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SOME QUADRATIC TRANSFORMATIONS AND PROJECTION FORMULAS

FOR BASIC HYPERGEOMETRIC SERIES

by

© Bassam G. Nassrallah, B.Sc. (Hon.), M.A.

A thesis submitted to the
Faculty of Graduate Studies
and Research in partial
fulfilment of the requirements
for the degree
of
Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

September, 1982

✓
The undersigned recommend to the
Faculty of Graduate Studies and Research
acceptance of the thesis

SOME QUADRATIC TRANSFORMATIONS AND PROJECTION FORMULAS
FOR BASIC HYPERGEOMETRIC SERIES

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ABSTRACT

A number of quadratic transformation formulas for very well-poised terminating basic hypergeometric series are obtained which, when $q \rightarrow 1^-$, approach known transformation formulas for nearly-poised ordinary hypergeometric series. These transformation theorems are used to obtain some summation formulas for basic hypergeometric series analogous to ordinary hypergeometric results and to prove q -analogues of the Cayley-Orr theorems.

Also a projection formula for recently found q -Wilson polynomials is obtained which is used to construct a reproducing kernel for these polynomials. This is generalized to give a connection relation with an eigenvalue expressed as a terminating and balanced ${}_4\phi_3$ series. q -analogues of known projection and connection formulas for Jacobi polynomials are obtained as special cases of this general formula.

To my family

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ABSTRACT

A number of quadratic transformation formulas for very well-poised terminating basic hypergeometric series are obtained which, when $q \rightarrow 1^-$, approach known transformation formulas for nearly-poised ordinary hypergeometric series. These transformation theorems are used to obtain some summation formulas for basic hypergeometric series analogous to ordinary hypergeometric results and to prove q -analogues of the Cayley-Orr theorems.

Also a projection formula for recently found q -Wilson polynomials is obtained which is used to construct a reproducing kernel for these polynomials. This is generalized to give a connection relation with an eigenvalue expressed as a terminating and balanced ${}_4\phi_3$ series. q -analogues of known projection and connection formulas for Jacobi polynomials are obtained as special cases of this general formula.

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Chapter 1

Introduction and Preliminaries

1. Hypergeometric series (Euler-Gauss Function). The theory of hypergeometric functions is a very old and involved one to which many great mathematicians have contributed. In 1303, a chinese algebraist Shih-Chieh Chu [27] gave, among other things, special cases of Vandermonde's and Pfaff-Saalschutz's summation theorems, (see Needham [48, p. 138], Askey [12, pp. 59-60], Takacs [62] and Carlitz [24]). The term 'hypergeometric', presumably used first by J. Wallis [65] in 1655, describes a series beyond the ordinary geometric series. However, it was Euler and Gauss who laid the foundations of the theory of these functions.

Following Gauss' notation [35], we define the hypergeometric series by

$${}_A F_B [a_1, \dots, a_A; b_1, \dots, b_B; z] = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_A)_k}{(b_1)_k \dots (b_B)_k} \cdot \frac{z^k}{k!}, \quad (1.1.1)$$

where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} a(a+1) \dots (a+k-1) & k > 0, \\ 1 & k = 0. \end{cases} \quad (1.1.2)$$

(1.1.1) is defined provided no zeros appear in the denominator and if any of the a_i 's is a negative integer $-n$, it reduces to a polynomial of degree n .

The case $A = 2$, $B = 1$ is what was originally studied by Euler [31, 32] who defined the infinite series

$${}_2F_1[a, b; c; z] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} z^k, \quad (1.1.3)$$

and proved it to satisfy the d.e.

$$z(1-z) \frac{d^2 y}{dz^2} + [c - (1+a+b)z] \frac{dy}{dz} - aby = 0, \quad (1.1.4)$$

(see Askey [12, p. 17]). Euler's extensive study of the gamma and beta functions is well known. However, in this context, we ought to mention his famous linear transformation

$${}_2F_1[a, b; c; z] = (1-z)^{c-a-b} {}_2F_1[c-a, c-b; c; z], \quad (1.1.5)$$

which has proved to be extremely useful.

In 1812, Gauss [35] gave a full discussion of the convergence of (1.1.3) along with his famous summation theorem

$${}_2F_1[a, b; c; 1] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}, \quad (1.1.6)$$

of which Chu-Vandermonde's theorem is the special case $a = -n$.

Next, in 1836, Kummer [45] showed that (1.1.4) has, in all, twenty four solutions of the form (1.1.3) or some $(\pm z)^\alpha (1-z)^\beta$ multiple of it, covering the entire complex plane. Thomae [63], later, worked

out the relations between these solutions, (see Erdélyi [30] or Slater [60]).

The cases $A > 2$, $B > 1$, which seem to be an obvious generalization were investigated by many people whose names became commonplace in the theory of hypergeometric functions, e.g. Dougall, Saalschutz, Dixon, Watson and many others. A special mention ought to be made of F.J.W. Whipple who, along with W.N. Bailey, pioneered the study of general transformations which helped simplify the proofs of many problems. Applications of such transformations will be seen in our present work.

2. Basic hypergeometric series (q-analogues). In the works of Euler, Gauss and Jacobi another type of series, involving a base, appeared. These examples were investigated by E. Heine who, in his book in 1898 [40], presented the following concept. If we define a basic number

$$a_q = \frac{1 - q^a}{1 - q},$$

with a and q being complex, then in the limit $q \rightarrow 1$ $a_q \rightarrow a$.

With this in mind, we can define the basic hypergeometric series by

$${}_{r+1}q_{r+j} \left[\begin{matrix} a_0, \dots, a_r \\ b_1, \dots, b_{r+j} \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_0; q)_k \dots (a_r; q)_k (-1)^{jk} q^{j \binom{k}{2}}}{(b_1; q)_k \dots (b_{r+j}; q)_k} \cdot \frac{z^k}{(q; q)_k} \quad (1.2.1)$$

where $j = 0, \pm 1, \pm 2, \dots$, $\binom{k}{2}$ is the binomial coefficient and

$$(a; q)_k = \begin{cases} (1-a)(1-aq) \dots (1-aq^{k-1}) & k \geq 1, \\ 1 & k = 0. \end{cases} \quad (1.2.2)$$

The definition of the basic series in (1.2.1) is basically the same as that in Bailey [21] and Slater [60] where they take $j=0$. But Slater [60, p. 93] mentions the difficulty of expressing some results in terms of basic series notation. Adopting this new notation that difficulty vanishes. The convergence of (1.2.1) can be checked easily. The series converges for all z when $|q| < 1$, $j=1, 2, \dots$, and no zeros appear in the denominator, and for $|z| < 1$ when $j=0$. If $j < 0$ and $|q| < 1$ the series diverges for all $z \neq 0$ unless it terminates, (see Askey and Wilson [18]).

It is understood that before taking the limit $q \rightarrow 1$ in (1.2.1) one first replaces each typical parameter a by q^a . Thus, it is easily seen that when $j=0$ and $r=1$ (1.2.1) reduces to (1.1.3) as $q \rightarrow 1$. Hence we see that (1.2.1) generalizes (1.1.1) and is called the basic or q -analogue of it.

For the remainder of this thesis we are going to deal with the case $j=0$. And since we are dealing with basic hypergeometric series, the Pochhammer symbol $(a)_k$ in (1.1.2) will be denoted by $\Gamma(a+k)/\Gamma(a)$, so that we can use the notation $(a)_k$ freely to indicate (1.2.2). That is from now on we have

$$(a)_k = (a; q)_k = \begin{cases} (1-a)(1-aq)\dots(1-aq^{k-1}) & k \geq 1, \\ 1 & k = 0, \end{cases} \quad (1.2.2')$$

unless the base is different from q in which case we use the notation (1.2.2) showing the base explicitly.

By late nineteenth century, it was realized that the basic functions have a theory parallel to that of the ordinary case. By 1894, L.J. Rogers [55, 56] had already done some work on the q -ultraspherical polynomials and studied some special cases of them, (see Askey and Wilson [18], Askey and Ismail [16], Rahman [53]).

Rogers was also led to a special type of results, rediscovered later by Ramanujan, known as the Rogers-Ramanujan identities, (Rogers [55], Rogers and Ramanujan [57], Bailey [21], Slater [60], Andrews [6], Andrews and Askey [9]).

Starting the twentieth century, a long journey into the q -world was started by F.W. Jackson. Over a period of fifty years, he extended most of the known summation theorems to q -analogous,

$$\begin{aligned}
& 10^{\frac{1}{2}} 9 \left[\begin{matrix} -q^a, q\sqrt{-q^a}, -q\sqrt{-q^a}, q^a, -q^b, q^b, -q^c, q^c, -q^{1+a-w}, q^{-m} \\ \sqrt{-q^a}, -\sqrt{-q^a}, -q, q^{1+a-b}, -q^{1+a-b}, q^{1+a-c}, -q^{1+a-c}, q^w, -q^{1+a+m} \end{matrix} ; q, q^R \right] \\
& = \frac{(q^{w-a})_m (-q^{a+1})_m}{(q^w)_m (-q)_m} 5^{\frac{1}{2}} 4 \left[\begin{matrix} p^{1+a-w}, p^{a/2}, p^{(a+1)/2}, p^{1+a-b-c}, p^{-m} \\ p^{1+a-b}, p^{1+a-c}, p^{(1+a-w-m)/2}, p^{1+(a-w-m)/2} \end{matrix} ; p, p \right],
\end{aligned}$$

(1.3.5)

where

$$R = a - 2b - 2c + w + m + 2, \quad (1.3.6)$$

and $p = q^2$. In addition, we shall generalize (1.3.5) and prove further results of similar nature along with corollaries that are q -analogues of known results.

In Chapter 3, we shall demonstrate the use of (1.3.5) and prove q -analogues of the Cayley-Orr theorems, in a manner similar to Whipple's. After we completed our work we discovered that Singh [59] also gave the same q -analogues, but he used a method different from ours, (also see [72]).

4. Projection formulas. The solutions to many problems can be seen more easily by reducing such problems into equivalent simple forms. The importance of projection formulas in this area has been pointed out by many people, (see Askey [10, 11, 12], Askey and Fitch [14], Askey and Gasper [15], Gasper [33, 34] and Szego [61]). These

Jackson's results is a very general summation theorem giving the sum of a very well-poised $8\phi_7$, namely

$$8\phi_7 \left[\begin{matrix} a, q/a, -q/a, b, c, d, e, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{1+n} \end{matrix} ; q, q \right] = \frac{(aq)_n (aq/cd)_n (aq/bd)_n (aq/bc)_n}{(aq/b)_n (aq/c)_n (aq/d)_n (aq/bcd)_n}, \quad (1.2.4)$$

provided $bcd = a^2 q^{n+1}$, n a +ve. integer, (Jackson [42], or see for example Bailey [21] or Slater [60]). This is the q -analogue of Dougall's ${}_7F_6$ summation theorem [29] (see Bailey [21]). Substituting for e its value and letting $n \rightarrow \infty$, one can easily obtain the sum of a very well-poised $6\phi_5$ known as the limiting case of Jackson's theorem [42], namely

$$6\phi_5 \left[\begin{matrix} a, q/a, -q/a, b, c, d \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix} ; q, \frac{aq}{bcd} \right] = \frac{(aq)_\infty (aq/cd)_\infty (aq/bd)_\infty (aq/bc)_\infty}{(aq/b)_\infty (aq/c)_\infty (aq/d)_\infty (aq/bcd)_\infty}, \quad (1.2.5)$$

provided $|\frac{aq}{bcd}| < 1$, (Slater [60], Andrews [4], Andrews and Askey [9]).

Talking about q -analogues of hypergeometric series one wonders about the purpose of finding them. Anyone who works in the area will find it hard to deny the beauty of how the results fall in to-

gether. Recently, applications of basic functions to physical problems have been found, however, it seems that their interesting applications remain in areas like partitions theory, number theory and combinatorial identities, (see Andrews [4, 5, 6, 7], Andrews and Askey [9]).

The story of the Rogers-Ramanujan identities and their rediscovery is well known (Slater [60], Andrews [6]). Among the many different proofs of them given at the time one, due to Watson [66], was the simplest, (see Bailey [21], Slater [60]). He proved them using his famous transformation between a very well-poised ${}_8\phi_7$ and a terminating balanced ${}_4\phi_3$, namely

$${}_8\phi_7 \left[\begin{matrix} a, q/a, -q/a, & b, & c, & d, & e, & f \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix} ; q, a^2 q^2 / bcdef \right]$$

$$= \frac{(aq)_\infty (aq/ef)_\infty (aq/fd)_\infty (aq/de)_\infty}{(aq/d)_\infty (aq/e)_\infty (aq/f)_\infty (aq/def)_\infty} {}_4\phi_3 \left[\begin{matrix} aq/bc, d, e, f \\ def/a, aq/b, aq/c \end{matrix} ; q, q \right],$$

(1.2.6)

provided d, e or f is of the form q^{-n} , n a positive integer. This is the q -analogue of Whipple's transform [67,68] connecting a very-well-poised ${}_7F_6$ with a balanced ${}_4F_3$. Recently other proofs of the Rogers-Ramanujan identities have been given. Andrews and Askey [9] proved these identities plus others as a result of solving the con-

nection coefficient problem for the little q -Jacobi polynomials. Baxter [22] used a model of phase-transitions to prove them along with other q -identities.

Watson's transformation (1.2.6) is the deepest result in this area and it contains other results also. Note that an interchange of the parameters b, c, d, e leaves the ${}_8\phi_7$ unchanged while changing the ${}_4\phi_3$. In effect we use the same transformation on itself, (see Bailey [21, § 7.2], while letting $f = q^{-n}$ to obtain

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix} ; q, q \right] = \frac{(de/bc)_n (df/bc)_n}{(e)_n (f)_n} \left(\frac{bc}{d} \right)^n {}_4\phi_3 \left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, de/bc, df/bc \end{matrix} ; q, q \right], \quad (1.2.7)$$

(see Rahman [53]). This extremely important transformation is due to Sears [58] and is valid provided the two series are balanced, that is $def = abc q^{1-n}$. Its importance is beyond doubt especially when dealing with the ${}_4\phi_3$ polynomials of Askey and Wilson [17, 18] and is reflected in the works of Rahman [53, 54] (see also Ismail and Wilson [41]).

Another result that can be found in (1.2.6) and (1.2.7) is Jackson's q -analogue [42] of Pfaff-Saalschutz summation theorem, namely

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, a, b \\ e, f \end{matrix} ; q, q \right] = b^n \frac{(e/b)_n (f/b)_n}{(e)_n (f)_n}, \quad (1.2.8)$$

provided $ef = ab q^{1-n}$. (1.2.8) follows from (1.2.7) when $c = d$ and from (1.2.6) when $f = q^{-n}$ and $c = 1$.

3. Quadratic transformations. An important and useful feature of hypergeometric functions is the existence of transformations. Of particular interest to us is the family of quadratic transformations. These transformations were first looked into by Gauss and as an example one can mention

$${}_2F_1(a, b; 1+a-b; z) = (1-z)^{-a} {}_2F_1\left(\frac{1}{2}a, \frac{1}{2}a + \frac{1}{2} - b; 1+a-b; \frac{-4z}{(1-z)^2}\right), \quad (1.3.1)$$

which is due to him, (see Bailey [21] and for a list of quadratic transformations see Erdélyi [30]). These transformations, though it is not the way they were found originally, can be seen easily through the differential equations that they satisfy. The basic functions are not known to satisfy any differential equations, though many of them satisfy q -difference equations, yet quadratic transformations for them are becoming familiar and useful, (see Askey and Wilson [18], Carlitz [23], Verma [64], Singh [59]).

In 1836, Kummer [45] compiled a list of these transformations which he used with Gauss' theorem (1.1.6) to evaluate sums of ${}_2F_1$'s at arguments different from 1, ((1.3.1) is used to show Kummer's summation theorem of a ${}_2F_1$ at argument -1 , see Bailey [21, §2.3]). Goursat [38] used higher order transformations to do the same thing. In a recently published paper Gessel and Stanton [36] proved summation theorems at arguments different from ± 1 using a change of variables technique equivalent to Lagrange inversion formula, which they also extended to the q -series [37].

To give an analytic proof to his addition formula for the Jacobi polynomials, Koornwinder [44] used a quadratic transformation along with (1.1.5). Another illustration of the use of these transformations; consider Whipple's quadratic transformation [69]

$${}_3F_2 \left[\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix} ; z \right] = (1-z)^{-a} {}_3F_2 \left[\begin{matrix} a/2, \frac{a+1}{2}, 1+a-b-c \\ 1+a-b, 1+a-c \end{matrix} ; \frac{-4z}{(1-z)^2} \right] \quad (1.3.2)$$

Let $a = -n$, and integrate from 0 to 1 with respect to $z^{d-1}(1-z)^{w-d-1}dz$, then set $d = -m$ in the result and revert $-n$ to a to obtain the following well known transformation due to Whipple [69]*

* This derivation was pointed out by the referee of [47].

$${}_4F_3 \left[\begin{matrix} a, b, c, -m \\ 1+a-b, 1+a-c, w \end{matrix} ; 1 \right] = \frac{\Gamma(w) \Gamma(w-a+m)}{\Gamma(w-a) \Gamma(w+m)}$$

$${}_5F_4 \left[\begin{matrix} 1+a-w, a/2, \frac{a+1}{2}, 1+a-b-c, -m \\ 1+a-b, 1+a-c, 1+\frac{a-w-m}{2}, \frac{1+a-w-m}{2} \end{matrix} ; 1 \right],$$

(1.3.3)

taking a nearly-poised ${}_4F_3$ of the second kind to a balanced ${}_5F_4$. Whipple [70] used (1.3.3) to prove Orr's [49] and Cayley's [25] theorems, (see also Bailey [21], Slater [60]).

Using Jackson's theorem (1.2.8), Carlitz [23] gave a q -analogue of (1.3.2), namely

$${}_3\phi_2 \left[\begin{matrix} a, b, c \\ aq/b, aq/c \end{matrix} ; q, \frac{aqx}{bc} \right] = \frac{(ax)_\infty}{(x)_\infty} {}_5\phi_4 \left[\begin{matrix} a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, aq/bc \\ aq/b, aq/c, ax, q/x \end{matrix} ; q, q \right].$$

(1.3.4)

This is true provided these series terminate, that is a has to be of the form q^{-n} , n a non-negative integer. Presumably, this transformation can be used in a similar way as above to find a q -analogue of (1.3.3)*. However, in Chapter 2 we shall prove an analogue of (1.3.3), using the method of Bailey's [21, Chapter 4], namely

* Also suggested by the referee of [47].

$$\begin{aligned}
& 10^{\frac{R}{2}} 9 \left[\begin{matrix} -q^a, q\sqrt{-q^a}, -q\sqrt{-q^a}, q^a, -q^b, q^b, -q^c, q^c, -q^{1+a-w}, q^{-m} \\ \sqrt{-q^a}, -\sqrt{-q^a}, -q, q^{1+a-b}, -q^{1+a-b}, q^{1+a-c}, -q^{1+a-c}, q^w, -q^{1+a+m} \end{matrix} ; q, q^R \right] \\
& = \frac{(q^{w-a})_m (-q^{a+1})_m}{(q^w)_m (-q)_m} 5^{\frac{R}{2}} 4 \left[\begin{matrix} p^{1+a-w}, p^{a/2}, p^{(a+1)/2}, p^{1+a-b-c}, p^{-m} \\ p^{1+a-b}, p^{1+a-c}, p^{(1+a-w-m)/2}, p^{1+(a-w-m)/2} \end{matrix} ; p, p \right], \\
& \hspace{25em} (1.3.5)
\end{aligned}$$

where

$$R = a - 2b - 2c + w + m + 2, \quad (1.3.6)$$

and $p = q^2$. In addition, we shall generalize (1.3.5) and prove further results of similar nature along with corollaries that are q -analogues of known results.

In Chapter 3, we shall demonstrate the use of (1.3.5) and prove q -analogues of the Cayley-Orr theorems, in a manner similar to Whipple's. After we completed our work we discovered that Singh [59] also gave the same q -analogues, but he used a method different from ours, (also see [72]).

4. Projection formulas. The solutions to many problems can be seen more easily by reducing such problems into equivalent simple forms. The importance of projection formulas in this area has been pointed out by many people, (see Askey [10, 11, 12], Askey and Fitch [14], Askey and Gasper [15], Gasper [33, 34] and Szego [61]). These

formulas are of the form

$$q_n(x) = \int p_n(y) d\gamma_x(y), \quad d\gamma_x(y) \geq 0, \quad (1.4.1)$$

and

$$q_n(x) = \sum \alpha_{n,k} p_k(x), \quad \alpha_{n,k} \geq 0, \quad (1.4.2)$$

where $q_n(x)$ and $p_n(x)$ are polynomials of degree n and not necessarily the same. Formulas for the measure $\gamma_x(y)$ have been known in the case $q_n(x)$ and $p_n(x)$ are the Jacobi polynomials with different sets of parameters, (see Askey [10, 11] and Szego [61]). Also formulas for $\alpha_{n,k}$ were found by Askey and Gasper [15] for the same polynomials. In a recently published paper, Laine [46] gave a new projection formula of type (1.4.1) which he used to prove Koornwinder's addition formula for the Jacobi polynomials.

We mentioned that Carlitz used Jackson's theorem (1.2.8) to prove (1.3.4). That theorem thus far has proved to be very important, in effect much more important than the ${}_4\phi_3$ extension of Dixon's theorem. Rahman [50] used (1.2.8) to project the discrete ${}_4\phi_3$ polynomials and write them as a sum of ${}_3\phi_2$ polynomials which he used in turn to find a very general family of reproducing kernels for these discrete polynomials. In Chapter 4 we shall do the same thing for the continuous ${}_4\phi_3$ polynomials where we shall find a

double integral kernel $K(x,y)$ for them. Limiting cases of this kernel will be considered which lead to projection formulas of type (1.4.1) analogous to known results for the Jacobi polynomials.

Chapter 2

q-Analogues of Some Transformations of
Nearly-Poised Hypergeometric Series

1. Introduction. The work in this chapter was motivated by Rahman's [53] extension of the linearization formula for the Jacobi polynomials [52] to a q-analogue. He needed a q-analogue of a special case of Whipple's transformation [69, (6.6)], (see Bailey [21, 4.7(1)]). To find such analogues one realizes that there is no obvious or routine procedure for doing that. Two different q-series with different properties can in effect reduce to the same hypergeometric limit. Consider the bi-basic series as defined in [1,2], namely

$$\begin{aligned}
 & {}_{r+s} \phi_{r+s-1} \left[\begin{matrix} a_1, \dots, a_r; & b_1, \dots, b_s \\ c_1, \dots, c_{r-1}; & d_1, \dots, d_s \end{matrix} ; q, q', z \right] \\
 &= \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k (b_1; q')_k \dots (b_s; q')_k}{(c_1)_k \dots (c_{r-1})_k (d_1; q')_k \dots (d_s; q')_k} \frac{z^k}{(q)_k},
 \end{aligned}
 \tag{2.1.1}$$

where q and q' are two different bases. As $q, q' \rightarrow 1^-$, this series will reduce to

$${}_{r+s}F_{r+s-1}(a_1, \dots, a_r, b_1, \dots, b_s; c_1, \dots, c_{r-1}, d_1, \dots, d_s; z)$$

defined in (1.1.1), as would the series

$${}_{r+s}F_{r+s-1} \left[\begin{matrix} a_1, \dots, a_r, & b_1, \dots, b_s \\ c_1, \dots, c_{r-1}, & d_1, \dots, d_s \end{matrix} ; q, z \right] \\ = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k (b_1)_k \dots (b_s)_k}{(c_1)_k \dots (c_{r-1})_k (d_1)_k \dots (d_s)_k} \frac{z^k}{(q)_k} \quad (2.1.2)$$

where $(a)_k$ and $(a; q)_k$ are as defined in (1.2.2) and (1.2.2').

Realizing this fact, we were able to prove the following theorems. The first one is a q -analogue of (1.3.3) or Bailey's [21, 4.5(1)].

Theorem 1. Let a, b, c, w be arbitrary parameters, real or complex. Let m be a non-negative integer and let $p \equiv q^2$. Then

$${}_6\phi_5 \left[\begin{matrix} p^a, -p^{1+a/2}, p^b, p^c; -q^{1+a-w}, q^{-m} \\ -p^{a/2}, p^{1+a-b}, p^{1+a-c}; q^w, -q^{1+a+m} \end{matrix} ; p, q, q^R \right] \\ = \frac{(q^{w-a})_m (-q^{a+1})_m}{(q^w)_m (-q)_m} {}_5\phi_4 \left[\begin{matrix} p^{1+a-w}, p^{a/2}, p^{(a+1)/2}, p^{1+a-b-c}, p^{-m} \\ p^{1+a-b}, p^{1+a-c}, p^{(1+a-w-m)/2}, p^{1+(a-w-m)/2} \end{matrix} ; p, p \right] \quad (2.1.3)$$

where

$$R = a - 2b - 2c + w + m + 2. \quad (2.1.3')$$

Using A-2, it is easy to see that the ${}_6\phi_5$ in (2.1.3) is the same as the ${}_{10}\phi_9$ in (1.3.5). However, we shall use the bi-basic notation mainly because of the space it saves.

In section 2 we shall prove theorem 1 along with the following four corollaries.

Corollary 1.1 (q-analogue of Bailey's [21, (1.1), p. 30]).

$$\begin{aligned}
 & {}_4\phi_3 \left[\begin{matrix} a, q^{1+a/2}, -q^{w-a/2-1}, q^{-m} \\ q^{a/2}, q^w, -q^{1+a/2-m} \end{matrix} ; q, q \right] \\
 &= \frac{(-q^{a/2})_m (q^{w-a})_m (q^{w-a-1-m}; q^2)_m (q^{2+a-w-m}; q^2)_m}{(-q^{-a/2})_m (q^w)_m (q^{1+a-w-m}; q^2)_m (q^{w-a-m}; q^2)_m} \quad (2.1.4)
 \end{aligned}$$

Corollary 1.2 (q-analogue of Bailey's [21, (1.2), p. 30]).

$$\begin{aligned}
 & {}_4\phi_3 \left[\begin{matrix} a, b, q^{1+b-m}, q^{-m} \\ q^{1+a-b}, q^{1+2b-m}, -q^{b-m} \end{matrix} ; q, q \right] \\
 &= \frac{(1+q^b)(1+q^{a/2-b+m})(q^{a-2b})_m (q^{1+a/2-b})_m (q^{-b})_m}{(1+q^{m-b})(1+q^{a/2-b})(q^{1+a-b})_m (q^{a/2-b})_m (q^{-2b})_m} \quad (2.1.5)
 \end{aligned}$$

Corollary 1.3 (q-analogue of Bailey's [21, (1.3), p. 30]).

$$\begin{aligned}
 & {}_6\phi_5 \left[\begin{matrix} p^a, p^{1+a/2}, -p^{1+a/2}, p^b; & -q^{a-2b+m}, q^{-m} \\ p^{a/2}, -p^{a/2}, p^{1+a-b}; & q^{1+2b-m}, -q^{1+a+m} \end{matrix} ; p, q, q \right] \\
 &= \frac{(-q^{a+1})_m (q^{a-2b})_m (p^{-b}; p)_m}{(-q)_m (q^{-2b})_m (p^{1+a-b}; p)_m} q^m.
 \end{aligned} \tag{2.1.6}$$

Corollary 1.4 (q-analogue of Bailey's [21, (1.4), p. 30]).

$$\begin{aligned}
 & {}_6\phi_5 \left[\begin{matrix} p^a, p^{1+a/2}, -p^{1+a/2}, p^b; & -q^{a-2b+m-1}, q^{-m} \\ p^{a/2}, -p^{a/2}, p^{1+a-b}; & q^{2+2b-m}, -q^{1+a+m} \end{matrix} ; p, q, p \right] \\
 &= \frac{(-q^{a+1})_m (q^{a-2b-1})_m (p^{a/2-b+1/2}; p)_m (p^{-b-1}; p)_m}{(-q)_m (q^{-2b-1})_m (p^{a/2-b-1/2}; p)_m (p^{1+a-b}; p)_m} q^m.
 \end{aligned} \tag{2.1.7}$$

In section 3, two general theorems will be proved, namely

Theorem 2 (q-analogue of Bailey's [21, (1), p. 24]). Let m be a non-negative integer and $a, b, c, a_1, \dots, a_s, b_1, \dots, b_{s+1}, Q$ be arbitrary parameters. Then

$$\begin{aligned}
& s+5 \begin{bmatrix} p^a, -p^{1+a/2}, p^b, p^c; q^1, \dots, q^s, q^{-m} \\ -p^{a/2}, p^{1+a-b}, p^{1+a-c}; q^1, \dots, q^s, q^{s+1} \end{bmatrix}; p, q, Q \\
& = \sum_{r=0}^m \frac{(p^a; p)_{2r} (-p^{1+a/2}; p)_r (p^{1+a-b-c}; p)_r (q^1)_r \dots (q^s)_r (q^{-m})_r}{(p; p)_r (-p^{a/2}; p)_r (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r (q^1)_r \dots (q^{s+1})_r} \\
& \cdot (-Q)^r q^{r(2b+2c-2a-1-r)}
\end{aligned}$$

$$s+3 \begin{bmatrix} p^{a+2r}, -p^{1+a/2+r}; q^{1+r}, \dots, q^{s+r}, q^{r-m} \\ -p^{a/2+r}; q^{1+r}, \dots, q^{s+1+r} \end{bmatrix}; p, q, Q q^{2(b+c-a-r-1)} \quad (2.1.8)$$

where $p = q^2$.

