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On Some Properties of the Diagonal Flip Adjacency Graphs of Near Triangulations on Surfaces

by

Jianyu Wang, B.Sc., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics

Carleton University

Ottawa, Ontario

Canada

December 1998

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Graphs of Near Triangulations on Surfaces**

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Jianyu Wang, B.Sc., M.Sc.

in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

Kenneth S. Williams

Director, School of Mathematics and Statistics

Z. Gao

Dr. Z. Gao, Thesis Supervisor

Bruce Rickard

External Examiner

Carleton University

December 1998

Abstract

A near triangulation on a surface is a map on the surface such that each face of the map, except possibly one face, has valence three. A diagonal flip adjacency graph of near triangulations on surfaces is defined as a graph in such a way that all near triangulations of some type, specified in applications, to be its vertices, two of which are connected by an edge if one near triangulation can be transformed into the other by a diagonal flip. In this thesis, we study some properties of these graphs when the near triangulations are 2-connected or simple.

Using generating function technique, we have found the total number of flippable edges in all rooted 2-connected (simple) planar near triangulations of the same type. From this, we obtain the total number of edges and average vertex degree of the corresponding diagonal flip adjacency graphs. We also find the probability that an edge is flippable in all rooted 2-connected (simple) planar near triangulations.

For any unlabelled, rooted or labelled simple (2-connected) planar near triangulations, we obtain the maximum number and minimum number of flippable edges in these triangulations. They are the same as the maximum and minimum vertex degrees in the corresponding diagonal flip adjacency graphs. We also investigate the diameters of these diagonal flip adjacency graphs. For unlabelled and rooted simple (2-connected) planar near triangulations of type $[n, m]$, we determine that the diameters of the corresponding diagonal flip adjacency graphs are both $O(n + m)$

and show that the order of this bound cannot be improved. In the labelled case, if the near triangulations are triangulations, i.e. $m = 3$, then the diameter of the corresponding diagonal flip adjacency graph is $O(n \log n)$. If $m \neq 3$, the corresponding diagonal flip adjacency graphs are not connected; however, the subgraphs consisting of all labelled near triangulations of type $[n, m]$ with the same labelling on their non-triangular faces have diameter $O(n \log n + m)$.

The exact enumeration of non-planar triangulations is usually very difficult. It is much more complicated than that of planar triangulations. Very few results on exact enumeration of triangulations on non-planar surfaces are known. In this thesis, we successfully enumerate rooted 2-connected triangulations on the Torus, rooted 2-connected triangulations on the Klein Bottle, and rooted 3-connected triangulations on the Projective Plane. In each case, we obtain a nice parametric expression for the generating function of the number of triangulations. In the language of the diagonal flip adjacency graph, we obtain the numbers of vertices in the corresponding diagonal flip adjacency graphs. We also enumerate simple Catalan triangulations on the Torus.

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Chapter 1

Definitions and Introduction

1.1 Definitions

1.1.1 Some Terminology of Maps

A *map* M on a surface Σ is a set V of points (called *vertices*) and a set E of open simple curves (called *edges*) on Σ , such that the end points (or point) of each edge are in V , no edges intersect each other except possibly at their ends, and each component of $\Sigma - V - E$ is a simple connected region (i.e. like a disk). These simply connected regions are called *faces* of the map. We use F to denote the set of all faces. Two maps M_1 and M_2 on the same surface Σ are considered the same if there is a homeomorphism from Σ to itself which transforms M_1 to M_2 . Maps on the sphere (or equivalently on a closed plane) are called *planar* maps. Maps on the surfaces which are not sphere are called *non-planar* maps.

For each face f of a map, we define a *cyclic sequence* Σ_f to be a closed walk along the boundary of f . If a vertex or an edge occurs more than once in Σ_f , it is

$\{1, 2, \dots, |V(M)|\}$ is a bijection, where $|V(M)|$ is the number of elements in the set $V(M)$. Two *labelled maps* on the same surface Σ are considered the same if there is a homeomorphism from Σ to itself which transforms one map to the other and preserves the labelling, i.e. it transforms the vertex with label i of one map into the vertex with label i of the other map, with i being any integer between 1 and $|V(M)|$.

A map is called *rooted* if a vertex of the map, together with an edge incident with this vertex and a side of the edge, are specified. The specified vertex, specified edge and the face corresponding to the side are called the *root vertex*, *root edge* and *root face* of the map, respectively. Two *rooted maps* on the same surface Σ are considered the same if there is a homeomorphism from Σ to itself which transforms one map into the other and preserves the rooting, i.e. it transforms the root vertex, root edge and root face of one map into the root vertex, root edge and root face of the other map, respectively.

The concept of rooting a map was first introduced by Tutte in the 1960s ([48], [51]); he proved that rooting a map destroys the possible symmetries, hence making the enumeration much easier. Since then, much work has been done on the field of enumerating rooted maps.

1.1.2 Triangulations and Their Diagonal Flips

A *triangulation* on a surface Σ is a map on Σ such that each face of the map has valence three. If Σ is a sphere or a closed plane, we call the triangulation a *planar triangulation*. A *near triangulation* on a surface Σ is a map on Σ such that each face of the map, except possibly one face, has valence three. If Σ is a sphere or a closed plane, we call this near triangulation a *planar near triangulation*.

A near triangulation with $n + m$ vertices is called of *type* $[n, m]$, if its only possible non-triangular face has valence m . For any near triangulation of type $[n, m]$, we will say it has n *interior vertices* and m *exterior vertices*. A triangulation with n vertices is a near triangulation of type $[n - 3, 3]$.

A triangulation is *rooted* if, as a map, it is rooted. A near triangulation of type $[n, m]$ with $m \neq 3$ is *rooted*, if, as a map, it is rooted and the root face is the only non-triangular face. If $n = 0$, then all its vertices are on the root face, and we call this near triangulation a *Catalan triangulation*. A near triangulation is called *labelled*, if, as a map, it is labelled. A near triangulation which is neither rooted nor labelled is called *unlabelled*.

Given a triangulation T , we select a point v_f in each face f and connect v_f with all vertices incident to f . The resulting triangulation is called the *subdivision triangulation* of T .

A *digon* of a near triangulation on a surface is a disk region, which is bounded by a pair of multiple edges. If the surface is the plane, we define a digon to be the interior region bounded by a pair of multiple edges. A digon is called a *separating digon* if there are vertices inside as well as outside the pair of multiple edges. It is clear that a digon in a triangulation is always a separating digon. A separating digon that contains no other separating digon is called a *minimal separating digon*. See Figure 1.1(a).

A triangle of a near triangulation on a surface is called a *separating triangle* if there are vertices inside as well as outside the triangle. A separating triangle that contains no other separating triangle is called a *minimal separating triangle*. See Figure 1.1(b).

Figure 1.1: Minimum separating digon and minimum separating triangle.

It was shown (see Gao [18] Lemma 1) that if T is a triangulation on a surface, then, the three conditions: T is loopless, T is nonsingular and T is 2-connected are equivalent. It was also shown that a 2-connected triangulation is 3-connected if and only if it has no multiple edges. See Gao [19] Lemma 2, for example. But for near triangulations on a surface, this statement is not true, since a 2-connected near triangulation without multiple edges is not necessary 3-connected.

A near triangulation is called *simple* if it is nonsingular and has no multiple edges. Hence for triangulations, simple and 3-connected are equivalent. In this thesis, we use “3-connected triangulation” and “simple triangulation” interchangeably. Notice that in some papers, a triangulation is defined to be simple if it is 3-connected and has no separating triangle. Some authors also call these triangulation 4-connected. See Tutte [48] and Gao [19], for example. It is clear that their definition is not the same as what we give here.

Let $v_i v_j$ be an edge in a triangulation T on a surface, and let $v_i v_j v_k$, $v_i v_j v_l$ be the triangular faces of T which are incident with $v_i v_j$. The operation *diagonal flip*

Figure 1.3: Flippable edges and unflippable edges in a 2-connected triangulation and a 3-connected triangulation.

For a rooted near triangulation, we consider all edges on the root face to be *unflippable*.

It is easy to see that if $v_i v_j$ is a flippable edge of a 2-connected (simple) near triangulation, then the map obtained by flipping $v_i v_j$ is still a 2-connected (simple) near triangulation of the same type. If $v_i v_j$ is a flippable edge of a rooted 2-connected (simple) near triangulation, then the map obtained by flipping $v_i v_j$ is still a rooted 2-connected (simple) near triangulation of the same type and having the same rooting.

For any planar near triangulation T of type $[n, m]$ with $m \neq 3$, we will always draw T on the plane in such a way that the only non-triangular face of T , i.e. the face with valence m , is the exterior face. For any rooted planar near triangulation, we will always let its root face to be the exterior face.

1.1.3 The Standard Forms of Planar Near Triangulations

Let T be an unlabelled planar simple near triangulation of type $[n, m]$. T is called the *standard form* of unlabelled simple planar near triangulations of type $[n, m]$ if there exist two vertices, v_a and v_b , on the exterior face of T , such that $\deg v_a = n + m - 1$ and $\deg v_b = n + 2$. We denote this standard form by $\Delta_{n,m}$. $\Delta_{n-3,3}$ is the standard form of unlabelled 3-connected planar triangulations with n vertices. For convenience, we also use Δ_n to denote $\Delta_{n-3,3}$. See Figure 1.4.

The *standard form* of rooted simple planar near triangulations of type $[n, m]$, denoted by $\Delta_{n,m}^r$, is defined to be the rooted simple planar near triangulation obtained by rooting $\Delta_{n,m}$ in the following way: choose vertex v_a as the root vertex, edge $v_a v_b$ as the root edge and the exterior face as the root face. $\Delta_{n-3,3}^r$ is the standard form of rooted 3-connected planar triangulations with n vertices. We also use Δ_n^r to denote $\Delta_{n-3,3}^r$. See Figure 1.4.

The *standard form* of labelled simple planar near triangulations of type $[n, m]$, denoted by $\Delta_{n,m}^l$, is defined to be the labelled simple planar near triangulation obtained by labelling $\Delta_{n,m}$ in the following way: two vertices v_a and v_b are labelled with v_1 and v_2 respectively; other vertices of the exterior face are labelled anti-clockwisely with $v_3, v_4, \dots, v_m$ respectively, and the path $\Delta_{n,m} - \{v_1, v_2, v_3, \dots, v_m\}$ is labelled with $v_{m+1}, v_{m+2}, \dots, v_{m+n-1}, v_{m+n}$ from top to bottom. $\Delta_{n-3,3}^l$ is the standard form of labelled 3-connected planar triangulations with n vertices. We also use Δ_n^l to denote $\Delta_{n-3,3}^l$. See Figure 1.4.

Since a simple planar near triangulation is also a 2-connected planar near triangulation of the same type, we define the standard form of unlabelled, rooted and labelled 2-connected planar near triangulations to be $\Delta_{n,m}$, $\Delta_{n,m}^r$ and $\Delta_{n,m}^l$, respec-

Figure 1.4: The standard forms of planar near triangulations.

tively.

1.1.4 The Diagonal Flip Adjacency Graphs of Near Triangulations on Surfaces

Let $\mathcal{T}_{n,m,kc}^\Sigma$, $\mathcal{T}_{n,m,kc}^{r,\Sigma}$ and $\mathcal{T}_{n,m,kc}^{l,\Sigma}$ be the set of all unlabelled, the set of all rooted, and the set of all labelled k -connected near triangulations of type $[n, m]$ on a surface Σ , respectively. Let $\mathcal{T}_{n,m,s}^\Sigma$, $\mathcal{T}_{n,m,s}^{r,\Sigma}$ and $\mathcal{T}_{n,m,s}^{l,\Sigma}$ be the set of all unlabelled, the set of all rooted, and the set of all labelled simple near triangulations of type $[n, m]$ on a surface Σ , respectively.

The *diagonal flip adjacency graph* of unlabelled (rooted, labelled) k -connected near triangulations of type $[n, m]$ on Σ , denoted by $GT_{n,m,kc}^\Sigma$ ($GT_{n,m,kc}^{r,\Sigma}$, $GT_{n,m,kc}^{l,\Sigma}$), is the graph with vertex set $\mathcal{T}_{n,m,kc}^\Sigma$ ($\mathcal{T}_{n,m,kc}^{r,\Sigma}$, $\mathcal{T}_{n,m,kc}^{l,\Sigma}$), and two vertices of this graph are connected by an edge if and only if one near triangulation can be transformed into the other by a single operation of diagonal flip on a flippable edge. Similarly, we define the diagonal flip adjacency graph of the unlabelled (rooted, labelled) simple near triangulations of type $[n, m]$ on Σ , denoted by $GT_{n,m,s}^\Sigma$ ($GT_{n,m,s}^{r,\Sigma}$, $GT_{n,m,s}^{l,\Sigma}$).

According to our definitions, $GT_{n-3,3,kc}^\Sigma$ is the diagonal flip adjacency graph of unlabelled k -connected triangulations with n vertices on Σ . We use the short notation $GT_{n,kc}^\Sigma$ to denote $GT_{n-3,3,kc}^\Sigma$. Similarly, we use $GT_{n,kc}^{r,\Sigma}$ and $GT_{n,kc}^{l,\Sigma}$ to denote $GT_{n-3,3,kc}^{r,\Sigma}$ and $GT_{n-3,3,kc}^{l,\Sigma}$, respectively.

We will only consider 2-connected and simple near triangulations in this thesis. We study some basic properties of different kinds of diagonal flip adjacency graphs on surfaces with non-negative Euler characteristic, such as the Sphere (or closed

Figure 1.5: The diagonal flip adjacency graph $GT_{3,3,S}^S$.

plane), the Projective Plane, the Torus and the Klein Bottle. In each of these cases, we use S (Sphere), P (Projective Plane), T (Torus) and K (Klein Bottle) instead of Σ .

Given two unlabelled (rooted, labelled) near triangulations of the same type on a surface, we say they are at distance d , if as vertices of the diagonal flip adjacency graph, the distance of these two near triangulations in the graph is d , i.e. one near triangulation can be transformed into the other by flipping d edges consecutively. The largest distance of two vertices in the diagonal flip adjacency graph is defined to be the *diameter* of the graph. In Figure 1.5, we give an example of the diagonal flip adjacency graph, $GT_{n,m,S}^S$, with $n = 3$, $m = 3$. It is a complete graph with three vertices and its diameter is 1.

In this language, the total number of vertices in a diagonal flip adjacency graph

$GT_{n,m,s}^\Sigma$ is the total number of unlabelled simple near triangulations of type $[n, m]$ on the surface Σ . The total vertex degree of $GT_{n,m,s}^\Sigma$ is the total number of flippable edges in all unlabelled simple triangulations of type $[n, m]$ on Σ . Using Euler's Lemma, we know the total number of edges in $GT_{n,m,s}^\Sigma$ is its total vertex degree divided by two. The diameter of $GT_{n,m,s}^\Sigma$ is the minimum number such that within this number of flips, any two unlabelled simple near triangulations of type $[n, m]$ on Σ can be transformed into each other. The minimum or maximum vertex degree in $GT_{n,m,s}^\Sigma$ is the minimum or maximum number of flippable edges in all simple near triangulations of type $[n, m]$ on Σ . For other diagonal flip adjacency graphs, for example, $GT_{n,m,s}^{r,\Sigma}$, $GT_{n,m,s}^{l,\Sigma}$, $GT_{n,m,2c}^\Sigma$ etc., we can describe all these concepts in a similar way, the only difference being that the near triangulations are rooted or labelled or only 2-connected. We will discuss these concepts in more detail, with Σ being the Sphere, the Projective Plane, the Torus and the Klein Bottle.

1.1.5 The Lagrange Inversion Formula

In this thesis, the following well-known Lagrange Inversion Formula will be used frequently.

Theorem 1.2 *Let $f(u)$ and $\phi(u)$ be formal power series in u , with $\phi(0) = 1$. Then there is a unique formal power series $u = u(t)$ which satisfies $u = t\phi(u)$. Furthermore, when expanded in a power series in t at $t = 0$, the value $f(u(t))$ of f at that root $u = u(t)$ satisfies*

$$[t^n]\{f(u(t))\} = \frac{1}{n}[u^{n-1}]\{f'(u)\phi(u)^n\}. \quad (1.1)$$

We will not prove this theorem here. For the proof of Theorem 1.2, refer to [58], for example.

1.2 A Brief Survey of the Known Results

In this section, we give references and very brief descriptions of the known results on the properties of the diagonal flip adjacency graphs of near triangulations on surfaces. For any near triangulation T on a surface and for any positive integer i , it is clear that if T is $(i + 1)$ -connected then it is i -connected. From now on, we will use 1-c, 2-c to denote 1-connected, 2-connected and so on.

1.2.1 Enumeration of Rooted Near Triangulations

Much work has been done in the enumeration of rooted maps since Tutte's breakthrough in the 1960s (see [47]–[52]). However, we will only refer to the results on the enumeration of near triangulations.

The number of rooted 1-c planar near triangulations of type $[n, m]$, i.e. the total number of vertices in $GT_{n,m,1c}^{r,S}$, has been determined by Gao [17]. The number of rooted 2-c near triangulations of type $[n, m]$, i.e. the total number of vertices in $GT_{n,m,2c}^{r,S}$, has been determined by Mullin [32] to be a simple expression involving factorials for $n \geq 0$ and $m \geq 3$. Brown [9] determined the number of rooted simple near triangulations of type $[n, m]$, i.e. the total number of vertices in $GT_{n,m,s}^{r,S}$. Tutte [48] obtained the number of rooted 3-c near triangulations of type $[n, m]$. Explicit formulas for the special cases, i.e. rooted 1-c, 2-c and 3-c triangulations with n vertices, have all been given and the asymptotics for fixed m as $n \rightarrow \infty$ have been

published in these papers, respectively. A sum of positive terms and an asymptotic formula for the number of 4-c rooted triangulations with n vertices have also been given by Tutte [48].

For non-planar rooted triangulations, very little work on enumeration has been done. In [17], Gao obtains a functional equation of the generating function for the number of rooted 1-c triangulations on a general surface. In the same paper, he gives simple parametric equations for the generating functions of the number of rooted 1-c triangulations on the Projective Plane and Torus, respectively. He also obtains a parametric expression for the generating function of the number of rooted 2-c triangulations on the Projective Plane in [18]. The asymptotic formulas for these numbers are also given in these papers. It seems to be very difficult to get the exact number of rooted triangulations on non-planar surfaces. On the other hand, the asymptotic results for the numbers of rooted 2-c triangulations and rooted 3-c triangulations on an arbitrary surface were obtained in [20] and [21], respectively.

1.2.2 Enumeration of Unlabelled Planar Near Triangulations

For the number of unlabelled near triangulations on a surface, there are very few results. This is because the symmetry in triangulations makes the enumeration very difficult.

There is the explicit result of Moon and Moser [34] on the triangular dissection of n -gons; there is also Brown's enumeration [9] of simple planar near triangulations of type $[m, n]$. Brown let $R_{n,m}$ denote the number, up to all homeomorphisms of

the disc (fixing the only possible non-triangular face), of simple planar near triangulations of type $[n, m]$, $G_{n,m}$ the number, up to rotational homeomorphisms, of rooted simple planar near triangulations of type $[n, m]$, $K_{n,m}$ the number of rooted simple planar near triangulations of type $[n, m]$ which have a reflectional symmetry and for which there is an isomorphism fixing the root edge, and $L_{n,m}$ the number of rooted simple planar near triangulations having a reflectional symmetry and for which there is an isomorphism fixing only the root vertex. Brown shows that

$$R_{n,m} = \frac{1}{2}G_{n,m} + \frac{1}{4}(K_{n,m} + L_{n,m})$$

and derives functional equations allowing the recursive computation of $G_{n,m}$, $K_{n,m}$ and $L_{n,m}$. He also shows that almost all simple planar triangulations are asymmetrical. In [43], Richmond and Wormald show that almost all k -connected triangulations on any surface are asymmetrical with $1 \leq k \leq 4$.

1.2.3 Catalan Triangulations on Surfaces

Counting the number of ways of triangulating a polygon with m vertices on the plane using $m - 3$ internal diagonals is a classical problem in combinatorial geometry, which goes back to Euler ([15]), and that led to the introduction of the Catalan number, which is used now nearly everywhere. See [23] for detail discussion. In our notation, it is actually counting the number of rooted simple planar near triangulations without interior vertices and with m exterior vertices, i.e. the number of simple planar Catalan triangulations with m vertices. It is clear in this case that each one of the $m - 3$ internal diagonals is flippable. The number of vertices in the

graph $GT_{0,m,s}^{r,S}$ is the Catalan number

$$C_m = \frac{1}{m-1} \binom{2m-4}{m-2}.$$

Notice each vertex of $GT_{0,m,s}^{r,S}$ has degree $m-3$. From Euler's Lemma, the number of edges in this graph is

$$\frac{1}{2}(m-3)C_m = \binom{2m-5}{m-1}.$$

It is also known that the graph $GT_{0,m,s}^{r,S}$ is connected. Moreover, Sleator, Tarjan and Thurston [46] proved that the diameter of $GT_{0,m,s}^{r,S}$ is $2m-10$ for all sufficiently large m .

In a very recent paper [16], Edelman and Reiner obtained the corresponding results for Catalan triangulations on the Möbius Band. They derived a generating function for Catalan triangulations with m vertices on the Möbius Band and showed that any two such triangulations are connected by a sequence of diagonal flips. Notice that the Möbius Band can be obtained by cutting off a disk from the Projective Plane. Thus the result in [16] gives the total number of vertices in the diagonal flip adjacency graph $GT_{0,m,s}^{r,P}$ and shows that the graph is connected.

We will discuss the corresponding result in the case of the Torus in Chapter 8.

1.2.4 The Connectivity of the Diagonal Flip Adjacency Graphs of Near Triangulations

A classical result of Wagner [54] states that any two planar unlabelled 3-c triangulations with the same number of vertices can be transformed into each other by a sequence of diagonal flips. Wagner's classical result implies that for any $n \geq 3$, the

graph $GT_{n,3c}^S$ is connected. Dewdney [14], Negami and Watanabe [37] have shown similar results for the Torus, the Projective Plane and the Klein Bottle, i.e. they have shown that $GT_{n,3c}^T$, $GT_{n,3c}^P$ and $GT_{n,3c}^K$ are all connected.

It is easy to see that Wagner's result extends to rooted and labelled 3-c planar triangulations, and to unlabelled (rooted, labelled) 2-c planar triangulations. However, it is not always true for 3-c labelled triangulations on a non-planar surface: for example, it is not true for the Projective Plane and the Torus, since there are different triangular embeddings of a labelled complete graph in each of these surfaces. In the rooted case, we can show that it is not true for rooted 3-c triangulations on the Torus and it is not true for rooted 2-c triangulations on the Klein Bottle either.

For triangulations in general surfaces, the diagonal flip adjacency graphs need not be connected even for unlabelled 3-c triangulations [31]. However, Negami [38] showed that for any surface Σ , there is a constant L , such that $GT_{n,3c}^{l,\Sigma}$ is connected for all $n \geq L$. Very recently, Komuro, Nakamoto and Negami ([29], [35]) obtained similar results for 3-c triangulations with minimum vertex degree at least four. Diagonal flips preserving some specified properties are discussed in [12].

1.2.5 Geometrical Triangulations

A closely related subject, the study of triangulations of point sets and polygons in the plane, has been studied intensely in the literature by different groups of researchers. See [1], [13], [25], [27] etc. This is because of their intrinsic beauty and for their use in many problems, such as image processing [45], mesh generation for finite element methods [59], scattered data interpolation [30] and many others such as computer graphics, solid modeling and geographical information systems.

Figure 1.6: A geometrical triangulation of a point set: edge pq is flippable, edge qr is not flippable.

See [40], [44], [57]. Since the meaning of the terminology “triangulation” used here is not the same as we described earlier, we will use “geometrical triangulation” to indicate “triangulation” in this area.

Let $P_n = \{v_1, v_2, \dots, v_n\}$ be a collection of points on the plane. A *geometrical triangulation* of P_n is a partitioning of the convex hull of P_n into a set of triangles such that the vertices of these triangles are elements of P_n , and no triangle of T contains an element of P_n in its interior. The edges of the triangles of T are *straight line segments* joining pairs of elements of P_n . See Figure 1.6. In this context, an edge of a geometrical triangulation is *flippable* if it is contained in the boundary of two triangles such that their union is a *convex quadrilateral* C . See [27]. By flipping edge pq we understand the operation of removing pq from a geometrical triangulation and replacing it by the other diagonal of C to obtain a new geometrical triangulation of P_n . See Figure 1.6, where edge pq is flippable, but edge qr is not flippable.

Notice the difference between the planar triangulation which we defined earlier

and the geometrical triangulation we just described. First, the edges used in geometrical triangulation must be straight line segments, hence we cannot have loops or multiple edges in geometrical triangulations, which means any geometrical triangulation is simple. Second, in a geometrical triangulation of a point set P_n , if the convex hull of P_n is not a triangle, then its exterior face is not a triangle, hence it actually should be called a geometrical near triangulation on the plane. If we require each edge of our simple planar near triangulation to be a straight line segment, then it is also a geometrical triangulation on the plane.

The graph of geometrical triangulations $GT(P_n)$ ($GT(Q_n)$) of a point set P_n (a polygon Q_n) with n vertices is the graph whose vertex set is the set of geometrical triangulations of P_n (Q_n). Two geometrical triangulations are adjacent if one can be obtained from the other by flipping an edge. It is known that the graph of geometrical triangulations of a point set is connected and that the diameter of $GT(Q_n)$ is $O(n^2)$. See [27]. In the same paper, it is shown that any geometrical triangulation of a point set contains at least $\lceil \frac{n-1}{2} \rceil$ flippable edges. It is easy to see that there should be more flippable edges in simple planar near triangulations than in geometrical triangulations. For example, in Figure 1.6, edge qr is flippable in the simple planar near triangulation, but it is unflippable in the geometrical triangulation.

In Chapter 4, we will prove that every 3-c planar triangulation with n vertices contains at least $n - 2$ flippable edges, and every simple planar near triangulation of type $[n, m]$ contains at least $\max(n - 1, m - 3)$ flippable edges.

in the graph $GT_{n,m,s}^{r,S}$ and the average number of flippable edges in a simple planar near triangulation. We also find the probability of a nonroot edge being flippable in a rooted simple planar near triangulation. For any nonroot face edge in a rooted simple planar triangulation, we show that the probability that it is flippable is asymptotically $3/4$.

From the fact that almost all k -connected ($1 \leq k \leq 4$) planar triangulations are asymmetrical (see [43]), we can say that the probability of an edge in an unlabelled (labelled) 2-c or simple planar triangulation being flippable is the same as that of a nonroot edge in a rooted 2-c or simple planar triangulation of the same type being flippable.

1.3.2 The Minimum and Maximum Vertex Degree in Diagonal Flip Adjacency Graphs of Planar Near Triangulations

In any planar near triangulation, there are a lot of flippable edges. One may ask the following question: what are the minimum and maximum number of flippable edges for all unlabelled (rooted, labelled) near triangulation of type $[n, m]$? That is the same question as: what are the the minimum vertex degree and maximum vertex degree in a diagonal flip adjacency graph? In Chapter 4, we answer this question completely for 2-c and simple planar near triangulations.

Finding the maximum vertex degrees of these graphs is not difficult, since there exist some unlabelled simple planar triangulations in which all edges are flippable, and there exist unlabelled simple planar near triangulations of type $[n, m]$, $m \neq$

3. such that all of their edges except the edges on the non-triangular face, are flippable. However, finding the minimum vertex degrees of these graphs is much more complicated and we need to consider different cases according to the total number of vertices in planar near triangulations. We observe that most of the unflippable edges are caused by degree three vertices in the planar triangulations. If the minimum vertex degree in the planar triangulations is at least four, we prove that the minimum number of flippable edges in these triangulations is essentially twice the minimum number of flippable edges in the general case.

1.3.3 Diameters of the Diagonal Flip Adjacency Graphs of Planar Near Triangulations

In a very long paper, Sleator, Tarjan and Thurston [46] proved that the diameter of the diagonal flip adjacency graph $GT_{0,m,s}^S$ of a planar m -gon C_m is $2m - 10$ for all sufficiently large m . Usually, it is very difficult to obtain the exact value of the diameter of a diagonal flip adjacency graph. If we only need to decide the order of the diameter, it is relatively easier. Ore's proof of Wagner's theorem in [41] gives a quadratic bound for the diameter of graph $GT_{n,3c}^S$. Komuro [28] improved it to be linear.

In Chapter 5, we prove that the diameter of $GT_{n,3c}^{r,S}$ is also $O(n)$ and show that the order of this bound cannot be improved. We also show that the diameter of $GT_{n,3c}^{l,S}$ is $O(n \log n)$, but we do not know if the order of this bound can be improved. In general, for simple planar near triangulations of type $[n, m]$, we prove that the diameters of $GT_{n,m,s}^S$ and $GT_{n,m,s}^{r,S}$ are $O(n + m)$, and show that the order of the

bound is tight. However, for labelled near triangulations, if $m \neq 3$, the diagonal flip adjacency graph $GT_{n,m,s}^{l,s}$ is not even connected. This is because if two near triangulations are of the same type, but the labellings of their non-triangular faces are not the same, then it is impossible to transform one into the other, since we are not allowed to flip any edge of the non-triangular face. So the diameter of $GT_{n,m,s}^{l,s}$ is infinite. Notice in this thesis, when we say the labelling of a face of one labelled near triangulation is the same as that of a face of the other labelled near triangulation, we mean that the two faces have the same cyclic labelling.

If we consider the subgraphs of $GT_{n,m,s}^{l,s}$, consisting of all 3-c labelled planar near triangulations of type $[n, m]$ and with the same labelling on the non-triangular face, we prove these subgraphs are connected and that their diameters are $O(n \log n + m)$. Again, in this case, we cannot show that the bound is tight.

For 2-c planar near triangulations, we obtain similar results.

1.3.4 The Enumeration of Rooted Triangulations on Non-planar Surfaces

In Chapters 6 and 7, we successfully enumerate rooted 2-c triangulations on the Torus and the Klein Bottle, and rooted simple triangulations on the Projective Plane, respectively. In Chapter 8, we enumerate simple Catalan triangulations on the Torus.

Chapter 6 focuses on the rooted 2-c triangulations on the Torus and Klein Bottle; the parametric equations of the generating function of the number of rooted 2-c triangulations are obtained in both cases.

In Chapter 7, we enumerate rooted simple triangulations on the Projective Plane, using two different approaches to achieve our goal. The first approach is the same as we used in Chapter 6. The second approach, which we call *functional decomposition*, is an approach which utilizes the result of the enumeration of rooted 2-c triangulations on the Projective Plane.

Technically, we can enumerate rooted simple triangulations on the Torus. However, the functional equations are too complicated, we cannot solve them even using MAPLE software. We do solve them in a simple case – simple Catalan triangulations; that is the task of Chapter 8.

1.4 Further Research on the Properties of the Diagonal Flip Adjacency Graphs

Although we have obtained a lot of results on diagonal flip adjacency graphs of near triangulations on surfaces, there are still many many open problems in this area. We will mention quite a few here. Among these problems, for some of them, we have some ideas for attacking them, but the results are far from satisfactory. Some problems are too far removed from the current research, and we need more efficient tools and more advanced technique to solve them.

1.4.1 The Total Number of Vertices and Edges in Diagonal Flip Adjacency Graphs

In the planar case, what is the total number of edges in $GT_{n,m,2c}^S$ and $GT_{n,m,s}^S$? To obtain the exact enumeration result on unlabelled or labelled near triangulations is really challenging, it is much harder than that of the rooted case.

For non-planar surfaces, we have known the total number of vertices in graphs $GT_{n,m,2c}^P$ and $GT_{n,m,s}^P$. Can we find the total number of edges in these graphs? For the Torus and Klein Bottle, what are the total number of vertices in $GT_{n,m,s}^T$ and $GT_{n,m,s}^K$? In general, are there any nice formulas for the generating functions of the number of vertices and edges in $GT_{n,m,2c}^\Sigma$ and $GT_{n,m,s}^\Sigma$? Here Σ is a non-planar surface. This kind of formulas should involve the genus of Σ .

