A THREE TERM ARITHMETIC FORMULA OF LIOUVILLE TYPE WITH APPLICATION TO SUMS OF SIX SQUARES

by
Erin McAfee
B. Math. (Honours), Carleton University

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School of Mathematics and Statistics
Ottawa-Carleton Institute for Mathematics and Statistics
Carleton University
Ottawa, Ontario, Canada

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Abstract

The French mathematician Joseph Liouville (1809-1882), in a series of eighteen papers published between 1858 and 1865, announced without proof a number of amazing elementary arithmetic formulae. From these formulae many arithmetic results, including results concerning the representation of positive integers as sums of squares and sums of triangular numbers, can be deduced. Typical of Liouville's formulae is the following result, where $d(n)$ denotes the number of divisors of n and $\sigma(n)$ the sum of the divisors of n :

Let n be a positive integer and let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function, then

$$\begin{aligned} & \sum_{ax+by=n} (F(a-b) - F(a+b)) \\ &= F(0)(\sigma(n) - d(n)) + \sum_{d|n} \left(1 - d + \frac{2n}{d}\right) F(d) - 2 \sum_{d|n} \left(\sum_{k=1}^d F(k) \right), \end{aligned}$$

where the summation on the left-hand side is over all 4-tuples (a, x, b, y) of positive integers satisfying the equation $ax + by = n$ and the summations on the right-hand side are over the positive divisors d of n .

We call this result “Liouville’s 2-term formula” since n is expressed as the sum

of two terms, namely “ ax ” and “ by ”. In this thesis we state and prove an analogous 3-term formula. Liouville never stated such a formula and no such formula seems to have been given in the mathematical literature since Liouville’s time. From our 3-term formula of Liouville type we are able to deduce the known formula for the number $r_6(n)$ of representations of a positive integer n as the sum of six squares. Since our 3-term formula is proved by entirely elementary means, our proof of the formula

$$r_6(n) = 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \quad (0.0.1)$$

is completely elementary, that is, only manipulation of finite sums is used, no limiting processes, no infinite sums or infinite products are needed. (Here $\left(\frac{-4}{*} \right)$ is the Legendre-Jacobi-Kronecker symbol.) The only elementary proof of (0.0.1) that we are aware of is based upon another identity of Liouville and a recurrence relation. An exposition of this proof is given in [91, pp. 436-439]. Our proof is perhaps the second arithmetic proof known of (0.0.1).

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Notation

$\mathbb{N}$ = set of positive integers.

$\mathbb{Z}$ = set of integers.

$\mathbb{R}$ = set of real numbers.

$\mathbb{C}$ = set of complex numbers.

$$\sigma_k(n) = \begin{cases} \sum_{d|n} d^k, & n \in \mathbb{N}, \\ 0, & n \notin \mathbb{N}. \end{cases}$$

$$d(n) = \sigma_0(n) = \sum_{d|n} 1.$$

$$\sigma(n) = \sigma_1(n).$$

$$\sigma^*(n) = \sum_{\substack{d|n \\ n/d \text{ odd}}} d.$$

$o(n)$ = odd part of $n \in \mathbb{N}$, that is, if $2^\alpha \parallel n$ then $o(n) = n/2^\alpha$.

$$\delta(n, k) = \begin{cases} 1, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n. \end{cases}$$

$r_k(n)$ = number of representations of n as the sum of k squares

$$= \text{card}\{(x_1, \dots, x_k) \in \mathbb{Z}^k \mid x_1^2 + \dots + x_k^2 = n\}.$$

$$T_k = \frac{k(k+1)}{2} = k^{\text{th}} \text{ triangular number } (k = 0, 1, 2, \dots).$$

$$\Delta = \{T_k \mid k \in \mathbb{N} \cup \{0\}\}.$$

$\delta_k(n)$ = number of representations of n as the sum of k triangular numbers

$$= \text{card}\{(y_1, \dots, y_k) \in \Delta^k \mid y_1 + \dots + y_k = n\}.$$

$$\left(\frac{d}{n}\right) = \text{Legendre-Jacobi-Kronecker symbol (see page 37)}.$$

$|A|$ = number of elements in the finite set A .

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Introduction

In his series of eighteen papers published between 1858 and 1865, Joseph Liouville (1809-1882) stated without proof several elementary arithmetic formulae.

One of the simplest of these is the following 2-term formula given by Liouville in his fifth article.

Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Then

$$\begin{aligned} \sum_{ax+by=n} (F(a-b) - F(a+b)) \\ = F(0) (\sigma(n) - d(n)) + \sum_{d|n} \left(1 - d + \frac{2n}{d}\right) F(d) - 2 \sum_{d|n} \sum_{1 \leq k \leq d} F(k), \end{aligned}$$

where the sum on the left-hand side is taken over all $(a, b, x, y) \in \mathbb{N}^4$ such that $ax + by = n$ and the sums on the right-hand side are taken over all positive divisors d of n .

Although a number of authors have given proofs of this formula, perhaps the simplest proof was given recently by Huard, Ou, Spearman and Williams in an unpublished manuscript. Since many of the ideas used in this thesis are based upon this proof, we describe briefly the ideas involved. The proof begins by collecting together

terms on the left-hand side having a common value of $F(k)$. When $k = 0$ there is just a single term, namely,

$$F(0) \sum_{\substack{ax+by=n \\ a=b}} 1,$$

and when $k \geq 1$ we have

$$F(k) \left(\sum_{\substack{ax+by=n \\ a-b=k}} 1 + \sum_{\substack{ax+by=n \\ a-b=-k}} 1 - \sum_{\substack{ax+by=n \\ a+b=k}} 1 \right) = F(k) \left(2 \sum_{\substack{ax+by=n \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a+b=k}} 1 \right),$$

as F is an even function.

Therefore we can rewrite the left-hand side of the identity as

$$\sum_{ax+by=n} (F(a-b) - F(a+b)) = F(0) \sum_{\substack{ax+by=n \\ a=b}} 1 + \sum_{k \geq 1} F(k) \left(2 \sum_{\substack{ax+by=n \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a+b=k}} 1 \right).$$

The first sum can be calculated very simply, but more consideration is required to show that

$$2 \sum_{\substack{ax+by=n \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a+b=k}} 1 = \left(1 - k + \frac{2n}{k} \right) \delta(n, k) - 2 \sum_{\substack{d|n \\ d \geq k}} 1,$$

as Liouville asserted. This is accomplished by determining relationships between the quantities

$$L_1 = \sum_{\substack{kx+ey=n \\ e < k}} 1, \quad L_2 = \sum_{\substack{kx+ey=n \\ e > k}} 1, \quad L_3 = \sum_{\substack{kx+ey=n \\ e=k}} 1,$$

$$M_1 = \sum_{\substack{kx+ey=n \\ x < y}} 1, \quad M_2 = \sum_{\substack{kx+ey=n \\ x > y}} 1, \quad M_3 = \sum_{\substack{kx+ey=n \\ x=y}} 1,$$

where k and n are fixed positive integers and the sums are taken over triples $(x, e, y) \in \mathbb{N}^3$ satisfying the stated conditions.

Although it is not immediately clear how these quantities relate to the above sum, the following calculation will provide some insight. We have

$$\sum_{\substack{ax+by=n \\ a-b=k}} 1 = \sum_{(b+k)x+by=n} 1 = \sum_{kx+b(x+y)=n} 1 = \sum_{\substack{kx+by'=n \\ y'>x}} 1 = M_1.$$

Through identities like this, and various relations between the quantities L_1, L_2, L_3, M_1, M_2 and M_3 , we achieve the desired result. Full details are given in Theorem 1.3.1.

This type of argument can be easily adapted to prove the following 2-term odd formula by Liouville, as well as other 2-term formulae stated by Liouville.

Let $n \in \mathbb{N}$ be even so that $n = 2^r m$ for $r, m \in \mathbb{N}$ with m odd. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Then

$$\sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} (F(a-b) - F(a+b)) = 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d (F(0) - F(2^r d)).$$

The proof of this formula is achieved by refining the definitions of L_1, L_2, L_3, M_1, M_2 and M_3 to specify the variables x, y and e as even or odd.

Liouville also stated without proof in his sixth article [64] the following formula, which we call Liouville's 3-term odd formula.

Let n be an odd positive integer. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function. Then

$$\begin{aligned}
& \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\
&= \frac{1}{8} \sum_{d|n} (d^2 - 1)F(d) - 3 \sum_{\substack{ax < n \\ a,x \text{ odd}}} \sigma(o(n-ax))F(a).
\end{aligned}$$

Although a more general 3-term formula, valid for all n , and not restricting a , b , c , x , y , z to be odd, was not stated by Liouville, nor by anyone else, we refer in honour of Liouville to the following formula as Liouville's 3-term formula.

Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function. Then

$$\begin{aligned}
& \sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\
&= \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) F(d) \\
&\quad + 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) F(k) - 6 \sum_{\substack{n=n_1+n_2 \\ n_1, n_2 \in \mathbb{N}}} \sigma(n_1) \sum_{d|n_2} F(d).
\end{aligned}$$

The main purpose of this thesis is to prove Liouville's 3-term odd formula and Liouville's 3-term formula by extending the above argument of Huard, Ou, Spearman and Williams to three variables. We begin in the same way, by collecting together the terms on the left-hand side having a common value of $F(k)$. We can ignore the case when $k = 0$ since for an odd function F , $F(0) = 0$. After some fairly simple manipulations, we obtain (as F is odd)

$$\begin{aligned} & \sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ &= \sum_{k \geq 1} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a+b+c=k}} 1 - 3 \sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1 + 3 \sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1 \right) \end{aligned}$$

where a, b, c, x, y, z are restricted to be odd in the case of Liouville's 3-term odd formula. In order to prove these 3-term formulae, we define for fixed positive integers n and k , quantities of the form

$$\sum_{kx+by+cz=n} 1,$$

where various inequalities involving the positive integral variables b, c, x, y, z are imposed. The 3-term odd formula is achieved by further specifying the variables b, c, x, y, z as even or odd in these quantities. By determining the relationships between these sums, this is much more complicated than in the 2-term case, we show that

$$\begin{aligned} & \sum_{\substack{ax+by+cz=n \\ a+b+c=k}} 1 - 3 \sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1 + 3 \sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1 \\ &= \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k \right) \right) + 3 \sum_{\substack{d|n \\ d > k}} \left(d - 2 \frac{n}{d} - k \right) - 6 \sum_{kx+cy=n} e \end{aligned}$$

and

$$\sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a, b, c, x, y, z \text{ odd}}} 1 - 3 \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ a, b, c, x, y, z \text{ odd}}} 1 + 3 \sum_{\substack{ax+by+cz=n \\ a-b-c=k \\ a, b, c, x, y, z \text{ odd}}} 1 = \frac{(k^2 - 1)}{8} \delta(n, k) - 3 \sum_{\substack{kx < n \\ x \text{ odd}}} \sigma(o(n - kx)),$$

thereby establishing Liouville's 3-term and 3-term odd formulae.

These 2-term and 3-term formulae by Liouville can be used to prove a variety of arithmetic results. As an application of Liouville's 3-term formula, we use it to give

an elementary arithmetic derivation of the formula for the number $r_6(n)$ of representations of n as the sum of six squares. As the number $r_2(m)$ of representations of $m \in \mathbb{N}$ as the sum of two squares is

$$r_2(m) = 4 \sum_{d|m} \left(\frac{-4}{d} \right),$$

where $\left(\frac{-4}{d} \right)$ is the Legendre-Jacobi-Kronecker symbol for discriminant -4 (see page 37), we have

$$\begin{aligned} r_6(n) &= \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \geq 0}} r_2(n_1)r_2(n_2)r_2(n_3) \\ &= \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \geq 1}} r_2(n_1)r_2(n_2)r_2(n_3) + 3r_2(n) + 3(r_4(n) - 2r_2(n)) \\ &= 64 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \geq 1}} \sum_{a|n_1} \left(\frac{-4}{a} \right) \sum_{b|n_2} \left(\frac{-4}{b} \right) \sum_{c|n_3} \left(\frac{-4}{c} \right) + 3r_4(n) - 3r_2(n) \\ &= 64 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \geq 1}} \sum_{a|n_1} \sum_{b|n_2} \sum_{c|n_3} \left(\frac{-4}{abc} \right) + 3r_4(n) - 3r_2(n) \\ &= 64 \sum_{\substack{ax+by+cz=n \\ a, b, c, x, y, z \in \mathbb{N}}} \left(\frac{-4}{abc} \right) + 3r_4(n) - 3r_2(n). \end{aligned}$$

We show that for the odd function $F(n) = -\frac{1}{4} \left(\frac{-4}{n} \right)$, we have

$$\left(\frac{-4}{abc} \right) = F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c).$$

Therefore by applying the above 3-term formulae, we are able to prove the formula for $r_6(n)$ when n is odd. A standard argument is used to extend this formula to the case when n is even, and thus obtain the formula for $r_6(n)$ for all n .

Chapter 1

Liouville's 2-term formulae

1.1 Liouville's eighteen articles

As mentioned in the Introduction, Liouville stated without proof numerous elementary arithmetic formulae in his articles [59]-[76]. Liouville discusses the ideas that led him to his results in [77]. These formulae can be used to prove a variety of arithmetic results. A summary of Liouville's formulae is given in *Dickson's History of the Theory of Numbers*, [19, Vol. II, Chap. XI, pp. 329-339]. Liouville's methods are discussed in [4, Chap. 8, pp. 365-433], [91], [110, p. 346], [111, Vol. I, p. 348] and [117].

Typical of one class of Liouville's formulae is the following result, which was given in his fifth article [63, p. 275].

Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Then

$$\begin{aligned}
\sum_{ax+by=n} (F(a-b) - F(a+b)) \\
= F(0) (\sigma(n) - d(n)) + \sum_{d|n} \left(1 - d + \frac{2n}{d}\right) F(d) - 2 \sum_{d|n} \sum_{1 \leq k \leq d} F(d),
\end{aligned}$$

where the sum on the left-hand side is taken over all $(a, b, x, y) \in \mathbb{N}^4$ such that $ax + by = n$ and the sums on the right-hand side are taken over all positive divisors d of n .

We call this result “Liouville’s 2-term formula” since there are two terms “ ax ” and “ by ” in the equation $ax + by = n$. Liouville did not publish a proof of this formula. The first proof was given by Piuma [97] in 1866. Other proofs have been given by Meissner [85] and Pepin [96, p. 93].

We present a proof of Liouville’s 2-term formula (Theorem 1.3.1) in Section 1.3. This proof was first given by Huard, Ou, Spearman and Williams in an unpublished manuscript. It is the simplest of the proofs of Liouville’s 2-term formula and is the prototype of all the proofs of the Liouville formulae in this thesis. An earlier less simple version of this proof is given in [112].

A similar formula, stated by Liouville in [59, p. 144], [60, p. 194], is the following result.

Let $n \in \mathbb{N}$ be even so that $n = 2^r m$ for $r, m \in \mathbb{N}$ with m odd. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Then

$$\sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} (F(a-b) - F(a+b)) = 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d (F(0) - F(2^r d)).$$

We call this result Liouville's 2-term odd formula since each of a , b , x and y must be an odd positive integer in the equation $ax + by = n$. As with Liouville's 2-term formula, Liouville did not give a proof of this result. H. J. S. Smith gave an elementary proof of this result in Part VI of his *Report on the Theory of Numbers to the British Association* in 1865 [110, pp. 346-348], [111, Vol. I, p. 348]. Baskakov [6, p. 344], Pepin [96, p. 94], Mathews [84] and Deltour [18, p. 123] gave elementary proofs of it in 1882, 1888, 1893 and 1911 respectively. Bougaief [9, p. 9] and Humbert [47] gave analytic proofs of it in 1885 and 1910 respectively. We give a proof in Section 1.5, see Theorem 1.5.1. Liouville also stated other formulae similar to these two. For example in his fourth article Liouville [62, p. 242] gave the following formula.

Let $n \in \mathbb{N}$. Set $n = 2^r m$, where r and m are integers such that $r \geq 0$ and $m \geq 1$ is odd. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Then

$$\sum_{\substack{ax+by=n \\ x,y \text{ odd}}} (F(a-b) - F(a+b)) = \sum_{d|m} \left(\frac{m}{d} - 2^r d \right) (F(2^r d) - F(0)),$$

where the summation on the left-hand side is over $(a, x, b, y) \in \mathbb{N}^4$ satisfying $ax + by = n$ and the summation on the right-hand side is over positive divisors d of n .

Pepin [95, p. 159] gave a proof of this formula in 1884. We give another proof in Section 1.6, see Theorem 1.6.1.

In his sixth article Liouville [64, p. 331] gave a formula for a sum over odd positive integers a, b, c, x, y, z satisfying $ax + by + cz = n$.

Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function. Then

$$\begin{aligned} \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ = \frac{1}{8} \sum_{d|n} (d^2 - 1)F(d) - 3 \sum_{\substack{ax < n \\ a,x \text{ odd}}} \sigma(o(n-ax))F(a), \end{aligned}$$

where the summation on the left-hand side is over $(a, b, c, x, y, z) \in \mathbb{N}^6$ such that $ax + by + cz = n$ with a, b, c, x, y, z odd.

We call this result Liouville's 3-term odd formula in analogy with Theorem 1.5.1. A search of the mathematical literature failed to find a proof of this result, although Nazimoff [92, p. 110] indicates how a proof can be given using elliptic functions but does not give the details. In Chapter 2 we give what we believe is the first proof of this theorem. The proof is completely elementary in nature and is based upon the ideas used in the proofs of Liouville's 2-term formulae (Theorems 1.3.1 and 1.5.1).

In Chapter 3 we prove an analog of Liouville's 2-term formula for the 3-term case. Although no such formula was stated by Liouville in his eighteen articles or appears in the mathematical literature, in honour of Liouville, we call our formula "Liouville's 3-term formula".

Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function. Then

$$\begin{aligned} \sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ = \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) F(d) \end{aligned}$$

$$+ 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) F(k) - 6 \sum_{\substack{n=n_1+n_2 \\ n_1, n_2 \in \mathbb{N}}} \sigma(n_1) \sum_{d|n_2} F(d),$$

where the summation on the left-hand side is over $(a, b, c, x, y, z) \in \mathbb{N}^6$ such that $ax + by + cz = n$.

In Chapter 4 we use Liouville's 3-term formula in conjunction with Theorem 1.6.1 to give an elementary proof of the formula for $r_6(n)$, the number of representations of a positive integer n as the sum of six squares. As far as we know this is only the second truly elementary proof of this result. The first makes use of a different type of elementary formula given by Liouville and involves a recursive relation, see [91, pp. 436-440].

1.2 The equation $kx + ey = n$ and the quantities L_1 ,

$$L_2, L_3, M_1, M_2, M_3$$

Let $k, n \in \mathbb{N}$. In this section we are interested in the number of solutions $(x, e, y) \in \mathbb{N}^3$ of the equation

$$kx + ey = n, \tag{1.2.1}$$

where k, n are fixed positive integers and the variables x, e, y are required to satisfy certain inequalities. Thus, for example, the sum

$$\sum_{\substack{kx+ey=n \\ e < k}} 1$$

counts the number of triples $(x, e, y) \in \mathbb{N}^3$ satisfying the equation $kx + ey = n$ with $e < k$ for fixed positive integers k and n .

Definition 1.2.1.

$$L_1 = \sum_{\substack{kx+ey=n \\ e < k}} 1, \quad L_2 = \sum_{\substack{kx+ey=n \\ e > k}} 1, \quad L_3 = \sum_{\substack{kx+ey=n \\ e=k}} 1.$$

Definition 1.2.2.

$$M_1 = \sum_{\substack{kx+ey=n \\ x < y}} 1, \quad M_2 = \sum_{\substack{kx+ey=n \\ x > y}} 1, \quad M_3 = \sum_{\substack{kx+ey=n \\ x=y}} 1.$$

The following result follows immediately from Definitions 1.2.1 and 1.2.2.

Lemma 1.2.1.

$$L_1 + L_2 + L_3 = M_1 + M_2 + M_3.$$

Next we evaluate L_3 .

Lemma 1.2.2.

$$L_3 = \delta(n, k) \left(\frac{n}{k} - 1 \right).$$

Proof. We have

$$L_3 = \sum_{\substack{kx+ey=n \\ e=k}} 1 = \sum_{x+y=n/k} 1 = \delta(n, k) \left(\frac{n}{k} - 1 \right).$$

□

M_3 can be determined in a similar manner.

Lemma 1.2.3.

$$M_3 = \sum_{\substack{d|n \\ d>k}} 1.$$

Proof. We have

$$M_3 = \sum_{\substack{kx+ey=n \\ x=y}} 1 = \sum_{(k+e)x=n} 1 = \sum_{\substack{d|n \\ d>k}} 1.$$

□

Our next lemma shows that L_2 and M_2 are equal.

Lemma 1.2.4.

$$L_2 = M_2.$$

Proof. We have

$$M_2 = \sum_{\substack{kx+ey=n \\ x>y}} 1 = \sum_{k(x+y)+ey=n} 1 = \sum_{kx+(k+e)y=n} 1 = \sum_{\substack{kx+ey=n \\ e>k}} 1 = L_2.$$

□

Lemma 1.2.5.

$$L_1 - M_1 = \sum_{\substack{d|n \\ d>k}} 1 - \delta(n, k) \left(\frac{n}{k} - 1 \right).$$

Proof. By Lemmas 1.2.1 and 1.2.4 we have

$$L_1 - M_1 = M_3 - L_3.$$

The asserted result now follows from Lemmas 1.2.2 and 1.2.3.

□

1.3 Proof of Liouville's 2-term formula

In this section we prove Liouville's 2-term formula. We begin with the following two lemmas.

Lemma 1.3.1. *Let $k, n \in \mathbb{N}$. Then*

$$\sum_{\substack{ax+by=n \\ a-b=k}} 1 = M_1.$$

Proof. We have

$$\sum_{\substack{ax+by=n \\ a-b=k}} 1 = \sum_{(b+k)x+by=n} 1 = \sum_{kx+b(x+y)=n} 1 = \sum_{\substack{kx+by'=n \\ y'>x}} 1 = M_1,$$

as required. □

Lemma 1.3.2. *Let $k, n \in \mathbb{N}$. Then*

$$\sum_{\substack{ax+by=n \\ a+b=k}} 1 = 2L_1 + (k-1)\delta(n, k).$$

Proof. We have

$$\begin{aligned} \sum_{\substack{ax+by=n \\ a+b=k}} 1 &= \sum_{\substack{ax+by=n \\ a+b=k \\ x < y}} 1 + \sum_{\substack{ax+by=n \\ a+b=k \\ x > y}} 1 + \sum_{\substack{ax+by=n \\ a+b=k \\ x=y}} 1 \\ &= 2 \sum_{\substack{ax+by=n \\ a+b=k \\ x < y}} 1 + \sum_{\substack{(a+b)x=n \\ a+b=k}} 1 \\ &= 2 \sum_{\substack{ax+b(x+y')=n \\ a+b=k}} 1 + (k-1)\delta(n, k) \\ &= 2 \sum_{\substack{(a+b)x+by=n \\ a+b=k}} 1 + (k-1)\delta(n, k) \end{aligned}$$

$$\begin{aligned}
 &= 2 \sum_{\substack{kx+by=n \\ b < k}} 1 + (k-1)\delta(n, k) \\
 &= 2L_1 + (k-1)\delta(n, k),
 \end{aligned}$$

as claimed. $\square$

We now prove Liouville's 2-term formula.

Theorem 1.3.1. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Then*

$$\begin{aligned}
 &\sum_{ax+by=n} (F(a-b) - F(a+b)) \\
 &= F(0) (\sigma(n) - d(n)) + \sum_{d|n} \left(1 - d + \frac{2n}{d} \right) F(d) - 2 \sum_{d|n} \sum_{1 \leq k \leq d} F(k),
 \end{aligned}$$

where the summation on the left-hand side is over all $(a, b, x, y) \in \mathbb{N}^4$ such that $ax + by = n$.

Proof. Using the fact that F is even, we observe that

$$\begin{aligned}
 &\sum_{ax+by=n} (F(a-b) - F(a+b)) \\
 &= F(0) \sum_{\substack{ax+by=n \\ a=b}} 1 + \sum_{k \geq 1} F(k) \left(\sum_{\substack{ax+by=n \\ a-b=k}} 1 + \sum_{\substack{ax+by=n \\ a-b=-k}} 1 - \sum_{\substack{ax+by=n \\ a+b=k}} 1 \right) \\
 &= F(0) (\sigma(n) - d(n)) + \sum_{k \geq 1} F(k) \left(2 \sum_{\substack{ax+by=n \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a+b=k}} 1 \right),
 \end{aligned}$$

as

$$\sum_{\substack{ax+by=n \\ a=b}} 1 = \sum_{a(x+y)=n} 1 = \sum_{d|n} \sum_{x+y=n/d} 1 = \sum_{d|n} \left(\frac{n}{d} - 1 \right) = \sum_{d|n} (d - 1) = \sigma(n) - d(n),$$

and by interchanging x with y and a with b we have

$$\sum_{\substack{ax+by=n \\ a-b=-k}} 1 = \sum_{\substack{ax+by=n \\ a-b=k}} 1.$$

Appealing to Lemmas 1.3.1, 1.3.2 and 1.2.5, we obtain

$$\begin{aligned} 2 \sum_{\substack{ax+by=n \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a+b=k}} 1 &= 2M_1 - 2L_1 - (k-1)\delta(n, k) \\ &= 2 \left(\delta(n, k) \left(\frac{n}{k} - 1 \right) - \sum_{\substack{d|n \\ d>k}} 1 \right) - (k-1)\delta(n, k) \\ &= \left(1 - k + \frac{2n}{k} \right) \delta(n, k) - 2 \sum_{\substack{d|n \\ d \geq k}} 1, \end{aligned}$$

so that

$$\begin{aligned} \sum_{k \geq 1} F(k) \left(2 \sum_{\substack{ax+by=n \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a+b=k}} 1 \right) &= \sum_{k \geq 1} F(k) \left(\left(1 - k + \frac{2n}{k} \right) \delta(n, k) - 2 \sum_{\substack{d|n \\ d \geq k}} 1 \right) \\ &= \sum_{d|n} \left(1 - d + \frac{2n}{d} \right) F(d) - 2 \sum_{d|n} \sum_{1 \leq k \leq d} F(k), \end{aligned}$$

as required. □

1.4 The equation $kx + ey = n$ and the quantities

$$L_1(a, b, c), L_2(a, b, c), L_3(a, b, c), M_1(a, b, c), M_2(a, b, c),$$

$$M_3(a, b, c)$$

Let $k, n \in \mathbb{N}$. In this section, as in Section 1.2, we are interested in the number of solutions $(x, e, y) \in \mathbb{N}^3$ of the equation

$$kx + ey = n, \quad (1.4.2)$$

where k, n are fixed positive integers and the unknowns x, e, y are required to satisfy certain congruences modulo 2, in addition to the inequalities of Definitions 1.2.1 and 1.2.2. All congruences are taken modulo 2. Thus, for example, the sum

$$\sum_{\substack{kx+ey=n \\ e < k \\ x \equiv a, e \equiv b, y \equiv c}} 1$$

counts the number of triples $(x, e, y) \in \mathbb{N}^3$ satisfying the equation $kx + ey = n$ with $e < k$ and $x \equiv a \pmod{2}, e \equiv b \pmod{2}, y \equiv c \pmod{2}$, where a, b, c are fixed integers and k, n are fixed positive integers.

Definition 1.4.1.

$$L_1(a, b, c) = \sum_{\substack{kx+ey=n \\ e < k \\ x \equiv a, e \equiv b, y \equiv c}} 1, \quad L_2(a, b, c) = \sum_{\substack{kx+ey=n \\ e > k \\ x \equiv a, e \equiv b, y \equiv c}} 1, \quad L_3(a, b, c) = \sum_{\substack{kx+ey=n \\ e = k \\ x \equiv a, e \equiv b, y \equiv c}} 1.$$

Definition 1.4.2.

$$M_1(a, b, c) = \sum_{\substack{kx+ey=n \\ x < y \\ x \equiv a, e \equiv b, y \equiv c}} 1, \quad M_2(a, b, c) = \sum_{\substack{kx+ey=n \\ x > y \\ x \equiv a, e \equiv b, y \equiv c}} 1, \quad M_3(a, b, c) = \sum_{\substack{kx+ey=n \\ x = y \\ x \equiv a, e \equiv b, y \equiv c}} 1.$$

The following result follows immediately from Definitions 1.4.1 and 1.4.2.

Lemma 1.4.1.

$$L_1(a, b, c) + L_2(a, b, c) + L_3(a, b, c) = M_1(a, b, c) + M_2(a, b, c) + M_3(a, b, c).$$

Next we evaluate $L_3(a, b, c)$. (Recall that all congruences are modulo 2.)

Lemma 1.4.2.

$$L_3(a, b, c) = \begin{cases} 0, & \begin{array}{l} \text{if } k \nmid n \\ \text{or } k \mid n, k \not\equiv b \\ \text{or } k \mid n, k \equiv b, \frac{n}{k} \not\equiv a + c, \end{array} \\ \frac{1}{2} \left(\frac{n}{k} - 1 \right) - \frac{1}{4} (-1)^a (1 + (-1)^{n/k}), & \text{if } k \mid n, k \equiv b, \frac{n}{k} \equiv a + c. \end{cases}$$

Proof. The evaluation of $L_3(a, b, c)$ follows easily from

$$L_3(a, b, c) = \sum_{\substack{kx+ey=n \\ e=k \\ x \equiv a, e \equiv b, y \equiv c}} 1 = \sum_{\substack{x+y=n/k \\ x \equiv a, k \equiv b, y \equiv c}} 1.$$

□

$M_3(a, b, c)$ can be calculated by a similar method.

Lemma 1.4.3.

$$M_3(a, b, c) = \begin{cases} 0, & \begin{array}{l} \text{if } a \not\equiv c \\ \text{or } a \equiv c, n \not\equiv a(k+b), \end{array} \\ \sum_{\substack{d \mid n, d > k \\ d \equiv k+b \\ n/d \equiv a}} 1, & \text{if } a \equiv c, n \equiv a(k+b). \end{cases}$$

Proof. If $a \not\equiv c$ then clearly $M_3(a, b, c) = 0$. If $a \equiv c$ the evaluation of $M_3(a, b, c)$ follows easily from

$$M_3(a, b, c) = \sum_{\substack{kx+ey=n \\ x=y \\ x \equiv a, e \equiv b}} 1 = \sum_{\substack{(k+e)x=n \\ x \equiv a, e \equiv b}} 1 = \sum_{\substack{(k+e)|n \\ e \equiv b \\ \frac{n}{k+e} \equiv a}} 1 = \sum_{\substack{d|n, d > k \\ d \equiv k+b \\ \frac{n}{d} \equiv a}} 1.$$

□

Our next lemma gives an important relationship between $L_2(a, b, c)$ and $M_2(a, b, c)$.

Lemma 1.4.4.

$$M_2(a, b, c) = L_2(a + c, k + b, c).$$

Proof. We have

$$\begin{aligned} M_2(a, b, c) &= \sum_{\substack{kx+ey=n \\ x > y \\ x \equiv a, e \equiv b, y \equiv c}} 1 \\ &= \sum_{\substack{k(x+y)+ey=n \\ x \equiv a+c, e \equiv b, y \equiv c}} 1 \\ &= \sum_{\substack{kx+(k+e)y=n \\ x \equiv a+c, e \equiv b, y \equiv c}} 1 \\ &= \sum_{\substack{kx+ey=n \\ e > k \\ x \equiv a+c, e \equiv k+b, y \equiv c}} 1 \\ &= L_2(a + c, k + b, c), \end{aligned}$$

as asserted. □

Lemma 1.4.5. *Let k and n be even positive integers. Then*

$$M_1(1, 1, 0) = L_1(1, 1, 0).$$

Proof. Applying Lemmas 1.4.2 and 1.4.3, we find that $L_3(1, 1, 0) = M_3(1, 1, 0) = 0$, and by Lemma 1.4.4, we obtain

$$L_2(1, 1, 0) = M_2(1, 1, 0).$$

Therefore by Lemma 1.4.1, we have

$$\begin{aligned} M_1(1, 1, 0) - L_1(1, 1, 0) &= L_2(1, 1, 0) + L_3(1, 1, 0) - M_2(1, 1, 0) - M_3(1, 1, 0) \\ &= (L_2(1, 1, 0) - M_2(1, 1, 0)) + L_3(1, 1, 0) - M_3(1, 1, 0) \\ &= 0 + 0 - 0 \\ &= 0, \end{aligned}$$

as required. □

Our last result of this section is a simple consequence of Lemmas 1.4.1, 1.4.2, and 1.4.3. It will prove to be very useful in the proof of Liouville's 3-term odd formula in Chapter 2.

Lemma 1.4.6. *If k and n are positive odd integers then*

$$L_1(1, 1, 0) - M_1(1, 1, 0) = L_2(1, 0, 0) - L_2(1, 1, 0) - \left(\frac{(n/k) - 1}{2} \right) \delta(n, k).$$

Proof. Taking $a = 1$, $b = 1$ and $c = 0$ in Lemma 1.4.1, we obtain

$$L_1(1, 1, 0) + L_2(1, 1, 0) + L_3(1, 1, 0) = M_1(1, 1, 0) + M_2(1, 1, 0) + M_3(1, 1, 0).$$

As $a \not\equiv c$ we have by Lemma 1.4.3

$$M_3(1, 1, 0) = 0.$$

If $k \nmid n$ we have by Lemma 1.4.2

$$L_3(1, 1, 0) = 0.$$

If $k \mid n$, as $k \equiv n \equiv 1$ we have $k \equiv b$ and $n/k \equiv a + c$, so that by Lemma 1.4.2

$$L_3(1, 1, 0) = \frac{(n/k) - 1}{2}.$$

Thus

$$L_3(1, 1, 0) = \left(\frac{(n/k) - 1}{2} \right) \delta(n, k).$$

By Lemma 1.4.4 we have

$$M_2(1, 1, 0) = L_2(1, 0, 0).$$

Putting this information together, we find that

$$L_1(1, 1, 0) + L_2(1, 1, 0) + \left(\frac{(n/k) - 1}{2} \right) \delta(n, k) = M_1(1, 1, 0) + L_2(1, 0, 0),$$

or equivalently

$$L_1(1, 1, 0) - M_1(1, 1, 0) = L_2(1, 0, 0) - L_2(1, 1, 0) - \left(\frac{(n/k) - 1}{2} \right) \delta(n, k).$$

□

1.5 Proof of Liouville's 2-term odd formula

In this section we prove Liouville's 2-term odd formula (Theorem 1.5.1). We require two simple lemmas.

Lemma 1.5.1. *Let n be an even positive integer. Set $n = 2^r m$, where r and m are positive integers with m odd. Then*

$$\sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a=b}} 1 = 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d.$$

Proof. We have

$$\begin{aligned} \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a=b}} 1 &= \sum_{\substack{a(x+y)=n \\ a,x,y \text{ odd}}} 1 \\ &= \sum_{\substack{a|n \\ a \text{ odd}}} \sum_{\substack{x+y=n/a \\ x,y \text{ odd}}} 1 \\ &= \sum_{\substack{a|n \\ a \text{ odd}}} \frac{n}{2a} \quad (\text{since } n/a \text{ is even}) \\ &= \sum_{a|m} 2^{r-1} \frac{m}{a} \quad (\text{since } n = 2^r m \text{ with } m \text{ odd}) \\ &= 2^{r-1} \sum_{a|m} a \\ &= 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d, \end{aligned}$$

as required. □

Lemma 1.5.2. *Let k and n be even positive integers. Then*

$$2 \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a+b=k}} 1 = \begin{cases} \frac{-k}{2}, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} & 2 \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a+b=k}} 1 \\ &= 2 \sum_{\substack{(b+k)x+by=n \\ b,x,y \text{ odd}}} 1 - \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a+b=k \\ x < y}} 1 - \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a+b=k \\ x > y}} 1 - \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a+b=k \\ x=y}} 1 \\ &= 2 \sum_{\substack{kx+b(x+y)=n \\ b,x,y \text{ odd}}} 1 - 2 \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a+b=k \\ x < y}} 1 - \sum_{\substack{(a+b)x=n \\ a,b,x \text{ odd} \\ a+b=k}} 1 \\ &= 2 \sum_{\substack{kx+by'=n \\ b,x \text{ odd}, y' \text{ even} \\ x < y'}} 1 - 2 \sum_{\substack{ax+b(x+y')=n \\ a,b,x \text{ odd}, y' \text{ even} \\ a+b=k}} 1 - \sum_{\substack{kx=n \\ a,b,x \text{ odd} \\ a+b=k}} 1 \\ &= 2M_1(1, 1, 0) - 2 \sum_{\substack{kx+by=n \\ b,x \text{ odd}, y \text{ even} \\ b < k}} 1 - \begin{cases} \frac{k}{2}, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases} \\ &= 2M_1(1, 1, 0) - 2L_1(1, 1, 0) - \begin{cases} \frac{k}{2}, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{-k}{2}, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

by Lemma 1.4.5. □

We are now ready to prove Liouville's 2-term odd formula.

Theorem 1.5.1. *Let n be an even positive integer. Set $n = 2^r m$, where $r, m \in \mathbb{N}$ with m odd. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Then*

$$\sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} (F(a-b) - F(a+b)) = 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d (F(0) - F(2^r d)),$$

where the summation on the left-hand side is over all $(a, b, x, y) \in \mathbb{N}^4$ such that $ax + by = n$ with a, b, x, y odd.

