Representation by Quaternary Quadratic Forms whose Coefficients are 1, 2, 7 or 14

by

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Abstract

We determine explicit formulae for the number of representations of a positive integer n by the quaternary quadratic forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, where $a_1, a_2, a_3, a_4 \in \{1, 2, 7, 14\}$ which satisfy the simplifying assumptions $a_1 \leq a_2 \leq a_3 \leq a_4$ and $\gcd(a_1, a_2, a_3, a_4) = 1$. We use a modular form approach. We then extend our work to determine explicit formulae for the number of representations of n by the octonary quadratic forms $x_1^2+x_2^2+x_3^2+x_4^2+7(x_5^2+x_6^2+x_7^2+x_8^2)$, $x_1^2+x_2^2+7(x_3^2+x_4^2+x_5^2+x_6^2+x_7^2+x_8^2)$ and $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + 7(x_7^2 + x_8^2)$.

Dedication

To my mother who was always supporting me and my husband Mohammed who never gave up.

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Contents

Chapter 1

Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers, real numbers and complex numbers respectively. Let $a_1, a_2, a_3, a_4 \in \mathbb{N}$, and $n \in \mathbb{N}_0$. Let $N(a_1, a_2, a_3, a_4; n)$ denote the number of representations of n by the quaternary quadratic form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, that is

$$N(a_1, a_2, a_3, a_4; n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2\}.$$

It is clear that $N(a_1, a_2, a_3, a_4; 0) = 1$. Since $N(a_1, a_2, a_3, a_4; n)$ is invariant under a permutation of a_1, a_2, a_3, a_4, we may suppose that

$$a_1 \leq a_2 \leq a_3 \leq a_4. \tag{1.0.1}$$

Note that if

$\gcd(a_1, a_2, a_3, a_4) = d$, then $N(a_1, a_2, a_3, a_4; n) = N(a_1/d, a_2/d, a_3/d, a_4/d; n/d)$. So we may also suppose that

$$\gcd(a_1, a_2, a_3, a_4) = 1. \tag{1.0.2}$$

Our first objective in this thesis is to determine explicit formulae for $N(a_1, a_2, a_3, a_4; n)$, where $a_1, a_2, a_3, a_4 \in \{1, 2, 7, 14\}$ which satisfy the simplifying assumptions (1.0.1) and (1.0.2).

We then extend our work to determine explicit formulae for the number of representations of n by the octonary quadratic forms $x_1^2+x_2^2+x_3^2+x_4^2+7(x_5^2+x_6^2+x_7^2+x_8^2)$, $x_1^2+x_2^2+7(x_3^2+x_4^2+x_5^2+x_6^2+x_7^2+x_8^2)$ and $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+7(x_7^2+x_8^2)$, which we denote by $N(1^4, 7^4; n)$, $N(1^2, 7^6; n)$ and $N(1^6, 7^2; n)$ respectively.

Over the years many people have worked on the problem of representations of integers by quadratic forms. In 1770 Lagrange [16] proved that every positive integer can be written as a sum of four integer squares.

Jacobi [12] gave formulae for $N(1, 1, 1, 1; n)$ as

$$N(1,1,1,1;n) = 8\sigma(n) - 32\sigma(n/4) = \begin{cases} 8\sigma(n) & \text{if } 4 \nmid n, \\ 24\sigma(n) & \text{if } 4 \mid n, \end{cases}$$

where $\sigma(n)$ is the sum of divisors function. See [26]. In 1860, Liouville [18] gave a formula for $N(1, 1, 2, 2; n)$ and in 1861 he gave two more formulae for $N(1, 1, 1, 2; n)$ and $N(1, 2, 2, 2; n)$. Also, Benz [4], Demuth [10], and Pepin [23] gave proofs for these formulae. Williams [29] gave a completely arithmetic proof of the Liouville formulae for $N(1, 1, 1, 2; n)$ and $N(1, 2, 2, 2; n)$.

In Chapter 2 we present some basic properties of modular groups and modular forms. In Chapter 3 we determine an explicit formula for $N(a_1, a_2, a_3, a_4; n)$ for each of the twenty-two quaternary quadratic forms given by

$$\begin{aligned}(a_1, a_2, a_3, a_4) = \; & (1,1,7,7),\ (2,2,7,7),\ (1,2,7,14),\ (1,1,14,14),\ (1,1,1,7), \\ & (1,2,2,7),\ (1,7,7,7),\ (1,1,2,14),\ (2,7,7,14),\ (1,7,14,14),\end{aligned}$$

$$(1,1,2,7),\ (2,2,2,7),\ (2,7,7,7),\ (1,1,1,14),\ (1,2,2,14),$$
$$(1,7,7,14),\ (2,7,14,14),\ (1,14,14,14),\ (1,2,7,7),\ (1,1,7,14),$$
$$(2,2,7,14),\ (1,2,14,14).$$

To the best of our knowledge, these are the only remaining diagonal quaternary quadratic forms with coefficients 1, 2, 7 and 14 for which explicit formulae for $N(a_1, a_2, a_3, a_4; n)$ have not been determined so far. In Chapter 4 we determine the number of representations of a positive integer n by the octonary quadratic forms $N(1^4, 7^4; n)$, $N(1^2, 7^6; n)$ and $N(1^6, 7^2; n)$. We conclude our thesis by indicating some further directions for our research.

Chapter 2

Basic Concepts

In this chapter we present some basic concepts for modular forms. For more information one can see [8], [9], [14], [15], [21], [22], [25], [27], and [28].

2.1 Modular Forms

Definition 2.1.1. The modular group $SL_2(\mathbb{Z})$ is defined as

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\},$$

which acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by the linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}, \text{ for } z \in \mathbb{H}.$$

Note that the modular group $SL_2(\mathbb{Z})$ is generated by two elements
$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Definition 2.1.2. Let $N \in \mathbb{N}$. We define the principal congruence subgroup $\Gamma(N)$ by

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1(\text{mod N}), b \equiv c \equiv 0(\text{mod N}) \right\}.$$

A subgroup Γ of $SL_2(\mathbb{Z})$ is called a congruence subgroup if it contains $\Gamma(N)$ for some positive integer N. The smallest such N is called the level of Γ. The two important congruence subgroups are

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0(\text{mod N}) \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid a \equiv d \equiv 1(\text{mod N}) \right\}.$$

Definition 2.1.3. A Dirichlet character $(\text{mod } N)$ is a function $\chi : \mathbb{Z} \to C$ satisfying
(i) $\chi(ab) = \chi(a)\chi(b)$ for any $a, b \in \mathbb{Z}$,
(ii) $\chi(a) \neq 0$ if $gcd(a, N) = 1$,
(iii)$\chi(a) = 0$ if $gcd(a, N) > 1$,
(iv) $\chi(a) = \chi(b)$ if $a \equiv b$ (mod N).

Definition 2.1.4. The trivial character is the Dirichlet character of modulus 1 and is denoted by χ_0.

Definition 2.1.5. The conductor of a Dirichlet character χ is the smallest positive integer M dividing its modulus such that there exists a Dirichlet character ψ of modulus M with $\chi(a) = \psi(a)$ for all $a \in \mathbb{Z}$ with $(a, N) = 1$. We say that a Dirichlet character modulo N is primitive if its conductor equals its modulus.

Definition 2.1.6. Let $k \in \mathbb{Z}$. A weakly modular function of weight k for a congruence subgroup Γ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k\ f(z),\ \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } z \in \mathbb{H}.$$

Definition 2.1.7. [14] Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$. We define the weight k operator $|[\gamma]_k$ on a function from $\mathbb{H}$ to $\mathbb{C}$ by

$$(f|[\gamma]_k)\ (z) = (cz+d)^{-k}\ f\left(\frac{az+b}{cz+d}\right).$$

Definition 2.1.8. [14] Let Γ be a congruence subgroup of level N in $SL_2(\mathbb{Z})$ and $k \in \mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k for Γ if it satisfies the following conditions

(i) f is weakly modular for Γ,

(ii) f is holomorphic on $\mathbb{H}$,

(iii) $f|[\alpha]_k$ is holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$.

Note that by condition (iii), $f|[\alpha]_k$ has a Fourier expansion of the form

$$f|[\alpha]_k = \sum_{n=0}^{\infty} a_n q_N^n, \qquad q_N = e^{2\pi i z/N}.$$

We say that f is a cusp form of weight k for Γ if $f|[\alpha]_k$ vanishes at ∞, which mean $a_0 = 0$ for every $\alpha \in SL_2(\mathbb{Z})$ in the Fourier expansion of $f|[\alpha]_k$.

Definition 2.1.9. [14] Let N be a positive integer and let χ be a Dirichlet character.

A function $f : \mathbb{H} \to \mathbb{C}$ which is holomorphic on $\mathbb{H}$ and $f|[\alpha]_k$ is holomorphic at ∞ for all α is a modular form of weight k for $\Gamma_0(N)$ with character χ if

$$f|[\gamma]_k = \chi(d)f(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight k and character χ, and $S_k(\Gamma_0(N), \chi)$ to denote the subspace of cusp forms of weight k and character χ.

Let $k \in \mathbb{Z}$. We write $E_k(\Gamma_0(N), \chi)$ to denote the subspace of Eisenstein series. It is known (see for example [28, p.83]) that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi). \tag{2.1.1}$$

Definition 2.1.10. The Dedekind eta function is defined on the upper half plane $\mathbb{H}$ by the product formula

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \;\; q = e^{2\pi i z}. \tag{2.1.2}$$

An eta quotient is defined as a finite product of the form

$$f(z) = \prod_{\delta} \eta^{r_\delta}(\delta z), \tag{2.1.3}$$

where δ runs through a finite set of positive integers and the exponents r_δ are non-zero integers. By taking N to be the least common multiple of the $\delta's$ we can write the eta quotient (2.1.3) as

$$f(z) = \prod_{1\leq\delta|N} \eta^{r_\delta}(\delta z), \tag{2.1.4}$$

where some of the exponents r_δ may be 0. When all exponents are non-negative, $f(z)$ is said to be an eta product . For $q \in \mathbb{C}$ with $|q| < 1$ we set

$$F(q) = \prod_{n=1}^{\infty}(1-q^n). \tag{2.1.5}$$

Appealing to (2.1.5) we can express the eta function (2.1.2) as

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = q^{1/24}F(q),\ q = e^{2\pi i z}.$$

Definition 2.1.11. For $q \in \mathbb{C}$ with $|q| < 1$ Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

We note that for quaternary quadratic forms $a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2$ $(a_1, a_2, a_3, a_4 \in \mathbb{N})$, we have

$$\sum_{n=0}^{\infty} N(a_1, a_2, a_3, a_4; n)q^n = \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}), \tag{2.1.6}$$

and for octonary quadratic forms $a_1x_1^2 + \cdots + a_8x_8^2$ $(a_1, \ldots, a_8 \in \mathbb{N})$, we have

$$\sum_{n=0}^{\infty} N(a_1, \ldots, a_8; n)q^n = \varphi(q^{a_1})\cdots\varphi(q^{a_8}). \tag{2.1.7}$$

The infinite product representation of $\varphi(q)$ is due to Jacobi ([3]),

$$\varphi(q) = \frac{F^5(q^2)}{F^2(q)F^2(q^4)}, \tag{2.1.8}$$

where $F(q)$ is given by (2.1.5). It follows from (2.1.2), (2.1.5) and (2.1.8) that

$$\eta(z) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}. \tag{2.1.9}$$

2.2 Eisenstein Series

Definition 2.2.1. Let χ and ψ be Dirichlet characters. For $n \in \mathbb{N}$ we define $\sigma_{(k-1,\chi,\psi)}(n)$ by

$$\sigma_{(k-1,\chi,\psi)}(n) := \sum_{1 \le m | n} \psi(m)\chi(n/m)m^{k-1}. \tag{2.2.1}$$

We set $\sigma_{(k-1,\chi,\psi)}(n) = 0$ for $n \notin \mathbb{N}$. If χ and ψ are trivial characters then $\sigma_{(k-1,\chi,\psi)}(n)$ becomes the sum of divisors function

$$\sigma_{k-1}(n) = \sum_{1 \le m | n} m^{k-1}.$$

Definition 2.2.2. Let ψ be a Dirichlet character of modulus N. We define the generalized Bernoulli numbers $\{B_{k,\psi}\}_{k \in \mathbb{N}}$ by the formal series

$$\sum_{a=1}^{N} \frac{xe^{ax}\psi(a)}{e^{Nx}-1} = \sum_{k=0}^{\infty} B_{k,\psi}\frac{x^k}{k!}.$$

Let χ and ψ be primitive Dirichlet characters with conductors L and M, respectively. We set

$$E_{k,\chi,\psi}(q) = c_0 + \sum_{n\geq 1}\Big(\sum_{m|n}\psi(m)\chi(n/m)m^{k-1}\Big)q^n, \tag{2.2.2}$$

where c_0 is written in terms of the generalized Bernoulli numbers defined by

$$c_0 = \begin{cases} 0 & \text{if } L > 1; \\ -\frac{B_{k,\psi}}{2k} & \text{if } L = 1. \end{cases}$$

If χ and ψ are trivial characters, then the Eisenstein series $E_{2,\chi_0,\chi_0}(q)$ and $E_{4,\chi_0,\chi_0}(q)$ become

$$L(q) := E_2(q) := E_{2,\chi_0,\chi_0}(q) = -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma(n)q^n, \tag{2.2.3}$$

and

$$E_4(q) := E_{4,\chi_0,\chi_0}(q) = \frac{1}{240} + \sum_{n=1}^{\infty}\sigma_3(n)q^n. \tag{2.2.4}$$

The following theorem can be found in [28].

Theorem 2.2.3. *Suppose t, k are positive integers. Let χ and ψ be Dirichlet characters with conductors L and M, respectively. The power series $E_{k,\chi,\psi}(q^t)$ with $LMt|N$ and $\chi\psi = \varepsilon$ form a basis for the Eisenstein subspace $E_k(\Gamma_0(N), \varepsilon)$. Except if $k = 2$, $\chi = \psi = 1$, $t > 1$ then $L(q) - tL(q^t)$ is a modular form of weight 2 in $M_2(\Gamma_0(t))$.*

2.3 Dimension Formulae

Let $N, k \in \mathbb{N}$ and χ a Dirichlet character. In this section we state formulae for the dimensions of $M_k(\Gamma_0(N), \chi)$, $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$. First, we state the trivial character case. Let

$$\mu_0(N) = N\prod_{p|N}(1+1/p),$$

$$\mu_{0,2}(N) = \begin{cases} 0 & \text{if } 4|N, \\ \prod_{p|N}(1+(\frac{-4}{p})) & \text{otherwise,} \end{cases}$$

$$\mu_{0,3}(N) = \begin{cases} 0 & \text{if } 2|N \textit{ or } 9|N, \\ \prod_{p|N}(1+(\frac{-3}{p})) & \text{otherwise,} \end{cases}$$

$$c_0(N) = \sum_{d|N}\phi(gcd(d,N/d)),$$

where ϕ is Euler totient function and p runs through the prime divisors of N. Also, let

$$g(N) := \quad 1 + \frac{\mu_0(N)}{12} - \frac{\mu_{0,2}(N)}{4} - \frac{\mu_{0,3}(N)}{4} - \frac{c_0(N)}{2}.$$

Then we have the following proposition which is taken from [28, Section 6.1, p. 93].

Proposition 2.3.1. *We have* $dimS_2(\Gamma_0(N)) = g(N)$, *and for* $k \geq 4$ *even,*

$$dimS_k(\Gamma_0(N)) = (k-1)\cdot(g(N)-1) + \left(\frac{k}{2}-1\right)\cdot c(N) + \mu_{0,2}(N)\left\lfloor\frac{k}{4}\right\rfloor + \mu_{0,3}(N)\left\lfloor\frac{k}{3}\right\rfloor,$$

where $\lfloor\ \rfloor$ *is the floor function. The dimension of the Eisenstein subspace is*

$$dimE_k(\Gamma_0(N)) = \begin{cases} c_0(N)-1 & \textit{if } k=2, \\ c_0(N) & \textit{if } k \neq 2. \end{cases}$$

Example 2.3.2. Let $N = 56$. We have $c(56) = 8$, $\mu_0(N) = 96$, $\mu_{0,2}(56) = \mu_{0,3}(56) = 0$. Hence

$$g(56) = 1 + \frac{96}{12} - \frac{8}{2} = 5.$$

Thus by Propostion 2.3.1 we have $\dim S_2(\Gamma_0(56)) = 5$ and $\dim E_2(\Gamma_0(56)) = 7$.

Second, we state the non-trivial character case. The formulae are taken from [28, Section 6.3, p. 98-100]. Let $v_p(N)$ denote the largest $r \in \mathbb{N}_0$ such that $p^r \mid \mathbb{N}$ and let c be the conductor of χ. We set

$$\lambda_{(p,N,v_p(c))} = \begin{cases} p^{\frac{r}{2}} + p^{\frac{r}{2}-1} & \text{if } 2 \cdot v_p(c) \leq r \text{ and } 2 \mid r, \\ 2 \cdot p^{\frac{r-1}{2}} & \text{if } 2 \cdot v_p(c) \leq r \text{ and } 2 \nmid r, \\ 2 \cdot p^{r - v_p(c)} & \text{if } 2 \cdot v_p(c) > r. \end{cases}$$

The rational numbers γ_3 and γ_4 are defined as follows

$$\gamma_3(k) = \begin{cases} -1/3 & \text{if } k \equiv 2 \pmod 3, \\ 0 & \text{if } k \equiv 1 \pmod 3, \\ 1/3 & \text{if } k \equiv 0 \pmod 3, \end{cases}$$

$$\gamma_4(k) = \begin{cases} -1/4 & \text{if } k \equiv 2 \pmod 4, \\ 0 & \text{if } k \; is \text{ odd}, \\ 1/4 & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

Let χ be a Dirichlet character of modulus N for which $\chi(-1) = (-1)^k$. Then we have

$$\dim S_k(\Gamma_0(N), \chi) - \dim M_{2-k}(\Gamma_0(N), \chi) = \frac{k-1}{12} \cdot \mu_0(N) - \frac{1}{2} \cdot \prod_{p|N} \lambda(p, N, v_p(c))$$
$$+\gamma_4(k) \cdot \sum_{x \in A_4(N)} \chi(x) + \gamma_3(k) \cdot \sum_{x \in A_3(N)} \chi(x) \ , \tag{2.3.1}$$

where

$$A_4(N) = \left\{x \in \mathbb{Z}/N\mathbb{Z} : x^2 + 1 = 0\right\} \text{ and } A_3(N) = \left\{x \in \mathbb{Z}/N\mathbb{Z} : x^2 + x + 1 = 0\right\}.$$

To compute $\dim M_k(\Gamma_0(N), \chi)$ for $k \geq 2$, we use the fact that $\dim S_k(\Gamma_0(N), \chi) = 0$ for $k \leq 0$. Then we have

$$\begin{aligned} \dim M_k(\Gamma_0(N), \chi) &= - \left(\dim S_{2-k}(\Gamma_0(N), \chi) - \dim M_k(\Gamma_0(N), \chi)\right) \\ &= - \Big(\frac{1-k}{12} \cdot \mu_0(N) - \frac{1}{2} \cdot \prod_{p|N} \lambda(p, N, v_p(c)) \\ &\quad + \gamma_4(2-k) \cdot \sum_{x \in A_4(N)} \chi(x) + \gamma_3(2-k) \cdot \sum_{x \in A_3(N)} \chi(x)\Big), \end{aligned} \tag{2.3.2}$$

and

$$\dim E_k(\Gamma_0(N), \chi) = \dim M_k(\Gamma_0(N), \chi) - \dim S_k(\Gamma_0(N), \chi). \tag{2.3.3}$$

