Dynamics and Control of Lunisolar Perturbations for Highly-Eccentric Earth-Orbiting Satellites

by

Matthew Bourassa

A thesis submitted to
the Faculty of Graduate and Postdoctoral Affairs
in partial fulfilment of
the requirements for the degree of

Doctor of Philosophy

in

Aerospace Engineering

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of Mechanical and Aerospace Engineering

Carleton University

Ottawa, Ontario

June 15, 2015

© Copyright

2015 - Matthew Bourassa

Abstract

Satellites in highly-eccentric orbits are unique in their ability to spend the majority of their period near the apogee. When these orbits are critically-inclined and have an argument of perigee equal to 270° , they become well-suited for Earth-observation missions for northern regions. However, their orbital characteristics result in a complex and dynamic set of perturbation forces acting on the satellite orbit due to the gravity of the Moon and the Sun, coupled with the perturbations caused by the oblateness of the Earth.

The focus of this thesis is on both the dynamics and the control of the lunisolar perturbations. The study of the dynamics involves using geometric and kinematic methods to analyze the conditions at which the rates of change of the orbital elements due to lunisolar perturbation forces are zero. The resulting analysis provides insight into the mechanisms of the various oscillations caused by the gravitational attraction of the Moon and Sun. A method to predict the future occurrences when the rates of change of the orbital elements are equal to zero is also developed.

Using the enhanced knowledge of the lunisolar perturbation forces acting on the highly-eccentric orbits, a control strategy is developed to decrease the amount of ΔV needed to maintain the eccentricity, inclination, and argument of perigee near their nominal values. The control strategy uses the grazing method to exploit the oscillations of the orbital elements caused by the Moon and Sun. The strategy, which uses analytical methods to compute the control requirements, is compared to an equivalent approach that uses numerical methods to exploit the lunisolar perturbations.

Acknowledgements

How does one thank everyone who helped them get through a Ph.D.? The truth is you can't. But, here goes my attempt to enumerate all those who've been there for me, in one way or another.

Without question, my thesis supervisors, Fred Afagh, Rob Langlois, and Bruce Burlton, deserve an exceptional amount of acknowledgement for their support of my work. Fred and Rob brought me to Carleton and allowed me find a project that I was truly happy with, even if it was outside their typical field of research. Bruce was kind enough to offer his expertise to develop my thesis and then became one of my co-supervisors. For this I am truly grateful. I am also thankful for the kinship provided by those who shared research space with me at Carleton, such as Nick and Jon, to name a few. During my Ph.D., I was fortunate enough to be hired by the Canadian Space Agency under the Research Affiliate Program. While there, my supervisors, Alfred Ng and Patrice Côté, were very accommodating by allowing me to spend time working on my thesis while at CSA. Without that, I would have required significantly more time to complete my Ph.D.

All of my family has always been very supportive of my desire to spend the majority of my 20's in university. I am thankful to them for motivating me to finish school by giving me a hard time for being in school for so long. To my all friends, those back in Alberta and the ones in Montreal and Ottawa, I appreciate all the kind things you have done for me over these past few years. And finally, I am thankful for Claire, for being there for me this past year and always being supportive.

The more you know, the more you know you know nothing.

Aristotle

List of Symbols

Symbol	Description
A	Direction cosine of the third-body effects
B	Direction cosine of the third-body effects
C	Direction cosine of the third-body effects
$C_{l,m}$	Gravitational coefficients of the Earth of degree l and order m
$F(i)$	The inclination function
F_R	The specific force in the radial direction
F_S	The specific force in the transverse direction
F_W	The specific force in the orbit normal direction
E	Amplitude of the long-period eccentricity oscillations using Kolyuka's method
G	The universal gravity constant
$G(e)$	The eccentricity function
$H(e)$	Hansen coefficients
$\hat{I}$	Inertial coordinate point towards the True-of-Date vernal equinox, defined in the equatorial plane (i.e., x-axis)
$\hat{J}$	Inertial coordinate completing the right hand rule, defined in the equatorial plane (i.e., y-axis)
J_2	Gravitational coefficient which models the oblateness of the Earth
J_l	The zonal gravitational coefficients of degree l
$\hat{K}$	Inertial coordinate pointing along the Earth polar axis
M	Mean anomaly
P	Keplerian period
P_Ω	Nodal period
$\vec{N}$	Ascending node vector

R_E	Equatorial radius of the Earth
R	The disturbing potential function
$\vec{R}$	Radius vector with respect to Earth
$\hat{R}$	Radial direction in the RSW coordinate system
$\hat{S}$	Transverse direction in the RSW coordinate system
$S_{l,m}$	Gravitational coefficients of the Earth of degree l and order m
U	Potential function
$\vec{V}$	Velocity vector of the spacecraft
$\hat{W}$	Orbit normal direction in the RSW coordinate system
a	Semi-major axis
e	Eccentricity
f	Weighting factor between the magnitude of the lunar and solar perturbations
$\vec{h}$	Angular momentum vector of an orbit
i	Inclination
m	The mass of a planetary body or satellite (not to be confused with the index value m)
n	Mean motion
p	Semi-latus rectum
r	Orbital radius w.r.t. Earth
r_a	Apogee radius
r_p	Perigee radius
t	Time
u	Argument of latitude
ΔV	Change in velocity of the spacecraft due to a manoeuvre
$\Delta \Omega$	Relative RAAN between the satellite and the third body
Δi	Difference between the inclinations of the satellite and the third body

Σi	Sum of the inclinations of the satellite and the third body
Ω	Right ascension of ascending node
α	the geocentric arc length between the satellite node vector and the third body position vector
β	the angle between the satellite plane and the plane defined by α
γ	The angle between the equatorial plane and the plane defined by α , measured CW
δ_e	The angle between the α -plane and the β -plane along the orbit of the third body
δ_i	The angle between the α -plane and the satellite orbit plane along the orbit of the third body
δ_Ω	The angle between the β -plane and the satellite orbit plane along the orbit of the third body
δ_ω	The angle between the ω -curve and the equatorial plane along the orbit of the third body
δ_α	The angle between the α -plane and the equatorial plane along the orbit of the third body
δ_β	The angle between the β -plane and the equatorial plane along the orbit of the third body
δ_{sat}	The angle between the satellite plane and the equatorial plane along the orbit of the third body
ζ	Geocentric arc length between the satellite and the third body
λ_0	Initial longitude of the ascending node of the satellite orbit
μ	Gravitational parameter
ν	True anomaly
ϕ	The satellite flight-path angle
$\phi_{\Delta V}$	The direction of the thrust vector for an impulsive manoeuvre

θ_{GMST}	The angle between the prime meridian and the inertial $\hat{I}$ vector
ω	Argument of perigee
ω_E	Rate of rotation of the Earth

Note: any variable with subscript 3 refers to the variable corresponding of a generic third body, whereas any with the subscript M or S refer to the Moon or Sun, respectively. Additionally, any rate of change variable with the subscript (M) or (S) is the rate of change due the the Moon (M) or Sun(S).

Acronyms

Acronyms	Definition
AOL	Argument of latitude
AOP	Argument of perigee
APC	Analytical prediction and control
CCW	Counter clockwise
CW	Clockwise
ESA	European Space Agency
GMST	Greenwich mean sidereal time
GPS	Global Positioning System
HEO	Highly eccentric orbit
HPOP	High precision orbit propagator
IC	Initial condition
LAN	Longitude of the ascending node
LEO	Low-Earth orbit
NASA	National Aeronautics and Space Administration
NPC	Numerical prediction and control
ODE	Ordinary differential equation
PCW	Polar Communication and Weather
RAAN	Right ascension of the ascending node
RMS	Root mean squared
STK	Systems Toolkit (formerly Satellite Toolkit)
swp	Switch point
TAP	Three-apogee
TOD	True-of-date

Contents

Abstract	ii
Acknowledgements	iii
List of Symbols	v
Acronyms	ix
Contents	x
List of Figures	xiii
List of Tables	xxi
1 Introduction	1
1.1 Motivation	2
1.2 Outline of Research	9
2 Dynamic Formulation	13
2.1 Orbital Elements and Coordinate Systems	13
2.2 Force Model of HEOs	22
2.2.1 Non-spherical Earth Gravitational Model	23
2.2.2 Lunar and Solar Perturbations	27

2.2.3	Force Model Validation	50
2.2.4	Variations of Elements over Time	54
2.3	Analytical Model of Lunisolar Perturbations	57
2.4	Truncation Error for Lunisolar Perturbations	61
3	Dynamic Analysis	63
3.1	Switch Point Development	69
3.1.1	Eccentricity	69
3.1.2	Inclination	90
3.1.3	RAAN	94
3.1.4	AOP	97
3.2	Switch Point Prediction	106
3.2.1	Eccentricity	107
3.2.2	Inclination	116
3.2.3	AOP	119
3.3	Validation of Switch Point Analysis	124
4	Orbit Control Strategy	130
4.1	Manoeuvre Sequence	142
4.1.1	Eccentricity Manoeuvres	143
4.1.2	Inclination Manoeuvres	149
4.1.3	AOP Manoeuvres	153
4.1.4	Ground-track Control Sequence	159
4.2	Sensitivity Analysis	160
4.3	Analytical Prediction and Control Strategy	163
4.3.1	Initial RAAN	165
4.3.2	Methodology	168

4.4	Numerical Prediction and Control and Basesline Control Strategies	174
4.5	Results	176
4.6	Validation of Orbit Control Strategy Using STK	192
5	Conclusions	217
	References	222
A	Constants	229
B	Lunar and Solar Orbital Elements	230
C	Supplemental Results	233
C.1	MATLAB Results	233
C.2	STK Results	240

