interaction of bridge and vehicle at any contact point, two conditions

should be fulfilled; these are:

- (i) Compatibility condition - displacements of the bridge deck and the vehicle are the same
- (ii) Equilibrium condition - the external loads on the bridge are equal and opposite to the reactions on the vehicle.

The final equations of motions of the bridge-vehicle dynamic system are derived in three steps. First, the dynamic equilibrium of vehicle sprung mass motion is considered. Second, the contact forces, which are the external forces acting on the bridge deck, are determined. Finally, the equations of motion for the bridge are formulated with due consideration to contact forces represented in terms of equivalent nodal loads.

4.3.1 Equations of Motion of the Sprung Mass

Figure 4.4 is a free-body diagram showing the forces and moments acting on the sprung mass of the vehicle including the forces and moments due to inertia. Forces F_1 , F_2 , F_3 , and F_4 comprising spring and damping forces are given by

$$F_1 = k_f \left[d_{13} - \alpha b d_{14} - a_2 L d_{15} - w_1 \left(\frac{1}{2} + \alpha \right) - w_2 \left(\frac{1}{2} - \alpha \right) \right] + c_f \left[\dot{d}_{13} - \alpha b \dot{d}_{14} - a_2 L \dot{d}_{15} - \dot{w}_1 \left(\frac{1}{2} + \alpha \right) - \dot{w}_2 \left(\frac{1}{2} - \alpha \right) \right] \quad (4.2)$$

$$F_2 = k_f \left[d_{13} + \alpha b d_{14} - a_2 L d_{15} - w_1 \left(\frac{1}{2} - \alpha \right) - w_2 \left(\frac{1}{2} + \alpha \right) \right] + c_f \left[\dot{d}_{13} + \alpha b \dot{d}_{14} - a_2 L \dot{d}_{15} - \dot{w}_1 \left(\frac{1}{2} - \alpha \right) - \dot{w}_2 \left(\frac{1}{2} + \alpha \right) \right] \quad (4.3)$$

$$F_3 = k_r \left[d_{13} - \alpha b d_{14} + a_1 L d_{15} - w_3 \left(\frac{1}{2} + \alpha \right) - w_4 \left(\frac{1}{2} - \alpha \right) \right] + c_r \left[\dot{d}_{13} - \alpha b \dot{d}_{14} + a_1 L \dot{d}_{15} - \dot{w}_3 \left(\frac{1}{2} + \alpha \right) - \dot{w}_4 \left(\frac{1}{2} - \alpha \right) \right] \quad (4.4)$$

$$F_4 = k_r \left[d_{13} + \alpha b d_{14} + a_1 L d_{15} - w_3 \left(\frac{1}{2} - \alpha \right) - w_4 \left(\frac{1}{2} + \alpha \right) \right] + c_r \left[\dot{d}_{13} + \alpha b \dot{d}_{14} + a_1 L \dot{d}_{15} - \dot{w}_3 \left(\frac{1}{2} - \alpha \right) - \dot{w}_4 \left(\frac{1}{2} + \alpha \right) \right] \quad (4.5)$$

where, k_f, k_r are the stiffnesses of the front and rear axle springs,

c_f, c_r are the viscous damping coefficients of the front and rear dash pots,
 L is the wheel base of the vehicle;
 b is the width of the vehicle;
 α, a_1, a_2 are defined in Fig. 4.3

Equilibrium of forces in the vertical direction, p_3 , gives

$$m_1 \ddot{d}_{13} + F_1 + F_2 + F_3 + F_4 = 0$$

or

$$\begin{aligned}
 m_1 \ddot{d}_{13} + 2(k_f + k_r)d_{13} + 2L(k_r a_1 - k_f a_2)d_{15} + 2(c_f + c_r)\dot{d}_{13} + 2L(c_r a_1 - c_f a_2)\dot{d}_{15} \\
 - k_f(w_1 + w_2) - k_r(w_3 + w_4) - c_f(\dot{w}_1 + \dot{w}_2) - c_r(\dot{w}_3 + \dot{w}_4) = 0
 \end{aligned} \quad (4.6)$$

The condition of Moment equilibrium about axis p_1 yields

$$I_{11} \ddot{d}_{14} + (F_2 - F_1 + F_4 - F_3)\alpha b = 0$$

or

$$\begin{aligned}
 I_{11} \ddot{d}_{14} + [2\alpha b(k_f + k_r)d_{14} + 2\alpha k_f(w_1 - w_2) + 2\alpha k_r(w_3 - w_4)]\alpha b \\
 + [2\alpha b(c_f + c_r)\dot{d}_{14} + 2\alpha c_f(\dot{w}_1 - \dot{w}_2) + 2\alpha c_r(\dot{w}_3 - \dot{w}_4)]\alpha b = 0
 \end{aligned}$$

or

$$\begin{aligned}
 \left(\frac{I_{11}}{b^2}\right) b \ddot{d}_{14} + [2b(k_f + k_r)d_{14} + 2k_f(w_1 - w_2) + 2k_r(w_3 - w_4)]\alpha^2 \\
 + [2b(c_f + c_r)\dot{d}_{14} + 2c_f(\dot{w}_1 - \dot{w}_2) + 2c_r(\dot{w}_3 - \dot{w}_4)]\alpha^2 = 0
 \end{aligned} \quad (4.7)$$

Finally moment equilibrium about axis p_2 gives

$$I_{12} \ddot{d}_{15} - (F_2 + F_1)a_2 L + (F_4 + F_3)a_1 L = 0$$

or

$$\begin{aligned}
 I_{12} \ddot{d}_{15} - (2k_f d_{13} - 2k_f a_2 L d_{15} - k_f(w_1 + w_2))a_2 L \\
 - (2c_f \dot{d}_{13} - 2c_f a_2 L \dot{d}_{15} - c_f(\dot{w}_1 + \dot{w}_2))a_2 L \\
 + (2k_r d_{13} + 2k_r a_1 L d_{15} - k_r(w_3 + w_4))a_1 L \\
 + (2c_r \dot{d}_{13} + 2c_r a_1 L \dot{d}_{15} - c_r(\dot{w}_3 + \dot{w}_4))a_1 L = 0
 \end{aligned}$$

or

$$\begin{aligned}
 \left(\frac{I_{12}}{L^2}\right) L \ddot{d}_{15} + 2(k_r a_1 - k_f a_2)d_{13} + 2L(k_f a_2^2 + k_r a_1^2)d_{15} + a_2 k_f(w_1 + w_2) \\
 - a_1 k_r(w_3 + w_4) + 2(c_r a_1 - c_f a_2)\dot{d}_{13} + 2L(c_f a_2^2 + c_r a_1^2)\dot{d}_{15}
 \end{aligned}$$

$$+a_2 c_f(\dot{w}_1 + \dot{w}_2) - a_1 c_r(\dot{w}_3 + \dot{w}_4) = 0 \quad (4.8)$$

Equations 4.6, 4.7 and 4.8 can be assembled in a matrix form ;

$$m\ddot{d} + c\dot{d} + kd + q = 0 \quad (4.9)$$

where,

$$m = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & \frac{I_{11}}{b^2} & 0 \\ 0 & 0 & \frac{I_{21}}{b^2} \end{bmatrix} \quad d = \begin{bmatrix} d_{13} \\ ad_{14} \\ Ad_{15} \end{bmatrix}$$

$$k = \begin{bmatrix} 2(k_r + k_f) & 0 & 2(a_1 k_r - a_2 k_f) \\ 0 & 2\alpha^2(k_f + k_r) & 0 \\ 2(a_1 k_r - a_2 k_f) & 0 & 2(a_2^2 k_f + a_1^2 k_r) \end{bmatrix}$$

$$c = \begin{bmatrix} 2(c_r + c_f) & 0 & 2(a_1 c_r - a_2 c_f) \\ 0 & 2\alpha^2(c_f + c_r) & 0 \\ 2(a_1 c_r - a_2 c_f) & 0 & 2(a_2^2 c_f + a_1^2 c_r) \end{bmatrix}$$

$$q = \begin{bmatrix} -k_f(w_1 + w_2) - k_r(w_3 + w_4) - c_f(\dot{w}_1 + \dot{w}_2) - c_r(\dot{w}_3 + \dot{w}_4) \\ 2\alpha^2 k_f(w_1 - w_2) + 2\alpha^2 k_r(w_3 - w_4) + 2\alpha^2 c_f(\dot{w}_1 - \dot{w}_2) + 2\alpha^2 c_r(\dot{w}_3 - \dot{w}_4) \\ a_2 k_f(w_1 + w_2) - a_1 k_r(w_3 + w_4) + a_2 c_f(\dot{w}_1 + \dot{w}_2) - a_1 c_r(\dot{w}_3 + \dot{w}_4) \end{bmatrix}$$

I_{11} and I_{12} are the mass moments of inertia of mass 1 about p_1 and p_2 axes respectively, I_{21} and I_{31} are the mass moments of inertia of masses 2 and 3 respectively about the p_1 axis, w_i is the vertical displacement of contact point i and overdots represent differentiation with respect to time.

4.3.2 Contact Forces

Considering equilibrium, the contact forces P_i at any contact point (i) can now be derived. Taking moments around point 2 in Fig. 4.5a,

$$P_1 b - \left(\frac{1}{2} + \alpha\right) b \left(F_1 - \frac{m_1 a_1 g}{2}\right) - \left(\frac{1}{2} - \alpha\right) b \left(F_2 - \frac{m_1 a_1 g}{2}\right) + \frac{m_2 b}{2} (\ddot{d}_{23} + g) - I_{21} \ddot{d}_{24} = 0$$

or

$$P_1 = \frac{1}{2} (F_1 + F_2) + \alpha (F_1 - F_2) - \frac{m_1 a_1 g}{2} - \frac{m_2 g}{2} - \frac{m_2 \ddot{d}_{23}}{2} + \left(\frac{I_{21}}{b^2}\right) (b \ddot{d}_{24})$$

Substituting for F1 and F2

$$\begin{aligned}
 P_1 = & [k_f \quad -2\alpha^2 k_f \quad -a_2 k_f] \begin{Bmatrix} d_{13} \\ b\dot{d}_{14} \\ L\dot{d}_{15} \end{Bmatrix} + [c_f \quad -2\alpha^2 c_f \quad -a_2 c_f] \begin{Bmatrix} \dot{d}_{13} \\ b\ddot{d}_{14} \\ L\ddot{d}_{15} \end{Bmatrix} \\
 & + \left(-\frac{1}{2}k_f - 2\alpha^2 k_f\right) w_1 + \left(-\frac{1}{2}k_f + 2\alpha^2 k_f\right) w_2 + \left(-\frac{1}{2}c_f - 2\alpha^2 c_f\right) \dot{w}_1 \\
 & + \left(-\frac{1}{2}c_f + 2\alpha^2 c_f\right) \dot{w}_2 - \frac{g}{2}(m_1 a_1 + m_2) - \frac{m_2 \ddot{d}_{23}}{2} + \left(\frac{I_{21}}{b^2}\right) (b\ddot{d}_{24})
 \end{aligned} \quad (4.10)$$

Taking moment around point 1 in Fig. 4.5a,

$$P_2 b - \left(\frac{1}{2} - \alpha\right) b \left(F_1 - \frac{m_1 a_1 g}{2}\right) - \left(\frac{1}{2} + \alpha\right) b \left(F_2 - \frac{m_1 a_1 g}{2}\right) + \frac{m_2 b}{2} (\ddot{d}_{23} + g) + I_{21} \ddot{d}_{24} = 0$$

or

$$P_2 = \frac{1}{2} (F_1 + F_2) - \alpha (F_1 - F_2) - \frac{m_1 a_1 g}{2} - \frac{m_2 g}{2} - \frac{m_2 \ddot{d}_{23}}{2} - \left(\frac{I_{21}}{b^2}\right) (b\ddot{d}_{24})$$

Substitution for F1 and F2 gives

$$\begin{aligned}
 P_2 = & [k_f \quad 2\alpha^2 k_f \quad -a_2 k_f] \begin{Bmatrix} d_{13} \\ b\dot{d}_{14} \\ L\dot{d}_{15} \end{Bmatrix} + [c_f \quad 2\alpha^2 c_f \quad -a_2 c_f] \begin{Bmatrix} \dot{d}_{13} \\ b\ddot{d}_{14} \\ L\ddot{d}_{15} \end{Bmatrix} \\
 & + \left(-\frac{1}{2}k_f + 2\alpha^2 k_f\right) w_1 + \left(-\frac{1}{2}k_f - 2\alpha^2 k_f\right) w_2 + \left(-\frac{1}{2}c_f + 2\alpha^2 c_f\right) \dot{w}_1 \\
 & + \left(-\frac{1}{2}c_f - 2\alpha^2 c_f\right) \dot{w}_2 - \frac{g}{2}(m_1 a_1 + m_2) - \frac{m_2 \ddot{d}_{23}}{2} - \left(\frac{I_{21}}{b^2}\right) (b\ddot{d}_{24})
 \end{aligned} \quad (4.11)$$

As stated earlier, the displacements and rotations of mass 2 and mass 3 are not independent.

Under the assumption of bridge-vehicle contact, the relations between the unsprung mass degrees of freedom and the bridge degrees of freedom are given by (Fig. 4.3b),

$$\begin{aligned}
 d_{23} &= \frac{1}{2}(w_1 + w_2) \\
 d_{24} &= \frac{1}{b}(w_2 - w_1)
 \end{aligned} \quad (4.12)$$

Substituting for d_{23} and d_{24} from Eq. 4.12, Eqs 4.10 and 4.11 reduce to

$$\begin{aligned}
 P_1 = & [k_f \quad -2\alpha^2 k_f \quad -a_2 k_f] \begin{Bmatrix} d_{13} \\ b\dot{d}_{14} \\ L\dot{d}_{15} \end{Bmatrix} + [c_f \quad -2\alpha^2 c_f \quad -a_2 c_f] \begin{Bmatrix} \dot{d}_{13} \\ b\ddot{d}_{14} \\ L\ddot{d}_{15} \end{Bmatrix} \\
 & + \left(-\frac{1}{2}k_f - 2\alpha^2 k_f\right) w_1 + \left(-\frac{1}{2}k_f + 2\alpha^2 k_f\right) w_2 + \left(-\frac{1}{2}c_f - 2\alpha^2 c_f\right) \dot{w}_1 \\
 & + \left(-\frac{1}{2}c_f + 2\alpha^2 c_f\right) \dot{w}_2 - \frac{g}{2}(m_1 a_1 + m_2) + \left(-\frac{m_2}{4} - \frac{I_{21}}{b^2}\right) \ddot{w}_1 + \left(-\frac{m_2}{4} + \frac{I_{21}}{b^2}\right) \ddot{w}_2
 \end{aligned} \quad (4.13)$$

$$\begin{aligned}
P_2 = & [k_f \quad 2\alpha^2 k_f \quad -a_2 k_f] \begin{Bmatrix} d_{13} \\ bd_{14} \\ Ld_{15} \end{Bmatrix} + [c_f \quad 2\alpha^2 c_f \quad -a_2 c_f] \begin{Bmatrix} \dot{d}_{13} \\ b\dot{d}_{14} \\ L\dot{d}_{15} \end{Bmatrix} \\
& + \left(-\frac{1}{2}k_f + 2\alpha^2 k_f\right) w_1 + \left(-\frac{1}{2}k_f - 2\alpha^2 k_f\right) w_2 + \left(-\frac{1}{2}c_f + 2\alpha^2 c_f\right) \dot{w}_1 \\
& + \left(-\frac{1}{2}c_f - 2\alpha^2 c_f\right) \dot{w}_2 - \frac{g}{2}(m_1 a_1 + m_2) + \left(-\frac{m_2}{4} + \frac{l_{21}}{b^2}\right) \ddot{w}_1 + \left(-\frac{m_2}{4} - \frac{l_{21}}{b^2}\right) \ddot{w}_2
\end{aligned} \quad (4.14)$$

Taking moments about point 4 in Fig. 4.5b

$$P_3 b - \left(\frac{1}{2} + \alpha\right) b \left(F_3 - \frac{m_1 a_2 g}{2}\right) - \left(\frac{1}{2} - \alpha\right) b \left(F_4 - \frac{m_1 a_2 g}{2}\right) + \frac{m_3 b}{2} (\ddot{d}_{33} + g) - l_{31} \ddot{d}_{34} = 0$$

or

$$P_3 = \frac{1}{2} (F_3 + F_4) + \alpha (F_3 - F_4) - \frac{m_1 a_2 g}{2} - \frac{m_3 g}{2} - \frac{m_3 \ddot{d}_{33}}{2} + \left(\frac{l_{31}}{b^2}\right) b \ddot{d}_{34}$$

Substitution for F3 and F4 gives

$$\begin{aligned}
P_3 = & \frac{1}{2} k_r [2d_{13} + 2a_1 Ld_{15} - w_3 - w_4] + \alpha k_r [-2\alpha b d_{14} - 2\alpha w_3 + 2\alpha w_4] \\
& + \frac{1}{2} c_r [2\dot{d}_{13} + 2a_1 L\dot{d}_{15} - \dot{w}_3 - \dot{w}_4] + \alpha c_r [-2\alpha b \dot{d}_{14} - 2\alpha \dot{w}_3 + 2\alpha \dot{w}_4] \\
& - \frac{m_1 a_2 g}{2} - \frac{m_3 g}{2} - \frac{m_3 \ddot{d}_{33}}{2} + \left(\frac{l_{31}}{b^2}\right) (b \ddot{d}_{34})
\end{aligned} \quad (4.15)$$

Also

$$\begin{aligned}
d_{33} &= \frac{1}{2}(w_3 + w_4) \\
d_{34} &= \frac{1}{b}(w_4 - w_3)
\end{aligned} \quad (4.16)$$

Equation 4.15 therefore reduces to

$$\begin{aligned}
P_3 = & [k_r \quad -2\alpha^2 k_r \quad a_1 k_r] \begin{Bmatrix} d_{13} \\ bd_{14} \\ Ld_{15} \end{Bmatrix} + [c_r \quad -2\alpha^2 c_r \quad a_1 c_r] \begin{Bmatrix} \dot{d}_{13} \\ b\dot{d}_{14} \\ L\dot{d}_{15} \end{Bmatrix} \\
& + \left(-\frac{1}{2}k_r - 2\alpha^2 k_r\right) w_3 + \left(-\frac{1}{2}k_r + 2\alpha^2 k_r\right) w_4 + \left(-\frac{1}{2}c_r - 2\alpha^2 c_r\right) \dot{w}_3 \\
& + \left(-\frac{1}{2}c_r + 2\alpha^2 c_r\right) \dot{w}_4 - \frac{g}{2}(m_1 a_2 + m_3) + \left(-\frac{m_3}{4} - \frac{l_{31}}{b^2}\right) \ddot{w}_3 + \left(-\frac{m_3}{4} + \frac{l_{31}}{b^2}\right) \ddot{w}_4
\end{aligned} \quad (4.17)$$

In a similar manner

$$\begin{aligned}
 P_4 = & [k_r \quad 2\alpha^2 k_r \quad a_1 k_r] \begin{Bmatrix} d_{13} \\ bd_{14} \\ Ld_{15} \end{Bmatrix} + [c_r \quad 2\alpha^2 c_r \quad a_1 c_r] \begin{Bmatrix} \dot{d}_{13} \\ b\dot{d}_{14} \\ L\dot{d}_{15} \end{Bmatrix} \\
 & + \left(-\frac{1}{2}k_r + 2\alpha^2 k_r\right) w_3 + \left(-\frac{1}{2}k_r - 2\alpha^2 k_r\right) w_4 + \left(-\frac{1}{2}c_r + 2\alpha^2 c_r\right) \dot{w}_3 \\
 & + \left(-\frac{1}{2}c_r - 2\alpha^2 c_r\right) \dot{w}_4 - \frac{g}{2}(m_1 a_2 + m_3) + \left(-\frac{m_3}{4} + \frac{I_{31}}{b^2}\right) \ddot{w}_3 + \left(-\frac{m_3}{4} - \frac{I_{31}}{b^2}\right) \ddot{w}_4
 \end{aligned} \quad (4.18)$$

The the x-coordinates of contact points are given by :

$$\begin{aligned}
 x_1 = x_2 &= v_0 t + \frac{1}{2} a_0 t^2 + a_2 L \\
 x_3 = x_4 &= v_0 t + \frac{1}{2} a_0 t^2 - a_1 L \\
 \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 &= v_0 + a_0 t \\
 \ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = \ddot{x}_4 &= a_0
 \end{aligned} \quad (4.19)$$

where v_0 is the velocity of the vehicle and a_0 is its acceleration. Now the vertical displacements of the contact points can be expressed in terms of the nodal degrees of freedom of the bridge as

$$w_i = N_i D \quad (4.20)$$

where N_i is an 1 by N matrix of the shape functions and D is the vector of the nodal d.o.f. of the bridge. In the *specialized shape function* row vector N_i , only terms which correspond to the d.o.f of the element in which wheel i is located are nonzero.