Proof. Using the fact that F is even, and appealing to Lemmas 1.5.1 and 1.5.2, we observe that

$$\begin{aligned} & \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} (F(a-b) - F(a+b)) \\ &= F(0) \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a=b}} 1 + \sum_{k \geq 1} F(k) \left(\sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a-b=k}} 1 + \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a-b=-k}} 1 - \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a+b=k}} 1 \right) \\ &= F(0) \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a=b}} 1 + \sum_{k \geq 1} F(k) \left(2 \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd} \\ a+b=k}} 1 \right) \\ &= F(0) 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d + \sum_{k \geq 1} F(k) \begin{cases} \frac{-k}{2}, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0, & \text{otherwise,} \end{cases} \\ &= 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d F(0) + \sum_{\substack{k|n \\ n/k \text{ odd}}} \frac{-k}{2} F(k) \\ &= 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d F(0) - \sum_{\substack{k=2^r d \\ d|n \\ d \text{ odd}}} \frac{k}{2} F(k) \end{aligned}$$

$$\begin{aligned}
 &= 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} dF(0) - \sum_{\substack{d|n \\ d \text{ odd}}} \frac{2^r d}{2} F(2^r d) \\
 &= 2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d(F(0) - F(2^r d)),
 \end{aligned}$$

as claimed. □

1.6 Another 2-term formula by Liouville

In this section we prove the third 2-term formula of Liouville given in Section 1.1.

Lemma 1.6.1. *Let $n, k \in \mathbb{N}$. Then*

$$\begin{aligned}
 &M_1(1, 1, 0) + M_1(1, 0, 0) - L_1(1, 1, 0) - L_1(1, 0, 0) \\
 &= \begin{cases} \frac{(n/k) - 1}{2}, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Proof. Applying Lemma 1.4.3, we find that $M_3(1, 1, 0) = M_3(1, 0, 0) = 0$, and by

Lemma 1.4.4 we obtain

$$L_2(1, 1, 0) = M_2(1, 1, 0), \quad L_2(1, 0, 0) = M_2(1, 0, 0), \quad \text{if } k \text{ is even,}$$

and

$$L_2(1, 1, 0) = M_2(1, 0, 0), \quad L_2(1, 0, 0) = M_2(1, 1, 0), \quad \text{if } k \text{ is odd.}$$

Therefore by Lemma 1.4.1 we have

$$M_1(1, 1, 0) + M_1(1, 0, 0) - L_1(1, 1, 0) - L_1(1, 0, 0)$$

$$\begin{aligned}
&= (M_1(1, 1, 0) - L_1(1, 1, 0)) + (M_1(1, 0, 0) - L_1(1, 0, 0)) \\
&= L_2(1, 1, 0) + L_3(1, 1, 0) - M_2(1, 1, 0) - M_3(1, 1, 0) \\
&\quad + L_2(1, 0, 0) + L_3(1, 0, 0) - M_2(1, 0, 0) - M_3(1, 0, 0) \\
&= L_3(1, 1, 0) + L_3(1, 0, 0).
\end{aligned}$$

But by Lemma 1.4.2 we have

$$L_3(1, 1, 0) = \begin{cases} \frac{(n/k) - 1}{2}, & \text{if } k \mid n, \text{ } k \text{ odd and } n/k \text{ odd,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$L_3(1, 0, 0) = \begin{cases} \frac{(n/k) - 1}{2}, & \text{if } k \mid n, \text{ } k \text{ even and } n/k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
&M_1(1, 1, 0) + M_1(1, 0, 0) - L_1(1, 1, 0) - L_1(1, 0, 0) \\
&= \begin{cases} \frac{(n/k) - 1}{2}, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

as required. □

Lemma 1.6.2. *Let $n \in \mathbb{N}$. Set $n = 2^r m$, where r is a nonnegative integer and m is a positive odd integer. Then*

$$\sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a=b}} 1 = \sum_{d \mid m} \left(2^r d - \frac{m}{d} \right).$$

Proof. We have

$$\begin{aligned}
 \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a=b}} 1 &= \sum_{\substack{a(x+y)=n \\ x,y \text{ odd}}} 1 \\
 &= \sum_{a|n} \sum_{\substack{x+y=n/a \\ x,y \text{ odd}}} 1 \\
 &= \sum_{2^t d | 2^r m} \sum_{\substack{x+y=2^{r-t}m/d \\ x,y \text{ odd}}} 1 \quad (a = 2^t d, \ t \geq 0, \ d \text{ odd}) \\
 &= \sum_{d|m} \sum_{0 \leq t < r} \sum_{\substack{x+y=2^{r-t}m/d \\ x,y \text{ odd}}} 1 \\
 &= \sum_{d|m} \sum_{0 \leq t < r} \frac{2^{r-t}m}{2d} \\
 &= \sum_{d|m} \frac{2^{r-1}m}{d} \sum_{0 \leq t < r} \left(\frac{1}{2}\right)^t \\
 &= \sum_{d|m} \frac{2^{r-1}m}{d} \left(\frac{1 - (1/2)^r}{1 - 1/2} \right) \\
 &= \sum_{d|m} \frac{2^r m}{d} (1 - (1/2)^r) \\
 &= \sum_{d|m} \left(\frac{2^r m}{d} - \frac{m}{d} \right) \\
 &= \sum_{d|m} 2^r d - \sum_{d|m} \frac{m}{d} \\
 &= \sum_{d|m} \left(2^r d - \frac{m}{d} \right),
 \end{aligned}$$

as claimed. □

Lemma 1.6.3. *Let $n, k \in \mathbb{N}$. Then*

$$2 \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a+b=k}} 1 = \begin{cases} \frac{n}{k} - k, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} & 2 \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a+b=k}} 1 \\ &= 2 \sum_{\substack{(k+b)x+by=n \\ x,y \text{ odd}}} 1 - \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a+b=k \\ x < y}} 1 - \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a+b=k \\ x > y}} 1 - \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a+b=k \\ x=y}} 1 \\ &= 2 \sum_{\substack{kx+b(x+y)=n \\ x,y \text{ odd}}} 1 - 2 \sum_{\substack{ax+b(x+y')=n \\ x \text{ odd}, y' \text{ even} \\ a+b=k}} 1 - \sum_{\substack{(a+b)x=n \\ x \text{ odd} \\ a+b=k}} 1 \\ &= 2 \sum_{\substack{kx+by'=n \\ x \text{ odd}, y' \text{ even} \\ x < y'}} 1 - 2 \sum_{\substack{kx+by'=n \\ x \text{ odd}, y' \text{ even} \\ b < k}} 1 - \sum_{\substack{kx=n \\ x \text{ odd} \\ a+b=k}} 1 \\ &= 2(M_1(1, 1, 0) + M_1(1, 0, 0)) - 2(L_1(1, 1, 0) + L_1(1, 0, 0)) - \delta(n, k) \sum_{\substack{n/k \text{ odd} \\ a+b=k}} 1 \\ &= \begin{cases} \left(\frac{n}{k} - 1\right) - (k - 1), & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0, & \text{otherwise,} \end{cases} \quad (\text{by Lemma 1.6.1}) \\ &= \begin{cases} \frac{n}{k} - k, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

as claimed. □

Theorem 1.6.1. *Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Set $n = 2^r m$,*

where r and m are integers such that $r \geq 0$ and $m \geq 1$ is odd. Then

$$\sum_{\substack{ax+by=n \\ x,y \text{ odd}}} (F(a-b) - F(a+b)) = \sum_{d|m} \left(\frac{m}{d} - 2^r d \right) (F(2^r d) - F(0)),$$

where the summation on the left-hand side is over all $(a, b, x, y) \in \mathbb{N}^4$ such that $ax + by = n$ with x, y odd.

Proof. Using the fact that F is even, we obtain using Lemma 1.6.3

$$\begin{aligned} & \sum_{\substack{ax+by=n \\ x,y \text{ odd}}} (F(a-b) - F(a+b)) \\ &= F(0) \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a=b}} 1 + \sum_{k \geq 1} F(k) \left(\sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a-b=k}} 1 + \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a-b=-k}} 1 - \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a+b=k}} 1 \right) \\ &= F(0) \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a=b}} 1 + \sum_{k \geq 1} F(k) \left(2 \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a-b=k}} 1 - \sum_{\substack{ax+by=n \\ x,y \text{ odd} \\ a+b=k}} 1 \right) \\ &= F(0) \sum_{d|m} \left(2^r d - \frac{m}{d} \right) + \sum_{k \geq 1} F(k) \begin{cases} \frac{n}{k} - k, & \text{if } k \mid n \text{ and } n/k \text{ odd,} \\ 0 & \text{otherwise.} \end{cases} \\ &= F(0) \sum_{d|m} \left(2^r d - \frac{m}{d} \right) + \sum_{\substack{k \mid n \\ n/k \text{ odd}}} \left(\frac{n}{k} - k \right) F(k) \\ &= F(0) \sum_{d|m} \left(2^r d - \frac{m}{d} \right) + \sum_{\substack{k=2^r d \\ d \mid n \\ d \text{ odd}}} \left(\frac{n}{k} - k \right) F(k) \\ &= F(0) \sum_{d|m} \left(2^r d - \frac{m}{d} \right) + \sum_{\substack{d \mid n \\ d \text{ odd}}} \left(\frac{n}{2^r d} - 2^r d \right) F(2^r d) \end{aligned}$$

$$= \sum_{d|m} \left(\frac{m}{d} - 2^r d \right) (F(2^r d) - F(0)),$$

as required. $\square$

Corollary 1.6.1. *Let n be an odd positive integer. Let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Then*

$$2 \sum_{\substack{ax+by=n \\ a,x,y \text{ odd}}} (F(a-b) - F(a+b)) = \sum_{d|n} \left(\frac{n}{d} - d \right) F(d).$$

Proof. Setting $r = 0$ in Theorem 1.6.1, we get

$$\sum_{\substack{ax+by=n \\ x,y \text{ odd}}} (F(a-b) - F(a+b)) = \sum_{d|n} \left(\frac{n}{d} - d \right) (F(d) - F(0)).$$

Since n is odd, at least one of a or b must be odd, therefore the left-hand side becomes

$$\begin{aligned} & \sum_{\substack{ax+by=n \\ a,x,y \text{ odd}}} (F(a-b) - F(a+b)) + \sum_{\substack{ax+by=n \\ b,x,y \text{ odd}}} (F(a-b) - F(a+b)) \\ &= 2 \sum_{\substack{ax+by=n \\ a,x,y \text{ odd}}} (F(a-b) - F(a+b)), \end{aligned}$$

since F is even. Simplifying the right-hand side, we obtain

$$\begin{aligned} \sum_{d|n} \left(\frac{n}{d} - d \right) (F(d) - F(0)) &= \sum_{d|n} \left(\frac{n}{d} - d \right) F(d) - F(0) \sum_{d|n} \left(\frac{n}{d} - d \right) \\ &= \sum_{d|n} \left(\frac{n}{d} - d \right) F(d), \end{aligned}$$

since $\sum_{d|n} d = \sum_{d|n} \frac{n}{d}$. Equating the left-hand side and right-hand side, we obtain the desired result. $\square$

1.7 Applications of Liouville's 2-term formulae

Our first application is the determination of the convolution sum $\sum_{m=1}^{n-1} \sigma(m)\sigma(n-m)$.

Theorem 1.7.1. *Let n be a positive integer. Then*

$$\sum_{m=1}^{n-1} \sigma(m)\sigma(n-m) = \frac{1}{12} (5\sigma_3(n) + (1-6n)\sigma(n)).$$

Proof. In order to prove this result, we apply Liouville's 2-term formula with $F(n) = n^2$. Then

$$F(a-b) - F(a+b) = (a-b)^2 - (a+b)^2 = -4ab.$$

The left-hand side of Liouville's 2-term formula gives

$$\begin{aligned} & \sum_{ax+by=n} (F(a-b) - F(a+b)) \\ &= -4 \sum_{ax+by=n} ab = -4 \sum_{n_1+n_2=n} \sigma(n_1)\sigma(n_2) = -4 \sum_{m=1}^{n-1} \sigma(m)\sigma(n-m). \end{aligned}$$

The right-hand side of Liouville's 2-term formula gives

$$\begin{aligned} & \sum_{d|n} \left(1 - d + \frac{2n}{d}\right) d^2 - 2 \sum_{d|n} \sum_{1 \leq k \leq d} k^2 \\ &= \sigma_2(n) - \sigma_3(n) + 2n\sigma(n) - 2 \sum_{d|n} \left(\frac{d(d+1)(2d+1)}{6}\right) \\ &= \sigma_2(n) - \sigma_3(n) + 2n\sigma(n) - \frac{1}{3} \sum_{d|n} (2d^3 + 3d^2 + d) \\ &= \sigma_2(n) - \sigma_3(n) + 2n\sigma(n) - \frac{2}{3}\sigma_3(n) - \sigma_2(n) - \frac{1}{3}\sigma(n) \\ &= -\frac{5}{3}\sigma_3(n) - \frac{1}{3}(1-2n)\sigma(n). \end{aligned}$$

Equating the left-hand side and right-hand side gives the desired result. $\square$

Theorem 1.7.1 originally appeared without proof in a letter from Besge to Liouville [8]. Dickson incorrectly attributes it to Lebesgue in [19, p. 338]. Rankin attributes it to Besge in [105, p. 115]. Lützen [83, p. 81] asserts that Besge/Besgue is a pseudonym for Liouville. The formula of Theorem 1.7.1 also appears in the work of Glaisher [34], [35], [36], Lahiri [55], Lehmer [57, p. 106], [58, p. 678], Ramanujan [99, Table IV], [100] and Skoruppa [108]. An elementary arithmetic proof of Theorem 1.7.1 has been given recently by Huard, Ou, Spearman and Williams [43, formula (3.10), p. 236].

Our second application is the evaluation of the convolution sum $\sum_{m=1}^{n-1} \sigma(m)\sigma_3(n-m)$.

Theorem 1.7.2.

$$\sum_{m=1}^{n-1} \sigma(m)\sigma_3(n-m) = \frac{1}{240}(21\sigma_5(n) + (10 - 30n)\sigma_3(n) - \sigma(n)).$$

Proof. In order to prove this result, we apply Liouville's 2-term formula with $F(n) = n^4$. Then

$$F(a-b) - F(a+b) = (a-b)^4 - (a+b)^4 = -8a^3b - 8ab^3.$$

The left-hand side of Liouville's 2-term formula gives

$$\begin{aligned} & \sum_{ax+by=n} (F(a-b) - F(a+b)) \\ &= -8 \sum_{ax+by=n} a^3b - 8 \sum_{ax+by=n} ab^3 = -16 \sum_{n_1+n_2=n} \sigma(n_1)\sigma_3(n_2) = -16 \sum_{m=1}^{n-1} \sigma(m)\sigma_3(n-m). \end{aligned}$$

The right-hand side of Liouville's 2-term formula gives

$$\begin{aligned}
 & \sum_{d|n} \left(1 - d + \frac{2n}{d}\right) d^4 - 2 \sum_{d|n} \sum_{1 \leq k \leq d} k^4 \\
 &= \sigma_4(n) - \sigma_5(n) + 2n\sigma_3(n) - 2 \sum_{d|n} \left(\frac{d(d+1)(2d+1)(3d^2+3d-1)}{30} \right) \\
 &= \sigma_4(n) - \sigma_5(n) + 2n\sigma_3(n) - \frac{1}{15} \sum_{d|n} (6d^5 + 15d^4 + 10d^3 - d) \\
 &= \sigma_4(n) - \sigma_5(n) + 2n\sigma_3(n) - \frac{1}{15} (6\sigma_5(n) + 15\sigma_4(n) + 10\sigma_3(n) - \sigma(n)) \\
 &= -\frac{1}{15} (21\sigma_5(n) + (10 - 30n)\sigma_3(n) - \sigma(n)).
 \end{aligned}$$

Equating the left-hand side and right-hand side gives the desired result. $\square$

Our next two theorems are applications of Liouville's 2-term odd formula, and are analogous to Theorems 1.7.1 and 1.7.2 respectively.

Theorem 1.7.3. *Let n be an even positive integer. Set $n = 2^r m$ where $r, m \in \mathbb{N}$ with m odd. Then*

$$\sum_{k=0}^{n/2-1} \sigma(2k+1)\sigma(n-2k-1) = 2^{3(r-1)}\sigma_3(m).$$

Proof. Setting $F(n) = n^2$ in the left-hand side of Liouville's 2-term odd formula gives

$$\begin{aligned}
 & \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} (F(a-b) - F(a+b)) = -4 \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} ab \\
 &= -4 \sum_{\substack{(2n_1+1)+(2n_2+1)=n \\ n_1, n_2 \geq 0}} \sigma(2n_1+1)\sigma(2n_2+1) = -4 \sum_{k=0}^{n/2-1} \sigma(2k+1)\sigma(n-2k-1).
 \end{aligned}$$

The right-hand side of Liouville's 2-term odd formula gives

$$2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d(-(2^r d)^2) = -2^{3r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d^3 = -2^{3r-1} \sum_{d|m} d^3 = -2^{3r-1} \sigma_3(m)$$

Equating the left-hand side and right-hand side gives the desired result. $\square$

This result for $n = 2N$ (N odd) was first stated by Liouville [59, p. 146] and proved for general n in [43, Cor. 1, p. 249].

Theorem 1.7.4. *Let n be an even positive integer. Set $n = 2^r m$ where $r, m \in \mathbb{N}$ with m odd. Then*

$$\sum_{k=0}^{n/2-1} \sigma(2k+1)\sigma_3(n-2k-1) = 2^{5(r-1)}\sigma_5(m).$$

Proof. In order to prove this result, we apply Liouville's 2-term odd formula with $F(n) = n^4$. The left-hand side of Liouville's 2-term formula gives

$$\begin{aligned} \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} (F(a-b) - F(a+b)) &= -8 \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} a^3b - 8 \sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} ab^3 \\ &= -16 \sum_{\substack{(2n_1+1)+(2n_2+1)=n \\ n_1, n_2 \geq 0}} \sigma(2n_1+1)\sigma_3(2n_2+1) \\ &= -16 \sum_{k=0}^{n/2-1} \sigma(2k+1)\sigma_3(n-2k-1). \end{aligned}$$

The right-hand side of Liouville's 2-term formula gives

$$2^{r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d(-(2^r d)^4) = -2^{5r-1} \sum_{\substack{d|n \\ d \text{ odd}}} d^5 = -2^{5r-1} \sum_{d|m} d^5 = -2^{5r-1} \sigma_5(m).$$

Equating the left-hand side and right-hand side gives the desired result. $\square$

Theorem 1.7.4 is an extension of a result of Liouville [59, p. 147].

The following theorem is an application of Liouville's 2-term formula in Theorem 1.6.1, and is analogous to Theorems 1.7.1 and 1.7.3.

Theorem 1.7.5. *Let n be an even positive integer. Set $n = 2^r m$ where r and m are integers such that $r \geq 0$ and $m \geq 1$ is odd. Then*

$$\sum_{k=1}^{n-1} \sigma^*(k) \sigma^*(n-k) = 2^{3r-2} \sigma_3(m) - 2^{2r-2} m \sigma(m),$$

$$\text{where } \sigma^*(k) = \sum_{\substack{d|k \\ k/d \text{ odd}}} d.$$

Proof. Setting $F(n) = n^2$ in the left-hand side of Liouville's 2-term odd formula gives

$$\begin{aligned} \sum_{\substack{ax+by=n \\ x,y \text{ odd}}} (F(a-b) - F(a+b)) &= -4 \sum_{\substack{ax+by=n \\ x,y \text{ odd}}} ab \\ &= -4 \sum_{n_1+n_2=n} \sum_{\substack{d_1|n_1 \\ n_1/d_1 \text{ odd}}} d_1 \sum_{\substack{d_2|n_2 \\ n_2/d_2 \text{ odd}}} d_2 = -4 \sum_{k=1}^{n-1} \sigma^*(k) \sigma^*(n-k). \end{aligned}$$

The right-hand side gives

$$\sum_{d|m} \left(\frac{m}{d} - 2^r d \right) (2^r d)^2 = 2^{2r} m \sum_{d|m} d - 2^{3r} \sum_{d|m} d^3 = 2^{2r} m \sigma(m) - 2^{3r} \sigma_3(m).$$

Equating the left-hand side and right-hand side gives the desired result. $\square$

1.8 The Legendre, Jacobi and Kronecker symbols

In this section, we define the Legendre, Jacobi and Kronecker symbols and discuss their properties. We focus primarily on those properties which are most relevant to our work.

Definition 1.8.1. Let p be an odd prime and a an integer. Then the Legendre symbol

$\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } p \nmid a, \text{ and } x^2 \equiv a \pmod{p} \text{ has a solution,} \\ -1, & \text{if } p \nmid a, \text{ and } x^2 \equiv a \pmod{p} \text{ does not have a solution,} \\ 0, & \text{if } p \mid a, \end{cases}$$

Definition 1.8.2. Let $m > 0$ be an odd integer and a an integer. Suppose m has the prime factorization $m = p_1 \cdots p_k$. The Jacobi symbol $\left(\frac{a}{m}\right)$ is defined as

$$\left(\frac{a}{m}\right) = \left(\frac{a}{p_1}\right) \cdots \left(\frac{a}{p_k}\right),$$

where the symbols on the right are Legendre symbols. Also

$$\left(\frac{a}{1}\right) = 1.$$

Definition 1.8.3. Let a be an integer. The Kronecker symbols $\left(\frac{a}{2}\right)$ and $\left(\frac{a}{-1}\right)$ are defined by

$$\left(\frac{a}{2}\right) = \begin{cases} 1, & \text{if } a \equiv 1 \pmod{8}, \\ 0, & \text{if } a \equiv 0 \pmod{4}, \\ -1, & \text{if } a \equiv 5 \pmod{8}, \\ \text{undefined,} & \text{otherwise,} \end{cases}$$

and

$$\left(\frac{a}{-1}\right) = \begin{cases} 1, & \text{if } a \geq 0, \\ -1, & \text{if } a < 0. \end{cases}$$

Definition 1.8.4. Let m and a be integers with $a \equiv 0, 1 \pmod{4}$. Suppose m has the prime factorization $m = \pm 2^\alpha p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $p_1, \dots, p_r$ are distinct odd primes, $\alpha, \alpha_1, \dots, \alpha_r$ are nonnegative integers and the plus sign holds if m is positive and the minus sign if m is negative. Then the Legendre-Jacobi-Kronecker symbol $\left(\frac{a}{m}\right)$ is defined to be

$$\left(\frac{a}{m}\right) = \left(\frac{a}{\pm 1}\right) \left(\frac{a}{2}\right)^\alpha \left(\frac{a}{p_1}\right)^{\alpha_1} \cdots \left(\frac{a}{p_r}\right)^{\alpha_r}.$$

The following properties follow immediately from the above definitions.

Lemma 1.8.1. Let a, b, m and n be integers with $m, n > 0$. Then

1. $\left(\frac{ab}{m}\right) = \left(\frac{a}{m}\right) \left(\frac{b}{m}\right),$
2. $\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right) \left(\frac{a}{n}\right),$
3. If $a \equiv b \pmod{m}$, then $\left(\frac{a}{m}\right) = \left(\frac{b}{m}\right).$

Lemma 1.8.2. Let d be an odd integer, then

$$\left(\frac{-4}{d}\right) = (-1)^{(d-1)/2}.$$

The following result will be used extensively in Chapter 4, see [19, p. 248].

Lemma 1.8.3. Let n be a positive integer, then the number $r_2(n)$ of representations of n as the sum of two squares is given by

$$r_2(n) = 4 \sum_{\substack{d|n \\ d \equiv 1 \pmod{2}}} (-1)^{(d-1)/2} = 4 \sum_{d|n} \left(\frac{-4}{d}\right).$$

Lemma 1.8.4. *Let n be an odd positive integer and let d be a positive divisor of n .*

Then

$$\left(\frac{-4}{n/d}\right) = (-1)^{(n-1)/2} \left(\frac{-4}{d}\right).$$

Proof. As n is odd and d divides n , d is odd. Thus $d^2 \equiv 1 \pmod{4}$ and so

$$\left(\frac{-4}{d^2}\right) = 1.$$

Finally

$$\begin{aligned} \left(\frac{-4}{n/d}\right) &= \left(\frac{-4}{n/d}\right) \left(\frac{-4}{d^2}\right) = \left(\frac{-4}{nd}\right) = \left(\frac{-4}{n}\right) \left(\frac{-4}{d}\right) \\ &= \left(\frac{-1}{n}\right) \left(\frac{-4}{d}\right) = (-1)^{(n-1)/2} \left(\frac{-4}{d}\right), \end{aligned}$$

as required. □

Lemma 1.8.5. *Let n be a positive integer satisfying $n \equiv 3 \pmod{4}$, then*

$$r_2(n) = 0.$$

Proof. We have

$$\sum_{d|n} \left(\frac{-4}{d}\right) = \sum_{d|n} \left(\frac{-4}{n/d}\right) = (-1)^{(n-1)/2} \sum_{d|n} \left(\frac{-4}{d}\right) = - \sum_{d|n} \left(\frac{-4}{d}\right),$$

by Lemma 1.8.4 together with the fact that $n \equiv 3 \pmod{4}$. Therefore

$$r_2(n) = 4 \sum_{d|n} \left(\frac{-4}{d}\right) = 0,$$

as required. □

Lemma 1.8.6. *Let d_1 be an odd integer and d_2 an even integer. Then*

$$\left(\frac{-4}{d_1 - d_2}\right) = \left(\frac{-4}{d_1 + d_2}\right)$$

and

$$\left(\frac{-4}{d_2 - d_1}\right) = -\left(\frac{-4}{d_2 + d_1}\right).$$

Proof. Since d_2 is even, we have $2d_2 \equiv 0 \pmod{4}$, or equivalently

$$d_2 \equiv -d_2 \pmod{4}.$$

It follows that

$$d_1 + d_2 \equiv d_1 - d_2 \pmod{4}$$

and

$$d_2 - d_1 \equiv -d_2 - d_1 \equiv -(d_2 + d_1) \pmod{4}.$$

Then by Lemma 1.8.1, we obtain the desired result. □

Chapter 2

Liouville's 3-term odd formula

In this chapter we give a proof of Liouville's 3-term odd formula (Theorem 2.4.1), which was stated but not proved by Liouville in [64, p. 331].

Let n be a positive odd integer, and let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function. Then

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ = \frac{1}{8} \sum_{d|n} (d^2 - 1)F(d) - 3 \sum_{\substack{ax < n \\ a,x \text{ odd}}} \sigma(o(n-ax))F(a),$$

where the sum on the left-hand side is taken over all $(a, b, c, x, y, z) \in \mathbb{N}^6$ such that $ax + by + cz = n$ with a, b, c, x, y, z odd, and the right-hand sum is over all $(a, x) \in \mathbb{N}^2$ with $ax < n$ and a, x odd.

We have been unable to locate a proof of this formula in the literature, and so it is possible that ours is the first proof of this result.

2.1 The equation $kx + ey = n$ and the quantities

$$R_1(a, b, c), R_2(a, b, c), R_3(a, b, c), S_1(a, b, c), S_2(a, b, c),$$

$$S_3(a, b, c)$$

In this section we introduce the quantities $R_1, R_2, R_3, S_1, S_2, S_3$ formed from $L_1, L_2, L_3, M_1, M_2, M_3$ respectively by taking the sums over e rather than 1. Throughout this section k and n are positive integers except in Lemma 2.1.5 where they are restricted to be odd. Also $a, b, c \in \{0, 1\}$.

Definition 2.1.1.

$$R_1(a, b, c) = \sum_{\substack{kx+ey=n \\ e < k \\ x \equiv a, e \equiv b, y \equiv c}} e, \quad R_2(a, b, c) = \sum_{\substack{kx+ey=n \\ e > k \\ x \equiv a, e \equiv b, y \equiv c}} e, \quad R_3(a, b, c) = \sum_{\substack{kx+ey=n \\ e = k \\ x \equiv a, e \equiv b, y \equiv c}} e.$$

Definition 2.1.2.

$$S_1(a, b, c) = \sum_{\substack{kx+ey=n \\ x < y \\ x \equiv a, e \equiv b, y \equiv c}} e, \quad S_2(a, b, c) = \sum_{\substack{kx+ey=n \\ x > y \\ x \equiv a, e \equiv b, y \equiv c}} e, \quad S_3(a, b, c) = \sum_{\substack{kx+ey=n \\ x = y \\ x \equiv a, e \equiv b, y \equiv c}} e.$$

We now examine relationships between $R_1, R_2, R_3, S_1, S_2, S_3$. These relationships are very similar to those we saw in Section 1.2 for $L_1, L_2, L_3, M_1, M_2, M_3$. Once again the following result follows immediately from the definitions.

Lemma 2.1.1.

$$R_1(a, b, c) + R_2(a, b, c) + R_3(a, b, c) = S_1(a, b, c) + S_2(a, b, c) + S_3(a, b, c).$$

Next we determine $R_3(a, b, c)$.

Lemma 2.1.2.

$$R_3(a, b, c) = \begin{cases} 0, & \text{if } k \nmid n \\ & \text{or } k \mid n, k \not\equiv b \\ & \text{or } k \mid n, k \equiv b, \frac{n}{k} \not\equiv a + c, \\ \frac{1}{2}(n - k) - \frac{k}{4}(-1)^a(1 + (-1)^{n/k}), & \text{if } k \mid n, k \equiv b, \frac{n}{k} \equiv a + c. \end{cases}$$

Proof. We have

$$R_3(a, b, c) = \sum_{\substack{kx+ey=n \\ e=k \\ x \equiv a, e \equiv b, y \equiv c}} e = \sum_{\substack{k(x+y)=n \\ x \equiv a, k \equiv b, y \equiv c}} k = k \sum_{\substack{k(x+y)=n \\ x \equiv a, k \equiv b, y \equiv c}} 1 = kL_3(a, b, c),$$

and the result follows from Lemma 1.4.2. $\square$

The next lemma evaluates $S_3(a, b, c)$.

Lemma 2.1.3.

$$S_3(a, b, c) = \begin{cases} 0, & \text{if } a \not\equiv c \\ & \text{or } a \equiv c, n \not\equiv a(k + b), \\ \sum_{\substack{d \mid n, d > k \\ d \equiv k+b \\ n/d \equiv a}} (d - k), & \text{if } a \equiv c, n \equiv a(k + b). \end{cases}$$

Proof. If $a \not\equiv c$ then $S_3(a, b, c) = 0$. If $a \equiv c$, we have

$$S_3(a, b, c) = \sum_{\substack{(k+e)x=n \\ x \equiv a, e \equiv b}} e = \sum_{\substack{k+e \mid n \\ \frac{n}{k+e} \equiv a \\ e \equiv b}} e = \sum_{\substack{d \mid n, d > k \\ d \equiv k+b \\ \frac{n}{d} \equiv a}} (d - k).$$

$\square$

The following lemma gives an important relationship between $R_2(a, b, c)$ and $S_2(a, b, c)$.

Lemma 2.1.4.

$$S_2(a, b, c) = R_2(a + c, k + b, c) - kL_2(a + c, k + b, c).$$

Proof. We have

$$\begin{aligned}
 S_2(a, b, c) &= \sum_{\substack{kx+ey=n \\ x>y \\ x\equiv a, e\equiv b, y\equiv c}} e \\
 &= \sum_{\substack{k(x'+y)+ey=n \\ x'\equiv a+c, e\equiv b, y\equiv c}} e \\
 &= \sum_{\substack{kx+(k+e)y=n \\ x\equiv a+c, e\equiv b, y\equiv c}} e \\
 &= \sum_{\substack{kx+e'y=n \\ e'>k \\ x\equiv a+c, e'\equiv k+b, y\equiv c}} (e' - k) \\
 &= \sum_{\substack{kx+ey=n \\ e>k \\ x\equiv a+c, e\equiv k+b, y\equiv c}} e - k \sum_{\substack{kx+ey=n \\ e>k \\ x\equiv a+c, e\equiv k+b, y\equiv c}} 1 \\
 &= R_2(a + c, k + b, c) - kL_2(a + c, k + b, c),
 \end{aligned}$$

as claimed. □

We conclude this section by noting the values of $R_3(1, 1, 0)$ and $S_3(1, 1, 0)$ when k and n are both odd.

Lemma 2.1.5. *If k and n are positive odd integers then*

$$R_3(1, 1, 0) = \left(\frac{n - k}{2} \right) \delta(n, k),$$

$$S_3(1, 1, 0) = 0.$$

Proof. These values follow easily from Lemmas 2.1.2 and 2.1.3. $\square$

2.2 Solutions of $kx + by + cz = n$ (k, n fixed odd integers)

Throughout this section, k and n are restricted to be positive odd integers. In this section, we are concerned with solutions $(x, b, y, c, z) \in \mathbb{N}^5$ of

$$kx + by + cz = n$$

with b, c, x odd and y, z even. For such solutions we have $b + c \neq k$, $b - c \neq k$ and $x \neq z$, so that

$$\sum_{\substack{kx+by+cz=n \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 = \sum_{\substack{kx+by+cz=n \\ b+c < k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c > k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1$$

and

$$\begin{aligned} \sum_{\substack{kx+by+cz=n \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 &= \sum_{\substack{kx+by+cz=n \\ b-c < k, x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b-c > k, x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\ &+ \sum_{\substack{kx+by+cz=n \\ b-c < k, x > z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b-c > k, x > z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1. \end{aligned}$$

With this in mind, we find it useful to define the following sums.

Definition 2.2.1.

$$A'_1 = \sum_{\substack{kx+by+cz=n \\ b+c < k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1, \quad A'_2 = \sum_{\substack{kx+by+cz=n \\ b+c > k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1.$$

Definition 2.2.2.

$$\begin{aligned}
 B'_1 &= \sum_{\substack{kx+by+cz=n \\ b-c < k, \, x < z \\ b,c,x \text{ odd}, \, y,z \text{ even}}} 1, & B'_2 &= \sum_{\substack{kx+by+cz=n \\ b-c > k, \, x < z \\ b,c,x \text{ odd}, \, y,z \text{ even}}} 1, \\
 B'_4 &= \sum_{\substack{kx+by+cz=n \\ b-c < k, \, x > z \\ b,c,x \text{ odd}, \, y,z \text{ even}}} 1, & B'_5 &= \sum_{\substack{kx+by+cz=n \\ b-c > k, \, x > z \\ b,c,x \text{ odd}, \, y,z \text{ even}}} 1.
 \end{aligned}$$

Before continuing we say a few words about the notation used in Definitions 2.2.1 and 2.2.2. The prime (') on each of $A'_1, A'_2, B'_1, B'_2, B'_4, B'_5$ indicates the condition “ b, c, x odd, y, z even.” In Chapter 3 the analogous sums $A_1, A_2, A_3, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9$ are defined without this condition (see Definitions 3.3.1 and 3.3.2). Thus for example we have

$$A_1 = \sum_{\substack{kx+by+cz=n \\ b+c < k}} 1, \quad A_2 = \sum_{\substack{kx+by+cz=n \\ b+c > k}} 1, \quad A_3 = \sum_{\substack{kx+by+cz=n \\ b+c=k}} 1,$$

and so on. We do not require $A'_3, B'_3, B'_6, B'_7, B'_8, B'_9$ in view of the comments at the beginning of the section.

Clearly

Lemma 2.2.1.

$$A'_1 + A'_2 = B'_1 + B'_2 + B'_4 + B'_5.$$

It is also convenient to define another sum.

Definition 2.2.3.

$$C'_1 = \sum_{\substack{kx+by+cz=n \\ x < y < z \\ b,c,x \text{ odd}, \, y,z \text{ even}}} 1.$$

We now prove various relations between A'_1 , A'_2 , B'_1 , B'_2 , B'_4 , B'_5 and C'_1 with the ultimate goal of determining $A'_1 - 2B'_1 + 2C'_1$, see Proposition 2.2.1.