List of Figures

1.1	Schematic comparing relative orbit sizes and shapes of Molniya, TAP, and Tundra orbits to common circular orbits.	7
1.2	Schematic comparing orbit orientations of Molniya, TAP, and Tundra orbits to common circular orbits.	8
1.3	Ground-track of a Molniya orbit.	9
1.4	Ground-track of a TAP orbit.	10
1.5	Ground-track of a Tundra orbit.	10
2.1	Two-dimensional diagram of an eccentric orbit.	15
2.2	Three-dimensional diagram of an eccentric orbit.	16
2.3	The RSW coordinate system projected onto a unit sphere.	20
2.4	The three body problem.	31
2.5	Comparison of the methods to model lunisolar perturbations using the STK and MATLAB propagators for a sample Tundra orbit.	48
2.6	Multiple oscillation modes of the eccentricity due to lunisolar perturbations.	49
2.7	Variations of the orbital elements using the MATLAB propagator for a sample TAP orbit.	56
2.8	Variations of the orbital elements using the MATLAB propagator for a sample Molniya orbit.	58

3.1	The rates of change of the orbital elements of a Molniya orbit due to solar perturbations as a function of the relative RAAN and the solar argument of latitude.	64
3.2	Unit sphere for visualizing the relative geometry of the Sun and satellite orbit.	71
3.3	Unit sphere for the switch point planes.	73
3.4	Variation of α over a solar period for a complete cycle of relative RAANs for an HEO.	74
3.5	Variation of α over a solar period for a smaller subset of relative RAANs for an HEO.	75
3.6	Maximum and minimum values of α over a solar period as a function of the relative RAAN.	76
3.7	Schematic of switch point angles γ and β when the relative RAAN is: (a) 0° , and (b) 180°	78
3.8	Variations of β over the solar period for a complete cycle of relative RAANs for an HEO.	80
3.9	Variations of β over the solar period for a smaller subset of relative RAANs for an HEO.	81
3.10	Variations of the amplitudes of direction cosines as a function of the relative RAAN.	82
3.11	Behaviour of α , β , and the relative RAAN for a Molniya orbit over time. .	84
3.12	Scatter plot of the rate of change of the eccentricity of a Molniya orbit due to the Sun as a function of the switch point angles α and β . The black lines signify when the rate of change is zero.	85
3.13	Unit sphere for δ angles.	86
3.14	Variation of the angles δ_α , δ_β , and δ_{sat} as a function of the relative RAAN.	87
3.15	Variation of the angles δ_e , δ_i , and δ_Ω as a function of the relative RAAN. .	88

3.16	Scatter plot showing the relationship between the rate of change of the eccentricity and δ_e . The upper curve is the maximum positive rate of change over a given solar period and the lower curve is the maximum negative. . .	89
3.17	Scatter plot of the rate of change of the inclination for a Molniya orbit due to the Sun as a function of the switch point angles α and β . The black lines signify when the rate of change is zero.	92
3.18	Scatter plot showing the relationship between the rate of change of the inclination and δ_i . The upper curve is the maximum positive rate of change over a given solar period and the lower curve is the maximum negative. . .	93
3.19	Scatter plot of the rate of change of the RAAN for a Molniya orbit due to the Sun as a function of the switch point angles α and β . The black lines signify when the rate of change is zero.	95
3.20	Scatter plot showing the relationship between the rate of change of the RAAN and δ_Ω . The upper curve is the maximum positive rate of change over a given solar period and the lower curve is the maximum negative. . .	96
3.21	Variation of α as a function of β for a constant value of $\Delta\Omega$ with respect to the ω curves.	100
3.22	Scatter plot of the rate of change of the AOP of a Molniya orbit due to the Sun as a function of the switch point angles α and β . The black lines signify when the rate of change is zero.	102
3.23	Unit sphere showing the various switch point planes: (a) <i>top view</i> ; (b) <i>front view</i> ; (c) <i>side view</i>	103
3.24	The unit sphere to define the angle δ_ω	104
3.25	Variation of the δ_ω angles as a function of the relative RAAN.	105
3.26	Prediction of the Sun crossing the α and β planes using the unit sphere. . .	108
3.27	Projection of the spherical triangles to model δ_α (a) and δ_β (b) onto a plane.	109

3.28	Variations of the rates of change of the δ angles per unit change of the relative RAAN as a function of the relative RAAN.	112
3.29	Comparison of the solar switch point with the combined switch point for an HEO.	114
3.30	Projection of the spherical triangle to model δ_{sat}	117
3.31	Discrepancy in the prediction of α as a function of β without accounting for the drift in the relative RAAN.	120
3.32	Projection of the spherical triangle to model δ_ω	121
3.33	Variation of the rate of change of δ_ω as a function of the relative RAAN.	123
3.34	Predicted combined switch point error for the eccentricity of an HEO.	127
3.35	Predicted combined switch point error for the inclination of an HEO.	128
3.36	Predicted combined switch point error for the AOP of an HEO.	129
4.1	Variation of the orbital elements for the Sirius-1 satellite.	140
4.2	Diagram of the RSW system in the orbit plane.	143
4.3	Diagram of the generalized Hohmann transfer.	145
4.4	The change in the eccentricity per unit ΔV as a function of the true anomaly and the direction of the manoeuvre. The black curve is the line such that $\Delta e/\Delta V$ is zero. The magenta line is flight path angle of the satellite in a Molniya orbit.	147
4.5	Variation of the change in orbital angles, i , ω , and Ω , per unit ΔV (in the normal direction) for a Molniya orbit.	150
4.6	Variation of the change in the AOP per unit ΔV_i and percent increase of the change in the inclination for manoeuvres located as determined by Equation 4.8 or 4.9 compared to nodal manoeuvres.	152

4.7	The change in AOP per unit ΔV as a function of the true anomaly and the direction of the manoeuvre. The green curve is the line such that $\Delta e/\Delta V$ is equal to zero. The black curve is the line such that $\Delta \omega/\Delta V$ is a maximum. The magenta line is flight path angle of the satellite in a Molniya orbit. . .	156
4.8	The change in semi-major axis per unit ΔV as a function of the true anomaly and the direction of the manoeuvre. The green curve is the line such that $\Delta e/\Delta V$ is equal to zero. The black curve is the line such that $\Delta \omega/\Delta V$ is a maximum. The magenta line is flight path angle of the satellite in a Molniya orbit.	158
4.9	Variation of the semi-major axis for the Molniya-193 satellite.	161
4.10	Variation of the node for the Molniya-193 satellite.	161
4.11	Histogram of the sensitivity of the eccentricity.	164
4.12	Histogram of the sensitivity of the inclination.	164
4.13	Histogram of the sensitivity of the AOP.	165
4.14	Schematic of the grazing method.	170
4.15	Block diagram of orbit control strategy.	172
4.16	Block diagram of the calculation of the desired eccentricity (e_{des}), inclination (i_{des}), or AOP (ω_{des}) using the APC method.	174
4.17	Control tolerances for the orbit control strategies.	178
4.18	The ΔV requirements using the Baseline control strategy for all three HEOs with various initial mission RAANs with the control tolerance set at $1/2 \times 2S$ (15-year mission).	181
4.19	The ΔV requirements using the Baseline control strategy for all three HEOs with various initial mission RAANs with the control tolerance set at $1/2 \times 2S$ (5-year mission).	183

4.20	Controlled variations of the orbital elements of a Tundra orbit over a 5-year period using the APC method with a tolerance size of $1/2 \times 2S$	185
4.21	ΔV results for a Tundra orbit comparing different orbit control strategies at various control tolerances for a 5-year mission. The numbers in brackets are the number of manoeuvres required.	186
4.22	ΔV results for a Tundra orbit comparing different orbit control strategies at various control tolerances for a 15-year mission. The numbers in brackets are the number of manoeuvres required.	187
4.23	Block diagram of the MATLAB-STK orbit control simulator.	194
4.24	Variation of the semi-major axis and LAN for a Tundra orbit using a $2S$ tolerance size.	196
4.25	Variation of the semi-major axis and LAN for a Tundra orbit using a $1/2 \times 2S$ tolerance size.	197
4.26	The ΔV results of the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Tundra orbit).	201
4.27	The mean required ΔV per year using the APC method from the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Tundra orbit).	210
4.28	Total required mission ΔV using the APC method for the Tundra and TAP orbits over various control tolerance sizes for 15-, 10-, and 5-year missions.	211
4.29	Total required mission ΔV using the APC method for the Molniya 3500 and 1000 orbits over various control tolerance sizes for 15-, 10-, and 5-year missions.	212
B.1	Variations of the RAAN of the Moon and Sun over a 25-year period. . . .	231
B.2	Variations of the inclination of the Moon and Sun over a 25-year period. . .	231

B.3	Variations of the argument of latitude of the Moon and Sun over a 1-year period.	232
B.4	Variations of the orbital radius of the Moon and Sun over a 1-year period.	232
C.1	Controlled variations of the orbital elements of a TAP orbit over a 5-year period using the APC method with a tolerance size of $1/2 \times 2S$	234
C.2	Controlled variations of the orbital elements of a Molniya orbit over a 5-year period using the APC method with a tolerance size of $1/2 \times 2S$	235
C.3	The ΔV results for a TAP orbit comparing different orbit control strategies at various control tolerances for a 15-year mission. The numbers in brackets are the number of manoeuvres required.	236
C.4	The ΔV results for a TAP orbit comparing different orbit control strategies at various control tolerances for a 5-year mission. The numbers in brackets are the number of manoeuvres required.	237
C.5	The ΔV results for a Molniya orbit comparing different orbit control strategies at various control tolerances for a 15-year mission. The numbers in brackets are the number of manoeuvres required.	238
C.6	The ΔV results for a Molniya orbit comparing different orbit control strategies at various control tolerances for a 5-year mission. The numbers in brackets are the number of manoeuvres required.	239
C.7	The ΔV results of the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (TAP orbit).	241
C.8	The ΔV results of the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Molniya 3500 orbit).	242