The time derivatives of w are given by

$$\dot{w} = \frac{\partial N}{\partial x} D \dot{x} + N \dot{D} \quad (4.21a)$$

$$\ddot{w} = \frac{\partial^2 N}{\partial x^2} D \dot{x}^2 + 2 \frac{\partial N}{\partial x} \dot{D} \dot{x} + \frac{\partial N}{\partial x} D \ddot{x} + N \ddot{D} \quad (4.21b)$$

On substituting for w , $\dot{w}$, and $\ddot{w}$ from Eqs. 4.20 and 4.21 in Eqs. 4.13, 4.14, 4.17, and 4.18 the contact forces can be expressed in terms of the d.o.f d and D and their derivatives. In a similar manner displacements w_i and their derivatives which enter Eqs. 4.9 through 4.18

can be expressed in terms of $\mathbf{D}$ and $\dot{\mathbf{D}}$. Substitution of Eqs. 4.21a and 4.21b in Eq. 4.13 gives

$$\begin{aligned}
 P_1 = & \left[\left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) (N_{1,xx} \dot{x}_1^2 + N_{1,x} \ddot{x}_1) + \left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) (N_{2,xx} \dot{x}_2^2 + N_{2,x} \ddot{x}_2) \right. \\
 & + \left(-\frac{1}{2} k_f - 2\alpha^2 k_f \right) N_1 + \left(-\frac{1}{2} k_f + 2\alpha^2 k_f \right) N_2 \\
 & + \left. \left(-\frac{1}{2} c_f - 2\alpha^2 c_f \right) N_{1,x} \dot{x}_1 + \left(-\frac{1}{2} c_f + 2\alpha^2 c_f \right) N_{2,x} \dot{x}_2 \right] \mathbf{D} \\
 & + \left[\left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) 2 N_{1,x} \dot{x}_1 + \left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) 2 N_{2,x} \dot{x}_2 \right. \\
 & + \left. \left(-\frac{1}{2} c_f - 2\alpha^2 c_f \right) N_1 + \left(-\frac{1}{2} c_f + 2\alpha^2 c_f \right) N_2 \right] \dot{\mathbf{D}} \\
 & + \left[\left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) N_1 + \left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) N_2 \right] \ddot{\mathbf{D}} \\
 & + [k_f \quad -2\alpha^2 k_f \quad -a_2 k_f] \mathbf{d} + [c_f \quad -2\alpha^2 c_f \quad -a_2 c_f] \dot{\mathbf{d}} \\
 & - \frac{g}{2} (m_2 + a_1 m_1)
 \end{aligned}$$

or

$$P_1 = \mathbf{f}_1 \mathbf{d} + \mathbf{g}_1 \dot{\mathbf{d}} + \mathbf{Z}_1 \mathbf{D} + \mathbf{G}_1 \dot{\mathbf{D}} + \mathbf{H}_1 \ddot{\mathbf{D}} - \frac{g}{2} (m_2 + a_1 m_1) \quad (4.22)$$

where,

$$\begin{aligned}
 \mathbf{f}_1 &= [k_f \quad -2\alpha^2 k_f \quad -a_2 k_f], \quad \mathbf{g}_1 = [c_f \quad -2\alpha^2 c_f \quad -a_2 c_f] \\
 \mathbf{Z}_1 &= \left[\left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) (N_{1,xx} \dot{x}_1^2 + N_{1,x} \ddot{x}_1) + \left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) (N_{2,xx} \dot{x}_2^2 + N_{2,x} \ddot{x}_2) \right. \\
 &+ \left(-\frac{1}{2} k_f - 2\alpha^2 k_f \right) N_1 + \left(-\frac{1}{2} k_f + 2\alpha^2 k_f \right) N_2 \\
 &+ \left. \left(-\frac{1}{2} c_f - 2\alpha^2 c_f \right) N_{1,x} \dot{x}_1 + \left(-\frac{1}{2} c_f + 2\alpha^2 c_f \right) N_{2,x} \dot{x}_2 \right] \\
 \mathbf{G}_1 &= \left[\left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) 2 N_{1,x} \dot{x}_1 + \left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) 2 N_{2,x} \dot{x}_2 \right. \\
 &+ \left. \left(-\frac{1}{2} c_f - 2\alpha^2 c_f \right) N_1 + \left(-\frac{1}{2} c_f + 2\alpha^2 c_f \right) N_2 \right] \\
 \mathbf{H}_1 &= \left[\left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) N_1 + \left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) N_2 \right]
 \end{aligned}$$

Note that $\mathbf{f}_1$ and $\mathbf{g}_1$ are each of size 1×3 and $\mathbf{Z}_1$, $\mathbf{G}_1$, and $\mathbf{H}_1$ are each of size $1 \times N$. Contact force P_2 is obtained in a similar manner

$$P_2 = \mathbf{f}_2 \mathbf{d} + \mathbf{g}_2 \dot{\mathbf{d}} + \mathbf{Z}_2 \mathbf{D} + \mathbf{G}_2 \dot{\mathbf{D}} + \mathbf{H}_2 \ddot{\mathbf{D}} - \frac{g}{2} (m_2 + a_1 m_1) \quad (4.23)$$

where,

$$\begin{aligned}
 \mathbf{f}_2 &= [k_f \quad 2\alpha^2 k_f \quad -a_2 k_f], \quad \mathbf{g}_2 = [c_f \quad 2\alpha^2 c_f \quad -a_2 c_f] \\
 \mathbf{Z}_2 &= \left[\left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) (\mathbf{N}_{1,ss} \dot{x}_1^2 + \mathbf{N}_{1,s} \ddot{x}_1) + \left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) (\mathbf{N}_{2,ss} \dot{x}_2^2 + \mathbf{N}_{2,s} \ddot{x}_2) \right. \\
 &\quad + \left(-\frac{1}{2} k_f + 2\alpha^2 k_f \right) \mathbf{N}_1 + \left(-\frac{1}{2} k_f - 2\alpha^2 k_f \right) \mathbf{N}_2 \\
 &\quad \left. + \left(-\frac{1}{2} c_f + 2\alpha^2 c_f \right) \mathbf{N}_{1,s} \dot{x}_1 + \left(-\frac{1}{2} c_f - 2\alpha^2 c_f \right) \mathbf{N}_{2,s} \dot{x}_2 \right] \\
 \mathbf{G}_2 &= \left[\left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) 2 \mathbf{N}_{1,s} \dot{x}_1 + \left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) 2 \mathbf{N}_{2,s} \dot{x}_2 \right. \\
 &\quad \left. + \left(-\frac{1}{2} c_f + 2\alpha^2 c_f \right) \mathbf{N}_1 + \left(-\frac{1}{2} c_f - 2\alpha^2 c_f \right) \mathbf{N}_2 \right] \\
 \mathbf{H}_2 &= \left[\left(-\frac{m_2}{4} + \frac{I_{21}}{b^2} \right) \mathbf{N}_1 + \left(-\frac{m_2}{4} - \frac{I_{21}}{b^2} \right) \mathbf{N}_2 \right]
 \end{aligned}$$

Similarly

$$P_3 = \mathbf{f}_3 \mathbf{d} + \mathbf{g}_3 \dot{\mathbf{d}} + \mathbf{Z}_3 \mathbf{D} + \mathbf{G}_3 \dot{\mathbf{D}} + \mathbf{H}_3 \ddot{\mathbf{D}} - \frac{g}{2}(m_3 + a_2 m_1) \quad (4.24)$$

where,

$$\begin{aligned}
 \mathbf{f}_3 &= [k_r \quad -2\alpha^2 k_r \quad a_1 k_r], \quad \mathbf{g}_3 = [c_r \quad -2\alpha^2 c_r \quad a_1 c_r] \\
 \mathbf{Z}_3 &= \left[\left(-\frac{m_3}{4} - \frac{I_{31}}{b^2} \right) (\mathbf{N}_{3,ss} \dot{x}_3^2 + \mathbf{N}_{3,s} \ddot{x}_3) + \left(-\frac{m_3}{4} + \frac{I_{31}}{b^2} \right) (\mathbf{N}_{4,ss} \dot{x}_4^2 + \mathbf{N}_{4,s} \ddot{x}_4) \right. \\
 &\quad + \left(-\frac{1}{2} k_r - 2\alpha^2 k_r \right) \mathbf{N}_3 + \left(-\frac{1}{2} k_r + 2\alpha^2 k_r \right) \mathbf{N}_4 \\
 &\quad \left. + \left(-\frac{1}{2} c_r - 2\alpha^2 c_r \right) \mathbf{N}_{3,s} \dot{x}_3 + \left(-\frac{1}{2} c_r + 2\alpha^2 c_r \right) \mathbf{N}_{4,s} \dot{x}_4 \right] \\
 \mathbf{G}_3 &= \left[\left(-\frac{m_3}{4} - \frac{I_{31}}{b^2} \right) 2 \mathbf{N}_{3,s} \dot{x}_3 + \left(-\frac{m_3}{4} + \frac{I_{31}}{b^2} \right) 2 \mathbf{N}_{4,s} \dot{x}_4 \right. \\
 &\quad \left. + \left(-\frac{1}{2} c_r - 2\alpha^2 c_r \right) \mathbf{N}_3 + \left(-\frac{1}{2} c_r + 2\alpha^2 c_r \right) \mathbf{N}_4 \right] \\
 \mathbf{H}_3 &= \left[\left(-\frac{m_3}{4} - \frac{I_{31}}{b^2} \right) \mathbf{N}_3 + \left(-\frac{m_3}{4} + \frac{I_{31}}{b^2} \right) \mathbf{N}_4 \right]
 \end{aligned}$$

and

$$P_4 = \mathbf{f}_4 \mathbf{d} + \mathbf{g}_4 \dot{\mathbf{d}} + \mathbf{Z}_4 \mathbf{D} + \mathbf{G}_4 \dot{\mathbf{D}} + \mathbf{H}_4 \ddot{\mathbf{D}} - \frac{g}{2}(m_3 + a_2 m_1) \quad (4.25)$$

where,

$$\begin{aligned}
 \mathbf{f}_4 &= [k_r \ 2\alpha^2 k_r \ a_1 k_r] , \quad \mathbf{g}_4 = [c_r \ 2\alpha^2 c_r \ a_1 c_r] \\
 \mathbf{Z}_4 &= \left[\left(-\frac{m_3}{4} + \frac{I_{31}}{b^2} \right) (\mathbf{N}_{3,xx} \dot{x}_3^2 + \mathbf{N}_{3,x} \ddot{x}_3) + \left(-\frac{m_3}{4} - \frac{I_{31}}{b^2} \right) (\mathbf{N}_{4,xx} \dot{x}_4^2 + \mathbf{N}_{4,x} \ddot{x}_4) \right. \\
 &\quad + \left(-\frac{1}{2} k_r + 2\alpha^2 k_r \right) \mathbf{N}_3 + \left(-\frac{1}{2} k_r - 2\alpha^2 k_r \right) \mathbf{N}_4 \\
 &\quad \left. + \left(-\frac{1}{2} c_r + 2\alpha^2 c_r \right) \mathbf{N}_{3,x} \dot{x}_3 + \left(-\frac{1}{2} c_r - 2\alpha^2 c_r \right) \mathbf{N}_{4,x} \dot{x}_4 \right] \\
 \mathbf{G}_4 &= \left[\left(-\frac{m_3}{4} + \frac{I_{31}}{b^2} \right) 2 \mathbf{N}_{3,x} \dot{x}_3 + \left(-\frac{m_3}{4} - \frac{I_{31}}{b^2} \right) 2 \mathbf{N}_{4,x} \dot{x}_4 \right. \\
 &\quad \left. + \left(-\frac{1}{2} c_r + 2\alpha^2 c_r \right) \mathbf{N}_3 + \left(-\frac{1}{2} c_r - 2\alpha^2 c_r \right) \mathbf{N}_4 \right] \\
 \mathbf{H}_4 &= \left[\left(-\frac{m_3}{4} + \frac{I_{31}}{b^2} \right) \mathbf{N}_3 + \left(-\frac{m_3}{4} - \frac{I_{31}}{b^2} \right) \mathbf{N}_4 \right]
 \end{aligned}$$

4.3.3 Equations of Motion of the Bridge

When the contact forces P_1 through P_4 are converted to equivalent nodal forces using the principle of virtual work, the equations of motion of the bridge are easily shown to be

$$\begin{aligned}
 \mathbf{M}\ddot{\mathbf{D}} + \mathbf{C}\dot{\mathbf{D}} + \mathbf{K}\mathbf{D} &= \mathbf{N}_1^T P_1 + \mathbf{N}_2^T P_2 + \mathbf{N}_3^T P_3 + \mathbf{N}_4^T P_4 \\
 &= \sum_{i=1}^4 \mathbf{N}_i^T P_i
 \end{aligned} \tag{4.26}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ the damping matrix, and $\mathbf{K}$ the stiffness matrix of the bridge structure. It should be noted that the P_i are themselves functions of $\mathbf{D}$ and $\mathbf{d}$.

4.3.4 Equations of Motion of The Bridge-Vehicle System

The coupling between the bridge and vehicle exists through the expressions for $\mathbf{q}$ in Eq. 4.9 which are dependent on $\mathbf{D}$ and $\dot{\mathbf{D}}$ and through P_i in Eq. 4.26 which are dependent on $\mathbf{d}$ and $\dot{\mathbf{d}}$. Substitution for P_1 through P_4 leads to the following

$$\begin{aligned}
 \mathbf{N}_i^T P_i &= \mathbf{N}_i^T \mathbf{f}_i \mathbf{d} + \mathbf{N}_i^T \mathbf{g}_i \dot{\mathbf{d}} + \mathbf{N}_i^T \mathbf{Z}_i \mathbf{D} + \mathbf{N}_i^T \mathbf{G}_i \dot{\mathbf{D}} + \mathbf{N}_i^T \mathbf{H}_i \ddot{\mathbf{D}} + \mathbf{N}_i^T Q_i \\
 &\quad \text{for } i = 1 \text{ to } 4
 \end{aligned}$$

where

$$\begin{aligned}
 Q_1 &= Q_2 = -\frac{g}{2}(m_2 + a_1 m_1) \\
 Q_3 &= Q_4 = -\frac{g}{2}(m_3 + a_2 m_1)
 \end{aligned}$$

Equation 4.26 gives

$$[0 \quad M + m^*] \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\mathbf{D}} \end{Bmatrix} + [c_v \quad C + c^*] \begin{Bmatrix} \dot{\mathbf{d}} \\ \dot{\mathbf{D}} \end{Bmatrix} + [k_v \quad K + k^*] \begin{Bmatrix} \mathbf{d} \\ \mathbf{D} \end{Bmatrix} = \{\mathbf{P}^*\} \quad (4.27)$$

where

$$\mathbf{m}^*_{(N \times N)} = - \sum_{i=1}^4 \mathbf{N}_i^T_{(N \times 1)} \mathbf{H}_{i(1 \times N)}$$

$$\mathbf{c}^*_{(N \times N)} = - \sum_{i=1}^4 \mathbf{N}_i^T_{(N \times 1)} \mathbf{G}_{i(1 \times N)}$$

$$\mathbf{k}^*_{(N \times N)} = - \sum_{i=1}^4 \mathbf{N}_i^T_{(N \times 1)} \mathbf{Z}_{i(1 \times N)}$$

$$\mathbf{c}_{v(N \times 3)} = - \sum_{i=1}^4 \mathbf{N}_i^T_{(N \times 1)} \mathbf{g}_{i(1 \times 3)}$$

$$\mathbf{k}_{v(N \times 3)} = - \sum_{i=1}^4 \mathbf{N}_i^T_{(N \times 1)} \mathbf{f}_{i(1 \times 3)}$$

$$\mathbf{P}^*_{(N \times 1)} = \sum_{i=1}^4 \mathbf{N}_i^T_{(N \times 1)} Q_i$$

It should be noted that $\mathbf{m}^*$, $\mathbf{c}^*$, $\mathbf{k}^*$ are very sparse matrices. By substituting Eqs. 4.20 and 4.21 in Eq. 4.9 we have

$$\begin{aligned} q_1 &= -k_f(w_1 + w_2) - k_r(w_3 + w_4) - c_f(\dot{w}_1 + \dot{w}_2) - c_r(\dot{w}_3 + \dot{w}_4) \\ &= -k_f(\mathbf{N}_1 + \mathbf{N}_2)\mathbf{D} - k_r(\mathbf{N}_3 + \mathbf{N}_4)\mathbf{D} \\ &\quad - c_f(\mathbf{N}_{1,x}\dot{x}_1 + \mathbf{N}_{2,x}\dot{x}_2)\mathbf{D} - c_f(\mathbf{N}_1 + \mathbf{N}_2)\dot{\mathbf{D}} \\ &\quad - c_r(\mathbf{N}_{3,x}\dot{x}_3 + \mathbf{N}_{4,x}\dot{x}_4)\mathbf{D} - c_r(\mathbf{N}_3 + \mathbf{N}_4)\dot{\mathbf{D}} \\ &= - \left[k_f(\mathbf{N}_1 + \mathbf{N}_2) + k_r(\mathbf{N}_3 + \mathbf{N}_4) + c_f(\mathbf{N}_{1,x}\dot{x}_1 + \mathbf{N}_{2,x}\dot{x}_2) + c_r(\mathbf{N}_{3,x}\dot{x}_3 + \mathbf{N}_{4,x}\dot{x}_4) \right] \mathbf{D} \\ &\quad - \left[c_f(\mathbf{N}_1 + \mathbf{N}_2) + c_r(\mathbf{N}_3 + \mathbf{N}_4) \right] \dot{\mathbf{D}} \end{aligned}$$

In a similar manner

$$\begin{aligned} q_2 &= 2\alpha^2 \left[k_f(\mathbf{N}_1 - \mathbf{N}_2) + k_r(\mathbf{N}_3 - \mathbf{N}_4) + c_f(\mathbf{N}_{1,x}\dot{x}_1 - \mathbf{N}_{2,x}\dot{x}_2) + c_r(\mathbf{N}_{3,x}\dot{x}_3 - \mathbf{N}_{4,x}\dot{x}_4) \right] \mathbf{D} \\ &\quad + 2\alpha^2 \left[c_f(\mathbf{N}_1 - \mathbf{N}_2) + c_r(\mathbf{N}_3 - \mathbf{N}_4) \right] \dot{\mathbf{D}} \end{aligned}$$

and

$$\begin{aligned} q_3 &= \left[a_2 k_f(\mathbf{N}_1 + \mathbf{N}_2) - a_1 k_r(\mathbf{N}_3 + \mathbf{N}_4) + a_2 c_f(\mathbf{N}_{1,x}\dot{x}_1 + \mathbf{N}_{2,x}\dot{x}_2) \right. \\ &\quad \left. - a_1 c_r(\mathbf{N}_{3,x}\dot{x}_3 + \mathbf{N}_{4,x}\dot{x}_4) \right] \mathbf{D} + \left[a_2 c_f(\mathbf{N}_1 + \mathbf{N}_2) - a_1 c_r(\mathbf{N}_3 + \mathbf{N}_4) \right] \dot{\mathbf{D}} \end{aligned}$$

We now define

$$\mathbf{k}_h(3 \times N) = \begin{bmatrix} -k_f(N_1 + N_2) - k_r(N_3 + N_4) \\ 2\alpha^2 k_f(N_1 - N_2) + 2\alpha^2 k_r(N_3 - N_4) \\ a_2 k_f(N_1 + N_2) - a_1 k_r(N_3 + N_4) \end{bmatrix} \quad (4.28)$$

$$+ \begin{bmatrix} -c_f(N_{1,x} \dot{x}_1 + N_{2,x} \dot{x}_2) - c_r(N_{3,x} \dot{x}_3 + N_{4,x} \dot{x}_4) \\ 2\alpha^2 c_f(N_{1,x} \dot{x}_1 - N_{2,x} \dot{x}_2) + 2\alpha^2 c_r(N_{3,x} \dot{x}_3 - N_{4,x} \dot{x}_4) \\ a_2 c_f(N_{1,x} \dot{x}_1 + N_{2,x} \dot{x}_2) - a_1 c_r(N_{3,x} \dot{x}_3 + N_{4,x} \dot{x}_4) \end{bmatrix}$$

$$\mathbf{c}_h(3 \times N) = \begin{bmatrix} -c_f(N_1 + N_2) - c_r(N_3 + N_4) \\ 2\alpha^2 c_f(N_1 - N_2) + 2\alpha^2 c_r(N_3 - N_4) \\ a_2 c_f(N_1 + N_2) - a_1 c_r(N_3 + N_4) \end{bmatrix} \quad (4.29)$$

Equation 4.9 can thus be written as

$$[\mathbf{m} \quad \mathbf{0}] \begin{Bmatrix} \dot{\mathbf{d}} \\ \dot{\mathbf{D}} \end{Bmatrix} + [\mathbf{c} \quad \mathbf{c}_h] \begin{Bmatrix} \dot{\mathbf{d}} \\ \dot{\mathbf{D}} \end{Bmatrix} + [\mathbf{k} \quad \mathbf{k}_h] \begin{Bmatrix} \mathbf{d} \\ \mathbf{D} \end{Bmatrix} = \{\mathbf{0}\} \quad (4.30)$$

Assembling of Eqs. 4.27 and 4.30 leads to the equations of motions for the bridge-vehicle system. These equations can be expressed in the form

$$\begin{bmatrix} \mathbf{m} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} + \mathbf{m}^* \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{d}} \\ \ddot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} \mathbf{c} & \mathbf{c}_h \\ \mathbf{c}_v & \mathbf{C} + \mathbf{c}^* \end{bmatrix} \begin{bmatrix} \dot{\mathbf{d}} \\ \dot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} \mathbf{k} & \mathbf{k}_h \\ \mathbf{k}_v & \mathbf{K} + \mathbf{k}^* \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{p}^* \end{bmatrix} \quad (4.31)$$

The matrices in Eq. 4.31 are of size $(N+3) \times (N+3)$ and are all unsymmetrical. Submatrices $\mathbf{m}$, $\mathbf{c}$, $\mathbf{k}$, $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$ are time invariant, however, $\mathbf{m}^*$, $\mathbf{c}^*$, $\mathbf{k}^*$, $\mathbf{c}_v$, $\mathbf{k}_v$, $\mathbf{k}_h$ and $\mathbf{c}_h$ vary with time because they contain the shape functions N_i and their derivatives which depend on the location of moving contact point i . Also, while submatrices $\mathbf{m}$, $\mathbf{c}$, $\mathbf{k}$, $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{C}$ are symmetric, $\mathbf{m}^*$, $\mathbf{c}^*$, $\mathbf{k}^*$ are not.