Example 2.3.3. For $N = 56$, $k = 2$, $\chi_3(m) = \left(\frac{28}{m}\right)$, $\chi_5(m) = \left(\frac{8}{m}\right)$ and $\chi_6(m) = \left(\frac{56}{m}\right)$, we have

$$\sum_{x \in A_4(N)} \chi(x) = \sum_{x \in A_3(N)} \chi(x) = 0.$$

Also, we have

χ	χ_3	χ_5	χ_6
$\prod_{p\mid 56} \lambda(p, 56, v_p(c))$	8	4	4

Thus by (2.3.1)–(2.3.3) we obtain

χ	$\dim M_2(\Gamma_0(56), \chi)$	$\dim S_2(\Gamma_0(56), \chi)$	$\dim E_2(\Gamma_0(56), \chi)$
χ_3	12	4	8
χ_5	10	6	4
χ_6	10	6	4

Chapter 3

Representations by Quaternary Quadratic Forms with Coefficients 1, 2, 7 and 14

3.1 Preliminaries

We recall that, for $a_1, a_2, a_3, a_4 \in \mathbb{N}$ and $n \in \mathbb{N}_0$, $N(a_1, a_2, a_3, a_4; n)$ denotes the number of representations of n by the quaternary form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, that is

$$N(a_1, a_2, a_3, a_4; n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2\}.$$

We also have the simplifying assumptions

$$a_1 \leq a_2 \leq a_3 \leq a_4, \tag{3.1.1}$$

Table 3.1.1

$M_2(\Gamma_0(56), \chi_0)$	$M_2(\Gamma_0(56), \chi_3)$	$M_2(\Gamma_0(56), \chi_5)$	$M_2(\Gamma_0(56), \chi_6)$
$(1,1,7,7)$	$(1,1,1,7)$	$(1,2,7,7)$	$(1,1,2,7)$
$(2,2,7,7)$	$(1,2,2,7)$	$(1,1,7,14)$	$(2,2,2,7)$
$(1,2,7,14)$	$(1,7,7,7)$	$(2,2,7,14)$	$(2,7,7,7)$
$(1,1,14,14)$	$(1,1,2,14)$	$(1,2,14,14)$	$(1,1,1,14)$
	$(2,7,7,14)$		$(1,2,2,14)$
	$(1,7,14,14)$		$(1,7,7,14)$
			$(2,7,14,14)$
			$(1,14,14,14)$

and

$$\gcd(a_1, a_2, a_3, a_4) = 1. \tag{3.1.2}$$

We also recall that χ_0 denotes the trivial character. For $m \in \mathbb{Z}$ we define six characters by

$$\chi_1(m) = \left(\frac{-7}{m}\right), \chi_2(m) = \left(\frac{-4}{m}\right), \chi_3(m) = \left(\frac{28}{m}\right), \tag{3.1.3}$$

$$\chi_4(m) = \left(\frac{-8}{m}\right), \chi_5(m) = \left(\frac{8}{m}\right), \chi_6(m) = \left(\frac{56}{m}\right). \tag{3.1.4}$$

Under the simplifying assumptions (3.1.1) and (3.1.2) there are twenty-six quaternary quadratic forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ for which $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi)$ where $\chi \in \{\chi_0, \chi_3, \chi_5, \chi_6\}$. Their coefficients (a_1, a_2, a_3, a_4) are listed in Table 3.1.1.

Formulae for the four quaternary quadratic forms $(1,1,1,1)$, $(1,1,2,2)$, $(1,2,2,2)$, $(1,1,1,2)$ appeared in [1], [29]. In this chapter we determine formulae for the remaining twenty-two quaternary quadratic forms listed in Table 3.1.1.

We use the following theorem to determine if an eta quotient $f(z) = \prod_{1\leq\delta|N} \eta^{r_\delta}(\delta z)$ is in $M_k(\Gamma_0(N), \chi)$. See [11], [13, Corollary 2.3, p. 37], [14, Theorem 5.7, p. 99] and [17].

Theorem 3.1.1. *(Ligozat) Let $N \in \mathbb{N}$ and let $f(z) = \prod_{1\leq\delta|N} \eta^{r_\delta}(\delta z)$ be an eta quotient which satisfies the following conditions:*

(L1) $\displaystyle\sum_{1\leq\delta|N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,

(L2) $\displaystyle\sum_{1\leq\delta|N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,

(L3) *for each $d \mid N$,* $\displaystyle\sum_{1\leq\delta|N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} \geq 0$.

Then $f(z)$ is in $M_k(\Gamma_0(N), \chi)$, where χ is given by

$$\chi(m) = \left(\frac{(-1)^k s}{m}\right),$$

with weight

$$k = \frac{1}{2} \sum_{1\leq\delta|N} r_\delta,$$

and

$$s = \prod_{1\leq\delta|N} \delta^{r_\delta}.$$

In addition to the above conditions if $f(z)$ also satisfies the condition

(L4) *for each $d \mid N$,* $\displaystyle\sum_{1\leq\delta|N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} > 0$,

then $f(z)$ is in $S_k(\Gamma_0(N), \chi)$.

3.2 The space $M_2(\Gamma_0(56), \chi_0)$

In this section we determine formulae for $N(a_1, a_2, a_3, a_4; n)$ for the quaternary quadratic forms listed in the first column of Table 3.1.1 in terms of $\sigma(n)$, $\sigma(n/2)$, $\sigma(n/4)$, $\sigma(n/7)$, $\sigma(n/8)$, $\sigma(n/14)$, $\sigma(n/28)$, $\sigma(n/56)$, and $a_k(n)$ $(1 \leq k \leq 5)$ defined by

$$A_1(q) = \sum_{n=1}^{\infty} a_1(n)q^n = \eta(2z)\eta(4z)\eta(14z)\eta(28z), \tag{3.2.1}$$

$$A_2(q) = \sum_{n=1}^{\infty} a_2(n)q^n = \frac{\eta^3(2z)\eta^3(28z)}{\eta(4z)\eta(14z)}, \tag{3.2.2}$$

$$A_3(q) = \sum_{n=1}^{\infty} a_3(n)q^n = \frac{\eta(z)\eta^3(4z)\eta(7z)\eta^3(28z)}{\eta(2z)\eta(8z)\eta(14z)\eta(56z)}, \tag{3.2.3}$$

$$A_4(q) = \sum_{n=1}^{\infty} a_4(n)q^n = \frac{\eta^3(2z)\eta(8z)\eta^3(14z)\eta(56z)}{\eta(z)\eta(4z)\eta(7z)\eta(28z)}, \tag{3.2.4}$$

$$A_5(q) = \sum_{n=1}^{\infty} a_5(n)q^n = \frac{\eta^4(4z)\eta^4(28z)}{\eta(2z)\eta(8z)\eta(14z)\eta(56z)}. \tag{3.2.5}$$

There is no linear relationship among the $A_k(q), 1 \leq k \leq 5$. The first fifty-six values of $a_k(n)$, are given in Table 3.2.1.

Table 3.2.1

n	$a_1(n)$	$a_2(n)$	$a_3(n)$	$a_4(n)$	$a_5(n)$	n	$a_1(n)$	$a_2(n)$	$a_3(n)$	$a_4(n)$	$a_5(n)$
1	0	0	1	0	0	29	0	−2	−2	2	0
2	1	0	−1	0	1	30	0	0	0	0	0
3	0	1	0	1	0	31	0	−2	0	2	0
4	−1	0	−1	1	1	32	1	0	1	−1	−1
5	0	−3	−2	−1	0	33	0	0	0	0	0
6	−2	0	2	0	−2	34	6	0	−6	0	6
7	0	1	1	0	0	35	0	−1	−2	−1	0
8	1	0	1	−1	−1	36	−1	0	−1	1	1
9	0	2	1	0	0	37	0	−2	−2	−2	0
10	0	0	0	0	0	38	2	0	−2	0	2
11	0	2	0	0	0	39	0	0	4	−2	0
12	2	0	2	−2	−2	40	0	0	0	0	0
13	0	−1	−2	1	0	41	0	−2	2	−2	0
14	1	0	−1	0	1	42	−2	0	2	0	−2
15	0	−4	−4	−2	0	43	0	6	8	0	0
16	−1	0	−1	1	1	44	0	0	0	0	0
17	0	2	2	−2	0	45	0	1	−2	−1	0

18	1	0	−1	0	1	46	0	0	0	0	0
19	0	−5	0	−1	0	47	0	2	−8	2	0
20	0	0	0	0	0	48	2	0	2	−2	−2
21	0	1	0	1	0	49	0	0	1	0	0
22	0	0	0	0	0	50	−5	0	5	0	−5
23	0	4	4	2	0	51	0	−2	−8	2	0
24	−2	0	−2	2	2	52	4	0	4	−4	−4
25	0	6	3	4	0	53	0	−8	−2	−4	0
26	−4	0	4	0	−4	54	4	0	−4	0	4
27	0	−2	0	−2	0	55	0	4	0	0	0
28	−1	0	−1	1	1	56	1	0	1	−1	−1

Theorem 3.2.1. *Let* (a_1, a_2, a_3, a_4) *be as in the first column of Table 3.1.1. Then* $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_0)$.

Proof. Appealing to (2.1.9) for each quadratic form, we then check conditions (L1), (L2) and (L3) of Theorem 3.1.1 for each form. We have $N = 56$. First we consider $(1, 1, 7, 7)$

$$\varphi^2(q)\varphi^2(q^7) = \frac{\eta^{10}(2z)\eta^{10}(14z)}{\eta^4(z)\eta^4(4z)\eta^4(7z)\eta^4(28z)}.$$

Then we have

Table 3.2.2(a)

δ	1	2	4	7	14	28
r_δ	−4	10	−4	−4	10	−4

It can be seen from Table 3.2.2(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.2.2(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1 \le \delta \mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	96/7	0	0	0	96	0	0

From Table 3.2.2(b) the condition $(L3)$ is also satisfied. Thus $\varphi^2(q)\varphi^2(q^7) \in M_2(\Gamma_0(56), \chi_0)$. Second we consider $(2, 2, 7, 7)$

$$\varphi^2(q^2)\varphi^2(q^7) = \frac{\eta^{10}(4z)\eta^{10}(14z)}{\eta^4(2z)\eta^4(7z)\eta^4(8z)\eta^4(28z)}.$$

Then we have

Table 3.2.3(a)

δ	2	4	7	8	14	28
r_δ	-4	10	-4	-4	10	-4

It can be seen from Table 3.2.3(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.2.3(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1 \le \delta \mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	12/7	24	0	0	84	24	0

From Table 3.2.3(b) the condition $(L3)$ is also satisfied. Thus $\varphi^2(q^2)\varphi^2(q^7) \in M_2(\Gamma_0(56), \chi_0)$. Third we consider $(1, 2, 7, 14)$

$$\varphi(q)\varphi(q^2)\varphi(q^7)\varphi(q^{14}) = \frac{\eta^3(2z)\eta^3(4z)\eta^3(14z)\eta^3(28z)}{\eta^2(z)\eta^2(7z)\eta^2(8z)\eta^2(56z)}.$$

Then we have

Table 3.2.4(a)

δ	1	2	4	7	8	14	28	56
r_δ	-2	3	3	-2	-2	3	3	-2

It can be seen from Table 3.2.4(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.2.4(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	48/7	96/7	0	0	48	96	0

From Table 3.2.4(b) the condition $(L3)$ is also satisfied. Thus $\varphi(q)\varphi(q^2)\varphi(q^7)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_0)$. Fourth we consider $(1, 1, 14, 14)$

$$\varphi^2(q)\varphi^2(q^{14}) = \frac{\eta^{10}(2z)\eta^{10}(28z)}{\eta^4(z)\eta^4(4z)\eta^4(14z)\eta^4(56z)}.$$

Then we have

Table 3.2.5(a)

δ	1	2	4	14	28	56
r_δ	-4	10	-4	-4	10	-4

It can be seen from Table 3.2.5(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.2.5(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	12	24/7	0	0	12	168	0

From Table 3.2.5(b) the condition $(L3)$ is also satisfied. Thus $\varphi^2(q)\varphi^2(q^{14}) \in M_2(\Gamma_0(56), \chi_0)$. ■

Theorem 3.2.2. *$A_k(q)$ $(1 \le k \le 5)$ given by* (3.2.1)–(3.2.5) *are in $S_2(\Gamma_0(56), \chi_0)$.*

Proof. We will check conditions (L1), (L2) and (L4) of Theorem 3.1.1. We have $N = 56$. First we consider

$$A_1(q) = \eta(2z)\eta(4z)\eta(14z)\eta(28z).$$

Then we have

Table 3.2.6(a)

δ	2	4	14	28
r_δ	1	1	1	1

It can be seen from Table 3.2.6(a) that conditions (L1) and (L2) are satisfied.

Table 3.2.6(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/7	24/7	48/7	6	48/7	24	48	48

From Table 3.2.6(b) the condition $(L4)$ is also satisfied. Thus $A_1(q) \in S_2(\Gamma_0(56), \chi_0)$.

Then

$$A_2(q) = \frac{\eta^3(2z)\eta^3(28z)}{\eta(4z)\eta(14z)}.$$

Then we have

Table 3.2.7(a)

δ	2	4	14	28
r_δ	3	-1	-1	3

It can be seen from Table 3.2.7(a) that conditions (L1) and (L2) are satisfied.

Table 3.2.7(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	9/7	36/7	24/7	3	24/7	12	72	72

From Table 3.2.7(b) the condition $(L4)$ is also satisfied. thus $A_2(q) \in S_2(\Gamma_0(56), \chi_0)$.

Then

$$A_3(q) = \frac{\eta(z)\eta^3(4z)\eta(7z)\eta^3(28z)}{\eta(2z)\eta(8z)\eta(14z)\eta(56z)}.$$

Then we have

Table 3.2.8(a)

δ	1	2	4	7	8	14	28	56
r_δ	1	−1	3	1	−1	−1	3	−1

It can be seen from Table 3.2.8(a) that conditions (L1) and (L2) are satisfied.

Table 3.2.8(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	9/7	12/7	72/7	9	24/7	12	72	24

From Table 3.2.8(b) the condition $(L4)$ is also satisfied. Thus $A_3(q) \in S_2(\Gamma_0(56), \chi_0)$.

Then

$$A_4(q) = \frac{\eta^3(2z)\eta(8z)\eta^3(14z)\eta(56z)}{\eta(z)\eta(4z)\eta(7z)\eta(28z)}.$$

Then we have

Table 3.2.9(a)

δ	1	2	4	7	8	14	28	56
r_δ	−1	3	−1	−1	1	3	−1	1

It can be seen from Table 3.2.9(a) that conditions (L1) and (L2) are satisfied.

Table 3.2.9(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	36/7	24/7	3	72/7	36	24	72

From Table 3.2.9(b) the condition $(L4)$ is also satisfied. Thus $A_4(q) \in S_2(\Gamma_0(56), \chi_0)$.

Then

$$A_5(q) = \frac{\eta^4(4z)\eta^4(28z)}{\eta(2z)\eta(8z)\eta(14z)\eta(56z)}.$$

Then we have

Table 3.2.10(a)

δ	2	4	8	14	28	56
r_δ	-1	4	-1	-1	4	-1

It can be seen from Table 3.2.10(a) that conditions (L1) and (L2) are satisfied.

Table 3.2.10(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	12/7	96/7	3	48/7	12	96	48

From Table 3.2.10(b) the condition $(L4)$ is also satisfied. Thus $A_5(q)$ $\in S_2(\Gamma_0(56), \chi_0)$. ■

Theorem 3.2.3. **(a)** $\{A_1(q), \ldots, A_5(q)\}$ *constitute a basis for* $S_2(\Gamma_0(56), \chi_0)$.
(b) $L(q) - tL(q^t)$ $(t = 2, 4, 7, 8, 14, 28, 56)$ *constitute a basis for* $E_2(\Gamma_0(56), \chi_0)$.
(c) $L(q) - tL(q^t)$ $(t = 2, 4, 7, 8, 14, 28, 56)$ *together with* $A_k(q)$ $(1 \le k \le 5)$ *constitute a basis for* $M_2(\Gamma_0(56), \chi_0)$.

Proof. **(a)** By Theorem 3.2.2, $A_k(q)$ $(1 \le k \le 5) \in S_2(\Gamma_0(56), \chi_0)$. There is no linear relationship among them. By Example 2.3.2, we have $\dim S_2(\Gamma_0(56), \chi_0) = 5$. Thus $A_k(q)$ $(1 \le k \le 5)$ constitute a basis for $S_2(\Gamma_0(56), \chi_0)$.
(b) By Example 2.3.2, we have $\dim E_2(\Gamma_0(56), \chi_0) = 7$. By Theorem 2.2.3, $L(q) - tL(q^t)$ $(t = 2, 4, 7, 8, 14, 28, 56)$ constitute a basis for $E_2(\Gamma_0(56), \chi_0)$.
(c) It follows from (a), (b) and (2.1.1) that the dimension of $M_2(\Gamma_0(56), \chi_0)$ is 12 and therefore $L(q) - tL(q^t)$ $(t = 2, 4, 7, 8, 14, 28, 56)$ together with $A_k(q)$ $(1 \le k \le 5)$ constitute a basis for $M_2(\Gamma_0(56), \chi_0)$. ■

Theorem 3.2.4.

$$\begin{aligned}\textbf{(a)}\ \varphi^2(q)\varphi^2(q^7) =& \frac{4}{3}\ L(q) - \frac{8}{3}\ L(q^2) + \frac{16}{3}\ L(q^4) - \frac{28}{3}\ L(q^7) + \frac{56}{3}\ L(q^{14}) \\ &- \frac{112}{3}\ L(q^{28}) + \frac{8}{3}A_3(q) - \frac{16}{3}A_4(q) + \frac{16}{3}A_5(q),\end{aligned}$$

$$\begin{aligned}(\mathbf{b})\ \varphi^2(q^2)\varphi^2(q^7) =& \frac{2}{3}\ L(q) - \frac{2}{3}\ L(q^2) - \frac{4}{3}\ L(q^4) - \frac{14}{3}\ L(q^7) + \frac{16}{3}\ L(q^8)\\ &+ \frac{14}{3}\ L(q^{14}) + \frac{28}{3}\ L(q^{28}) - \frac{112}{3}\ L(q^{56}) - \frac{10}{3}A_1(q) + 4A_2(q)\\ &- \frac{2}{3}A_3(q) - \frac{20}{3}A_4(q) + \frac{16}{3}A_5(q),\end{aligned}$$

$$\begin{aligned}(\mathbf{c})\ \varphi(q)\varphi(q^2)\varphi(q^7)\varphi(q^{14}) =& \frac{2}{3}\ L(q) - \frac{2}{3}\ L(q^2) - \frac{4}{3}\ L(q^4) - \frac{14}{3}\ L(q^7)\\ &+ \frac{16}{3}\ L(q^8) + \frac{14}{3}\ L(q^{14}) + \frac{28}{3}\ L(q^{28}) - \frac{112}{3}\ L(q^{56})\\ &+ \frac{2}{3}A_1(q) + \frac{4}{3}A_3(q) + \frac{4}{3}A_4(q) + \frac{4}{3}A_5(q),\end{aligned}$$

$$\begin{aligned}(\mathbf{d})\ \varphi^2(q)\varphi^2(q^{14}) =& \frac{2}{3}\ L(q) - \frac{2}{3}\ L(q^2) - \frac{4}{3}\ L(q^4) - \frac{14}{3}\ L(q^7) + \frac{16}{3}\ L(q^8)\\ &+ \frac{14}{3}\ L(q^{14}) + \frac{28}{3}\ L(q^{28}) - \frac{112}{3}\ L(q^{56}) + \frac{2}{3}A_1(q) - 4A_2(q)\\ &+ \frac{10}{3}A_3(q) + \frac{4}{3}A_4(q) + \frac{16}{3}A_5(q).\end{aligned}$$

Proof. Let (a_1, a_2, a_3, a_4) be one of the quadratic forms listed in the first column of Table 3.1.1. By Theorem 3.2.1 and Theorem 3.2.3 (c), $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combinations of $L(q) - tL(q^t)$ $(t = 2, 4, 7, 8, 14, 28, 56)$ and $A_k(q)$ $(1 \leq k \leq 5)$, namely

$$\begin{aligned}\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) &= x_1\ (L(q) - 2L(q^2)) + x_2\ (L(q) - 4L(q^4))\\ &+x_3\ (L(q) - 7L(q^7)) + x_4\ (L(q) - 8L(q^8)) + x_5\ (L(q) - 14L(q^{14}))\\ &+x_6\ (L(q) - 28L(q^{28})) + x_7\ (L(q) - 56L(q^{56})) + y_1A_1(q) + y_2A_2(q)\\ &+y_3A_3(q) + y_4A_4(q) + y_5A_5(q).\end{aligned}$$

We equate the first 60 coefficients of q^n on both sides of the equation above to obtain a system of linear equations with the unknowns $x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_1, y_2, y_3, y_4, y_5$. Then, using MAPLE we solve the system to find the asserted coefficients. ∎

We now give an explicit formulae for $N(a_1, a_2, a_3, a_4; n)$ for the quadratic forms (a_1, a_2, a_3, a_4) in Theorem 3.2.4 in terms of $\sigma(n/d)$ $(d = 1, 2, 4, 7, 14, 28, 56)$ and $a_k(n)$ $(1 \leq k \leq 5)$.