1.4.2 The Diameters of Diagonal Flip Adjacency Graphs

In the planar case, we have known that the diameter of $GT_{n,m,s}^S$ and $GT_{n,m,s}^{r,S}$ is $O(n+m)$. But what are the exact diameters of $GT_{n,m,s}^S$ and $GT_{n,m,s}^{r,S}$? In the labelled case, is $O(n \log n)$ a tight bound of the diameter of $GT_{n,3c}^{l,S}$?

To find the diameter of the diagonal flip adjacency graphs of near triangulations on non-planar surfaces is really challenging. We could start from Catalan triangulations. What is the diameter of the diagonal flip adjacency graph of simple Catalan triangulations with n vertices on the Projective Plane? We could give a bound $O(n^2)$ but we cannot show it is tight. In general, can we find a bound for the diameters of the diagonal flip adjacency graphs of simple Catalan triangulations on a surface with genus g ?

1.4.3 The Graphic Properties of the Diagonal Flip Adjacency Graphs

In our thesis, we focus on the enumeration properties of diagonal flip adjacency graphs. How about the graphic properties? For example, are these graphs always connected?

In the triangulation case, Negami [38] showed that for any surface Σ , there is a constant L , such that $GT_{n,3c}^{l,\Sigma}$ is connected for all $n \geq L$. We have already mentioned earlier that this statement is not true for labelled near triangulation case. Can this statement be extended to the unlabelled or rooted near triangulations case? If the answer for this question is YES, can we find the constant L ? or more specifically, the minimum L ?

We can show that for the Projective Plane, $GT_{n,m,2c}^{r,P}$ and $GT_{n,m,s}^{r,P}$ are both connected; for the Torus, $GT_{n,m,2c}^{r,T}$ is connected but $GT_{n,m,s}^{r,T}$ is not connected; while for the Klein Bottle, even $GT_{n,m,2c}^{r,K}$ is not connected. In fact, there are three different rooted 2-c triangulations with three vertices on the Klein Bottle; they cannot transform into each other by diagonal flips.

A more general question is: can we find any kind of characteristic properties that can define the diagonal flip adjacency graphs of near triangulations on surfaces? i.e. with what kind of properties can we claim a graph is the diagonal flip adjacency graph of near triangulations on some surface?

Chapter 2

Number of Flippable edges in Rooted 2-c Planar Near Triangulations

2.1 Introduction

In this chapter, we find the total number of flippable edges in all rooted 2-c planar near triangulations of type $[n, m]$, which is the same as the total vertex degree of the diagonal flip adjacency graph $GT_{n,m,2c}^{r,S}$. The total number of edges in $GT_{n,m,2c}^{r,S}$ is exactly this number divided by two. Since the number of rooted 2-c planar near triangulations of type $[n, m]$ was obtained by Mullin [32], we get the average vertex degree of graph $GT_{n,m,2c}^{r,S}$, i.e. the average number of flippable edges in a rooted 2-c planar near triangulation of type $[n, m]$. For fixed m , we find the probability of any non-root face edge being flippable is asymptotically $5/6$.

In map enumeration, the most powerful tool is the generating function methodology. We will carry on this method in this chapter. The main steps are as follows: first, we define a generating function of the variables in which we are interested, then derive a functional equation for this generating function, which can be done by erasing the root edge of a rooted triangulation and decomposing the triangulation. The decomposition makes the rooted triangulation into one or several parts. However, the maps of these parts are not necessarily triangulations, since the root face of these parts may not have valence three. So we need to introduce a new parameter – the valence of the root face, in our generating function. That is why in the area of enumeration of triangulations, people always start from near triangulations.

Starting with a rooted 2-c planar near triangulation, we decompose it and root the maps of the derived parts. Then each map of the derived part becomes a rooted 2-c planar near triangulation. From this we obtain a functional equation of the generating function, which normally involves at least two variables; in our case it involves three variables: the total number of vertices, the root face valence and the total number of flippable edges. After this, we can use Brown's Quadratic method ([10]) to solve this functional equation and get the parametric equations of the generating function. If we are lucky enough, from the parametric equations, we can obtain an explicit expression of the coefficients using Language Inversion Formula. In our case it is the total number of flippable edges. If we are not that lucky, we can still obtain the asymptotic result by applying asymptotic analysis to the parametric equations. See [3].

Let f_{ijk} be the number of rooted 2-c planar near triangulations, having root face valence j , total number of vertices $(i + j)$ and total number of flippable edges k . If

$i = 0$, then all vertices are on the root face; we are actually dealing with 2-c planar Catalan triangulations.

Let us define the generating function $f(x, y, z)$ of f_{ijk} as

$$f(x, y, z) = \sum_{i,j,k} f_{ijk} x^i y^j z^k = \sum_j f_j(x, z) y^j.$$

where $f_j(x, z)x^j$ is the generating functions of the numbers of rooted 2-c planar near triangulations with specific number of interior vertices, specific number of flippable edges, and root face valence j . $f_3(x, z)x^3$ is the generating function of the number of rooted 2-c planar triangulations with parameters of total number of vertices and total number of flippable edges.

In next section, we derive a functional equation for $f(x, y, z)$. In Section 3 we use Brown's Quadratic method to solve this functional equation and obtain algebraic equations for $f(x, y, z)$ and $f_3(x, z)$. The generating functions of the total number of flippable edges in rooted 2-c planar triangulations and rooted 2-c planar near triangulations can be obtained as follows: taking the derivative of $f_3(x, z)$ and $f(x, y, z)$ with respect to z , and then setting $z = 1$ in the derived equations. We discuss this in detail in Section 4.

The two main results in this chapter are:

Theorem 2.1 *The generating function $f_3(x, z)$ is given by*

$$f_3(x, z) = \frac{g^3(1 - 2xg^3)}{z^3}, \quad (2.1)$$

where g is the unique power series satisfying $G(x, z, g) = 0$ with

$$G = 2zx^2(1 - z)g^5 - 2zxg^3 - 2zx(1 - z)g^2 + (1 - zx + 2z^2x - z^3x)g - z. \quad (2.2)$$

Theorem 2.2 *The total number of flippable edges in all rooted 2-c planar triangulations with $n + 3$ vertices is $\frac{45n \times 2^n}{(n+2)(3n+2)} \binom{3n+2}{n-1}$. The average number of flippable edges in a rooted 2-c planar triangulation is $\frac{15n^2}{2(3n+2)}$. In general, the average number of flippable edges in a rooted 2-c planar near triangulation of type $[n, m]$ is*

$$\frac{44m - 55nm - 24m^2 + 15mn^2 + 14nm^2 - 24 + 4m^3 - 15n^2 + 39n}{2(m-1)(3n+2m-4)}.$$

2.2 A Functional Equation for $f(x, y, z)$

In this section we obtain a functional equation for $f(x, y, z)$. For simplicity, we use f , f_2 and f_3 to denote $f(x, y, z)$, $f_2(x, z)$ and $f_3(x, z)$, respectively.

Lemma 2.3 *$f(x, y, z)$ satisfies the following functional equation:*

$$\begin{aligned} f = & y^2 + 2yz(f - y^2) + y^{-1}z^2(f - y^2)^2 + y^3 + 2xz^2(f_2 - 1)((f - y^2)z + y^2) \\ & + x((f - y^2)z + y^2) + y^{-1}xz^2(f - f_2y^2 - 2yz(f_2 - 1)((f - y^2)z + y^2) \\ & - y((f - y^2)z + y^2)). \end{aligned} \quad (2.3)$$

Proof: The degenerate case is there are only two vertices in a rooted 2-c planar near triangulation; the contribution to f in this case is counted by y^2 . Now, we assume our near triangulations have at least three vertices, i.e. $i + j \geq 3$.

For a 2-c or simple near triangulation T with the root vertex v_1 on a surface, the root edge v_1v_2 of T belongs to exactly two faces: the root face and a nonroot triangular face $v_1v_2v_3$. We *decompose* T as follows: if v_3 is on the root face, removing v_1v_2 and merging the two faces incident with it, we obtain a new face F . There is a simple closed loop which begins at v_3 stays always in F and returning to v_3 on the

opposite from which it departed. Cutting the surface along this loop produces one or two surfaces with boundary. By attaching a disk to each boundary, we produce one or two near triangulations on the surface. If v_3 is not on the root face, we just remove the root edge and merge the two faces incident with it, and produce another near triangulation on the surface.

Notice this definition of the decomposition is suitable for all rooted 2-c or simple near triangulations on a surface. In the following chapters of this thesis, we will not repeat saying this process rather we simply say “decomposing a rooted near triangulation”.

After the decomposition, we have several cases and subcases:

Case A: v_3 is on the root face (see Figure 2.1).

T is decomposed into an ordered pair of 2-c planar near triangulations: T_1 and T_2 . Let us root T_1 and T_2 by choosing edge v_1v_3 , v_3v_2 to be their root edges, respectively, v_1 and v_3 to be their root vertices, respectively, and the exterior face of each near triangulation to be the root face. Then T_1 and T_2 become rooted 2-c planar near triangulations. There are four subcases:

Subcase A_1 : T_1 and T_2 are both degenerate.

In this subcase, T_1 is only the edge v_1v_3 and T_2 is only the edge v_3v_2 .

The contribution in this subcase is y^3 .

Subcase A_2 : T_1 is degenerate, but T_2 is not.

The contribution in this subcase is $y^{-1}y^2z(f - y^2) = yz(f - y^2)$.

It can be explained as follows: T_2 cannot be a single edge v_3v_2 since otherwise T only has three vertices, hence we have term $f - y^2$. y^{-1} appeared because we split v_3 into two copies; z appeared because v_3v_2 is flippable in T but it becomes

Figure 2.1: A Rooted 2-c planar near triangulation with v_3 being an exterior vertex.

unflippable in T_2 , since it is on the root face of T_2 .

Subcase A_3 : T_2 is degenerate, but T_1 is not.

The contribution is the same as in subcase A_2 , i.e. $yz(f - y^2)$.

Subcase A_4 : Neither T_1 nor T_2 is degenerate.

The contribution is $y^{-1}z^2(f - y^2)^2$.

It can be explained as follows: T_1 and T_2 can be counted by $z(f - y^2)$. z appeared since v_1v_3 (or v_2v_3) is flippable in T , but it is unflippable in T_1 (or T_2). We need to multiply by y^{-1} since v_3 is separated into two copies.

Case B: v_3 is an interior vertex (see Figure 2.2).

Removing the root edge v_1v_2 , we obtain another 2-c planar near triangulation T' . Let us root T' by choosing v_1 to be the root vertex, v_1v_3 to be the root edge, and its exterior face to be the root face. It is clear that the root face valence of T' is one more than that of T , while the number of interior vertices of T' is one less than that of T . Hence T' has root face valence at least three.

We have the following subcases:

Figure 2.2: A Rooted 2-c planar near triangulation with v_3 being an interior vertex.

Subcase B_1 : In T , edge v_1v_3 is unflippable and edge v_2v_3 is flippable (Figure 2.2(a)).

In this subcase, we have a digon v_2v_3 . The contribution in this case is

$$y^{-1}x(y^{-1}(f_2y^2 - y^2)z^2((f - y^2)z + y^2)) = xz^2(f_2 - 1)((f - y^2)z + y^2).$$

It can be explained as follows: $y^{-1}x$ appeared because of the difference of the root face valence and number of interior vertices between T and T' . $f_2y^2 - y^2$ is the contribution from the digon v_2v_3 . Notice it cannot be a single edge (otherwise v_2v_3 is unflippable), so we need to subtract y^2 from f_2y^2 . z^2 appeared since the pair of multiple edges v_2v_3 are flippable in T . The expression $(f - y^2)z + y^2$ is the contribution of the shadow area other than the digon v_2v_3 in Figure 2.2(a); if this shadow area has only two vertices, the contribution from that is y^2 , otherwise it is $(f - y^2)z$ since the interior edge v_1v_2 of T becomes a root face edge of the near triangulation of the shadow area.

Subcase B_2 : In T , edge v_2v_3 is unflippable and edge v_1v_3 is flippable (Figure 2.2(b)).

The contribution to f in this case is the same as in subcase B_1 .

Subcase B_3 : In T , edge v_1v_3 and edge v_2v_3 are all unflippable (Figure 2.2(c)).

Similar to the subcase B_1 , we obtain the contribution from this subcase. It is

$$y^{-1}xy((f - y^2)z + y^2) = x((f - y^2)z + y^2).$$

Subcase B_4 : Both edge v_1v_3 and edge v_2v_3 are flippable.

The contribution in this case is

$$y^{-1}xz^2((f - f_2y^2) - 2y^{-1}(f_2y^2 - y^2)z((f - y^2)z + y^2) - y((f - y^2)z + y^2)).$$

This contribution is obtained by subtracting all the contributions of the other subcases from the total $f - f_2y^2$.

Combining all these cases and subcases, we obtain Equation (2.3). $\square$

2.3 The Proof of Theorem 2.1

Now, we use Brown's Quadratic Method ([10]) to solve Equation (2.3) and prove Theorem 2.1. To simplify the expression, let $\bar{f}(x, y, z) = (f(x, y, z) - y^2)z + y^2$, then

$$\bar{f}_2(x, z) = [y^2]\bar{f}(x, y, z) = (f_2 - 1)z + 1,$$

$$\bar{f}_3(x, z) = [y^3]\bar{f}(x, y, z) = zf_3.$$

Rewriting (2.3) with $\bar{f}$ and $\bar{f}_2$, we obtain the following functional equation for $\bar{f}$:

$$\bar{f}^2 + (xy + xz - z^{-1}y - 2xyz + xyz^2 + 2xyz\bar{f}_2(1 - z))\bar{f} = xzy^2\bar{f}_2 - z^{-1}y^3. \quad (2.4)$$

Rearrange (2.4) as

$$A(x, y, z)^2 = B(x, y, z)$$

with

$$A = \bar{f} + xyz(\bar{f}_2 - 1) - xyz^2\bar{f}_2 + (xy + xz + xyz^2 - z^{-1}y)/2,$$

$$B = xzy^2\bar{f}_2 - z^{-1}y^3 + (xyz(\bar{f}_2 - 1) - xyz^2\bar{f}_2 + (xy + xz + xyz^2 - z^{-1}y)/2)^2.$$

Let $h(x, z) = \sum_{i \geq 0} h_{ik} x^i z^k$ be a power series solution to $A(x, h, z) = 0$. Then

$$B(x, h, z) = 0, \text{ and } \left. \frac{\partial B}{\partial y} \right|_{y=h(x,z)} = 0.$$

Let us solve these equations for $\bar{f}_2$. We have

$$\bar{f}_2 = \frac{h}{xz^2} - \frac{h^4}{x^3 z^5} \quad (2.5)$$

$$\text{and } (xhz - xhz^2)\bar{f}_2 + \frac{1}{2}(xh + xz + xhz^2 - z^{-1}h) = xhz - \frac{h^3}{xz^2}. \quad (2.6)$$

Let $h = xzg$. Then

$$\bar{f}_2 = \frac{g(1 - xg^3)}{z}; \quad (2.7)$$

Plugging $\bar{f}_2$ in (2.7) into (2.6) and using $h = xzg$, we obtain $G(x, z, g) = 0$ with

$$G = 2zx^2(1 - z)g^5 - 2zxg^3 - 2zx(1 - z)g^2 + (1 - zx + 2z^2x - z^3x)g - z;$$

this is (2.2) in Theorem 2.1.

(2.7) is an algebraic equation for $\bar{f}_2$ with g defined by $G = 0$. Using Equation (2.4), we can obtain an algebraic equation for $\bar{f}(x, y, z)$. Notice there are two solutions for $\bar{f}$ from this equation; we need to take the one which has the negative square root since $\bar{f}$ must be a formal power series of x, y, z .

$$\begin{aligned} \bar{f} = & \frac{1}{2z}(y - xyz - xz^2 - xyz^3 + 2xyz^2 - 2xyzg(1 - xg^3)(1 - z) \\ & - ((y - xyz - xz^2 - xyz^3 + 2xyz^2 - 2xyzg(1 - xg^3)(1 - z))^2 \\ & + 4y^2z(xzg(1 - xg^3) - y))^{1/2}). \end{aligned}$$

From $f = (\bar{f} - y^2)/z + y^2$, we obtain

$$\begin{aligned} f = & \frac{1}{2z^2}(2y^2z^2 - 2y^2z + y - xyz - xz^2 - xyz^3 + 2xyz^2 - 2xyzg(1 - xg^3)(1 - z) \\ & - ((y - xyz - xz^2 - xyz^3 + 2xyz^2 - 2xyzg(1 - xg^3)(1 - z))^2 \\ & + 4y^2z(xzg(1 - xg^3) - y))^{1/2}). \end{aligned} \quad (2.8)$$

In the next section, we use equation (2.8) to obtain the total number of flippable edges in rooted 2-c planar near triangulations.

If we take the coefficient of $[y^3]$ on both sides of (2.8), and use equation (2.2) to simplify the expression, we have

$$f_3 = \frac{g^3(1 - 2xg^3)}{z^3}. \quad (2.9)$$

We have proven Theorem 2.1 completely.

Using MAPLE, we obtain the first few terms of the Taylor series expansion of f_3 .

$$\begin{aligned} f_3 = & 1 + (3 + z^2)zx + 6(1 + z + z^2 + z^4)z^2x^2 + 2(5 + 15z + 18z^2 + 6z^3 + 27z^4 \\ & + 3z^5 + 14z^6)z^3x^3 + (15 + 90z + 182z^2 + 138z^3 + 300z^4 + 174z^5 + 354z^6 \\ & + 30z^7 + 173z^8)z^4x^4 + \dots \end{aligned} \quad (2.10)$$

Notice g is a function of x and z , and g is defined by $G(x, z, g) = 0$. If we can get the explicit expression of g , plugging it into (2.9), we can obtain the explicit solution for f_3 .

Figure 2.3 shows all rooted 2-c planar triangulations with total number of vertices at most five. In these triangulations, we fix the root face to be the exterior face. The arrows tell us the root vertex and root edge, they are always from the root vertex

Figure 2.3: Rooted 2-c planar triangulations with number of vertices ≤ 5 .

to the other vertex of the root edge. If there are several arrows in a triangulation, it means that triangulation has several different rootings. The flippable edges in each triangulation are specified by darker edges.

2.4 Number of Flippable Edges in Rooted 2-c Planar Near Triangulations

Let us consider rooted 2-c planar triangulations first.

Lemma 2.4 *Let $q_{i,3}$ be the number of flippable edges in all rooted 2-c planar triangulations with i interior vertices, then $q_3(x) = \sum_{i=0}^{\infty} q_{i,3}x^i$ is given by*

$$q_3(x) = \left. \frac{\partial f_3}{\partial z} \right|_{z=1}.$$

Proof: Since $f_3(x, z) = \sum_{i,k} f_{i,3,k} x^i z^k$, then

$$\left. \frac{\partial f_3}{\partial z} \right|_{z=1} = \sum_{i,k} f_{i,3,k} k x^i z^{k-1} \Big|_{z=1} = \sum_{i,k} k f_{i,3,k} x^i.$$

Notice $q_{i,3} = \sum_k k f_{i,3,k}$; Lemma 2.4 follows. $\square$

Notice g is an implicit function of x and z , and it is defined by $G(x, z, g) = 0$.

Then

$$\frac{\partial f_3}{\partial z} = \frac{\partial f_3(x, z, g)}{\partial z} + \frac{\partial f_3(x, z, g)}{\partial g} \times \frac{\partial g}{\partial z}.$$

Using $\frac{\partial g}{\partial z} = -\frac{\partial G}{\partial z} / \frac{\partial G}{\partial g}$ we get

$$\left. \frac{\partial g}{\partial z} \right|_{z=1} = \frac{1 - 2xg^2 + 2xg^3 + 2x^2g^5}{1 - 6xg^2}.$$

Also, from $G(x, z, g)|_{z=1} = 0$ we have

$$x = \frac{g-1}{2g^3}.$$

Hence

$$\left. \frac{\partial f_3}{\partial z} \right|_{z=1} = \frac{3g^2}{2}(5g-3)(g-1).$$

Similarly, we get

$$\left. \frac{\partial f}{\partial z} \right|_{z=1} = \frac{(-3 + 4g - 5g^2 + 4g^2y)y}{4g^2} - \frac{(3 - 4g + 5g^2 - 12g^3y)y}{4g^2\sqrt{(1 - 4yg^2)}}. \quad (2.11)$$

To apply the Lagrange Inversion Formula, let us change variable with $g = 1 + t$. Then

$$x = \frac{t}{2(1+t)^3} \quad (2.12)$$

$$\text{and } \left. \frac{\partial f_3}{\partial z} \right|_{z=1} = \frac{3t}{2}(2 + 5t)(1 + t)^2. \quad (2.13)$$

These are parametric equations of the generating function of the total number of flippable edges in all rooted 2-c planar triangulations. Now let us use the Lagrange Inversion Formula to get this number.

Notice

$$q_3(x) = \left. \frac{\partial f_3}{\partial z} \right|_{z=1} = \frac{3t}{2}(2 + 5t)(1 + t)^2$$

and $t = xA(t)$

with $A(t) = 2(1 + t)^3$ (see (2.12)). Applying Lagrange Inversion Formula, we have

$$\begin{aligned} q_{n,3} &= [x^n]q_3(x) = \frac{1}{n}[t^{n-1}]\left(\frac{dq_3}{dt}A(t)^n\right) \\ &= \frac{3}{2n}[t^{n-1}]((10t + 2)(1 + t)^2 + 2(5t^2 + 2t)(1 + t))2^n(1 + t)^{3n} \\ &= \frac{45n \times 2^n}{(n + 2)(3n + 2)} \binom{3n + 2}{n - 1}. \end{aligned}$$

Notice the total number of rooted 2-c planar triangulations with n interior vertices is (see Mullin [32])

$$t_{n,3} = \frac{2^{n+2}}{(2n + 4)(2n + 3)} \binom{3n + 3}{n + 1}.$$

Hence the average number of flippable edges in rooted 2-c planar triangulation with n interior vertices is

$$\frac{q_{n,3}}{t_{n,3}} = \frac{\frac{45n \times 2^n}{(n+2)(3n+2)} \binom{3n+2}{n-1}}{\frac{2^{n+2}}{(2n+4)(2n+3)} \binom{3n+3}{n+1}} = \frac{15n^2}{2(3n+2)}.$$

The total number of edges in a rooted 2-c planar triangulation with n interior vertices is $e = 3(n+3) - 6 = 3n+3$, which can be easily obtained by Euler's Lemma in Chapter 1. Since the edges of the root face are always unflippable, the total number of possible flippable edges is $(3n+3-3) = 3n$. While in average, we have $\frac{15n^2}{2(3n+2)}$ flippable edges. Hence the probability that a nonroot face edge is flippable is:

$$\frac{15n^2}{2(3n+2)3n} \sim \frac{5}{6}.$$

Similarly, we can obtain the total number of flippable edges in rooted 2-c planar near triangulations.

Let $q(x, y) = \frac{\partial f}{\partial z}|_{z=1}$. Then from (2.11),

$$q(x, y) = \frac{(-3 + 4g - 5g^2 + 4g^2y)y}{4g^2} - \frac{(3 - 4g + 5g^2 - 12g^3y)y}{4g^2\sqrt{(1 - 4yg^2)}} \quad (2.14)$$

with $x = \frac{g-1}{2g^3}$.

Similar to Lemma 2.4, we can show $q_m(x) = [y^m]q(x, y)$ is the generating function of the number of flippable edges in all rooted 2-c planar near triangulations with root face valence m .

Taking the coefficient of $[y^m]$ on both sides of (2.14), we have

$$\begin{aligned} q_m(x) &= \frac{1}{4}(3 - 4g + 5g^2)g^{2m-4} \binom{2m-2}{m-1} - 3g^{2m-3} \binom{2m-4}{m-2} \\ &= \frac{5}{4} \binom{2m-2}{m-1} g^{2m-2} - \left(\binom{2m-2}{m-1} + 3 \binom{2m-4}{m-2} \right) g^{2m-3} + \frac{3}{4} \binom{2m-2}{m-1} g^{2m-4}. \end{aligned}$$

Let us use Language Inversion Formula to obtain the total number of flippable edges in all rooted 2-c planar near triangulations of type $[n, m]$. Again, letting $g = 1 + t$, we have $t = xA(t)$ with $A(t) = 2(1 + t)^3$. Let $q_{n,m} = [x^n]q_m(x)$. Then

$$\begin{aligned} q_{n,m} &= \frac{1}{n} [t^{n-1}] \left(\frac{dq_m}{dt} A(t)^n \right) \\ &= \frac{2^{n-1} \binom{2m-2}{m-1} (3n + 2m - 5)!}{n! (2n + 2m - 2)!} (44m - 55nm - 24m^2 + 15mn^2 + 14nm^2 \\ &\quad - 24 + 4m^3 - 15n^2 + 39n). \end{aligned}$$

Since the total number of rooted 2-c planar near triangulations of type $[n, m]$ is (see Mullin [32])

$$t_{n,m} = \frac{2^{n+1} (2m - 3)! (3n + 2m - 4)!}{(m - 2)!^2 n! (2n + 2m - 2)!},$$

the average number of flippable edges in any rooted 2-c planar near triangulation of type $[n, m]$ is

$$\frac{q_{n,m}}{t_{n,m}} = \frac{44m - 55nm - 24m^2 + 15mn^2 + 14nm^2 - 24 + 4m^3 - 15n^2 + 39n}{2(m - 1)(3n + 2m - 4)}.$$

If $n = 0$, this number is $m - 3$, which is the number of nonroot face edges in a planar Catalan triangulation with m vertices in. Obviously these edges are all flippable.

Since the total number of edges in rooted 2-c planar near triangulations of type $[n, m]$ is $e = 3n + 2m - 3$, which can be obtained from $n + m - e + f = 2$ and $3(f - 1) + m = 2e$ (Euler's Lemma), the number of possible flippable edges is $3n + 2m - 3 - m = 3n + m - 3$. Thus the probability of any possible flippable edge in a rooted 2-c planar near triangulation of type $[n, m]$ being flippable is

$$p_{n,m} = \frac{44m - 55nm - 24m^2 + 15mn^2 + 14nm^2 - 24 + 4m^3 - 15n^2 + 39n}{2(m - 1)(3n + m - 3)(3n + 2m - 4)}.$$

If m is fixed and $n \rightarrow \infty$,

$$p_{n,m} \rightarrow \frac{5}{6}.$$

If n is fixed and $m \rightarrow \infty$,

$$p_{n,m} \rightarrow 1.$$

If $n \rightarrow \infty$ and $m \rightarrow \infty$ with $n \sim m$,

$$p_{n,m} \rightarrow \frac{33}{40}.$$

Chapter 3

Number of Flippable Edges in Rooted Simple Planar Near Triangulations

3.1 Introduction

In this chapter, we solve the following problem: How many edges are there in the diagonal flip adjacency graph $GT_{n,m,s}^{r,S}$ of the rooted simple planar near triangulations?

Similar to Chapter 2, we will solve this problem by finding the total number of flippable edges in all rooted simple planar near triangulations of type $[n, m]$, i.e. the total vertex degree of graph $GT_{n,m,s}^{r,S}$. This number is exactly twice of the number of edges in $GT_{n,m,s}^{r,S}$. Brown [9] enumerated the rooted simple planar near triangulations, which gives the total number of vertices in $GT_{n,m,s}^{r,S}$. From these results, we obtain the

and number of the flippable edges.

The main results in this chapter are:

Theorem 3.1 *The generating function $F_3(x, z)$ is given by*

$$F_3 = \frac{g(2xzg^4 + (12xz^2(1-z) - 1)g^2 + z(1+x-3xz^2+2xz^3)g - 3z(1-z))}{xz^3g^3(1+x-3xz^2+2xz^3) + 9xz^3(1-z)g^2 - 3z^2(1-z)} \quad (3.1)$$

where g is defined by $K(x, z, g) = 0$ with

$$\begin{aligned} K = & x^2zg^7 + 12x^2z^2(1-z)g^5 - xz(1+x-3xz^2+2xz^3)g^4 + xz(36z^2x \\ & - 78z^3x + 4x^2z^7 + 2xz - 6x^2z^3 + 9x^2z^5 + 4x^2z^4 - 12x^2z^6 + 16z \\ & + 40xz^4 + x^2z - 15)g^3 + 12xz^2(1-z)(1+x-3xz^2+2xz^3)g^2 \\ & + 3(1-z)(1-3xz^2+3xz^3)g - 3z(1-z)(1+x-3xz^2+2xz^3). \end{aligned} \quad (3.2)$$

Theorem 3.2 *The total number of flippable edges in all rooted simple planar triangulations with $n+3$ vertices is $\frac{18}{n+1} \binom{4n}{n-2}$; the average number of flippable edges in any triangulation of this kind is $\frac{9n(n-1)}{4n+1}$. In general, the average number of flippable edges in any rooted simple planar near triangulation of type $[n, m]$ is*

$$\frac{4m^3 + 14m^2n - 28m^2 + 18mn^2 - 75nm - 27n^2 + 63m + 72n - 45}{(2m-3)(4n+2m-5)}.$$

The rest of this chapter is organized as follows: Section 2 deals with a special kind of rooted simple planar near triangulations and its generating function $H(x, y, z, t)$; in Section 3, we obtain a functional equation for $F(x, y, z)$, which involves with $H(x, y, z, 1)$; in section 4 and 5, we prove Theorem 3.1 and Theorem 3.2 respectively.

3.2 A Special Kind of Rooted Simple Planar Near Triangulation

First, we define a special kind of rooted simple planar near triangulation.

Let T be a rooted simple planar near triangulation with the root vertex v_1 , the root edge v_1v_2 and the root face the exterior face. Furthermore, we require the root edge v_1v_2 to lie on a nonroot face $\Delta v_1v_2v_3$ with v_3 being an interior vertex. If we erase v_1v_2 , we obtain another simple planar near triangulation T' . Let us root T' by choosing v_1 to be the root vertex, v_1v_3 to be the root edge, and the exterior face of T' to be the root face. T' becomes a rooted simple planar near triangulation with root face valence at least four. Obviously, v_1 and v_2 are not connected by an edge in T' (see Figure 3.1).

We define the set of a special kind of rooted simple planar near triangulations to be the set $\mathcal{T}'$, which consists of all rooted simple planar near triangulations like T' .

Let $T' \in \mathcal{T}'$ and suppose T' is obtained from a rooted simple planar near triangulation T by erasing its root edge v_1v_2 . For any edge v_iv_j of T' , we call it a *t-class edge* of T' if v_iv_j is flippable in T' , but is unflippable in T .

It is easy to see that v_iv_j is a *t-class edge* of T' if and only if it is a flippable edge of T' and flipping v_iv_j generates edge v_1v_2 . That means $v_iv_jv_1$ and $v_iv_jv_2$ are two faces with which v_iv_j is incident. For example, in Figure 3.1, edge ab is a *t-class edge*.

Let h_{ijkl} be the number of rooted simple planar near triangulations in $\mathcal{T}'$ with number of interior vertices i , root face valence j , number of flippable edges $k+l$ and number of *t-class edges* l . Hence the number of flippable edges which are not *t-class*

Let $T' \in \mathcal{T}'$, and suppose T' has at least one t -class edge. Then we can always find a vertex b of T' such that all t -class edges of T' are inside of the quadrangle $v_1v_3v_2b$. Let ab be a t -class edge incident with vertex b , we call edge ab *the last t -class edge* of T' . See Figure 3.1. Notice if $a = v_3$, T' has only one t -class edge. i.e. edge v_3b .

The following lemma gives a functional equation for $H(x, y, z, t)$.