TABLE 4.1 FREE VIBRATIONS ANALYSIS OF RECTANGULAR PLATES

Boundary Conditions :
simple-simple-simple-simple edges

thickness $h = 0.125$ "
Young's modulus $E = 10.0E6$ psi
Poisson's ratio $\nu = 0.333$
Density $\rho = 9.75E-2$ lb mass/in

Mode Number	Frequency (Hz.)		Percentage Difference (%)	Mode Configuration
	Theoretical	Using HCT		
1	172.8	173.3	0.29	M = 1 N 1
2	332.2	334.2	0.60	2
3	531.6	535.7	0.77	1
4	598.0	603.8	0.97	3
5	691.0	699.3	1.20	2
6	970.1	973.3	0.33	4

Material Properties :Modulus of Elasticity $E = 1.0E5$ Poisson's Ratio $\nu = 0.30$ Thickness $= 1.00$ Mass Density $\rho = 0.10$ **FIGURE 4.1 CANTILEVER BEAM ANALYSIS****TABLE 4.2 CANTILEVER BEAM STATIC ANALYSIS RESULTS**

Element Type	BEAM#1				BEAM#2			
	Displacement at j		Stress at j		Displacement at j		Stress at j	
	Load A	Load B	Load A	Load B	Load A	Load B	Load A	Load B
THEORY	0.135	0.202	90.00	90.00	1.000	1.000	16650.0	11112.0
QUAD4	0.069	0.103	47.80	50.40	0.733	0.670	10278.0	8231.1
QUAD4SR	0.123	0.184	85.50	90.00	0.928	0.910	13875.0	11112.0
QUAD6	0.135	0.202	85.51	90.00	1.035	1.000	13937.0	11112.0
SHELLN	0.116	0.167	62.65	65.94	0.857	0.913	16654.0	11117.0
SHELLO	0.058	0.086	49.98	52.60	0.467	0.607	16654.0	11117.0
QUAD83	0.134	0.201	89.00	90.00	1.062	0.988	15475.0	11147.0
QUAD82	0.134	0.201	90.00	90.00	1.073	0.989	15929.5	11111.0

**TABLE 4.3 FREE VIBRATIONS ANALYSIS OF
CANTILEVER BEAM#1**

Mode Number	Frequency (rad/sec)				
	Theory	QUAD6	QUAD4SR	SHELLN	SHELLO
1	4.510 - LF1	4.478 - LF1	4.696 - LF1	4.922 - LF1	6.835 - LF1
2	28.27 - LF2	27.30 - LF2	28.69 - LF2	29.49 - LF2	41.00 - LF2
3	79.16 - LF3	73.86 - LF3	77.87 - LF3	77.89 - LF3	105.0 - AX1
4	155.1 - LF4	104.9 - AX1	104.9 - AX1	104.9 - AX1	108.4 - LF3
5	256.4 - LF5	138.7 - LF4	146.9 - LF4	142.0 - LF4	198.0 - LF4
6	383.1 - LF6	218.7 - LF5	232.6 - LF5	216.4 - LF5	302.3 - LF5
7	535.1 - LF7	310.3 - LF6	312.0 - AX2	296.1 - LF6	312.9 - AX2
8	711.9 - LF8	312.0 - AX2	331.7 - LF6	312.0 - AX2	414.5 - LF6
9	915.5 - LF9	410.3 - LF7	440.8 - LF7	376.5 - LF7	512.0 - AX3
10	1143. - LF1	511.1 - AX3	511.3 - AX3	452.7 - LF8	528.0 - LF7

**TABLE 4.4 FREE VIBRATIONS ANALYSIS OF
CANTILEVER BEAM#2**

Mode Number	Frequency (rad/sec)				
	Theory	QUAD6	QUAD4SR	SHELLN	SHELLO
1	112.8 - LF1	98.94 - LF1	105.6 - LF1	6.555 - AX1	123.8 - LF1
2	706.7 - LF2	422.7 - AX1	481.1 - AX1	6.581 - AX2	195.4 - AX1
3	1979. - LF3	520.9 - AX2	521.2 - AX2	6.606 - AX3	489.0 - LF2
4	3878. - LF4	883.1 - LF2	1036. - LF2	98.98 - LF1	533.5 - AX2
5	6411. - LF5	1192. - AX3	1261. - AX3	370.4 - LF2	542.6 - AX3
6	9576. - LF6	1216. - AX4	1411. - AX4	542.6 - AX4	728.8 - AX4

AX - Axial Mode

LF - Longitudinal Flexure

FIGURE 4.2 FLAT SHELL ELEMENT

(a)

(b)

FIGURE 4.3 DYNAMIC MODEL OF THE VEHICLE

FIGURE 4.4 FREE BODY DIAGRAM OF THE SPRUNG MASS

(a) FRONT AXLE

(b) REAR AXLE

FIGURE 4.5 FREE BODY DIAGRAMS OF THE UNSPRUNG MASSES

5.1 General

Natural frequencies and mode shapes are unique characteristics of a bridge model. These characteristics strongly influence the response of the bridge and their knowledge permits a better understanding of the bridge behaviour under moving loads. The calculation of fundamental frequency is important for studying the frequency ratio of the bridge and vehicle. On the other hand, the frequency of the highest significant mode must be known in order to determine the time increment to be used in numerical integration in order to obtain a reasonable accuracy. This chapter is devoted to the determination of the natural frequencies and mode shapes of the bridge models and to an examination of the effect of several parameters on these characteristics of the structure. In later chapters, the mode shapes and natural frequencies will be used as efficient tools in the analysis of the forced vibration response.

5.2 Natural Frequencies of the Bridge

In the previous chapter the equations of motion of the bridge-vehicle system were derived. If no external disturbance is present on the bridge and damping is neglected, Eq. 4.31 reduces to

$$M\ddot{D} + KD = 0 \quad (5.1)$$

The solution of Eq. 5.1 can be assumed as (Bathe and Wilson, 1976)

$$D = \Phi \sin(\omega t + \Theta) \quad (5.2)$$

where Θ is the phase angle and Φ is the *modal matrix of the structure*. Substituting Eq. 5.2 in Eq. 5.1

$$(\mathbf{K} - \omega^2 \mathbf{M})\Phi \sin(\omega t + \Theta) = 0 \quad (5.3)$$

which is true for all values of t and therefore can be written as

$$\mathbf{K}\Phi = \omega^2 \mathbf{M}\Phi \quad (5.4)$$

Equation 5.3 is commonly known as a *linearized eigenvalue problem* which, in general, gives N different eigenvalues ω . Each eigenvalue is associated with a particular shape or mode in which the ratios of various nodal displacements are fixed but not their magnitudes.

Many numerical schemes (Bathe and Wilson, 1976 and Humar, 1990) exist for the determination of the eigenvalues and eigenvectors. In this study two different solution techniques (Bathe et al, 1974) are used. The *determinant search technique* (Bathe and Wilson, 1973) is employed when the analysis encounters large systems with matrices of a small band width. The *subspace iteration technique* (Bathe and Wilson, 1973) is used when the system has large order and bandwidth. In the first technique the solution algorithm combines triangular factorization and vector inverse iteration in an optimum way to obtain the required eigenvalues and eigenvectors. The second technique applies the *Ritz method* (Bathe and Wilson, 1976 and Humar, 1990) repeatedly until convergence to the required number of eigenvalues and eigenvectors is obtained. The two techniques apply whether a lumped mass or a consistent mass matrix is used. In the present study the lumped mass matrix approach is used. This approach minimizes the required computer storage without any significant loss in the accuracy (Tong et al, 1971 and Hinton et al, 1976). The diagonal mass matrix used has diagonal zero elements corresponding to the rotational degrees of freedom. Hence a mass condensation procedure is performed on the mass matrix before the final solution. In the computer programs developed for this study, the algorithms and the source code for eigenvalue solutions using the *determinant search* and *subspace iteration* techniques were taken from program SAP IV (Bathe et al., 1974).

5.3 Parametric Study

Finite element analyses are carried out on the model shown in Fig. 5.1 to determine the effect of the following parameters on the frequencies and mode shapes of the bridge.

1. Aspect ratio (span/width ratio)
2. Number of diaphragms within boxes
3. Bracings between boxes

The aspect ratio was varied from 1 to 4 corresponding to spans ranging from 9.35 m to 37.40 m. Solid plate diaphragms of thickness 12mm are provided at both ends of the bridge within each box. A sample analysis on a model using no diaphragms indicated that the first mode was always a distortion mode of the boxes with a relatively low frequency. The end diaphragms are therefore introduced in each model that was analyzed. Several trial runs are carried out to determine the convergence of the results with refinement of mesh. Based on the results of these studies, 112 finite elements are used to model the bridge of aspect ratio 1, 284 elements are used for the bridge of aspect ratio 2, 336 elements are used for the bridge of aspect ratio 3, and 448 elements are used for the bridge of aspect ratio 4. In all cases the bridge was assumed to be supported by four pin supports located under the left hand end of the box girders as shown in Fig. 5.1a and four roller supports similarly located at the right hand end.

5.3.1 Effect of the Aspect Ratio

The first twenty frequencies of the four bridges with aspect ratio varying from 1 to 4 are shown in Table 5.1. Also included is a brief description of the type of mode shape.

The first and the second mode shapes for the bridge with aspect ratio 1 are shown in Fig. 5.2. The fourth mode shape for this bridge is shown in Fig. 5.3. The first, second, and seventh mode shapes for bridge with aspect ratio 3 are shown in Fig. 5.4. The sixteenth mode shape of bridge with aspect ratio 4 is presented in Fig. 5.5.

The results presented in Table 5.1 and Figs. 5.2 through 5.5 show that for a bridge with low aspect ratio, the lower modes are torsional or lateral flexural or a combination of the two. For an aspect ratio of 1, the flexural mode is fourth in order and even then it is coupled with transverse flexure. The second flexural mode occurs at order 11 and is again mixed with transverse flexure. The axial mode is nineteenth in order. It is mixed with transverse flexure and has a relatively high frequency.

As would be expected, an increase in the span of the bridge results in a significant decrease in the frequency when the same cross-sectional dimension is maintained. From

Table 5.1, it is seen that the frequency of the first flexural mode for a bridge of aspect ratio 3 is about half that for one with aspect ratio 2. The frequency of a bridge with an aspect ratio 4 is 60% of that with an aspect ratio 3. Also, for aspect ratio greater than 1 the fundamental mode changes to one of longitudinal flexure. Thus, for bridges with aspect ratios 2, 3, and 4 the first mode is always flexural, followed by a combined torsional and transverse mode in case of aspect ratio 2 and a pure torsional mode for the other two aspect ratios. Also, the flexural and torsional frequencies are fairly close. As observed from Table 5.1, no axial mode was captured in the first twenty modes for bridges with aspect ratios 2, 3, and 4.

5.3.2 Effect of Diaphragms

The effect of diaphragms within the boxes is studied for the following diaphragm locations. The bridge models studied have aspect ratios 1 and 3.

- a. Diaphragms at each end of each box girders
- b. Diaphragms at ends and at midspan
- c. Diaphragms at ends, at midspan, and at two quarter span points.

The results for diaphragms located at each ends are presented in Table 5.1. The first twenty frequencies for three and five diaphragms are shown in Table 5.2. Figures 5.6 and 5.7 show the effect of increasing the number of diaphragms on the dynamic response of bridges with different aspect ratios. The second mode shapes of bridge with aspect ratio 1 using three and five diaphragms are shown in Fig. 5.6a and 5.6b respectively. Figures 5.7a and 5.7b display the second mode shapes of bridge with aspect ratio 3 when using three and five diaphragms respectively.

The results presented in Table 5.2 show a significant improvement in the torsional resistance of the bridge as a result of the presence of diaphragms. For an aspect ratio of 1, the addition of diaphragms at the midspan results in a shift of the flexural mode from the fourth to the third position. The first mode remains as a combination of torsion and transverse bending with almost the same frequency. With five diaphragms the first flexural mode appears as the second mode and the second flexural as the tenth mode.

For an aspect ratio of 3 and with diaphragms located at the midspan and ends, the pure torsion mode moves from the second to the third position. With five diaphragms the

second flexural mode appears in the fourth position and a third flexural mode in the eighth. In all cases, the first mode shape, which is flexural, is not affected.

5.3.3 Effect of Bracing

The effect of bracing between the boxes is studied for the following brace locations

- a. A pair of braces at each support
- b. A pair of braces at each support and at the mid span
- c. A pair of braces at each support, at midspan, and at two quarter span points.

When bracings are placed as indicated above no, diaphragms are used anywhere within the boxes. Bridge models with two different aspect ratios 1 and 3 are analyzed to study the effect of braces. The frequency results are summarized in Tables 5.3 and 5.4 and some of the mode shapes of the two bridge models are shown in Figs. 5.8 through 5.10.

For an aspect ratio of 1 and with braces located only at the ends, the first natural frequency is 50.0% lower than that for the same bridge but with diaphragms instead of braces at the ends. The first mode is, in fact, one involving lateral distortion of the boxes. The second and third modes are quite similar to the second and third modes of the bridge with end diaphragms; also the corresponding frequencies are close to each other. The first axial mode occurs at the twelfth position with a frequency lower than that associated with the first axial mode for the bridge with end diaphragms.

For an aspect ratio of 3, the presence of bracings does not affect the first mode which is purely flexural. The frequencies of the subsequent modes are increased somewhat although not a great deal. It is also noticed that the provision of five braces suppresses the pure torsional mode and the torsional motion is always accompanied by transverse flexure.

5.4 Summary and Conclusions

A series of analytical studies have been carried out on 3 dimensional models of simple span box girder bridges to determine their natural frequencies and mode shapes. Some conclusions can be drawn as an outcome of this study.

1. Bridges of low aspect ratio have their lower modes as combined torsional and transverse flexural.

2. With an increase in the span, the fundamental mode becomes flexural. Such a mode is followed by a torsional mode.
3. Distortion mode of a very small frequency is likely to occur in simply supported box girder bridges if no diaphragms are provided at both ends of the bridge.

**TABLE 5.1. NATURAL FREQUENCIES OF BRIDGE MODELS
WITH DIFFERENT ASPECT RATIOS**

	Aspect ratio 1.0		Aspect ratio 2.0		Aspect ratio 3.0		Aspect ratio 4.0	
Mode No.	Frequency Hz.	Type of Mode	Frequency Hz.	Type of Mode	Frequency Hz.	Type of Mode	Frequency Hz.	Type of Mode
1	22.52	TO-TF	10.29	LF	5.075	LF	2.976	LF
2	23.85	TF	11.11	TO-TF	5.723	TO	3.495	TO
3	25.34	TF	11.77	TF	6.867	TF	5.129	TF
4	28.35	LF-TF	14.90	TF	10.19	TO-TF	7.093	TO-TF
5	29.25	TF	17.09	TO-TF	12.47	TO-TF	10.10	TO-TF
6	30.47	TF	23.57	TF	15.39	TF	10.17	LF
7	31.17	TF	23.64	TF	15.65	LF	10.37	TF
8	32.23	TF	25.06	LF-TF	16.10	TO-TF	10.65	TO-TF
9	38.55	TF	25.91	TF	17.92	TF	13.38	TF
10	38.72	TF	25.98	TF	18.37	TO-TF	15.14	TO-TF
11	42.27	LF-TF	28.72	TF	20.33	TF	15.42	LF
12	45.04	TF	29.84	TF	20.96	LF	15.57	TO-TF
13	46.43	TF	30.54	TF	23.19	TO-TF	17.97	TF
14	50.50	TF	30.86	TF	24.56	TF	18.33	TF
15	51.23	TF	31.34	TF	24.68	TO-TF	18.59	TO-TF
16	52.06	TF	31.67	TF	27.02	LF	21.36	LF
17	52.16	TF	32.62	LF-TF	28.20	TF	22.96	TF
18	55.90	TF	33.98	TF	28.56	TF	23.63	TF
19	59.13	AX-TF	35.36	TF	29.75	TF	23.71	TF
20	61.06	TF	38.37	TF	30.51	TF	26.34	TF

TO - Torsion TF - Transverse flexure LF - Longitudinal flexure

AX - Axial DS - Distortion

**TABLE 5.2. NATURAL FREQUENCIES OF BRIDGE MODELS
WITH VARYING NO. OF DIAPHRAGMS.**

Mode No.	Aspect ratio 1.0				Aspect ratio 3.0			
	Three Diaphragms		Five diaphragms		Three Diaphragms		Five Diaphragms	
	Frequency Hz.	Type of Mode	Frequency Hz.	Type of Mode	Frequency Hz.	Type of Mode	Frequency Hz.	Type of Mode
1	22.43	TO-TF	22.34	TO-TF	5.049	LF	5.020	LF
2	25.36	TF	25.52	LF-TF	7.818	TO-TF	8.089	TF-TO
3	28.27	LF-TF	27.97	TF	10.59	TO	10.68	TO
4	29.01	TF	29.40	TF	14.83	TF	15.50	LF
5	29.26	TF	29.83	TF	15.63	LF	16.77	TF
6	30.43	TF	30.59	TF	15.81	TF-TO	17.88	TO-TF
7	31.21	TF	31.32	TF	15.94	TF-TO	19.25	TO-TF
8	38.24	TF	38.31	TF	17.85	TF	20.77	LF
9	39.71	TF	39.73	TF	20.02	TF-TO	21.70	TO-TF
10	41.04	TF	41.43	LF-TF	20.92	LF	25.04	TF
11	41.97	LF-TF	43.01	TF	22.06	TF	26.34	TF
12	44.88	TF	45.09	TF	22.40	TF-TO	26.46	TO-TF
13	46.48	TF	48.35	TF	23.30	TO-TF	26.88	TO-TF
14	51.23	TF	51.23	TF	27.01	TF	28.06	TF
15	51.80	TF	52.23	TF	27.10	TF	28.67	TF
16	54.00	TF	53.99	TF	27.58	TF	29.14	TO-TF
17	55.88	TF	55.84	TF	28.19	TF	31.07	LF-TF
18	59.27	LF-TF	59.35	LF-TF	28.57	TF	31.85	TO-TF
19	62.05	TF	63.23	TF	30.16	TF	32.48	TF
20	63.57	TO-TF	63.66	TO-TF	31.21	TF	32.62	TO-TF

**TABLE 5.3. NATURAL FREQUENCIES OF BRIDGE MODELS
WITH BRACING BETWEEN BOXES - ASPECT RATIO = 1.0**

Two bracings			Three bracings		Five bracings	
Mode no.	Frequency Hz	Type of mode	Frequency Hz	Type of mode	Frequency Hz	Type of mode
1	10.96	DS-TF	11.96	TO-TF	12.89	TO-TF
2	22.93	TF	23.00	TF	23.06	TO-TF
3	22.97	TF	23.04	TF	23.09	TF
4	26.46	LF-TF	26.42	LF-TF	26.37	LF-TF
5	26.49	TF	28.50	TF	29.04	TO-TF
6	29.10	TF	29.11	TF	29.13	TF
7	29.92	TF	30.55	TF	31.56	TF
8	31.44	TF	31.58	TF	31.69	TF
9	33.57	TF	34.14	TO-TF	35.03	TO-TF
10	38.44	TF	38.46	TF	38.46	TF
11	38.45	TF	38.72	TF	38.84	TF
12	39.38	AX	39.35	AX	39.29	AX
13	44.64	TF	44.66	TF	44.73	TF
14	49.09	TO-TF	49.31	TO-TF	50.05	TO-TF
15	49.88	TO-TF	50.63	TF	51.08	TF
16	50.87	TF	51.05	TF	51.37	TO-TF
17	51.12	TF	51.28	TF	51.39	TF
18	51.91	TF	52.05	TF	52.30	TF
19	52.57	LF-TF	52.57	LF-TF	52.49	LF-TF
20	59.58	TO-TF	60.15	TF	60.15	TF

**TABLE 5.4. NATURAL FREQUENCIES OF BRIDGE MODELS
WITH BRACING BETWEEN BOXES - ASPECT RATIO = 3.0**

Two bracings			Three bracings		Five bracings	
Mode no.	Frequency Hz	Type of mode	Frequency Hz	Type of mode	Frequency Hz	Type of mode
1	5.060	LF	5.056	LF	5.052	LF
2	5.670	TO	8.262	TO-TF	9.027	TO-TF
3	6.864	TF	9.286	TF	10.41	TF
4	10.57	TF-TO	10.63	TF-TO	10.78	TF-TO
5	13.06	TF-TO	13.13	TF-TO	13.30	TF-TO
6	15.30	TF	15.37	TF	15.58	LF
7	15.63	LF	15.64	LF	16.64	TO-TF
8	15.86	TO-TF	15.97	TO-TF	17.08	TF
9	17.70	TF	20.07	TF	20.82	LF
10	18.27	TO-TF	20.82	LF	22.81	TO-TF
11	20.07	TO-TF	21.00	TF	23.06	TF
12	20.83	LF	22.44	TO-TF	24.50	TO-TF
13	24.48	TF	24.55	TF	25.66	TF
14	24.57	TF	24.93	TO-TF	25.84	TF-TO
15	25.82	TO-TF	25.83	TO-TF	28.02	TF
16	27.59	TF	27.60	TF	28.53	TO-TF
17	28.39	TO-TF	28.39	TO-TF	28.99	TF
18	28.84	TF	29.69	TF	30.44	TF
19	29.81	TF	30.07	TF	31.29	TF
20	30.66	TF	30.73	TF	31.76	TF

