1

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

35509

National Library
of CanadaBibliothèque nationale
du CanadaCANADIAN THESES
ON MICROFICHETHÈSES CANADIENNES
SUR MICROFICHE

NAME OF AUTHOR/NOM DE L'AUTEUR

A. BOSE

TITLE OF THESIS/TITRE DE LA THÈSE

Brownian Measure Processes

UNIVERSITY/UNIVERSITÉ

Carleton University

DEGREE FOR WHICH THESIS WAS PRESENTED/

GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE

Ph. D

YEAR THIS DEGREE CONFERRED/ANNÉE D'OBTENTION DE CE DEGRÉ

1977

NAME OF SUPERVISOR/NOM DU DIRECTEUR DE THÈSE

Prof. D. A. Dawson

Permission is hereby granted to the NATIONAL LIBRARY OF
CANADA to microfilm this thesis and to lend or sell copies
of the film.

L'autorisation est, par la présente, accordée à la BIBLIOTHÈ-
QUE NATIONALE DU CANADA de microfilmer cette thèse et
de prêter ou de vendre des exemplaires du film.

The author reserves other publication rights, and neither the
thesis nor extensive extracts from it may be printed or other-
wise reproduced without the author's written permission.

L'auteur se réserve les autres droits de publication; ni la
thèse ni de longs extraits de celle-ci ne doivent être imprimés
ou autrement reproduits sans l'autorisation écrite de l'auteur.

DATED/DATE

Aug 11, 77

SIGNED/SIGNÉ

Amitara Bose

PERMANENT ADDRESS/RÉSIDENCE FIXE

4135 Kensington Ave
Montreal

National Library of Canada

Cataloguing Branch
Canadian Theses Division

Ottawa, Canada
K1A 0N4

Bibliothèque nationale du Canada

Direction du catalogage
Division des thèses canadiennes

NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

**THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED**

AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

**LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS L'AVONS REÇUE**

Brownian Measure Processes

by

Amitava Bose

A thesis submitted to the Faculty of Graduate
Studies and Research in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

Department of Mathematics

Carleton University

Ottawa, Canada

May, 1977

©

Amitava Bose

1977

The undersigned recommend to the Faculty of Graduate
Studies and Research acceptance of the thesis

"Brownian Measure Process"

submitted by Amitava Bose in partial fulfillment of the
requirements for the degree of Doctor of Philosophy.

D. A. Dawson
.....
Dr. D. A. Dawson, Thesis Supervisor
and Chairman, Department of Mathematics

W. H. Fleming
.....
Dr. W. H. Fleming, External Examiner

Carleton University

May, 1977

Abstract

In recent years, the idea of a diffusion approximation has been extended to space-time models in population biology. This type of stochastic models can be best described as solutions of stochastic evolution equations.

Here, we propose Brownian measure processes - a class of measure valued stochastic processes - as solutions to a class of homogeneous stochastic evolution equations.

In Chapter I, we discuss the Radon measures and the vague topology, random measures and Prohorov's theorem on the uniqueness of characteristic functional, completely random measures and Kingman's representation and prove a few results of technical nature to be used in Chapter III.

In the first half of Chapter 2, we give a self-contained exposition of the theory of singular diffusions developed by Feller. In the remaining part of this chapter we establish an entrance law for singular diffusions.

In Chapter 3, we show the existence of Brownian measure processes. Our main tool here is the entrance law developed in the last chapter. In fact, Chapman-Kolmogorov relation is obtained from the entrance law.

In the fourth chapter, a few properties of Brownian measure process is discussed. Exploiting Doob's h-path process interpretation for the entrance law, the stochastic continuity of Brownian measure process is established. We also indicate how to extend our model to allow for immigrations.

Table of Contents.

Acknowledgements	iv
Introduction	v
Chapter I Measure-valued Markov processes	1
§1.0 Introduction	1
§1.1 Radon measures	3
§1.2 Random measures	11
§1.3 Completely random measures	17
§1.4 Measure-valued Markov processes	24
§1.5 Local processes	27
Chapter II Singular diffusions	31
§2.0 Introduction	31
§2.1 Feller processes	35
§2.2 Theory of McKean	52
§2.3 Consequences of Feller-McKean theory : Entrance law	55
Chapter III Brownian measure processes: Existence	76
§3.0 Introduction	76
§3.1 Atomic initial condition	77
§3.2 Non-atomic initial condition	88

Chapter IV	Brownian measure processes: Properties	103
4.0	Introduction	103
4.1	Calculation of mean	103
4.2	Stochastic Continuity	106
4.3	Multiplicativity	108
4.4	Immigration	112
Final Remark		114
References		115

Acknowledgement

It has been an honour and privilege to be a student of Professor D. A. Dawson without whose constructive criticism and constant advice this work would not have seen the light of day.

I would like to thank S. Mizuno for pointing out a proof (in Japanese!) of Professor S. Watanabe for a theorem of Skorokhod on random measures.

The financial support from the National Research Council of Canada in the form of a postgraduate scholarship is gratefully acknowledged.

Special thanks to Mrs. Murray for her expert typing on very short notice.

Introduction

In a pioneering study [8], Feller considered diffusion processes as approximations to stochastic population models such as branching processes and birth and death processes. The main motivation behind such an approach is best described in Feller's own words appearing in the introduction of the aforementioned paper.

"We are concerned with mathematical models of population growth. Relatively small populations require discrete model, but for large population it is possible to apply continuous approximations, and this leads to processes of diffusion type. It is known that an essential part of Wright's theory is mathematically equivalent to assuming a certain diffusion equation for the gene frequency. This equation is ... obtained ... by a limiting process which is actually implied in Wright's theory."

In recent years, the idea of a diffusion approximation has been extended to space-time models in population biology. A few examples are: (i) the theory of branching diffusion in R^n , due to Sawyer [16], (ii) birth, death and migration models for spatially distributed populations of Bailey [2], (iii) measure diffusion models for geographically structured population, introduced by Dawson [6].

Here, we consider a particular problem arising in the example (iii). In this model, the "population density" at time t and location x is assumed to satisfy a partial stochastic

differential equation. If one ignores the spatial effects, this assumption is valid as shown by Feller. Apart from implications in biology, the space-time stochastic model is also a mathematically desirable extension of the Itô-theory of stochastic differential equation.

A partial stochastic differential equation obtained as a limit of discrete-state-continuous-time stochastic population model is called a stochastic evolution equation. A generic description of stochastic evolution equations in terms of qualitative behaviors, reflecting all possible attributes of the underlying population process, could be found in Dawson [6; pp.3]. To circumvent various difficulties encountered in the example (iii), the notion of a measure-valued Markov process [6; pp.32] is proposed and a class of solvable stochastic evolution equation is presented.

Here, we propose Brownian measure processes as solutions to a larger class of homogeneous stochastic evolution equations. In some special cases, we can accommodate the non-homogeneous term by interpreting it as an immigration. Before giving an informal description of Brownian measure processes, we need to consider the nature of diffusion processes obtained by Feller in his study of population biology.

For models insensitive to spatial effects, Feller obtained a diffusion representing the time evolution of biological

mass. That is, this diffusion has as state space the positive half of the real axis and it is the so-called continuous (state) branching process as it was obtained from a discrete (state) branching process by a limiting procedure. Let us now note Feller's observations of the nature of the continuous branching process.

"Now this equation is of a particular type and it should be realized that the limiting process in question is but one in a family of possible processes. ... This diffusion equation (as well as others occurring in population theory) are of a singular type and lead to new types of boundary conditions and mathematical problem which have not yet been investigated. In physical diffusion theory the solution depends not only on the initial distribution, but also on boundary conditions. By contrast, our equations are of a singular type with the [diffusion] coefficient $b(x)$ vanishing on the boundaries. This leads to a new phenomenon, namely natural boundaries where no conditions need, or can, be imposed."

In a series of papers Feller developed the theory of singular diffusions which now carries his name.

A Brownian measure process at location x behaves like a singular diffusion, that is, the local mass evolution is governed by a singular diffusion. At time t , a Brownian measure process is a completely random measure on R^d (non-negative generalized random process with independent values at every location, in the terminology of Gelfand). It is known that a completely random measure on R_+^1 is a Lévy process. In fact, Kingman's representation of the characteristic functional of a completely random measure on R_+^1 is the Ito-representation of the characteristic functional of

a Levy process. Apart from being valid on an arbitrary locally compact, separable, Hausdorff space, Kingman's representation allows for a separation of space and time dependencies. In this dissertation we exploit this separation of variables to construct a class of Markov processes, namely, Brownian measure processes.

In chapter I, we discuss the Radon measures and the vague topology, random measures and Prohorov's theorem on the uniqueness of characteristic functional, completely random measures and Kingman's representation and prove a few results of technical nature to be used in chapter III.

In the first half of chapter 2, we give a self-contained exposition of the theory of singular diffusions developed by Feller. In the remaining part of this chapter we establish an entrance law for singular diffusions. This proof depends on a few technical details, to be found in a paper by McKean [11], about the fundamental solutions of singular diffusions.

In chapter 3 we show the existence of Brownian measure processes. Technical constraint dictates that we consider separately two cases of initial conditions; atomic and non-atomic initial measures. Our main tool here is the entrance law developed in the last chapter. In fact, Chapman-Kolmogorov relation is obtained from the entrance law.

In the fourth chapter, a few properties of Brownian measure processes is discussed. Exploiting Doob's h-path process [7] interpretation for the entrance law, the stochastic continuity of Brownian measure processes is established. A characterizing property of continuous branching process, multiplicativity, is introduced and the connection to Brownian measure processes is established. We also indicate how to extend our model to allow for immigrations.

We conclude by pointing out two problems of interest.

Because of the nature of this dissertation, we identify any result found in a reference, either stated or proved, as a proposition.

Chapter I Measure-valued Markov Processes

§1.0 Introduction

In this chapter, we introduce the notion of a measure-valued Markov process and show how to construct such processes from transition functions.

The space of positive Radon measures is the state space of a measure-valued Markov process. In the first section, we explain the vague topology on the space of Radon measures. In the construction of the transition function, we will need to use the fact that the sets of atomic and non-atomic positive Radon measures are Borel. The proofs of these results will be provided.

A measurable map from an abstract probability space into the space of positive Radon measures is called a random measure. In the second section, various properties of random measures, including the weak convergence of random measures via the method of characteristic functionals, will be discussed. A Lévy-Cramer type theorem can be deduced easily from the characteristic functional techniques.

An interesting subclass - named completely random measure by Kingman - will be discussed next, in the third section. This family, which corresponds to the family of independent increment processes on the positive half-line, has been completely identified by Kingman using characteristic functionals techniques.

A family of random measures (indexed by the time parameter)

with Markovian time-evolution is said to be a measure-valued Markov process. The existence of such processes will be inferred from the existence of transition functions defined on the space of positive Radon measures.

A Brownian measure process is defined as a measure-valued Markov process consisting of completely random measures. In a later chapter, we construct the transition function corresponding to a Brownian measure process.

§1.1. Radon Measures

Following Choquet [5] we introduce the positive Radon measures as the positive linear forms on the space of continuous functions with compact supports. We define the vague topology on the space of measures and show that the set of purely atomic and non-atomic measures are Borel. But we need to state a few topological facts.

Notation 1.1.1.

By S we denote a locally compact, Hausdorff space with a countable base for the topology. It will also be assumed that S has no isolated points.

Remark 1.1.2

It is well-known that the topology of S is metrizable, σ -compact and there exists a sequence $\{U_n\}$ of relatively compact subsets of S such that

- i) $\overline{U_n} \subset U_{n+1}$ for each n .
- ii) $\bigcup_{n=1}^{\infty} U_n = S$.

Notation 1.1.3

$B(S)$ = Borel field of S .

$\text{supp}(f)$ = support of $f \equiv \overline{\{x : f(x) \neq 0\}}$

$C_c(S)$ = vector space of real-valued continuous functions with compact supports.

$C_k(S)$ = vector subspace of $C_c(S)$ with support in the given compact set $K \subset S$.

$$C_c^+(S) = \{f \in C_c(S) \mid f \geq 0\}.$$

Remark 1.1.4

Because of remark 1.1.2, we may introduce the inductive limit topology on $C_c(S)$, i.e., the finest topology making all the inclusion maps

$$C_{\overline{U}_n}(S) \longrightarrow C_c(S)$$

continuous. From now on, the space $C_c(S)$ will denote the vector space $C_c(S)$ with the inductive limit topology.

Definition 1.1.5

The topological dual of $C_c(S)$ is called the space of Radon measures on S , denoted as $M(S)$. Let us denote the continuous linear form on $C_c(S)$ induced by $\mu \in M(S)$ as

$$\begin{aligned} \langle \mu, \cdot \rangle : C_c(S) &\longrightarrow {}^nR^1 \\ f &\longrightarrow \langle \mu, f \rangle. \end{aligned}$$

A Radon measure μ is said to be positive if

$$\langle \mu, f \rangle \geq 0 \text{ for } f \in C_c^+(S).$$

The cone of positive Radon measures will be denoted as $M_+(S)$.

At the beginning of this section, we introduced the positive Radon measures as the continuous linear forms on $C_c(S)$. However, it is a consequence of the topological property of $C_c(S)$ that every

positive linear form on $C_c(S)$ is automatically continuous.

Proposition 1.1.6

Any positive linear form on $C_c(S)$ is a positive Radon measure.

Proof: Choquet [5; vol. I. p. 186].

The relationship between the positive linear forms on $C_c(S)$ and the positive set functions on $B(S)$ is provided by the Riesz representation theorem.

Proposition 1.1.7

There is a 1-1 correspondence between $M_+(S)$ and the family of positive, regular, tight, σ -finite, complete measures on $B(S)$.

Definition 1.1.8

The vague topology on $M(S)$ is the weakest topology making all the maps on $M(S)$

$$\mu \longrightarrow \langle \mu, f \rangle, \quad f \in C_c(S)$$

continuous. We will denote these maps by $\langle \cdot, f \rangle$. Let us define the Borel field of $M(S)$, $B[M(S)]$, as the σ -field generated by the vague topology.

Proposition 1.1.9

$$B[M(S)] = \sigma\{\langle \cdot, 1_A \rangle \mid A \in B(S)\}$$

where 1_A denotes the indicator function of the Borel set A in S .

6

Proof: Jagers [10; p.187].

Remark 1.1.10

A real-valued measurable map on $M(S)$

$$F : M(S) \longrightarrow \mathbb{R}^1$$

can be expressed as $F = F_n + F$ where $F_n = \sum_{j=1}^n c_j \langle \cdot, 1_{A_j} \rangle$ with A_j , $j=1, \dots, n$ disjoint and $c_j \in \mathbb{R}^1$.

It is an interesting fact that, even though the vague topology of $M(S)$ is not metrizable, the vague topology of $M_+(S)$ is.

Proposition 1.1.11

$M_+(S)$ with the vague topology is metrizable and separable.

It is also a vaguely closed subset of $M(S)$.

Proof: Choquet [5; pp. 219].

We will encounter several examples of complete metric spaces with a countable dense set. So we have

Definition 1.1.12

A metrizable topological space which is complete and separable is called a Polish space.

Remark 1.1.13

The importance of Polish spaces is due to the fact that the set of probability measures on such spaces also form (in the topology of weak convergence) a Polish space. Moreover, Prohorov's tightness

criterion for relative compactness holds in Polish spaces.. We will have more to say on these topics in the next section. But before that, we introduce

Notation 1.1.14

$$B[M_+(S)] \equiv M_+(S) \cap B[M(S)]$$

An element $\mu \in M_+(S)$ is called

non-atomic if $\langle \mu, 1_{\{x\}} \rangle = 0 \quad x \in S$

atomic if $\{x_n : n \in \mathbb{Z}^+\} \subset S$ such that

$\langle \mu, 1_{\{x_n\}} \rangle > 0, \quad n \in \mathbb{Z}^+ \quad \text{and} \quad \mu \equiv 0 \quad \text{outside}$
 $\{x_n : n \in \mathbb{Z}^+\}.$

We denote the family of positive atomic Radon measure by A , and the family of non-atomic positive Radon measure by $N.A.$

In the next two theorems we will establish that the sets of non-atomic and atomic measures are measurable in $B[M_+(S)]$.

Theorem 1.1.15

The set of non-atomic measures in $M_+(S)$ is Borel.

Proof: We establish that A the complement of non-atomic measures, i.e., the set of measures with at least one atom is Borel.

For every $r \in \mathbb{Q}^+$, the positive rationals in $\mathbb{R}^1$, denote by $A(r)$ the class of all measures with at least one atom of size $\geq r$.

Remembering that S is a σ -compact space

$$A(r) = \bigcup_{n=1}^{\infty} A(r, K_n)$$

where $A(r, K_n)$ denotes the class of all measures with at least one atom, in the compact set K_n , of size $\geq r$.

We show that for each r and n , $A(r, K_n)$ is vaguely closed.

Let $\mu_j \in A(r, K_n)$ and $\mu_j \xrightarrow{v} \mu$.

Thus there exists a sequence $\{x_j\} \subset K_n$, $\mu_j(\{x_j\}) \geq r$ and $x_j \rightarrow x_0 \in K_n$.

Now for V open and $x_0 \in V$, there is a relatively compact open V_1 such that $x_0 \in V_1 \subset \bar{V}_1 \subset V$ (local compactness)

and there exists M such that $m \geq M$ implies $x_m \in V_1$.

Consider $f \in C_c(s)$, $0 \leq f \leq 1$ such that $f \equiv 1$ on $\bar{V}_1$ and $\text{supp}(f) \subset V$.

From the vague convergence $\lim_{j \rightarrow \infty} \langle \mu_j, f \rangle = \langle \mu, f \rangle$.

But $\langle \mu_j, f \rangle \geq \langle \mu_j, 1_{V_1} \rangle$ and $\langle \mu_m, 1_{V_1} \rangle \geq r$ for $m \geq M$

since $x_m \in V_1$, $m \geq M$ and $\langle \mu_m, 1_{V_1} \rangle \geq \langle \mu_m, 1_{\{x_m\}} \rangle \geq r$.

Hence $\langle \mu, f \rangle \geq r$. But $\langle \mu, f \rangle \leq \langle \mu, 1_V \rangle$

consequently $\langle \mu, 1_V \rangle \geq r$.

That is, for every open neighbourhood V of x_0 we have $\langle \mu, 1_V \rangle \geq r$.

But μ is regular (prop. 1.1.7), i.e.,

$$\langle \mu, l_{\{x_0\}} \rangle = \inf \{ \langle \mu, l_V \rangle \mid x_0 \in V, V \text{ open} \}.$$

Hence $\langle \mu, l_{\{x_0\}} \rangle \geq r$, so that μ has at least one atom of size $\geq r$, thus proving that $A(r, K_n)$ is vaguely closed, hence Borel.

It follows that $A = \bigcup_{r \in \mathbb{Q}^+} \bigcup_{n=1}^{\infty} A(r, K_n)$ is Borel.

The next theorem is based on a simple idea due to S. Mizuno.

Theorem 1.1.16

The set of atomic measures (see notation 1.1.14) in $M_+(S)$ is Borel.

Proof:

Step (1). Let $K \subset S$ be compact. Construct a (finite) partition P_1 of K by sets of diameter ≤ 1 . Denote this partition by $\{\Delta_1^1, \dots, \Delta_{m(1)}^1\}$. Let P_2 denote a refinement of P_1 by sets of diameter $\leq \frac{1}{2}$. Thus $P_2 = \{\Delta_1^2, \dots, \Delta_{m(2)}^2\}$. Similarly, $P_n = \{\Delta_1^n, \dots, \Delta_{m(n)}^n\}$ is a refinement of P_{n-1} by sets of diameter $\leq \frac{1}{n}$.

Step (2). For a positive integer M define

$$\phi_k^M(n) : M_+(S) \longrightarrow M_+(S)$$

$$\phi_k^M(n) : \mu \longrightarrow \sum_{j=1}^{m(n)} \mu(\Delta_j^n) \delta_{x_j^n}$$

$$\mu(\Delta_j^n) \geq M^{-1}$$

where $x_j^n \in \Delta_j^n$.

$\phi_K^M(n)$ would be measurable if $f \in C_c(S)$

$$f \circ \phi_K^M(n) : M_+(S) \rightarrow \mathbb{R}^1$$

is measurable.

$$\text{But } (f \circ \phi_K^M(n))\mu = \sum_{j=1}^{m(n)} \mu(\Delta_j^n) f(x_j^n)$$

$$\mu(\Delta_j^n) \geq M^{-1}$$

is measurable.

Step 3. $\phi_K^M \equiv \lim_{n \rightarrow \infty} \phi_K^M(n)$ is then a measurable map $M_+(S) \rightarrow M_+(S)$.

Thus one obtains via a measurable operation the locations of atoms of μ in the set K of mass $\geq M^{-1}$.

$$\text{Step 4. } \phi_K \equiv \lim_{M \rightarrow \infty} \phi_K^M$$

$$\phi \equiv \lim \phi_K \quad (S \text{ is } \sigma\text{-compact})$$

Hence ϕ is a measurable map $M_+(S) \rightarrow M_+(S)$.

Step 5. The set of atomic measures is the inverse image of 0 under the map $I - \phi$, where $I : M_+(S) \rightarrow M_+(S)$ is the identity map.

§1.2 Random measures

Definition 1.2.1

Let $(\Omega, \mathcal{F}, P^*)$ be a given probability space. A measurable map

$$X : \Omega \rightarrow M_+(S)$$

will be called a random measure on $(\Omega, \mathcal{F}, P^*)$.

Remark 1.2.2

(i) If X is a random measure on Ω , then for each $f \in C_c(S)$

$$\langle X(\cdot), f \rangle : \omega \longrightarrow \langle X(\omega), f \rangle$$

is a real-valued random variable on Ω . For $H \in \mathcal{B}(R^1)$, note

$$\{\omega : \langle X(\omega), f \rangle \in H\} = X^{-1}\{\mu \vdash \mu \in M_+(S), \langle \mu, f \rangle \in H\}.$$

(ii) A random measure X induces a probability measure on $M_+(S)$, namely P^*X^{-1} . We call this probability measure the law of X and denote it by P_X . It is also clear that given such a probability measure P , one can always find $(\Omega, \mathcal{F}, P^*)$ and a random measure X on Ω such that $P_X = P$. Simply take $\Omega = M_+(S)$.

Notation 1.2.3

Denote the set of all probability measures on $M_+(S)$ by $M^1[M_+(S)]$. Similarly, $M^1(R^1)$ will denote the probability measures on R^1 .

Remark 1.2.4

We have seen that $M_+(S)$ is Polish [prop. 1.1.11 and defn. 1.1.12]. It is also known that the family of probability measures on a Polish space consists of regular and tight probability measures. In the usual manner, one defines the weak topology on $M^1[M_+(S)]$. It can also be shown that the weak topology of $M^1[M_+(S)]$ is again Polish.