Lemma 2.2.2.

$$B'_2 = \sum_{\substack{kx+by+cz=n \\ y < x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1.$$

Proof. We have

$$\begin{aligned} \sum_{\substack{kx+by+cz=n \\ y < x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 &= \sum_{\substack{k(y+x')+by+c(y+z')=n \\ x' < z' \\ b,c,x' \text{ odd}, y,z' \text{ even}}} 1 \quad (x = y + x', z = y + z') \\ &= \sum_{\substack{kx+(k+b+c)y+cz=n \\ x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \quad (x = x', z = z') \\ &= \sum_{\substack{kx+b'y+cz=n \\ x < z, b' > k+c \\ b',c,x \text{ odd}, y,z \text{ even}}} 1 \quad (b' = k + b + c) \\ &= \sum_{\substack{kx+by+cz=n \\ x < z, b-c > k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \quad (b = b') \\ &= B'_2, \end{aligned}$$

as required. □

Lemma 2.2.3.

$$2C'_1 = B'_1 - \frac{1}{2}S_1(1, 0, 0).$$

Proof. We have

$$\begin{aligned}
 2C'_1 &= 2 \sum_{\substack{kx+by+cz=n \\ x < y < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ x < y < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ x < z < y \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 - \sum_{\substack{kx+by+cz=n \\ y < x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 - \sum_{\substack{kx+by+cz=n \\ x < y = z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ b-c < k, x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b-c > k, x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 - B'_2 - \sum_{\substack{kx+(b+c)y=n \\ x < y \\ b,c,x \text{ odd}, y \text{ even}}} 1 \quad (\text{by Lemma 2.2.2}) \\
 &= B'_1 + B'_2 - B'_2 - \sum_{\substack{kx+ey=n \\ x < y, e=b+c \\ b,c,x \text{ odd}, e,y \text{ even}}} 1 \quad (\text{by Definition 2.2.2}) \\
 &= B'_1 - \sum_{\substack{kx+ey=n \\ x < y \\ x \text{ odd}, e,y \text{ even}}} \frac{e}{2} \\
 &= B'_1 - \frac{1}{2} S_1(1, 0, 0),
 \end{aligned}$$

as there are $e/2$ pairs (b, c) of positive odd integers with $b + c = e$ when e is even. $\square$

Lemma 2.2.4.

$$B'_5 = \sum_{\substack{kx+by+cz=n \\ y < z < x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1.$$

Proof. We have

$$\sum_{\substack{kx+by+cz=n \\ y < z < x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 = \sum_{\substack{k(y+x')+by+c(y+z')=n \\ z' < x' \\ b,c,x' \text{ odd}, y,z' \text{ even}}} 1 \quad (x = y + x', z = y + z')$$

$$\begin{aligned}
 &= \sum_{\substack{kx+(k+b+c)y+cz=n \\ z < x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \quad (x = x', z = z') \\
 &= \sum_{\substack{kx+b'y+cz=n \\ z < x, b' > k+c \\ b',c,x \text{ odd}, y,z \text{ even}}} 1 \quad (b' = k + b + c) \\
 &= \sum_{\substack{kx+by+cz=n \\ z < x, b-c > k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \quad (b = b') \\
 &= B'_5,
 \end{aligned}$$

as required. □

Lemma 2.2.5.

$$B'_2 + B'_5 = B'_4 - \frac{1}{2}S_2(1, 0, 0).$$

Proof. By Definition 2.2.2 we have

$$\sum_{\substack{kx+by+cz=n \\ x > z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 = \sum_{\substack{kx+by+cz=n \\ x > z, b-c < k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ x > z, b-c > k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 = B'_4 + B'_5.$$

Also

$$\begin{aligned}
 &\sum_{\substack{kx+by+cz=n \\ x > z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ y < z < x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ z < y < x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ z < x < y \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ y = z < x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ y < z < x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ y < z < x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ y < x < z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+(b+c)y=n \\ y < x \\ b,c,x \text{ odd}, y \text{ even}}} 1
 \end{aligned}$$

$$\begin{aligned}
 &= B'_5 + B'_5 + B'_2 + \sum_{\substack{kx+ey=n \\ y < x \\ x \text{ odd}, e, y \text{ even}}} \frac{e}{2} \quad (\text{by Lemmas 2.2.2 and 2.2.4}) \\
 &= 2B'_5 + B'_2 + \frac{1}{2}S_2(1, 0, 0).
 \end{aligned}$$

Hence

$$B'_4 + B'_5 = 2B'_5 + B'_2 + \frac{1}{2}S_2(1, 0, 0).$$

It follows that

$$B'_2 + B'_5 = B'_4 - \frac{1}{2}S_2(1, 0, 0),$$

which is the assertion of the lemma. □

Lemma 2.2.6.

$$A'_2 = 2B'_4 + \frac{1}{2}R_2(1, 0, 0).$$

Proof. From Definition 2.2.1 we obtain

$$\begin{aligned}
 A'_2 &= \sum_{\substack{kx+by+cz=n \\ b+c > k \\ b, c, x \text{ odd}, y, z \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ b+c > k, y > z \\ b, c, x \text{ odd}, y, z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c > k, y < z \\ b, c, x \text{ odd}, y, z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c > k, y = z \\ b, c, x \text{ odd}, y, z \text{ even}}} 1 \\
 &= 2 \sum_{\substack{kx+by+cz=n \\ b+c > k, y > z \\ b, c, x \text{ odd}, y, z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c > k, y = z \\ b, c, x \text{ odd}, y, z \text{ even}}} 1,
 \end{aligned}$$

where we used the transformation $(b, y, c, z) \rightarrow (c, z, b, y)$ in the last step. We now examine the above two sums. Firstly, we have

$$\begin{aligned}
 \sum_{\substack{kx+by+cz=n \\ b+c>k, y=z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 &= \sum_{\substack{kx+(b+c)y=n \\ b+c>k \\ b,c,x \text{ odd}, y \text{ even}}} 1 \\
 &= \sum_{e \text{ even}} \frac{e}{2} \sum_{\substack{kx+ey=n \\ e>k \\ x \text{ odd}, y \text{ even}}} 1 \\
 &= \frac{1}{2} \sum_{\substack{kx+ey=n \\ e>k \\ x \equiv 1, e \equiv 0, y \equiv 0}} e \\
 &= \frac{1}{2} R_2(1, 0, 0),
 \end{aligned}$$

by Definition 2.1.1, and secondly,

$$\begin{aligned}
 \sum_{\substack{kx+by+cz=n \\ b+c>k, y>z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 &= \sum_{\substack{kx+b(z+y')+cz=n \\ b+c>k \\ b,c,x \text{ odd}, y',z \text{ even}}} 1 \quad (y = z + y') \\
 &= \sum_{\substack{kx+by+(b+c)z=n \\ b+c>k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \quad (y = y') \\
 &= \sum_{\substack{kx+by+ez=n \\ e=b+c, e>k \\ b,c,x \text{ odd}, e,y,z \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+ez=n \\ e>b, e>k \\ b,x \text{ odd}, e,y,z \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+(k+c')z=n \\ k+c'>b \\ b,c',x \text{ odd}, y,z \text{ even}}} 1 \quad (e = k + c')
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{k(x+z)+by+cz=n \\ b-c < k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \quad (c = c') \\
&= \sum_{\substack{kx'+by+cz=n \\ b-c < k, x' > z \\ b,c,x' \text{ odd}, y,z \text{ even}}} 1 \quad (x' = x + z) \\
&= B'_4,
\end{aligned}$$

by Definition 2.2.2. This completes the proof of the lemma. $\square$

Putting together all that we have proven so far, we are able to prove the following proposition.

Proposition 2.2.1.

$$A'_1 - 2B'_1 + 2C'_1 = -\frac{1}{2}R_2(1, 0, 0) - \frac{1}{2}S_1(1, 0, 0) - \frac{1}{2}S_2(1, 0, 0).$$

Proof. Appealing to Lemmas 2.2.3, 2.2.1, 2.2.5 and 2.2.6, we obtain

$$\begin{aligned}
A'_1 - 2B'_1 + 2C'_1 &= A'_1 - 2B'_1 + B'_1 - \frac{1}{2}S_1(1, 0, 0) \\
&= A'_1 - B'_1 - \frac{1}{2}S_1(1, 0, 0) \\
&= -A'_2 + B'_2 + B'_4 + B'_5 - \frac{1}{2}S_1(1, 0, 0) \\
&= -A'_2 + 2B'_4 - \frac{1}{2}S_2(1, 0, 0) - \frac{1}{2}S_1(1, 0, 0) \\
&= -\frac{1}{2}R_2(1, 0, 0) - \frac{1}{2}S_2(1, 0, 0) - \frac{1}{2}S_1(1, 0, 0),
\end{aligned}$$

as asserted. $\square$

2.3 Solutions of $ax+by+cz = n$ (n fixed odd integer)

Throughout this section k and n are positive odd integers. We are interested in the solutions $(a, x, b, y, c, z) \in \mathbb{N}^6$ of the equation

$$ax + by + cz = n$$

with a, b, c, x, y, z odd and $a + b + c = k$ or $a + b - c = k$ or $a - b - c = k$. We denote the number of such solutions by U' , V' and W' respectively.

Definition 2.3.1.

$$U' = \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd}}} 1, \quad V' = \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1, \quad W' = \sum_{\substack{ax+by+cz=n \\ a-b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1.$$

In the next three lemmas we relate U' , V' , W' to quantities of the previous two sections and Section 1.4.

Lemma 2.3.1.

$$U' = \frac{(k^2 - 1)}{8} \delta(n, k) + \frac{3}{2} (kL_1(1, 1, 0) - R_1(1, 1, 0)) + 3A'_1.$$

Proof. We have

$$U' = \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd}}} 1 = \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ x \neq y, y \neq z, z \neq x}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ \text{exactly 2 of } x,y,z \text{ equal}}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ x=y=z}} 1.$$

We begin by considering the first sum on the right-hand side of the above expression.

The condition $x \neq y, y \neq z, z \neq x$, comprises six possibilities, namely, $x < y < z$,

$x < z < y$, $y < x < z$, $y < z < x$, $z < x < y$ and $z < y < x$. Simple transformations of the summation variables show that these six cases yield the same contribution to the sum. Thus

$$\begin{aligned}
 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ x \neq y, y \neq z, z \neq x}} 1 &= 6 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x < y < z \\ a,b,c,x,y,z \text{ odd}}} 1 \\
 &= 6 \sum_{\substack{(k-b-c)x+by+cz=n \\ b+c < k \\ x < y < z \\ b,c,x,y,z \text{ odd}}} 1 \\
 &= 6 \sum_{\substack{kx+b(y-x)+c(z-x)=n \\ b+c < k \\ x < y < z \\ b,c,x,y,z \text{ odd}}} 1 \\
 &= 6 \sum_{\substack{kx+by'+cz'=n \\ b+c < k \\ y' < z' \\ b,c,x \text{ odd}, y',z' \text{ even}}} 1 \quad (y' = y - x, z' = z - x) \\
 &= \frac{6}{2} \sum_{\substack{kx+by+cz=n \\ b+c < k \\ y \neq z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\
 &= 3 \sum_{\substack{kx+by+cz=n \\ b+c < k \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 - 3 \sum_{\substack{kx+by+cz=n \\ b+c < k \\ y=z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\
 &= 3A'_1 - 3 \sum_{\substack{kx+(b+c)y=n \\ b+c < k \\ b,c,x \text{ odd}, y \text{ even}}} 1 \quad (\text{by Definition 2.2.1}) \\
 &= 3A'_1 - 3 \sum_{\substack{kx+ey=n \\ e < k \\ x \text{ odd}, y,e \text{ even}}} \frac{e}{2}
 \end{aligned}$$

$$= 3A'_1 - \frac{3}{2}R_1(1, 0, 0),$$

by Definition 2.1.1. Next we consider the second sum on the right-hand side of the expression for U' . There are six different cases in which exactly two of x , y , and z are equal. Simple transformations of the summation variables show that the cases $x = y < z$, $x = z < y$, and $y = z < x$, lead to the same sum, and similarly for the cases $x < y = z$, $y < x = z$, and $z < y = x$. Therefore

$$\begin{aligned} \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ \text{exactly 2 of } x,y,z \text{ equal}}} 1 &= 3 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x=y < z \\ a,b,c,x,y,z \text{ odd}}} 1 + 3 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x < y=z \\ a,b,c,x,y,z \text{ odd}}} 1 \\ &= 3 \sum_{\substack{(a+b)x+cz=n \\ a+b+c=k \\ x < z \\ a,b,c,x,z \text{ odd}}} 1 + 3 \sum_{\substack{ax+(b+c)y=n \\ a+b+c=k \\ x < y \\ a,b,c,x,y \text{ odd}}} 1 \\ &= 3 \sum_{\substack{ex+cz=n \\ e+c=k \\ x < z \\ e=a+b \\ a,b,c,x,z \text{ odd}, e \text{ even}}} 1 + 3 \sum_{\substack{ax+ey=n \\ a+e=k \\ x < y \\ e=b+c \\ a,b,c,x,y \text{ odd}, e \text{ even}}} 1 \\ &= 3 \sum_{\substack{ex+cz=n \\ e+c=k \\ x < z \\ c,x,z \text{ odd}, e \text{ even}}} \frac{e}{2} + 3 \sum_{\substack{ax+ey=n \\ a+e=k \\ x < y \\ a,x,y \text{ odd}, e \text{ even}}} \frac{e}{2} \\ &= 3 \sum_{\substack{ex+(k-e)z=n \\ x < z \\ e < k \\ x,z \text{ odd}, e \text{ even}}} \frac{e}{2} + 3 \sum_{\substack{(k-e)x+ey=n \\ x < y \\ e < k \\ x,y \text{ odd}, e \text{ even}}} \frac{e}{2} \\ &= 3 \sum_{\substack{ex+(k-e)(x+z')=n \\ e < k \\ x \text{ odd}, e, z' \text{ even}}} \frac{e}{2} + 3 \sum_{\substack{(k-e)x+e(x+y')=n \\ e < k \\ x \text{ odd}, e, y' \text{ even}}} \frac{e}{2} \end{aligned}$$

$$\begin{aligned}
&= 3 \sum_{\substack{kx+(k-e)z=n \\ e < k \\ x \text{ odd}, e, z \text{ even}}} \frac{e}{2} + 3 \sum_{\substack{kx+ey=n \\ e < k \\ x \text{ odd}, e, y \text{ even}}} \frac{e}{2} \\
&= 3 \sum_{\substack{kx+e'z=n \\ e' < k \\ x, e' \text{ odd}, z \text{ even}}} \frac{k-e'}{2} + \frac{3}{2} R_1(1, 0, 0) \\
&= \frac{3}{2} (kL_1(1, 1, 0) - R_1(1, 1, 0) + R_1(1, 0, 0)),
\end{aligned}$$

by Definitions 1.4.1 and 2.1.1. Finally, we consider the third sum on the right-hand side of the expression for U' . We have

$$\begin{aligned}
\sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a, b, c, x, y, z \text{ odd} \\ x=y=z}} 1 &= \sum_{\substack{(a+b+c)x=n \\ a+b+c=k \\ a, b, c, x \text{ odd}}} 1 \\
&= \sum_{\substack{kx=n \\ a+b+c=k \\ a, b, c, x \text{ odd}}} 1 \\
&= \delta(n, k) \sum_{\substack{a+b+c=k \\ a, b, c \text{ odd}}} 1 \\
&= \delta(n, k) \left(\frac{k-1}{2} + \frac{k-3}{2} + \cdots + \frac{k-(k-2)}{2} \right) \\
&= \delta(n, k) \left(\frac{k^2-1}{8} \right).
\end{aligned}$$

Putting these three sums together, we obtain

$$\begin{aligned}
U' &= 3A'_1 - \frac{3}{2} R_1(1, 0, 0) + \frac{3}{2} (kL_1(1, 1, 0) - R_1(1, 1, 0) + R_1(1, 0, 0)) \\
&\quad + \delta(n, k) \left(\frac{k^2-1}{8} \right) \\
&= \frac{(k^2-1)}{8} \delta(n, k) + \frac{3}{2} (kL_1(1, 1, 0) - R_1(1, 1, 0)) + 3A'_1,
\end{aligned}$$

as asserted. □

Lemma 2.3.2.

$$V' = \frac{1}{2}(kM_1(1, 1, 0) + S_1(1, 1, 0)) + 2B'_1.$$

Proof. We have

$$\begin{aligned}
 V' &= \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 = \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x>y \\ a,b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x<y \\ a,b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x=y \\ a,b,c,x,y,z \text{ odd}}} 1 \\
 &= 2 \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x<y \\ a,b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x=y \\ a,b,c,x,y,z \text{ odd}}} 1 \\
 &= 2 \sum_{\substack{(k-b+c)x+by+cz=n \\ b-c<k \\ x<y \\ b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{(a+b)x+cz=n \\ a+b-c=k \\ a,b,c,x,z \text{ odd}}} 1 \\
 &= 2 \sum_{\substack{kx+b(y-x)+c(z+x)=n \\ b-c<k \\ x<y \\ b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{ex+cz=n \\ e-c=k \\ e=a+b \\ a,b,c,x,z \text{ odd}, e \text{ even}}} 1 \\
 &= 2 \sum_{\substack{kx+by'+c(z+x)=n \\ b-c<k \\ b,c,x,z \text{ odd}, y' \text{ even}}} 1 + \sum_{\substack{ex+cz=n \\ e=c+k \\ c,x,z \text{ odd}, e \text{ even}}} \frac{e}{2} \\
 &= 2 \sum_{\substack{kx+by+cz'=n \\ b-c<k, z'>x \\ b,c,x \text{ odd}, y,z' \text{ even}}} 1 + \sum_{\substack{(c+k)x+cz=n \\ c,x,z \text{ odd}}} \frac{c+k}{2} \\
 &= 2 \sum_{\substack{kx+by+cz=n \\ b-c<k, z>x \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+cy=n \\ y>x \\ c,x \text{ odd}, y \text{ even}}} \frac{k+c}{2} \\
 &= 2B'_1 + \frac{k}{2} \sum_{\substack{kx+ey=n \\ x<y \\ e,x \text{ odd}, y \text{ even}}} 1 + \frac{1}{2} \sum_{\substack{kx+ey=n \\ x<y \\ e,x \text{ odd}, y \text{ even}}} e \\
 &= 2B'_1 + \frac{1}{2}(kM_1(1, 1, 0) + S_1(1, 1, 0)),
 \end{aligned}$$

as required. $\square$

Lemma 2.3.3.

$$W' = \frac{1}{2}S_1(1, 0, 0) + 2C'_1.$$

Proof. We have

$$\begin{aligned}
 W' &= \sum_{\substack{ax+by+cz=n \\ a-b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 \\
 &= \sum_{\substack{(k+b+c)x+by+cz=n \\ b,c,x,y,z \text{ odd}}} 1 \\
 &= \sum_{\substack{kx+b(x+y)+c(x+z)=n \\ b,c,x,y,z \text{ odd}}} 1 \\
 &= \sum_{\substack{kx+by'+cz'=n \\ y'>x, z'>x \\ b,c,x \text{ odd}, y',z' \text{ even}}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ y>x, z>x, y<z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ y>x, z>x, y>z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ y>x, z>x, y=z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 \\
 &= 2 \sum_{\substack{kx+by+cz=n \\ x<y<z \\ b,c,x \text{ odd}, y,z \text{ even}}} 1 + \sum_{\substack{kx+(b+c)y=n \\ y>x \\ b,c,x \text{ odd}, y \text{ even}}} 1 \\
 &= 2C'_1 + \sum_{\substack{kx+ey=n \\ y>x \\ e=b+c \\ b,c,x \text{ odd}, e,y \text{ even}}} 1 \quad (\text{by Definition 2.2.3}) \\
 &= 2C'_1 + \sum_{\substack{kx+ey=n \\ x<y \\ x \text{ odd}, e,y \text{ even}}} \frac{e}{2} \\
 &= 2C'_1 + \frac{1}{2}S_1(1, 0, 0),
 \end{aligned}$$

as claimed. $\square$

Lemma 2.3.4.

$$U' - 3V' + 3W' = \frac{(k^2 - 1)}{8} \delta(n, k) - 3 \sum_{\substack{kx < n \\ x \text{ odd}}} \sigma(o(n - kx)).$$

Proof. Applying Lemmas 2.3.1, 2.3.2 and 2.3.3 we find

$$\begin{aligned} & U' - 3V' + 3W' \\ &= \frac{(k^2 - 1)}{8} \delta(n, k) + \frac{3}{2} (kL_1(1, 1, 0) - R_1(1, 1, 0)) + 3A'_1 \\ &\quad - 3 \left(\frac{1}{2} (kM_1(1, 1, 0) + S_1(1, 1, 0)) + 2B'_1 \right) + 3 \left(\frac{1}{2} S_1(1, 0, 0) + 2C'_1 \right) \\ &= \frac{(k^2 - 1)}{8} \delta(n, k) + \frac{3}{2} \left(k(L_1(1, 1, 0) - M_1(1, 1, 0)) - R_1(1, 1, 0) - S_1(1, 1, 0) + \right. \\ &\quad \left. S_1(1, 0, 0) \right) + 3(A'_1 - 2B'_1 + 2C'_1). \end{aligned}$$

Appealing to Lemmas 1.4.6 and 2.2.1, we obtain

$$\begin{aligned} & U' - 3V' + 3W' \\ &= \frac{(k^2 - 1)}{8} \delta(n, k) + \frac{3}{2} \left(k(L_2(1, 0, 0) - L_2(1, 1, 0) - \left(\frac{(n/k) - 1}{2} \right) \delta(n, k)) \right. \\ &\quad \left. - R_1(1, 1, 0) - S_1(1, 1, 0) + S_1(1, 0, 0) \right) \\ &\quad - \frac{3}{2} \left(R_2(1, 0, 0) + S_2(1, 0, 0) + S_1(1, 0, 0) \right) \\ &= \frac{(k^2 - 1)}{8} \delta(n, k) + \frac{3}{2} \left(- (S_2(1, 0, 0) + kL_2(1, 1, 0)) - (R_2(1, 0, 0) \right. \\ &\quad \left. - kL_2(1, 0, 0)) - \left(\frac{n - k}{2} \right) \delta(n, k) - R_1(1, 1, 0) - S_1(1, 1, 0) \right) \end{aligned}$$

Appealing to Lemmas 2.1.4 and 2.1.5 we obtain

$$U' - 3V' + 3W'$$

$$\begin{aligned}
 &= \frac{(k^2 - 1)}{8} \delta(n, k) + \frac{3}{2} \left(-R_2(1, 1, 0) - S_2(1, 1, 0) - R_3(1, 1, 0) \right. \\
 &\quad \left. - R_1(1, 1, 0) - S_1(1, 1, 0) \right) \\
 &= \frac{(k^2 - 1)}{8} \delta(n, k) - \frac{3}{2} \left(R_1(1, 1, 0) + R_2(1, 1, 0) + R_3(1, 1, 0) \right. \\
 &\quad \left. + S_1(1, 1, 0) + S_2(1, 1, 0) + S_3(1, 1, 0) \right) \\
 &= \frac{(k^2 - 1)}{8} \delta(n, k) - \frac{3}{2} \sum_{\substack{kx+ey=n \\ x \equiv 1, e \equiv 1, y \equiv 0}} e - \frac{3}{2} \sum_{\substack{kx+ey=n \\ x \equiv 1, e \equiv 1, y \equiv 0}} e \\
 &= \frac{(k^2 - 1)}{8} \delta(n, k) - 3 \sum_{\substack{kx+ey=n \\ x \equiv 1, e \equiv 1, y \equiv 0}} e \\
 &= \frac{(k^2 - 1)}{8} \delta(n, k) - 3 \sum_{\substack{kx < n \\ x \text{ odd}}} \sum_{\substack{ey = n - kx \\ e \text{ odd}, y \text{ even}}} e \\
 &= \frac{(k^2 - 1)}{8} \delta(n, k) - 3 \sum_{\substack{kx < n \\ x \text{ odd}}} \sum_{\substack{e | n - kx \\ e \text{ odd}}} e \\
 &= \frac{(k^2 - 1)}{8} \delta(n, k) - 3 \sum_{\substack{kx < n \\ x \text{ odd}}} \sigma(o(n - kx)),
 \end{aligned}$$

as asserted. □

2.4 Proof of Liouville's 3-term odd formula

We now use the results of Sections 2.1-2.3 to prove Liouville's 3-term odd formula.

Theorem 2.4.1. *Let n be a positive odd integer, and let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c))$$

$$= \frac{1}{8} \sum_{d|n} (d^2 - 1) F(d) - 3 \sum_{\substack{ax < n \\ a, x \text{ odd}}} \sigma(o(n - ax)) F(a),$$

where the sum on the left-hand side is taken over all $(a, b, c, x, y, z) \in \mathbb{N}^6$ such that $ax + by + cz = n$ with a, b, c, x, y, z odd, and the right-hand sum is over all $(a, x) \in \mathbb{N}^2$ with $ax < n$ and a, x odd.

Proof. First we have

$$\begin{aligned} & \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ &= \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a+b+c) + \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a-b-c) \\ & \quad - \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a+b-c) - \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a-b+c). \end{aligned}$$

Next we look at each of these sums individually. We have

$$\begin{aligned} \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a+b+c) &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd}}} 1 \\ &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) U', \\ \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a-b-c) &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a-b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 - \sum_{\substack{ax+by+cz=n \\ a-b-c=-k \\ a,b,c,x,y,z \text{ odd}}} 1 \right) \quad (\text{as } F \text{ is odd}) \\ &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a-b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 - \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) (W' - V'), \\
 \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a+b-c) &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 - \sum_{\substack{ax+by+cz=n \\ a+b-c=-k \\ a,b,c,x,y,z \text{ odd}}} 1 \right) \\
 &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 - \sum_{\substack{ax+by+cz=n \\ a-b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 \right) \\
 &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) (V' - W'), \\
 \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a-b+c) &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a-b+c=k \\ a,b,c,x,y,z \text{ odd}}} 1 - \sum_{\substack{ax+by+cz=n \\ a-b+c=-k \\ a,b,c,x,y,z \text{ odd}}} 1 \right) \\
 &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 - \sum_{\substack{ax+by+cz=n \\ a-b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1 \right) \\
 &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) (V' - W').
 \end{aligned}$$

Therefore

$$\begin{aligned}
 &\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\
 &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) (U' + (W' - V') - (V' - W') - (V' - W')) \\
 &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) (U' - 3V' + 3W').
 \end{aligned}$$

Appealing to Lemma 2.3.4, we obtain

$$\begin{aligned}
 & \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\
 &= \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} F(k) \left(\frac{(k^2-1)}{8} \delta(n, k) - 3 \sum_{\substack{kx < n \\ x \text{ odd}}} \sigma(o(n-kx)) \right) \\
 &= \sum_{d|n} \frac{(d^2-1)}{8} F(d) - 3 \sum_{\substack{ax < n \\ x, a \text{ odd}}} \sigma(o(n-ax)) F(a),
 \end{aligned}$$

as required. $\square$

We remark that

$$\sum_{\substack{ax < n \\ x, a \text{ odd}}} \sigma(o(n-ax)) F(a) = \sum_{\substack{n=n_1+n_2 \\ n_1 \text{ odd}}} \sum_{d_1|n_1} \sum_{d_2|o(n_2)} F(d_1) d_2.$$

2.5 Applications of Liouville's 3-term odd formula

Theorem 2.5.1. *Let n be a positive odd integer. Then*

$$\sum_{\substack{ax < n \\ a, x \text{ odd}}} a \sigma(o(n-ax)) = \frac{1}{24} \sigma_3(n) - \frac{1}{24} \sigma(n),$$

where the sum on the left-hand side is taken over all $(a, x) \in \mathbb{N}^2$ with $ax < n$ and a, x odd.

Proof. Let $F(n) = n$, then

$$\begin{aligned}
 & F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c) \\
 &= (a+b+c) + (a-b-c) - (a+b-c) - (a-b+c)
 \end{aligned}$$

$$= 0.$$

Then, by Theorem 2.4.1, we have

$$\begin{aligned} \sum_{\substack{ax < n \\ a, x \text{ odd}}} a\sigma(o(n - ax)) &= \frac{1}{3} \sum_{d|n} \left(\frac{d^2 - 1}{8} \right) d \\ &= \frac{1}{24} \sum_{d|n} d^3 - \frac{1}{24} \sum_{d|n} d \\ &= \frac{1}{24} \sigma_3(n) - \frac{1}{24} \sigma(n). \end{aligned}$$

□

Theorem 2.5.2. *Let n be an odd positive integer. Then*

$$\sum_{\substack{n_1 + n_2 + n_3 = n \\ n_1, n_2, n_3 \text{ (odd)} \in \mathbb{N}}} \sigma(n_1)\sigma(n_2)\sigma(n_3) = \frac{1}{192}\sigma_5(n) - \frac{1}{192}\sigma_3(n) - \frac{1}{8} \sum_{\substack{n_1 + 2^s n_2 = n \\ s \geq 1, n_1, n_2 \text{ (odd)} \in \mathbb{N}}} \sigma(n_2)\sigma_3(n_1).$$

Proof. In order to prove this result, we apply Liouville's 3-term odd formula with

$F(n) = n^3$. Then

$$\begin{aligned} &F(a + b + c) + F(a - b - c) - F(a + b - c) - F(a - b + c) \\ &= (a + b + c)^3 + (a - b - c)^3 - (a + b - c)^3 - (a - b + c)^3 \\ &= 24abc. \end{aligned}$$

Therefore, the left-hand side of Liouville's 3-term odd formula gives

$$\begin{aligned} &\sum_{\substack{ax + by + cz = n \\ a, b, c, x, y, z \text{ odd}}} (F(a + b + c) + F(a - b - c) - F(a + b - c) - F(a - b + c)) \\ &= \sum_{\substack{ax + by + cz = n \\ a, b, c, x, y, z \text{ odd}}} 24abc \end{aligned}$$

$$\begin{aligned}
 &= 24 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \text{ odd}}} \sum_{a|n_1} a \sum_{b|n_2} b \sum_{c|n_3} c \\
 &= 24 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \text{ odd}}} \sigma(n_1)\sigma(n_2)\sigma(n_3).
 \end{aligned}$$

The right-hand side gives

$$\begin{aligned}
 &\frac{1}{8} \sum_{d|n} (d^2 - 1)d^3 - 3 \sum_{\substack{ax < n \\ a, x \text{ odd}}} \sigma(o(n - ax))a^3 \\
 &= \frac{1}{8}\sigma_5(n) - \frac{1}{8}\sigma_3(n) - 3 \sum_{\substack{n_1+2^s n_2=n \\ s \geq 1, n_1, n_2 \text{ odd}}} \sigma(n_2)\sigma_3(n_1).
 \end{aligned}$$

Equating the left-hand side and the right-hand side of Liouville's 3-term formula gives

$$192 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \text{ odd}}} \sigma(n_1)\sigma(n_2)\sigma(n_3) + 24 \sum_{\substack{n_1+2^s n_2=n \\ s \geq 1, n_1, n_2 \text{ odd}}} \sigma(n_2)\sigma_3(n_1) = \sigma_5(n) - \sigma_3(n),$$

as claimed. $\square$

Theorem 2.5.3. *Let n be an odd positive integer. Then*

$$1920 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \text{ (odd)} \in \mathbb{N}}} \sigma_3(n_1)\sigma(n_2)\sigma(n_3) + 24 \sum_{\substack{n_1+2^s n_2=n \\ s \geq 1, n_1, n_2 \text{ (odd)} \in \mathbb{N}}} \sigma(n_2)\sigma_5(n_1) = \sigma_7(n) - \sigma_5(n).$$

Proof. In order to prove this result, we apply Liouville's 3-term odd formula with

$F(n) = n^5$. Then

$$\begin{aligned}
 &F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c) \\
 &= (a+b+c)^5 + (a-b-c)^5 - (a+b-c)^5 - (a-b+c)^5 \\
 &= 80abc^3 + 80a^3bc + 80ab^3c.
 \end{aligned}$$

Therefore, the left-hand side of Liouville's 3-term odd formula gives

$$\begin{aligned}
 & \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\
 &= \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} 80abc^3 + 80a^3bc + 80ab^3c \\
 &= 80 \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \text{ odd}}} \sum_{a|n_1} a \sum_{b|n_2} b \sum_{c|n_3} c^3 + 80 \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \text{ odd}}} \sum_{a|n_1} a^3 \sum_{b|n_2} b \sum_{c|n_3} c \\
 &\quad + 80 \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \text{ odd}}} \sum_{a|n_1} a \sum_{b|n_2} b^3 \sum_{c|n_3} c \\
 &= 80 \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \text{ odd}}} \sigma(n_1)\sigma(n_2)\sigma_3(n_3) + 80 \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \text{ odd}}} \sigma_3(n_1)\sigma(n_2)\sigma(n_3) \\
 &\quad + 80 \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \text{ odd}}} \sigma(n_1)\sigma_3(n_2)\sigma_3(n_3) \\
 &= 240 \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \text{ odd}}} \sigma_3(n_1)\sigma(n_2)\sigma(n_3).
 \end{aligned}$$

The right-hand side gives

$$\begin{aligned}
 & \frac{1}{8} \sum_{d|n} (d^2 - 1)d^5 - 3 \sum_{\substack{ax < n \\ a,x \text{ odd}}} \sigma(o(n - ax))a^5 \\
 &= \frac{1}{8} \sigma_7(n) - \frac{1}{8} \sigma_5(n) - 3 \sum_{\substack{n_1+2^s n_2=n \\ s \geq 1, n_1, n_2 \text{ odd}}} \sigma(n_2)\sigma_5(n_1).
 \end{aligned}$$

Equating the left-hand side and the right-hand side of Liouville's 3-term odd formula gives

$$1920 \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \text{ odd}}} \sigma_3(n_1)\sigma(n_2)\sigma(n_3) + 24 \sum_{\substack{n_1+2^s n_2=n \\ s \geq 1, n_1, n_2 \text{ odd}}} \sigma(n_2)\sigma_5(n_1) = \sigma_7(n) - \sigma_5(n),$$

as claimed. $\square$

In our next application, we use Liouville's 3-term odd formula in conjunction with an identity involving theta functions to derive a formula for the number of representations of a positive integer n with $n \equiv 6 \pmod{8}$ as the sum of six odd squares. This application is the only place in this thesis where we use analytic rather than elementary methods. The formula for the number of representations of a positive integer $n \equiv 2 \pmod{4}$ as a sum of six odd squares is proved in an elementary manner in Lemma 4.4.4.

Jacobi's theta functions $\theta_1(x, q)$, $\theta_2(x, q)$, $\theta_3(x, q)$ and $\theta_4(x, q)$ are defined for $|q| < 1$ by

$$\begin{aligned}\theta_1(x, q) &= 2 \sum_{m=0}^{\infty} (-1)^m q^{\frac{(2m+1)^2}{4}} \sin(2m+1)x, \\ \theta_2(x, q) &= 2 \sum_{m=0}^{\infty} q^{\frac{(2m+1)^2}{4}} \cos(2m+1)x, \\ \theta_3(x, q) &= 1 + 2 \sum_{m=1}^{\infty} q^{m^2} \cos 2mx, \\ \theta_4(x, q) &= 1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cos 2mx,\end{aligned}$$

see for example [118, p. 464].

The following identity is well-known, see for example [47, p. 29]

$$\theta_2(0, q)\theta_3(0, q)\frac{\theta_1(x, q)}{\theta_4(x, q)} = 4 \sum_{m=0}^{\infty} \frac{q^{\frac{2m+1}{2}}}{1 - q^{2m+1}} \sin(2m+1)x. \quad (2.5.1)$$

Theorem 2.5.4. *Let n be a positive integer satisfying $n \equiv 6 \pmod{8}$. Set $n = 2N$ so that $N \equiv 3 \pmod{4}$. Then*

$$r_6^{0,6}(n) = -8 \sum_{d|N} \left(\frac{-4}{d} \right) d^2,$$

where $r_6^{0,6}(n)$ counts the number of representations of n as the sum of six odd squares.

(The more general notation $r_6^{a,6-a}(n)$ is defined in Definition 4.4.1.)