FIGURE 5.1 SINGLE SPAN BOX GIRDER BRIDGE

(a) FIRST MODE SHAPE

(b) SECOND MODE SHAPE

**FIGURE 5.2 BRIDGE WITH ASPECT RATIO $R = 1.0$
MIDDLE CROSS SECTIONS**

**FIGURE 5.3 BRIDGE WITH ASPECT RATIO $R = 1.0$
FOURTH MODE SHAPE - ISOMETRIC VIEW**

(a) FIRST MODE SHAPE - ISOMETRIC VIEW

FIGURE 5.4 BRIDGE WITH ASPECT RATIO $R = 3.0$

(b) SECOND MODE SHAPE

(c) SEVENTH MODE SHAPE - ISOMETRIC VIEW

FIGURE 5.4 BRIDGE WITH ASPECT RATIO $R=3.0$

FIGURE 5.5 BRIDGE WITH ASPECT RATIO $R=4.0$
SIXTEENTH MODE SHAPE - ISOMETRIC VIEW

**(a) SECOND MODE SHAPE - MIDDLE CROSS SECTION
THREE DIAPHRAGMS**

**(b) SECOND MODE SHAPE - ISOMETRIC VIEW
FIVE DIAPHRAGMS**

FIGURE 5.6 BRIDGE WITH ASPECT RATIO $R = 1.0$

(a) SECOND MODE SHAPE - THREE DIAPHRAGMS

(b) SECOND MODE SHAPE - FIVE DIAPHRAGMS

**FIGURE 5.7 BRIDGE WITH ASPECT RATIO $R = 3.0$
MIDDLE CROSS SECTIONS**

(a) FIRST MODE SHAPE - TWO BRACINGS

**(b) TWELVETH MODE SHAPE - ISOMETRIC VIEW
TWO BRACINGS**

FIGURE 5.8 BRIDGE WITH ASPECT RATIO $R = 1.0$

(a) SECOND MODE SHAPE - THREE BRACINGS

(b) SECOND MODE SHAPE - FIVE BRACINGS

**FIGURE 5.9 BRIDGE WITH ASPECT RATIO $R = 1.0$
MIDDLE CROSS SECTIONS**

(a) SECOND MODE SHAPE - TWO BRACINGS

(b) SECOND MODE SHAPE - THREE BRACINGS

(c) SECOND MODE SHAPE - FIVE BRACINGS

**FIGURE 5.10 BRIDGE WITH ASPECT RATIO $R \approx 3.0$
MIDDLE CROSS SECTIONS**

6.1 General

In this chapter a procedure for a three dimensional analysis of dynamic response of bridges to a moving vehicle is presented. The method is based on the application of finite element discretization of the bridge superstructure and is quite general in nature. The formulation of the equations for a 3D analysis of box girder bridges is presented in Chapter 4. However, the final equations of motion obtained are applicable for almost all types of bridge superstructures, including solid slabs, slab on girder and box girder. The nature of finite elements to be used in each case may differ. As mentioned in Chapter 4, the resulting equations of motion of the bridge-vehicle system are a set of nonlinear differential equations. The nonlinearity of these equation arises from the consideration of the vehicle mass. Thus, the stiffness, damping, and inertia properties of the bridge-vehicle system keep varying with time, hence the natural frequencies and mode shapes of the combined system keep changing. As a result, the *modal method* of solution using the modes of the combined system is not applicable in this case and the bridge dynamic response can not be expressed as a combination of such mode shapes. Standard literature on the subject (Newmark, 1959 and Oran and Veletsos, 1961) offers another approach to solving the resulting time dependent equations of motion of the bridge-vehicle system, that is by using numerical integration. The *step-integration methods* have been widely used in the analysis of vibration response of bridges under the influence of moving traffic because of the generality of the solution procedure. Out of these integration methods, the *Newmark's β -method* was chosen for the present study. The *average acceleration method*, which corresponds to $\beta = \frac{1}{4}$ in Newmark's integration scheme, provides an unconditionally stable scheme. In this scheme, the governing differential equations are transformed to a set of algebraic equations for the

unknown displacements at time t_{n+1} . The unknown displacements are solved by *Gaussian-elimination method*, and the accelerations and velocities are obtained from the calculated displacements. The stresses and reactions are also obtained from the displacements.

The coefficient matrices involved in Eq. 4.31 are neither banded nor symmetric. Hence, a general *Gaussian-elimination method* for non-symmetric and non-banded coefficient matrices must be used. This requirement has an adverse impact on the speed of the solution as well as on the storage requirements.

The above discussion emphasizes the special nature of the equations of motion of the bridge-vehicle system which imposes several restrictions on the solution procedure to be used for the dynamic problem. Numerous attempts have been made in the past to overcome the obstacles associated with the solution of problems involving large, sparse, but unsymmetric and nonlinear systems of equations. These attempts can be categorized into two main groups, *modified versions of Gauss-elimination methods* and the *iteration methods*. The variations of *Gauss-elimination methods* attempt to increase the efficiency of the computations and reduce the required storage. Under this category lies the *frontal solution method* first devised by B.M. Irons (1970) to solve linear equations with a significant reduction in the storage requirements. P. Hood (1976) later modified the method to handle unsymmetric matrices. The adoption of the *iterative method* in solving the linear equations serves the purpose of reducing the storage space, however, its effectiveness may be severely limited in case of large order structural problems. In these cases the equations are frequently ill-conditioned and hence the convergence rate of the iteration may be very slow. This poor rate of convergence will adversely affect both the computations time and the degree of accuracy of the solution. Tuff et al (1973) employed both concepts, the elimination and iteration to solve non-linear equations. In their investigation, they aimed to improve the convergence, especially in cases where ill conditioned equations were encountered.

To determine the suitability of an equation solver for the present study, two techniques of solving the equations of motion are investigated. The following sections contain the details of each approach. Depending on the performance of the two approaches, a decision is to be made as to which method should be adopted for the present study.

6.2 Proposed methods for the Solution of the Equations

Before presenting the details of each proposed approach, it will be useful to summarize the *average acceleration method* since it will be used in both of the two proposed methods.

6.3 Average Acceleration Method

In the *Newmark's direct integration scheme* the following expressions are used to relate the velocities and displacements at time $t + \Delta t$ to their historic values

$$\dot{D}_{t+\Delta t} = \dot{D}_t + [(1 - \epsilon)\ddot{D}_t + \epsilon\ddot{D}_{t+\Delta t}]\Delta t$$

$$D_{t+\Delta t} = D_t + \dot{D}_t\Delta t + [(\frac{1}{2} - \nu)\ddot{D}_t + \nu\ddot{D}_{t+\Delta t}]\Delta t^2$$

where ϵ and ν are parameters that are selected to obtain integration accuracy and stability. When $\epsilon = \frac{1}{2}$ and $\nu = \frac{1}{4}$ the above relations reduce to the *Constant Average Acceleration method* which is an unconditionally stable scheme. In this method, the nodal accelerations during a small interval of time are assumed to be constant and to be given by

$$\ddot{D}(\tau) = \frac{1}{2}(\ddot{D}_t + \ddot{D}_{t+\Delta t}) \quad (6.1a)$$

where $\ddot{D}(\tau)$ represents acceleration at the lapse of time τ measured from time t . Integration of Eq. 6.1a with respect to time once and a second time gives the nodal velocities and displacements respectively as

$$\dot{D}(\tau) = \dot{D}_t + \left(\frac{1}{2}\ddot{D}_t + \frac{1}{2}\ddot{D}_{t+\Delta t}\right)\tau \quad (6.1b)$$

$$D(\tau) = D_t + \dot{D}_t\Delta t + \left(\frac{1}{4}\ddot{D}_t + \frac{1}{4}\ddot{D}_{t+\Delta t}\right)\tau^2 \quad (6.1c)$$

Thus the assumption of constant accelerations during the time interval results in linear and quadratic variations of the velocities and the displacements respectively. Evaluating these latter expressions at the end of the interval ($\tau = \Delta t$) leads to the following equations for the increments of velocities and displacements :

$$\Delta\dot{D} = \left(\ddot{D}_t + \frac{1}{2}\Delta\ddot{D}\right)\Delta t \quad (6.2a)$$

$$\Delta D = \dot{D}_t\Delta t + \frac{1}{2}\ddot{D}_t\Delta t^2 + \frac{1}{4}\Delta\ddot{D}\Delta t^2 \quad (6.2b)$$

It is convenient to use the incremental displacements as the basic variable of the analysis; hence Eq. 6.2b is solved for the incremental accelerations, and this expression is substituted into Eq. 6.2a to obtain

$$\Delta \ddot{\mathbf{D}} = a_0 \Delta \mathbf{D} - a_2 \dot{\mathbf{D}}_t - a_3 \ddot{\mathbf{D}}_t \quad (6.3a)$$

$$\Delta \dot{\mathbf{D}} = a_1 \Delta \mathbf{D} - a_3 \dot{\mathbf{D}}_t \quad (6.3b)$$

where

$$\begin{aligned} \Delta \mathbf{D} &= \mathbf{D}_{t+\Delta t} - \mathbf{D}_t \\ a_0 &= \frac{4}{\Delta t^2}, \quad a_1 = \frac{2}{\Delta t}, \quad a_2 = \frac{4}{\Delta t}, \quad a_3 = 2 \end{aligned}$$

The equations of motion are now written in an incremental form giving

$$\mathbf{M} \Delta \ddot{\mathbf{D}} + \mathbf{C} \Delta \dot{\mathbf{D}} + \mathbf{K} \Delta \mathbf{D} = \Delta \mathbf{p} \quad (6.4)$$

Substitution of Eq. 6.3 in Eq. 6.4 gives

$$(a_0 \mathbf{M} + a_1 \mathbf{C} + \mathbf{K}) \Delta \mathbf{D} = \Delta \mathbf{P} + \mathbf{M}(a_2 \dot{\mathbf{D}}_t + a_3 \ddot{\mathbf{D}}_t) + a_3 \mathbf{C} \dot{\mathbf{D}}_t \quad (6.5)$$

Having obtained the incremental displacements from the equations of motion, the displacements, the velocities, and the accelerations at the end of this time increment can be calculated from Eqs. 6.3

$$\begin{aligned} \mathbf{D}_{t+\Delta t} &= \mathbf{D}_t + \Delta \mathbf{D} \\ \dot{\mathbf{D}}_{t+\Delta t} &= \dot{\mathbf{D}}_t + \Delta \dot{\mathbf{D}} \\ \ddot{\mathbf{D}}_{t+\Delta t} &= \ddot{\mathbf{D}}_t + \Delta \ddot{\mathbf{D}} \end{aligned} \quad (6.6)$$

When Eqs. 6.6 are evaluated, the analysis for this time increment is completed, and the entire process may be repeated for the next time interval for which parameter values at $t + \Delta t$, as given by Eqs. 6.6, will represent the initial conditions.

6.3.1 Incremental Equations of Equilibrium

The equations of motion 4.31 derived in Chapter 4 are valid at any time t . For non-linear analysis these equations should be applied in an incremental form so that during the incremental time step, which is small, the stiffness, damping, and mass coefficient matrices can be assumed to be constant. In doing so Eq. 4.31 will be rewritten in the form;

$$\begin{bmatrix} \mathbf{m} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} + \mathbf{m}^* \end{bmatrix} \begin{bmatrix} \Delta \ddot{\mathbf{d}} \\ \Delta \ddot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} \mathbf{c} & \mathbf{c}_h \\ \mathbf{c}_v & \mathbf{C} + \mathbf{c}^* \end{bmatrix} \begin{bmatrix} \Delta \dot{\mathbf{d}} \\ \Delta \dot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} \mathbf{k} & \mathbf{k}_h \\ \mathbf{k}_v & \mathbf{K} + \mathbf{k}^* \end{bmatrix} \begin{bmatrix} \Delta \mathbf{d} \\ \Delta \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \Delta \mathbf{p}^* \end{bmatrix} \quad (6.7)$$

Decomposition of Eq. 6.7 gives

$$m\Delta\ddot{d} + c\Delta\dot{d} + k\Delta d + c_h\Delta\dot{D} + k_h\Delta D = 0 \quad (6.8)$$

and,

$$(M + m^*)\Delta\ddot{D} + (C + c^*)\Delta\dot{D} + (K + k^*)\Delta D + c_v\Delta\dot{d} + k_v\Delta d = \Delta p^* \quad (6.9)$$

Equations 6.8 and 6.9 represent the general set of equations for the moving mass problem of the bridge-vehicle system. The limiting case, the moving force problem, can be easily obtained by suppressing the moving mass d.o.f's and eliminating the dynamic properties of the vehicle. At each time increment the matrices m^* , c^* , k^* , c_v , k_v , k_h , and c_h are to be calculated with reference to the vehicle position on the bridge. As mentioned before, these matrices are very sparse but unsymmetrical. Thus, while writing the **FORTRAN** code of the computer program, advantage can be taken of storing only the non-zero terms associated with their locations in the global mass, damping, and stiffness matrices of the bridge-vehicle system. This will help in reducing the memory space required, but the difficulty of solving N equations at each time step still represents a severe restriction on the analysis. This restriction will affect the finite element discretization procedure as well as the time step, and thus affect the accuracy of the solution obtained.

6.4 Iterative Method

The first approach proposed here combines the concepts of the *step-integration* and the *iteration* techniques in such a way that both the banded and symmetrical nature of the coefficient matrices in Eqs. 6.8 and 6.9 are maintained. As a result, the storage requirements are reduced significantly. The strategy of this method is based upon treating the two Eqs. 6.8 and 6.9 as separate equations. Equation 6.8 represents the equations of motion of the vehicle and contains the vehicle displacements Δd , velocities $\Delta\dot{d}$, and accelerations $\Delta\ddot{d}$ as unknowns. The bridge parameters ΔD , $\Delta\dot{D}$, and $\Delta\ddot{D}$ in this equation are considered to be known external variables. Equation 6.9 represents the equations of motion of the bridge and expresses the unknown bridge variables ΔD , $\Delta\dot{D}$, and $\Delta\ddot{D}$ in terms of the vehicle parameters which are treated as known values. Substituting from Eqs. 6.3, Eqs. 6.8 and 6.9 can be rewritten in the forms;

$$m(a_0\Delta d - a_2\dot{d}_t - a_3\ddot{d}_t) + c(a_1\Delta d - a_3\dot{d}_t) + k\Delta d + c_h\Delta\dot{D} + k_h\Delta D = 0 \quad (6.10)$$

and,

$$(M + m^*)(a_0\Delta D - a_2\dot{D}_t - a_3\ddot{D}_t) + (C + c^*)(a_1\Delta D - a_3\dot{D}_t) + (K + k^*)\Delta D + c_v\Delta\dot{d} + k_v\Delta d = \Delta P^* \quad (6.11)$$

Collecting the known terms in Eqs. 6.10 and 6.11 and placing them on the right hand side results in

$$(a_0m + a_1c + k)\Delta d^{i+1} = m(a_2\dot{d}_t + a_3\ddot{d}_t) + ca_3\dot{d}_t + c_h a_3\dot{D}_t - (c_h a_1 + k_h)\Delta D^i \quad (6.12)$$

where superscript i denotes value at the end of the i^{th} iteration, and

$$(a_0M + a_1C + K)\Delta D^{i+1} = M(a_2\dot{D}_t + a_3\ddot{D}_t) + Ca_3\dot{D}_t + m^*(a_2\dot{D}_t + a_3\ddot{D}_t) + c^*a_3\dot{D}_t + \Delta P^* + a_3c_v\dot{d}_t - (a_1c_v + k_v)\Delta d^{i+1} - (a_0m^* + a_1c^* + k^*)\Delta D^i \quad (6.13)$$

In Eq. 6.11 it is found useful to transfer the matrices m^* , c^* , k^* to the right hand side along with the other known values. The reason for this procedure is to maintain both the banded and symmetrical nature of the left hand side during the solution of Eqs. 6.13. Also by having these matrices on the right side, only multiplication operations needs to be carried out rather than inversion of sparse matrices. At each time step a number of iterations are carried out on the two Eqs 6.12 and 6.13 until convergence to a desired accuracy is attained. The solution procedure can be summarized as follows;

1. During any time step the values of all the terms are treated as constant except for response parameters of the vehicle and the bridge, namely the incremental displacements, incremental velocities, and incremental accelerations.
2. During the first time step no iterations are required since the matrices m^* , c^* , k^* , k_h , k_v , c_h , and c_v are all null at $t = 0$ and the front wheels of the vehicle are at the edge of the bridge. As a result, only the equations of motion of the bridge, Eq. 6.13, are to be solved at this time step, the initial conditions of the bridge being taken as zero. Having solved the equations of motion, the bridge nodal displacements D are determined and consequently the velocities and accelerations. These values serve as the initial conditions for the next time step.
3. At the new time step the equations of motion of the vehicle are to be solved first assuming that all the right hand side terms are known. At the beginning of the iteration, the values of ΔD^i can be set as zeros. Thus the values of Δd^{i+1} will be obtained.

4. The equations of motion of the bridge are solved next to determine the values of $\Delta \mathbf{D}^{i+1}$. In doing this, the values of $\Delta \mathbf{d}^{i+1}$ are taken as those obtained from step 3. The $\Delta \mathbf{D}^{i+1}$ values, obtained as above, are considered as $\Delta \mathbf{D}^i$ vector for the next iteration in order to calculate the new values of $\Delta \mathbf{d}^{i+1}$.
5. Steps 3 and 4 are repeated until convergence to a specified accuracy is achieved for both $\Delta \mathbf{d}$ and $\Delta \mathbf{D}$.
6. Steps 3 through 5 are to be repeated at each new time step.
7. After the vehicle has traversed the bridge, the bridge vibrates freely and no iteration is required to determine the response.

Comments on the iterative method:

In practice, it is found that the method proposed above is not successful. Matrix $\mathbf{m}^*$ is a very sparse but unsymmetric matrix of size $N \times N$. The bridge mass matrix $\mathbf{M}$ is a diagonal matrix — with zeros on the diagonal corresponding to rotary degree of freedoms. To obtain accurate results, the time step should be of relatively small value. Thus the values of the integration constants, a_0 , a_1 , and a_2 , especially a_0 become relatively large. Although these constants appear on both sides of Eq. 6.13, yet their effect on the matrix $\mathbf{m}^*$, which contains more terms on and off the diagonal than the matrix $\mathbf{M}$, is more pronounced. This causes instability in the calculated displacement values which in turn increase unboundedly.

6.5 Combined Modal Method

This approach combines two concepts, the *modal method* (Humar 1990) and the *step-integration*. In this method the free vibrations analysis is first performed for both the bridge structure and the vehicle system in order to calculate their characteristics namely, the natural frequencies and the mode shapes. Both the bridge and vehicle nodal displacements can be expressed in terms of modal coordinates by means of a transformation.

$$\mathbf{D} = \Phi_b \mathbf{Y}_b \quad \text{and} \quad \mathbf{d} = \Phi_v \mathbf{Y}_v$$

or

$$\Delta \mathbf{D} = \Phi_b \Delta \mathbf{Y}_b \quad \text{and} \quad \Delta \mathbf{d} = \Phi_v \Delta \mathbf{Y}_v$$

where Φ_b is an $N \times N$ matrix, referred to as the *modal matrix* of the bridge system. This matrix comprises the N *mass-orthonormal* mode shapes ϕ_b placed side by side in the

following form

$$\Phi = [\phi_1 \ \phi_2 \ \dots \ \phi_N]$$

matrix Φ_v is the *modal matrix* of the vehicle and is of size 3×3 . The orthogonality property of the mode shapes gives

$$\Phi_b^T M \Phi_b = I \quad \text{and} \quad \Phi_v^T m \Phi_v = I$$

$$\Phi_b^T K \Phi_b = \Lambda_b \quad \text{and} \quad \Phi_v^T k \Phi_v = \Lambda_v$$

$$\Phi_b^T C \Phi_b = \Psi_b \quad \text{and} \quad \Phi_v^T c \Phi_v = \Psi_v$$

where Λ_b and Λ_v are the *spectral matrices* of the bridge and the vehicle respectively. They are diagonal matrices of the eigenvalues λ . Also Ψ_b and Ψ_v are the transformed damping matrices of the bridge and vehicle respectively. The i^{th} term in these matrices is $2\zeta_i\omega_i$ where ζ_i and ω_i are the i^{th} mode damping ratio and frequency respectively.