Definition 1.2.5

Let $P, P_n \in M^1[M_+(S)]$. The sequence P_n converges weakly to P , denoted $P_n \xrightarrow{w} P$, if for each $F \in C_B(M_+(S), R^1)$

$$\int_{M_+(S)} F(\mu) P_n(d\mu) \longrightarrow \int_{M_+(S)} F(\mu) P(d\mu)$$

where $C_B(M_+(S), R^1) \equiv \{F \mid F : M_+(S) \rightarrow R^1, \text{ continuous and bounded}\}$.

We will denote the weak convergence on $M^1(R^1)$ as $\xrightarrow{w}$.

A sequence of random measures converges weakly to a random measure if the corresponding sequence of laws converges weakly to the law of the limit random measure.

$\{P_n\}_{n=1}^\infty \subset M^1[M_+(S)]$ is tight if for all $\epsilon > 0$ there is a compact $\Gamma_\epsilon \subset M_+(S)$ such that for all n

$$P_n(\Gamma_\epsilon) > 1 - \epsilon.$$

A sequence of random measures is tight if the corresponding sequence of laws is tight.

Proposition 1.2.6 [Prohorov]

A sequence of random measures $\{X_n\}_{n=1}^{\infty}$ is tight

For each $f \in C_c(S)$, the sequence of random variables $\{\langle X_n, f \rangle\}_{n=1}^{\infty}$ is tight.

Proof: Jagers [10].

We define the characteristic functional of a random measure and following Prohorov state the result connecting the weak convergence of law and the convergence of characteristic functionals.

Definition 1.2.7

Let $P \in M^1_+[M_+(S)]$. The characteristic functional of P , $L(P)$, is the map

$$f \rightarrow \int_{M_+(S)} e^{i\langle \mu, f \rangle} P(d\mu)$$

defined for $f \in C_c(S)$.

The characteristic functional L_X of a random measure X on (Ω, F, P^*) is the characteristic functional of its law P_X :

$$L_X(f) = L(P_X, f) = \int_{\Omega} e^{i\langle X(\omega), f \rangle} P^*(d\omega) = E[e^{i\langle X(\omega), f \rangle}].$$

Proposition 1.2.8 [Prohorov]

(a) $P \in M^1_+[M_+(S)]$ is uniquely determined by its characteristic functional $L(P)$.

(b) Let $\{X_n\}_{n=1}^{\infty}$ and X be random measures.

$$X_n \xrightarrow{w} X$$

Then,

$$L_{X_n}(f) \xrightarrow{\quad} L_X(f) \quad \text{for each } f \in C_c(S)$$

$$\Uparrow$$

$$\langle X_n, f \rangle \xrightarrow{\quad} \langle X, f \rangle \quad \text{for each } f \in C_c(S)$$

Proof: Jagers [10].

Let us now state and prove a Lévy-Cramer type theorem for random measures.

Theorem 1.2.9

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random measures with $\{P_n\}_{n=1}^{\infty}$ and $\{L_n\}_{n=1}^{\infty}$, respectively, the corresponding sequences of laws and characteristic functionals. Suppose

(i) for each $f \in C_c(S)$

$L_n(\theta f)$ converges and defines a limit function

$L_{\infty}(\theta f)$ for $\theta \in \mathbb{R}^1$.

(ii) $L_{\infty}(\theta f)$ is continuous at $\theta = 0$.

Then there exists

$P \in M^1[M_+(S)]$ such that

a) $P_n \xrightarrow{w} P$

b) L_{∞} is the characteristic functional of P .

Proof: We may consider $L_n(\theta f)$ to be the characteristic function of the real-valued random variable $\langle X_n, f \rangle$ with $\theta \in \mathbb{R}^1$ as the parameter.

So the assumptions (i), (ii) are exactly those of Lévy-Cramer Theorem. Thus there exists a weak limit [in $M^1(\mathbb{R}^1)$] of $\{\langle X_n, f \rangle\}_{n=1}^\infty$. So $\{\langle X_n, f \rangle\}_{n=1}^\infty$ is tight [in $\mathbb{R}^1$] for each $f \in C_c(S)$. Hence $\{X_n\}_{n=1}^\infty$ is tight [in $M^1[M_+(S)]$]. So $\{P_n\}_{n=1}^\infty$ is tight, thus

weakly relative compact and consequently has a weakly convergent subsequence, say $P_{n_j} \xrightarrow{w} P$. Then $L_{n_j}(f) \rightarrow L(P, f)$ for each $f \in C_c(S)$.

Putting $\theta = 1$, we have from hypothesis (i)₀ that $L_{n_j}(f) \rightarrow L_\infty(f)$ for each $f \in C_c(S)$. Hence $L(P, f) = L_\infty(f)$. From the uniqueness of characteristic functional of a random measure we may conclude (a), (b).

Remark 1.2.10

In proposition 1.2.9, if we assume in addition that $\{X_n\}_{n=1}^\infty$ are random measures on $M_+(S)$, i.e., $\Omega = M_+(S)$, then we can conclude $X_n \xrightarrow{w} X$ where X is determined by P . See remark 1.2.2(ii).

We now introduce the notion of conditional expectations for random measures.

Definition 1.2.11

A random measure X on $(\Omega, \mathcal{F}, P^*)$ has finite expectation,

denoted $X \in L_1(P^*)$, if the real-valued random variable $\langle X, f \rangle$ has finite expectations for each $f \in C_c(S)$.

$$E[\langle X(\omega), f \rangle] \equiv \int_{\Omega} \langle X(\omega), f \rangle P^*(d\omega).$$

Note that $E[\langle X(\omega), \cdot \rangle]$ is a positive linear form on $C_c(S)$. Thus (proposition 1.1.6)

$$\mu(\cdot) \equiv E[\langle X(\omega), \cdot \rangle] \in M_+(S).$$

Let X be an $L_1(P^*)$ random measure on (Ω, F) and G be a sub σ -field of F . Then the conditional expectation of X with respect to G , denoted $E[X|G]$, is a random measure on (Ω, F, P^*) such that

$$(i) \quad \langle E[X|G], f \rangle = E[\langle X, f \rangle | G]$$

for each $f \in C_c(S)$.

$$(ii) \quad E[X|G] \text{ is } P^* - \text{a.s. unique.}$$

Taking the condition (i) as the defining relation for conditional expectations, it has been shown by S. Mizuno [14] that all the usual properties of conditional expectation remain valid in the present set-up.

§1.3 Completely Random measures

An interesting class of random measures, the family of completely random measures, was introduced by Kingman. If the space $S = \mathbb{R}_+^1$, then this family corresponds to the independent increment processes on $[0, \infty)$. We present a representation of characteristic functionals for these random measures, due to Kingman, analogous to Lévy-Itô representation of characteristic functions of independent increment.

Definition 1.3.1

A random measure X on $(\Omega, \mathcal{F}, P^*)$ is completely random if the random variables $\langle X, f \rangle$ and $\langle X, g \rangle$ are independent (with respect to P^*) whenever $\text{supp}(f) \cap \text{supp}(g) = \emptyset$.

A $P \in \mathcal{M}^1[M_+(S)]$ is said to be a completely random law if the identity map $X : M_+(S) \rightarrow M_+(S)$ is a completely random measure on $(M_+(S), \mathcal{B}[M_+(S)], P)$.

An important example of completely random measure is the Poisson random measure. It will be shown that every completely random measure - modulo the pathological ones - can be obtained as a weak limit of a sequence of Poisson random measures.

Definition 1.3.2

Denote by $N(S)$ the integer-valued measures on S ,

$$N(S) = \{\mu \mid \mu \in M_+(S), \mu : S \rightarrow \mathbb{Z}^+\}$$

where Z^+ is the set of positive integers.

It is known [Jagers p.200] that $N(S)$ is a vaguely closed subset of $M_+(S)$.

Suppose $\mu \in M_+(S)$ is non-atomic. For each $B \in \mathcal{B}(S)$ define a measure on $N(S)$ as

$$P_B(n) = \frac{e^{-\mu(B)}}{n!} [\mu(B)]^n \quad n \in Z^+. \quad (1.3.1)$$

Under some consistency conditions [Jagers 10;p.193,197,201,212], one can construct a P on $M_+(S)$ with $P[N(S)] = 1$ such that the finite dimensional distributions are given by equation (1.3.1). Moreover the random measure X on $M_+(S)$ with the law P is completely random and has the property

$$E(X) = \mu.$$

The random measure X so obtained is called the Poisson random measure and P_X is known as the Poisson random law.

Remark 1.3.3

The characteristic functional of a Poisson random measure X with expectation μ is given by,

$$L_X(f) = \exp \left[\int_S (e^{if(x)} - 1) \mu(dx) \right] \quad (1.3.2)$$

for $f \in C_c(S)$.

Proposition 1.3.4 [Kingman]

A completely random measure is the sum of a random measure with fixed atoms, a non-random measure and a generalized Poisson random measure which is the weak- $M^1[M_+(S)]$ limit of completely random measures where the location of atoms are first chosen by a Poisson random measure, and then independently given a random mass.

Proof: Jagers [10; p.218-226].

We will only outline the major steps of Kingman's proof.

Let X denote a completely random measure on $(\Omega, \mathcal{F}, P^*)$

Step 1. $A \equiv \{x \in S : P^*[\omega : \langle X(\omega), 1_{\{x\}} \rangle > 0] > 0\}$.

Then A is at most countable.

Step 2. There exists a family of independent real-valued random variables $\{Z_x : x \in S\}$ such that

$$X = \sum_{x \in A} Z_x \delta_x + X_0$$

where δ_x denotes the usual δ -measure at x and

$$X_0 \equiv X - \sum_{x \in A} Z_x \delta_x.$$

Moreover, the family $\{Z_x : x \in A\}$ is independent of the random measure X_0 .

Step 3. For each $f \in C_c(S)$, the random variable $\langle X_0, f \rangle$ is infinitely divisible.

Step 4. There exists a non-atomic Radon measure M on $S \times R_+^1$ and a non-atomic $\mu \in M_+(S)$ such that

$$E[e^{i\langle X_0, f \rangle}] = \exp \left[\int_{S \times R_+^1} (e^{iyf(x)} - 1) M(dx, dy) + i \int_S f(x) \mu(dx) \right].$$

Step 5. Thus far, we have obtained the fixed atomic component (Step 2) and the non-random component (Step 4) of the completely random measure. It remains now to represent $\exp \left[\int_{S \times R_+^1} (e^{iyf(x)} - 1) M(dx, dy) \right]$ as the characteristic functional of a completely random measure obtained as a weak limit of a sequence of completely random measures.

Define a measure μ on $B(S)$ by

$$\mu(A) = M(A \times R_+^1), \quad A \in B(S).$$

For any fixed $B \in B(R_+^1)$, the measure $M(\cdot \times B)$ on $B(S)$ is absolutely continuous with respect to μ .

Two cases may arise

Case (i) μ is σ -finite.

Then there exists a measurable function $x \mapsto \nu(x, B)$ such that

$$M(A \times B) = \int_A \nu(x, B) \mu(dx).$$

Moreover, for fixed x , $\nu(x, \cdot)$ can be shown to be a probability measure on R_+^1 . Hence,

$$\exp \left[\int_{S \times R_+^1} (e^{iyf(x)} - 1) M(dx, dy) \right] = \exp \left[\int_S \mu(dx) \int_{R_+^1} (e^{iyf(x)} - 1) \nu(x, dy) \right] \quad (1.3.3)$$

The expression on the right hand side of (1.3.3) is the characteristic functional of a completely random measure obtained as a Poisson [with expectation μ] random sum of a family of independent positive random variables $\{Z_x : x \in S\}$ with law $v(x, \cdot)$. Moreover, $\{Z_x : x \in S\}$ is also independent of the Poisson random measure.

Thus, the Poisson random measure determines the location of the atoms and Z_x denotes the mass at x .

Case (ii). If μ is not σ -finite, still $\mu_n(\cdot) \equiv M_n(\cdot \times \mathbb{R}_+^1)$ is, where $M_n(A \times B) \equiv M_n(A \times \{B \cap [\frac{1}{n}, \infty)\})$.

Then define $v_n(x, \cdot)$ to be such that

$$M_n(A \times B) = \int_A v_n(x, B) \mu_n(dx)$$

Proceed then, as in case (i), to construct a Poisson (with expectation μ_n) random sum of a family of independent positive random variables with law $v_n(x, \cdot)$. Now letting $n \rightarrow \infty$, we get the desired result and the equation (1.3.3) remains valid.

Definition 1.3.5

The random measure with the characteristic functional (1.3.3) will be called the random atomic component of X .

Remark 1.3.6

During the course of this remark only, we assume that μ is σ -finite. Define

$$A(\omega) = \{x \in S : \langle \xi_\mu(\omega), 1_{\{x\}} \rangle = 1\}$$

where ξ_μ is the Poisson random measure with expectation μ . In terms of $A(\omega)$, the random atomic component is

$$\sum_{x \in A(\omega)} Z_x \delta_x \quad (1.3.4)$$

We call $A(\omega)$ the random support of the random atomic component. As in step (1), one can show that $A(\omega)$ is a.s. countable. It also follows from (1.3.4) and the independence of the family $\{Z_x\}$ that the random atomic component is necessarily completely random.

Remark 1.3.7

Remembering M is a non-atomic Radon measure on $S \times \mathbb{R}_+^1$ (step 4), we may define (defn. 1.3.2) a Poisson Random measure χ with expectation M on $S \times \mathbb{R}_+^1$ (which is still Polish). Define ξ_0 as follows

$$\langle \xi_0, 1_A \rangle = \int_{A \times \mathbb{R}_+^1} y \chi(dx, dy), \quad A \in \mathcal{B}(S)$$

extend it to $f \in C_c(S)$ in the usual way

$$\langle \xi_0, f \rangle = \int_{S \times \mathbb{R}_+^1} y f(x) \chi(dx, dy) \quad (1.3.5)$$

By a simple computation, we can establish that the characteristic functional of ξ_0 is given by (1.3.3). Thus the random atomic component is expressible as an integral with respect to a Poisson random measure. However, this fact should not be surprising in view of (1.3.4). The present approach - integration with respect to a

Poisson random measure - is also found in Itô's derivation of characteristic function of an independent increment process. (Breiman [4, ch. 14]).

The measure $\nu(x, \cdot)$ appearing in (1.3.3) will be referred to as the Lévy measure of a completely random measure.

§1.4 Measure-valued Markov processes

Definition 1.4.1

A family of random measure on $(\Omega, \mathcal{F}, P^*)$ indexed by $t \in T \equiv [0, \infty)$, $\{X_t : t \in T\}$, is a measure-valued Markov process on $(\Omega, \mathcal{F}, P^*)$ if, for all: F

$$E[F \circ X_{t_1+t_2} | \mathcal{F}_{t_2}] = E[F \circ X_{t_1+t_2} | \sigma(X_{t_2})] \quad (1.4.1)$$

where

$F : M_+(S) \rightarrow \mathbb{R}^1$ is bounded (or positive) measurable

$$\sigma(X_{t_2}) \equiv \sigma\{\langle X_{t_2}, 1_B \rangle : B \in \mathcal{B}(S)\}$$

$$\mathcal{F}_{t_2} \equiv \bigvee_{u \leq t_2} \sigma(X_u)$$

Definition 1.4.2

A map $T \times M_+(S) \times \mathcal{B}[M_+(S)] \rightarrow [0,1]$

$$(t, \mu, \Gamma) \rightarrow P_t^\mu(\Gamma)$$

is said to be a transition function on $M_+(S)$ if

(i) for $t \in T$, and $\mu \in M_+(S)$

$$P_t^\mu(\cdot) : \mathcal{B}[M_+(S)] \rightarrow [0,1]$$

is a probability measure, i.e., $P_t^\mu \in M^1[M_+(S)]$.

(ii) for $t \in T$ and $\Gamma \in B[M_+(S)]$

$$P_t^\cdot(\Gamma) : M_+(S) \rightarrow [0,1]$$

is measurable.

(iii): for $\mu \in M_+(S)$ and $\Gamma \in B[M_+(S)]$

$$P_0^\mu(\Gamma) = \delta_\mu(\Gamma) .$$

That is, if $\mu \in \Gamma$, then $P_0^\mu(\Gamma) = 1$, zero otherwise.

(iv) Chapman-Kolmogorov condition

$$P_{t_1+t_2}^\mu(\Gamma) = \int_{M_+(S)} P_{t_1}^\cdot(d\eta) \int_{\Gamma} P_{t_2}^\eta(d\xi) .$$

for $\Gamma \in B[M_+(S)]$.

Proposition 1.4.3

Given a transition function $\{P_t^\cdot(\Gamma) : t \in T, \mu \in M_+(S), \Gamma \in B[M_+(S)]\}$, one can construct a measure-valued Markov process $\{X_t : t \in T\}$ on $(M_+(S)^T, B[M_+(S)]^T, P^*)$

where $M_+(S)^T$ is the product space with the product σ -field $B[M_+(S)]^T$, and P^* is the projective limit of finite dimensional distributions induced by the transition function.

Moreover, for $\Gamma \in B[M_+(S)]$

$$P\{X_{t_1+t_2} \in \Gamma \mid F_{t_1}^{X_{t_1}}\} = P_{t_2}^{X_{t_1}}(\Gamma), \quad P^* - \text{a.s.} \quad (1.4.2)$$

where $\mathcal{F}_{t_1}$ is the σ -field generated by random measures upto time t_1 .

The existence of conditional probability in (1.4.2), is part of the conclusion.

Proof: The usual proof for the real-valued processes remains valid here. One may find a proof of this proposition in Bauer [3; pp.319-322 for the existence of conditional probability, p. 360-362 for the existence and uniqueness of the projective limit, p. 378-379 for the construction of the finite dimensional distribution, p. 383-385 for the proof of (1.4.2)]..

The conclusions of this proposition remain valid whenever the state space (here it is $M_+(S)$) is Polish and has the Borel σ -field.

Definition 1.4.4

A measure-valued Markov process $\{X_t : t \in T\}$ is a Brownian measure process if, for $t \in T$, X_t is a completely random measure on S and for $x \in S$, $\langle X_t, l_{\{x\}} \rangle$ is a diffusion with state space $[0, \infty]$. Thus, "locally", the evolution of mass is determined by a diffusion.

§1.5 Local processes

Recall that, to every random measure X one can associate a real-valued random variable $\langle X, 1_B \rangle$ for $B \in \mathcal{B}(S)$. [Remark 1.2.2(ii)].

In a similar fashion a real-valued Markov process can be obtained from a measure-valued Markov process. However, one has to define the conditioning σ -fields with care. With this in mind, we introduce

Definition 1.5.1

A family of random measures $\{X_t : t \in T\}$ is said to be a B -local process if for some $B \in \mathcal{B}(S)$ one has, for all f

$$E[f \circ \langle X_{t_1+t_2}, 1_B \rangle | F_{t_2}(B)] = E[f \circ \langle X_{t_1+t_2}, 1_B \rangle | \sigma(X_{t_2}, B)] \quad (1.5.1)$$

where

$f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is bounded (or positive) measurable.

$$\sigma(X_{t_2}, B) \equiv \sigma\{\langle X_{t_2}, 1_B \rangle : B \in \mathcal{B}(B)\}$$

$$F_{t_2}(B) \equiv \bigvee_{u \leq t_2} \sigma(X_u, B).$$

One sees that the σ -fields appearing in (1.5.1) depends only on the given Borel set B , in other words, the future behavior of X_t on B depends only on the local environment. This should clarify the nomenclature.

Definition 1.5.2

A family of random measures is local if it is local for every $B \in \mathcal{B}(S)$.

We now proceed to show that the definitions 1.4.1 and 1.5.1 are equivalent for a family of random measures $\{X_t : t \in T\}$ satisfying the assumption of independence:

For $B_1, B_2 \in \mathcal{B}(S)$ with $B_1 \cap B_2 = \emptyset$, the random variables $\langle X_{t_1}, 1_{B_1} \rangle$ and $\langle X_{t_2}, 1_{B_2} \rangle$ are independent for all $t_1, t_2 \in T$. (*)

Note that the condition (*) implies that the family $\{X_t : t \in T\}$ consists of completely random measures. However, it should be obvious that a family of completely random measures need not satisfy (*).

Theorem 1.5.3

Let $\{X_t : t \in T\}$ be a family of random measures with the property (*).

This family is a measure-valued Markov process

it is a local process.

Proof: ($\Downarrow$)

Let f be a positive (or bounded) Borel measurable function $\mathbb{R}^1 \rightarrow \mathbb{R}^1$. Set $F = f \circ \langle \cdot, 1_A \rangle$ for $A \in \mathcal{B}(S)$. Then $F : M_+(S) \rightarrow \mathbb{R}^1$ is measurable. So the equation (1.4.1) becomes

$$E[f \circ \langle X_{t_1+t_2}, 1_A \rangle | F_{t_2}] = E[f \circ \langle X_{t_1+t_2}, 1_A \rangle | \sigma(X_{t_2})]$$

Condition both sides of the above equality by $F_{t_2}(A)$ to obtain

$$E[f \circ \langle X_{t_1+t_2}, 1_A \rangle | F_{t_2}(A)] = E[E[f \circ \langle X_{t_1+t_2}, 1_A \rangle | \sigma(X_{t_2})] | F_{t_2}(A)] \quad (1.5.2)$$

$$\sigma(X_{t_2}) = \sigma(X_{t_2}, A) \vee \sigma(X_{t_2}, CA), \quad CA \text{ denoting the complement of } A.$$

From the property (*), $\langle X_{t_1+t_2}, 1_A \rangle$ is independent of (X_{t_2}, CA) .

So the right hand side of (1.5.2) equals $E[f \circ \langle X_{t_1+t_2}, 1_A \rangle | \sigma(X_{t_2}, A)]$, as $\sigma(X_{t_2}, A) \subset F_{t_2}(A)$.

Thus equation (1.5.1) is established.

(||) It suffices to show that (see remark 1.1.10) the equation (1.4.1) holds for

a) $F = 1_{m_+}(s).$

b) $F = \prod_{i \in \alpha} 1_{H_i} \circ \langle \cdot, 1_{A_i} \rangle$

where α is finite, $H_i \in \mathcal{B}(R^1)$, $A_i \in \mathcal{B}(s)$ and for distinct i, j $A_i \cap A_j = \emptyset$.

c) Let F_n satisfy the representation in (b) and be increasing. Put $F = \lim_{n \rightarrow \infty} F_n$.