Theorem 3 (q-analogue of Bailey's [21, (6), p. 26]). Let the parameters be as in theorem 2, and set $K = 1 + 2a - b - c - d$. Then

$$\begin{aligned}
& s+6 \begin{bmatrix} p^a, -p^{1+a/2}, p^b, p^c, p^d; q^1, \dots, q^s, q^{-m} \\ -p^{a/2}, p^{1+a-b}, p^{1+a-c}, p^{1+a-d}; q^1, \dots, q^s, q^{s+1} \end{bmatrix}; p, q, Q \\
& = \sum_{\ell=0}^m \frac{(p^a; p)_{2\ell} (-p^{1+a/2}; p)_\ell (p^K; p)_\ell (p^{1+K/2}; p)_\ell (-p^{1+K/2}; p)_\ell (p^{K+b-a}; p)_\ell}{(p; p)_\ell (-p^{a/2}; p)_\ell (p^{K+1}; p)_{2\ell} (p^{K/2}; p)_\ell (-p^{K/2}; p)_\ell (p^{1+a-b}; p)_\ell} \\
& \cdot \frac{(p^{K+c-a}; p)_\ell (p^{K+d-a}; p)_\ell (q^1)_\ell \dots (q^s)_\ell (q^{-m})_\ell (Qp^{a-K})_\ell}{(p^{1+a-c}; p)_\ell (p^{1+a-d}; p)_\ell (q^1)_\ell \dots (q^{s+1})_\ell} \quad (2.1.9)
\end{aligned}$$

$$s+4 \begin{matrix} \phi \\ s+3 \end{matrix} \left[\begin{matrix} p^{a+2\ell}, p^{1+a/2+\ell}, p^{a-K}; q^{a_1+\ell}, \dots, q^{a_s+\ell}, \\ -p^{a/2+\ell}, p^{1+K+2\ell}; q^{b_1+\ell}, \dots, q^{b_{s+1}+\ell}; p, q, Q \end{matrix} \right].$$

Using these theorems and the corollaries to theorem 1 we shall derive q -analogues of the transformations of nearly-poised series listed on pp. 30-33 of Bailey's [21]. This will be done in sections 4 and 5.

2. Proof of theorem 1 and corollaries. The proof of theorem 1 depends on Jackson's theorem (1.2.8) and the terminating version of (1.2.5), namely A-1 (in the appendix). To start with, we have for a non-negative k

$$3 \begin{matrix} \phi \\ 2 \end{matrix} \left[\begin{matrix} p^{-k}, p^{a+k}, p^{1+a-b-c} \\ p^{1+a-b}, p^{1+a-c}; p, p \end{matrix} \right] = \frac{(p^c; p)_k (p^b; p)_k}{(p^{1+a-b}; p)_k (p^{1+a-c}; p)_k} p^{(1+a-b-c)k} \quad (2.2.1)$$

by (1.2.8). Thus the l.h.s. of (2.1.3) becomes

$$\begin{aligned} S_m &= \sum_{k=0}^m \frac{(p^a; p)_k (-p^{1+a/2}; p)_k (-q^{1+a-w})_k (q^{-m})_k q^{kR}}{(p; p)_k (-p^{a/2}; p)_k (q^w)_k (-q^{1+a+m})_k} \cdot \frac{(p^b; p)_k (p^c; p)_k}{(p^{1+a-b}; p)_k (p^{1+a-c}; p)_k} \\ &= \sum_{k=0}^m \frac{(p^a; p)_k (-p^{1+a/2}; p)_k (-q^{1+a-w})_k (q^{-m})_k}{(p; p)_k (-p^{a/2}; p)_k (q^w)_k (-q^{1+a+m})_k} q^{kR} \cdot p^{(b+c-a-1)k} \\ &\quad \cdot \sum_{r=0}^k \frac{(p^{-k}; p)_r (p^{a+k}; p)_r (p^{1+a-b-c}; p)_r}{(p; p)_r (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r} p^r \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^m \sum_{r=0}^k \frac{(q^{-m})_k (-q^{1+a-w})_k (-p^{1+a/2}; p)_{r+k} (p^a; p)_{r+k} (p^{1+a-b-c}; p)_r (p^{-k}; p)_r}{(p; p)_{r+k} (q^w)_k (-q^{1+a+m})_k (-p^{a/2}; p)_k (p; p)_r (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r} \\
&\quad \cdot q^{kR} p^{r+(b+c-a-1)k} \quad (2.2.2)
\end{aligned}$$

The next step is to change the order of summation by making the variable transform $k \rightarrow r+k$, hence

$$\begin{aligned}
S_m &= \sum_r \sum_k \frac{(p^a; p)_{2r+k} (-p^{1+a/2}; p)_{r+k} (p^{1+a-b-c}; p)_r (p^{-r-k}; p)_r}{(p; p)_{r+k} (p; p)_r (-p^{a/2}; p)_{r+k} (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r} \\
&\quad \cdot \frac{(-q^{1+a-w})_{r+k} (q^{-m})_{r+k} q^{(r+k)R}}{(q^w)_{r+k} (-q^{1+a+m})_{r+k}} p^{r+(b+c-a-1)(r+k)} \\
&= \sum_{r=0}^m \frac{(p^a; p)_{2r} (-p^{1+a/2}; p)_r (p^{1+a-b-c}; p)_r (-q^{1+a-w})_r (q^{-m})_r (-q^u)^r}{(p; p)_r (-p^{a/2}; p)_r (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r (q^w)_r (-q^{1+a+m})_r} \\
&\quad \cdot {}_6\phi_5 \left[\begin{matrix} -q^{a+2r}, q\sqrt{-q^{a+2r}}, -q\sqrt{-q^{a+2r}}, q^{a+2r}, -q^{1+a-w+r}, q^{r-m} \\ \sqrt{-q^{a+2r}}, -\sqrt{-q^{a+2r}}, -q, q^{w+r}, -q^{1+a+m+r} \end{matrix} ; q, q^{w-a+m-2r} \right] \quad (2.2.3)
\end{aligned}$$

where $u = R + 2b + 2c - 2a - (r+1) = w - a + m + 1 - r$. To obtain the last equality in (2.2.3) use was made of A-2 and A-5. The ${}_6\phi_5$ in (2.2.3) satisfies the conditions for summation by A-1. Thus it is equal to

$$\frac{(-q^{1+a+2r})_{m-r} (q^{w-a-r})_{m-r}}{(-q)_{m-r} (q^{w+r})_{m-r}}$$

Simplifying this expression by using A-4, A-6 and A-7 we get

$$\frac{(-q^{l+a+2r})_{m-r}}{(q^{w+r})_{m-r}} = \frac{(-q^{l+a})_{m+r} (q^w)_r}{(-q^{l+a})_{2r} (q^w)_m} = \frac{(-q^{l+a})_m (-q^{l+a+m})_r (q^w)_r}{(q^w)_m (-q^{l+a})_{2r}}, \text{ by A-4.}$$

$$\frac{(q^{w-a-r})_{m-r}}{(-q)_{m-r}} = \frac{(q^{w-a})_m (q^{l+a-w})_r (-q^{-m})_r}{(q^{l+a-w-m})_{2r} (-q)_m} (-1)^r q^{r^2 + (l+a-w-m)r},$$

by A-6 and A-7. Therefore the sum of ${}_6\phi_5$ becomes

$$\frac{(-q^{a+1})_m (q^{w-a})_m (-q^{a+1+m})_r (q^w)_r (q^{l+a-w})_r (-q^{-m})_r q^{r^2 + (l+a-w-m)r}}{(-q)_m (q^w)_m (-q^{a+1})_{2r} (q^{l+a-w-m})_{2r} (-1)^r} \quad (2.2.4)$$

A-8 gives us

$$(q^{l+a-w-m})_{2r} = (p^{(l+a-w-m)/2}; p)_r (p^{l+(a-w-m)/2}; p)_r,$$

and from A-9 we have

$$\frac{(p^a; p)_{2r} (-p^{l+a/2}; p)_r}{(-q^{a+1})_{2r} (-p^{a/2}; p)_r} = (p^{a/2}; p)_r (p^{l/2+a/2}; p)_r.$$

Finally by A-2, $(-q^{l+a-w})_r (q^{l+a-w})_r = (p^{l+a-w}; p)_r$.

Using the above expressions in (2.2.3) we thus get

$$S_m = \frac{(-q)^{a+1}_m (q^{w-a})_m}{(-q)_m (q^w)_m} \sum_{r=0}^m \frac{(p^{a/2}; p)_r (p^{1/2+a/2}; p)_r (p^{1+a-b-c}; p)_r (p^{-m}; p)_r}{(p; p)_r (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r (p^{(1+a-w-m)/2}; p)_r} \cdot \frac{(p^{1+a-w}; p)_r p^r}{(p^{1+(a-w-m)/2}; p)_r}, \quad (2.2.5)$$

which is the r.h.s. of (2.1.3).

One thing that we notice about the ${}_5\phi_4$ in (2.1.3) is that it is balanced. Hence if we can reduce it to a ${}_3\phi_2$ we can sum it using Jackson's (1.2.8). This will be done by choosing special values for the parameters, which in turn will help us prove the corollaries 1.1-1.4.

Proof of corollary 1.1. Let us choose $b = 1 + a/2$, $c = \frac{1+a}{2}$.

Then the ${}_5\phi_4$ in (2.1.3) reduces to

$${}_3\phi_2 \left[\begin{matrix} p^{-m}, p^{-\frac{1}{2}}, p^{1+a-w} \\ p^{(1+a-w-m)/2}, p^{1+(a-w-m)/2} \end{matrix} ; p, p \right] = \frac{(p^{-(1+a-w+m)/2}; p)_m (p^{1+(a-w-m)/2}; p)_m}{(p^{(1+a-w-m)/2}; p)_m (p^{-(a-w+m)/2}; p)_m}. \quad (2.2.6)$$

Thus we have

$${}_5\phi_4 \left[\begin{matrix} p^a, -p^{1+a/2}, p^{1+a/2}; -q^{1+a-w}, q^{-m} \\ -p^{a/2}, p^{a/2}; q^w, -q^{1+a+m} \end{matrix} ; p, q, q^{w-a-1+m} \right] = \frac{(q^{w-a})_m (-q)^{a+1}_m (p^{-(1+a-w+m)/2}; p)_m (p^{1+(a-w-m)/2}; p)_m}{(q^w)_m (-q)_m (p^{(1+a-w-m)/2}; p)_m (p^{-(a-w+m)/2}; p)_m}. \quad (2.2.7)$$

If we take the limit $q \rightarrow 1^-$ and simplify then we get Bailey's [21, (1.1), p. 30]. However, we need a transformation to show that (2.2.7) is actually Corollary 1.1. The ${}_5\phi_4$ in (2.2.7) is bi-basic, using A-2 it becomes

$${}_8\phi_7 \left[\begin{matrix} -q^a, q\sqrt{-q^a}, -q\sqrt{-q^a}, q^a, q\sqrt{q^a}, -q\sqrt{q^a}, -q^{1+a-w}, q^{-m} \\ \sqrt{-q^a}, -\sqrt{-q^a}, -q, -\sqrt{q^a}, \sqrt{q^a}, q^w, -q^{1+a+m}; q, q^{w-a-1+m} \end{matrix} \right] \quad (2.2.8)$$

which can be transformed by Watson's theorem (1.2.6) to

$$\frac{(-q^{a+1})_m (-q^{-a/2})_m}{(-q)_m (-q^{a/2})_m} {}_4\phi_3 \left[\begin{matrix} q^a, q^{1+a/2}, -q^{w-a/2-1}, q^{-m} \\ q^{a/2}, q^w, -q^{a/2+1-m}; q, q \end{matrix} \right]. \quad (2.2.9)$$

This leads to corollary 1.1.

Proof of Corollary 1.2. To prove this result we choose

$c = w = a - 2b + m$. Then the l.h.s. of (2.1.3) becomes

$$\begin{aligned} & {}_8\phi_7 \left[\begin{matrix} -q^a, q\sqrt{-q^a}, -q\sqrt{-q^a}, q^a, q^b, -q^b, -q^{a-2b+m}, q^{-m} \\ \sqrt{-q^a}, -\sqrt{-q^a}, -q, -q^{1+a-b}, q^{1+a-b}, q^{2b+1-m}, -q^{1+a+m}; q, q^2 \end{matrix} \right] \\ &= \frac{(-q^{1+a})_m (-q^{1-b})_m}{(-q)_m (-q^{1+a-b})_m} {}_4\phi_3 \left[\begin{matrix} q^a, q^b, -q^{1+b-m}, q^{-m} \\ q^{1+a-b}, q^{1+2b-m}, -q^{b-m}; q, q \end{matrix} \right] \end{aligned} \quad (2.2.10)$$

by (1.2.6), while the r.h.s. gives

$$\begin{aligned}
& \frac{(q^{m-2b})_m (-q^{1+a})_m}{(q^{a-2b+m})_m (-q)_m} {}_3\phi_2 \left[\begin{matrix} p^{a/2}, p^{(a+1)/2}, p^{-m} \\ p^{1+a-b}, p^{1/2+b-m} \end{matrix} ; p, p \right] \\
&= \frac{(q^{m-2b})_m (-q^{1+a})_m (p^{(a+1)/2-b}; p)_m (p^{1+a/2-b}; p)_m}{(q^{a-2b+m})_m (-q)_m (p^{1+a-b}; p)_m (p^{1/2-b}; p)_m} \quad (2.2.11)
\end{aligned}$$

Equating the above two expressions we get

$$\begin{aligned}
& {}_4\phi_3 \left[\begin{matrix} q^a, q^b, -q^{1+b-m}, q^{-m} \\ q^{1+a-b}, q^{1+2b-m}, -q^{b-m} \end{matrix} ; q, q \right] \\
&= \frac{(-q^{1+a-b})_m (q^{m-2b})_m (q^{a+1-2b}; q^2)_m (q^{2+a-2b}; q^2)_m}{(-q^{1-b})_m (q^{a-2b+m})_m (q^{2+2a-2b}; q^2)_m (q^{1-2b}; q^2)_m} \quad (2.2.12)
\end{aligned}$$

If we simplify we get

$$\begin{aligned}
\frac{(q^{m-2b})_m}{(q^{a-2b+m})_m} &= \frac{(q^{-2b})_{2m} (q^{a-2b})_m}{(q^{-2b})_m (q^{a-2b})_{2m}} = \frac{(q^{-2b}; q^2)_m (q^{1-2b}; q^2)_m (q^{a-2b})_m}{(q^{a-2b}; q^2)_m (q^{1+a-2b}; q^2)_m (q^{-2b})_m} \\
&= \frac{(-q^{-b})_m (q^{-b})_m (q^{1-2b}; q^2)_m (q^{a-2b})_m}{(q^{a/2-b})_m (-q^{a/2-b})_m (q^{1+a-2b}; q^2)_m (q^{-2b})_m}
\end{aligned}$$

where we have used A-4, A-8 and A-2.

Therefore (2.2.12) becomes

$$\begin{aligned}
 {}_4\phi_3[] &= \frac{(-q^{1+a-b})_m (-q^{-b})_m (q^{-b})_m (q^{a-2b})_m (q^{1+a/2-b})_m (-q^{1+a/2-b})_m}{(-q^{1-b})_m (q^{a/2-b})_m (-q^{a/2-b})_m (q^{-2b})_m (q^{1+a-b})_m (-q^{1+a-b})_m} \\
 &= \frac{(1+q^{-b})(1+q^{a/2-b+m})(q^{a-2b})_m (q^{1+a/2-b})_m (q^{-b})_m}{(1+q^{m-b})(1+q^{a/2-b})(q^{1+a-b})_m (q^{a/2-b})_m (q^{-2b})_m}
 \end{aligned}
 \tag{2.2.13}$$

which is corollary 1.2.

Proof of Corollary 1.3. Choose $c = 1+a/2$, $w = 1+2b-m$.

In a similar manner as before, the l.h.s. of (2.1.3) remains as

a bi-basic ${}_6\phi_5$ whereas the r.h.s. becomes

$$\begin{aligned}
 {}_3\phi_2 \left[\begin{matrix} p^{-m}, p^{a-2b+m}, p^{1/2+a/2} \\ p^{1+a-b}, p^{1/2+a/2-b} \end{matrix} ; p, p \right] \\
 = \frac{(p^{-b}; p)_m q^{(1+a)m}}{(p^{1+a-b}; p)_m},
 \end{aligned}
 \tag{2.2.14}$$

by (1.2.8). Also

$$(q^{w-a})_m / (q^w)_m = \frac{(q^{1+2b-a-m})_m}{(q^{1+2b-m})_m} = \frac{(q^{a-2b})_m}{(q^{-2b})_m} q^{-ma},$$

by A-5. Thus putting things together we get corollary 1.3.

Proof of Corollary 1.4. Set $c = 1+a/2$, $w = 2+2b-m$. The proof would be exactly like that of Corollary 1.3.

3. Proofs of theorems 2 and 3. We note that the parameters on the l.h.s. of (2.1.8) with base p are the same as the ones on the l.h.s. of (2.1.3). Hence the proof of theorem 2 will be the same as that of theorem 1. In fact, theorem 1 can be seen as a corollary to theorem 2 if we set $s = 1$, $q^{a_1} = -q^{l+a-w}$, $q^{b_1} = q^w$, $q^{b_2} = -q^{l+a+m}$ and $Q = q^R$, R being defined in (2.1.3').

However, the proof of theorem 3 is a little different, and to start with it depends on Jackson's theorem (1.2.4), which, in the form required for our proof, reads

$$8 \frac{1}{7} \left[\begin{matrix} K, & l+K/2, & -p^{l+K/2}, & p^{K+b-a}, & p^{K+c-a}, & p^{K+d-a}, & p^{a+l}, & p^{-l} \\ p, & p, & -p, & p, & p, & p, & p, & p \end{matrix} ; p, p \right]$$

$$= \frac{(p^{K+1}; p)_\ell (p^{l+2a-b-c-K}; p)_\ell (p^{l+2a-c-d-K}; p)_\ell (p^{l+2a-d-b-K}; p)_\ell}{(p^{l+a-b}; p)_\ell (p^{l+a-c}; p)_\ell (p^{l+a-d}; p)_\ell (p^{l+3a-b-c-d-2K}; p)_\ell}, \quad (2.3.1)$$

where K is some given parameter and ℓ is a non-negative integer.

If we set

$$K = l + 2a - b - c - d, \quad (2.3.2)$$

then (2.3.1) gives us

$$\begin{aligned}
& \frac{(p^{K+1}; p)_\ell (p^b; p)_\ell (p^c; p)_\ell (p^d; p)_\ell}{(p^{a-K}; p)_\ell (p^{1+a-b}; p)_\ell (p^{1+a-c}; p)_\ell (p^{1+a-d}; p)_\ell} = \sum_{i=0}^{\ell} \frac{(p^K; p)_i (p^{1+K/2}; p)_i}{(p; p)_i (p^{K/2}; p)_i} \\
& \cdot \frac{(-p^{1+K/2}; p)_i (p^{K+b-a}; p)_i (p^{K+c-a}; p)_i (p^{K+d-a}; p)_i (p^{a+i}; p)_i (p^{-i}; p)_i}{(-p^{K/2}; p)_i (p^{1+a-b}; p)_i (p^{1+a-c}; p)_i (p^{1+a-d}; p)_i (p^{1+K-a-i}; p)_i (p^{1+K+i}; p)_i} p^i.
\end{aligned}
\tag{2.3.3}$$

Denoting the series on the l.h.s. of (2.1.9) by T_m , and using (2.3.3)

we have:

$$\begin{aligned}
T_m &= \sum_{\ell=0}^m \frac{(q^{-m})_\ell (q^{a_1})_\ell \dots (q^{a_s})_\ell (-p^{1+a/2}; p)_\ell (p^a; p)_\ell (p^b; p)_\ell (p^c; p)_\ell (p^d; p)_\ell}{(q^{b_1})_\ell \dots (q^{b_{s+1}})_{\ell+1} (-p^{a/2}; p)_\ell (p; p)_\ell (p^{1+a-b}; p)_\ell (p^{1+a-c}; p)_\ell (p^{1+a-d}; p)_\ell} Q^\ell \\
&= \sum_{\ell=0}^m \sum_{i=0}^{\ell} \frac{(q^{-m})_\ell (q^{a_1})_\ell \dots (q^{a_s})_\ell (-p^{1+a/2}; p)_\ell (p^{a-K}; p)_\ell}{(q^{b_1})_\ell \dots (q^{b_{s+1}})_{\ell+1} (p; p)_\ell (-p^{a/2}; p)_\ell} Q^\ell \\
&\quad \cdot \frac{(p^K; p)_i (p^{1+K/2}; p)_i (-p^{1+K/2}; p)_i (p^{K+b-a}; p)_i (p^{K+c-a}; p)_i}{(p; p)_i (p^{K/2}; p)_i (-p^{K/2}; p)_i (p^{1+a-b}; p)_i (p^{1+a-c}; p)_i} \\
&\quad \cdot \frac{(p^{K+d-a}; p)_i (p^{-i}; p)_i (p^a; p)_{\ell+1}}{(p^{1+a-d}; p)_i (p^{1+K}; p)_{\ell+1} (p^{1+K-a-i}; p)_i} p^i.
\end{aligned}
\tag{2.3.4}$$

Since $m \geq \ell \geq i$ we make a variable transformation $\ell \rightarrow \ell+1$ so that we can interchange the summation. Also observing that

$$\frac{(p^{a-K}; p)_{\ell+1} (p^{-\ell-i}; p)_i}{(p^{K+1-a-\ell-i}; p)_i (p; p)_{\ell+1}} = \frac{(p^{a-K}; p)_\ell p^{(a-K-1)_i}}{(p; p)_\ell}$$

by A-3 and A-6. Thus we obtain

$$T_m = \sum_{i=0}^m \frac{(p^a; p)_{2i} (-p^{1+a/2}; p)_i (p^K; p)_i (p^{1+K/2}; p)_i (-p^{1+K/2}; p)_i}{(p; p)_i (-p^{a/2}; p)_i (p^{K+1}; p)_{2i} (p^{K/2}; p)_i (-p^{K/2}; p)_i}$$

$$\cdot \frac{(p^{K+b-a}; p)_i (p^{K+c-a}; p)_i (p^{K+d-a}; p)_i (q^a)_i \dots (q^a)_i (q^{-m})_i}{(p^{1+a-b}; p)_i (p^{1+a-c}; p)_i (p^{1+a-d}; p)_i (q^b)_i \dots (q^{b_{s+1}})_i}$$

$$\cdot (Q p^{a-K})^i \sum_{\ell=0}^{m-i} \frac{(p^{a+2i}; p)_\ell (-p^{1+a/2+i}; p)_\ell (p^{a-K}; p)_\ell (q^{i-m})_\ell}{(p; p)_\ell (-p^{a/2+i}; p)_\ell (p^{1+K+2i}; p)_\ell}$$

$$\cdot \frac{(q^{a_1+i})_\ell \dots (q^{a_s+i})_\ell Q^\ell}{(q^{b_1+i})_\ell \dots (q^{b_{s+1}+i})_\ell} \quad (2.3.5)$$

This is Theorem 3 where $i \rightarrow \ell$.

4. q-analogues of transformations of well-poised and series of the second kind. As in the case of ordinary series the transformation formulas in theorems 2 and 3 unless the parameters can be chosen so that the ϕ series

r.h. sides are summable in terms of ratios of products. Consider the series in theorem 2. Since there is no known summation theorem for a series higher than an $8\phi_7$ let us set $s = 3$ in (2.1.8) and rewrite the theorem

$$8\phi_7 \left[\begin{matrix} p^a, -p^{1+a/2}, p^b, p^c; q^{a_1}, q^{a_2}, q^{a_3}, q^{-m} \\ -p^{a/2}, p^{1+a-b}, p^{1+a-c}; q^{b_1}, q^{b_2}, q^{b_3}, q^{b_4} \end{matrix} ; p, q, Q \right]$$

$$= \sum_{r=0}^m \frac{(p^a; p)_{2r} (-p^{1+a/2}; p)_r (p^{1+a-b-c}; p)_r (q^{a_1})_r (q^{a_2})_r (q^{a_3})_r (q^{-m})_r}{(p; p)_r (-p^{a/2}; p)_r (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r (q^{b_1})_r \dots (q^{b_4})_r} (-Q)^r$$

$$q^{r(2b+2c-2a-1-r)} 6\phi_5 \left[\begin{matrix} p^{a+2r}, -p^{1+a/2+r}, q^{a_1+r}, q^{a_2+r}, q^{a_3+r}, q^{r-m} \\ -p^{a/2+r}, q^{b_1+r}, q^{b_2+r}, q^{b_3+r}, q^{b_4+r} \end{matrix} ; p, q, Qq^{2(b+c-a-r)} \right]$$

(2.4.1)

If we take $q^{a_1} = q^{a_2} = -q^{a_3} = -q^{-m}$, $q^{b_1} = -q^{1+a-d} = -q^{b_2}$ and $q^{b_3} = q^{1+a+m} = -q^{b_4}$, the $6\phi_5$ reduces to a $4\phi_3$ with the single base p and the $8\phi_7$ becomes $6\phi_5$, that is we have

$$6\phi_5 \left[\begin{matrix} p^a, -p^{1+a/2}, p^b, p^c, p^d, p^{-m} \\ -p^{a/2}, p^{1+a-b}, p^{1+a-c}, p^{1+a-d}, p^{1+a+m} \end{matrix} ; p, Q \right]$$

$$= \sum_{r=0}^m \frac{(p^a; p)_{2r} (-p^{1+a/2}; p)_r (p^{1+a-b-c}; p)_r (p^d; p)_r (p^{-m}; p)_r}{(p; p)_r (-p^{a/2}; p)_r (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r (p^{1+a-d}; p)_r (p^{1+a+m}; p)_r} (-Q)^r$$

$$p^{r(r+1)/2 + r(b+c-a)} 4\phi_3 \left[\begin{matrix} p^{a+2r}, -p^{1+a/2+r}, p^{d+r}, p^{r-m} \\ -p^{a/2+r}, p^{1+a-d+r}, p^{1+a+r+m} \end{matrix} ; p, Qp^{b+c-a-1-r} \right]$$

(2.4.2)

It is possible to sum the ${}_4\phi_3$ by Dixon's analogue A-12, provided

Q is chosen to satisfy that theorem. Thus Q must be equal to

$p^{2+(3a-2b-2c-2d)/2+m}$, and the ${}_4\phi_3$ is equal to

$$\frac{(p^{1+a+2r}; p)_{m-r} (p^{1+a/2-d}; p)_{m-r}}{(p^{1+a/2+r}; p)_{m-r} (p^{1+a-d+r}; p)_{m-r}} \\ = \frac{(p^{1+a}; p)_m (p^{1+a/2-d}; p)_m (p^{1+a/2}; p)_r (p^{1+a+m}; p)_r (p^{1+a-d}; p)_r p^{\frac{r}{2}(r+1)}}{(p^{1+a/2}; p)_m (p^{1+a-d}; p)_m (p^{1+a}; p)_{2r} (p^{d-a/2-m}; p)_r (-p^{m+1+a/2-d})^r}$$

by A-10 and A-6. Substitute this in (2.4.2) and notice that

$$\frac{(p^{1+a/2}; p)_r (p^a; p)_{2r} (-p^{1+a/2}; p)_r}{(p^{1+a}; p)_{2r} (-p^{a/2}; p)_r} = \frac{(p^{a/2}; p)_r (p^{1/2+a/2}; p)_r (p^{1+a/2}; p)_r}{(q^{a+1})_{2r}} = (p^{a/2}; p)_r,$$

by A-2, A-9 and A-8. Thus we obtain

$${}_6\phi_5 \left[\begin{matrix} p^a, -p^{1+a/2}, p^b, p^c, p^d, p^{-m} \\ -p^{a/2}, p^{1+a-b}, p^{1+a-c}, p^{1+a-d}, p^{1+a+m} \end{matrix} ; p, p^{m+2+(3a-2b-2c-2d)/2} \right] \\ = \frac{(p^{1+a}; p)_m (p^{1+a/2-d}; p)_m}{(p^{1+a/2}; p)_m (p^{1+a-d}; p)_m} {}_4\phi_3 \left[\begin{matrix} p^{1+a-b-c}, p^{a/2}, p^d, p^{-m} \\ p^{1+a-b}, p^{1+a-c}, p^{d-a/2-m} \end{matrix} ; p, p \right]$$

(2.4.3)

This is the q -analogue of Bailey's [21, (2), p. 25]. The ${}_4\phi_3$ in (2.4.3) is balanced. Hence it can be summed provided it reduces to a ${}_3\phi_2$. For example, if $b = 1+a/2$, (2.4.3) leads to $A-1$.

One can set a higher value for s in (2.1.8) and derive Watson's formula (1.2.6). However, since the result is known we refrain from this exercise and use (2.4.1) to find a q -analogue of a known formula transforming a nearly-poised series of the second kind.