Proof. We begin by applying Liouville's 3-term odd formula with $F(n) = n^{2m+1}$, for $m = 0, 1, 2, 3, \dots$. We obtain

$$\begin{aligned} \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} ((a+b+c)^{2m+1} + (a-b-c)^{2m+1} - (a+b-c)^{2m+1} - (a-b+c)^{2m+1}) \\ = \frac{1}{8} \sum_{d|n} (d^2 - 1) d^{2m+1} - 3 \sum_{\substack{ax < n \\ a,x \text{ odd}}} \sigma(o(n-ax)) a^{2m+1}. \end{aligned} \quad (2.5.2)$$

We recall that for all u , we have

$$\sin u = \sum_{m=0}^{\infty} (-1)^m \frac{u^{2m+1}}{(2m+1)!}.$$

Multiplying (2.5.2) by $(-1)^m \frac{u^{2m+1}}{(2m+1)!}$ and summing over m , we get

$$\begin{aligned} \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} \sum_{m=0}^{\infty} (-1)^m \left((a+b+c)^{2m+1} + (a-b-c)^{2m+1} \right. \\ \left. - (a+b-c)^{2m+1} - (a-b+c)^{2m+1} \right) \frac{u^{2m+1}}{(2m+1)!} \\ = \frac{1}{8} \sum_{d|n} \sum_{m=0}^{\infty} (-1)^m (d^{2m+3} - d^{2m+1}) \frac{u^{2m+1}}{(2m+1)!} - 3 \sum_{\substack{ax < n \\ a,x \text{ odd}}} \sigma(o(n-ax)) \sum_{m=0}^{\infty} (-1)^m \frac{(au)^{2m+1}}{(2m+1)!}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (\sin(a+b+c)u + \sin(a-b-c)u - \sin(a+b-c)u - \sin(a-b+c)u) \\ = \frac{1}{8} \sum_{d|n} d^2 \sin du - \frac{1}{8} \sum_{d|n} \sin du - 3 \sum_{\substack{ax < n \\ a,x \text{ odd}}} \sigma(o(n-ax)) \sin au. \end{aligned}$$

From the addition and subtraction formulae for $\sin u$ and $\cos u$, we have

$$\begin{aligned} & \sin(a+b+c)u + \sin(a-b-c)u - \sin(a+b-c)u - \sin(a-b+c)u \\ &= -4 \sin au \sin bu \sin cu. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (-4 \sin au \sin bu \sin cu) \\ &= \frac{1}{8} \sum_{d|n} d^2 \sin du - \frac{1}{8} \sum_{d|n} \sin du - 3 \sum_{\substack{ax < n \\ a,x \text{ odd}}} \sigma(o(n-ax)) \sin au. \quad (2.5.3) \end{aligned}$$

We are able to use this identity to simplify the following sum. We obtain

$$\begin{aligned} & \left(\sum_{l=0}^{\infty} \frac{q^{2l+1}}{1-q^{4l+2}} \sin(2l+1)u \right)^3 \\ &= \left(\sum_{l=0}^{\infty} q^{2l+1} \sin(2l+1)u \sum_{h=0}^{\infty} (q^{4l+2})^h \right)^3 \\ &= \left(\sum_{l,h=0}^{\infty} q^{(2l+1)(2h+1)} \sin(2l+1)u \right)^3 \\ &= \sum_{l,m,n,h,k,j=0}^{\infty} q^{(2l+1)(2h+1)+(2m+1)(2k+1)+(2n+1)(2j+1)} \sin(2l+1)u \sin(2m+1)u \sin(2n+1)u \\ &= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} q^N \sum_{\substack{l,m,n,h,k,j=0 \\ (2l+1)(2h+1)+(2m+1)(2k+1)+(2n+1)(2j+1)=N}} \sin(2l+1)u \sin(2m+1)u \sin(2n+1)u \\ &= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} q^N \sum_{\substack{ax+by+cz=N \\ a,b,c,x,y,z \text{ odd}}} \sin au \sin bu \sin cu \\ &= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} q^N \left(-\frac{1}{32} \sum_{d|N} d^2 \sin du + \frac{1}{32} \sum_{d|N} \sin du + \frac{3}{4} \sum_{\substack{ax < N \\ a,x \text{ odd}}} \sigma(o(N-ax)) \sin au \right). \end{aligned}$$

We now take $u = \pi/2$. Then, for m odd, we have

$$\sin mu = \sin m(\pi/2) = (-1)^{(m-1)/2}.$$

Thus

$$\begin{aligned} & \left(\sum_{l=0}^{\infty} \frac{(-1)^l q^{2l+1}}{1 - q^{4l+2}} \right)^3 \\ &= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} q^N \left(-\frac{1}{32} \sum_{d|N} (-1)^{(d-1)/2} d^2 + \frac{1}{32} \sum_{d|N} (-1)^{(d-1)/2} \right. \\ & \quad \left. + \frac{3}{4} \sum_{\substack{ax < N \\ a, x \text{ odd}}} \sigma(o(N - ax)) (-1)^{(a-1)/2} \right). \end{aligned}$$

Setting $u = \pi/2$ in (2.5.1) and replacing q by q^2 , we obtain

$$\begin{aligned} & \sum_{l=0}^{\infty} \frac{(-1)^l q^{2l+1}}{1 - q^{4l+2}} \\ &= \frac{1}{4} \theta_2(0, q^2) \theta_3(0, q^2) \frac{\theta_1(\pi/2, q^2)}{\theta_4(\pi/2, q^2)} \\ &= \frac{\frac{1}{4} \left(2 \sum_{l=0}^{\infty} q^{\frac{(2l+1)^2}{2}} \right) \left(1 + 2 \sum_{l=1}^{\infty} q^{2l^2} \right) \left(2 \sum_{l=0}^{\infty} q^{\frac{(2l+1)^2}{2}} (-1)^l (-1)^l \right)}{\left(1 + 2 \sum_{l=1}^{\infty} q^{2l^2} (-1)^l \cos \pi l \right)} \\ &= \frac{\frac{1}{4} \left(2 \sum_{l=0}^{\infty} q^{\frac{(2l+1)^2}{2}} \right) \left(1 + 2 \sum_{l=1}^{\infty} q^{2l^2} \right) \left(2 \sum_{l=0}^{\infty} q^{\frac{(2l+1)^2}{2}} \right)}{\left(1 + 2 \sum_{l=1}^{\infty} q^{2l^2} \right)} \\ &= \left(\sum_{l=0}^{\infty} q^{\frac{(2l+1)^2}{2}} \right)^2. \end{aligned}$$

Therefore

$$\begin{aligned}
 \left(\sum_{l=0}^{\infty} q^{\frac{(2l+1)^2}{2}} \right)^6 &= \left(\sum_{l=0}^{\infty} \frac{(-1)^l q^{2l+1}}{1 - q^{4l+2}} \right)^3 \\
 &= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} q^N \left(-\frac{1}{32} \sum_{d|N} (-1)^{(d-1)/2} d^2 + \frac{1}{32} \sum_{d|N} (-1)^{(d-1)/2} \right. \\
 &\quad \left. + \frac{3}{4} \sum_{\substack{ax < N \\ a, x \text{ odd}}} \sigma(o(N - ax)) (-1)^{(a-1)/2} \right). \quad (2.5.4)
 \end{aligned}$$

The left-hand side (*LHS*) of (2.5.4) gives

$$\begin{aligned}
 LHS &= \sum_{l_1, \dots, l_6=0}^{\infty} q^{\frac{(2l_1+1)^2 + \dots + (2l_6+1)^2}{2}} \\
 &= \sum_{n=1}^{\infty} q^{\frac{n}{2}} \sum_{\substack{a_1^2 + \dots + a_6^2 = n \\ a_1, \dots, a_6 \text{ (odd)} > 0}} 1 \\
 &= \sum_{\substack{n=1 \\ n \equiv 6 \pmod{8}}}^{\infty} q^{\frac{n}{2}} \left(\frac{r_6^{0,6}(n)}{2^6} \right) \\
 &= \frac{1}{64} \sum_{\substack{N=1 \\ N \equiv 3 \pmod{4}}}^{\infty} q^N r_6^{0,6}(2N).
 \end{aligned}$$

In the above calculation we divided by 2^6 since $a_1, \dots, a_6$ are required to be positive, whereas $r_6^{0,6}(n)$ counts both positive and negative a_i in $a_1^2 + \dots + a_6^2 = n$.

The right-hand side (*RHS*) of (2.5.4) gives

$$\begin{aligned}
 RHS &= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} q^N \left(-\frac{1}{32} \sum_{d|N} (-1)^{(d-1)/2} d^2 + \frac{1}{32} \sum_{d|N} (-1)^{(d-1)/2} \right. \\
 &\quad \left. + \frac{3}{4} \sum_{\substack{ax < N \\ a, x \text{ odd}}} \sigma(o(N - ax)) (-1)^{(a-1)/2} \right) \\
 &= \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} q^N \left(-\frac{1}{32} \sum_{d|N} \left(\frac{-4}{d} \right) d^2 + \frac{1}{32} \sum_{d|N} \left(\frac{-4}{d} \right) + \frac{3}{4} \sum_{\substack{n_1+n_2=N \\ n_1 \text{ odd}, n_2 \text{ even}}} \sigma(o(n_2)) \left(\frac{-4}{d_1} \right) \right).
 \end{aligned}$$

When $N \equiv 3 \pmod{4}$,

$$\begin{aligned}
& -\frac{1}{32} \sum_{d|N} \left(\frac{-4}{d}\right) d^2 + \frac{1}{32} \sum_{d|N} \left(\frac{-4}{d}\right) + \frac{3}{4} \sum_{\substack{n_1+n_2=N \\ n_1 \text{ odd}, n_2 \text{ even}}} \sigma(o(n_2)) \left(\frac{-4}{d_1}\right) \\
& = -\frac{1}{32} \sum_{d|N} \left(\frac{-4}{d}\right) d^2 + \frac{r_2(N)}{128} - \frac{3}{32} \sum_{d|n} \left(\frac{-4}{d}\right) d^2 \quad (\text{by Lemma 4.2.2}) \\
& = -\frac{1}{8} \sum_{d|N} \left(\frac{-4}{d}\right) d^2.
\end{aligned}$$

We note that Lemma 4.2.2 is an elementary lemma which is a consequence of Corollary

1.6.1. Equating the left-hand side and the right-hand side, we obtain

$$\begin{aligned}
& \frac{1}{64} \sum_{\substack{N=1 \\ N \equiv 3 \pmod{4}}}^{\infty} q^N r_6^{0,6}(2N) = \sum_{\substack{N=1 \\ N \equiv 3 \pmod{4}}}^{\infty} q^N \left(\frac{-1}{8} \sum_{d|N} \left(\frac{-4}{d}\right) d^2 \right) \\
& + \sum_{\substack{N=1 \\ N \equiv 1 \pmod{4}}}^{\infty} q^N \left(-\frac{1}{32} \sum_{d|N} \left(\frac{-4}{d}\right) d^2 + \frac{1}{32} \sum_{d|N} \left(\frac{-4}{d}\right) + \frac{3}{4} \sum_{\substack{n_1+n_2=N \\ n_1 \text{ odd}, n_2 \text{ even}}} \sigma(o(n_2)) \left(\frac{-4}{d_1}\right) \right).
\end{aligned}$$

Therefore

$$-\frac{1}{32} \sum_{d|N} \left(\frac{-4}{d}\right) d^2 + \frac{1}{32} \sum_{d|N} \left(\frac{-4}{d}\right) + \frac{3}{4} \sum_{\substack{n_1+n_2=N \\ n_1 \text{ odd}, n_2 \text{ even}}} \sigma(o(n_2)) \left(\frac{-4}{d_1}\right) = 0,$$

for $N \equiv 1 \pmod{4}$, and

$$r_6^{0,6}(n) = r_6^{0,6}(2N) = -8 \sum_{d|N} \left(\frac{-4}{d}\right) d^2, \quad n \equiv 6 \pmod{8},$$

as required. □

2.6 Sums of triangular numbers

The triangular numbers are the nonnegative integers

$$T_k = \frac{1}{2}k(k+1), \quad k \in \mathbb{N} \cup \{0\},$$

so that

$$T_0 = 0, T_1 = 1, T_2 = 3, T_3 = 6, T_4 = 10, T_5 = 15, \dots$$

Definition 2.6.1. *Let*

$$\Delta = \{T_k \mid k \in \mathbb{N} \cup \{0\}\},$$

and for $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, let

$$\delta_m(n) = \text{card}\{(t_1, \dots, t_m) \in \Delta^m \mid n = t_1 + \dots + t_m\}.$$

Then $\delta_m(n)$ counts the number of representations of n as the sum of m triangular numbers.

It is an easily proved classical result that

$$\delta_2(n) = \sum_{a|4n+1} \left(\frac{-4}{a} \right), \quad n \in \mathbb{N} \cup \{0\},$$

see for example Adiga [1], Ewell [31], [32], Ono, Robins and Wahl [94]. Gauss showed that every nonnegative integer is the sum of three triangular numbers and a somewhat complicated formula for $\delta_3(n)$ is known, see for example Ewell [30], Ono, Robins and Wahl [94]. The evaluation of $\delta_4(n)$ was known to Legendre, namely,

$$\delta_4(n) = \sigma(2n + 1), \quad n \in \mathbb{N} \cup \{0\}.$$

Ono, Robins and Wahl [94, p. 80] have proved this result using modular forms. An elementary proof has been given by Huard, Ou, Spearman and Williams [43, p. 259].

Ono, Robins and Wahl [94] have used modular forms to give formulae for $\delta_k(n)$ for $k = 6, 8, 10, 12$ and 24 . Huard, Ou, Spearman and Williams have given elementary proofs of the formulae for $\delta_6(n)$ and $\delta_8(n)$ [43, pp. 262, 265]. In a paper to appear Huard and Williams [44] treat $\delta_{16}(n)$ and $\delta_{24}(n)$. For a related result on triangular numbers see Williams [120]

Using Theorem 2.5.2 and the above formula for $\delta_4(n)$, we obtain a new evaluation of $\delta_{12}(n)$.

Theorem 2.6.1. *Let n be a positive integer. Then*

$$\delta_{12}(n) = \frac{1}{192}\sigma_5(2n+3) - \frac{1}{192}\sigma_3(2n+3) - \frac{1}{8} \sum_{\substack{n_1+2^s n_2=2n+3 \\ s \geq 1, n_1, n_2 \text{ odd}}} \sigma_3(n_1)\sigma(n_2).$$

Proof. As

$$\delta_4(n) = \sigma(2n+1), \quad n \in \mathbb{N} \cup \{0\},$$

we have

$$\begin{aligned} \delta_{12}(n) &= \sum_{\substack{m_1+m_2+m_3=n \\ m_1, m_2, m_3 \geq 0}} \delta_4(m_1)\delta_4(m_2)\delta_4(m_3) \\ &= \sum_{\substack{m_1+m_2+m_3=n \\ m_1, m_2, m_3 \geq 0}} \sigma(2m_1+1)\sigma(2m_2+1)\sigma(2m_3+1) \\ &= \sum_{\substack{n_1+n_2+n_3=2n+3 \\ n_1, n_2, n_3 \text{ odd}}} \sigma(n_1)\sigma(n_2)\sigma(n_3). \end{aligned}$$

Appealing to Theorem 2.5.2 we obtain the assertion of the theorem. $\square$

Chapter 3

Liouville's 3-term formula

In this chapter we state and prove a 3-term formula of Liouville type, analogous to Liouville's 3-term odd formula (Theorem 2.4.1) but without the restriction that the variables a, b, c, x, y, z in $ax + by + cz = n$ be odd.

We prove the following result.

Let n be a positive integer, and let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function. Then

$$\begin{aligned} & \sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ &= \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) F(d) \\ & \quad + 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) F(k) - 6 \sum_{\substack{n=n_1+n_2 \\ n_1, n_2 \in \mathbb{N}}} \sigma(n_1) \sum_{d|n_2} F(d), \end{aligned}$$

where the sum on the left-hand side is taken over all $(a, b, c, x, y, z) \in \mathbb{N}^6$ such that $ax + by + cz = n$.

This formula was not stated by Liouville, nor as far as we can determine does it

appear in the literature, so we believe it to be new.

3.1 The equation $kx + ey = n$ and the quantities R_1 ,

$$R_2, R_3, S_1, S_2, S_3$$

In this section we introduce the quantities $R_1, R_2, R_3, S_1, S_2, S_3$ formed from $L_1, L_2, L_3, M_1, M_2, M_3$ respectively by taking the sums over y rather than 1. We note that $R_1, R_2, R_3, S_1, S_2, S_3$ are not quite analogous to $R_1(a, b, c), R_2(a, b, c), R_3(a, b, c), S_1(a, b, c), S_2(a, b, c), S_3(a, b, c)$ as the former are sums over y whereas the latter are sums over e .

Definition 3.1.1.

$$R_1 = \sum_{\substack{kx+ey=n \\ e < k}} y, \quad R_2 = \sum_{\substack{kx+ey=n \\ e > k}} y, \quad R_3 = \sum_{\substack{kx+ey=n \\ e=k}} y.$$

Definition 3.1.2.

$$S_1 = \sum_{\substack{kx+ey=n \\ x < y}} y, \quad S_2 = \sum_{\substack{kx+ey=n \\ x > y}} y, \quad S_3 = \sum_{\substack{kx+ey=n \\ x=y}} y.$$

The following result follows immediately from Definitions 3.1.1 and 3.1.2.

Lemma 3.1.1.

$$R_1 + R_2 + R_3 = S_1 + S_2 + S_3.$$

Next we evaluate R_3 .

Lemma 3.1.2.

$$R_3 = \delta(n, k) \frac{(n-k)n}{2k^2}.$$

Proof. We have

$$\begin{aligned} R_3 &= \sum_{\substack{kx+ey=n \\ e=k}} y = \sum_{k(x+y)=n} y = \delta(n, k) \left(1 + 2 + \cdots + \left(\frac{n}{k} - 1\right)\right) \\ &= \delta(n, k) \frac{\left(\frac{n}{k} - 1\right) \left(\frac{n}{k}\right)}{2} \\ &= \delta(n, k) \frac{(n-k)n}{2k^2}. \end{aligned}$$

□

S_3 can be calculated by a similar method.

Lemma 3.1.3.

$$S_3 = \sum_{\substack{d|n \\ d>k}} \frac{n}{d}.$$

Proof. We have

$$S_3 = \sum_{\substack{kx+ey=n \\ x=y}} y = \sum_{(k+e)x=n} x = \sum_{\substack{d|n \\ d>k}} \frac{n}{d}.$$

□

Our next lemma shows that R_2 and S_2 are equal.

Lemma 3.1.4.

$$R_2 = S_2.$$

Proof. We have

$$S_2 = \sum_{\substack{kx+ey=n \\ x>y}} y = \sum_{k(x+y)+ey=n} y = \sum_{kx+(k+e)y=n} y = \sum_{\substack{kx+ey=n \\ e>k}} y = R_2.$$

□

3.2 The equation $kx + ey = n$ and the quantities P_1 ,

$$P_2, P_3, Q_1, Q_2, Q_3$$

In this section we introduce the quantities $P_1, P_2, P_3, Q_1, Q_2, Q_3$ formed from $L_1, L_2, L_3, M_1, M_2, M_3$ respectively by taking the sums over x rather than 1.

Definition 3.2.1.

$$P_1 = \sum_{\substack{kx+ey=n \\ e<k}} x, \quad P_2 = \sum_{\substack{kx+ey=n \\ e>k}} x, \quad P_3 = \sum_{\substack{kx+ey=n \\ e=k}} x.$$

Definition 3.2.2.

$$Q_1 = \sum_{\substack{kx+ey=n \\ x<y}} x, \quad Q_2 = \sum_{\substack{kx+ey=n \\ x>y}} x, \quad Q_3 = \sum_{\substack{kx+ey=n \\ x=y}} x.$$

The following result follows immediately from Definitions 3.2.1 and 3.2.2.

Lemma 3.2.1.

$$P_1 + P_2 + P_3 = Q_1 + Q_2 + Q_3.$$

Next we evaluate P_3 .

Lemma 3.2.2.

$$P_3 = \delta(n, k) \frac{(n-k)n}{2k^2}.$$

Proof. We have

$$P_3 = \sum_{\substack{kx+ey=n \\ e=k}} x = \sum_{k(x+y)=n} x = \delta(n, k) \frac{\left(\frac{n}{k} - 1\right) \left(\frac{n}{k}\right)}{2} = \delta(n, k) \frac{(n-k)n}{2k^2}.$$

□

Q_3 can be calculated by a similar method.

Lemma 3.2.3.

$$Q_3 = \sum_{\substack{d|n \\ d>k}} \frac{n}{d}.$$

Proof. We have

$$Q_3 = \sum_{\substack{kx+ey=n \\ x=y}} x = \sum_{(k+e)x=n} x = \sum_{\substack{d|n \\ d>k}} \frac{n}{d}.$$

□

Our next lemma gives an important relationship between P_2 and Q_2 .

Lemma 3.2.4.

$$Q_2 = P_2 + R_2.$$

Proof. We have

$$Q_2 = \sum_{\substack{kx+ey=n \\ x>y}} x = \sum_{k(x+y)+ey=n} (x+y) = \sum_{kx+(k+e)y=n} (x+y)$$

$$= \sum_{\substack{kx+ey=n \\ e>k}} (x+y) = P_2 + R_2.$$

□

Our final lemma of this section evaluates the quantity $P_1 - Q_1 - S_2$, which will be needed in Section 3.5.

Lemma 3.2.5.

$$P_1 - Q_1 - S_2 = \sum_{\substack{d|n \\ d>k}} \frac{n}{d} - \delta(n, k) \left(\frac{(n-k)n}{2k^2} \right).$$

Proof. We have

$$\begin{aligned} P_1 - Q_1 - S_2 &= P_1 - Q_1 - R_2 && \text{(by Lemma 3.1.4)} \\ &= P_1 - Q_1 - (Q_2 - P_2) && \text{(by Lemma 3.2.4)} \\ &= P_1 + P_2 - Q_1 - Q_2 \\ &= Q_3 - P_3 && \text{(by Lemma 3.2.1)} \\ &= \sum_{\substack{d|n \\ d>k}} \frac{n}{d} - \delta(n, k) \left(\frac{(n-k)n}{2k^2} \right), \end{aligned}$$

by Lemmas 3.2.2 and 3.2.3.

□

3.3 Solutions of $kx + by + cz = n$ (k, n fixed integers)

Throughout this section k and n are positive integers that can be either even or odd.

We are concerned with solutions $(x, b, y, c, z) \in \mathbb{N}^5$ of the equation

$$kx + by + cz = n.$$

For such solutions, we do not have the simplifications of the previous chapter. We begin with the following definition, analogous to Definition 2.2.1. Once again, our ultimate goal will be to determine $A_1 - 2B_1 + 2C_1$, see Proposition 3.3.1.

Definition 3.3.1.

$$A_1 = \sum_{\substack{kx+by+cz=n \\ b+c < k}} 1, \quad A_2 = \sum_{\substack{kx+by+cz=n \\ b+c > k}} 1, \quad A_3 = \sum_{\substack{kx+by+cz=n \\ b+c=k}} 1.$$

Our next lemma evaluates A_3 in terms of P_1 and L_1 .

Lemma 3.3.1.

$$A_3 = 2(P_1 - L_1) + (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k).$$

Proof. We have

$$\begin{aligned} A_3 &= \sum_{\substack{kx+by+cz=n \\ b+c=k}} 1 \\ &= \sum_{\substack{kx+by+cz=n \\ b+c=k \\ y < z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c=k \\ y > z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c=k \\ y=z}} 1 \\ &= \sum_{\substack{kx+(k-c)y+cz=n \\ c < k, y < z}} 1 + \sum_{\substack{kx+by+(k-b)z=n \\ b < k, y > z}} 1 + \sum_{\substack{kx+(b+c)y=n \\ b+c=k}} 1 \\ &= \sum_{\substack{k(x+y)+c(z-y)=n \\ c < k, y < z}} 1 + \sum_{\substack{k(x+z)+b(y-z)=n \\ b < k, y > z}} 1 + \sum_{k(x+y)=n} (k-1) \\ &= \sum_{\substack{k(x+y)+cz'=n \\ c < k}} 1 + \sum_{\substack{k(x+z)+bz'=n \\ b < k}} 1 + (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) \\ &= \sum_{\substack{kx'+cz'=n \\ c < k}} (x' - 1) + \sum_{\substack{kx'+bz'=n \\ b < k}} (x' - 1) + (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) \end{aligned}$$

$$\begin{aligned}
 &= 2 \sum_{\substack{kx+cy=n \\ e < k}} (x-1) + (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) \\
 &= 2(P_1 - L_1) + (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k),
 \end{aligned}$$

by Definition 1.2.1 and 3.2.1. □

The following definition is analogous to Definition 2.2.2, but in this case we may have $b - c = k$ and $x = z$.

Definition 3.3.2.

$$B_1 = \sum_{\substack{kx+by+cz=n \\ b-c < k, x < z}} 1, \quad B_2 = \sum_{\substack{kx+by+cz=n \\ b-c > k, x < z}} 1, \quad B_3 = \sum_{\substack{kx+by+cz=n \\ b-c = k, x < z}} 1,$$

$$B_4 = \sum_{\substack{kx+by+cz=n \\ b-c < k, x > z}} 1, \quad B_5 = \sum_{\substack{kx+by+cz=n \\ b-c > k, x > z}} 1, \quad B_6 = \sum_{\substack{kx+by+cz=n \\ b-c = k, x > z}} 1,$$

$$B_7 = \sum_{\substack{kx+by+cz=n \\ b-c < k, x = z}} 1, \quad B_8 = \sum_{\substack{kx+by+cz=n \\ b-c > k, x = z}} 1, \quad B_9 = \sum_{\substack{kx+by+cz=n \\ b-c = k, x = z}} 1.$$

The following three lemmas give important relationships for the quantities $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9$.

Lemma 3.3.2.

$$B_3 = Q_1 - M_1.$$

Proof. We have

$$\begin{aligned}
 B_3 &= \sum_{\substack{kx+by+cz=n \\ b-c=k, \, x < z}} 1 \\
 &= \sum_{\substack{kx+(c+k)y+cz=n \\ x < z}} 1 \\
 &= \sum_{\substack{k(x+y)+c(y+z)=n \\ x < z}} 1 \\
 &= \sum_{\substack{kx'+cy'=n \\ x' < y', \, y < x'}} 1 \quad (x' = x + y, \, y' = y + z) \\
 &= \sum_{\substack{kx+cy=n \\ x < y}} (x - 1) \\
 &= Q_1 - M_1,
 \end{aligned}$$

as asserted. □

Lemma 3.3.3.

$$B_6 = S_2 - M_2.$$

Proof. We have

$$\begin{aligned}
 B_6 &= \sum_{\substack{kx+by+cz=n \\ b-c=k, \, x > z}} 1 \\
 &= \sum_{\substack{kx+(c+k)y+cz=n \\ x > z}} 1 \\
 &= \sum_{\substack{k(x+y)+c(y+z)=n \\ x > z}} 1
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{kx'+cy'=n \\ x'>y', y<y'}} 1 \quad (x' = x + y, y' = y + z) \\
 &= \sum_{\substack{kx+cy=n \\ x>y}} (y - 1) \\
 &= S_2 - M_2,
 \end{aligned}$$

as asserted. □

Lemma 3.3.4.

$$A_2 = 2B_4 + \sum_{\substack{kx+ey=n \\ e>k}} e - L_2.$$

Proof. We have

$$\begin{aligned}
 A_2 &= \sum_{\substack{kx+by+cz=n \\ b+c>k}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ b+c>k, y>z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k, y<z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k, y=z}} 1 \\
 &= 2 \sum_{\substack{kx+by+cz=n \\ b+c>k, y>z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k, y=z}} 1.
 \end{aligned}$$

The last step is obtained by observing the symmetry between b and c , and between y and z . By manipulating the two above sums, we are able to complete the proof of this lemma. Firstly,

$$\sum_{\substack{kx+by+cz=n \\ b+c>k, y=z}} 1 = \sum_{\substack{kx+(b+c)y=n \\ b+c>k}} 1$$

$$\begin{aligned}
&= \sum_{\substack{kx+ey=n \\ e>k \\ e=b+c}} 1 \\
&= \sum_{\substack{kx+ey=n \\ e>k}} (e-1) \\
&= \sum_{\substack{kx+ey=n \\ e>k}} e - L_2,
\end{aligned}$$

and secondly,

$$\begin{aligned}
B_4 &= \sum_{\substack{kx+by+cz=n \\ b-c < k, \ x > z}} 1 \quad (x = z + x') \\
&= \sum_{\substack{k(z+x')+by+cz=n \\ b-c < k}} 1 \\
&= \sum_{\substack{kx+by+(k+c)z=n \\ b < k+c}} 1 \quad (e = k + c) \\
&= \sum_{\substack{kx+by+ez=n \\ b < e, \ e > k}} 1 \quad (e = b + c') \\
&= \sum_{\substack{kx+by+(b+c')z=n \\ b+c' > k}} 1 \\
&= \sum_{\substack{kx+b(y+z)+cz=n \\ b+c > k}} 1 \quad (y' = y + z) \\
&= \sum_{\substack{kx+by'+cz=n \\ b+c > k, \ y' > z}} 1 \\
&= \sum_{\substack{kx+by+cz=n \\ b+c > k, \ y > z}} 1.
\end{aligned}$$

This completes the proof of the lemma. □

In the following definition, C_1 is analogous to C'_1 in Definition 2.2.3. As we will see, the quantities C_2 through C_{13} are necessary for the proof of Proposition 3.3.1.

Definition 3.3.3.

$$C_1 = \sum_{\substack{kx+by+cz=n \\ x < y < z}} 1, \quad C_2 = \sum_{\substack{kx+by+cz=n \\ x < z < y}} 1, \quad C_3 = \sum_{\substack{kx+by+cz=n \\ y < x < z}} 1,$$

$$C_4 = \sum_{\substack{kx+by+cz=n \\ y < z < x}} 1, \quad C_5 = \sum_{\substack{kx+by+cz=n \\ z < x < y}} 1, \quad C_6 = \sum_{\substack{kx+by+cz=n \\ z < y < x}} 1,$$

$$C_7 = \sum_{\substack{kx+by+cz=n \\ x < y = z}} 1, \quad C_8 = \sum_{\substack{kx+by+cz=n \\ y < x = z}} 1, \quad C_9 = \sum_{\substack{kx+by+cz=n \\ z < x = y}} 1,$$

$$C_{10} = \sum_{\substack{kx+by+cz=n \\ x = y < z}} 1, \quad C_{11} = \sum_{\substack{kx+by+cz=n \\ x = z < y}} 1, \quad C_{12} = \sum_{\substack{kx+by+cz=n \\ y = z < x}} 1,$$

$$C_{13} = \sum_{\substack{kx+by+cz=n \\ x=y=z}} 1.$$

The following lemma follows immediately from Definitions 3.3.1, 3.3.2 and 3.3.3.

Lemma 3.3.5.

$$\sum_{i=1}^3 A_i = \sum_{i=1}^9 B_i = \sum_{i=1}^{13} C_i.$$

The equalities in the next lemma follow by symmetry.

Lemma 3.3.6.

$$C_1 = C_2,$$

$$C_3 = C_5,$$

$$C_4 = C_6,$$

$$C_8 = C_9,$$

$$C_{10} = C_{11}.$$

Lemma 3.3.7.

$$B_2 = C_3.$$

Proof. We have

$$\begin{aligned}
 C_3 &= \sum_{\substack{kx+by+cz=n \\ y < x < z}} 1 \quad (x = y + x', \quad z = y + z') \\
 &= \sum_{\substack{k(y+x')+by+c(y+z')=n \\ x' < z'}} 1 \\
 &= \sum_{\substack{kx+(b+c+k)y+cz=n \\ x < z}} 1 \quad (b' = k + b + c) \\
 &= \sum_{\substack{kx+b'y+cz=n \\ x < z, \quad b' > k+c}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ x < z, \quad b-c > k}} 1 \\
 &= B_2,
 \end{aligned}$$

as claimed. □

Lemma 3.3.8.

$$B_5 = C_4.$$

Proof. We have

$$\begin{aligned}
 C_4 &= \sum_{\substack{kx+by+cz=n \\ y < z < x}} 1 \quad (x = y + x', z = y + z') \\
 &= \sum_{\substack{k(y+x')+by+c(y+z')=n \\ z' < x'}} 1 \\
 &= \sum_{\substack{kx+(b+c+k)y+cz=n \\ z < x}} 1 \quad (b' = k + b + c) \\
 &= \sum_{\substack{kx+b'y+cz=n \\ z < x, b' > k+c}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ z < x, b-c > k}} 1 \\
 &= B_5,
 \end{aligned}$$

as asserted. □

Lemma 3.3.9.

$$C_7 = \sum_{\substack{kx+cy=n \\ x < y}} e - M_1.$$

Proof. We have

$$C_7 = \sum_{\substack{kx+by+cz=n \\ x < y=z}} 1$$

$$\begin{aligned}
 &= \sum_{\substack{kx+(b+c)y=n \\ x < y}} 1 \\
 &= \sum_{e=b+c} \sum_{\substack{kx+ey=n \\ x < y}} 1 \\
 &= \sum_e (e-1) \sum_{\substack{kx+ey=n \\ x < y}} 1 \\
 &= \sum_{\substack{kx+ey=n \\ x < y}} e - \sum_{\substack{kx+ey=n \\ x < y}} 1 \\
 &= \sum_{\substack{kx+ey=n \\ x < y}} e - M_1,
 \end{aligned}$$

by Definition 1.2.2.

□

Lemma 3.3.10.

$$C_{12} = \sum_{\substack{kx+ey=n \\ x > y}} e - M_2.$$

Proof. We have

$$\begin{aligned}
 C_{12} &= \sum_{\substack{kx+by+cz=n \\ y=z < x}} 1 \\
 &= \sum_{\substack{kx+(b+c)y=n \\ x > y}} 1 \\
 &= \sum_{e=b+c} \sum_{\substack{kx+ey=n \\ x > y}} 1 \\
 &= \sum_e (e-1) \sum_{\substack{kx+ey=n \\ x > y}} 1 \\
 &= \sum_{\substack{kx+ey=n \\ x > y}} e - \sum_{\substack{kx+ey=n \\ x > y}} 1
 \end{aligned}$$

$$= \sum_{\substack{kx+ey=n \\ x>y}} e - M_2,$$

by Definition 1.2.2. □

Lemma 3.3.11.

$$C_{13} = \sum_{\substack{d|n \\ d>k}} (d - k - 1).$$

Proof. We have

$$\begin{aligned} C_{13} &= \sum_{\substack{kx+by+cz=n \\ x=y=z}} 1 \\ &= \sum_{(k+b+c)x=n} 1 \\ &= \sum_{\substack{(k+e)x=n \\ e=b+c}} 1 \\ &= \sum_{(k+e)x=n} (e - 1) \\ &= \sum_{\substack{d|n \\ d=k+e}} (e - 1) \\ &= \sum_{\substack{d|n \\ d>k}} (d - k - 1), \end{aligned}$$

as asserted. □

Lemma 3.3.12.

$$B_7 + B_8 + B_9 = C_8 + C_{10} + C_{13}.$$

Proof. We begin by noting that by Definition 3.3.2

$$\sum_{\substack{kx+by+cz=n \\ x=z}} 1 = \sum_{\substack{kx+by+cz=n \\ b-c < k \\ x=z}} 1 + \sum_{\substack{kx+by+cz=n \\ b-c > k \\ x=z}} 1 + \sum_{\substack{kx+by+cz=n \\ b-c = k \\ x=z}} 1 = B_7 + B_8 + B_9.$$

and also by Definition 3.3.3

$$\sum_{\substack{kx+by+cz=n \\ x=z}} 1 = \sum_{\substack{kx+by+cz=n \\ x=z < y}} 1 + \sum_{\substack{kx+by+cz=n \\ y < x=z}} 1 + \sum_{\substack{kx+by+cz=n \\ x=y=z}} 1 = C_{10} + C_8 + C_{13},$$

Therefore

$$B_7 + B_8 + B_9 = C_8 + C_{10} + C_{13}$$

as required. $\square$

Proposition 3.3.1.

$$\begin{aligned} A_1 - 2B_1 + 2C_1 = & - \sum_{\substack{kx+ey=n \\ e > k}} e - \sum_{\substack{kx+ey=n \\ x \neq y}} e - (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) \\ & + \sum_{\substack{d|n \\ d > k}} \left(d - 2\frac{n}{d} - k - 1 \right) + \delta(n, k) \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} \right) - M_1 + 2L_1. \end{aligned}$$

Proof. By Lemmas 3.3.2, 3.3.3, 3.3.7, 3.3.8 and 3.3.12, we obtain

$$\sum_{i=1}^9 B_i = B_1 + C_3 + Q_1 - M_1 + B_4 + C_4 + S_2 - M_2 + C_8 + C_{10} + C_{13}.$$

By Lemma 3.3.6 we have

$$\sum_{i=1}^{13} C_i = 2C_1 + 2C_3 + 2C_4 + 2C_8 + 2C_{10} + C_7 + C_{12} + C_{13}.$$

From Lemma 3.3.5 we deduce

$$\sum_{i=1}^3 A_i - 2 \sum_{i=1}^9 B_i + \sum_{i=1}^{13} C_i = 0.$$

Substituting the expressions obtained above for $\sum_{i=1}^9 B_i$ and $\sum_{i=1}^{13} C_i$ into this equation, we obtain after some cancellation

$$\begin{aligned} A_1 + A_2 + A_3 - 2B_1 - 2Q_1 + 2M_1 - 2B_4 - 2S_2 + 2M_2 \\ + 2C_1 + C_7 + C_{12} - C_{13} = 0. \end{aligned}$$

Rearranging this equation, we obtain

$$\begin{aligned} A_1 - 2B_1 + 2C_1 \\ = (-A_2 + 2B_4) - A_3 - C_7 - C_{12} + C_{13} + 2Q_1 - 2M_1 + 2S_2 - 2M_2. \end{aligned}$$

Then, appealing to Lemmas 3.3.1, 3.3.4, 3.3.9, 3.3.10 and 3.3.11, we obtain

$$\begin{aligned} A_1 - 2B_1 + 2C_1 \\ = - \sum_{\substack{kx+ey=n \\ e>k}} e + L_2 - 2(P_1 - L_1) - (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) - \sum_{\substack{kx+ey=n \\ x<y}} e + M_1 \\ - \sum_{\substack{kx+ey=n \\ x>y}} e + M_2 + \sum_{\substack{d|n \\ d>k}} (d - k - 1) + 2Q_1 - 2M_1 + 2S_2 - 2M_2 \\ = - \sum_{\substack{kx+ey=n \\ e>k}} e - \sum_{\substack{kx+ey=n \\ x \neq y}} e - (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) + \sum_{\substack{d|n \\ d>k}} (d - k - 1) \\ - 2(P_1 - Q_1 - S_2) - M_1 + L_2 - M_2 + 2L_1 \\ = - \sum_{\substack{kx+ey=n \\ e>k}} e - \sum_{\substack{kx+ey=n \\ x \neq y}} e - (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) + \sum_{\substack{d|n \\ d>k}} (d - k - 1) \\ - 2 \left(\sum_{\substack{d|n \\ d>k}} \frac{n}{d} - \frac{1}{2} \delta(n, k) \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} \right) \right) - M_1 + 2L_1, \end{aligned}$$

by Lemmas 1.2.4 and 3.2.5. Simplifying, we obtain

$$\begin{aligned}
 & A_1 - 2B_1 + 2C_1 \\
 &= - \sum_{\substack{kx+ey=n \\ e>k}} e - \sum_{\substack{kx+ey=n \\ x \neq y}} e - (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) + \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k - 1 \right) \\
 & \quad + \delta(n, k) \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} \right) - M_1 + 2L_1,
 \end{aligned}$$

as required. □

3.4 Solutions of $ax + by + cz = n$ (n fixed integer)

Throughout this section k and n are fixed positive integers. We are interested in the solutions $(a, x, b, y, c, z) \in \mathbb{N}^6$ of the equation

$$ax + by + cz = n$$

with $a + b + c = k$ or $a + b - c = k$ or $a - b - c = k$. We denote the number of such solutions by U , V and W respectively.