Case (a). This is trivial. Put $f \equiv 1$ and $B = \mathcal{S}$ in (1.5.1) and obtain (1.4.1) by noting that $F_{t_2}(s) = F_{t_2}$ and $\sigma(X_{t_2}, s) = \sigma(X_{t_2})$.

Case (b). Assume α consists of two elements.

$$\begin{aligned}
 & E\left[\left(\prod_{i=1}^2 I_{H_i} \circ \langle \cdot, 1_{A_i} \rangle\right) X_{t_1+t_2} \mid F_{t_2}\right] \\
 &= E\left[\prod_{i=1}^2 I_{H_i} \circ \langle X_{t_1+t_2}, 1_{A_i} \rangle \mid F_{t_2}(A_1) \vee F_{t_2}(A_2)\right] \\
 &= E\left[\prod_{i=1}^2 E[I_{H_i} \circ \langle X_{t_1+t_2}, 1_{A_i} \rangle \mid F_{t_2}(A_i)]\right] \\
 &= E\left[\prod_{i=1}^2 I_{H_i} \circ \langle X_{t_1+t_2}, 1_{A_i} \rangle \mid \sigma(X_{t_2}, A_i)\right] \\
 &= E\left[\prod_{i=1}^2 I_{H_i} \circ \langle X_{t_1+t_2}, 1_{A_i} \rangle \mid \sigma(X_{t_2}, A_1 \cup A_2)\right] \\
 &= E\left[\left(\prod_{i=1}^2 I_{H_i} \circ \langle \cdot, 1_{A_i} \rangle\right) X_{t_1+t_2} \mid \sigma(X_{t_2})\right]
 \end{aligned}$$

Now one can extend easily the above argument for finite α .

Case (c) follows from the continuity property of conditional expectation.

Remark 1.5.4

Even for a local process, $X_t(\cdot, A)$ is not a real-valued Markov process with respect to the usual σ -field generated by $\{X_t(\cdot, A) : t \in T\}$.

Chapter II Singular Diffusions

§2.0 Introduction

In the last chapter, we have introduced the notion of a measure-valued Markov process. The existence of such a process was deduced from the existence of a transition function. In this chapter and the next one, we construct a family of transition functions corresponding to Brownian measure processes.

Recall that a Brownian measure process, X_t , should have the following properties:

- (i) for each $t \in T$, $X_t(\cdot, \cdot)$ is a completely random measure on S .
- (ii) for each $x \in S$, $X_t(\cdot, \{x\})$ is a diffusion $Z_x(t)$ on $[0, \infty)$.
- (iii) time evolution of $X_t(\cdot, \cdot)$ is Markovian.

Remember now, a completely random measure is determined by its Lévy measure. Denote by ν_t the Lévy measure of $X_t(\cdot, \cdot)$. So one would expect that the Markovian nature of $X_t(\cdot, \cdot)$ is related to some property of ν_t . In turn, the Lévy measure should reflect that the evolution of mass at a given location is Markovian. These considerations lead us to identify a class of diffusions.

Denote by T_t the semigroup defined by Z_t . It will be shown in chapter three that the time evolution of $X_t(\cdot, \cdot)$ is indeed Markovian if one has

$$v_{t+s} = T_t v_s \quad (2.0.1)$$

Definition 2.0.1

Any family of (positive) measures $\{v_t : t \in T\}$ satisfying equation (2.0.1) is called an entrance law (loi d'entree - Meyer [13; p.7]) for the diffusion semigroup T_t .

Let us consider two examples of entrance laws.

Example 2.0.2 [Continuous State Branching]

Consider a diffusion process whose forward Kolmogorov equation is

$$\frac{\partial}{\partial t} u(t,x) = \beta \frac{\partial^2}{\partial x^2} \{xu(t,x)\} - \alpha \frac{\partial}{\partial x} \{xu(t,x)\}, \quad 0 < x < \infty \quad (2.0.2)$$

where α, β are constants.

If the population size at $t = 0$ is ξ , then [Feller 8; p.235]

$$u(t,x; \xi) = \frac{\xi \alpha^2 e^{\alpha t}}{\beta^2 (e^{\alpha t} - 1)^2} \exp\left[\frac{-\alpha(\xi e^{\alpha t} + x)}{(e^{\alpha t} - 1)}\right] \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left[\frac{\alpha(\xi x e^{\alpha t})}{\beta(e^{\alpha t} - 1)}\right]^{2n}.$$

We now propose to show that $\lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} u(t,x; \xi)$ is again a solution of the equation (2.0.2).

So rewrite the expression for $u(t,x; \xi)$ as follows

$$u(t, x; \xi) = \frac{\alpha^2 e^{\alpha t}}{\beta^2 (e^{\alpha t} - 1)^2} \exp \left[\frac{-\alpha x}{\beta (e^{\alpha t} - 1)} \right]$$

$$\times \left\{ \xi \exp \left[\frac{-\alpha \xi e^{\alpha t}}{\beta (e^{\alpha t} - 1)} \right] \sum_{n=0}^{\infty} \frac{1}{n! (n+1)!} \left[\frac{\alpha}{\beta (e^{\alpha t} - 1)} \right]^{2n} (x e^{\alpha t})^n \xi^n \right\}.$$

Then it can easily be checked that

$$\lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} u(t, x; \xi) = \frac{\alpha^2 e^{\alpha t}}{\beta^2 (e^{\alpha t} - 1)^2} \exp \left[\frac{-\alpha x}{\beta (e^{\alpha t} - 1)} \right].$$

Set $f(t, x) \equiv \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} u(t, x; \xi)$

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\alpha^3 e^{\alpha t}}{\beta^2 (e^{\alpha t} - 1)^2} \exp \left[-\frac{\alpha x}{\beta (e^{\alpha t} - 1)} \right] \left[-\frac{\alpha x e^{\alpha t}}{\beta (e^{\alpha t} - 1)^2} - \frac{1 + e^{\alpha t}}{e^{\alpha t} - 1} \right] \\ &= -\alpha \frac{\partial}{\partial x} (x f) + \beta \frac{\partial^2}{\partial x^2} (x f). \end{aligned}$$

Thus $\lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} u(t, x; \xi)$ is an entrance law for the semigroup defined by the equation (2.0.2). Similar results also hold for $\alpha = 0$. For $\alpha = 0$ (zero drift) we obtain a martingale.

Example 2.0.3 [Brownian motion absorbed at the origin]

The transition density function for the absorbed Brownian motion is

$$u(t, x; \xi) = \frac{1}{\sqrt{2\pi t}} \left[e^{-(x-\xi)^2/2t} - e^{-(x+\xi)^2/2t} \right]$$

where ξ denotes the initial location ($0 < \xi < \infty$).

We now show that $\lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} u(t, x; \xi)$ satisfies

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}.$$

$$\lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} u(t, x; \xi) = \sqrt{\frac{2}{\pi}} \frac{x}{t^{3/2}} \exp\left(-\frac{x^2}{2t}\right).$$

$$\text{Set } f(x, t) = \sqrt{\frac{2}{\pi}} \frac{x}{t^{3/2}} \exp\left(-\frac{x^2}{2t}\right).$$

$$\text{Then } \frac{\partial f}{\partial t} = \frac{1}{\sqrt{2\pi}} \frac{x}{t^{5/2}} \left[-3 + \frac{x^2}{t}\right] \exp\left(-\frac{x^2}{2t}\right)$$

$$= \frac{1}{2} \frac{\partial^2 f}{\partial x^2}.$$

So, again we have an entrance law.

§2.1 Feller processes

The unifying points of two instances of diffusions considered in the last section are:

(i) the origin of the state space $[0, \infty)$ is, in the terminology of Feller, an accessible boundary

(ii) both are martingales (take $\alpha = 0$ in example 2.0.2).

In this section we introduce the boundary classifications of one-dimensional diffusions in the context of Feller processes. Our exposition is based on a series of papers by Feller. For the final version of the theory see [9] and the references therein. In the next section, we exhibit non-degenerate entrance laws for diffusions with an accessible left boundary. Our method yields, however, only the trivial entrance law for diffusions with an inaccessible left boundary.

Notation 2.1.1

Let us start by assuming that the reader is familiar with the basic theory of diffusion e.g. Breiman [4; Chapter 16].

So let Z_t denote a diffusion in natural scale, with the speed measure m , i.e., the infinitesimal generator looks like $\frac{d}{dm} \frac{d}{dx}$ on the state space $(\beta_1, \beta_2) \subset \mathbb{R}^1$. The speed measure m is defined on $B[(\beta_1, \beta_2)]$ with $m(B) < \infty$ for B bounded and $\bar{B} \subset (\beta_1, \beta_2)$. In terms of speed measure m , the Kolmogorov backward equation for a diffusion process in scale is:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{d}{dm} \frac{d}{dx} u.$$

Analytically, m is a strictly increasing right continuous function on (β_1, β_2) , not necessarily bounded at the end points:

$$m^+ , m(x) = \lim_{h \downarrow 0} m(x+h).$$

The difference $m(y) - m(x)$ ($x < y$) should be thought of as a measure of $(x, y]$.

Let us introduce the left and right derivatives of a real-valued function f :

$$\text{left derivative } f^-(x) = \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x} = \frac{d^-}{dx} f(x)$$

$$\text{right derivative } f^+(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x} = \frac{d^+}{dx} f(x).$$

f is differentiable at x if $f^-(x) = f^+(x) \equiv f'(x)$.

If m is continuous at a point, differentiation with respect to m , denoted $\frac{d}{dm}$, is defined in the usual manner. For a discontinuity point a little care is needed.

Let $I \subset (\beta_1, \beta_2)$ be an interval. If g is a function defined on I such that the right and left limits $g(x^+)$ and $g(x^-)$ exist for each $x \in I$, we define the right derivative of g with respect to m by

$$\frac{d^+}{dm} g(x) = \lim_{y \downarrow x} \frac{g(y^+) - g(x^-)}{m(y^+) - m(x^-)}, \quad (2.1.1)$$

provided the limit exists. The left derivative $\frac{d^-}{dm} g(x)$ is defined symmetrically.

At a point of discontinuity of m .

$$\frac{d^+}{dm} g(x) = \frac{g(x^+) - g(x^-)}{m(x^+) - m(x^-)} = \frac{d^-}{dm} g(x)$$

and a differentiable g is continuous at each point of continuity of m .

Definition 2.1.2

(a) We say that a function f is in the domain $\mathcal{D}(U, I)$ of the operator $U = \frac{d}{dm} \frac{d}{dx}$ in the interval $I \subset (\beta_1, \beta_2)$ if:

(i) the right and left derivatives f^+, f^- exist and

$$f^+(x) = \lim_{y \downarrow x} f^+(y) = \lim_{y \downarrow x} f^-(y)$$

$$f^-(x) = \lim_{y \uparrow x} f^+(y) = \lim_{y \uparrow x} f^-(y)$$

for each $x \in I$.

(ii) $\frac{d}{dm} f^+$ exists and is continuous in I .

The definition (2.1.1) implies $\frac{d}{dm} \frac{d^+}{dx} = \frac{d}{dm} \frac{d^-}{dx}$.

(b) $f \in \mathcal{D}(U, (\beta_1, \beta_2))$ if $f \in \mathcal{D}(U, I)$ for $I \subset (\beta_1, \beta_2)$.

We also note for future reference $\int_{x^+}^{y^-} Uf \, dm = f^-(y) - f^+(x)$.

Definition 2.1.3

Boundary classifications for processes in scale on state space

$$(\beta_1, \beta_2) \cdot \beta \in (\beta_1, \beta_2) \cdot$$

The right end point β_2 is said to be

regular if $\beta_2 < \infty$, $m(\beta_2^-) < \infty$

exit if $\beta_2 < \infty$, $m(\beta_2^-) = \infty$ and $\int_{\beta}^{\beta_2} (\beta_2 - x) dm(x) < \infty$

entrance if $\beta_2 < \infty$, and $\int_{\beta}^{\beta_2} x dm(x) < \infty$

natural $\begin{cases} \text{if } \beta_2 < \infty, \text{ and } \int_{\beta}^{\beta_2} (\beta_2 - x) dm(x) = \infty \\ \text{if } \beta_2 = \infty, \text{ and } \int_{\beta}^{\beta_2} x dm(x) \neq \infty \end{cases}$

One introduces similar classifications for the left end point β_1 .

Regular and exit boundaries are called accessible, whereas the class of inaccessible boundaries consists of entrance and natural ones.

Note that in the case of a regular boundary at β_2 we may still have $m(\beta_2^-) = \infty$. This type of regular boundary will be called a regular absorbing boundary. However, we still have

$$\int_{\beta}^{\beta_2} (\beta_2 - x) dm(x) \equiv \lim_{y \uparrow \beta_2} \int_{\beta}^y (\beta_2 - x) dm(x) < \infty. \quad (2.1.2)$$

The difference between the exit and regular absorbing boundary is:

In the exit case, the behavior of the speed measure in the interior of (β_1, β_2) determines uniquely the value of m at β_2 .

However, for the case of a regular absorbing boundary at β_2 , the speed measure has an atom at β_2 , whose size $m(\beta_2)$ is not determined by the knowledge of properties of the speed measure in the interior of (β_1, β_2) .

Proposition 2.1.4

The eigen equation $Uu = \lambda u$, $x \in (\beta_1, \beta_2)$, $\lambda > 0$ has two continuous solutions $u_1(x, \lambda)$, $u_2(x, \lambda)$ such that

$$(i) \quad u_i(x, \lambda) > 0 \quad (i = 1, 2) \quad x \in (\beta_1, \beta_2).$$

(ii) $u_1(x, \lambda)$ is strictly increasing and $u_2(x, \lambda)$ is strictly decreasing.

(iii) at accessible boundaries b , $u_i(x, \lambda)$ ($i = 1, 2$) have finite limits as $x \rightarrow b$

at inaccessible right boundary β_2 , $u_1(x, \lambda) \rightarrow \infty$ as $x \rightarrow \beta_2$
left boundary β_1 , $u_2(x, \lambda) \rightarrow \infty$ as $x \rightarrow \beta_1$

(iv) any other continuous solution of the eigen equation is a linear combination of $u_1(x, \lambda)$, $u_2(x, \lambda)$ $x \in (\beta_1, \beta_2)$.

Proof: Breiman [4; p.380]

We now proceed to show that $u_i(x, \lambda)$ ($i = 1, 2$) are actually strictly convex for $x \in (\beta_1, \beta_2)$.

Proposition 2.1.5

If $h(x)$ is a continuous function in some neighbourhood of $x_0 \in (\beta_1, \beta_2)$ and $h \in \mathcal{D}(U, \{x_0\})$ and if $h(x)$ has a local minimum,

at x_0 then

$$(Uh)x_0 \geq 0.$$

Proof: Breiman [4; p.380].

Corollary 2.1.6

Every continuous positive function f on (β_1, β_2) with $f \in \mathcal{D}(U, (\beta_1, \beta_2))$ and $Uf > 0$ on (β_1, β_2) , must be convex on (β_1, β_2) .

Proof: Let c_1, c_2 be given constants and define

$$h(x) = f(x) - c_1 x - c_2.$$

Then $(Uh)x > 0$, $x \in (\beta_1, \beta_2)$.

From the last proposition we know that, at a point of local maximum h must be ≤ 0 . Thus h can not have a local maximum in (β_1, β_2) . So, if x_1, x_2 are two zeroes of h , then h must be strictly negative in the interval (x_1, x_2) .

Hence f is strictly convex on (β_1, β_2) .

Corollary 2.1.7

The functions $u_1(x, \lambda)$, $u_2(x, \lambda)$ described in the proposition 2.1.4 are strictly convex on (β_1, β_2) .

Remark 2.1.8

The properties of positivity and monotonicity of $u_1(x, \lambda)$

imply

$$\lim_{x \rightarrow \beta_1} u_1(x, \lambda) \geq 0, \quad \lim_{x \rightarrow \beta_2} u_1(x, \lambda) \leq \infty$$

$$\lim_{x \rightarrow \beta_1} u_2(x, \lambda) \leq \infty, \quad \lim_{x \rightarrow \beta_2} u_2(x, \lambda) \geq 0$$

We want inequalities, similar to above ones, involving the derivatives of $u_i(x, \lambda)$ ($i = 1, 2$).

Coming to the relation between a convex function and its derivatives, we state a known result.

Proposition 2.1.9

Let $f : (\beta_1, \beta_2) \rightarrow \mathbb{R}^1$ be a convex (strictly convex) function.

Then

- (a) f satisfies a Lipschitz condition on any closed interval contained in the interior of (β_1, β_2) .
- (b) the right and left derivatives f^+, f^- exist on (β_1, β_2)
- (c) $f^-(x) \leq f^+(x) \leq f^-(y) \leq f^+(y)$ for $x, y \in (\beta_1, \beta_2)$ with $x < y$.
(Strict inequality).
- (d) $\lim_{x \rightarrow y} f^+(x) = f^+(y)$ and $\lim_{x \rightarrow y} f^-(x) = f^-(y)$, $y \in (\beta_1, \beta_2)$
- (e) the limits in (d) are also valid at β_1 and β_2 , respectively, provided f is defined and continuous there.
- (f) the derivative of f , f' , may fail to exist only at countably many points.

Remark 2.1.10

From the convexity and monotonicity we have

$$0 \leq \lim_{x \rightarrow \beta_1} u_1^+(x, \lambda) < \infty, \quad -\infty < \lim_{x \rightarrow \beta_2} u_2^+(x, \lambda) \leq 0$$

$$0 < \lim_{x \rightarrow \beta_1} u_2^+(x, \lambda) \leq \infty, \quad 0 < \lim_{x \rightarrow \beta_2} u_1^+(x, \lambda) \leq \infty.$$

We now show how the boundary classifications (definition 2.1.3) allow one to replace the above inequalities by strict inequality or equality.

Lemma 2.1.11

$$\int_{\beta}^{\beta_2} (\beta_2 - y) dm(y) < \infty \Rightarrow \int_{\beta}^{\beta_2} [m(y) - m(\beta)] dy < \infty.$$

Proof:

$$\begin{aligned} \int_{\beta}^x (\beta_2 - y) dm(y) &= \beta_2 [m(x) - m(\beta)] - y m(y) \Big|_{\beta}^x + \int_{\beta}^x m(y) dy \\ &= \beta_2 [m(x) - m(\beta)] - x m(x) + \beta m(\beta) + \int_{\beta}^x m(y) dy \\ &= \beta_2 [m(x) - m(\beta)] - x [m(x) - m(\beta)] + [\beta m(\beta) - x m(\beta)] + \int_{\beta}^x m(y) dy \\ &= (\beta_2 - x) [m(x) - m(\beta)] + \int_{\beta}^x [m(y) - m(\beta)] dy. \end{aligned}$$

Note that $\int_{\beta}^x (\beta_2 - x) dm(y) \leq \int_{\beta}^x (\beta_2 - y) dm(y).$

Consequently from the hypothesis, we conclude

$$\int_{\beta}^{\beta_2} [m(y) - m(\beta)] dy < \infty.$$

Proposition 2.1.12

Let $u(\cdot, \lambda)$ be an increasing solution of $Uu = \lambda u$ such that $u(\beta) = 1$, $u^+(\beta) = 0$ with $\beta \in (\beta_1, \beta_2)$. Then

$$(i) \quad \int_{\beta}^{\beta_2} [m(y) - m(\beta)] dy < \infty \iff \lim_{x \uparrow \beta_2} u(x, \lambda) < \infty$$

$$(ii) \quad \int_{\beta}^{\beta_2} [m(y) - m(\beta)] dy < \infty, \quad m(\beta_2^-) \equiv \lim_{x \uparrow \beta_2} m(x) < \infty$$

$$\iff \lim_{x \uparrow \beta_2} u(x, \lambda) < \infty, \quad \lim_{x \uparrow \beta_2} u^+(x, \lambda) < \infty$$

Proof:

(i) ($\Rightarrow$) Integrating the given eigen equation with respect to m , we obtain

$$u^+(x, \lambda) = u^+(\beta, \lambda) + \lambda \int_{\beta}^x u(y, \lambda) dm(y)$$

$$= \lambda \int_{\beta}^x u(y, \lambda) dm(y). \quad (2.1.3)$$

Remembering that $u(\cdot, \lambda)$ is increasing and is bounded below by 1 on $[\beta, \beta_2)$, so

$$u^+(x, \lambda) \leq \lambda u(x, \lambda) [m(x) - m(\beta)] \quad \text{for } x \in [\beta, \beta_2). \quad (2.1.4)$$

Hence

$$\int_{\beta}^x \frac{u^+(y, \lambda)}{u(y, \lambda)} dy \leq \lambda \int_{\beta}^x [m(y) - m(\beta)] dy$$

so the conclusion follows by noting that $\ln u(x, \lambda)$ and hence $u(x, \lambda)$ remains bounded as $x \uparrow \beta_2$.

(i) ($\Leftarrow$) From equation (2.1.3) we also get (using monotonicity of u)

$$\begin{aligned} u^+(x, \lambda) &\geq \lambda u(\beta, \lambda) [m(x) - m(\beta)] \\ &= \lambda [m(x) - m(\beta)] \end{aligned} \quad (2.1.5)$$

Integrating the above inequality we get,

$$u(x, \lambda) - u(\beta, \lambda) \geq \lambda \int_{\beta}^x [m(y) - m(\beta)] dy$$

so the conclusion follows.

(ii) ($\Downarrow$) We need only to establish $\lim_{x \uparrow \beta_2} u^+(x, \lambda) < \infty$.

But it follows from the inequality (2.1.4).

(iii) ($\Uparrow$) Follows from the inequality (2.1.5).

Coming to the decreasing solution we have a simple

Proposition 2.1.13

For each positive decreasing solution u of $Uu = \lambda u$ and each $\beta \in (\beta_1, \beta_2)$

$$\int_{\beta}^{\beta_2} u(y, \lambda) dm(y) < \infty.$$

Proof:

Integrating the eigen equation

$$u^+(x, \lambda) = u^+(\beta, \lambda) + \lambda \int_{\beta}^x u(y, \lambda) \cdot dm(y) .$$

For a positive decreasing u , left side is < 0 , while the integrand is positive.

Thus,

$$\int_{\beta}^x u(y, \lambda) dm(y) < - \frac{u^+(\beta, \lambda)}{\lambda}$$

and the conclusion follows by letting $x \uparrow \beta_2$.

Proposition 2.1.14

Let β_2 be either regular absorbing or exit. Then

- (i) $\lim_{x \uparrow \beta_2} u_2(x, \lambda) = 0$,
- (ii) $-\infty < \lim_{x \uparrow \beta_2} u_2^+(x, \lambda) < 0$.