Lemma 3.4 *The generating function $H(x, y, z, t)$ satisfies*

$$\begin{aligned} H(x, y, z, t) = & H(x, y, z, 0) + xt z^2 (1 + x z^2 [y^4] (H(x, y, z, t))) H(x, y, z, 0) \\ & + t(1 + x z^2 [y^4] H(x, y, z, t)) (y^2 + zF)^2. \end{aligned} \quad (3.5)$$

Proof: Let $T' \in \mathcal{T}'$. We do case analysis as follows:

Case A: *there is no t -class edge in T' at all.*

The contribution in this case is: $H(x, y, z, 0)$

Case B: *there is at least one t -class edge in T' .*

Let edge ab to be the last t -class edge in T' . According to the position of b , we have two subcases:

Subcase B_1 : *vertex b is an interior vertex. See Figure 3.1(a).*

If $a = v_3$, the contribution is $xt z^2 H(x, y, z, 0)$; otherwise, the contribution from the near triangulation of the quadrangle $v_1v_3v_2a$ is $[y^4] H(x, y, z, t)$, the contribution from the near triangulation of the other shaded area is $H(x, y, z, 0)$, since there is no t -class edge there. Hence the total contribution from this subcase is

$$xt z^2 H(x, y, z, 0) + x^2 t z^4 [y^4] H(x, y, z, t) H(x, y, z, 0).$$

Subcase B_2 : *vertex b is an exterior vertex. See Figure 3.1(b).*

If $a = v_3$, the contribution is $t(y^2 + zF)^2$; otherwise, the contribution is

$$xtz^2[y^4]H(x, y, z, t)(y^2 + zF)^2.$$

In total, the contribution in this subcase is

$$t(1 + xz^2[y^4]H(x, y, z, t))(y^2 + zF)^2.$$

Lemma 3.4 follows by combining all these cases and subcases. $\square$

We will need the expression of $H(x, y, z, 1)$ in next section. It can be seen as the generating function of the number of all near triangulations in $\mathcal{T}'$ with assuming all t -class edges to be unflippable.

The next lemma gives the expression of $H(x, y, z, 1)$. We use F_3 , F_4 and F to denote $F_3(x, z)$, $F_4(x, z)$ and $F(x, y, z)$, respectively, and use H_0 and H_4 to denote $H(x, y, z, 0)$ and $[y^4]H(x, y, z, t)$, respectively.

Lemma 3.5 *$H(x, y, z, 1)$ satisfies the following equation:*

$$H(x, y, z, 1) = \frac{F - yzF_3F - y^3F_3 + (1 - z)(y^2 + zF)^2(1 + xz^2(F_4 - zF_3^2))}{1 - xz^2(1 - z)(1 + xz^2(F_4 - zF_3^2))} \quad (3.6)$$

Proof: Plugging $t = z$ in (3.5) and using (3.3), we have

$$F - yzF_3F - y^3F_3 = H_0 + xz^3(1 + z^2x(F_4 - zF_3^2))H_0 + z(1 + z^2x(F_4 - zF_3^2))(y^2 + zF)^2.$$

Hence

$$H_0 = \frac{F - yzF_3F - z(1 + xz^2(F_4 - zF_3^2))(y^2 + zF)^2}{1 + xz^3(1 + xz^2(F_4 - zF_3^2))}. \quad (3.7)$$

Taking the coefficient of $[y^4]$ on both sides of (3.7), we have

$$[y^4]H_0 = \frac{(1 - xz^3)(F_4 - zF_3^2) - z}{1 + xz^3(1 + xz^2(F_4 - zF_3^2))}.$$

Now, taking the coefficient of $[y^4]$ on both sides of equation (3.5) and solving for H_4 , we have

$$H_4 = \frac{t - z + (1 + xtz^2 - xz^3)(F_4 - zF_3^2)}{1 - xtz^2 + xz^3 - x^2z^4(t - z)(F_4 - zF_3^2)}.$$

Since we have obtained the expressions for H_4 and H_0 , plugging them in (3.5), we obtain the expression of $H(x, y, z, t)$, which involves F_3 , F_4 and F . Setting $t = 1$ in the expression of $H(x, y, z, t)$ and simplifying it, we obtain (3.6).

Lemma 3.5 has been proven. $\square$

3.3 A Functional Equation for $F(x, y, z)$

The following lemma gives a functional equation for F .

Lemma 3.6 *F satisfies the relation*

$$\begin{aligned} & (y^2 + zF)^2 zu + (y^2 + zF)(xz^2(1 - yzF_3)u + xz^2y(1 - z)(1 + xz^2(F_4 - zF_3^2))) \\ & - y(y^2 + zF) + y^3(1 - xz^2(1 - z)(1 + xz^2(F_4 - zF_3^2))) - xy^2z^2u = 0 \end{aligned} \quad (3.8)$$

with $u = 1 + x - xz^2 + 2xz^2(1 - z)(F_3 - 1)$.

Proof: Let T be a rooted simple planar near triangulation with v_1 being the root vertex and v_1v_2 being the root edge. v_1v_2 lies on a nonroot triangular face $v_1v_2v_3$. Decomposing T , we have the following cases and subcases:

Case A: v_3 is a root face vertex.

We decompose T into an ordered pair of simple planar near triangulations T_1 and T_2 . Let us root T_1 and T_2 by choosing v_1 and v_3 to be the root vertices, v_1v_3

Figure 3.2: Rooted simple planar near triangulations with v_3 being an interior vertex.

Let $\Delta v_1 v_3 v_4$ be a triangular face in T , we have two subsubcases:

Subsubcase B_{11} : v_4 is an interior vertex. See Figure 3.2(a).

The contribution in this situation is

$$F_{211} = y^{-1} x^2 z^4 (F_3 - 1) H(x, y, z, 1).$$

Here $(F_3 - 1)$ is the contribution from the separating triangle $\Delta v_3 v_2 v_4$, we need to subtract 1 from F_3 since the separating triangle must contain some vertex inside, otherwise edge $v_2 v_3$ is unflippable. $H(x, y, z, 1)$ is the contribution from the near triangulations on the shadow area other than the separating triangle $\Delta v_3 v_2 v_4$ in Figure 3.2(a); these near triangulations are in set $\mathcal{T}'$ and here we note the t -class edges of these near triangulations are unflippable.

Subsubcase B_{12} : v_4 is an exterior vertex. See Figure 3.2(b).

The contribution in this situation is

$$F_{212} = x z^2 (F_3 - 1) y^3 + 2 x y z^3 (F_3 - 1) F + y^{-1} x z^4 (F_3 - 1) F^2.$$

Here $x z^2 (F_3 - 1) y^3$ is the contribution from the case that $v_1 v_4$ and $v_2 v_4$ both are the root face edges; $2 x y z^3 (F_3 - 1) F$ is the contribution from the case that one of $v_1 v_4$ and $v_2 v_4$ is the root face edge; the last term in F_{212} is the contribution from the case that neither edge $v_1 v_4$ nor edge $v_2 v_4$ is on the root face.

The total contribution of subcase B_1 is

$$F_{21} = F_{211} + F_{212}.$$

Subcase B_2 : In T , $v_2 v_3$ is unflippable and $v_1 v_3$ is flippable. See Figure 3.2(c) and (d).

Let the contribution in this subcase is F_{22} . F_{22} is the same as F_{21} of the subcase B_1 .

Subcase B_3 : In T , v_1v_3 and v_2v_3 are both unflippable. See Figure 3.2(e) and (f).

We need to consider two different situations in this subcase. Namely, v_4 is an interior vertex or v_4 is an exterior vertex. The contribution F_{23} of this subcase can be obtained in a similar way as we obtain F_{21} in subcase B_1 .

$$F_{23} = y^{-1}x^2z^2H(x, y, z, 1) + xy^3 + 2xyzF + y^{-1}xz^2F^2.$$

Subcase B_4 : v_1v_3 and v_2v_3 are both flippable.

The contribution F_{24} of this subcase can be obtained through subtracting all the contributions of the other subcases from $H(x, y, z, 1)$. Hence

$$\begin{aligned} F_{24} = & y^{-1}xz^2(H(x, y, z, 1) - 2((F_3 - 1)H(x, y, z, 1)xz^3 + (F_3 - 1)z(y^2 + zF)^2) \\ & - (xz^2H(x, y, z, 1) + (y^2 + zF)^2)). \end{aligned}$$

Combining all these cases and subcases, we have

$$F = F_1 + 2(F_{211} + F_{212}) + F_{23} + F_{24}.$$

Notice it is a recursion for F . Plugging the expression of $H(x, y, z, 1)$ (see equation (3.6)) in this recursion and simplify it, we obtain the functional equation (3.8).

Lemma 3.6 has been proven. $\square$

3.4 The Proof of Theorem 3.1

In this section we use Brown's Quadratic method ([10]) to solve the functional equation (3.8) and prove Theorem 3.1.

Let $\bar{F} = y^2 + zF$. Then

$$\begin{aligned}\bar{F}_3 &= [y^3]\bar{F} = zF_3, \\ \bar{F}_4 &= [y^4]\bar{F} = zF_4, \\ F - yzF_3F - y^3F_3 &= \frac{\bar{F} - y^2 - y\bar{F}_3\bar{F}}{z}.\end{aligned}$$

Equation (3.8) can be written as

$$\begin{aligned}\bar{F}^2zu + \bar{F}(xz^2(1 - y\bar{F}_3)u + xyz^2(1 - z)(1 + xz(\bar{F}_4 - \bar{F}_3^2)) - y) \\ + y^3(1 - xz^2(1 - z)(1 + xz(\bar{F}_4 - \bar{F}_3^2))) - xy^2z^2u = 0\end{aligned}\quad (3.9)$$

with

$$u = 1 + x - xz^2 + 2xz(1 - z)(\bar{F}_3 - z). \quad (3.10)$$

In (3.9), there are three unknown variables: $\bar{F}$, $\bar{F}_4$ and $\bar{F}_3$; we cannot apply Brown's Quadratic method directly. Taking the coefficient of $[y^4]$ on both sides of this equation, we have

$$1 + xz(\bar{F}_4 - \bar{F}_3^2) = \frac{\bar{F}_3}{uz + xz^2(1 - z)\bar{F}_3}. \quad (3.11)$$

Plugging this relation back into (3.9), we get a functional equation for $\bar{F}$ which only involves $\bar{F}$ and $\bar{F}_3$. Then we can solve it by Quadratic method.

Rearrange (3.9) as $A(x, y, z)^2 = B(x, y, z)$ with

$$A = \bar{F}uz + (xz^2(1 - y\bar{F}_3)u - y + xz^2y(1 - z)(1 + xz(\bar{F}_4 - \bar{F}_3^2)))/2 \quad (3.12)$$

$$\begin{aligned}
B = & ((xz^2(1 - y\bar{F}_3)u - y + xz^2y(1 - z)(1 + xz(\bar{F}_4 - \bar{F}_3^2)))/2)^2 \\
& - uz(y^3 - xz^2y^3(1 - z)(1 + xz(\bar{F}_4 - \bar{F}_3^2)) - xy^2z^2u), \tag{3.13}
\end{aligned}$$

while u is given in (3.10), $\bar{F}_4$ and $\bar{F}_3$ satisfy the (3.11).

Let $h(x, z) = \sum_{i \geq 0} h_{ik} x^i z^k$ be a power series solution to $A(x, h(x, z), z) = 0$.

Then

$$B(x, h(x, z), z) = 0, \text{ and } \left. \frac{\partial B}{\partial y} \right|_{y=h(x,z)} = 0.$$

Let $h = xzg$. The previous equations can be simplified as

$$\begin{aligned}
(xz)^2 g^3 \bar{F}_3^2 - (4xzg^2 - 1)\bar{F}_3 + g(4xzg^2 - 1) &= 0 \\
z(1 - xzg\bar{F}_3)(u + xz(1 - z)\bar{F}_3) &= g - 2xzg^3.
\end{aligned}$$

Plugging $\bar{F}_3 = zF_3$ back into these equations, we have

$$xz^4 g^3 F_3^2 - (4xz^2 g^2 - z)F_3 + g(4xzg^2 - 1) = 0 \tag{3.14}$$

$$z(1 - xz^2 g F_3)(u + xz^2(1 - z)F_3) = g - 2xzg^3. \tag{3.15}$$

Here u is given by equation (3.10).

Now, it is easy to verify that equations (3.14) and (3.15) are equivalent with equations (3.1) and $K(x, z, g) = 0$ with $K(x, z, g)$ being defined by (3.2).

We have proven Theorem 3.1 completely.

Using the quadratic equation $A^2 = B$ and (3.1), we can solve for $\bar{F}$. Using the relation $\bar{F} = y^2 + zF$, we can obtain an algebraic equation for F . Notice there are two solutions for $\bar{F}$ when we solve the quadratic equation; we need to take the solution which has the negative square root since $\bar{F}$ is a formal power series of x, y, z .

$$F = -\frac{\sqrt{B} + y^2 zu}{z^2 u} + \frac{y - xz^2(1 - y\bar{F}_3)u - xyz^2(1 - z)(1 + xz(\bar{F}_4 - \bar{F}_3^2))}{2z^2 u}. \tag{3.16}$$

Here B is given by (3.13), u is given in (3.10), $\bar{F}_4$ and $\bar{F}_3$ satisfy (3.11). If we plug the expression of F_3 in (3.1) into (3.16), we obtain an expression for F . However, the expression for F is too complicated to be displayed here.

Using MAPLE software, we obtain the first few terms of the Taylor series expansion of $F_3(x, z)$.

$$\begin{aligned} F_3 = & 1 + x + 3z^2x^2 + (3 + 9z + z^6)z^3x^3 + (1 + 27z^2 + 27z^3 + 7z^6 + 3z^8 + 3z^9)z^3x^4 \\ & + 3(4 + 6z + 54z^2 + 27z^3 + 7z^4 + 14z^6 + 11z^7 + 3z^8 + 3z^9 + 4z^{10})z^5x^5 \\ & + \dots \end{aligned} \tag{3.17}$$

Figure 3.3 shows all the rooted simple planar triangulations with number of vertices not more than six. In these triangulations, we fix the root face to be the exterior face, the arrows on the edges indicate the root vertex and the root edge. If there are several arrows in a triangulation, it means that triangulation has several different rooting. The flippable edges in each triangulation are specified by darker edges.

3.5 Number of Flippable Edges in Rooted Simple Planar Near Triangulations

In this section we prove Theorem 3.2.

First we consider the rooted simple planar triangulations. The following lemma is easy to prove.

Lemma 3.7 *Let $Q_{i,3}$ be the number of flippable edges in all rooted simple planar triangulations with i interior vertices, then the generating function $Q_3(x) = \sum_{i=0}^{\infty} Q_{i,3}x^i$*

Figure 3.3: Rooted simple planar triangulations with number of vertex ≤ 6 .

is given by

$$Q_3(x) = \left. \frac{\partial F_3}{\partial z} \right|_{z=1}.$$

Notice $K(x, z, g) = 0$ with K being defined by (3.2). which defines g as an implicit function of x and z , i.e. $g = g(x, z)$. From $\frac{\partial g}{\partial z} = -\frac{\partial G}{\partial z} / \frac{\partial G}{\partial g}$ we get

$$\left. \frac{\partial g}{\partial z} \right|_{z=1} = \frac{12x^2g^5 - x^2g^7 - 17xg^3 + 12xg^2 + xg^4 - 3 + 3g}{xg^2(7xg^4 + 3 - 4g)}.$$

Also from $K(x, z = 1, g) = 0$, we get

$$x = \frac{g - 1}{g^4}.$$

Using (3.1), (3.16), and

$$\frac{\partial F_3}{\partial z} = \frac{\partial F_3(x, z, g)}{\partial z} + \frac{\partial F_3(x, z, g)}{\partial g} \times \frac{\partial g}{\partial z},$$

$$\frac{\partial F}{\partial z} = \frac{\partial F(x, z, g)}{\partial z} + \frac{\partial F(x, z, g)}{\partial g} \times \frac{\partial g}{\partial z},$$

we can obtain

$$\begin{aligned} \left. \frac{\partial F_3}{\partial z} \right|_{z=1} &= 6g(g-1)^2. \\ \left. \frac{\partial F}{\partial z} \right|_{z=1} &= \frac{(-3 + 5g - 3g^2 + g^3y)y}{g^3} + \frac{(3 - 5g + 3g^2 - 6g^2y + 9g^3y - 6g^4y)y}{g^3\sqrt{(1-4yg^2)}}. \end{aligned}$$

Let $g = 1 + t$. Then

$$x = \frac{t}{(1+t)^4} \quad (3.18)$$

$$\text{and } \left. \frac{\partial F_3}{\partial z} \right|_{z=1} = 6t^2(1+t). \quad (3.19)$$

These are the parametric equations of the generating function of the total number of the flippable edges in rooted simple planar triangulations.

Notice

$$\begin{aligned} Q_3(x) &= \left. \frac{\partial F_3(x, z)}{\partial z} \right|_{z=1} = 6t^2(1+t), \\ t &= x\psi(t) \end{aligned}$$

with $\psi(t) = (1+t)^4$. Using Lagrange Inversion Formula, we have

$$\begin{aligned} Q_{n,3} &= [x^n]Q_3(x) = \frac{1}{n}[t^{n-1}]\left(\frac{dQ_3}{dt}\psi(t)^n\right) \\ &= \frac{1}{n}[t^{n-1}]((12t + 18t^2)(1+t)^{4n}) \\ &= \frac{12}{n}[t^{n-2}]((1+t)^{4n}) + \frac{18}{n}[t^{n-3}]((1+t)^{4n}) \\ &= \frac{18}{n+1} \binom{4n}{n-2}. \end{aligned}$$

Since the total number of rooted simple planar triangulations with n interior vertices is obtained by Brown (see [9]), which is

$$T_{n,3} = \frac{\binom{4(n+3)-9}{n+1}}{\binom{4(n+3)-9}{2}}.$$

From this, we can obtain the average number of flippable edges in rooted simple planar triangulations with n interior vertices, i.e.

$$\frac{Q_{n,3}}{T_{n,3}} = \frac{\frac{18}{n+1} \binom{4n}{n-2} \binom{4n+3}{2}}{\binom{4n+3}{n+1}} = \frac{9n(n-1)}{4n+1}.$$

The total number of edges in a simple planar triangulation with n interior vertices is $e = 3n + 3$. Since we are not allowed to flip edges on the root face, the total number of possible flippable edges is $(3n + 3 - 3) = 3n$. In average, we have $\frac{9n(n-1)}{4n+1}$ flippable edges. Hence the probability that a non-root face edge is flippable is

$$\frac{\frac{9n(n-1)}{4n+1}}{3n} \sim \frac{3}{4}.$$

Since almost all simple planar triangulations are asymmetrical, we can say that the probability of an edge in an unlabelled (labelled) simple planar triangulation being flippable is also asymptotically $3/4$.

Similarly, we can find the total number of flippable edges in all rooted simple planar near triangulations of type $[n, m]$.

Let $Q(x, y) = \left. \frac{\partial F}{\partial z} \right|_{z=1}$. Then

$$Q(x, y) = \frac{(-3 + 5g - 3g^2 + g^3y)y}{g^3} + \frac{(3 - 5g + 3g^2 - 6g^2y + 9g^3y - 6g^4y)y}{g^3\sqrt{(1 - 4yg^2)}} \quad (3.20)$$

with $x = (g - 1)/(g^4)$.

Let $Q_m(x) = [y^m]Q(x, y)$. It is clear that $Q_m(x)$ is the generating function of the number of flippable edges in all rooted simple planar near triangulations with root face valence m . Taking the coefficient of $[y^m]$ on both sides of (3.20), we have

$$\begin{aligned} Q_m(x) &= (3 - 5g + 3g^2)g^{2m-5} \binom{2m-2}{m-1} - (6 - 9g + 6g^2)g^{2m-5} \binom{2m-4}{m-2} \\ &= \frac{6(m-2) \binom{2m-4}{m-2}}{m-1} g^{2m-5} - \frac{(11m-21) \binom{2m-4}{m-2}}{m-1} g^{2m-4} + \frac{6(m-2) \binom{2m-4}{m-2}}{m-1} g^{2m-3}. \end{aligned}$$

With this equation and

$$x = \frac{g-1}{g^4},$$

we can obtain the total number of flippable edges in all rooted simple planar near triangulations of type $[n, m]$.

Let $g = 1 + t$. Then $t = x\psi(t)$ with $\psi(t) = (1+t)^4$. Let $Q_{n,m} = [x^n]Q_m(x)$. $Q_{n,m}$ is the total number of flippable edges in all rooted simple planar near triangulations of type $[n, m]$. Using Lagrange Inversion Formula, we have

$$\begin{aligned} Q_{n,m} &= \frac{1}{n} [t^{n-1}] \left(\frac{dQ_m}{dt} \psi(t)^n \right) \\ &= \frac{2 \binom{2m-4}{m-1} (4n+2m-6)!}{n!(3n+2m-3)!} (63m - 75nm - 28m^2 + 18mn^2 + 14nm^2 \\ &\quad - 45 + 4m^3 - 27n^2 + 72n). \end{aligned}$$

Notice the total number of rooted simple planar near triangulations of type $[n, m]$ is (see Brown [9])

$$T_{n,m} = \frac{2(2m-3)!(4n+2m-5)!}{(m-3)!(m-1)!n!(3n+2m-3)!}.$$

The average number of flippable edges in a rooted simple planar near triangulation of type $[n, m]$ is

$$\frac{Q_{n,m}}{T_{n,m}} = \frac{4m^3 + 14m^2n - 28m^2 + 18mn^2 - 75nm - 27n^2 + 63m + 72n - 45}{(2m-3)(4n+2m-5)}.$$

Chapter 4

Minimum and Maximum Vertex Degrees of the Diagonal Flip Adjacency Graphs of Simple and 2-c Planar Near Triangulations

4.1 Introduction

In Chapters 2 and 3, we have obtained the number of flippable edges in rooted 2-c planar near triangulations of type $[n, m]$ and the number of flippable edges in rooted simple planar near triangulations of type $[n, m]$, respectively. They are the total vertex degrees in graph $GT_{n,m,2c}^{r,S}$ and graph $GT_{n,m,s}^{r,S}$ respectively. The average vertex degrees in there graphs are also determined. In this chapter, we consider: what are the minimum and maximum number of flippable edges in all rooted simple

(2-c) planar near triangulation of type $[n, m]$? That is, what is the minimum and maximum vertex degrees in $GT_{n,m,2c}^{r,S}$ and $GT_{n,m,s}^{r,S}$? We will solve this problem not only in rooted near triangulation case, but also in the unlabelled and labelled cases. Notice some results in this chapter are already given in [22].

When we consider the minimum vertex degree and maximum vertex degree in the diagonal flip adjacency graphs of near triangulations, we do not need to distinguish the unlabelled case and the labelled case: they are always the same. If near triangulations are not triangulations, we even do not need to distinguish all three cases: the unlabelled case, the labelled case and the rooted case. But if near triangulations are triangulations, the rooted case is different from the unlabelled (labelled) case, since a root face edge is always unflippable in a rooted triangulation, while as an edge of the corresponding unlabelled triangulation, it may be flippable. Hence we need to consider the minimum (maximum) vertex degree of the diagonal flip adjacency graphs of unlabelled near triangulations, and that of the diagonal flip adjacency graphs of rooted triangulations.

For $m \neq 3$, let $D_{n,m,s}^S$ and $d_{n,m,s}^S$ be the maximum vertex degree and the minimum vertex degree of $GT_{n,m,s}^S$ respectively; let $D_{n,m,2c}^S$ and $d_{n,m,2c}^S$ be the maximum vertex degree and the minimum vertex degree of $GT_{n,m,2c}^S$ respectively. For $m = 3$, let $D_{n,3c}^S$, $D_{n,3c}^{r,S}$, $D_{n,2c}^S$ and $D_{n,2c}^{r,S}$ be the maximum vertex degrees of $GT_{n,3c}^S$, $GT_{n,3c}^{r,S}$, $GT_{n,2c}^S$ and $GT_{n,2c}^{r,S}$ respectively; let $d_{n,3c}^S$, $d_{n,3c}^{r,S}$, $d_{n,2c}^S$ and $d_{n,2c}^{r,S}$ be the minimum vertex degrees of $GT_{n,3c}^S$, $GT_{n,3c}^{r,S}$, $GT_{n,2c}^S$ and $GT_{n,2c}^{r,S}$ respectively.

The maximum vertex degrees of these graphs are easy to obtain. In the case of $m = 3$, there exist some unlabelled simple planar triangulations with n vertices in which each edge is flippable. In the case of $m \neq 3$, there exists some unlabelled

simple planar near triangulations of type $[n, m]$ whose edges except those on the non-triangular face are all flippable. We discuss this in detail in Section 2. In Section 3, we obtain $d_{n,3c}^S$ and $d_{n,3c}^{r,S}$. Section 4 deals with $d_{n,2c}^S$ and $d_{n,2c}^{r,S}$. In the last section of this chapter, we discuss $d_{n,m,s}^S$ and $d_{n,m,2c}^S$.

In [27], Hurt, Noy and Urrutia proved that every geometrical triangulation of a point set P_n of n points on the plane contains at least $\lceil \frac{n-1}{2} \rceil$ flippable edges and the bound is tight. In this chapter, we prove that every simple planar triangulation with n vertices contains at least $n - 2$ flippable edges, and every simple planar near triangulation of type $[n, m]$ with $m \neq 3$ contains at least $\max(n - 1, m - 3)$ flippable edges. These bounds are tight.

4.2 Maximum Number of Flippable Edges

It is easy to show that if a simple planar triangulation T has no separating triangle, then each edge of T is flippable. Tutte [48] obtained the numbers of all rooted simple planar triangulations without a separating triangle and with n vertices for any n ; these numbers are all positive. Hence it shows the existence of simple planar triangulations in which every edge is flippable. Thus the maximum number of flippable edges in an unlabelled simple planar triangulation is the total number of edges in the triangulation.

In this section, for each n , we construct an unlabelled simple planar triangulation with n vertices in which every edge is flippable. Notice our examples have separating triangles.

We consider three different cases:

Figure 4.1: Triangulations with $n = 3k$ or $n = 3k+1$ vertices, all edges are flippable.

Case 1: $n = 3k$ and k is an integer with $k \geq 2$.

We construct a simple planar triangulation T with n vertices in Figure 4.1(a). In T , $\triangle v_{3i-2}v_{3i-1}v_{3i}$ ($2 \leq i \leq k-1$) is a separating triangle. $\triangle v_{3i-2}v_{3i-1}v_{3i}$ is contained in $\triangle v_{3j-2}v_{3j-1}v_{3j}$ for all i, j with $1 \leq j \leq i \leq k$.

Case 2: $n = 3k+1$ and k is an integer with $k \geq 2$.

We construct a triangulation with n vertices in Figure 4.1(b). The most interior face $\triangle v_{3k-2}v_{3k-1}v_{3k}$ of Figure 4.1(a) has been substituted by a quadrangle $v_{3k-2}v_{3k-1}v_{3k}v_{3k+1}$, with v_{3k-1} and v_{3k+1} are connected by an edge.

Case 3: $n = 3k+2$ and k is an integer with $k \geq 2$.

We construct a triangulation with n vertices in Figure 4.2. Edge $v_{3k-1}v_{3k+1}$ in Figure 4.1(b) has been deleted, a vertex v_{3k+2} has been inserted inside the quadrangle $v_{3k-2}v_{3k-1}v_{3k}v_{3k+1}$ and v_{3k+2} is connected to all four vertices of the quadrangle.

It is easy to verify that each edge of the three triangulations of Figure 4.1 and Figure 4.2 is flippable. If we root these triangulations in an arbitrary way, the three

Figure 4.2: A triangulation with $n = 3k + 2$ vertices, all edges are flippable.

root face edges will become unflippable, other edges are still flippable.

Theorem 4.1 For $n \geq 6$, the maximum vertex degrees of the graphs $GT_{n,3c}^S$ and $GT_{n,3c}^{r,S}$ are given by $D_{n,3c}^S = 3n - 6$, $D_{n,3c}^{r,S} = 3n - 9$. For $n = 5$, $D_{n,3c}^S = 3$, $D_{n,3c}^{r,S} = 2$. For $n = 3$ and $n = 4$, $D_{n,3c}^S = D_{n,3c}^{r,S} = 0$.

Proof: For any $n \geq 6$, Figure 4.1 and Figure 4.2 show there exists an unlabelled simple planar triangulation with n vertices in which each edge is flippable. Hence there exists a rooted simple planar triangulation with n vertices in which each edge, except the three root face edges, is flippable. Since the total number of edges in these triangulations is $3n - 6$, the result follows.

For $n = 3, 4$ and 5 , we can verify the result directly from Figure 3.3 or the Taylor series of $F_3(x, z)$ in (3.17). □

The next result follows by noticing that every simple planar triangulation is also a 2-c planar triangulation.

Figure 4.3: A near triangulation of type $[6,6]$, all edges other than edges on the non-triangular face are flippable.

Corollary 4.2 For $n \geq 4$, the maximum vertex degrees of the graphs $GT_{n,2c}^S$ and $GT_{n,2c}^{r,S}$ are given by $D_{n,2c}^S = 3n - 6$, $D_{n,2c}^{r,S} = 3n - 9$. For $n = 3$, $D_{n,3c}^S = D_{n,3c}^{r,S} = 0$.

Proof: We only need to prove the result for the cases $n = 3, 4$ and 5 . These can be verified directly from Figure 2.3 or the Taylor series of f_3 in (2.10). $\square$

For a simple planar near triangulation of type $[n, m]$, if $m \neq 3$, then all the edges of the non-triangular face are unflippable. In Figure 4.3, we show the existence of an unlabelled simple planar triangulation of type $[n, m]$ (with $n = m = 6$) in which each edge except the edge on the non-triangular face is flippable. In general, we can obtain this kind of graph for any $n \geq 0$ and $m > 3$ by putting all interior vertices inside a triangle whose three vertices are exterior vertices, then constructing a triangulation inside this triangle as in Figure 4.1 and Figure 4.2.

The following theorem is easy to prove.

Theorem 4.3 For any $n \geq 0$ and $m > 3$, the maximum vertex degrees of the graphs

$GT_{n,m,s}^S$ and $GT_{n,m,2c}^S$ are given by $D_{n,m,s}^S = D_{n,m,2c}^S = 3n + m - 3$. For $m = 2$ and $n \geq 2$, $D_{n,2,2c}^S = 3n - 1$. For $m = 2$ and $n = 0, 1$, $D_{0,2,2c}^S = D_{1,2,2c}^S = 0$.

4.3 Minimum Number of Flippable Edges in 3-c Planar Triangulations

In this section, we give a tight lower bound on the number of flippable edges in simple planar triangulations. The following result has been proven in [22].

Theorem 4.4 *For $n > 4$, every unlabelled simple planar triangulation T with n vertices contains at least $n - 2$ flippable edges.*

Proof: Two edges of a triangulation are called *cofacial* if they belong to the boundary of a face of the triangulation. Let F be the set of flippable edges in T and $\bar{F}$ be the set of unflippable edges in T . Define a relation $\mathcal{R} \subseteq \bar{F} \times F$ as follows:

$$(e, f) \in \mathcal{R} \iff e \in \bar{F}, f \in F, \text{ and } e, f \text{ are cofacial.}$$

We claim that each unflippable edge is related to at least two flippable edges. Let $v_i v_j$ be any unflippable edge in T , and let $\Delta v_i v_j v_k$ and $\Delta v_i v_j v_l$ be the two triangular faces of T incident with $v_i v_j$. Since $v_i v_j$ is unflippable, v_k and v_l are adjacent in T . Since T has more than four vertices, vertices v_i and v_j cannot both have degree three. If vertex v_i has degree at least four, then both edges $v_i v_k$ and $v_i v_l$ are flippable; if vertex v_j has degree at least four, then both edges $v_j v_k$ and $v_j v_l$ are flippable. Hence $v_i v_j$ is related to at least two flippable edges.

On the other hand, each flippable edge is related to at most four unflippable edges. This is obvious since each flippable edge is incident with exactly two triangular faces.

Therefore

$$2|\bar{F}| \leq |\mathcal{R}| \leq 4|F|,$$

where $|F|$, $|\bar{F}|$ and $|\mathcal{R}|$ are the number of elements in the set F , $\bar{F}$ and $\mathcal{R}$ respectively. Hence

$$|\bar{F}| \leq 2|F|.$$

Since the total number of edges in T is $3n - 6$, it follows that the number of flippable edges is at least $(3n - 6)/3 = n - 2$. $\square$

From Theorem 4.4, we have $d_{n,3c}^S \geq n - 2$. To obtain the exact value of $d_{n,3c}^S$, we prove the following two lemmas first.