In (2.4.1) let

$$q^{a_1} = q^{1+a/2} = -q^{a_2}, \quad q^{a_3} = -q^{1+a-w}, \quad q^{b_1} = -q^{a/2} = -q^{b_2}$$

$$q^{b_3} = q^w \quad \text{and} \quad q^{b_4} = -q^{1+a+m}.$$

Then the ${}_6\phi_5$ on the r.h.s. of (2.4.1) becomes

$$\begin{aligned} A_r &\equiv {}_5\phi_4 \left[\begin{matrix} p^{a+2r}, p^{1+a/2+r}, -p^{1+a/2+r}, -q^{1+a-w+r}, q^{r-m} \\ -p^{a/2+r}, p^{a/2+r}, q^{w+r}, -q^{1+a+m+r} \end{matrix} ; p, q, Qq^{2(b+c-a-1-r)} \right] \\ &= {}_8\phi_7 \left[\begin{matrix} -q^{a+2r}, q\sqrt{-q^{a+2r}}, -q\sqrt{-q^{a+2r}}, q^{a+2r}, q^{1+a/2+r}, -q^{1+a/2+r}, \\ \sqrt{-q^{a+2r}}, -\sqrt{-q^{a+2r}}, -q, -q^{a/2+r}, q^{a/2+r}, \\ -q^{1+a-w+r}, q^{r-m} \\ q^{w+r}, -q^{1+a+m+r} \end{matrix} ; q, Qq^{2(b+c-a-1-r)} \right], \end{aligned} \quad (2.4.4)$$

by A-2. This ${}_8\phi_7$ can be transformed into a balanced ${}_4\phi_3$ using Watson's theorem (1.2.6) provided Q is suitably chosen to satisfy the conditions. For this purpose we choose Q to equal $q^{m+w+a-2b-2c+1}$. Thus

$$A_r = \frac{(-q^{a+1+2r})_{m-r} (-q^{-a/2-r})_{m-r}}{(-q)_{m-r} (-q^{a/2+r})_{m-r}} {}_4\phi_3 \left[\begin{matrix} q^{a+2r}, q^{1+a/2+r}, -q^{2-a/2-1}, q^{r-m} \\ q^{a/2+r}, q^{w+r}, -q^{1+a/2-m+2r} \end{matrix} ; q, q \right]. \quad (2.4.5)$$

This ${}_4\phi_3$ has the right form to be summed by the formula in corollary 1.1, where $a \rightarrow a+2r$, $w \rightarrow w+r$, $m \rightarrow m-r$. Thus we have

$$\begin{aligned} A_r &= \frac{(-q^{a+1+2r})_{m-r} (-q^{-a/2-r})_{m-r} (-q^{a/2+r})_{m-r} (q^{w-a-r})_{m-r} (q^{w-a-1-m}; q^2)_{m-r}}{(-q)_{m-r} (-q^{a/2+r})_{m-r} (-q^{-a/2-r})_{m-r} (q^{w+r})_{m-r} (q^{1+a-w-m+2r})_{m-r}} \\ &\quad \frac{(q^{2+a-w-m+2r}; q^2)_{m-r}}{(q^{w-a-m}; q^2)_{m-r}} \\ &= \frac{(-q^{a+1})_m (q^{w-a})_m (q^{2+a-w-m}; q^2)_m (q^{w-a-1-m}; q^2)_m}{(-q)_m (q^w)_m (q^{1+a-w-m}; q^2)_m (q^{w-a-m}; q^2)_m} \\ &= \frac{(-q^{a+1+m})_r (-q^{-m})_r (q^{1+a-w})_r (q^w)_r (q^{1+a-w-m}; q^2)_r (-1)^r q^{(2+a-w+r-m)r}}{(-q^{a+1})_{2r} (q^{1+a-w-m})_{2r} (q^{3+a-w-m}; q^2)_{2r}} \end{aligned} \quad (2.4.6)$$

The last line was obtained by using A-10, A-7 and A-6 and simplifying. Therefore (2.4.1) becomes

$$7^{\frac{a}{2}} \left[\begin{matrix} p^a, p^{1+a/2}, -p^{1+a/2}, p^b, p^c; -q^{1+a-w}, q^{-m} \\ p^{a/2}, -p^{a/2}, p^{1+a-b}, p^{1+a-c}; q^w, -q^{1+a+m} \end{matrix} \right] ; p, q, q^{a-2b-2c+w+m+1}$$

$$= \frac{(-q^{a+1})_m (q^{w-a})_m (q^{2+a-w-m}; q^2)_m (q^{w-a-1-m}; q^2)_m}{(-q)_m (q^w)_m (q^{1+a-w-m}; q^2)_m (q^{w-a-m}; q^2)_m} \cdot U_m, \quad (2.4.7)$$

where

$$U_m = \sum_{r=0}^m \frac{(p^a; p)_{2r} (-p^{1+a/2}; p)_r (p^{1+a-b-c}; p)_r (p^{1+a/2}; p)_r (-q^{1+a-w})_r (q^{-m})_r}{(p; p)_r (-p^{a/2}; p)_r (p^{1+a-b}; p)_r (p^{1+a-c}; p)_r (p^{a/2}; p)_r (q^w)_r (-q^{1+a+m})_r}$$

$$\cdot \frac{(-q^{1+a+m})_r (-q^{-m})_r (q^{1+a-w})_r (q^w)_r (q^{1+a-w-m}; p)_r}{(-q^{a+1})_{2r} (q^{1+a-w-m})_{2r} (q^{3+a-w-m}; p)_r} p^r. \quad (2.4.8)$$

By A-9 we know that

$$\frac{(p^a; p)_{2r} (-p^{1+a/2}; p)_r}{(-q^{a+1})_{2r} (-p^{a/2}; p)_r (p^{a/2}; p)_r} = (p^{1/2+a/2}; p)_r,$$

and

$$\frac{(q^{1+a-w-m}; p)_r}{(q^{1+a-w-m})_{2r} (q^{3+a-w-m}; p)_r} = \frac{1}{(q^{2+a-w-m}; p)_r (q^{3+a-w-m}; p)_r}$$

Using the above expressions in (2.4.7) we get

$$\begin{aligned}
& 7\phi_6 \left[\begin{matrix} p^a, p^{1+a/2}, -p^{1+a/2}, p^b, p^c; -q^{1+a-w}, q^{-m} \\ p^{a/2}, -p^{a/2}, p^{1+a-b}, p^{1+a-c}; q^w, -q^{1+a+m}; p, q, q^{a-2b-2c+w+m+1} \end{matrix} \right] \\
&= \frac{(-q^{a+1})_m (q^{w-a})_m (q^{2+a-w-m}; q^2)_m (q^{w-a-1-m}; q^2)_m}{(-q)_m (q^w)_m (q^{1+a-w-m}; q^2)_m (q^{w-a-m}; q^2)_m} \\
& \cdot S_4^{\phi} \left[\begin{matrix} p^{1+a/2}, p^{1/2+a/2}, p^{1+a-b-c}, p^{1+a-w}, p^{-m} \\ p^{1+a-b}, p^{1+a-c}, p^{1+(a-w-m)/2}, p^{(3+a-w-m)/2} \end{matrix} ; p, p \right] . \\
& \hspace{25em} (2.4.9)
\end{aligned}$$

This is a q -analogue of Bailey's [21, (2), p. 30].

The last application of theorem 2 is the derivation of a q -analogue of Whipple's transformation of a nearly-poised series of the second kind [69, (6.6)] or Bailey's [21, 4.7(1), p. 33]. We can easily see that (2.1.8) remains true for any value of m provided we choose a to be a negative integer. Let $a = -n$, $-m = d$ and consider the case $s = 1$ with

$$\begin{matrix} a_1 & b_1 & b_2 \\ q & q & q \end{matrix} = \begin{matrix} -q^{1-w-n} & q^w & -q^{1-n-d} \end{matrix} .$$

With these values, the r.h.s. of (2.1.8) becomes

$$\begin{aligned}
s_n &\equiv \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(p^{-n}; p)_{2r} (-p^{1-n/2}; p)_r (p^{1-n-b-c}; p)_r (-q^{1-w-n})_r (q^d)_r}{(p; p)_r (-p^{-n/2}; p)_r (p^{1-n-b}; p)_r (p^{1-n-c}; p)_r (q^w)_r (-q^{1-n-d})_r} \\
&\quad \cdot (-Q)^r q^{r(2b+2c+2n-1-r)} \\
&\quad \cdot {}_6\phi_5 \left[\begin{matrix} -q^{-n+2r}, q\sqrt{-q^{-n+2r}}, -q\sqrt{-q^{-n+2r}}, q^{-n+2r}, -q^{1-w-n+r}, q^{r+d} \\ \sqrt{-q^{-n+2r}}, -\sqrt{-q^{-n+2r}}, -q, q^{w+r}, -q^{1-n-d+r} \end{matrix} ; q, q^v \right]
\end{aligned}
\tag{2.4.10}$$

where

$$q^v = Q q^{2(b+c+n-r-1)}.$$

This ${}_6\phi_5$ can be summed by A-1 provided $q^v = q^{n-2r+w-d}$ which implies that

$$Q = q^{2-n+w-d-2b-2c} \tag{2.4.11}$$

Thus

$${}_6\phi_5[] = \frac{(-q^{1-n+2r})_{n-2r} (q^{w-d})_{n-2r}}{(q^{w+r})_{n-2r} (-q^{1-d+r-n})_{n-2r}}.$$

With the help of A-10, A-6 and A-11, this becomes

$${}_6\phi_5[] = \frac{(-q^{1-n})_n (q^{w-d})_n (q^w)_r (q^{1-w-n})_r (-q^{1-d-n})_r (-q^d)_r}{(q^w)_n (-q^{1-d-n})_n (-q^{1-n})_{2r} (q^{1+d-w-n})_{2r} (-1)^r} q^{(w+n-d-1-r)r}.
\tag{2.4.12}$$

Thus

$$S_n = \frac{(-q^{1-n})_n (q^{w-d})_n}{(q^w)_n (-q^{1-d-n})_n} \sum_r \frac{(p^{-n}; p)_{2r} (-p^{1-n/2}; p)_r (p^{1-n-b-c}; p)_r}{(p; p)_r (-p^{-n/2}; p)_r (p^{1-n-b}; p)_r} \cdot \frac{(-q^{1-w-n})_r (q^{1-w-n})_r (q^d)_r (-q^d)_r}{(p^{1-n-c}; p)_r (-q^{1-n})_{2r} (q^{1+d-w-n})_{2r}} q^{2r} \quad (2.4.13)$$

By A-9

$$\frac{(p^{-n}; p)_{2r} (-p^{1-n/2}; p)_r}{(-q^{1-n})_{2r} (-p^{-n/2}; p)_r} = (p^{-n/2}; p)_r (p^{1/2-n/2}; p)_r,$$

and by A-8

$$(q^{1+d-w-n})_{2r} = (q^{1+d-w-n}; q^2)_r (q^{2+d-w-n}; q^2)_r.$$

This reduces (2.4.13) to

$$S_n = \frac{(-q^{1-n})_n (q^{w-d})_n}{(q^w)_n (-q^{1-d-n})_n} \sum_r \frac{(p^{-n/2}; p)_r (p^{1/2-n/2}; p)_r (p^{1-n-b-c}; p)_r}{(p; p)_r (p^{1-n-b}; p)_r (p^{1-n-c}; p)_r} \cdot \frac{(p^{1-w-n}; p)_r (p^d; p)_r}{(p^{(1+d-w-n)/2}; p)_r (p^{1+(d-w-n)/2}; p)_r} p^r \quad (2.4.14)$$

This is equal to the l.h.s. of (2.1.8), namely

$$\begin{aligned}
& {}_6\phi_5 \left[\begin{matrix} p^{-n}, -p^{1-n/2}, p^b, p^c; -q^{1-w-n}, q^d \\ -p^{-n/2}, p^{1-n-b}, p^{1-n-c}; q^w, -q^{1-n-d} \end{matrix} ; p, q, q^{2-n+w-d-2b-2c} \right] \\
&= \frac{(-q^{1-n})_n (q^{w-d})_n}{(q^w)_n (-q^{1-d-n})_n} {}_5\phi_4 \left[\begin{matrix} p^{-n/2}, p^{1/2-n/2}, p^{1-n-b-c}, p^{1-w-n}, p^d \\ p^{1-n-b}, p^{1-n-c}, p^{(1+d-w-n)/2}, p^{1+(d-w-n)/2} \end{matrix} ; p, p \right] \\
&\quad (2.4.15)
\end{aligned}$$

which is the required transformation.

We now turn to the application of theorem 3. As it stands, the theorem is too general unless the bi-basic series on the r.h.s. of (2.1.9) can be reduced to a series that can be summed. We first look at the case when $s = 1$. Then (2.1.9) becomes

$$\begin{aligned}
& {}_7\phi_6 \left[\begin{matrix} p^a, -p^{1+a/2}, p^b, p^c, p^d; q^{a_1}, q^{-m} \\ -p^{a/2}, p^{1+a-b}, p^{1+a-c}, p^{1+a-d}; q^{b_1}, q^{b_2} \end{matrix} ; p, q, Q \right] \\
&= \sum_{\ell=0}^m \frac{(p^a; p)_{2\ell} (-p^{1+a/2}; p)_\ell (p^K; p)_\ell (p^{1+K/2}; p)_\ell (-p^{1+K/2}; p)_\ell (p^{K+b-a}; p)_\ell}{(p; p)_\ell (-p^{a/2}; p)_\ell (p^{K+1}; p)_{2\ell} (p^{K/2}; p)_\ell (-p^{K/2}; p)_\ell (p^{1+a-b}; p)_\ell} \\
&\quad \cdot \frac{(p^{K+c-a}; p)_\ell (p^{K+d-a}; p)_\ell (q^{a_1})_\ell (q^{-m})_\ell}{(p^{1+a-c}; p)_\ell (p^{1+a-d}; p)_\ell (q^{b_1})_\ell (q^{b_2})_\ell} (Q p^{a-K})^\ell \\
&\quad \cdot {}_5\phi_4 \left[\begin{matrix} p^{a+2\ell}, -p^{1+a/2+\ell}, p^{a-K}; q^{a_1+\ell}, q^{\ell-m} \\ -p^{a/2+\ell}, p^{K+1+2\ell}; q^{b_1+\ell}, q^{b_2+\ell} \end{matrix} ; p, q, Q \right], \\
&\quad (2.4.16)
\end{aligned}$$

where as before $K = 1 + 2a - b - c - d$.

At this stage it is convenient to write the ${}_5\phi_4$ on the r.h.s. of (2.4.16) as an ${}_8\phi_7$ with a single base. This will make it easier to realize the possibilities of choosing the free parameters a_1, b_1, b_2 and Q . Denoting this series by $U_{m,l}$ we have

$$\begin{aligned}
 U_{m,l} &= {}_5\phi_4 \left[\begin{matrix} p^{a+2l}, -p^{1+a/2+l}, p^{a-K}; q^{a_1+l}, q^{l-m} \\ -p^{a/2+l}, p^{K+1+2l}; q^{b_1+l}, q^{b_2+l} \end{matrix} ; p, q, Q \right] \\
 &= {}_8\phi_7 \left[\begin{matrix} -q^{a+2l}, q^{\sqrt{-q^{a+2l}}}, -q^{\sqrt{-q^{a+2l}}}, q^{a+2l}, q^{a-K}, \\ \sqrt{-q^{a+2l}}, -\sqrt{-q^{a+2l}}, -q, -q^{K+1+2l}, \\ -q^{a-K}, q^{a_1+l}, q^{l-m} \\ q^{K+1+2l}, q^{b_1+l}, q^{b_2+l} \end{matrix} ; q, Q \right].
 \end{aligned}
 \tag{2.4.17}$$

This ${}_8\phi_7$ series can be either transformed by Watson's theorem (1.2.6) or summed by Jackson's (1.2.4). In either case it has to be very well-poised and with an appropriate value for Q . To be very well-poised implies that

$$q^{b_1} = -q^{a+1-a_1} \quad \text{and} \quad q^{b_2} = -q^{a+1+m}. \tag{2.4.17'}$$

This will give rise to various cases.

Case I. To be able to use Jackson's summation theorem (1.2.4) we must choose $Q = q$, and the series has to be balanced, i.e. the product of the denominator parameters equals q times the product of the numerator ones. This will require

$${}_q a_1 = -q^{2K-a+1+m}.$$

With this choice of values, Jackson's theorem gives

$$U_{m,l} = \frac{(-q^{a+1+2l})_{m-l} (-q^{K+1-a})_{m-l} (q^{2K+1-a+2l})_{m-l} (q^{K+1-a})_{m-l}}{(-q)_{m-l} (-q^{K+1+2l})_{m-l} (q^{K+1+2l})_{m-l} (q^{2K+1-2a})_{m-l}},$$

which upon simplifying by A-6 and A-10 gives

$$U_{m,l} = \frac{(-q^{a+1})_m (q^{2K+1-a})_m (p^{K+1-a}; p)_m (-q^{a+1+m})_l (q^{2K+1-a+m})_l}{(-q)_m (q^{2K+1-2a})_m (p^{K+1}; p)_m (q^{a-K-m})_l (-q^{a-K-m})_l} \cdot \frac{(-q^{-m})_l (q^{2a-2K-m})_l (p^{K+1}; p)_{2l}}{(-q^{a+1})_{2l} (q^{2K-a+1})_{2l} (p^{K+1+m}; p)_l}.$$

(2.4.18)

Substituting this into (2.4.16) gives

$$7^{\frac{1}{2}} 6 \left[\begin{matrix} p^a, -p^{1+a/2}, p^b, p^c, p^d; -q^{2K-a+m+l}, q^{-m} \\ -p^{a/2}, p^{1+a-b}, p^{1+a-c}, p^{1+a-d}; q^{2a-2K-m}, -q^{a+m+l} \end{matrix} ; p, q, q \right]$$

$$= \frac{(-q^{a+1})_m (q^{2K+1-a})_m (p^{K+1-a}; p)_m}{(-q)_m (q^{2K+1-2a})_m (p^{K+1}; p)_m} \sum_{\ell=0}^m \frac{(p^a; p)_{2\ell} (-p^{1+a/2}; p)_{\ell} (p^K; p)_{\ell}}{(p; p)_{\ell} (-p^{a/2}; p)_{\ell} (p^{K+1}; p)_{2\ell}}$$

$$\frac{(p^{1+K/2}; p)_{\ell} (-p^{1+K/2}; p)_{\ell} (p^{K+b-a}; p)_{\ell} (p^{K+c-a}; p)_{\ell} (p^{K+d-a}; p)_{\ell}}{(p^{K/2}; p)_{\ell} (-p^{K/2}; p)_{\ell} (p^{1+a-b}; p)_{\ell} (p^{1+a-c}; p)_{\ell} (p^{1+a-d}; p)_{\ell}}$$

$$\frac{(-q^{2K-a+m+1})_{\ell} (q^{-m})_{\ell} (qp^{a-K})_{\ell} (-q^{a+1+m})_{\ell} (q^{2K+1-a+m})_{\ell}}{(q^{2a-2K-m})_{\ell} (-q^{a+m+l})_{\ell} (q^{a-K-m})_{\ell} (-q^{a-K-m})_{\ell}}$$

$$\frac{(-q^{-m})_{\ell} (q^{2a-2K-m})_{\ell} (p^{K+1}; p)_{2\ell}}{(-q^{a+1})_{2\ell} (q^{2K-a+1})_{2\ell} (p^{K+1+m}; p)_{\ell}}$$

$$= \frac{(-q^{a+1})_m (q^{2K+1-a})_m (p^{K+1-a}; p)_m}{(-q)_m (q^{2K+1-2a})_m (p^{K+1}; p)_m}$$

$$\cdot 10^{\frac{1}{2}} 9 \left[\begin{matrix} K, p^{1+K/2}, -p^{1+K/2}, p^{a/2}, p^{1/2+a/2}, p^{K+b-a}, p^{K+c-a}, \\ p^{K/2}, -p^{K/2}, p^{K-a/2+1}, p^{K-a/2+1/2}, p^{1+a-b}, p^{1+a-c}, \\ p^{K+d-a}, p^{2K-a+1+m}, p^{-m}, \\ p^{1+a-d}, p^{a-K-m}, p^{K+1+m} \end{matrix} ; p, p^{a-K+1/2} \right]$$

(2.4.19)

The last equality is obtained by using A-2, A-8 and A-9, a very simple exercise. (2.4.19) is a q -analogue of Bailey's [21, (3), p. 31].

Another analogue was obtained by Bailey [20, eq. (1)] that transforms a balanced nearly-poised ${}_5\phi_4$ into a very well-poised balanced

${}_{12}\phi_{11}$.

The other possibility for (2.4.17) and (2.4.17') is to transform it using Watson's theorem (1.2.6). This will imply that we choose

$$Q = -q^{2K-a+2+m-a_1}.$$

Doing that, we get

$$U_{m,\ell} = \frac{(-q^{a+1+2\ell})_{m-\ell} (-q^{K-a+1})_{m-\ell}}{(-q)_{m-\ell} (-q^{K+1+2\ell})_{m-\ell}} {}_4\phi_3 \left[\begin{matrix} q^{a+2\ell}, q^{a-K}, q^{K+1-a_1+\ell}, q^{\ell-m} \\ q^{K+1+2\ell}, -q^{a-a_1+1+\ell}, -q^{a-K-m+\ell} \end{matrix} ; q, q \right]. \quad (2.4.20)$$

Case II. We can see that the above ${}_4\phi_3$ is of the form of the ${}_4\phi_3$ found in corollary 1.2 provided that a_1 is appropriately chosen. This can be seen if we make the correspondence $a \rightarrow a+2\ell$, $b \rightarrow a-K$ and $m \rightarrow m-\ell$. Hence to be able to use corollary 1.2 a_1 must be chosen such that $q^{K+1-a_1+\ell} = -q^{1+a-K-m+\ell}$ or

$$q^{a_1} = -q^{2K-a+m},$$

and hence $Q = q^2$. Therefore

$$\begin{aligned}
U_{m,\ell} &= \frac{(-q^{a+1+2\ell})_{m-\ell} (-q^{K-a+1})_{m-\ell}}{(-q)_{m-\ell} (-q^{K+1+2\ell})_{m-\ell}} \\
& {}_4\phi_3 \left[\begin{matrix} q^{a+2\ell}, q^{a-K}, -q^{a-K+1-m+\ell}, q^{\ell-m} \\ q^{K+1+2\ell}, q^{2a-2K+1-m+\ell}, -q^{a-K-m+\ell} \end{matrix} ; q, q \right] \\
&= \frac{(-q^{a+1+2\ell})_{m-\ell} (-q^{K+1-a})_{m-\ell} (1+q^{K-a})(1+q^{K-a/2+m})}{(-q)_{m-\ell} (-q^{K+1+2\ell})_{m-\ell} (1+q^{K-a+m-\ell})(1+q^{K-a/2+\ell})} \\
& \quad \cdot \frac{(q^{2K-a+2\ell})_{m-\ell} (q^{1+K-a/2+\ell})_{m-\ell} (q^{K-a})_{m-\ell}}{(q^{K+1+2\ell})_{m-\ell} (q^{K-a/2+\ell})_{m-\ell} (q^{2K-2a})_{m-\ell}} \\
&= \frac{(-q^{a+1})_m (q^{2K-a})_m (-q^{1+K-a/2})_m (q^{1+K-a/2})_m (-q^{K-a})_m (q^{K-a})_m}{(-q)_m (q^{2K-2a})_m (-q^{K+1})_m (q^{K+1})_m (-q^{K-a/2})_m (q^{K-a/2})_m} \\
& \quad \cdot \frac{(-q^{K-a/2})_\ell (q^{K-a/2})_\ell (-q^{K+1})_{2\ell} (q^{K+1})_{2\ell} (q^{2K-a+m})_\ell (q^{1-2K+2a-m})_\ell}{(-q^{1+K-a/2})_\ell (q^{1+K-a/2})_\ell (-q^{K+1+m})_\ell (q^{K+1+m})_\ell (q^{1+a-K-m})_\ell} \\
& \quad \cdot \frac{(-q^{1+a+m})_\ell (-q^{-m})_\ell}{(-q^{1+a-K-m})_\ell (q^{2K-a})_{2\ell} (-q^{a+1})_{2\ell}}
\end{aligned}$$

(2.4.21)

This was obtained by realizing that

$$\frac{(1+q^{K-a})(1+q^{K-a/2+m})}{(1+q^{K-a+m-\ell})(1+q^{K-a/2+\ell})} = \frac{(-q^{K-a})_{m-\ell} (-q^{1+K-a/2})_m (-q^{K-a/2})_\ell}{(-q^{1+K-a})_{m-\ell} (-q^{1+K-a/2})_\ell (-q^{K-a/2})_m},$$

and then using A-6 and A-10. Next, we use A-2 and get

$$U_{m,\ell} = \frac{(-q^{a+1})_m (q^{2K-a})_m (p^{1+K-a/2}; p)_m (p^{K-a}; p)_m}{(-q)_m (q^{2K-2a})_m (p^{K+1}; p)_m (p^{K-a/2}; p)_m}$$

$$\cdot \frac{(p^{K-a/2}; p)_\ell (p^{K+1}; p)_{2\ell} (q^{2K-a+m})_\ell (q^{1+2a-2K-m})_\ell}{(p^{1+K-a/2}; p)_\ell (p^{K+1+m}; p)_\ell (p^{1+a-K-m}; p)_\ell (q^{2K-a})_{2\ell}}$$

$$\cdot \frac{(-q^{1+a+m})_\ell}{(-q^{a+1})_{2\ell}} (-q^{-m})_\ell \quad (2.4.22)$$

We now substitute this in (2.4.16) and obtain

$$7^{\Phi_6} \left[\begin{matrix} p^a, -p^{1+a/2}, & p^b, & p^c, & p^d; & -q^{2K-a+m}, q^{-m} \\ & -p^{a/2}, p^{1+a-b}, & p^{1+a-c}, & p^{1+a-d}; & q^{2a-2K+1-m}, -q^{1+a-m} \end{matrix} ; p, q, p \right]$$

$$= \frac{(-q^{a+1})_m (q^{2K-a})_m (p^{1+K-a/2}; p)_m (p^{K-a}; p)_m}{(-q)_m (q^{2K-2a})_m (p^{K+1}; p)_m (p^{K-a/2}; p)_m}$$

$$\sum_{\ell=0}^m \frac{(p^a; p)_{2\ell} (-p^{1+a/2}; p)_{\ell} (p^K; p)_{\ell} (p^{1+K/2}; p)_{\ell} (-p^{1+K/2}; p)_{\ell} (p^{K+b-a}; p)_{\ell}}{(p; p)_{\ell} (-p^{a/2}; p)_{\ell} (p^{K/2}; p)_{\ell} (-p^{K/2}; p)_{\ell} (p^{1+a-b}; p)_{\ell} (p^{1+a-c}; p)_{\ell}}$$

$$\frac{(p^{K+c-a}; p)_{\ell} (p^{K+d-a}; p)_{\ell} (-q^{2K-a+m})_{\ell} (q^{-m})_{\ell} (q^{2K-a+m})_{\ell} (p^{K-a/2}; p)_{\ell}}{(p^{1+a-d}; p)_{\ell} (p^{1+K-a/2}; p)_{\ell} (p^{1+K+m}; p)_{\ell} (p^{1+a-K-m}; p)_{\ell} (p^{K-a/2}; p)_{\ell}}$$

$$\cdot \frac{(-q^{-m})_{\ell}}{(p^{K+1/2-a/2}; p)_{\ell} (q^{a+1})_{2\ell}} (p^{1+a-K})_{\ell} \quad (2.4.23)$$

Using A-9, the following transformation follows

$$7^{\frac{1}{2}} 6 \left[\begin{matrix} a, & 1+a/2, & b, & c, & d, & -q^{2K-a+m}, & q^{-m} \\ p, & -p, & p, & p, & p, & -q^{2K-a+m}, & q^{-m} \end{matrix} ; p, q, p \right]$$

$$= \frac{(-q^{a+1})_m (q^{2K-a})_m (p^{1+K-a/2}; p)_m (p^{K-a}; p)_m}{(-q)_m (q^{2K-2a})_m (p^{K+1}; p)_m (p^{K-a/2}; p)_m}$$

$$10^{\frac{1}{2}} 9 \left[\begin{matrix} K, & 1+K/2, & -p^{1+K/2}, & p^{a/2}, & p^{(a+1)/2}, & p^{K+b-a}, & p^{K+c-a}, \\ p, & p, & -p, & p, & p, & p^{K+b-a}, & p^{K+c-a}, \end{matrix} \right]$$

$$\left[\begin{matrix} K+d-a, & 2K-a+m, & p^{-m} \\ p, & p, & p^{-m} \end{matrix} ; p, p^{a-K+1} \right], \quad (2.4.24)$$

where $K = 1 + 2a - b - c - d$ as before. This is a q -analogue of Bailey's [21, (5), p. 31].

So far we have used corollaries 1.1 and 1.2 in deriving (2.4.9) and (2.4.24) respectively. In order to take advantage of corollaries 1.3 and 1.4, we set $s = 3$ in theorem 3. We also take

$$\begin{aligned} q^{a_1} &= q^{1+a/2} = -q^{a_2}, & -q^{b_1} &= q^{a/2} = q^{b_2} \\ q^{b_3} &= -q^{1+a-a_3}, & q^{b_4} &= q^{1+a+m} \end{aligned}$$

With these values (2.1.9) of theorem 3 becomes

$$8^{\frac{6}{7}} \left[\begin{matrix} a, & 1+a/2, & -p^{1+a/2}, & b, & c, & d; & q^{a_3}, & q^{-m} \\ p, & p, & -p, & p, & p, & p; & & \end{matrix} ; p, q, Q \right]$$

$$\left[\begin{matrix} a/2, & -p^{a/2}, & p^{1+a-b}, & p^{1+a-c}, & p^{1+a-d}; & -q^{1+a-a_3}, & -q^{1+a+m} \\ p, & -p, & p, & p, & p; & & \end{matrix} \right]$$

$$= \sum_{\ell=0}^m \frac{(p^a; p)_{2\ell} (p^{1+a/2}; p)_{\ell} (-p^{1+a/2}; p)_{\ell} (p^K; p)_{\ell} (p^{1+K/2}; p)_{\ell} (-p^{1+K/2}; p)_{\ell}}{(p; p)_{\ell} (-p^{a/2}; p)_{\ell} (-p^{a/2}; p)_{\ell} (p^{K+1}; p)_{2\ell} (p^{K/2}; p)_{\ell} (-p^{K/2}; p)_{\ell}}$$

$$\frac{(p^{K+b-a}; p)_{\ell} (p^{K+c-a}; p)_{\ell} (p^{K+d-a}; p)_{\ell} (q^{a_3})_{\ell} (q^{-m})_{\ell} (Qp^{a-K})_{\ell}}{(p^{1+a-b}; p)_{\ell} (p^{1+a-c}; p)_{\ell} (p^{1+a-d}; p)_{\ell} (-q^{1+a-a_3})_{\ell} (-q^{1+a+m})_{\ell}}$$

$$6^{\frac{6}{5}} \left[\begin{matrix} a+2\ell, & 1+a/2+\ell, & -p^{1+a/2+\ell}, & p^{a-K}, & q^{a_3+\ell}, & q^{\ell-m} \\ p, & p, & -p, & p; & & \end{matrix} ; p, q, Q \right]$$

$$\left[\begin{matrix} a/2+\ell, & -p^{a/2+\ell}, & p^{K+1+2\ell}; & -q^{1+a-a_3+\ell}, & -q^{1+a+m+\ell} \\ p, & -p, & p; & & \end{matrix} \right]$$

(2.4.25)

Case III. To use corollary 1.3 on the above $6\phi_5$, we must choose $Q = q$ and $q^{a_3} = -q^{2K-a+m}$. Thus the series becomes summable and equal to

$$\begin{aligned}
 & 6\phi_5 \left[\begin{matrix} p^{a+2\ell}, p^{1+a/2+\ell}, -p^{1+a/2+\ell}, p^{a-K}; -q^{2K-a+m+\ell}, q^{\ell-m} \\ p^{a/2+\ell}, -p^{a/2+\ell}, p^{1+K+2\ell}; q^{1+2a-2K-m+\ell}, -q^{1+a+m+\ell} \end{matrix} ; p, q, q \right] \\
 &= \frac{(-q^{a+1+2\ell})_{m-\ell} (q^{2K-a+2\ell})_{m-\ell} (p^{K-a}; p)_{m-\ell}}{(-q)_{m-\ell} (q^{2K-2a})_{m-\ell} (p^{K+1+2\ell}; p)_{m-\ell}} q^{m-\ell} \\
 &= \frac{(-q^{a+1})_m (q^{2K-a})_m (p^{K-a}; p)_m (-q^{a+1+m})_\ell (q^{2K-a+m})_\ell (q^{1+2a-2K-m})_\ell}{(-q)_m (q^{2K-2a})_m (p^{K+1}; p)_m (-q^{a+1})_{2\ell} (q^{2K-a})_{2\ell} (p^{1+a-K-m}; p)_\ell} \\
 &= \frac{(-q^{-m})_\ell (p^{K+1}; p)_{2\ell}}{(p^{1+K+m}; p)_\ell} q^m, \quad (2.4.26)
 \end{aligned}$$

by A-6 and A-10. This, when substituted in (2.4.25), gives

$$\begin{aligned}
 & 8\phi_7 \left[\begin{matrix} p^a, p^{1+a/2}, -p^{1+a/2}, p^b, p^c, p^d; -q^{2K-a+m}, q^{-m} \\ p^{a/2}, -p^{a/2}, p^{1+a-b}, p^{1+a-c}, p^{1+a-d}; q^{2a-2K-m+1}, -q^{1+a+m} \end{matrix} ; p, q, q \right] \\
 &= \frac{(-q^{a+1})_m (q^{2K-a})_m (p^{K-a}; p)_m q^m}{(-q)_m (q^{2K-2a})_m (p^{K+1}; p)_m} \sum_{\ell=0}^m \frac{(p^a; p)_{2\ell} (p^{1+a/2}; p)_\ell (-p^{1+a/2})_\ell (p^K; p)_\ell}{(p; p)_\ell (p^{a/2}; p)_\ell (-p^{a/2}; p)_\ell (-q^{a+1})_{2\ell}}
 \end{aligned}$$

$$\begin{aligned}
& \frac{(p^{1+K/2}; p)_\ell (-p^{1+K/2}; p)_\ell (p^{K+b-a}; p)_\ell (p^{K+c-a}; p)_\ell (p^{K+d-a}; p)_\ell}{(p^{K/2}; p)_\ell (-p^{K/2}; p)_\ell (p^{1+a-b}; p)_\ell (p^{1+a-c}; p)_\ell (p^{1+a-d}; p)_\ell} \\
& \cdot \frac{(-q^{2K-a+m})_\ell (q^{-m})_\ell (-q^{a+1+m})_\ell (q^{2K-a+m})_\ell (q^{1+2a-2K-m})_\ell}{(q^{2a-2K+1-m})_\ell (-q^{1+a+m})_\ell (q^{2K-a})_{2\ell} (p^{1+a-K-m}; p)_\ell (p^{1+K+m}; p)_\ell} \\
& \cdot (-q^{-m})_\ell (p^{1/2+a-K})_\ell \\
& = \frac{(-q^{a+1})_m (q^{2K-a})_m (p^{K-a}; p)_m}{(-q)_m (q^{2K-2a})_m (p^{K+1}; p)_m} q^m \\
& 10^{\frac{5}{9}} \left[\begin{array}{c} p^K, p^{1+K/2}, -p^{1+K/2}, p^{(a+1)/2}, p^{1+a/2}, p^{K+b-a}, \\ p^{K/2}, -p^{K/2}, p^{K-a/2+1/2}, p^{K-a/2}, p^{1+a-b}, \\ p^{K+c-a}, p^{K+d-a}, p^{2K-a+m}, p^{-m} \\ ; p, p^{a-K+1/2} \\ p^{1+a-c}, p^{1+a-d}, p^{1+a-K-m}, p^{1+K+m} \end{array} \right] \quad (2.4.27)
\end{aligned}$$

by A-9 and A-8. This a q -analogue of Bailey's [21, (4), p. 31].