Proof: (i) Remember that $u_2(x, \lambda)$ is positive, decreasing. So suppose

$$\lim_{x \uparrow \beta_2} u_2(x, \lambda) = \epsilon > 0 .$$

Then $\int_{\beta}^{\beta_2} u_2(x, \lambda) dm(x) \geq \epsilon [m(\beta_2) - m(x)]$.

For the kind of β_2 that satisfies the hypothesis one has $m(\beta_2) = \infty$. Thus we have a contradiction to proposition 2.1.11.

Hence, $\lim_{x \uparrow \beta_2} u_2(x, \lambda) = 0$.

(ii) Remember, in the present situation $\beta_2 < \infty$.

Because of convexity of $u_2(\cdot, \lambda)$ we have, for $x > \beta$

$$u_2(x, \lambda)(\beta_2 - x)^{-1} < -u_2^+(x, \lambda) < -u_2^+(\beta, \lambda)$$

or, $u_2(x, \lambda) < -u_2^+(\beta, \lambda)(\beta_2 - x)$.

But,

$$\begin{aligned} u_2^+(x, \lambda) &= u_2^+(\beta, \lambda) + \lambda \int_{\beta}^x u_2(y, \lambda) dm(y) \\ &< u_2^+(\beta, \lambda) - u_2^+(\beta, \lambda) \int_{\beta}^x (\beta_2 - y) dm(y). \end{aligned}$$

Thus,

$$-u_2^+(x, \lambda) > -u_2^+(\beta, \lambda) [1 - \lambda \int_{\beta}^x (\beta_2 - y) dm(y)].$$

Recall (e.g. definition 2.1.2)

$$\int_{\beta}^{\beta_2} (\beta_2 - y) dm(y) < \infty.$$

So that we may choose β close to β_2 such that

$$\int_{\beta}^{\beta_2} (\beta_2 - y) dm(y) < \frac{1}{2\lambda}.$$

Thus, $-u_2^+(x, \lambda) > -\frac{1}{2} u_2^+(\beta, \lambda)$ for $x > \beta$

$$-\lim_{x \uparrow \beta_2} u_2^+(x, \lambda) \geq -\frac{1}{2} u_2^+(\beta, \lambda) > 0.$$

We have already noted in remark 2.1.9, $-\infty < \lim_{x \uparrow \beta_2} u_2^+(x, \lambda)$.

Remark 2.1.15

It can also be shown [Feller 9] that if β_2 is an entrance boundary then,

$$0 < \lim_{x \uparrow \beta_2} u_2(x, \lambda) < \infty, \quad \lim_{x \uparrow \beta_2} u_2^+(x, \lambda) = 0$$

if β_2 is a natural boundary then

$$\lim_{x \uparrow \beta_2} u_2(x, \lambda) = 0 = \lim_{x \uparrow \beta_2} u_2^+(x, \lambda).$$

Remember [prop. 2.1.4(iv)] that every continuous solution of the eigen equation is a linear combination of $u(x, \lambda)$ [of prop. 2.1.12] and $u_2(x, \lambda)$, so the behavior of $u_1(x, \lambda)$ and its derivative $u_1^+(x, \lambda)$ near β_2 can be obtained from the corresponding properties of $u(x, \lambda)$ and $u_2(x, \lambda)$. Moreover, from the symmetry one can also deduce the behavior of $u_1(x, \lambda)$ and $u_1^+(x, \lambda)$ near β_1 . Summarizing we have

Table 2.1.16

	Regular Absorbing	Exit	Entrance	Natural
$u_1(\beta_2)$	$< \infty$	$< \infty$	$< \infty$	$< \infty$
$u_1^+(\beta_2)$	$< \infty$	$= +\infty$	$< \infty$	$= +\infty$
$u_2(\beta_2)$	$= 0$	$= 0$	> 0	$= 0$
$u_2^+(\beta_2)$	< 0	< 0	$= 0$	$= 0$
$u_1(\beta_1)$	$= 0$	$= 0$	> 0	$= 0$
$u_1^+(\beta_1)$	> 0	> 0	$= 0$	$= 0$
$u_2(\beta_1)$	$< \infty$	$< \infty$	$= +\infty$	$= +\infty$
$u_2^+(\beta_1)$	$> -\infty$	$= -\infty$	$> -\infty$	$= -\infty$

Remark 2.1.17

If the left boundary β_1 is regular absorbing or exit then

$\lim_{x \rightarrow \beta_1} u_1(x, \lambda) = 0$. Remembering that u_1 is convex we get from

prop. 2.1.8(e)

$$\lim_{x \rightarrow \beta_1} u_1^+(x, \lambda) = \lim_{x \rightarrow \beta_1} \frac{u_1(x, \lambda)}{x - \beta_1}.$$

Definition 2.1.18

Define the Wronskian, W , of u_1, u_2 as

$$W(x) = u_2(x) u_1'(x) - u_1(x) u_2'(x)$$

where $u_1'(x) = \frac{d}{dx} u_1(x)$.

From prop. 2.1.8(f) we know that $W(x)$ may fail to exist only at a countably many points.

Proposition 2.1.19

W is constant except possibly where $u_1'(x)$ ($i = 1, 2$) do not exist.

Proof: $\alpha < \beta$;

From the eigen equation we have

$$u_2 \frac{d}{dm} u_1' - u_1 \frac{d}{dm} u_2' = 0.$$

$$\text{Hence, } 0 = \int_{\alpha}^{\beta} (u_2 \frac{d}{dm} u_1' - u_1 \frac{d}{dm} u_2') dm$$

$$= u_2 u_1' - u_1 u_2' \Big|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} (u_2' \frac{du_1}{dx} - u_1' \frac{du_2}{dx}) dx.$$

Thus $W(\alpha) = W(\beta)$ as required.

Definition 2.1.20

The Green function, $G(x, y, \lambda)$ of u_1 's, is defined as follows

$$G(x, y, \lambda) = \begin{cases} u_1(x, \lambda) u_2(y, \lambda), & x < y \\ u_1(y, \lambda) u_2(x, \lambda), & x > y \end{cases}$$

usually the expression for the Green function should contain the Wronskian $W(x)$. But because of prop. 2.1.19 we may assume that $W(x) \equiv 1$, whenever it exists.

However, the definition of Green function is not complete without the side conditions. From the table 2.1.15, the appropriate side conditions are:

$$\begin{aligned} f(\beta_1) &= 0 & \beta_1 & \text{regular absorbing} \\ f(\beta_1) &= 0 & & \text{exit} \\ f^+(\beta_1) &= 0 & & \text{entrance} \\ f(\beta_1) &= 0 = f^+(\beta_1) & & \text{natural} \end{aligned}$$

Remark 2.1.21

Define the Green operator as follows

$$G_\lambda f(x) = \int_{\beta}^{\beta_2} G(x, y, \lambda) f(y) d\mu(y).$$

The domain of G_λ consists of those f for which the integral exists in the sense of absolute convergence. It can be shown [Feller 9; p. 482] that the Green operator identity holds, i.e.,

$$G_\mu - G_\lambda + (\mu - \lambda) G_\mu G_\lambda = 0$$

This equation in turn implies that $G(x, y, \cdot)$ is completely monotonic,

$$- (1)^n \frac{\partial^n}{\partial \lambda^n} G(x, y, \lambda) > 0, \quad n \geq 0 \quad (2.1.6)$$

Proof of this result could be found in McKean [11].

§2.2 Theory of McKean

In a paper [11] McKean has shown the existence of $p(x, t, y)$ such that

$$G(x, y, \lambda) = \int_0^{\infty} e^{-\lambda t} p(x, t, y) dt.$$

It has also been established that

1. $p(\cdot, t, \cdot)$ is positive, symmetric on $(\beta_1, \beta_2) \times (\beta_1, \beta_2)$ for each $t > 0$, and p is continuous on $(\beta_1, \beta_2) \times (0, \infty) \times (\beta_1, \beta_2)$.
2. $p(x, 0, y) = 0$ for $x, y \in (\beta_1, \beta_2)$, $x \neq y$.
3. The derivatives $\frac{\partial^n}{\partial t^n} p(\cdot, t, y)$ belong to the side conditions (defn. 2.1.20) and

$$\frac{\partial^n}{\partial t^n} p(\cdot, t, y) = U^n p(\cdot, t, y), \quad t > 0, \quad y \in (\beta_1, \beta_2), \quad n > 0.$$

4. $\int_{\beta_1}^{\beta_2} p(x, t, y) dm(y) \leq 1$; $t > 0$, $x \in (\beta_1, \beta_2)$.
5. $p(x, t_1 + t_2, y) = \int_{\beta_1}^{\beta_2} p(x, t_1, z) p(z, t_2, y) dm(z)$.
6. The map $(t, x) \mapsto \int_{\beta_1}^{\beta_2} p(x, t, y) dm(y)$ is continuous on $(0, \infty) \times (\beta_1, \beta_2)$.

He also established the following

Table 2.2.1 $i = 1, 2$

	Regular Absorbing	exit	entrance	natural
$\lim_{x \rightarrow \beta_1} p(x, t, y)$	$= 0$	$= 0$	> 0	$= 0$
$(-1)^{\frac{1}{2}} \lim_{x \rightarrow \beta_1} \frac{\partial^+}{\partial x} p(x, t, y)$	< 0	< 0	$= 0$	$= 0$

Remark 2.2.2

We see that for an entrance or natural boundary at β_1 , our method will yield only the degenerate entrance law. In the next section we will show for a regular absorbing or exit boundary,

$\lim_{x \rightarrow \beta_1} \frac{\partial^+}{\partial x} p(x, t, y)$ satisfies the equation (2.0.1), that is, it is an entrance law.

We need to mention another result which allows us to calculate the Laplace transforms of

$$\lim_{x \rightarrow \beta_1} p(x, t, y) \quad \text{and} \quad \lim_{x \rightarrow \beta_1} \frac{\partial^+}{\partial x} p(x, t, y)$$

in terms of $u_1(x, \lambda)$.

Proposition 2.2.3 [Feller-McKean]

Given $x, y \in (\beta_1, \beta_2)$, there exists a right continuous increasing function $\phi(x, \cdot, y) : [0, \infty) \rightarrow \mathbb{R}^1$ such that

$$(1) \quad \phi(x, 0, y) = 0$$

$$(2) \int_0^{\infty} e^{-\lambda t} d\phi_t(x, t, y) = \begin{cases} u_1(x, \lambda)/u_1(y, \lambda) & x < y \\ u_2(x, \lambda)/u_2(y, \lambda) & x > y \end{cases} \quad \lambda > 0$$

$$(3) \phi(x, \cdot, y) \leq 1 \text{ on } [0, \infty)$$

$$(4) \phi(\cdot, t, y) \text{ is convex increasing on } (\beta_1, y)$$

$$\text{convex decreasing on } (y, \beta_2)$$

$$(5) \phi(x, t, y) = o(t^n) \text{ as } t \rightarrow 0 \text{ for } n > 0, x \neq y$$

Proof: McKean [11; p. 435-437].

52.3 Consequences of Feller-McKean theory: Entrance law

In this section we assume

- (I) the state space of the diffusion is $[0, \infty)$, with a regular absorbing or exit boundary at $\{0\}$.
- (II) the diffusion is a martingale.

The monotonicity of the function $\phi(\cdot, t, y)$ allows easy interchanges of limit and integration. We first obtain some information about its Laplace transformation. Then we will establish a formula connecting the functions ϕ and p .

Proposition 2.3.1

$$(i) \quad \int_0^{\infty} e^{-\lambda t} \phi(x, t, \xi) dt = \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} d\phi_t(x, t, \xi)$$

$$(ii) \quad \lim_{x \rightarrow 0} \phi(x, t, \xi) = 0, \quad 0 < x < \xi.$$

$$(iii) \quad \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} \phi(x, t, \xi) = \lim_{x \rightarrow 0} \frac{\phi(x, t, \xi)}{x}, \quad 0 < x < \xi.$$

$$(iv) \quad \text{On } [0, \xi), \quad \frac{\partial^+}{\partial x} \phi(x, t, \xi) \text{ is a bounded function in } t, \\ \text{and it decreases as } x \text{ decreases.}$$

$$(v) \quad \int_0^{\infty} e^{-\lambda t} \lim_{x \rightarrow 0} \frac{\phi(x, t, \xi)}{x} dt = \lim_{x \rightarrow 0} \int_0^{\infty} e^{-\lambda t} \frac{\phi(x, t, \xi)}{x} dt, \quad 0 < x < \xi.$$

Proof:

$$(i) \int_0^{\infty} e^{-\lambda t} d\phi_t(x, t, \xi) = \phi(x, t, \xi) e^{-\lambda t} \Big|_0^{\infty} + \lambda \int_0^{\infty} e^{-\lambda t} \phi(x, t, \xi) dt$$

now remember that $0 \leq \phi(x, \cdot, \xi) \leq 1$.

(ii) If $x < \xi$, then

$$\int_0^{\infty} e^{-\lambda t} \phi(x, t, \xi) dt = \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} d\phi_t(x, t, \xi) = \frac{u_1(x, \lambda)}{\lambda u_1(\xi, \lambda)}.$$

$$\text{Then } \lim_{x \rightarrow 0} \int_0^{\infty} e^{-\lambda t} \phi(x, t, \xi) dt = 0.$$

But $\phi(\cdot, t, \xi)$ is convex increasing on $(0, \xi)$ and $\phi(x, \cdot, \xi) \leq 1$. So, by bounded convergence,

$$\int_0^{\infty} e^{-\lambda t} \lim_{x \rightarrow 0} \phi(x, t, \xi) dt = 0.$$

But $\lim_{x \rightarrow 0} \phi(x, t, \xi) \geq 0$; conclude then $\lim_{x \rightarrow 0} \phi(x, t, \xi) = 0$ (a.s.t.).

From monotonicity of $\phi(x, \cdot, \xi)$, we have

$$\lim_{x \rightarrow 0} \phi(x, t_1, \xi) \leq \lim_{x \rightarrow 0} \phi(x, t_2, \xi) \quad (t_1 < t_2)$$

So, $\lim_{x \rightarrow 0} \phi(x, t, \xi) = 0$ for all t .

(iii) Extend the domain of definition of ϕ by continuity, so set

$\phi(0, t, \xi) = 0$, and obtain a convex increasing function on $[0, \xi)$.

So the right derivative $\frac{\partial^+}{\partial x} \phi(x, t, \xi)$ is positive and right continuous on $[0, \xi)$. (Lemma 2.1.8(e)).

From these observations,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} \phi(x, t, \xi) &= \frac{\partial^+}{\partial x} \phi(x, t, \xi) \Big|_{x=0} \\ &= \lim_{x \rightarrow 0} \frac{\phi(x, t, \xi) - \phi(0, t, \xi)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\phi(x, t, \xi)}{x} \end{aligned}$$

(iv) Remembering, $\frac{\partial^+}{\partial x} \phi(x, t, \xi) + \frac{\partial^+}{\partial x} \phi(x, t, \xi) \Big|_{x=0}$ as $x \rightarrow 0$, it suffices to show

for $0 < x < \xi$, $\frac{\partial^+}{\partial x} \phi(x, t, \xi)$ is a bounded function of t .

From prop. 2.1.8(a), we know that $\phi(\cdot, t, \xi)$ is Lipschitz on every $[a, b] \subset (0, \xi)$. We want to show that the Lipschitz constant is independent of t for every $[a, b]$.

Let δ be such that $[a, b+\delta] \subset (0, \xi)$. Suppose $a < x < y < b$. Put $z = y + \frac{\delta}{|y-x|} (y-x)$ then $z \in [a, b+\delta]$

and $\alpha = \frac{|y-x|}{\delta + |y-x|}$, then $\alpha z + (1-\alpha)x = y$

$$\phi(y, t, \xi) \leq \alpha[\phi(z, t, \xi) - \phi(x, t, \xi)] + \phi(x, t, \xi).$$

Consequently, $\phi(y, t, \xi) - \phi(x, t, \xi) \leq \alpha$, $0 \leq \phi(w, \cdot, \xi) \leq 1$ for $w \in (0, \xi)$.

or,
$$\frac{\phi(y, t, \xi) - \phi(x, t, \xi)}{y - x} \leq \frac{1}{\delta}$$

So, $\frac{\partial^+}{\partial x} \phi(x, t, \xi)$ is a bounded function of t on $[0, \xi - \sigma]$ for fixed $\delta > 0$.

$$\phi^+(0, t, \xi) \equiv \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} \phi(x, t, \xi).$$

(v) From (iv),

$$\begin{aligned} & \int_0^\infty e^{-\lambda t} \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} \phi(x, t, \xi) dt \\ &= \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} \int_0^\infty e^{-\lambda t} \phi(x, t, \xi) dt \\ &= \frac{u_1^+(0, \lambda)}{\lambda u_2(\xi, \lambda)} \end{aligned}$$

$$\begin{aligned} & \int_0^\infty e^{-\lambda t} \lim_{x \rightarrow 0} \frac{\phi(x, t, \xi)}{x} dt = \int_0^\infty e^{-\lambda t} \phi^+(0, t, \xi) dt \\ &= \frac{u_1^+(0, \lambda)}{\lambda u_2(\xi, \lambda)} \end{aligned} \quad (2.3.1)$$

$$= \lim_{x \rightarrow 0} \frac{u_1(x, \lambda)}{x} \frac{1}{\lambda u_2(\xi, \lambda)} \quad [\text{Remark 2.1.17}]$$

$$= \lim_{x \rightarrow 0} \int_0^\infty e^{-\lambda t} \frac{\phi(x, t, \xi)}{x} dt.$$

Coming to the connexion between the function ϕ and the fundamental solution p , we have

Proposition 2.3.2

Let $\xi \in (0, \infty)$ be fixed. Then, for $x < \xi < y$ or $x > \xi > y$ one has

$$p(x, t, y) = \int_0^t \phi(x, \sigma, \xi) \frac{\partial}{\partial t} p(\xi, t - \sigma, y) d\sigma$$

Proof:

Step 1. Consider the convolution $\Psi(t) = \int_0^t d\phi_\sigma(x, \sigma, \xi) \cdot$

$\int_0^{t-\sigma} p(\xi, \tau, y) d\tau$. The function $\Psi(t)$ is continuous in t . So let

us calculate its Laplace transform:

$$\int_0^\infty e^{-\lambda t} d\Psi(t) = \int_0^\infty e^{-\lambda t} d\phi_t(x, t, \xi) \cdot \int_0^\infty e^{-\lambda t} p(\xi, t, y) dt$$

[product rule for convolution]

$$= \begin{cases} \frac{u_1(x, \lambda)}{u_1(\xi, \lambda)} \cdot u_1(\xi, \lambda) u_2(y, \lambda) & (x \leq \xi < y) \\ \frac{u_2(x, \lambda)}{u_2(\xi, \lambda)} \cdot u_1(\xi, \lambda) u_1(y, \lambda) & (x < \xi < y) \end{cases}$$

$$= G(x, y, \lambda) = \int_0^\infty e^{-\lambda t} p(x, t, y) dt$$

From continuity of $p(x, \cdot, y)$ and $\Psi(\cdot)$ we may conclude [Widder p.3]

$$\int_0^t p(x, \tau, y) d\tau = \Psi(t) = \int_0^t d\phi_\sigma(x, \sigma, \xi) \int_0^{t-\sigma} p(\xi, \tau, y) d\tau. \quad (2.3.2)$$

Step 2

We want to differentiate both sides of (2.3.2) with respect to t . The left hand side presents no problem. For the right hand side let us start by estimating $\int_0^t p(x, \tau, y) d\tau$.

$$\text{Recall } G(\xi, y, \lambda) = \int_0^\infty e^{-\lambda t} p(\xi, t, y) dt$$

and that $G(\xi, y, \cdot)$ is a completely monotonic function [(2.1.6)].

So, from Bernstein's theorem [Widder 1945 p.154], it follows that

$\int_0^t p(\xi, \tau, y) d\tau$ is, as a function of t , bounded by say $K(\xi, y)$.

$$\begin{aligned} \text{Thus from (2.3.2), } \int_0^t p(x, \tau, y) d\tau &\leq K(\xi, y) \int_0^t d\phi_\sigma(x, \sigma, \xi) \\ &= K(\xi, y) \phi(x, t, \xi) \\ &\leq O(t^n) \text{ as } t \rightarrow 0. \end{aligned} \quad (2.3.3)$$

Step 3

$$\text{To establish: } p(x, t, y) = \int_0^t d\phi_\sigma(x, \sigma, \xi) p(\xi, t-\sigma, y) \quad (2.3.4)$$

$$\begin{aligned} \Psi(t+h) &= \int_0^{t+h} d\phi_\sigma(x, \sigma, \xi) \int_0^{t+h-\sigma} p(\xi, \tau, y) d\tau \\ &= \int_0^t d\phi_\sigma(x, \sigma, \xi) \int_0^{t+h-\sigma} p(\xi, \tau, y) d\tau + \int_t^{t+h} d\phi_\sigma(x, \sigma, \xi) \int_0^{t+h-\sigma} p(\xi, \tau, y) d\tau \end{aligned}$$

$$\begin{aligned} \frac{\Psi(t+h) - \Psi(t)}{h} &= \frac{1}{h} \int_0^t d\phi_\sigma(x, \sigma, \xi) \left[\int_0^{t+h-\sigma} p(\xi, \tau, y) d\tau - \int_0^{t-\sigma} p(\xi, \tau, y) d\tau \right] \\ &\quad + \frac{1}{h} \int_t^{t+h} d\phi_\sigma(x, \sigma, \xi) \int_0^{t+h-\sigma} p(\xi, \tau, y) d\tau \end{aligned}$$

$$\frac{1}{h} \int_{t-\sigma}^{t+h-\sigma} p(\xi, \tau, y) d\tau \xrightarrow{h \rightarrow 0} p(\xi, t-\sigma, y).$$

$$\frac{1}{h} \int_t^{t+h} d\phi_\sigma(x, \sigma, \xi) \int_0^{t+h-\sigma} p(\xi, \tau, y) d\tau$$

$$\leq \frac{1}{h} \int_t^{t+h} d\phi_\sigma(x, \sigma, \xi) \int_0^h p(\xi, \tau, y) d\tau \quad \left\{ \begin{array}{l} t \leq \sigma \leq t+h \\ t+h-\sigma \leq t+h-t \leq h \end{array} \right.$$

$$= \frac{1}{h} \int_0^h p(\xi, \tau, y) d\tau \cdot \int_t^{t+h} d\phi_\sigma(x, \sigma, \xi)$$

$$\leq 2 \frac{1}{h} \int_0^h p(\xi, \tau, y) d\tau$$

$$\longrightarrow 0.$$

See (2.3.3)

So we have established the equation (2.3.4).