The application of normal coordinate transformation to Eqs. 6.8 and 6.13, yields

$$\hat{m}\Delta\ddot{Y}_v + \hat{c}\Delta\dot{Y}_v + \hat{k}\Delta Y_v + \hat{c}_b\Delta\dot{Y}_b + \hat{k}_b\Delta Y_b = 0 \quad (6.14)$$

and,

$$\hat{M}\Delta\ddot{Y}_b + \hat{C}\Delta\dot{Y}_b + \hat{K}\Delta Y_b + \hat{c}_v\Delta\dot{Y}_v + \hat{k}_v\Delta Y_v = \hat{P} \quad (6.15)$$

where

$$\hat{m}_{3 \times 3} = I_{3 \times 3}$$

$$\hat{c}_{3 \times 3} = \Psi_{v3 \times 3}$$

$$\hat{k}_{3 \times 3} = \Lambda_{v3 \times 3}$$

$$\hat{M}_{J \times J} = \Phi_b^T J \times N \ m^*_{N \times N} \ \Phi_{bN \times J} + I_{J \times J}$$

$$\hat{C}_{J \times J} = \Phi_b^T J \times N \ c^*_{N \times N} \ \Phi_{bN \times J} + \Psi_{bJ \times J}$$

$$\hat{K}_{J \times J} = \Phi_b^T J \times N \ k^*_{N \times N} \ \Phi_{bN \times J} + \Lambda_{bJ \times J}$$

$$\hat{c}_{b3 \times J} = \Phi_v^T 3 \times 3 \ c_{b3 \times N} \ \Phi_{bN \times J}$$

$$\hat{c}_{vJ \times 3} = \Phi_b^T J \times N \ c_{vN \times 3} \ \Phi_{v3 \times 3}$$

$$\hat{k}_{b3 \times J} = \Phi_v^T 3 \times 3 \ k_{b3 \times N} \ \Phi_{bN \times J}$$

$$\hat{k}_{vJ \times 3} = \Phi_b^T J \times N \ k_{vN \times 3} \ \Phi_{v3 \times 3}$$

$$\hat{P}_{J \times 1} = \Phi_b^T J \times N \ \Delta P^*_{N \times 1}$$

and J is the number of the lowest mode shapes of the bridge included in the analysis. Equations 6.14 and 6.15 can be combined into one equation given by

$$\mathbf{M}_T \Delta \ddot{\mathbf{U}} + \mathbf{C}_T \Delta \dot{\mathbf{U}} + \mathbf{K}_T \Delta \mathbf{U} = \mathbf{P}_T \quad (6.16)$$

where

$$\mathbf{M}_T = \begin{bmatrix} \hat{\mathbf{m}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{M}} \end{bmatrix} \quad \mathbf{C}_T = \begin{bmatrix} \hat{\mathbf{c}} & \mathbf{c}_h \\ \mathbf{c}_v & \hat{\mathbf{C}} \end{bmatrix} \quad \mathbf{K}_T = \begin{bmatrix} \hat{\mathbf{k}} & \mathbf{k}_h \\ \mathbf{k}_v & \hat{\mathbf{K}} \end{bmatrix}$$

and

$$\Delta \mathbf{U} = \begin{bmatrix} \Delta \mathbf{Y}_v \\ \Delta \mathbf{Y}_b \end{bmatrix} \quad \mathbf{P}_T = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{P}} \end{bmatrix}$$

Thus the application of the *modal method* results in a reduced set of transformed Eqs. 6.16 that are coupled. They can be solved by a simultaneous direct integration of the whole set of reduced equations. Therefore, at any time step $t + 1$ the implementation of the *average acceleration method* gives (see Eq. 6.5)

$$(a_0 \mathbf{M}_T + a_1 \mathbf{C}_T + \mathbf{K}_T) \Delta \mathbf{U} = \mathbf{M}_T (a_2 \dot{\mathbf{U}}_t + a_3 \ddot{\mathbf{U}}_t) + a_3 \mathbf{C}_T \dot{\mathbf{U}}_t + \mathbf{P}_T \quad (6.17)$$

Equations 6.17 are solved for $\Delta \mathbf{U}$. The nodal displacements of the bridge-vehicle system are then obtained as follows

$$\begin{bmatrix} \Delta \mathbf{d} \\ \Delta \mathbf{D} \end{bmatrix} = \begin{bmatrix} \Phi_v & \mathbf{0} \\ \mathbf{0} & \Phi_b \end{bmatrix} \begin{bmatrix} \Delta \mathbf{Y}_v \\ \Delta \mathbf{Y}_b \end{bmatrix}$$

$$\mathbf{d}_{t+1} = \mathbf{d}_t + \Delta \mathbf{d} \quad \text{and} \quad \mathbf{D}_{t+1} = \mathbf{D}_t + \Delta \mathbf{D}$$

Also the initial conditions for the next time step are

$$\mathbf{Y}_{v,t+1} = \mathbf{Y}_{v,t} + \Delta \mathbf{Y}_v \quad \text{and} \quad \mathbf{Y}_{b,t+1} = \mathbf{Y}_{b,t} + \Delta \mathbf{Y}_b$$

Comments on the combined modal method

- At the beginning of each time step matrices $\mathbf{M}_T$, $\mathbf{C}_T$, and $\mathbf{K}_T$ have the same values as at the end of the previous time step. Therefore, at the first time step matrices $\mathbf{m}^*$, $\mathbf{c}^*$, $\mathbf{k}^*$, $\mathbf{k}_h$, $\mathbf{k}_v$, $\mathbf{c}_h$, and $\mathbf{c}_v$ are all null. Consequently

$$\Delta U_i = \frac{\hat{P}_i}{a_0 + 2a_1 \zeta_i \omega_i + \omega_i^2} \quad i = 3, 4, \dots, J + 3$$

- After the vehicle has traversed the bridge, the bridge will vibrate freely and the generalized displacements can be determined from

$$\Delta U_i = \frac{(a_2 + 2a_3\zeta_i\omega_i)\dot{U}_i + a_3\ddot{U}_i}{a_0 + 2a_1\zeta_i\omega_i + \omega_i^2} \quad i = 3, 4, \dots, J + 3$$

- The most expensive phase in this method is the evaluation of the eigenvalues and the eigenvectors. Therefore, in the program once the eigensolution has been obtained, the frequencies and mode shapes together with the element information are stored on permanent storage. At the beginning of any subsequent analysis, this information is retrieved first. This also makes it convenient for the analyst to study the convergence of the solution with a smaller time step or a lower number of mode shapes without affecting the cost of the analysis significantly.
- In this method, after performing the free vibrations analysis and before conducting the forced response, the bridge mode shapes should be carefully classified. In modeling the bridge, if the end diaphragms or braces are not accounted for, there will exist very low frequency modes related to the distortion of the cross-section. In practice, these modes do not exist because diaphragms or braces are invariably provided. Such modes should not therefore be included in the analysis, particularly because, even when not excited, they may affect the stability of the structure.

6.6 Numerical Examples

Computers programs have been developed for the two proposed methods mentioned above. The first approach namely, *the iterative method*, showed instability even with a fairly small time step. Therefore, this approach was rejected. In this section, several illustrative examples are pursued to prove the validity of the second approach, namely, the *combined modal method*. Response values are compared with the results of other investigators and also with exact solutions wherever possible.

6.6.1 Beam Problem

Figure 6.1 shows a simply supported beam discretized into four finite elements. The magnitude of the moving force is one pound and it travels with a constant speed equal to 4912 in./sec. The travel time is equal to the fundamental period of the beam. The first

phase of the analysis was to calculate the natural frequencies and mode shapes of the beam. It should be noted that the beam model in Fig. 6.1 can only predict the first three modes as it has three lumped masses. To determine whether these three modes are sufficient to obtain a reasonably accurate solution, the first seven modes using an eight element model were also obtained. They are plotted in Figs. 6.2 and 6.3. It was found that the first three modes gave a solution that was virtually identical to one obtained by using seven modes. The response curve is plotted in Fig. 6.4 in which the ordinate represents the central displacement divided by the static maximum deflection and the abscissa is the real time divided by the travel time. It is seen from Fig. 6.4 that the maximum dynamic deflection occurs at time equal to $0.66 T_1$ and has a value of 1.706 compared to 1.700 obtained by Yoshida (1971) using a consistent mass idealization, 1.680 obtained by Filho (1966) using a lumped mass discrete model, and 1.707 the exact solution (Eichmann 1953).

The same beam was tested for the moving mass problem using different mass ratios, that is, the ratio of the mass of vehicle to the mass of the beam. The values of the mass ratios used in the analysis were 0.30, 0.50, 1.0, 2.0, 4.72, and 10.0. The response curves corresponding to the different mass ratios together with one for the moving force case are shown in Fig. 6.5. The maximum dynamic deflections for the mass ratios are 1.932, 2.071, 2.386, 2.962, 6.665, and 13.743 respectively. For mass ratios of 0.50, 1.00, 2.00, and, 4.72, the maximum dynamic deflections obtained by Filho (1978) were 2.047, 2.493, 3.397, and 6.252. It is noted from Fig. 6.5 that the time at peak displacement increases as the mass ratio increases.

6.6.2 Plate Problem

The response analysis of square plate simply supported on all sides was carried out to demonstrate the accuracy of the *modal approach*. Figure 6.7 shows the characteristics and the finite element discretization of the plate. Taking advantage of the symmetry, only half of the plate was analyzed. Again the dynamic characteristics of the plate were obtained. The lowest eleven modes are presented in Table 6.1. The dynamic response was first obtained for a force of two pounds moving at the centerline of the plate. The velocity of the traveling force was taken as 4210 in./sec, therefore the travel time is equal to the fundamental period of the plate. The peak dynamic magnification factor, defined as the ratio of dynamic to maximum static deflection at the center of the plate, occurred at $0.64T_1$ and was 1.513 compared with 1.568 obtained by Yoshida (1971) using consistent mass

procedure and 1.510 obtained by Wilson and Tsirk (1967) using lumped mass approach. Figure 6.6 demonstrates the dynamic response curve of the plate. It can be noted from Figs. 6.4 and 6.7 that the linear dynamic responses of the beam and plate models subjected to moving forces are quite similar, only the magnitudes of the dynamic amplifications are less in case of the plate model.

The plate model shown in Fig. 6.6 was used again to obtain the dynamic response under a moving mass. The mass ratios were taken as 0.30, 1.0, 3.0, and 10.0 and the travel speed was set to be 4120.0 in./sec. In Fig. 6.8 the dynamic responses for these mass ratios are plotted. The peak dynamic amplifications were 2.086, 3.105, 8.404, and, 19.27 respectively. Figure 6.9 shows the dynamic response curves for a travel speed of 1029 in./sec for the same mass ratios as above. the maximum dynamic deflection are 1.077, 0.946, 1.492, and 2.871 respectively compared to 1.125, 1.111, 1.505, and, "extremely large" obtained by Yoshida (1971).

The problems solved above in Sections 6.7.1 and 6.7.2 establish the reliability of the *combined modal method* developed in this work, in obtaining the dynamic response of the structures subjected to a moving force or a moving vehicle.

TABLE 6.1. PLATE EXAMPLE, LOWEST ELEVEN MODE SHAPES

Mode Number	Frequency (Cycles/Sec.)	Period (Sec.)
1	0.1047E+04	0.9553E-03
2	0.2659E+04	0.3761E-03
3	0.5174E+04	0.1933E-03
4	0.5174E+04	0.1933E-03
5	0.6276E+04	0.1593E-03
6	0.7408E+04	0.1350E-03
7	0.2368E+05	0.4223E-04
8	0.3159E+05	0.3165E-04
9	0.3161E+05	0.3163E-04
10	0.3510E+05	0.2849E-04
11	0.3565E+05	0.2805E-04

Beam physical properties :

Beam cross-sectional dimension = $\frac{1}{4} \times \frac{1}{4}$

$E = 30E6$ psi. , $\mu = .001$ lb-sec/in.⁴

$A = .0625$ in.² , $I = 3.255E-4$ in.⁴

Fundamental Period: $T1 = 8.149E-4$ sec.

FIGURE 6.1 BEAM EXAMPLE

FIGURE 6.2 BEAM MODE SHAPES 1 TO 4 **FIGURE 6.3 BEAM MODE SHAPES 5 TO 7**

FIGURE 6.4 BEAM RESPONSE TO MOVING FORCE

FIGURE 6.5 BEAM RESPONSE TO MOVING MASS

Plate physical properties :

$E = 30E6$ psi. , $\mu = .001$ lb-sec/in.

$\nu = 0.30$, $t = 0.1$ in.

Fundamental Period: $T1 = 9.556E-4$ sec.

FIGURE 6.6 PLATE EXAMPLE

FIGURE 6.7 PLATE RESPONSE TO MOVING FORCE

**FIGURE 6.8 PLATE RESPONSE TO MOVING MASS
TRAVEL SPEED 4120 in./sec.**

**FIGURE 6.9 PLATE RESPONSE TO MOVING MASS
TRAVEL SPEED 1029 in./sec.**

FORCED VIBRATION ANALYSIS OF BOX-GIRDER BRIDGES

7.1 General

This chapter is devoted to a study of the forced vibration response of a 3D model of a box girder bridge traversed by a vehicle. The vehicle itself is represented by a sophisticated model which has three degrees of freedom corresponding to the bounce, pitch, and roll of the vehicle. The 3D idealization is required for an adequate representation of a box girder bridge. Such a model was used in Chapter 5 to study the mode shapes and frequencies of vibration of box girder bridges. A general finite element computer program is developed for the analysis of the response of 3D model. This program is capable of analyzing almost all types of bridge superstructures, including solid slab, slab on girder and box girder. The computer program can deal with any type of supports and can be used to analyze continuous bridges as well.

The vehicle model that will be used in the present chapter was described earlier in Chapter 4. The thin shell finite element used in the program was also described in Chapter 4. The equation solver developed in Chapter 6 is utilized for solving the nonlinear equations associated with the analysis of the complex bridge-vehicle system. The analysis is focussed on parametric study of the response of simple span box girder bridges. The study performed in the present chapter is an extension of the study presented in Chapter 5 and includes the effect of vehicle in the dynamic analysis of bridges. The control parameters governing the dynamic response of 1D and 2D models, as described in Chapters 2 and 3, are used again in defining the type of parameters to be considered here.

7.2 Comparison between the Responses of Different Models of a Bridge Traversed by a Sprung Mass

This section investigates the validity of using three different models, 1D, 2D, and 3D, to predict the response of a box girder bridge. A bridge with an aspect ratio of 2 is used in the investigation. Values of α , κ , and ϕ are kept the same for the three types of models. An isotropic plate idealization is used for the 2D model. Figure 7.1a displays a comparison of the responses of the three type of models to a sprung mass moving along the center line of the bridge for values of $\alpha = 0.30$ and $\kappa = 0.50$. Normalization is done by dividing the maximum dynamic deflection at mid-point of the bridge model by the maximum static deflection at the same point. The figure shows close comparison between beam and plate models – as was demonstrated earlier in Chapter 3, (Fig. 3.4). The response of the 3D model is quite different from those of 1D and 2D models. The results presented here show that even if the values of the parameters ϕ , κ , α , and aspect ratio are kept the same, a beam or an isotropic plate model is incapable of representing the behaviour of box girder bridges.

Comparison is also made for the case of a sprung mass moving along the center line of an offset lane. The response results obtained for $\alpha = 0.30$ and $\kappa = 0.50$ are plotted in Fig. 7.1b. The maximum dynamic deflections at four different locations across the middle cross section of the bridge model, shown in Fig. 7.1, are determined and then divided by the maximum static deflection at location 2 due to a load traveling along the center line of lane. The normalized values so calculated are then plotted against the parameter ϕ . Figure 7.1b shows large differences in the nature of behaviour as well as in the values of normalized deflection of the plate model and the 3D model. In general, the response of a 3D model is lower than that of a 2D model. Also, in case of the 2D model the torsional mode is excited by an off-center vehicle. Consequently the bridge cross section rotates as a rigid body so that location 1 experiences greater deflection than location 2. On the other hand, the behaviour of the 3D model is governed by a combination of transverse flexural and torsional modes. This is because of the fact that a box girder bridge has a greater torsional resistance. Again, an isotropic plate model proves to be inadequate for representing the behaviour of a box girder bridge.

7.3 Characterizing Parameters

In Chapters 2 and 3 we have presented the governing parameters for 1D and 2D models of a bridge respectively. Both analytical studies and numerical examples were used to show that the parameters identified governed the dynamic response of bridge models. In the case of a 3D model of a bridge, it is difficult to determine such governing parameters analytically. However, the validity of using the controlling parameters associated with an isotropic plate model, namely ϕ , κ , α , and aspect ratio, to characterize the response of a 3D model will be investigated further through different examples in the following paragraph.

A bridge model (Br1) with dimensions $9.35 \times 9.35\text{m}$, aspect ratio 1, and a cross section as shown in Fig 5.1 is analyzed for its response to a sprung mass moving along either the center line of the bridge (Fig.7.2a) or the center line of a lane (Fig.7.2b). The normalized deflections are plotted against the parameter ϕ in Fig. 7.2 for values of $\alpha = 0.30$ and $\kappa = 0.50$. Again the normalization is done in the same manner as in Fig. 7.1. Figure 7.2 displays also the normalized deflections of a bridge model (Br2) with dimensions $18.7 \times 18.7\text{m}$ having the same cross section dimensions as bridge model (Br1) except that the width of the two steel box girders is twice those in bridge model (Br1). The first mode shape of bridge model (Br2) is symmetric transverse flexural with a frequency of 6.91 Hz. as compared to a frequency of 22.52 Hz. for the first mode shape of bridge model (Br1) which is mainly torsion. In fact the lower thirteen mode shapes of bridge model (Br2) are all transverse flexural modes revealing the fact that bridge model (Br2) is weaker in the transverse directions than bridge model (Br1). The fourteenth mode of (Br2) is torsional mixed with transverse flexural with a frequency of 15.7 Hz. Thus the horizontal mapping of bridge (Br1) into bridge (Br2) results in changing of the free vibration behaviour of the bridge. Consequently its forced vibration response will also change.

For a central load (Fig. 7.2a), the normalized deflections at the mid-point for the two models are somewhat close. In this case the excited mode is predominantly longitudinal flexural mixed with a transverse flexural one and the response is not affected by the torsional rigidity of the bridge cross section. This will be evident from Figs. 7.3a and 7.3c which present the static and dynamic deflections of a cross section across the mid-span of the two bridge models. Values of $\alpha = 0.30$, $\kappa = 0.50$, and $\phi = 0.60$ are selected to draw Fig. 7.3. The

dynamic deflection curves in Fig. 7.3a and 7.3c are drawn at the time when the maximum dynamic deflection occurs at the mid-point of the bridge.

For an off-center load (Fig. 7.2b), the responses of the two bridge models are not as close as for the case of central load. In general, the values of dynamic amplifications of bridge model (Br1) are lower than those of bridge model (Br2). Figures 7.3b and 7.3d present the static and dynamic deflections of a cross section across the mid-span of the two bridge models. The dynamic deflection curves are drawn at the time of occurrence of the maximum dynamic deflection at a point on the center line of the lane on which the load is traveling. In this case the excited mode is mainly torsional for the bridge model (Br1) and transverse flexural for the bridge model (Br2). This explains the wide difference in the response of the two bridge models for an off-center load.

Another bridge model (Br3) with dimensions $9.35 \times 28.05\text{m}$, aspect ratio 3, and a cross section as shown in Fig 5.1 is analyzed for its response to a sprung mass moving along either the center line of the bridge (Fig.7.4a) or the center line of a lane (Fig.7.4b). The normalized deflections are plotted against the parameter ϕ in Fig. 7.4 for values of $\alpha = 0.30$ and $\kappa = 0.50$. Again the normalization is done in the same manner as in Fig. 7.1. Figure 7.4 also displays the normalized deflections of a bridge model (Br4) with dimensions $18.7 \times 56.1\text{m}$ which is a horizontal mapping of bridge model (Br3) obtained in the same manner as for the two bridge models (Br1) and (Br2). The first mode shape of bridge model (Br4) is longitudinal flexural with a frequency of 1.357 Hz. as compared to a frequency of 5.075 Hz. for the first mode shape of bridge model (Br3) which is also longitudinal flexural. The second mode shapes of the two bridge models (Br3) and (Br4) are torsional modes with frequencies 1.651 Hz. and 5.723 Hz. respectively. In this case the horizontal mapping does not significantly change the type of mode shapes of the two models and therefore one will expect that there will be similarity between the responses of the two bridge models.

Figures 7.4a and 7.4b present the responses of the two models for a central load and an off-center load respectively. In this case the responses of the two models are fairly close for both paths of the moving load. Figure 7.5 shows the static and dynamic deflections of a cross section across the mid-span of the two bridge models. Values of $\alpha = 0.30$, $\kappa = 0.50$, and $\phi = 0.60$ are selected to draw Fig. 7.5. The dynamic deflection curves in Fig. 7.5a and 7.5c are drawn at the time when the maximum dynamic deflection occurs at the mid-point of the bridge. In Figs. 7.5b and 7.5d, the dynamic deflection curves are drawn at the time

of maximum dynamic deflection at a point on the center line of the lane on which the load is traveling.

From the above discussion, it is evident that the governing parameters of box girder bridges are not simply ϕ , κ , α , and aspect ratio. The parameters in this case should include also the relative ratio of the flexural and torsional rigidities. In other words, two bridge models will have the same response if the parameters ϕ , κ , and α are the same and the dimensions of the bridges cross sections are such that the bridge models have the same mode shape types and the same values of natural frequencies to within a scaling factor. For bridge models in which the type of mode shapes, especially those which are excited by the moving loads, are not altered when the bridge dimensions are varied keeping the aspect ratio unchanged, the four parameters, namely ϕ , κ , α , and aspect ratio, can be considered as the governing parameters for all practical purposes. Such will be the case for bridges of larger aspect ratios where the longitudinal flexural mode dominates the response.

7.4 Parametric Study of the Bridge Response under a moving vehicle

Finite element analyses are performed on the model shown in Fig. 5.1 to determine the effect of the following parameters on the forced vibration response of the bridge.

Parameters related to the bridge model:

1. Aspect ratio (span/width ratio)
2. Number of diaphragms within boxes
3. Bracings between boxes

Parameters related to the vehicle model:

1. Spring stiffness of the vehicle
2. Path of travel of the vehicle on the bridge

Parameters related to the bridge-vehicle system:

1. The speed ratio, α
2. The mass ratio, κ
3. The frequency ratio, ϕ

The physical properties of the vehicle used in the parametric study are shown in Fig. 7.6 (Chaudhuri 1975). The physical characteristics of the three-axle vehicle model is shown in Fig. 7.6c. The characteristics of the two-axle model, obtained by neglecting the tractor contribution, (Fig. 7.6d) are obtained from those of the three-axle model by keeping almost the same total mass of the vehicle while maintaining both the bounce and pitch natural frequencies of the vehicle. The adequacy of the two-axle model to represent the actual three-axle model is demonstrated in appendix A through numerical examples.

7.4.1 Effect of the Aspect Ratio

The aspect ratio, defined earlier in Chapter 5, is varied from 1 to 4. The corresponding spans range from 9.35 m to 37.40 m. For each bridge model, two diaphragms located at each ends are used to avoid the distortion modes detected in Chapter 5 for box girder bridges without end diaphragms.