Theorem 3.2.5. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
\textbf{(a)}\ N(1,1,7,7;n) =& \frac{4}{3}\sigma(n) - \frac{8}{3}\sigma(n/2) + \frac{16}{3}\sigma(n/4) - \frac{28}{3}\sigma(n/7) + \frac{56}{3}\sigma(n/14) \\
& - \frac{112}{3}\sigma(n/28) + \frac{8}{3}a_3(n) - \frac{16}{3}a_4(n) + \frac{16}{3}a_5(n), \\
\textbf{(b)}\ N(2,2,7,7;n) =& \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8) \\
& + \frac{14}{3}\sigma(n/14) + \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) - \frac{10}{3}a_1(n) \\
& + 4a_2(n) - \frac{2}{3}a_3(n) - \frac{20}{3}a_4(n) + \frac{16}{3}a_5(n), \\
\textbf{(c)}\ N(1,2,7,14;n) =& \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8) \\
& + \frac{14}{3}\sigma(n/14) + \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) + \frac{2}{3}a_1(n) \\
& + \frac{4}{3}a_3(n) + \frac{4}{3}a_4(n) + \frac{4}{3}a_5(n), \\
\textbf{(d)}\ N(1,1,14,14;n) =& \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8) \\
& + \frac{14}{3}\sigma(n/14) + \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) + \frac{2}{3}a_1(n) \\
& - 4a_2(n) + \frac{10}{3}a_3(n) + \frac{4}{3}a_4(n) + \frac{16}{3}a_5(n).
\end{aligned}$$

Proof. From (2.1.6), (2.2.3) and Theorem 3.2.4, we obtain

$$\begin{aligned}
\textbf{(a)}\ & \sum_{n=0}^{\infty} N(1,1,7,7;n)q^n = \varphi^2(q)\varphi^2(q^7) \\
& = 1 + \sum_{n=1}^{\infty} \Big(\frac{4}{3}\sigma(n) - \frac{8}{3}\sigma(n/2) + \frac{16}{3}\sigma(n/4) - \frac{28}{3}\sigma(n/7) + \frac{56}{3}\sigma(n/14) \\
& \quad - \frac{112}{3}\sigma(n/28) + \frac{8}{3}a_3(n) - \frac{16}{3}a_4(n) + \frac{16}{3}a_5(n) \Big) q^n,
\end{aligned}$$

(b) $$\sum_{n=0}^{\infty} N(2,2,7,7;n)q^n = \varphi^2(q^2)\varphi^2(q^7)$$

$$= 1 + \sum_{n=1}^{\infty} \Big(\frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8) + \frac{14}{3}\sigma(n/14)$$
$$+ \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) - \frac{10}{3}a_1(n) + 4a_2(n) - \frac{2}{3}a_3(n) - \frac{20}{3}a_4(n)$$
$$+ \frac{16}{3}a_5(n)\Big) q^n,$$

(c) $$\sum_{n=0}^{\infty} N(1,2,7,14;n)q^n = \varphi(q)\varphi(q^2)\varphi(q^7)\varphi(q^{14})$$

$$= 1 + \sum_{n=1}^{\infty} \Big(\frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8)$$
$$+ \frac{14}{3}\sigma(n/14) + \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) + \frac{2}{3}a_1(n) + \frac{4}{3}a_3(n) + \frac{4}{3}a_4(n)$$
$$+ \frac{4}{3}a_5(n)\Big) q^n,$$

(d) $$\sum_{n=0}^{\infty} N(1,1,14,14;n)q^n = \varphi^2(q)\varphi^2(q^{14})$$

$$= 1 + \sum_{n=1}^{\infty} \Big(\frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) - \frac{4}{3}\sigma(n/4) - \frac{14}{3}\sigma(n/7) + \frac{16}{3}\sigma(n/8) + \frac{14}{3}\sigma(n/14)$$
$$+ \frac{28}{3}\sigma(n/28) - \frac{112}{3}\sigma(n/56) + \frac{2}{3}a_1(n) - 4a_2(n) + \frac{10}{3}a_3(n) + \frac{4}{3}a_4(n)$$
$$+ \frac{16}{3}a_5(n)\Big) q^n.$$

Equating the coefficients of q^n on both sides of equations (a)–(d) yields the results. ∎

For $(a_1, a_2, a_3, a_4) = (1,1,7,7),\ (1,1,14,14),\ (2,2,7,7),\ (1,2,7,14)$, the values of $N(a_1, a_2, a_3, a_4; n)$ for $1 \le n \le 20$ are given in Table 3.2.11. One can verify them by using Table 3.2.1.

Table 3.2.11

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$N(1,1,7,7;n)$	4	4	0	4	8	0	4	20	20	8	16	32	8	4	32	36	40	20	32	40
$N(2,2,7,7;n)$	0	4	0	4	0	0	4	4	16	8	16	0	0	4	16	20	32	20	0	8
$N(1,2,7,14;n)$	2	2	4	2	0	4	2	6	10	8	8	4	8	2	8	18	12	18	12	8

$N(1,1,14,14;n)$	4	4	0	4	8	0	0	4	4	8	0	0	8	4	16	20	8	20	32	8

For example, using Table 3.2.1, we obtain

$$\begin{aligned} N(2,2,7,7;16) &= \frac{2}{3}\sigma(16) - \frac{2}{3}\sigma(8) - \frac{4}{3}\sigma(4) - \frac{14}{3}\sigma(16/7) + \frac{16}{3}\sigma(2) \\ &\quad + \frac{14}{3}\sigma(16/14) + \frac{28}{3}\sigma(16/28) - \frac{112}{3}\sigma(16/56) - \frac{10}{3}a_1(16) \\ &\quad + 4a_2(16) - \frac{2}{3}a_3(16) - \frac{20}{3}a_4(16) + \frac{16}{3}a_5(16) \\ &= \frac{2}{3}(31) - \frac{2}{3}(15) - \frac{4}{3}(7) + \frac{16}{3}(3) - \frac{10}{3}(-1) - \frac{2}{3}(-1) - \frac{20}{3} + \frac{16}{3} \\ &= 20, \end{aligned}$$

which agrees with the value of $N(2,2,7,7;16)$ in Table 3.2.11.

3.3 The space $M_2(\Gamma_0(56), \chi_3)$

Let χ_0 be the trivial character and χ_1, χ_2, χ_3 as in (3.1.3). We define the Eisenstein series

$$E_{2,\chi_3,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_3,\chi_0)}(n) q^n, \tag{3.3.1}$$

$$E_{2,\chi_0,\chi_3}(q) = -4 + \sum_{n=1}^{\infty} \sigma_{(\chi_0,\chi_3)}(n) q^n, \tag{3.3.2}$$

$$E_{2,\chi_1,\chi_2}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_1,\chi_2)}(n) q^n, \tag{3.3.3}$$

$$E_{2,\chi_2,\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_2,\chi_1)}(n) q^n. \tag{3.3.4}$$

We determine $N(a_1, a_2, a_3, a_4; n)$ for the six quaternary quadratic forms listed in the second column of Table 3.1.1 in terms of $\sigma_{(\chi,\psi)}(n)$, where $\chi, \psi \in \{\chi_0, \chi_1, \chi_2, \chi_3\}$, and

$b_k(n)$ $(1 \leq k \leq 4)$ defined by

$$B_1(q) = \sum_{n=1}^{\infty} b_1(n)q^n = \frac{\eta^2(2z)\eta^3(7z)}{\eta(z)}, \tag{3.3.5}$$

$$B_2(q) = \sum_{n=1}^{\infty} b_2(n)q^n = \frac{\eta^3(8z)\eta^2(28z)}{\eta(56z)}, \tag{3.3.6}$$

$$B_3(q) = \sum_{n=1}^{\infty} b_3(n)q^n = \frac{\eta^2(4z)\eta^3(56z)}{\eta(8z)}, \tag{3.3.7}$$

$$B_4(q) = \sum_{n=1}^{\infty} b_4(n)q^n = \frac{\eta^3(z)\eta^2(14z)}{\eta(7z)}. \tag{3.3.8}$$

There is no linear relationship among the $B_k(q), 1 \leq k \leq 4$. The first fifty-six values of $b_k(n)$, $1 \leq k \leq 4$, are given in Table 3.3.1.

Table 3.3.1

n	$b_1(n)$	$b_2(n)$	$b_3(n)$	$b_4(n)$	n	$b_1(n)$	$b_2(n)$	$b_3(n)$	$b_4(n)$
1	1	1	0	1	29	−2	−2	0	−2
2	1	0	0	−3	30	0	0	0	0
3	0	0	0	0	31	0	0	0	0
4	1	0	0	5	32	5	0	0	9
5	0	0	0	0	33	0	0	0	0
6	0	0	0	0	34	0	0	0	0
7	1	0	1	−7	35	0	0	0	0
8	−3	0	0	1	36	−3	0	0	−15
9	−3	−3	0	−3	37	6	6	0	6
10	0	0	0	0	38	0	0	0	0
11	−2	0	−2	14	39	0	0	0	0
12	0	0	0	0	40	0	0	0	0
13	0	0	0	0	41	0	0	0	0
14	−3	0	0	−7	42	0	0	0	0
15	0	0	0	0	43	−2	0	−2	14
16	1	0	0	−11	44	−10	0	0	14
17	0	0	0	0	45	0	0	0	0
18	−3	0	0	9	46	−6	0	0	−14
19	0	0	0	0	47	0	0	0	0
20	0	0	0	0	48	0	0	0	0
21	0	0	0	0	49	−7	−7	0	−7
22	6	0	0	14	50	5	0	0	−15
23	2	0	2	−14	51	0	0	0	0
24	0	0	0	0	52	0	0	0	0
25	5	5	0	5	53	−10	−10	0	−10
26	0	0	0	0	54	0	0	0	0
27	0	0	0	0	55	0	0	0	0

28	5	0	0	−7	56	1	0	0	21

Theorem 3.3.1. *Let (a_1, a_2, a_3, a_4) be as in the second column of Table 3.1.1. Then $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_3)$.*

Proof. We appeal to (2.1.9) for each of the six quadratic forms and then check the conditions (L1), (L2) and (L3) of Theorem 3.1.1 for each quadratic form. We have $N = 56$. First we consider $(1, 1, 1, 7)$

$$\varphi^3(q)\varphi(q^7) = \frac{\eta^{15}(2z)\eta^5(14z)}{\eta^6(z)\eta^6(4z)\eta^2(7z)\eta^2(28z)}.$$

Then we have

Table 3.3.2(a)

δ	1	2	4	7	14	28
r_δ	−6	15	−6	−2	5	−2

It can be seen from Table 3.3.2(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.3.2(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	132/7	0	0	0	60	0	0

From Table 3.3.2(b) the condition $(L3)$ is also satisfied. Thus $\varphi^3(q)\varphi(q^7)$ $\in M_2(\Gamma_0(56), \chi_3)$. Now for the form $(1, 2, 2, 7)$ we have

$$\varphi(q)\varphi^2(q^2)\varphi(q^7) = \frac{\eta(2z)\eta^8(4z)\eta^5(14z)}{\eta^2(z)\eta^2(7z)\eta^4(8z)\eta^2(28z)}.$$

Then we have

Table 3.3.3(a)

δ	1	2	4	7	8	14	28
r_δ	-2	1	8	-2	-4	5	-2

It can be seen from Table 3.3.3(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 3.3.3(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	48/7	24	0	0	48	24	0

From Table 3.3.3(b) the condition ($L3$) is also satisfied. Thus $\varphi(q)\varphi^2(q^2)\varphi(q^7) \in M_2(\Gamma_0(56), \chi_3)$. Then for the form $(1, 7, 7, 7)$ we have

$$\varphi(q)\varphi^3(q^7) = \frac{\eta^5(2z)\eta^{15}(14z)}{\eta^2(z)\eta^2(4z)\eta^6(7z)\eta^6(28z)}.$$

Then we have

Table 3.3.4(a)

δ	1	2	4	7	14	28
r_δ	-2	5	-2	-6	15	-6

It can be seen from Table 3.3.4(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 3.3.4(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	60/7	0	0	0	132	0	0

From Table 3.3.4(b) the condition ($L3$) is also satisfied. Thus $\varphi(q)\varphi^3(q^7) \in M_2(\Gamma_0(56), \chi_3)$. Now for the form $(1, 1, 2, 14)$ we have

$$\varphi^2(q)\varphi(q^2)\varphi(q^{14}) = \frac{\eta^8(2z)\eta(4z)\eta^5(28z)}{\eta^4(z)\eta^2(8z)\eta^2(14z)\eta^2(56z)}.$$

Then we have

Table 3.3.5(a)

δ	1	2	4	8	14	28	56
r_δ	-4	8	1	-2	-2	5	-2

It can be seen from Table 3.3.5(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.3.5(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\displaystyle\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	12	96/7	0	0	12	96	0

From Table 3.3.5(b) the condition $(L3)$ is also satisfied. Thus $\varphi^2(q)\varphi(q^2)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_3)$. Now for the form $(2, 7, 7, 14)$ we have

$$\varphi(q^2)\varphi^2(q^7)\varphi(q^{14}) = \frac{\eta^5(4z)\eta^8(14z)\eta(28z)}{\eta^2(2z)\eta^4(7z)\eta^2(8z)\eta^2(56z)}.$$

Then we have

Table 3.3.6(a)

δ	2	4	7	8	14	28	56
r_δ	-2	5	-4	-2	8	1	-2

It can be seen from Table 3.3.6(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.3.6(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\displaystyle\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	12/7	96/7	0	0	84	96	0

From Table 3.3.6(b) the condition $(L3)$ is also satisfied. Thus $\varphi(q^2)\varphi^2(q^7)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_3)$. Now for the form $(1, 7, 14, 14)$ we have

$$\varphi(q)\varphi(q^7)\varphi^2(q^{14}) = \frac{\eta^5(2z)\eta(14z)\eta^8(28z)}{\eta^2(z)\eta^2(4z)\eta^2(7z)\eta^4(56z)}.$$

Then we have

Table 3.3.7(a)

δ	1	2	4	7	14	28	56
r_δ	-2	5	-2	-2	1	8	-4

It can be seen from Table 3.3.7(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.3.7(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	48/7	24/7	0	0	48	168	0

From Table 3.3.7(b) the condition $(L3)$ is also satisfied. Thus $\varphi(q)\varphi(q^7)\varphi^2(q^{14}) \in M_2(\Gamma_0(56), \chi_3)$. ■

Theorem 3.3.2. *$B_k(q)$ $(1 \leq k \leq 4)$ given by* (3.3.5)–(3.3.8) *are in* $S_2(\Gamma_0(56), \chi_3)$.

Proof. We will check conditions (L1),(L2) and (L4) of Theorem 3.1.1. We have $N = 56$. First we consider

$$B_1(q) = \frac{\eta^2(2z)\eta^3(7z)}{\eta(z)}.$$

Then we have

Table 3.3.8(a)

δ	1	2	7
r_δ	-1	2	3

It can be seen from Table 3.3.8(a) that conditions (L1) and (L2) are satisfied.

Table 3.3.8(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	24/7	24/7	21	24/7	24	24	24

From Table 3.3.8(b) the condition $(L4)$ is also satisfied. Thus $B_1(q) \in S_2(\Gamma_0(56), \chi_3)$. Secondly we consider

$$B_2(q) = \frac{\eta^3(8z)\eta^2(28z)}{\eta(56z)}.$$

Then we have

Table 3.3.9(a)

δ	8	28	56
r_δ	3	2	-1

It can be seen from Table 3.3.9(a) that conditions (L1) and (L2) are satisfied.

Table 3.3.9(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	12/7	48/7	3	24	12	48	24

From Table 3.3.9(b) the condition $(L4)$ is also satisfied. Thus $B_2(q) \in S_2(\Gamma_0(56), \chi_3)$. Thirdly we consider

$$B_3(q) = \frac{\eta^2(4z)\eta^3(56z)}{\eta(8z)}.$$

Then we have

Table 3.3.10(a)

δ	4	8	56
r_δ	2	-1	3

It can be seen from Table 3.3.10(a) that conditions (L1) and (L2) are satisfied.

Table 3.3.10(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	12/7	48/7	3	24/7	12	48	168

From Table 3.3.10(b) the condition $(L4)$ is also satisfied. Thus $B_3(q) \in S_2(\Gamma_0(56), \chi_3)$.

Fourthly we consider

$$B_4(q) = \frac{\eta^3(z)\eta^2(14z)}{\eta(7z)}.$$

Then we have

Table 3.3.11(a)

δ	1	7	14
r_δ	3	-1	2

It can be seen from Table 3.3.11(a) that conditions (L1) and (L2) are satisfied.

Table 3.3.11(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3	24/7	24/7	3	24/7	24	24	24

From Table 3.3.11(b) the condition $(L4)$ is also satisfied. Thus $B_4(q) \in S_2(\Gamma_0(56), \chi_3)$. ∎

Theorem 3.3.3. **(a)** $\{B_1(q), B_2(q), B_3(q), B_4(q)\}$ *is a basis for* $S_2(\Gamma_0(56), \chi_3)$.
(b) $\{E_{2,\chi_3,\chi_0}(q^t),\ E_{2,\chi_0,\chi_3}(q^t),\ E_{2,\chi_1,\chi_2}(q^t),\ E_{2,\chi_2,\chi_1}(q^t) \mid t = 1, 2\}$ *is a basis for* $E_2(\Gamma_0(56), \chi_3)$.
(c) $\{E_{2,\chi_3,\chi_0}(q^t),\ E_{2,\chi_0,\chi_3}(q^t),\ E_{2,\chi_1,\chi_2}(q^t),\ E_{2,\chi_2,\chi_1}(q^t) \mid t = 1, 2\}$ *together with* $B_k(q)$ $(1 \le k \le 4)$ *constitute a basis for* $M_2(\Gamma_0(56), \chi_3)$.