Lemma 4.5 *For any unlabelled simple planar triangulation T with $n \geq 5$ vertices, if T has only $n - 2$ flippable edges, then each face of T is incident with exactly one flippable edge.*

Proof: Notice in the proof of Theorem 4.4, when we prove each unflippable edge is related to at least two flippable edges, we have already shown that each face of T must be incident with at least one flippable edge. To show it can be incident with at most one flippable edge, assume there exists a face incident with at least two flippable edges, then

$$2|\bar{F}| \leq |\mathcal{R}| \leq 4(|F| - 2) + 3 \times 2$$

since the two flippable edges of this face can be related to at most three unflippable edges. Hence

$$|\bar{F}| \leq 2|F| - 1.$$

It contradicts with $|F| = n - 2$ and $|\bar{F}| = 2n - 4$. $\square$

Lemma 4.6 *Let T be an unlabelled simple planar triangulation with $n \geq 5$ vertices, then T has only $n - 2$ flippable edges if and only if T is the subdivision of some other simple planar triangulation T' .*

Proof: Suppose T has only $n - 2$ flippable edges. Let x be the number of degree three vertices in T . Deleting all degree three vertices in T , we obtain a triangulation T' with $k = n - x$ vertices. The number of faces in T' is $2k - 4$. To show T is the subdivision of T' , we only need to show $x = 2k - 4$.

That $x \leq 2k - 4$ is obvious. If $x < 2k - 4$, there is a face $v_i v_j v_k$ of T' which is also a face of T . By Lemma 4.5, we know there is exactly one flippable edge and two unflippable edges among the three edges of the face $v_i v_j v_k$. Without loss of generality, let $v_i v_j$ be flippable and $v_i v_k, v_j v_k$ be unflippable. This implies vertex v_k is a degree three vertex in T and cannot be a vertex of T' . Contradiction! Thus $x = 2k - 4$.

Now assume T is the subdivision of T' with T' having k vertices. Then the number of vertices in T is $n = k + 2k - 4 = 3k - 4$. It is clear that the only edges of T that are flippable are exactly the edges of T' , which has $3k - 6 = n - 2$ edges. $\square$

Figure 4.4: Simple planar triangulations with minimum number of flippable edges.

Theorem 4.7 For any $n \geq 5$, the minimum vertex degree of $GT_{n,3c}^S$ is given by

$$d_{n,3c}^S = \begin{cases} n-2, & \text{if } n = 3k-4, \\ n-1, & \text{if } n = 3k-3 \text{ or } 3k-2; \end{cases}$$

while $d_{n,3c}^S = 0$ for $n = 3$ and $n = 4$.

Proof: The case $n = 3k - 4$ has been proven in Lemma 4.6. We construct an unlabelled simple planar triangulation T with n vertices, which is the subdivision of some other unlabelled simple planar triangulation with k vertices ($k \geq 3$). See Figure 4.4(a). In Figure 4.4, the white vertices belong to the original triangulation T' , a black vertex is inserted into each face of T' .

If $n = 3k - 2$ and $n = 3k - 3$, by Lemma 4.6, an unlabelled simple planar triangulation T with n vertices must have at least $n - 1$ flippable edges. We need to show there exist unlabelled simple planar triangulations T with n vertices such that T has only $n - 1$ flippable edges.

For $n = 3k - 2$, let T' be an unlabelled simple planar triangulation with l vertices ($l \geq 4$). Except the exterior face of T' , in the middle of each face, we insert

a vertex adjacent to all the vertices of the face. See Figure 4.4(b). Hence T have $n = l + 2l - 5 = 3(l - 1) - 2$ vertices and the number of flippable edges in it is $3l - 6 = n - 1$. The result follows by letting $k = l - 1$.

For $n = 3k - 3$, let T' be a simple planar triangulation with $l \geq 4$ vertices and T' has at least one unflippable edge v_1v_2 . In the middle of each face of T' except the two faces incident with v_1v_2 , we insert a vertex adjacent to all the vertices of the face. See Figure 4.4(c). The resulting triangulation T has $n = l + 2l - 6 = 3l - 6 = 3(l - 1) - 3$ vertices and the number of flippable edges in T is $3l - 6 - 1 = n - 1$, since v_1v_2 is still unflippable in T . The result follows by letting $k = l - 1$.

The case for $n = 3$ and $n = 4$ is obvious. $\square$

Theorem 4.8 *For any $n \geq 3$, the minimum vertex degree of $GT_{n,3c}^{r,S}$ is given by*

$$d_{n,3c}^{r,S} = \begin{cases} n - 4, & \text{if } n = 3k - 2, \\ n - 3, & \text{if } n = 3k - 3 \text{ or } 3k - 4. \end{cases}$$

Proof: Let T^r be a rooted simple planar triangulation with n vertices and T be the corresponding unlabelled triangulation of T^r . First we assume $n \geq 5$. Notice rooting an unlabelled triangulation makes at most three flippable edges become unflippable. Hence from Theorem 4.7, we have

$$d_{n,3c}^{r,S} \geq \begin{cases} n - 5, & \text{if } n = 3k - 4, \\ n - 4, & \text{if } n = 3k - 3 \text{ or } 3k - 2. \end{cases}$$

For the case $n = 3k - 4$, if T has at least n flippable edges, T^r has at least $n - 3$ flippable edges. We claim if T has only $n - 1$ flippable edges, T^r also has at least $n - 3$ flippable edges. To show this, it is enough to prove that each face of T can be incident with at most two flippable edges.

Assume there is a face of T which is incident with three flippable edges. Inserting a degree three vertex inside this face, we obtain a triangulation T_1 which has $n+1 = 3k-3$ vertices and $n-1$ flippable edges. This Contradicts to Theorem 4.7.

If T has only $n-2$ flippable edges, from Lemma 4.5 we know any face of T is incident with exactly one flippable edge. Thus T^r has at least $n-3$ flippable edges.

So we have shown $d_{n,3c}^{r,S} \geq n-3$ for $n = 3k-4$. To prove the equality holds, we root the unlabelled simple planar triangulation which has only $n-2$ flippable edges in an arbitrary way. The resulting rooted simple planar triangulation have only $n-3$ flippable edges.

For the case $n = 3k-3$, from Theorem 4.7, an unlabelled triangulation T with n vertices has at least $n-1$ flippable edges. If T has only $n-1$ flippable edges, the same argument used in the previous paragraph can show that each face of T is incident with at most two flippable edges. Hence $d_{3k-3,3c}^{r,S} \geq n-3$. To show $d_{3k-3,3c}^{r,S} \leq n-3$, let T be the unlabelled triangulation with $n = 3k-3$ vertices and with $n-1$ flippable edges given in Figure 4.4(c). We root it by choosing the exterior face to be the root face. The resulting rooted simple planar triangulation has $n = 3k-3$ vertices and only $n-3$ flippable edges.

For the case $n = 3k-2$, we already have $d_{n,3c}^{r,S} \geq n-4$; so we only need to show there exists a rooted simple planar triangulation with only $n-4$ flippable edges. Let T be the unlabelled triangulation with $n = 3k-2$ vertices and with $n-1$ flippable edges given in Figure 4.4(b). We root T by choosing the exterior face to be the root face. The resulting rooted triangulation has $n = 3k-2$ vertices and only $n-4$ flippable edges.

The case for $n = 3$ and $n = 4$ is obvious.

Theorem 4.8 has been proven. $\square$

Most of the unflippable edges in simple planar triangulations are caused by degree three vertices. If a planar triangulation has its minimum vertex degree at least four, then it has many more flippable edges. The following result is from [22].

Theorem 4.9 *In an unlabelled planar simple triangulation with n vertices and minimum vertex degree at least four, the number of flippable edges is at least $\min(2n + 3, 3n - 6)$. In an rooted simple planar triangulation with n vertices and the minimum vertex degree at least four, the number of flippable edges is at least $\min(2n, 3n - 9)$. Our bounds are tight.*

Proof: We consider the unlabelled case first. Let T be an unlabelled simple planar triangulation with n vertices and minimum vertex degree at least four. It is clear that if T has more than four vertices and has an unflippable edge, then T contains a separating triangle. Thus if T contains no separating triangles, then all edges of T are flippable. Hence T has at least $\min(2n + 3, 3n - 6)$ flippable edges.

Now assume T contains a separating triangle. Since T has minimum vertex degree at least four, the argument used in the proof of Theorem 4.4 shows that each unflippable edge is related to exactly four flippable edges, i.e. $4|\bar{F}| = |\mathcal{R}|$. Also, if $\Delta v_i v_j v_k$ is a face of T such that $v_i v_j$ is flippable, then at least one of $v_i v_k$ and $v_j v_k$ is flippable, since otherwise v_k has degree three. Hence each flippable edge is related to at most two unflippable edges.

Now we show that T contains at least eighteen flippable edges which are not related to any unflippable edge, and there are at least three extra flippable edges which are related to at most one unflippable edge.

Let $T(v_i v_j v_k)$ be a minimum separating triangle in triangulation T . Since T has no vertex of degree three, $T(v_i v_j v_k)$ contains at least six vertices. Since $T(v_i v_j v_k)$ contains no separating triangle, all edges of $T(v_i v_j v_k)$ are flippable in T . Therefore, each edge inside $\Delta v_i v_j v_k$ (there are at least nine such edges) is not related to any unflippable edges. Similarly, T contains at least nine flippable edges outside $\Delta v_i v_j v_k$, each of which is not related to any unflippable edge. Notice also that each of the three edges $v_i v_j$, $v_j v_k$, $v_i v_k$ is related to at most one unflippable edge. Thus

$$4|\bar{F}| = |\mathcal{R}| \leq 2(|F| - 18 - 3) + 3.$$

Using $|\bar{F}| + |F| = 3n - 6$, we obtain $|F| \geq \min(2n + 2 + 1/2, 3n - 6)$. i.e. $|F| \geq \min(2n + 3, 3n - 6)$.

Triangulations that achieve this bound can be obtained as follows: Let T' be the standard form of unlabelled simple planar triangulations with $n - 6$ vertices ($n \geq 9$) and with $\deg(v_a) = \deg(v_b) = n - 7$. Inserting a triangle in each of the two faces which are incident with $v_a v_b$ in such a way that the degree of the six new vertices is four. See Figure 4.5. We can easily verify that the resulting triangulation achieves the previous bound.

If T^r is a rooted simple planar triangulation with n vertices and minimum vertex degree at least four, then the number of flippable edges in T^r is at least $\min(2n + 3, 3n - 6) - 3 = \min(2n, 3n - 9)$. To show the bound is tight, we root the triangulation in Figure 4.5 by choosing the exterior face to be the root face. The resulting rooted simple planar triangulation has n vertices and has only $\min(2n, 3n - 9)$ flippable edges. $\square$

Figure 4.5: A simple planar triangulation with n vertices, with minimum vertex degree four and with $2n + 3$ flippable edges.

4.4 Minimum Number of Flippable Edges in 2-c Planar Triangulations

In this section, we discuss the minimum number of flippable edges in unlabelled and rooted 2-c planar triangulations.

Theorem 4.10 *For $n \geq 4$, the minimum vertex degrees $d_{n,2c}^S$ of $GT_{n,2c}^S$ is given by $d_{n,2c}^S = n - 2$. For $n = 3$, $d_{3,2c}^S = 0$.*

Proof: We follow the same steps as in the proof of Theorem 4.4. Let F be the set of flippable edges in T and $\bar{F}$ be the set of unflippable edges in T . We define the relation $\mathcal{R}$ as in the proof of Theorem 4.4 and can show

$$2|\bar{F}| \leq |\mathcal{R}| \leq 4|F|.$$

Hence $|F| \geq |\bar{F}|/2$. Thus at least $(3n - 6)/3 = n - 2$ edges are flippable. An unlabelled 2-c planar triangulation with n vertices and only $(n - 2)$ flippable edges is given in Figure 4.6(a). The case $n = 3$ is obvious. $\square$

Lemma 4.11 *For any 2-c planar triangulation T with n vertices, if there is a face of T such that all three edges on this face are flippable, then T contains at least n flippable edges.*

Proof: We prove this lemma by induction on n . For $n = 3$ and 4, Lemma 4.11 is obvious. Suppose the lemma is true for $n = k$. We now prove it is also true for $n = k + 1$.

Let T be a 2-c planar triangulation with $k + 1$ vertices and $v_1v_2v_3$ be a face of T such that v_1v_2 , v_1v_3 and v_2v_3 are all flippable. Let $v_1v_2v_4$ be the other face in T with which v_1v_2 is incident. Then $v_3 \neq v_4$, since otherwise v_1v_2 is unflippable. If at least one of the edges v_1v_4 and v_2v_4 is flippable, then we have

$$2|\bar{F}| \leq |\mathcal{R}| \leq 4(|F| - 4) + 2 \times 2 + 3 + 1 = 4|F| - 8.$$

Hence $|\bar{F}| \leq 2|F| - 4$ and thus

$$|F| = 3(k + 1) - 6 - |\bar{F}| \geq 3(k + 1) - 6 - 2|F| + 4.$$

Hence $|F| \geq k + 1 - 2/3$ and which implies $|F| \geq k + 1$.

If both v_1v_4 and v_2v_4 are unflippable, then v_4 has vertex degree two and we have multiple edges v_1v_2 . Now deleting vertex v_4 and the edge v_1v_2 of the face $v_1v_2v_3$, we get another 2-c planar triangulation T' with k vertices. T' also has a face $v_1v_2v_3$ such that all its three edges are flippable. By induction hypothesis, T' has at least

Figure 4.6: Two 2-c triangulations with n vertices: the first has $n - 2$ flippable edges. the second has minimum vertex degree three and has $2n$ flippable edges.

k flippable edges. These flippable edges of T' are also flippable edges of T . Hence T has at least $k + 1$ flippable edges. Our result follows by induction. $\square$

Theorem 4.12 *For any $n \geq 3$, the minimum vertex degree $d_{n,2c}^{r,S}$ of $GT_{n,2c}^{r,S}$ is $n - 3$.*

Proof: First we prove $d_{n,2c}^{r,S} \geq n - 3$. Let T^r be a rooted 2-c planar triangulation with n vertices and T be the corresponding unlabelled triangulation of T^r . If T has at least n flippable edges, then T^r has at least $n - 3$ flippable edges. The same argument used in Lemma 4.5 can show that if T only has $n - 2$ flippable edges, then any face of T is incident with exactly one flippable edge. From Lemma 4.11, if T has only $n - 1$ flippable edges, then any face of T is incident with at most two flippable edges. Hence T^r has at least $n - 3$ flippable edges.

To show $d_{n,2c}^{r,S} \leq n - 3$, we root the unlabelled 2-c planar triangulation with n vertices and $n - 2$ flippable edges in Figure 4.6(a) in an arbitrary way. The resulting rooted 2-c planar triangulation has n vertices and only $n - 3$ flippable edges. $\square$

In 2-c planar triangulations, most unflippable edges are caused by degree two vertices. If a 2-c planar triangulation has minimum vertex degree at least three, then it has many more flippable edges.

Theorem 4.13 *In an unlabelled 2-c planar triangulation with n vertices and minimum vertex degree at least three, the number of the flippable edges is at least $\min(2n, 3n - 6)$. In an rooted 2-c planar triangulation with n vertices and minimum vertex degree at least three, the number of flippable edges is at least $\min(2n - 3, 3n - 9)$. Our bounds are tight.*

Proof: Let T be an unlabelled 2-c planar triangulation with n vertices and minimum vertex degree at least three. If T has an unflippable edge, T contain a separating digon. Thus if T contains no separating digon, then each edge of T is flippable; hence it has at least $\min(2n, 3n - 6)$ flippable edges.

Now we assume T contains a separating digon. Since T has minimum vertex degree at least three, each unflippable edge of T is related to exactly four flippable edges, i.e. $4|\bar{F}| = |\mathcal{R}|$. If $\triangle v_i v_j v_k$ is a face of T such that edge $v_i v_j$ is flippable, then at least one of the edges $v_i v_k$ and $v_j v_k$ is flippable, since otherwise v_k has degree two. Hence each flippable edge is related to at most two unflippable edges.

Now we show T contains at least ten flippable edges which are not related to any unflippable edge, and there are at least two extra flippable edges which are related to at most one unflippable edge. Let $T_{v_i v_j}$ be a minimum separating digon in T . Since T has no vertex of degree two, $T(v_i v_j)$ contains at least four vertices. All edges of $T(v_i v_j)$ are flippable. Therefore, all edges inside $T_{v_i v_j}$ (there are at least five such edges) are not related to any unflippable edges. Similarly, T contains at

least five flippable edges outside $T_{v_i v_j}$, which are not related to any unflippable edge. Notice also that each of the multiple edges $v_i v_j$ of $T(v_i v_j)$ is related to at most one unflippable edge. We obtain

$$4|\bar{F}| = |\mathcal{R}| \leq 2(|F| - 10 - 2) + 2.$$

Hence $|F| \geq \min(2n - 1/3, 3n - 6)$, i.e. $|F| \geq \min(2n, 3n - 6)$.

A triangulation that achieves this bound is given in Figure 4.6(b).

For an rooted 2-c planar triangulation T^r with n vertices and minimum vertex degree at least three, the number of flippable edges in T^r is at least $\min(2n, 3n - 6) - 3 = \min(2n - 3, 3n - 9)$. If we root the unlabelled 2-c planar triangulation in Figure 4.6(b) by choosing the exterior face to be the root face, the resulting rooted 2-c planar triangulation achieves this bound. $\square$

4.5 Minimum Number of Flippable Edges in 2-c and Simple Planar Near Triangulations

In this section, we discuss $d_{n,m,s}^S$ and $d_{n,m,2c}^S$. Notice in the case $m \neq 3$, each edge of the non-triangular face is unflippable. We do not need to distinguish the unlabelled case with the rooted case. We consider simple planar near triangulations first.

Theorem 4.14 *For $m \geq 3$ and $n \geq 0$, every rooted simple planar near triangulation of type $[n, m]$ contains at least $m - 3$ flippable edges.*

Proof: Let T be a rooted simple planar near triangulation of type $[n, m]$ with $m \geq 3$ and $n \geq 0$. We use induction on n to prove the result.

If $n = 0$, T is a simple planar Catalan triangulation. There are $m - 3$ diagonals in T ; each of them is flippable. Now suppose for all $n < k$ ($k > 0$) the result is true. We prove it is also true for $n = k$. To prove this, we use induction on m .

The case $m = 3$ is obvious. Assume the result is true for $m < t$. we prove it is also true for $m = t$. Let v_1v_2 be the root edge and $v_1v_2v_3$ be a nonroot face of T . If v_3 is an exterior vertex, decomposing T by erasing v_1v_2 and separating v_3 into two copies, we obtain two near triangulations T_1 and T_2 . See Figure 2.1 in Chapter 2. Suppose T_1 is of type $[n_1, m_1]$ and T_2 is of type $[n_2, m_2]$, then $n_1 + n_2 = k$, $m_1 + m_2 = t + 1$ and $2 \leq m_1, m_2 < t$. By induction, T_1 has at least $m_1 - 3$ flippable edges and T_2 has at least $m_2 - 3$ flippable edges. Notice v_1v_3 and v_2v_3 are flippable in T . Hence T has at least $m_1 - 3 + m_2 - 3 + 2 = t - 3$ flippable edges.

If v_3 is an interior vertex, deleting v_1v_2 , we obtain another simple planar near triangulation T' which has $k - 1$ interior vertices and $t + 1$ exterior vertices. Hence T' contains at least $t + 1 - 3 = t - 2$ flippable edges. If T' contains no t -class edges, then all the flippable edges of T' are flippable in T and T has at least $t - 2$ flippable edges. If T' has t -class edges, there exists a vertex b of T' such that all t -class edges of T' are inside the quadrangle $v_1v_3v_2b$. See Figure 3.1 in Chapter 3. If b is an exterior vertex, the same argument used in the previous paragraph can be used here and prove the result. If b is an interior vertex, deleting all edges of quadrangle $v_1v_3v_2b$ except edge v_1b and edge v_2b , we obtain another simple planar triangulation T'' which has $t + 1$ exterior vertices and the number of interior vertices is less than $k - 1$. Hence T'' contain at least $t + 1 - 3 = t - 2$ flippable edges. All of these edges are flippable edges of T .

By induction, Theorem 4.14 has been proven. □

Theorem 4.15 *For $m \geq 3$ and $n \geq 0$, every rooted simple planar near triangulation of type $[n, m]$ contains at least $n - 1$ flippable edges.*

Proof: Similar to the proof of Theorem 4.4, let F be the set of flippable edges in T and $\bar{F}$ be the set of unflippable edges in a rooted simple planar near triangulation T of type $[n, m]$. Define a relation $\mathcal{R} \subseteq \bar{F} \times F$ as follows:

$$(e, f) \in \mathcal{R} \iff e \in \bar{F}, f \in F, \text{ and } e, f \text{ are cofacial.}$$

Then each flippable edge is related to at most four unflippable edges. Each unflippable edge, except the exterior edges and the edges which are cofacial with the exterior edges, is related to at least two flippable edges. For an unflippable interior edge (an edge is not on the exterior face is called an interior edge) which is cofacial with exterior edges, if we assume exterior edges are flippable, then it is also related to at least two flippable edges. Since each exterior edge can be cofacial with at most two interior edges, it is counted at most twice as the flippable edges. Hence

$$2(|\bar{F}| - m) - 2m \leq |\mathcal{R}| \leq 4|F|.$$

Thus $2|F| \geq |\bar{F}| - 2m$. From $|F| + |\bar{F}| = 3n + 2m - 3$, we have

$$2|F| \geq 3n + 2m - 3 - |F| - 2m,$$

which gives $|F| \geq n - 1$. □

From Theorem 4.14 and Theorem 4.15, we have

Corollary 4.16 *For $m \geq 4$, $n \geq 0$, we have $d_{n,m,s}^S = d_{n,m,s}^{r,S} \geq \max(n - 1, m - 3)$.*

Note: If $n - 1 = m - 3$, i.e. near triangulations are of type $[n, n + 2]$, the total number of vertices in these near triangulations is $N = 2n + 2$. The lower bound of the number of flippable edges is $n - 1 = \frac{N-4}{2}$. This worst case bound is the same as that of the geometrical triangulations case. See [27]. But the number of flippable edges in a geometrical triangulation of type $[n, m]$ (i.e. the geometrical triangulation has n interior vertices and m exterior vertices) does not have lower bound $\max(n-1, m-3)$. See Figure 10 in [27]; there we have a geometrical triangulation of type $[3m - 2, m]$ has only $2m - 3$ flippable edges.

Lemma 4.17 *For an unlabelled simple planar near triangulation T of type $[n, m]$ with $m \geq 4$ and $n \geq m - 2$, if T has only $n - 1$ flippable edges, then $n = m - 2 + 3k$ for some $k \geq 0$.*

Proof: As in the proof of Lemma 4.6, let x be the number of degree three interior vertices in T . Deleting all degree three interior vertices of T , we obtain a near triangulation T' of type $[k, m]$ with $k = n - x$. The number of faces in T' is $2k + m - 1$, the number of triangular faces in T' is $2k + m - 2$. We show $x = 2k + m - 2$.

That $x \leq 2k + m - 2$ is obvious. Suppose $x < 2k + m - 2$. There is a face $v_i v_j v_k$ of T' which is also a face of T . If all three vertices v_i, v_j and v_k are exterior vertices, adding a degree three vertex in the face $v_i v_j v_k$, we obtain a simple planar near triangulation with $n + 1$ interior vertices but it has only $n - 1$ flippable edges, contradicting Theorem 4.15. Hence at least one vertex of the face $v_i v_j v_k$ is an interior vertex, say v_i .

If $v_j v_k$ is an exterior edge, from the proof of Theorem 4.15, $v_i v_j$ and $v_i v_k$ must be unflippable. That implies the degree of v_i in T is only three and it cannot be a

vertex of T' .

If $v_j v_k$ is not an exterior edge, the same argument used in Lemma 4.5 can show that there are exactly one flippable edge and two unflippable edges among the three edges of face $v_i v_j v_k$. That imply one of the vertices of the face $v_i v_j v_k$ has degree three. If v_j and v_k both are exterior vertices, then $v_j v_k$ is an flippable edge since it is a diagonal. Hence $v_i v_j$ and $v_i v_k$ must be unflippable edges and that force $\deg v_i = 3$. If at least one of v_j and v_k is an interior vertex, then the only possible exterior vertex has at least degree four (it connect to at least two exterior vertices and two interior vertices). That will imply one of the interior vertices of the face $v_i v_j v_k$ has degree three. Contradiction again!

Hence $x = 2k + m - 2$, i.e. $n = m - 2 + 3k$. □

Theorem 4.18 *For $m \geq 4$, $k \geq 0$, the minimum vertex degree of $GT_{n,m,s}^S$ is*

$$d_{n,m,s}^S = \begin{cases} m - 3, & \text{if } n \leq m - 2, \\ n - 1, & \text{if } n = m - 2 + 3k, \\ n, & \text{if } n = m - 2 + 3k + 1 \text{ or } n = m - 2 + 3k + 2. \end{cases}$$

Proof: For the case $n \leq m - 2$, we only need to show there exists an unlabelled simple planar near triangulation T of type $[n, m]$ with $m \geq 4$, such that T only has $m - 3$ flippable edges. T can be constructed as follows: First, let an exterior vertex v_1 connect to all the other exterior vertices of T . Then, inside each triangular face, we insert at most one vertex of degree three until we insert n vertices. The resulting near triangulation is of type $[n, m]$ with $m \geq 4$ and $0 \leq n \leq m - 2$, which has only $m - 3$ flippable edges. See Figure 4.7(a) with $m = 6$ and $n = 3$.

Figure 4.7: Two simple planar near triangulations: the first is of type $[3, 6]$ and has three flippable edges, the second is of type $[10, 6]$ and has nine flippable edges.

For the case $n = m - 2 + 3k$, we only need to show there exists an unlabelled simple planar near triangulation T of type $[n, m]$ with $m \geq 4$, such that T only has $n - 1$ flippable edges. T can be constructed similarly as in the case $n \leq m - 2$: After we insert an interior degree three vertices in each triangular face (there are totally $m - 2$ such faces), we insert all the other $3k$ interior vertices into a separating triangle $T(v_i v_j v_k)$ (with v_i, v_j and v_k are exterior vertices) in the following way: each time we add three degree three vertices in each face except its exterior face of $T(v_i v_j v_k)$. The resulting near triangulation is of type $[n, m]$ with $m \geq 4$ and $n = m - 2 + 3k$, which has only $m - 3 + 3k = n - 1$ flippable edges. See Figure 4.7(b) with $m = 6$ and $k = 2$.

For $n = m - 2 + 3k + 1$ and $n = m - 2 + 3k + 2$, we only need to show there exist unlabelled simple planar near triangulations T of type $[n, m]$ with $n = m - 2 + 3k + 1$ or $n = m - 2 + 3k + 2$ such that T has only n flippable edges.

Figure 4.8: Two simple planar near triangulations: the first is of type $[11, 6]$ and has eleven flippable edges, the second is of type $[12, 6]$ and has twelve flippable edges.

For $n = m - 2 + 3k + 1$, T can be constructed as follows: suppose T' is a near triangulation of type $[m - 2 + 3k, m]$ only having $m - 3 + 3k$ flippable edges as we just constructed in Figure 4.7(b). Adding one vertex of degree three in a face which has an exterior edge, we obtain a near triangulation of type $[m - 2 + 3k + 1, m]$ with $m \geq 4$, which has only $m - 3 + 3k + 2$ flippable edges. See Figure 4.8(a) for $m = 6$, $k = 2$.

For the case $n = m - 2 + 3k + 2$, we insert two more vertices of degree three in T' such that only three unflippable edges of T' are changed to flippable. See Figure 4.8(b) for $m = 6$ and $k = 2$. $\square$

Now we consider 2-c planar near triangulations.

Theorem 4.19 *For $m \geq 3$ and $n \geq 0$, any rooted 2-c planar near triangulation of type $[n, m]$ contains at least $n + m - 3$ flippable edges.*

Proof: We prove Theorem 4.19 by induction on n .

If $n = 0$, Theorem 4.19 is trivial. Suppose it is true for any $n < k$. We prove it is also true for $n = k$. To prove this, we use induction on m .

The case $m = 3$ has been proven in Theorem 4.12. Assume the result is true for $m < t$. we prove it is also true for $m = t$. Let v_1v_2 be the root edge and $v_1v_2v_3$ be a nonroot face of T . If v_3 is an exterior vertex, decomposing T by erasing v_1v_2 and separating v_3 into two copies, we obtain two near triangulations T_1 and T_2 . Suppose T_1 is of type $[n_1, m_1]$ and T_2 is of type $[n_2, m_2]$ with $n_1 + n_2 = k$, $m_1 + m_2 = t + 1$ and $2 \leq m_1, m_2 < t$. By induction, it is easy to show that the number of flippable edges in T is at least

$$n_1 + m_1 - 3 + n_2 + m_2 - 3 + 2 = k + t - 3.$$

If v_3 is an interior vertex, deleting v_1v_2 , we obtain another simple planar near triangulation T' of type $[k - 1, t + 1]$. T' contains at least $k - 1 + t + 1 - 3 = k + t - 3$ flippable edges. Since each flippable edge of T' is also flippable in T , T has at least $k + t - 3$ flippable edges. Hence the result is true for $m = t$.

That proves the case $n = k$. Our result follows. $\square$

Theorem 4.20 *For $m \geq 2$, $m \neq 3$ and $n \geq 0$, the minimum vertex degree $d_{n,m,2c}^S$ of $GT_{n,m,2c}^S$ is $n + m - 3$.*

Proof: For $m \geq 4$, we only need to show there exists an unlabelled 2-c planar near triangulation T of type $[n, m]$, such that T has only $n + m - 3$ flippable edges. T can be constructed as follows: Let v_1 be a vertex of an m -gon. We connect v_1 to all other $m - 1$ vertices and insert n vertices of degree two into these triangular faces. See Figure 4.9(a) for $m = 6$ and $n = 4$.

Figure 4.9: Two 2-c planar near triangulations: the first is of type $[4, 6]$ and has seven flippable edges, the second is of type $[5, 2]$ and has four flippable edges.

For $m = 2$, we also show there exists an unlabelled 2-c planar near triangulation of type $[n, 2]$, which has only $n - 1$ flippable edges. It is constructed by letting all interior vertices having degree two. See Figure 4.9(b) for $n = 5$. To show an unlabelled 2-c planar triangulation T of type $[n, 2]$ has at least $n - 1$ flippable edges, we use induction. For $n = 0, 1$, the result is obvious. Now assume $n > 1$. Let v_1 and v_2 be its exterior vertices and $v_1v_2v_3$ is a face. Deleting edge v_1v_2 , we obtain an unlabelled 2-c planar triangulation T' with $n + 2$ vertices. We root it by choosing the exterior face to be root face. By Theorem 4.12, T' has at least $n - 1$ flippable edges. Hence T has at least $n + m - 3 = n - 1$ flippable edges. $\square$

Chapter 5

Diameters of Diagonal Flip Adjacency Graphs of Simple and 2-c Planar Near Triangulations

5.1 Introduction

In this chapter, we obtain bounds for the diameters of the diagonal flip adjacency graphs of unlabelled (rooted, labelled) simple planar near triangulations and 2-c planar near triangulations. Some results in this chapter are already given in [22].

Classically, Wagner [54] proved that any two unlabelled simple planar triangulations with the same number of vertices can be transformed into each other by a sequence of diagonal flips. Dewdney [14], and Negami and Watanabe [37] have shown the similar results for the Torus, the Projective Plane and the Klein Bottle. In the language of the diagonal flip adjacency graph, these results imply that graphs

$GT_{n,3c}^S$, $GT_{n,3c}^T$, $GT_{n,3c}^P$ and $GT_{n,3c}^K$ are all connected for any n .

The proof of Wagner's theorem found in Ore's book [41] gives a quadratic bound on the diameter of $GT_{n,3c}^S$. It is not hard to show that this proof can be extended to the rooted triangulation case, since neither one of the three edges of the exterior face is flipped in the proof. Hence the bound of the diameter of $GT_{n,3c}^{r,S}$ is also quadratic. A more recent result obtained by Hideo Komuro ([28]) shows that any two unlabelled planar triangulations with n vertices can be transformed into each other by at most $8n - 54$ diagonal flips if $n \geq 12$ and by at most $8n - 48$ diagonal flips if $n \geq 6$. However, his proof cannot be extended to the rooted case directly.

In this chapter we prove that the diameters of the graphs $GT_{n,3c}^{r,S}$, $GT_{n,2c}^S$ and $GT_{n,2c}^{r,S}$ are all $O(n)$, while the diameters of the graphs $GT_{n,3c}^{l,S}$ and $GT_{n,2c}^{l,S}$ are both $O(n \log n)$. For near triangulations, we prove the diameters of $GT_{n,m,s}^S$, $GT_{n,m,s}^{r,S}$, $GT_{n,m,2c}^S$ and $GT_{n,m,2c}^{r,S}$ are all $O(n + m)$. However, in the labelled case, if $m > 3$, neither $GT_{n,m,s}^{l,S}$ nor $GT_{n,m,2c}^{l,S}$ is connected.