Definition 3.4.1.

$$U = \sum_{\substack{ax+by+cz=n \\ a+b+c=k}} 1, \quad V = \sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1, \quad W = \sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1.$$

In the next three lemmas we relate U , V and W to quantities of the previous four sections.

Lemma 3.4.1.

$$U = \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} \right) - 3 \sum_{\substack{kx+ey=n \\ e < k}} e + 3(k-1)L_1 + 3A_1.$$

Proof. We have

$$U = \sum_{\substack{ax+by+cz=n \\ a+b+c=k}} 1 = \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x \neq y, y \neq z, z \neq x}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ \text{exactly 2 of } x, y, z \text{ equal}}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x=y=z}} 1.$$

We begin by considering the first sum on the right-hand side of the above expression for U . The condition $x \neq y, y \neq z, z \neq x$, comprises six possibilities, namely, $x < y < z$, $x < z < y$, $y < x < z$, $y < z < x$, $z < x < y$ and $z < y < x$. Simple transformations of the summation variables show that these six cases yield the same contribution to the sum. Thus

$$\begin{aligned} \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x \neq y, y \neq z, z \neq x}} 1 &= 6 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x < y < z}} 1 \\ &= 6 \sum_{\substack{(k-b-c)x+by+cz=n \\ b+c < k \\ x < y < z}} 1 \\ &= 6 \sum_{\substack{kx+b(y-x)+c(z-x)=n \\ b+c < k \\ x < y < z}} 1 \\ &= 6 \sum_{\substack{kx+by'+cz'=n \\ b+c < k \\ y' < z'}} 1 \quad (y' = y - x, z' = z - x) \\ &= \frac{6}{2} \sum_{\substack{kx+by+cz=n \\ b+c < k \\ y \neq z}} 1 \end{aligned}$$

$$\begin{aligned}
 &= 3 \sum_{\substack{kx+by+cz=n \\ b+c < k}} 1 - 3 \sum_{\substack{kx+by+cz=n \\ b+c < k \\ y=z}} 1 \\
 &= 3A_1 - 3 \sum_{\substack{kx+(b+c)y=n \\ b+c < k}} 1 \quad (\text{by Definition 3.3.1}) \\
 &= 3A_1 - 3 \sum_{\substack{kx+ey=n \\ e < k}} (e-1) \\
 &= 3A_1 + 3L_1 - 3 \sum_{\substack{kx+ey=n \\ e < k}} e,
 \end{aligned}$$

by Definition 1.2.1. Next we consider the second sum on the right-hand side of the expression for U . There are six different cases in which exactly two of x , y , and z are equal. Simple transformations of the summation variables show that the cases $x = y < z$, $x = z < y$, and $y = z < x$, lead to the same sum, and similarly for the cases $x < y = z$, $y < x = z$, and $z < y = x$. Therefore

$$\begin{aligned}
 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ \text{exactly 2 of } x,y,z \text{ equal}}} 1 &= 3 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x=y < z}} 1 + 3 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x < y=z}} 1 \\
 &= 3 \sum_{\substack{(a+b)x+cz=n \\ a+b+c=k \\ x < z}} 1 + 3 \sum_{\substack{ax+(b+c)y=n \\ a+b+c=k \\ x < y}} 1 \\
 &= 3 \sum_{\substack{ex+cz=n \\ e+c=k \\ x < z \\ e=a+b}} 1 + 3 \sum_{\substack{ax+ey=n \\ a+e=k \\ x < y \\ e=b+c}} 1 \\
 &= 3 \sum_{\substack{ex+cz=n \\ e+c=k \\ x < z}} (e-1) + 3 \sum_{\substack{ax+ey=n \\ a+e=k \\ x < y}} (e-1)
 \end{aligned}$$

$$\begin{aligned}
 &= 3 \sum_{\substack{ex+(k-e)z=n \\ x < z \\ e < k}} (e-1) + 3 \sum_{\substack{(k-e)x+ey=n \\ x < y \\ e < k}} (e-1) \\
 &= 3 \sum_{\substack{ex+(k-e)(x+z')=n \\ e < k}} (e-1) + 3 \sum_{\substack{(k-e)x+e(x+y')=n \\ e < k}} (e-1) \\
 &= 3 \sum_{\substack{kx+(k-e)z=n \\ e < k}} (e-1) + 3 \sum_{\substack{kx+ey=n \\ e < k}} (e-1) \\
 &= 3 \sum_{\substack{kx+e'z=n \\ e' < k}} (k-e'-1) + 3 \sum_{\substack{kx+ey=n \\ e < k}} (e-1) \\
 &= 3(k-2)L_1,
 \end{aligned}$$

by Definition 1.2.1. Finally, we consider the third sum on the right-hand side of the expression for U . We have

$$\begin{aligned}
 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x=y=z}} 1 &= \sum_{\substack{(a+b+c)x=n \\ a+b+c=k}} 1 \\
 &= \sum_{\substack{kx=n \\ a+b+c=k}} 1 \\
 &= \delta(n, k) \sum_{a+b+c=k} 1 \\
 &= \delta(n, k) ((k-2) + (k-3) + (k-4) + \cdots + 1) \\
 &= \delta(n, k) \left(\frac{(k-1)(k-2)}{2} \right) \\
 &= \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} \right).
 \end{aligned}$$

Putting these three sums together, we obtain

$$\begin{aligned}
 U &= 3A_1 + 3L_1 - 3 \sum_{\substack{kx+ey=n \\ e < k}} e + 3(k-2)L_1 + \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} \right) \\
 &= \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} \right) - 3 \sum_{\substack{kx+ey=n \\ e < k}} e + 3(k-1)L_1 + 3A_1,
 \end{aligned}$$

as asserted. $\square$

Lemma 3.4.2.

$$V = 2B_1 + (k-1)M_1 + \sum_{\substack{kx+ey=n \\ x < y}} e.$$

Proof. We have

$$\begin{aligned}
 V &= \sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1 = \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x > y}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x < y}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x=y}} 1 \\
 &= 2 \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x < y}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x=y}} 1 \\
 &= 2 \sum_{\substack{(k-b+c)x+by+cz=n \\ b-c < k \\ x < y}} 1 + \sum_{\substack{(a+b)x+cz=n \\ a+b-c=k}} 1 \\
 &= 2 \sum_{\substack{kx+b(y-x)+c(z+x)=n \\ b-c < k \\ x < y}} 1 + \sum_{\substack{ex+cz=n \\ e-c=k \\ e=a+b}} 1 \\
 &= 2 \sum_{\substack{kx+by'+c(z+x)=n \\ b-c < k}} 1 + \sum_{\substack{ex+cz=n \\ e=c+k}} (e-1) \\
 &= 2 \sum_{\substack{kx+by+cz'=n \\ b-c < k, z' > x}} 1 + \sum_{(c+k)x+cz=n} (c+k-1)
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \sum_{\substack{kx+by+cz=n \\ b-c < k, z > x}} 1 + \sum_{\substack{kx+cy=n \\ y > x}} (k+c-1) \\
 &= 2B_1 + (k-1) \sum_{\substack{kx+ey=n \\ x < y}} 1 + \sum_{\substack{kx+ey=n \\ x < y}} e \\
 &= 2B_1 + (k-1)M_1 + \sum_{\substack{kx+ey=n \\ x < y}} e,
 \end{aligned}$$

by Definitions 1.2.2 and 3.3.2. □

Lemma 3.4.3.

$$W = 2C_1 + \sum_{\substack{kx+ey=n \\ x < y}} e - M_1.$$

Proof. We have

$$\begin{aligned}
 W &= \sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1 \\
 &= \sum_{(k+b+c)x+by+cz=n} 1 \\
 &= \sum_{kx+b(x+y)+c(x+z)=n} 1 \\
 &= \sum_{\substack{kx+by'+cz'=n \\ y' > x, z' > x}} 1 \\
 &= \sum_{\substack{kx+by+cz=n \\ y > x, z > x, y < z}} 1 + \sum_{\substack{kx+by+cz=n \\ y > x, z > x, y > z}} 1 + \sum_{\substack{kx+by+cz=n \\ y > x, z > x, y=z}} 1 \\
 &= 2 \sum_{\substack{kx+by+cz=n \\ x < y < z}} 1 + \sum_{\substack{kx+(b+c)y=n \\ y > x}} 1
 \end{aligned}$$

$$\begin{aligned}
 &= 2C_1 + \sum_{\substack{kx+ey=n \\ y>x}} (e-1) \quad (\text{by Definition 3.3.3}) \\
 &= 2C_1 + \sum_{\substack{kx+ey=n \\ x<y}} e - M_1, \quad (\text{by Definition 1.2.2})
 \end{aligned}$$

as claimed. $\square$

Lemma 3.4.4.

$$\begin{aligned}
 &U - 3V + 3W \\
 &= \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k \right) \right) + 3 \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k \right) - 6 \sum_{kx+ey=n} e.
 \end{aligned}$$

Proof. Applying Lemmas 3.4.1, 3.4.2 and 3.4.3, we find

$$\begin{aligned}
 &U - 3V + 3W \\
 &= \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} \right) - 3 \sum_{\substack{kx+ey=n \\ e<k}} e + 3(k-1)L_1 + 3A_1 \\
 &\quad - 3 \left(2B_1 + (k-1)M_1 + \sum_{\substack{kx+ey=n \\ x<y}} e \right) + 3 \left(2C_1 + \sum_{\substack{kx+ey=n \\ x<y}} e - M_1 \right) \\
 &= \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} \right) - 3 \sum_{\substack{kx+ey=n \\ e<k}} e + 3(k-1)L_1 - 3kM_1 \\
 &\quad + 3(A_1 - 2B_1 + 2C_1).
 \end{aligned}$$

Appealing to Proposition 3.3.1, we obtain

$$\begin{aligned}
 &U - 3V + 3W \\
 &= \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} \right) - 3 \sum_{\substack{kx+ey=n \\ e<k}} e + 3(k-1)L_1 - 3kM_1
 \end{aligned}$$

$$\begin{aligned}
& + 3 \left(- \sum_{\substack{kx+ey=n \\ e>k}} e - \sum_{\substack{kx+ey=n \\ x \neq y}} e - (k-1) \left(\frac{n}{k} - 1 \right) \delta(n, k) \right. \\
& \left. + \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k - 1 \right) + \delta(n, k) \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} \right) - M_1 + 2L_1 \right) \\
& = \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k + 1 \right) \right) - 3 \sum_{\substack{kx+ey=n \\ e \neq k}} e - 3 \sum_{\substack{kx+ey=n \\ x \neq y}} e \\
& \quad + 3(k+1)(L_1 - M_1) + 3 \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k - 1 \right) \\
& = \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k + 1 \right) \right) - 6 \sum_{kx+ey=n} e + 3 \sum_{(k+e)x=n} e \\
& \quad + 3 \sum_{k(x+y)=n} k + 3(k+1) \left(\sum_{\substack{d|n \\ d>k}} 1 - \delta(n, k) \left(\frac{n}{k} - 1 \right) \right) + 3 \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k - 1 \right) \\
& \hspace{15em} \text{(by Lemma 1.2.5)} \\
& = \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - 2k \right) \right) - 6 \sum_{kx+ey=n} e + 3 \sum_{\substack{d|n \\ d>k}} (d - k) \\
& \quad + 3\delta(n, k)k \left(\frac{n}{k} - 1 \right) + 3 \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} \right) \\
& = \delta(n, k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k \right) \right) + 3 \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k \right) - 6 \sum_{kx+ey=n} e,
\end{aligned}$$

as required. $\square$

3.5 Proof of Liouville's 3-term formula

We now use the results of Sections 3.1-3.4 to prove Liouville's 3-term formula.

Theorem 3.5.1. *Let n be a positive integer, and let $F : \mathbb{Z} \rightarrow \mathbb{C}$ be an odd function.*

Then

$$\begin{aligned} \sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ = \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) F(d) \\ + 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) F(k) - 6 \sum_{\substack{n=n_1+n_2 \\ n_1, n_2 \in \mathbb{N}}} \sigma(n_1) \sum_{d|n_2} F(d), \end{aligned}$$

where the sum on the left-hand side is taken over all $(a, b, c, x, y, z) \in \mathbb{N}^6$ such that $ax + by + cz = n$.

Proof. First we have

$$\begin{aligned} \sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\ = \sum_{ax+by+cz=n} F(a+b+c) + \sum_{ax+by+cz=n} F(a-b-c) \\ - \sum_{ax+by+cz=n} F(a+b-c) - \sum_{ax+by+cz=n} F(a-b+c). \end{aligned}$$

Next we look at each of these sums individually. We have

$$\begin{aligned} \sum_{ax+by+cz=n} F(a+b+c) &= \sum_{k \in \mathbb{N}} F(k) \sum_{\substack{ax+by+cz=n \\ a+b+c=k}} 1 \\ &= \sum_{k \in \mathbb{N}} F(k) U, \\ \sum_{ax+by+cz=n} F(a-b-c) &= \sum_{k \in \mathbb{N}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1 - \sum_{\substack{ax+by+cz=n \\ a-b-c=-k}} 1 \right) \quad (\text{as } F \text{ is odd}) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k \in \mathbb{N}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1 - \sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1 \right) \\
 &= \sum_{k \in \mathbb{N}} F(k) (W - V), \\
 \sum_{ax+by+cz=n} F(a+b-c) &= \sum_{k \in \mathbb{N}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1 - \sum_{\substack{ax+by+cz=n \\ a+b-c=-k}} 1 \right) \\
 &= \sum_{k \in \mathbb{N}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1 - \sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1 \right) \\
 &= \sum_{k \in \mathbb{N}} F(k) (V - W), \\
 \sum_{ax+by+cz=n} F(a-b+c) &= \sum_{k \in \mathbb{N}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a-b+c=k}} 1 - \sum_{\substack{ax+by+cz=n \\ a-b+c=-k}} 1 \right) \\
 &= \sum_{k \in \mathbb{N}} F(k) \left(\sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1 - \sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1 \right) \\
 &= \sum_{k \in \mathbb{N}} F(k) (V - W).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 &\sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\
 &= \sum_{k \in \mathbb{N}} F(k) (U + (W - V) - (V - W) - (V - W)) \\
 &= \sum_{k \in \mathbb{N}} F(k) (U - 3V + 3W).
 \end{aligned}$$

Appealing to Lemma 3.4.4, we obtain

$$\sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c))$$

$$\begin{aligned}
 &= \sum_{k \in \mathbb{N}} F(k) \left(\delta(n, k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k \right) \right) \right. \\
 &\quad \left. + 3 \sum_{\substack{d|n \\ d > k}} \left(d - 2\frac{n}{d} - k \right) F(k) - 6 \sum_{kx+ey=n} e \right) \\
 &= \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) F(d) \\
 &\quad + 3 \sum_{d|n} \sum_{1 \leq k < d} \left(d - 2\frac{n}{d} - k \right) F(k) - 6 \sum_{kx+ey=n} e F(k) \\
 &= \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) F(d) \\
 &\quad + 3 \sum_{d|n} \sum_{1 \leq k < d} \left(d - 2\frac{n}{d} - k \right) F(k) - 6 \sum_{n_1+n_2=n} \sum_{e|n_1} e \sum_{d|n_2} F(d) \\
 &= \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) F(d) \\
 &\quad + 3 \sum_{d|n} \sum_{1 \leq k < d} \left(d - 2\frac{n}{d} - k \right) F(k) - 6 \sum_{n_1+n_2=n} \sigma(n_1) \sum_{d|n_2} F(d),
 \end{aligned}$$

as required. □

3.6 Applications of Liouville's 3-term formula

In this section we prove various identities by applying our 3-term formula of Liouville type with different odd functions F . We begin by giving another proof of Theorem 1.7.1.

Theorem 1.7.1. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n-1} \sigma(k) \sigma(n-k) = \frac{5}{12} \sigma_3(n) + \sigma(n) \left(\frac{1}{12} - \frac{n}{2} \right).$$

Second Proof. In order to prove this result, we apply Liouville's 3-term formula with

$F(n) = n$. Then

$$F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c) = 0.$$

Also

$$\begin{aligned} \sum_{n_1+n_2=n} \sigma(n_1) \sum_{d|n_2} F(d) &= \sum_{n_1+n_2=n} \sigma(n_1) \sum_{d|n_2} d \\ &= \sum_{n_1+n_2=n} \sigma(n_1) \sigma(n_2) \\ &= \sum_{k < n} \sigma(k) \sigma(n-k). \end{aligned}$$

Therefore, applying Liouville's 3-term formula gives

$$\begin{aligned} 0 &= \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) d \\ &\quad + 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) k - 6 \sum_{k < n} \sigma(k) \sigma(n-k). \end{aligned}$$

However

$$\begin{aligned} &\sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) d \\ &= \frac{1}{2} \sum_{d|n} d^3 - \frac{3}{2} \sum_{d|n} d^2 + \sum_{d|n} d + 3 \sum_{d|n} \left(\frac{n^2}{d} - n - nd + d^2 \right) \\ &= \frac{1}{2} \sigma_3(n) - \frac{3}{2} \sigma_2(n) + \sigma(n) + 3 \sum_{d|n} (nd - n - nd + d^2) \\ &= \frac{1}{2} \sigma_3(n) + \frac{3}{2} \sigma_2(n) + \sigma(n) - 3n\tau(n), \end{aligned}$$

and

$$3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) k$$

$$\begin{aligned}
&= -6 \sum_{d|n} \frac{n}{d} \sum_{1 \leq k < d} k + 6 \sum_{d|n} d \sum_{1 \leq k < d} k - 3 \sum_{d|n} \sum_{1 \leq k < d} k^2 \\
&= -6 \sum_{d|n} \frac{n}{d} \left(\frac{d(d-1)}{2} \right) + 6 \sum_{d|n} d \left(\frac{d(d-1)}{2} \right) - 3 \sum_{d|n} \frac{(d-1)d(2d-1)}{6} \\
&= -3n \sum_{d|n} (d-1) + 3 \sum_{d|n} (d^3 - d^2) - \frac{1}{2} \sum_{d|n} (2d^3 - 3d^2 + d) \\
&= -3n\sigma(n) + 3n\tau(n) + 3\sigma_3(n) - 3\sigma_2(n) - \sigma_3(n) + \frac{3}{2}\sigma_2(n) - \frac{1}{2}\sigma(n).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{k < n} \sigma(k)\sigma(n-k) \\
&= \frac{1}{6} \left(\frac{1}{2}\sigma_3(n) + \frac{3}{2}\sigma_2(n) + \sigma(n) - 3n\tau(n) - 3n\sigma(n) + 3n\tau(n) + 3\sigma_3(n) \right. \\
&\quad \left. - 3\sigma_2(n) - \sigma_3(n) + \frac{3}{2}\sigma_2(n) - \frac{1}{2}\sigma(n) \right) \\
&= \frac{1}{6} \left(\frac{5}{2}\sigma_3(n) + \left(\frac{1}{2} - 3n \right) \sigma(n) \right) \\
&= \frac{5}{12}\sigma_3(n) + \sigma(n) \left(\frac{1}{12} - \frac{n}{2} \right),
\end{aligned}$$

as asserted. □

Theorem 3.6.1. *Let n be a positive integer. Then*

$$\begin{aligned}
&\sum_{\substack{n=n_1+n_2+n_3 \\ n_1, n_2, n_3 \in \mathbb{N}}} \sigma(n_1)\sigma(n_2)\sigma(n_3) \\
&= \frac{1}{192} \left((1 - 12n + 24n^2)\sigma(n) + (10 - 30n)\sigma_3(n) + 7\sigma_5(n) \right).
\end{aligned}$$

Proof. In order to prove this result, we apply Liouville's 3-term formula with $F(n) = n^3$. Then

$$F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)$$

$$\begin{aligned}
 &= (a+b+c)^3 + (a-b-c)^3 - (a+b-c)^3 - (a-b+c)^3 \\
 &= 24abc.
 \end{aligned}$$

Therefore, the left-hand side of Liouville's 3-term formula gives

$$\begin{aligned}
 &\sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)) \\
 &= \sum_{ax+by+cz=n} 24abc \\
 &= 24 \sum_{n_1+n_2+n_3=n} \sum_{a|n_1} a \sum_{b|n_2} b \sum_{c|n_3} c \\
 &= 24 \sum_{n_1+n_2+n_3=n} \sigma(n_1)\sigma(n_2)\sigma(n_3).
 \end{aligned}$$

The right-hand side gives

$$\begin{aligned}
 &\sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) d^3 \\
 &+ 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) k^3 - 6 \sum_{\substack{n=n_1+n_2 \\ n_1, n_2 \in \mathbb{N}}} \sigma(n_1) \sum_{d|n_2} d^3,
 \end{aligned}$$

where

$$\begin{aligned}
 &\sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) d^3 \\
 &= \sum_{d|n} \left(\frac{d^5}{2} - \frac{3d^4}{2} + d^3 + 3n^2d - 3nd^2 - 3nd^3 + 3d^4 \right) \\
 &= \frac{1}{2} \sigma_5(n) - \frac{3}{2} \sigma_4(n) + \sigma_3(n) + 3n^2 \sigma(n) - 3n \sigma_2(n) - 3n \sigma_3(n) + 3 \sigma_4(n) \\
 &= \frac{1}{2} \sigma_5(n) + \frac{3}{2} \sigma_4(n) + (1 - 3n) \sigma_3(n) - 3n \sigma_2(n) + 3n^2 \sigma(n), \\
 &3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) k^3
 \end{aligned}$$

$$\begin{aligned}
&= 6 \sum_{d|n} d \sum_{1 \leq k < d} k^3 - 6 \sum_{d|n} \frac{n}{d} \sum_{1 \leq k < d} k^3 - 3 \sum_{d|n} \sum_{1 \leq k < d} k^4 \\
&= 6 \sum_{d|n} d \left(\frac{(d-1)d}{2} \right)^2 - 6 \sum_{d|n} \frac{n}{d} \left(\frac{(d-1)d}{2} \right)^2 \\
&\quad - 3 \sum_{d|n} \frac{1}{30} (d-1)d(2d-1)(3d^2-3d-1) \\
&= \frac{6}{4} \sum_{d|n} d (d^4 - 2d^3 + d^2) - \frac{6}{4} n \sum_{d|n} \left(\frac{d^4 - 2d^3 + d^2}{d} \right) \\
&\quad - \frac{3}{30} \sum_{d|n} (6d^5 - 15d^4 + 10d^3 - d) \\
&= \frac{3}{2} (\sigma_5(n) - 2\sigma_4(n) + \sigma_3(n)) - \frac{3}{2} n (\sigma_3(n) - 2\sigma_2(n) + \sigma(n)) \\
&\quad - \frac{1}{10} (6\sigma_5(n) - 15\sigma_4(n) + 10\sigma_3(n) - \sigma(n)) \\
&= \frac{9}{10} \sigma_5(n) - \frac{3}{2} \sigma_4(n) + \left(\frac{1}{2} - \frac{3n}{2} \right) \sigma_3(n) + 3n\sigma_2(n) + \left(\frac{1}{10} - \frac{3n}{2} \right) \sigma(n)
\end{aligned}$$

and

$$\begin{aligned}
-6 \sum_{n=n_1+n_2} \sigma(n_1) \sum_{d_2|n_2} d_2^3 &= -6 \sum_{n=n_1+n_2} \sigma(n_1) \sigma_3(n_2) \\
&= \frac{-6}{240} (21\sigma_5(n) + (10 - 30n)\sigma_3(n) - \sigma(n)) \quad (\text{Theorem 1.7.2}) \\
&= \frac{-21}{40} \sigma_5(n) + \left(\frac{3n}{4} - \frac{1}{4} \right) \sigma_3(n) + \frac{1}{40} \sigma(n).
\end{aligned}$$

Equating the left-hand side and the right-hand side of Liouville's 3-term formula gives

$$\begin{aligned}
24 \sum_{n_1+n_2+n_3=n} \sigma(n_1) \sigma(n_2) \sigma(n_3) \\
&= \frac{1}{2} \sigma_5(n) + \frac{3}{2} \sigma_4(n) + (1 - 3n) \sigma_3(n) - 3n\sigma_2(n) + 3n^2 \sigma(n) \\
&\quad + \frac{9}{10} \sigma_5(n) - \frac{3}{2} \sigma_4(n) + \left(\frac{1}{2} - \frac{3n}{2} \right) \sigma_3(n) + 3n\sigma_2(n) + \left(\frac{1}{10} - \frac{3n}{2} \right) \sigma(n)
\end{aligned}$$

$$\begin{aligned}
 & -\frac{21}{40}\sigma_5(n) + \left(\frac{3n}{4} - \frac{1}{4}\right)\sigma_3(n) + \frac{1}{40}\sigma(n) \\
 &= \frac{7}{8}\sigma_5(n) + \left(\frac{5-15n}{4}\right)\sigma_3(n) + \left(\frac{24n^2-12n+1}{8}\right)\sigma(n).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \sum_{n=n_1+n_2+n_3} \sigma(n_1)\sigma(n_2)\sigma(n_3) \\
 = \frac{1}{192} ((1-12n+24n^2)\sigma(n) + (10-30n)\sigma_3(n) + 7\sigma_5(n)),
 \end{aligned}$$

as claimed. $\square$

This result is due to Lahiri [55, eqn. (5.2)].

Theorem 3.6.2. *Let n be a positive integer. Then*

$$\begin{aligned}
 \sum_{\substack{n=n_1+n_2+n_3 \\ n_1, n_2, n_3 \in \mathbb{N}}} \sigma_3(n_1)\sigma(n_2)\sigma(n_3) \\
 = \frac{1}{2880} (10\sigma_7(n) + 21(1-2n)\sigma_5(n) - 6n(5-6n)\sigma_3(n) - (1-6n)\sigma(n)).
 \end{aligned}$$

Proof. In order to prove this result, we apply Liouville's 3-term formula with $F(n) = n^5$. Then

$$\begin{aligned}
 & F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c) \\
 &= (a+b+c)^5 + (a-b-c)^5 - (a+b-c)^5 - (a-b+c)^5 \\
 &= 80abc^3 + 80a^3bc + 80ab^3c.
 \end{aligned}$$

Therefore, the left-hand side of Liouville's 3-term formula gives

$$\sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c))$$

$$\begin{aligned}
 &= \sum_{ax+by+cz=n} 80abc^3 + 80a^3bc + 80ab^3c \\
 &= 80 \sum_{n_1+n_2+n_3=n} \sum_{a|n_1} a \sum_{b|n_2} b \sum_{c|n_3} c^3 + 80 \sum_{n_1+n_2+n_3=n} \sum_{a|n_1} a^3 \sum_{b|n_2} b \sum_{c|n_3} c \\
 &\quad + 80 \sum_{n_1+n_2+n_3=n} \sum_{a|n_1} a \sum_{b|n_2} b^3 \sum_{c|n_3} c \\
 &= 80 \sum_{n_1+n_2+n_3=n} \sigma(n_1)\sigma(n_2)\sigma_3(n_3) + 80 \sum_{n_1+n_2+n_3=n} \sigma_3(n_1)\sigma(n_2)\sigma(n_3) \\
 &\quad + 80 \sum_{n_1+n_2+n_3=n} \sigma(n_1)\sigma_3(n_2)\sigma_3(n_3) \\
 &= 240 \sum_{n_1+n_2+n_3=n} \sigma_3(n_1)\sigma(n_2)\sigma(n_3)
 \end{aligned}$$

The right-hand side gives

$$\begin{aligned}
 &\sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) d^5 \\
 &\quad + 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) k^5 - 6 \sum_{\substack{n=n_1+n_2 \\ n_1, n_2 \in \mathbb{N}}} \sigma(n_1) \sum_{d|n_2} d^5,
 \end{aligned}$$

where

$$\begin{aligned}
 &\sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) d^5 \\
 &= \sum_{d|n} \left(\frac{d^7}{2} - \frac{3d^6}{2} + d^5 + 3n^2d^3 - 3nd^4 - 3nd^5 + 3d^6 \right) \\
 &= \frac{1}{2}\sigma_7(n) - \frac{3}{2}\sigma_6(n) + \sigma_5(n) + 3n^2\sigma_3(n) - 3n\sigma_4(n) - 3n\sigma_5(n) + 3\sigma_6(n) \\
 &= \frac{1}{2}\sigma_7(n) + \frac{3}{2}\sigma_6(n) + (1 - 3n)\sigma_5(n) - 3n\sigma_4(n) + 3n^2\sigma_3(n), \\
 &3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) k^5
 \end{aligned}$$

$$\begin{aligned}
&= 6 \sum_{d|n} d \sum_{1 \leq k < d} k^5 - 6 \sum_{d|n} \frac{n}{d} \sum_{1 \leq k < d} k^5 - 3 \sum_{d|n} \sum_{1 \leq k < d} k^6 \\
&= 6 \sum_{d|n} d \left(\frac{-d^2}{12} + \frac{5d^4}{12} + \frac{d^6}{6} - \frac{d^5}{2} \right) - 6 \sum_{d|n} \frac{n}{d} \left(\frac{-d^2}{12} + \frac{5d^4}{12} + \frac{d^6}{6} - \frac{d^5}{2} \right) \\
&\quad - 3 \sum_{d|n} \left(\frac{d}{42} - \frac{d^3}{6} + \frac{d^5}{2} + \frac{d^7}{7} - \frac{d^6}{2} \right) \\
&= \sum_{d|n} \left(\frac{-d^3}{2} + \frac{5d^5}{2} + d^7 - 3d^6 \right) - n \sum_{d|n} \left(\frac{-d}{2} + \frac{5d^3}{2} + d^5 - 3d^4 \right) \\
&\quad - \sum_{d|n} \left(\frac{d}{14} - \frac{d^3}{2} + \frac{3d^5}{2} + \frac{3d^7}{7} - \frac{3d^6}{2} \right) \\
&= \sum_{d|n} \left(\frac{4}{7}d^7 - \frac{3}{2}d^6 + (1-n)d^5 + 3nd^4 - \frac{5n}{2}d^3 + \left(\frac{7n-1}{14} \right) d \right) \\
&= \frac{4}{7}\sigma_7(n) - \frac{3}{2}\sigma_6(n) + (1-n)\sigma_5(n) + 3n\sigma_4(n) - \frac{5n}{2}\sigma_3(n) + \left(\frac{7n-1}{14} \right) \sigma(n)
\end{aligned}$$

and

$$\begin{aligned}
-6 \sum_{n=n_1+n_2} \sigma(n_1) \sum_{d_2|n_2} d_2^5 &= -6 \sum_{n=n_1+n_2} \sigma(n_1) \sigma_5(n_2) \\
&= -\frac{1}{84} (20\sigma_7(n) + (21-42n)\sigma_5(n) + \sigma(n)).
\end{aligned}$$

This last step was done using the known result

$$\sum_{n=n_1+n_2} \sigma(n_1) \sigma_5(n_2) = \frac{1}{504} (20\sigma_7(n) + (21-42n)\sigma_5(n) + \sigma(n)).$$

Huard, Ou, Spearman and Williams proved this result in [43, p. 239] in an elementary manner.

Equating the left-hand side and right-hand side of Liouville's 3-term formula gives

$$240 \sum_{n_1+n_2+n_3=n} \sigma_3(n_1) \sigma(n_2) \sigma(n_3)$$

$$\begin{aligned}
&= \frac{1}{2}\sigma_7(n) + \frac{3}{2}\sigma_6(n) + (1-3n)\sigma_5(n) - 3n\sigma_4(n) + 3n^2\sigma_3(n) + \frac{4}{7}\sigma_7(n) \\
&\quad - \frac{3}{2}\sigma_6(n) + (1-n)\sigma_5(n) + 3n\sigma_4(n) - \frac{5n}{2}\sigma_3(n) + \left(\frac{7n-1}{14}\right)\sigma(n) \\
&\quad - \frac{1}{84}(20\sigma_7(n) + (21-42n)\sigma_5(n) + \sigma(n)) \\
&= \frac{5}{6}\sigma_7(n) + \frac{7}{4}(1-2n)\sigma_5(n) - \frac{n}{2}(5-6n)\sigma_3(n) - \frac{1}{12}(1-6n)\sigma(n).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{n_1+n_2+n_3=n} \sigma_3(n_1)\sigma(n_2)\sigma(n_3) \\
&= \frac{1}{2880} (10\sigma_7(n) + 21(1-2n)\sigma_5(n) - 6n(5-6n)\sigma_3(n) - (1-6n)\sigma(n)),
\end{aligned}$$

as required. □

This result is due to Lahiri [55, eqn. (7.3)]. Lahiri's results on convolution sums given in [55] were proved in an elementary manner in [43]. Those in [56] require Ramanujan's tau function for their evaluation.