Step 4

$$p(x, t, y) = \int_0^t d\phi_\sigma(x, \sigma, \xi) p(\xi, t-\sigma, y)$$

$$= \phi(x, \sigma, \xi) p(\xi, t-\sigma, y) \Big|_{\sigma=0}^{\sigma=t} + \int_0^t \phi(x, \sigma, \xi) \frac{\partial}{\partial t} p(\xi, t-\sigma, y) d\sigma$$

$$= \int_0^t \phi(x, \sigma, \xi) \frac{\partial}{\partial t} p(\xi, t-\sigma, y) d\sigma$$

since $\phi(x, 0, \xi) = 0 = p(\xi, 0, y)$.

Combining propositions 2.3.1 and 2.3.2, we obtain, in the next three lemmas, some properties of the fundamental solution.

Lemma 2.3.3

$$i) \quad \lim_{x \rightarrow 0} p(x, t, y) = 0$$

$$ii) \quad \lim_{x \rightarrow 0} \frac{p(x, t, y)}{x} = \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} p(x, t, y)$$

Proof:

i). Let $\xi \in (0, \infty)$ be fixed and $0 < x < \xi < y$. Then from proposition 2.3.2

$$\begin{aligned} \lim_{x \rightarrow 0} p(x, t, y) &= \lim_{x \rightarrow 0} \int_0^t \phi(x, \sigma, \xi) \frac{\partial}{\partial t} p(\xi, t - \sigma, y) d\sigma \\ &= \int_0^t \lim_{x \rightarrow 0} \phi(x, \sigma, \xi) \frac{\partial}{\partial t} p(\xi, t - \sigma, y) d\sigma \\ &\quad \left\{ \begin{array}{l} \text{boundedness of } \phi(x, \cdot, \xi) \\ \text{monotonicity of } \phi(\cdot, \sigma, \xi) \end{array} \right. \\ &= 0 \quad [\text{proposition 2.3.1 (ii)}]. \end{aligned}$$

ii) Again from proposition 2.3.2 we get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} p(x, t, y) &= \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} \int_0^t \phi(x, \sigma, \xi) \frac{\partial}{\partial t} p(\xi, t - \sigma, y) d\sigma \\ &= \int_0^t \lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} \phi(x, \sigma, \xi) \frac{\partial}{\partial t} p(\xi, t - \sigma, y) d\sigma \\ &= \int_0^t \lim_{x \rightarrow 0} \frac{\phi(x, \sigma, \xi)}{x} \frac{\partial}{\partial t} p(\xi, t - \sigma, y) d\sigma \\ &\quad [\text{Justification by prop. 2.3.1 (iv).}] \\ &= \lim_{x \rightarrow 0} \int_0^t \frac{\phi(x, \sigma, \xi)}{x} \frac{\partial}{\partial t} p(\xi, t - \sigma, y) d\sigma \quad (2.3.5). \end{aligned}$$

This interchange is justified by lemma 2.3.4.

$$= \lim_{x \rightarrow 0} \frac{p(x, t, y)}{x} \quad [\text{prop. 2.3.2}]$$

Lemma 2.3.4

For $x \in [0, \xi)$, $\frac{\phi(x, t, \xi)}{x}$ decreases as x decreases, and remains bounded as a function of t .

Proof: Let $0 \leq x < y < \xi$. Then every $z \in (x, y)$ can be written as

$$z = (1-\alpha)x + \alpha y \quad \text{for } \alpha = \frac{z-x}{y-x}.$$

Hence

$$\phi(z, t, \xi) \leq \phi(x, t, \xi) + \alpha[\phi(y, t, \xi) - \phi(x, t, \xi)]$$

or

$$\phi(z, t, \xi) - \phi(x, t, \xi) \leq z-x \left[\frac{\phi(y, t, \xi) - \phi(x, t, \xi)}{y-x} \right]$$

And we get the desired result by putting $x = 0$. Boundedness in t follows using an argument similar to prop. 2.3.1(iv).

Lemma 2.3.5

$$\begin{aligned} & \int_0^\infty e^{-\lambda t} \lim_{x \rightarrow 0} \frac{p(x, t, y)}{x} dt \\ &= u_1^+(0, \lambda) u_2(y, \lambda) \quad (2.3.6) \\ &= \lim_{x \rightarrow 0} \int_0^\infty e^{-\lambda t} \frac{p(x, t, y)}{x} dt. \end{aligned}$$

Proof: For $0 < x < \xi < y$ we get from prop. 2.3.2 by the convolution rule

$$\int_0^{\infty} e^{-\lambda t} p(x, t, y) dt = \left[\int_0^{\infty} e^{-\lambda t} \phi(x, \sigma, \xi) dt \right] \cdot \left[\int_0^{\infty} e^{-\lambda t} \frac{\partial}{\partial t} p(\xi, t, y) dt \right].$$

Remembering prop. 2.3.1(1), for $\xi < y$

$$\int_0^{\infty} e^{-\lambda t} \frac{\partial}{\partial t} p(\xi, t, y) dt = \lambda u_1(\xi, \lambda) u_2(y, \lambda). \quad (2.3.7)$$

From (2.3.5) similarly we get

$$\int_0^{\infty} e^{-\lambda t} \lim_{x \rightarrow 0} \frac{p(x, t, y)}{x} dt = \left[\int_0^{\infty} e^{-\lambda t} \lim_{x \rightarrow 0} \frac{\phi(x, t, \xi)}{x} dt \right] \cdot$$

$$\left[\int_0^{\infty} e^{-\lambda t} \frac{\partial}{\partial t} p(\xi, t, y) dt \right]$$

$$= \frac{u_1^+(0, \lambda)}{\lambda u_1(\xi, \lambda)} \cdot \lambda u_1(\xi, \lambda) u_2(y, \lambda) \quad \left\{ \begin{array}{l} \text{using (2.3.1)} \\ \text{and (2.3.7)} \end{array} \right.$$

$$= u_1^+(0, \lambda) u_2(y, \lambda)$$

$$= \lim_{x \rightarrow 0} \int_0^{\infty} e^{-\lambda t} \frac{p(x, t, y)}{x} dt.$$

Finally we are ready to show that $\lim_{x \rightarrow 0} \frac{\partial^+}{\partial x} p(x, t, y)$, or what

is the same thing, $\lim_{x \rightarrow 0} \frac{p(x, t, y)}{x}$ is an entrance law.

Theorem 2.3.6

$$v_{t_1+t_2}(y) = \int_0^{\infty} v_{t_1}(z) p(z, t_2, y) dm(z) \quad (2.3.8)$$

where $v_t(y) \equiv \lim_{x \rightarrow 0} \frac{p(x, t, y)}{x}$.

Proof:

Step 1. First we establish the equality (2.3.8) a.s. t_1 .

From
$$p(x, t_1 + t_2, y) = \int_0^\infty p(x, t_1, z) p(z, t_2, y) dm(z)$$

noting that everything is positive, we get by Fatou's lemma

$$v_{t_1+t_2}(y) \geq \int_0^\infty v_{t_1}(z) p(z, t_2, y) dm(z) \quad (2.3.9)$$

Then,

$$\int_0^\infty e^{-\lambda t_1} v_{t_1+t_2}(y) dt_1 \geq \int_0^\infty e^{-\lambda t_1} \left[\int_0^\infty v_{t_1}(z) p(z, t_2, y) dm(z) \right] dt_1 \quad (2.3.10)$$

Again because of positivity, via Tonelli

$$\begin{aligned} & \int_0^\infty e^{-\lambda t_1} \left[\int_0^\infty p(x, t_1, z) p(z, t_2, y) dm(z) \right] dt_1 \\ &= \int_0^\infty p(z, t_2, y) \left[\int_0^\infty e^{-\lambda t_1} p(x, t_1, z) dt_1 \right] dm(z) \\ &= \int_0^x p(z, t_2, y) \left[\int_0^\infty e^{-\lambda t_1} p(x, t_1, z) dt_1 \right] dm(z) \\ &\quad + \int_x^\infty p(z, t_2, y) \left[\int_0^\infty e^{-\lambda t_1} p(x, t_1, z) dt_1 \right] dm(z) \\ &= \int_0^x p(z, t_2, y) [u_1(z, \lambda) u_2(x, \lambda)] dm(z) + \int_x^\infty p(z, t_2, y) [u_1(x, \lambda) u_2(z, \lambda)] dm(z) \end{aligned}$$

Now $u_1 \uparrow, u_2 \uparrow$, so for $z < x$

$$u_1(z, \lambda) \leq u_1(x, \lambda)$$

$$u_2(x, \lambda) \leq u_2(z, \lambda).$$

$$\begin{aligned} &\leq \int_0^\infty p(z, t_2, y) [u_1(x, \lambda) u_2(z, \lambda)] dm(z) + \int_x^\infty p(z, t_2, y) [u_1(x, \lambda) u_2(z, \lambda)] dm(z) \\ &= \int_0^\infty u_1(x, \lambda) u_2(z, \lambda) p(z, t_2, y) dm(z) \end{aligned}$$

$$\text{Hence, } \int_0^\infty e^{-\lambda t_1} p(x, t_1 + t_2, y) dt_1 \leq \int_0^\infty u_1(x, \lambda) u_2(z, \lambda) p(z, t_2, y) dm(z) \quad (*)$$

Thus by Fatou,

$$\begin{aligned} &\int_0^\infty e^{-\lambda t_1} v_{t_1 + t_2}(y) dt_1 \\ &\leq \lim_{x \rightarrow 0} \int_0^\infty e^{-\lambda t_1} \frac{p(x, t_1 + t_2, y)}{x} dt_1 \\ &\leq \lim_{x \rightarrow 0} \frac{u_1(x, \lambda)}{x} \int_0^\infty u_2(z, \lambda) p(z, t_2, y) dm(z) \quad [\text{From } (*)] \\ &= u_1^+(0, \lambda) \int_0^\infty u_2(z, \lambda) p(z, t_2, y) dm(z) \quad [\text{Remark 2.1.17}] \\ &= \int_0^\infty p(z, t_2, y) \left[\int_0^\infty e^{-\lambda t_1} \left[\lim_{x \rightarrow 0} \frac{p(x, t_1, z)}{x} \right] dt_1 \right] dm(z) \quad [\text{lemma 2.3.5}] \\ &= \int_0^\infty e^{-\lambda t_1} \left[\int_0^\infty v_{t_1}(z) p(z, t_2, y) dm(z) \right] dt_1 \quad [\text{Tonelli}] \end{aligned}$$

Rewriting,
$$\int_0^{\infty} e^{-\lambda t_1} v_{t_1+t_2}(y) dt_1 \leq \int_0^{\infty} e^{-\lambda t_1} \left[\int_0^{\infty} v_{t_1}(z) p(z, t_2, y) dm(z) \right] dt_1$$

Remembering (2.3.10) we get

$$\int_0^{\infty} e^{-\lambda t_1} v_{t_1+t_2}(y) dt_1 = \int_0^{\infty} e^{-\lambda t_1} \left[\int_0^{\infty} v_{t_1}(z) p(z, t_2, y) dm(z) \right] dt_1.$$

Finally from (2.3.9) one has,

$$v_{t_1+t_2}(y) = \int_0^{\infty} v_{t_1}(z) p(z, t_2, y) dm(z) \quad \text{a.s. } t_1. \quad (2.3.11)$$

Step 2. Till now, we have not used the assumption that the diffusions involved are all martingales. (See assumption II at the beginning of this section.) We will use the following consequence of the assumption II.

$$\int_0^{\infty} y p(x, t, y) dm(y) = x$$

Note that the speed measure m is not σ -finite on $[0, \infty)$. It has an atom of infinite mass at the origin. So Tonelli's theorem is not applicable on $[0, \infty)$. We avoid this problem by recalling that the speed measure m is σ -finite on $(0, \infty)$. [See notation 2.1.12.]

$$\text{If } v_{t_1+t_2}(y) = \int_0^{\infty} v_{t_1}(z) p(z, t_2, y) dm(z)$$

$$\text{then, } \int_0^{\infty} y v_{t_1+t_2}(y) dm(y) = \int_0^{\infty} z v_{t_1}(z) dm(z).$$

Proof: To begin with, note $\int_0^\infty y v_t(y) dm(y) \leq 1$.

Recall $\int_0^N y dm(y) < \infty$ for an accessible absorbing boundary at $\{0\}$.

So, $\int_0^\infty y v_{t_1+t_2}(y) dm(y)$

$$= \lim_{N \rightarrow \infty} \int_0^N y v_{t_1+t_2}(y) dm(y)$$

$$= \lim_{N \rightarrow \infty} \left[\int_0^N y \left[\int_0^\infty v_{t_1}(z) p(z, t_2, y) dm(z) \right] dm(y) \right]$$

$$= \lim_{N \rightarrow \infty} \left[\int_0^N y \lim_{\substack{\delta_1 \rightarrow 0 \\ \delta_2 \rightarrow \infty}} \left\{ \int_{\delta_1}^{\delta_2} v_{t_1}(z) p(z, t_2, y) dm(z) \right\} \right] dm(y)$$

$$= \lim_{N \rightarrow \infty} \lim_{\substack{\delta_1 \rightarrow 0 \\ \delta_2 \rightarrow \infty}} \int_0^N y dm(y) \left\{ \int_{\delta_1}^{\delta_2} v_{t_1}(z) p(z, t_2, y) dm(z) \right\} dm(y)$$

{because of positivity, via monotone convergence.

$$= \lim_{N \rightarrow \infty} \lim_{\substack{\delta_1 \rightarrow 0 \\ \delta_2 \rightarrow \infty}} \left[\int_{\delta_1}^{\delta_2} v_{t_1}(z) dm(z) \left[\int_0^N y p(z, t_2, y) dm(y) \right] \right] \quad (\text{Tonelli})$$

$$\left\{ \text{note } \int_0^N y p(z, t_2, y) dm(y) \leq \int_0^\infty y p(z, t_2, y) dm(y) = z \right.$$

$$= \lim_{N \rightarrow \infty} \int_0^\infty v_{t_1}(z) dm(z) \int_0^N y p(z, t_2, y) dm(y)$$

$$= \int_0^\infty v_{t_1}(z) dm(z) \int_0^\infty y p(z, t_2, y) dm(y) \quad (\text{dominated convergence})$$

$$= \int_0^\infty z v_{t_1}(z) dm(z).$$

Step 3. In step 1, we have established the equation (2.3.8) a.s. t_1 . Now let us prove it for all t_1 .

Suppose (t_1, t_2) such that

$$v_{t_1+t_2}(y) > \int_0^\infty v_{t_1}(z) p(z, t_2, y) dm(z). \quad (2.3.12)$$

$$\text{Then, } \int_0^\infty y v_{t_1+t_2}(y) dm(y) > \int_0^\infty y dm(y) \int_0^\infty v_{t_1}(z) p(z, t_2, y) dm(z).$$

From step 2 we get

$$\begin{aligned} & \int_0^\infty y dm(y) \int_0^\infty v_{t_1}(z) p(z, t_2, y) dm(z) \\ &= \int_0^\infty v_{t_1}(z) \left[\int_0^\infty y p(z, t_2, y) dm(y) \right] dm(z) \\ &= \int_0^\infty z v_{t_1}(z) dm(z). \end{aligned}$$

The last equality follows from (2.3.11). Combining we obtain,

$$\int_0^\infty y v_{t_1+t_2} dm(y) > \int_0^\infty z v_{t_1}(z) dm(z). \quad (2.3.13)$$

Let $\delta > 0$.

Then from equation (2.3.9) we obtain, following the same steps as we did to conclude (2.3.13) from (2.3.12),

$$\int_0^\infty y v_{t_1+t_2+\delta}(y) dm(y) \geq \int_0^\infty z v_{t_1+t_2}(z) dm(z). \quad (2.3.14)$$

So the inequalities (2.3.13) and (2.3.14) together give, for $\delta > 0$

$$\int_0^{\infty} y v_{t_1+t_2+\delta}(y) dm(y) > \int_0^{\infty} z v_{t_1}(z) dm(z).$$

Similarly, for $0 < \epsilon < t_1$ we get

$$\int_0^{\infty} y v_{t_1+t_2+\delta}(y) dm(y) > \int_0^{\infty} z v_{\epsilon}(z) dm(z). \quad (2.3.15)$$

Now we show that if (u, r) is such that $u > t_1+t_2 > t_1 > r$

$$\text{and} \quad \int_0^{\infty} y v_u(y) dm(y) > \int_0^{\infty} y v_r(y) dm(y) \quad (2.3.16)$$

then one must have

$$v_u(y) > \int_0^{\infty} v_r(z) p(z, u-r, y) dm(z). \quad (2.3.17)$$

An equality in (2.3.17) will contradict (2.3.16).

From (2.3.15) then we obtain

$$v_{t_1+t_2+\delta}(y) > \int_0^{\infty} v_{\epsilon}(z) p(z, t_1+t_2+\delta-\epsilon, y) dm(z)$$

for $\delta > 0$ and $0 < \epsilon < t_1$. This clearly contradicts the conclusion of the step 1, that is, equation (2.3.11).

Hence, there does not exist any (t_1, t_2) satisfying (2.3.12).

Thus the equation (2.3.11) holds for all t_1 .

In the next theorem we show that the function v_t has one of the properties of a Lévy measure namely,

$$\int_0^{\infty} y v_t(y) dm(y) = 1.$$

This equality will allow us to prove in the chapter III the stochastic continuity of the Brownian measure process.

Theorem 2.3.9

$$(A) \quad \lim_{\xi \rightarrow 0} \lim_{x \rightarrow 0} \int_0^{\xi} z \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z) = 0.$$

$$(B) \quad \int_0^{\infty} z v_{t_1}(z) dm(z) = 1.$$

Proof:

(A) We have already established in theorem 2.3.6 that

$$v_{t_1+t_2}(y) = \int_0^{\infty} v_{t_1}(z) p(z, t_2, y) dm(z) \quad (2.3.18)$$

Let us calculate $v_{t_1+t_2}(y)$ in a different way to obtain the desired result.

Choose $\xi \in (0, \infty)$ close to 0.

$$\begin{aligned} p(x, t_1+t_2, y) &= \int_0^{\infty} p(x, t_1, z) p(z, t_2, y) dm(z) \\ &= \int_0^{\xi} p(x, t_1, z) p(z, t_2, y) dm(z) + \int_{\xi}^{\infty} p(x, t_1, z) p(z, t_2, y) dm(z) \end{aligned} \quad (2.3.19)$$

But remembering prop. 2.3.2 we write for $x < \xi < z$

$$\begin{aligned}
 & \int_{\xi}^{\infty} \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z) \\
 &= \int_{\xi}^{\infty} \left\{ \int_0^{t_1} \frac{\phi(x, \sigma, \xi)}{x} \frac{\partial}{\partial t} p(\xi, t_1 - \sigma, z) d\sigma \right\} p(z, t_2, y) dm(z) \\
 &= \int_{\xi}^{\infty} \left\{ \int_0^{t_1} \frac{\phi(x, \sigma, \xi)}{x} \left[\frac{\partial}{\partial t} p(\xi, t_1 - \sigma, z) \right]_+ d\sigma - \int_0^{t_1} \frac{\phi(x, \sigma, \xi)}{x} \right. \\
 &\quad \left. \left[\frac{\partial}{\partial t} p(\xi, t_1 - \sigma, z) \right]_- d\sigma \right\} p(z, t_2, y) dm(z)
 \end{aligned}$$

where the subscript $+$ ($-$) denotes the positive (negative) part of $\frac{\partial}{\partial t} p(\xi, t_1 - \sigma, z)$ as a function of σ .

$$\text{So put } G_{\pm}(x, z) = \int_0^{t_1} \frac{\phi(x, \sigma, \xi)}{x} \left[\frac{\partial}{\partial t} p(\xi, t_1 - \sigma, z) \right]_{\pm} d\sigma$$

Remembering that $\frac{\phi(x, \sigma, \xi)}{x}$ decreases as x decreases to zero, we may conclude that $G_{\pm}(x, z)$ decreases as x decreases to zero.

Hence

$$\lim_{x \rightarrow 0} \int_{\xi}^{\infty} G_{\pm}(x, z) p(z, t_2, y) dm(z) = \int_{\xi}^{\infty} \lim_{x \rightarrow 0} G_{\pm}(x, z) p(z, t_2, y) dm(z).$$

Thus,

$$\lim_{x \rightarrow 0} \int_{\xi}^{\infty} \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z) = \int_{\xi}^{\infty} \lim_{x \rightarrow 0} \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z)$$

So going back to equation (2.3.19) we have

$$\lim_{\xi \rightarrow 0} \lim_{x \rightarrow 0} \frac{p(x, t_1 + t_2, y)}{x} = \lim_{\xi \rightarrow 0} \lim_{x \rightarrow 0} \int_0^{\xi} \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z) \\ + \lim_{\xi \rightarrow 0} \int_{\xi}^{\infty} \lim_{x \rightarrow 0} \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z)$$

or,

$$v_{t_1+t_2}(y) = \lim_{\xi \rightarrow 0} \lim_{x \rightarrow 0} \int_0^{\xi} \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z) \\ + \int_0^{\infty} v_{t_1}(z) p(z, t_2, y) dm(z)$$

Comparing with eqn. 2.3.18 we obtain

$$\lim_{\xi \rightarrow 0} \lim_{x \rightarrow 0} \int_0^{\xi} \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z) = 0$$

(B) To prove (B), first we need an auxiliary result.

From Fatou's lemma

$$\lim_{\xi \rightarrow 0} \int_0^{\xi} v_{t_1}(z) p(z, t_2, y) dm(z) \leq \lim_{\xi \rightarrow 0} \lim_{x \rightarrow 0} \int_0^{\xi} \frac{p(x, t_1, z)}{x} p(z, t_2, y) dm(z) \\ = 0$$

Using positivity we get

$$\lim_{\xi \rightarrow 0} \int_0^{\xi} v_{t_1}(z) p(z, t_2, y) dm(z) = 0 \quad (2.3.20)$$

But $p(z, t_2, y)$ is infinitely differentiable, [see §2.2].

$$\text{Then } p(z, t_2, y) = p(0, t_2, y) + z \frac{\partial}{\partial z} p(z, t_2, y) \Big|_{z=z^*}, \quad 0 < z^* < z$$

$$= z \frac{\partial}{\partial z} p(z, t_2, y) \Big|_{z=z^*}$$

$p(0, t_2, y) = 0$ since $\{0\}$ is accessible, absorbing (see Table 2.2.1).

Thus,

$$\lim_{\xi \rightarrow 0} \int_0^{\xi} z v_{t_1}(z) \frac{\partial}{\partial z} p(z, t_2, y) \Big|_{z=z^*} dm(z) = 0. \quad (2.3.21)$$

Now we claim that for z near zero, $\frac{\partial}{\partial z} p(z, t_2, y)$ is, as a function of z , bounded below by a positive quantity.