Figures 7.7 through 7.10 display the response of bridges having different aspect ratios. The response is plotted as a function of the frequency parameter, ϕ , defined here as the ratio between the bounce frequency of the vehicle and the fundamental frequency of the bridge. The figures are drawn for value of $\alpha = 0.30$ and two values of κ , namely 0.50, 1.50. The response values shown are normalized mid-point deflections produced by a vehicle moving along the center line of the bridge model. The normalization is done with reference to the static mid-point deflection of the same bridge model due to a vehicle stationed at the position which causes a maximum deflection at mid-point.

Figure 7.7 shows the results for a value of aspect ratio 1.0. In general, the response corresponding to a value of $\kappa = 1.5$ is greater than that corresponding to a value of 0.50. The relationship $DI_{max} - \phi$ shows a peak within a range of ϕ values studied, that is 0 to 1.2. For a value of $\kappa = 0.50$, the peak response occurs at a value of $\phi = 0.60$. When κ takes the value 1.50, the peak response takes place at $\phi = 0.50$.

For all of the remaining three aspect ratios, namely 2, 3 and 4, the relationship $DI_{max} - \phi$ (Figs 7.8 through 7.10) displays a trough within the value of ϕ studied. The location of the trough depends on the values of κ and aspect ratio. In general, the location of the trough moves towards higher values of ϕ as the value of aspect ratio increases. In most cases, the response corresponding to value of $\kappa = 1.50$ is larger than that corresponding to $\kappa = 0.50$. Also for a given aspect ratio, the trough corresponding to $\kappa = 0.50$ occurs at

a higher value of ϕ than that corresponding to $\kappa = 1.50$ with the value of DI_{max} at the trough always larger for $\kappa = 0.50$. The higher the aspect ratio, the greater is the difference between the values of DI_{max} at troughs corresponding to the two values of κ .

Figure 7.11 shows the effect of aspect ratio on bridge response for different values of ϕ . Figure 7.11a is drawn for values of $\alpha = 0.30$, $\kappa = 0.50$. It is clear from the figure that for all bridges that have a flexural mode as their fundamental mode shape, the maximum impact factors increase with an increase in the aspect ratio. This difference in response decreases as the value of ϕ increases. For aspect ratio 1, where the fundamental mode is torsional, the variation of response with ϕ is quite different from those exhibited by bridges having a flexural fundamental mode. Similar features are observed in Fig. 7.11b which is drawn for values of $\alpha = 0.30$, $\kappa = 1.50$. However, in this case for higher values of ϕ , the response of bridge models which exhibit a flexural first mode shape is different, the response for higher aspect ratios being lower than that for a smaller aspect ratio.

Figure 7.12 shows the static and dynamic deflections of a cross section across the mid-span of the two bridge models. Values of $\alpha = 0.30$, $\kappa = 0.50$, and $\phi = 0.60$ are selected to draw Fig. 7.12. In Figs. 7.12a and 7.12c, the maximum dynamic deflections observed at certain points across the middle cross section of the bridge are shown; such dynamic deflections may occur at different times (method 1). The curves in Fig. 7.12b and 7.12d show deflections observed at the time when the maximum dynamic deflection occurs at the mid-point of the bridge (method 2). For aspect ratio of 1, the excited mode is a combination of transverse flexural and longitudinal flexural and therefore there is large difference between Fig. 7.12a and 7.12b obtained by the two different methods. For higher aspect ratio of 3, the excited mode is mainly longitudinal flexural and therefore there is only a small difference between Fig. 7.12c and 7.12d.

7.4.2 Effect of Path of travel of the vehicle

In this section the response results of two bridge models with different aspect ratios, namely 1 and 3, traversed by a vehicle moving at the center line of a lane which is off-set from the center line of the bridge (Fig. 7.13b) are presented. Both bridge models have diaphragms located at each end (Fig. 7.14a) so that the distortion modes are suppressed. Maximum values of displacements are calculated on a cross-section at the mid-span of the bridge. Figure 7.15 shows the maximum normalized deflections at several locations on the

transverse section at mid-span plotted as a function of ϕ for $\kappa = 0.50$, $\alpha = 0.30$ and two values of aspect ratio, namely 1 and 3. The deflections are normalized with respect to the maximum static mid-span deflection at the center line of a lane for a vehicle moving along the center line of the same lane. For the bridge model with aspect ratio 1, the maximum dynamic amplification factors are experienced at the center line of the lane along which the vehicle is traveling. At locations farthest away from the vehicle, the amplification factors are the lowest. For the bridge model with aspect ratio 3, the largest deflections caused by an off-center vehicle are at the bridge edge closer to the running vehicle. Comparison between the two figures, Figs. 7.15a and 7.15b, indicates that amplification factors in case of a bridge model with aspect ratio 3 are generally lower than those of a bridge model with aspect ratio 1.

7.4.3 Effect of Diaphragms

As in Chapter 5, the effect of diaphragms within the boxes is studied for the following diaphragm locations (Fig. 7.14). The bridge model studied has an aspect ratio of 3.

- a. Diaphragms at each end of each box girder
- b. Diaphragms at ends and at midspan
- c. Diaphragms at ends, at midspan, and at two quarter span points.

The results for diaphragms located at each end are presented in Fig. 7.15b. Figures 7.16a and 7.16b show the dynamic response of two bridges each with an aspect ratio of 3, the first one having three diaphragms and the second one five diaphragms. The dynamic response is obtained for a vehicle moving along the center line of the lane and for values of $\kappa = 0.50$ and $\alpha = 0.30$. The deflections are normalized with respect to the maximum static mid-span deflection at the center line of a lane for a vehicle moving along the center line of the same lane.

When the results presented in Figs. 7.16a and 7.16b are compared with those presented in Fig. 7.15b, it is seen that the presence of diaphragms improves the torsional resistance of the bridge. For an aspect ratio of 3 and with diaphragms located at the midspan and ends, the edge closer to the vehicle exhibits lower amplification factors than those observed in the bridge with diaphragms only at the ends. The center lines of the bridge and of the lane display almost the same response as those in a bridge with diaphragms at each end of each

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OF/DE

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PM-1 3 1/2 x 4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

box girder. On the other hand, the edge farther-away from the vehicle experiences higher amplification factors than in the bridge with diaphragms only at the ends. The reason for this may be attributed to the fact that the bridge structure section with diaphragms at the mid-span behaves more rigidly. The increase in the number of diaphragms from 3 to 5 does not result in any significant improvement in the torsional resistance of the bridge.

7.4.4 Effect of Bracing

The effect bracings between the box s have on the forced vibrations response is studied for the following brace locations (Fig. 7.11)

- a. A pair of braces at each support
- b. A pair of braces at each support and at the mid span
- c. A pair of braces at each support, at midspan, and at two quarter span points.

When bracings are placed as indicated above, no diaphragms are used anywhere within the boxes. Bridge model with an aspect ratio of 3 is analyzed to study the effect of braces on its dynamic response for a vehicle moving along the center line of the lane. Values of $\kappa = 0.50$ and $\alpha = 0.30$ are chosen for this analysis. The deflections are normalized with respect to the maximum static mid-span deflection at the center line of a lane for a vehicle moving along the center line of the same lane.

The results obtained for two, three, and five pairs of braces are presented in Figs. 7.17a, 7.17b, and 7.17c respectively. Comparison between Figs. 7.15b and 7.17a, indicates that the provision of diaphragms with 12mm thickness at each end of each box girders and a pair of braces – X-shape of rectangular cross section of 50×100 mm – at each support produce approximately the same deflection response. The results presented in Fig. 7.17b show an improvement in the torsional resistance of the bridge when three pairs of bracings are used. Again, the increase of braces from three to five does not yield any further improvement as seen by comparing Figs. 7.17b and 7.17c.

7.5 Comparison between the Deflection and Amplification Factors and Moment Amplification Factors

Figures 7.18a and 7.18b show the normalized longitudinal moments, M_x , as a function of ϕ for bridge models with aspect ratios 1 and 3 respectively traversed by a vehicle moving

along the center line of the bridge. Values of $\kappa = 0.50$ and $\alpha = 0.30$ have been used in the analyses. Maximum values are calculated on a cross-section at the mid-span of the bridge. The normalization is done by dividing the maximum dynamic moment response at mid-element (location 3) by the static response at location 3. Figures 7.18a displays the response of a bridge with an aspect ratio of 1. In general, the moment amplification factors are higher than the deflection amplification factors for the range of ϕ used in the analysis, from 0.1 to 1.2. For bridge with an aspect ratio of 3 (Fig. 7.18b), the deflection amplification factors are higher than the moment amplification factors in the range of ϕ between 0.0 and 0.75. In the range of values of ϕ between 0.75 and 1.20, the moment amplification factors values are always higher than the values of the deflection amplification factors.

The results produced by a vehicle moving along the center line of a lane for a bridge with aspect ratio 1 and having diaphragms at the ends are presented in Fig. 7.19. Values of $\kappa = 0.50$ and $\alpha = 0.30$ have been used in the analyses. The normalization is done by dividing the maximum dynamic moment response at any location by the static response at location 2 under the path of the vehicle. Comparing Figs. 7.15a and 7.19, one can observe that the moment amplification factors (Figs. 7.19) are, in general, higher than deflection amplification factors (Figs. 7.15a).

Figures 7.20a through 7.20d display the normalized longitudinal moments, M_x , as a function of ϕ for a bridge with aspect ratio 3. The results produced by a vehicle moving along the center line of a lane for a bridge having diaphragms are presented in Figs. 7.20a and 7.20b, whereas those produced in a bridge with bracings are shown in Figs. 7.20c and 7.20d. Values of $\kappa = 0.50$ and $\alpha = 0.30$ have been used in the analyses. Maximum values are calculated on a cross-section at the mid-span of the bridge. The normalization is done by dividing the maximum dynamic moment response at any location by the static response at location 2 under the path of the vehicle.

Figure 7.20a is for a bridge with two diaphragms while Fig. 7.20b is for a bridge with three diaphragms. Comparison of the two figures shows that the normalized response at location 2, that is under the path of the load, is quite similar for the two bridges. On the other hand, the responses at the edge closer to the vehicle, location 1, and at the mid-point, location 3, are smaller for the bridge with three diaphragms than those for a bridge with two diaphragms. The farther edge, location 4, experiences somewhat higher responses in the case of bridge with three diaphragms than in the case of bridge with two diaphragms.

Figures 7.20c and 7.20d show the effect of introducing two and three pairs of braces respectively on the dynamic moment response at four different locations across the mid-span cross section for an off-center moving vehicle. At all locations, namely 1, 2, 3, and 4, the normalized response in the bridge with three pairs of braces is higher than that in the bridge with two pairs of braces.

On comparing the normalized moment responses for a bridge with diaphragms at each end of the bridge and a bridge with two pairs of braces, Figs. 7.20a and 7.20c respectively, one can observe the difference in the responses of the two types of bridges. The responses at locations 1, 3, and 4 is almost the same in either case. The responses at location 2 in a bridge with braces is smaller than those in a bridge with diaphragms. Similar comparison can be made between the normalized moment responses of bridges with three diaphragms and three pairs of braces, Figs. 7.20b and 7.20d respectively. In this case, the responses at all locations are higher in a bridge with braces than in a bridge with diaphragms.

On comparing Figs. 7.15b through 7.17a, which show the normalized deflections, with Figs. 7.20a through 7.20d, which show the normalized moments, one can observe the following for a bridge with aspect ratio 3: (1) the normalized deflections and moments at location 2 are almost the same for all values of ϕ ; (2) at all other locations and for all values of ϕ ; the normalized deflections are higher than the normalized moments; and (3) the normalized deflections and moments at location 3, the mid-point of the bridge, are always greater than those at location 4. The above comparison is made for bridges having two or three diaphragms. Similar comparison can be made for bridges with braces by examining Figs. 7.8a, and 7.8b drawn for deflection responses and Figs. 7.20c, and 7.20d drawn for moment responses. It is found that in this case the normalized deflections at all locations are, in general, larger than the normalized moments at the same locations.

7.6 Transverse Distributions

The results presented in the previous sections can be further explained by monitoring the distribution of deflections and longitudinal moments, M_x , across a transverse section of the bridge.

7.6.1 Transverse Distributions of Deflections

Figures 7.21a through 7.21d present the distribution of the maximum static and dy-

dynamic deflections across the middle cross section of a simply supported bridge model of aspect ratio 1 having diaphragms at each end and subjected to a moving vehicle at the center line of a lane. The response is drawn for values of $\kappa = 0.50$, $\alpha = 0.30$, and two values of ϕ , namely 0.10 and 1.20. The maximum static distribution is obtained by placing the vehicle at the center line of a lane. The dynamic values are obtained by moving the vehicle across the center line of the lane and recording the deflections along the middle cross section at all times. The values of deflections are scanned for the entire period during which the vehicle is traveling across the bridge and the maximum values are selected for all locations along the cross-section (method 1). These values are plotted in Figs 7.21a and 7.21b. As an alternative the value of deflections at various locations along the cross-section are obtained at the instant when the deflection is maximum at location 2 (method 2). These values are plotted in Figs. 7.21c and 7.21d. The passage of a vehicle along the center line of a lane excites both the transverse flexural and the torsional modes. Consequently, there is a large difference between the dynamic deflections obtained by method 1 (Figs. 7.21a and 7.21b) and those obtained by method 2 (Figs. 7.21c and 7.21d).

Comparison between the distributions of the static and dynamic deflections across the mid cross section, shown in Figs. 7.21a through 7.21d, reflects two important characteristics. The dynamic distribution is more localized than the static distribution for both values of ϕ . Also, since the torsional resistance of a bridge model with aspect ratio 1 is very large, the mode shape excited by an off-center vehicle is a combination of transverse flexural and torsion (Table 5.1).

Figures 7.22a and 7.22b display the distributions of deflections across the middle cross section of a bridge model of aspect ratio 3 with diaphragms at the end supports for a vehicle moving along the center line of a lane for values of $\alpha = 0.30$, $\kappa = 0.50$, and two values of $\phi = 0.10$ and 1.20. The dynamic values are the maximum observed and the maxima at different locations along the cross-section may not occur at the same time (method 1). Figures 7.22c and 7.22d show the dynamic deflections observed at the time when the maximum dynamic deflection occurs at location 2 (method 2). In this case the main excited mode is the torsional and therefore the dynamic values obtained by the two different methods are the almost the same. It is also interesting to note that when the vehicle is moving along the center line of an off-center lane and only two diaphragms are used, the torsional mode of vibration is readily excited and the bridge deflections, are dominated by such a mode.

Also, the dynamic distribution is quite similar to the static distribution. Upon increasing the number of diaphragms to three (Figs. 7.23a and 7.23b), the torsional resistance of bridge cross section increases and a combination of transverse flexural and torsional modes are excited (Table 5.2). Comparing the responses of the bridge models with two and three diaphragms, it is seen that the type of the excited mode is different for the two cases. Also, the dynamic and static deflections for a three diaphragm bridge are smaller than those for a two diaphragm bridge. The increase in the number of diaphragms from three to five does not change either the static or the dynamic response significantly (Figs. 7.24a and 7.24b).

The deflection responses of the bridge model with an aspect ratio of 3 are again investigated to study the effect of providing braces. Figures 7.25a through 7.27b present the results of this investigation. The figures are drawn for the same parameters as for bridge with diaphragms. Studying Figs. 7.25a and 7.25b, one can observe that the dominant mode shape for both the dynamic and static responses is the torsion mode. Figures 7.25a and 7.25b are quite similar to Figs. 7.20a and 7.20b. The type of deflection response is not much affected by using three or five pairs of braces (Figs. 7.26 and 7.27 respectively), that is, the responses are still dominated by the torsion mode. This observation indicates that the provision of braces does not produce as much torsional resistance as the provision of diaphragms. The previous statement is valid for the dimensions of braces and diaphragms used. To appreciate the difference of stiffness provided by each provision, namely X-shape braces and diaphragms, a simple analysis is carried out to obtain the flexibility of a cross section having the same dimensions as in Fig. 5.1. The analysis is carried out twice, once by using two diaphragms of thickness 12 mm and second by using X-shape braces with dimensions 50×100 mm. In each case a unit force is applied to the cross section as shown in Fig. 7.28. Figures 7.28a and 7.28b show that the use of diaphragms produces lateral stiffness which is as much as six times the stiffness produced by using X-shape braces at the middle of the mid-span cross section.

7.6.2 Transverse Distributions of Moments

The present section investigates the effect of diaphragms and braces on the transverse distribution of maximum dynamic and static moments across the middle cross section of the bridge deck with an aspect ratio of 3 subjected to a vehicle moving at the center line of a lane. The response is obtained for values of $\kappa = 0.50$, $\alpha = 0.30$, and two values of ϕ ,

namely 0.10 and 1.20. As before, the maximum static distribution is obtained by placing the vehicle at the center line of a lane, while the dynamic values are obtained by moving the vehicle across the center line of the lane and recording the moments along the middle cross section at all times. The maximum values of moments during the entire period in which the vehicle is traveling across the bridge are then selected for all locations along the cross-section.

Figures 7.29a and 7.29b are for the bridge model with diaphragms at ends. The figures reveal two important features of the moments responses. First, the dynamic distribution is in general similar to the static distribution. Second, the moment distributions, both dynamic and static, are more peaked than deflection distributions (Figs. 7.20c and 7.20d). Figures 7.30a and 7.30b display transverse distribution of maximum dynamic and static moments across the middle cross section of the bridge deck when extra diaphragms are used at the middle cross section of the bridge in addition to the diaphragms at the ends. The observations as made for bridge model with two diaphragms are also valid for a model with three diaphragms.

Figures 7.31a and 7.31b are for the bridge model with braces at ends. The distribution of moments across the middle cross section of the bridge deck in these figures are fairly similar to those in Figs. 7.29a and 7.29b. The effect of the transverse flexural mode comes in the picture when extra braces are used at the middle cross section of the bridge in addition to the braces at the ends (Figs. 7.32a and 7.32b).

7.7 Summary and Conclusions

A series of analytical studies are carried out on 3 dimensional models of simple span box girder bridges to determine their forced responses. The effect of several parameters on the forced vibration response of the bridge is studied. Damping in the bridge and the vehicle is ignored. The response is measured in terms of the normalized dynamic deflections as well moments. Some conclusions that can be drawn as a result of this study are as follows

1. A beam or an isotropic plate model is incapable of representing the behaviour of box girder bridges.
2. the governing parameters of box girder bridges are not simply ϕ , κ , α , and aspect ratio. The parameters in this case should include also the relative values of the flexural and

torsional rigidities. It is evident that the response is governed by the mode shapes. Hence for two bridges models to have identical normalized response, they should have identical mode shapes. The mode shapes, in general, depend in a complicated manner on the cross sectional dimensions of the bridge and the relative values of the flexural and torsional rigidities. It is difficult to identify simple characterizing parameters.

3. For bridge models in which the type of mode shapes, especially those which are excited by the moving loads, are not altered by changing bridge dimensions while keeping the aspect ratio unchanged, the four parameters, namely ϕ , κ , α , and aspect ratio, can be considered as the governing parameters for all practical purposes.
4. The relationship $DI_{max} - \phi$ shows a peak within the range of ϕ values between 0 and 1.2 for aspect ratio 1, where the fundamental mode is a torsional mode.
5. For all bridges that have a flexural mode as their fundamental mode shape, the relationship $DI_{max} - \phi$ displays a a trough within the range of ϕ values between 0 and 1.2. In general, the maximum amplification factors of such bridge models increase with an increase in the aspect ratio.
6. The amplification factors in case of a bridge model with aspect ratio 3 is generally lower than that of a bridge model with aspect ratio 1 when the bridge is subjected to a vehicle moving along the center line of a lane.
7. The increase in the number of diaphragms from three to five does not give much improvement in the torsional resistance of the bridge.
8. The increase in the number of braces along the span from three to five does not give much improvement in the torsional resistance of the bridge.
9. The forced response of bridges of low aspect ratio is dominated by combined torsional and transverse flexural.
10. For a bridge of high aspect ratio, the response is dominated by longitudinal flexural mode when the vehicle is moving along the center line and by a torsional mode or a combination of transverse flexural and torsional mode when the vehicle is moving along the center line.
11. In general, the normalized deflections are higher than the normalized moments for bridge models that have a flexural mode as their fundamental mode shape. For bridge

models where the fundamental mode is a torsional mode the normalized deflections are, in general, lower than the normalized moments.