Lemma 3.6.1. *Let n be a positive integer, and set $n = 2^r m$, where $r, m \in \mathbb{N}$ and m is odd. Then*

$$\sum_{d|n} (-1)^d d^k = \left(-2 + \frac{2^{k(r+1)} - 1}{2^k - 1} \right) \sigma_k(m),$$

for any positive integer k . If $k = 0$ then

$$\sum_{d|n} (-1)^d = (r-1)\tau(m).$$

Proof. We have

$$\begin{aligned} \sum_{d|n} (-1)^d d^k &= - \sum_{d|m} d^k + \sum_{d|m} (2d)^k + \sum_{d|m} (2^2 d)^k + \cdots + \sum_{d|m} (2^r d)^k \\ &= (-1 + 2^k + 2^{2k} + \cdots + 2^{rk}) \sum_{d|m} d^k. \end{aligned}$$

If $k = 0$, then

$$\sum_{d|n} (-1)^d = (r-1) \sum_{d|m} 1 = (r-1)\tau(m),$$

and if $k \geq 1$, then

$$\sum_{d|n} (-1)^d d^k = \left(-2 + \sum_{l=0}^r (2^k)^l \right) \sum_{d|m} d^k = \left(-2 + \frac{2^{k(r+1)} - 1}{2^k - 1} \right) \sigma_k(m),$$

as required. □

Lemma 3.6.2. *Let d be a positive integer. Then*

$$\sum_{1 \leq k < d} (-1)^k k = \begin{cases} -\frac{d}{2}, & \text{if } d \text{ is even,} \\ \frac{d-1}{2}, & \text{if } d \text{ is odd.} \end{cases}$$

Proof. We have

$$\begin{aligned} \sum_{1 \leq k < d} (-1)^k k &= \begin{cases} (-1+2) + (-3+4) + \cdots - (d-1), & \text{if } d \text{ is even,} \\ (-1+2) + (-3+4) + \cdots + (d-1), & \text{if } d \text{ is odd,} \end{cases} \\ &= \begin{cases} \frac{d-2}{2} - (d-1), & \text{if } d \text{ is even,} \\ \frac{d-1}{2}, & \text{if } d \text{ is odd,} \end{cases} \\ &= \begin{cases} -\frac{d}{2}, & \text{if } d \text{ is even,} \\ \frac{d-1}{2}, & \text{if } d \text{ is odd,} \end{cases} \end{aligned}$$

as required. □

Lemma 3.6.3. *Let d be a positive integer. Then*

$$\sum_{1 \leq k < d} (-1)^k k^2 = (-1)^d d \left(\frac{1-d}{2} \right).$$

Proof. We have

$$\begin{aligned} \sum_{1 \leq k < d} (-1)^k k^2 &= \sum_{1 \leq 2k < d} (2k)^2 - \sum_{1 \leq 2k-1 < d} (2k-1)^2 \\ &= \begin{cases} \sum_{k=1}^{d/2-1} 4k^2 - \sum_{k=1}^{d/2} (4k^2 - 4k + 1), & \text{if } d \text{ is even,} \\ \sum_{k=1}^{(d-1)/2} 4k^2 - \sum_{k=1}^{(d-1)/2} (4k^2 - 4k + 1), & \text{if } d \text{ is odd,} \end{cases} \\ &= \begin{cases} 4 \sum_{k=1}^{d/2} k - \sum_{k=1}^{d/2} 1 - d^2, & \text{if } d \text{ is even,} \\ 4 \sum_{k=1}^{(d-1)/2} k - \sum_{k=1}^{(d-1)/2} 1, & \text{if } d \text{ is odd,} \end{cases} \\ &= \begin{cases} 2 \left(\frac{d}{2} \right) \left(\frac{d}{2} + 1 \right) - \frac{d}{2} - d^2, & \text{if } d \text{ is even,} \\ 2 \left(\frac{d-1}{2} \right) \left(\frac{d-1}{2} + 1 \right) - \frac{d-1}{2}, & \text{if } d \text{ is odd,} \end{cases} \\ &= \begin{cases} d \left(\frac{1-d}{2} \right), & \text{if } d \text{ is even,} \\ -d \left(\frac{1-d}{2} \right), & \text{if } d \text{ is odd,} \end{cases} \\ &= (-1)^d d \left(\frac{1-d}{2} \right), \end{aligned}$$

as asserted. □

Theorem 3.6.3. *Let n and k be positive integers. Set $n = 2^r m$, where r is a non-negative integer and m is an odd positive integer. Then*

$$\sum_{k < n} \sum_{d|k} (-1)^d d \sigma(n-k) = \left(\frac{2^{3(r+1)} - 15}{84} \right) \sigma_3(m) + \left(-n(2^r - 1) + \frac{2^r}{3} - \frac{1}{4} \right) \sigma(m).$$

Proof. In order to prove this result, we apply Liouville's 3-term formula with $F(n) = n(-1)^n$. We note that

$$F(-n) = (-n)(-1)^{-n} = (-n)(-1)^n = -n(-1)^n = -F(n).$$

As

$$a \pm b \pm c \equiv a + b + c \pmod{2},$$

we have

$$\begin{aligned} & F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c) \\ &= (-1)^{a+b+c}((a+b+c) + (a-b-c) - (a+b-c) - (a-b+c)) \\ &= 0. \end{aligned}$$

Also

$$\sum_{n_1+n_2=n} \sigma(n_1) \sum_{d|n_2} F(d) = \sum_{n_1+n_2=n} \sigma(n_1) \sum_{d|n_2} (-1)^d d = \sum_{k < n} \sum_{d|k} (-1)^d d \sigma(n-k).$$

Therefore, applying Liouville's 3-term formula gives

$$\begin{aligned} 0 &= \sum_{d|n} (-1)^d \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) d \\ &\quad + 3 \sum_{d|n} \sum_{1 \leq k < d} (-1)^k \left(2d - \frac{2n}{d} - k \right) k - 6 \sum_{k < n} \sum_{d|k} (-1)^d d \sigma(n-k). \end{aligned}$$

However

$$\sum_{d|n} (-1)^d \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) d$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{d|n} (-1)^d d^3 - \frac{3}{2} \sum_{d|n} (-1)^d d^2 + \sum_{d|n} (-1)^d d + 3 \sum_{d|n} (-1)^d \left(\frac{n^2}{d} - n - nd + d^2 \right) \\
&= \frac{1}{2} \sum_{d|n} (-1)^d d^3 + \frac{3}{2} \sum_{d|n} (-1)^d d^2 + \sum_{d|n} (-1)^d d - 3n \sum_{d|n} (-1)^d \\
&\quad + 3n \sum_{d|n} (-1)^{n/d} d - 3n \sum_{d|n} (-1)^d d \\
&= \frac{1}{2} \sum_{d|n} (-1)^d d^3 + \frac{3}{2} \sum_{d|n} (-1)^d d^2 + \sum_{d|n} (-1)^d d - 3n \sum_{d|n} (-1)^d \\
&\quad + 6n \sum_{\substack{d|n \\ n/d \text{ even}, d \text{ odd}}} d - 6n \sum_{\substack{d|n \\ n/d \text{ odd}, d \text{ even}}} d \\
&= \frac{1}{2} \sum_{d|n} (-1)^d d^3 + \frac{3}{2} \sum_{d|n} (-1)^d d^2 + \sum_{d|n} (-1)^d d - 3n \sum_{d|n} (-1)^d + 6n(1 - 2^r) \sum_{\substack{d|n \\ d \text{ odd}}} d
\end{aligned}$$

and

$$\begin{aligned}
&3 \sum_{d|n} \sum_{1 \leq k < d} (-1)^k \left(2d - \frac{2n}{d} - k \right) k \\
&= -6 \sum_{d|n} \frac{n}{d} \sum_{1 \leq k < d} (-1)^k k + 6 \sum_{d|n} d \sum_{1 \leq k < d} (-1)^k k - 3 \sum_{d|n} \sum_{1 \leq k < d} (-1)^k k^2 \\
&= 3n \sum_{\substack{d|n \\ d \text{ even}}} 1 - 3 \sum_{\substack{d|n \\ d \text{ odd}}} \frac{n(d-1)}{d} - 3 \sum_{\substack{d|n \\ d \text{ even}}} d^2 + 3 \sum_{\substack{d|n \\ d \text{ odd}}} d(d-1) - 3 \sum_{d|n} (-1)^d d \left(\frac{1-d}{2} \right) \\
&= 3n \sum_{\substack{d|n \\ d \text{ even}}} 1 - 3n \sum_{\substack{d|n \\ d \text{ odd}}} 1 + 3 \sum_{\substack{d|n \\ d \text{ odd}}} \frac{n}{d} - 3 \sum_{\substack{d|n \\ d \text{ even}}} d^2 + 3 \sum_{\substack{d|n \\ d \text{ odd}}} d^2 - 3 \sum_{\substack{d|n \\ d \text{ odd}}} d \\
&\quad - \frac{3}{2} \sum_{d|n} (-1)^d d + \frac{3}{2} \sum_{d|n} (-1)^d d^2 \\
&= 3n \sum_{\substack{d|n \\ d \text{ even}}} (-1)^d + 3n \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^d + 3 \cdot 2^r \sum_{\substack{d|n \\ d \text{ odd}}} d - 3 \sum_{\substack{d|n \\ d \text{ even}}} (-1)^d d^2 - 3 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^d d^2 \\
&\quad - 3 \sum_{\substack{d|n \\ d \text{ odd}}} d - \frac{3}{2} \sum_{d|n} (-1)^d d + \frac{3}{2} \sum_{d|n} (-1)^d d^2
\end{aligned}$$

$$= 3n \sum_{d|n} (-1)^d - \frac{3}{2} \sum_{d|n} (-1)^d d^2 + 3(2^r - 1) \sum_{\substack{d|n \\ d \text{ odd}}} d - \frac{3}{2} \sum_{d|n} (-1)^d d.$$

Therefore

$$\begin{aligned} & 6 \sum_{k < n} \sum_{d|k} (-1)^d d \sigma(n - k) \\ &= \frac{1}{2} \sum_{d|n} (-1)^d d^3 + \frac{3}{2} \sum_{d|n} (-1)^d d^2 + \sum_{d|n} (-1)^d d - 3n \sum_{d|n} (-1)^d + 6n(1 - 2^r) \sum_{\substack{d|n \\ d \text{ odd}}} d \\ & \quad + 3n \sum_{d|n} (-1)^d - \frac{3}{2} \sum_{d|n} (-1)^d d^2 + 3(2^r - 1) \sum_{\substack{d|n \\ d \text{ odd}}} d - \frac{3}{2} \sum_{d|n} (-1)^d d \\ &= \frac{1}{2} \sum_{d|n} (-1)^d d^3 + 3(2^r - 1)(1 - 2n) \sum_{\substack{d|n \\ d \text{ odd}}} d - \frac{1}{2} \sum_{d|n} (-1)^d d \\ &= \frac{1}{2} \left(-2 + \frac{2^{3(r+1)} - 1}{2^3 - 1} \right) \sigma_3(m) + 3(2^r - 1)(1 - 2n) \sigma(m) - \frac{1}{2} \left(-2 + \frac{2^{(r+1)} - 1}{2 - 1} \right) \sigma(m) \\ &= \left(\frac{2^{3(r+1)} - 15}{14} \right) \sigma_3(m) + \left(-6n(2^r - 1) + 2^{r+1} - \frac{3}{2} \right) \sigma(m). \end{aligned}$$

Hence

$$\sum_{k < n} \sum_{d|k} (-1)^d d \sigma(n - k) = \left(\frac{2^{3(r+1)} - 15}{84} \right) \sigma_3(m) + \left(-n(2^r - 1) + \frac{2^r}{3} - \frac{1}{4} \right) \sigma(m),$$

as asserted. □

Chapter 4

Sums of six squares

In this chapter we apply our 3-term formula of Liouville type (Theorem 3.5.1), together with formulae of Liouville (Theorems 2.4.1 and Corollary 1.6.1), to give an elementary arithmetic proof of Jacobi's formula for the number $r_6(n)$ of representations of a positive integer n as the sum of six squares, namely

$$r_6(n) = 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2,$$

where $\left(\frac{-4}{*} \right)$ is the Kronecker symbol.

The origins of this formula lie in Jacobi's monumental work on elliptic functions [51], [53, Vol. I, pp. 49-239]. Implicit in this work is the identity

$$\left(\sum_{m=-\infty}^{\infty} q^{m^2} \right)^6 = 1 + 16 \sum_{m=1}^{\infty} \frac{m^2 q^m}{1 + q^{2m}} - 4 \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} (-1)^{\frac{m-1}{2}} \frac{m^2 q^m}{1 - q^m},$$

see [51, Section 40], [53, Vol. I, pp. 159-164], from which the formula for $r_6(n)$ is obtained by equating the coefficients of q^n on both sides. Jacobi never explicitly

stated this formula. He did however give explicit formulae in the case of four squares [50], [51], [53, Vol. I, pp. 247], [53, Vol. VI, pp. 245-251].

Eisenstein [23, p. 136], [24, Vol. I, p. 501] asserted that he had an arithmetic proof of the formula for $r_6(n)$ (n odd) but did not give it. The only arithmetic proof that we are aware of uses another identity of Liouville and a recurrence relation. An excellent exposition of this approach is given in [91, pp. 436-440]. Many authors have given analytic proofs, see for example [12], [17], [37]. We believe that the proof presented in this chapter is only the second truly elementary arithmetic proof of Jacobi's formula for $r_6(n)$.

4.1 Sums of squares

Let n and k be positive integers. The number of representations of n as the sum of k squares is denoted by $r_k(n)$, that is

$$r_k(n) = \text{card}\{(x_1, \dots, x_k) \in \mathbb{Z}^k \mid n = x_1^2 + \dots + x_k^2\}.$$

The determination of formulae for $r_k(n)$ has a long history going back to Fermat, who gave a formula for $r_2(n)$. Dickson's *History of the Theory of Numbers* [19, Vol. II, Chaps. 6-9] summarizes the work on the evaluation of $r_k(n)$ up to 1920. Grosswald's book [38] gives a comprehensive treatment of sums of squares up to 1985. There is a huge literature on the determination of $r_k(n)$ for small values of k , usually for k between 2 and 20, see for example [2], [3], [4, pp. 365-423], [5], [7], [10], [11], [12]-[17],

[20], [21], [22, Vol. II, pp. 89-96, pp. 201-208], [25]-[29], [32], [33], [37], [39], [40], [41], [42], [44], [45], [48], [49, Vol. 2, pp. 5-7], [52], [79], [80], [81], [82], [88], [89], [90], [98], [101], [102], [103], [104], [106], [107], [109], [110, p. 306], [111, p. 306], [113], [114], [115, pp. 417-419], [116], [119]. Recently, interest in the problem of representing integers as sums of squares has been rekindled by the work of Kac and Wakimoto [54], Milne [86], [87], Zagier [121], and Ono [93]. This work culminated in Ono's formulae for $r_k(n)$ when $k = 4s^2$ and $4s^2 + 4s$ in terms of convolution sums of divisor functions.

Generally proofs of formulae for $r_k(n)$ are non-elementary in nature, that is they involve complex analysis, summation formulae involving infinite series and/or infinite products, modular forms, etc. Liouville's approach was somewhat different. His view was that whenever there was an elementary formula for $r_k(n)$, that is one just involving sums of divisors of functions, then the formula should be provable by elementary means. Our approach in this thesis reflects this philosophy of Liouville.

4.2 Arithmetic proof of Jacobi's six squares theorem for $n \equiv 3 \pmod{4}$

In this section we prove the six squares theorem in the special case when $n \equiv 3 \pmod{4}$ (Theorem 4.2.1). In this case the formula for $r_6(n)$ can be simplified.

By Lemma 1.8.4, we have

$$\sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 = - \sum_{d|n} \left(\frac{-4}{d} \right) d^2,$$

so we must show that

$$r_6(n) = -20 \sum_{d|n} \left(\frac{-4}{d} \right) d^2.$$

In order to prove this result we apply Liouville's 3-term odd formula (Theorem 2.4.1) with the odd function

$$F(n) = -\frac{1}{4} \left(\frac{-4}{n} \right).$$

The following lemma provides an important relation involving this function.

Lemma 4.2.1. *For all $a, b, c \in \mathbb{Z}$ we have*

$$\left(\frac{-4}{abc} \right) = F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)$$

with

$$F(n) = -\frac{1}{4} \left(\frac{-4}{n} \right), \quad n \in \mathbb{Z}.$$

Proof. We consider four cases:

1. When a, b, c are even, then abc is even, so

$$\left(\frac{-4}{abc} \right) = 0.$$

Similarly $a \pm b \pm c$ is even so

$$F(a \pm b \pm c) = -\frac{1}{4} \left(\frac{-4}{a \pm b \pm c} \right) = 0.$$

2. When one of a, b, c is odd and two of a, b, c are even, then again abc is even so

$$\left(\frac{-4}{abc}\right) = 0.$$

In this case $a \pm b \pm c$ is odd, therefore

$$\begin{aligned} & F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c) \\ &= -\frac{1}{4} \left((-1)^{\frac{a+b+c-1}{2}} + (-1)^{\frac{a-b-c-1}{2}} - (-1)^{\frac{a+b-c-1}{2}} - (-1)^{\frac{a-b+c-1}{2}} \right) \\ &= -\frac{1}{4} (-1)^{\frac{a-b-c-1}{2}} ((-1)^{b+c} + 1 - (-1)^b - (-1)^c) \\ &= -\frac{1}{4} (-1)^{\frac{a-b-c-1}{2}} ((-1)^b - 1)((-1)^c - 1) \\ &= 0, \end{aligned}$$

as either b or c is even in this case.

3. When two of a, b, c are odd and one of a, b, c is even, then again abc is even so

$$\left(\frac{-4}{abc}\right) = 0.$$

In this case $a \pm b \pm c$ is even so

$$F(a \pm b \pm c) = -\frac{1}{4} \left(\frac{-4}{a \pm b \pm c} \right) = 0.$$

4. When a, b, c are odd, then abc is odd, so

$$\begin{aligned} \left(\frac{-4}{abc}\right) &= \left(\frac{-1}{abc}\right) = \left(\frac{-1}{a}\right) \left(\frac{-1}{b}\right) \left(\frac{-1}{c}\right) = (-1)^{\frac{a-1}{2}} (-1)^{\frac{b-1}{2}} (-1)^{\frac{c-1}{2}} \\ &= (-1)^{\frac{a+b+c-3}{2}} = (-1)^{\frac{a+b+c+1}{2}}. \end{aligned}$$

In this case $a \pm b \pm c$ is odd, so as in part 2

$$\begin{aligned}
 & F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c) \\
 &= -\frac{1}{4}(-1)^{\frac{a-b-c-1}{2}} ((-1)^b - 1) ((-1)^c - 1) \\
 &= -\frac{1}{4}(-1)^{\frac{a-b-c-1}{2}} (-2)(-2) \quad (\text{as } b, c \text{ odd}) \\
 &= -(-1)^{\frac{a-b-c-1}{2}} \\
 &= (-1)^{\frac{a-b-c+1}{2}} \\
 &= (-1)^{\frac{a+b+c+1}{2}},
 \end{aligned}$$

as $b+c$ is even. □

Lemma 4.2.2. *Let n be a positive integer with $n \equiv 3 \pmod{4}$. Then*

$$\sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1 | n_1} \left(\frac{-4}{d_1} \right) \sigma(n_2) = -\frac{1}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d^2.$$

Proof. In order to prove this, we apply Corollary 1.6.1 with the function

$$F(d) = \left(\frac{-4}{d} \right) d.$$

It is easy to see that this function is even since

$$F(-d) = \left(\frac{-4}{-d} \right) (-d) = - \left(\frac{-4}{d} \right) (-d) = \left(\frac{-4}{d} \right) d = F(d).$$

We begin by calculating the left-hand-side (LHS) of Corollary 1.6.1. We have

$$LHS = 2 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1 | n_1} \sum_{d_2 | n_2} \left(\left(\frac{-4}{d_1 - 2^s d_2} \right) (d_1 - 2^s d_2) - \left(\frac{-4}{d_1 + 2^s d_2} \right) (d_1 + 2^s d_2) \right).$$

As $n \equiv 3 \pmod{4}$ we have

$$\begin{cases} n_1 \equiv 1 \pmod{4}, & \text{if } s = 1, \\ n_1 \equiv 3 \pmod{4}, & \text{if } s \geq 2, \end{cases}$$

so that $s \geq 1$. Thus

$$2^s \equiv -2^s \pmod{4}, \quad \text{and so} \quad d_1 + 2^s d_2 \equiv d_1 - 2^s d_2 \pmod{4}.$$

Therefore

$$\begin{aligned} LHS &= 2 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \sum_{d_2|n_2} \left(\left(\frac{-4}{d_1 + 2^s d_2} \right) (d_1 - 2^s d_2) - \left(\frac{-4}{d_1 + 2^s d_2} \right) (d_1 + 2^s d_2) \right) \\ &= -4 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \sum_{d_2|n_2} 2^s d_2 \left(\frac{-4}{d_1 + 2^s d_2} \right) \\ &= -4 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} \sum_{d_1|n_1} \sum_{d_2|n_2} 2^s d_2 \left(\frac{-4}{d_1} \right) - 4 \sum_{\substack{n=n_1+2n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \sum_{d_2|n_2} 2d_2 \left(\frac{-4}{d_1 + 2d_2} \right). \end{aligned}$$

Since d_2 is odd, we have $d_2 = 2x + 1$ for some integer x , so that

$$\left(\frac{-4}{d_1 + 2d_2} \right) = \left(\frac{-4}{d_1 + 4x + 2} \right) = \left(\frac{-4}{d_1 + 2} \right) = - \left(\frac{-4}{d_1} \right).$$

Thus

$$\begin{aligned} LHS &= -4 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} \sum_{d_1|n_1} \sum_{d_2|n_2} 2^s d_2 \left(\frac{-4}{d_1} \right) + 8 \sum_{\substack{n=n_1+2n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \sum_{d_2|n_2} d_2 \left(\frac{-4}{d_1} \right) \\ &= -4 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd} \\ n_1 \equiv 3 \pmod{4}}} \sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \sum_{d_2|n_2} 2^s d_2 + 8 \sum_{\substack{n=n_1+2n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \sum_{d_2|n_2} d_2. \end{aligned}$$

As

$$\sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) = 0 \quad \text{for} \quad n_1 \equiv 3 \pmod{4}$$

we deduce that

$$\begin{aligned} LHS &= 8 \sum_{\substack{n=n_1+2n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \sigma(n_2) \\ &= 8 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \sigma(n_2). \end{aligned}$$

Now we consider the right-hand-side of Corollary 1.6.1. We have

$$\begin{aligned} RHS &= \sum_{d|n} \left(\frac{n}{d} - d \right) \left(\frac{-4}{d} \right) d \\ &= n \sum_{d|n} \left(\frac{-4}{d} \right) - \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \\ &= - \sum_{d|n} \left(\frac{-4}{d} \right) d^2, \end{aligned}$$

since $n \equiv 3 \pmod{4}$. Equating the left-hand-side and right-hand-side, we obtain the desired result. $\square$

We now have all the tools we need to prove the six squares theorem for $n \equiv 3 \pmod{4}$.

Theorem 4.2.1. *Let n be an odd positive integer such that $n \equiv 3 \pmod{4}$. Then*

$$r_6(n) = -20 \sum_{d|n} \left(\frac{-4}{d} \right) d^2.$$

Proof. We prove this result by applying Liouville's 3-term odd formula (Theorem 2.4.1) with the function $F(n) = -\frac{1}{4} \left(\frac{-4}{n} \right)$. We begin by considering the left-hand-side of the formula. We write $\#(n = x^2 + y^2)$ for the number of pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ one odd, one even

satisfying $n = x^2 + y^2$ with one of x and y odd and one of them even. We have

$$\begin{aligned}
 & 8 \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b-c)) \\
 &= 8 \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} \left(\frac{-4}{abc} \right) \quad (\text{by Lemma 4.2.1}) \\
 &= 8 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \text{ odd}}} \sum_{d_1|n_1} \sum_{d_2|n_2} \sum_{d_3|n_3} \left(\frac{-4}{d_1 d_2 d_3} \right) \\
 &= 8 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \text{ odd}}} \left(\sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \right) \left(\sum_{d_2|n_2} \left(\frac{-4}{d_2} \right) \right) \left(\sum_{d_3|n_3} \left(\frac{-4}{d_3} \right) \right) \\
 &= 8 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \text{ odd}}} \frac{1}{4} \#(n_1 = x_1^2 + x_2^2) \frac{1}{4} \#(n_2 = x_3^2 + x_4^2) \frac{1}{4} \#(n_3 = x_5^2 + x_6^2) \\
 &\quad \text{one odd, one even} \quad \text{one odd, one even} \quad \text{one odd, one even} \\
 &= 8 \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \text{ odd}}} \frac{1}{2} \#(n_1 = x_1^2 + x_2^2) \frac{1}{2} \#(n_2 = x_3^2 + x_4^2) \frac{1}{2} \#(n_3 = x_5^2 + x_6^2) \\
 &\quad x_1 \text{ odd, } x_2 \text{ even} \quad x_3 \text{ odd, } x_4 \text{ even} \quad x_5 \text{ odd, } x_6 \text{ even} \\
 &= \#(n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) \\
 &\quad x_1, x_3, x_5 \text{ odd, } x_2, x_4, x_6 \text{ even} \\
 &= \frac{r_6(n)}{20},
 \end{aligned}$$

since the only way to write $n \equiv 3 \pmod{4}$ as the sum of six squares is with 3 of the squares even and 3 of them odd, and there are $\binom{6}{3} = 20$ ways to choose which x_i are even and which x_i are odd.

Next we consider the right-hand-side of Liouville's 3-term odd formula, again with

$$F(n) = -\frac{1}{4} \left(\frac{-4}{n} \right). \text{ We have}$$

$$\sum_{d|n} \left(-\frac{1}{4} \right) \left(\frac{-4}{d} \right) (d^2 - 1) - 24 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \left(-\frac{1}{4} \right) \left(\frac{-4}{d_1} \right) \sigma(n_2)$$

$$\begin{aligned}
&= -\frac{1}{4} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + \frac{1}{4} \sum_{d|n} \left(\frac{-4}{d} \right) + 6 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \sigma(n_2) \\
&= -\frac{1}{4} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + 6 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \sigma(n_2) \quad (\text{since } n \equiv 3 \pmod{4}) \\
&= -\frac{1}{4} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 - \frac{6}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \quad (\text{by Lemma 4.2.2}) \\
&= -\sum_{d|n} \left(\frac{-4}{d} \right) d^2.
\end{aligned}$$

Therefore

$$r_6(n) = -20 \sum_{d|n} \left(\frac{-4}{d} \right) d^2,$$

as required. □

4.3 Arithmetic proof of Jacobi's six squares theorem for $n \equiv 1 \pmod{4}$

In this section we prove the six squares theorem for the special case when $n \equiv 1 \pmod{4}$ (Theorem 4.3.1).

First we apply Lemma 1.8.4 to simplify the formula for $r_6(n)$. For $n \equiv 1 \pmod{4}$ we have

$$\sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 = \sum_{d|n} \left(\frac{-4}{d} \right) d^2,$$

so we must show that

$$r_6(n) = 12 \sum_{d|n} \left(\frac{-4}{d} \right) d^2.$$

The following four lemmas evaluate sums that arise in the proof of Theorem 4.3.1.

Lemma 4.3.1. *Let n be an odd positive integer, then*

$$\sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k = \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d} \right) (1 - d).$$

Proof. Since n is odd, the divisors d of n must also be odd. Therefore, we have

$$\begin{aligned} & \sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k \\ &= \sum_{d|n} \begin{cases} (1-3) + (5-7) + \cdots + ((d-4) - (d-2)), & \text{if } d \equiv 1 \pmod{4}, \\ (1-3) + (5-7) + \cdots + ((d-6) - (d-4)) + (d-2), & \text{if } d \equiv 3 \pmod{4}. \end{cases} \\ &= \sum_{d|n} \begin{cases} -2 \left(\frac{d-1}{4} \right), & \text{if } d \equiv 1 \pmod{4}, \\ -2 \left(\frac{d-3}{4} \right) + (d-2), & \text{if } d \equiv 3 \pmod{4}. \end{cases} \\ &= \sum_{d|n} \begin{cases} \left(\frac{1-d}{2} \right), & \text{if } d \equiv 1 \pmod{4}, \\ -\left(\frac{1-d}{2} \right), & \text{if } d \equiv 3 \pmod{4}. \end{cases} \\ &= \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d} \right) (1 - d), \end{aligned}$$

as claimed. □

Lemma 4.3.2. *Let n be a positive integer with $n \equiv 1 \pmod{4}$, then*

$$\sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} 2^s r_2(n_1) \sigma(n_2) = \sum_{d|n} \left(\frac{-4}{d} \right) d^2 - \frac{n}{4} r_2(n).$$

Proof. We prove this result in the same way that we proved Lemma 4.2.2, that is by applying Corollary 1.6.1 to the function

$$F(d) = \left(\frac{-4}{d} \right) d,$$

except that here we have $n \equiv 1 \pmod{4}$ rather than $n \equiv 3 \pmod{4}$. From the proof of Lemma 4.2.2, we can immediately conclude that

$$\begin{aligned}
LHS &= -4 \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} \sum_{d_1|n_1} \sum_{d_2|n_2} 2^s d_2 \left(\frac{-4}{d_1} \right) + 8 \sum_{\substack{n=n_1+2n_2 \\ n_1, n_2 \text{ odd}}} \sum_{d_1|n_1} \sum_{d_2|n_2} d_2 \left(\frac{-4}{d_1} \right) \\
&= - \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} 2^s \cdot 4 \sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \sum_{d_2|n_2} d_2 + 8 \sum_{\substack{n=n_1+2n_2 \\ n_1, n_2 \text{ odd} \\ n_1 \equiv 3 \pmod{4}}} \sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \sum_{d_2|n_2} d_2 \\
&= - \sum_{\substack{n=n_1+2^s n_2 \\ n_1, n_2 \text{ odd}, s \geq 2}} 2^s r_2(n_1) \sigma(n_2).
\end{aligned}$$

Now we consider the right-hand-side of Corollary 1.6.1. We have

$$\begin{aligned}
RHS &= \sum_{d|n} \left(\frac{n}{d} - d \right) \left(\frac{-4}{d} \right) d \\
&= n \sum_{d|n} \left(\frac{-4}{d} \right) - \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \\
&= \frac{n}{4} r_2(n) - \sum_{d|n} \left(\frac{-4}{d} \right) d^2.
\end{aligned}$$

Equating the left-hand-side and right-hand-side, we get the desired result. $\square$

Lemma 4.3.3. *Let n be a positive integer with $n \equiv 1 \pmod{4}$. Then*

$$\sum_{\substack{n=n_1+2^{s_2} n_2 \\ n_1, n_2 \text{ odd}, s_2 \geq 1}} r_2(n_2) \sigma(n_1) = \frac{r_6(n)}{12} - \frac{r_4(n)}{8}.$$

Proof. It is well known that the number $r_4(m)$ of ways to write a positive integer m as the sum of four squares, is given by

$$r_4(m) = 8(\sigma(m) - 4\sigma(m/4)),$$

where $\sigma(m/4)$ is understood to be zero if $m/4$ is not an integer, see for example [113], where an elementary arithmetic proof of this formula is given. In particular, if m is odd, we have

$$r_4(m) = 8\sigma(m). \quad (4.3.1)$$

Let $n \equiv 1 \pmod{4}$. If $n = x_1^2 + \cdots + x_6^2$ then exactly one of $x_1, \dots, x_6$ is even or exactly one of $x_1, \dots, x_6$ is odd. Set

$$t_0(n) = \text{card}\{(x_1, \dots, x_6) \in \mathbb{Z}^6 \mid n = x_1^2 + \cdots + x_6^2, x_1 \text{ even}, x_2, \dots, x_6 \text{ odd}\}$$

and

$$t_1(n) = \text{card}\{(x_1, \dots, x_6) \in \mathbb{Z}^6 \mid n = x_1^2 + \cdots + x_6^2, x_1 \text{ odd}, x_2, \dots, x_6 \text{ even}\}$$

so that

$$r_6(n) = 6t_0(n) + 6t_1(n).$$

Next

$$t_0(n) = \sum_{\substack{1 \leq n_1 \leq n \\ n_1 \equiv 3 \pmod{4}}} \sum_{\substack{x_1, x_2, x_3, x_4 \in \mathbb{Z} \\ x_1 \text{ even}, x_2, x_3, x_4 \text{ odd} \\ n_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2}} 1 \sum_{\substack{x_5, x_6 \in \mathbb{Z} \\ x_5, x_6 \text{ odd} \\ n - n_1 = x_5^2 + x_6^2}} 1$$

and

$$t_1(n) = \sum_{\substack{1 \leq n_1 \leq n \\ n_1 \equiv 1 \pmod{4}}} \sum_{\substack{x_1, x_2, x_3, x_4 \in \mathbb{Z} \\ x_1 \text{ odd}, x_2, x_3, x_4 \text{ even} \\ n_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2}} 1 \sum_{\substack{x_5, x_6 \in \mathbb{Z} \\ x_5, x_6 \text{ even} \\ n - n_1 = x_5^2 + x_6^2}} 1.$$

Hence

$$t_0(n) = \sum_{\substack{1 \leq n_1 < n \\ n_1 \equiv 3 \pmod{4}}} \frac{1}{4} r_4(n_1) r_2(n - n_1) = \frac{1}{4} \sum_{\substack{1 \leq n_1 < n \\ n_1 \equiv 3 \pmod{4}}} r_4(n_1) r_2(n - n_1)$$

and

$$t_1(n) = \sum_{\substack{1 \leq n_1 \leq n \\ n_1 \equiv 1 \pmod{4}}} \frac{1}{4} r_4(n_1) r_2(n - n_1) = \frac{1}{4} \sum_{\substack{1 \leq n_1 < n \\ n_1 \equiv 1 \pmod{4}}} r_4(n_1) r_2(n - n_1) + \frac{1}{4} r_4(n).$$

Thus

$$\begin{aligned} \frac{r_6(n)}{6} &= t_0(n) + t_1(n) = \frac{1}{4} \sum_{1 \leq n_1 < n} r_4(n_1) r_2(n - n_1) + \frac{1}{4} r_4(n) \\ &= 2 \sum_{1 \leq n_1 < n} \sigma(n_1) r_2(n - n_1) + \frac{1}{4} r_4(n) \\ &= 2 \sum_{\substack{n_1 + 2^s n_2 = n \\ n_1, n_2 \text{ odd}, s \geq 1}} \sigma(n_1) r_2(2^s n_2) + \frac{1}{4} r_4(n) \\ &= 2 \sum_{\substack{n_1 + 2^s n_2 = n \\ n_1, n_2 \text{ odd}, s \geq 1}} \sigma(n_1) r_2(n_2) + \frac{1}{4} r_4(n). \end{aligned}$$

Finally

$$\sum_{\substack{n_1 + 2^s n_2 = n \\ n_1, n_2 \text{ odd}, s \geq 1}} \sigma(n_1) r_2(n_2) = \frac{r_6(n)}{12} - \frac{r_4(n)}{8},$$

as claimed. □

Lemma 4.3.4. *Let n be a positive integer with $n \equiv 1 \pmod{4}$. Then*

$$\sum_{\substack{n = 2^{s_1} n_1 + n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} r_2(n_2) \sigma(n_1) = \frac{1}{6} \left(\sum_{d|n} \left(\frac{-4}{d} \right) d + \sum_{d|n} \left(\frac{-4}{d} \right) d^2 - 2 \sum_{d|n} \sum_{k=1}^d \left(\frac{-4}{k} \right) k \right).$$

Proof. In order to prove this lemma, we apply Liouville's two term identity with the function

$$F(n) = n \left(\frac{-4}{n} \right).$$

We begin by considering the left-hand-side. We have

$$\begin{aligned} LHS &= \sum_{n=n_1+n_2} \sum_{d_1|n_1} \sum_{d_2|n_2} \left((d_1 - d_2) \left(\frac{-4}{d_1 - d_2} \right) - (d_1 + d_2) \left(\frac{-4}{d_1 + d_2} \right) \right) \\ &= \sum_{n=n_1+n_2} \left(\sum_{\substack{d_1|n_1, d_2|n_2 \\ 2|d_1, 2|d_2}} \left((d_1 - d_2) \left(\frac{-4}{d_1 - d_2} \right) - (d_1 + d_2) \left(\frac{-4}{d_1 + d_2} \right) \right) \right. \\ &\quad \left. + \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 2 \nmid d_2}} \left((d_1 - d_2) \left(\frac{-4}{d_1 - d_2} \right) - (d_1 + d_2) \left(\frac{-4}{d_1 + d_2} \right) \right) \right), \end{aligned}$$

since if d_1 and d_2 are both even, or both odd then $d_1 \pm d_2$ is even, so $\left(\frac{-4}{d_1 \pm d_2} \right) = 0$.

We continue to simplify the LHS further. We have

$$\begin{aligned} LHS &= \sum_{n=n_1+n_2} \left(\sum_{\substack{d_1|n_1, d_2|n_2 \\ 2|d_1, 2|d_2}} \left((d_1 - d_2) \left(\frac{-4}{d_1 - d_2} \right) - (d_1 + d_2) \left(\frac{-4}{d_1 - d_2} \right) \right) \right. \\ &\quad \left. + \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 2 \nmid d_2}} \left((d_1 - d_2) \left(\frac{-4}{d_1 - d_2} \right) + (d_1 + d_2) \left(\frac{-4}{d_1 - d_2} \right) \right) \right) \\ &= \sum_{n=n_1+n_2} \left(\sum_{\substack{d_1|n_1, d_2|n_2 \\ 2|d_1, 2|d_2}} (-2d_2) \left(\frac{-4}{d_1 - d_2} \right) + \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 2 \nmid d_2}} (2d_1) \left(\frac{-4}{d_1 - d_2} \right) \right) \\ &= \sum_{n=n_1+n_2} \left(\sum_{\substack{d_1|n_1, d_2|n_2 \\ 2|d_1, 2|d_2}} (-2d_2) \left(\frac{-4}{d_1 - d_2} \right) + \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 2|d_2}} (2d_2) \left(\frac{-4}{d_2 - d_1} \right) \right) \\ &= \sum_{n=n_1+n_2} \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 2|d_2}} (-4d_2) \left(\frac{-4}{d_1 - d_2} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=n_1+n_2} \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 2||d_2}} (-4d_2) \left(\frac{-4}{d_1 - d_2} \right) + \sum_{n=n_1+n_2} \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 4|d_2}} (-4d_2) \left(\frac{-4}{d_1 - d_2} \right) \\
&= \sum_{n=n_1+n_2} \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 2||d_2}} (4d_2) \left(\frac{-4}{d_1} \right) + \sum_{n=n_1+n_2} \sum_{\substack{d_1|n_1, d_2|n_2 \\ 2 \nmid d_1, 4|d_2}} (-4d_2) \left(\frac{-4}{d_1} \right) \\
&= 4 \sum_{n=n_1+n_2} \sum_{\substack{d_1|n_1 \\ 2 \nmid d_1}} \left(\frac{-4}{d_1} \right) \sum_{\substack{d_2|n_2 \\ 2||d_2}} d_2 - 4 \sum_{n=n_1+n_2} \sum_{\substack{d_1|n_1 \\ 2 \nmid d_1}} \left(\frac{-4}{d_1} \right) \sum_{\substack{d_2|n_2 \\ 4|d_2}} d_2 \\
&= \sum_{n=n_1+n_2} r_2(n_1) \sum_{\substack{d_2|n_2 \\ 2||d_2}} d_2 - \sum_{n=n_1+n_2} r_2(n_1) \sum_{\substack{d_2|n_2 \\ 4|d_2}} d_2.
\end{aligned}$$

If n_2 is odd this expression is zero. If $2 \parallel n_2$ then the second sum is zero. Thus

$$\begin{aligned}
LHS &= \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 1}} r_2(n_1) \sum_{\substack{d_2|2^s N_2 \\ 2||d_2}} d_2 - \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) \sum_{\substack{d_2|2^s N_2 \\ 4|d_2}} d_2 \\
&= \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 1}} r_2(n_1) \sum_{e_2|N_2} 2e_2 \quad (d_2 = 2e_2, e_2 \text{ odd}) \\
&\quad - \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) \sum_{e_2|2^{s-2} N_2} 4e_2 \quad (d_2 = 4e_2, e_2 \text{ odd}) \\
&= 2 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 1}} r_2(n_1) \sigma(N_2) - 4 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) \sigma(2^{s-2} N_2) \\
&= 2 \sum_{\substack{n=n_1+2N_2 \\ n_1, N_2 \text{ odd}}} r_2(n_1) \sigma(N_2) + 2 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) (1 - 2\sigma(2^{s-2})) \sigma(N_2) \\
&= 2 \sum_{\substack{n=n_1+2N_2 \\ n_1 \equiv 3 \pmod{4}}} r_2(n_1) \sigma(N_2) + 2 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) (1 - 2(2^{s-1} - 1)) \sigma(N_2) \\
&= 2 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) (3 - 2^s) \sigma(N_2) \\
&= 6 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) \sigma(N_2) - 2 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} 2^s r_2(n_1) \sigma(N_2)
\end{aligned}$$

$$= 6 \sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) \sigma(N_2) - 2 \left(\sum_{d|n} \left(\frac{-4}{d} \right) d^2 - \frac{n}{4} r_2(n) \right) \quad (\text{by Lemma 4.3.2}).$$

Next we consider the right-hand side of the Liouville 2-term identity. We have

$$\begin{aligned} RHS &= \sum_{d|n} \left(1 - d + \frac{2n}{d} \right) d \left(\frac{-4}{d} \right) - 2 \sum_{d|n} \sum_{1 \leq k \leq d} \left(\frac{-4}{k} \right) k \\ &= \sum_{d|n} \left(\frac{-4}{d} \right) d - \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + 2n \sum_{d|n} \left(\frac{-4}{d} \right) - 2 \sum_{d|n} \sum_{1 \leq k \leq d} \left(\frac{-4}{k} \right) k \\ &= \sum_{d|n} \left(\frac{-4}{d} \right) d - \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + n \frac{r_2(n)}{2} - 2 \sum_{d|n} \sum_{1 \leq k \leq d} \left(\frac{-4}{k} \right) k. \end{aligned}$$

Equating the *LHS* and *RHS* we obtain

$$\begin{aligned} &\sum_{\substack{n=n_1+2^s N_2 \\ n_1, N_2 \text{ odd}, s \geq 2}} r_2(n_1) \sigma(N_2) \\ &= \frac{1}{6} \left(2 \left(\sum_{d|n} \left(\frac{-4}{d} \right) d^2 - \frac{n}{4} r_2(n) \right) + \sum_{d|n} \left(\frac{-4}{d} \right) d - \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \right. \\ &\quad \left. + n \frac{r_2(n)}{2} - 2 \sum_{d|n} \sum_{1 \leq k \leq d} \left(\frac{-4}{k} \right) k \right) \\ &= \frac{1}{6} \left(\sum_{d|n} \left(\frac{-4}{d} \right) d + \sum_{d|n} \left(\frac{-4}{d} \right) d^2 - 2 \sum_{d|n} \sum_{k=1}^d \left(\frac{-4}{k} \right) k \right), \end{aligned}$$

as required. □

Theorem 4.3.1. *Let n be a positive integer such that $n \equiv 1 \pmod{4}$. Then*

$$r_6(n) = 12 \sum_{d|n} \left(\frac{-4}{d} \right) d^2.$$

Proof. We prove this by applying Liouville's 3-term formula (Theorem 3.5.1) with the function $F(n) = -\frac{1}{4} \left(\frac{-4}{n} \right)$. We begin by considering the left-hand-side of the

formula. We have

$$\begin{aligned}
 & \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \in \mathbb{N}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b-c)) \\
 &= \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \in \mathbb{N}}} \left(\frac{-4}{abc} \right) \quad (\text{by Lemma 4.2.1}) \\
 &= \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \in \mathbb{N}}} \sum_{d_1|n_1} \sum_{d_2|n_2} \sum_{d_3|n_3} \left(\frac{-4}{d_1 d_2 d_3} \right) \\
 &= \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \in \mathbb{N}}} \left(\sum_{d_1|n_1} \left(\frac{-4}{d_1} \right) \right) \left(\sum_{d_2|n_2} \left(\frac{-4}{d_2} \right) \right) \left(\sum_{d_3|n_3} \left(\frac{-4}{d_3} \right) \right) \\
 &= \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \in \mathbb{N}}} \left(\frac{r_2(n_1)}{4} \right) \left(\frac{r_2(n_2)}{4} \right) \left(\frac{r_2(n_3)}{4} \right) \\
 &= \frac{1}{64} \sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \in \mathbb{N}}} r_2(n_1) r_2(n_2) r_2(n_3) \\
 &= \frac{1}{64} \left(\sum_{\substack{n_1+n_2+n_3=n \\ n_1, n_2, n_3 \in \mathbb{N} \cup \{0\}}} r_2(n_1) r_2(n_2) r_2(n_3) - \binom{3}{1} \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \in \mathbb{N} \cup \{0\}}} r_2(n_1) r_2(n_2) + \binom{3}{2} \sum_{\substack{n_1=n \\ n_1 \in \mathbb{N} \cup \{0\}}} r_2(n_1) \right) \\
 &= \frac{r_6(n) - 3r_4(n) + 3r_2(n)}{64}.
 \end{aligned}$$

Next we consider the right-hand-side of Liouville's 3-term formula (Theorem 3.5.1).