Case IV. To use corollary 1.4, Q must equal $q^2 = p$ and $q^{a_3} = -q^{2K-a+m-1}$. In this case the $6\frac{5}{9}$ in (2.4.25) becomes

$$\begin{aligned}
& {}_6\phi_5 \left[\begin{matrix} p^{a+2\ell}, p^{1+a/2+\ell}, -p^{1+a/2+\ell}, p^{a-K}; -q^{2K-a+m-1+\ell}, q^{\ell-m} \\ p^{a/2+\ell}, -p^{a/2+\ell}, p^{1+K+2\ell}; q^{2a+2-2K-m+\ell}, -q^{1+a+m+\ell} \end{matrix} ; p, q, p \right] \\
&= \frac{(-q^{a+1})_{m-\ell} (q^{2K-a-1+2\ell})_{m-\ell} (p^{K-a/2+1/2+\ell}; p)_{m-\ell} (p^{K-a-1}; p)_{m-\ell}}{(-q)_{m-\ell} (q^{2K-2a})_{m-\ell} (p^{K-a/2-1/2+\ell}; p)_{m-\ell} (p^{K+1+2\ell}; p)_{m-\ell}} q^{m-\ell} \\
&= \frac{(-q^{a+1})_m (q^{2K-a-1})_m (p^{K-a/2+1/2}; p)_m (p^{K-a-1}; p)_m (-q^{-m})_\ell q^m}{(-q)_m (q^{2K-2a-1})_m (p^{K-a/2-1/2}; p)_m (p^{K+1}; p)_m (-q^{a+1})_{2\ell}} \\
&= \frac{(q^{2+2a-2K-m})_\ell (-q^{a+1+m})_\ell (q^{2K-a-1+m})_\ell (p^{K-a/2-1/2}; p)_\ell (p^{K+1}; p)_{2\ell}}{(q^{2K-a-1})_{2\ell} (p^{K-a/2+1/2}; p)_\ell (p^{2+a-K-m}; p)_\ell (p^{K+1+m}; p)_\ell} q^\ell \\
& \quad (2.4.28)
\end{aligned}$$

by A-6 and A-10. When we introduce this in (2.4.25) we get

$$\begin{aligned}
& {}_8\phi_7 \left[\begin{matrix} p^a, p^{1+a/2}, -p^{1+a/2}, p^b, p^c, p^d; -q^{2K-a-1+m}, q^{-m} \\ p^{a/2}, -p^{a/2}, p^{1+a-b}, p^{1+a-c}, p^{1+a-d}; q^{2a-2K+2-m}, -q^{1+a+m} \end{matrix} ; p, q, p \right] \\
&= \frac{(-q^{a+1})_m (q^{2K-a-1})_m (p^{K-a/2+1/2}; p)_m (p^{K-a-1}; p)_m}{(-q)_m (q^{2K-2a})_m (p^{K-a/2-1/2}; p)_m (p^{K+1}; p)_m} q^m \\
&= \sum_{\ell=0}^m \frac{(p^a; p)_{2\ell} (p^{1+a/2}; p)_\ell (-p^{1+a/2}; p)_\ell (p^K; p)_\ell (p^{1+K/2}; p)_\ell (-p^{1+K/2}; p)_\ell}{(p; p)_\ell (p^{a/2}; p)_\ell (-p^{a/2}; p)_\ell (p^{K/2}; p)_\ell (-p^{K/2}; p)_\ell (-q^{a+1})_{2\ell}}
\end{aligned}$$

$$\begin{aligned}
& \frac{(p^{K+b-a}; p)_\ell (p^{K+c-a}; p)_\ell (p^{K+d-a}; p)_\ell (-q^{2K-a-1+m})_\ell (q^{-m})_\ell (-q^{-m})_\ell}{(p^{1+a-b}; p)_\ell (p^{1+a-c}; p)_\ell (p^{1+a-d}; p)_\ell (q^{2K-a-1})_{2\ell} (p^{K-a/2+1/2}; p)_\ell} \\
& \frac{(q^{2K-a-1+m})_\ell (p^{K-a/2-1/2}; p)_\ell}{(p^{2+a-K-m}; p)_\ell (p^{K+1+m}; p)_\ell} (q p^{a-K})^\ell.
\end{aligned}
\tag{2.4.29}$$

Simplifying the terms by A-2, A-8 and A-9 we get

$$\begin{aligned}
& 8^{\frac{\phi}{2}} 7 \left[\begin{matrix} p^a, p^{1+a/2}, -p^{1+a/2}, p^b, p^c, p^d; -q^{2K-a-1+m}, q^{-m} \\ p^{a/2}, -p^{a/2}, p^{1+a-b}, p^{1+a-c}, p^{1+a-d}; q^{2a-2K+2-m}, -q^{1+a+m} \end{matrix} ; p, q, p \right] \\
& = \frac{(-q^{a+1})_m (q^{2K-a-1})_m (p^{K-a-1}; p)_m (1-p^{K-a/2-1/2+m})}{(-q)_m (q^{2K-2a-1})_m (p^{K+1}; p)_m (1-p^{K-a/2-1/2})} q^m \\
& \cdot 10^{\frac{\phi}{2}} 9 \left[\begin{matrix} p^K, p^{1+K/2}, -p^{1+K/2}, p^{(a+1)/2}, p^{1+a/2}, p^{K+b-a}, p^{K+c-a}, \\ p^{K/2}, -p^{K/2}, p^{K-a/2+1/2}, p^{K-a/2}, p^{1+a-b}, p^{1+a-c}, \\ p^{K+d-a}, p^{2K-a-1+m}, p^{-m} \\ p^{1+a-d}, p^{2+a-K-m}, p^{1+K+m} \end{matrix} ; p, p^{a-K+3/2} \right].
\end{aligned}
\tag{2.4.30}$$

This is a q -analogue of Bailey's [21, (6), p. 32]. It may be compared with Bailey's transformation [20, eq. (2)] which transforms a very

nearly-poised ${}_{7\phi_6}$ to a very well-poised balanced ${}_{12\phi_{11}}$.

In closing this section it would be appropriate to point out that the series on the left-hand sides of (2.4.9), (2.4.24), (2.4.27) and (2.4.30) are very well-poised, even though in the limit $q \rightarrow 1$ they all approach nearly-poised series of the second kind. It is conceivable that by starting with other extensions of Whipple's transformation (1.3.3) one may obtain transformations between well-poised series rather than very well-poised.

5. q-analogues of transformations of nearly-poised series of the

first kind. When a series is terminating the terms can be rearranged without changing the sum of that series. If a terminating series has $m+1$ terms $a_0, a_1, \dots, a_m$, then if we reverse the order of summation, the r th term becomes the $(m+1-r)$ th one.

Therefore to reverse this series we make the variable correspondence $r \rightarrow m-r$. One way of obtaining transformations of nearly-poised series of the first kind is to reverse the series in transformations of nearly-poised ones of the second kind, (see Bailey [21, Ch. 4]). To do this in the case of q -series, we have in general

$$\begin{aligned}
& A+B \begin{bmatrix} a_1, \dots, a_A; & b_1, \dots, b_{B-1}, q^{-m} \\ c_1, \dots, c_{A-1}; & d_1, \dots, d_{B-1}, d_B \end{bmatrix} ; p, q, z \\
&= \sum_{r=0}^m \frac{(a_1; p)_r \dots (a_A; p)_r (b_1; q)_r \dots (b_{B-1}; q)_r (q^{-m}; q)_r}{(p; p)_r (c_1; p)_r \dots (c_{A-1}; p)_r (d_1; q)_r \dots (d_B; q)_r} z^r \\
&= \sum_{r=0}^m \frac{(a_1; p)_{m-r} \dots (a_A; p)_{m-r} (b_1; q)_{m-r} \dots (b_{B-1}; q)_{m-r} (q^{-m}; q)_{m-r}}{(p; p)_{m-r} (c_1; p)_{m-r} \dots (c_{A-1}; p)_{m-r} (d_1; q)_{m-r} \dots (d_B; q)_{m-r}} z^{m-r} \\
&= \frac{(a_1; p)_m \dots (a_A; p)_m (b_1; q)_m \dots (b_{B-1}; q)_m (q^{-m}; q)_m}{(p; p)_m (c_1; p)_m \dots (c_{A-1}; p)_m (d_1; q)_m \dots (d_B; q)_m} z^m \\
&= \sum_{r=0}^m \frac{(q^{1-m}/d_1; q)_r \dots (q^{1-m}/d_B; q)_r (p^{1-m}/c_1; p)_r \dots (p^{1-m}/c_{A-1}; p)_r (p^{-m}; p)_r}{(q; q)_r (q^{1-m}/b_1; q)_r \dots (q^{1-m}/b_{B-1}; q)_r (p^{1-m}/a_1; p)_r \dots (p^{1-m}/a_A; p)_r} \\
&= \left[\frac{c_1 \cdot c_2 \dots c_{A-1} \cdot d_1 \dots d_B p q^m}{a_1 \cdot a_2 \dots a_A \cdot b_1 \dots b_{B-1} z} \right]^r
\end{aligned}$$

(2.5.1)

by A-6. Using A-3 we have

$$(q^{-m})_m = (-1)^m q^{-\frac{m}{2}(m+1)} (q)_m.$$

Thus the reversal formula is

$$\begin{aligned}
 & {}_{A+B} \phi_{A+B-1} \left[\begin{matrix} a_1, \dots, q_A; b_1, \dots, b_{B-1}, q^{-m} \\ c_1, \dots, c_{A-1}; d_1, \dots, d_B \end{matrix} ; p, q, z \right] \\
 &= \frac{(a_1; p)_m \dots (a_A; p)_m (b_1; q)_m \dots (b_{B-1}; q)_m (q; q)_m (-1)^m q^{-\frac{m}{2}(m+1)}}{(p; p)_m (c_1; p)_m \dots (c_{A-1}; p)_m (d_1; q)_m \dots (d_B; q)_m} z^m \\
 & \cdot {}_{A+B} \phi_{A+B-1} \left[\begin{matrix} q^{1-m}/d_1, \dots, q^{1-m}/d_B; p^{1-m}/c_1, \dots, p^{1-m}/c_{A-1}, p^{-m} \\ q^{1-m}/b_1, \dots, q^{1-m}/b_{B-1}; p^{1-m}/a_1, \dots, p^{1-m}/a_A \end{matrix} ; q, p, \frac{c_1 \dots c_{A-1} \cdot d_1 \dots d_B p q^m}{a_1 \dots a_A \cdot b_1 \dots b_{B-1} z} \right] \\
 & \quad (2.5.2)
 \end{aligned}$$

Next, we apply (2.5.2) to (2.1.3). Doing that to both sides of the transformation results in

$$\begin{aligned}
 & \frac{(p^a; p)_m (-p^{1+a/2}; p)_m (p^b; p)_m (p^c; p)_m (-q^{1+a-w})_m (q)_m (-q^R)_m q^{-\frac{m}{2}(m+1)}}{(p; p)_m (-p^{a/2}; p)_m (p^{1+a-b}; p)_m (p^{1+a-c}; p)_m (q^w)_m (-q^{1+a+m})_m} \\
 & \cdot {}_6 \phi_5 \left[\begin{matrix} q^{1-w-m}, -q^{-a-2m}; p^{b-a-m}, p^{c-a-m}, -p^{1-a/2-m}, p^{-m} \\ -q^{w-a-m}; p^{1-b-m}, p^{1-c-m}, -p^{-a/2-m}, p^{1-a-m} \end{matrix} ; q, p, q^R \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(p^{1+a-w}; p)_m (p^{a/2}; p)_m (p^{(a+1)/2}; p)_m (p^{1+a-b-c}; p)_m (-p)^m p^{-\frac{m}{2}(m+1)}}{(p^{1+a-b}; p)_m (p^{1+a-c}; p)_m (p^{(1+a-w-m)/2}; p)_m (p^{1+(a-w-m)/2}; p)_m} \\
&\quad \frac{(q^{w-a})_m (-q^{a+1})_m}{(q^w)_m (-q)_m} {}_5\phi_4 \left[\begin{matrix} p^{b-a-m}, p^{c-a-m}, p^{(w-a-m+1)/2}, p^{(w-a-m)/2}, p^{-m} \\ p^{w-a-m}, p^{1-a/2-m}, p^{1/2-a/2-m}, p^{b+c-a-m} \end{matrix} ; p, p \right].
\end{aligned}$$

(2.5.3)

To simplify the coefficients of these series we have

$$\frac{(p; p)_m}{(q)_m (-q)_m} = 1,$$

by A-2,

$$\begin{aligned}
&\frac{(-q^{1+a+m})_m (-p^{a/2}; p)_m (p^{a/2}; p)_m (p^{(a+1)/2}; p)_m (-q^{a+1})_m}{(-p^{1+a/2}; p)_m} \\
&= \frac{(q^a)_{2m} (1+p^{a/2}) (-q^{a+1})_{2m}}{(1+p^{a/2+m})} = (q^a)_{2m} (-q^a)_{2m} = (p^a; p)_{2m},
\end{aligned}$$

by A-4, A-8 and A-2, and

$$\begin{aligned}
&\frac{(p^{1+a-w}; p)_m (q^{w-a})_m}{(p^{(1+a-w-m)/2}; p)_m (p^{1+(a-w-m)/2}; p)_m (-q^{1+a-w})_m} = \frac{(q^{1+a-w})_m (q^{w-a})_m}{(q^{1+a-w-m})_{2m}} \\
&= \frac{(q^{1+a-w})_m (q^{w-a})_m}{(q^{1+a-w-m})_m (q^{1+a-w})_m} = \frac{(q^{w-a})_m q^{\frac{m}{2}(m+1)}}{(q^{w-a})_m (-q^{1+a-w})_m} = (-1)^m q^{\frac{m}{2}(m+1) - (1+a-w)m}
\end{aligned}$$

by A-2, A-8, A-4 and A-3.

Plugging these terms in (2.5.3) we get

$$\begin{aligned}
 & {}_6\phi_5 \left[\begin{matrix} 1-w-m, -q^{-a-2m}, p^{b-a-m}, p^{c-a-m}, -p^{1-a/2-m}, p^{-m} \\ -q^{w-a-m}, p^{1-b-m}, p^{1-c-m}, -p^{-a/2-m}, p^{1-a-m} \end{matrix} ; q, p, q^R \right] \\
 &= \frac{(p^a; p)_{2m} (p^{1+a-b-c}; p)_m (-1)^m}{(p^a; p)_m (p^b; p)_m (p^c; p)_m} p^{m(b+c-a) - \frac{m}{2}(m+1)} \\
 &\cdot {}_5\phi_4 \left[\begin{matrix} p^{b-a-m}, p^{c-a-m}, p^{(w-a-m+1)/2}, p^{(w-a-m)/2}, p^{-m} \\ p^{w-a-m}, p^{1-a/2-m}, p^{1/2-a/2-m}, p^{b+c-a-m} \end{matrix} ; p, p \right], \\
 & \hspace{15em} (2.5.4)
 \end{aligned}$$

where $R = a - 2b - 2c + w + m + 2$ as before. If we set $1 - a - 2m = \lambda$, $1 - w - m = A$, $b - a - m = B$, $c - a - m = C$, the coefficient of the ${}_5\phi_4$ on the r.h.s. of (2.5.4) becomes

$$\begin{aligned}
 & \frac{(p^{\lambda-B-C}; p)_m (p^{1-\lambda-m}; p)_m}{(p^{B+1-\lambda-m}; p)_m (p^{C+1-\lambda-m}; p)_m} (-1)^m p^{m(B+C+1-\lambda) - \frac{m}{2}(m+1)} \\
 &= \frac{(p^{\lambda-B-C}; p)_m (p^{\lambda}; p)_m}{(p^{\lambda-B}; p)_m (p^{\lambda-C}; p)_m},
 \end{aligned}$$

by A-3. Substituting this in (2.5.4) leads to

$$\begin{aligned}
& {}_6\phi_5 \left[\begin{matrix} -q^{\lambda-1}, q^A; p^B, p^C, -p^{1/2+\lambda/2}, p^{-m} \\ -q^{\lambda-A}; p^{\lambda-B}, p^{\lambda-C}, -p^{\lambda/2}, p^{\lambda+1+m} \end{matrix} ; q, p, q^{3\lambda+2m-A-2B-2C} \right] \\
&= \frac{(p^{\lambda-B-C}; p)_m (p^{\lambda}; p)_m}{(p^{\lambda-B}; p)_m (p^{\lambda-C}; p)_m} {}_5\phi_4 \left[\begin{matrix} p^{(\lambda-A)/2}, p^{(\lambda-A+1)/2}, p^B, p^C, p^{-m} \\ p^{\lambda-A}, p^{\lambda/2}, p^{1/2+\lambda/2}, p^{B+C-\lambda+1-m} \end{matrix} ; p, p \right] \\
& \quad (2.5.5)
\end{aligned}$$

This is a q -analogue of Bailey's [21, (1), p. 32]. If we let $m \rightarrow \infty$ then

$$\begin{aligned}
\lim_{m \rightarrow \infty} q^{2mk} \frac{(p^{-m}; p)_k}{(p^{\lambda+1+m}; p)_k} &= \lim_{m \rightarrow \infty} p^{mk} (-1)^k \frac{(1-p^m) \dots (p^{k-1} - p^m) p^{-mk}}{(p^{\lambda+1+m}; p)_k} \\
&= p^{\frac{k}{2}(k-1)} (-1)^k,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{m \rightarrow \infty} \frac{(p^{-m}; p)_k p^k}{(p^{B+C-\lambda+1-m}; p)_k} &= \lim_{m \rightarrow \infty} \frac{p^{-mk+k} (1-p^m) \dots (p^{k-1} - p^m)}{(p^{B+C-\lambda+1-m}; p)_k \dots (p^{B+C-\lambda+1-k-1-m}; p)_k p^{-mk}} \\
&= \frac{p^k}{(p^{B+C-\lambda+1})^k} = p^{(\lambda-B-C)k}.
\end{aligned}$$

Therefore if we let $m \rightarrow \infty$, (2.5.5) gives us

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(q^A)_k (-q^{\lambda-1})_k (p^B; p)_k (p^C; p)_k (-p^{1/2+\lambda/2}; p)_k}{(q^{\lambda-A})_k (-q^{\lambda-A})_k (p^{\lambda-B}; p)_k (p^{\lambda-C}; p)_k (-p^{\lambda/2}; p)_k} (-1)^k \frac{p^{\frac{k}{2}(k-1)}}{p^{k(B+C+A/2-3\lambda/2)}} \\
&= \frac{(p^{\lambda}; p)_{\infty} (p^{\lambda-B-C}; p)_{\infty}}{(p^{\lambda-B}; p)_{\infty} (p^{\lambda-C}; p)_{\infty}} {}_4\phi_3 \left[\begin{matrix} p^{(\lambda-A)/2}, p^{(\lambda-A+1)/2}, p^B, p^C \\ p^{\lambda-A}, p^{\lambda/2}, p^{1/2+\lambda/2} \end{matrix} ; p, p^{\lambda-B-C} \right] \\
& \quad (2.5.6)
\end{aligned}$$

which is a q -analogue of Bailey's [21, (3), p. 33].

Finally, we reverse the series on both sides of (2.4.9). In a manner similar to what we have done in deriving (2.5.4) we obtain

$$\begin{aligned}
 & 7\phi_6 \left[\begin{matrix} -q^\lambda, q^A; p^{1+\lambda}/2, -p^{1+\lambda}/2, p^B, p^C \\ -q^{\lambda+1-A}, p^\lambda/2, -p^\lambda/2, p^{1+\lambda-B}, p^{1+\lambda-C} \end{matrix} \right] \\
 & \quad p^{-m} \\
 & \quad ; q, p, q^{3\lambda+2m-A-2B-2C+2} \\
 & \quad p^{1+\lambda+m} \left. \vphantom{\begin{matrix} -q^\lambda, q^A; p^{1+\lambda}/2, -p^{1+\lambda}/2, p^B, p^C \\ -q^{\lambda+1-A}, p^\lambda/2, -p^\lambda/2, p^{1+\lambda-B}, p^{1+\lambda-C} \end{matrix}} \right] \\
 & = \frac{(p^{\lambda+1}; p)_m (p^{\lambda+1-B-C}; p)_m}{(p^{\lambda+1-B}; p)_m (p^{\lambda+1-C}; p)_m} 5\phi_4 \left[\begin{matrix} p^{(\lambda-A)/2}, p^{(\lambda-A+1)/2}, p^B, p^C, p^{-m} \\ p^{\lambda-A+1}, p^\lambda/2, p^{(\lambda+1)/2}, p^{B+C-\lambda-m} \end{matrix} ; p, p \right], \\
 & \hspace{15em} (2.5.7)
 \end{aligned}$$

where we have set $\lambda = -a - 2m$, $A = 1 - w - m$, $B = b - a - m$ and

$C = c - a - m$. This is a q -analogue of Bailey's [21, (2), p. 32].

Chapter 3

q-analogues of Cayley-Orr's Theorems

1. Introduction. In Chapter 1 section 3 we mentioned that Whipple [70] used (1.3.3) to prove Orr's and hence Cayley's theorems in a rather simple way. That was by comparing coefficients of powers of z on both sides of the identities. Let's recall those theorems.

Theorem A (Cayley). If

$$(1-z)^{\alpha+\beta-\gamma} {}_2F_1(2\alpha, 2\beta; 2\gamma; z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$${}_2F_1(\alpha, \beta; \gamma + \frac{1}{2}; z) {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma + \frac{1}{2}; z) = \frac{\Gamma(\gamma + \frac{1}{2})}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + n)}{\Gamma(\gamma + \frac{1}{2} + n)} a_n z^n.$$

Theorem B (Orr). If

$$(1-z)^{\alpha+\beta-\gamma-\frac{1}{2}} {}_2F_1(2\alpha, 2\beta; 2\gamma; z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$${}_2F_1(\alpha, \beta; \gamma; z) {}_2F_1(\gamma - \alpha + \frac{1}{2}, \gamma - \beta + \frac{1}{2}; \gamma + 1; z) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \frac{1}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + \frac{1}{2} + n)}{\Gamma(\gamma + 1 + n)} a_n z^n.$$

Theorem C (Orr). If

$$(1-z)^{\alpha+\beta-\gamma-\frac{1}{2}} {}_2F_1(2\alpha-1, 2\beta; 2\gamma-1; z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$${}_2F_1(\alpha, \beta; \gamma; z) {}_2F_1(\gamma - \alpha + \frac{1}{2}, \gamma - \beta - \frac{1}{2}; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \frac{1}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma - \frac{1}{2} + n)}{\Gamma(\gamma + n)} a_n z^n.$$

The q -analogues of these theorems, as can be expected from (2.1.3), involve two bases q and $p = q^2$. The q -analogues of the binomial theorems involved here turn out to be in base p . In the next section we shall prove the following theorems.

Theorem 1 (q -analogue of Cayley's theorem). If

$$\frac{(zp^{\gamma-\alpha-\beta}; p)_{\infty}}{(z; p)_{\infty}} {}_2\phi_1 \left[\begin{matrix} q^{2\alpha}, q^{2\beta} \\ q^{2\gamma} \end{matrix}; q, zq^{2\gamma-2\alpha-2\beta} \right] = \sum_{n=0}^{\infty} a_n z^n,$$

then

$${}_2\phi_1 \left[\begin{matrix} p^{\gamma-\alpha}, p^{\gamma-\beta} \\ p^{\gamma+\frac{1}{2}} \end{matrix}; p, z \right] {}_2\phi_1 \left[\begin{matrix} p^{\alpha}, p^{\beta} \\ p^{\gamma+\frac{1}{2}} \end{matrix}; p, zp^{\gamma+\frac{1}{2}-\alpha-\beta} \right] = \sum_{n=0}^{\infty} \frac{(p^{\gamma}; p)_n}{(p^{\gamma+\frac{1}{2}}; p)_n} a_n z^n.$$

We may remark here that the q -binomial theorem is defined by

$$\frac{(az)_{\infty}}{(z)_{\infty}} = {}_1\phi_0(a; -; q, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n},$$

which goes by the name Heine's theorem (Slater [60, (IV.11), p. 248]).

Theorem 2 (q-analogue of theorem B). If

$$\frac{(zp^{\frac{1}{2}+\gamma-\alpha-\beta}; p)_{\infty}}{(z; p)_{\infty}} {}_2\phi_1 \left[\begin{matrix} q^{2\alpha}, q^{2\beta} \\ q^{2\gamma} \end{matrix} ; q, zq^{2\gamma-2\alpha-2\beta} \right] = \sum_{n=0}^{\infty} a_n z^n,$$

then

$${}_2\phi_1 \left[\begin{matrix} p^{\alpha}, p^{\beta} \\ p^{\gamma} \end{matrix} ; p, zp^{\gamma-\alpha-\beta} \right] {}_2\phi_1 \left[\begin{matrix} p^{\gamma-\alpha+\frac{1}{2}}, p^{\gamma-\beta+\frac{1}{2}} \\ p^{\gamma+1} \end{matrix} ; p, q \right] = \sum_{n=0}^{\infty} \frac{(p^{\gamma+\frac{1}{2}}; p)_n}{(p^{\gamma+1}; p)_n} a_n z^n.$$

Theorem 3 (q-analogue of theorem C). If

$$\frac{(zp^{\frac{1}{2}+\gamma-\alpha-\beta}; p)_{\infty}}{(z; p)_{\infty}} {}_2\phi_1 \left[\begin{matrix} q^{2\alpha-1}, q^{2\beta} \\ q^{2\gamma-1} \end{matrix} ; q, zq^{2\gamma-2\alpha-2\beta} \right] = \sum_{n=0}^{\infty} a_n z^n,$$

then

$${}_2\phi_1 \left[\begin{matrix} p^{\alpha}, p^{\beta} \\ p^{\gamma} \end{matrix} ; p, zp^{\gamma-\alpha-\beta} \right] {}_2\phi_1 \left[\begin{matrix} p^{\gamma-\alpha+\frac{1}{2}}, p^{\gamma-\beta+\frac{1}{2}} \\ p^{\gamma} \end{matrix} ; p, z \right] = \sum_{n=0}^{\infty} \frac{(p^{\alpha-\frac{1}{2}}; p)_n}{(p^{\gamma}; p)_n} a_n z^n.$$

Singh [59] proved these theorems as a consequence of a quadratic transformation that he gave. Our method is to use (2.1.3) to show that coefficients of similar powers on both sides are equal. This will imply three identities which are q-analogues of similar ones in the ordinary case. Another transformation that takes a very well-poised balanced ${}_{10}\phi_9$ to a balanced ${}_4\phi_3$ is also shown, namely,

$$\begin{aligned}
& 10^{\Phi} 9 \left[\begin{array}{cccccc} -q^{\alpha+\beta+\gamma-\frac{1}{2}}, & q\sqrt{-q^{\alpha+\beta+\gamma-\frac{1}{2}}}, & -q\sqrt{-q^{\alpha+\beta+\gamma-\frac{1}{2}}}, & q^{2\beta}, & q^{\gamma}, & -q^{\gamma}, \\ & \sqrt{-q^{\alpha+\beta+\gamma+\frac{1}{2}}}, & -\sqrt{-q^{\alpha+\beta+\gamma+\frac{1}{2}}}, & -q^{\frac{1}{2}+\alpha-\beta+\gamma}, & -q^{\frac{1}{2}+\alpha+\beta}, & q^{\frac{1}{2}+\alpha+\beta} \end{array} \right. \\
& \left. \begin{array}{cccc} q^{2\alpha}, & -q^{\frac{1}{2}+\alpha+\beta-\gamma}, & -q^{m+2\gamma}, & q^{-m} \\ -q^{\frac{1}{2}+\beta+\gamma-\alpha}, & q^{2\gamma}, & q^{\frac{1}{2}+\alpha+\beta-\gamma-m}, & -q^{\frac{1}{2}+\alpha+\beta+\gamma+m} \end{array} \right] ; q, q \\
& = \frac{(p^{\frac{1}{2}+\gamma-\alpha}; p)_m (p^{\frac{1}{2}+\gamma-\beta}; p)_m (-q^{\frac{1}{2}+\alpha+\beta+\gamma})_m}{(p^{\gamma+\frac{1}{2}}; p)_m (q^{\frac{1}{2}-\alpha-\beta+\gamma})_m (-q^{\frac{1}{2}+\alpha+\gamma-\beta})_m (-q^{\frac{1}{2}-\alpha+\beta+\gamma})_m} \\
& \cdot 4^{\Phi} 3 \left[\begin{array}{cccc} p^{-m}, & p^{\alpha}, & p^{\beta}, & p^{\frac{1}{2}+\alpha+\beta-2\gamma-m} \\ p^{\frac{1}{2}+\alpha+\beta}, & p^{\frac{1}{2}+\alpha-\gamma-m}, & p^{\frac{1}{2}+\beta-\gamma-m} & \end{array} \right] ; p, p
\end{aligned}$$

(3.1.1)

This is a q -analogue of Bailey's [21, 10.2(1), p. 86].