Since $v_{t_2}(y) \equiv \lim_{z \rightarrow 0} \frac{\partial}{\partial z} p(z, t_2, y)$ is strictly positive, one can extend the domain of definition of $\frac{\partial}{\partial z} p(z, t_2, y)$ by continuity to include zero. From continuity, there exists an ϵ such that $\frac{\partial}{\partial z} p(z, t_2, y)$ is strictly positive on $[0, \epsilon]$. Define

$$M_1(t_2, y, \epsilon) = \inf_{z \in [0, \epsilon]} \frac{\partial}{\partial z} p(z, t_2, y).$$

Then

$$M_1(t_2, y, \epsilon) > 0. \quad (2.3.22)$$

Consequently from equation (2.3.21) we have for $\xi < \epsilon$ and $\delta > 0$

$$M_1(t_2, y, \epsilon) \int_0^{\xi} z v_{t_1}(z) dm(z) < \delta.$$

Remembering inequality (2.3.22) we obtain

$$\lim_{\xi \rightarrow 0} \int_0^{\xi} z v_{t_1}(z) dm(z) \leq 0.$$

From positivity of the integrand we conclude

$$\lim_{\xi \rightarrow 0} \int_0^{\xi} z v_{t_1}(z) dm(z) = 0.$$

Now,

$$\begin{aligned} (P) \quad 1 &= \int_0^{\infty} y \frac{P(x, t, y)}{x} dm(y) \\ &= \int_0^{\xi} y \frac{P(x, t, y)}{x} dm(y) + \int_{\xi}^{\infty} y \frac{P(x, t, y)}{x} dm(y) . \end{aligned}$$

Now using the representation given by prop. 2.3.2 we get as in part (A) that

$$\lim_{\xi \rightarrow 0} \lim_{x \rightarrow 0} \int_{\xi}^{\infty} y \frac{P(x, t, y)}{x} dm(y) = \int_0^{\infty} y v_t(y) dm(y)$$

From part (A) we know $\lim_{\xi \rightarrow 0} \lim_{x \rightarrow 0} \int_0^{\xi} y \frac{P(x, t, y)}{x} dm(y) = 0$.

So we obtain the conclusion (B).

Chapter III. Brownian Measure Processes: Existence

§3.0 Introduction

Now we proceed to show the existence of Brownian measure processes. Our construction will be based on prop. 1.4.3. That is, we will produce a transition function on $M_+(S)$. Moreover, the nature of the initial measure will determine the method of construction.

If the initial point is a non-atomic measure, then the time evolution will be determined by a diffusion on $(0, \infty)$ with a regular absorbing or exit boundary at the origin. In this case, the completely random measures will have only the random atomic component. [defn. 1.3.6].

On the other hand, atomic initial point does not put any restriction on the nature of the underlying diffusion. But the completely random measures, in this case, will have only fixed atoms at the locations determined by the initial measure.

§3.1 Atomic initial condition

Assumption I: 3.1.1

Suppose $(\Omega, \mathcal{F}, P^*)$ is a given probability space on which lives a family (indexed by $x \in S$) of independent diffusions in scale, $\{Z_x(t) : t \in T\}$ which are also martingales (in $t \in T$).
 $x \in S$

For each $x \in S$, the state space of the process $\{Z_x(t) : t \in T\}$ is $[0, \infty)$. This assumption will be in force throughout this chapter.

Notation 3.1.2

To specify a diffusion $Z_x(t)$ that starts from $a \in (0, \infty)$ we write $Z_x(t; a)$. Moreover, for $A \in \mathcal{B}(R_+^1)$

$$P^*[Z_x(t; a) \in A] = \int_A p(a, t, y) \, dm(y).$$

That is, the transition probability and the speed measure do not depend upon location $x \in S$.

Assumption II: 3.1.3

In the construction of a Brownian measure process with atomic initial condition, we do not appeal to the existence of non-degenerate entrance laws. Consequently, we assume nothing about the nature of boundaries. This particular assumption will be in force throughout this section.

Let μ be an atomic Radon measure [see notation 1.1.14],
 i.e., $\mu \in \mathcal{A}$ with the representation:

$$\mu = \sum_{j=1}^{\infty} a_j \delta_{x_j}, \quad a_j \geq 0 \text{ and } x_j \in S$$

such that for every compact $K \subset S$

$$\langle \mu, 1_K \rangle = \sum_{j=1}^{\infty} a_j 1_K(x_j) < \infty \quad (3.1.1)$$

Or, equivalently, for every $f \in C_c(S)$,

$$\langle \mu, f \rangle = \sum_{j=1}^{\infty} a_j f(x_j) < \infty$$

Define
$$X_t^{\mu}(\omega) \equiv \sum_{j=1}^{\infty} Z_{x_j}(\omega, t; a_j) \delta_{x_j} \quad (3.1.2)$$

In the next two theorems, we establish that for each $t \in T$, X_t^{μ} is a completely random measure on $(\Omega, \mathcal{F}, P^*)$ with fixed atomic component only.

Theorem 3.1.4

(a) For compact $K \subset S$,

$$\sup_{t \in T} \langle X_t^{\mu}(\omega), 1_K \rangle < \infty \text{ a.s. } P^*$$

(b) For each $f \in C_c(S)$

$$\sup_{t \in T} \langle X_t^{\mu}(\omega), f \rangle < \infty \text{ a.s. } P^*$$

Proof:

Step 1: We show that $\sum_{j=1}^{\infty} Z_{x_j}(t; a_j)$ is a positive martingale with respect to the natural σ -fields.

$$\begin{aligned}
& E\left[\sum_{j=1}^{\infty} Z_{x_j}(t_1+t_2; a_j) \mid \sigma\{Z_{x_l}(u; a_l) : l \in Z^+, u \leq t_1\}\right] \\
&= \sum_{j=1}^{\infty} E[Z_{x_j}(t_1+t_2; a_j) \mid \sigma\{Z_{x_l}(u; a_l) : l \in Z^+, u \leq t_1\}] \quad \left\{ \begin{array}{l} \text{positivity and} \\ \text{continuity} \end{array} \right. \\
&= \sum_{j=1}^{\infty} E[Z_{x_j}(t_1+t_2; a_j) \mid \sigma\{Z_{x_j}(u; a_j) : u \leq t_1\}] \quad (\text{independence}) \\
&= \sum_{j=1}^{\infty} Z_{x_j}(t_1; a_j), \text{ as required.}
\end{aligned}$$

Remembering that Z_x is a positive martingale we get for compact $K \subset S$,

$$E[\langle X_t^H(\omega), 1_K \rangle] = \sum_{j=1}^{\infty} a_j 1_K(x_j) < \infty.$$

That is, for each compact $K \subset S$, $\{\langle X_t^H, 1_K \rangle : t \in T\}$ is an $L_1(P^*)$ -martingale.

Step 2: Using positivity we establish now the conclusion (a).

$$\text{Set } Y_t^n(\omega) = \sum_{j=1}^n Z_{x_j}(\omega, t; a_j) 1_K(x_j).$$

Since Z 's are positive, $Y_t^n(\omega)$ is increasing in n .

Moreover Z 's are martingales with continuous sample paths. Thus

$$\langle X_t(\omega), 1_K \rangle = \sup_n Y_t^n(\omega).$$

So the paths of the process $\langle X_t(\omega), 1_K \rangle$ are then ass. right continuous and free of oscillatory discontinuities. [See Meyer 12; p.99 T 16]. That is, $\langle X_t(\omega), 1_K \rangle$ is a separable martingale.

Using a theorem of Neveu [15; p.133 prop. IV.5.2] we may conclude

$$\sup_{t \in T} \langle X_t^\mu(\omega), 1_K \rangle < \infty \quad \text{a.s.}$$

Step 3: Similar arguments establish (b).

Theorem 3.1.5.

For each $t \in T$, X_t^μ is a completely random measure on $(\Omega, \mathcal{F}, P^*)$ with fixed atomic component only.

Proof:

Step 1: For $\omega \in \Omega$, $X_t^\mu(\omega)$ is a measure on S .

Because of prop. 1.1.6, it suffices to establish that $X_t^\mu(\omega)$ is positive and linear on $C_c(S)$. Linearity follows from the theorem 3.1.4 and the definition of X_t^μ . Positivity follows from (3.1.2).

Step 2: For $t \in T$, the map

$$X_t^\mu : \Omega \rightarrow M_+(S)$$

is measurable.

Note that for $f \in C_c(S)$, $\langle X_t^\mu(\omega), f \rangle : \Omega \rightarrow \mathbb{R}^1$ is measurable.

Remember $Z_x(\omega)$ is a real-valued random variable. Now use prop. 1.1.9.

Step 3: For $t \in T$, X_t^μ is a complete random measure.

Take $f, g \in C_c(S)$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$. Then $\langle X_t^\mu, f \rangle$ and $\langle X_t^\mu, g \rangle$ are independent, since $\{Z_x : x \in S\}$ is an independent family.

Step 4: X_t^μ has fixed atomic component.

This is obvious from equation (3.1.2).

Remark 3.1.6

Thus, for each $\mu \in \Omega$ we have obtained a family of completely random measures $\{X_t^\mu : t \in T\}$. Denote by Q_t^μ the law of X_t^μ .

Remembering that A is a measurable subset of $M_+(S)$ [theorem 1.1.6] we set $Q_t^\mu \equiv 0$ for $\mu \in N.A.$ (See notation 1.1.14).

Let us calculate the characteristic functional of X_t^μ .

$$\begin{aligned} E[e^{i\langle X_t^\mu, f \rangle}] &= \prod_{j=1}^{\infty} E[e^{if(x_j)Z_{x_j}(t; a_j)}] \\ &= \prod_{j=1}^{\infty} \phi_t(f(x_j), a_j) = \exp \left[\sum_{j=1}^{\infty} \log \phi_t(f(x_j), a_j) \right] \end{aligned} \quad (3.1.3)$$

where $\phi_t(f(x_j), a_j)$

$$= E[e^{if(x_j)Z_{x_j}(t; a_j)}]$$

$$= \begin{cases} \int_0^{\infty} e^{if(x_j)y_j} p(a_j, t, y_j) dm(y_j), & \{0\} \text{ is inaccessible} \\ \int_0^{\infty} (e^{if(x_j)y_j} - 1) p(a_j, t, y_j) dm(y_j) + 1, & \{0\} \text{ is accessible, absorbing} \end{cases} \quad (3.1.4)$$

Next we establish the Chapman-Kolmogorov condition for the family $\{Q_t^\mu\}$.

Theorem 3.1.7

$$\int_{M_+(S)} e^{i\langle \eta, f \rangle} \int_{M_+(S)} Q_{t_1}^\mu(d\lambda) Q_{t_2}^\lambda(d\eta) = \int_{M_+(S)} e^{i\langle \eta, f \rangle} Q_{t_1+t_2}^\mu(d\eta) \quad \text{for } f \in \mathcal{C}_c(S).$$

Proof:

Assume μ has the representation $\sum_{j=1}^{\infty} a_j \delta_{x_j}$

Case I: $\{0\}$ is an inaccessible boundary.

Then, $M_+(S) e^{i\langle \eta, f \rangle} \int_{M_+(S)} Q_{t_1}^{\mu}(d\lambda) \cdot Q_{t_2}^{\lambda}(dn)$

$$= \int_{M_+(S)} Q_{t_1}^{\mu}(d\lambda) \int_{M_+(S)} e^{i\langle \eta, f \rangle} Q_{t_2}^{\lambda}(dn)$$

$$= E[\exp \{i \sum_{j=1}^{\infty} (-i) \log \phi_{t_2}[f(x_j), Z_{x_j}(t_1; a_j)]\}]$$

$$= \prod_{j=1}^{\infty} E\{\phi_{t_2}[f(x_j), Z_{x_j}(t_1; a_j)]\} \text{ (independence of } Z_x) \text{ (3.1.5)}$$

$$= \prod_{j=1}^{\infty} \left[\int_0^{\infty} \phi_{t_2}(f(x_j), b_j) p(a_j, t_1, b_j) dm(b_j) \right]$$

$$= \prod_{j=1}^{\infty} \left\{ \int_0^{\infty} e^{if(x_j)y_j} p(b_j, t_2, y_j) dm(y_j) \right\} p(a_j, t_1, b_j) dm(b_j) \\ \text{(definition of } \phi_{t_2}(f(x_j), b_j) \text{ see (3.1.4))}$$

$$= \prod_{j=1}^{\infty} \left\{ \int_0^{\infty} e^{-f(x_j)y_j} dm(y_j) \int_0^{\infty} p(a_j, t_1, b_j) p(b_j, t_2, y_j) dm(b_j) \right\}$$

$$= \prod_{j=1}^{\infty} \left\{ \int_0^{\infty} e^{if(x_j)y_j} p(a_j, t_1+t_2, y_j) dm(y_j) \right\}$$

$$= \prod_{j=1}^{\infty} \phi_{t_1+t_2}(f(x_j), a_j)$$

$$= \int_{M_+(S)} e^{i\langle \eta, f \rangle} Q_{t_1+t_2}^{\mu}(dn)$$

To justify the above interchange proceed as follows:

$$\begin{aligned}
 & \int_0^\infty \left[\int_0^\infty e^{if(x_j)y_j} p(b_j, t_2, y_j) dm(y_j) \right] p(a_j, t_1, b_j) dm(b_j) \\
 &= \int_0^\infty \left[\lim_{\substack{\delta_1 \rightarrow 0 \\ \delta_2 \rightarrow \infty}} \int_{\delta_1}^{\delta_2} e^{if(x_j)y_j} p(b_j, t_2, y_j) dm(y_j) \right] p(a_j, t_1, b_j) dm(b_j) \\
 &= \lim_{\substack{\delta_1 \rightarrow 0, 0 \\ \delta_2 \rightarrow \infty}} \int_{\delta_1}^{\delta_2} \int_0^\infty e^{if(x_j)y_j} p(b_j, t_2, y_j) dm(y_j) p(a_j, t_1, b_j) dm(b_j) \quad (\text{dominated convergence}) \\
 &= \lim_{\substack{\delta_1 \rightarrow 0 \\ \delta_2 \rightarrow \infty}} \int_{\delta_1}^{\delta_2} e^{if(x_j)y_j} dm(y_j) \int_0^\infty p(b_j, t_2, y_j) p(a_j, t_1, b_j) dm(b_j) \quad (\text{Tonelli}) \\
 &= \lim_{\substack{\delta_1 \rightarrow 0 \\ \delta_2 \rightarrow \infty}} \int_{\delta_1}^{\delta_2} e^{if(x_j)y_j} dm(y_j) p(a_j, t_1 + t_2, y_j) \\
 &= \int_0^\infty e^{if(x_j)y_j} p(a_j, t_1 + t_2, y_j) dm(y_j)
 \end{aligned}$$

Case II: $\{0\}$ is an accessible, absorbing boundary

The difference between these two cases lies in the form of characteristic function of Z_x . So let us start from equation (3.1.5).

$$\begin{aligned}
 & \int_{M_+(S)} e^{i\langle \eta, f \rangle} \int_{M_+(S)} Q_{t_1}^\mu(d\lambda) Q_{t_2}^\lambda(d\eta) \\
 &= \prod_{j=1}^\infty E\{e^{i\langle f(x_j), Z_{x_j}(t_1; a_j) \rangle}\} \\
 &= \prod_{j=1}^\infty \left[\int \phi_{t_2}(f(x_j), b_j) p(a_j, t_1, b_j) dm(b_j) + \phi_{t_2}(f(x_j), 0) \right. \\
 &\quad \left. (1 - \int_0^\infty p(a_j, t_1, b_j) dm(b_j)) \right] \quad (\text{but } \phi_{t_2}^\leftarrow(f(x_j), 0) = 1, \{0\} \text{ is absorbing.})
 \end{aligned}$$

2 OF/DE 2

OF/DE

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

$$\begin{aligned}
&= \prod_{j=1}^{\infty} \left\{ 1 + \int_0^{\infty} [\phi_{t_2}(f(x_j), b_j) - 1] p(a_j, t_1, b_j) dm(b_j) \right\} \\
&= \prod_{j=1}^{\infty} \left\{ 1 + \int_0^{\infty} \left[\int_0^{\infty} (e^{if(x_j)y_j} - 1) p(b_j, t_2, y_j) dm(y_j) + 1 - 1 \right] p(a_j, t_1, b_j) dm(b_j) \right\} \\
&= \prod_{j=1}^{\infty} \left\{ 1 + \int_0^{\infty} (e^{if(x_j)y_j} - 1) \left[\int_0^{\infty} p(a_j, t_1, b_j) p(b_j, t_2, y_j) dm(b_j) \right] dm(y_j) \right\} \\
&\quad \text{[justifications as in case (I)]} \\
&= \prod_{j=1}^{\infty} \left\{ 1 + \int_0^{\infty} (e^{if(x_j)y_j} - 1) p(a_j, t_1+t_2, y_j) dm(y_j) \right\} \\
&= \prod_{j=1}^{\infty} \phi_{t_1+t_2}(f(x_j), a_j) \\
&= \int_{m_+(s)} e^{i\langle n, f \rangle} Q_{t_1+t_2}^u(dn) .
\end{aligned}$$

Theorem 3.1.8

For $\Gamma \in B[M_+(S)]$,

$$\int_{m_+(S)} Q_{t_1}^u(d\lambda) Q_{t_2}^\lambda(\Gamma) = Q_{t_1+t_2}^u(\Gamma) .$$

Proof:

It follows from the uniqueness of characteristic functional of random measures and theorem 3.1.7.

Theorem 3.1.9

For $\mu \in A$, $\mu \rightarrow E[e^{i\theta \langle X_t^u, 1 \rangle_{B_j}}]$

is measurable for $B \in B(S)$ and $\theta \in \mathbb{R}^1$.

Proof: It suffices to establish the required measurability for K compact in S .

Suppose $\mu = \sum_{j=1}^{\infty} a_j \delta_{x_j}$, then from equation (3.1.3) we have

$$E[e^{i\theta \langle X_t^\mu, 1_K \rangle}] = \prod_{j=1}^{\infty} \phi_t(1_K(x_j), a_j)$$

$$= \lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} \prod_{l=1}^{m(n)} \int_0^{\infty} e^{i u_j} p(\mu(\Delta_l^n), t, y_j) dm(y_j)$$

$\mu(\Delta_l^n) \leq M^{-1}$

[Using the notations of thm.1.1.16]

$$\text{But } \mu \rightarrow \mu(\Delta_l^n) \rightarrow \int_0^{\infty} e^{i y_j} p(\mu(\Delta_l^n), t, y_j) dm(y_j)$$

is a measurable map. The first map is measurable follows from the definition of $B[M_+(S)]$. The second map is continuous.

Similar arguments establish the required measurability if $\{Q\}$ is an accessible absorbing boundary.

Theorem 3.1.10

$$\text{If } \mu \rightarrow E[e^{i\theta \langle X_t, 1_B \rangle}]$$

is measurable for every $B \in \mathcal{B}(S)$, then

$$\mu \rightarrow Q_t^\mu(\Gamma)$$

is measurable for every $\Gamma \in B[M_+(S)]$.

Proof: Let $B \in \mathcal{B}(S)$ and $H = (a-h, a+h) \subset \mathbb{R}^1$ and put

$$\Gamma_o^H = \{ \eta : \eta \in M_+(S), \langle \eta, 1_B \rangle \in H \}$$

Then

$$\begin{aligned} Q_t^\mu(\Gamma_o^H) &= P^* \{ X_t^\mu \in \Gamma_o^H \} = P^* \{ \langle X_t^\mu, 1_B \rangle \in H \} \\ &= \lim_{N \rightarrow \infty} \int_{-N}^N \frac{\sin h\theta}{\theta} e^{-ia\theta} E[e^{i\theta \langle X_t^\mu, 1_B \rangle}] d\theta \end{aligned}$$

Then $\mu \rightarrow Q_t^\mu(\Gamma_o^H)$ is measurable.

Let us establish now

$\mathcal{B} \equiv \{ H : H \subset \mathbb{R}^1, \mu \rightarrow Q_t^\mu(\Gamma_o^H) \text{ is measurable} \}$ contains the σ -field of Borel sets in $\mathbb{R}^1$.

(i) $\mathbb{R}^1 \in \mathcal{B}$

(ii) We have already seen that the intervals are in $\mathcal{B}$.

(iii) So suppose $H_j \in \mathcal{B}$, $H_j \subset H_{j+1}$; $j = 1, 2, \dots$

Then $\{ \langle X_t^\mu, 1_B \rangle \in H_j \} \subseteq \{ \langle X_t^\mu, 1_B \rangle \in H_{j+1} \}$.

Thus, $Q_t^\mu(\Gamma_o^{\bigcup_j H_j}) = \lim_{j \rightarrow \infty} Q_t^\mu(\Gamma_o^{H_j})$. Hence $\bigcup_j H_j \in \mathcal{B}$.

(iv) $H_1, H_2 \in \mathcal{B}$, $H_1 \subset H_2$

$$Q_t^\mu(\Gamma_o^{H_2 - H_1}) = Q_t^\mu(\Gamma_o^{H_2}) - Q_t^\mu(\Gamma_o^{H_1}). \text{ Then } H_2 - H_1 \in \mathcal{B}.$$

(v) finite disjoint union.

Consider $H_j \in \mathcal{B}$ $j = 1, \dots, n$.

Then $Q_t^\mu(\bigcup_{j=1}^n H_j) = \sum_{j=1}^n Q_t^\mu(H_j)$. Then $\bigcup_{j=1}^n H_j \in \mathcal{B}$.

$$\{ \langle X_t^\mu, 1_B \rangle \in H_l \} \cap \{ \langle X_t^\mu, 1_B \rangle \in H_m \} = \emptyset, \quad m \neq n.$$

Thus $\mathcal{B}$ contains the Borel field $\mathcal{B}(R^1)$.

Now we show that

$$\tilde{\mathcal{B}} = \{ \Gamma : \Gamma \subset M_+(S), \mu \mapsto Q_t^\mu(\Gamma) \text{ is measurable} \}$$

contains the Borel field $\mathcal{B}[M_+(S)]$.

We already know that the sets of the form

$$\Gamma = \{ \eta : \langle \eta, 1_B \rangle \in H \}, \text{ where } B \in \mathcal{B}(S), H \in \mathcal{B}(R^1)$$

are in $\tilde{\mathcal{B}}$.

The proof follows exactly the same steps as before.

Thus we have shown that for all $\Gamma \in \mathcal{B}[M_+(S)]$

$$\mu \mapsto Q_t^\mu(\Gamma)$$

is measurable.

Theorem 3.1.11

For every atomic $\mu \in M_+(S)$, there exists a probability measure Q_* on $(M_+(S)^T, \mathcal{B}[M_+(S)^T])$.