Again with $F(n) = -\frac{1}{4} \left(\frac{-4}{n} \right)$, we have

$$\begin{aligned}
 RHS &= \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) \left(-\frac{1}{4} \right) \left(\frac{-4}{d} \right) \\
 &+ 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) \left(-\frac{1}{4} \right) \left(\frac{-4}{k} \right) - 6 \sum_{\substack{n=n_1+n_2 \\ n_1, n_2 \in \mathbb{N}}} \sigma(n_1) \sum_{d|n_2} \left(-\frac{1}{4} \right) \left(\frac{-4}{d} \right).
 \end{aligned}$$

The right-hand side can be broken up into three sums. We begin by examining the

first sum. We have

$$\begin{aligned}
 & -\frac{1}{4} \sum_{d|n} \left(\frac{d^2 - 3d + 2}{2} + 3 \left(\frac{n}{d} - 1 \right) \left(\frac{n}{d} - d \right) \right) \left(\frac{-4}{d} \right) \\
 &= -\frac{1}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + \frac{3}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d - \frac{1}{4} \sum_{d|n} \left(\frac{-4}{d} \right) - \frac{3}{4} \sum_{d|n} \left(\frac{-4}{d} \right) \left(\frac{n}{d} \right)^2 \\
 & \quad + \frac{3}{4} n \sum_{d|n} \left(\frac{-4}{d} \right) + \frac{3}{4} \sum_{d|n} \left(\frac{-4}{d} \right) \frac{n}{d} - \frac{3}{4} \sum_{d|n} \left(\frac{-4}{d} \right) d \\
 &= -\frac{1}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + \frac{3}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d - \frac{1}{16} r_2(n) - \frac{3}{4} \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 \\
 & \quad + \frac{3}{16} n r_2(n) + \frac{3}{4} \sum_{d|n} \left(\frac{-4}{n/d} \right) d - \frac{3}{4} \sum_{d|n} \left(\frac{-4}{d} \right) d \\
 &= -\frac{7}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 - \frac{r_2(n)}{16} (1 - 3n) + \frac{3}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d,
 \end{aligned}$$

by Lemma 1.8.4.

Next we consider the second sum. We have

$$\begin{aligned}
 & 3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) \left(-\frac{1}{4} \right) \left(\frac{-4}{k} \right) \\
 &= -\frac{3}{2} \sum_{d|n} d \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) + \frac{3}{2} \sum_{d|n} \frac{n}{d} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) + \frac{3}{4} \sum_{d|n} \sum_{1 \leq k < d} k \left(\frac{-4}{k} \right).
 \end{aligned}$$

Recalling that d is odd, we conclude that

$$\sum_{1 \leq k < d} \left(\frac{-4}{k} \right) = \sum_{k=1}^{d-1} \left(\frac{-4}{k} \right) = \begin{cases} 1 - 1 + 1 - 1 + \cdots - 1 = 0, & \text{if } d \equiv 1 \pmod{4}, \\ 1 - 1 + 1 - 1 + \cdots + 1 = 1, & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

Therefore

$$\sum_{d|n} \frac{n}{d} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right)$$

$$\begin{aligned}
 &= \sum_{d|n} \frac{n}{d} \begin{cases} 0, & \text{if } d \equiv 1 \pmod{4}, \\ 1, & \text{if } d \equiv 3 \pmod{4}. \end{cases} \\
 &= \sum_{d|n} d \begin{cases} 0, & \text{if } n/d \equiv 1 \pmod{4}, \\ 1, & \text{if } n/d \equiv 3 \pmod{4}. \end{cases} \\
 &= \sum_{d|n} d \begin{cases} 0, & \text{if } d \equiv 1 \pmod{4}, \\ 1, & \text{if } d \equiv 3 \pmod{4}. \end{cases} \quad (\text{as } n \equiv 1 \pmod{4}) \\
 &= \sum_{d|n} d \sum_{1 \leq k < d} \left(\frac{-4}{k} \right).
 \end{aligned}$$

Therefore, the second sum is

$$3 \sum_{d|n} \sum_{1 \leq k < d} \left(2d - \frac{2n}{d} - k \right) \left(-\frac{1}{4} \right) \left(\frac{-4}{k} \right) = \frac{3}{4} \sum_{d|n} \sum_{1 \leq k < d} k \left(\frac{-4}{k} \right).$$

Finally, we consider the third sum,

$$-6 \sum_{\substack{n=n_1+n_2 \\ n_1, n_2 \in \mathbb{N}}} \sigma(n_1) \sum_{d|n_2} \left(-\frac{1}{4} \right) \left(\frac{-4}{d} \right),$$

which we can rewrite as

$$\frac{3}{2} \sum_{\substack{n=2^{s_1}n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}}} \sigma(2^{s_1}n_1) \sum_{d|n_2} \left(\frac{-4}{d} \right),$$

where n_1 and n_2 are positive odd integers. We also note that since n is odd, we must either have $s_1 = 0$ or $s_2 = 0$. This expression can be simplified as follows.

$$\begin{aligned}
 &\frac{3}{2} \sum_{\substack{n=2^{s_1}n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}}} \sigma(2^{s_1}n_1) \sum_{d|n_2} \left(\frac{-4}{d} \right) \\
 &= \frac{3}{2} \sum_{\substack{n=2^{s_1}n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}}} \frac{r_2(n_2)}{4} \sigma(2^{s_1}) \sigma(n_1)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{8} \sum_{\substack{n=2^{s_1}n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}}} r_2(n_2)(2^{s_1+1} - 1)\sigma(n_1) \\
&= \frac{3}{4} \sum_{\substack{n=2^{s_1}n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}}} 2^{s_1}r_2(n_2)\sigma(n_1) - \frac{3}{8} \sum_{\substack{n=2^{s_1}n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}}} r_2(n_2)\sigma(n_1) \\
&= \frac{3}{4} \sum_{\substack{n=2^{s_1}n_1+n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} 2^{s_1}r_2(n_2)\sigma(n_1) + \frac{3}{4} \sum_{\substack{n=n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}, s_2 \geq 1}} r_2(n_2)\sigma(n_1) \\
&\quad - \frac{3}{8} \sum_{\substack{n=2^{s_1}n_1+n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} r_2(n_2)\sigma(n_1) - \frac{3}{8} \sum_{\substack{n=n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}, s_2 \geq 1}} r_2(n_2)\sigma(n_1) \\
&= \frac{3}{4} \sum_{\substack{n=2^{s_1}n_1+n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} 2^{s_1}r_2(n_2)\sigma(n_1) + \frac{3}{8} \sum_{\substack{n=n_1+2^{s_2}n_2 \\ n_1, n_2 \text{ odd}, s_2 \geq 1}} r_2(n_2)\sigma(n_1) \\
&\quad - \frac{3}{8} \sum_{\substack{n=2^{s_1}n_1+n_2 \\ n_1, n_2 \text{ odd}, s_1 \geq 1}} r_2(n_2)\sigma(n_1).
\end{aligned}$$

Appealing to Lemmas 4.3.2, 4.3.3 and 4.3.4 we simplify the third sum as follows.

$$\begin{aligned}
&\frac{3}{2} \sum_{n=2^{s_1}n_1+2^{s_2}n_2} \sigma(2^{s_1}n_1) \sum_{d|n_2} \left(\frac{-4}{d}\right) \\
&= \frac{3}{4} \left(\sum_{d|n} \left(\frac{-4}{d}\right) d^2 - \frac{n}{4} r_2(n) \right) + \frac{3}{8} \left(\frac{r_6(n)}{12} - \frac{r_4(n)}{8} \right) \\
&\quad - \frac{3}{8} \cdot \frac{1}{6} \left(\sum_{d|n} \left(\frac{-4}{d}\right) d + \sum_{d|n} \left(\frac{-4}{d}\right) d^2 - 2 \sum_{d|n} \sum_{k=1}^d \left(\frac{-4}{k}\right) k \right) \\
&= \frac{1}{32} r_6(n) - \frac{3}{64} r_4(n) - \frac{3n}{16} r_2(n) - \frac{1}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d + \frac{11}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d^2 \\
&\quad + \frac{1}{8} \sum_{d|n} \sum_{k=1}^d \left(\frac{-4}{k}\right) k \\
&= \frac{1}{32} r_6(n) - \frac{3}{64} r_4(n) - \frac{3n}{16} r_2(n) + \frac{1}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d + \frac{11}{16} \sum_{d|n} \left(\frac{-4}{d}\right) d^2
\end{aligned}$$

$$+ \frac{1}{8} \sum_{d|n} \sum_{k=1}^{d-1} \left(\frac{-4}{k} \right) k.$$

Therefore

$$\begin{aligned} RHS &= \left(-\frac{7}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 - \frac{r_2(n)}{16} (1 - 3n) + \frac{3}{8} \sum_{d|n} \left(\frac{-4}{d} \right) d \right) \\ &\quad + \left(\frac{3}{4} \sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k \right) + \left(\frac{1}{32} r_6(n) - \frac{3}{64} r_4(n) - \frac{3n}{16} r_2(n) \right) \\ &\quad + \frac{1}{16} \sum_{d|n} \left(\frac{-4}{d} \right) d + \frac{11}{16} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + \frac{1}{8} \sum_{d|n} \sum_{k=1}^{d-1} \left(\frac{-4}{k} \right) k \\ &= \frac{1}{32} r_6(n) - \frac{3}{64} r_4(n) - \frac{1}{16} r_2(n) + \frac{7}{16} \sum_{d|n} \left(\frac{-4}{d} \right) d - \frac{3}{16} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \\ &\quad + \frac{7}{8} \sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k. \end{aligned}$$

Equating the *LHS* and *RHS*, and multiplying both sides by 64, we get

$$\begin{aligned} r_6(n) - 3r_4(n) + 3r_2(n) &= 2r_6(n) - 3r_4(n) - 4r_2(n) + 28 \sum_{d|n} d \left(\frac{-4}{d} \right) \\ &\quad - 12 \sum_{d|n} d^2 \left(\frac{-4}{d} \right) + 56 \sum_{d|n} \sum_{1 \leq k < d} k \left(\frac{-4}{k} \right). \end{aligned}$$

Solving for r_6 , we get

$$\begin{aligned} r_6(n) &= 12 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + 7r_2(n) - 28 \sum_{d|n} \left(\frac{-4}{d} \right) d - 56 \sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k \\ &= 12 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + 28 \sum_{d|n} \left(\frac{-4}{d} \right) - 28 \sum_{d|n} \left(\frac{-4}{d} \right) d - 56 \sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k \\ &= 12 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + 28 \left(\sum_{d|n} \left(\frac{-4}{d} \right) (1 - d) - 2 \sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k \right) \\ &= 12 \sum_{d|n} \left(\frac{-4}{d} \right) d^2, \end{aligned}$$

by Lemma 4.3.1, as required. □

4.4 Arithmetic proof of Jacobi's six squares theorem for $n \equiv 0 \pmod{2}$

In this section we prove the six squares theorem in the special case when $n \equiv 0 \pmod{2}$. We prove the three subcases $n \equiv 2 \pmod{4}$ (Theorem 4.4.1), $n \equiv 4 \pmod{8}$ (Theorem 4.4.2) and $n \equiv 0 \pmod{8}$ (Theorem 4.4.3) separately. In order to prove these theorems, we use standard arguments relating $r_6(n)$ for n even to $r_6(n)$ for n odd, which we already have formulae for.

Definition 4.4.1. For $a \in \{0, 1, 2, 3, 4, 5, 6\}$, we define $r_6^{a, 6-a}(n)$ to be the number of representations $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6$ with $n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$ and a of $x_1, x_2, x_3, x_4, x_5, x_6$ even and $6 - a$ of $x_1, x_2, x_3, x_4, x_5, x_6$ odd.

Lemma 4.4.1. Let n be a positive integer. Then

$$r_6^{6,0}(n) = 0, \quad \text{if } n \not\equiv 0 \pmod{4},$$

$$r_6^{5,1}(n) = 0, \quad \text{if } n \not\equiv 1 \pmod{4},$$

$$r_6^{4,2}(n) = 0, \quad \text{if } n \not\equiv 2 \pmod{4},$$

$$r_6^{3,3}(n) = 0, \quad \text{if } n \not\equiv 3 \pmod{4},$$

$$r_6^{2,4}(n) = 0, \quad \text{if } n \not\equiv 0 \pmod{4},$$

$$r_6^{1,5}(n) = 0, \quad \text{if } n \not\equiv 1 \pmod{4},$$

$$r_6^{0,6}(n) = 0, \quad \text{if } n \not\equiv 6 \pmod{8}.$$

Proof. Suppose

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$$

where a of $x_1, x_2, x_3, x_4, x_5, x_6$ are even and $6 - a$ of $x_1, x_2, x_3, x_4, x_5, x_6$ are odd. Then

$$n \equiv \underbrace{1 + \cdots + 1}_{6-a} \equiv 6 - a \equiv 2 - a \pmod{4}.$$

Therefore if $n \not\equiv 2 - a \pmod{4}$ then $r_6^{a,6-a}(n) = 0$. This proves the cases when $a = 1, 2, 3, 4, 5, 6$. When $a = 0$, we have

$$n \equiv \underbrace{1 + \cdots + 1}_6 \equiv 6 \pmod{8}.$$

Therefore if $n \not\equiv 6 \pmod{8}$ then $r_6^{0,6}(n) = 0$. □

Lemma 4.4.2. *Let n be a positive integer. Then*

$$r_6(n) = \begin{cases} r_6^{2,4}(n) + r_6^{6,0}(n), & \text{if } n \equiv 0 \pmod{4}, \\ r_6^{1,5}(n) + r_6^{5,1}(n), & \text{if } n \equiv 1 \pmod{4}, \\ r_6^{0,6}(n) + r_6^{4,2}(n), & \text{if } n \equiv 2 \pmod{4}, \\ r_6^{3,3}(n), & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Proof. Clearly

$$r_6(n) = \sum_{a=0}^6 r_6^{a,6-a}(n).$$

The lemma follows by Lemma 4.4.1. □

Lemma 4.4.2 is the basis for our proof of the formula for $r_6(n)$ when n is even.

The following three lemmas address the case when $n \equiv 2 \pmod{4}$.

Lemma 4.4.3. *Let n be a positive integer satisfying $n \equiv 6 \pmod{8}$. Then*

$$r_6^{0,6}(n) = \frac{2}{5} r_6\left(\frac{n}{2}\right).$$

Proof. Suppose

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2,$$

where x_1, x_2, x_3, x_4, x_5 and x_6 are odd. For $i, j \in \{1, 2, 3, 4, 5, 6\}$, we observe that either

$$x_i \equiv x_j \pmod{4} \quad \text{or} \quad x_i \equiv -x_j \pmod{4}.$$

We define A to be the set of solutions $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{N}^6$ to $n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$ with $x_1, x_2, x_3, x_4, x_5, x_6$ odd and $x_1 \equiv x_2 \pmod{4}$, $x_3 \equiv x_4 \pmod{4}$ and $x_5 \equiv x_6 \pmod{4}$. Then

$$r_6^{0,6}(n) = 8 |A|.$$

Next we define B to be the set of solutions $(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{N}^6$ to $n/2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2$ with y_1, y_3, y_5 odd, and y_2, y_4, y_6 even. Then

$$r_6^{3,3}(n/2) = \binom{6}{3} |B| = 20 |B|.$$

Let $\Phi : A \rightarrow B$ be defined by

$$\Phi(x_1, x_2, x_3, x_4, x_5, x_6) = \left(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}, \frac{x_3 + x_4}{2}, \frac{x_3 - x_4}{2}, \frac{x_5 + x_6}{2}, \frac{x_5 - x_6}{2} \right).$$

Then Φ is clearly one-to-one, and is onto since $(2y_1 + 2y_2 + 1, 2y_1 - 2y_2 + 1, 2y_3 + 2y_4 + 1, 2y_3 - 2y_4 + 1, 2y_5 + 2y_6 + 1, 2y_5 - 2y_6 + 1) \in A$, and

$$\Phi(2y_1 + 2y_2 + 1, 2y_1 - 2y_2 + 1, 2y_3 + 2y_4 + 1, 2y_3 - 2y_4 + 1, 2y_5 + 2y_6 + 1, 2y_5 - 2y_6 + 1)$$

$$= (2y_1 + 1, 2y_2, 2y_3 + 1, 2y_4, 2y_5 + 1, 2y_6) \in B.$$

It follows that $|A| = |B|$. Therefore, by Lemma 4.4.2, we have

$$r_6^{0,6}(n) = 8 |A| = 8 |B| = \frac{8}{20} r_6^{3,3} \left(\frac{n}{2} \right) = \frac{2}{5} r_6 \left(\frac{n}{2} \right),$$

as required. □

Lemma 4.4.4. *Let n be a positive integer satisfying $n \equiv 2 \pmod{4}$. Let $N = o(n)$.*

Then

$$r_6^{0,6}(n) = \left(4 \left(\frac{-4}{N} \right) - 4 \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2.$$

Proof. If $n \equiv 2 \pmod{8}$, then by Lemma 4.4.1, $r_6^{0,6}(n) = 0$. For $n \equiv 2 \pmod{8}$, we have $N \equiv 1 \pmod{4}$, so that

$$\left(\frac{-4}{N} \right) = 1.$$

Then

$$\left(4 \left(\frac{-4}{N} \right) - 4 \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2 = 0,$$

as required.

When $n \equiv 6 \pmod{8}$, we have $N = \frac{n}{2} \equiv 3 \pmod{4}$, so that

$$\left(\frac{-4}{N} \right) = -1.$$

Then

$$\left(4 \left(\frac{-4}{N} \right) - 4 \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2$$

$$\begin{aligned}
 &= -8 \sum_{d|N} \left(\frac{-4}{d} \right) d^2 \\
 &= \frac{2}{5} \left(-20 \sum_{d|n/2} \left(\frac{-4}{d} \right) d^2 \right) \\
 &= \frac{2}{5} r_6 \left(\frac{n}{2} \right) \quad (\text{by Theorem 4.2.1}) \\
 &= r_6^{0,6}(n),
 \end{aligned}$$

by Lemma 4.4.3. □

Lemma 4.4.5. *Let n be a positive integer satisfying $n \equiv 2 \pmod{4}$. Then*

$$r_6^{4,2}(n) = \begin{cases} 5r_6 \left(\frac{n}{2} \right), & \text{if } n \equiv 2 \pmod{8}, \\ 3r_6 \left(\frac{n}{2} \right), & \text{if } n \equiv 6 \pmod{8}. \end{cases}$$

Proof. Let A be the set of solutions $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{N}^6$ to

$$n = (2x_1)^2 + (2x_2)^2 + (2x_3)^2 + (2x_4)^2 + (2x_5 + 1)^2 + (2x_6 + 1)^2,$$

then

$$r_6^{4,2}(n) = \binom{6}{2} |A| = 15 |A|.$$

If $x_5 \not\equiv x_6 \pmod{2}$, we replace x_5 by $-x_5 - 1$. Then $-x_5 - 1 \equiv x_6 \pmod{2}$, and the sum is unaffected since

$$(2(-x_5 - 1) + 1)^2 = (-2x_5 - 1)^2 = (2x_5 + 1)^2.$$

Next we define

$$B = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in A \mid x_5 \equiv x_6 \pmod{2}\},$$

so that

$$|A| = 2|B|.$$

In order to calculate the cardinality of B , we define

$$B_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \equiv x_2 \pmod{2}, x_3 \equiv x_4 \pmod{2}\},$$

$$B_2 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \not\equiv x_2 \pmod{2}, x_3 \equiv x_4 \pmod{2}\},$$

$$B_3 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \equiv x_2 \pmod{2}, x_3 \not\equiv x_4 \pmod{2}\},$$

$$B_4 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \not\equiv x_2 \pmod{2}, x_3 \not\equiv x_4 \pmod{2}\}.$$

Then clearly

$$|B| = |B_1| + |B_2| + |B_3| + |B_4|.$$

Let C be the set of solutions $(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{N}^6$ such that $n/2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2$, and define

$$C_1 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_1, y_2, y_3, y_4, y_6 \text{ even}, y_5 \text{ odd}\},$$

$$C_2 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_3, y_4, y_6 \text{ even}, y_1, y_2, y_5 \text{ odd}\},$$

$$C_3 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_1, y_2, y_6 \text{ even}, y_3, y_4, y_5 \text{ odd}\},$$

$$C_4 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_6 \text{ even}, y_1, y_2, y_3, y_4, y_5 \text{ odd}\}.$$

Then clearly

$$|C_1| = \frac{r_6^{5,1}\left(\frac{n}{2}\right)}{6}, \quad |C_2| = \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20}, \quad |C_3| = \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20}, \quad |C_4| = \frac{r_6^{1,5}\left(\frac{n}{2}\right)}{6}.$$

Let $\Phi_i : B_i \rightarrow C_i$ be defined by

$$\Phi_i(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 + x_2, x_1 - x_2, x_3 + x_4, x_3 - x_4, x_5 + x_6 + 1, x_5 - x_6).$$

Then Φ_i is clearly one-to-one, and we show that it is onto for each $i \in \{1, 2, 3, 4\}$.

Let $(2y_1, 2y_2, 2y_3, 2y_4, 2y_5 + 1, 2y_6) \in C_1$, $(2y_1 + 1, 2y_2 + 1, 2y_3, 2y_4, 2y_5 + 1, 2y_6) \in C_2$, $(2y_1, 2y_2, 2y_3 + 1, 2y_4 + 1, y_3 - y_4, 2y_5 + 1, 2y_6) \in C_3$ and $(2y_1 + 1, 2y_2 + 1, 2y_3 + 1, 2y_4 + 1, 2y_5 + 1, 2y_6) \in C_4$, then

$$\begin{aligned} \Phi_1(y_1 + y_2, y_1 - y_2, y_3 + y_4, y_3 - y_4, y_5 + y_6, y_5 - y_6) \\ = (2y_1, 2y_2, 2y_3, 2y_4, 2y_5 + 1, 2y_6) \end{aligned}$$

$$\begin{aligned} \Phi_2(y_1 + y_2 + 1, y_1 - y_2, y_3 + y_4, y_3 - y_4, y_5 + y_6, y_5 - y_6) \\ = (2y_1 + 1, 2y_2 + 1, 2y_3, 2y_4, 2y_5 + 1, 2y_6) \end{aligned}$$

$$\begin{aligned} \Phi_3(y_1 + y_2, y_1 - y_2, y_3 + y_4 + 1, y_3 - y_4, y_5 + y_6, y_5 - y_6) \\ = (2y_1, 2y_2, 2y_3 + 1, 2y_4 + 1, 2y_5 + 1, 2y_6) \end{aligned}$$

$$\begin{aligned} \Phi_4(y_1 + y_2 + 1, y_1 - y_2, y_3 + y_4 + 1, y_3 - y_4, y_5 + y_6, y_5 - y_6) \\ = (2y_1 + 1, 2y_2 + 1, 2y_3 + 1, 2y_4 + 1, 2y_5 + 1, 2y_6) \end{aligned}$$

where $(y_1 + y_2, y_1 - y_2, y_3 + y_4, y_3 - y_4, y_5 + y_6, y_5 - y_6) \in B_1$, $(y_1 + y_2 + 1, y_1 - y_2, y_3 + y_4, y_3 - y_4, y_5 + y_6, y_5 - y_6) \in B_2$, $(y_1 + y_2, y_1 - y_2, y_3 + y_4 + 1, y_3 - y_4, y_5 + y_6, y_5 - y_6) \in B_3$ and $(y_1 + y_2 + 1, y_1 - y_2, y_3 + y_4 + 1, y_3 - y_4, y_5 + y_6, y_5 - y_6) \in B_4$.

It follows that

$$r_6^{4,2}(n) = 15 \mid A \mid = 30 \mid B \mid$$

$$\begin{aligned}
 &= 30(|B_1| + |B_2| + |B_3| + |B_4|) \\
 &= 30(|C_1| + |C_2| + |C_3| + |C_4|) \\
 &= 30 \left(\frac{r_6^{5,1}\left(\frac{n}{2}\right)}{6} + \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20} + \frac{r_6^{3,3}\left(\frac{n}{2}\right)}{20} + \frac{r_6^{1,5}\left(\frac{n}{2}\right)}{6} \right) \\
 &= 5r_6^{5,1}\left(\frac{n}{2}\right) + 3r_6^{3,3}\left(\frac{n}{2}\right) + 5r_6^{1,5}\left(\frac{n}{2}\right) \\
 &= \begin{cases} 5 \left(r_6^{5,1}\left(\frac{n}{2}\right) + r_6^{1,5}\left(\frac{n}{2}\right) \right), & \text{if } \frac{n}{2} \equiv 1 \pmod{4}, \\ 3r_6^{3,3}\left(\frac{n}{2}\right), & \text{if } \frac{n}{2} \equiv 3 \pmod{4}, \end{cases} \quad (\text{by Lemma 4.4.1}) \\
 &= \begin{cases} 5r_6\left(\frac{n}{2}\right), & \text{if } n \equiv 2 \pmod{8}, \\ 3r_6\left(\frac{n}{2}\right), & \text{if } n \equiv 6 \pmod{8}, \end{cases} \quad (\text{by Lemma 4.4.2})
 \end{aligned}$$

as required. □

Lemma 4.4.6. *Let n be a positive integer satisfying $n \equiv 2 \pmod{4}$. Let $N = o(n)$.*

Then

$$r_6^{4,2}(n) = 60 \left(\frac{-4}{N} \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2.$$

Proof. Appealing to Lemma 4.4.5, and Theorems 4.2.1 and 4.3.1, we obtain

$$\begin{aligned}
 r_6^{4,2}(n) &= \begin{cases} 5r_6\left(\frac{n}{2}\right), & \text{if } n \equiv 2 \pmod{8}, \\ 3r_6\left(\frac{n}{2}\right), & \text{if } n \equiv 6 \pmod{8}, \end{cases} \\
 &= \begin{cases} 5r_6(N), & \text{if } N \equiv 1 \pmod{4}, \\ 3r_6(N), & \text{if } N \equiv 3 \pmod{4}, \end{cases} \\
 &= \begin{cases} 60 \sum_{d|N} \left(\frac{-4}{d} \right) d^2, & \text{if } N \equiv 1 \pmod{4}, \\ -60 \sum_{d|N} \left(\frac{-4}{d} \right) d^2, & \text{if } N \equiv 3 \pmod{4}, \end{cases}
 \end{aligned}$$

$$= 60 \left(\frac{-4}{N} \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2,$$

as claimed. □

Lemma 4.4.7. *Let $n \in \mathbb{N}$ be even. Set $n = 2^k N$ with $k, N \in \mathbb{N}$, N odd. Suppose $m = 2^l N$ for a nonnegative integer $l \leq k$. Then*

$$\sum_{d|m} \left(\frac{-4}{m/d} \right) d^2 = 2^{2(l-k)} \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2.$$

Proof. By changing d , where $d | m$, to c , where $c | n$, by setting $c = 2^{k-l}d$, we have

$$\sum_{d|m} \left(\frac{-4}{m/d} \right) d^2 = \sum_{c|n} \left(\frac{-4}{(2^{k-l}m/c)} \right) \left(\frac{c}{2^{k-l}} \right)^2 = 2^{2(l-k)} \sum_{c|n} \left(\frac{-4}{n/c} \right) c^2,$$

as required. □

We are now ready to prove the formula for $r_6(n)$ when $n \equiv 2 \pmod{4}$.

Theorem 4.4.1. *Let n be a positive integer with $n \equiv 2 \pmod{4}$. Let $N = o(n)$.*

Then

$$r_6(n) = 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2.$$

Proof. Appealing to Lemmas 4.4.2, 4.4.4, and 4.4.6, we obtain

$$\begin{aligned} r_6(n) &= r_6^{0,6}(n) + r_6^{4,2}(n) \\ &= \left(4 \left(\frac{-4}{N} \right) - 4 \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2 + 60 \left(\frac{-4}{N} \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2 \\ &= \left(64 \left(\frac{-4}{N} \right) - 4 \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2 \end{aligned}$$

$$\begin{aligned}
 &= 64 \sum_{d|N} \left(\frac{-4}{Nd} \right) d^2 - 4 \sum_{d|N} \left(\frac{-4}{d} \right) d^2 \\
 &= 64 \sum_{d|N} \left(\frac{-4}{N/d} \right) \left(\frac{-4}{d^2} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \\
 &= 64 \sum_{d|N} \left(\frac{-4}{N/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \\
 &= 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2,
 \end{aligned}$$

by Lemma 4.4.7. □

The following two lemmas address the case when $n \equiv 0 \pmod{4}$.

Lemma 4.4.8. *Let n be a positive integer satisfying $n \equiv 0 \pmod{4}$. Then*

$$r_6^{6,0}(n) = r_6 \left(\frac{n}{4} \right).$$

Proof. This result follows immediately from the fact that

$$n = (2x_1)^2 + (2x_2)^2 + (2x_3)^2 + (2x_4)^2 + (2x_5)^2 + (2x_6)^2$$

if and only if

$$\frac{n}{4} = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2.$$

□

Lemma 4.4.9. *Let n be a positive integer satisfying $n \equiv 0 \pmod{4}$. Let $N = o(n)$*

and $n = 2^k N$. Then

$$r_6^{2,4}(n) = 15 \cdot 2^{2k} \left(\frac{-4}{N} \right) \sum_{d|N} \left(\frac{-4}{d} \right) d^2.$$

Proof. Let A to be the set of solutions $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{N}^6$ to

$$n = (2x_1)^2 + (2x_2)^2 + (2x_3 + 1)^2 + (2x_4 + 1)^2 + (2x_5 + 1)^2 + (2x_6 + 1)^2,$$

so that

$$r_6^{2,4}(n) = \binom{6}{2} |A| = 15 |A|.$$

Similarly to the proof of Lemma 4.4.5, we can choose x_3, x_4, x_5, x_6 so that $x_3 \equiv x_4 \pmod{2}$ and $x_5 \equiv x_6 \pmod{2}$. We define

$$B = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in A \mid x_3 \equiv x_4 \pmod{2}, x_5 \equiv x_6 \pmod{2}\},$$

so that

$$|A| = 4 |B|.$$

In order to calculate the cardinality of B , we define

$$B_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \equiv x_2 \pmod{2}\},$$

$$B_2 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in B \mid x_1 \not\equiv x_2 \pmod{2}\}.$$

Then clearly

$$|B| = |B_1| + |B_2|.$$

Let C be the set of solutions $(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{N}^6$ to $n/2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2$, and define

$$C_1 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_1, y_2, y_4, y_6 \text{ even, } y_3, y_5 \text{ odd}\},$$

$$C_2 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in C \mid y_4, y_6 \text{ even, } y_1, y_2, y_3, y_5 \text{ odd}\}.$$

Then clearly

$$|C_1| = \frac{r_6^{4,2}\left(\frac{n}{2}\right)}{15} \quad \text{and} \quad |C_2| = \frac{r_6^{2,4}\left(\frac{n}{2}\right)}{15}.$$

Let $\Phi_i : B_i \rightarrow C_i$ ($i = 1, 2$) be defined by

$$\Phi_i(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 + x_2, x_1 - x_2, x_3 + x_4 + 1, x_3 - x_4, x_5 + x_6 + 1, x_5 - x_6).$$

Then Φ_i is clearly one-to-one, and we show that it is onto for $i = 1, 2$.

Let $(2y_1, 2y_2, 2y_3 + 1, 2y_4, 2y_5 + 1, 2y_6) \in C_1$ and $(2y_1 + 1, 2y_2 + 1, 2y_3 + 1, 2y_4, 2y_5 + 1, 2y_6) \in C_2$, then

$$\begin{aligned} & \Phi_1(y_1 + y_2, y_1 - y_2, y_3 + y_4, y_3 - y_4, y_5 + y_6, y_5 - y_6) \\ &= (2y_1, 2y_2, 2y_3 + 1, 2y_4, 2y_5 + 1, 2y_6) \\ & \Phi_2(y_1 + y_2 + 1, y_1 - y_2, y_3 + y_4, y_3 - y_4, y_5 + y_6, y_5 - y_6) \\ &= (2y_1 + 1, 2y_2 + 1, 2y_3 + 1, 2y_4, 2y_5 + 1, 2y_6) \end{aligned}$$

where $(y_1 + y_2, y_1 - y_2, y_3 + y_4, y_3 - y_4, y_5 + y_6, y_5 - y_6) \in B_1$ and $(y_1 + y_2 + 1, y_1 - y_2, y_3 + y_4, y_3 - y_4, y_5 + y_6, y_5 - y_6) \in B_2$.

It follows that

$$\begin{aligned} r_6^{2,4}(n) &= 15 |A| = 60 |B| = 60(|B_1| + |B_2|) \\ &= 60(|C_1| + |C_2|) \\ &= 60 \left(\frac{r_6^{4,2}\left(\frac{n}{2}\right)}{15} + \frac{r_6^{2,4}\left(\frac{n}{2}\right)}{15} \right) \end{aligned}$$

$$= 4 \begin{cases} r_6^{2,4}\left(\frac{n}{2}\right), & \text{if } n \equiv 0 \pmod{8}, \\ r_6^{4,2}\left(\frac{n}{2}\right), & \text{if } n \equiv 4 \pmod{8}, \end{cases}$$

by Lemma 4.4.1.

Now set $n = 2^k N$, where k is an integer with $k \geq 2$ and N is an odd positive integer. Then

$$\begin{aligned} r_6^{2,4}(n) &= r_6^{2,4}(2^k N) \\ &= 2^2 r_6^{2,4}(2^{k-1} N) \\ &\dots \\ &= 2^{2(k-2)} r_6^{2,4}(4N) \\ &= 2^{2(k-1)} r_6^{4,2}(2N) \\ &= 2^{2(k-1)} \cdot 60 \left(\frac{-4}{N}\right) \sum_{d|N} \left(\frac{-4}{d}\right) d^2 \quad (\text{by Lemma 4.4.6}) \\ &= 15 \cdot 2^{2k} \left(\frac{-4}{N}\right) \sum_{d|N} \left(\frac{-4}{d}\right) d^2, \end{aligned}$$

as required. □

In the following two theorems, we use the fact that when n is odd,

$$r_6(n) = 16 \sum_{d|n} \left(\frac{-4}{n/d}\right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d}\right) d^2,$$

as proven in an elementary manner in Sections 4.2 and 4.3.