2. Proof of theorems. To prove Theorem 1 we need Theorems 2 and 3.

Hence we start with proving Theorem 2. The implied identity to

prove is

$$\begin{aligned}
& 4^{\Phi} 3 \left[\begin{array}{cccc} q^{-m}, & q^{2\alpha}, & q^{2\beta}, & -q^{-m} \\ q^{2\gamma}, & q^{\frac{1}{2}+\alpha+\beta-\gamma-m}, & -q^{\frac{1}{2}+\alpha+\beta-\gamma-m} & \end{array} \right] ; q, q \\
& = \frac{(p^{\frac{1}{2}+\gamma-\alpha}; p)_m (p^{\frac{1}{2}+\gamma-\beta}; p)_m}{(p^{\frac{1}{2}-\alpha-\beta+\gamma}; p)_m (p^{\gamma+\frac{1}{2}}; p)_m} 4^{\Phi} 3 \left[\begin{array}{cccc} p^{-m}, & p^{\alpha}, & p^{\beta}, & p^{-m-\gamma} \\ p^{\gamma}, & p^{\frac{1}{2}+\alpha-\gamma-m}, & p^{\frac{1}{2}+\beta-\gamma-m} & \end{array} \right] ; p, p
\end{aligned}$$

(3.2.1)

This is a q -analogue of [21, 10.1(2)]. Let $c = \frac{a+1}{2}$ in (2.1.3), thus upon doing the following transformation $a \rightarrow 2\alpha$, $b \rightarrow \frac{1}{2} + \alpha + \beta - \gamma$ and $w \rightarrow 1 + 2\alpha - 2\gamma - m$ we get

$$\begin{aligned}
 & {}_5\phi_4 \left[\begin{matrix} p^{2\alpha}, -p^{1+\alpha}, p^{\frac{1}{2}+\alpha+\beta-\gamma}, -q^{2\gamma+m}, q^{-m} \\ -p^\alpha, p^{\frac{1}{2}+\alpha-\beta+\gamma}, : q^{1+2\alpha-2\gamma-m}, -q^{1+2\alpha+m} \end{matrix} ; p, q, q^{1-2\beta} \right] \\
 &= \frac{(q^{1-2\gamma-m})_m (-q^{2\alpha+1})_m}{(q^{1+2\alpha-2\gamma-m})_m (-q)_m} {}_4\phi_3 \left[\begin{matrix} p^{2\gamma+m}, p^\alpha, p^{\gamma-\beta}, p^{-m} \\ p^{\frac{1}{2}+\alpha-\beta+\gamma}, p^\gamma, p^{\frac{1}{2}+\gamma} \end{matrix} ; p, p \right] \\
 &= \frac{(q^{2\gamma})_m (-q^{2\alpha+1})_m (p^{\frac{1}{2}+\gamma-\alpha}; p)_m (p^{\frac{1}{2}+\gamma-\beta}; p)_m}{(q^{2\gamma-2\alpha})_m (-q)_m (p^{\frac{1}{2}+\gamma}; p)_m (p^{\frac{1}{2}+\alpha-\beta+\gamma}; p)_m} \\
 & \cdot {}_4\phi_3 \left[\begin{matrix} p^{-m}, p^\alpha, p^\beta, p^{-m-\gamma} \\ p^\gamma, p^{\frac{1}{2}+\alpha-\gamma-m}, p^{\frac{1}{2}+\beta-\gamma-m} \end{matrix} ; p, p \right].
 \end{aligned}
 \tag{3.2.2}$$

To obtain the last equality, use of (1.2.7) was made.

Next, if we write the ${}_5\phi_4$ on the l.h.s. of (3.2.2) as an ${}_8\phi_7$ using only one base, then applying Watson's theorem (1.2.6) we obtain

$$\begin{aligned}
& 8^{\frac{1}{2}} 7 \left[\begin{array}{cccccc} -q^{2\alpha}, & q\sqrt{-q}^{2\alpha}, & -q\sqrt{-q}^{2\alpha}, & q^{2\alpha}, & q^{\frac{1}{2}+\alpha+\beta-\gamma}, & -q^{\frac{1}{2}+\alpha+\beta-\gamma}, \\ & \sqrt{-q}^{2\alpha}, & -\sqrt{-q}^{2\alpha}, & -q, & -q^{\frac{1}{2}+\alpha-\beta+\gamma}, & q^{\frac{1}{2}+\alpha-\beta+\gamma}, \\ & & & -q^{2\gamma+m}, & q^{-m} & \\ & & & & & q^{1-2\beta} \\ & & & q^{1+2\alpha-2\gamma-m}, & -q^{1+2\alpha+m} & \end{array} \right] \\
& = \frac{(-q^{1+2\alpha})_m (q^{1-2\gamma-m})_m}{(-q)_m (q^{1-2\gamma+2\alpha-m})_m} 4^{\frac{1}{2}} 3 \left[\begin{array}{cccc} q^{-m}, & q^{2\alpha}, & -q^{2\gamma+m}, & q^{2\gamma-2\beta} \\ & & & \\ 2\gamma, & q^{\frac{1}{2}+\alpha-\beta+\gamma}, & -q^{\frac{1}{2}+\alpha-\beta+\gamma} & \end{array} ; q, q \right] \\
& = \frac{(-q^{1+2\alpha})_m (q^{2\gamma})_m (q^{\frac{1}{2}+\alpha-\beta+\gamma})_m (-q^{\frac{1}{2}-\alpha-\beta+\gamma})_m}{(-q)_m (q^{2\gamma-2\alpha})_m (q^{\frac{1}{2}+\alpha-\beta+\gamma})_m (-q^{\frac{1}{2}+\alpha-\beta+\gamma})_m} \\
& \cdot 4^{\frac{1}{2}} 3 \left[\begin{array}{cccc} q^{-m}, & q^{2\alpha}, & q^{2\beta}, & -q^{-m} \\ & & & \\ 2\gamma, & -q^{\frac{1}{2}+\alpha+\beta-\gamma-m}, & q^{\frac{1}{2}+\alpha+\beta-\gamma-m} & \end{array} ; q, q \right] .
\end{aligned}$$

(3.2.3)

To derive the last equality use of (1.2.7) has been made again with some simplifications.

Using (3.2.2) and (3.2.3) we obtain (3.2.1), and hence Theorem 2.

If in (3.2.1) we make the transform $\alpha \rightarrow \alpha - \frac{1}{2}$, $\gamma \rightarrow \gamma - \frac{1}{2}$, the following identity is obtained:

$$\begin{aligned}
& {}_4\phi_3 \left[\begin{matrix} q^{-m}, q^{2\alpha-1}, q^{2\beta}, -q^{-m} \\ q^{2\gamma-1}, q^{\frac{1}{2}+\alpha+\beta-\gamma-m}, -q^{\frac{1}{2}+\alpha+\beta-\gamma-m} \end{matrix} ; q, q \right] \\
&= \frac{(p^{\frac{1}{2}+\gamma-\alpha}; p)_m (p^{\gamma-\beta}; p)_m}{(p^{\frac{1}{2}-\alpha-\beta+\gamma}; p)_m (p^{\gamma}; p)_m} {}_4\phi_3 \left[\begin{matrix} p^{-m}, p^{\beta}, p^{\frac{1}{2}-\gamma-m}, p^{\alpha-\frac{1}{2}} \\ p^{\gamma-\frac{1}{2}}, p^{\frac{1}{2}+\alpha-\gamma-m}, p^{1+\beta-\gamma-m} \end{matrix} ; p, p \right] \\
&= \frac{(p^{\frac{1}{2}+\gamma-\alpha}; p)_m (p^{\gamma-\beta-\frac{1}{2}}; p)_m}{(p^{\frac{1}{2}-\alpha-\beta+\gamma}; p)_m (p^{\gamma-\frac{1}{2}}; p)_m} {}_4\phi_3 \left[\begin{matrix} p^{-m}, p^{\beta}, p^{\alpha}, p^{1-\gamma-m} \\ p^{\gamma}, p^{\frac{1}{2}+\alpha-\gamma-m}, p^{3/2+\beta-\gamma-m} \end{matrix} ; p, p \right].
\end{aligned}
\tag{3.2.4}$$

Use of (1.2.7) was made to derive the last equality. (3.2.4) is

a q -analogue of [21, 10.1(3)], which in turn proves Theorem 3.

Next, let us consider (3.2.1) and (3.2.4). Multiply (3.2.1) by

$$\frac{q^{2\alpha}}{(1-q^{2\alpha})}, \text{ and in (3.2.4) make the transform } a \rightarrow \alpha+1, \gamma \rightarrow \gamma+1,$$

then multiply by $\frac{q^{2\gamma}}{(1-q^{2\gamma})}$ and subtract the two resulting identities.

Thus we obtain

$$\begin{aligned}
& \sum_{r=0}^m \frac{(q^{-m})_r (q^{2\beta})_r (-q^{-m})_r}{(q)_r (q^{\frac{1}{2}+\alpha+\beta-\gamma-m})_r (-q^{\frac{1}{2}+\alpha+\beta-\gamma-m})_r} q^r \left[\frac{q^{2\alpha} (q^{2\alpha})_r}{(1-q^{2\alpha}) (q^{2\gamma})_r} - \frac{q^{2\gamma} (q^{2\alpha+1})_r}{(1-q^{2\gamma}) (q^{2\gamma+1})_r} \right] \\
&= \frac{(p^{\frac{1}{2}+\gamma-\alpha}; p)_m (p^{\frac{1}{2}+\gamma-\beta}; p)_m}{(p^{\frac{1}{2}-\alpha-\beta+\gamma}; p)_m (p^{\gamma+\frac{1}{2}}; p)_m} \sum_{r=0}^m \frac{(p^{-m}; p)_r (p^{\beta}; p)_r (p^{-\gamma-m}; p)_r}{(p; p)_r (p^{\frac{1}{2}+\alpha-\gamma-m}; p)_r (p^{\frac{1}{2}+\beta-\gamma-m}; p)_r} p^r \\
&\quad \cdot \left[\frac{q^{2\alpha} (p^{\alpha}; p)_r}{(1-q^{2\alpha}) (p^{\gamma}; p)_r} - \frac{q^{2\gamma} (p^{\alpha+1}; p)_r}{(1-q^{2\gamma}) (p^{\gamma+1}; p)_r} \right],
\end{aligned}$$

which upon simplifying gives

$$\begin{aligned}
 & \sum_{r=0}^m \frac{(q^{-m})_r (q^{2\alpha})_r (q^{2\beta})_r (-q^{-m})_r}{(q)_r (q^{2\gamma+1})_r (q^{\frac{1}{2}+\alpha+\beta-\gamma-m})_r (-q^{\frac{1}{2}+\alpha+\beta-\gamma-m})_r} q^r \\
 &= \frac{(p^{\frac{1}{2}+\gamma-\alpha}; p)_m (p^{\frac{1}{2}+\gamma-\beta}; p)_m}{(p^{\frac{1}{2}-\alpha-\beta+\gamma}; p)_m (p^{\gamma+\frac{1}{2}}; p)_m} \sum_{r=0}^m \frac{(p^{-m}; p)_r (p^{\alpha}; p)_r (p^{\beta}; p)_r (p^{-\gamma-m}; p)_r p^r}{(p; p)_r (p^{\frac{1}{2}+\alpha-\gamma-m}; p)_r (p^{\frac{1}{2}+\beta-\gamma-m}; p)_r (p^{\gamma+1}; p)_r} \\
 & \hspace{15em} (3.2.5)
 \end{aligned}$$

Replace γ by $\gamma - \frac{1}{2}$, and we get

$$\begin{aligned}
 & {}_4\phi_3 \left[\begin{matrix} q^{-m}, & q^{2\alpha}, & q^{2\beta}, & -q^{-m} \\ q^{2\gamma}, & q^{1+\alpha+\beta-\gamma-m}, & -q^{1+\alpha+\beta-\gamma-m} \end{matrix} ; q, q \right] \\
 &= \frac{(p^{\gamma-\alpha}; p)_m (p^{\gamma-\beta}; p)_m}{(p^{\gamma}; p)_m (p^{\gamma-\alpha-\beta}; p)_m} {}_4\phi_3 \left[\begin{matrix} p^{-m}, & p^{\alpha}, & p^{\beta}, & p^{\frac{1}{2}-\gamma-m} \\ p^{\frac{1}{2}+\gamma}, & p^{1+\alpha-\gamma-m}, & p^{1+\beta-\gamma-m} \end{matrix} ; p, p \right], \\
 & \hspace{15em} (3.2.6)
 \end{aligned}$$

which is a q -analogue of [21, 10.1(1)], which in turn proves Theorem 1.

If we go back to (2.1.3) and let $a \rightarrow 2\alpha$, $b \rightarrow \frac{1}{2} + \alpha + \beta - \gamma$,

$c \rightarrow \frac{1}{2} + \alpha - \beta$, $w = 1 - m + 2\alpha - 2\gamma$, we obtain

$$\begin{aligned}
& {}_6\phi_5 \left[\begin{matrix} 2\alpha, -p^{1+\alpha}, p^{\frac{1}{2}+\alpha+\beta-\gamma}, p^{\frac{1}{2}+\alpha-\beta} : -q^{2\gamma+m}, q^{-m} \\ -p^\alpha, p^{\frac{1}{2}+\alpha-\beta+\gamma}, p^{\frac{1}{2}+\alpha+\beta} : q^{1+2\alpha-2\gamma-m}, -q^{1+2\alpha+m} \end{matrix} ; p, q, q \right] \\
&= \frac{(q^{1-2\gamma-m})_m (-q^{2\alpha+1})_m}{(q^{1+2\alpha-2\gamma-m})_m (-q)_m} {}_4\phi_3 \left[\begin{matrix} p^{-m}, p^\alpha, p^{2\gamma+m}, p^{\alpha+\frac{1}{2}} \\ p^{\frac{1}{2}+\alpha+\beta}, p^{\frac{1}{2}+\alpha-\beta+\gamma}, p^{\gamma+\frac{1}{2}} \end{matrix} ; p, p \right] \\
&= \frac{(q^{2\gamma})_m (-q^{2\alpha+1})_m (p^{\frac{1}{2}+\gamma-\alpha}; p)_m (p^{\frac{1}{2}+\gamma-\beta}; p)_m}{(q^{2\gamma-2\alpha})_m (-q)_m (p^{\frac{1}{2}+\alpha-\beta+\gamma}; p)_m (p^{\gamma+\frac{1}{2}}; p)_m} \\
& {}_4\phi_3 \left[\begin{matrix} p^{-m}, p^\alpha, p^\beta, p^{\frac{1}{2}+\alpha+\beta-2\gamma-m} \\ p^{\frac{1}{2}+\alpha+\beta}, p^{\frac{1}{2}+\alpha-\gamma-m}, p^{\frac{1}{2}+\beta-\gamma-m} \end{matrix} ; p, p \right].
\end{aligned} \tag{3.2.7}$$

The last equality was obtained with the use of (1.2.7) with some simplification of the coefficients.

On the other hand the ${}_6\phi_5$ on the l.h.s. of (3.2.7) can be written as a ${}_{10}\phi_9$ using the one base notation. Hence we get

$$\begin{aligned}
{}_6\phi_5 &= {}_{10}\phi_9 \left[\begin{matrix} -q^{2\alpha}, q\sqrt{-q^{2\alpha}}, -q\sqrt{-q^{2\alpha}}, q^{\frac{1}{2}+\alpha+\beta-\gamma}, q^{\frac{1}{2}+\alpha-\beta}, -q^{\frac{1}{2}+\alpha-\beta}, \\ \sqrt{-q^{2\alpha}}, -\sqrt{-q^{2\alpha}}, -q^{\frac{1}{2}+\alpha+\gamma-\beta}, -q^{\frac{1}{2}+\alpha+\beta}, q^{\frac{1}{2}+\alpha+\beta} \end{matrix} \right. \\
& \left. \begin{matrix} q^{2\alpha}, -q^{\frac{1}{2}+\alpha+\beta-\gamma}, -q^{m+2\gamma}, q^{-m} \\ -q, q^{\frac{1}{2}+\alpha+\gamma-\beta}, q^{1+2\alpha-2\gamma-m}, -q^{1+2\alpha+m} \end{matrix} ; q, q \right]
\end{aligned}$$

$$\frac{(-q)^{1+2\alpha} (q^{\frac{1}{2}-\alpha-\beta+\gamma})_m (q^{1-2\gamma-m})_m (-q^{\frac{1}{2}+\alpha-\beta-\gamma-m})_m}{(-q)_m (q^{\frac{1}{2}+\alpha-\beta+\gamma})_m (q^{1+2\alpha-2\gamma-m})_m (-q^{\frac{1}{2}-\alpha-\beta-\gamma-m})_m}$$

$$\begin{aligned} & 10^{\frac{1}{2}} 9 \left[\begin{array}{c} K, q\sqrt{K}, -q\sqrt{K}, q^{2\beta}, q^\gamma, -q^\gamma, q^{2\alpha}, \\ \sqrt{K}, -\sqrt{K}, -q^{\frac{1}{2}+\alpha-\beta+\gamma}, -q^{\frac{1}{2}+\alpha+\beta}, q^{\frac{1}{2}+\alpha+\beta}, -q^{\frac{1}{2}+\beta+\gamma-\alpha}, \\ -q^{\frac{1}{2}+\alpha+\beta-\gamma}, -q^{m+2\gamma}, q^{-m} \\ q^{2\gamma}, q^{\alpha+\beta-\gamma+\frac{1}{2}-m}, -q^{\frac{1}{2}+\alpha+\beta+\gamma+m} \end{array} ; q, q \right], \\ & (3.2.8) \end{aligned}$$

where $K = -q^{\alpha+\beta+\gamma-\frac{1}{2}}$. (3.2.8) was obtained by using Jackson's transform between two very well-poised $10^{\frac{1}{2}} 9$, namely

$$\begin{aligned} & 10^{\frac{1}{2}} 9 \left[\begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, g, h, j \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h, aq/j \end{array} ; q, q \right] \\ & = \frac{(aq)_\infty (aq/fg)_\infty (aq/fh)_\infty (aq/fj)_\infty (aq/gh)_\infty (aq/gj)_\infty (aq/jh)_\infty (aq/fgjh)_\infty}{(aq/f)_\infty (aq/g)_\infty (aq/h)_\infty (aq/j)_\infty (aq/ghj)_\infty (aq/fhj)_\infty (aq/fg h)_\infty (aq/fgj)_\infty} \end{aligned}$$

$$10^{\frac{1}{2}} 9 \left[\begin{array}{c} K, q\sqrt{K}, -q\sqrt{K}, Kc/a, Kd/a, Ke/a, f, g, h, j \\ \sqrt{K}, -\sqrt{K}, aq/e, aq/d, aq/e, Kq/f, Kq/g, Kq/h, Kq/j \end{array} ; q, q \right]$$

where $K = a^2 q / cde$, $a^3 q^2 = cdefghj$ and f, g, h or j of the form q^{-N} , N a positive integer, (see Slater [60, (3.4.2.4)]). Upon simplifying the coefficients of the $10\phi_9$ on the r.h.s. of (3.2.8) and going back to (3.2.7), we obtain (3.1.1).

3. Special cases. If in Theorems B and C we let $\gamma \rightarrow \alpha + \beta - \frac{1}{2}$, then we get the q -analogues of Bailey [21, 10.1(5) and (6)], namely

$$\begin{aligned}
 {}_2\phi_1 \left[\begin{matrix} p^\alpha, p^\beta \\ p^{\alpha+\beta-\frac{1}{2}} \end{matrix} ; p, zp^{-\frac{1}{2}} \right] {}_2\phi_1 \left[\begin{matrix} p^\alpha, p^\beta \\ p^{\alpha+\beta+\frac{1}{2}} \end{matrix} ; p, z \right] \\
 = {}_4\phi_3 \left[\begin{matrix} q^{\alpha+\beta}, -q^{\alpha+\beta}, q^{2\alpha}, q^{2\beta} \\ q^{\alpha+\beta+\frac{1}{2}}, -q^{\alpha+\beta+\frac{1}{2}}, q^{2\alpha+2\beta-1} \end{matrix} ; q, zq^{-1} \right],
 \end{aligned}
 \tag{3.3.1}$$

and

$$\begin{aligned}
 {}_2\phi_1 \left[\begin{matrix} p^\alpha, p^\beta \\ p^{\alpha+\beta-\frac{1}{2}} \end{matrix} ; p, zp^{-\frac{1}{2}} \right] {}_2\phi_1 \left[\begin{matrix} p^\alpha, p^\beta \\ p^{\alpha+\beta-\frac{1}{2}} \end{matrix} ; p, z \right] \\
 = {}_4\phi_3 \left[\begin{matrix} q^{\alpha+\beta-1}, -q^{\alpha+\beta-1}, q^{2\beta}, q^{2\alpha-1} \\ q^{\alpha+\beta-\frac{1}{2}}, -q^{\alpha+\beta-\frac{1}{2}}, q^{2\alpha+2\beta-2} \end{matrix} ; q, zq^{-1} \right].
 \end{aligned}
 \tag{3.3.2}$$

Similarly, letting $\gamma = \alpha + \beta$ in theorem A, we get

$${}_2\phi_1 \left[\begin{matrix} p^\alpha, p^\beta \\ p^{\alpha+\beta+\frac{1}{2}} \end{matrix} ; p, z \right] {}_2\phi_1 \left[\begin{matrix} p^\alpha, p^\beta \\ p^{\alpha+\beta+\frac{1}{2}} \end{matrix} ; p, zp^{\frac{1}{2}} \right] = {}_4\phi_3 \left[\begin{matrix} q^{2\alpha}, q^{2\beta}, q^{\alpha+\beta}, -q^{\alpha+\beta} \\ q^{\alpha+\beta+\frac{1}{2}}, -q^{\alpha+\beta+\frac{1}{2}}, q^{2\alpha+2\beta} \end{matrix} ; q, z \right],
 \tag{3.3.3}$$

which is Jackson's [43] q -analogue of Clausen's formula [28] or

Bailey's [21, 10.1(4)].

Chapter 4

A Reproducing Kernel and Projection

Formulas for the Continuous q -Wilson Polynomials

1. Introduction. In [50,51], Rahman looked into the problem of constructing reproducing kernels for the Jacobi polynomials. These kernels, of an iterative nature, were obtained by introducing a shift in the variables within the weight function with respect to which the polynomials are orthogonal.

To extend these results to their q -analogues one encounters many possibilities and different methods. q -analogues of the Jacobi polynomials have been found in different forms. There are the ${}_2\phi_1$ polynomials of Hahn [39], for which reproducing kernels were found by Al-Salam and Ismail [3], which are a special case of Rahman's [54]. Other q -analogues are the ${}_3\phi_2$ polynomials studied by Andrews and Askey [9(a)]. Askey and Wilson investigated a very general set of ${}_4\phi_3$ polynomials which extend the classical polynomials. The discrete case was discussed in [17], and recently they presented an extensive study of the general case [18].

The work of Rahman seems to extend rather nicely using these ${}_4\phi_3$ polynomials (q -Wilson polynomials), particularly because they

are balanced which in turn, with the use of Sears' (1.2.7), provide a great maneuverability. In [54], Rahman worked the discrete case for these polynomials, in which he found a very general family of double sum reproducing kernels. In what follows we extend his results to the continuous case.

The polynomials we are talking about are defined by

$$p_n(x) = p_n(x; a, b, c, d) = {}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right] \quad (4.1.1)$$

where $x = \cos \theta$. They satisfy the orthogonality relation

$$\int_{-1}^1 dx w(x) p_n(x) p_m(x) = h_n \delta_{mn}, \quad (4.1.2)$$

where $w(x)$ is the weight function defined by

$$w(x) = w(x; a, b, c, d) = (1-x^2)^{-\frac{1}{2}} \frac{h(x, 1)h(x, \sqrt{q})h(x, -1)h(x, -\sqrt{q})}{h(x, a)h(x, b)h(x, c)h(x, d)}, \quad (4.1.3)$$

with

$$h(x, a) = \prod_{k=0}^{\infty} (1 - 2xaq^k + a^2q^{2k}), \quad (4.1.4)$$

and

$$h_n = h_0 \frac{a^{2n}(q)_n (cd)_n (bd)_n (bc)_n (1-abcdq^{-1})}{(abcdq^{-1})_n (ab)_n (ac)_n (ad)_n (1-abcdq^{2n-1})}. \quad (4.1.5)$$

h_0 is the value of (4.1.2) when $m = n = 0$ and was evaluated by Askey and Wilson [18] and Rahman [53] to be

$$\int_{-1}^1 dx w(x) = \frac{2\pi(abcd)_\infty}{(q)_\infty (ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty} \quad (4.1.6)$$

(4.1.2) and (4.1.6) are valid when $\max(|a|, |b|, |c|, |d|) < 1$.

The integral (4.1.6) furnishes the most general q -extension known, with an absolutely continuous integrand, of the beta integral

$$\int_{-c}^d \left(1 - \frac{x}{d}\right)^{\beta-1} \left(1 + \frac{x}{c}\right)^{\alpha-1} dx = \frac{(c+d)^{\alpha+\beta-1}}{c^{\alpha-1} d^{\beta-1}} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (4.1.7)$$

(see Andrews and Askey [8], Askey and Wilson [18]). In [18] the authors wonder about the existence of four free parameters in that integral and they mention that special assignments to some of these parameters would make the polynomials look like the Jacobi polynomials, as was also investigated by Rahman [53]. This feature would be seen to be the key to the present work.

It is easily seen from (4.1.1) and (4.1.6) that

$$\begin{aligned} & \int_{-1}^1 dx w(x; a, b, c, d) p_n(x; a, b, c', d') \\ &= \sum_{k=0}^n \frac{(q^{-n})_k (q^{n-1} abc'd')_k}{(q)_k (ab)_k (ac')_k (ad')_k} q^k \int_{-1}^1 dx w(x; a, b, c, d) (ae^{i\theta})_k (ae^{-i\theta})_k \\ & \quad (4.1.8) \end{aligned}$$

But, from (4.1.3) and (4.1.4) we can check that

$$\begin{aligned}
 \int_{-1}^1 dx w(x; a, b, c, d) (ae^{i\theta})_k (ae^{-i\theta})_k &= \int_{-1}^1 dx w(x; aq^k, b, c, d) \\
 &= \frac{2\pi (abcd q^k)_\infty}{(q)_\infty (abq^k)_\infty (acq^k)_\infty (adq^k)_\infty (bc)_\infty (bd)_\infty (cd)_\infty} \\
 &= \frac{(ab)_k (ac)_k (ad)_k h_0}{(abcd)_k} \quad (4.1.9)
 \end{aligned}$$

Using (4.1.9) and (4.1.8) with $c = \lambda e^{i\phi}$, $d = \lambda e^{-i\phi}$ we get

$$\begin{aligned}
 \int_{-1}^1 dx w(x; a, b, \lambda e^{i\phi}, \lambda e^{-i\phi}) p_n(x; a, b, c', d') \\
 = K {}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{n-1} abc'd', a\lambda e^{i\phi}, a\lambda e^{-i\phi} \\ ac', ad', ab\lambda^2 \end{matrix} ; q, q \right] \quad (4.1.10)
 \end{aligned}$$

where

$$K = \frac{2\pi (ab\lambda^2)_\infty}{(ab)_\infty (a\lambda e^{i\phi})_\infty (a\lambda e^{-i\phi})_\infty (b\lambda e^{i\phi})_\infty (b\lambda e^{-i\phi})_\infty (\lambda^2)_\infty} \quad (4.1.11)$$

Dropping the dashes, (4.1.10) leads to the following projection formula, namely

$$p_n(y; a\lambda^{-1}, b\lambda^{-1}, c\lambda^{-1}, d\lambda^{-1}) = \frac{1}{K} \int_{-1}^1 dx w(x; a, b, \lambda e^{i\phi}, \lambda e^{-i\phi}) \cdot p_n(x; a, b, c, d) \quad (4.1.12)$$

where $y = \cos \phi$ and $|\lambda| < 1$.