Proof. **(a)** By Theorem 3.3.2, $B_k(q)$ $(1 \le k \le 4) \in S_2(\Gamma_0(56), \chi_3)$. There is no linear relationship among them. By Example 2.3.3, we have $\dim S_2(\Gamma_0(56), \chi_3) = 4$. Therefore, $B_k(q)$ $(1 \le k \le 4)$ constitute a basis for $S_2(\Gamma_0(56), \chi_3)$.
(b) By Example 2.3.3, we have $\dim E_2(\Gamma_0(56), \chi_3) = 8$. By Theorem 2.2.3, $\{E_{2,\chi_3,\chi_0}(q^t),\ E_{2,\chi_0,\chi_3}(q^t),\ E_{2,\chi_1,\chi_2}(q^t),\ E_{2,\chi_2,\chi_1}(q^t) \mid t = 1, 2\}$ is a basis for $E_2(\Gamma_0(56), \chi_3)$.
(c) By Example 2.3.3, we have $\dim M_2(\Gamma_0(56), \chi_3) = 12$. Therefore, by (2.1.1)

$\{E_{2,\chi_3,\chi_0}(q^t),\ E_{2,\chi_0,\chi_3}(q^t),\ E_{2,\chi_1,\chi_2}(q^t),\ E_{2,\chi_2,\chi_1}(q^t) \mid t=1,2\}$ together with $B_k(q)$ $(1 \leq k \leq 4)$ constitute a basis for $M_2(\Gamma_0(56),\chi_3)$. ∎

Theorem 3.3.4. *Let χ_0 be the trivial character and χ_1,χ_2,χ_3 be as in* (3.1.3). *Then*

$$\begin{aligned}
\textbf{(a)}\ \varphi^3(q)\varphi(q^7) =& \frac{7}{2}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q) + \frac{7}{4}E_{2,\chi_1,\chi_2}(q) - \frac{1}{2}E_{2,\chi_2,\chi_1}(q)\\
&+ 3B_2(q) - 21B_3(q) - \frac{3}{2}B_4(q),\\
\textbf{(b)}\ \varphi(q)\varphi^2(q^2)\varphi(q^7) =& \frac{7}{4}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q^2) + \frac{7}{4}E_{2,\chi_1,\chi_2}(q^2) - \frac{1}{4}E_{2,\chi_2,\chi_1}(q)\\
&- \frac{7}{8}B_1(q) + \frac{3}{2}B_2(q) - \frac{21}{2}B_3(q) - \frac{1}{8}B_4(q),\\
\textbf{(c)}\ \varphi(q)\varphi^3(q^7) =& \frac{1}{2}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q) - \frac{1}{4}E_{2,\chi_1,\chi_2}(q) + \frac{1}{2}E_{2,\chi_2,\chi_1}(q)\\
&- \frac{3}{2}B_1(q) + 3B_2(q) + 3B_3(q),\\
\textbf{(d)}\ \varphi^2(q)\varphi(q^2)\varphi(q^{14}) =& \frac{7}{4}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q^2) + \frac{7}{4}E_{2,\chi_1,\chi_2}(q^2) - \frac{1}{4}E_{2,\chi_2,\chi_1}(q)\\
&+ \frac{21}{8}B_1(q) - \frac{1}{2}B_2(q) + \frac{7}{2}B_3(q) + \frac{3}{8}B_4(q),\\
\textbf{(e)}\ \varphi(q^2)\varphi^2(q^7)\varphi(q^{14}) =& \frac{1}{4}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q^2) - \frac{1}{4}E_{2,\chi_1,\chi_2}(q^2) + \frac{1}{4}E_{2,\chi_2,\chi_1}(q)\\
&+ \frac{3}{8}B_1(q) - \frac{1}{2}B_2(q) - \frac{1}{2}B_3(q) - \frac{3}{8}B_4(q),\\
\textbf{(f)}\ \varphi(q)\varphi(q^7)\varphi^2(q^{14}) =& \frac{1}{4}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q^2) - \frac{1}{4}E_{2,\chi_1,\chi_2}(q^2) + \frac{1}{4}E_{2,\chi_2,\chi_1}(q)\\
&- \frac{1}{8}B_1(q) + \frac{3}{2}B_2(q) + \frac{3}{2}B_3(q) + \frac{1}{8}B_4(q).
\end{aligned}$$

Proof. Let (a_1,a_2,a_3,a_4) be one of the quadratic forms listed in the second column of Table 3.1.1. By Theorem 3.3.1 we have $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56),\chi_3)$. Therefore, by Theorem 3.3.3 (c), $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $\{E_{2,\chi_3,\chi_0}(q^t),\ E_{2,\chi_0,\chi_3}(q^t),\ E_{2,\chi_1,\chi_2}(q^t),\ E_{2,\chi_2,\chi_1}(q^t) \mid t=1,2\}$ and $B_k(q)$ $(1 \leq$

$k \leq 4$), namely

$$\begin{aligned}\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) =& x_1 E_{2,\chi_3,\chi_0}(q) + x_2 E_{2,\chi_3,\chi_0}(q^2) + x_3 E_{2,\chi_0,\chi_3}(q) \\ &+ x_4 E_{2,\chi_0,\chi_3}(q^2) + x_5 E_{2,\chi_1,\chi_2}(q) + x_6 E_{2,\chi_1,\chi_2}(q^2) \\ &+ x_7 E_{2,\chi_2,\chi_1}(q) + x_8 E_{2,\chi_2,\chi_1}(q^2) + y_1 B_1(q) + y_2 B_2(q) \\ &+ y_3 B_3(q) + y_4 B_4(q).\end{aligned}$$

We equate the first twenty coefficients of q^n on both sides of the equation above to obtain a system of linear equations with the unknowns $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ and y_1, y_2, y_3, y_4. Then, using MAPLE we solve the system to find the asserted coefficients. ■

Theorem 3.3.5. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in* (2.2.1) *for $i, j \in \{0,1,2,3\}$. Then*

$$\begin{aligned}\textbf{(a) } N(1,1,1,7;n) =& \frac{7}{2}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n) - \frac{1}{2}\sigma_{\chi_2,\chi_1}(n) + 3b_2(n) \\ &- 21b_3(n) - \frac{3}{2}b_4(n),\end{aligned}$$

$$\begin{aligned}\textbf{(b) } N(1,2,2,7;n) =& \frac{7}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n/2) - \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) \\ &- \frac{7}{8}b_1(n) + \frac{3}{2}b_2(n) - \frac{21}{2}b_3(n) - \frac{1}{8}b_4(n),\end{aligned}$$

$$\begin{aligned}\textbf{(c) } N(1,7,7,7;n) =& \frac{1}{2}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n) + \frac{1}{2}\sigma_{\chi_2,\chi_1}(n) - \frac{3}{2}b_1(n) \\ &+ 3b_2(n) + 3b_3(n),\end{aligned}$$

$$\begin{aligned}\textbf{(d) } N(1,1,2,14;n) =& \frac{7}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n/2) - \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) \\ &+ \frac{21}{8}b_1(n) - \frac{1}{2}b_2(n) + \frac{7}{2}b_3(n) + \frac{3}{8}b_4(n),\end{aligned}$$

$$\begin{aligned}\textbf{(e) } N(2,7,7,14;n) =& \frac{1}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n/2) + \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) \\ &+ \frac{3}{8}b_1(n) - \frac{1}{2}b_2(n) - \frac{1}{2}b_3(n) - \frac{3}{8}b_4(n),\end{aligned}$$

$$\textbf{(f)}\ N(1,7,14,14;n) = \frac{1}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n/2) + \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) - \frac{1}{8}b_1(n) + \frac{3}{2}b_2(n) + \frac{3}{2}b_3(n) + \frac{1}{8}b_4(n).$$

Proof. From (2.1.6), (3.3.1)-(3.3.4) and Theorem 3.3.4, we obtain

$$\textbf{(a)}\ \sum_{n=0}^{\infty} N(1,1,1,7;n)q^n = \varphi^3(q)\varphi(q^7)$$
$$= \frac{7}{2}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q) + \frac{7}{4}E_{2,\chi_1,\chi_2}(q) - \frac{1}{2}E_{2,\chi_2,\chi_1}(q) + 3B_2(q) - 21B_3(q) - \frac{3}{2}B_4(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{7}{2}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n) - \frac{1}{2}\sigma_{\chi_2,\chi_1}(n) + 3b_2(n) - 21b_3(n) - \frac{3}{2}b_4(n)\Big)q^n,$$

$$\textbf{(b)}\ \sum_{n=0}^{\infty} N(1,2,2,7;n)q^n = \varphi(q)\varphi^2(q^2)\varphi(q^7)$$
$$= \frac{7}{4}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q^2) + \frac{7}{4}E_{2,\chi_1,\chi_2}(q^2) - \frac{1}{4}E_{2,\chi_2,\chi_1}(q) - \frac{7}{8}B_1(q) + \frac{3}{2}B_2(q) - \frac{21}{2}B_3(q) - \frac{1}{8}B_4(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{7}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n/2) - \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) - \frac{7}{8}b_1(n) + \frac{3}{2}b_2(n) - \frac{21}{2}b_3(n) - \frac{1}{8}b_4(n)\Big)q^n,$$

$$\textbf{(c)}\ \sum_{n=0}^{\infty} N(1,7,7,7;n)q^n = \varphi(q)\varphi^3(q^7)$$
$$= \frac{1}{2}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q) - \frac{1}{4}E_{2,\chi_1,\chi_2}(q) + \frac{1}{2}E_{2,\chi_2,\chi_1}(q) - \frac{3}{2}B_1(q) + 3B_2(q) + 3B_3(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(- \frac{1}{2}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n) + \frac{1}{2}\sigma_{\chi_2,\chi_1}(n) - \frac{3}{2}b_1(n) + 3b_2(n) + 3b_3(n)\Big)q^n,$$

(d) $\displaystyle\sum_{n=0}^{\infty} N(1,1,2,14;n)q^n = \varphi^2(q)\varphi(q^2)\varphi(q^{14})$

$$= \frac{7}{4}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q^2) + \frac{7}{4}E_{2,\chi_1,\chi_2}(q^2) - \frac{1}{4}E_{2,\chi_2,\chi_1}(q) + \frac{21}{8}B_1(q)$$
$$- \frac{1}{2}B_2(q) + \frac{7}{2}B_3(q) + \frac{3}{8}B_4(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{7}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(n/2) - \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) + \frac{21}{8}b_1(n)$$
$$- \frac{1}{2}b_2(n) + \frac{7}{2}b_3(n) + \frac{3}{8}b_4(n)\Big)q^n,$$

(e) $\displaystyle\sum_{n=0}^{\infty} N(2,7,7,14;n)q^n = \varphi(q)\varphi^2(q^7)\varphi(q^{14})$

$$=\frac{1}{4}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q^2) - \frac{1}{4}E_{2,\chi_1,\chi_2}(q^2) + \frac{1}{4}E_{2,\chi_2,\chi_1}(q) + \frac{3}{8}B_1(q) - \frac{1}{2}B_2(q)$$
$$- \frac{1}{2}B_3(q) - \frac{3}{8}B_4(q) - \frac{1}{2}D_1(q) - \frac{7}{5}D_3(q) - \frac{12}{5}D_4(q) + \frac{1}{2}D_5(q) + \frac{7}{5}D_6(q),$$
$$=1 + \sum_{n=1}^{\infty}\Big(\frac{1}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n/2) + \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) + \frac{3}{8}b_1(n)$$
$$- \frac{1}{2}b_2(n) - \frac{1}{2}b_3(n) - \frac{3}{8}b_4(n)\Big)q^n,$$

(f) $\displaystyle\sum_{n=0}^{\infty} N(1,7,14,14;n)q^n = \varphi(q)\varphi(q^7)\varphi^2(q^{14})$

$$= \frac{1}{4}E_{2,\chi_3,\chi_0}(q) - \frac{1}{4}E_{2,\chi_0,\chi_3}(q^2) - \frac{1}{4}E_{2,\chi_1,\chi_2}(q^2) + \frac{1}{4}E_{2,\chi_2,\chi_1}(q) - \frac{1}{8}B_1(q) + \frac{3}{2}B_2(q)$$
$$+ \frac{3}{2}B_3(q) + \frac{1}{8}B_4(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{1}{4}\sigma_{\chi_3,\chi_0}(n) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(n/2) - \frac{1}{4}\sigma_{\chi_1,\chi_2}(n/2) + \frac{1}{4}\sigma_{\chi_2,\chi_1}(n) - \frac{1}{8}b_1(n)$$
$$+ \frac{3}{2}b_2(n) + \frac{3}{2}b_3(n) + \frac{1}{8}b_4(n)\Big)q^n.$$

Equating the coefficients of q^n on both sides of equations (a)–(f) yields the results. ∎

The values of $N(a_1, a_2, a_3, a_4; n)$ for $1 \le n \le 20$ for the quadratic forms (a_1, a_2, a_3, a_4) in Theorem 3.3.5 are given in Table 3.3.12. One can verify them by using Table 3.3.1.

Table 3.3.12

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$N(1,1,1,7;n)$	6	12	8	6	24	24	2	24	54	40	36	56	72	48	24	66	96	84	40	72
$N(1,2,2,7;n)$	2	4	8	6	8	8	2	16	18	24	36	24	24	16	24	26	32	60	40	40
$N(1,7,7,7;n)$	2	0	0	2	0	0	6	12	2	0	12	0	0	12	24	14	0	24	0	0
$N(1,1,2,14;n)$	4	6	8	12	8	8	16	6	12	24	8	24	24	2	24	24	32	54	40	40
$N(2,7,7,14;n)$	0	2	0	0	0	0	4	2	8	0	0	0	0	6	8	12	0	2	0	0
$N(1,7,14,14;n)$	2	0	0	2	0	0	2	4	2	0	4	0	0	4	8	6	0	8	0	0

For example, by substituting $n = 16$ in Theorem 3.3.5(b), we obtain

$$\begin{aligned} N(1,2,2,7;16) &= \frac{7}{4}\sigma_{\chi_3,\chi_0}(16) - \frac{1}{4}\sigma_{\chi_0,\chi_3}(8) + \frac{7}{4}\sigma_{\chi_1,\chi_2}(8) - \frac{1}{4}\sigma_{\chi_2,\chi_1}(16) \\ &\quad -\frac{7}{8}b_1(16) + \frac{3}{2}b_2(16) - \frac{21}{2}b_3(16) - \frac{1}{8}b_4(16). \end{aligned}$$

Then appealing to (2.2.1) and (3.1.3), we obtain

$$\sigma_{\chi_3,\chi_0}(16) = 16, \quad \sigma_{\chi_0,\chi_3}(8) = 1, \quad \sigma_{\chi_1,\chi_2}(8) = 1, \quad \sigma_{\chi_2,\chi_1}(16) = 16.$$

From Table 3.3.1, we have

$$b_1(16) = 1, \quad b_2(16) = b_3(16) = 0, \quad b_4(16) = -11.$$

Thus we have

$$N(1,2,2,7;16) = \frac{7}{4}(16) - \frac{1}{4} + \frac{7}{4} - \frac{1}{4}(16) - \frac{7}{8} - \frac{1}{8}(-11) = 26,$$

which agrees with the value of $N(1,2,2,7;16)$ in Table 3.3.12.

3.4 The space $M_2(\Gamma_0(56), \chi_5)$

Let χ_0 be the trivial character and χ_5 as in (3.1.4). We define the Eisenstein series

$$E_{2,\chi_5,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_5,\chi_0)}(n) q^n, \tag{3.4.1}$$

$$E_{2,\chi_0,\chi_5}(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \sigma_{(\chi_0,\chi_5)}(n) q^n. \tag{3.4.2}$$

We determine $N(a_1, a_2, a_3, a_4; n)$ for the quaternary quadratic forms listed in the third column of Table 3.1.1 in terms of $\sigma_{(\chi,\psi)}(n)$, where $\chi, \psi \in \{\chi_0, \chi_5\}$, and $c_k(n)$ $(1 \leq k \leq 6)$ defined by

$$C_1(q) = \sum_{n=1}^{\infty} c_1(n) q^n = \frac{\eta^3(2z)\eta(7z)\eta^2(8z)\eta(28z)}{\eta(z)\eta^2(4z)}, \tag{3.4.3}$$

$$C_2(q) = \sum_{n=1}^{\infty} c_2(n) q^n = \frac{\eta(2z)\eta^2(7z)\eta(8z)\eta^3(28z)}{\eta^2(14z)\eta(56z)}, \tag{3.4.4}$$

$$C_3(q) = \sum_{n=1}^{\infty} c_3(n) q^n = \frac{\eta^2(z)\eta^3(4z)\eta(14z)\eta(56z)}{\eta^2(2z)\eta(8z)}, \tag{3.4.5}$$

$$C_4(q) = \sum_{n=1}^{\infty} c_4(nq^n) = \frac{\eta^6(2z)\eta(8z)\eta^4(28z)}{\eta^2(z)\eta^3(4z)\eta(14z)\eta(56z)}, \tag{3.4.6}$$

$$C_5(q) = \sum_{n=1}^{\infty} c_5(n) q^n = \frac{\eta^4(2z)\eta(7z)\eta^6(28z)}{\eta(z)\eta(4z)\eta^3(14z)\eta^2(56z)}, \tag{3.4.7}$$

$$C_6(q) = \sum_{n=1}^{\infty} c_6(n) q^n = \frac{\eta^4(4z)\eta^6(14z)\eta(56z)}{\eta(2z)\eta^2(7z)\eta(8z)\eta^3(28z)}. \tag{3.4.8}$$

There is no linear relationship among the $C_k(q), 1 \leq k \leq 6$. The first fifty-six values of $c_k(n)$, $1 \leq k \leq 6$, are given in Table 3.4.1.

Table 3.4.1

n	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$	$c_5(n)$	$c_6(n)$	n	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$	$c_5(n)$	$c_6(n)$
1	0	1	0	0	1	0	29	0	2	−2	0	4	4
2	1	0	0	1	1	1	30	0	4	−4	0	4	4
3	1	−1	1	2	−2	0	31	−4	−4	0	−4	−8	4

4	−1	0	−2	−1	−1	1	32	3	−2	2	5	−5	3
5	0	−1	1	−2	0	0	33	0	0	−4	0	−4	0
6	1	0	−2	2	−2	−2	34	−2	0	0	−6	2	−6
7	0	0	1	0	1	0	35	−1	1	−1	0	0	−2
8	−1	−2	2	−3	−1	−1	36	1	0	−2	5	−7	−5
9	−2	−1	2	−2	−1	2	37	4	−2	2	8	−4	4
10	−1	2	0	0	4	0	38	1	4	2	2	6	−2
11	0	2	−2	2	2	2	39	−2	2	4	−2	4	2
12	0	2	−2	−2	2	−2	40	2	−4	0	4	−4	−4
13	−2	1	−1	−2	0	−4	41	4	2	4	4	10	−4
14	0	0	0	1	−1	1	42	−1	−2	0	−2	−2	2
15	2	2	0	2	4	−2	43	−4	2	−2	−6	2	−6
16	−1	2	2	1	3	−1	44	−2	0	0	−2	2	−6
17	0	−2	−4	0	−6	0	45	2	−5	5	2	−8	−4
18	1	−4	4	3	−5	−1	46	−4	4	−4	−8	4	−4
19	−1	−1	1	−2	−2	−4	47	0	0	0	0	0	0
20	0	−2	2	0	−4	4	48	−2	0	0	−2	2	10
21	0	1	−1	0	2	2	49	0	1	0	0	1	0
22	0	−4	0	−2	−6	2	50	−1	4	−4	1	1	5
23	2	−2	−4	2	−4	−2	51	−2	−2	2	−4	−4	−8
24	0	4	0	−2	10	2	52	4	2	−2	8	4	4
25	2	1	2	2	5	−2	53	−4	4	−4	−4	4	−4
26	1	2	4	0	4	0	54	−4	−4	−4	−4	−4	4
27	2	0	0	0	4	8	55	−4	4	0	−4	0	4
28	1	−2	0	1	−3	−1	56	−1	2	−2	−1	1	−3

Theorem 3.4.1. *Let* (a_1, a_2, a_3, a_4) *be as in the third column of Table 3.1.1.Then* $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_5)$.