C.9	The ΔV results of the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Molniya 1000 orbit).	243
C.10	The mean required ΔV per year using the APC method from the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (TAP orbit).	244
C.11	The mean required ΔV per year using the APC method from the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Molniya 3500 orbit).	245
C.12	The mean required ΔV per year using the APC method from the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Molniya 1000 orbit).	246
C.13	Comparison of the AOP variations for a 15- and 5-year mission using the APC method.	247

List of Tables

1.1	Orbital elements of various HEOs.	6
2.1	Error between variations of the orbital elements between MATLAB and STK.	47
2.2	Results of force model validation for a Molniya orbit.	52
2.3	Results of force model validation for a TAP orbit.	53
2.4	Results of force model validation for a Tundra orbit.	54
2.5	RMS error of simulations after truncation of ICs.	62
3.1	Summary of values for angles α and β such that a switch point occurs.	106
3.2	Mean solar switch time error (hours)	126
4.1	Type of Hohmann transfers: the resulting change in the eccentricity and the temporary change in the semi-major axis.	146
4.2	Summary of the inclination manoeuvre sequence.	152
4.3	Summary of the AOP manoeuvre sequence.	157
4.4	Control tolerances and parameters for a Tundra orbit mission.	186
4.5	Various tolerances sizes implemented with the orbit control strategies.	199
C.1	Control tolerances and parameters for a TAP orbit mission.	233
C.2	Control tolerances and parameters for a Molniya orbit mission.	236
C.3	Control tolerances and parameters for a Molniya 3500 orbit mission.	240

Chapter 1

Introduction

Satellites in highly-eccentric orbits around the Earth experience a complex and dynamic set of perturbations which affect their trajectories in a significantly different manner compared to satellites in low-Earth orbits or geostationary orbits. Because of the low perigee and high apogee radii, over the course of a given orbit the source of the dominant perturbations changes. Near the perigee, the non-spherical geopotential is the dominant perturbation source, whereas at the high altitudes near apogee, the gravitational attractions of the Sun and Moon become dominant [1]. It is this unique environment that makes the study of the perturbations, as well as methods to compensate for these perturbations, such a challenging research topic.

Highly-inclined, highly-eccentric orbits (HEOs) are well known for their suitability for coverage of Northern latitudes as well as Arctic regions. For approximately 60 years, Russia has used one particular type of HEO called the Molniya orbit to enable cross-country communication [2]. In the context of this research project, it is desired to place a satellite in an HEO such that it can provide communications support as well as Earth observation data through both general imagery and meteorological payloads to support Canadian in-

terests.

An HEO is well suited for these types of missions because its large eccentricity places the apogee of the orbit past the geostationary altitude; while the perigee can be anywhere from GPS altitude to low Earth orbit (LEO) altitude. Other special aspects of the HEOs under study in this thesis are the high inclination and argument of perigee (AOP)¹. The AOP is set to 270° which places the apogee at the northern-most point possible. This allows satellites in HEOs to ‘hover’ with respect to the Earth when it is close to the apogee², thereby providing ample opportunity to cover the Arctic regions.

1.1 Motivation

This project began as a study of atmospheric drag effects on a Molniya orbit for the potential upcoming Canadian PCW (Polar Communication and Weather) mission. There are two primary objectives for PCW [3]. The first is to provide near-24 hour communication services to the northern population of Canada that is unable to be serviced by geostationary communications due to the high latitudes of the northern settlements. Second, meteorological measurements will be performed for the high-latitude regions for support of weather prediction and climate monitoring. At this point in the mission, PCW is planned as a constellation of two satellites placed in a single HEO to enable nearly-constant coverage of the Arctic region.

The initial study of the problem of modelling drag perturbations for an HEO with

---

¹See Section 2.1 for the definitions of the orbital elements.

²The satellite hovers near apogee since its velocity slows down due to the trade-off between the kinetic and potential energy of the satellite.

a sufficiently-low perigee radius led to the multiple literature sources which cited drag as an often secondary perturbation compared to sources such as the non-spherical Earth gravitational field and the gravitational attraction of the Moon and Sun [4]. Furthermore, simulations performed in the industry-standard orbit simulation software Systems Toolkit (STK) showed that indeed the effects due to atmospheric drag are quite small and that there are much-more-significant variations due to the lunisolar perturbations which are coupled with the oblateness of the Earth.

The first part of this thesis involved the study of the Moon and Sun acting on critically-inclined³ HEOs to understand the fundamental behaviour of the interaction between the gravity of the third body⁴ and a satellite's orbit. From there, using the detailed study of the lunisolar perturbations, a method to exploit the perturbations acting on the HEO was developed as part of an orbit control strategy. This strategy maintains the eccentricity, inclination, and AOP within some prescribed tolerances and estimates the ΔV ⁵ requirements. Additionally, the ground-track of the HEO is controlled. While the PCW mission was used as inspiration, the methodologies developed in thesis were kept intentionally generic so that they can be applied to other missions as well.

The combination of the perturbations from oblateness of the Earth and the gravity of the Moon and Sun causes large variations in the orbital elements of an HEO [4]. Depending on the satellite mission, these variations may interfere with the objectives imposed by the type of payload on-board. The lunisolar perturbations also affect the lifetime of the satellite orbit if the initial perigee altitude is set low enough, as is the case for certain

---

³A critically-inclined orbit has an inclination of 63.4° . This will be discussed in more detail in subsequent chapters.

⁴In this thesis, a 'third body' is defined as an extra-terrestrial object of sufficient mass to be able to affect the orbit of an artificial Earth satellite, a.k.a. the Moon or Sun.

⁵The term ΔV is the change in velocity of the satellite using a propulsion system used to control the orbital elements. The parameter is an analog for on-board fuel requirements.

Molniya orbits. As a result, for missions such as the upcoming PCW, a control strategy must be implemented to maintain some of the orbit parameters.

A control strategy that exploits the lunisolar perturbations to reduce the required ΔV to maintain the eccentricity, inclination, AOP, and ground-track was developed. The control strategy was designed to perform a set of impulsive manoeuvres to maintain the orbital elements within a specified set of limits. It is used as a mission planning tool in order to estimate the total mission ΔV for an HEO. The lunisolar perturbations are exploited using the grazing method, which accounts for the change in sign of the variations and applies a manoeuvre to adjust a given orbital element such that the time when the slope of the corresponding variations is zero coincides with the time when the orbital element reaches the edge of its allowable range. The change in sign of the rate of change of the variations causes the element to then drift back towards the nominal value.

This novel strategy for controlling the orbital elements of an HEO depends on two predictions of the variations: the time when the slope is zero and the value of the orbital element at that time. The control strategy developed in this thesis employs an analytical method to predict these values. As outlined in References [5] and [6], there exist conditions based on the relative geometry of the third body and satellite orbit whereby it is known that the perturbations due to the third body for a given orbital element are equal to zero. The times of occurrence of these conditions can be predicted using the common assumption of circular orbits of the Moon with respect to the Earth and the Earth with respect to the Sun [7]. This time is called the ‘switch point’ time as it indicates when the sign of the perturbation switches. The value of the orbital element at the switch point time is predicted using a set of analytical equations developed for this thesis. The equations, which are set as functions of time, model the effects of the Moon and Sun on HEOs

for periods of time less than one half year. The orbital control strategy is thus able to analytically predict the required ΔV for each manoeuvre in order to maintain the orbital elements near some nominal value such that the natural perturbations caused by the Moon and Sun are exploited.

There are three separate types of HEOs that are under study in this thesis. The first is the Molniya⁶ orbit. The Molniya orbit is characterized by its orbital period equal to one half of a sidereal day⁷ which equates to approximately 11h 58m [8]. This particular orbit, as a result, will complete two revolutions of its period before repeating its ground-track. The perigee altitude for the particular type of Molniya orbit studied here is 1000 km. The initial elements for the Molniya orbit are shown in Table 1.1. As with all the HEOs discussed here, the inclination is set to the critical value of 63.4° and the AOP is set to 270° . The initial value of the right ascension of the ascending node (RAAN) of the HEO is not given here because it is mission dependent and will be calculated in Chapter 4 as part of the control strategy. In 1965, the first satellite was launched into a Molniya orbit and there have been over 100 satellites launched into the orbit since then, and these have been primarily Russian communications satellites [9].

The second HEO type is called the Three APogee (TAP) orbit. The TAP orbit has its period set such that it will repeat its ground-track after three orbit revolutions over two sidereal days [10]. As a result, it has a period of approximately 16 hours. The orbital elements of the TAP orbit are shown in Table 1.1. The inclination and AOP are set to the same values as the Molniya orbit and the RAAN has not been defined yet. No satellite has flown in a TAP orbit for Arctic observation as of yet, but it has been proposed as a

---

⁶The word *Molniya* is Russian for lightning.

⁷A sidereal day is the time required for an observer on the Earth to revolve once and observe the stars in the same direction [8].

potential orbit for the PCW mission [10]. One of the many advantages of the TAP orbit over the Molniya orbit is the satellite's avoidance of the more harsh regions of the Earth's radiation belt.