If we use Sears' transform (1.2.7), we see that

$$p_n(y; a\lambda, b\lambda, c\lambda^{-1}, d\lambda^{-1}) = \frac{(bc)_n (cd\lambda^{-2})_n (a\lambda^2 c^{-1})_n}{(ad)_n (ab\lambda^2)_n} p_n(y; c\lambda^{-1}, d\lambda^{-1}, a\lambda, b\lambda). \quad (4.1.13)$$

Assuming $|c\lambda^{-1}|, |d\lambda^{-1}| < 1$ we obtain from (4.1.12) the formula

$$\begin{aligned} \int_{-1}^1 dy w(y; c\lambda^{-1}, d\lambda^{-1}, \lambda e^{i\psi}, \lambda e^{-i\psi}) p_n(y; c\lambda^{-1}, d\lambda^{-1}, a\lambda, b\lambda) \\ = \frac{2\pi (cd)_\infty}{(q)_\infty (cd\lambda^{-2})_\infty (\lambda^2)_\infty | (ce^{i\psi})_\infty (de^{i\psi})_\infty |^2} p_n(z; c, d, a, b), \end{aligned} \quad (4.1.14)$$

where $z = \cos \psi$. Thus

$$\begin{aligned} \int_{-1}^1 dy w(y; c\lambda^{-1}, d\lambda^{-1}, \lambda e^{i\psi}, \lambda e^{-i\psi}) | (a\lambda e^{i\phi})_\infty (b\lambda e^{i\phi})_\infty |^2 \\ \int_{-1}^1 dx w(x; a, b, \lambda e^{i\phi}, \lambda e^{-i\phi}) p_n(x; a, b, c, d) \\ = \frac{4\pi^2 (ab\lambda^2)_\infty (cd)_\infty}{[(q)_\infty]^2 (ab)_\infty (\lambda^2)_\infty (cd\lambda^{-2})_\infty (\lambda^2)_\infty | (ce^{i\psi})_\infty (de^{i\psi})_\infty |^2} \\ \frac{(bc)_n (cd\lambda^{-2})_n}{(ad)_n (ab\lambda^2)_n} \left(\frac{a\lambda^2}{c} \right) p_n(z; c, d, a, b). \end{aligned} \quad (4.1.15)$$

But by Sears' (1.14), again, we have

$$p_n(z; c, d, a, b) = \frac{(ad)_n (ab)_n}{(bc)_n (cd)_n} \left(\frac{c}{a}\right)^n p_n(z; a, b, c, d). \quad (4.1.16)$$

Thus we have

$$\int_{-1}^1 K(x, y) p_n(y) dy = \mu_n p_n(x), \quad (4.1.17)$$

where

$$K(x, y) = \frac{(ab)_\infty (\lambda^2)_\infty (\lambda^2)_\infty (cd \lambda^{-2})_\infty}{4\pi^2 (ab \lambda^2)_\infty (cd)_\infty} [(q)_\infty]^2 |(ce^{i\theta})_\infty (de^{i\theta})_\infty|^2$$

$$\cdot \int_{-1}^1 dz w(z; c\lambda^{-1}, d\lambda^{-1}, \lambda e^{i\theta}, \lambda e^{-i\theta}) w(y; a, b, \lambda e^{i\psi}, \lambda e^{-i\psi})$$

$$\cdot |(a\lambda e^{i\psi})_\infty (b\lambda e^{i\psi})_\infty|^2, \quad (4.1.18)$$

with $z = \cos \psi$, and

$$\mu_n = \frac{(ab)_n (cd \lambda^{-2})_n}{(cd)_n (ab \lambda^2)_n} \lambda^{2n}. \quad (4.1.19)$$

To generalize this result, use of projection formulas of the type

$$p_n(x) = \sum_{k=0}^n a_k q_k(x)$$

has been made. Thus we were able to prove

$$\int_{-1}^1 dy K(x,y) p_n(y; a,b,c,d) = \lambda_n p_n(x; a\lambda^{-1}\beta, b\lambda^{-1}\beta, c\lambda\beta^{-1}, d\lambda\beta^{-1}), \quad (4.1.20)$$

where the kernel $K(x,y)$ is a double integral and is given by

$$\begin{aligned} K(x,y) = & c_1^{-1}(\theta) c_2^{-1}(\theta) \int_{-1}^1 dz_1 w(z_1; \mu\lambda\beta^{-1}e^{i\theta}, \mu\lambda\beta^{-1}e^{-i\theta}, c\mu^{-1}, d\mu^{-1}) \\ & \cdot \int_{-1}^1 dz_2 w(z_2; a\lambda^{-1}, b\lambda^{-1}, \beta e^{i\theta}, \beta e^{-i\theta}) \\ & \cdot C^{-1}(\psi_1, \psi_2) w(y; \mu e^{i\psi_1}, \mu e^{-i\psi_1}, \lambda e^{-i\psi_2}, \lambda e^{-i\psi_2}) \end{aligned} \quad (4.1.21)$$

with

$$c_1(\theta) = \frac{2\pi (ab\lambda^{-2}\beta^2)_\infty}{(q)_\infty (ab\lambda^{-2})_\infty (\beta^2)_\infty |(a\lambda^{-1}\beta e^{i\theta})_\infty (b\lambda^{-1}\beta e^{-i\theta})_\infty|^2} \quad (4.1.22)(a)$$

$$c_2(\theta) = \frac{2\pi (cd\lambda^2\beta^{-2})_\infty}{(q)_\infty (\mu\lambda^2\beta^{-2})_\infty (cd)_\infty |(c\lambda\beta^{-1}e^{i\theta})_\infty (d\lambda\beta^{-1}e^{-i\theta})_\infty|^2} \quad (4.1.22)(b)$$

and

$$C(\psi_1, \psi_2) = \frac{2\pi (\mu\lambda^2)_\infty}{(q)_\infty (\mu^2)_\infty (\lambda^2)_\infty |(\mu\lambda e^{i\psi_1+1\psi_2})_\infty (\mu\lambda e^{i\psi_2-1\psi_1})_\infty|^2} \quad (4.1.22)(c)$$

and where

$$\lambda_n = {}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{n-1}abcd, \lambda^2, \mu\lambda^2\beta^{-2} \\ ab, \lambda^2\mu^2, cd\lambda^2\beta^{-2} \end{matrix} ; q, q \right] \quad (4.1.23)$$

It is understood that $y = \cos \phi$, $x = \cos \theta$, $z_1 = \cos \psi_1$,
 $z_2 = \cos \psi_2$ and

$$\max(|\mu\lambda\beta|^{-1}, |c\mu|^{-1}, |d\mu|^{-1}, |a\lambda|^{-1}, |b\lambda|^{-1}, |\beta|, |\mu|, |\lambda|) < 1.$$

In section 2 we shall establish (4.1.20)-(4.1.23). In section 3 we look at special cases of these kernels. We also notice that the polynomials defined by (4.1.1) form a closed set in the $L^2(-1,1)$ space, (Szegő [61, Th. (3.1.5), p. 40]). Using that, it is easy to show that

$$\sum_{n=0}^{\infty} \frac{p_n(y) p_n(x)}{h_n} = \frac{\delta(x-y)}{w(x)}, \quad (4.1.24)$$

where δ is the Dirac delta function. This is an orthogonality relation dual to (4.1.2). We shall establish (4.1.24) in Section 4, using straightforward computations in which we demonstrate the use of the transformations of Watson and Sears (1.2.6) and (1.2.7) along with the summation theorem (1.2.5).

2. Construction of the kernel $K(x,y)$. The construction of the reproducing kernel involves three major steps, which are mainly three integral evaluations. Use of projection formulae is needed in the first and last phases. To start with, we have defined the $4\frac{1}{3}$ polynomials as

$$p_n(y) = p_n(y; a, b, c, d) = {}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\phi}, ae^{-i\phi} \\ ab, ac, ad \end{matrix} ; q, q \right] \quad (4.2.1)$$

with $y = \cos \phi$. Using Jackson's theorem (1.2.8) we can show that

$$(ae^{i\phi})_r (ae^{-i\phi})_r = (a\lambda e^{i\psi/2})_r (a\lambda^{-1} e^{-i\psi/2})_r \quad (4.2.2)$$

$${}_3\phi_2 \left[\begin{matrix} q^{-r}, \lambda e^{i\psi/2+i\phi}, \lambda e^{i\psi/2-i\phi} \\ a\lambda e^{i\psi/2}, a^{-1}\lambda e^{i\psi/2} q^{1-r} \end{matrix} ; q, q \right]$$

Substituting (4.2.2) into (4.2.1), we get the following projection formula

$$p_n(y) = \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1}abcd)_r (a\lambda e^{i\psi/2})_r (a\lambda^{-1} e^{-i\psi/2})_r}{(q)_r (ab)_r (ac)_r (ad)_r} q^r$$

$$\sum_{k=0}^r \frac{(q^{-r})_k (\lambda e^{i\psi/2+i\phi})_k (\lambda e^{i\psi/2-i\phi})_k}{(q)_k (a\lambda e^{i\psi/2})_k (a^{-1}\lambda e^{i\psi/2} q^{1-r})_k} q^k \quad (4.2.3)$$

Consider the integral

$$I_1 = \int_{-1}^1 dy w(y; \mu e^{i\psi/2}, \mu e^{-i\psi/2}, \lambda e^{i\psi/2}, \lambda e^{-i\psi/2}) p_n(y) \quad (4.2.4)$$

Substituting the values (4.2.3) and (4.1.3) and simplifying we get

$$\begin{aligned}
I_1 &= \sum_{r=0}^R \frac{(q^{-n})_r (q^{n-1}abcd)_r (a\lambda e^{i\psi_2})_r (a\lambda^{-1}e^{-i\psi_2})_r}{(q)_r (ab)_r (ac)_r (ad)_r} q^r \\
&\cdot \sum_{k=0}^r \frac{(q^{-r})_k}{(q)_k (a\lambda e^{i\psi_2})_k (a^{-1}\lambda e^{i\psi_2} q^{1-r})_k} q^k \\
&\cdot \int_{-1}^1 dy w(y; \mu e^{i\psi_1}, \mu e^{-i\psi_1}, \lambda q^k e^{i\psi_2}, \lambda e^{-i\psi_2})
\end{aligned} \quad (4.2.5)$$

Using (4.1.6) the integral on the r.h.s. of (4.2.5) can be evaluated to give

$$\begin{aligned}
&\frac{2\pi(\mu^2 \lambda^2 q^k)_\infty}{(q)_\infty (\mu^2)_\infty (\mu\lambda e^{i\psi_2+i\psi_1} q^k)_\infty (\mu\lambda e^{i\psi_1-i\psi_2})_\infty (\mu\lambda e^{i\psi_2-i\psi_1} q^k)_\infty} \\
&\cdot \frac{1}{(\mu\lambda e^{-i\psi_1-i\psi_2})_\infty (\lambda^2 q^k)_\infty} \\
&= \frac{2\pi(\mu^2 \lambda^2)_\infty}{(q)_\infty (\mu^2)_\infty (\lambda^2)_\infty |(\mu\lambda e^{i\psi_2+i\psi_1})_\infty (\mu\lambda e^{i\psi_2-i\psi_1})_\infty|^2} \\
&\cdot \frac{(\lambda^2)_k (\mu\lambda e^{i\psi_2+i\psi_1})_k (\mu\lambda e^{i\psi_2-i\psi_1})_k}{(\mu^2 \lambda^2)_\infty}
\end{aligned} \quad (4.2.6)$$

Putting this value into (4.2.5), results in

$$I_1 = C(\psi_1, \psi_2) \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r (a\lambda e^{i\psi_2})_r (a\lambda^{-1} e^{-i\psi_2})_r}{(q)_r (ab)_r (ac)_r (ad)_r} q^r$$

$$4\phi_3 \left[\begin{matrix} q^{-r}, \lambda^2, \mu\lambda e^{i\psi_2+i\psi_1}, \mu\lambda e^{i\psi_2-i\psi_1} \\ a\lambda e^{i\psi_2}, a^{-1}\lambda e^{i\psi_2} q^{1-r}, \mu^2 \lambda^2 \end{matrix} ; q, q \right] \quad (4.2.7)$$

where $C(\psi_1, \psi_2)$ is defined by (4.1.22)(c).

The $4\phi_3$ series in (4.2.7) is a balanced one to which we can apply Sears' (1.2.7). This along with simplifying the coefficients will give us

$$I_1 = C(\psi_1, \psi_2) \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r (a\lambda^{-1} e^{-i\psi_2})_r (a\lambda^{-1} e^{i\psi_2})_r (\mu^2)_r}{(q)_r (ab)_r (ac)_r (ad)_r (\mu^2 \lambda^2)}$$

$$= (\lambda^2 q)^r \cdot 4\phi_3 \left[\begin{matrix} q^{-r}, \lambda^2, a^{-1} \mu q^{1-r} e^{-i\psi_1}, a^{-1} \mu q^{1-r} e^{i\psi_1} \\ a^{-1} \lambda e^{i\psi_2} q^{1-r}, \mu^{-2} q^{1-r}, a^{-1} \lambda e^{-i\psi_2} q^{1-r} \end{matrix} ; q, q \right] \quad (4.2.8)$$

Using A-6, we notice that

$$\frac{(a\lambda^{-1} e^{-i\psi_2})_r (a\lambda^{-1} e^{i\psi_2})_r (\mu^2)_r}{(a^{-1} \lambda e^{i\psi_2} q^{1-r})_k (\mu^{-2} q^{1-r})_k (a^{-1} \lambda e^{-i\psi_2} q^{1-r})_k}$$

$$= (\mu^2)_{r-k} (a\lambda^{-1} e^{-i\psi_2})_{r-k} (a\lambda^{-1} e^{i\psi_2})_{r-k} (-a^2 \lambda^{-2} \mu^2)^k q^{3rk - \frac{3}{2}k(k+1)} \quad (4.2.9)$$

Putting this in (4.2.8) leads to the following expression for I_1 :

$$\begin{aligned}
 I_1 = C(\psi_1, \psi_2) \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r}{(q)_r (ab)_r (ac)_r (ad)_r (\mu^2 \lambda^2)_r} (\lambda^2 q)^r \\
 \cdot \sum_{k=0}^r \frac{(q^{-r})_k (\lambda^2)_k (a^{-1} \mu^{-1} q^{1-r} e^{-i\psi_1})_k (a^{-1} \mu^{-1} q^{1-r} e^{i\psi_1})_k}{(q)_k} \\
 \cdot (\mu^2)_{r-k} (a \lambda^{-1} e^{-i\psi_2})_{r-k} (a \lambda^{-1} e^{i\psi_2})_{r-k} (-a^2 \lambda^{-2} \mu^2 q)^k \\
 \cdot q^{3rk - \frac{3}{2} k(k+1)} \quad (4.2.10)
 \end{aligned}$$

The next step is to consider the following integral

$$I_2 = \int_{-1}^1 dz_2 w(z_2; a \lambda^{-1}, b \lambda^{-1}, \beta e^{i\theta}, \beta e^{-i\theta}) c^{-1}(\psi_1, \psi_2) I_1, \quad (4.2.11)$$

where as before $z_2 = \cos \psi_2$. Hence if we substitute (4.2.10) in (4.2.11) and simplify as before, we are led to

$$\begin{aligned}
 I_2 = \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r}{(q)_r (ab)_r (ac)_r (ad)_r (\mu^2 \lambda^2)_r} (\lambda^2 q)^r \\
 \cdot \sum_{k=0}^r \frac{(q^{-r})_k (\lambda^2)_k (a^{-1} \mu^{-1} q^{1-r} e^{-i\psi_1})_k (a^{-1} \mu^{-1} q^{1-r} e^{i\psi_1})_k}{(q)_k} \\
 \cdot (\mu^2)_{r-k} (-a^2 \lambda^{-2} \mu^2 q)^k \cdot (q^{3rk - \frac{3}{2} k(k+1)}) \cdot J_2, \quad (4.2.12)
 \end{aligned}$$

where

$$\begin{aligned}
 J_2 &= \int_{-1}^1 dz_2 w(z_2; a\lambda^{-1} q^{r-k}, b\lambda^{-1}, \beta e^{i\theta}, \beta e^{-i\theta}) \\
 &= c_1(\theta) \frac{(ab\lambda^{-2})_{r-k} (a\lambda^{-1}\beta e^{i\theta})_{r-k} (a\lambda^{-1}\beta e^{-i\theta})_{r-k}}{(ab\lambda^{-2}\beta^2)_{r-k}} \quad (4.2.13)
 \end{aligned}$$

This integral was evaluated using (4.1.6) as before, and the coefficient $c_1(\theta)$ is defined as in (4.1.22)(a). Thus, using (4.2.12), and (4.2.13) results in

$$\begin{aligned}
 I_2 &= c_1(\theta) \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r}{(q)_r (ab)_r (ac)_r (ad)_r (\mu^2 \lambda^2)_r} (\lambda^2 q)^r \\
 &\quad \sum_{k=0}^r \frac{(q^{-r})_k (\lambda^2)_k (\mu^2)_{r-k} (ab\lambda^{-2})_{r-k}}{(q)_k (ab\lambda^{-2}\beta^2)_{r-k}} \\
 &\quad (a\lambda^{-1}\beta e^{i\theta})_{r-k} (a\lambda^{-1}\beta e^{-i\theta})_{r-k} (-a^2 \lambda^{-2} \mu^2 q)^k \\
 &\quad q^{3rk - \frac{3}{2}k(k+1)} (a^{-1} \mu^{-1} q^{1-r} e^{-i\theta})_k (a^{-1} \mu^{-1} q^{1-r} e^{i\theta})_k \quad (4.2.14)
 \end{aligned}$$

The final step of this construction is the most involved. It contains very messy but rather straightforward calculations. In this step we consider the integral

$$I_3 = \int_{-1}^1 dz_1 w(z_1; \mu \lambda \beta^{-1} e^{i\theta}, \mu \lambda \beta^{-1} e^{-i\theta}, c_\mu^{-1}, d_\mu^{-1}) \quad (4.2.15)$$

$$= c_1^{-1}(\theta) I_2,$$

with $z_1 = \cos \psi_1$.

Before substituting the values in, we make use once again of Jackson's (1.2.8) to get

$$(a^{-1} q^{-1} e^{-i\psi_1})_k (a^{-1} \mu^{-1} q^{-1} e^{+i\psi_1})_k = (a^{-1} \lambda \beta^{-1} q^{-1} e^{i\theta})_k (a^{-1} \mu^{-2} \lambda^{-1} \beta q^{-1} e^{-i\theta})_k$$

$$= 3^{\frac{1}{2}} \begin{bmatrix} -k, \mu \lambda \beta^{-1} e^{i\theta+i\psi_1}, \mu \lambda \alpha^{-1} e^{i\theta-i\psi_1} \\ a^{-1} \lambda \beta^{-1} q^{-1} e^{i\theta}, a \mu^2 \lambda^{-1} e^{i\theta} q^{-1} \end{bmatrix} \quad (4.2.16)$$

Hence, using (4.2.14) and (4.2.16) in (4.2.15), I_3 becomes:

$$I_3 = \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r}{(q)_r (ab)_r (ac)_r (ad)_r (\mu^2 \lambda^2)_r} (\lambda^2 q)^r$$

$$= \sum_{k=0}^r \frac{(q^{-r})_k (\lambda^2)_k (\mu^2)_{r-k} (ab \lambda^{-2})_{r-k} (a \lambda^{-1} \beta e^{i\theta})_{r-k}}{(q)_k (ab \lambda^{-2} \beta^2)_{r-k}}$$

$$= (a \lambda^{-1} \beta e^{i\theta})_{r-k} (-a^2 \lambda^{-2} \mu^2 q)^k \cdot q^{3rk - \frac{3}{2}k(k+1)}$$

$$\begin{aligned}
& : (a^{-1} \lambda \beta^{-1} q^{1-r} e^{i\theta})_k (a^{-1} \mu^{-2} \lambda^{-1} \beta q^{1-r} e^{-i\theta})_k \\
& \cdot \sum_{\ell=0}^k \frac{(q^{-k})_\ell}{(q)_\ell (a^{-1} \lambda \beta^{-1} q^{1-r} e^{i\theta})_\ell (a \mu^2 \lambda \beta^{-1} e^{i\theta} q^{r-k})_\ell} q^\ell \\
& \cdot J_1, \tag{4.2.17}
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int_{-1}^1 dz_1 w(z_1; \mu \lambda \beta^{-1} q^\ell e^{i\theta}, \mu \lambda \beta^{-1} e^{-i\theta}, c \mu^{-1}, d \mu^2) \\
&= c_2(\theta) \frac{(\mu^2 \lambda \beta^{-2})_\ell (c \lambda \beta^{-1} e^{i\theta})_\ell (d \lambda \beta^{-1} e^{i\theta})_\ell}{(cd \lambda^2 \beta^{-2})_\ell}, \tag{4.2.18}
\end{aligned}$$

by (4.1.6) and where $c_2(\theta)$ is defined by (4.1.22)(b). Introducing (4.2.18) into (4.2.17) gives us

$$\begin{aligned}
I_3 &= c_2(\theta) \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r}{(q)_r (ab)_r (ac)_r (ad)_r (\mu^2 \lambda^2)_r} (\lambda^2 q)^r \\
&\cdot \sum_{k=0}^r \frac{(q^{-r})_k (\lambda^2)_k (\mu^2)_{r-k} (ab \lambda^{-2})_{r-k} (a \lambda^{-1} \beta e^{i\theta})_{r-k}}{(q)_k (ab \lambda^2 \beta^2)_{r-k}} \\
&\cdot (a \lambda^{-1} \beta e^{-i\theta})_{r-k} (-a^2 \lambda^{-2} \mu^2 q)^k \cdot q^{3rk - \frac{3}{2} k(k+1)} \\
&\cdot (a^{-1} \lambda \beta^{-1} q^{1-r} e^{i\theta})_k (a^{-1} \mu^{-2} \lambda^{-1} \beta q^{1-r} e^{-i\theta})_k
\end{aligned}$$

$$4^{\Phi}_3 \left[\begin{matrix} q^{-k}, \mu^2 \lambda \theta^{-2}, c \lambda \theta^{-1} e^{i\theta}, d \lambda \theta^{-1} e^{i\theta} \\ a^{-1} \lambda \theta^{-1} q^{1-r} e^{i\theta}, a \lambda \mu \theta^{-2} e^{i\theta} q^{r-k}, cd \lambda \theta^{-2} \end{matrix} ; q, q \right] \quad (4.2.19)$$

This 4^{Φ}_3 is a balanced series to which we can apply (1.2.7), as many times as we please. Do that twice and we obtain upon simplifications

$$4^{\Phi}_3 = \frac{(a^{-1} c^{-1} q^{1-r})_k (a^{-1} \mu^{-2} c q^{1-r})_k}{(a^{-1} \lambda \theta^{-1} q^{1-r} e^{i\theta})_k (a^{-1} \mu^{-2} \lambda \theta^{-1} q^{1-r} e^{-i\theta})_k} \cdot 4^{\Phi}_3 \left[\begin{matrix} q^{-k}, cd \mu^{-2}, c \lambda \theta^{-1} e^{i\theta}, c \lambda \theta^{-1} e^{-i\theta} \\ cd \lambda \theta^{-2}, ac q^{r-k}, a^{-1} \mu^{-2} c q^{1-r} \end{matrix} ; q, q \right]$$

$$= \frac{(a^{-1} c^{-1} q^{1-r})_k (a^{-1} d^{-1} q^{1-r})_k (\mu^2 \lambda \theta^{-2})_k (cd \mu^{-2})^k}{(a^{-1} \lambda \theta^{-1} q^{1-r} e^{i\theta})_k (a^{-1} \mu^{-2} \lambda \theta^{-1} q^{1-r} e^{-i\theta})_k (cd \lambda \theta^{-2})_k}$$

$$4^{\Phi}_3 \left[\begin{matrix} q^{-k}, cd \mu^{-2}, a \lambda^{-1} \theta q^{r-k} e^{-i\theta}, a \lambda^{-1} \theta q^{r-k} e^{i\theta} \\ ac q^{r-k}, ad q^{r-k}, \mu^{-2} \lambda \theta^{-2} q^{1-k} \end{matrix} ; q, q \right]$$

(4.2.20)

Therefore

$$\begin{aligned}
 I_3 &= c_2(\theta) \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r}{(q)_r (ab)_r (ac)_r (ad)_r (\mu^2 \lambda^2)_r} (\lambda^2 q)^r \\
 &\quad \cdot \sum_{k=0}^r \frac{(q^{-r})_k (\lambda^2)_k (\mu^2)_{r-k} (ab \lambda^{-2})_{r-k} (a \lambda^{-1} \beta e^{i\theta})_{r-k}}{(q)_k (ab \lambda^{-2} \beta^2)_{r-k} (cd \lambda^2 \beta^{-2})_k} \\
 &\quad \cdot (a \lambda^{-1} \beta e^{-i\theta})_{r-k} (\mu^2 \lambda^2 \beta^{-2})_k (a^{-1} c^{-1} q^{1-r})_k \\
 &\quad \cdot (a^{-1} d^{-1} q^{1-r})_k (-a^2 cd \lambda^{-2} q)^k q^{3rk - \frac{3}{2}k(k+1)} \\
 &\quad \cdot {}_4\phi_3 \left[\begin{matrix} q^{-k}, cd \mu^{-2}, a \lambda^{-1} \beta q^{r-k} e^{-i\theta}, a \lambda^{-1} \beta q^{r-k} e^{i\theta} \\ ac q^{r-k}, ad q^{r-k}, \mu^{-2} \lambda^2 \beta^{-2} q^{1-k} \end{matrix} ; q, q \right]
 \end{aligned}
 \tag{4.2.21}$$

By A-6 and A-4 we can show that

$$\begin{aligned}
 \frac{(a^{-1} d^{-1} q^{1-r})_k (a^{-1} c^{-1} q^{1-r})_k}{(ac q^{r-k})_\ell (ad q^{r-k})_\ell} &= \frac{(ac)_r (ad)_r (a^2 cd)^k q^{-2kr+k(k+1)}}{(ac)_{r-k} (ad)_{r-k} (ac q^{r-k})_\ell (ad q^{r-k})_\ell} \\
 &= \frac{(ac)_r (ad)_r (a^2 cd)^k q^{-2kr+k(k+1)}}{(ac)_{r-k+\ell} (ad)_{r-k+\ell}}
 \end{aligned}
 \tag{4.2.22}$$

and

$$(q^{-r})_k = \frac{(q)_r}{(q)_{r-k}} (-1)^k q^{-kr + \frac{k}{2}(k-1)}
 \tag{4.2.23}$$

Substituting (4.2.22) and (4.2.23) into (4.2.21), with some obvious simplifications, results in

$$I_3 = c_2(\theta) \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1}abcd)_r (\lambda^2 q)^r}{(ab)_r (\mu^2 \lambda^2)_r} \sum_{k=0}^r \sum_{\ell=0}^{r-k} \frac{(\lambda^2)_k (\mu^2)_{r-k}}{(q)_{r-k} (q)_k (q)_\ell} \\ \cdot \frac{(ab\lambda^{-2})_{r-k} (\mu^2 \lambda^2 \beta^{-2})_k (q^{-k})_\ell (cd\mu^{-2})_\ell (a\lambda^{-1} \beta e^{i\theta})_{r-k+\ell}}{(\beta \mu^2 \lambda^{-2} q^{1-k})_\ell (cd\lambda^2 \beta^{-2})_k (ab\lambda^{-2} \beta^2)_{r-k} (ac)_{r-k+\ell}} \\ \cdot \frac{(a\lambda^{-1} \beta e^{-i\theta})_{r-k+\ell}}{(ad)_{r-k+\ell}} (\lambda^{-2})^k q^\ell, \quad (4.2.24)$$

which upon replacing k by $r-k$ gives

$$I_3 = c_2(\theta) \sum_{r=0}^n \frac{(q^{-n})_r (q^{n-1}abcd)_r}{(ab)_r (\mu^2 \lambda^2)_r} q^r \sum_{k=0}^r \sum_{\ell=0}^{r-k} \frac{(\lambda^2)_{r-k} (\mu^2)_k}{(q)_k (q)_{r-k}} \\ \cdot \frac{(ab\lambda^{-2})_k (\mu^2 \lambda^2 \beta^{-2})_{r-k} (q^{-r+k})_\ell (cd\mu^{-2})_\ell (a\lambda^{-1} \beta e^{i\theta})_{k+\ell}}{(q)_\ell (cd\lambda^2 \beta^{-2})_{r-k} (ab\lambda^{-2} \beta^2)_k (\beta \mu^2 \lambda^{-2} q^{1-r+k})_\ell (ac)_{k+\ell}} \\ \cdot \frac{(a\lambda^{-1} \beta e^{-i\theta})_{k+\ell}}{(ad)_{k+\ell}} (\lambda^2)^k q^\ell. \quad (4.2.25)$$

Set $\ell = s - k$ so that we can change the order of summation and hence obtain

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$$c_2^{-1}(\theta) I_3 = \sum_{s=0}^n \frac{(a\lambda^{-1} e^{i\theta})_s (a\lambda^{-1} \beta e^{-i\theta})_s}{(ac)_s (ad)_s} U_{n,s}, \quad (4.2.26)$$

where

$$U_{n,s} = \sum_{r=s}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r}{(ab)_r (\mu^2 \lambda^2)_r} q^r \sum_{k=0}^r \frac{(\lambda^2)_{r-k} (\mu^2)_k (ab \lambda^{-2})_k}{(q)_{r-k} (q)_k (ab \beta^2 \lambda^{-2})_k} \cdot \frac{(\mu^2 \lambda^2 \beta^{-2})_{r-k} (q^{k-r})_{s-k} (cd \mu^{-2})_{s-k}}{(cd \lambda^2 \beta^{-2})_{r-k} (q)_{s-k} (\beta^2 \mu^{-2} \lambda^{-2} q^{1-r+k})_{s-k}} (\lambda^2)^k q^{s-k}. \quad (4.2.27)$$

The second summation on the r.h.s. of (4.2.27) should be actually up to s because of the existence of the term $(q)_{s-k}$ in the denominator. Replace k by $s-k$ and we get

$$U_{n,s} = \sum_{r=s}^n \frac{(q^{-n})_r (q^{n-1} abcd)_r}{(ab)_r (\mu^2 \lambda^2)_r} q^r \sum_{k=0}^s \frac{(\lambda^2)_{r+k-s} (\mu^2)_{s-k}}{(q)_{s-k} (q)_{r+k-s} (ab \lambda^{-2} \beta^2)_{s-k}} \cdot \frac{(ab \lambda^{-2})_{s-k} (\mu^2 \lambda^2 \beta^{-2})_{r+k-s} (q^{s-r-k})_k (cd \mu^{-2})_k}{(cd \lambda^2 \beta^{-2})_{r+k-s} (q)_k (\beta^2 \mu^{-2} \lambda^{-2} q^{1-r+s-k})_k (\lambda^{-2})_{s-k}} q^k. \quad (4.2.28)$$