Proof. We have $N = 56$. First we consider quadratic form $(1, 2, 7, 7)$. By (2.1.9) we have

$$\varphi(q)\varphi(q^2)\varphi^2(q^7) = \frac{\eta^3(2z)\eta^3(4z)\eta^{10}(14z)}{\eta^2(z)\eta^4(7z)\eta^2(8z)\eta^4(28z)}.$$

Then we have Table 3.4.2(a) and Table 3.4.2(b).

Table 3.4.2(a)

δ	1	2	4	7	8	14	28
r_δ	−2	3	3	−4	−2	10	−4

It can be seen from Table 3.4.2(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.4.2(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	54/7	12	0	0	90	12	0

From Table 3.4.2(b) the condition ($L3$) is also satisfied. Thus $\varphi(q)\varphi(q^2)\varphi^2(q^7) \in M_2(\Gamma_0(56), \chi_5)$. Secondly we consider the form $(1, 1, 7, 14)$. By (2.1.9) we have

$$\varphi^2(q)\varphi(q^7)\varphi(q^{14}) = \frac{\eta^{10}(2z)\eta^3(14z)\eta^3(28z)}{\eta^4(z)\eta^4(4z)\eta^2(7z)\eta^2(56z)}.$$

Then we have

Table 3.4.3(a)

δ	1	2	4	7	14	28	56
r_δ	-4	10	-4	-2	3	3	-2

It can be seen from Table 3.4.3(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 3.4.3(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	90/7	12/7	0	0	54	84	0

From Table 3.4.3(b) the condition ($L3$) is also satisfied. Thus $\varphi^2(q)\varphi(q^7)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_5)$. Thirdly we consider the form $(2, 2, 7, 14)$. We have

$$\varphi^2(q^2)\varphi(q^7)\varphi(q^{14}) = \frac{\eta^{10}(4z)\eta^3(14z)\eta^3(28z)}{\eta^4(2z)\eta^2(7z)\eta^4(8z)\eta^2(56z)}.$$

Then we have

Table 3.4.4(a)

δ	2	4	7	8	14	28	56
r_δ	-4	10	-2	-4	3	3	-2

It can be seen from Table 3.4.4(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 3.4.4(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6/7	180/7	0	0	42	108	0

From Table 3.4.4(b) the condition ($L3$) is also satisfied. Thus $\varphi^2(q^2)\varphi(q^7)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_5)$. Fourthly we consider the form $(1, 2, 14, 14)$. We have

$$\varphi(q)\varphi(q^2)\varphi^2(q^{14}) = \frac{\eta^3(2z)\eta^3(4z)\eta^{10}(28z)}{\eta^2(z)\eta^2(8z)\eta^4(14z)\eta^4(56z)}.$$

Then we have

Table 3.4.5(a)

δ	1	2	4	8	14	28	56
r_δ	-2	3	3	-2	-4	10	-4

It can be seen from Table 3.4.5(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 3.4.5(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6	108/7	0	0	6	180	0

From Table 3.4.5(b) the condition ($L3$) is also satisfied. Thus $\varphi(q)\varphi(q^2)\varphi^2(q^{14}) \in M_2(\Gamma_0(56), \chi_5)$. ■

Theorem 3.4.2. *$C_k(q)$ $(1 \le k \le 6)$ given by (3.4.3)–(3.4.8) are in $S_2(\Gamma_0(56), \chi_5)$.*

Proof. We will check conditions (L1), (L2) and (L4) of Theorem 3.1.1. We have $N = 56$. First we consider

$$C_1(q) = \frac{\eta^3(2z)\eta(7z)\eta^2(8z)\eta(28z)}{\eta(z)\eta^2(4z)}.$$

Then we have

Table 3.4.6(a)

δ	1	2	4	7	8	28
r_δ	-1	3	-2	1	2	1

It can be seen from Table 3.4.6(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.4.6(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	30/7	12/7	9	96/7	18	36	48

From Table 3.4.6(b) the condition $(L4)$ is also satisfied. Thus $C_1(q) \in S_2(\Gamma_0(56), \chi_5)$.

Secondly we consider

$$C_2(q) = \frac{\eta(2z)\eta^2(7z)\eta(8z)\eta^3(28z)}{\eta^2(14z)\eta(56z)}.$$

Then we have

Table 3.4.7(a)

δ	2	7	8	14	28	56
r_δ	1	2	1	-2	3	-1

It can be seen from Table 3.4.7(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.4.7(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/7	18/7	36/7	12	72/7	6	60	24

From Table 3.4.7(b) the condition $(L4)$ is also satisfied. Thus $C_2(q) \in S_2(\Gamma_0(56), \chi_5)$.

Thirdly we consider

$$C_3(q) = \frac{\eta^2(z)\eta^3(4z)\eta(14z)\eta(56z)}{\eta^2(2z)\eta(8z)}.$$

Then we have

Table 3.4.8(a)

δ	1	2	4	8	14	56
r_δ	2	−2	3	−1	1	1

It can be seen from Table 3.4.8(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.4.8(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	12/7	6/7	60/7	6	24/7	18	36	72

From Table 3.4.8(b) the condition $(L4)$ is also satisfied. Thus $C_3(q) \in S_2(\Gamma_0(56), \chi_5)$.

Fourthly we consider

$$C_4(q) = \frac{\eta^6(2z)\eta(8z)\eta^4(28z)}{\eta^2(z)\eta^3(4z)\eta(14z)\eta(56z)}.$$

Then we have

Table 3.4.9(a)

δ	1	2	4	8	14	28	56
r_δ	−2	6	−3	1	−1	4	−1

It can be seen from Table 3.4.9(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.4.9(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	54/7	12/7	3	48/7	18	84	48

From Table 3.4.9(b) the condition $(L4)$ is also satisfied. Thus $C_4(q) \in S_2(\Gamma_0(56), \chi_5)$.

Fifthly we consider

$$C_5(q) = \frac{\eta^4(2z)\eta(7z)\eta^6(28z)}{\eta(z)\eta(4z)\eta^3(14z)\eta^2(56z)}.$$

Then we have

Table 3.4.10(a)

δ	1	2	4	7	14	28	56
r_δ	-1	4	-1	1	-3	6	-2

It can be seen from Table 3.4.10(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.4.10(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/7	6	36/7	6	24/7	6	108	24

From Table 3.4.10(b) the condition $(L4)$ is also satisfied. Thus $C_5(q) \in S_2(\Gamma_0(56), \chi_5)$.

Sixthly we consider

$$C_6(q) = \frac{\eta^4(4z)\eta^6(14z)\eta(56z)}{\eta(2z)\eta^2(7z)\eta(8z)\eta^3(28z)}.$$

Then we have

Table 3.4.11(a)

δ	2	4	7	8	14	28	56
r_δ	-1	4	-2	-1	6	-3	1

It can be seen from Table 3.4.11(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.4.11(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	18/7	12	3	48/7	54	12	48

From Table 3.4.11(b) the condition $(L4)$ is also satisfied. Thus $C_6(q) \in S_2(\Gamma_0(56), \chi_5)$. ∎

Theorem 3.4.3. **(a)** $\{C_1(q), \ldots, C_6(q)\}$ *is a basis for* $S_2(\Gamma_0(56), \chi_5)$.

(b) $\{E_{2,\chi_5,\chi_0}(q^t), E_{2,\chi_0,\chi_5}(q^t) \mid t = 1, 7\}$ *is a basis for* $E_2(\Gamma_0(56), \chi_5)$.

(c) $\{E_{2,\chi_5,\chi_0}(q^t), E_{2,\chi_0,\chi_5}(q^t) \mid t = 1, 7\}$ *together with* $C_k(q)$ $(1 \le k \le 6)$ *constitute a basis for* $M_2(\Gamma_0(56), \chi_5)$.

Proof. **(a)** By Theorem 3.4.2, $C_k(q)$ $(1 \le k \le 6) \in S_2(\Gamma_0(56), \chi_5)$. There is no linear relationship among them. By Example 2.3.3, we have $\dim S_2(\Gamma_0(56), \chi_5) = 6$. Thus $C_k(q)$ $(1 \le k \le 6)$ constitute a basis for $S_2(\Gamma_0(56), \chi_5)$.

(b) By Example 2.3.3, we have $\dim E_2(\Gamma_0(56), \chi_5) = 4$. By Theorem 2.2.3, $\{E_{2,\chi_5,\chi_0}(q^t), E_{2,\chi_0,\chi_5}(q^t) \mid t = 1, 7\}$ constitute a basis for $E_2(\Gamma_0(56), \chi_5)$.

(c) By Example 2.3.3, we have $\dim M_2(\Gamma_0(56), \chi_5) = 10$. Therefore, by (2.1.1) $\{E_{2,\chi_5,\chi_0}(q^t), E_{2,\chi_0,\chi_5}(q^t) \mid t = 1, 7\}$ together with $C_k(q)$ $(1 \le k \le 6)$ constitute a basis for $M_2(\Gamma_0(56), \chi_5)$. ∎

Theorem 3.4.4. *Let* χ_0 *be the trivial character and* χ_5 *be as in* (3.1.4). *Then*

$$\begin{aligned}\textbf{(a)}\ \varphi(q)\varphi(q^2)\varphi^2(q^7) =& \frac{4}{3}E_{2,\chi_5,\chi_0}(q) - \frac{28}{3}E_{2,\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{2,\chi_0,\chi_5}(q) - \frac{7}{3}E_{2,\chi_0,\chi_5}(q^7)\\ &- 4C_1(q) - \frac{2}{3}C_2(q) + \frac{4}{3}C_3(q) + 3C_4(q) + C_5(q) - C_6(q),\\ \textbf{(b)}\ \varphi^2(q)\varphi(q^7)\varphi(q^{14}) =& \frac{4}{3}E_{2,\chi_5,\chi_0}(q) - \frac{28}{3}E_{2,\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{2,\chi_0,\chi_5}(q) - \frac{7}{3}E_{2,\chi_0,\chi_5}(q^7)\\ &+ 4C_1(q) + \frac{10}{3}C_2(q) + \frac{4}{3}C_3(q) - 3C_4(q) - C_5(q) + C_6(q),\\ \textbf{(c)}\ \varphi^2(q^2)\varphi(q^7)\varphi(q^{14}) =& \frac{2}{3}E_{2,\chi_5,\chi_0}(q) - \frac{14}{3}E_{2,\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{2,\chi_0,\chi_5}(q) - \frac{7}{3}E_{2,\chi_0,\chi_5}(q^7)\\ &- 2C_1(q) - 3C_2(q) - C_3(q) + \frac{5}{3}C_4(q) + 2C_5(q) + \frac{2}{3}C_6(q),\\ \textbf{(d)}\ \varphi(q)\varphi(q^2)\varphi^2(q^{14}) =& \frac{2}{3}E_{2,\chi_5,\chi_0}(q) - \frac{14}{3}E_{2,\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{2,\chi_0,\chi_5}(q) - \frac{7}{3}E_{2,\chi_0,\chi_5}(q^7)\\ &+ 2C_1(q) + 3C_2(q) + C_3(q) - \frac{1}{3}C_4(q) - 2C_5(q) + \frac{2}{3}C_6(q).\end{aligned}$$

Proof. Let (a_1, a_2, a_3, a_4) be one of the quadratic forms listed in the third column of Table 3.1.1. By Theorem 3.4.1, $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in (M_2(\Gamma_0(56), \chi_5)$. Therefore by Theorem 3.4.3 (c), $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $\{E_{2,\chi_5,\chi_0}(q^t), E_{2,\chi_0,\chi_5}(q^t) \mid t = 1, 7\}$ and $C_k(q)$ $(1 \le k \le 6)$, namely

$$\begin{aligned}\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) =& x_1 E_{2,\chi_5,\chi_0}(q) + x_2 E_{2,\chi_5,\chi_0}(q^7) + x_3 E_{2,\chi_0,\chi_5}(q)\\ &+ x_4 E_{2,\chi_0,\chi_5}(q^7) + y_1 C_1(q) + y_2 C_2(q) + y_3 C_3(q) + y_4 C_4(q)\\ &+ y_5 C_5(q) + y_6 C_6(q).\end{aligned}$$

We equate the first fourteen coefficients of q^n on both sides of the equation above to obtain a system of linear equations with the unknowns $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, y_5, y_6$. Then, using MAPLE we solve the system to find the asserted coefficients. ∎

Theorem 3.4.5. *Let* $n \in \mathbb{N}$. *Let* $\sigma_{\chi_i,\chi_j}(n)$ *be as in* (2.2.1) *for* $i, j \in \{0, 5\}$. *Then*

$$\begin{aligned}\textbf{(a)}\ N(1,2,7,7;n) =& \frac{4}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{28}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\ &- 4c_1(n) - \frac{2}{3}c_2(n) + \frac{4}{3}c_3(n) + 3c_4(n) + c_5(n) - c_6(n),\\ \textbf{(b)}\ N(1,1,7,14;n) =& \frac{4}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{28}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\ &+ 4c_1(n) + \frac{10}{3}c_2(n) + \frac{4}{3}c_3(n) - 3c_4(n) - c_5(n) + c_6(n),\\ \textbf{(c)}\ N(2,2,7,14;n) =& \frac{2}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{14}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\ &- 2c_1(n) - 3c_2(n) - c_3(n) + \frac{5}{3}c_4(n) + 2c_5(n) + \frac{2}{3}c_6(n),\\ \textbf{(d)}\ N(1,2,14,14;n) =& \frac{2}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{14}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\ &+ 2c_1(n) + 3c_2(n) + c_3(n) - \frac{1}{3}c_4(n) - 2c_5(n) + \frac{2}{3}c_6(n).\end{aligned}$$

Proof. Appealing to (2.1.6), (3.4.1), (3.4.2) and Theorem 3.4.4, we obtain

$$
\begin{aligned}
&\textbf{(a)}\ \sum_{n=0}^{\infty} N(1,2,7,7;n)q^n = \varphi(q)\varphi(q^2)\varphi^2(q^7)\\
&= \frac{4}{3}E_{2,\chi_5,\chi_0}(q) - \frac{28}{3}E_{2,\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{2,\chi_0,\chi_5}(q) - \frac{7}{3}E_{2,\chi_0,\chi_5}(q^7) - 4C_1(q)\\
&\quad - \frac{2}{3}C_2(q) + \frac{4}{3}C_3(q) + 3C_4(q) + C_5(q) - C_6(q),\\
&= 1 + \sum_{n=1}^{\infty}\Big(\frac{4}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{28}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\
&\quad - 4c_1(n) - \frac{2}{3}c_2(n) + \frac{4}{3}c_3(n) + 3c_4(n) + c_5(n) - c_6(n)\Big)q^n,
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{(b)}\ \sum_{n=0}^{\infty} N(1,1,7,14;n)q^n = \varphi^2(q)\varphi(q^7)\varphi(q^{14})\\
&= \frac{4}{3}E_{2,\chi_5,\chi_0}(q) - \frac{28}{3}E_{2,\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{2,\chi_0,\chi_5}(q) - \frac{7}{3}E_{2,\chi_0,\chi_5}(q^7) + 4C_1(q)\\
&\quad + \frac{10}{3}C_2(q) + \frac{4}{3}C_3(q) - 3C_4(q) - C_5(q) + C_6(q),\\
&= 1 + \sum_{n=1}^{\infty}\Big(\frac{4}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{28}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\
&\quad + 4c_1(n) + \frac{10}{3}c_2(n) + \frac{4}{3}c_3(n) - 3c_4(n) - c_5(n) + c_6(n)\Big)q^n,
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{(c)}\ \sum_{n=0}^{\infty} N(2,2,7,14;n)q^n = \varphi^2(q^2)\varphi(q^7)\varphi(q^{14})\\
&= \frac{2}{3}E_{2,\chi_5,\chi_0}(q) - \frac{14}{3}E_{2,\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{2,\chi_0,\chi_5}(q) - \frac{7}{3}E_{2,\chi_0,\chi_5}(q^7)\\
&\quad - 2C_1(q) - 3C_2(q) - C_3(q) + \frac{5}{3}C_4(q) + 2C_5(q) + \frac{2}{3}C_6(q),\\
&= 1 + \sum_{n=1}^{\infty}\Big(\frac{2}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{14}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)\\
&\quad - 2c_1(n) - 3c_2(n) - c_3(n) + \frac{5}{3}c_4(n) + 2c_5(n) + \frac{2}{3}c_6(n)\Big)q^n,
\end{aligned}
$$

$$\textbf{(d)} \sum_{n=0}^{\infty} N(1,2,14,14;n)q^n = \varphi(q)\varphi(q^2)\varphi^2(q^{14})$$
$$= \frac{2}{3}E_{2,\chi_5,\chi_0}(q) - \frac{14}{3}E_{2,\chi_5,\chi_0}(q^7) + \frac{1}{3}E_{2,\chi_0,\chi_5}(q) - \frac{7}{3}E_{2,\chi_0,\chi_5}(q^7)$$
$$+ 2C_1(q) + 3C_2(q) + C_3(q) - \frac{1}{3}C_4(q) - 2C_5(q) + \frac{2}{3}C_6(q),$$
$$= 1 + \sum_{n=1}^{\infty} \Big(\frac{2}{3}\sigma_{\chi_5,\chi_0}(n) - \frac{14}{3}\sigma_{\chi_5,\chi_0}(n/7) + \frac{1}{3}\sigma_{\chi_0,\chi_5}(n) - \frac{7}{3}\sigma_{\chi_0,\chi_5}(n/7)$$
$$+ 2c_1(n) + 3c_2(n) + c_3(n) - \frac{1}{3}c_4(n) - 2c_5(n) + \frac{2}{3}c_6(n)\Big)q^n.$$

Equating the coefficients of q^n on both sides of equations (a)–(d) yields the results. ■

The values of $N(a_1, a_2, a_3, a_4; n)$ for $1 \le n \le 20$ for the quadratic forms (a_1, a_2, a_3, a_4) in Theorem 3.4.5 are given in Table 3.4.12.

Table 3.4.12

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$N(1,2,7,7;n)$	2	2	4	2	0	4	4	10	14	16	12	4	16	4	16	34	20	30	20	16
$N(1,1,7,14;n)$	4	4	0	4	8	0	2	12	12	8	8	16	8	2	16	20	24	12	16	24
$N(2,2,7,14;n)$	0	4	0	4	0	0	2	4	8	8	8	0	0	2	8	12	16	12	0	8
$N(1,2,14,14;n)$	2	2	4	2	0	4	0	2	6	0	4	4	0	4	8	10	20	14	4	16

Again one can verify the values in Table 3.4.12 by using (2.2.1), (3.1.4) and Table 3.4.1.