If two labelled near triangulations is of the same type, but the labelling of their non-triangular faces are not the same, then it is impossible to transform one into the other by diagonal flips, since we are not allowed to flip any edge of the non-triangular face. Notice when we say the labelling of a face of one labelled near triangulation is the same as that of a face of the other labelled near triangulation, it means these two faces have the same cyclic labelling. If we consider the subgraphs of $GT_{n,m,s}^{l,S}$ ($GT_{n,m,2c}^{l,S}$) which consist of all simple (2-c) labelled planar near triangulations of type $[n, m]$ and with the same labelling on the non-triangular face, we prove that these subgraphs are connected and their diameters are $O(n \log n + m)$.

We begin with simple planar triangulations and simple planar near triangula-

tions. By showing any 2-c planar near triangulation of type $[n, m]$ can be transformed into a simple planar near triangulation of type $[n, m]$ within $O(n + m)$ diagonal flips, we prove that all the results for the simple triangulations case can be extended to the 2-c case.

In Section 2, we prove that the diameter of the graph $GT_{n,3c}^{r,S}$ is $O(n)$ and show this bound is tight. Section 3 deals with the labelled case; it shows that the diameter of the graph $GT_{n,3c}^{l,S}$ is $O(n \log n)$, but we are not able to show this bound is tight. In Section 4 we prove the results on simple planar near triangulations. In the last section of this chapter, we prove the corresponding results for 2-c case.

5.2 The Diameter of $GT_{n,3c}^{r,S}$

Theorem 5.1 *The diagonal flip adjacency graph $GT_{n,3c}^{r,S}$ of rooted simple planar triangulations with n vertices is connected and has diameter at most $8n - 32$.*

Proof: It is enough to prove that any rooted simple planar triangulation with n vertices can be transformed into Δ_n^r by at most $4n - 16$ diagonal flips. Let T be a rooted simple planar triangulation with n vertices. Let $\Delta_{v_i v_j v_k}$ be the root face, $v_i v_j$ be the root edge and v_i be the root vertex. We define the *potential* $p_T(v_i, v_j)$ of T by

$$p_T(v_i, v_j) = \deg(v_i) + 3\deg(v_j)$$

and show that if T is not the standard form Δ_n^r , then by performing some diagonal flips we can always increase $p_T(v_i, v_j)$. Note that $p_T(v_i, v_j) \leq 4(n - 1)$ and equality holds only when $\deg(v_i) = \deg(v_j) = n - 1$, i.e. T is Δ_n^r .

Let $v_i = v_{\sigma(0)}, v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(l-1)}, v_{\sigma(l)} = v_k$ be the neighbors of v_j in T lying around v_j in the anticlockwise order. Let m be the largest integer such that $v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(m)}$ are all adjacent to v_i in T , and each triangle $v_i v_{\sigma(t-1)} v_{\sigma(t)}$ bounds a face for $t = 1, 2, \dots, m$. If $m = l$, then T is already the rooted standard form Δ_n^r and no diagonal flip is needed. Otherwise let $v_i v_{\sigma(m)} u$ be the other face incident with the edge $v_i v_{\sigma(m)}$ with $u \neq v_{\sigma(m+1)}$. We distinguish the following two cases:

Case 1: u is not a neighbor of v_j .

If $m = 1$, we can flip the edge $v_i v_{\sigma(1)}$. After this flipping, $\deg(v_j)$ is increased by one and $\deg(v_i)$ is decreased by one. Overall, $p_T(v_i, v_j)$ is increased by two with one diagonal flip.

If $m > 1$, we can flip edge $v_i v_{\sigma(m)}$ and then flip edge $v_{\sigma(m-1)} v_{\sigma(m)}$. After these flippings, $\deg(v_j)$ is increased by one and $\deg(v_i)$ is decreased by one. Overall, $p_T(v_i, v_j)$ is increased by two with two diagonal flips.

Case 2: $u = v_{\sigma(s)}$ for some $m + 2 \leq s \leq l$.

In this case edge $v_{\sigma(m)} v_{\sigma(s)}$ can be flipped. After this flipping, $\deg(v_i)$ is increased by one and $p_T(v_1, v_3)$ is increased by one with one diagonal flip.

By iterating the above process, we can transform T into Δ_n^r and the total number of diagonal flips involved does not exceed $4(n - 1) - p_T(v_i, v_j)$. Now our theorem follows from the fact $p_T(v_i, v_j) \geq 12$ for any T . $\square$

Note: We could define the *potential* $P_T(v_i, v_j)$ of T by

$$p_T(v_i, v_j) = \deg(v_i) + c \deg(v_j)$$

with c being any real number and $c > 1$. Using this definition, we can still prove that within $O(n)$ diagonal flips, T can be transformed into Δ_n^r . However, it is not

hard to show that $c = 3$ is the best choice in the sense of considering the exact number of diagonal flips needed, if we have no further information about T .

Let T be a rooted simple planar triangulation with v_i being the root vertex and $v_i v_j$ being the root edge. The proof of Theorem 5.1 actually shows that we can transform T into Δ_n^r within $4(n-1) - \deg(v_i) - 3\deg(v_j)$ diagonal flips. We will use this result as the induction basis when we prove the corresponding result for rooted simple near triangulations.

It is easy to verify that we need at least a linear number of diagonal flips to transform the triangulations in Figure 4.1 and Figure 4.2 into the unlabelled standard form Δ_n . Hence the diameter of the $GT_{n,3c}^S$ is at least linear. The diameter of $GT_{n,3c}^{r,S}$ cannot be smaller than that of $GT_{n,3c}^S$, so it is also at least linear.

5.3 The Diameter of $GT_{n,3c}^{l,S}$

Let T be a labelled simple planar triangulation with n vertices obtained by labelling the unlabelled standard form Δ_n with $v_1, v_2, v_3, \dots, v_n$. Let v_i and v_j be the vertices which have vertex degree $n-1$ and $v_i v_j v_k$ be the exterior face of T . We call T a $\Delta_n(i, j, k)$ simple planar triangulation. Notice $\Delta_n(i, j, k) - \{v_i, v_j, v_k\}$ is a path P . Assuming the vertices of P are labelled with $\{v_{a(1)}, v_{a(2)}, \dots, v_{a(n-3)}\}$ from bottom to top, we say T is *sorted* if $a(1) > a(2) > \dots > a(n-3)$.

Let T' be another $\Delta_n(i, j, k)$ labelled simple planar triangulation. T' is called a *transpose* of T if $T' - \{v_i, v_j, v_k\}$ is a path P' which can be obtained from P by *transposing* two consecutive vertices of P . We can obtain T' from T by four diagonal flips. For example, by flipping edges $v_i v_{a(3)}$, $v_j v_{a(2)}$, $v_{a(1)} v_{a(2)}$ and $v_{a(3)} v_{a(4)}$

Figure 5.1: A $\Delta_7(i, j, k)$ labelled simple planar triangulation and a transpose of it.

of the labelled triangulation in Figure 5.1(a), we obtain its transpose, the labelled triangulation in Figure 5.1(b). Hence we have

Lemma 5.2 *Let T be a $\Delta_n(i, j, k)$ labelled simple planar triangulation, and T' be a transpose of T . Then T' can be obtained from T by at most four diagonal flips.*

Using our notation, the labelled standard form Δ_n^l is exactly a sorted $\Delta_n(1, 2, 3)$ triangulation.

Lemma 5.3 *Within $O(n)$ diagonal flips, every labelled simple planar triangulation T with n vertices can be transformed into a not necessary sorted $\Delta_n(1, 2, 3)$ triangulation.*

Proof: Let v_i be a neighbour of v_1 in T and $\Delta_{v_1 v_i v_j}$ be a face of T . Rooting T by choosing v_1 to be the root vertex, $v_1 v_i$ to be the rooted edge and $\Delta_{v_1 v_i v_j}$ to be the root face, we obtain a rooted triangulation T^r . By Theorem 5.1, within $O(n)$ diagonal flips, we can transform T^r into the rooted standard form Δ_n^r . We denote it by T' .

Figure 5.2: A $2\text{-}\Delta_n(i, j, k)$ triangulation and the merged triangulations with $n = 10$.

Since v_2 is adjacent to v_1 in T' , re-rooting T' by choosing $v_1 v_2$ to be the root edge and a face incident with $v_1 v_2$ to be the root face, we obtain a rooted triangulation T'' . Using Theorem 5.1 again, within $O(n)$ diagonal flips, we can transform this T'' into a $\Delta_n(1, 2, s)$ triangulation with $3 \leq s \leq n$. If $s \neq 3$, using Lemma 5.2, we can perform a linear number of transposes until $v_1 v_2 v_3$ is the exterior face. Our result follows. $\square$

Notice that using this lemma, together with Lemma 5.2, we can prove that any labelled simple planar triangulation with n vertices can be transformed into the labelled standard form Δ_n^l within a quadratic number of diagonal flips. Now we proceed to show that we can accomplish this in at most $O(n \log n)$ flips.

A $2\text{-}\Delta_n(i, j, k)$ triangulation is a triangulation consisting of two sorted triangulations $\Delta_{n_1}(i, s, k)$ and $\Delta_{n_2}(s, j, k)$ with $n_1 + n_2 = n + 2$, having two common vertices, v_s and v_k , glued along the edge joining v_s to v_k , and having an edge joining v_i to v_j . See Figure 5.2(a) with $n = 10$ and $n_1 = n_2 = 6$.

The following merging lemma is crucial in estimating the diameter of $GT_{n,3c}^{l,S}$.

Lemma 5.4 *Within $7n$ diagonal flips, every $2\text{-}\Delta_n(i, j, k)$ triangulation can be transformed into a sorted $\Delta_n(i, j, k)$ triangulation. Moreover, the exterior face $v_i v_j v_k$ can be kept in the whole process.*

Proof Let T be a $2\text{-}\Delta_n(i, j, k)$ triangulation. Let $\Delta_{n_1}(i, s, k)$ and $\Delta_{n_2}(s, j, k)$ be the sorted triangulations forming T with $n_1 + n_2 = n + 2$. Let $v_{a(1)}, v_{a(2)}, \dots, v_{a(n_1)}$ be the interior vertices of $\Delta_{n_1}(i, s, k)$ and $v_{b(1)}, v_{b(2)}, \dots, v_{b(n_2)}$ be the interior vertices of $\Delta_{n_2}(s, j, k)$.

Without loss of generality, let $a(1) > b(1)$. Then by two diagonal flips, first flipping $v_i v_s$, then flipping $v_{a(1)} v_{a(2)}$, we obtain a triangulation in which $v_{a(1)}$ is adjacent to v_i , v_j and v_s . See Figure 5.2(b). It is now easy to see that using three diagonal flips at a time, we can move the remaining interior vertices of $\Delta_{n_1}(i, s, k)$ and $\Delta_{n_2}(s, j, k)$ so that in the end we get an almost sorted $\Delta_n(i, j, k)$ triangulation. For example, in Figure 5.2(b), if $a(2) < b(1)$, we flip $v_s v_j$, $v_s v_{a(1)}$ and $v_{b(1)} v_{b(2)}$, then $v_{b(1)}$ is moved into the separating triangle $v_i v_j v_s$. See Figure 5.2(c). The total number of flips we have used is at most $3n$. The only possible vertex out of place is v_s . This can be fixed using Lemma 5.2 by performing a linear number of diagonal flips until it is moved to its correct position. The total number of flips for this is at most $4n$. It is clear that in the whole process, the exterior face $v_i v_j v_k$ is fixed. Our result follows. $\square$

Now we define a special type of labelled simple planar triangulations. A labelled simple planar triangulations T with n vertices is called a *binary triangulation*, denoted by $b\Delta_n(i, j, k)$, if:

1. $v_i v_j v_k$ is the exterior face of T .

Figure 5.3: A binary triangulation.

2. The dual graph of $T - \{v_k\}$ (excluding the vertex corresponding to the only face of $T - \{v_k\}$ that is not a triangular face) is an almost balanced binary tree, i.e. it is a tree obtained by removing some leaves from a balanced binary tree. See Figure 5.3.

Lemma 5.5 *A binary triangulation $b\Delta_n(i, j, k)$ can be transformed into a sorted $\Delta_n(i, j, k)$ triangulation within $24n \log n$ diagonal flips which keep the exterior face.*

Proof: We prove the lemma by induction. It is easy to verify that our lemma is true for $3 \leq n \leq 8$. Suppose that our result is true for $n < m$ ($m > 8$). We now show that it is still true for $n = m$.

Let T be a binary triangulation $b\Delta_n(i, j, k)$ with $n = m$ vertices. Notice that T splits into two binary triangulations T' and T'' with n_1 and n_2 vertices and

$$\lceil \frac{m}{2} \rceil \leq n_1, n_2 \leq \lceil \frac{m}{2} \rceil + 1.$$

By induction, T' and T'' can be transformed into a sorted $\Delta_{n_1}(i, s, k)$ triangulation and a sorted $\Delta_{n_2}(s, j, k)$ triangulation in $24n_1 \log n_1$ and $24n_2 \log n_2$ diagonal flips, respectively, and with their exterior faces kept. By Lemma 5.4, we can transform the resulting $2\text{-}\Delta_m(i, j, k)$ triangulation into a sorted $\Delta_m(i, j, k)$ triangulation within $7m$ diagonal flips. It is clear that the exterior face $v_i v_j v_k$ is kept in the whole process. The total number of diagonal flips we used is at most $24(n_1 \log n_1 + n_2 \log n_2) + 7m$. Since $m > 8$, it is easy to show

$$24(n_1 \log n_1 + n_2 \log n_2) + 7m \leq 24m \log m.$$

Hence the lemma is true for $n = m$. Our result follows. $\square$

Lemma 5.6 *Any labelled simple planar triangulation with n vertices and with exterior face $v_i v_j v_k$ can be transform into a sorted $\Delta_n(i, j, k)$ triangulation within $O(n \log n)$ diagonal flips. Moreover, the exterior face can be kept in the whole process.*

Proof: Let T be a labelled simple planar triangulation with n vertices and with exterior face $v_i v_j v_k$. By Theorem 5.1, we can transform T into a binary triangulation $b\Delta_n(i, j, k)$ using only $O(n)$ diagonal flips. By Lemma 5.5, we can transform $b\Delta_n(i, j, k)$ into a sorted $\Delta_n(i, j, k)$ triangulation within $O(n \log n)$ diagonal flips. All these diagonal flips keep the exterior face of T intact. Our result follows. $\square$

Theorem 5.7 *The diameter of the diagonal flip adjacency graph $GT_{n,3c}^{l,S}$ of the labelled simple planar triangulation is $O(n \log n)$.*

Proof: Let T be a labelled simple planar triangulation with n vertices. It is enough to show that T can be transformed into the labelled standard form Δ_n^l (i.e. the sorted $\Delta_n(1, 2, 3)$ triangulation) by $O(n \log n)$ diagonal flips. By Lemma 5.3, within $O(n)$ diagonal flips we can transform T into a $\Delta_n(1, 2, 3)$ triangulation T' . By Lemma 5.6, we can transform T' into a sorted $\Delta_n(1, 2, 3)$ triangulation by $O(n \log n)$ diagonal flips. Our result follows. $\square$

However, we cannot show that $O(n \log n)$ is also the lower bound for the diameter. In other words, we cannot find two labelled simple planar triangulations which need at least $O(n \log n)$ diagonal flips to be transformed into each other.

5.4 Diameter of the Diagonal Flip Adjacency Graph of Simple Planar Near Triangulations

In this section, we prove similar results for simple planar near triangulations. We prove the diameters of $GT_{n,m,s}^S$ and $GT_{n,m,s}^{r,S}$ are $O(n + m)$. If $m \geq 4$, $GT_{n,m,s}^{l,S}$ is not connected. Let $GT_{n,m,s}^{l',l,S}$ be the subgraph of $GT_{n,m,s}^{l,S}$, consisting of all simple labelled planar near triangulations of type $[n, m]$ with the same labelling l' on the non-triangular face of labelled near triangulations. We prove the diameter of $GT_{n,m,s}^{l',l,S}$ is $O(n \log n + m)$.

First we consider the unlabelled case. Using degree sequences, we can construct two unlabelled simple planar near triangulations of type $[n, m]$ such that to transform one into the other requires $O(n + m)$ diagonal flips. We construct an unlabelled simple planar near triangulation $T_{n,m}$ of type $[n, m]$ as follows: First we ignore the

Figure 5.4: A near triangulation of type $[6, 8]$ with maximum vertex degree 6.

interior vertices, and triangulate the m -polygon in such a way that its maximum vertex degree is four; then we put all interior vertices into one triangular face of this polygon triangulation and triangulate that face as we did in Figure 4.1 or Figure 4.2. See Figure 5.4. The maximum vertex degree of $T_{n,m}$ is at most six. Since the maximum vertex degree of the unlabelled standard form $\Delta_{n,m}$ is $n + m - 1$, we need at least $O(n + m)$ diagonal flips to transform $T_{n,m}$ into $\Delta_{n,m}$.

Since the diameter of $GT_{n,m,s}^{r,S}$ is not smaller than the diameter of $GT_{n,m,s}^S$, to prove the diameters of $GT_{n,m,s}^S$ and $GT_{n,m,s}^{r,S}$ are $O(n + m)$, it is enough to show that any rooted simple planar near triangulation of type $[n, m]$ can be transformed into the standard form $\Delta_{n,m}^r$ by at most $O(n + m)$ diagonal flips.

Let $\mathcal{T}_{n,v_1v_2\cdots v_m}$ be the set of rooted simple planar near triangulations of type $[n, m]$ with the root face $v_1v_2\cdots v_m$, the root edge v_1v_2 and the root vertex v_1 . Furthermore, the exterior vertices $v_1, v_2, v_3, \dots, v_m$ are in anti-clockwise order.

Figure 5.5: A $T_{7,6}^3$ rooted simple planar near triangulation.

Let $T_{n,m}^x \in \mathcal{T}_{n,v_1v_2\ldots v_m}$. Assume v_2v_m is an edge in $T_{n,m}^x$. Furthermore, the rooted triangulation $T_{v_1v_2v_m}$, which consists of the triangle $v_1v_2v_m$ and x interior vertices of $T_{n,m}^x$ inside it, with v_1 being the root vertex and v_1v_2 being the root edge, is the root standard form Δ_{x+3}^r . Here $0 \leq x \leq n$. See Figure 5.5. We define $\mathcal{T}_{n,m}^x$ to be the set of all rooted simple planar near triangulations of type $[n, m]$ like $T_{n,m}^x$.

The *potential* of an rooted near triangulation T with v_1 being the root vertex and v_1v_2 being the root edge is also defined as

$$P_T(v_1, v_2) = \deg(v_1) + 3\deg(v_2).$$

Lemma 5.8 *Any rooted simple planar near triangulation $T \in \mathcal{T}_{n,v_1v_2\ldots v_m}$ can be transformed into a $T_{n,m}^x \in \mathcal{T}_{n,m}^x$ within $P_{T_{n,m}^x}(v_1, v_2) - P_T(v_1, v_2)$ diagonal flips.*

Proof: We use induction to prove this lemma. The case $m = 3$ is already proved

in Theorem 5.1. Assuming the lemma is true for $m < k$, we prove it is also true for $m = k$.

Let

$$N = \{v_1 = v_{a(0)}, v_{a(1)}, v_{a(2)}, \dots, v_{a(l-1)}, v_{a(l)} = v_3\}$$

be the set of neighbors of v_2 in T lying around v_2 in clockwise order. Let v_y be the first exterior vertex in the set $N - \{v_1\} = \{v_{a(1)}, v_{a(2)}, \dots, v_{a(l-1)}, v_{a(l)} = v_3\}$ ($1 \leq i \leq l$) in this order. We distinguish two cases:

Case 1: $N - \{v_1, v_3\}$ contains at least one exterior vertex v_y .

Separating T by splitting the diagonal v_2v_y , we obtain two near triangulations T_1 and T_2 . Let T_1 be the near triangulation which includes vertex v_1 and T_2 be the other. Rooting T_1 by choosing v_1 to be the root vertex, the edge v_1v_2 to be the root edge, and the exterior face to be its root faces. Now T_1 is a rooted simple planar near triangulation. Suppose T_1 is of type $[n_1, m_1]$ with $m_1 < k$. By induction, T_1 can be transformed into a T_{n_1, m_1}^x within $P_{T_{n_1, m_1}^x}(v_1, v_2) - P_{T_1}(v_1, v_2)$ diagonal flips. Combining T_{n_1, m_1}^x and T_2 along the edge v_2v_y , we obtain a near triangulation $T_{n, m}^x$ which is in the set $\mathcal{T}_{n, m}^x$. Notice

$$P_{T_{n, m}^x}(v_1, v_2) = P_{T_{n_1, m_1}^x}(v_1, v_2) + 3\deg_{T_2}(v_2) - 3,$$

$$P_T(v_1, v_2) = P_{T_1}(v_1, v_2) + 3\deg_{T_2}(v_2) - 3.$$

Hence to transform T into $T_{n, m}^x$ we use at most

$$P_{T_{n, m}^x}(v_1, v_2) - P_T(v_1, v_2) = P_{T_{n_1, m_1}^x}(v_1, v_2) - P_{T_1}(v_1, v_2)$$

diagonal flips. We have proven the case $m = k$ for case 1.

Case 2: $N - \{v_1, v_3\}$ contains no exterior vertex at all.

Using the same argument we used in the proof of Theorem 5.1, we can increase $P_T(v_1, v_2)$ either one with one diagonal flip, or two within two diagonal flips. We keep increase the potential until v_2 is first connected to an exterior vertex of T , except v_1 and v_3 , say it is v_y . Let the resulting near triangulation to be $\bar{T}$. We use at most $P_{\bar{T}}(v_1, v_2) - P_T(v_1, v_2)$ diagonal flips to transform T into $\bar{T}$.

The same as in case 1, we separate $\bar{T}$ by splitting the diagonal v_2v_y and obtain two near triangulations T_1 and T_2 and root T_1 as in case 1. Suppose T_1 is of type $[n_1, m_1]$ with $m_1 < k$. By induction, T_1 can be transformed into a T_{n_1, m_1}^x within $P_{T_{n_1, m_1}^x}(v_1, v_2) - P_{T_1}(v_1, v_2)$ diagonal flips. Combining T_{n_1, m_1}^x and T_2 along the edge v_2v_y , we obtain a near triangulation $T_{n, m}^x$ which is in the set $\mathcal{T}_{n, m}^x$. To transform $\bar{T}$ into $T_{n, m}^x$, we use at most

$$P_{T_{n_1, m_1}^x}(v_1, v_2) - P_{T_1}(v_1, v_2) = P_{T_{n, m}^x}(v_1, v_2) - P_{\bar{T}}(v_1, v_2)$$

diagonal flips. In total, we use at most

$$P_{\bar{T}}(v_1, v_2) - P_T(v_1, v_2) + P_{T_{n_1, m_1}^x}(v_1, v_2) - P_{T_1}(v_1, v_2) = P_{T_{n, m}^x}(v_1, v_2) - P_T(v_1, v_2)$$

diagonal flips. That proves the case $m = k$ for case 2.

Lemma 5.8 has been proven. □

Lemma 5.9 *Within $4n + 3(m - 1 - \deg_T(v_2))$ diagonal flips, a rooted simple planar near triangulation $T \in \mathcal{T}_{n, v_1 v_2 \dots v_m}$ can be transformed into a rooted simple planar near triangulation $T' \in \mathcal{T}_{n, v_1 v_2 \dots v_m}$ such that $\deg_{T'}(v_2) = n + m - 1$.*

Proof: We prove this lemma by induction on m .

If $m = 3$, it is already proved in Theorem 5.1. Assuming the result is true for $m < k$, we prove it is still true for $m = k$.

Using Lemma 5.8, we know T can be transformed into a $T_{n,k}^x \in \mathcal{T}_{n,k}^x$ within $P_{T_{n,k}^x}(v_1, v_2) - P_T(v_1, v_2)$ diagonal flips. Let $\deg_{T_{n,k}^x}(v_2) = z$. Notice

$$\begin{aligned} P_{T_{n,k}^x}(v_1, v_2) - P_T(v_1, v_2) &= 3(z - \deg_T(v_2)) + (x + 2 - \deg_T(v_1)) \\ &\leq 3(z - \deg_T(v_2)) + x. \end{aligned}$$

Let us separate $T_{n,k}^x$ into two near triangulations by splitting edge v_2v_k . Let T_1 be the near triangulation which includes v_1 and T_2 be the other. Then $\deg_{T_2}(v_2) = z - x - 1$ and T_2 is a near triangulation of type $[n - x, k - 1]$. By induction, we can transform T_2 into a near triangulation T_3 such that in T_3 , v_2 is connected to all other vertices. The number of diagonal flips needed to transform T_2 into T_3 is at most

$$4(n - x) + 3(k - 1 - 1 - (z - x - 1)) = 4(n - x) + 3(k - 1 - z + x).$$

Now combining T_3 with T_1 , we obtain a simple planar near triangulation T' in which v_2 is connected to all other vertices. Hence we have transformed T into a rooted near triangulation T' with $\deg_{T'}v_2 = n + k - 1$. The number of diagonal flips we used is at most

$$3(z - \deg_T(v_2)) + x + 4(n - x) + 3(k - 1 - z + x) = 4n + 3(k - 1 - \deg_T(v_2)).$$

This proves the result for $m = k$.

By induction, Lemma 5.9 has been proven. $\square$

Figure 5.6(a) shows a rooted simple planar near triangulation of type $[7, 6]$ and $\deg(v_2) = 7 + 6 - 1 = 12$. If the rooted simple planar near triangulation T is a Catalan triangulation, then we have a better bound than the bound given in Lemma 5.9.

Lemma 5.10 *Let T_m be a Catalan triangulation with m vertices $v_1, v_2, \dots, v_m$ lying anti-clockwisely. Within at most $m - 1 - \deg_{T_m}(v_1)$ diagonal flips, we can transform T_m into T'_m such that $\deg_{T'_m}(v_1) = m - 1$.*

Proof: We prove this lemma by induction. The case $m = 3$ is trivial. Suppose the lemma is true for $m < k$. Now we prove it is still true for $m = k$.

Case 1: $\deg_{T_m}(v_1) > 2$, i.e. there is a vertex v_i such that v_1v_i is an edge of T_m .

Splitting T_m along v_1v_i , we obtain two Catalan triangulations T_{m_1} and T_{m_2} with $m_1 < m$ and $m_2 < m$. T_{m_1} can be transformed into T'_{m_1} within $m_1 - 1 - \deg_{T_{m_1}}(v_1)$ diagonal flips, T_{m_2} can be transformed into T'_{m_2} within $m_2 - 1 - \deg_{T_{m_2}}(v_1)$ diagonal flips, with $\deg_{T'_{m_1}}(v_1) = m_1 - 1$ and $\deg_{T'_{m_2}}(v_1) = m_2 - 1$. Combining T'_{m_1} and T'_{m_2} along edge v_1v_i , we obtain T'_m . The total number of flips is at most

$$m_1 - 1 - \deg_{T_{m_1}}(v_1) + m_2 - 1 - \deg_{T_{m_2}}(v_1) = m - 1 - \deg_{T_m}(v_1).$$

Case 2: $\deg_{T_m}(v_1) = 2$, i.e. v_2v_m is an edge of T_m .

Flipping v_2v_m , we obtain a Catalan triangulation $\bar{T}_m$ which has $\deg_{\bar{T}_m}(v_1) > 2$. Using the same argument given in case 1, we can prove the result.

Lemma 5.10 is proven. □

Theorem 5.11 *The diameters of $GT_{n,m,s}^{r,S}$ and $GT_{n,m,s}^S$ are at most $10(n + m) - 32$.*

Proof: It is enough to show that any $T \in \mathcal{T}_{n,v_1v_2\dots v_m}$ can be transformed into $\Delta_{n,m}^r$ by at most $5(n + m) - 16$ diagonal flips.

By Lemma 5.9, within $4n + 3(m - 1 - \deg_T(v_2))$ diagonal flips, we can transform T into a near triangulation T' with $\deg_{T'}(v_2) = n + m - 1$. See Figure 5.6(a).

Figure 5.6: Illustrating Theorem 5.11 with $n = 7$ and $m = 6$.

Flipping all the edges v_2v_i ($4 \leq i \leq m$) in T' , we obtain a near triangulation T'' in which v_2 is connected to all interior vertices, but not connected to any exterior vertices other than v_1 and v_3 . See Figure 5.6(b). Notice $T'' - v_2$ is a Catalan triangulation with $n + m - 1$ vertices. By Lemma 5.10, within $n + m - 4$ diagonal flips, we can transform $T'' - v_2$ into a Catalan triangulation T_{n+m-1} such that $\deg_{T_{n+m-1}}(v_1) = n + m - 2$. If we put vertex v_2 back with T_{n+m-1} , we obtain the rooted standard form $\Delta_{n,m}^r$. See Figure 5.6(c). The total number of diagonal flips we used is at most

$$4n + 3(m - 1 - \deg_T(v_2)) + (m - 3) + (n + m - 4) \leq 5(n + m) - 16$$

Hence the diameter of $GT_{n,m,s}^{r,S}$ is at most $10(n + m) - 32$. $\square$

Now let us deal with labelled simple planar near triangulations of type $[n, m]$ with $m \neq 3$. Notice in this case, if two near triangulations are of the same type, but the labelling of their exterior faces are not the same, then it is impossible to transform one into the other. Hence the diagonal flip adjacency graph $GT_{n,m,s}^{l,S}$ is not even

connected. So the diameter of $GT_{n,m,s}^{l,S}$ is infinite. But if we consider the subgraph $GT_{n,m,s}^{l',l,S}$ of $GT_{n,m,s}^{l,S}$ which consists of all labelled simple planar near triangulations of type $[n, m]$ with the same labelling l' on their exterior faces, we can prove these subgraphs are connected with the diameter $O(n \log n + m)$.

Theorem 5.12 *The diameter of $GT_{n,m,s}^{l',l,S}$ is $O(n \log n + m)$.*

Proof: We only prove the case that l' is the same labelling of the exterior face of the labelled standard form $\Delta_{n,m}^l$. The other cases can be proven similarly.

Let T be a labelled simple planar near triangulation of type $[n, m]$ with exterior vertices $v_1, v_2, \dots, v_m$ and interior vertices $v_{m+1}, v_{m+2}, \dots, v_{m+n}$. We only need to show that T can be transformed into the labelled standard form $\Delta_{n,m}^l$ within $O(n \log n + m)$ diagonal flips.

We root T with v_1 being the root vertex, $v_1 v_2$ being the root edge, and the exterior face being the root face. Using Theorem 5.11, after at most $O(n + m)$ diagonal flips, we can transform T into the rooted standard form $\Delta_{n,m}^r$. $\Delta_{n,m}^r$ contains a triangulation $T_{v_1 v_2 v_3}$ which consists of the triangle $v_1 v_2 v_3$ and everything inside this triangle. If we root $T_{v_1 v_2 v_3}$ by choosing the root face to be the exterior face $v_1 v_2 v_3$, the root edge to be $v_1 v_2$ and the root vertex to be v_1 , then it becomes the rooted standard form of simple planar triangulations with $n + 3$ vertices. Using Lemma 5.6, we can transform it into a labelled standard form within $O(n \log n)$ diagonal flips and we can keep the root face $v_1 v_2 v_3$ fixed in this whole process. After these diagonal flips, T is transformed into the labelled standard form $\Delta_{n,m}^l$. The total number of diagonal flips is $O(n + m) + O(n \log n) = O(n \log n + m)$. Our result follows. $\square$

However, as the case of the labelled simple planar triangulations, we cannot prove $O(n \log n + m)$ is a tight bound of the diameter of $GT_{n,m,s}^{l',l,S}$.