Let $r \rightarrow r-s$, hence

$$U_{n,s} = \sum_{r=0}^{n-s} \frac{(q^{-n})_{r+s} (q^{n-1} abcd)_{r+s}}{(ab)_{r+s} (\mu^2 \lambda^2)_{r+s}} q^{r+s} \sum_{k=0}^s \frac{(\lambda^2)_{r+k} (\mu^2)_{s-k}}{(q)_{s-k} (q)_{r+k}}$$

$$\begin{aligned}
& \frac{(ab\lambda^{-2})_{s-k} (\mu^2 \lambda^2 \beta^{-2})_{r+k} (q^{-r-k})_k (cd\mu^{-2})_k (\lambda^2)^{s-k}}{(ab\lambda^{-2} \beta^2)_{s-k} (cd\lambda^2 \beta^{-2})_{r+k} (q)_k (\beta^2 \mu^{-2} \lambda^{-2} q^{1-r-k})_k} q^k \\
&= \frac{(q^{-n})_s (q^{n-1} abcd)_s (\mu^2)_s (ab\lambda^{-2})_s}{(q)_s (ab)_s (\mu^2 \lambda^2)_s (ab\lambda^{-2} \beta^2)_s} (\lambda^2 q)^s \\
& \sum_{r=0}^{n-s} \frac{(q^{-n+s})_r (q^{n-1+s} abcd)_r (\lambda^2)_s (ab\lambda^2 \beta^{-2})_r}{(q)_r (cd\lambda^2 \beta^{-2})_r (abq^s)_r (\mu^2 \lambda^2 q^s)_r} q^r \\
& \sum_{k=0}^s \frac{(\lambda^2 q^r)_k (a^{-1} b^{-1} \lambda^2 \beta^{-2} q^{1-s})_k (cd\mu^{-2})_k (q^{-s})_k}{(q)_k (\mu^{-2} q^{1-s})_k (a^{-1} b^{-1} \lambda^2 q^{1-s})_k (cd\lambda^2 \beta^{-2} q^r)_k} q^k
\end{aligned}$$

(4.2.29)

The last equality was obtained by straight simplifications using A-4 and A-6. The last series on the r.h.s. is a balanced one to which we can apply Sears' (1.2.7). Thus we have

$$\begin{aligned}
\sum_{k=0}^s &= \frac{(\lambda^2 \mu^2 q^r)_s (abq^r)_s}{(\mu^2)_s (ab\lambda^{-2} \beta^2)_s} (\lambda^2 q^r)^s \\
& \cdot {}_4\phi_3 \left[\begin{matrix} q^{-s}, \lambda^2 q^r, abcdq^{r+s-1}, \lambda^2 \mu^2 \beta^{-2} q^r \\ cd\lambda^2 \beta^{-2} q^r, \lambda^2 \mu^2 q^r, abq^r \end{matrix} ; q, q \right]
\end{aligned}$$

(4.2.30)

Using the identity A-4 again we can show that

$$\frac{(\lambda \mu q^r)_s}{(\mu \lambda^2)_s (\mu \lambda q^s)_r (\lambda \mu q^r)_\ell} = \frac{1}{(\lambda \mu^2)_{r+\ell}},$$

$$\frac{(abq^r)_s}{(ab)_s (abq^s)_r (abq^r)_\ell} = \frac{1}{(ab)_{r+\ell}},$$

and

$$(cd abq^{r+s-1})_\ell = \frac{(abcd q^{s-1})_{r+\ell}}{(abcd q^{s-1})_{r+\ell}}.$$

Substituting these identities along with (4.2.30) in (4.2.29) results in

$$U_{n,s} = \frac{(q^{-n})_s (q^{n-1} abcd)_s q^s}{(q)_s (ab \lambda^{-2} \beta^2)_s} \sum_{r=0}^{n-s} \sum_{\ell=0}^s \frac{(q^{s-n})_r (q^{s+n-1} abcd)_r}{(q)_r (q)_\ell (abcd q^{s-1})_r} \cdot \frac{(q^{-s})_\ell (abcd q^{s-1})_{r+\ell} (\mu \lambda \beta^{-2})_{r+\ell} (\lambda^2)_{r+\ell}}{(ab)_{r+\ell} (cd \lambda^2 \beta^{-2})_{r+\ell} (\lambda \mu^2)_{r+\ell}} q^{r+\ell-rs}, \quad (4.2.31)$$

which by setting $r+\ell = m$ enables us to change the order of summation. This leads to

$$\begin{aligned}
 U_{n,s} &= \frac{(q^{-n})_s (q^{n-1} abcd)_s}{(q)_s (ab \lambda^{-2} \beta^2)_s} q^s \sum_{m=0}^n \frac{(abcd q^{s-1})_m (\mu^2 \lambda^2 \beta^{-2})_m (\lambda^2)_m}{(cd \lambda^2 \beta^{-2})_m (ab)_m (\mu^2 \lambda^2)_m} q^m \\
 &\quad \sum_r \frac{(q^{-s})_{m-r} (q^{n+s-1} abcd)_r (q^{s-n})_r}{(q)_r (q)_{m-r} (q^{s-1} abcd)_r} q^{-rs} \\
 &\quad (4.2.32)
 \end{aligned}$$

Noting that by A-6 we have

$$\frac{(q^{-s})_{m-r}}{(q)_{m-r}} = \frac{(q^{-s})_m (q^{-m})_r}{(q)_m (q^{1+s-m})_r} (q^{1+s})^r,$$

the last series on the r.h.s. of (4.2.32) reduces to

$$\begin{aligned}
 \sum_r &= \frac{(q^{-s})_m}{(q)_m} {}_3\phi_2 \left[\begin{matrix} q^{-m}, q^{s-n}, q^{s+n-1} abcd \\ q^{1+s-m}, q^{s-1} abcd \end{matrix} ; q, q \right] \\
 &= \frac{(q^{-s})_m (q^{1+n-m})_m (q^{n-1} abcd)_m}{(q)_m (q^{1+s-m})_m (q^{s-1} abcd)_m} (q^{s-n})^m, \text{ by (1.2.8)} \\
 &= \frac{(q^{-n})_m (q^{n-1} abcd)_m}{(q)_m (q^{s-1} abcd)_m} \\
 &\quad (4.2.33)
 \end{aligned}$$

Therefore

$$\begin{aligned}
 U_{n,s} &= \frac{(q^{-n})_s (q^{n-1} abcd)_s}{(q)_s (ab \lambda^{-2} \beta^2)_s} q^s {}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{n-1} abcd, \lambda^2, \mu^2 \lambda^2 \beta^{-2} \\ ab, \mu^2 \lambda^2, cd \lambda^2 \beta^{-2} \end{matrix} ; q, q \right], \\
 &\quad (4.2.34)
 \end{aligned}$$

and hence, going back to (4.2.26), we have

$$c_2^{-1}(\theta) I_3 = 4^{\frac{1}{2}}_3 \begin{bmatrix} q^{-n}, & q^{n-1} abcd, & \lambda^2, & \mu^2 \lambda^2 \beta^{-2} \\ & ab, & \lambda^2 \mu^2, & cd \lambda^2 \beta^{-2} \end{bmatrix} ; q, q$$

$$\cdot 4^{\frac{1}{2}}_3 \begin{bmatrix} q^{-n}, & q^{n-1} abcd, & a \lambda^{-1} \beta e^{i\theta}, & a \lambda^{-1} \beta e^{-i\theta} \\ & ac, & ad, & a \beta^2 b \lambda^{-2} \end{bmatrix} ; q, q$$

Or

(4.2.35)

$$c_2^{-1}(\theta) I_3 = \lambda_n p_n(x; a \lambda^{-1} \beta, b \lambda^{-1} \beta, c \lambda \beta^{-1}, d \lambda \beta^{-1}).$$

(4.2.36)

This, upon tracing back, gives (4.1.20)-(4.1.23).

3. Special cases. Let's write (4.1.21) in the expanded form.

Thus it looks like

$$K(x, y) = \left(\frac{(q)_\infty}{2\pi} \right)^3 \frac{(\mu^2 \lambda^2 \beta^{-2})_\infty (cd \mu^{-2})_\infty (ab \lambda^{-2})_\infty (\beta^2)_\infty (\mu^2)_\infty (\lambda^2)_\infty}{(cd \lambda^2 \beta^{-2})_\infty (ab \lambda^{-2} \beta^2)_\infty (\mu^2 \lambda^2)_\infty}$$

$$\cdot |(c \lambda \beta^{-1} e^{i\theta})_\infty (d \lambda \beta^{-1} e^{i\theta})_\infty (a \lambda^{-1} \beta e^{i\theta})_\infty (b \lambda^{-1} \beta e^{i\theta})_\infty (e^{2i\theta})_\infty|^2$$

$$\cdot \int_0^\pi d\psi_1 \left| \frac{(e^{2i\psi_1})_\infty}{(\mu \lambda \beta^{-1} e^{i\psi_1 + i\theta})_\infty (\mu \lambda \beta^{-1} e^{i\psi_1 - i\theta})_\infty (\mu e^{i\psi_1 + i\theta})_\infty (\mu e^{i\psi_1 - i\theta})_\infty} \right|^2$$

$$\cdot \frac{1}{|(c \mu^{-1} e^{i\psi_1})_\infty (d \mu^{-1} e^{i\psi_1})_\infty|^2} \cdot \int_0^\pi \frac{d\psi_2}{|(a \lambda^{-1} e^{i\psi_2})_\infty (b \lambda^{-1} e^{i\psi_2})_\infty|^2}$$

$$\left| \frac{(e^{2i\psi_2})_\infty}{(\beta e^{i\psi_2+i\theta})_\infty (\beta e^{i\psi_2-i\theta})_\infty (\lambda e^{i\psi_2+i\varphi})_\infty (\lambda e^{i\psi_2-i\varphi})_\infty} \right|^2$$

$$\cdot \left| (\mu \lambda e^{i\psi_1+i\psi_2})_\infty (\mu \lambda e^{i\psi_2-i\psi_1})_\infty \right|^2.$$

(4.3.1)

We know that

$$\lim_{\lambda \rightarrow 1-} \frac{(e^{\lambda^2})_\infty}{\left| (\lambda e^{i\psi_2-i\varphi})_\infty \right|^2} = \frac{2\pi}{(q)_\infty} \delta(\psi_2 - \varphi),$$

see (4.4.15). Using this in (4.3.1) we get

$$\lim_{\lambda \rightarrow 1-} K(x, y) = \frac{(\mu \beta^{-2})_\infty (cd \mu^{-2})_\infty (ab)_\infty (\beta^2)_\infty}{(cd \beta^{-2})_\infty (ab \beta^2)_\infty} \left(\frac{(q)_\infty}{2\pi} \right)^2$$

$$\cdot \left| (c\beta^{-1} e^{i\theta})_\infty (d\beta^{-1} e^{i\theta})_\infty (a\beta e^{i\theta})_\infty (d\beta e^{i\theta})_\infty (e^{2i\varphi})_\infty \right|^2$$

$$\cdot \frac{(e^{2i\psi_1})_\infty^2}{\int_0^\pi d\psi_1 \left| (\mu \beta^{-1} e^{-i\psi_1+i\theta})_\infty (\mu \beta^{-1} e^{i\psi_1-i\theta})_\infty (c\mu^{-1} e^{i\psi_1})_\infty (d\mu^{-1} e^{i\psi_1})_\infty \right|^2}$$

$$\cdot \frac{1}{\left| (\mu e^{i\psi_1+i\varphi})_\infty (\mu e^{i\psi_1-i\varphi})_\infty \right|^2} \int_0^\pi d\psi_2 \frac{\delta(\varphi - \psi_2) (e^{2i\psi_2})_\infty^2}{\left| (e^{i\psi_2-i\varphi})_\infty \right|^2}$$

$$\cdot \frac{\left| (\mu e^{i\psi_1+i\psi_2})_\infty (\mu e^{i\psi_2-i\psi_1})_\infty \right|^2}{\left| (ae^{i\psi_2})_\infty (be^{i\psi_2})_\infty (\beta e^{i\psi_2+i\theta})_\infty (\beta e^{i\psi_2-i\theta})_\infty \right|^2}$$

$$= \left(\frac{(q)_\infty}{2\pi} \right)^2 \frac{(\mu^2 \beta^{-2})_\infty (cd\mu^{-2})_\infty (ab)_\infty (\beta^2)_\infty}{(cd\beta^{-2})_\infty (ab\beta^2)_\infty}$$

$$\frac{w(y; a, b, \beta e^{i\theta}, \beta e^{-i\theta})}{w(x; a\beta, b\beta, c\beta^{-1}, d\beta^{-1})} |(e^{2i\theta})_\infty|^2$$

$$\int_{-1}^1 dz_1 w(z_1; \mu\beta^{-1}e^{i\theta}, \mu\beta^{-1}e^{-i\theta}, c\mu^{-1}, d\mu^{-1})$$

$$= \frac{(q)_\infty (ab)_\infty (\beta^2)_\infty |(a\beta e^{i\theta})_\infty (b\beta e^{i\theta})_\infty|^2}{2\pi (ab\beta^2)_\infty} \cdot w(y; a, b, \beta e^{i\theta}, \beta e^{-i\theta})$$

(4.3.2)

Hence (4.1.20) becomes

$$\int_{-1}^1 dy w(y; a, b, \beta e^{i\theta}, \beta e^{-i\theta}) p_n(y)$$

$$= \frac{2\pi (ab\beta^2)_\infty}{(q)_\infty (ab)_\infty (\beta^2)_\infty |(a\beta e^{i\theta})_\infty (b\beta e^{i\theta})_\infty|^2} p_n(x; a\beta, b\beta, c\beta^{-1}, d\beta^{-1})$$

(4.3.3)

This is the same as (4.1.12) where β is replaced by λ .

Askey and Wilson [18] define the continuous q -Jacobi polynomials

by

$$p_n^{(\alpha, \beta)}(x|q) = \frac{(q^{\alpha+1})_n}{(q)_n} p_n(x; q^{\frac{2\alpha+1}{4}}, q^{\frac{2\alpha+3}{4}}, -q^{\frac{2\beta+1}{4}}, -q^{\frac{2\beta+3}{4}})$$

So if we set $\beta = q^{\mu/2}$ and

$$a = q^{\frac{2\alpha+1}{4}}, \quad b = q^{\frac{2\alpha+3}{4}}, \quad c = -q^{\frac{2\beta+1}{4}}, \quad d = -q^{\frac{2\beta+3}{4}},$$

then (4.3.3) becomes

$$\begin{aligned} & \int_{-1}^1 dy \, w(y; q^{\frac{2\alpha+1}{4}}, q^{\frac{2\alpha+3}{4}}, q^{\mu/2} e^{i\theta}, q^{\mu/2} e^{-i\theta}) \\ & \cdot p_n^{(\alpha, \beta)}(y|q) \cdot \frac{(q)_n}{(q^{\alpha+1})_n} \\ & = \frac{2\pi (q^{\alpha+1+\mu})_\infty}{(q)_\infty (q^{\alpha+1})_\infty (q^\mu)_\infty | (q^{\frac{2\mu+2\alpha+1}{4}} e^{i\theta})_\infty (q^{\frac{2\mu+2\alpha+3}{4}} e^{i\theta})_\infty |^2} \\ & \cdot p_n^{(\alpha+\mu, \beta-\mu)}(x|q) \cdot \frac{(q)_n}{(q^{\alpha+\mu+1})_n} \end{aligned} \quad (4.3.4)$$

This is the q -analogue of the projection formula for Jacobi polynomials, (see Askey [12]), namely

$$(1-x)^{\alpha+\mu} \frac{p_n^{(\alpha+\mu, \beta-\mu)}(x)}{p_n^{(\alpha+\mu, \beta-\mu)}(1)} = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 (1-y)^\alpha \frac{p_n^{(\alpha, \beta)}(y)}{p_n^{(\alpha, \beta)}(1)} (y-x)^{\mu-1} dy, \quad (4.3.5)$$

where $p_n^{(\alpha, \beta)}(x)$ is the ordinary Jacobi polynomial.

In a similar manner we let $\beta \rightarrow 1 -$ in (4.3.1). Again

$$\lim_{\beta \rightarrow 1-} \frac{(\beta^2)_\infty}{|(\beta e^{i\psi_2 - i\theta})_\infty|^2} = \frac{2\pi}{(q)_\infty} \delta(\psi_2 - \theta).$$

Hence

$$\lim_{R \rightarrow 1-} K(x, y) = \left(\frac{(q)_\infty}{2\pi} \right)^2 \frac{(cd\mu^{-2})_\infty (\mu^2)_\infty (\lambda^2)_\infty}{(cd\lambda^2)_\infty}$$

$$\left| \frac{(c\lambda e^{i\theta})_\infty (d\lambda e^{i\theta})_\infty (e^{2i\varphi})_\infty}{(\lambda e^{i\theta+i\varphi})_\infty (\lambda e^{i\theta-i\varphi})_\infty} \right|^2$$

$$\int_0^\pi d\psi_1 \left| \frac{(e^{2i\psi_1})_\infty}{(\mu e^{i\psi_1+i\varphi})_\infty (\mu e^{i\psi_1-i\varphi})_\infty (c\mu^{-1} e^{i\psi_1})_\infty (d\mu^{-1} e^{i\psi_1})_\infty} \right|^2$$

$$= \frac{[(q)_\infty]^2 (cd\mu^{-2})_\infty (\mu^2)_\infty (\lambda^2)_\infty}{(2\pi)^2 (cd\lambda^2)_\infty}$$

$$\left| \frac{(c\lambda e^{i\theta})_\infty (d\lambda e^{i\theta})_\infty (e^{2i\varphi})_\infty}{(\lambda e^{i\theta+i\varphi})_\infty (\lambda e^{i\theta-i\varphi})_\infty} \right|^2$$

$$\frac{2\pi (cd)_\infty}{(q)_\infty (\mu^2)_\infty (cd\mu^{-2})_\infty | (ce^{i\varphi})_\infty (de^{i\varphi})_\infty |^2}$$

$$= \frac{(q)_\infty (cd)_\infty (\lambda^2)_\infty |(c\lambda e^{i\theta})_\infty (d\lambda e^{i\theta})_\infty|^2}{2\pi (cd\lambda^2)}$$

$$w(y; c, d, \lambda e^{i\theta}, \lambda e^{-i\theta}). \quad (4.3.6)$$

Also when $\beta \rightarrow 1^-$, the eigen-values λ_n in (4.1.23) become

$$\lambda_n = 3^{\frac{n}{2}} \left[\begin{matrix} q^{-n}, q^{n-1}abcd, \lambda^2 \\ ab, cd\lambda^2 \end{matrix} ; q, q \right] = \frac{(cd)_n (ab\lambda^{-2})_n}{(ab)_n (cd\lambda^2)_n} \lambda^{2n}, \quad (4.3.7)$$

by (1.2.8).

Substituting (4.3.7) and (4.3.6) into (4.1.20) we get

$$\begin{aligned} & \frac{(q)_\infty (cd)_\infty (\lambda^2)_\infty}{2\pi (cd\lambda^2)} |(c\lambda e^{i\theta})_\infty (d\lambda e^{i\theta})_\infty|^2 \\ & \cdot \int_{-1}^1 dy w(y; c, d, \lambda e^{i\theta}, \lambda e^{-i\theta}) p_n(y) \\ & = \frac{(cd)_n (ab\lambda^{-2})_n}{(ab)_n (cd\lambda^2)_n} \lambda^{2n} p_n(x; a\lambda^{-1}, b\lambda^{-1}, c\lambda, d\lambda). \end{aligned} \quad (4.3.8)$$

Again if we set $\lambda = q^{\mu/2}$, $a = q^{\frac{2\alpha+1}{4}}$, $b = q^{\frac{2\alpha+3}{4}}$, $c = -q^{\frac{2\beta+1}{4}}$,
 $\frac{2\beta+3}{4}$

and $d = -q^{\frac{2\beta+3}{4}}$ where $\mu > 0$, then (4.3.8) becomes the q -analogue of another well-known projection formula for Jacobi polynomials,

namely,

$$(1+x)^{\beta+\mu} \frac{p_n^{(\alpha-\mu, \beta+\mu)}(x)}{p_n^{(\beta+\mu, \alpha-\mu)}(1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^x (1+y)^{\beta} \frac{p_n^{(\alpha, \beta)}(y)}{p_n^{(\beta, \alpha)}(1)} (x-y)^{\mu-1} dy, \quad (4.3.9)$$

(see Askey [12]).

4. The dual orthogonality of the ${}_4\phi_3$ polynomials. In this section we shall establish the dual orthogonality relation (4.1.24) using general transformations of ϕ series.

To start with, we apply Sears' transformation (1.2.7) to $p_n(y)$ to obtain

$$\begin{aligned} p_n(y) &= \frac{(de^{-1\phi})_n (ce^{-1\phi})_n (ae^{1\phi})^n}{(ac)_n (ad)_n} \\ &\quad \cdot {}_4\phi_3 \left[\begin{matrix} q^{-n}, c^{-1}d^{-1}q^{1-n}, ae^{1\phi}, be^{1\phi} \\ ab, d^{-1}e^{1\phi}q^{1-n}, c^{-1}e^{1\phi}q^{1-n} \end{matrix} ; q, q \right] \\ &= \frac{(de^{-1\phi})_n (ce^{-1\phi})_n (bc^{-1}q)_\infty (ac^{-1}q)_\infty (abcde^{1\phi}q)^n (de^{-1\phi}q^n)_\infty}{(ac)_n (ad)_n (abc^{-1}e^{1\phi}q^n)_\infty (bdq^n)_\infty (adq^n)_\infty (c^{-1}e^{-1\phi}q)_\infty} \\ &\quad \cdot (ae^{1\phi})^n {}_8\phi_7 \left[\begin{matrix} abc^{-1}e^{1\phi}, q\sqrt{abc^{-1}e^{1\phi}}, -q\sqrt{abc^{-1}e^{1\phi}}, ae^{1\phi}, be^{1\phi}, \\ \sqrt{abc^{-1}e^{1\phi}}, -\sqrt{abc^{-1}e^{1\phi}}, bc^{-1}q, ac^{-1}q, \end{matrix} \right] \end{aligned}$$

$$\left[\begin{array}{l} abq^n, qc^{-1}e^{i\phi}, c^{-1}d^{-1}q^{1-n} \\ c^{-1}e^{i\phi}q^{1-n}, ab, abde^{i\phi}q^n \end{array} ; q, de^{-i\phi} \right] \quad (4.4.1)$$

The last equality is obtained upon applying Watson's (1.2.6).

On the other hand if we take v such that $|v| < 1$ and define

$$p_n^v(x) \equiv p_n(x; av, bv, cv^{-1}, dv^{-1}) = {}_4\phi_3 \left[\begin{array}{l} q^{-n}, q^{n-1}abcd, ave^{i\theta}, ave^{i\theta} \\ abv^2, ac, ad \end{array} ; q, q \right] \quad (4.4.2)$$

then, using (1.2.6) we get

$$p_n^v(x) = \frac{(dv^{-1}e^{-i\theta})_n (bc)_n (ave^{i\theta})^n}{(abcve^{i\theta})_n (ad)_n}$$

$${}_8\phi_7 \left[\begin{array}{l} abcve^{i\theta}q^{-1}, q\sqrt{abcve^{i\theta}q^{-1}}, -q\sqrt{abcve^{i\theta}q^{-1}}, cv^{-1}e^{i\theta}, \\ \sqrt{abcve^{i\theta}q^{-1}}, -\sqrt{abcve^{i\theta}q^{-1}}, abv^2, \\ bve^{i\theta}, ave^{i\theta}q^{n-1}abcd, q^{-n} \end{array} ; q, qd^{-1}ve^{-i\theta} \right] \quad (4.4.3)$$

Using (4.4.1) and (4.4.3) in the following expression leads

to

$$f(x, y) \equiv \sum_{n=0}^{\infty} h_n^{-1} p_n^v(x) p_n(y)$$

$$= \frac{(bc^{-1}q)_{\infty} (ac^{-1}q)_{\infty} (de^{-i\phi})_{\infty} (abde^{i\phi})_{\infty}}{(abc^{-1}e^{i\phi}q)_{\infty} (c^{-1}e^{-i\phi}q)_{\infty} (ad)_{\infty} (bd)_{\infty}} h_0^{-1}$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^n \sum_{k=0}^{\infty} \frac{(abcdq^{-1})_n (q\sqrt{abcdq^{-1}})_n (-q\sqrt{abcdq^{-1}})_n (ab)_n}{(q)_n (\sqrt{abcdq^{-1}})_n (-\sqrt{abcdq^{-1}})_n (cd)_n}$$

$$\frac{(dv^{-1}e^{-i\theta})_n (ce^{-i\phi})_n (ve^{i\phi+i\theta})^n (abcve^{i\theta}q^{-1})_r (q\sqrt{abcve^{i\theta}q^{-1}})_r}{(abcve^{i\theta}v)_n (abd e^{i\phi})_n (q)_r (\sqrt{abcve^{i\theta}q^{-1}})_r}$$

$$\frac{(-q\sqrt{abcve^{i\theta}q^{-1}})_r (cv^{-1}e^{i\theta})_r (bve^{i\theta})_r (ave^{i\theta})_r (q^{n-1}abcd)_r}{(-\sqrt{abcve^{i\theta}q^{-1}})_r (abv^2)_r (ac)_r (bc)_r (d^{-1}ve^{i\theta}q^{1-n})_r}$$

$$\frac{(q^{-n})_r (d^{-1}vqe^{i\theta})^r (abc^{-1}e^{i\phi})_k (q\sqrt{abc^{-1}e^{i\phi}})_k}{(abcve^{i\theta}q^n)_r (q)_k (\sqrt{abc^{-1}e^{i\phi}})_k}$$

$$\frac{(-q\sqrt{abc^{-1}e^{i\phi}})_k (ae^{i\phi})_k (be^{i\phi})_k (c^{-1}e^{i\phi}q)_k (c^{-1}d^{-1}q^{1-n})_k (abq^n)_k (de^{-i\phi})^k}{(-\sqrt{abc^{-1}e^{i\phi}})_k (bc^{-1}q)_k (ac^{-1}q)_k (ab)_k (abde^{i\phi}q^n)_k (c^{-1}e^{i\phi}q^{1-n})_k}$$

$$= \frac{(bc^{-1}q)_\infty (ac^{-1}q)_\infty (de^{-1\phi})_\infty (abde^{1\phi})_\infty}{(abc^{-1}e^{1\phi}q)_\infty (c^{-1}e^{-1\phi}q)_\infty (ad)_\infty (bd)_\infty} h_0^{-1}$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^n \sum_{k=0}^{\infty} \frac{(abc dq^{-1})_{n+r} (q\sqrt{abcd} q^{-1})_n (-q\sqrt{abcd} q^{-1})_n (ab)_{n+k}}{(q)_{n-r} (\sqrt{abcd} q^{-1})_n (-\sqrt{abcd} q^{-1})_n (cd q^{-k})_n}$$

$$\frac{(ce^{-1\phi} q^{-k})_n (dv^{-1} e^{-1\theta})_{n-r} (ve^{1\phi+1\theta})^n (abc ve^{1\theta} q^{-1})_r}{(abde^{1\phi})_{n+k} (abc ve^{1\theta})_{n+r} (q)_r}$$

$$\frac{(q\sqrt{abc ve^{1\theta} q^{-1}})_r (-q\sqrt{abc ve^{1\theta} q^{-1}})_r (cv^{-1} e^{1\theta})_r (bve^{1\theta})_r (ave^{1\theta})_r (a^{-21\theta})^r}{(\sqrt{abc ve^{1\theta} q^{-1}})_r (-\sqrt{abc ve^{1\theta} q^{-1}})_r (abv^2)_r (ac)_r (bc)_r}$$

$$\frac{(abc^{-1} e^{1\phi})_k (q\sqrt{abc^{-1} e^{1\phi}})_k (-q\sqrt{abc^{-1} e^{1\phi}})_k (ae^{1\phi})_k (be^{1\phi})_k}{(q)_k (\sqrt{abc^{-1} e^{1\phi}})_k (-\sqrt{abc^{-1} e^{1\phi}})_k (bc^{-1} q)_k (ac^{-1} q)_k (ab)_k} (de^{-1\phi})^k$$

(4.4.4)

Replace $n-r$ by n and simplify to obtain, after interchanging the order of summation,

$$f(x, y) = \frac{(bc^{-1}q)_\infty (ac^{-1}q)_\infty (de^{-1\phi})_\infty (abde^{1\phi})_\infty}{(abc^{-1}e^{1\phi}q)_\infty (c^{-1}e^{-1\phi}q)_\infty (ad)_\infty (bd)_\infty} h_0^{-1}$$

$$\sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(abcd q^{-1})_{2r} (q\sqrt{abcd} q^{-1})_r (-q\sqrt{abcd} q^{-1})_r (ab q^k)_r}{(q)_r (\sqrt{abcd} q^{-1})_r (-\sqrt{abcd} q^{-1})_r (cd q^{-k})_r}$$

- [69] _____, Some transformations of generalized hypergeometric series, Proc. London Math. Soc. (2), 26 (1927), 257-72.
- [70] _____, On a formula implied in Orr's theorems concerning the products of hypergeometric series, J. London Math. Soc., 4 (1929), 48-50.
- [71] A.H. Zemanian, Distribution Theory and Transformation Analysis, McGraw-Hill Book Company, 1965.
- *[72] R.P. Agarwal, Generalized Hypergeometric Series, Uttar Pradesh, Scientific Research Committee Monograph #6, (1963).