3.5 The space $M_2(\Gamma_0(56), \chi_6)$

Let χ_0 be the trivial character and χ_1, χ_4, χ_6 as in (3.1.3) and (3.1.4). We define the Eisenstein series

$$E_{2,\chi_4,\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_4,\chi_1)}(n)q^n, \qquad (3.5.1)$$

$$E_{2,\chi_1,\chi_4}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_1,\chi_4)}(n)q^n, \tag{3.5.2}$$

$$E_{2,\chi_6,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_6,\chi_0)}(n)q^n, \tag{3.5.3}$$

$$E_{2,\chi_0,\chi_6}(q) = -10 + \sum_{n=1}^{\infty} \sigma_{(\chi_0,\chi_6)}(n)q^n. \tag{3.5.4}$$

We determine the representation number for the quaternary quadratic forms listed in the fourth column of Table 3.1.1 in terms of $\sigma_{(\chi,\psi)}(n)$, where $\chi, \psi \in \{\chi_0, \chi_1, \chi_4, \chi_6\}$, and $d_k(n)$ $(1 \leq k \leq 6)$ defined by

$$D_1(q) = \sum_{n=1}^{\infty} d_1(n) = \frac{\eta^4(2z)\eta^2(4z)\eta(14z)\eta(56z)}{\eta^2(z)\eta(8z)\eta(28z)}, \tag{3.5.5}$$

$$D_2(q) = \sum_{n=1}^{\infty} d_2(n)q^n = \frac{\eta^2(2z)\eta^4(4z)\eta(7z)\eta(28z)}{\eta(z)\eta^2(8z)\eta(14z)}, \tag{3.5.6}$$

$$D_3(q) = \sum_{n=1}^{\infty} d_3(n)q^n = \frac{\eta^3(2z)\eta(7z)\eta^2(56z)}{\eta(z)\eta(4z)}, \tag{3.5.7}$$

$$D_4(q) = \sum_{n=1}^{\infty} d_4(n)q^n = \frac{\eta^3(4z)\eta^2(7z)\eta(56z)}{\eta(2z)\eta(8z)}, \tag{3.5.8}$$

$$D_5(q) = \sum_{n=1}^{\infty} d_5(n)q^n = \frac{\eta(2z)\eta(8z)\eta^4(14z)\eta^2(28z)}{\eta(4z)\eta^2(7z)\eta(56z)}, \tag{3.5.9}$$

$$D_6(q) = \sum_{n=1}^{\infty} d_6(n)q^n = \frac{\eta(z)\eta^2(8z)\eta^3(14z)}{\eta(7z)\eta(28z)}. \tag{3.5.10}$$

There is no linear relationship among the $D_k(q), 1 \leq k \leq 6$. The first fifty-six values of $d_k(n)$, $1 \leq k \leq 6$, are given in Table 3.5.1.

Table 3.5.1

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$	$d_5(n)$	$d_6(n)$	n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$	$d_5(n)$	$d_6(n)$
1	0	1	0	0	0	1	29	8	12	0	0	0	0
2	1	1	0	0	1	-1	30	4	-4	0	0	0	6
3	2	0	0	1	0	-1	31	4	-8	2	2	0	0
4	1	1	0	0	-1	0	32	-3	7	0	0	-1	4

5	2	−4	1	1	0	0	33	4	0	0	2	0	−2
6	0	−4	1	0	0	1	34	0	8	−2	0	0	−2
7	−4	1	−1	−1	0	0	35	−4	0	0	−1	2	−5
8	1	−5	0	0	−1	2	36	7	−5	0	0	1	0
9	−2	3	0	0	2	−3	37	−12	−4	0	0	−4	0
10	−4	4	−1	−2	0	1	38	0	4	−1	0	0	−1
11	2	−4	0	0	−2	2	39	2	−4	0	0	2	0
12	−4	8	−2	−2	0	0	40	0	−8	2	0	0	2
13	−2	4	−1	−1	0	0	41	0	0	0	0	0	0
14	1	−1	1	2	1	−2	42	−8	8	−1	−2	0	−5
15	−2	4	0	0	−2	0	43	−6	12	0	0	6	−6
16	−5	−3	0	0	1	−4	44	−8	4	0	0	0	0
17	−4	0	0	−2	0	2	45	−6	12	−3	−3	0	0
18	3	7	0	0	−1	3	46	2	−2	0	0	−2	−4
19	−2	0	0	−1	0	1	47	−4	8	−2	−2	0	0
20	4	0	0	2	0	−2	48	−8	0	0	−4	0	4
21	4	−8	1	1	2	0	49	8	−7	0	2	−4	3
22	−4	−8	0	0	0	−2	50	−5	−9	0	0	−1	−1
23	6	2	0	0	2	0	51	4	0	0	0	−4	12
24	8	−8	2	4	0	−2	52	−4	0	0	−2	0	2
25	2	−5	0	0	−2	1	53	−8	−12	0	0	0	0
26	4	−4	1	2	0	−1	54	0	0	0	0	0	0
27	0	0	0	0	0	0	55	4	−8	2	2	0	0
28	−1	7	0	−2	1	4	56	7	9	−2	0	1	−4

Theorem 3.5.1. *Let* (a_1, a_2, a_3, a_4) *be as in the fourth column of Table 3.1.1. Then* $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_6)$.

Proof. We appeal to (2.1.9) for each quadratic forms and then check the conditions (L1), (L2) and (L3) of Theorem 3.1.1 for each quadratic form. We have $N = 56$. First we consider $(1, 1, 2, 7)$

$$\varphi^2(q)\varphi(q^2)\varphi(q^7) = \frac{\eta^8(2z)\eta(4z)\eta^5(14z)}{\eta^4(z)\eta^2(7z)\eta^2(8z)\eta^2(28z)}.$$

Then we have

Table 3.5.2(a)

δ	1	2	4	7	8	14	28
r_δ	−4	8	1	−2	−2	5	−2

It can be seen from Table 3.5.2(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.2(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	90/7	12	0	0	54	12	0

From Table 3.5.2(b) the condition $(L3)$ is also satisfied. Thus $\varphi^2(q)\varphi(q^2)\varphi(q^7) \in M_2(\Gamma_0(56), \chi_6)$. Secondly we consider $(2,2,2,7)$. We have

$$\varphi^3(q^2)\varphi(q^7) = \frac{\eta^{15}(4z)\eta^5(14z)}{\eta^6(2z)\eta^2(7z)\eta^6(8z)\eta^2(28z)}.$$

Then we have

Table 3.5.3(a)

δ	2	4	7	8	14	28
r_δ	-6	15	-2	-6	5	-2

It can be seen from Table 3.5.3(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.3(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6/7	36	0	0	42	36	0

From Table 3.5.3(b) the condition $(L3)$ is also satisfied. Thus $\varphi^3(q^2)\varphi(q^7) \in M_2(\Gamma_0(56), \chi_6)$. Thirdly we consider $(2,7,7,7)$. We have

$$\varphi(q^2)\varphi^3(q^7) = \frac{\eta^5(4z)\eta^{15}(14z)}{\eta^2(2z)\eta^6(7z)\eta^2(8z)\eta^6(28z)}.$$

Then we have

Table 3.5.4(a)

δ	2	4	7	8	14	28
r_δ	-2	5	-6	-2	15	-6

It can be seen from Table 3.5.4(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.4(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2\cdot r_\delta}{\delta}$	0	18/7	12	0	0	126	12	0

From Table 3.5.4(b) the condition $(L3)$ is also satisfied. Thus $\varphi(q^2)\varphi^3(q^7) \in M_2(\Gamma_0(56), \chi_6)$. Fourthly we consider $(3, 0, 0, 1)$. We have

$$\varphi^3(q)\varphi(q^{14}) = \frac{\eta^{15}(2z)\eta^5(28z)}{\eta^6(z)\eta^6(4z)\eta^2(14z)\eta^2(56z)}.$$

Then we have

Table 3.5.5(a)

δ	1	2	4	14	28	56
r_δ	-6	15	-6	-2	5	-2

It can be seen from Table 3.5.5(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.5(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2\cdot r_\delta}{\delta}$	0	18	12/7	0	0	18	84	0

From Table 3.5.5(b) the condition $(L3)$ is also satisfied. Thus $\varphi^3(q)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_6)$. Fifthly we consider $(1, 2, 2, 14)$. We have

$$\varphi(q)\varphi^2(q^2)\varphi(q^{14}) = \frac{\eta(2z)\eta^8(4z)\eta^5(28z)}{\eta^2(z)\eta^4(8z)\eta^2(14z)\eta^2(56z)}.$$

Then we have

Table 3.5.6(a)

δ	1	2	4	8	14	28	56
r_δ	-2	1	8	-4	-2	5	-2

It can be seen from Table 3.5.6(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.6(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6	180/7	0	0	6	108	0

From Table 3.5.6(b) the condition $(L3)$ is also satisfied. Thus $\varphi(q)\varphi^2(q^2)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_6)$. Sixthly we consider $(1, 7, 7, 14)$. We have

$$\varphi(q)\varphi^2(q^7)\varphi(q^{14}) = \frac{\eta^5(2z)\eta^8(14z)\eta(28z)}{\eta^2(z)\eta^2(4z)\eta^4(7z)\eta^2(56z)}.$$

Then we have

Table 3.5.7(a)

δ	1	2	4	7	14	28	56
r_δ	-2	5	-2	-4	8	1	-2

It can be seen from Table 3.5.7(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.7(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	54/7	12/7	0	0	90	84	0

From Table 3.5.7(b) the condition $(L3)$ is also satisfied. Thus $\varphi(q)\varphi^2(q^7)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_6)$. Seventhly we consider $(2, 7, 14, 14)$. Then

$$\varphi(q^2)\varphi(q^7)\varphi^2(q^{14}) = \frac{\eta^5(4z)\eta(14z)\eta^8(28z)}{\eta^2(2z)\eta^2(7z)\eta^2(8z)\eta^4(56z)}.$$

Then we have

Table 3.5.8(a)

δ	2	4	7	8	14	28	56
r_δ	-2	5	-2	-2	1	8	-4

It can be seen from Table 3.5.8(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.8(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6/7	108/7	0	0	42	180	0

From Table 3.5.8(b) the condition $(L3)$ is also satisfied. Thus $\varphi(q)\varphi(q^2)\varphi(q^7)\varphi(q^{14}) \in M_2(\Gamma_0(56), \chi_6)$. Eighthly we consider $(1, 14, 14, 14)$. We have

$$\varphi(q)\varphi^3(q^{14}) = \frac{\eta^5(2z)\eta^{15}(28z)}{\eta^2(z)\eta^2(4z)\eta^6(14z)\eta^6(56z)}.$$

Then we have

Table 3.5.9(a)

δ	1	2	4	14	28	56
r_δ	-2	5	-2	-6	15	-6

It can be seen from Table 3.5.9(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.9(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6	36/7	0	0	6	252	0

From Table 3.5.9(b) the condition $(L3)$ is also satisfied. Thus $\varphi(q)\varphi^3(q^{14}) \in M_2(\Gamma_0(56), \chi_6)$. ∎

Theorem 3.5.2. *$D_k(q)$ $(1 \le k \le 6)$ given by (3.5.5)–(3.5.10) are in $S_2(\Gamma_0(56), \chi_6)$.*

Proof. We will check conditions (L1), (L2) and (L4) of Theorem 3.1.1. We have $N = 56$. First we consider

$$D_1(q) = \frac{\eta^4(2z)\eta^2(4z)\eta(14z)\eta(56z)}{\eta^2(z)\eta(8z)\eta(28z)}.$$

Then we have

Table 3.5.10(a)

δ	1	2	4	8	14	28	56
r_δ	-2	4	2	-1	1	-1	1

It can be seen from Table 3.5.10(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 3.5.10(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2\cdot r_\delta}{\delta}$	3/7	54/7	12	3	48/7	18	12	48

From Table 3.5.10(b) the condition ($L4$) is also satisfied. Thus $D_1(q) \in S_2(\Gamma_0(56), \chi_6)$.

Secondly we consider

$$D_2(q) = \frac{\eta^2(2z)\eta^4(4z)\eta(7z)\eta(28z)}{\eta(z)\eta^2(8z)\eta(14z)}.$$

Then we have

Table 3.5.11(a)

δ	1	2	4	7	8	14	28
r_δ	-1	2	4	1	-2	-1	1

It can be seen from Table 3.5.11(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 3.5.11(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2\cdot r_\delta}{\delta}$	6/7	6	108/7	6	24/7	6	36	24

From Table 3.5.11(b) the condition ($L4$) is also satisfied. Thus $D_2(q) \in S_2(\Gamma_0(56), \chi_6)$.

Thirdly we consider

$$D_3(q) = \frac{\eta^3(2z)\eta(7z)\eta^2(56z)}{\eta(z)\eta(4z)}.$$

Then we have

Table 3.5.12(a)

δ	1	2	4	7	56
r_δ	-1	3	-1	1	2

It can be seen from Table 3.5.12(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.12(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	30/7	12/7	9	24/7	18	36	120

From Table 3.5.12(b) the condition $(L4)$ is also satisfied. Thus $D_3(q) \in S_2(\Gamma_0(56), \chi_6)$.

Fourthly we consider

$$D_4(q) = \frac{\eta^3(4z)\eta^2(7z)\eta(56z)}{\eta(2z)\eta(8z)}.$$

Then we have

Table 3.5.13(a)

δ	2	4	7	8	56
r_δ	-1	3	2	-1	1

It can be seen from Table 3.5.13(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.13(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\le\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	6/7	60/7	15	24/7	18	36	72

From Table 3.5.13(b) the condition $(L4)$ is also satisfied. Thus $D_4(q) \in S_2(\Gamma_0(56), \chi_6)$.

Fifthly we consider

$$D_5(q) = \frac{\eta(2z)\eta(8z)\eta^4(14z)\eta^2(28z)}{\eta(4z)\eta^2(7z)\eta(56z)}.$$

Then we have

Table 3.5.14(a)

δ	2	4	7	8	14	28	56
r_δ	1	-1	-2	1	4	2	-1

It can be seen from Table 3.5.14(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.14(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/7	18/7	12/7	3	48/7	54	84	48

From Table 3.5.14(b) the condition $(L4)$ is also satisfied. Thus $D_5(q) \in S_2(\Gamma_0(56), \chi_6)$.

Sixthly we consider

$$D_6(q) = \frac{\eta(z)\eta^2(8z)\eta^3(14z)}{\eta(7z)\eta(28z)}.$$

Then we have

Table 3.5.15(a)

δ	1	7	8	14	28
r_δ	1	-1	2	3	-1

It can be seen from Table 3.5.15(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 3.5.15(b)

$d \mid 56$	1	2	4	7	8	14	28	56
$\sum_{1\leq\delta\mid 56} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	9/7	18/7	36/7	3	120/7	30	12	24

From Table 3.5.15(b) the condition $(L4)$ is also satisfied. Thus $D_6(q) \in S_2(\Gamma_0(56), \chi_6)$. ∎

Theorem 3.5.3. **(a)** $\{D_1(q), \ldots, D_6(q)\}$ *is a basis for* $S_2(\Gamma_0(56), \chi_6)$.

(b) $\{E_{2,\chi_4,\chi_1}(q), E_{2,\chi_1,\chi_4}(q), E_{2,\chi_6,\chi_0}(q), E_{2,\chi_0,\chi_6}(q)\}$ *is a basis for* $E_2(\Gamma_0(56), \chi_6)$.

(c) *$\{E_{2,\chi_4,\chi_1}(q), E_{2,\chi_1,\chi_4}(q), E_{2,\chi_6,\chi_0}(q), E_{2,\chi_0,\chi_6}(q)\}$ together with $D_k(q)$ $(1 \le k \le 6)$ constitute a basis for $M_2(\Gamma_0(56), \chi_6)$.*

Proof. **(a)** By Theorem 3.5.2, $D_k(q)$ $(1 \le k \le 6) \in S_2(\Gamma_0(56), \chi_6)$. There is no linear relationship among them. By Example 2.3.3, we have $\dim S_2(\Gamma_0(56), \chi_6) = 6$. Thus $D_k(q)$ $(1 \le k \le 6)$ constitute a basis for $S_2(\Gamma_0(56), \chi_6)$.

(b) By Example 2.3.3, we have $\dim E_2(\Gamma_0(56), \chi_6) = 4$. By Theorem 2.2.3, $\{E_{2,\chi_4,\chi_1}(q), E_{2,\chi_1,\chi_4}(q), E_{2,\chi_6,\chi_0}(q), E_{2,\chi_0,\chi_6}(q)\}$ constitute a basis for $E_2(\Gamma_0(56), \chi_6)$.

(c) By Example 2.3.3, we have $\dim M_2(\Gamma_0(56), \chi_6) = 10$. Therefore, by (2.1.1) $\{E_{2,\chi_4,\chi_1}(q), E_{2,\chi_1,\chi_4}(q), E_{2,\chi_6,\chi_0}(q), E_{2,\chi_0,\chi_6}(q)\}$ together with $D_k(q)$ $(1 \le k \le 6)$ constitute a basis for $M_2(\Gamma_0(56), \chi_6)$. ∎

Theorem 3.5.4. *Let χ_0 be the trivial character and χ_1, χ_4, χ_6 be as in* (3.1.3) *and* (3.1.4). *Then*

$$\begin{aligned}
\textbf{(a)}\ \varphi^2(q)\varphi(q^2)\varphi(q^7) = &-\frac{2}{5}E_{2,\chi_4,\chi_1}(q) + \frac{7}{10}E_{2,\chi_1,\chi_4}(q) + \frac{14}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) \\
&- \frac{1}{10}D_1(q) + \frac{6}{5}D_2(q) - \frac{7}{5}D_3(q) - \frac{7}{10}D_5(q) - \frac{1}{5}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(b)}\ \varphi^3(q^2)\varphi(q^7) = &-\frac{1}{5}E_{2,\chi_4,\chi_1}(q) + \frac{7}{10}E_{2,\chi_1,\chi_4}(q) + \frac{7}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) \\
&+ \frac{51}{20}D_1(q) + \frac{9}{10}D_2(q) + \frac{63}{10}D_3(q) - \frac{63}{5}D_4(q) - \frac{63}{20}D_5(q) - \frac{27}{10}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(c)}\ \varphi(q^2)\varphi^3(q^7) = &\frac{2}{5}E_{2,\chi_4,\chi_1}(q) - \frac{1}{10}E_{2,\chi_1,\chi_4}(q) + \frac{2}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) \\
&- \frac{3}{2}D_1(q) + \frac{3}{5}D_3(q) + \frac{12}{5}D_4(q) + \frac{3}{2}D_5(q) - \frac{3}{5}D_6(q),
\end{aligned}$$

$$\begin{aligned}
\textbf{(d)}\ \varphi^3(q)\varphi(q^{14}) = &-\frac{2}{5}E_{2,\chi_4,\chi_1}(q) + \frac{7}{10}E_{2,\chi_1,\chi_4}(q) + \frac{14}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) \\
&+ \frac{9}{10}D_1(q) + \frac{6}{5}D_2(q) + \frac{63}{5}D_3(q) + \frac{63}{10}D_5(q) + \frac{9}{5}D_6(q),
\end{aligned}$$

$$\begin{aligned}\textbf{(e)}\ \varphi(q)\varphi^2(q^2)\varphi(q^{14}) =& -\frac{1}{5}E_{2,\chi_4,\chi_1}(q) + \frac{7}{10}E_{2,\chi_1,\chi_4}(q) + \frac{7}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q)\\ &+ \frac{21}{20}D_1(q) - \frac{1}{10}D_2(q) - \frac{7}{10}D_3(q) + \frac{7}{5}D_4(q) + \frac{7}{20}D_5(q)\\ &+ \frac{3}{10}D_6(q),\end{aligned}$$

$$\begin{aligned}\textbf{(f)}\ \varphi(q)\varphi^2(q^7)\varphi(q^{14}) =& \frac{2}{5}E_{2,\chi_4,\chi_1}(q) - \frac{1}{10}E_{2,\chi_1,\chi_4}(q) + \frac{2}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q)\\ &- \frac{1}{2}D_1(q) - \frac{7}{5}D_3(q) + \frac{12}{5}D_4(q) + \frac{1}{2}D_5(q) + \frac{7}{5}D_6(q),\end{aligned}$$

$$\begin{aligned}\textbf{(g)}\ \varphi(q^2)\varphi(q^7)\varphi^2(q^{14}) =& \frac{1}{5}E_{2,\chi_4,\chi_1}(q) - \frac{1}{10}E_{2,\chi_1,\chi_4}(q) + \frac{1}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q)\\ &+ \frac{1}{20}D_1(q) - \frac{1}{10}D_2(q) - \frac{3}{10}D_3(q) - \frac{1}{5}D_4(q) + \frac{27}{20}D_5(q)\\ &- \frac{1}{10}D_6(q),\end{aligned}$$

$$\begin{aligned}\textbf{(h)}\ \varphi(q)\varphi^3(q^{14}) =& \frac{1}{5}E_{2,\chi_4,\chi_1}(q) - \frac{1}{10}E_{2,\chi_1,\chi_4}(q) + \frac{1}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q)\\ &- \frac{9}{20}D_1(q) + \frac{9}{10}D_2(q) + \frac{27}{10}D_3(q) + \frac{9}{5}D_4(q) - \frac{3}{20}D_5(q)\\ &+ \frac{9}{10}D_6(q).\end{aligned}$$