The third and final HEO is the Tundra orbit. This orbit has a period equal to a single sidereal day and will repeat its ground-track after a single revolution. This period is the same as the geosynchronous orbit, 23h 56m. The resulting semi-major axis and eccentricity are shown in Table 1.1. Tundra orbits are not as common as Molniya orbits but have seen use in the commercial industry. Sirius Satellite Radio operates three satellites in Tundra orbits as part of their global constellation to provide coverage for the majority of the planet [11]. These satellites use their on-board propulsion systems to maintain their orbital elements based on their coverage requirements.

Table 1.1: Orbital elements of various HEOs.

Orbit type	Semi-major axis axis (km)	Eccentricity	Inclination (deg)	RAAN (deg)	AOP (deg)
Molniya	26561.1	0.722	63.4	Mission Dependent	270
TAP	32175.0	0.550	63.4	Mission Dependent	270
Tundra	42164.1	0.357	63.4	Mission Dependent	270

Figure 1.1 depicts a schematic of the relative shapes and sizes of the three HEO types compared to common circular orbits such as a geostationary orbit, a GPS orbit, and a satellite in LEO. For all three HEOs, the apogee altitude extends beyond that of the geostationary orbit. The perigee altitudes of the HEOs differ considerably however. For the Molniya orbit the perigee altitude is set at the edge of the LEO region, whereas the Tundra orbit's perigee altitude is near the GPS satellite orbit. Figure 1.2 shows a schematic of the orientations of the HEOs compared to the common circular orbits from Figure 1.1. The figure demonstrates how the inclination and AOP of the HEOs enables a satellite in those

Figure 1.1: Schematic comparing relative orbit sizes and shapes of Molniya, TAP, and Tundra orbits to common circular orbits.

orbits to be well-suited for Arctic observations.

The ground-track of an orbit is the line on the surface of the Earth where the satellite passes directly overhead. Since the ground-track itself is not a measurable parameter of the orbit as is the eccentricity, the discussion of ground-track control refers to maintaining the longitude of the ascending node (LAN), λ , with respect to the prime meridian. Since the periods of the HEOs are equal to one-half, two-thirds, or a full sidereal day, under ideal conditions their LANs do not vary and their ground-track patterns remain constant. In reality, however, various perturbation sources affect the semi-major axis and hence the LAN and the ground-track may vary outside acceptable margins. Figures 1.3 through 1.5

Figure 1.2: Schematic comparing orbit orientations of Molniya, TAP, and Tundra orbits to common circular orbits.

Figure 1.3: Ground-track of a Molniya orbit.

show a sample ground-track for each of the HEOs. In Figure 1.3, the Molniya orbit is shown to reach its apogee twice (the point of highest latitude) as it completes its ground-track revolution. The naming of the TAP orbit is demonstrated in Figure 1.4. A satellite in a TAP orbit passes its apogee three times before it completes a single ground-track revolution. With a period equal to a single sidereal day, the Tundra orbit's ground-track, shown in Figure 1.5, is a closed curve that only covers a specific longitude region of the Earth as it progresses through its orbit.

1.2 Outline of Research

Chapter 2 - Dynamic Formulation

In this chapter, the formulation of the dynamic forces involved in modelling the trajectory of a satellite in an HEO is presented. Before the dynamics are discussed in detail, however, the orbital elements used to represent an orbit are introduced as well as the coor-

Figure 1.4: Ground-track of a TAP orbit.

Figure 1.5: Ground-track of a Tundra orbit.

dinate systems used and the frames of reference. The next section in Chapter 2 deals with the discussion of the force model acting on an HEO. The methods to model the perturbations due to the Earth's non-spherical gravitational field are presented first. Following that, a literature review of the methods to model lunisolar perturbations is conducted. Using numerical simulations performed in STK, the choice of the force model for simulating a satellite in an HEO is then validated. After the force model discussion, a new method to analytically model the lunisolar perturbations as well as the oblateness of the Earth is developed. Finally, a study on the sensitivity of the numerical simulations executed in this research is performed to confirm that the system of modelling lunisolar perturbations on the aforementioned HEOs is not overly sensitive to truncation error of the initial conditions.

Chapter 3 - Dynamic Analysis

Chapter 3 focuses on the development of a geometric method to determine the conditions, in terms of the relative HEO and third body geometry, when the rates of change of the orbital elements due to third-body perturbations are zero. These occurrences are called switch points. For each orbital element the general development of the method to determine the conditions for the occurrence of a switch point is presented. Furthermore, a method to predict future occurrences of switch points is developed for the to-be-controlled orbital elements: the eccentricity, the inclination, and the AOP.

Chapter 4 - Control Strategy

The development of the orbit control strategy, which exploits the lunisolar perturbations, is presented in Chapter 4. The purpose of the strategy is to maintain the eccentricity, inclination, AOP, as well as the ground-track position by utilizing the correspondingly pre-

dicted switch points from Chapter 3. Chapter 4 begins with a literature review of existing methods to control HEOs. The method to exploit the lunisolar perturbations is based on the grazing method. The analytical equations developed in Chapter 2 are evaluated at the predicted switch point times. The result is then used to determine the amount of ΔV required to exploit the future changes in sign of the perturbations. This is called the Analytical Prediction and Control (APC) method. Two alternate control methods are also developed: the Numerical Prediction and Control (NPC) method, and the Baseline control method. The NPC method also exploits the lunisolar perturbations just as the APC method; however, it performs an extra set of numerical simulations to predict the future perturbations acting on the HEO and to find the switch points based on numerical data. The NPC method is used to demonstrate that the analytical methods of the APC method to compute the ΔV to exploit the perturbations are valid. The Baseline method performs a simple correction of eccentricity, inclination, and AOP to set them back to their nominal values. It has no future knowledge of the perturbations and acts as a comparison to the APC and NPC methods to quantify the reduction of the mission ΔV by exploiting the lunisolar perturbations.

Chapter 5 - Conclusion

To conclude this thesis, a summary of the accomplishments of this research is presented. The future work that could be performed is then discussed. The final conclusions are then made with regards to the nature of the lunisolar perturbations on HEOs and the ability to exploit these effects in a control strategy.

Chapter 2

Dynamic Formulation

In this chapter, the focus is on the dynamic formulation of the perturbing forces acting on the HEOs. The dynamic formulation is the development of the methods to model the perturbations acting on a satellite. Utilizing these dynamic models, numerical integration can be used to simulate the behaviour of a satellite in an HEO under the primary perturbation forces. An independent analytical method for modelling the primary perturbations was also derived for this thesis. This analytical dynamic model is used as part of the orbit control strategy to predict the future uncontrolled variations of the orbital elements in order to compute the required ΔV to exploit the lunisolar perturbations rather than resist them.

2.1 Orbital Elements and Coordinate Systems

To model the orbit of a satellite, a set of six variables is required. Depending on the orbit type and the method of analysis performed for the orbit, there exist many variable sets to choose from. The most common set is called Keplerian orbital elements, also known as the classical orbital elements. Other element sets that exist include equinoctial

elements, Delaunay elements, or the three-dimensional position and velocity vectors of the satellite [8]. For this thesis, Keplerian elements are used because the variables are based mostly on physical parameters that can be visualized easily. As such, the other element sets that were mentioned shall not be discussed in this thesis.

The Keplerian elements set is based on the simple two-body motion of a point-mass, the satellite, orbiting another point-mass, the Earth or another planetary body¹. In this type of motion, the orbit of the satellite is modelled using conic sections as stated by Kepler's laws of astrodynamics. Since this thesis involves the study of HEOs, an ellipse can be used to model the satellite's orbit. The **semi-major axis** (a) is one half of the major axis length and the **eccentricity** (e) defines the shape of the orbit and the type of conic section [12]. Eccentric orbits have an eccentricity between 0 and 1. Figure 2.1 shows an elliptical orbit of a satellite with respect to a planetary body. The periapsis (or perigee when referring to the Earth) is the point of orbit closest to the planetary body; the apoapsis (apogee) is the point furthest from the body. The remaining four orbital elements define the orientation of the orbit with respect to a coordinate frame as well as the location of the satellite within the orbit and are shown in Figure 2.2. The **inclination** (i) is the angle between the orbit plane and the equatorial plane of the Earth, the **RAAN** (Ω) is the angle between the unit vector $\hat{I}$ (which is directed towards the vernal equinox²) and the ascending node of the orbit ($\vec{N}$) measured in the equatorial or reference plane [8]. The ascending node is the location on the equatorial plane where the satellite crosses from the southern hemisphere to the northern hemisphere. The **AOP** (ω) is the angle between the ascending node vector and the perigee of the satellite's orbit, in the orbit plane. The

---

¹Two-body motion is defined as a system of two bodies where the mass of one body is negligible compared to the mass of the other [8].

²The vernal equinox is the direction when the Earth crosses the ecliptic plane of the Sun from the southern hemisphere to the northern hemisphere [8]. More details of this parameter are provided later in this section.

true anomaly, identified as ν , is the angle between the satellite's position vector, $\vec{R}$, and the perigee, in the plane of the satellite's orbit, measured counter-clockwise when viewed from the northern hemisphere [8]. An alternate element for the location of the satellite within the orbit is the mean anomaly, M . The mean anomaly is the angle measured from the perigee, that corresponds to the equivalent uniform circular motion on a circle with a radius equal to the semi-major axis [8]. A third element that can be used to define the position of a satellite or planetary body within its orbit is the argument of latitude (AOL). The AOL, $u = \omega + \nu$, is the angle between the radius vector and the ascending node vector and is calculated by summing the true anomaly and the AOP.