* The author is grateful to the examining board for pointing out the omission of this reference.

$$\begin{aligned}
& \frac{(abcd)_\infty (de^{1\phi})_\infty (abve^{1\phi+1\theta})_\infty (cve^{1\theta})_\infty (abcve^{1\theta})_{2r} (abde^{1\phi})_{k+r}}{(abcve^{1\theta})_\infty (ve^{1\phi+1\theta})_\infty (abde^{1\phi})_\infty (cd)_\infty (abcd)_{2r} (abve^{1\phi+1\theta})_{k+r}} \\
& \frac{(cdq^{-k})_r (c^{-1}v^{-1}e^{-1\theta}q)_k}{(cve^{1\theta}q^{-k})_r (c^{-1}d^{-1}q)_k}
\end{aligned}$$

which when inserted into (4.4.5) gives

$$\begin{aligned}
f(x,y) &= \frac{(q)_\infty (ab)_\infty (ac)_\infty (bc)_\infty (bc^{-1}q)_\infty (ac^{-1}q)_\infty (abve^{1\phi+1\theta})_\infty}{2\pi (abc^{-1}e^{1\phi}q)_\infty (c^{-1}e^{-1\phi}q)_\infty (abcve^{1\theta})_\infty (ve^{1\phi+1\theta})_\infty} \\
& \cdot (cve^{1\theta})_\infty | (de^{1\phi})_\infty |^2 \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(abcve^{1\theta}q^{-1})_r (q\sqrt{abcve^{1\theta}q^{-1}})_r}{(q)_r (\sqrt{abcve^{1\theta}q^{-1}})_r} \\
& \frac{(-q\sqrt{abcve^{1\theta}q^{-1}})_r (abq^k)_r (ce^{-1\phi}q^{-k})_r (cv^{-1}e^{1\theta})_r (bve^{1\theta})_r}{(-\sqrt{abcve^{1\theta}q^{-1}})_r (cve^{1\theta}q^{-k})_r (abve^{1\phi+1\theta})_{k+r} (abv^2)_r (ac)_r} \\
& \frac{(ave^{1\theta})_r (ve^{1\phi-1\theta})^r (abc^{-1}e^{1\phi})_k (q\sqrt{abc^{-1}e^{1\phi}})_k (-q\sqrt{abc^{-1}e^{1\phi}})_k}{(bc)_r (q)_k (\sqrt{abc^{-1}e^{1\phi}})_k (-\sqrt{abc^{-1}e^{1\phi}})_k} \\
& \frac{(ae^{1\phi})_k (be^{1\phi})_k}{(bc^{-1}q)_k (ac^{-1}q)_k} (c^{-1}v^{-1}e^{-1\theta}q)_k (ve^{1\theta-1\phi})^k
\end{aligned}$$

$$\begin{aligned}
&= \frac{(q)_\infty (ab)_\infty (ac)_\infty (bc)_\infty (bc^{-1}q)_\infty (ac^{-1}q)_\infty (abve^{i\phi+i\theta})_\infty (cve^{i\theta})_\infty}{2\pi (abc^{-1}e^{i\phi}q)_\infty (c^{-1}e^{-i\phi}q)_\infty (abcve^{i\theta})_\infty (ve^{i\phi+i\theta})_\infty} \\
&\quad |(de^{i\phi})_\infty|^2 \sum_{r=0}^{\infty} \frac{(abcve^{i\theta}q^{-1})_r (q\sqrt{abcve^{i\theta}q^{-1}})_r (-q\sqrt{abcve^{i\theta}q^{-1}})_r (cv^{-1}e^{i\theta})_r}{(q)_r (\sqrt{abcve^{i\theta}q^{-1}})_r (-\sqrt{abcve^{i\theta}q^{-1}})_r (abv^2)_r} \\
&\quad \frac{(bve^{i\theta})_r (ave^{i\theta})_r (ce^{-i\phi})_r (ab)_r (ve^{i\phi-i\theta})_r}{(ac)_r (bc)_r (abve^{i\phi+i\theta})_r (cve^{i\theta})_r} \\
&\quad \cdot {}_8\phi_7 \left[\begin{matrix} abc^{-1}e^{i\phi}, & q\sqrt{abc^{-1}e^{i\phi}}, & -q\sqrt{abc^{-1}e^{i\phi}}, & ae^{i\phi}, & be^{i\phi}, & c^{-1}e^{-i\phi}q, \\ & \sqrt{abc^{-1}e^{i\phi}}, & -\sqrt{abc^{-1}e^{i\phi}}, & bc^{-1}q, & ac^{-1}q, & ab, \end{matrix} \right. \\
&\quad \left. \begin{matrix} c^{-1}v^{-1}e^{-i\theta}q^{1-r}, & abq^r \\ abve^{i\phi+i\theta}q^r, & c^{-1}e^{i\phi}q^{1-r} \end{matrix} ; q, ve^{i\theta-i\phi} \right].
\end{aligned}$$

(4.4.6)

The above ${}_8\phi_7$ is a non-terminating series, but using a sequence of transformations on it starting with Watson (1.2.6), followed by Sears (1.2.7) and finally (1.2.6) again we have the following steps that lead to a terminating ${}_8\phi_7$, namely,

$$8^{\frac{1}{2}} 7^{\frac{1}{2}} = \frac{(abc^{-1} e^{i\frac{1}{2}} q)_{\infty} (c^{-1} e^{-i\frac{1}{2}} q)_{\infty} (ave^{i\theta} q^r)_{\infty} (bve^{i\theta} q^r)_{\infty}}{(ac^{-1} q)_{\infty} (bc^{-1} q)_{\infty} (abve^{i\frac{1}{2}+i\theta} q^r)_{\infty} (ve^{i\theta-i\frac{1}{2}})_{\infty}}$$

$$4^{\frac{1}{2}} 3 \left[\begin{matrix} q^{-r}, & c^{-1} v^{-1} e^{-i\theta} q^{1-r}, & ae^{i\frac{1}{2}}, & be^{i\frac{1}{2}} \\ ab, & v^{-1} e^{i\frac{1}{2}-i\theta} q^{1-r}, & c^{-1} e^{i\frac{1}{2}} q^{1-r} \end{matrix} ; q, q \right]$$

$$= \frac{(abc^{-1} e^{i\frac{1}{2}} q)_{\infty} (c^{-1} e^{-i\frac{1}{2}} q)_{\infty} (ave^{i\theta})_{\infty} (bve^{i\theta})_{\infty} (abve^{i\frac{1}{2}+i\theta})_r (ac)_r (a^{-1} e^{-i\frac{1}{2}})^r}{(ac^{-1} q)_{\infty} (bc^{-1} q)_{\infty} (abve^{i\frac{1}{2}+i\theta})_{\infty} (ve^{i\theta-i\frac{1}{2}})_{\infty} (bve^{i\theta})_r (ce^{-i\frac{1}{2}})_r}$$

$$4^{\frac{1}{2}} 3 \left[\begin{matrix} q^{-r}, & q^{r-1} abc ve^{i\theta}, & ae^{i\frac{1}{2}}, & ae^{-i\frac{1}{2}} \\ ab, & ac, & ave^{i\theta} \end{matrix} ; q, q \right]$$

$$= \frac{(abc^{-1} e^{i\frac{1}{2}} q)_{\infty} (c^{-1} e^{-i\frac{1}{2}} q)_{\infty} (ave^{i\theta})_{\infty} (bve^{i\theta})_{\infty} (abve^{i\frac{1}{2}+i\theta})_r (ac)_r}{(ac^{-1} q)_{\infty} (bc^{-1} q)_{\infty} (abve^{i\frac{1}{2}+i\theta})_{\infty} (ve^{i\theta-i\frac{1}{2}})_{\infty} (bve^{i\theta})_r (ce^{-i\frac{1}{2}})_r}$$

$$\frac{(ve^{i\theta-i\frac{1}{2}})_r (bc)_r}{(abce^{i\frac{1}{2}})_r (ave^{i\theta})_r} \cdot 8^{\frac{1}{2}} 7 \left[\begin{matrix} abce^{i\frac{1}{2}} q^{-1}, & q \sqrt{abce^{i\frac{1}{2}} q^{-1}}, & -q \sqrt{abce^{i\frac{1}{2}} q^{-1}}, \\ \sqrt{abce^{i\frac{1}{2}} q^{-1}}, & -\sqrt{abce^{i\frac{1}{2}} q^{-1}}, \end{matrix} \right]$$

$$\left[\begin{matrix} ce^{i\frac{1}{2}}, & be^{i\frac{1}{2}}, & ae^{i\frac{1}{2}}, & q^{r-1} abc ve^{i\theta}, & q^{-r} \\ ab, & ac, & bc, & v^{-1} e^{i\frac{1}{2}-i\theta} q^{1-r}, & abce^{i\frac{1}{2}} q^r \end{matrix} ; q, q v^{-1} e^{-i\theta-i\frac{1}{2}} \right]$$

Insert this into (4.4.6) to get upon some straight forward simplifications

$$\begin{aligned}
 f(x,y) &= \frac{(q)_\infty (ab)_\infty (ac)_\infty (bc)_\infty (cve^{i\theta})_\infty (ave^{i\theta})_\infty (bve^{i\theta})_\infty |(de^{i\phi})_\infty|^2}{2\pi (abcve^{i\theta})_\infty (ve^{i\theta-i\phi})_\infty (ve^{i\phi+i\theta})_\infty} \\
 &= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(abcve^{i\theta}q^{-1})_{r+k} (q\sqrt{abcve^{i\theta}q^{-1}})_r (-q\sqrt{abcve^{i\theta}q^{-1}})_r}{(q)_{r-k} (\sqrt{abcve^{i\theta}q^{-1}})_r (-\sqrt{abcve^{i\theta}q^{-1}})_r} \\
 &\quad \frac{(cv^{-1}e^{i\theta})_r (ab)_r (ve^{i\theta-i\phi})_{r-k} (ve^{i\phi-i\theta})^r (abce^{i\phi}q^{-1})_k}{(abv^2)_r (cve^{i\theta})_r (abce^{i\phi})_{r+k} (q)_k} \\
 &\quad \frac{(q\sqrt{abcve^{i\phi}q^{-1}})_k (-q\sqrt{abcve^{i\phi}q^{-1}})_k (ae^{i\phi})_k (be^{i\phi})_k (ce^{i\phi})_k (e^{-2i\phi})^k}{(\sqrt{abcve^{i\phi}q^{-1}})_k (-\sqrt{abcve^{i\phi}q^{-1}})_k (bc)_k (ac)_k (ab)_k}
 \end{aligned}
 \tag{4.4.9}$$

which, upon replacing $r-k$ by r and simplifying, gives

$$\begin{aligned}
 f(x,y) &= \frac{(q)_\infty (ab)_\infty (ac)_\infty (bc)_\infty (cve^{i\theta})_\infty (ave^{i\theta})_\infty (bve^{i\theta})_\infty |(de^{i\phi})_\infty|^2}{2\pi (abcve^{i\theta})_\infty (ve^{i\theta-i\phi})_\infty (ve^{i\phi+i\theta})_\infty} \\
 &= \sum_{k=0}^{\infty} \frac{(abcve^{i\theta}q^{-1})_{2k} (q\sqrt{abcve^{i\theta}q^{-1}})_k (-q\sqrt{abcve^{i\theta}q^{-1}})_k (cv^{-1}e^{i\theta})_k}{(q)_k (\sqrt{abcve^{i\theta}q^{-1}})_k (-\sqrt{abcve^{i\theta}q^{-1}})_k (abv^2)_k}
 \end{aligned}$$

$$\frac{(ve^{-1\phi-1\theta})^k (abce^{1\phi}q^{-1})_k (q\sqrt{abce^{1\phi}q^{-1}})_k (-q\sqrt{abce^{1\phi}q^{-1}})_k}{(cve^{1\theta})_k (abce^{1\phi})_{2k} (\sqrt{abce^{1\phi}q^{-1}})_k (-\sqrt{abce^{1\phi}q^{-1}})_k}$$

$$\frac{(ae^{1\phi})_k (be^{1\phi})_k (ce^{1\phi})_k}{(bc)_k (ac)_k}$$

$${}_6\phi_5 \left[\begin{matrix} abc ve^{1\theta} q^{2k-1}, & q\sqrt{abc ve^{1\theta} q^{2k-1}}, & -q\sqrt{abc ve^{1\theta} q^{2k-1}}, & abq^k, \\ & \sqrt{abc ve^{1\theta} q^{2k-1}}, & -\sqrt{abc ve^{1\theta} q^{2k-1}}, & cve^{1\theta} q^k, \\ & & & cv^{-1} e^{1\theta} q^k, & ve^{1\theta-1\phi} \\ & & & & ; q, ve^{1\phi-1\theta} \\ & & & & abv^2 q^k, & abce^{1\phi} q^{2k} \end{matrix} \right] \quad (4.4.10)$$

Once again the ${}_6\phi_5$ above is summable by (1.2.5) and the sum

is:

$$\frac{(abc ve^{1\theta})_{\infty} (v^2)_{\infty} (ce^{1\theta})_{\infty} (abve^{1\phi-1\theta})_{\infty} (cve^{1\theta})_k (abv^2)_k (abce^{1\phi})_{2k}}{(ve^{1\phi-1\theta})_{\infty} (cve^{1\theta})_{\infty} (abv^2)_{\infty} (abce^{1\phi})_{\infty} (abcve^{1\theta})_{2k} (ce^{1\phi})_k (abve^{1\phi-1\theta})_k}$$

Thus we have

$$f(x,y) = \frac{(q)_\infty (ab)_\infty (ac)_\infty (bc)_\infty (ave^{1\theta})_\infty (bve^{1\theta})_\infty (v^2)_\infty (ce^{1\phi})_\infty}{2\pi |(ve^{1\phi-1\theta})_\infty|^2 (ve^{1\phi+1\theta})_\infty (abv^2)_\infty (abce^{1\phi})_\infty}$$

$$\cdot (abve^{1\phi-1\theta})_\infty |(de^{1\phi})_\infty|^2$$

$$6^{\phi}_5 \left[\begin{array}{cccc} abce^{1\phi} q^{-1}, & q \sqrt{abce^{1\phi} q^{-1}}, & -q \sqrt{abce^{1\phi} q^{-1}}, & cv^{-1} e^{1\theta}, \\ & \sqrt{abce^{1\phi} q^{-1}}, & -\sqrt{abce^{1\phi} q^{-1}}, & abve^{1\phi-1\theta}, \end{array} \right]$$

$$\left[\begin{array}{c} be^{1\phi}, ae^{1\phi} \\ ; q, ve^{-1\phi-1\theta} \\ ac, bc \end{array} \right]$$

$$= \frac{(q)_\infty (ab)_\infty |(ave^{1\theta})_\infty (bve^{1\theta})_\infty (ce^{1\phi})_\infty (de^{1\phi})_\infty|^2 (v^2)_\infty}{2\pi |(ve^{1\phi-1\theta})_\infty (ve^{1\phi+1\theta})_\infty|^2 (abv^2)_\infty} \quad (4.4.11)$$

The last line was obtained by summing the 6^{ϕ}_5 using (1.2.5) once more.

Now consider the terms

$$\begin{aligned}
\frac{(v^2)_\infty(q)_\infty}{|(ve^{i\phi-i\theta})_\infty|^2} &= \frac{(v^2q)_\infty(1-v^2)(q)_\infty}{|(vqe^{i\phi-i\theta})_\infty|^2(1-ve^{i\phi-i\theta})(1-ve^{-i\phi+i\theta})} \\
&= \frac{(v^2q)_\infty(q)_\infty}{|(vqe^{i\phi-i\theta})_\infty|^2} \cdot \frac{(1-v^2)}{2u} \left[\frac{1}{u+it} + \frac{1}{u-it} \right],
\end{aligned}$$

where $u = 1 - v \cos(\phi - \theta)$ and $t = v \sin(\phi - \theta)$.

Borrowing the following result from the theory of distributions, (see [71 p. 67]), namely

$$\lim_{u \rightarrow 0+} \frac{1}{u \pm it} = \pi \delta(t) \mp i \text{PV} \frac{1}{t}, \quad (4.4.13)$$

where $\delta(t)$ is Dirac's delta function and PV is the Cauchy principal value, we have

$$\lim_{u \rightarrow 0+} \frac{1}{u+it} + \frac{1}{u-it} = 2\pi \delta(t). \quad (4.4.14)$$

But $u \rightarrow 0+$ when $\phi = \theta$ and $v \rightarrow 1$. Therefore (4.4.12) gives

$$\lim_{v \rightarrow 1-} \frac{(v^2)_\infty(q)_\infty}{|(ve^{i\phi-i\theta})_\infty|^2} = 2\pi \delta(\phi - \theta), \quad (4.4.15)$$

and

$$\lim_{v \rightarrow 1-} f(x,y) = \sum_{n=0}^{\infty} h_n^{-1} p_n(x) p_n(y) = \frac{\delta(\phi-\theta)}{w(x)} \quad (4.4.16)$$

as required. Here $w(x)$ is given in the alternative form

$$w(x) = \left| \frac{(e^{2i\theta})_{\infty}}{(ae^{i\theta})_{\infty} (be^{i\theta})_{\infty} (ce^{i\theta})_{\infty} (de^{i\theta})_{\infty}} \right|^2, \quad (4.4.17)$$

where $x = \cos \theta$, (see Askey & Wilson [18]).

APPENDIX

The terminating version of the limiting case of Jackson's theorem (1.2.5), that is when $d = q^{-m}$, may be written as

$$\begin{aligned}
 \text{A-1} \quad & {}_6\phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-m} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1+m} \end{matrix} ; q, \frac{aq^{1+m}}{bc} \right] \\
 & = \frac{(aq)_m (aq/bc)_m}{(aq/b)_m (aq/c)_m}
 \end{aligned}$$

In addition to that, use of the following identities will be made:

If $p = q^2$, then

$$\text{A-2} \quad (p^A; p)_k = (q^{2A}; q^2)_k = (q^A)_k (-q^A)_k, \quad (\text{Slater [60, (II.16), p. 242]}).$$

$$\text{A-3} \quad (aq^{-k})_k = (-a)^k q^{\frac{k}{2}(k+1)} (q/a)_k, \quad (\text{Slater [60, (II.5), p. 241]}).$$

$$\text{A-4} \quad (a)_{k+r} = (a)_k (aq^k)_r = (a)_r (aq^r)_k.$$

A-3 and A-4 combine to give

$$\text{A-5} \quad (q^{-k-r})_r = (-1)^r q^{-kr - \frac{r}{2}(r+1)} \frac{(q)_{k+r}}{(q)_k}.$$

$$A-6 \quad (a)_{n-1} = \frac{(a)_N (-1)^i q^{\frac{1}{2}(i+1)-N1}}{(q^{1-N}/a)_i a^i}, \quad (\text{Slater [60, (II.9), p.241]}).$$

$$A-7 \quad (aq^{-1})_{N-1} = \frac{(a)_N (q/a)_i (-1)^i q^{\frac{1}{2}(3i+1)-2N1}}{(q^{1-N}/a)_{2i} a^i}, \quad (\text{Slater [60, (II.12), p.241]}).$$

$$A-8 \quad (a)_{2i} = (a; q^2)_i (aq; q^2)_i, \quad (\text{Slater [60, (II.17), p.242]}).$$

Use of A-2 and A-8 gives

$$A-9 \quad \frac{(p^a; p)_{2r} (-p^{1+a/2}; p)_r}{(-q^{a+1})_{2r} (-p^{a/2}; p)_r} = \frac{(q^a)_{2r} (-q^a)_{2r} (-p^{1+a/2}; p)_r}{(-q^{a+1}; p)_r (-q^{a+2}; p)_r (-p^{a/2}; p)_r}$$

$$= \frac{(q^a)_{2r} (-q^a)_{2r} (-q^{a+2}; p)_r}{(-q^{a+1}; q^2)_r (-q^a; q^2)_r (-q^a; p)_r} = \frac{(q^a)_{2r} (-q^a)_{2r}}{(-q^a)_{2r}} = (q^a)_{2r}$$

$$= (p^{a/2}; p)_r (p^{1/2+a/2}; p)_r$$

where $p \equiv q^2$.

$$A-10 \quad (aq^{kn})_{N-n} = \frac{(a)_N (aq^N) (k-1)_n}{(a)_{kn}}, \quad \text{Slater [60, (II.11, p. 241)]}.$$

$$A-11 \quad (aq^n)_{N-2N} = \frac{(a)_N (-1)^n q^{\frac{1}{2}(n+1)n - nN}}{(a)_n (q^{1-N}/a)_n a^n}, \quad (\text{Slater [60, (II.15)], p. 242}).$$

Dixon's q-analogue (special case of A-1, $c = \sqrt{a}$).

$$A-12 \quad {}_4\phi_3 \left[\begin{matrix} a, -q\sqrt{a}, b, q^{-m} \\ -\sqrt{a}, aq/b, aq^{1+m} \end{matrix} ; q, q^{1+m}\sqrt{a}/b \right]$$

$$= \frac{(aq)_m (q\sqrt{a}/b)_m}{(q\sqrt{a})_m (aq/b)_m}, \quad (\text{Slater [60, (IV.5)], p.247}).$$

REFERENCES

- [1] R.P. Agarwal and A. Verma, Generalized basic hypergeometric series with unconnected bases, Proc. Cambridge Philos. Soc. 63 (1967), 727-34.
- [2] _____, Generalized basic hypergeometric series with unconnected bases (ii), Quart. J. Math. Oxford (2), 18 (1967), 181-192.
- [3] W.A. Al-Salam and M.E.-H. Ismail, Reproducing kernels for q -Jacobi polynomials, Proc. AMS., 67 #1 (1977), 105-10.
- [4] G.E. Andrews, Applications of basic hypergeometric series, SIAM Rev. 16 (1974), 441-84.
- [5] _____, Problems and prospects for basic hypergeometric functions, "Theory and Applications of Special Functions", (R.A. Askey, Editor), Academic Press, N.Y., 1975.
- [6] _____, The Theory of Partitions, Encyclopedia of Math. and its Applications, Vol. 2, Addison-Wesley, Reading, Mass., 1977.
- [7] _____, Connection coefficient problem of partitions, AMS Proc. Symp. Pure Math. 34 (1979), edited by Ray-Chaudhuri, 1-24.
- [8] G.E. Andrews and R.A. Askey, Another q -extension of the beta function, Proc. AMS., 81 #1 (1981), 97-100.

- [9] _____, Enumeration of partitions: The role of Eulerian series and q -orthogonal polynomials, Higher Combinatorics, ed. M. Aigner, D. Reidel, Dordrecht, Holland, 1977, 3-26.
- [9a] _____, The classical and discrete orthogonal polynomials and their q -analogues, to appear.
- [10] R.A. Askey, Orthogonal polynomials and positivity, "Studies in Applied Mathematics, 6, Special Functions and Wave Propagation", edited by D. Ludwig and F. Olver, SIAM, Philadelphia (1970), 64-85.
- [11] _____, Summability of Jacobi series, Trans. AMS. 179 (1973), 71-84.
- [12] _____, Orthogonal Polynomials and Special Functions, "Regional Conference Series in Applied Mathematics", Vol. 21, SIAM, Philadelphia (1975).
- [13] _____, The q -gamma and q -beta functions, Applicable Analysis, 8 (1978), 125-41.
- [14] R.A. Askey and J. Fitch, Integral representations for Jacobi polynomials and some applications, J. Math. Anal. Appl., 26 (1969), 411-37.
- [15] R.A. Askey and G. Gasper, Jacobi polynomial expansions of Jacobi polynomials with non-negative coefficients, Proc. Cambridge Philos. Soc., 70 (1971), 243-55.

- [16] R.A. Askey and M.E.-H. Ismail, A generalization of ultraspherical polynomials, "Technical Summary Report #1851", Math. Res. Center, Univ. of Wisconsin, Madison, Wis., 1978.
- [17] R.A. Askey and J. Wilson, A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols, SIAM J. Math. Anal. 10 (1979), 1008-16.
- [18] _____, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, to appear.
- [19] W.N. Bailey, Some theorems concerning products of hypergeometric series, Proc. London Math. Soc. (2), 38 (1935), 377-84.
- [20] _____, Transformations of nearly-poised basic hypergeometric series, J. London Math. Soc. 22 (1947), 237-40.
- [21] _____, Generalized Hypergeometric Series, Etchert-Hafner Service Agency N.Y., 1964.
- [22] R.J. Baxter, Rogers-Ramanujan identities in the hard hexagon model, J. Stat. Phys., to appear.
- [23] L. Carlitz, Some formulas of F.H. Jackson, Monat. für Math., 73 (1969), 193-98.
- [24] _____, On a problem in the history of Chinese mathematics, Mat. Lapok., 6 (1955), 219-20.

- [25] A. Cayley, On a theorem relating to hypergeometric series, Phil. Mag. (4), 16 (1858), 356-57. [Collected papers Vol. 3 268-69].
- [26] T.W. Chaundy, Frank Hilton Jackson, J. London Math. Soc. 37 (1962), 126-8.
- [27] Shih-Chieh Chu, Ssu Yuan Yu Chien (Previous Mirror of the Four Elements), 1303 (Chinese).
- [28] T. Clausen, Über die Falle Wenn die Reighe $y = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \dots$ ein quadratron der Form $z = 1 + \frac{\alpha' \cdot \beta'}{1 \cdot \gamma'} + \dots$ hat, J. für Math., 3 (1828), 89-95.
- [29] J. Dougall, On Vandermonde's theorem and some more general expansions, Proc. Edinburgh Math. Soc., 25 (1907), 114-32.
- [30] A. Erdélyi, Higher Transcendental Functions, Vol. 1, McGraw-Hill, N.Y., 1953.
- [31] L. Euler, Introductio in Analysis Infinitum, (1748), Lausanne, Vol. 1.
- [32] _____, Specimen transformationis singularis serierum, Nova Acta Academiae Scientiarum Petropolitanae, 92 (1794), 1801, 58-70; reproduced in Opera Omnia, 16 (part 2) (1935), 41-55.
- [33] G. Gasper, Positivity and special functions, "Theory and Applications of Special Functions", (R. Askey, ed.), Academic Press, (1975), 375-433.

- [34] _____, Projection formulas for orthogonal polynomials of a discrete variable, J. Math. Anal. Appl., 45 (1974), 176-98.
- [35] C.F. Gauss, Disquisitiones Generales Circa Seriem Infinitam, Thesis (1812) Gottingen; published in Ges. Werke Gottingen (1866), Vol. II, 437-45; III, 123-63; III, 207-29; III, 446-60.
- [36] I. Gessel and D. Stanton, Strange evaluations of hypergeometric series, SIAM J. Math. Anal. Vol. 13, 2 (1982), 295-308.
- [37] _____, Applications of q-Lagrange inversions to basic hypergeometric series, to appear.
- [38] E. Goursat, Sur l'equation différentielle linéaire qui admet pour intégral la série hypergéométrique, Ann. Sci. École Norm. Sup. 10 (1881), 3-142.
- [39] W. Hahn, Über Orthogonalpolynome, die q-Differenzengleichungen genügen, Math. Nach. 2 (1949), 263-78.
- [40] E. Heine, Handbuck die Kugelfunctionen (1898).
- [41] M.E.-H. Ismail and J.A. Wilson, Asymptotic and generating relations for the q-Jacobi and ${}_4\phi_3$ polynomials, to appear.
- [42] F.H. Jackson, Summation of q-hypergeometric series, Mess. Math. 50 (1921), 101-112.

- [43] _____, The q^θ -equations whose solutions are products of solutions of q^θ -equations of lower order, Quart. J. of Math., Oxford, 11 (1940), 1-17.
- [44] T. Koornwinder, Jacobi polynomials, III. An analytic proof of the addition formula, SIAM J. Math. Anal., Vol. 6, 3 (1975), 533-43.
- [45] E.E. Kummer, Über die hypergeometrische reihe, J. für Math. 15 (1836), 39-83 and 127-72.
- [46] T.P. Laine, Projection formulas and a new proof of the addition formula for the Jacobi polynomials, SIAM J. Math. Anal. Vol. 13, 2 (1982), 324-30.
- [47] B. Nassrallah and M. Rahman, On the q -analogues of some transformations of nearly-poised hypergeometric series, Trans. AMS, Vol. 268, 1 (1981), 211-29.
- [48] J. Needham, Science and Civilization in China, Vol. 3, Mathematics and the Sciences of the Heavens and the Earth, Cambridge University Press, N.Y., 1959.
- [49] W. McF.Orr, Theorems relating to the product of two hypergeometric series, Trans. Camb. Phil. Soc., 17 (1899), 1-15.
- [50] M. Rahman, Construction of a family of positive kernel from Jacobi polynomials, SIAM J. Math. Anal., Vol. 7, 1 (1976), 92-116.

- [51] _____, A five-parameters family of positive kernels from Jacobi polynomials, SIAM J. Math. Anal., 7 (1976), 386-413.
- [52] _____, A non-negative representation of the linearization coefficients of the product of Jacobi polynomials, Can. J. Math., Vol. XXXIII, 4 (1981), 915-28.
- [53] _____, The linearization of the product of continuous q -Jacobi polynomials, Can. J. Math., Vol. XXXIII, 4 (1981), 961-87.
- [54] _____, Reproducing kernels and bilinear sums for q -Racah and q -Wilson polynomials, Trans. AMS., to appear.
- [55] L.T. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc., 26 (1894), 318-43.
- [56] _____, Third memoir on the expansion of certain infinite products, Proc. London Math. Soc. 26 (1895), 15-32.
- [57] L.T. Rogers and S. Ramanujan, Proof of certain identities in combinatory analysis (with a prefatory note by G.H. Hardy), Proc. Camb. Phil. Soc., 19 (1919), 211-16.
- [58] D.B. Sears, On the transformation theory of basic hypergeometric functions, Proc. London Math. Soc. (2), 53 (1951), 158-180.
- [59] V.N. Singh, The basic analogues of identities of the Cayley-Orr type, J. London Math. Soc., 34 (1959), 15-22.

- [60] L.T. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, 1966.
- [61] G. Szego, Orthogonal Polynomials, AMS Colloq. publ., Vol. 23, Fourth edition, AMS. Providence, R.I., 1975.
- [62] L. Takács, On an identity of Shih-Chieh Chu, Acta Sci. Math. (Szeged), 34 (1973), 383-91.
- [63] J. Thomae, Über die Funktionen welche ... , J. für Math. (Crelle), 87, 26-73.
- [64] Arun Verma, A quadratic transformations of a basic hypergeometric series, SIAM J. Math. Anal., 11 (1980), 425-27.
- [65] J. Wallis, Arithmetica Infinitorum, London (1655).
- [66] G.N. Watson, A new proof of the Rogers-Ramanujan identities, J. London Math. Soc., 4 (1929), 4-9.
- [67] F. Whipple, On well-poised series, generalized hypergeometric series, having parameters in pairs, each pair with the same sum, Proc. London Math. Soc. (2), 24 (1926), 247-63.
- [68] _____, A fundamental relation between generalized hypergeometric series, J. London Math. Soc., 1 (1926), 138-45.

- [69] _____, Some transformations of generalized hypergeometric series, Proc. London Math. Soc. (2), 26 (1927), 257-72.
- [70] _____, On a formula implied in Orr's theorems concerning the products of hypergeometric series, J. London Math. Soc., 4 (1929), 48-50.
- [71] A.H. Zemanian, Distribution Theory and Transformation Analysis, McGraw-Hill Book Company, 1965.
- *[72] R.P. Agarwal, Generalized Hypergeometric Series, Uttar Pradesh, Scientific Research Committee Monograph #6, (1963).

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