Proof: It suffices to satisfy the conditions of prop. 1.4.3.

But we have exactly done that in theorems 3.1.8, 3.1.9, and 3.1.10.

3.2 Non-atomic initial condition:

Let the initial point be μ , a non-atomic measure. In this case, contrary to the previous one, one cannot introduce the time evolution in any obvious way. So we construct a sequence of completely random measures of random atomic type $\{X_0^\mu(n)\}_{n=1}^\infty$ converging weakly in $M^1[M_+(S)]$ to μ . Now that $\{X_0^\mu(n)\}_{n=1}^\infty$ are of random atomic type, one may introduce the time evolution at each atom via a diffusion to obtain again a sequence of completely random measures $\{X_t^\mu(n)\}_{n=1}^\infty$. The weak limit in $M^1[M_+(S)]$ (as $n \rightarrow \infty$) then gives a completely random measure X_t . The relationship between μ and X_t can be described as follows:

$$\begin{array}{ccc} X_0^\mu(n) & \xrightarrow{\quad} & X_t^\mu(n) \\ \uparrow & & \downarrow \\ \mu & & X_t \end{array}$$

Theorem 3.2.1

Suppose $\mu \in M_+(S)$ is non-atomic. Then there exists a sequence of completely random measures $\{X_0^\mu(n)\}_{n=1}^\infty$ converging weakly in $M^1[M_+(S)]$ (hence in probability) to μ .

Proof:

Denote by X_0^μ the Poisson random measure with intensity $n\mu$.

We know the existence of a probability space (Ω_1, A, P^*) such that

$$X_0^\mu : \Omega_1 \rightarrow M_+(S)$$

is measurable. From definition 1.3.2 we actually have, (Ω_1, A) is the canonical space $(M_+(S), B[M_+(S)])$ and P^* is the law of X_0^μ on $M_+(S)$.

For $\omega_1 \in \Omega_1$, define $A_n(\omega_1) = \{x \in S : X_0^\mu(\omega_1, \{x\}) = 1\}$.

From the property of Poisson random measure (step 1 of prop. 1.3.5),

$A_n(\omega_1)$ is almost surely countable and for any compact $K \subset S$,

$A_n(\omega_1) \cap K$ is almost surely finite.

So the random measure X_0^μ can be represented as

$$X_0^\mu(\omega_1) = \sum_{x \in A_n(\omega_1)} \delta_x.$$

By putting a mass of $\frac{1}{n}$ at the location of each jump of every sample path of X_0^μ , we obtain another completely random measure, say,

$X_0^\mu(n)$. We can write X_0^μ as $\frac{1}{n} X_0^\mu(n)$ or in terms of ω_1 as follows

$$X_0^\mu(n) = \sum_{x \in A_n(\omega_1)} \frac{1}{n} \delta_x \quad (3.2.1)$$

where $A_n(\omega_1)$ denotes the random support of X_0^μ .

Let us calculate the characteristic functional of $X_0^\mu(n)$. Take f

to be $\sum_{\ell=1}^m a_\ell I_{K_\ell}$ where $\{K_\ell : \ell=1, \dots, m\}$ are disjoint relatively compact subsets of S .

$$\begin{aligned} E[\exp i \langle X_0^\mu(n), f \rangle] &= \prod_{\ell=1}^m E[\exp i a_\ell \langle X_0^\mu(n), I_{K_\ell} \rangle] \\ &= \prod_{\ell=1}^m \sum_{j=0}^{\infty} \exp(i a_\ell \frac{1}{n}) \frac{[\mu(K_\ell)n]^j}{j!} \exp(-\mu(K_\ell)n) \end{aligned}$$

$$\begin{aligned}
&= \prod_{l=1}^m \exp(-\mu(K_l)n) \exp[\mu(K_l)n \exp \frac{ia_l}{n}] \\
&= \prod_{l=1}^m \exp[\mu(K_l)n \{\exp \frac{ia_l}{n} - 1\}] \\
&= \exp \left[\int_S \mu(dx) n \left\{ \exp \frac{if(x)}{n} - 1 \right\} \right].
\end{aligned}$$

Now use monotone convergence to establish the above equality for $f \in C_c^+(S)$, and remembering that $X_0(n)$ is linear on $C_c(S)$ we conclude for $f \in C_c(S)$

$$E[\exp i \langle X_0^\mu(n), f \rangle] = \exp \left[\int_S \mu(dx) \left\{ \exp \frac{if(x)}{n} - 1 \right\} n \right]. \quad (3.2.2)$$

$$\begin{aligned}
\text{Then, } \lim_{n \rightarrow \infty} E[\exp i \langle X_0^\mu(n), f \rangle] &= \exp \left[\int_S i f(x) \mu(dx) \right] \\
&= \exp [i \langle \mu, f \rangle]. \quad (3.2.3)
\end{aligned}$$

The above interchange is justified by the following observation

$$n \left| \exp \frac{if(x)}{n} - 1 \right| \leq n \frac{|f(x)|}{n} = |f(x)|.$$

Theorem 3.2.2.

Let $\mu \in M_+(S)$ be non-atomic. Suppose $p(y, t, z)$ denotes the transition density function with respect to the speed measure m of a family $\{Z_x(t) : x \in S\}$ of independent diffusions on $[0, \infty)$ with an absorbing boundary at the origin. Then there exists a family of completely random measures $\{X_t^\mu : t \in T\}$ with characteristic functional

$$L_t^\mu(f) = \exp \left[\int_S \mu(dx) \int_0^\infty (e^{if(x)z} - 1) v_t(z) dm(z) \right] \quad (3.2.4)$$

where $f \in C_c(S)$ and $v_t(z) \equiv \lim_{y \rightarrow 0} \frac{p(y, t, z)}{y}$.

Remark 3.2.3

The restriction on the nature of boundary at $\{0\}$ is needed, since $v_t(z)$ is degenerate for entrance or natural boundaries at $\{0\}$. See table 2.2.1 and lemma 2.3.3(ii).

Proof:

Note that the sequence of completely random measures $\{X_0^\mu(n)\}_{n=1}^\infty$ consists of random jump types of measures. One can introduce time evolution for such measures.

So define

$$X_t^\mu(\underline{\omega}, n) = \sum_{x \in A_n(\omega_1)} Z_x(\omega_2, t; \frac{1}{n}) \delta_x \quad (3.2.5)$$

where $\underline{\omega} = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$.

$A_n(\omega_1)$ is the atmost countable support of $X_0^\mu(\omega_1)$, where X_0^μ is the Poisson process with intensity $\mu(\cdot)$.

$\{Z_x(\omega_2, t) : x \in S\}$ is a family of independent diffusions on Ω_2 - of the type indicated in the statement of the theorem. As before $Z_x(\omega_2, t; a)$ with $a \in \mathbb{R}_+^1$ denotes the diffusion such that $Z_x(\omega_2, 0) = a$.

We shall assume that $\{Z_x : x \in S\}$ is independent of the Poisson random measure X_0^μ .

Hence for each $n \in \mathbb{Z}^+$, $X_t^\mu(\omega, n)$ is a family of completely random measures indexed by $t \in T$. The proof of this fact follows the proof of theorem 3.1.5, and hence shall be omitted.

Step 1. We have a sequence of completely random measures $\{X_t^\mu(\cdot, n)\}_{n=1}^\infty$. Let us proceed to calculate the characteristic functional of $X_t^\mu(\cdot, n)$.

Take f to be $\sum_{l=1}^m a_l 1_{K_l}$ with disjoint relatively compact K_l , $l = 1, \dots, m$.

$$E[\exp i \langle X_t^\mu(n), f \rangle] = \prod_{l=1}^m E[\exp i a_l \langle X_t^\mu(n), 1_{K_l} \rangle] \quad (\text{independence})$$

$$\text{But, } E[\exp i a_l \langle X_t^\mu(n), 1_{K_l} \rangle]$$

$$= \sum_{j=0}^{\infty} E[\exp i a_l \langle X_t^\mu(n), 1_{K_l} \rangle \mid |A(\omega_1) \cap K_l| = j] \cdot P^*[|A(\omega_1) \cap K_l| = j]$$

$$= \sum_{j=0}^{\infty} E[(\exp\{i a_l Z(t; \frac{1}{n})\})^j [n\mu(K_l)]^j \frac{e^{-n\mu(K_l)}}{j!}]$$

If $|A(\omega_1) \cap K_l| = j$, then $X_0^\mu(n)$ has j atoms in K_l . But the diffusions are independent amongst themselves and the spatial parameter, and thus the last equality.

$$\text{Put } \psi_t(a_l, \frac{1}{n}) = E[\exp i a_l Z(t; \frac{1}{n})]. \text{ Thus}$$

$$E[\exp i \langle X_t^\mu(n), f \rangle] = \prod_{l=1}^m \sum_{j=0}^{\infty} [\psi_t(a_l, \frac{1}{n})]^j [n\mu(K_l)]^j \frac{e^{-n\mu(K_l)}}{j!}$$

$$\begin{aligned}
&= \prod_{l=1}^m e^{-n\mu(K_l)} \sum_{j=0}^{\infty} \frac{[n\mu(K_l) \psi_t(a_l, \frac{1}{n})]^j}{j!} \\
&= \prod_{l=1}^m \exp[n\mu(K_l) \{\psi_t(a_l, \frac{1}{n}) - 1\}] \quad (3.2.6)
\end{aligned}$$

But $\psi_t(a_l, \frac{1}{n}) = \int_0^{\infty} (e^{ia_l y} - 1) p(\frac{1}{n}, t, y) dm(y) + 1$.

[See equation (3.1.4).]

So that from equation (3.2.6)

$$\begin{aligned}
E[\exp i \langle X_t^{\mu}(n), f \rangle] &= \prod_{l=1}^m \exp[n\mu(K_l) \int_0^{\infty} (e^{ia_l y} - 1) p(\frac{1}{n}, t, y) dm(y)] \\
&= \exp \left\{ n \sum_{l=1}^m \mu(K_l) \int_0^{\infty} (e^{ia_l y} - 1) p(\frac{1}{n}, t, y) dm(y) \right\} \\
&= \exp \left\{ \int_S \mu(dx) n \int_0^{\infty} (e^{if(x)y} - 1) p(\frac{1}{n}, t, y) dm(y) \right\}.
\end{aligned}$$

Reasoning as in the proof of the theorem 3.2.1, we conclude that the above formula for the characteristic functional holds for $f \in C_c^0(S)$.

Step 2. To show that

$$\lim_{n \rightarrow \infty} E[\exp i \langle X_t^{\mu}(n), f \rangle] = \exp \left\{ \int_S \mu(dx) \int_0^{\infty} (e^{if(x)y} - 1) \nu_t(y) dm(y) \right\}.$$

We need to justify two interchanges.

$$\begin{aligned}
1) \quad \lim_{n \rightarrow \infty} \int_S \mu(dx) n \int_0^{\infty} (e^{if(x)y} - 1) p(\frac{1}{n}, t, y) dm(y) \\
= \int_S \mu(dx) \lim_{n \rightarrow \infty} \int_0^{\infty} (e^{if(x)y} - 1) np(\frac{1}{n}, t, y) dm(y)
\end{aligned}$$

Note that,

$$\begin{aligned} & \left| n \int_0^{\infty} (e^{if(x)y} - 1) p\left(\frac{1}{n}, t, y\right) dm(y) \right| \\ & \leq n |f(x)| \int_0^{\infty} y p\left(\frac{1}{n}, t, y\right) dm(y) \\ & = |f(x)| \left[\text{using } \int_0^{\infty} y p\left(\frac{1}{n}, t, y\right) dm(y) = \frac{1}{n} \right] \end{aligned}$$

So the interchange follows from the dominated convergence theorem.

ii) To show that

$$\lim_{n \rightarrow \infty} \int_0^{\infty} (e^{if(x)y} - 1) n p\left(\frac{1}{n}, t, y\right) dm(y) = \int_0^{\infty} (e^{if(x)y} - 1) \lim_{n \rightarrow \infty} [n p\left(\frac{1}{n}, t, y\right)] dm(y).$$

Fix $\xi \in (0, \infty)$, set $z < \xi$, and write

$$\begin{aligned} \int_0^{\infty} (e^{if(x)y} - 1) \frac{p(z, t, y)}{z} dm(y) &= \int_0^{\xi} (e^{if(x)y} - 1) \frac{p(z, t, y)}{z} dm(y) \\ &+ \int_{\xi}^{\infty} (e^{if(x)y} - 1) \frac{p(z, t, y)}{z} dm(y) \quad (3.2.7) \end{aligned}$$

We proceed to show that

$$\lim_{\xi \rightarrow 0} \lim_{z \rightarrow 0} \int_0^{\xi} (e^{if(x)y} - 1) \frac{p(z, t, y)}{z} dm(y) = 0.$$

Then it suffices to show that

$$\lim_{\xi \rightarrow 0} \lim_{z \rightarrow 0} \int_0^{\xi} y \frac{p(z, t, y)}{z} dm(y) = 0.$$

But this conclusion is obtained (actually implied by) in thm.2.3.9(A).

It remains to establish

$$\lim_{\xi \rightarrow 0} \lim_{z \rightarrow 0} \int_{\xi}^{\infty} (e^{if(x)y} - 1) \frac{p(z, t, y)}{z} dm(y) = \int_0^{\infty} (e^{if(x)y} - 1) v_t(y) dm(y).$$

For $z < \xi < y$ we have the representation for $p(z, t, y)$ given by the prop. 2.3.2 i.e.,

$$p(z, t, y) = \int_0^t \phi(z, \sigma, \xi) \frac{\partial}{\partial t} p(\xi, t - \sigma, y) d\sigma.$$

So define $G_{\pm}(z, y) = \int_0^t \frac{\phi(z, \sigma, \xi)}{z} \left[\frac{\partial}{\partial t} p(\xi, t - \sigma, y) \right]_{\pm} d\sigma$

where the subscript $\pm$ denotes the positive(negative) part of $\frac{\partial}{\partial t} p(\xi, t - \sigma, y)$ considered as a function of σ .

Moreover, $e^{if(x)y} - 1 = [\cos(f(x)y) - 1] + i\{[\sin(f(x)y)]_+ - [\sin(f(x)y)]_-\}$ since $f(x)$ is a constant during this argument the subscript $\pm$ denotes the positive and negative parts of $\sin(f(x)y)$ as a function of y .

$$\begin{aligned} \text{Thus, } & \int_{\xi}^{\infty} (e^{if(x)y} - 1) \frac{p(z, t, y)}{z} dm(y) \\ &= \int_{\xi}^{\infty} [\cos(f(x)y) - 1] \frac{G_+(z, y)}{z} dm(y) - \int_{\xi}^{\infty} [\cos(f(x)y) - 1] \frac{G_-(z, y)}{z} dm(y) \\ &+ i \int_{\xi}^{\infty} [\sin(f(x)y)]_+ \frac{G_+(z, y)}{z} dm(y) - i \int_{\xi}^{\infty} [\sin(f(x)y)]_+ \frac{G_-(z, y)}{z} dm(y) \\ &- i \int_{\xi}^{\infty} [\sin(f(x)y)]_- \frac{G_+(z, y)}{z} dm(y) + i \int_{\xi}^{\infty} [\sin(f(x)y)]_- \frac{G_-(z, y)}{z} dm(y). \end{aligned}$$

Now each integrand is a monotone function of z , since $\frac{G_{\pm}(z, y)}{z}$ are, which follows from the corresponding property of

$\frac{\phi(z, \sigma, \xi)}{z}$, so we have

$$\begin{aligned}
& \lim_{\xi \rightarrow 0} \left[\lim_{z \rightarrow 0} \int_{\xi}^{\infty} (e^{if(x)y-1}) \frac{p(z,t,y)}{z} dm(y) \right] \\
&= \lim_{\xi \rightarrow 0} \int_{\xi}^{\infty} (e^{if(x)y-1}) v_t(y) dm(y) \\
&= \int_0^{\infty} (e^{if(x)y-1}) v_t(y) dm(y) \quad \text{since} \quad \int_0^{\infty} y v_t(y) dm(y) = 1.
\end{aligned}$$

Step 3. We now proceed to show that there exists for $t \in T$, a random measure, say, X_t^μ such that for $f \in C_c(S)$

$$E[\exp i \langle X_t^\mu, f \rangle] = \exp \left[\int_S \mu(dx) \int_0^{\infty} (e^{if(x)y-1}) v_t(y) dm(y) \right].$$

We have shown that the sequence of characteristic functions of the random variables $\{\langle X_t^\mu(n), f \rangle\}_{n=1}^{\infty}$, $E[\exp i \theta \cdot \langle X_t^\mu(n), f \rangle] = F_t(f, \theta, n)$ converges everywhere for $\theta \in \mathbb{R}^1$ and defines a limit function, namely,

$$F_t^\circ(f, \theta) \equiv \exp \left[\int_S \mu(dx) \int_0^{\infty} (e^{if(x)\theta z-1}) v_t(z) dm(z) \right].$$

Now we observe that $F_t(f, \theta)$ is continuous at $\theta = 0$.

$$\begin{aligned}
& \left| \int_0^{\infty} (e^{if(x)\theta_1 z-1}) v_t(z) dm(z) - \int_0^{\infty} (e^{if(x)\theta_2 z-1}) v_t(z) dm(z) \right| \\
&\leq \int_0^{\infty} z |f(x)| |\theta_1 - \theta_2| v_t(z) dm(z) \\
&\leq |\theta_1 - \theta_2| |f(x)| \quad \text{using} \quad \int_0^{\infty} z v_t(z) dm(z) = 1.
\end{aligned}$$

Thus by prop. 1.2.9, we conclude that there exists for $t \in T$, a random measure, X_t^μ , with the given characteristic functional.

Step 4. We have established that X_t^μ is a random measure.

Now we want to show that X_t^μ is a completely random measure.

Let $f_j \in C_c(S)$ ($j=1,2$) with $\text{supp}(f_1) \cap \text{supp}(f_2) = \emptyset$.

Remembering that $X_t(n)$ is completely random, we have for each n ,

$$E[e^{i\langle X_t^\mu(n), \theta_1 f_1 + \theta_2 f_2 \rangle}] = \prod_{j=1}^2 E[e^{i\langle X_t^\mu(n), \theta_j f_j \rangle}].$$

$$\text{But } \lim_{n \rightarrow \infty} E[e^{i\langle X_t^\mu(n), \theta_1 f_1 + \theta_2 f_2 \rangle}] = E[e^{i\langle X_t^\mu, \theta_1 f_1 + \theta_2 f_2 \rangle}].$$

$$\text{Moreover, } \lim_{n \rightarrow \infty} \prod_{j=1}^2 E[e^{i\langle X_t^\mu(n), \theta_j f_j \rangle}] = \prod_{j=1}^2 E[e^{i\langle X_t^\mu, \theta_j f_j \rangle}].$$

Thus, by linearity we have,

$$E[e^{i\{\theta_1 \langle X_t^\mu, f_1 \rangle + \theta_2 \langle X_t^\mu, f_2 \rangle\}}] = \prod_{j=1}^2 E[e^{i\theta_j \langle X_t^\mu, f_j \rangle}]$$

so that the random variables $\{\langle X_t^\mu, f_j \rangle\}_{j=1}^2$ are independent, and thereby proving the complete randomness.

Step 5. X_t^μ is of random jump type.

It suffices to show that X_t^μ does not have any fixed atom. Then from Kingman's representation theorem [prop. 1.3.4] we may conclude that X_t^μ does not have any non-atomic non-random component either.

$$\begin{aligned} E[e^{i\theta \langle X_t^\mu, 1_{\{x\}} \rangle}] &= \exp\left[\mu(1_{\{x\}}) \int_0^\infty (e^{i\theta z} - 1) \nu_t(z) m(dz)\right] \\ &= 1, \text{ since } \mu \text{ is non-atomic,} \end{aligned}$$

Thus for every $x \in S$,

$$E[e^{i\langle X_t, 1_{\{x\}} \rangle}] \equiv 1.$$

That is, $\langle X_t^H, 1_{\{x\}} \rangle = 0$ a.e.

This proves theorem 3.2.2.

Remark 3.2.4

Thus far, we have not said anything about the time evolution of random measure. We now proceed to do just that. Combining Remark 1.3.8 and the theorem 3.2.2, we note that the characteristic functional of X_t can be expressed as,

$$\exp\left[\int_{S \times \mathbb{R}_+^1} (e^{izf(x)} - 1) M_t(dx, dz)\right]$$

where M_t is a non-atomic Radon measure on $S \times \mathbb{R}_+^1$ with the representation

$$M_t(dx, dz) = \mu(dx) \nu_t(z) dm(z).$$

Suppose χ_t is a Poisson random measure with intensity M_t on $S \times \mathbb{R}_+^1$ then ξ_t defined by

$$\langle \xi_t, f \rangle \equiv \int_{S \times \mathbb{R}_+^1} z f(x) \chi_t(dx dz)$$

has the same characteristic functional as X_t^H and is also completely random.

Thus, one writes

$$\langle X_t^H, f \rangle = \int_{S \times \mathbb{R}_+^1} f(x) \chi_t(dx dz). \quad (3.2.10)$$

Theorem 3.2.5

We want to show that Y defined by

$$\langle Y(\omega), f \rangle \equiv \int_{S \times R_+^1} f(x) Z_x(\omega_2, t_2; z) \chi_{t_1}(\omega_1, dx dz)$$

(1) is a completely random measure.

(2) has the same characteristic functional as $X_{t_1+t_2}^\mu$.