Figure 2.1: Two-dimensional diagram of an eccentric orbit.

Aside from the standard six orbital elements, there exist other orbit parameters which are functions of the Keplerian elements but that are sometimes needed in the analysis of the dynamics and control of HEOs. The flight-path angle is the angle between the velocity vector of the satellite and the 'local horizontal' vector, which is a vector perpendicular to the radius vector of the satellite in the orbital plane [8]. The equations to solve for the

Figure 2.2: Three-dimensional diagram of an eccentric orbit.

flight-path angle (ϕ) are:

$$\sin \phi = \frac{e \sin \nu}{\sqrt{1 + 2e \cos \nu + e^2}} \quad (2.1)$$

$$\cos \phi = \frac{1 + e \cos \nu}{\sqrt{1 + 2e \cos \nu + e^2}} \quad (2.2)$$

The period of an orbit, assuming two-body motion, is quite straightforward to calculate since it is based on the semi-major axis³, $P = 2\pi\sqrt{a^3/\mu}$. It is also a function of the gravitational parameter, μ , of the planetary body being orbited. However, there is a more practical period that takes into account the large perturbations due to the oblateness of the Earth. The nodal period (P_Ω) is the time interval between two successive equator crossings (either at ascending or descending nodes). This period is important for HEOs that require a commensurability⁴ with the rotation of the Earth in order to maintain a

³This is commonly called the Keplerian period and is based on Kepler's third law [8].

⁴A commensurable relationship is one where two bodies have an integer value of their relative periods.

constant ground-track such as those under study in this thesis. The nodal period is based on the Keplerian period but also includes the effects of the Earth's oblateness on the AOP and the mean anomaly which are discussed in Section 2.2.1. The equation for the nodal period is defined as:

$$P_{\Omega} = \frac{P}{1 + \xi [4 + 2\sqrt{1 - e^2} - (5 + 3\sqrt{1 - e^2}) \sin^2 i]} \quad (2.3)$$

where $\xi = 3R_E^2 J_2 / (4p^2)$ [13].

As mentioned in the previous chapter, the control of the LAN will enable the orbit to maintain its repeat ground-track coverage of the Earth. While the LAN is not a Keplerian element, it is a function of these elements. The LAN of the satellite is the longitude with respect to the prime meridian when the satellite reaches its ascending node, i.e., when the AOL is equal to zero. The drift of the LAN is a function of the aforementioned nodal period. As a result, the variations of the LAN are not caused by the lunisolar perturbation forces but instead by the non-spherical Earth gravity model, discussed in Section 2.2.1. It will be demonstrated later in this chapter that the lunisolar perturbation forces have no long-term effects on the semi-major axis. Thus, the development of the control strategy using switch points is restricted to only the eccentricity, inclination, and AOP. Control of the LAN will be performed as a part of the manoeuvres to control the eccentricity and AOP. This is discussed in much greater detail in Chapter 4.

Since the orbit is defined in terms of the Keplerian elements, to model the forces acting on HEOs, six equations of motion are necessary. There are two main methods to capture the effects of the perturbations using ordinary differential equations (ODEs): the

Lagrange planetary equations and the Gauss planetary equations [8]. Both approaches use the method of variations of parameters to create a set of equations that compute the rate of change of the orbital elements due to some disturbance from two-body motion. The Lagrange equations requires that the calculations of the perturbations be in the form of a partial derivative of the disturbing function. Therefore, only conservative forces can be included in this method. The Gauss method requires that the perturbations be modelled in terms of a specific force and can thus accept both conservative and non-conservative forces [8]. Since the control of a satellite orbit requires a non-conservative thrust to adjust any given orbit element, the following equations, based on the Gauss method, are used:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}} \left\{ e \sin(\nu) F_R + \frac{p}{r} F_S \right\} \quad (2.4)$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na} \left\{ \sin(\nu) F_R + \left(\cos(\nu) + \frac{e + \cos(\nu)}{1 + e \cos(\nu)} \right) F_S \right\} \quad (2.5)$$

$$\frac{di}{dt} = \frac{r \cos(u)}{na^2 \sqrt{1-e^2}} F_W \quad (2.6)$$

$$\frac{d\Omega}{dt} = \frac{r \sin(u)}{na^2 \sqrt{1-e^2} \sin(i)} F_W \quad (2.7)$$

$$\begin{aligned} \frac{d\omega}{dt} = & \frac{\sqrt{1-e^2}}{nae} \left\{ -\cos(\nu) F_R + \sin(\nu) \left(1 + \frac{r}{p} \right) F_S \right\} \\ & - \frac{r \cot(i) \sin(i)}{h} F_W \end{aligned} \quad (2.8)$$

$$\frac{dM}{dt} = n + \frac{1}{na^2 e} \{ (p \cos(\nu) - 2er) F_R - (p + r) \sin(\nu) F_S \} \quad (2.9)$$

The Gauss planetary equations are a set of nonlinear equations for the rates of change

of the Keplerian elements. The perturbations acting on the satellite orbit are modelled as a specific force vector $\vec{F} = F_R\hat{R} + F_S\hat{S} + F_W\hat{W}$ in the RSW coordinate system where F_R is the radial component, F_S is the transverse component, and F_W is the orbit normal component. Figure 2.3 shows the schematic of the RSW coordinate system. The forces in this satellite-centred coordinate system can be derived from a disturbing function (R) using the following equations:

$$F_R = \frac{\partial R}{\partial r} \quad (2.10)$$

$$F_S = \frac{1}{r} \frac{\partial R}{\partial u} \quad (2.11)$$

$$F_W = \frac{1}{r \sin u} \frac{\partial R}{\partial i} \quad (2.12)$$

where r is the magnitude of the satellite radius vector with respect to Earth.

Integrating the ODEs using numerical methods results in the variations of the orbital elements as a function of time. In this thesis, there are four classifications to identify primary effects of the perturbation forces on the orbital elements: secular (non-periodic), long-periodic, medium-periodic, and short-periodic. The short-period oscillations are those with a period equal to or less than the orbital period of the satellite. The long-period classification is applied to oscillations modes with a period equal to or greater than the order of magnitude of the mission duration, which for these particular HEO missions ranges from five to fifteen years. Medium-period oscillations are thus any oscillation mode with a period greater than the satellite orbit period, but less than the mission period. Since the primary purpose of the orbital control strategy is to provide general mission planning

Figure 2.3: The RSW coordinate system projected onto a unit sphere.

capabilities for an HEO, only the medium- and long-period oscillations, as well as the secular variations, are of interest. Therefore, the ODEs used to model the perturbation forces must be averaged to eliminate the short-period effects.

The use of these so-called mean elements is very common in the study of orbital dynamics as a method to focus primarily on the most significant effects. Rather than using the ‘osculating’⁵ Keplerian elements, which are the true time-varying orbital elements that include all types of variations, mean elements work well for predicting the satellite’s long-term behaviour [8]. To transform equations with osculating elements to mean elements, a fast-variable that is responsible for the short-period effects is identified, such as the mean anomaly of the satellite, and the equations are averaged over the period of that variable.

---

⁵This term comes from the Latin word *osculate*, which means to kiss [12].

The conversion typically takes the form of $\bar{x} = \frac{1}{2\pi} \int x dM$, where x is any orbital element. The resulting $\bar{x}$ now has the same long-term trends but with the high-frequency short-periodic variations removed. This process is also called ‘single-averaging’ [8]. In STK, there exist several methods to compute the mean elements. For the purposes of this thesis, the Kozai-Izsak mean elements are used which only consider averaging the short-period effects involving the Earth’s oblateness [14].

In Figure 2.2, the orbital angles were defined in an Earth-centred theoretically-inertial coordinate system based on the Earth’s equatorial plane, its north polar axis $\hat{K}$, and the line of equinoxes $\hat{I}$. In reality, these vectors are not inertial but can be thought of as pseudo-inertial. The gravitational effect of the Sun and Moon on the Earth itself causes a precession effect in the polar axis and the planetary gravitational forces cause a precession of the ecliptic plane [8]. There is also a small nutation effect on the polar axis which is due to a torque produced by the Moon acting on the bulge of the equator. The ‘true-of-date’ (TOD) system models the planetary and lunisolar precession of the equinox as well as the nutation effects on the equator on a given date and it is used in this study. This is because the methods to model the orbits of the Moon and Sun, discussed in more detail in Section 2.2.2, are output in the TOD system. There exist many other systems such as the mean-equinox-of-date, which does not take the nutation into account; the ‘true-of-epoch’ system which fixes the orientation of the coordinate system to an external date [8]; and the ‘J2000’ system which uses the mean equator and mean equinox values set at January 1, 2000 at noon GMT. These alternatives have their advantages depending on the applications of the orbit analysis involved and have been introduced by scientists for the purposes of standardization. For long-term mission analysis, any system would work equally well as long as there is commonality among all the elements defined by the coordinate systems.