12. The dynamic distribution of the moment along the middle cross section of a bridge is, in general, similar to the static distribution. Also, the moment distributions, both dynamic and static, are more peaked than deflection distributions

Fig. 7.1a Central Load

Fig. 7.1b Off-center Load

**FIGURE 7.1 NORMALIZED DEFLECTIONS OF DIFFERENT BRIDGE MODELS
SUBJECTED TO A SPRUNG MASS MOVING ALONG DIFFERENT PATHS
ASPECT RATIO = 2.0 & TWO DIAPHRAGMS
 $\alpha = 0.30$ & $\kappa = 0.50$**

Fig. 7.2a Central Load

Fig. 7.2b Off-center Load

FIGURE 7.2 NORMALIZED DEFLECTIONS OF BRIDGE MODELS WITH ASPECT RATIO = 1.0 SUBJECTED TO A SPRUNG MASS MOVING ALONG DIFFERENT PATHS, 2-DIAPHRAGMS ; $\alpha = 0.30$ & $\kappa = 0.50$

Fig. 7.3c Central load --Br2

Fig. 7.3d Off-center load --Br2

Fig. 7.3a Central load --Br1

Fig. 7.3 b Off-center load --Br1

FIGURE 7.3 MID-SPAN DEFLECTIONS OF BRIDGE MODELS WITH
ASPECT RATIO 1.0 --- TWO DIAPHRAGMS
 $\phi = 0.60$ & $\alpha = 0.30$ & $\kappa = 0.50$

Fig. 7.4a Central Load

Fig. 7.4b Off-centre Load

FIGURE 7.4 NORMALIZED DEFLECTIONS OF BRIDGE MODELS WITH ASPECT RATIO = 3.0 SUBJECTED TO A SPRUNG MASS MOVING ALONG DIFFERENT PATHS, 2-DIAPHRAGMS,

$$\alpha = 0.30 \text{ \& } \kappa = 0.50$$

FIGURE 7.5 MID-SPAN DEFLECTIONS OF BRIDGE MODELS WITH ASPECT RATIO 3.0 --- TWO DIAPHRAGMS
 $\phi = 0.60$ & $\alpha = 0.30$ & $\kappa = 0.50$

Physical characteristics of FWHA test vehicle
[three-axle tractor and semi-trailer loaded to simulate an
HS20-44 design loading (Chaudhuri, 1975)]

Gross vehicle weight = 77,300 lbs (343.81 kN)
 • Tractor sprung weight ratio = 0.08 (27.505 kN)
 • Tractor front axle weight ratio = 0.03 (10.314 kN)
 • Tractor sprung weight ratio = 0.80 (275.048 kN)
 • Tractor front axle weight ratio = 0.05 (17.191 kN)
 • Tractor rear axle weight ratio = 0.04 (13.752 kN)

• Wheel base of the tractor = 156.0 inches (3.96 m)
 • Wheel base of the trailer = 243.6 inches (6.19 m)
 • Width of the trailer = 72.0 inches (1.83 m)
 • Centeroidal distance parameters : $a_1 = 0.57$ & $a_2 = 0.43$

• Front axle stiffness K_f = 6500 lbs/in (1138.3 kN/m)
 • Rear axle stiffness K_r = 6500 lbs/in (1138.3 kN/m)
 • Viscous damping ratio : $C_r = C_f = 0.0$
 • Dynamic Indices : $i_{11}, i_{12}, i_{21}, i_{31} = 1.0$

Mass moments of inertia :

$I_{11} = 23.448 \text{ tone.m}^2$ & $I_{12} = 263.052 \text{ tone.m}^2$
 $I_{21} = 4.982 \text{ tone.m}^2$ & $I_{31} = 0.879 \text{ tone.m}^2$

(c) Three-axle model

FIGURE 7.6 IDEALIZATION OF THE VEHICLE

Gross vehicle weight	= 75,793 lbs (337.1 kN)
** Trailer sprung weight ratio	= 0.86 (295.85 kN)
** Trailer front axle weight ratio	= 0.08 (27.504 kN)
** Trailer rear axle weight ratio	= 0.04 (13.752 kN)

** Wheel base of the trailer	= 243.6 inches (6.19 m)
** Width of the trailer	= 72.0 inches (1.83 m)
** Centeroidal distance parameters	: a1 = 0.57 & a2 = 0.43

** Front axle stiffness Kf	= 6500 lbs/in (1138.3 kN/m)
** Rear axle stiffness Kr	= 6500 lbs/in (1138.3 kN/m)
** Viscous damping ratio	: Cr = Cf = 0.0
** Dynamic Indices	: I ₁₁ , I ₁₂ , I ₂₁ , I ₂₂ = 1.0

Mass moments of inertia :	
I ₁₁ = 23.448 tone.m ²	& I ₂₂ = 263.052 tone.m ²
I ₂₁ = 4.982 tone.m ²	& I ₁₂ = 0.879 tone.m ²

(d) Two-axle model

FIGURE 7.6 IDEALIZATION OF THE VEHICLE

**FIGURE 7.7 RESPONSE OF SIMPLE SPAN BOX GIRDER BRIDGES
SUBJECTED TO A VEHICLE MOVING ALONG THE CENTER LINE
OF THE BRIDGE**

ASPECT RATIO = 1.0 & $\alpha = 0.30$ & 2 DIAPHRAGMS

**FIGURE 7.8 RESPONSE OF SIMPLE SPAN BOX GIRDER BRIDGES
SUBJECTED TO A VEHICLE MOVING ALONG THE CENTER LINE
OF THE BRIDGE**

ASPECT RATIO = 2.0 & $\alpha = 0.30$ & 2 DIAPHRAGMS

**FIGURE 7.9 RESPONSE OF SIMPLE SPAN BOX GIRDER BRIDGES
SUBJECTED TO A VEHICLE MOVING ALONG THE CENTER LINE OF
THE BRIDGE
ASPECT RATIO = 3.0 & $\alpha = 0.30$ & 2 DIAPHRAGMS**

**FIGURE 7.10 RESPONSE OF SIMPLE SPAN BOX GIRDER BRIDGES
SUBJECTED TO A VEHICLE MOVING ALONG THE CENTER LINE OF
THE BRIDGE
ASPECT RATIO = 4.0 & $\alpha = 0.30$ & 2 DIAPHRAGMS**

Fig. 7.11a $\alpha = 0.30$ & $\kappa = 0.50$ Fig. 7.11b $\alpha = 0.30$ & $\kappa = 1.50$

FIGURE 7.11 RESPONSE OF SIMPLE SPAN BOX GIRDER BRIDGES WITH DIFFERENT ASPECT RATIOS SUBJECTED TO A CENTRAL VEHICLE

Fig. 7.12c Aspect ratio = 3.0 & method1

Fig. 7.12d Aspect ratio = 3.0 & method2

Fig. 7.12a Aspect ratio = 1.0 & method1

Fig. 7.12b Aspect ratio = 1.0 & method2

FIGURE 7.12 MID-SPAN DEFLECTIONS OF BRIDGE MODEL WITH DIFFERENT ASPECT RATIOS DUE TO CENTRAL LOAD -- TWO DIAPHRAGMS

 $\kappa = 0.50$ & $\alpha = 0.30$ & $\phi = 0.60$

(a) Position of the Truck in the center line of the bridge

(b) Position of the truck in center line of a lane

FIGURE 7. 13 TRANSVERSE POSITIONS OF THE VEHICLE ON THE BRIDGE

(a) Two diaphragms

(b) Three diaphragms

(c) Five diaphragms

FIGURE 7. 14 DIAPHRAGMS OR BRACES POSITIONS ACROSS THE SPAN OF THE BRIDGE

Fig. 7.15a Aspect ratio = 1.0 & Two diaphragms
 $\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.15b Aspect ratio = 3.0 & Two diaphragms
 $\kappa = 0.50$ & $\alpha = 0.30$

FIGURE 7.15 NORMALIZED MID-SPAN DEFLECTIONS FOR BRIDGE MODELS WITH DIFFERENT ASPECT RATIOS UNDER AN OFF-CENTER VEHICLE

Fig. 7.16a Three diaphragms & $\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.16b Five diaphragms & $\kappa = 0.50$ & $\alpha = 0.30$

FIGURE 7.16 NORMALIZED MID-SPAN DEFLECTIONS FOR BRIDGE MODEL WITH ASPECT RATIO = 3.0 UNDER AN OFF-CENTER VEHICLE

FIGURE 7.17 NORMALIZED MID-SPAN DEFLECTIONS FOR BRIDGE MODEL WITH ASPECT RATIO = 3.0 UNDER AN OFF-CENTER VEHICLE

Fig. 7.18a Aspect ratio = 1.0

Fig. 7.18b Aspect ratio = 3.0

FIGURE 7.18 NORMALIZED MID-SPAN DEFLECTIONS AND MOMENTS FOR BRIDGE MODELS WITH DIFFERENT ASPECT RATIOS UNDER A CENTRAL VEHICLE
 $\kappa = 0.50$ & $\alpha = 0.30$

**FIGURE 7.19 NORMALIZED MID-SPAN MOMENTS FOR
BRIDGE MODELS WITH ASPECT RATIO= 1.0 UNDER AN
OFF-CENTER VEHICLE
 $\kappa = 0.50$ & $\alpha = 0.30$**

Fig. 7.20a Two diaphragms

Fig. 7.20c Two braces

Fig. 7.20b Three diaphragms

Fig. 7.20d Three braces

FIGURE 7.20 NORMALIZED MID-SPAN MOMENTS FOR BRIDGE MODEL
WITH ASPECT RATIO = 3.0 UNDER AN OFF-CENTER VEHICLE
 $\kappa = 0.50$ & $\alpha = 0.30$

**FIGURE 7.21 MID-SPAN DEFLECTIONS OF BRIDGE
MODEL WITH ASPECT RATIO 1.0 - TWO DIAPHRAGMS**
 $\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.22a $\phi = 0.10$
METHOD 1

Fig. 7.22b $\phi = 1.20$
METHOD 1

Fig. 7.22c $\phi = 0.10$
METHOD 2

Fig. 7.22d $\phi = 1.20$
METHOD 2

FIGURE 7.22 MID-SPAN DEFLECTIONS OF BRIDGE MODEL WITH ASPECT RATIO 3.0 -- TWO DIAPHRAGMS

$\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.23a $\phi = 0.10$ Fig. 7.23b $\phi = 1.20$

FIGURE 7.23 MID-SPAN DEFLECTIONS OF BRIDGE MODEL WITH ASPECT RATIO 3.0 -- THREE DIAPHRAGMS

$\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.24a $\phi = 0.10$ Fig. 7.24b $\phi = 1.20$

FIGURE 7.24 MID-SPAN DEFLECTIONS OF BRIDGE MODEL WITH ASPECT RATIO 3.0 -- FIVE DIAPHRAGMS

$\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.25a $\phi = 0.10$ Fig. 7.25b $\phi = 1.20$

FIGURE 7.25 MID-SPAN DEFLECTIONS OF BRIDGE MODEL WITH ASPECT RATIO 3.0 -- TWO PAIRS OF BRACES

$\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.26a $\phi = 0.10$ Fig. 7.26b $\phi = 1.20$

FIGURE 7.26 MID-SPAN DEFLECTIONS OF BRIDGE MODEL WITH ASPECT RATIO 3.0 -- THREE PAIRS OF BRACES
 $\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.27a $\phi = 0.10$ Fig. 7.27b $\phi = 1.20$

FIGURE 7.27 MID-SPAN DEFLECTIONS OF BRIDGE MODEL WITH ASPECT RATIO 3.0 -- FIVE PAIRS BRACES
 $\kappa = 0.50$ & $\alpha = 0.30$

Fig. 7.28a Response of Cross Section with Diaphragms to a Unit Force

Fig. 2.28b Response of Cross Section with Braces to a Unit Force

FIGURE 7.28 COMPARISONS OF RESPONSES OF CROSS SECTIONS WITH DIFFERENT TYPE OF LATERAL BRACINGS

Fig. 7.29a $\phi = 0.10$

Fig. 7.29b $\phi = 1.20$

Fig. 7.30a $\phi = 0.10$

Fig. 7.30b $\phi = 1.20$

FIGURE 7.29 MID-SPAN LONGITUDINAL MOMENTS FOR A BRIDGE MODEL WITH ASPECT RATIO 3.0 TWO DIAPHRAGMS $\kappa = 0.50$ & $\alpha = 0.30$

FIGURE 7.30 MID-SPAN LONGITUDINAL MOMENTS FOR A BRIDGE MODEL WITH ASPECT RATIO 3.0 THREE DIAPHRAGMS $\kappa = 0.50$ & $\alpha = 0.30$

FIGURE 7.31 MID-SPAN LONGITUDINAL MOMENTS FOR A BRIDGE MODEL WITH ASPECT RATIO 3.0 TWO PAIRS OF BRACES $\kappa = 0.50$ & $\alpha = 0.30$

FIGURE 7.32 MID-SPAN LONGITUDINAL MOMENTS FOR A BRIDGE MODEL WITH ASPECT RATIO 3.0 THREE PAIRS OF BRACES $\kappa = 0.50$ & $\alpha = 0.30$

8.1 Summary

The main objective of the current research is to investigate the nature and mechanics of the complex dynamic system of vehicle moving on a bridge. The first step in the study involves the investigation of different parameters that govern the vehicle-bridge interaction and consequently the bridge dynamic response.

In order to clearly identify the important parameters, the bridge and the vehicle are at first represented by highly idealized models that are still believed to capture their essential dynamic characteristics. The bridge deck is modelled by a uniform prismatic beam. The beam is assumed to be simply supported. The vibration shapes of such a model are sinusoidal or harmonic and the first harmonic dominates the response. Analytical studies are therefore carried out assuming that the beam vibrates in the first sinusoidal mode. The effect of some higher modes — up to three — is then included and the resultant response is compared with that obtained by using the fundamental mode. As an alternative, the beam is modelled by a series of interconnected prismatic beam elements and the standard finite element method is used to analyze such a model. The vehicle is modelled by a single unsprung mass and a sprung mass supported by a spring and a dash-pot, the latter representing a viscous damping mechanism. The vehicle is assumed to remain in contact with the surface of the deck as it rolls along the deck. Several supporting computer programs are developed for the analysis of beam models using sinusoidal mode shapes. The characterizing parameters are determined. The practical range of values in which the parameters lie are discussed, and the effect of the parameters on the response of the beam model is analyzed. A brief description of the derivation of the equations of motion using the finite element technique is also presented, followed by a comparison between the response obtained by using the finite element method and that obtained using three sinusoidal mode shapes.

The effects of initial conditions and vehicle damping on the response of the beam model are also studied. The results of this study show that the displacement history of the vehicle influences the response of a bridge insignificantly. The development of a design procedure based on a single axle vehicle is described. The response under two axle vehicle is then presented and design charts are generated. A comparison between the responses under two single sprung vehicles in sequence and a two axle vehicle is carried out.

The second stage of the present work is concerned with a study of the dynamic response of 2D bridge models. The bridge is modeled either as an orthotropic plates or as an isotropic plates. For each type of idealization, the governing differential equations are presented and the controlling parameters for both free and forced vibrations are identified. The effect of vehicle path on the response is studied. The response of the bridge under the passage of two vehicles traversing the bridge along different paths is determined and the results are analyzed. The transverse distributions of deflections and moments at the middle cross-section arising from the passage of a single vehicle are discussed. A comparative study is made between the response of orthotropic plate model and that of isotropic plate model.

In the next stage of the work a general formulation of the equations of motion for a 3D model of the bridge traversed by a two axle vehicle is derived. In this formulation, the bridge is modelled by using quadrilateral thin shell elements. The vehicle is idealized as a two axle, four wheel system with sprung and unsprung masses. A comparative study is carried out to help in the selection of the finite element that can adequately simulate the behaviour of the bridge superstructure.

The 3D analysis is focussed on the 3 dimensional models of simple span box girder bridges. A parametric study is conducted to determine the natural frequencies and mode shapes of such a bridge and the effect of different parameters such as span-to-width ratio, the presence of diaphragms, and the existence of bracings on the frequencies and mode shapes is investigated.

Different methods are proposed for the solution of the nonlinear differential equations of motion for the vehicle-bridge system. Two equation solvers are discussed and the performance of each of them is investigated. Selection is made of the solver to be used in the present study based on criteria of stability and efficiency. A general computer program is developed for the analysis of the different bridge models for their dynamic response to var-

ious moving vehicle idealizations. The program is capable of obtaining the static response due to vehicle load as well as the free vibration response. Several numerical examples are presented to support the validity of the developed computer program.

A parametric study is then conducted to investigate the effect on the forced dynamic response of box girder bridges of several parameters such as span-to-width ratio, the presence of diaphragms, vehicle speed and weight, the ratio between the flexural and torsional rigidities, and the transverse position of the vehicle on the bridge.

8.2 Conclusions

In this work, each chapter is followed by a brief summary and conclusions. The main conclusions are rewritten in the following paragraphs for ease of reference.

Conclusions pertaining to 1D analysis

1. Bridge-vehicle interaction is a very important consideration in determining the response of a bridge under the action of a moving vehicle.
2. The dynamic response of the bridge is independent of the absolute value of the vehicle sprung mass frequency ω_v .
3. Besides the initial conditions of the vehicle on entry to the bridge, only three other parameters control the dynamic response of a bridge, namely the weight ratio, κ , the frequency, ϕ , and the speed parameter, α .
4. When the initial conditions are not accounted for and ϕ and κ are small, the response of the bridge depends mainly on α .
5. The relationship between the dynamic amplification factor, DI_{max} , and the frequency ratio, ϕ , does not always exhibit a peak within the range of practical values of ϕ , and even if there is a peak, it does not occur at $\phi = 1.0$. The maximum values of response may correspond to ϕ values ranging from 0.40 to more than 1. Also, the response increases with both κ and α .
6. The response of the bridge to a moving oscillating force of amplitude A_m can be obtained by using a moving sprung mass model with $\kappa = 0.0$ and initial conditions as follows

$$\text{vertical displacement} = -A_m g / \omega_v^2,$$

vertical velocity = 0.0 , and

vertical acceleration = $A_m g$

7. Initial conditions of the vehicle significantly affect the response. For $\kappa = 0.0$, the response exhibits a marked peak at $\phi = 1.0$ when initial conditions are included in the analysis. For higher values of κ there are no distinct peaks and the effect of initial conditions is less pronounced, especially for higher values of α .
8. In general, the inclusion of vehicle damping reduces the response of the bridge.
9. The dynamic response of a bridge traversed by a two-axle vehicle model is governed by six parameters, namely, κ , ϕ , α , the wheel spacing, S/L , the location of the center of mass of vehicle, a_1 , and the mass moment of inertia of the vehicle represented by $\hat{I}$. The effect of $\hat{I}$ on the response of the bridge is however very small and can be neglected.
10. The response of a bridge to a moving two-axle vehicle model is less than that to a moving single axle vehicle model of equivalent mass and spring stiffnesses.
11. In general, higher values of S/L parameter produce smaller responses especially for higher values of α .
12. For $S/L \leq 0.05$, the two axle vehicle produces approximately the same response as an equivalent single axle vehicle.
13. Based on the results of the analytical investigations, a series of design envelope curves are derived. These curves relate the dynamic amplification factors to parameters κ and α . As in the case of bridges designed according to the current Ontario Highway Bridge Design code, a preliminary design must be carried out first to obtain values of κ and α . Using these values the design curves can be entered to obtain the dynamic amplification factor. The design charts are drawn for both a single-axle vehicle model and a two-axle model.
14. In general, the response due to two vehicles in series is less than that due to a single vehicle of equivalent characteristics.

Conclusions pertaining to 2D analysis

1. The characterizing parameters that govern the dynamic response of a bridge structure modelled by an isotropic plate are: aspect ratio, the speed parameter α , the frequency ratio ϕ , and the mass ratio κ . Two different bridges that have identical values of these

parameters will have identical responses.

2. The maximum responses of two different bridges with identical values of α , ϕ , κ but different aspect ratio are not exactly the same but still close enough.
3. The response produced by two vehicles, each of mass m_v , traveling abreast of each other along paths symmetrically located with respect to the center line, is similar to that produced by a single vehicle of mass $2m_v$ traveling along the center line.
4. Response produced by two vehicles each of mass m_v traveling in sequence is less than that produced by a single vehicle of mass $2m_v$.
5. Response produced by two vehicles each of mass m_v traveling along parallel paths but one lagging the other is significantly less than that produced by the same vehicles traveling abreast of each other.
6. For isotropic plate idealization, the transverse distribution of longitudinal moments are flatter than static one. Also, the normalized moments are always less than the normalized deflections for the range of parameters studied.
7. The effect of aspect ratio on the normalized deflection is very small.
8. As the aspect ratio increases, the normalized moment increases and the variation with ϕ becomes very similar to that of the normalized deflection.
9. The characterizing parameters that govern the dynamic response of a bridge structure modelled by an orthotropic plate are: χ , θ , the speed parameter α , the frequency ratio ϕ , and the mass ratio κ . Two different bridges that have identical values of these parameters will have identical responses.

Conclusions pertaining to 3D analysis

Free vibrations analysis

1. Bridges of low aspect ratio have their lower modes as combined torsional and transverse flexural.
2. With an increase in the span, the fundamental mode becomes flexural. Such a mode is followed by a torsional mode.
3. Distortion mode of a very small frequency is likely to occur in simply supported box girder bridges if no diaphragms are provided at both ends of the bridge.

Forced vibrations analysis

1. A beam or an isotropic plate model is incapable of representing the behaviour of box girder bridges.
2. The governing parameters of box girder bridges are not simply ϕ , κ , α , and aspect ratio. The response is, in this case, also influenced by the relative value of the flexural and torsional rigidities and their distribution.
3. For bridge models in which the type of mode shapes, especially those which are excited by the moving loads, are not altered by changing bridge dimensions but keeping the aspect ratio unchanged, the four parameters, namely ϕ , κ , α , and aspect ratio, can be considered as the governing parameters for all practical purposes.
4. The relationship $DI_{max} - \phi$ exhibits a peak for ϕ values in the range of 0.10 to 1.20 for aspect ratio 1, where the fundamental mode is a torsional mode.
5. For all bridges that have a flexural mode as their fundamental mode shape, the relationship $DI_{max} - \phi$ displays a trough in the range of ϕ values from 0.10 to 1.20. In general, the maximum amplification factors of such bridge models increase with an increase in the aspect ratio.
6. The increase in the number of diaphragms or X-shape bracings from two – at the ends of the bridge – to three – by providing a third diaphragm or X-shape at the midspan – improves the torsional resistance of the bridge.
7. The increase in the number of diaphragms from three to five does not give much improvement in the torsional resistance of the bridge.
8. The increase in the number of braces along the span from three to five does not give much improvement in the torsional resistance of the bridge.
9. The forced response of bridges of low aspect ratio is dominated by combined torsional and transverse flexural.
10. For bridges with higher aspect ratios, the response of the bridge is dominated by a longitudinal flexural mode when the vehicle is moving along the center line of and by a torsional mode or a combination of transverse flexural and torsional mode when the vehicle is moving along the center line of a lane.