5.5 Corresponding Results for 2-c Planar Near Triangulations

In this section we prove analogous results for 2-c planar near triangulations. We show, for any $n \geq 0$ and $m \geq 2$, the diameters of $GT_{n,2c}^S$ and $GT_{n,2c}^{r,S}$ are $O(n)$, the diameter of $GT_{n,2c}^{l,S}$ is $O(n \log n)$, and the diameters of $GT_{n,m,2c}^S$ and $GT_{n,m,2c}^{r,S}$ are $O(n + m)$. If $m \neq 3$, let $GT_{n,m,2c}^{l',l,S}$ be the subgraph of $GT_{n,m,2c}^{l,S}$, which consists of all labelled 2-c planar near triangulations of type $[n, m]$ with the same labelling l' on their exterior face. We prove that the diameter of $GT_{n,m,2c}^{l',l,S}$ is $O(n \log n + m)$. First we prove the following lemma.

Lemma 5.13 *Any unlabelled (rooted, labelled) 2-c planar near triangulation of type $[n, m]$ ($m \geq 3$) can be transformed into an unlabelled (rooted, labelled) simple planar near triangulation of the same type by $O(n + m)$ diagonal flips. Moreover, the exterior face can be kept in the whole process.*

Proof: Let T be a unlabelled (rooted, labelled) 2-c planar near triangulation of type $[n, m]$ with $m \geq 3$. If T does not have multiple edges, T is already a simple planar near triangulation. If T does have multiple edges, let D be a minimal separating digon in T . The boundary of D is a pair of multiple edges $u1v$ and $u2v$.

It is clear that at least one of the edges $u1v$ and $u2v$ is not on the exterior face. Assume it is $u1v$. Notice every interior multiple edge is flippable and flipping it does

not create new multiple edge. Hence we can flip edge $u1v$. The total number of multiple edges decrease one after this flipping. Since the total number of edges in T is $(3n + 2m - 3)$, with at most this many diagonal flips, T can be transformed into a simple unlabelled (rooted, labelled) planar near triangulation. $\square$

Theorem 5.14 *The diameters of $GT_{n,2c}^S$ and $GT_{n,2c}^{r,S}$ are $O(n)$; the diameters of $GT_{n,2c}^{l,S}$ and $GT_{n,2,2c}^{l,S}$ are $O(n \log n)$. In general, the diameters of $GT_{n,m,2c}^S$ and $GT_{n,m,2c}^{r,S}$ are $O(n + m)$; the diameter of $GT_{n,m,2c}^{l,l,S}$ is $O(n \log n + m)$.*

Proof: We prove the general result. Let T be an unlabelled (rooted, labelled) 2-c planar near triangulation of type $[n, m]$. We only need to show that within $O(n + m)$ ($O(n + m)$, $O(n \log n + m)$) diagonal flips, we can transform T into the corresponding standard form. If T has no multiple edges, it is actually a simple near triangulation. We have already proven this result.

If T does have multiple edges and $m \geq 3$, using Lemma 5.13, we can transform T into a simple unlabelled (rooted, labelled) planar near triangulation by $O(n + m)$ diagonal flips. The result follows.

If $m = 2$, notice that there is an one to one correspondence between the set of unlabelled (rooted, labelled) 2-c planar triangulations of type $[n, 2]$ and the set of unlabelled (rooted, labelled) 2-c planar triangulations with $n + 2$ vertices. In fact, deleting one exterior edges of an unlabelled (rooted, labelled) 2-c planar near triangulation of type $[n, 2]$, we obtain an unlabelled (rooted, labelled) 2-c planar triangulation with $n + 2$ vertices. All unlabelled (rooted, labelled) 2-c planar triangulations can be obtained in this way. Hence our result follows. $\square$

Since all simple near triangulations are 2-c near triangulations, it is clear that in the unlabelled case and rooted case, the bounds we give in Theorem 5.14 are tight. But for the labelled case, we cannot prove that the bound is tight.

Chapter 6

Exact Enumeration of Rooted 2-c Triangulations on the Torus and the Klein Bottle

6.1 Introduction

In previous chapters, we have studied some properties of the diagonal flip adjacency graphs of the 2-c and simple planar near triangulations. From this chapter, we begin to enumerate non-planar triangulations. For planar map enumeration, much work has been done by Tutte, Brown and Mullin ([9], [32], [33], [47], [48], [49]) after Tutte's breakthrough in the 1960s; they have counted various types of rooted planar maps. For non-planar map enumeration, Bender, Canfield, Gao, Richmond, Robinson and Wormald etc. have done a lot of work. See [4], [6], [7], [11], [17], [18], [21], [22], [55]. In [53], Visentin reduces the combinatorial problem of counting

maps to the algebraic problem of multiplying central elements of the group algebra of the symmetric group. It seems to be much more difficult to determine the exact numbers of non-planar maps than that of planar maps.

For exact enumeration of the triangulations on surfaces, Gao [17] obtained parametric expressions of the generating functions of the exact number of rooted 1-c triangulations on the Sphere, the Projective Plane and the Torus. Mullin [32] determined the exact number of rooted 2-c planar triangulations: Gao [18] obtained the corresponding result on the Projective Plane. In this chapter, we enumerate rooted 2-c triangulations on the Torus and the Klein Bottle, the parametric expressions of the generating functions of these numbers are obtained in each case.

Let b_{ij} , p_{ij} , t_{ij} , k_{ij} be the numbers of rooted 2-c near triangulations on the Sphere, the Projective Plane, the Torus and the Klein Bottle respectively, having $(i + j)$ vertices and root face valence j . If $i = 0$, these near triangulations are 2-c Catalan triangulations on these surfaces. Define

$$B(x, y) = \sum_{i,j} b_{i,j} x^i y^j = \sum_j B_j(x) y^j$$

$$P(x, y) = \sum_{i,j} p_{i,j} x^i y^j = \sum_j P_j(x) y^j$$

$$T(x, y) = \sum_{i,j} t_{i,j} x^i y^j = \sum_j T_j(x) y^j$$

$$K(x, y) = \sum_{i,j} k_{i,j} x^i y^j = \sum_j K_j(x) y^j$$

Then $B_3(x)x^3$, $P_3(x)x^3$, $T_3(x)x^3$ and $K_3(x)x^3$ are the generating functions of the numbers of rooted 2-c triangulations on Sphere, the Projective Plane, the Torus and the Klein Bottle respectively. The main result in this chapter is

Theorem 6.1 *The generating functions of the numbers of rooted 2-c triangulations on the Torus and Klein Bottle are given by*

$$T_3(x)x^3 = \frac{t^3(1 + 15t - 45t^2)}{(1 - 3t)^4(1 - 6t)^2} = x^3 + 51x^4 + \dots \quad (6.1)$$

$$\begin{aligned} K_3(x)x^3 &= \frac{3(2 - 38t + 297t^2 - 1226t^3 + 2815t^4 - 3372t^5 + 1620t^6)}{2t(1 - 6t)^2(1 - 3t)^4} \\ &\quad - \frac{3(2 - 14t + 27t^2)(1 - 2t)^{3/2}}{2t(1 - 3t)^2(1 - 6t)^{3/2}} \\ &= 3x^3 + 153x^4 + \dots \end{aligned} \quad (6.2)$$

Here t is a power series of x given by $x = t(1 - 2t)^2$ and $t(0) = 0$.

To prove this theorem, first we find functional equations for $T(x, y)$ and $K(x, y)$, and then solve these equations. The functional equations of $T(x, y)$ and $K(x, y)$ will involve the generating functions of the number of the rooted 2-c near triangulations on the *Cylinder* and on the Möbius Band.

A *Cylinder* is a surface whose boundary has two disjoint parts. It can be obtained by cutting off a disk D_1 inside a disk D . When we talk about rooted near triangulations on the Cylinder, we always choose the root vertex and the root edge on the exterior boundary, i.e. the boundary of the disk D . The root face is the exterior face. We call the exterior boundary the *root-boundary*. The other part of the boundary is called the *nonroot-boundary*. The vertices which are not on the boundary are called interior vertices. Let $c_{n,i,j}$ be the number of 2-c rooted triangulation on the Cylinder, having n interior vertices, i root-boundary vertices and j nonroot-boundary vertices. Defining

$$C(x, u, v) = \sum_{n \geq 0, i \geq 2, j \geq 2} c_{n,i,j} x^n u^i v^j = \sum_{i \geq 2} C_i(x, v) u^i,$$

we will obtain a functional equation for $C(x, u, v)$ in next section. If $n = 0$, the corresponding triangulations are 2-c Catalan triangulations on the Cylinder.

The Möbius Band is a surface obtained by cutting off a disk from the Projective Plane. A rooted triangulation on the Möbius Band is obtained from a rooted near triangulation on the Projective Plane by cutting off its root face.

To simplify the expressions, we use B, P, C, T and K to denote $B(x, y), P(x, y), C(x, u, v), T(x, y)$ and $K(x, y)$, respectively, and use B_j, P_j, C_j, T_j and K_j to denote $B_j(x), P_j(x), C_j(x, v), T_j(x)$ and $K_j(x)$, respectively.

The rest of this chapter is organized as follow: In Section 2, we obtain functional equations for B, P and C . Section 3 deals with a special kind of rooted triangulations on the Cylinder and on the Möbius Band. We obtain functional equations for T and K in Section 4. In Section 5, we use the results of the previous sections to complete the proof of Theorem 6.1.

6.2 Functional Equations for B, P and C

In this section, we obtain functional equations for B, P and C . The following lemma gives the functional equations of B and P .

Lemma 6.2 *The generating functions B and P satisfy the following equations:*

$$B = y^2 + y^{-1}B^2 + y^{-1}x(B - B_2y^2), \quad (6.3)$$

$$P = 2y^{-1}BP + L + y^{-1}x(P - P_2y^2). \quad (6.4)$$

Where $B_y(x, y)$ is the partial derivative of B with respect to y and

$$L = y^{-1}(yB_y(x, y) - 3B + B_2y^2 - (B - B_2y^2)^2/(B_2y^2)). \quad (6.5)$$

Figure 6.1: The decomposition creates two near triangulations on the Sphere and the Projective Plane.

Proof: Let T be a rooted 2-c near triangulation on the Sphere (the Projective Plane). If a near triangulation on the Sphere only has two vertices, the contribution to B is y^2 . There is no rooted 2-c near triangulation with only two vertices on the Projective Plane. Now, we assume our near triangulations have at least three vertices, i.e. $i + j \geq 3$.

Let v_1v_2 be the root edge of T . v_1v_2 lies on a non-root face $v_1v_2v_3$. Decomposing all 2-c near triangulations as we described in Chapter 2, according to the position of v_3 , we have the following cases and subcases:

Case A: v_3 is an exterior vertex.

After the decomposition, we have two subcases:

Subcase A_1 : The decomposition creates two near triangulations.

The contributions are:

B (Figure 6.1(a)): $y^{-1}B^2$.

P (Figure 6.1(b) and (c)): $y^{-1}BP + y^{-1}PB = 2y^{-1}BP$.

Subcase A_2 : *The decomposition destroys a cross cap.*

This only happens for near triangulations on the Projective Plane. After the decomposition, the Projective Plane becomes a disk. See Figure 6.2. The contributions in this subcase are:

B : Nothing.

P : $L = y^{-1}(yB_y(x, y) - 3B + B_2y^2 - (B - B_2y^2)^2/(B_2y^2))$.

Let the contribution from rooted 2-c planar near triangulations with a distinguished vertex v_3'' other than v_1 , v_2 and v_3' to be L_1 . Then

$$L_1 = \sum_{n \geq 0, m \geq 4} (m-3)b_{nm}x^n y^m = yB_y(x, y) - 3B(x, y) + B_2(x)y^2.$$

See Figure 6.2(a). However, this contribution includes the case that v_3' and v_3'' are connected by an edge, which will introduce a loop in the original near triangulation T . Let J denote the contribution of that case. See Figure 6.2(b). To obtain L , we need to subtract J from L_1 . We show

$$J = (B - B_2y^2)^2/(B_2y^2).$$

Let e_1 be the most left edge joining v_3' and v_3'' in near triangulations counted by J . If we split e_1 , the disk is separated into two pieces. We root left piece and right piece by choosing v_1 and v_3'' to be the root vertex respectively, v_1v_3' and $v_3''v_3'$ to be the root edge respectively. The left piece becomes a rooted 2-c planar near triangulation with no edges joining v_3' and v_3'' other than e_1 . The right piece is a 2-c planar near triangulation with at least three exterior vertices. Let Δ be the generating function of the 2-c near triangulations on the left piece, then

$$J = y^{-2}\Delta(B - B_2y^2).$$

Figure 6.2: The decomposition destroys a cross cap of the Projective Plane.

Now we need to show $\Delta = (B - B_2y^2)/B_2$. Notice the edge splitting process we just described splits any near triangulation counted by $B - B_2y^2$ into a near triangulation counted by Δ and a near triangulation counted by B_2y^2 . Hence

$$B - B_2y^2 = y^{-2}\Delta B_2y^2 = \Delta B_2.$$

We obtain Δ , J and L as we expected.

Case B: v_3 is an interior vertex.

After the decomposition, we got another 2-c rooted near triangulation which has root face valence at least three. The contributions are

B (Figure 6.3(a)): $y^{-1}x(B - B_2y^2)$.

P (Figure 6.3(b)): $y^{-1}x(P - P_2y^2)$.

Lemma 6.2 follows by combining all these cases and subcases. $\square$

The following lemma gives the functional equation of C .

Lemma 6.3

$$C = 2u^{-1}B(x, u)C + v^{-1}(\Gamma - \Lambda) + u^{-1}x(C - C_2u^2) \quad (6.6)$$

Figure 6.3: Rooted 2-c triangulations on the Sphere and the Projective Plane with v_3 being an interior vertex.

with Γ and Λ being given by

$$\Gamma = \frac{u^2 B(x, v)}{v(v-u)} - \frac{v^3 B(x, u)}{u^2(v-u)} + v(v+u)B_2 + uv^2 B_3. \quad (6.7)$$

$$\Lambda = \frac{(B(x, v) - v^2 B_2)(B(x, u) - u^2 B_2 - u^3 B_3)}{u^2 B_2} \quad (6.8)$$

Proof: Similar to the proof of the previous lemma, let $v_1 v_2$ be the root edge of a 2-c triangulation T on the Cylinder, $v_1 v_2$ lies on a non-root triangle face $v_1 v_2 v_3$. Decomposing all 2-c triangulations on the Cylinder, according to the position of vertex v_3 , we have following cases:

Case A: v_3 is on the root-boundary.

The contribution is: $u^{-1}(B(x, u)C + CB(x, u)) = 2u^{-1}B(x, u)C$.

Case B: v_3 is on the nonroot-boundary.

The contribution is (Figure 6.4): $v^{-1}(\Gamma - \Lambda)$.

Figure 6.4: A triangulation on the Cylinder: v_3 is on nonroot-boundary.

The analysis of the contribution is similar to the subcase A_2 of the proof of the previous lemma. Here Γ is given by

$$\begin{aligned} \Gamma &= \sum_{n \geq 0, m \geq 5} b_{nm} x^n (u^2 v^{m-2} + u^3 v^{m-3} + \cdots + u^{m-3} v^3) \\ &= \frac{u^2 B(x, v)}{v(v-u)} - \frac{v^3 B(x, u)}{u^2(v-u)} + v(v+u)B_2 + uv^2B_3, \end{aligned}$$

which counts the contribution from all rooted 2-c planar near triangulations with a distinguished vertex v_3'' on the exterior face, and with two different indexes u and v keeping track of the boundary vertices. However, the contribution Γ includes the case that v_3' and v_3'' are connected by an edge, which will introduce a loop in T . So we need to subtract the contribution of that case from Γ . Let Λ be this contribution. It can be shown that (see Figure 6.4(b))

$$\Lambda = \frac{(B(x, v) - v^2 B_2)(B(x, u) - u^2 B_2 - u^3 B_3)}{u^2 B_2}.$$

Case C: v_3 is an interior vertex.

The contribution is: $v^{-1}(C - C_2 u^2)$.

Lemma 6.3 follows by combining all these cases. $\square$

Figure 6.5: A special kind of triangulations on the Cylinder and the Möbius Band.

6.3 A Special Kind of Rooted 2-c Triangulation on the Cylinder and the Möbius Band

In this section, we consider special kinds of rooted 2-c triangulations on the Cylinder and the Möbius Band, respectively. These are defined as follows: two edges on boundary are specified: the root edge v_1v_3' and another edge v_2v_3'' , and v_1 is the root vertex. Furthermore, v_3' and v_3'' cannot share an edge. In the case of the Cylinder, the root-boundary vertices are indicated by index u and the nonroot-boundary vertices are indicated by index v , and we require that v_3' is on the root-boundary and v_3'' is on the nonroot-boundary. See Figure 6.5(a). In the case of Möbius Band, we only need one index y to keep track of the boundary vertices. See Figure 6.5(b).

We consider the generating functions $E(x, u, v)$ and $H(x, y)$, which are defined to be the generating functions of the number of these special kinds of rooted 2-c triangulations on the Cylinder and the Möbius Band, respectively.

Let us analyze $E(x, u, v)$ first.

Figure 6.6: A special kind of triangulations on the Cylinder: cutting along edge e_1 .

First let us allow v'_3 and v''_3 to share edges in the triangulations. Then

$$\sum_{n \geq 0, i \geq 2, j \geq 2} j C_{n,i,j} x^n u^i v^j = v C_v(x, u, v)$$

is the generating function of the number of all rooted 2-c triangulations on the Cylinder with a distinguished vertex v''_3 on the nonroot-boundary. To obtain $E(x, u, v)$, we need to subtract the contribution of those triangulations which do have edge $v'_3 v''_3$ from $v C_v(x, u, v)$. Let Φ be the contribution.

Now we analyze Φ . Starting from the root edge $v_1 v'_3$, we move clockwise around vertex v'_3 . Let e_1 be the first edge $v'_3 v''_3$ we meet. Cutting the Cylinder along e_1 , we obtain a rooted near triangulation on a disk in which there are four types of edges forbidden. We label these edges by e_a , e_b , e_c and e_d respectively. See Figure 6.6. The dot lines in this figure indicated those edges which are not allowed.

First we ignore e_d and consider the situation that the 2-c planar near triangulations with three forbidden types of edges e_a , e_b and e_c . Let Θ be the contribution of this situation.

We use the principle of inclusion and exclusion to obtain Θ .

Let Θ_0 be the generating function of the 2-c planar near triangulations in which the exterior vertices are indicated by two indexes: u and v , and with at least three index u vertices and at least three index v vertices. Let Θ_a be the contribution from those near triangulations counted by Θ_0 with edge e_a being present; Θ_b be the contribution from those near triangulations counted by Θ_0 with edge e_b being present and edge e_a not being present; Θ_c be the contribution from those near triangulations counted by Θ_0 with edge e_c being present and neither edge e_a nor edge e_b being present. Then

$$\Theta = \Theta_0 - \Theta_a - \Theta_b - \Theta_c.$$

With some analysis, we can obtain the expressions of Θ_0 , Θ_a , Θ_b and Θ_c ; they are given by the following equations. We will not give the detailed analysis for these equations. It becomes quite routine as long as we draw a figure for each case and pay special attention to the forbidden edges. See Figure 6.7(a) for Θ_a , Figure 6.7(b) for Θ_b and Figure 6.7(c) for Θ_c .

$$\begin{aligned} \Theta_0 &= \sum_{n \geq 0, m \geq 6} b_{n,m} x^n (u^3 v^{m-3} + u^4 v^{m-4} + \dots + u^{m-3} v^3) \\ &= \frac{u^3 B(x, v)}{v^2(v-u)} - \frac{v^3 B(x, u)}{u^2(v-u)} + (v^2 + vu + u^2)B_2 + uv(u+v)B_3 + u^2 v^2 B_4 \quad (6.9) \\ \Theta_a &= \frac{(B(x, u) - B_2 u^2)(B(x, v) - B_2 v^2 - B_3 v^3 - B_4 v^4)}{B_2 v^2} \\ \Theta_b &= \frac{(B(x, v) - B_2 v^2)(B(x, u) - B_2 u^2 - B_3 u^3 - B_4 u^4 - (B(x, u) - B_2 u^2)B_4 u^2)}{B_2^2 u^2} \\ \Theta_c &= \frac{(B(x, u) - B_2 u^2 - u B_3 B(x, u)/B_2)(B(x, v) - B_2 v^2 - v B_3 B(x, v)/B_2)}{B_2 uv}. \end{aligned}$$

Figure 6.7: Triangulations on a disk with forbidden edges.

Now we consider the forbidden edge e_d . There is a very simple relation between Θ and Φ .

$$\Theta = y^{-2}\Phi B_2 y^2 = \Phi B_2.$$

Hence we obtain the following expression of $E(x, u, v)$, which involves B .

$$E(x, u, v) = vC_v(x, u, v) - \Phi = vC_v(x, u, v) - (\Theta_0 - \Theta_a - \Theta_b - \Theta_c)/B_2. \quad (6.10)$$

Similarly, We analyze $H(x, y)$.

If we allow v'_3 and v''_3 to share an edge in the rooted triangulations of the Möbius Band counted by $H(x, y)$, we count exactly the number of all rooted 2-c triangulations on the Möbius Band with a distinguished boundary vertex v''_3 . Let the generating function of this number be H_0 . Then

$$H_0 = \sum_{n \geq 0, m \geq 4} p_{n,m} x^n y^m (m-3) = yP_y(x, y) - 3P + P_2 y^2.$$

Here $p_{n,m}$ is the number of rooted 2-c near triangulations of type $[n, m]$ on the Projective Plane.

To obtain $H(x, y)$, we need to subtract the contribution of those triangulations which do have edge $v'_3 v''_3$ from H_0 . Notice there are three different types of edge $v'_3 v''_3$. Let us name them type a , type b and type c , respectively. See Figure 6.8.

Figure 6.8: Triangulations on the Möbius Band with forbidden edge $v'_3v''_3$.

Let H_1 be the contribution from those rooted 2-c triangulations in which edge $v'_3v''_3$ of type a is present. Let H_2 be the contribution from those rooted 2-c triangulations in which edge $v'_3v''_3$ of type b is present but edge $v'_3v''_3$ of type a is not present. H_3 be the contribution from those rooted triangulations in which edge $v'_3v''_3$ of type c is present but neither edge $v'_3v''_3$ of type a nor edge $v'_3v''_3$ of type b is present. Then

$$H(x, y) = H_0 - H_1 - H_2 - H_3. \quad (6.11)$$

We need to find the expressions for H_1 , H_2 and H_3 .

In the case of H_1 and H_3 , we start from edge $v_1v'_3$ and move around v'_3 clockwise. In the case of H_2 , we move around v_3 anti-clockwisely. Let e_x be the first edge $v'_3v''_3$ of type x we meet when we move ($x \in \{a, b, c\}$). Cutting the Möbius band along edge e_x , we can obtain the following equations. See Figure 6.9(I) for H_1 , Figure 6.9(II) for H_2 and Figure 6.9(III) for H_3 . In these figures, the dot lines always mean that those edges cannot be present in the near triangulations.

$$H_1 = \frac{1}{B_2y^2}(\Theta_0|_{u=v=y} - \frac{2(B - B_2y^2 - B_3y^3)^2}{B_2y^2})$$

$$H_2 = \frac{B - B_2y^2}{B_2y^2}(P - P_2y^2 - \frac{B - B_2y^2 - B_3y^3 - B_4y^4 - 2yB_3(B - B_2y^2 - B_3y^3)}{B_2y^2})$$

Figure 6.9: Triangulations on the Möbius Band: Cutting forbidden edges $v'_3 v''_3$.

$$H_3 = \frac{B - B_2y^2}{B_2y^2} \left(P - P_2y^2 - \frac{B - B_2y^2 - B_3y^3 - B_4y^4 - 2yB_3(B - B_2y^2 - B_3y^3)}{B_2y^2} \right. \\ \left. - \frac{P_2(B - B_2y^2)}{B_2} + \frac{(B - B_2y^2)(B_4 - 2B_3^2)}{B_2^2} \right).$$

Here Θ_0 is given by (6.9).

Hence we obtain the expression of $H(x, y)$, which involves with B and P .

We will need $E(x, u, v)$ and $H(x, y)$ in next section where we find functional equations for $T(x, y)$ and $K(x, y)$.

6.4 Functional Equations for $T(x, y)$ and $K(x, y)$

Now, we are ready to enumerate rooted 2-c near triangulations on the Torus and the Klein Bottle.

Lemma 6.4 *The generating functions of T and K satisfy*

$$T = 2y^{-1}BT + y^{-1}E(x, y) + y^{-1}x(T - T_2y^2) \quad (6.12)$$

$$K = 2y^{-1}BK + y^{-1}P^2 + y^{-1}(E(x, y) + H(x, y)) + y^{-1}x(K - K_2y^2) \quad (6.13)$$

where $E(x, y) = E(x, u, v)|_{u=v=y}$. $E(x, u, v)$ and $H(x, y)$ are given by (6.10) and (6.11) of the previous section.

Proof: Let v_1v_2 be the root edge of a 2-c near triangulation T_1 on the Torus or the Klein Bottle. v_1v_2 lies on a non-root face $v_1v_2v_3$. Decomposing all the 2-c near triangulations, according to the position of vertex v_3 , we have the following cases and subcases:

Case A: v_3 is an exterior vertex.

Figure 6.10: A triangulation on the Klein Bottle: the decomposition creates two Möbius Bands.

We have two subcases:

Subcase A_1 : *The decomposition creates two near triangulations.*

The contribution in the Torus case is:

$$T : y^{-1}BT + y^{-1}TB = 2y^{-1}BT.$$

In the case of Klein Bottle, there is one more term, since the decomposition of a the Klein Bottle may create two Möbius Bands. See Figure 6.10. The contribution for the Klein Bottle in this case is:

$$K : y^{-1}BK + y^{-1}KB + y^{-1}P^2 = 2y^{-1}BK + y^{-1}P^2.$$

Subcase A_2 : *The decomposition creates a Cylinder.*

For the Torus, see Figure 6.11. In this Cylinder, the root-boundary vertex v_3' and the nonroot-boundary v_3'' cannot share an edge, since otherwise there will be a loop in the original near triangulations T_1 . In the previous section, we have obtained the generating function $E(x, u, v)$. If we do not distinguish the index of

Figure 6.11: A triangulation on the Torus: the decomposition creates a Cylinder.

root-boundary vertices with the index of nonroot-boundary vertices, $E(x, u, v)$ is exactly the contribution of this subcase. Hence the contribution in this subcase for the Torus is

$$T : y^{-1} E(x, y) = y^{-1} E(x, u, v)|_{u=v=y}.$$

For the Klein Bottle, we have the same situation. See Figure 6.12(a). The contribution in this subsubcase is also

$$K : y^{-1} E(x, y) = y^{-1} E(x, u, v)|_{u=v=y}.$$

Subcase A_3 : *The decomposition destroys a cross cap.*

This only happens for the Klein Bottle. After the decomposition, we obtain a Möbius Band. Two boundary vertices v'_3 and v''_3 of the Möbius Band cannot be connected by an edge, otherwise there will be a loop in the original triangulation T_1 . See Figure 6.12(b). We have already studied this situation in the previous section. The contribution in this subcase is

$$T : \text{nothing}.$$

Figure 6.12: Triangulations on the Klein Bottle: the decomposition creates a Cylinder or a Möbius Band.

$$K : y^{-1}H(x, y).$$

Case B: v_3 is an interior vertex.

After removing the root edge v_1v_2 , we obtain another 2-c near triangulation which has at least three vertices on the root face. The contributions are:

$$T : y^{-1}x(T - T_2y^2),$$

$$K : y^{-1}x(K - K_2y^2).$$

Combining all these cases and subcases, we obtain functional equations (6.12) and (6.13). $\square$

6.5 Parametric Expressions

In this section we prove Theorem 6.1. Notice $B(x, y)$ was obtained by Mullin [33]. We can obtain the same result by solving (6.3) using Quadratic Method. Let us summarize this result as a lemma.

Lemma 6.5 *The generating function $B(x, y)$ is given by*

$$B = \frac{1}{2}(y - t(1 - 2t)^2 - (y - t(1 - 2t))\sqrt{(1 - 2t)^2 - 4y}) \quad (6.14)$$

with $B_2 = \frac{1-3t}{(1-2t)^2}$, $B_3 = \frac{1-4t}{(1-2t)^4}$ and $B_4 = \frac{2-9t}{(1-2t)^6}$. Here t is the power series of x defined by $x = t(1 - 2t)^2$ and $t(0) = 0$.

We first rewriting (6.4) and (6.6) as

$$P(1 - 2y^{-1}B - y^{-1}x) = L - xyP_2, \quad (6.15)$$

$$C(1 - 2u^{-1}B - u^{-1}x) = v^{-1}(\Gamma - \Lambda) - xuC_2. \quad (6.16)$$

Here L is given by (6.5) in Lemma 6.2, and Γ and Λ are given by (6.7) and (6.8) in Lemma 6.3, respectively.

Let $t = t(x)$ be the power series satisfying $x = t(1 - 2t)^2$ and let $f(x) = t - 2t^2$. It is routine to verify that

$$1 - 2B(x, f(x))/f(x) - x/f(x) = 0.$$

Hence

$$(L - xyP_2)|_{y=t-2t^2} = 0.$$

$$(v^{-1}(\Gamma - \Lambda) - xuC_2)|_{u=t-2t^2} = 0.$$

Thus

$$P_2 = \frac{L}{xy} \Big|_{y=t-2t^2}, \quad (6.17)$$

$$C_2 = \frac{\Gamma - \Lambda}{xuv} \Big|_{u=t-2t^2}. \quad (6.18)$$

Then from equations (6.15) and (6.16), we have

$$P = \frac{L - xyP_2}{1 - 2y^{-1}B - y^{-1}x}, \quad (6.19)$$

$$C = \frac{v^{-1}(\Gamma - \Lambda) - xuC_2}{1 - 2u^{-1}B(x, u) - u^{-1}x}. \quad (6.20)$$

Plugging the parametric expressions of B , B_2 and B_3 as in Lemma 6.5 into the expressions of L , Γ and Λ , which are given in (6.5), (6.7) and (6.8) respectively, we can obtain the parametric expressions of L , Γ and Λ . Then from equations (6.17), (6.18), (6.19) and (6.20), we can obtain the parametric expressions for P_2 , C_2 , P and L . The calculations were carried out using MAPLE. We will not display the parametric expressions of P and C here since they are too complicated. The

parametric expressions of P_2 and C_2 are given in the following lemma. Notice P_2 is already obtained in [18].

Lemma 6.6

$$P_2 = \frac{(1-2t)(1-5t) - (1-3t)\sqrt{(1-2t)(1-6t)}}{2t^2(1-3t)(1-2t)^3}, \quad (6.21)$$

$$C_2 = \frac{1-2t - \sqrt{(1-2t)^2 - 4v}}{2(1-2t)v^2} - \frac{(1-4t)(1-2t) + t\sqrt{(1-2t)^2 - 4v}}{(1-2t)^3(1-3t)v} - \frac{2-3t - \sqrt{(1-2t)^2 - 4v}}{(1-3t)(1-2t)^4}. \quad (6.22)$$

Here t is the unique power series of x defined by $x = t(1-2t)^2$ and $t(0) = 0$.

In order to solve the functional equations (6.12) and (6.13), we need the expressions of $E(x, y)$ and $H(x, y)$. Notice these expressions only involve B , C and P . Plugging the parametric expressions of B , C and P into $E(x, y)$ and $H(x, y)$, we obtain the parametric expressions of $E(x, y)$ and $H(x, y)$. These calculations are also carried out using Maple. The parametric expressions are too complicated, we will not display them here either.

Now, we are ready to solve functional equations of $T(x, y)$ and $K(x, y)$. We rewrite (6.12) and (6.13) as

$$T(1 - 2y^{-1}B - y^{-1}x) = y^{-1}E(x, y) - xyT_2, \quad (6.23)$$

$$K(1 - 2y^{-1}B - y^{-1}x) = y^{-1}(E(x, y) + H(x, y) + P^2) - xyK_2. \quad (6.24)$$

Using the same argument as we used to solve (6.15) and (6.16), we have

$$(y^{-1}E(x, y) - xyT_2)|_{y=t-2t^2} = 0,$$

$$(y^{-1}(E(x, y) + H(x, y) + P^2) - xyK_2)|_{y=t-2t^2} = 0.$$

Hence

$$T_2 = \frac{E(x, y)}{xy^2} \Big|_{y=t-2t^2} \quad (6.25)$$

$$K_2 = \frac{E(x, y) + H(x, y) + P^2}{xy^2} \Big|_{y=t-2t^2}. \quad (6.26)$$

Plugging the parametric expressions of $E(x, y)$ and $H(x, y)$ into (6.25) and (6.26), we obtain parametric expressions of T_2 and K_2 . Plugging the parametric expressions of T_2 and K_2 into (6.23) and (6.24) respectively, we obtain parametric expressions of T and K .

$$T = \frac{y^{-1}E(x, y) - xyT_2}{(1 - 2y^{-1}B - y^{-1}x)}, \quad (6.27)$$

$$K = \frac{y^{-1}(E(x, y) + H(x, y) + P^2) - xyK_2}{(1 - 2y^{-1}B - y^{-1}x)}. \quad (6.28)$$

The parametric expressions of T and K are too complicated to be displayed here. Since $T_2 = xT_3$, we have $T_3(x) = T_2/x$. Similarly, $K_3(x) = K_2/x$. Hence we can obtain parametric expressions of T_3x^3 and K_3x^3 ; they are given in Theorem 6.1.