2.2 Force Model of HEOs

Each orbit regime, from LEO to geostationary orbits, will have different primary perturbation forces that must absolutely be included to accurately model the variations of the orbital elements. For Earth-orbiting spacecraft, there are four major perturbation forces that can potentially be a primary effect: the non-spherical gravitational model of the Earth, lunar and solar gravity, atmospheric drag, and solar radiation pressure. The non-spherical gravity model is due to the non-symmetric distribution of mass of the Earth. These Earth gravity effects are commonly modelled as a series of harmonic terms which are used to define the Earth's shape. Depending on the orbit type, there could be multiple harmonics that are classified as a primary perturbation. Section 2.2.1 reviews a standard method to model the Earth's gravity. Lunisolar perturbation forces, which are technically two independent perturbation sources, are the forces due to the gravitational attractions of the Moon and the Sun. These perturbation forces are superimposed since the method to model these effects is often the same. Since part of the focus of this thesis is the study of the dynamics of lunisolar perturbations, in Section 2.2.2, multiple methods to model these perturbation forces are presented and discussed in an attempt to ensure a complete understanding of the effects of the Moon and the Sun. Since the effective Earth atmosphere that affects satellite orbits is negligible above approximately 1000 km, the HEOs that are under study here are not affected by atmospheric drag and it can be neglected. Even the Molniya orbit, having its perigee set at 1000 km, is not affected by atmospheric drag enough to warrant its inclusion [15]. For this study, the final primary perturbation force, solar radiation pressure, is also considered a secondary effect and not included in the force model, due to its negligibly small effects compared to the much larger geopotential and lunisolar perturbations [8, 15].

Following the development of Earth gravitational and lunisolar perturbation models in this chapter, the selected force model is validated by performing simulations in STK with various perturbation models. This is performed to confirm the level of detail needed to accurately model the variations of the HEOs for mission planning purposes. Finally, the variations of the orbital elements of the HEOs are presented to demonstrate the magnitude of the perturbations acting on the HEOs and to provide a sense of the control requirements for offsetting these perturbations.

2.2.1 Non-spherical Earth Gravitational Model

The non-spherical shape of the Earth and its uneven mass distribution create a non-uniform gravitational field. Modelling the gravitational field is a geodetic problem that is dealt with in geodesy [16, 17]. The common practice is to model the conservative field using an infinite harmonic potential series since the gradient of the potential produces acceleration. This is accomplished by using solid spherical harmonics⁶ according to the Sturm-Liouville theorem to constitute the gravitational model [8]. There are three types of harmonic terms: zonal, sectorial, and tesseral. Zonal harmonics are longitude-independent and have their field symmetric about the polar axis. Sectorial harmonics are latitude-independent and are symmetric about the equatorial plane. Tesseral harmonics are functions of both latitude and longitude and model specific regions on the Earth. Using this method, the disturbing potential⁷ due to the non-spherical gravity of the Earth is modelled as [17]:

---

⁶The spherical harmonics used here are the division of a sphere into sections to model a non-uniform potential field.

⁷The disturbing potential models the Earth gravitational attraction that is not accounted for by an idealized Earth point mass or by a spherically-symmetric distribution of mass.

$$R = \sum_{l=2}^{\infty} \sum_{m=0}^l \sum_{p=0}^l \sum_{q=-\infty}^{\infty} \frac{\mu}{a} \left(\frac{R_E}{a} \right)^l F_{lmp}(i) G_{lpq}(e) S_{lmpq}(\omega, M, \Omega, \theta_{GMST}) \quad (2.13)$$

where,

$$S_{lmpq}(\omega, M, \Omega, \theta_{GMST}) = \begin{cases} C_{l,m} \cos(\Theta_{lmpq}) + S_{l,m} \sin(\Theta_{lmpq}) \\ \quad \text{[if (l-m) is even]} \\ -S_{l,m} \cos(\Theta_{lmpq}) + C_{l,m} \sin(\Theta_{lmpq}) \\ \quad \text{[if (l-m) is odd]} \end{cases} \quad (2.14)$$

and

$$\Theta_{lmpq} = (l - 2p)\omega + (l - 2p + q)M + m(\Omega - \theta_{GMST}) \quad (2.15)$$

Here, $C_{l,m}$ and $S_{l,m}$ are gravitational coefficients which model the non-spherical potential of the Earth's gravity based on satellite measurements [8]. The terms $F_{lmp}(i)$ and G_{lpq} are known as the inclination function and eccentricity function, respectively, and their values can be determined using tables in Reference [16].

It is the Earth's oblateness that represents its most significant deviation from a sphere. As such, a zonal harmonic term is used to account for this gravitational perturbation force. When $l = 2$ and $m = 0$, Equation 2.13 can be simplified to only account for the perturbation due to the Earth's oblateness. Based on the definitions of $C_{l,m}$ and $S_{l,m}$, the coefficient $S_{l,0}$ is always zero [18]. Therefore, the gravitational coefficient to model the Earth's oblateness is equal to $C_{2,0}$. It is common for the zonal harmonics to be symbolized

by J , where $J_l = -C_{l,0}$ [8]. Since the oblateness is the largest contributor to the non-spherical shape of the Earth, the value for J_2 is the largest gravitational coefficient by a factor of approximately 400 (see Appendix A). As such, the ‘ J_2 effect’ is often used to model the effect of the oblateness of the Earth on the orbital elements.

While J_2 is by far the most-dominant contributor to the non-spherical Earth gravity perturbation force, there exist higher-order gravitational coefficients that can also significantly affect the satellite’s orbit depending on the type of orbit. However, since the disturbing potential is a function of $(R_E/a)^l$, as the index l increases, the magnitude of the disturbing function for a given value of l decreases. Geostationary satellites must take into account the effects of J_2 as well as the sectorial harmonics $C_{2,2}$ and $S_{2,2}$ in order to ensure that the Earth’s slight triaxiality⁸ is modelled which affects its longitude drift [19]. Satellites in LEO must also account for many higher-order harmonics compared to satellites in higher orbits since the $(R_E/a)^l$ term in Equation 2.13 is much closer to unity for these orbits.

In the mid 1990’s, a series of papers was published by T. Ely and K. Howell regarding occurrences of resonance using highly-inclined HEOs due to the tesseral harmonic terms in Equation 2.14 [9, 20]. The authors state that this resonance is due to the commensurability of the orbital period of the satellite with that of the Earth’s rotational speed. Using Hamiltonian dynamics, the authors show that the tesseral harmonics can create highly-complex and sometimes chaotic behaviour in the variation of the semi-major axis. From these results, Ely goes on to mention that as a result of this dynamic behaviour, classical control strategies to maintain the satellite’s orbit may no longer be useful. However, in his thesis only the ground-track is controlled; there is no mention of control of the other

---

⁸Triaxiality refers the elliptic shape of the cross-section of the Earth’s equator [19].

orbital elements such as the eccentricity, inclination, or AOP.

It is useful to reduce Equation 2.13 to consider only the secular effects due to J_2 and combine them with the Gauss ODEs to quantify the magnitude of the effects on the orbital elements. The resulting derivations show that there is no secular effect on the semi-major axis, eccentricity, or inclination. There is, however, a measurable effect on the RAAN, AOP, and mean anomaly [8]. The resulting effects are modelled as:

$$\dot{\Omega}_{sec} = -\frac{3nR_E^2 J_2}{2p^2} \cos(i) \quad (2.16)$$

$$\dot{\omega}_{sec} = \frac{3nR_E^2 J_2}{4p^2} \{4 - 5 \sin^2(i)\} \quad (2.17)$$

$$\dot{M}_{sec} = n - \frac{3nR_E^2 J_2 \sqrt{1-e^2}}{4p^2} \{3 \sin^2(i) - 2\} \quad (2.18)$$

As mentioned in Chapter 1, the HEOs have a critical inclination of 63.4° ⁹. This is the result of trying to minimize the variations on the AOP by eliminating the secular effects of J_2 by setting the inclination such that $\dot{\omega}_{sec} = 0$. Equation 2.17 is used to solve for these critical values. The effects from J_2 must also be accounted for to accurately compute the time between nodal crossings. Equations 2.17 and 2.18 are also used in the derivation of the nodal period of an orbit from Equation 2.3 [13].

Since the inclination is set to minimize the rate of change of the AOP due to J_2 , the largest perturbation force acting on the HEO due to J_2 is on the RAAN. Even compared to the third-body perturbations on the RAAN, the J_2 effect on the RAAN is the most

---

⁹An inclination of 116.6° is also technically a critical inclination.

dominant. As mentioned in Chapter 1, and explained further in the next section as well as in Chapter 3, there is a coupling between the variations of the RAAN (primarily due to J_2) and the lunisolar perturbation forces. Each HEO has a significantly different secular rate of change of the RAAN, which affects the period of the oscillations of the long-period lunisolar perturbations. Based on the initial conditions for the HEOs in Table 1.1, for a Molniya orbit, the rate of change of the RAAN¹⁰ is approximately $-0.13^\circ/\text{day}$. For the TAP and Tundra orbits, the rates of change were determined to be approximately $-0.032^\circ/\text{day}$ and $-0.0079^\circ/\text{day}$, respectively.

2.2.2 Lunar and Solar Perturbations

The study of third-body perturbations¹¹ began as part of lunar theory, i.e., studying the gravitational effect of the Sun on the orbit of the Moon. In the dawn of the satellite age in the mid-20th century, these studies evolved to trying to model the effects of both the Moon and Sun on artificial Earth satellites.

In 1959, L. Blitzer made efforts to analyze the influence of the Sun on a close Earth satellite by averaging the solar gravity over the entirety of its apparent orbit which he phrases as “equivalent to the Sun being spread out into a ring of matter” [21]. From there he goes on to compute that the perturbations on a satellite due to the Moon is approximately 2.2 times greater than that of the Sun. As with many investigations of the time, to analyze the radial acceleration on the satellite orbit, the assumption was made that its orbit was circular.

---

¹⁰These rates of change are due to the combined effect of J_2 and the secular component of the lunisolar perturbation forces.

¹¹Lunar and solar perturbations are a specific type of third-body effect.

Also in 1959, E. Upton et al. specifically discuss the effects of the lunar and solar gravity on HEOs and mention the large oscillations induced into the perigee height, and therefore eccentricity [22]. This paper also hints at the relationship between the long-period oscillations in the eccentricity and the initial RAAN of the HEO, stating that the launch conditions (i.e., time of launch) may have a significant effect.