8.3 Recommendations for Future Research

1. Extensive statistical field studies may be conducted to determine the range of values that the magnitude of the dimensionless amplitude, A_m , may take. This may be then incorporated as input data for the computer programs developed in this study to develop design charts that take into account the initial conditions of the vehicles.
2. The results of the parametric studies for 1D and 2D models and the design charts produced for the single-axle and two-axle vehicle models developed in the present study may be verified by conducting field tests or/and experiments and comparing the results.
3. Additional analytical investigations based on the concepts developed here may be carried out on the effect of bridge vibrations governing human response.
4. The effect of the forces caused by braking and acceleration on the dynamic response of a bridge may be investigated.
5. The research work may be extended to investigate free and forced vibration problems of box girder bridges with cross section different from those studied here.
6. The present studies may be extended to continuous bridges, skew bridges and curved bridges.

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APPENDIX A

RESPONSE OF 2D BRIDGE MODELS UNDER A THREE AXLE VEHICLE

A.1 Vehicle Model

The vehicle model used in Chapter 7 is a three-axle tractor and semi-trailer combination. Earlier in Chapter 2 the formulation of equations of motion of the bridge-vehicle system are derived utilizing a two-axle four wheels vehicle model. To determine the adequacy of using the two-axle vehicle model instead of a three-axle model, the following study is carried out.

A.2 Equations of Motions of a Three-axle Three Wheels Vehicle Model

To simplify the assessment of the adequacy of two-axle four wheels model, the equations of motion of a three-axle three wheel vehicle, idealized as 2D system, are derived. In this manner, the complexity of deriving the equations of motion of three dimensional idealization are avoided without invalidating the generality of the assessment.

The three-axle vehicle model has three degrees of freedom , D.O.F., (Fig. A1); q_1 bouncing of the trailer, q_2 pitching of the trailer, and q_3 pitching of the tractor. Referring to Fig. A1, the following geometric relationships can be written;

$$y_1 = q_1 - a_1 L_1 q_2 = d_1 - a_1 d_2 \quad (A1a)$$

$$y_2 = q_1 + a_2 L_1 q_2 = d_1 + a_2 d_2 \quad (A1b)$$

$$y_3 = q_1 + (a_2 L_1 + a_5 L_2) q_2 = d_1 + \left(a_2 + a_5 \frac{L_2}{L_1} \right) d_2 \quad (A1c)$$

$$y_4 = y_3 + (a_3 L_2 - a_5 L_2) q_3 = d_1 + \left(a_2 + a_5 \frac{L_2}{L_1} \right) d_2 + (a_3 - a_5) d_3 \quad (A1d)$$

$$y_5 = y_3 + (L_2 - a_5 L_2) q_3 = d_1 + \left(a_2 + a_5 \frac{L_2}{L_1} \right) d_2 + (1 - a_5) d_3 \quad (A1e)$$

where

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ L_1 q_2 \\ L_2 q_3 \end{Bmatrix}$$

Now we define;

$$a_6 = a_2 + a_5 \frac{L_2}{L_1} \quad (A2a)$$

$$a_7 = 1 - a_5 \quad (A2b)$$

$$a_8 = a_3 - a_5 \quad (A2c)$$

Also

$$\begin{array}{lll} \frac{\partial y_1}{\partial d_1} = 1 & \frac{\partial y_1}{\partial d_2} = -a_1 & \frac{\partial y_1}{\partial d_3} = 0 \\ \frac{\partial y_2}{\partial d_1} = 1 & \frac{\partial y_2}{\partial d_2} = a_2 & \frac{\partial y_2}{\partial d_3} = 0 \\ \frac{\partial y_4}{\partial d_1} = 1 & \frac{\partial y_4}{\partial d_2} = a_6 & \frac{\partial y_4}{\partial d_3} = a_8 \\ \frac{\partial y_5}{\partial d_1} = 1 & \frac{\partial y_5}{\partial d_2} = a_6 & \frac{\partial y_5}{\partial d_3} = a_7 \end{array}$$

Before calculating the total virtual work in the system, the different forces in the system must be identified. Forces due to damping and stiffness of the springs F_1 , F_2 , F_3 are given by

$$F_1 = k_1(y_1 - N_1 D) + c_1(\dot{y}_1 - \dot{x} N_{1,x} D - N_1 \dot{D}) \quad (A3a)$$

$$F_2 = k_2(y_2 - N_2 D) + c_2(\dot{y}_2 - \dot{x} N_{2,x} D - N_2 \dot{D}) \quad (A3b)$$

$$F_3 = k_3(y_5 - N_3 D) + c_3(\dot{y}_5 - \dot{x} N_{3,x} D - N_3 \dot{D}) \quad (A3c)$$

Now the total virtual work done is given by

$$F_1 \delta y_1 + F_2 \delta y_2 + F_3 \delta y_5 + m_1 \ddot{q}_1 \delta q_1 + m_2 \ddot{y}_4 \delta y_4 + I_1 \ddot{q}_2 \delta q_2 + I_2 \ddot{q}_3 \delta q_3 = 0 \quad (A4)$$

Substituting for the forces F_1 , F_2 , F_3 from Eqs. A3 into Eq. A4 gives

$$\begin{aligned} & k_1(d_1 - a_1 d_2 - N_1 D)(\delta d_1 - a_1 \delta d_2) \\ & + k_2(d_1 + a_2 d_2 - N_2 D)(\delta d_1 + a_2 \delta d_2) \end{aligned}$$

$$\begin{aligned}
& +k_3(d_1 + a_6d_2 + a_7d_3 - N_3D)(\delta d_1 + a_6\delta d_2 + a_7\delta d_3) \\
& +c_1(\dot{d}_1 - a_1\dot{d}_2 - N_1\dot{D} - \dot{x}N_{1,x}D)(\delta\dot{d}_1 - a_1\delta\dot{d}_2) \\
& +c_2(\dot{d}_1 + a_2\dot{d}_2 - \dot{x}N_{2,x}D - N_2\dot{D})(\delta\dot{d}_1 + a_2\delta\dot{d}_2) \\
& +c_3(\dot{d}_1 + a_6\dot{d}_2 + a_7\dot{d}_3 - \dot{x}N_{3,x}D - N_3\dot{D})(\delta\dot{d}_1 + a_6\delta\dot{d}_2 + a_7\delta\dot{d}_3) \\
& +m_1\ddot{d}_1\dot{d}_1 + \frac{I_1}{L_1^2}\ddot{d}_2\delta d_2 + \frac{I_2}{L_2^2}\ddot{d}_3\delta d_3 + m_2(\ddot{d}_1 + a_6\ddot{d}_2 + a_7\ddot{d}_3)(\delta d_1 + a_6\delta d_2 + a_7\delta d_3) = 0
\end{aligned} \tag{A5}$$

Rearranging of Eq. A5 results in

$$\begin{aligned}
& ((k_1 + k_2 + k_3)d_1 + (-a_1k_1 + a_2k_2 + a_6k_3)d_2 + a_7k_3d_3)\delta d_1 \\
& + \left((-a_1k_1 + a_2k_2 + a_6k_3)\dot{d}_1 + (a_1^2k_1 + a_2^2k_2 + a_6^2k_3)d_2 \right)\delta d_2 \\
& + \left(a_7k_3d_1 + a_6a_7k_3d_2 + a_7^2k_3d_3 \right)\delta d_3 \\
& + \left((c_1 + c_2 + c_3)\dot{d}_1 + (-a_1c_1 + a_2c_2 + a_6c_3)\dot{d}_2 + a_7c_3\dot{d}_3 \right)\delta d_1 \\
& + \left((-a_1c_1 + a_2c_2 + a_6c_3)\dot{d}_1 + (a_1^2c_1 + a_2^2c_2 + a_6^2c_3)\dot{d}_2 \right)\delta d_2 \\
& + \left(a_7c_3\dot{d}_1 + a_6a_7c_3\dot{d}_2 + a_7^2c_3\dot{d}_3 \right)\delta d_3 \\
& + \left((m_1 + m_2)\ddot{d}_1 + a_6m_2\ddot{d}_2 + a_8m_2\ddot{d}_3 \right)\delta d_1 \\
& + \left(a_6m_2\ddot{d}_1 + \left(\frac{I_1}{L_1^2} + a_6^2m_2 \right)\ddot{d}_2 + a_6a_8m_2\ddot{d}_3 \right)\delta d_2 \\
& + \left(a_8m_2\ddot{d}_1 + a_6a_8m_2\ddot{d}_2 + \left(\frac{I_2}{L_2^2} + a_8^2m_2 \right)\ddot{d}_3 \right)\delta d_3 \\
& -(k_1N_1 + k_2N_2 + k_3N_3 + \dot{x}c_1N_{1,x} + \dot{x}c_2N_{2,x} + \dot{x}c_3N_{3,x})D\delta d_1 \\
& - (c_1N_1 + c_2N_2 + c_3N_3)\dot{D}\delta d_1 \\
& + \left(\sum_{i=1}^3 \beta_i k_i N_i + \sum_{i=1}^3 \beta_i c_i N_{i,x} \right) D + \left(\left(\sum_{i=1}^3 \beta_i c_i N_i \right) \dot{D} \right) \delta d_2 \\
& - (a_7k_3N_3 + a_7\dot{x}c_3N_{3,x})D\delta d_3 - a_7c_3N_3\dot{D}\delta d_3 = 0
\end{aligned} \tag{A6}$$

where

$$\begin{aligned}
\beta_i &= a_1 & \text{for } i &= 1 \\
&= -a_2 & &= 2 \\
&= -a_6 & &= 3
\end{aligned}$$

Since δd_1 , δd_2 , δd_3 are all virtual quantities, Eq. A6 can be written in a matrix form

$$\mathbf{m}\ddot{\mathbf{d}} + \mathbf{c}\dot{\mathbf{d}} + \mathbf{k}\mathbf{d} + \mathbf{k}_h\mathbf{D} + \mathbf{c}_h\dot{\mathbf{D}} = \mathbf{0} \quad (\text{A7})$$

where

$$\begin{aligned} \mathbf{m}_{(3 \times 3)} &= \begin{bmatrix} m_1 + m_2 & a_6 m_2 & a_6 m_2 \\ a_6 m_2 & \frac{I_1}{L_1^2} + a_6^2 m_2 & a_8 a_6 m_2 \\ a_8 m_2 & a_8 a_6 m_2 & \frac{I_2}{L_2^2} + a_8^2 m_2 \end{bmatrix} & \mathbf{d} &= \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \\ \mathbf{k}_{(3 \times 3)} &= \begin{bmatrix} k_1 + k_2 + k_3 & \sum_{i=1}^3 \beta_i k_i & a_7 k_3 \\ \sum_{i=1}^3 \beta_i k_i & a_1^2 k_1 + a_2^2 k_2 + a_6^2 k_3 & a_6 a_7 k_3 \\ a_7 k_3 & a_6 a_7 k_3 & a_7^2 k_3 \end{bmatrix} \\ \mathbf{c}_{(3 \times 3)} &= \begin{bmatrix} c_1 + c_2 + c_3 & \sum_{i=1}^3 \beta_i c_i & a_7 c_3 \\ \sum_{i=1}^3 \beta_i c_i & a_1^2 c_1 + a_2^2 c_2 + a_6^2 c_3 & a_6 a_7 c_3 \\ a_7 c_3 & a_6 a_7 c_3 & a_7^2 c_3 \end{bmatrix} \\ \mathbf{k}_{h(3 \times N)} &= \begin{bmatrix} -\left(\sum_{i=1}^3 k_i N_i + \dot{x} c_i N_{i,x}\right) \\ \sum_{i=1}^3 \beta_i k_i N_i + \beta_i \dot{x} c_i N_{i,x} \\ -a_7 k_3 - a_7 \dot{x} c_3 N_{3,x} \end{bmatrix} & \mathbf{c}_{h(3 \times N)} &= \begin{bmatrix} \sum_{i=1}^3 c_i N_i \\ \sum_{i=1}^3 \beta_i c_i N_i \\ -a_7 c_3 N_3 \end{bmatrix} \end{aligned}$$

A.2.1 Contact Forces

Figure A1 shows the contact forces between the vehicle and bridge surface P_1 , P_2 , and P_3 . It can easily be shown that the contact forces can be expressed in the form

$$P_1 = \mathbf{f}_1 \dot{\mathbf{d}} + \mathbf{g}_1 \dot{\mathbf{d}} + \mathbf{Z}_1 \mathbf{D} + \mathbf{G}_1 \dot{\mathbf{D}} + \mathbf{H}_1 \ddot{\mathbf{D}} + Q_1 \quad (\text{A8})$$

where,

$$\begin{aligned} \mathbf{f}_1 &= [k_1 \quad -a_1 k_1 \quad 0], \quad \mathbf{g}_1 = [c_1 \quad -a_1 c_1 \quad 0] \\ \mathbf{Z}_1 &= -(k_1 N_1 + N_{1,x} \dot{x} c_1 + (N_{1,xx} \dot{x}^2 + N_{1,x} \dot{x}_1) m_{t1}) \\ \mathbf{G}_1 &= -(c_1 N_1 + 2N_{1,x} \dot{x} m_{t1}) \\ \mathbf{H}_1 &= -m_{t1} N_1 \\ Q_1 &= -(a_2 m_1 + m_{t1}) g \end{aligned}$$

Note that $\mathbf{f}_1$ and $\mathbf{g}_1$ are each of size 1×3 and $\mathbf{Z}_1$, $\mathbf{G}_1$, and $\mathbf{H}_1$ are each of size $1 \times N$. Similarly

$$P_2 = \mathbf{f}_2 \mathbf{d} + \mathbf{g}_2 \dot{\mathbf{d}} + \mathbf{Z}_2 \mathbf{D} + \mathbf{G}_2 \dot{\mathbf{D}} + \mathbf{H}_2 \ddot{\mathbf{D}} + Q_2 \quad (A9)$$

where,

$$\begin{aligned} \mathbf{f}_2 &= [k_2 \quad a_2 k_2 \quad 0], \quad \mathbf{g}_2 = [c_2 \quad a_2 c_2 \quad 0] \\ \mathbf{Z}_2 &= -(k_2 \mathbf{N}_2 + \mathbf{N}_{2,r} \dot{c}_2 + (\mathbf{N}_{2,rr} \dot{x}^2 + \mathbf{N}_{2,r} \ddot{x}) m_{t2}) \\ \mathbf{G}_2 &= -(c_2 \mathbf{N}_2 + 2\mathbf{N}_{2,r} \dot{x} m_{t2}) \\ \mathbf{H}_2 &= -m_{t2} \mathbf{N}_2 \\ Q_2 &= -(a_2 m_2 + (1 - a_5) a_1 m_1 + m_{t2}) g \end{aligned}$$

and

$$P_3 = \mathbf{f}_3 \mathbf{d} + \mathbf{g}_3 \dot{\mathbf{d}} + \mathbf{Z}_3 \mathbf{D} + \mathbf{G}_3 \dot{\mathbf{D}} + \mathbf{H}_3 \ddot{\mathbf{D}} + Q_3 \quad (A10)$$

where,

$$\begin{aligned} \mathbf{f}_3 &= [k_3 \quad a_6 k_3 \quad a_7 k_3], \quad \mathbf{g}_3 = [c_3 \quad a_6 c_3 \quad a_7 c_3] \\ \mathbf{Z}_3 &= -(k_3 \mathbf{N}_3 + \mathbf{N}_{3,r} \dot{c}_3 + (\mathbf{N}_{3,rr} \dot{x}^2 + \mathbf{N}_{3,r} \ddot{x}) m_{t3}) \\ \mathbf{G}_3 &= -(c_3 \mathbf{N}_3 + 2\mathbf{N}_{3,r} \dot{x} m_{t3}) \\ \mathbf{H}_3 &= -m_{t3} \mathbf{N}_3 \\ Q_3 &= -(a_3 m_2 + a_5 a_1 m_1 + m_{t3}) g \end{aligned}$$

In a final step the equations of motion for bridge-vehicle system are to be derived in the same manner as earlier in Chapters 2 and 3 and can be expressed in the form;

$$\begin{bmatrix} \mathbf{m} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} + \mathbf{m}^* \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{d}} \\ \ddot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} \mathbf{c} & \mathbf{c}_h \\ \mathbf{c}_v & \mathbf{C} + \mathbf{c}^* \end{bmatrix} \begin{bmatrix} \dot{\mathbf{d}} \\ \dot{\mathbf{D}} \end{bmatrix} + \begin{bmatrix} \mathbf{k} & \mathbf{k}_h \\ \mathbf{k}_v & \mathbf{K} + \mathbf{k}^* \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{p}^* \end{bmatrix} \quad (A11)$$

where

$$\begin{aligned}
 \mathbf{m}^*_{(N \times N)} &= - \sum_{i=1}^3 \mathbf{N}_i^T_{(N \times 1)} \mathbf{H}_{i(1 \times N)} \\
 \mathbf{c}^*_{(N \times N)} &= - \sum_{i=1}^3 \mathbf{N}_i^T_{(N \times 1)} \mathbf{G}_{i(1 \times N)} \\
 \mathbf{k}^*_{(N \times N)} &= - \sum_{i=1}^3 \mathbf{N}_i^T_{(N \times 1)} \mathbf{Z}_{i(1 \times N)} \\
 \mathbf{c}_{v(N \times 3)} &= - \sum_{i=1}^3 \mathbf{N}_i^T_{(N \times 1)} \mathbf{g}_{i(1 \times 3)} \\
 \mathbf{k}_{v(N \times 3)} &= - \sum_{i=1}^3 \mathbf{N}_i^T_{(N \times 1)} \mathbf{f}_{i(1 \times 3)} \\
 \mathbf{p}^*_{(N \times 1)} &= \sum_{i=1}^3 \mathbf{N}_i^T_{(N \times 1)} Q_i
 \end{aligned}$$

A.3 Equations of Motions of the Two-axle Two-wheel Vehicle Model

The equations of motion of a two-axle two-wheels vehicle model can easily be derived from those of the three-axle vehicle model by setting $m_2, I_2, m_{t3}, k_3, c_3, a_3, a_4$, and a_5 to zero in Eq. A11. To derive the physical characteristics of the two-axle vehicle model from those of the three-axle model, the following criteria are followed. Both the total mass and the lowest two natural frequencies of the vehicle, bounce and pitching of the trailer, are kept almost the same while all the other vehicle characteristics may be altered. The vehicle model (FWHA) used in Chapter 7, which can be loaded to simulate an HS20-44 design loading, is adopted for the current study. The description of this vehicle is shown in Fig. 7.6.

A.4 Numerical Examples

A computer program has been developed to consider the two dimensional representation of the vehicle, either three or two-axle model. In this section, several illustrative examples are presented to show the adequacy of the two-axle model. In each example the response of the bridge structure under the passage of the traversing two-axle and three-axle models are compared. For that purpose, a single span, multigirder highway bridge of composite

construction (steel girders and concrete deck action monolithically) is considered. The specifications of the bridge and its idealized model are taken from Gupta (1980) who selected the bridge from the Standard Plans for Highway Bridges.

A.4.1 Beam Problems

Properties of example bridge idealized as beams are as follow;

Mass density $\dot{m}$	$= 6.037 \text{ Mg/m}^3$
Uniform cross section area	$= 1.0 \text{ m}^2$
Flexural rigidity EI	$= 1.942 \times 10^6 \text{ kN.m}^2$
Equivalent torsional rigidity GJ	$= 1.353 \times 10^6 \text{ kN.m}^2/\text{m}$
Vehicle speed	$= 15.24 \text{ m/sec.}$
Bridge span	$= 18.288 \text{ m}$
Natural frequencies ω_1	$= 2.664 \text{ Hz.}$
ω_2	$= 10.65 \text{ Hz.}$
ω_3	$= 24.01 \text{ Hz.}$
ω_4	$= 42.52 \text{ Hz.}$

Figure A2 shows the normalized central deflection versus the normalized time for both models of vehicle. It is seen from Fig. A2 that the responses for the two idealizations of the vehicle are almost matching, the difference in phase between the two vehicles as they enter the bridge (the length of tractor / vehicle speed) is taken into consideration. It is also seen from Fig. A2 that the amplification factors for the two-axle idealization are slightly higher than those for the three-axle idealization.

A.4.2 Orthotropic Plate Problems

Properties of example bridge idealized as an orthotropic plate are as follow;

Mass density m	$= 3.709 \text{ Mg/m}^3$
Bridge width	$= 9.144 \text{ m}$
Bridge thickness	$= 0.178 \text{ m}$

$$\begin{aligned}
D_{xx} &= 2.2095 \times 10^5 \text{ kN.m}^2/\text{m} \\
D_{yy} &= 1.2420 \times 10^4 \text{ kN.m}^2/\text{m} \\
D_t &= 2.6193 \times 10^4 \text{ kN.m}^2/\text{m} \\
D_{xy} &= 1863 \text{ kN.m}^2/\text{m} \\
\text{Vehicle speed} &= 15.24 \text{ m/sec} \\
\text{Bridge span} &= 18.288 \text{ m} \\
\text{Natural frequencies} & \\
\omega_1 &= 2.739(FL_1) \text{ Hz.} \\
\omega_2 &= 4.673(TO) \text{ Hz.} \\
\omega_3 &= 10.04(TF) \text{ Hz.} \\
\omega_4 &= 10.69(FL_2) \text{ Hz.}
\end{aligned}$$

In this case, results are obtained for the vehicle moving along the center-line of the plate. The same observations as made for the beam example are still valid. Also, strong resemblance between the responses of both beam and plates is readily seen from Figs. A2 and A3.

The problems solved above prove the adequacy of the two-axle vehicle idealization to represent an actual three-axle vehicle.

FIGURE A.1 THREE-AXLE VEHICLE MODEL

FIGURE A.2 BEAM PROBLEM **FIGURE A.3 PLATE PROBLEM**

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