Proof. Let (a_1, a_2, a_3, a_4) be one of the quadratic forms listed in the fourth column of Table 3.1.1. By Theorem 3.5.1, $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(56), \chi_6)$. Therefore, by Theorem 3.5.3(c), $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $E_{2,\chi_1,\chi_4}(q), E_{2,\chi_4,\chi_1}(q), E_{2,\chi_6,\chi_0}(q), E_{2,\chi_0,\chi_6}(q)$ and $D_k(q)$ $(1 \le k \le 6)$, namely

$$\begin{aligned}\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) =& x_1E_{2,\chi_4,\chi_1}(q) + x_2E_{2,\chi_1,\chi_4}(q) + x_3E_{2,\chi_6,\chi_0}(q) + x_4E_{2,\chi_0,\chi_6}(q)\\ &+ y_1D_1(q) + y_2D_2(q) + y_3D_3(q) + y_4D_4(q) + y_5D_5(q)\\ &+ y_6D_6(q).\end{aligned}$$

We equate the first fourteen coefficients of q^n on both sides of the equation above to obtain a system of linear equations with the unknowns $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, y_5, y_6$. Then, using MAPLE we solve the system to find the asserted coefficients. ■

Theorem 3.5.5. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in (2.2.1) for $i, j \in \{0, 1, 4, 6\}$. Then*

$$\begin{aligned}
\textbf{(a)}\ N(1,1,2,7;n) =& -\frac{2}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{14}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) \\
& - \frac{1}{10}d_1(n) + \frac{6}{5}d_2(n) - \frac{7}{5}d_3(n) - \frac{7}{10}d_5(n) - \frac{1}{5}d_6(n),
\end{aligned}$$

$$\begin{aligned}
\textbf{(b)}\ N(2,2,2,7;n) =& -\frac{1}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{7}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) \\
& + \frac{51}{20}d_1(n) + \frac{9}{10}d_2(n) + \frac{63}{10}d_3(n) - \frac{63}{5}d_4(n) - \frac{63}{20}d_5(n) \\
& - \frac{27}{10}d_6(n),
\end{aligned}$$

$$\begin{aligned}
\textbf{(c)}\ N(2,7,7,7;n) =& \frac{2}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{2}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) \\
& - \frac{3}{2}d_1(n) + \frac{3}{5}d_3(n) + \frac{12}{5}d_4(n) + \frac{3}{2}d_5(n) - \frac{3}{5}d_6(n),
\end{aligned}$$

$$\begin{aligned}
\textbf{(d)}\ N(1,1,1,14;n) =& -\frac{2}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{14}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) \\
& + \frac{9}{10}d_1(n) + \frac{6}{5}d_2(n) + \frac{63}{5}d_3(n) + \frac{63}{10}d_5(n) + \frac{9}{5}d_6(n),
\end{aligned}$$

$$\begin{aligned}
\textbf{(e)}\ N(1,2,2,14;n) =& -\frac{1}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{7}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) \\
& + \frac{21}{20}d_1(n) - \frac{1}{10}d_2(n) - \frac{7}{10}d_3(n) + \frac{7}{5}d_4(n) + \frac{7}{20}d_5(n) \\
& + \frac{3}{10}d_6(n),
\end{aligned}$$

$$\begin{aligned}
\textbf{(f)}\ N(1,7,7,14;n) =& \frac{2}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{2}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) \\
& - \frac{1}{2}d_1(n) - \frac{7}{5}d_3(n) + \frac{12}{5}d_4(n) + \frac{1}{2}d_5(n) + \frac{7}{5}d_6(n),
\end{aligned}$$

$$\begin{aligned}
\textbf{(g)}\ N(2,7,14,14;n) =& \frac{1}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{1}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) \\
& + \frac{1}{20}d_1(n) - \frac{1}{10}d_2(n) - \frac{3}{10}d_3(n) - \frac{1}{5}d_4(n) + \frac{27}{20}d_5(n) \\
& - \frac{1}{10}d_6(n),
\end{aligned}$$

$$(\mathbf{h})\ N(1,14,14,14;n) = \frac{1}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{1}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) - \frac{9}{20}d_1(n) + \frac{9}{10}d_2(n) + \frac{27}{10}d_3(n) + \frac{9}{5}d_4(n) - \frac{3}{20}d_5(n) + \frac{9}{10}d_6(n).$$

Proof. Appealing to (2.1.6), (3.5.1)-(3.5.4) and Theorem 3.5.4, we have

$$(\mathbf{a})\ \sum_{n=0}^{\infty} N(1,1,2,7;n)q^n = \varphi^2(q)\varphi(q^2)\varphi(q^7)$$
$$= -\frac{2}{5}E_{2,\chi_4,\chi_1}(q) + \frac{7}{10}E_{2,\chi_1,\chi_4}(q) + \frac{14}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) - \frac{1}{10}D_1(q) + \frac{6}{5}D_2(q) - \frac{7}{5}D_3(q) - \frac{7}{10}D_5(q) - \frac{1}{5}D_6(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(-\frac{2}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{14}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) - \frac{1}{10}d_1(n) + \frac{6}{5}d_2(n) - \frac{7}{5}d_3(n) - \frac{7}{10}d_5(n) - \frac{1}{5}d_6(n)\Big)q^n,$$

$$(\mathbf{b})\ \sum_{n=0}^{\infty} N(2,2,2,7;n)q^n = \varphi^3(q^2)\varphi(q^7)$$
$$= -\frac{1}{5}E_{2,\chi_4,\chi_1}(q) + \frac{7}{10}E_{2,\chi_1,\chi_4}(q) + \frac{7}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) + \frac{51}{20}D_1(q) + \frac{9}{10}D_2(q) + \frac{63}{10}D_3(q) - \frac{63}{5}D_4(q) - \frac{63}{20}D_5(q) - \frac{27}{10}D_6(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(-\frac{1}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{7}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) + \frac{51}{20}d_1(n) + \frac{9}{10}d_2(n) + \frac{63}{10}d_3(n) - \frac{63}{5}d_4(n) - \frac{63}{20}d_5(n) - \frac{27}{10}d_6(n)\Big)q^n,$$

(c) $$\sum_{n=0}^{\infty} N(2,7,7,7;n)q^n = \varphi(q^2)\varphi^3(q^7)$$
$$= \frac{2}{5}E_{2,\chi_4,\chi_1}(q) - \frac{1}{10}E_{2,\chi_1,\chi_4}(q) + \frac{2}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) - \frac{3}{2}D_1(q)$$
$$+ \frac{3}{5}D_3(q) + \frac{12}{5}D_4(q) + \frac{3}{2}D_5(q) - \frac{3}{5}D_6(q),$$
$$= 1 + \sum_{n=1}^{\infty} \Big(\frac{2}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{2}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n)$$
$$- \frac{3}{2}d_1(n) + \frac{3}{5}d_3(n) + \frac{12}{5}d_4(n) + \frac{3}{2}d_5(n) - \frac{3}{5}d_6(n)\Big)q^n,$$

(d) $$\sum_{n=0}^{\infty} N(1,1,1,14;n)q^n = \varphi^3(q)\varphi(q^{14})$$
$$= -\frac{2}{5}E_{2,\chi_4,\chi_1}(q) + \frac{7}{10}E_{2,\chi_1,\chi_4}(q) + \frac{14}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) + \frac{9}{10}D_1(q)$$
$$+ \frac{6}{5}D_2(q) + \frac{63}{5}D_3(q) + \frac{63}{10}D_5(q) + \frac{9}{5}D_6(q),$$
$$= 1 + \sum_{n=1}^{\infty} \Big(- \frac{2}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{14}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) + \frac{9}{10}d_1(n)$$
$$+ \frac{6}{5}d_2(n) + \frac{63}{5}d_3(n) + \frac{63}{10}d_5(n) + \frac{9}{5}d_6(n)\Big)q^n,$$

(e) $$\sum_{n=0}^{\infty} N(1,2,2,14;n)q^n = \varphi(q)\varphi^2(q^2)\varphi(q^{14})$$
$$= -\frac{1}{5}E_{2,\chi_4,\chi_1}(q) + \frac{7}{10}E_{2,\chi_1,\chi_4}(q) + \frac{7}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q)$$
$$+ \frac{21}{20}D_1(q) - \frac{1}{10}D_2(q) - \frac{7}{10}D_3(q) + \frac{7}{5}D_4(q) + \frac{7}{20}D_5(q) + \frac{3}{10}D_6(q),$$
$$= 1 + \sum_{n=1}^{\infty} \Big(- \frac{1}{5}\sigma_{\chi_4,\chi_1}(n) + \frac{7}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{7}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n)$$
$$+ \frac{21}{20}d_1(n) - \frac{1}{10}d_2(n) - \frac{7}{10}d_3(n) + \frac{7}{5}d_4(n) + \frac{7}{20}d_5(n) + \frac{3}{10}d_6(n)\Big)q^n,$$

(f) $\displaystyle\sum_{n=0}^{\infty} N(1,7,7,14;n)q^n = \varphi(q)\varphi^2(q^7)\varphi(q^{14})$

$$= \frac{2}{5}E_{2,\chi_4,\chi_1}(q) - \frac{1}{10}E_{2,\chi_1,\chi_4}(q) + \frac{2}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) - \frac{1}{2}D_1(q)$$
$$- \frac{7}{5}D_3(q) + \frac{12}{5}D_4(q) + \frac{1}{2}D_5(q) + \frac{7}{5}D_6(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{2}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{2}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) - \frac{1}{2}d_1(n)$$
$$- \frac{7}{5}d_3(n) + \frac{12}{5}d_4(n) + \frac{1}{2}d_5(n) + \frac{7}{5}d_6(n)\Big)q^n,$$

(g) $\displaystyle\sum_{n=0}^{\infty} N(2,7,14,14;n)q^n = \varphi(q^2)\varphi(q^7)\varphi^2(q^{14})$

$$= \frac{1}{5}E_{2,\chi_4,\chi_1}(q) - \frac{1}{10}E_{2,\chi_1,\chi_4}(q) + \frac{1}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q)$$
$$+ \frac{1}{20}D_1(q) - \frac{1}{10}D_2(q) - \frac{3}{10}D_3(q) - \frac{1}{5}D_4(q) + \frac{27}{20}D_5(q) - \frac{1}{10}D_6(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{1}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{1}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n)$$
$$+ \frac{1}{20}d_1(n) - \frac{1}{10}d_2(n) - \frac{3}{10}d_3(n) - \frac{1}{5}d_4(n) + \frac{27}{20}d_5(n) - \frac{1}{10}d_6(n)\Big)q^n,$$

(h) $\displaystyle\sum_{n=0}^{\infty} N(1,14,14,14;n)q^n = \varphi(q)\varphi^3(q^{14})$

$$= \frac{1}{5}E_{2,\chi_4,\chi_1}(q) - \frac{1}{10}E_{2,\chi_1,\chi_4}(q) + \frac{1}{5}E_{2,\chi_6,\chi_0}(q) - \frac{1}{10}E_{2,\chi_0,\chi_6}(q) - \frac{9}{20}D_1(q)$$
$$+ \frac{9}{10}D_2(q) + \frac{27}{10}D_3(q) + \frac{9}{5}D_4(q) - \frac{3}{20}D_5(q) + \frac{9}{10}D_6(q),$$
$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{1}{5}\sigma_{\chi_4,\chi_1}(n) - \frac{1}{10}\sigma_{\chi_1,\chi_4}(n) + \frac{1}{5}\sigma_{\chi_6,\chi_0}(n) - \frac{1}{10}\sigma_{\chi_0,\chi_6}(n) - \frac{9}{20}d_1(n)$$
$$+ \frac{9}{10}d_2(n) + \frac{27}{10}d_3(n) + \frac{9}{5}d_4(n) - \frac{3}{20}d_5(n) + \frac{9}{10}d_6(n)\Big)q^n.$$

Equating the coefficients of q^n on both sides of equations (a)–(h) yields the results. ∎

The values of $N(a_1, a_2, a_3, a_4; n)$ for $1 \le n \le 20$ for the quadratic forms (a_1, a_2, a_3, a_4) in Theorem 3.5.5 are given in Table 3.5.16.

Table 3.5.16

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$N(1,1,2,7;n)$	4	6	8	12	8	8	18	14	24	40	32	40	40	32	28	36	64	46	72	72
$N(2,2,2,7;n)$	0	6	0	12	0	8	2	6	12	24	24	24	16	0	12	12	48	30	48	24
$N(2,7,7,7;n)$	0	2	0	0	0	0	6	2	12	0	0	0	0	12	12	24	0	2	0	0
$N(1,1,1,14;n)$	6	12	8	6	24	24	0	12	30	24	24	8	24	50	12	30	64	48	72	72
$N(1,2,2,14;n)$	2	4	8	6	8	8	0	12	10	8	24	8	8	18	4	14	32	24	40	40
$N(1,7,7,14;n)$	2	0	0	2	0	0	4	8	2	0	8	0	0	6	12	10	0	12	0	0
$N(2,7,14,14;n)$	0	2	0	0	0	0	2	2	4	0	0	0	0	4	4	8	0	2	0	0
$N(1,14,14,14;n)$	2	0	0	2	0	0	0	0	2	0	0	0	0	6	12	2	0	12	0	0

Again one can verify the values in Table 3.5.16 by using (2.2.1), (3.1.3), (3.1.4) and Table 3.5.1.

Chapter 4

Representations by Octonary Quadratic Forms and Future Work

In this chapter we extend our work to determine explicit formulae for the number of representations of n by the octonary quadratic forms $x_1^2+x_2^2+x_3^2+x_4^2+7(x_5^2+x_6^2+x_7^2+x_8^2)$, $x_1^2+x_2^2+7(x_3^2+x_4^2+x_5^2+x_6^2+x_7^2+x_8^2)$ and $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+7(x_7^2+x_8^2)$, which we denote by $N(1^4,7^4;n)$, $N(1^2,7^6;n)$ and $N(1^6,7^2;n)$ respectively. A formula for $N(1^8;n)$ is given in [2] and [30].

4.1 Representations by Octonary Quadratic Forms with Coefficients 1 and 7

For $a_1,\ldots,a_8 \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a_1,\ldots,a_8;n) := \text{card}\{(x_1,\ldots,x_8) \in \mathbb{Z}^8 \mid n = a_1x_1^2+\cdots+a_8x_8^2\}.$$

For $i, j \in \mathbb{N}_0$ with $i + j = 8$ we define

$$N(1^i, 7^j; n) := N(\underbrace{1, \ldots, 1}_{i}, \underbrace{7, \ldots, 7}_{j}; n).$$

We give formulae for $N(1^i, 7^j; n)$ in the cases $(i, j) = (4, 4), (6, 2)$ and $(2, 6)$ in terms of $\sigma_3(n)$, $\sigma_3(n/2)$, $\sigma_3(n/4)$, $\sigma_3(n/7)$, $\sigma_3(n/14)$, $\sigma_3(n/28)$ and $v_k(n)$ $(1 \leq k \leq 9)$ defined by

$$V_1(q) = \sum_{n=1}^{\infty} v_1(n) q^n = \eta^2(z)\eta^2(2z)\eta^2(7z)\eta^2(14z), \tag{4.1.1}$$

$$V_2(q) = \sum_{n=1}^{\infty} v_2(n) q^n = \eta^2(2z)\eta^2(4z)\eta^2(14z)\eta^2(28z), \tag{4.1.2}$$

$$V_3(q) = \sum_{n=1}^{\infty} v_3(n) q^n = \eta^2(z)\eta^2(7z)\eta^4(14z), \tag{4.1.3}$$

$$V_4(q) = \sum_{n=1}^{\infty} v_4(n) q^n = \eta^4(2z)\eta^2(4z)\eta^2(28z), \tag{4.1.4}$$

$$V_5(q) = \sum_{n=1}^{\infty} v_5(n) q^n = \eta^2(z)\eta^4(2z)\eta^2(7z), \tag{4.1.5}$$

$$V_6(q) = \sum_{n=1}^{\infty} v_6(n) q^n = \eta^2(4z)\eta^4(14z)\eta^2(28z), \tag{4.1.6}$$

$$V_7(q) = \sum_{n=1}^{\infty} v_7(n) q^n = \eta(z)\eta(4z)\eta(7z)\eta^4(14z)\eta(28z), \tag{4.1.7}$$

$$V_8(q) = \sum_{n=1}^{\infty} v_8(n) q^n = \eta(z)\eta^4(2z)\eta(4z)\eta(7z)\eta(28z), \tag{4.1.8}$$

$$V_9(q) = \sum_{n=1}^{\infty} v_9(n) q^n = \frac{\eta(z)\eta(2z)\eta^9(7z)}{\eta^3(14z)}. \tag{4.1.9}$$

Note that

$$V_2(q) = V_1(q^2),$$

so that

$$v_1(2n) = v_2(n).$$

There is no linear relationship among the $V_k(q), 1 \leq k \leq 9$. The first twenty-eight values of $v_k(n)$, are given in Table 4.4.1.

Table 4.4.1

n	$v_1(n)$	$v_2(n)$	$v_3(n)$	$v_4(n)$	$v_5(n)$	$v_6(n)$	$v_7(n)$	$v_8(n)$	$v_9(n)$
1	0	0	0	0	1	0	0	0	1
2	1	0	0	0	−2	0	0	1	−1
3	−2	0	1	1	−5	0	0	−1	−2
4	−3	1	−2	0	10	0	1	−5	1
5	6	0	−1	−4	7	1	−1	4	0
6	2	−2	2	0	−10	0	−1	5	2
7	0	0	1	0	0	0	0	0	1
8	−1	−3	2	0	−22	0	−1	11	−9
9	−12	0	−2	16	−1	−2	2	−16	9
10	4	6	−2	0	38	0	1	−19	16
11	4	0	2	−10	−6	0	0	10	−8
12	−2	2	0	0	8	0	0	−4	−2
13	2	0	−3	−16	−15	−1	1	16	−20
14	−7	0	−2	0	0	0	1	0	−9
15	16	0	−4	0	52	0	0	0	32
16	17	−1	6	0	−58	0	−3	29	−31
17	−4	0	−2	0	−44	2	−2	0	−42
18	1	−12	12	0	−38	0	−6	19	23
19	−38	0	11	39	−31	−4	4	−39	18
20	−24	4	−12	0	132	0	6	−66	80
21	14	0	−5	0	49	1	−1	0	30
22	−16	4	−12	0	−12	0	6	6	−56
23	32	0	4	0	76	8	−8	0	48
24	18	−2	4	0	−12	0	−2	6	18
25	12	0	−6	−32	−11	2	−2	32	−29
26	24	2	−6	0	−70	0	3	35	−60
27	4	0	14	−70	−166	4	−4	70	−156
28	21	−7	10	0	−98	0	−5	49	−39

Theorem 4.1.1. *For* $(i, j) = (4, 4), (6, 2), (2, 6),$ *we have* $\varphi^i(q)\varphi^j(q^7) \in M_4(\Gamma_0(28))$.