Gao [18] list all the rooted 2-c triangulations on the Projective Plane with three and four vertices. See [18] Fig 4.1. From Theorem 6.1, we have $t_{0,3} = 1$, $t_{1,3} = 51$, and $k_{0,3} = 3$. We list all the rooted 2-c triangulations on the Torus and on the Klein Bottle with three vertices in Figure 6.13 and all the rooted 2-c triangulations on the Torus with four vertices in Figure 6.14. In these figures, the smaller arrows indicate that we need glue edges according that directions to obtain the Torus or the Klein Bottle. The larger arrows indicate the root vertex and root edge of each rooting.

Since $k_{0,3} = 3$, we have three different minimal rooted 2-c triangulations on the Klein Bottle. It is easy to check that they are not connected by diagonal flips since

Figure 6.13: Triangulations on the Torus and the Klein Bottle with three vertices.

all of their edges are unflippable. Hence the diagonal flip adjacency graph $GT_{n,2c}^{r,K}$ is not connected for any n .

Total number of triangulations: $3+12+12+1+3+8+12 = 51$.

Figure 6.14: Triangulations on the Torus with four vertices.

Chapter 7

Exact Enumeration of Rooted Simple Triangulations on the Projective Plane

7.1 Introduction

In this chapter, we count rooted simple triangulations on the Projective Plane. Not too much work has been done on exact enumeration of simple near triangulations on surfaces since Brown's result [9]; he obtained the exact number of rooted simple planar near triangulations. It is very difficult to obtain the exact enumeration result of non-planar simple triangulations. The only result we have seen in this area is Edelman and Reiner's paper [16]. They determined the exact number of simple Catalan triangulations of the Möbius Band. Since the Möbius Band can be obtained by cutting off a disk from the Projective Plane, they actually deal with the rooted

simple near triangulations of the Projective Plane without interior vertices. That is a very special case of our result here.

Let $a_{i,j}$, $\bar{p}_{i,j}$ be the numbers of rooted simple near triangulations on the Sphere and the Projective Plane respectively, having $(i + j)$ vertices and root face valence j . Let

$$A(x, y) = \sum_{i,j} a_{i,j} x^i y^j = \sum_j A_j(x) y^j,$$

$$\bar{P}(x, y) = \sum_{i,j} \bar{p}_{i,j} x^i y^j = \sum_j \bar{P}_j(x) y^j.$$

Then $A_j(x)x^j$ and $\bar{P}_j(x)x^j$ are the generating functions of the number of rooted simple near triangulations with root face valence j on the Sphere and the Projective Plane, respectively. $A_3(x)x^3$ and $\bar{P}_3(x)x^3$ is the generating function of the number of rooted simple triangulations on the Sphere and the Projective Plane respectively.

The main result in this chapter is:

Theorem 7.1 *The generating function of the number of rooted simple triangulations of the projective plane is given by*

$$\begin{aligned} \bar{P}_3(x)x^3 &= \frac{1}{2}(1-t)^3(1-3t-(1-t)\sqrt{1-4t}) - \frac{t^3(1-5t+12t^2-14t^3+4t^4+3t^5)}{(1-2t)(1-t)^3} \\ &= x^6 + 33x^7 + 615x^8 + \dots, \end{aligned} \tag{7.1}$$

where t is the power series of x defined by $x = t(1-t)^3$ and $t(0) = 0$.

In the following sections, we will use two different approaches to obtain the parametric expression of $\bar{P}_3(x)x^3$. The first one is the same as the approach we have already used for several times in the previous chapters. That is, decomposing all rooted simple triangulations, we analyze each case and derive a functional equation.

The second approach is called *functional decomposition*. The main idea here is using the result of the number of rooted 2-c triangulations on the Projective Plane. In each rooted 2-c triangulation T , we “close” all 2-cycles, then T becomes a simple triangulation. Here a *2-cycle* is a cycle of length two. “*closing a 2-cycle*” means deleting everything inside the 2-cycle and merging two multiple edges on the boundary of the 2-cycle. All rooted simple triangulations on the Projective Plane can be generated in this way. Hence we can derive a relation between the generating function of the number of rooted 2-c triangulations and the generating function of the number of rooted simple triangulations. We will discuss this in detail in the last section of this chapter.

For simplicity, we use $A, \bar{P}$ to denote $A(x, y), \bar{P}(x, y)$ respectively, and use $A_j, \bar{P}_j$ to denote $A_j(x), \bar{P}_j(x)$ respectively.

7.2 Two Special Kinds of Rooted Simple Planar Triangulations

In order to get the functional equation of $\bar{P}(x, y)$, we need to consider two special kinds of rooted simple planar near triangulations.

The first special kind of rooted simple planar near triangulations is defined as follows: It is the set of all rooted simple planar near triangulations in which two vertices, the root vertex v'_1 and the other exterior vertex v''_1 , can neither share an edge nor both adjacent by an edge to a common vertex. Let $\mathcal{R}$ denote this set. Actually, we have already seen the similar situation in Chapter 6. But now we are dealing with simple triangulations, there are more restrictions. See Figure 7.1(a).

Figure 7.1: Two special kinds of simple planar near triangulations.

The second special kind of rooted simple planar near triangulations is defined as the set of all rooted simple planar near triangulations in which we specify two root face edges: the root edge $v_1'v_2'$ and another root face edge $v_1''v_2''$. v_1' is the root vertex. v_1' and v_1'' can neither share an edge nor both adjacent by an edge to a common vertex. v_2' and v_2'' can neither share an edge nor both adjacent by an edge to a common vertex. v_1' , v_2' , v_1'' and v_2'' appear on the root face anti-clockwisely in this order. Let $\mathcal{S}$ denote this set. See Figure 7.1(b).

Let $R \in \mathcal{R}$ and $S \in \mathcal{S}$. We use two indexes, u and v , to indicate the exterior vertices in the near triangulations R and S . Index u indicates the exterior vertices of the left side between v_1' and v_1'' exclusively; index v indicates the exterior vertices of the right side between v_1' and v_1'' inclusively. See Figure 7.1.

Let $r_{i,j,k}$ and $s_{i,j,k}$ be the numbers of simple planar near triangulations with i interior vertices, j exterior vertices of index u and k exterior vertices of index v in

R and S , respectively. The generating functions of these numbers are defined by

$$R(x, u, v) = \sum_{i,j,k} r_{i,j,k} x^i u^j v^k = \sum_j R_j(x, v) u^j,$$

$$S(x, u, v) = \sum_{i,j,k} s_{i,j,k} x^i u^j v^k = \sum_j S_j(x, v) u^j.$$

We will use the principle of inclusion and exclusion to find the expressions of $R(x, u, v)$ and $S(x, u, v)$.

First let us consider $R(x, u, v)$.

Let T be a near triangulation in set $\mathcal{R}$. There is no edge $v'_1 v''_1$ in T , neither is there any path of length two which connects v'_1 and v''_1 . According to the position of the middle vertex v_2 which is connected to v'_1 and v''_1 , we have three different kinds of paths of length two between v'_1 and v''_1 . Namely, v_2 is an exterior vertex of index u , or is an exterior vertex of index v , or is an interior vertex. Hence totally there are following four forbidden situations:

- (1) There is an edge $v'_1 v''_1$.
- (2) There is a path $v'_1 v_2 v''_1$ of length two and v_2 is an exterior vertex of index u .
- (3) There is a path $v'_1 v_2 v''_1$ of length two and v_2 is an exterior vertex of index v .
- (4) There is a path $v'_1 v_2 v''_1$ of length two and v_2 is an interior vertex.

If we ignore these forbidden situations, we actually deal with rooted simple planar near triangulations with the root vertex v_1 , with another distinguished exterior vertex v''_1 , and with two indexes u and v indicating the exterior vertices. There are at least one index u vertex and three index v vertices in these near triangulations.

Figure 7.2: The first special kind of simple planar near triangulations: the forbidden situations (1), (2) and (3).

Let R_0 be the contribution of these simple planar near triangulations. Then

$$\begin{aligned}
 R_0 &= \sum_{n \geq 0, m \geq 4} a_{n,m} x^n (uv^{m-1} + u^2v^{m-2} + \cdots + u^{m-3}v^3) \\
 &= \frac{u}{v-u} (A(x, v) - \frac{v^3}{u^3} A(x, u)) + v^2.
 \end{aligned}$$

To obtain $R(x, u, v)$, we need to subtract the contribution of the four forbidden situations from R_0 . Let R_i ($i = 1, 2, 3, 4$) denote the case that the i th forbidden situation appears, and for any $j < i$, the j th forbidden situation does not appear. If $i = 1$, we ignore j . We obtain R_1 , R_2 and R_3 as follows. See Figure 7.2.

$$R_1 = u^{-2} (A(x, u) - u^2) (A(x, v) - v^2)$$

$$R_2 = u^{-3} v^{-1} A(x, u)^2 (A(x, v) - v^2 - v A_3 A(x, v))$$

$$R_3 = u^{-3} v^{-1} A(x, v)^2 (A(x, u) - u^2 - u A_3 A(x, u) - A(x, u)^2 (A_4 - A_3^2)).$$

Figure 7.3: The first special kind of simple planar near triangulations: the forbidden situation (4).

To obtain the expression of R_4 , we need to do a little bit more. First we find the most right path $v_1'v_2v_1''$ with v_2 being an interior vertex, say this path is p_1 . Splitting the near triangulation along p_1 , we get two planar near triangulations. Let the contribution of the left piece be R_{41} and the contribution of the right piece be R_{42} . See Figure 7.3. We have

$$R_4 = u^{-3}v^{-1}xR_{41}R_{42} \quad (7.2)$$

$$R_{41} = A(x, u) - u^2 - uA_3A(x, u) - A(x, u)^2(A_4 - A_3^2) \quad (7.3)$$

$$R_{42} = R_{41}|_{u=v} - xR_{42}(A_4 - A_3^2). \quad (7.4)$$

Equations (7.2) and (7.3) are easy to understand. To understand (7.4), first we ignore the possible forbidden path $v_1'v_3v_1''$ of the right piece, it then becomes the same as the left piece except that the index of exterior vertices is changed from u to v . Notice the contribution of the right piece with path $v_1'v_3v_1''$ existing is

$xR_{42}(A_4 - A_3^2)$. Hence we have (7.4). Thus

$$R_{42} = \frac{R_{41}|_{u=v}}{1 + x(A_4 - A_3^2)}.$$

We have obtained the expressions of R_{41} , R_{42} and R_4 ; they involve A , A_3 and A_4 .

From

$$R(x, u, v) = R_0 - R_1 - R_2 - R_3 - R_4,$$

we obtain the expression of $R(x, u, v)$, which involves A , A_3 and A_4 .

Now we consider $S(x, u, v)$.

It is more complicated than $R(x, u, v)$, since we need consider more forbidden situations. We use the expression of $R(x, u, v)$ to get the expression of $S(x, u, v)$.

A near triangulation T in $\mathcal{S}$ is also a near triangulation in $\mathcal{R}$, and with some restrictions. The restrictions which can make a near triangulation in $\mathcal{R}$ to be a near triangulation in $\mathcal{S}$ is: vertices v'_2 and v''_2 can neither share an edge nor both adjacent to a common vertex by an edge.

Let us make it more clear. Let $T \in \mathcal{R}$ with two edges, $v'_1v'_2$ and $v''_1v''_2$, both being on the root face. Then $T \in \mathcal{S}$ if the follow four situations do not happen in T :

- (1') There is an edge $v'_2v''_2$.
- (2') There is a path $v'_2v_3v''_2$ of length two and v_3 is an exterior vertex of index u .
- (3') There is a path $v'_2v_3v''_2$ of length two and v_3 is an exterior vertex of index v .
- (4') There is a path $v'_2v_3v''_2$ of length two and v_3 is an interior vertex.

To obtain $S(x, u, v)$, we need to subtract the contributions of these four forbidden situations from $R(x, u, v)$. Let $S_{i'}$ ($i = 1, 2, 3, 4$) denote the case that the i' th

Figure 7.4: The second special kind of simple planar near triangulations: the forbidden situations (1'), (2') and (3').

forbidden situation appears, and for any $j < i$, the j 'th forbidden situation does not appear. We can obtain $S_{1'}$, $S_{2'}$ and $S_{3'}$ as follows. See Figure 7.4.

$$\begin{aligned}
 S_{1'} &= u^{-2}(A(x, u) - u^2 - uA_3A(x, u))(A(x, v) - v^2 - vA_3A(x, v)) \\
 S_{2'} &= u^{-3}v^{-1}A(x, u)(A(x, u) - u^2)(A(x, v) - v^2 - vA_3A(x, v) - v^2(A_4 - A_3^2)A(x, v)) \\
 &\quad - u^{-2}A_3^2A(x, u)^2(A(x, v) - v^2 - vA_3A(x, v)) \\
 S_{3'} &= u^{-3}v^{-1}A(x, v)(A(x, v) - v^2)(A(x, u) - u^2 - uA_3A(x, u) - A(x, u)^2(A_4 - A_3^2)) \\
 &\quad - u^{-2}A_3^2A(x, v)^2(A(x, u) - u^2 - uA_3A(x, u)).
 \end{aligned}$$

To obtain the expression of $S_{4'}$, just like R_4 , we need to do a little bit more. First we find the most right path $v_2'v_3v_2''$, say it is $p_{1'}$. Splitting near triangulations along the path $p_{1'}$, we get two planar near triangulations. See Figure 7.5. Let the contribution of the left one be S_{41} and the contribution of the right one be S_{42} . Then

$$S_{4'} = u^{-3}v^{-1}xS_{41}S_{42} - xA_3^2S_{1'}, \quad (7.5)$$

Figure 7.5: The second special kind of simple planar near triangulations: the forbidden situation (4').

$$S_{41} = A(x, v) - v^2 - vA_3A(x, v) - A(x, v)^2(A_4 - A_3^2), \quad (7.6)$$

$$S_{42} = S_{41}|_{v=u} - u^{-4}xS_{42}(A_4 - A_3^2)u^4. \quad (7.7)$$

Notice in (7.5), we need to subtract $xA_3^2S_{1'}$, since we are not allowed to have edge v_1v_3 in the left piece and v_1v_3 in the right piece at the same time. (7.7) can be explained in the same way as we did for R_{42} in (7.4). Hence

$$S_{42} = \frac{S_{41}|_{v=u}}{1 + x(A_4 - A_3^2)}.$$

We have obtained the expressions of S_{41} , S_{42} and $S_{4'}$.

From

$$S(x, u, v) = R(x, u, v) - S_{1'} - S_{2'} - S_{3'} - S_{4'},$$

we obtain the expression of $S(x, u, v)$, which involves A , A_3 , A_4 and A_5 .

Figure 7.6: The decomposition destroys a cross cap of the Projective Plane.

Subcase A_2 : *The decomposition destroys a cross cap.*

This can only happen on the Projective Plane. See Figure 7.6(a). The contributions are:

$$P : y^{-1}(R - y^{-2}(A - y^2 - yA_3A)^2).$$

This can be explained as follows: $R = R(x, u, v)|_{u=v=y}$ counts the contribution of the first special kind of rooted simple planar near triangulations, with using only one index y to indicate the exterior vertices. We need to subtract the contribution of those near triangulations which have edge v_1v_2 from R . That can be counted by

$$y^{-2}(A - y^2 - yA_3A)^2.$$

The reason is: Cutting along edge v_1v_2 and splitting it into two edges, we obtain two separate pieces. Each piece is a simple near triangulation with root face at least three and have one forbidden edge (the left piece can not have edge v_1v_3'' and the right piece can not have edge v_2v_3'). See Figure 7.6(b).

Case B: v_3 is an interior vertex.

After the decomposition, we get another simple near triangulation T' . We root T' by choosing v_1v_3 to be the root edge and v_1 to be the root vertex, the face

Figure 7.7: Simple Near triangulations on the Sphere and the Projective Plane with v_3 being an interior vertex.

obtained by merging the root face of T and the face $v_1v_2v_3$ to be the root face. T' has root face valence at least three and cannot have edge v_1v_2 . The contributions in this cases are:

$$A : \quad y^{-1}x(A - y^2 - yA_3A),$$

$$P : \quad y^{-1}x(\bar{P} - yA_3\bar{P} - yA\bar{P}_3 - y^{-2}S_1).$$

For planar near triangulations, since the root face valence is at least three, we need to subtract y^2 from A . We also need to subtract the contribution of those near triangulations which have edge v_1v_2 , that is yA_3A . See Figure 7.7(a).

For near triangulations on the Projective Plane, there are three different types of edge v_1v_2 . Let edges e_a , e_b and e_c be these three different types of edge v_1v_2 . See Figure 7.7(b). We need to subtract the contributions of all these three cases from $\bar{P}$. The contributions of the case that there is an edge e_a and the case that there is an edge e_b are $yA_3\bar{P}$ and $yA\bar{P}_3$, respectively. For the case that there is an edge e_c in the near triangulation, we split the near triangulation along e_c . Then we obtain a

second special kind of planar near triangulation with only one index u vertex, and with index y indicating all exterior vertices. Hence the contribution from this is counted by

$$y^{-2}([u]S(x, u, v)u)|_{u=v=y} = y^{-2}S_1.$$

Lemma 7.2 follows by combining all these cases and subcases. $\square$

7.4 A Parametric Expression for $\bar{P}_3(x, y)$

Using Brown's Quadratic method ([10]), we can solve the functional equation (7.8) and obtain the parametric equation for $A(x, y)$. Notice this result was obtained by Brown [9].

Lemma 7.3 *The generating function of the number of rooted simple planar near triangulations is given by*

$$A = \frac{1}{2}(y(1-t)(1+2t) - t(1-t)^3 - (y - t(1-t)^2)\sqrt{(1-t)^2 - 4y}) \quad (7.10)$$

with $A_3 = \frac{1-2t}{(1-t)^3}$, $A_4 = \frac{2-5t}{(1-t)^5}$ and $A_5 = \frac{5-14t}{(1-t)^7}$, where t is a power series of x defined by $x = t(1-t)^3$ and $t(0) = 0$.

Now we are ready to solve the functional equation (7.9) and obtain a parametric expression for $\bar{P}$. Rewriting (7.9) as

$$\bar{P}(1 - 2y^{-1}A - y^{-1}x + xA_3) = y^{-1}(R - y^{-2}(A - y^2 - yA_3A)^2) - y^{-3}xS_1 - xA\bar{P}_3.$$

Let $t = t(x)$ be the power series satisfying $x = t(1-t)^3$ and let $f(x) = t(1-t)^2$. It is routine to verify that

$$1 - 2A(x, f(x))/f(x) - x/f(x) + xA_3 = 0.$$

Figure 7.8: All triangulations on the Projective Plane with six and seven vertices.

Hence

$$\bar{P}_3 = \frac{R - y^{-2}(A - y^2 - yA_3A)^2 - y^{-2}xS_1}{xyA} \Big|_{y=t(1-t)^2}. \quad (7.11)$$

Using MAPLE software, we obtain a parametric expression of $\bar{P}_3(x)x^3$, which is given in Theorem 7.1.

Since $\bar{p}_{3,3} = 1$, $\bar{p}_{4,3} = 33$. So there is only 1 rooted simple triangulation with 6 vertices and there are 33 different rooted simple triangulations with 7 vertices on the Projective Plane. We list all these different rooted simple triangulations in Figure

7.8. The number under each map is the number of different rooting of that map. The arrows indicate the root vertex and root edge of each rooting. The Antipodal points on the circles are identified to form the cross-cap of the Projective Plane.

7.5 Functional Decomposition: From 2-c Triangulations to Simple Triangulations

In this section, we prove Theorem 7.1 using a different approach.

Let $B(x, y)$, $P(x, y)$ be the generating functions of the numbers of rooted 2-c triangulations on the Sphere and on the Projective Plane, respectively. In Chapter 6, we have obtained parametric expressions of $P_2(x)$ and B . From $P_2x^2 = P_3x^3$, we can get an parametric expression of P_3x^3 . We use P_3 and $\bar{P}_3$ to denote $P_3(x)x^3$ and $\bar{P}_3(x)x^3$, respectively.

Observation 7.4 $\bar{P}_3$ can be obtained from P_3 by subtracting the contribution of all those rooted 2-c triangulations which do have multiple edges.

Let T be a rooted 2-c triangulation on the Projective Plane. We call a cycle in T *contractable* if it encloses a disk, otherwise it is called *noncontractable*.

We need three steps to achieve our goal.

Step 1: *Closing all contractable 2-cycles which do not enclose the root face. See Figure 7.9.*

After step 1, T is transformed into another triangulation T_1 . Let $g_{1,n}$ be the number of triangulations with n vertices which has no contractable 2-cycles like T_1

Figure 7.9: Functional decomposition: Closing all contractable 2-cycles which do not enclose the root face.

and let $G_1 = \sum_n q_{1,n} x^n$. Then

$$P_3 = \sum_n g_{1,n} x^n (B_2)^e.$$

Here e is the number of edges in the triangulations on the Projective Plane with n vertices. This equation simply describes the following fact: all rooted 2-c triangulations can be obtain by replacing each edge of a rooted triangulation like T_1 with a rooted 2-c near triangulation having root face valence two.

From $n - e + f = 1$ and $3f = 2e$, we have $e = 3(n - 1)$. Hence

$$P_3 = B_2^{-3} G_1(x B_2^3).$$

Let $X = x B_2^3$ and $u = \frac{t}{1-2t}$. Then

$$\begin{aligned} X &= u(1-u)^3, \\ G_1(X) &= \frac{1}{2}(1+2u)(1-u)^2(1-3u-(1-u)\sqrt{1-4u}). \end{aligned} \quad (7.12)$$

If T_1 does not have any contractable 2-cycle enclosing the root face, we do not

Figure 7.10: Functional decomposition: Cutting along C_{max} and C_{min} .

need to proceed the following step 2 and we can go to step 3 directly. Now we assume that T_1 has at least one contractable 2-cycle which enclose the root face.

Step 2: *Considering the contractable 2-cycles which enclose the root face of T_1 .*

We define the *maximal 2-cycle* of T_1 to be the largest 2-cycle enclosing the root face in T_1 and denote it by C_{max} , and the *minimal 2-cycle* of T_1 to be the smallest 2-cycle enclosing the root face in T_1 and denote it by C_{min} .

Cutting T_1 along C_{max} , we separate T_1 into two parts. See Figure 7.10(a).

Let T_2 be the part which contains the root face of T_1 and T_3 be the other part.

Let G_2 be the generating function of the number of planar near triangulations like T_2 , and G_3 be the generating function of the number of near triangulations like T_3 . We root T_3 by choosing the root face to be the face enclosed by C_{max} , and either one of the vertices and edges on this face to be the root vertex and the root edge. T_3 becomes a rooted near triangulation. If we delete the nonroot edge of the root face in T_3 , then T_3 becomes a rooted triangulation with no contractable 2-cycle. Hence

$$G_1 = G_3 + x^{-2}G_3G_2.$$

If we cut T_2 along C_{min} , it is also separated into two parts. See Figure 7.10(b). Let T_N be the part which contains the root face of T_2 . If we delete one edge of the exterior face of T_N , T_N becomes a rooted simple planar triangulation with a distinguished edge. Let N be the generating function of the number of this kind of planar triangulations. For the other part, if we delete the same edge which we have deleted in T_N , that part becomes a planar near triangulation just like T_2 . Hence

$$G_2 = N + x^{-2}NG_2,$$

$$N = e_1 A_3 x^3.$$

Here e_1 is the number of edges in the rooted simple planar triangulation with n interior vertices. From $(n+3) - e_1 + f = 2$ and $3f = 2e_1$, we have $e_1 = 3(n+1)$. Thus

$$N = 3(n+1)A_3x^3 = 3(n+1) \sum_{n \geq 0} a_{3n}x^n x^3 = 3x^3 \left(x \frac{dA_3}{dx} + A_3 \right). \quad (7.13)$$

Using the parametric expression of A_3 given in Lemma 7.3, we obtain the parametric expressions of N , G_2 and G_3 , while G_1 is given by (7.12).

$$G_2 = \frac{N}{1 - x^{-2}N},$$

Figure 7.11: Four different kinds of simple planar near triangulations.

$$G_3 = \frac{G_1}{1 + x^{-2}G_2} = \frac{G_1(x^2 - N)}{x^2}. \quad (7.14)$$

If T_3 does not have any noncontractable 2-cycles, then T_3 is already a rooted simple triangulation on the Projective Plane and we do not need to process the following step 3. Now we assume T_3 have at least one noncontractable 2-cycle.

Step 3: *Considering the noncontractable 2-circles in T_3 .*

Let G_4 be the generating function of the number of rooted 2-c triangulations like T_3 , and with at least one noncontractable 2-cycle. Then

$$G_3 = G_4 + \bar{P}_3.$$

If we can obtain G_4 , we obtain $\bar{P}_3$. In order to find G_4 , we discuss four different kinds of simple planar near triangulations.

Let W_1 , W_2 , W_3 and W_4 be the generating functions of the number of rooted simple planar near triangulations on Figure 7.11(1)–(4) respectively. The dot line and dot path indicate that edge or that path are not present. We have the following result:

Lemma 7.5 *The generating functions W_i ($1 \leq i \leq 4$) are given by*

$$W_1 = x^4 A_3^2, \quad (7.15)$$

$$W_2 = \frac{(A_4 - A_3^2)x^4}{1 + (A_4 - A_3^2)x}, \quad (7.16)$$

$$W_3 = (A_4 - 2A_3^2)x^4 - 2(A_4 - A_3^2)xW_2 + A_3^4x^5. \quad (7.17)$$

$$W_4 = x^3(2x \frac{dA_3}{dx} + A_3). \quad (7.18)$$

Proof: Equation (7.15) is obvious. For W_2 , if we ignore the forbidden path $v_i v_k v'_i$, we get contribution $(A_4 - A_3^2)x^4$. Using the similar argument as we obtain R_{42} in (7.4) or S_{42} in (7.7), we obtain

$$W_2 = (A_4 - A_3^2)x^4 - xW_2(A_4 - A_3^2).$$

Hence we have (7.16).

We use the principle of inclusion and exclusion to obtain W_3 . The contribution of near triangulations in which neither edge $v_i v_{i'}$ nor edge $v_j v_{j'}$ appears is $(A_4 - 2A_3^2)x^4$. From that, we need subtract the contribution of the near triangulations which have path $v_i v_k v_{i'}$ or path $v_j v_l v_{j'}$, which is counted by

$$2x^{-3}(A_4 - A_3^2)x^4 W_2 = 2x(A_4 - A_3^2)W_2.$$

Since path $v_i v_k v_{i'}$ and path $v_j v_l v_{j'}$ can happen at the same time (i.e. $v_k = v_l$. In this case the quadrangle is separated into four triangles), we need add the contribution from this situation, which is $A_3^4 x^5$. Hence we have (7.17).

W_4 is the generating function of the rooted planar triangulations with a distinguished interior face. Hence

$$W_4 = (f - 1) \sum_n a_{3,n} x^{n+3} = (2n + 1) \sum_n a_{3,n} x^{n+3} = x^3(2x \frac{dA_3}{dx} + A_3).$$

Figure 7.12: There is only one noncontractable 2-cycle in T_3 .

Notice $f - 1 = 2n + 1$, which is obtained by $n + 3 - e + f = 2$ and $3f = 2e$. $\square$

Now we go back to G_4 . We need to distinguish three different cases:

Case A: *There is only one noncontractable 2-cycle in triangulations like T_3 .*

Let G_{4a} be the contribution of this case. Let $v_i v_j v_i$ be the unique noncontractable 2-cycle in T_3 . Cutting along this 2-cycle, we obtain an “almost” rooted simple triangulation T_a which has four forbidden situations. See Figure 7.12. We say it is “almost” because the exterior face has valence four. Notice the root face of T_a is not the exterior face. If we set another rooting on the exterior face, T_a becomes a double rooted simple planar near triangulation. It can be seen as a near triangulation in Figure 7.11(3), with a distinguished interior face and a distinguished edge on that face. Hence

$$G_{4a} = 3(f - 1)x^{-2}W_3/4.$$

From $n - e + f = 2$ and $3(f - 1) + 4 = 2e$, we have $f - 1 = 2n - 6$. Let $W_3 = \sum_{n \geq 0} w_{3,n}x^n$. Then

$$G_{4a} = \frac{3}{4}x^{-2}\left(\sum_{n \geq 0} w_{3,n}x^n(2n - 6)\right) = \frac{3}{2}x^{-2}\left(x\frac{dW_3}{dx} - 3W_3\right). \quad (7.19)$$

Figure 7.13: There are at least two noncontractable 2-cycles in T_3 , all of them share a common vertex v_i .

Case B: *There are at least two noncontractable 2-cycles in triangulations like T_3 , all of them share a common vertex v_i .*

Let G_{4b} be the contribution of this case. We assume all these 2-cycles are between the 2-cycle $v_i v_j v_i$ and the 2-cycle $v_i v_k v_i$. Cutting along these two 2-cycles (see Figure 7.13), we can get the following relation:

$$G_{4b} = 2x^{-5} \left(\frac{3}{2} \left(x \frac{dW_2}{dx} - 3W_2 \right) (A_4 - A_3^2) x^4 - \frac{3}{2} \left(x \frac{dW_1}{dx} - 3W_1 \right) W_1 \right). \quad (7.20)$$

Here $\frac{3}{2} \left(x \frac{dW_2}{dx} - 3W_2 \right)$ and $\frac{3}{2} \left(x \frac{dW_1}{dx} - 3W_1 \right)$ are the contributions of Figure 7.13(c)

Figure 7.14: There are exactly three noncontractable 2-cycles in T_3 . They separate the Projective Plane into four disk regions.

and Figure 7.13(e), respectively. They are obtained using the same method as we obtain G_{4a} . While $(A_4 - A_3^2)x^4$ is corresponding to Figure 7.13(d) and W_1 is corresponding to Figure 7.13(f). We need to multiple by 2 since the root face in the near triangulation of Figure 7.13(b) can be in either side of 2-cycle $v_i v_k v_i$.

Case C: *There are exactly three noncontractable 2-cycles in triangulations like T_3 . These three 2-cycles share three vertices and separate the Projective Plane into four disk regions. See Figure 7.14.*

Let G_{4c} be the contribution in this case. Then

$$G_{4c} = W_4 A_3^3. \quad (7.21)$$

Combining the contributions from all three cases, we obtain the parametric expression of G_4 .

$$G_4 = G_{4a} + G_{4b} + G_{4c}.$$

Here G_{4a} , G_{4b} and G_{4c} are given in equations (7.19), (7.20) and (7.21), respectively. Hence we obtain a parametric expression of G_4 . Notice G_3 is given by (7.14). Hence we obtain a parametric expression of $\bar{P}_3$.

$$\bar{P}_3 = G_3 - G_4.$$

Using MAPLE software, we obtain the parametric expression of $\bar{P}_3 = \bar{P}_3(x)x^3$, which is given in Theorem 7.1.

Chapter 8

Counting Simple Catalan Triangulations on the Torus

8.1 Introduction

We have enumerated rooted 2-c triangulations on the Torus and on the Klein Bottle. We have also enumerated rooted simple (or 3-c) triangulations on the Projective Plane. The next step is to enumerate rooted simple (or 3-c) triangulations on the Torus. We can write down functional equations for the generating function of the number of rooted simple near triangulations on the Torus, but the expressions are so complicated that we cannot solve these equations using MAPLE. However, we can enumerate the simple Catalan triangulations on the Torus. Notice Edelman and Reiner [16] obtained the exact number of simple Catalan triangulations on the Projective Plane (or equivalently, the Möbius Band). In this chapter, we obtain the generating function of the number of simple Catalan triangulations on the Torus.