Proof:

Calculation of characteristic functional:

Noting that χ_{t_1} is a two dimensional δ -function we get

$$\begin{aligned} E[\exp i \langle Y, f \rangle] &= E[\exp i \sum_{(x,z) \in \chi_{t_1}(\omega_1)} f(x) Z_x(\omega_2, t_2; z)] \\ &= E[\Pi E[\exp i f(x) Z_x(\omega_2, t_2; z) | \chi_{t_1}]] \\ &= E[\Pi \phi_{t_2}(f(x), z)] \text{ where } \phi_{t_2}(f(x), z) = E[\exp i f(x) Z_x(t_2; z)] \\ &= E[\exp \sum \log \phi_{t_2}(f(x), z)] \\ &= E[\exp \int_{S \times R_+^1} \log \phi_{t_2}(f(x), z) \chi_{t_1}(dx dz)] \\ &= E[\exp i \langle X_t^\mu, -i \log \phi_{t_2}(f(x), z) \rangle] \text{ [from (3.2.10)]} \\ &= \exp \int_S \mu(dx) \left\{ \int_0^\infty (\exp[i \{-i \log \phi_{t_2}(f(x), z)\}] - 1) v_{t_1}(z) dm(z) \right\} \\ &\quad \text{[from (3.1.4)]} \\ &= \exp \int_S \mu(dx) \left\{ \int_0^\infty (e^{if(x)z} - 1) p(z, t_2, y) dm(y) \right\} v_{t_1}(z) dm(z). \end{aligned}$$

$$\begin{aligned}
\text{Now, } & \int_0^\infty \left\{ \int_0^\infty (e^{if(x)y} - 1) p(z, t_2, y) dm(y) \right\} v_{t_1}(z) dm(z) \\
&= \int_0^\infty \left\{ \lim_{N \rightarrow \infty} \int_0^N (e^{if(x)y} - 1) p(z, t_2, y) dm(y) \right\} v_{t_1}(z) dm(z) \\
&= \lim_{N \rightarrow \infty} \int_0^\infty \left\{ \int_0^N (e^{if(x)y} - 1) p(z, t_2, y) dm(y) \right\} v_{t_1}(z) dm(z) \\
&\quad \text{[using dominated convergence]} \\
&= \lim_{N \rightarrow \infty} \left[\int_0^N (e^{if(x)y} - 1) dm(y) \int_0^\infty v_{t_1}(z) p(z, t_2, y) dm(z) \right].
\end{aligned}$$

Applying Tonelli to $[\cos(f(x)y) - 1]$ and $[\sin(f(x)y)]_+$

$$\begin{aligned}
&= \lim_{N \rightarrow \infty} \int_0^N (e^{if(x)y} - 1) v_{t_1+t_2}(y) dm(y) \\
&= \int_0^\infty (e^{if(x)y} - 1) v_{t_1+t_2}(y) dm(y).
\end{aligned}$$

$$\text{Thus, } E[\exp i \langle Y, f \rangle] = E[\exp i \langle X_{t_1+t_2}^\mu, f \rangle].$$

Notation 3.2.6

Denote the law of X_t^μ on $M_+(S)$ by P_t^μ .

Theorem 3.2.6

For every non-atomic $\mu \in M_+(S)$, there exists a probability measure P_*^μ on $(M_+(S))^T, \mathcal{B}[M_+(S)]^T$ such that for $\Gamma \in \mathcal{B}[M_+(S)]$

$$P\{X_{t_1+t_2} \in \Gamma \mid \mathcal{F}_{t_1}\} = P_{t_2}^{X_{t_1}}(\Gamma), P_*^\mu \text{ a.s.}$$

Proof:

It suffices to show that $\{P_t^\mu : t \in T\}$ with $P_0^\mu \equiv \delta_\mu$ form a transition function on $M_+(S)$. See prop. 1.4.3.

(i) By construction, for each non-atomic μ and $t \in T$, $P_t^\mu(\cdot)$ is a probability measure on $B[M_+(S)]$. Note that for $t > 0$, P_t^μ is a measure on the set of atomic measures on S , i.e., $P_t^\mu \in M_+^1(A)$. [See thm. 3.2.2 step 5.]

(ii) For each $\Gamma \in B[M_+(S)]$ and $t \in T$, $P_t^\mu(\Gamma)$ is a real-valued measurable map on $M_+(S)$.

In view of theorem 3.1.10, it suffices to show that

$$\mu \mapsto E[\exp i\theta \langle X_t, 1_B \rangle]$$

is measurable for every $B \in B(S)$.

$$\text{But } E[\exp i\theta \langle X_t, 1_B \rangle] = \exp [\mu(B) \int_0^\infty (e^{i\theta y} - 1) \nu_t(y) dm(y)].$$

Recall that the map $\mu \mapsto \mu(B)$ defined on $M_+(S)$ is measurable, moreover the set of non-atomic measures is $B[M_+(S)]$ measurable.

(iii) Chapman-Kolmogorov relation

Because of theorem 3.1.8, it suffices to establish for

$$f \in C_c(S)$$

$$\int_{M_+(S)} e^{i\langle n, f \rangle} \int_{M_+(S)} P_{t_1}^\mu(d\lambda) P_{t_2}^\lambda(d\eta) = \int_{M_+(S)} e^{i\langle n, f \rangle} P_{t_1+t_2}^\mu(d\eta).$$

$$\text{L.H.S.} = \int_{M_+(S)} P_{t_1}^\mu(d\lambda) \int_{M_+(S)} e^{i\langle n, f \rangle} P_{t_2}^\lambda(d\eta)$$

$$\text{But } \int_{M_+(S)} e^{i\langle n, f \rangle} P_{t_2}^\lambda(d\eta) = e^{i\langle \lambda, -1 \log \phi_{t_2}(f(x), z) \rangle}$$

Now remember λ is atomic, say, $\lambda = \sum_j a_j \delta_{x_j}$. So interpret

$$\langle\langle \lambda, \log \phi_{t_2}(f(x), z) \rangle\rangle = \sum_j \log \phi_{t_2}(f(x_j), a_j)$$

where $\phi_{t_2}(f(x), z) = \int_0^\infty (e^{if(x)y} - 1) p(z, t_2, y) dm(y) + 1$ (see eqn. 3.1.4)

So,

$$\text{L.H.S.} = \int_{M_+(S)} p_{t_1}^\mu(d\lambda) e^{i\langle\langle \lambda, -i \log \phi_{t_2}(f(x), z) \rangle\rangle}$$

$$= E[\exp i\langle X_{t_1}^\mu, -i \log \phi_{t_2}(f(x), z) \rangle]$$

$$= \exp \left[\int_S \mu(dx) \int_0^\infty [\exp \{i(-i \log \phi_{t_2}(f(x), z))\} - 1] v_{t_1}(z) dm(z) \right]$$

$$= \exp \left[\int_S \mu(dx) \left\{ \int_0^\infty \left[\int_0^\infty (e^{if(x)y} - 1) p(z, t_2, y) dm(y) \right] v_{t_1}(z) dm(z) \right\} \right]$$

$$= \exp \int_S \mu(dx) \int_0^\infty (e^{if(x)y} - 1) v_{t_1+t_2}(y) dm(y)$$

[justification as in Thm. 3.2.5]

$$= \int_{M_+(S)} e^{i\langle n, f \rangle} p_{t_1+t_2}^\mu(dn)$$

Chapter IV Brownian measure processes: Properties

§4.0 Introduction

In this chapter we establish a few properties of Brownian measure processes constructed from non-atomic initial measures. (See §3.2). We also elaborate on the relationships between Brownian measure processes, continuous branching processes (example 2.0.2) and local processes (§1.5).

§4.1 Calculations of mean.

Here we shall employ the notations of §3.2 to establish that X_t^μ has mean μ . We already know that the approximating sequence $X_t^\mu(n)$ has this property.

Theorem 4.1.1

For $f \in C_c(S)$ and $t \geq 0$

$$E[\langle X_t^\mu, f \rangle] = \langle \mu, f \rangle.$$

Proof:

(1) First we prove the existence of the first moment for the random variable $\langle X_t^\mu, f \rangle$.

Recall that X_t was obtained as a weak limit of $X_t^\mu(n)$. Thus

$$\langle X_t^\mu(n), f \rangle \xrightarrow{w} \langle X_t^\mu, f \rangle, \text{ as } n \rightarrow \infty. \quad (4.1.1)$$

For $A > 0$, define g_A on R_+^1 as

$$g_A(y) = \begin{cases} y & \text{if } y \leq A \\ A & \text{if } y > A \end{cases}$$

Then g_A is a continuous bounded function on $\mathbb{R}_+^1$ and from (4.1.1) one gets

$$\int_0^\infty g_A(y) dF_n(y) \rightarrow \int_0^\infty g_A(y) dF(y), \text{ as } n \rightarrow \infty \quad (4.1.2)$$

where F, F_n are respectively the distribution functions of random variables $\langle X_t^\mu, f \rangle$ and $\langle X_t^\mu(n), f \rangle$.

Recall also, for each n

$$\int_0^\infty y dF_n(y) = E[\langle X_t^\mu(n), f \rangle] = \langle \mu, f \rangle.$$

Then, for each n , $\int_0^\infty g_A(y) dF_n(y) \leq \langle \mu, f \rangle$. (4.1.3)

Consequently, $\int_0^\infty y dF(y)$

$$= \lim_{A \rightarrow \infty} \int_0^\infty g_A(y) dF(y) \quad (\text{monotone convergence})$$

$$= \lim_{A \rightarrow \infty} \lim_{n \rightarrow \infty} \int_0^\infty g_A(y) dF_n(y) \quad (\text{using (4.1.2)})$$

$$\leq \langle \mu, f \rangle \quad (\text{using (4.1.3)}).$$

Thus, the first moment of the random variable $\langle X_t^\mu, f \rangle$ exists.

(ii) Now we proceed to evaluate the first moment from the characteristic function of $\langle X_t^\mu, f \rangle$, $E[\exp i \theta \langle X_t^\mu, f \rangle]$.

$$E[\exp i\theta \langle X_t^\mu, f \rangle] = \exp \left[\int_S \mu(dx) \int_0^\infty (e^{i\theta f(x)y} - 1) v_t(y) dm(y) \right]$$

So that,

$$\begin{aligned} & \frac{\partial}{\partial \theta} E[\exp i\theta \langle X_t^\mu, f \rangle] \Big|_{\theta=0} \\ &= \lim_{\theta \rightarrow 0} \int_S \mu(dx) \int_0^\infty \left(\frac{e^{i\theta f(x)y} - 1}{\theta} \right) v_t(y) dm(y) \\ &= \int_S \mu(dx) \int_0^\infty \lim_{\theta \rightarrow 0} \left(\frac{e^{i\theta f(x)y} - 1}{\theta} \right) v_t(y) dm(y) . \end{aligned}$$

The last two interchanges are justified by the dominated convergence theorem.

$$\text{Hence, } E[\langle X_t^\mu, f \rangle] = -i \frac{\partial}{\partial \theta} E[\exp i\theta \langle X_t^\mu, f \rangle] \Big|_{\theta=0}$$

$$= \int_S f(x) \mu(dx)$$

$$= \langle \mu, f \rangle .$$

§4.2 Stochastic Convergence

In this section we establish the stochastic continuity of X_t^μ as t goes to zero. To do this, we show that the characteristic functional of X_t^μ goes to that of δ_μ - the indicator function of μ - as t tends to zero. Thus one may conclude that X_t^μ converges to μ in probability.

The trick is to rewrite the expression for the characteristic functional so that the interchange of limit and integral does not pose any problem. At the same time we obtain some added information about the Lévy measure ν_t .

$$\text{Put } q_t(z, dy) = \frac{y}{z} p(z, t, y) dm(y).$$

Then q_t denotes the transition probability of a new diffusion process, called the h-path process of the original diffusion with the transition function $p(z, t, y)$. See Doob [7] for a discussion on h-path processes.

Note in particular,

$$q_t(0, dy) = y \nu_t(y) dm(y).$$

Thus, we have succeeded in obtaining a new interpretation for the Lévy measure ν_t .

$$\text{Hence, } \lim_{t \downarrow 0} E[\exp i \langle X_t^\mu, f \rangle]$$

$$= \lim_{t \downarrow 0} \exp \left[\int_S \mu(dx) \int_0^\infty \left(\frac{e^{if(x)y} - 1}{y} \right) q_t(0, dy) \right]$$

$$= \exp \left[\int_S \mu(dx) \lim_{t \downarrow 0} \int_0^\infty \left(\frac{e^{if(x)y} - 1}{y} \right) q_t(0, dy) \right].$$

But q_t is the transition probability of a diffusion and $(e^{if(x)y}-1)/y$ is a bounded continuous function in y , consequently

$$\begin{aligned} & \lim_{t \rightarrow 0} \int_0^\infty \left(\frac{e^{if(x)y}-1}{y} \right) q_t(0, dy) \\ &= \lim_{y \rightarrow 0} \frac{e^{if(x)y}-1}{y} \\ &= if(x) \end{aligned}$$

Hence, $\lim_{t \rightarrow 0} E[\exp i \langle X_t, f \rangle] = \exp i \int_S f(x) \mu(dx) = e^{i \langle \mu, f \rangle}$.

The desired result now follows from the properties of characteristic functionals. So we have proved

Theorem 4.2.1

$$X_t^\mu \xrightarrow{P} \mu \quad \text{as } t \rightarrow 0 \dots$$

§4.3 Multiplicativity

In this section we discuss the notion of a multiplicative process introduced by several authors. Our interest in multiplicative process lies in the fact that the continuous branching process is the only multiplicative diffusion. [See Athreya and Ney [1; pp.257]]. The multiplicativity of continuous branching processes gives rise to an important non-linear semigroup. This semigroup has been exploited by Watanabe [17] and Dawson [6] to derive important properties of measure-valued processes.

In our construction of Brownian measure processes in §3.2, if the singular diffusion $\{Z_x(t) : x \in S\}$ on $[0, \infty)$ with $\{0\}$ as an exit boundary is multiplicative, that is $\{Z_x(t)\}$ is continuous branching, one obtains the non-linear semigroup mentioned above. It has also been pointed out by Watanabe [17] that in this case only the measure-valued process is both multiplicative and a diffusion, i.e., has continuous sample path on the state space of measures.

Definition 4.3.1

A real-valued process is said to be multiplicative if for $t \geq 0$

$$E_{z_1+z_2} [e^{1\theta Z(t)}] = E_{z_1} [e^{1\theta Z(t)}] E_{z_2} [e^{1\theta Z(t)}] \quad (4.3.1)$$

where $\theta \in \mathbb{R}^1$ and $z_j \in \mathbb{R}_+^1$, ($j = 1, 2$) and

$$E_z [e^{1\theta Z(t)}] \equiv E[e^{1\theta Z(t)} \mid Z(0) = z].$$

One reinterprets the parameters in equation (4.3.1) to define a measure-valued multiplicative process.

Theorem 4.3.2

If $\{Z_x(t) : x \in S\}$ is an independent family of real-valued singular multiplicative diffusions with $\{0\}$ as an absorbing boundary then the map

$$\Psi_t : f \longrightarrow -i \int_0^\infty (e^{if(x)y} - 1) \nu_t(y) \, dm(y)$$

defined on $f \in C_c(S)$, satisfies

- (a) $\Psi_{t_1+t_2} = \Psi_{t_2}(\Psi_{t_1})$
- (b) $\lim_{t \rightarrow 0} \Psi_t = \text{Identity (pointwise)}.$

Proof: (a) The equation (4.3.1) implies

$$E_z[e^{if(x)Z(t_1)}] = e^{iz\Psi_t(f(x))} \quad (4.3.2)$$

where Ψ_t is some function of $f(x)$. But

$$E_z[e^{if(x)Z(t_1)}] = \int_0^\infty (e^{if(x)y} - 1) p(z, t_1, y) \, dm(y) + 1$$

Thus,

$$\int_0^\infty (e^{if(x)y} - 1) p(z, t_1, y) \, dm(y) = e^{iz\Psi_t(f(x))} - 1. \quad (4.3.3)$$

Dividing through by z and letting $z \rightarrow 0$,

$$\int_0^\infty (e^{if(x)y} - 1) \nu_{t_1}(y) \, dm(y) = i \Psi_t(f(x)).$$

For justification, see theorem 3.2.2 step II (ii),

$$\text{or, } \Psi_{t_1}(f(x)) = -i \int_0^\infty (e^{if(x)y} - 1) v_{t_1}(y) dm(y) \quad (4.3.4)$$

$$\begin{aligned} \text{Now, } \Psi_{t_2}[\Psi_{t_1}(f(x))] &= -i \int_0^\infty (e^{i \Psi_{t_1}(f(x))z} - 1) v_{t_2}(z) dm(z) \\ &= -i \int_0^\infty \left[\int_0^\infty (e^{if(x)y} - 1) p(z, t_1, y) dm(y) \right] v_{t_2}(z) dm(z) \\ &\quad \text{[using eqn. (4.3.3)]} \\ &= -i \int_0^\infty (e^{if(x)y} - 1) \left[\int_0^\infty p(z, t_1, y) v_{t_2}(z) dm(z) \right] dm(y) \\ &\quad \text{[see theorem 3.2.5]} \\ &= -i \int_0^\infty (e^{if(x)y} - 1) v_{t_1+t_2}(y) dm(y) \\ &= \Psi_{t_1+t_2}[f(x)] . \end{aligned}$$

(b) We want to show

$$\lim_{t \rightarrow 0} \Psi_t f(x) = f(x) .$$

$$\begin{aligned} \text{But } \lim_{t \rightarrow 0} \Psi_t f(x) &= -i \lim_{t \rightarrow 0} \int_0^\infty (e^{if(x)y} - 1) v_t(y) dm(y) \\ &= -i [if(x)] \quad \text{[see thm. 4.3.2]} \\ &= f(x) . \end{aligned}$$

Remark 4.3.3

(i) For μ non-atomic, we find that X_t^μ , constructed in §3.2, is multiplicative regardless of the singular diffusions.

$\{Z_x(t) : x \in S\}$. However, we cannot conclude that X_t^μ is multiplicative for non-atomic initial measure μ . Thus a measure-valued multiplicative process should satisfy the equation (4.3.1) for all initial measures.

(ii) If, however, the local evolution is described by the continuous branching process then the corresponding Brownian measure process is multiplicative. In this case the expression for the characteristic functional of X_t^μ , with μ atomic, does satisfy equation (4.3.1).

(iii) The Brownian measure process, with the continuous branching diffusion describing the local evolution, has yet another property: $A \in \mathcal{B}(S)$, $X_t^\mu(\cdot, A)$ is a real-valued Markov process with respect to σ -fields generated by the random variables $\{X_t(\cdot, A) : t \in T\}$. For a comparison, see local processes described in §1.5.

In fact, one can show that for a measure-valued Markov process, which is multiplicative and local, $X_t^\mu(\cdot, A)$ is a real-valued Markov process in the sense described above.

§4.4 Immigration

So far we have considered mass evolution taking place in isolation, governed locally by a singular diffusion. A useful and realistic modification is to consider the possibility of an immigration from an outside source. In the terminology of singular measure-valued Markov process (Dawson [6; pp.32]), immigration can be viewed as a non-homogeneous component. For an account of continuous branching processes with immigration, using an extension of the non-linear semigroup described in the theorem 4.3.2, see Watanabe [18]. Here, we show that $\int_0^t v_\tau(y) d\tau$ can be viewed as a particular case of continuous immigration.

Let us proceed by showing that

$$n_t(y) \equiv \int_0^t v_\tau(y) d\tau$$

exists.

From equation (2.3.3) we have

$$\int_0^t p(z, \tau, y) d\tau \leq K(\xi, y) \phi(z, t, \xi)$$

where $\xi \in (z, y)$ and is fixed.

Then, by Fatou's lemma, we get

$$\int_0^t v_\tau(y) d\tau \leq K(\xi, y) \lim_{z \downarrow 0} \frac{\phi(z, t, \xi)}{z}$$

and, for an absorbing boundary at $\{0\}$, $\lim_{z \downarrow 0} \frac{\phi(z, t, \xi)}{z}$ exists (prop. 2.3.1).

We next note that

$$\int_0^{\infty} y \eta_t(y) \, dm(y) = \int_0^t d\tau \int_0^{\infty} y v_{\tau}(y) \, dm(y) \\ = t.$$

The characteristic functional of the Brownian measure process with immigration is

$$\exp\left[\int_S \mu(dx) \int_0^{\infty} (e^{if(x)y} - 1) v_t(y) dm(y) + \int_S L(dx) \int_0^{\infty} (e^{if(x)y} - 1) \eta_t(y) dm(y)\right]$$

where L is some fixed measure on S . It could be taken as the Lebesgue measure if $S = \mathbb{R}^d$.

Final Remark

The continuous branching process has as its discrete state analogue the branching process. It will be of interest to identify those discrete state processes which could be approximated by singular diffusions on $[0, \infty)$ with $\{0\}$ acting as an absorbing boundary. The continuous branching process model cannot incorporate any "local interaction", a fact manifested by the property of multiplicativity. However, any discrete analogue for any singular diffusions with $\{0\}$ absorbing, other than the continuous branching process, must be of "local interaction" type. So that we may denote this class of singular diffusions as the "zero-range interaction" type.

A useful and desirable model would allow for particle motion on the space S . This model has already been worked out by Dawson [6] when the mass evolution is governed by the continuous branching diffusion. So one would like to have an extension of Brownian measure processes which would allow particle motion.

References

- [1] Athreya, K. B. and Ney, P. E., Branching processes, Springer-Verlag, Berlin, 1972.
- [2] Bailey, N. T. J., Stochastic birth, death and migration processes for spatially distributed populations, *Biometrika* 55 (1963), 189-198.
- [3] Bauer, H., Probability theory and elements of measure theory, Holt, Rinehart and Winston, New York, 1972.
- [4] Breiman, L., Probability, Addison-Wesley, Reading, Mass., 1968.
- [5] Choquet, G., Lectures in Analysis V.I, W. A. Benjamin, New York, 1969.
- [6] Dawson, D. A., Stochastic evolution equation and related measure processes, *J. Multivariate Anal.* 5 (1975), 1-52.
- [7] Doob, J. L., Conditional Brownian motion and the boundary limits of harmonic functions, *Bull. Soc. Math. France* 85 (1957), 431-458.
- [8] Feller, W., Diffusion processes in genetics, *Proc. Second Berkeley Symp., Univ. of Calif. Press, Berkeley, Calif.*, pp. 227-246, 1951.
- [9] ———, Generalized second order differential operators and their lateral conditions, *Illinois J. Math.* 1 (1957), 459-504.
- [10] Jagers, P., Aspects of random measures and point processes, In "Advances in probability theory V.3", (Ney, P.E. and Port, S., Eds.) Marcel Dekker, New York, 1974.
- [11] McKean, H. P., Elementary solutions for certain parabolic partial differential equations, *Trans. Amer. Math. Soc.* 82 (1956), 519-548.
- [12] Meyer, P.A., Probability and potentials, Blaisdell, Walt., Mass., 1966.
- [13] ———, Processus de Markov, Lecture notes in Math. #139, Springer-Verlag, Berlin.

- [14] Mizuno, S., Critical measure diffusion processes, Ph.D. dissertation, Dept. of Math., Carleton Univ., in preparation.
- [15] Neveu, J., Mathematical foundations of the calculus of probability, Holden-Day, San Francisco, 1965.
- [16] Sawyer, S., Branching diffusion processes in population genetics, Adv. Appl. Prob. 8 (1976), 659-689..
- [17] Watanabe, S., A limit theorem of branching processes and continuous state branching processes, J. Math. Kyoto Univ., 8 (1968), 141-167.
- [18] ----- and Kawazu, K., Branching processes with immigration and related limit theorems, Th. Prob. Appl. 16 (1971), 36-54.
- [19] Widder, D. V., The Laplace transform, Princeton Univ. Press, Princeton, N.J., 1946.

END

4

08

78

FIN