Theorem 4.4.2. *Let n be a positive integer satisfying $n \equiv 4 \pmod{8}$. Then*

$$r_6(n) = 16 \sum_{d|n} \left(\frac{-4}{n/d}\right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d}\right) d^2.$$

Proof. As $n \equiv 4 \pmod{8}$, we set $n = 4N$ for an odd positive integer N . Then

$$\begin{aligned}
r_6(n) &= r_6^{2,4}(4N) + r_6^{6,0}(4N) \quad (\text{by Lemmas 4.4.2}) \\
&= 15 \cdot 2^4 \sum_{d|N} \left(\frac{-4}{N/d} \right) d^2 + r_6(N) \quad (\text{by Lemmas 4.4.8 and 4.4.9}) \\
&= 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 + r_6(N) \quad (\text{by Lemma 4.4.7}) \\
&= 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 + 16 \sum_{d|N} \left(\frac{-4}{N/d} \right) d^2 - 4 \sum_{d|N} \left(\frac{-4}{d} \right) d^2 \\
&= 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \quad (\text{by Lemma 4.4.7})
\end{aligned}$$

as claimed. □

Theorem 4.4.3. *Let n be a positive integer satisfying $n \equiv 0 \pmod{8}$. Then*

$$r_6(n) = 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2.$$

Proof. As $n \equiv 0 \pmod{8}$, we have $n = 2^k N$ for a positive integer $k \geq 3$ and an odd positive integer N . If $k = 3$ then we have

$$\begin{aligned}
&r_6(n) \\
&= r_6^{2,4}(8N) + r_6^{6,0}(8N) \quad (\text{by Lemma 4.4.2}) \\
&= 15 \cdot 2^6 \sum_{d|N} \left(\frac{-4}{N/d} \right) d^2 + r_6(2N) \quad (\text{by Lemmas 4.4.8 and 4.4.9}) \\
&= 15 \cdot 2^6 \sum_{d|N} \left(\frac{-4}{N/d} \right) d^2 + 16 \sum_{d|2N} \left(\frac{-4}{2N/d} \right) d^2 - 4 \sum_{d|2N} \left(\frac{-4}{d} \right) d^2 \quad (\text{by Theorem 4.4.1}) \\
&= 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 + \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \quad (\text{by Lemma 4.4.7})
\end{aligned}$$

$$= 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2,$$

as required. If $k \geq 4$ then we have

$$\begin{aligned} r_6(n) &= r_6^{2,4}(2^k N) + r_6^{6,0}(2^k N) \quad (\text{by Lemma 4.4.2}) \\ &= r_6^{2,4}(2^k N) + r_6(2^{k-2} N) \quad (\text{by Lemma 4.4.8}) \\ &= r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + r_6^{6,0}(2^{k-2} N) \quad (\text{by Lemma 4.4.2}) \\ &= r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + r_6(2^{k-4} N) \quad (\text{by Lemma 4.4.8}) \\ &\dots \\ &= \begin{cases} r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + \dots + r_6^{2,4}(4N) + r_6^{6,0}(4N), & \text{if } k \equiv 0 \pmod{2}, \\ r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + \dots + r_6^{2,4}(8N) + r_6^{6,0}(8N), & \text{if } k \equiv 1 \pmod{2}, \end{cases} \\ &= \begin{cases} r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + \dots + r_6^{2,4}(4N) + r_6(N), & \text{if } k \equiv 0 \pmod{2}, \\ r_6^{2,4}(2^k N) + r_6^{2,4}(2^{k-2} N) + \dots + r_6^{2,4}(8N) + r_6(2N), & \text{if } k \equiv 1 \pmod{2}. \end{cases} \end{aligned}$$

Appealing to Lemmas 4.4.9 and 4.4.7 we obtain

$$r_6^{2,4}(2^l N) = 15 \cdot 2^{2l} \sum_{d|N} \left(\frac{-4}{N/d} \right) d^2 = 15 \cdot 2^{2(l-k)} \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2,$$

for $l \in \mathbb{N}$, $1 \leq l \leq k$. Therefore

$$\begin{aligned} r_6(n) &= \begin{cases} 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 (1 + 2^{2(-2)} + \dots + 2^{2(2-k)}) + r_6(N), & \text{if } k \equiv 0 \pmod{2}, \\ 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 (1 + 2^{2(-2)} + \dots + 2^{2(3-k)}) + r_6(2N), & \text{if } k \equiv 1 \pmod{2}, \end{cases} \\ &= \begin{cases} 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 \sum_{i=0}^{(k-2)/2} 2^{-4i} + r_6(N), & \text{if } k \equiv 0 \pmod{2}, \\ 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 \sum_{i=0}^{(k-3)/2} 2^{-4i} + r_6(2N), & \text{if } k \equiv 1 \pmod{2}, \end{cases} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 \left(\frac{2^{-2k} - 1}{2^{-4} - 1} \right) + r_6(N), & \text{if } k \equiv 0 \pmod{2}, \\ 15 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 \left(\frac{2^{-2k+2} - 1}{2^{-4} - 1} \right) + r_6(2N), & \text{if } k \equiv 1 \pmod{2}, \end{cases} \\
&= \begin{cases} 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 (1 - 2^{-2k}) + r_6(N), & \text{if } k \equiv 0 \pmod{2}, \\ 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 (1 - 2^{-2k+2}) + r_6(2N), & \text{if } k \equiv 1 \pmod{2}. \end{cases}
\end{aligned}$$

But

$$\begin{aligned}
r_6(N) &= 16 \sum_{d|N} \left(\frac{-4}{N/d} \right) d^2 - 4 \sum_{d|N} \left(\frac{-4}{d} \right) d^2 \\
&= 16 \cdot 2^{-2k} \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2,
\end{aligned}$$

and

$$\begin{aligned}
r_6(2N) &= 16 \sum_{d|2N} \left(\frac{-4}{2N/d} \right) d^2 - 4 \sum_{d|2N} \left(\frac{-4}{d} \right) d^2 \\
&= 16 \cdot 2^{-2k+2} \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2,
\end{aligned}$$

by Lemma 4.4.7 and Theorem 4.4.1. Therefore

$$r_6(n) = 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2,$$

as required. □

4.5 Final formula for $r_6(n)$

The following theorem summarizes what we have proven throughout this chapter.

Theorem 4.5.1. *Let n be a positive integer. Then*

$$r_6(n) = 16 \sum_{d|n} \left(\frac{-4}{n/d} \right) d^2 - 4 \sum_{d|n} \left(\frac{-4}{d} \right) d^2.$$

Proof. This result follows immediately from Theorems 4.2.1 and 4.3.1 when $n \equiv 1 \pmod{2}$, and from Theorems 4.4.1, 4.4.2 and 4.4.3 when $n \equiv 0 \pmod{2}$. $\square$

4.6 Sums of six triangular numbers

In this section, we prove the known formula for $\delta_6(n)$, the number of representations of n as the sum of six triangular numbers, in an elementary manner.

Theorem 4.6.1. *Let n be a nonnegative integer. Then*

$$\delta_6(n) = -\frac{1}{8} \sum_{d|4n+3} \left(\frac{-4}{d} \right) d^2.$$

Proof. If n is the sum of six triangular numbers, then we have

$$n = \frac{m_1(m_1 + 1)}{2} + \cdots + \frac{m_6(m_6 + 1)}{2},$$

where m_1, m_2, m_3, m_4, m_5 and m_6 are nonnegative integers. Equivalently, we have

$$8n + 6 = (2m_1 + 1)^2 + \cdots + (2m_6 + 1)^2.$$

Therefore $\delta_6(n)$ is equal to the number of representations of $8n + 6$ as the sum of six positive odd squares. It follows that

$$\delta_6(n) = \frac{1}{2^6} r_6^{0,6}(8n + 6)$$

$$\begin{aligned}
 &= \frac{1}{64} \left(4 \left(\frac{-4}{4n+3} \right) - 4 \right) \sum_{d|4n+3} \left(\frac{-4}{d} \right) d^2 \quad (\text{by Lemma 4.4.4}) \\
 &= \frac{1}{64} (-8) \sum_{d|4n+3} \left(\frac{-4}{d} \right) d^2 \\
 &= -\frac{1}{8} \sum_{d|4n+3} \left(\frac{-4}{d} \right) d^2,
 \end{aligned}$$

as required. $\square$

For a proof of this formula for $\delta_6(n)$ using modular forms see [94, Theorem 4, p. 81]. For a different elementary proof see [43, p. 262]. An analytic proof of this result was given recently by Liu [78, Theorem 8, p. 416].

4.7 Elementary evaluation of $\sum_{ax+ky=n} \left(\frac{-4}{a} \right) b$ for $k = 1, 2, 4$

In this section, as a final application of our results, we use our elementary evaluation of $r_6(n)$ in conjunction with the elementary identity of Huard, Ou, Spearman and Williams [43] to determine in an elementary way the evaluation of the sums

$$\begin{aligned}
 A(n) &= \sum_{ax+by=n} \left(\frac{-4}{a} \right) b, \\
 B(n) &= \sum_{ax+2by=n} \left(\frac{-4}{a} \right) b, \\
 C(n) &= \sum_{ax+4by=n} \left(\frac{-4}{a} \right) b,
 \end{aligned}$$

where $n \in \mathbb{N}$ and the sums are taken over all positive integers a, b, x, y satisfying the stated equality. Our formulae for $A(n)$, $B(n)$ and $C(n)$ (Theorem 4.7.1) have been

conjectured by Huard and Williams.

For $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ we define

$$J_i(n) = \sum_{d|n} \left(\frac{-4}{d} \right) d^i,$$

and

$$K_i(n) = \sum_{d|n} \left(\frac{-4}{n/d} \right) d^i.$$

We note that

$$J_i(2n) = \sum_{d|2n} \left(\frac{-4}{d} \right) d^i = \sum_{d|n} \left(\frac{-4}{d} \right) d^i = J_i(n),$$

and

$$K_i(2n) = \sum_{d|2n} \left(\frac{-4}{2n/d} \right) d^i = \sum_{\substack{d|2n \\ d=2c}} \left(\frac{-4}{n/c} \right) (2c)^i = 2^i \sum_{c|n} \left(\frac{-4}{n/c} \right) c^i = 2^i K_i(n).$$

The following lemmas are needed to evaluate the sums $A(n)$, $B(n)$ and $C(n)$.

Lemma 4.7.1. *Let n be a positive integer. Then*

$$A(n) = B(2n).$$

Proof. We have

$$\begin{aligned} B(2n) &= \sum_{ax+2by=2n} \left(\frac{-4}{a} \right) b = \sum_{\substack{ax+2by=2n \\ ax \text{ even}}} \left(\frac{-4}{a} \right) b \\ &= \sum_{\substack{ax+2by=2n \\ x \text{ even}}} \left(\frac{-4}{a} \right) b = \sum_{\substack{2ax'+2by=2n \\ x'=x/2}} \left(\frac{-4}{a} \right) b = \sum_{ax'+by=n} \left(\frac{-4}{a} \right) b = A(n), \end{aligned}$$

as claimed. □

Lemma 4.7.2. *Let n be a positive integer. Then*

$$B(n) = C(2n).$$

Proof. We have

$$\begin{aligned} C(2n) &= \sum_{ax+4by=2n} \left(\frac{-4}{a}\right) b = \sum_{\substack{ax+4by=2n \\ ax \text{ even}}} \left(\frac{-4}{a}\right) b \\ &= \sum_{\substack{ax+4by=2n \\ x \text{ even}}} \left(\frac{-4}{a}\right) b = \sum_{\substack{2ax'+4by=2n \\ x'=x/2}} \left(\frac{-4}{a}\right) b = \sum_{ax'+2by=n} \left(\frac{-4}{a}\right) b = B(n), \end{aligned}$$

as claimed. □

Lemma 4.7.3. *Let n be a positive integer. Then*

$$8A(n) - 32C(n) = 4K_2(n) - J_2(n) - J_0(n) - 2\sigma(n) + 8\sigma\left(\frac{n}{4}\right).$$

Proof. Clearly

$$r_6(n) = \sum_{k=0}^n r_2(k)r_4(n-k) = r_2(n) + r_4(n) + \sum_{k=1}^{n-1} r_2(k)r_4(n-k).$$

Using the following formulae for $r_2(m)$, $r_4(m)$ and $r_6(m)$ ($m \in \mathbb{N}$), we obtain the desired result. We have

$$r_2(m) = 4 \sum_{d|m} \left(\frac{-4}{d}\right) = 4J_0(m),$$

$$r_4(m) = 8\sigma(m) - 32\sigma\left(\frac{m}{4}\right),$$

$$r_6(m) = 16 \sum_{d|m} \left(\frac{-4}{m/d}\right) d^2 - 4 \sum_{d|m} \left(\frac{-4}{d}\right) d^2 = 16K_2(m) - 4J_2(m),$$

where $m \in \mathbb{N}$. All of these formulae have elementary proofs. Therefore

$$\begin{aligned}
 \sum_{k=1}^{n-1} r_2(k)r_4(n-k) &= \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \in \mathbb{N}}} r_2(n_1)r_4(n_2) \\
 &= \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \in \mathbb{N}}} 4 \sum_{a|n_1} \left(\frac{-4}{a} \right) \left(8\sigma(n_2) - 32\sigma\left(\frac{n_2}{4}\right) \right) \\
 &= 32 \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \in \mathbb{N}}} \sum_{a|n_1} \left(\frac{-4}{a} \right) \sum_{b|n_2} b - 128 \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \in \mathbb{N}}} \sum_{a|n_1} \left(\frac{-4}{a} \right) \sum_{b|(n_2/4)} b \\
 &= 32 \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \in \mathbb{N}}} \sum_{a|n_1} \sum_{b|n_2} \left(\frac{-4}{a} \right) b - 128 \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \in \mathbb{N}}} \sum_{a|n_1} \sum_{4b|n_2} \left(\frac{-4}{a} \right) b \\
 &= 32 \sum_{\substack{ax+by=n \\ a, b, x, y \in \mathbb{N}}} \left(\frac{-4}{a} \right) b - 128 \sum_{\substack{ax+4by=n \\ a, b, x, y \in \mathbb{N}}} \left(\frac{-4}{a} \right) b \\
 &= 32A(n) - 128C(n).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 8A(n) - 32C(n) &= \frac{1}{4} \left(16K_2(n) - 4J_2(n) - 4J_0(n) - 8\sigma(n) + 32\sigma\left(\frac{n}{4}\right) \right) \\
 &= 4K_2(n) - J_2(n) - J_0(n) - 2\sigma(n) + 8\sigma\left(\frac{n}{4}\right),
 \end{aligned}$$

as claimed. □

Lemma 4.7.4. *Let n be an even positive integer. Then*

$$\sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k = \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 2 \pmod{4}}} d - \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d + \frac{1}{2} J_0(n) - \frac{1}{2} J_1(n).$$

Proof. Consider

$$\sum_{\substack{d|n \\ d \text{ even}}} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k$$

$$\begin{aligned}
 &= \sum_{\substack{d|n \\ d \text{ even}}} \begin{cases} (1-3) + (5-7) + \cdots + ((d-3) - (d-1)), & \text{if } d \equiv 0 \pmod{4}, \\ (1-3) + (5-7) + \cdots + ((d-5) - (d-3)) + (d-1), & \text{if } d \equiv 2 \pmod{4}, \end{cases} \\
 &= \sum_{\substack{d|n \\ d \text{ even}}} \begin{cases} -2 \left(\frac{d}{4} \right), & \text{if } d \equiv 0 \pmod{4}, \\ -2 \left(\frac{d-2}{4} \right) + (d-1), & \text{if } d \equiv 2 \pmod{4}, \end{cases} \\
 &= \sum_{\substack{d|n \\ d \text{ even}}} \begin{cases} -\frac{d}{2}, & \text{if } d \equiv 0 \pmod{4}, \\ \frac{d}{2}, & \text{if } d \equiv 2 \pmod{4}, \end{cases} \\
 &= \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 2 \pmod{4}}} d - \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 &\sum_{d|n} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k \\
 &= \sum_{\substack{d|n \\ d \text{ even}}} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k + \sum_{\substack{d|n \\ d \text{ odd}}} \sum_{1 \leq k < d} \left(\frac{-4}{k} \right) k \\
 &= \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 2 \pmod{4}}} d - \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d + \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d} \right) (1-d) \quad (\text{by Lemma 4.3.1}) \\
 &= \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 2 \pmod{4}}} d - \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d + \frac{1}{2} J_0(n) - \frac{1}{2} J_1(n),
 \end{aligned}$$

as required. □

For our next two lemmas we need the following elementary identity of Huard, Ou, Spearman and Williams [43, p. 230].

Proposition 4.7.1. *Let $F : \mathbb{Z}^4 \rightarrow \mathbb{C}$ be such that*

$$F(a, b, x, y) - F(x, y, a, b) = F(-a, -b, x, y) - F(x, y, -a, -b)$$

for all integers a, b, x and y . Then

$$\begin{aligned}
 & \sum_{ax+by=n} (F(a, b, x, -y) - F(a, -b, x, y) + F(a, a-b, x+y, y) \\
 & \quad - F(a, a+b, y-x, y) + F(b-a, b, x, x+y) - F(a+b, b, x, x-y)) \\
 &= \sum_{d|n} \sum_{x < d} (F(0, n/d, x, d) + F(n/d, 0, d, x) + F(n/d, n/d, d-x, -x) \\
 & \quad - F(x, x-d, n/d, n/d) - F(x, d, 0, n/d) - F(d, x, n/d, 0))
 \end{aligned}$$

where the sum on the left-hand side is over all positive integers a, b, x and y satisfying $ax + by = n$, the inner sum on the right-hand side is over all positive integers x satisfying $x < d$, and the outer sum on the right-hand side is over all positive integers d dividing n .

Lemma 4.7.5. For $n \in \mathbb{N}$, we have

$$8A(n) - 32B(n) = 4nJ_0(n) - J_2(n) - J_0(n) - 2\sigma(n) + 8\sigma\left(\frac{n}{2}\right).$$

Proof. In order to prove this result, we let $F(a, b, x, y) = \left(\frac{-4}{a}\right)b$ in Proposition 4.7.1, and suppose that n is even. The left-hand side gives

$$\begin{aligned}
 LHS &= \sum_{ax+by=n} \left(\left(\frac{-4}{a}\right)b - \left(\frac{-4}{a}\right)(-b) + \left(\frac{-4}{a}\right)(a-b) - \left(\frac{-4}{a}\right)(a+b) \right. \\
 & \quad \left. + \left(\frac{-4}{b-a}\right)b - \left(\frac{-4}{a+b}\right)b \right) \\
 &= \sum_{ax+by=n} \left(\left(\frac{-4}{b-a}\right)b - \left(\frac{-4}{a+b}\right)b \right) \\
 &= \sum_{\substack{ax+by=n \\ a \text{ odd}, b \text{ even}}} \left(\left(\frac{-4}{b-a}\right)b - \left(\frac{-4}{a+b}\right)b \right) + \sum_{\substack{ax+by=n \\ a \text{ even}, b \text{ odd}}} \left(\left(\frac{-4}{b-a}\right)b - \left(\frac{-4}{a+b}\right)b \right)
 \end{aligned}$$

$$\begin{aligned}
&= -2 \sum_{\substack{ax+by=n \\ a \text{ odd}, b \text{ even}}} \left(\frac{-4}{a+b} \right) b \quad (\text{by Lemma 1.8.6}) \\
&= -2 \sum_{\substack{ax+by=n \\ a \text{ odd}, 2||b}} \left(\frac{-4}{a+b} \right) b - 2 \sum_{\substack{ax+by=n \\ a \text{ odd}, 4|b}} \left(\frac{-4}{a+b} \right) b \\
&= 2 \sum_{\substack{ax+by=n \\ a \text{ odd}, 2||b}} \left(\frac{-4}{a} \right) b - 2 \sum_{\substack{ax+by=n \\ a \text{ odd}, 4|b}} \left(\frac{-4}{a} \right) b \\
&= 2 \sum_{\substack{ax+2b'y=n \\ a, b' \text{ odd}}} \left(\frac{-4}{a} \right) 2b' - 2 \sum_{\substack{ax+4b'y=n \\ a \text{ odd}}} \left(\frac{-4}{a} \right) 4b' \\
&= 4 \sum_{\substack{ax+2b'y=n \\ b' \text{ odd}}} \left(\frac{-4}{a} \right) b' - 8 \sum_{ax+4b'y=n} \left(\frac{-4}{a} \right) b' \\
&= 4 \sum_{ax+2b'y=n} \left(\frac{-4}{a} \right) b' - 4 \sum_{\substack{ax+2b'y=n \\ b' \text{ even}}} \left(\frac{-4}{a} \right) b' - 8 \sum_{ax+4b'y=n} \left(\frac{-4}{a} \right) b' \\
&= 4B(n) - 4 \sum_{ax+4b''y=n} \left(\frac{-4}{a} \right) 2b'' - 8C(n) \\
&= 4B(n) - 16C(n) \\
&= 4A\left(\frac{n}{2}\right) - 16B\left(\frac{n}{2}\right),
\end{aligned}$$

by Lemmas 4.7.1 and 4.7.2. The right-hand side gives

$$\begin{aligned}
RHS &= \sum_{d|n} \sum_{x < d} \left(\left(\frac{-4}{0} \right) \frac{n}{d} + \left(\frac{-4}{n/d} \right) 0 + \left(\frac{-4}{n/d} \right) \frac{n}{d} \right. \\
&\quad \left. - \left(\frac{-4}{x} \right) (x-d) - \left(\frac{-4}{x} \right) d - \left(\frac{-4}{d} \right) x \right) \\
&= \sum_{d|n} \sum_{x < d} \left(\left(\frac{-4}{n/d} \right) \frac{n}{d} - \left(\frac{-4}{x} \right) x - \left(\frac{-4}{d} \right) x \right) \\
&= \sum_{d|n} \left(\frac{-4}{n/d} \right) \frac{n}{d} \sum_{x < d} 1 - \sum_{d|n} \sum_{x < d} \left(\frac{-4}{x} \right) x - \sum_{d|n} \left(\frac{-4}{d} \right) \sum_{x < d} x
\end{aligned}$$

$$\begin{aligned}
&= \sum_{d|n} \left(\frac{-4}{n/d} \right) \frac{n}{d} (d-1) - \sum_{d|n} \sum_{x < d} \left(\frac{-4}{x} \right) x - \sum_{d|n} \left(\frac{-4}{d} \right) \frac{d(d-1)}{2} \\
&= n \sum_{d|n} \left(\frac{-4}{n/d} \right) - \sum_{d|n} \left(\frac{-4}{n/d} \right) \frac{n}{d} - \left(\frac{1}{2} \sum_{\substack{d|n \\ d \equiv 2 \pmod{4}}} d - \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d + \frac{1}{2} J_0(n) \right. \\
&\quad \left. - \frac{1}{2} J_1(n) \right) - \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 + \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d} \right) d \quad (\text{by Lemma 4.7.4}) \\
&= nJ_0(n) - J_1(n) - \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 2 \pmod{4}}} d + \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d - \frac{1}{2} J_0(n) \\
&\quad + \frac{1}{2} J_1(n) - \frac{1}{2} J_2(n) + \frac{1}{2} J_1(n) \\
&= nJ_0(n) - \frac{1}{2} J_2(n) - \frac{1}{2} J_0(n) + \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d - \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 2 \pmod{4}}} d.
\end{aligned}$$

Equating the left-hand side and the right-hand side we obtain

$$4A\left(\frac{n}{2}\right) - 16B\left(\frac{n}{2}\right) = nJ_0(n) - \frac{1}{2}J_2(n) - \frac{1}{2}J_0(n) + \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 0 \pmod{4}}} d - \frac{1}{2} \sum_{\substack{d|n \\ d \equiv 2 \pmod{4}}} d,$$

for n even. Therefore, for any $n \in \mathbb{N}$ we have

$$\begin{aligned}
8A(n) - 32B(n) &= 4nJ_0(2n) - J_2(2n) - J_0(2n) + \sum_{\substack{d|2n \\ d \equiv 0 \pmod{4}}} d - \sum_{\substack{d|2n \\ d \equiv 2 \pmod{4}}} d \\
&= 4nJ_0(n) - J_2(n) - J_0(n) + \sum_{\substack{d|2n \\ d=4d'}} d - \sum_{\substack{d|2n \\ d=2d', d' \text{ odd}}} d \\
&= 4nJ_0(n) - J_2(n) - 4J_0(n) + 4 \sum_{d'|n/2} d' - 2 \sum_{\substack{d'|n \\ d' \text{ odd}}} d' \\
&= 4nJ_0(n) - J_2(n) - J_0(n) + 4\sigma\left(\frac{n}{2}\right) - 2 \sum_{d'|n} d' + 4 \sum_{\substack{d'|n \\ d'=2d''}} d''
\end{aligned}$$

$$\begin{aligned}
 &= 4nJ_0(n) - J_2(n) - J_0(n) + 4\sigma\left(\frac{n}{2}\right) - 2\sigma(n) + 4 \sum_{d''|n/2} d'' \\
 &= 4nJ_0(n) - J_2(n) - J_0(n) + 8\sigma\left(\frac{n}{2}\right) - 2\sigma(n),
 \end{aligned}$$

as required. $\square$

Lemma 4.7.6. *Let n be an odd positive integer. Then*

$$4B(n) - 16C(n) = nJ_0(n) - \frac{1}{2}J_2(n) - \frac{1}{2}J_0(n).$$

Proof. In order to prove this result, we let $F(a, b, x, y) = \left(\frac{-4}{a}\right)b$ in Proposition 4.7.1, and suppose that n is odd. As in the proof of the previous lemma, we have

$$LHS = 4B(n) - 16C(n).$$

The right-hand side gives

$$\begin{aligned}
 RHS &= \sum_{d|n} \sum_{x < d} \left(\left(\frac{-4}{0}\right) \frac{n}{d} + \left(\frac{-4}{n/d}\right) 0 + \left(\frac{-4}{n/d}\right) \frac{n}{d} \right. \\
 &\quad \left. - \left(\frac{-4}{x}\right) (x-d) - \left(\frac{-4}{x}\right) d - \left(\frac{-4}{d}\right) x \right) \\
 &= \sum_{d|n} \sum_{x < d} \left(\left(\frac{-4}{n/d}\right) \frac{n}{d} - \left(\frac{-4}{x}\right) x - \left(\frac{-4}{d}\right) x \right) \\
 &= \sum_{d|n} \left(\frac{-4}{n/d}\right) \frac{n}{d} \sum_{x < d} 1 - \sum_{d|n} \sum_{x < d} \left(\frac{-4}{x}\right) x - \sum_{d|n} \left(\frac{-4}{d}\right) \sum_{x < d} x \\
 &= \sum_{d|n} \left(\frac{-4}{n/d}\right) \frac{n}{d} (d-1) - \sum_{d|n} \sum_{x < d} \left(\frac{-4}{x}\right) x - \sum_{d|n} \left(\frac{-4}{d}\right) \frac{d(d-1)}{2} \\
 &= n \sum_{d|n} \left(\frac{-4}{n/d}\right) - \sum_{d|n} \left(\frac{-4}{n/d}\right) \frac{n}{d} - \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d}\right) (1-d) - \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d}\right) d^2 \\
 &\quad + \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d}\right) d \quad (\text{by Lemma 4.3.1})
 \end{aligned}$$

$$\begin{aligned}
&= n \sum_{d|n} \left(\frac{-4}{d} \right) - \sum_{d|n} \left(\frac{-4}{d} \right) d - \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d} \right) + \sum_{d|n} \left(\frac{-4}{d} \right) d - \frac{1}{2} \sum_{d|n} \left(\frac{-4}{d} \right) d^2 \\
&= nJ_0(n) - \frac{1}{2}J_0(n) - \frac{1}{2}J_2(n).
\end{aligned}$$

Equating the left-hand side and the right-hand side gives the desired result. $\square$

Lemma 4.7.7. *Let n be an odd positive integer. Then*

$$\begin{aligned}
A(n) &= \frac{1}{24} (J_0(n) + J_2(n) + 16K_2(n) - 12nJ_0(n) - 6\sigma(n)), \\
B(n) &= \frac{1}{24} (J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n)), \\
C(n) &= \frac{1}{24} (J_0(n) + J_2(n) + K_2(n) - 3nJ_0(n)).
\end{aligned}$$

Proof. We prove this result by treating the equations in Lemmas 4.7.3, 4.7.5 and 4.7.6 as a system of three equations in the three unknowns $A(n)$, $B(n)$, $C(n)$. By subtracting the equation in Lemma 4.7.5 from that in Lemma 4.7.3, we obtain as n is odd

$$4B(n) - 4C(n) = \frac{1}{2}K_2(n) - \frac{n}{2}J_0(n). \quad (4.7.2)$$

Next we calculate (4.7.2) minus the equation in Lemma 4.7.6 to obtain

$$C(n) = \frac{1}{24} (J_0(n) + J_2(n) + K_2(n) - 3nJ_0(n)).$$

Then from (4.7.2) we obtain

$$B(n) = \frac{1}{24} (4K_2(n) - 6nJ_0(n) + J_2(n) + J_0(n)).$$

Finally from Lemma 4.7.5, we obtain

$$A(n) = \frac{1}{24} (J_0(n) + J_2(n) + 16K_2(n) - 12nJ_0(n) - 6\sigma(n)),$$

as required. $\square$

Theorem 4.7.1. *Let n be a positive integer. Then*

$$\begin{aligned} A(n) &= \frac{1}{24} (J_0(n) + J_2(n) + 16K_2(n) - 12nJ_0(n) - 6\sigma(n)), \\ B(n) &= \frac{1}{24} \left(J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n) - 6\sigma\left(\frac{n}{2}\right) \right), \\ C(n) &= \frac{1}{24} \left(J_0(n) + J_2(n) + K_2(n) - 3nJ_0(n) - 6\sigma\left(\frac{n}{4}\right) \right). \end{aligned}$$

Proof. Appealing to Lemmas 4.7.1 and 4.7.5 we obtain for any $n \in \mathbb{N}$

$$B(2n) - 4B(n) = \frac{1}{2}nJ_0(n) - \frac{1}{8}J_2(n) - \frac{1}{8}J_0(n) - \frac{1}{4}\sigma(n) + \sigma\left(\frac{n}{2}\right). \quad (4.7.3)$$

Let $n = 2^k N$ where k and N are nonnegative integers and N is odd. Using this identity, and Lemma 4.7.7 we perform induction on k to prove

$$B(n) = \frac{1}{24} \left(J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n) - 6\sigma\left(\frac{n}{2}\right) \right).$$

When $k = 1$ we have

$$\begin{aligned} B(n) &= B(2N) = \frac{1}{2}NJ_0(N) - \frac{1}{8}J_2(N) - \frac{1}{8}J_0(N) - \frac{1}{4}\sigma(N) + \sigma\left(\frac{N}{2}\right) + 4B(N) \\ &= \frac{1}{2}NJ_0(N) - \frac{1}{8}J_2(N) - \frac{1}{8}J_0(N) - \frac{1}{4}\sigma(N) \\ &\quad + \frac{1}{6} (J_0(N) + J_2(N) + 4K_2(N) - 6NJ_0(N)) \\ &= -\frac{1}{2}NJ_0(N) + \frac{1}{24}J_0(N) + \frac{1}{24}J_2(N) - \frac{1}{4}\sigma(N) + \frac{2}{3}K_2(N) \\ &= -\frac{1}{4}nJ_0(n) + \frac{1}{24}J_0(n) + \frac{1}{24}J_2(n) - \frac{1}{4}\sigma\left(\frac{n}{2}\right) + \frac{1}{6}K_2(n) \\ &= \frac{1}{24} \left(J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n) - 6\sigma\left(\frac{n}{2}\right) \right). \end{aligned}$$

Now suppose that

$$B(n) = \frac{1}{24} \left(J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n) - 6\sigma\left(\frac{n}{2}\right) \right),$$

for $n = 2^k N$. Then $2^{k+1}N = 2n$ and by (4.7.3) and the inductive hypothesis we obtain

$$\begin{aligned} B(2n) &= \frac{1}{2}nJ_0(n) - \frac{1}{8}J_2(n) - \frac{1}{8}J_0(n) - \frac{1}{4}\sigma(n) + \sigma\left(\frac{n}{2}\right) + 4B(n) \\ &= \frac{1}{2}nJ_0(n) - \frac{1}{8}J_2(n) - \frac{1}{8}J_0(n) - \frac{1}{4}\sigma(n) + \sigma\left(\frac{n}{2}\right) \\ &\quad + \frac{1}{6} \left(J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n) - 6\sigma\left(\frac{n}{2}\right) \right) \\ &= -\frac{1}{2}nJ_0(n) + \frac{1}{24}J_2(n) + \frac{1}{24}J_0(n) + \frac{2}{3}K_2(n) - \frac{1}{4}\sigma(n) \\ &= \frac{1}{24}J_0(2n) + \frac{1}{24}J_2(2n) + \frac{1}{6}K_2(2n) - \frac{1}{4}(2n)J_0(2n) - \frac{1}{4}\sigma(n) \\ &= \frac{1}{24} (J_0(2n) + J_2(2n) + 4K_2(2n) - 6(2n)J_0(2n) - 6\sigma(n)), \end{aligned}$$

as required. Therefore, by the principle of mathematical induction, we have

$$B(n) = \frac{1}{24} \left(J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n) - 6\sigma\left(\frac{n}{2}\right) \right),$$

for all $n \in \mathbb{N}$. Then from Lemma 4.7.1 we obtain

$$A(n) = B(2n) = \frac{1}{24} (J_0(n) + J_2(n) + 16K_2(n) - 6nJ_0(n) - 6\sigma(n)),$$

and from Lemma 4.7.2 we deduce

$$C(n) = B\left(\frac{n}{2}\right) = \frac{1}{24} \left(J_0(n) + J_2(n) + K_2(n) - 3nJ_0(n) - 6\sigma\left(\frac{n}{4}\right) \right),$$

as claimed. □

Conclusion

We conclude this thesis by mentioning some possible directions for future research.

Theorem 2.4.1 is the 3-term analogue of Theorem 1.5.1, and Theorem 3.5.1 is the 3-term analogue of Theorem 1.3.1. We have not investigated a 3-term analogue of Theorem 1.6.1. The results of this thesis suggest that there should be an evaluation of the 3-term sum

$$\sum_{\substack{ax+by+cz=n \\ x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)),$$

where $F : \mathbb{Z} \rightarrow \mathbb{C}$ is odd. This remains to be investigated.

Liouville's formula for the sum

$$\sum_{\substack{ax+by=n \\ a,b,x,y \text{ odd}}} (F(a+b) - F(a-b)),$$

where n is a positive integer and F is an even function (Theorem 1.5.1), and his formula for

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) - F(a+b-c) - F(a-b+c) + F(a-b-c)),$$

where n is a positive integer and F is an odd function (Theorem 2.4.1), suggest that there might be a formula for a sum of the type

$$\begin{aligned} & \sum_{\substack{a_1 x_1 + \dots + a_s x_s = n \\ a_1, \dots, a_s, x_1, \dots, x_s \text{ odd}}} (F(a_1 + a_2 + \dots + a_s) - F(a_1 - a_2 + a_3 + \dots + a_s) \\ & - F(a_1 + a_2 - a_3 + \dots + a_s) - \dots - F(a_1 + a_2 + a_3 + \dots - a_s) \\ & + F(a_1 - a_2 - a_3 + a_4 + \dots + a_s) + \dots + F(a_1 + a_2 + \dots - a_{s-1} - a_s) \\ & - \dots + (-1)^{s-1} F(a_1 - a_2 - \dots - a_s)), \end{aligned}$$

where $s, n \in \mathbb{N}$ and $F(-x) = (-1)^s F(x)$. However to find such a formula and to prove it seems to be a very difficult problem. There may be a relationship with Zagier's formula [121]

$$\sum_{\substack{a_1 x_1 + \dots + a_s x_s = n \\ a_1, \dots, a_s, x_1, \dots, x_s \text{ odd}}} \prod_{i=1}^s a_i \prod_{1 \leq i < j \leq s} (a_i^2 - a_j^2)^2 = 4^{s(s-1)} s! \prod_{j=1}^{2s-1} j! \delta_{4s^2} \left(\frac{n - s^2}{2} \right).$$

Liouville [63, p. 284] has stated, and other mathematicians have proved, the following generalization of Theorem 1.3.1.

Let $n \in \mathbb{N}$. Let $F : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$ satisfy

$$F(x, y) = F(-x, y) = F(x, -y)$$

for all $x, y \in \mathbb{Z}$. Then

$$\begin{aligned} & \sum_{ax+by=n} (F(a-b, x+y) - F(a+b, x-y)) \\ & = \sum_{d|n} (d-1) (F(0, d) - F(d, 0)) + 2 \sum_{d|n} \sum_{e < n/d} (F(d, e) - F(e, d)). \end{aligned}$$

Theorem 1.3.1 is obtained by taking $F(x, y)$ to be a function of x only. It would be interesting to determine whether our Theorem 3.5.1 has a 2-variable analogue of this.

As regards to Theorem 4.7.1 there is the question of whether the sum

$$\sum_{ax+8by=n} \left(\frac{-4}{a} \right) b$$

has a similar evaluation, and whether the sums

$$\sum_{ax+ky=n} \left(\frac{-3}{a} \right) b$$

can be determined in a like manner for $k = 1, 3, 9$.

Much remains to be investigated and discovered.

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