Let $\mathcal{A}_n^c$, $\mathcal{P}_n^c$ and $\mathcal{T}_n^c$ be the set of simple Catalan triangulations on the Sphere, the Projective Plane and the Torus with n vertices, respectively. Let a_n^c , p_n^c and t_n^c be the cardinality of $\mathcal{A}_n^c$, $\mathcal{P}_n^c$ and $\mathcal{T}_n^c$, respectively. It is well known that a_n is the Catalan number $\frac{1}{n-1} \binom{2n-4}{n-2}$ (we use the convention $a_2^c = 1$). Hence

Lemma 8.1 *The generating function $A^c(x) = \sum_{n \geq 2} a_n^c x^n$ is given by*

$$\begin{aligned} A^c(x) &= \frac{2x^2}{1 + \sqrt{1 - 4x}} \\ &= x^2 + x^3 + 2x^4 + 5x^5 + \dots \end{aligned} \quad (8.1)$$

The following result is from [16].

Lemma 8.2 *The generating function $P^c(x) = \sum_{n \geq 5} p_n^c x^n$ is given by*

$$\begin{aligned} P^c(x) &= \frac{x^2((2 - 5x - 4x^2) + \sqrt{1 - 4x}(-2 + x + 2x^2))}{(1 - 4x)(1 - 4x + 2x^2 + \sqrt{1 - 4x}(1 - 2x))} \\ &= x^5 + 14x^6 + 113x^7 + \dots \end{aligned} \quad (8.2)$$

In this chapter, we prove

Theorem 8.3 *The generating function $T^c(x) = \sum_{n \geq 6} t_n^c x^n$ is given by*

$$\begin{aligned} T^c(x) &= \frac{40 - 560x + 2900x^2 - 6512x^3 + 4936x^4 + 1664x^5 - 1640x^6 + 176x^7 - 960x^8}{x^4(1 - 4x)^2(1 + \sqrt{1 - 4x})^2} \\ &\quad - \frac{4\sqrt{1 - 4x}(10 - 120x + 505x^2 - 818x^3 + 228x^4 + 336x^5 - 24x^6 + 100x^7 - 48x^8)}{x^4(1 - 4x)^2(1 + \sqrt{1 - 4x})^2} \\ &= 2x^6 + 68x^7 + 1070x^8 + 11060x^9 + 89740x^{10} + \dots \end{aligned} \quad (8.3)$$

Edelman and Reiner [16] show that simple Catalan triangulations on the Projective Plane are connected under diagonal flips. Here we show that the simple Catalan triangulations on the Torus are not connected under diagonal flips.

8.2 Two Special Kind of Planar Simple Catalan Triangulations

We consider the following two special kinds of planar simple Catalan triangulations.

The first special kind of planar simple Catalan triangulation is defined as follows: It is the set of all simple Catalan triangulations on the disk in which two vertices, root vertex v'_1 and the other vertex v''_1 , can neither share an edge nor both adjacent to a common vertex. See Figure 8.1(a). It is the same figure as Figure 7.1(a) but here we only consider simple Catalan triangulations. Let $\mathcal{R}^c$ denote this set.

The second special kind of simple Catalan triangulations on the disk is defined as the set of all simple Catalan triangulations on the disk in which we specify two edges on the boundary, edges $v'_2v'_1$ and edge $v''_2v''_1$, with v'_1 being the root vertex. v'_1 and v''_1 can neither share an edge nor both adjacent by an edge to a common vertex; v'_2 and v''_2 can neither share an edge nor both adjacent by an edge to a common vertex; v'_2 , v'_1 , v''_1 and v''_2 appear on the boundary of the disk anti-clockwisely in this order. See Figure 8.1(b). It is similar to Figure 7.1(b), but the order of the four vertices v'_2 , v'_1 , v''_1 and v''_2 appeared on the boundary is different, and here we consider simple Catalan triangulations. Let $\mathcal{K}^c$ denote this set.

For any $R \in \mathcal{R}^c$ and any $K \in \mathcal{K}^c$, we use two indexes, u and v , to indicate the vertices of the simple Catalan triangulations R and K . Index u indicates the vertices of the left side between v'_1 and v''_1 exclusively, index v indicates the vertices of the right side between v'_1 and v''_1 inclusively. See Figure 8.1.

Let $r_{i,j}$ and $k_{i,j}$ be the numbers of simple Catalan triangulations which have i vertices of index u and j vertices of index v in R and K , respectively. The generating

Figure 8.1: Two special kinds of planar simple Catalan triangulations.

functions of these numbers are defined by

$$R(u, v) = \sum_{i,j} r_{i,j} u^i v^j = \sum_i R_i(v) u^i,$$

$$K(u, v) = \sum_{i,j} k_{i,j} u^i v^j = \sum_i K_i(v) u^i.$$

First let us consider $R(u, v)$. Notice in Chapter 7, we have obtained $R(x, u, v)$. It is clear that

$$R(u, v) = R(x, u, v)|_{x=0}. \quad (8.4)$$

Now we consider $K(u, v)$.

A simple Catalan triangulation in $\mathcal{K}^c$ is also a simple Catalan triangulation in $\mathcal{R}^c$ with some restrictions. The restrictions which can make a simple Catalan triangulation in $\mathcal{R}^c$ to be a simple Catalan triangulation in $\mathcal{K}^c$ is: v_2' and v_2'' can neither share an edge nor both adjacent to a common vertex by an edge, and there are at least three vertices of index u in the simple Catalan triangulation.

Figure 8.2: The second special kinds of planar simple Catalan triangulations: the three forbidden situations.

Let us make it more clear. Let T be a simple Catalan triangulation in $\mathcal{R}^c$ with two edges, $v_2'v_1'$ and $v_1''v_2''$, in root face. T is also a simple Catalan triangulation in $\mathcal{K}^c$ if the following three situations do not appear:

- (1) There is an edge $v_2'v_2''$.
- (2) There is a path $v_2'v_3v_2''$ of length two and v_3 is an exterior vertex of index u .
- (3) There is a path $v_2'v_3v_2''$ of length two and v_3 is an exterior vertex of index v .

To obtain $K(u, v)$, we need to subtract the contribution of these three forbidden situations from K_0 , while K_0 is given by

$$K_0 = R(u, v) - [u^2]R(u, v)u^2.$$

Here we notice that any element of $\mathcal{K}^c$ has at least three index u vertices. Let K_i ($i = 1, 2, 3$) be the contributions of the case that the i th forbidden situation appears, and for any $j < i$, the j th forbidden situation does not appear. We can obtain K_1 ,

K_2 and K_3 as follows. See Figure 8.2.

$$\begin{aligned} K_1 &= (A(u) - u^2)[u^2]R(u, v) \\ K_2 &= u^{-4}A(u)^2([u^3]R(u, v)u^3 - u[u^2]B(u, v)u^2) \\ K_3 &= u^{-1}(A(u) - u^2 - uA(u) - A(u)^2)((A(v) - v^2)^2v^{-3} - A(v)^2v^{-1}) \end{aligned}$$

From

$$K(u, v) = K_0 - K_1 - K_2 - K_3, \quad (8.5)$$

we obtain the expression of $K(u, v)$.

8.3 Simple Catalan Triangulations on the Cylinder

Let $c_{i,j}$ be the number of simple Catalan triangulations on the Cylinder, having i root-boundary vertices and j nonroot-boundary vertices. We use index u to indicate the root-boundary vertices and index v to indicate the nonroot-boundary vertices. Let

$$C(u, v) = \sum_{i \geq 3, j \geq 3} c_{i,j} u^i v^j = \sum_{i \geq 3} C_i(v) u^i,$$

The following lemma gives the functional equation for $C(u, v)$.

Lemma 8.4 *The generating function $C(u, v)$ satisfies the following equation:*

$$C(u, v) = 2u^{-1}A(u)C(u, v) + v^{-1}(R(u, v) - A(u)[u^2]R(u, v)). \quad (8.6)$$

Here $R(u, v)$ is given in (8.4), $[u^2]R(u, v)$ is the coefficient of u^2 in $R(u, v)$.

Figure 8.3: A simple Catalan triangulation on the Cylinder: v_3 is a root-boundary vertex.

Proof: Let v_1v_2 be the root edge of any simple Catalan triangulation T on the Cylinder. v_1v_2 lies on a nonroot triangular face $v_1v_2v_3$. We decompose the simple Catalan triangulations on the Cylinder. According to the position of vertex v_3 , we have two cases:

Case A: v_3 is a root boundary vertex. See Figure 8.3.

The decomposition creates two maps, one is a simple Catalan triangulation on the disk and the other is a simple Catalan triangulation on the cylinder. The contribution in this case is:

$$u^{-1}(A(u)C(u, v) + C(u, v)A(u)) = 2u^{-1}A(u)C(u, v).$$

Case B: v_3 is a nonroot-boundary vertex. See Figure 8.4.

In this case the contribution is:

$$v^{-1}(R(u, v) - A(u)[u^2]R(u, v)).$$

This is explained as follows: After the decomposition, we obtain a first special kind of planar simple Catalan triangulation. Since the set of simple Catalan triangula-

Figure 8.4: A simple Catalan triangulation on the Cylinder: v_3 is a nonroot-boundary vertex.

tions counted by $R(u, v)$ includes those triangulations which have edge v_1v_2 , that will introduce multiple edges in the original simple Catalan triangulations on the Cylinder, so we need to subtract the contribution of those simple Catalan triangulations having edge v_1v_2 from $R(u, v)$. That is counted by $A(u)([u^2]R(u, v))$.

Lemma 8.4 follows by combining all these cases and subcases. $\square$

From Lemma 8.4, we obtain

$$C(u, v) = \frac{R(u, v) - [u^2]R(u, v)}{v - 2A(u)v/u}. \quad (8.7)$$

8.4 Two Special Kinds of Simple Catalan Triangulations on the Cylinder

In this section, we consider two special kinds of simple Catalan triangulations on the Cylinder.

Figure 8.5: Two special kinds of simple Catalan triangulations on the Cylinder.

The first special kind of simple Catalan triangulations is the set of the simple Catalan triangulations on the Cylinder in which two vertices, root-boundary vertex v_3' and nonroot-boundary vertex v_3'' , cannot share an edge, neither can they connect to a common vertex. Moreover, v_1 is the root vertex; v_1v_3' is the root edge; v_2v_3'' is an edge on the nonroot-boundary. See Figure 8.5(a). We have considered this situation in Chapter 6, but here we deal with simple Catalan triangulations.

The second special kind of simple Catalan triangulations is the set of those elements in the first special kind which have an extra condition: v_1 and v_2 are connected by an edge. See Figure 8.5(b).

Let $e_{i,j}$ and $d_{i,j}$ be the number of the first and the second special kind of simple Catalan triangulations on the Cylinder with i vertices of index u and j vertices of index v , respectively. Let $E(u, v)$ be the generating function of the number of the first special kind of simple Catalan triangulations on the Cylinder. Let $D(u, v)$ be the generating function of the number of the second special kind of simple Catalan

Figure 8.6: Case 1 of the enumeration of $D(u, v)$.

Figure 8.7: Case 2 of the enumeration of $D(u, v)$.

Figure 8.8: Case A of the enumeration of $E(u, v)$.

Let the contribution in this case be D_2 . Then

$$\begin{aligned}
 D_2 = & v^{-3}(A(v) - v^2 - vA(v) - v^2A(v))\left(\sum_{m \geq 4} a_m(u^2v^{m-2} + \dots + u^{m-2}v^2)\right) \\
 & - v^{-2}A(u)(A(v) - v^2 - v^3) - 2u^{-1}v^{-1}(A(u) - u^2 - uA(u))(A(v) - v^2)) \\
 & - u^{-2}v^{-2}A(v)(A(v) - v^2 - vA(v))(A(u) - u^2 - u^3 - 2u(A(u) - u^2)).
 \end{aligned}$$

Noticing

$$\sum_{m \geq 4} a_m(u^2v^{m-2} + \dots + u^{m-2}v^2) = \frac{v^2A(u)}{u(u-v)} - \frac{u^2A(v)}{v(u-v)} + uv, \quad (8.8)$$

we obtain the expression for $D(u, v)$.

$$D(u, v) = D_1 + D_2. \quad (8.9)$$

Now we consider $E(u, v)$. After decomposing simple Catalan triangulations of the first kind on the Cylinder, according to the position of vertex v_4 , we have the following cases and subcases:

Case A: v_4 is on the root-boundary and edge $v_3'v_4$ separates v_1 with v_2 .

The decomposition creates two maps; one is a planar simple Catalan triangulation and the other is still a first special kind of simple Catalan triangulation on the

and let F_b be the contribution of the simple Catalan triangulations which have the edge b but do not have the edge a . Then

$$F(u, v) = F_0 - F_a - F_b$$

with

$$F_0 = \sum_{m,n} n C_{m,n} u^m v^n = v C_v(u, v).$$

F_a can be enumerated easily. See Figure 8.10(1). That is

$$F_a = u^{-1} v^{-1} K(u, v)$$

with $K(u, v)$ given by equation (8.5) of the previous sections. F_b is a little complicated. Let F_{b1} and F_{b2} be the contributions of simple Catalan triangulations with one of the two possible edges $v_1 v_3''$. See Figure 8.10(2). Then

$$F_b = u^{-1} v^{-1} (K(u, v) - F_{b1} - F_{b2})$$

with

$$F_{b1} = (A(u) - u^2 - uA(u))[u^2]R(u, v).$$

$$F_{b2} = u(R(u, v) - A(u)[u^2]R(u, v)).$$

Here $R(u, v)$ is given by equation (8.4). Hence we have F_b and $F(u, v)$. Thus we obtain E_2 .

Case C: v_4 is on the nonroot-boundary.

Let the contribution in this case be E_3 . We consider the triangular face which incident with the edge $v_3' v_4$ other than face $\Delta v_1 v_4 v_3'$, say it is $\Delta v_3' v_4 v_5$. According to the position of v_5 , we have the following subcases:

Figure 8.10: Case B of the enumeration of $E(u, v)$: Enumerating F_a and F_b .

Figure 8.11: Case C of the enumeration of $E(u, v)$: Subcase C_1 .

Subcase C_1 : v_5 is on root-boundary. See Figure 8.11.

Let the contribution from this subcase be E_{31} . Then

$$E_{31} = u^{-1}v^{-1}A(u)E_{3a} \quad (8.10)$$

where E_{3a} is the contribution of the first special kind of planar simple Catalan triangulations with a distinguished vertex of index v and four forbidden edges: e_1 , e_2 , e_3 and e_4 . Let the contribution from the case of ignoring the forbidden edges be E_{a0} and the contribution from the case having edge e_i but not having edge e_j be E_{ai} ($1 \leq i \leq 4$, $j < i$). Then

$$E_{3a} = E_{a0} - E_{a1} - E_{a2} - E_{a3} - E_{a4}. \quad (8.11)$$

These E_{ai} ($0 \leq i \leq 4$) are enumerated as follows:

$$E_{a0} = \sum_{m \geq 4} r_{n,m} u^n v^m (m-3) = vR_v(u, v) - 3R(u, v).$$

$$E_{a1} = v^{-2}A(v)R(u, v).$$

$$E_{a2} = v^{-2}(A(v) - v^2)R(u, v)$$

$$E_{a3} = v^{-2}(A(v) - v^2 - vA(v))\left(\sum_{m \geq 5} a_m(u^2v^{m-2} + \dots + u^{m-3}v^3)\right) \\ - u^{-1}v^{-1}(A(u) - u^2)(A(v) - v^2 - v^3) - u^{-2}(A(v) - v^2)(A(u) - u^2 - uA(u)).$$

$$E_{a4} = v^{-2}(A(v) - v^2 - vA(v))\left(\sum_{m \geq 4} a_m(u^2v^{m-2} + \dots + u^{m-2}v^2)\right) \\ - u^{-1}v^{-1}(A(u) - u^2)(A(v) - v^2) - u^{-2}A(v)(A(u) - u^2 - uA(u)) \\ - u^{-1}v^{-1}(A(u) - u^2)(A(v) - v^2 - vA(v)).$$

Using

$$\sum_{m \geq 5} a_m(u^2v^{m-2} + \dots + u^{m-3}v^3) = \frac{v^3A(u)}{u^2(u-v)} - \frac{u^2A(v)}{v(u-v)} + v(u+v) + uv^2 \quad (8.12)$$

Figure 8.12: Case C of the enumeration of $E(u, v)$: Subcase C_2 .

and (8.8), we obtain the expression of E_{3a} . From (8.10), we obtain E_{31} .

Subcase C_2 : v_5 is on nonroot-boundary and it is between v_4 and v_2 anticlockwisely. See Figure 8.12.

Let the contribution from this subcase to be E_{32} . Then

$$E_{32} = v^{-2} A(v) E_{3b}, \quad (8.13)$$

where E_{3b} is the contribution of the simple Catalan triangulations given in Figure 8.12, which have five forbidden edges: l_1, l_2, l_3, l_4 and l_5 . Let the contribution from the case which ignores the forbidden edges be E_{b0} and the contribution from the case that the triangulation has edge l_i but does not have edge l_j be E_{bi} ($1 \leq i \leq 5$,

$j < i$). Then

$$E_{3b} = E_{b0} - E_{b1} - E_{b2} - E_{b3} - E_{b4} - E_{b5}.$$

These E_{bi} ($0 \leq i \leq 5$) are enumerated as follows:

$$E_{b0} = uv^{-1} \sum_{n,m} r_{n,m} v^m (uv^{n-1} + u^2 v^{n-2} + \cdots + u^{n-1} v) = \frac{uR(u,v)}{u-v} - \frac{u^2 R(v,v)}{(u-v)v}$$

$$E_{b1} = v^{-2} A(v) R(u, v)$$

$$E_{b2} = uv^{-3} (A(v) - v^2 - vA(v)) R(u, v)$$

$$E_{b3} = u^{-2} (A(u) - u^2 - u^3) \left(\sum_{m \geq 4} a_m v^m (m-3) - v^{-2} A(v) (A(v) - v^2 - v^3) \right. \\ \left. - v^{-2} (A(v) - v^2) (A(v) - v^2 - vA(v)) \right)$$

$$E_{b4} = u^{-1} v^{-1} (A(u) - u^2) \left(R(v, v) - v \left(\sum_{m \geq 4} a_m v^m (m-3) - v^{-2} A(v) (A(v) - v^2 - v^3) \right) \right. \\ \left. - v^{-2} (A(v) - v^2) (A(v) - v^2 - vA(v)) \right)$$

$$E_{b5} = v^{-2} A(u) (R(v, v) - A(v)[u^2] R(u, v) - v R(v, v) - v^2 \left(\sum_{m \geq 4} a_m v^m (m-3) \right. \\ \left. - v^{-2} (A(v) - v^2)^2 - v^{-2} (A(v) - v^2 - vA(v)) \right)$$

Using

$$\sum_{m \geq 4} a_m v^m (m-3) = vA_v(v) - 3A(v) + v^2,$$

We obtain the expression of E_{3b} and from (8.13), we obtain E_{32} .

Subcase C_3 : v_5 is on nonroot-boundary and it is between v_3'' and v_4 anticlockwisely. See Figure 8.13.

Let the contribution from this subcase be E_{33} . Then

$$E_{33} = v^{-2} \left(\sum_{m \geq 3} a_m v^m (m-2) - v^{-2} A(v) (A(v) - v^2) - v^{-2} A(v) (A(v) - v^2 - vA(v)) \right. \\ \left(\sum_{m \geq 4} a_m (u^2 v^{m-2} + \cdots + u^{m-2} v^2) - u^{-2} A(v) (A(u) - u^2 - u^3) - v^{-2} A(u) \right. \\ \left. (A(v) - v^2 - v^3 - 2v^2 A(v)) - u^{-1} v^{-1} (A(u) - u^2 - uA(u)) (A(v) - v^2 - vA(v)) \right).$$

 Figure 8.13: Case C of the enumeration of $E(u, v)$: Subcase C_3 .

Using

$$\sum_{m \geq 3} a_m v^m (m-2) = v A_v(v) - 2A(v)$$

and (8.8), we obtain the expression of E_{33} .

We have obtained E_{31} , E_{32} and E_{33} . The total contribution of case C is

$$E_3 = E_{31} + E_{32} + E_{33}.$$

Combining all these three cases, we have

$$E(u, v) = E_1 + E_2 + E_3 = u^{-1} A(u) E(u, v) + E_2 + E_3.$$

Hence

$$E(u, v) = \frac{E_2 + E_3}{1 - A(u)/u}. \quad (8.14)$$

We have obtained the expressions of $E(u, v)$ and $D(u, v)$.

8.5 Enumerating Simple Catalan Triangulations on the Torus

In this section we find the expression for $T^c(x)$ and prove Theorem 8.3.

Lemma 8.5 *The generating function $T^c(x)$ satisfies the follow equation:*

$$T^c(x) = 2x^{-1}A(x)T^c(x) + x^{-1}(E(x, x) - D(x, x)), \quad (8.15)$$

where $E(x, x) = E(u, v)|_{u=x, v=x}$ and $D(x, x) = D(u, v)|_{u=x, v=x}$. $E(u, v)$ and $D(u, v)$ are given by (8.14) and (8.9) in the previous section.

Proof: Let v_1v_2 be the root edge of the simple Catalan triangulation T on the Torus. v_1v_2 lies on a nonroot triangular face $v_1v_2v_3$. We decompose all simple Catalan triangulations according to the position of v_3 . There are two cases:

Case A: *The decomposition creates two maps. See Figure 8.14.*

The contribution in this case is:

$$T_1 = x^{-1}A(x)T^c(x) + x^{-1}T^c(x)A(x) = 2x^{-1}A(x)T^c(x).$$

Case B: *The decomposition creates a Cylinder.*

There are two vertices v'_3 and v''_3 in this Cylinder which cannot share an edge, neither can they both connect to the same vertex. v'_3 is on the root-boundary and v''_3 is on the nonroot-boundary. We have obtained the generating function $E(u, v)$ and $D(u, v)$ in the previous section. It is clear that the contribution from this case is

$$T_2 = (E(u, v) - D(u, v))|_{u=v=x} = E(x, x) - D(x, x).$$

Figure 8.14: A Simple Catalan Triangulation on the Torus: the decomposition creates two maps.

Figure 8.15: A Simple Catalan Triangulation on the Torus: the decomposition creates a Cylinder.

Notice here we do not need to distinguish the root-boundary vertex and the nonroot-boundary vertex. See Figure 8.15.

Lemma 8.5 follows by combining all these cases and subcases. $\square$

From equation 8.15, we obtain the expression of $T^c(x)$.

$$T^c(x) = \frac{E(x, x) - D(x, x)}{x - A(x)}. \quad (8.16)$$

Figure 8.16: Two Simple Catalan Triangulations on the Torus with 6 vertices.

Using MAPLE software, we obtain the expression of $T^c(x)$ as given in Theorem 8.3. The first few terms of its Taylor series are also given in Theorem 8.3.

Since $t_6^c = 2$, there are two different simple Catalan triangulations with 6 vertices on the Torus. Notice we need at least 6 vertices on the Torus to construct a simple Catalan triangulation. We draw these two minimal simple Catalan triangulations on the Torus in Figure 8.16. It is clear these two minimal simple Catalan triangulations are not connected by diagonal flips. In fact, any diagonal flip on them will result multiple edges. Hence simple Catalan triangulations on the Torus cannot be connected by diagonal flips, although, as proven in [16], the corresponding result on the Projective Plane is true.

If we insert a vertex v inside the root face of each of these two minimal simple Catalan triangulations, and connect v with all vertices, then we obtain two different rooted simple triangulations on the Torus with 7 vertices. It is clear they are not connected by diagonal flips. Hence the diagonal flip adjacency graph of rooted simple triangulations on the Torus with 7 vertices is not connected.

Bibliography

- [1] B. Aronov, R. Seidel and D. Souvaine, On compatible triangulations of simple polygons, *Computational Geometry: Theory and Application* 3 (1993), 27–35.
- [2] D. Barnette, Generating the triangulations of the projective plane, *J. Combin. Theory Ser. B* 54 (1992), 37–63.
- [3] E. A. Bender, Asymptotic methods in enumeration, *SIAM Review*, 16 (1974), 485–515.
- [4] E. A. Bender and E. R. Canfield, The asymptotic number of rooted maps on a surface, *J. Combin. Theory Ser. A* 43 (1986), 244–257.
- [5] E. A. Bender and L. B. Richmond, A survey of the asymptotic behaviour of maps, *J. Combin. Theory Ser. B* 40 (1986), 297–329.
- [6] E. A. Bender, E. R. Canfield and R. W. Robinson, The enumeration of maps on the torus and the projective plane, *Canad. Math. Bull.* 31 (1988), 257–271.
- [7] E. A. Bender and N. C. Wormald, The asymptotic number of rooted non-separable maps on a surface, *J. Combin. Theory Ser. A* 49 (1988), 370–380.

- [8] E. A. Bender, Z. C. Gao, L. R. Richmond and N. C. Wormald, Asymptotic properties of rooted 3-connected maps on surfaces, *J. Austral. Math. Soc. (Series A)* 60 (1996), 31–41.
- [9] W. G. Brown, Enumeration of triangulations of the disk. *Proc. Lond. Math. Soc.*(3) 14 (1964), 746–768.
- [10] W. G. Brown, An algebraic technique for solving certain problems in the theory of graphs, in “Theory of Graphs: Proc. Colloquium, Tihany, Hungary, 1966”(P. Erdos and G. Katona, Eds), Academic Press, New York, 1968, 57–60.
- [11] W. G. Brown, On the enumeration of non-planar maps. *Mem. Amer. Math. Soc.* 65 (1966).
- [12] R. Brunet, A. Nakamoto and S. Negami, Diagonal flips of triangulations on closed surfaces preserving specified properties, *J. Combin. Theory ser. B* 68 (1996) no.2, 295–309.
- [13] B. Chazelle, Triangulating a simple polygon in linear time. *Discrete and Computational Geometry* 6 (1991) 485–524.
- [14] A. K. Dewdney, Wagner’s theorem for the torus graph, *Discrete Math.* 4 (1973), 139–149.
- [15] H. Dorrie, 100 Great Problems of Elementary Mathematics. Dover, 1965.
- [16] P. H. Edelman and V. Reiner, Catalan triangulations of the Mobius band, *Graphs and Combinatorics*, 13 (1997), no.3, 231–243.

- [17] Z. C. Gao, The number of rooted triangular maps on a surface, *J. Combin. Theory Ser. B* 52 (1991), 236–249.
- [18] Z. C. Gao, The number of rooted 2-connected triangular maps on the projective plane, *J. Combin. Theory Ser. B* 53 (1991), 130–142.
- [19] Z. C. Gao, Lecture notes on map enumeration. Dept. of Combinatorics and Optimization, Univ. of Waterloo, 1989.
- [20] Z. C. Gao, The asymptotic number of rooted 2-connected triangular maps on a surface, *J. Combin. Theory Ser. B* 54 (1992), 102–112.
- [21] Z. C. Gao, A pattern for the asymptotic number of rooted maps on surfaces, *J. Combin. Theory ser. A* 64 (1993), 246–264.
- [22] Z. C. Gao, J. Urrutia and J. Wang, Diagonal flips in labelled planar triangulations. Preprint, 1997.
- [23] H. W. Gould, Research bibliography on two special number sequences, *Mathematica Monongaliae* 12 (1971).
- [24] I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, John Wiley & Sons, Inc. 1983.
- [25] L. Guibas, J. Hershberger, D. Levem, M. Sharir and R. Tarjan, Algorithms for visibility and shortest path problems inside triangulated simple polygons, *Algorithmica*, 2 (1987), 200–237.
- [26] A. Heck, *Introduction to maple*, second edition, Springer-Verlag New York, Inc. 1996.

- [27] F. Hurtado, M. Noy and J. Urrutia, Flipping edges on triangulation. Proceeding 12th ACM Symposium on Computational Geometry, May 24–26, 1996. 214–223.
- [28] H. Komuro, The diagonal flips of triangulations on the sphere. *Yokohama Math. J.* 44 (1997), 115–122.
- [29] H. Komuro and S. Negami, Diagonal flips in triangulations on the sphere with minimum degree at least 4, preprint.
- [30] M. J. Lai and L. L. Schumaker, Scattered data interpolation using C^2 piecewise polynomials of degree six, Third workshop on Proximity Graphs. Mississippi State University, 1994.
- [31] S. Lawrencenko, S. Negami and A. T. White, Three nonisomorphic triangulations of an orientable surface with the same complete graph, *Discrete Math.* 135 (1994) 367–369.
- [32] R. C. Mullin, Enumeration of rooted triangular maps. *Amer. Math. Monthly*, 71 (1964), 1007–1010.
- [33] R. C. Mullin, On counting rooted triangular maps. *Canad. J. Math.* 17 (1965), 373–382.
- [34] J. W. Moon and L. Moser, Triangular dissections of n -gons, *Canad. Math. Bull.* 6 (1963), 175–178.
- [35] A. Nakamoto and S. Negami, Diagonal flips in triangulation on closed surfaces with minimum degree at least 4, preprint.

- [36] A. Nakamoto and K. Ota, Note on irreducible triangulations of surfaces, *J. Graph Theory* 20 (1995), 227–233.
- [37] S. Negami and S. Watanabe, Diagonal transformations of triangulations on surfaces, *Tsukuba J. Math* 14 (1990), 155–166.
- [38] S. Negami, Diagonal flips in triangulations of surfaces, *Discrete Mathematics* 135 (1994), 225–232.
- [39] S. Negami and M. Nakamoto, Diagonal transformations of graphs on closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec I, No. 40*, 1993.
- [40] A. Okabe, B. Boots and K. Sugihara, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*. John Wiley and Sons, 1992.
- [41] O. Ore, *The four-color problem*, Academic Press, New York, 1967
- [42] L. B. Richmond and N. C. Wormald, Random triangulations of the plane, *European J. Combin.* 9 (1988), 61–71.
- [43] L. B. Richmond and N. C. Wormald, Almost all maps are asymmetric, *J. Combin. Theory ser B* 63 (1995), 1–7.
- [44] A. Saalfeld, Joint triangulations and triangulated maps, *Proc. 3rd Annual ACM Symposium in Computational Geometry*, 1987, 195–204.
- [45] M. Shapira and A. Rappoport, Shape blending using the star-skeleton representation, to appear in *IEEE Computer Graphics and Applications*.

- [46] D. Sleator, R. Tarjan and W. Thurston, Rotation distance, triangulations, and hyperbolic geometry, *J. Amer. Math. Soc.* 1 (1988), 647–681.
- [47] W. T. Tutte, The enumerative theory of planar maps. in “A Survey of Combinatorial Theory” (J. N. Srivastava, Ed), North-Holland. Amsterdam. 1973. 437–448.
- [48] W. T. Tutte, A census of planar triangulations. *Canad. J. Math.* 14 (1962), 21–38.
- [49] W. T. Tutte, A census of planar maps. *Canad. J. Math.* 15 (1963), 249–271.
- [50] W. T. Tutte, On the enumeration of convex polyhedra, *J. Combin. Theory ser B* 28 (1980), 105–126.
- [51] W. T. Tutte, A census of slicing, *Canad. J. Math.* 14 (1962), 708–722.
- [52] W. T. Tutte, What is map? in “New Directions in the Theory of Graphs” (F. Harary Ed.). Academic Press, New York, 1973, 309–325.
- [53] T. I. Visentin, A character theoretic approach to the study of combinatorial properties of maps in orientable surfaces, Ph. D Thesis, University of Waterloo, 1989.
- [54] K. Wagner, Bemerkungem zum Vierfarbenproblem, *Jber. Deutsch. Math-Verein* 46 (1936), 126–132
- [55] T. R. S. Walsh and A. B. Lehman, Counting rooted maps by genus, I, *J. Combin. Theory Ser. B* 13 (1972), 122–141.

- [56] T. R. S. Walsh and A. B. Lehman, Counting rooted maps by genus, *J. Combin. Theory Ser. B* 13 (1972), 192–218.
- [57] D. F. Watson and G. M. Philips, Systematic triangulations. *Computer Vision, Graphics and Image Processing* 26 (1984), 217–223.
- [58] H. S. Wilf, *Generatingfunctionology*, second edition, Academic Press, Inc. 1994.
- [59] O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method*. McGraw-Hill, 1989.

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