Proof. Appealing to (2.1.9) we then check conditions (L1), (L2) and (L3) of Theorem 3.1.1 for each quadratic form. We have $N = 28$. First

$$\varphi^4(q)\varphi^4(q^7) = \frac{\eta^{20}(2z)\eta^{20}(14z)}{\eta^8(z)\eta^8(4z)\eta^8(7z)\eta^8(28z)}.$$

Then we have

Table 4.4.2(a)

δ	1	2	4	7	14	28
r_δ	-8	20	-8	-8	20	-8

It can be seen from Table 4.4.2(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 4.4.2(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\le\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	192/7	0	0	192	0

From Table 4.4.2(b) the condition ($L3$) is also satisfied. Thus $\varphi^4(q)\varphi^4(q^7) \in M_4(\Gamma_0(28))$.

Secondly we have

$$\varphi^2(q)\varphi^6(q^7) = \frac{\eta^{10}(2z)\eta^{30}(14z)}{\eta^4(z)\eta^4(4z)\eta^{12}(7z)\eta^{12}(28z)}.$$

Then we have

Table 4.4.3(a)

δ	1	2	4	7	14	28
r_δ	-4	10	-4	-12	30	-12

It can be seen from Table 4.4.3(a) that conditions ($L1$) and ($L2$) are satisfied.

Table 4.4.3(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\leq\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	120/7	0	0	264	0

From Table 4.4.3(b) the condition $(L3)$ is also satisfied. Thus $\varphi^2(q)\varphi^6(q^7) \in M_4(\Gamma_0(28))$.

Thirdly we have

$$\varphi^6(q)\varphi^2(q^7) = \frac{\eta^{30}(2z)\eta^{10}(14z)}{\eta^{12}(z)\eta^{12}(4z)\eta^4(7z)\eta^4(28z)}.$$

Then we have

Table 4.4.4(a)

δ	1	2	4	7	14	28
r_δ	-12	30	-12	-4	10	-4

It can be seen from Table 4.4.4(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.4(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\leq\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	264/7	0	0	120	0

From Table 4.4.4(b) the condition $(L3)$ is also satisfied. Thus $\varphi^6(q)\varphi^2(q^7) \in M_4(\Gamma_0(28))$. ∎

Theorem 4.1.2. *$V_k(q)$ $(1 \leq k \leq 9)$ given by* (4.1.1)–(4.1.9) *are in* $S_4(\Gamma_0(28))$.

Proof. We will check conditions (L1), (L2) and (L4) of Theorem 3.1.1. We have $N = 28$. Firstly we have

$$V_1(q) = \eta^2(z)\eta^2(2z)\eta^2(7z)\eta^2(14z).$$

Then we have

Table 4.4.5(a)

δ	1	2	7	14
r_δ	2	2	2	2

It can be seen from Table 4.4.5(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.5(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\le\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	24/7	48/7	48/7	24	48	48

From Table 4.4.5(b) the condition $(L4)$ is also satisfied. Thus $V_1(q) \in S_4(\Gamma_0(28))$. Secondly we consider

$$V_2(q) = \eta^2(2z)\eta^2(4z)\eta^2(14z)\eta^2(28z).$$

Then we have

Table 4.4.6(a)

δ	2	4	14	28
r_δ	2	2	2	2

It can be seen from Table 4.4.6(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.6(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\le\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	12/7	48/7	96/7	12	48	96

From Table 4.4.6(b) the condition $(L4)$ is also satisfied. Thus $V_2(q) \in S_4(\Gamma_0(28))$. Thirdly we consider

$$V_3(q) = \eta^2(z)\eta^2(7z)\eta^4(14z).$$

Then we have

Table 4.4.7(a)

δ	1	7	14
r_δ	2	2	4

It can be seen from Table 4.4.7(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.7(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\le\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	18/7	24/7	24/7	30	72	72

From Table 4.4.7(b) the condition $(L4)$ is also satisfied. Thus $V_3(q) \in S_4(\Gamma_0(28))$.

Fourthly we consider

$$V_4(q) = \eta^4(2z)\eta^2(4z)\eta^2(28z).$$

Then we have

Table 4.4.8(a)

δ	2	4	28
r_δ	4	2	2

It can be seen from Table 4.4.8(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.8(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\le\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	18/7	72/7	120/7	6	24	72

From Table 4.4.8(b) the condition $(L4)$ is also satisfied. Thus $V_4(q) \in S_4(\Gamma_0(28))$.

Fifthly we consider

$$V_5(q) = \eta^2(z)\eta^4(2z)\eta^2(7z).$$

Then we have

Table 4.4.9(a)

δ	1	2	7
r_δ	2	4	2

It can be seen from Table 4.4.9(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.9(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\leq\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	30/7	72/7	72/7	18	24	24

From Table 4.4.9(b) the condition $(L4)$ is also satisfied. Thus $V_5(q) \in S_4(\Gamma_0(28))$.

Sixthly we consider

$$V_6(q) = \eta^2(4z)\eta^4(14z)\eta^2(28z).$$

Then we have

Table 4.4.10(a)

δ	4	14	28
r_δ	2	4	2

It can be seen from Table 4.4.10(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.10(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\leq\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/7	24/7	72/7	18	72	120

From Table 4.4.10(b) the condition $(L4)$ is also satisfied. Thus $V_6(q) \in S_4(\Gamma_0(28))$.

Seventhly we have

$$V_7(q) = \eta(z)\eta(4z)\eta(7z)\eta^4(14z)\eta(28z).$$

Then we have

Table 4.4.11(a)

δ	1	4	7	14	28
r_δ	1	1	1	4	1

It can be seen from Table 4.4.11(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.11(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\le\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	12/7	24/7	48/7	24	72	96

From Table 4.4.11(b) the condition $(L4)$ is also satisfied. Thus $V_7(q) \in S_4(\Gamma_0(28))$. Eighthly we consider

$$V_8(q) = \eta(z)\eta^4(2z)\eta(4z)\eta(7z)\eta(28z).$$

Then we have

Table 4.4.12(a)

δ	1	2	4	7	28
r_δ	1	4	1	1	1

It can be seen from Table 4.4.12(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.12(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\le\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	24/7	72/7	96/7	12	24	24

From Table 4.4.12(b) the condition $(L4)$ is also satisfied. Thus $V_8(q) \in S_4(\Gamma_0(28))$. Ninthly we consider

$$V_9(q) = \frac{\eta(z)\eta(2z)\eta^9(7z)}{\eta^3(14z)}.$$

Then we have

Table 4.4.13(a)

δ	1	2	7	14
r_δ	1	1	9	-3

It can be seen from Table 4.4.13(a) that conditions $(L1)$ and $(L2)$ are satisfied.

Table 4.4.13(b)

$d \mid 28$	1	2	4	7	14	28
$\sum_{1\leq\delta\mid 28} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	18/7	24/7	24/7	54	24	24

From Table 4.4.13(b) the condition $(L4)$ is also satisfied. Thus $V_9(q) \in S_4(\Gamma_0(28))$. ■

Theorem 4.1.3. **(a)** $\{V_1(q), \ldots, V_9(q)\}$ *constitute a basis for* $S_4(\Gamma_0(28))$.

(b) $E_4(q^k)$ $(k = 1, 2, 4, 7, 14, 28)$ *constitute a basis for* $E_4(\Gamma_0(28))$.

(c) $E_4(q^k)$ $(k = 1, 2, 4, 7, 14, 28)$ *together with* $V_k(q)$ $(1 \leq k \leq 9)$ *constitute a basis for* $M_4(\Gamma_0(28))$.

Proof. **(a)** By Theorem 4.1.2, $V_k(q)$ $(1 \leq k \leq 9) \in S_4(\Gamma_0(28))$. There is no linear relationship among them. By Proposition 2.3.1, we have $\dim S_4(\Gamma_0(28)) = 9$. Thus $V_k(q)$ $(1 \leq k \leq 9)$ constitute a basis for $S_4(\Gamma_0(28))$.

(b) By Proposition 2.3.1, we have $\dim E_4(\Gamma_0(28)) = 6$. By Theorem 2.2.3, $E_4(q^k)$ $(k = 1, 2, 4, 7, 14, 28)$ constitute a basis for $E_4(\Gamma_0(28))$.

(c) It follows from (a), (b) and (2.1.1) that the dimension of $M_4(\Gamma_0(28))$ is 15 and therefore $E_4(q^k)$ $(k = 1, 2, 4, 7, 14, 28)$ together with $V_k(q)$ $(1 \leq k \leq 9)$ constitute a basis for $M_4(\Gamma_0(28))$. ■

Theorem 4.1.4.

(**a**) $$\begin{aligned}\varphi^4(q)\varphi^4(q^7) =&\frac{8}{25}E_4(q) - \frac{16}{25}E_4(q^2) + \frac{128}{25}E_4(q^4) + \frac{392}{25}E_4(q^7) - \frac{784}{25}E_4(q^{14})\\ &+ \frac{6272}{25}E_4(q^{28}) - \frac{448}{25}V_1(q) + \frac{3712}{25}V_2(q) - \frac{544}{5}V_3(q) - \frac{768}{25}V_4(q)\\ &- \frac{1184}{25}V_5(q) + \frac{5376}{25}V_6(q) + \frac{1376}{25}V_9(q),\end{aligned}$$

(**b**) $$\begin{aligned}\varphi^2(q)\varphi^6(q^7) =&\frac{4}{75}E_4(q) - \frac{64}{75}E_4(q^4) - \frac{1204}{75}E_4(q^7) + \frac{19264}{75}E_4(q^{28}) - \frac{104}{25}V_1(q)\\ &+ \frac{416}{25}V_2(q) + \frac{728}{75}V_3(q) + \frac{1184}{75}V_4(q) + \frac{296}{75}V_5(q) + \frac{2912}{75}V_6(q)\\ &+ \frac{2224}{75}V_7(q) + \frac{1168}{75}V_8(q),\end{aligned}$$

(**c**) $$\begin{aligned}\varphi^6(q)\varphi^2(q^7) =&\frac{172}{75}E_4(q) - \frac{2752}{75}E_4(q^4) - \frac{1372}{75}E_4(q^7) + \frac{21952}{75}E_4(q^{28}) + \frac{728}{25}V_1(q)\\ &- \frac{2912}{25}V_2(q) + \frac{14504}{75}V_3(q) + \frac{2912}{75}V_4(q) + \frac{728}{75}V_5(q)\\ &+ \frac{58016}{75}V_6(q) + \frac{57232}{75}V_7(q) + \frac{2224}{75}V_8(q).\end{aligned}$$

Proof. By Theorem 4.1.1, we have $\varphi^{2i}(q)\varphi^{2j}(q^7) \in M_4(\Gamma_0(28))$. Therefore by Theorem 4.1.3 $\varphi^{2i}(q)\varphi^{2j}(q^7)$ must be a linear combination of $E_4(q^k)$ $(k = 1, 2, 4, 7, 14, 28)$ and $V_k(q)$ $(1 \le k \le 9)$, namely

$$\begin{aligned}\varphi^i(q)\varphi^j(q^7) =&x_1E_4(q) + x_2E_4(q^2) + x_3E_4(q^4) + x_4E_4(q^7) + x_5E_4(q^{14}) + x_6E_4(q^{28})\\ &+ y_1V_1(q) + y_2V_2(q) + y_3V_3(q) + y_4V_4(q) + y_5V_5(q) + y_6V_6(q)\\ &+ y_7V_7(q) + y_8V_8(q) + y_9V_9(q).\end{aligned}$$

We equate the first twenty-eight coefficients of q^n on both sides of the equation above to obtain a system of linear equations with the unknowns $x_1, x_2, x_3, x_4, x_5, x_6$ and $y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9$. Then, using MAPLE we solve the system to find the asserted coefficients. ∎

Theorem 4.1.5. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
(\mathbf{a})\ N(1^4,7^4;n) =& \frac{8}{25}\sigma_3(n) - \frac{16}{25}\sigma_3(n/2) + \frac{128}{25}\sigma_3(n/4) + \frac{392}{25}\sigma_3(n/7) \\
&- \frac{784}{25}\sigma_3(n/14) + \frac{6272}{25}\sigma_3(n/28) - \frac{448}{25}v_1(n) + \frac{3712}{25}v_2(n) \\
&- \frac{544}{5}v_3(n) - \frac{768}{25}v_4(n) - \frac{1184}{25}v_5(n) + \frac{5376}{25}v_6(n) + \frac{1376}{25}v_9(n),
\end{aligned}$$

$$\begin{aligned}
(\mathbf{b})\ N(1^2,7^6;n) =& \frac{4}{75}\sigma_3(n) - \frac{64}{75}\sigma_3(n/4) - \frac{1204}{75}\sigma_3(n/7) + \frac{19264}{75}\sigma_3(n/28) \\
&- \frac{104}{25}v_1(n) + \frac{416}{25}v_2(n) + \frac{728}{75}v_3(n) + \frac{1184}{75}v_4(n) + \frac{296}{75}v_5(n) \\
&+ \frac{2912}{75}v_6(n) + \frac{2224}{75}v_7(n) + \frac{1168}{75}v_8(n),
\end{aligned}$$

$$\begin{aligned}
(\mathbf{c})\ N(1^6,7^2;n) =& \frac{172}{75}\sigma_3(n) - \frac{2752}{75}\sigma_3(n/4) - \frac{1372}{75}\sigma_3(n/7) + \frac{21952}{75}\sigma_3(n/28) \\
&+ \frac{728}{25}v_1(n) - \frac{2912}{25}v_2(n) + \frac{14504}{75}v_3(n) + \frac{2912}{75}v_4(n) + \frac{728}{75}v_5(n) \\
&+ \frac{58016}{75}v_6(n) + \frac{57232}{75}v_7(n) + \frac{2224}{75}v_8(n).
\end{aligned}$$

Proof. Appealing (2.1.7), (2.2.4) and Theorem 4.1.4, we obtain

$$\begin{aligned}
(\mathbf{a})\ &\sum_{n=0}^{\infty} N(1^4,7^4;n)q^n = \varphi^4(q)\varphi^4(q^7) \\
&= \frac{8}{25}E_4(q) - \frac{16}{25}E_4(q^2) + \frac{128}{25}E_4(q^4) + \frac{392}{25}E_4(q^7) - \frac{784}{25}E_4(q^{14}) \\
&\quad + \frac{6272}{25}E_4(q^{28}) - \frac{448}{25}V_1(q) + \frac{3712}{25}V_2(q) - \frac{544}{5}V_3(q) - \frac{768}{25}V_4(q) \\
&\quad - \frac{1184}{25}V_5(q) + \frac{5376}{25}V_6(q) + \frac{1376}{25}V_9(q), \\
&= 1 + \sum_{n=1}^{\infty}\Big(\frac{8}{25}\sigma_3(n) - \frac{16}{25}\sigma_3(n/2) + \frac{128}{25}\sigma_3(n/4) + \frac{392}{25}\sigma_3(n/7) \\
&\quad - \frac{784}{25}\sigma_3(n/14) + \frac{6272}{25}\sigma_3(n/28) - \frac{448}{25}v_1(n) + \frac{3712}{25}v_2(n) - \frac{544}{5}v_3(n) \\
&\quad - \frac{768}{25}v_4(n) - \frac{1184}{25}v_5(n) + \frac{5376}{25}v_6(n) + \frac{1376}{25}v_9(n)\Big)q^n,
\end{aligned}$$

(**b**) $\sum_{n=0}^{\infty} N(1^2, 7^6; n)q^n = \varphi^2(q)\varphi^6(q^7)$

$$= \frac{4}{75}E_4(q) - \frac{64}{75}E_4(q^4) - \frac{1204}{75}E_4(q^7) + \frac{19264}{75}E_4(q^{28}) - \frac{104}{25}V_1(q) + \frac{416}{25}V_2(q) + \frac{728}{75}V_3(q) + \frac{1184}{75}V_4(q) + \frac{296}{75}V_5(q) + \frac{2912}{75}V_6(q) + \frac{2224}{75}V_7(q) + \frac{1168}{75}V_8(q),$$

$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{4}{75}\sigma_3(n) - \frac{64}{75}\sigma_3(n/4) - \frac{1204}{75}\sigma_3(n/7) + \frac{19264}{75}\sigma_3(n/28) - \frac{104}{25}v_1(n) + \frac{416}{25}v_2(n) + \frac{728}{75}v_3(n) + \frac{1184}{75}v_4(n) + \frac{296}{75}v_5(n) + \frac{2912}{75}v_6(n) + \frac{2224}{75}v_7(n) + \frac{1168}{75}v_8(n)\Big)q^n,$$

(**c**) $\sum_{n=0}^{\infty} N(1^6, 7^2; n)q^n = \varphi^6(q)\varphi^2(q^7)$

$$= \frac{172}{75}E_4(q) - \frac{2752}{75}E_4(q^4) - \frac{1372}{75}E_4(q^7) + \frac{21952}{75}E_4(q^{28}) + \frac{728}{25}V_1(q) - \frac{2912}{25}V_2(q) + \frac{14504}{75}V_3(q) + \frac{2912}{75}V_4(q) + \frac{728}{75}V_5(q) + \frac{58016}{75}V_6(q) + \frac{57232}{75}V_7(q) + \frac{2224}{75}V_8(q),$$

$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{172}{75}\sigma_3(n) - \frac{2752}{75}\sigma_3(n/4) - \frac{1372}{75}\sigma_3(n/7) + \frac{21952}{75}\sigma_3(n/28) + \frac{728}{25}v_1(n) - \frac{2912}{25}v_2(n) + \frac{14504}{75}v_3(n) + \frac{2912}{75}v_4(n) + \frac{728}{75}v_5(n) + \frac{58016}{75}v_6(n) + \frac{57232}{75}v_7(n) + \frac{2224}{75}v_8(n)\Big)q^n.$$

Equating the coefficients of q^n on both sides of equations (a)–(c) yields the results. ■

4.2 Conclusion and Future Work

In this thesis we have determined formulae for the number of representations of positive integers by quadratic forms by using a modular form approach. We worked on quaternary and octonary quadratic forms with certain coefficients.

We first found the number of representations of positive integers by quaternary quadratic form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, where $a_1, a_2, a_3, a_4 \in \{1, 2, 7, 14\}$. We then extended our work to octonary quadratic forms, and determined explicit formulae for $N(1^i, 7^j; n)$, where $(i, j) = (4, 4), (6, 2)$ and $(2, 6)$. We plan to extend our work to the remainning octonary quadratic forms with coefficients 1, 2, 7 and 14.

It would be natural to extend our work to find the number of representations of positive integers by quadratic forms with an odd number of variables. Our first step would be to start with ternary, quinary or septenary quadratic forms with coefficients 1, 2 and 7. For some recent work on this subject, one can see [5], [6] and [7].

Finally it would be interesting to find all the eta quotients in spaces $M_2(\Gamma_0(56), \chi)$ and $M_4(\Gamma_0(28), \chi)$, and study the properties of their Fourier coefficients.

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