P. Musen published two papers in 1961 which studied the ‘long-period’¹² lunisolar effects on artificial Earth satellites [23, 24]. To study the medium- and long-period effects, the author averaged the perturbations with respect to the positions of the Moon and the satellite. Therefore, the lunar effects only included long-period oscillations whereas the solar effects included both medium- and long-period effects. This is similar to the analytical model that is developed in Section 2.3.

While it is known that the magnitude of the lunar effects on a close Earth satellite are generally twice as large as the solar effects, in 1962 D.E. Smith showed that the cumulative effect of the medium-period oscillations due to the Sun is larger than the medium-period oscillations of the Moon [25]. He states this is due to the Moon’s shorter period since the apparent orbit period of the Sun relative to the Earth is longer than the orbit period of the Moon around the Earth. Smith also goes on to discuss how for a given lunar or solar orbital period there will be two maximum and minimum rates of change of the orbital elements due to third-body perturbations. This concept led to the development of the switch point analysis and is discussed further in Chapter 3.

The coupling effect of J_2 and the lunisolar perturbations is mentioned in papers written

---

¹²The word long-period is put in quotations to differentiate it from the definition used in this thesis. In the papers, long-period oscillations represent any oscillations that have periods greater than the orbital period of the satellite.

by R.R. Allan and by B.A. Shute and J. Chiville [26, 27]. In Reference [26], Allan states how the third-body perturbations are highly dependent on the orientation of the orbit, and that they can change sign as the orbit rotates about the polar axis. Shute and Chiville specifically state how near-Earth HEOs exhibit large changes in the RAAN which affect the behaviour of the lunisolar perturbations. Since the RAAN is primarily affected by the oblateness of the Earth, it is said that J_2 has an indirect effect on the medium- and long-period oscillations of the orbital elements.

The coupling effect is demonstrated to be significant in D.G. King-Hele's 1975 paper where he studies the orbital lifetimes of Molniya satellites [2]. In the paper, the author develops an analytical method for estimating the lifetime of Molniya orbits with perigee altitudes low enough that the long-period oscillations of lunisolar effects can cause the satellite to prematurely re-enter the Earth's atmosphere. The result of the paper is an empirical relationship between the initial RAAN of the orbit and its lifetime assuming no orbit control is available. While this analysis further emphasizes the sensitivity of the initial RAAN on the behaviour of the lunisolar perturbations, the lifetime estimations are not relevant to this thesis since an orbit control system is employed to maintain the eccentricity.

Resonance is mentioned as a result of the lunisolar perturbations by many papers. It depends on the conditions of the orbit such as when the rate of change of the RAAN or the AOP is equal to zero [28, 29, 30, 31]. Since the inclination of the HEOs in this study is set to the critical value, the rate of change on the AOP due to the oblateness effects from J_2 is near zero¹³. These resonance effects, along with those due to commensurability of the satellite orbital period with the rotational period of the Earth, may cause large variations in the orbital elements [9].

---

¹³There are secular and medium- and long-period effects of the AOP due to third-body effects so the rate of change is not exactly zero [8].

In the late 1980's, a satellite program was conceived by the European Space Agency (ESA) to launch a constellation of satellites in a Molniya or Tundra orbit to improve the land-mobile communications in Europe [32]. Even though the project was eventually cancelled, several papers were published on the subject of analyzing the perturbations caused by the gravity of the Moon and Sun. In F. Delhaise's paper [33], which studied lunisolar effects at the critical inclination, the author used Hamiltonian dynamics to demonstrate the perturbation effects on the orbits. To complete this study, the author made a few assumptions such as using only the first term in an infinite series summation equation to describe the disturbing potential of the lunar and solar gravity, as well as assuming that the orbital eccentricity of the Moon and Sun with respect to the Earth are both zero. Just as with King-Hele, Delhaise's paper stated that the effects of the lunisolar system are strong enough to prematurely reduce the perigee altitude such that the satellite re-enters the atmosphere, and that these effects are sensitive to the initial conditions of the satellite.

Many methods exist to model the effects of the Moon and Sun on Earth-orbiting satellites. Each method involves slightly different assumptions relating to the orbit type or to the manner in which the perturbations are analyzed. The following subsections presents various (but certainly not all) methods to model the lunisolar perturbations.

Newton's Gravitational Method

In Newton's gravitational method, the development of the equations to model the third-body perturbation effects begins with Newton's Law of Gravitation. If three bodies are placed in an inertial coordinate system, as shown in Figure 2.4, the perturbing third-body effects acting on one body can be computed with respect to another. This is accomplished

by summing the gravitational accelerations for Bodies 1 and 2 in the inertial frame, then adding the acceleration vector of Body 2 (m_2) from Body 1 (m_1). The acceleration due to gravity of all three bodies acting on Body 1 relative to Body 2, can be expressed as [12]:

$$\ddot{\vec{R}}_{12} = -\frac{G(m_1 + m_2)}{\|\vec{R}_{12}\|^3} \vec{R}_{12} - G \sum_{j=3}^n m_j \left(\frac{\vec{R}_{j2}}{\|\vec{R}_{j2}\|^3} - \frac{\vec{R}_{j1}}{\|\vec{R}_{j1}\|^3} \right) \quad (2.19)$$

where m_i are the masses of the bodies and $\vec{R}_{13}$ is the radius vector from Body 1 to Body 3. The first term on the right hand side is the primary acceleration of Body 1 orbiting Body 2, while the second term represents the perturbations acting on Body 1 due to third-body gravitational forces of Body 3.

Figure 2.4: The three body problem.

Equation 2.19 can be numerically integrated along with $\dot{\vec{R}}_{12} = \vec{V}_{12}$ to model the third-body perturbations, bypassing the need to use the Gauss ODEs. While this purely nu-

merical method is an extremely simple model for the perturbations, modelling the effects in terms of Cartesian values is not desirable. Firstly, not having the functions in terms of orbital elements of the satellite or the third body inhibits the ability to develop any meaningful understanding as to the nature of the third-body effects and how the initial conditions of the satellite can strongly affect the behaviour of the perturbations as described earlier in this subsection. Secondly, as previously mentioned, for long-period analysis of the orbit, the short-period oscillations of the satellite are desired to be averaged out. This is impossible when integrating Equations 2.19. As a result, an alternate method to model the third-body perturbations is desired; one which is defined in terms of Keplerian elements of the satellite and third body.

Cook's Method

In 1962, G.E. Cook developed a method for determining the disturbing forces acting on a satellite due to the gravity of a third body [28]. Using a method developed by W.M. Smart in 1953, the disturbing function was developed in terms of the Cartesian components of the distances between the three bodies [34]. The transformation was accomplished by taking the partial derivatives of $1/r_{13}$ and $x_{21}x_{23}/r_{23}^3$ and substituting them into Equation 2.19, where Bodies 1, 2, and 3 are the satellite, Earth, and Moon or Sun, respectively. This method is possible because the third-body gravitational forces are conservative, which means they can be expressed in the form of a potential [35]. The disturbing function then can be written as [28, 34]:

$$R = \mu_3 \left(\frac{1}{r_{13}} - \frac{x_{21}x_{23} + y_{21}y_{23} + z_{21}z_{23}}{r_{23}^3} \right) \quad (2.20)$$

where, (x_{21}, y_{21}, z_{21}) represents the Cartesian position of the the satellite with respect to the Earth and (x_{23}, y_{23}, z_{23}) is the Cartesian position of a third body with respect to the

Earth, such that $r_{13}^2 = r_{21}^2 + r_{23}^2 - 2r_{21}r_{23}\cos(\zeta)$. The symbol ζ refers to the geocentric angle between the satellite and the third body, and μ_3 is the gravitational parameter of the third body.

The next step is to take the partial derivatives of the disturbing function with respect to x_{21} , y_{21} , and z_{21} , resulting in:

$$\begin{aligned}\frac{\partial R}{\partial x_{21}} &= -\mu_3 \left(\frac{x_{21} - x_{23}}{r_{13}^3} + \frac{x_{23}}{r_{23}^3} \right) \\ \frac{\partial R}{\partial y_{21}} &= -\mu_3 \left(\frac{y_{21} - y_{23}}{r_{13}^3} + \frac{y_{23}}{r_{23}^3} \right) \\ \frac{\partial R}{\partial z_{21}} &= -\mu_3 \left(\frac{z_{21} - z_{23}}{r_{13}^3} + \frac{z_{23}}{r_{23}^3} \right)\end{aligned}\tag{2.21}$$

The term $1/r_{13}^3$ is then factored outside the brackets and expanded as a Taylor series about $r_{21}/r_{23} = 0$. The series is limited to the first two derivatives in the Taylor series since subsequent terms will become smaller as the power term in the ratio r_{21}/r_{23} grows. This assumes that the size of the orbit of the satellite is small compared to that of the third body.

Now that the disturbing potential has been developed, the disturbing forces can be determined in the RSW coordinate system in terms of the Keplerian elements. The RSW components of the disturbing force due to third-body effects is determined by multiplying the partial derivatives of the disturbing function with the direction cosines of the RSW unit vectors. The resulting final equations set is:

$$\begin{aligned}
F_R = & -\frac{\mu_3 r}{r_3^3} \left[1 - \frac{3}{2}(A^2 + B^2) - 3AB \sin(2u) - \frac{3}{2}(A^2 - B^2) \cos(2u) \right. \\
& \left. + \frac{3r}{2r_3} (A \cos(u) + B \sin(u))(3 - 5(A \cos(u) - B \sin(u))^2) \right] \quad (2.22)
\end{aligned}$$

$$\begin{aligned}
F_S = & \frac{3\mu_3 r}{r_3^3} \left[AB \cos(2u) - \frac{1}{2}(A^2 - B^2) \sin(2u) \right. \\
& \left. + \frac{r}{2r_3} (A \sin(u) - B \cos(u))(1 - 5(A \cos(u) + B \sin(u))^2) \right] \quad (2.23)
\end{aligned}$$

$$F_W = \frac{3\mu_3 r C}{r_3^3} \left[A \cos(u) + B \sin(u) - \frac{r}{2r_3} \{1 - 5(A \cos(u) + B \sin(u))^2\} \right] \quad (2.24)$$

where r has replaced r_{21} , and r_3 has replaced r_{23} . The variables A , B , and C are called the direction cosines¹⁴ in Cook's original paper and are given by:

$$A = \cos(\Omega - \Omega_3) \cos u_3 + \cos i_3 \sin u_3 \sin(\Omega - \Omega_3) \quad (2.25)$$

$$\begin{aligned}
B = & \cos i \left[-\sin(\Omega - \Omega_3) \cos u_3 + \cos i_3 \sin u_3 \cos(\Omega - \Omega_3) \right] \\
& + \sin i \sin i_3 \sin u_3 \quad (2.26)
\end{aligned}$$

$$\begin{aligned}
C = & \sin i \left[\sin(\Omega - \Omega_3) \cos u_3 - \cos i_3 \sin u_3 \cos(\Omega - \Omega_3) \right] \\
& + \cos i \sin i_3 \sin u_3 \quad (2.27)
\end{aligned}$$

Combining Equations 2.22 through 2.24 with Equations 2.4 through 2.9 provides a model

---

¹⁴Only A is a direction cosine in the traditional sense.

of the lunisolar perturbations acting on a satellite in terms of the Keplerian elements. In this form, the relationship between the direction of the lunisolar perturbations and the RAAN of the orbit with respect to the RAAN of the third body, i.e., $\Omega - \Omega_3$, is easily observable since the direction cosines are all functions of this parameter.

Cook's Averaged Method

Cook's equations for the specific forces, as well as the Gauss ODEs, still include terms for the short-period oscillations, u and ν , that are undesirable for long-term mission analysis. As such, in his paper, Cook substitutes the specific force equations, Equations 2.22 through 2.24, into the set of ODEs based on the Gauss method, and integrates them over the orbital period of the satellite [28]. The result is a set of six single-averaged ODEs:

$$\frac{da}{dt} = 0 \quad (2.28)$$

$$\frac{de}{dt} = -\frac{15}{2n}e\sqrt{1-e^2}\frac{\mu_3}{r_3^3}\left[AB\cos 2\omega - \frac{1}{2}(A^2 - B^2)\sin 2\omega\right] \quad (2.29)$$

$$\frac{di}{dt} = \frac{3C}{4n\sqrt{1-e^2}}\frac{\mu_3}{r_3^3}\left[A(2 + 3e^2 + 5e^2\cos 2\omega) + 5Be^2\sin 2\omega\right] \quad (2.30)$$

$$\frac{d\Omega}{dt} = \frac{3C}{4n\sqrt{1-e^2}\sin i}\frac{\mu_3}{r_3^3}\left[5Ae^2\sin 2\omega + B(2 + 3e^2 - 5e^2\sin 2\omega)\right] \quad (2.31)$$

$$\begin{aligned} \frac{d\omega}{dt} = & \frac{3\sqrt{1-e^2}}{2n}\frac{\mu_3}{r_3^3}\left[5\left(AB\sin(2\omega) + \frac{1}{2}(A^2 - B^2)\cos 2\omega\right) - 1 + \frac{3}{2}(A^2 + B^2)\right. \\ & \left. + \frac{5a}{2er_3}\left(1 - \frac{5}{4}(A^2 + B^2)\right)(A\cos \omega + B\sin \omega)\right] - \frac{d\Omega}{dt}\cos i \end{aligned} \quad (2.32)$$

$$\frac{dM}{dt} = 0 \quad (2.33)$$

Equations 2.28 through 2.33 are now in a form that can be directly integrated by combining them with Equations 2.16 through 2.18 to model the major perturbations acting on an HEO¹⁵. It is important to note that the resulting equations show that the lunisolar perturbations have no medium- or long-period effects on the semi-major axis or the mean anomaly after the averaging process.

---

¹⁵The third-body ODEs must be calculated twice: once for the Moon and once for the Sun.

Cook's Double-Averaged Method

In addition to eliminating the dependency of the lunisolar perturbations on the position of the satellite, for some of the analysis involved in solving for the switch points in Chapter 3, it is desired also to eliminate the medium-period positional effects of the Moon. This is achieved by averaging the product of the direction cosines seen in Equations 2.25 through 2.27 with respect to the position of the Moon, M_M [7]. The orbit of the Moon is also assumed to be sufficiently circular so that its mean anomaly can be approximated by the true anomaly [7]. From there, u_M is replaced by M_M . The averaging process produced the following equations:

$$\overline{AB} = \frac{\sin \Delta\Omega}{2} (\cos \Delta\Omega \cos i \cos^2 i_3 + \sin i \sin i_3 \cos i_3 - \cos \Delta\Omega \cos i) \quad (2.34)$$

$$\overline{A^2} = \frac{\pi - \pi \sin^2 \Delta\Omega \sin^2 i_3}{2\pi} \quad (2.35)$$

$$\begin{aligned} \overline{B^2} = & \frac{\cos^2 \Delta\Omega \cos^2 i \cos^2 i_3}{2} - \frac{\cos^2 \Delta\Omega \cos^2 i}{2} + \sin i \sin i_3 \cos \Delta\Omega \cos i \cos i_3 \\ & + \frac{\cos^2 i \cos^2 i_3}{2} - \frac{\cos^2 i_3}{2} + \frac{1}{2} \end{aligned} \quad (2.36)$$

$$\overline{AC} = \frac{\sin \Delta\Omega}{2} (-\cos \Delta\Omega \sin i \cos^2 i_3 + \cos i \sin i_3 \cos i_3 - \cos \Delta\Omega \sin i) \quad (2.37)$$

$$\begin{aligned} \overline{BC} = & -\frac{\sin i \cos^2 \Delta\Omega \cos i \cos^2 i_3}{2} + \frac{\cos^2 \Delta\Omega \sin i \cos i}{2} \\ & + \sin i_3 \cos \Delta\Omega \cos^2 i \cos i_3 - \frac{\cos \Delta\Omega \sin i_3 \cos i_3}{2} - \frac{\sin i \cos i \cos^2 i_3}{2} \end{aligned} \quad (2.38)$$

The averaged direction cosines were then inserted into Equations 2.28 through 2.32 to produce a set of double-averaged ODEs¹⁶.

Another set of equations to model lunisolar perturbations is created based on the ODEs in Equations 2.28 through 2.32. By assuming that the AOP is equal to 270° or that it remains relatively close to that value, which is valid especially if a control strategy is employed, Cook's single-averaged equations can be greatly simplified. As a result, any term multiplied by $\sin 2\omega$ becomes zero. Additionally, to simplify Equation 2.32 further for solar perturbations, the term $\frac{5a}{2e r_s}$ is quite small and can be considered negligible. The resulting new set of simplified ODEs is:

$$\frac{de}{dt} = \frac{15}{2n} e \sqrt{1-e^2} \frac{\mu_3}{r_3^3} [AB] \quad (2.39)$$

$$\frac{di}{dt} = \frac{3\sqrt{1-e^2}}{2n} \frac{\mu_3}{r_3^3} [AC] \quad (2.40)$$

$$\frac{d\Omega}{dt} = \frac{3(1+4e^2)}{2n\sqrt{1-e^2} \sin i} \frac{\mu_3}{r_3^3} [BC] \quad (2.41)$$

$$\frac{d\omega}{dt} = \frac{3\sqrt{1-e^2}}{2n} \frac{\mu_3}{r_3^3} [(4B^2 - A^2 - 1) + \chi BC] \quad (2.42)$$

where the parameter χ is defined as:

$$\chi = -\frac{\cot i(1+4e^2)}{1-e^2} \quad (2.43)$$

The double-averaged direction cosines can also be implemented into these new ODEs when

---

¹⁶The averaged values of A and B are equal to zero.

modelling the effects of the Moon and/or Sun to eliminate the dependency of the position of the third body, if desired. These ODEs form the basis of the switch point analysis that is discussed in the next chapter as well as the analytical model developed in Section 2.3.

Kaula's Method

In the same year that Cook published his method to model lunisolar perturbations, W.M. Kaula published a paper which models the perturbations using a more thorough method [36]. His development began with the disturbing function for a third body. The function was derived from equating $\ddot{\vec{R}} = \nabla U_{2-body} + \nabla R_{3-body}$ to Equation 2.19, where ∇U_{2-body} is the gradient of the two-body potential function of a central body and ∇R_{3-body} is the gradient of the disturbing potential for a third body. According to A. Cayley [37], the disturbing function to model the perturbations of the Sun on the orbit of the Moon was presented by P.A. Hansen in his book *Fundamenta Nova*. The third-body disturbing function acting on the satellite can be shown to be [38]:

$$R_{3-body} = \frac{\mu_3}{r_3} \sum_{l=2}^{\infty} \left(\frac{r}{r_3} \right)^l P_l[\cos(\zeta)] \quad (2.44)$$

where $P_l[\cos(\zeta)]$ is the l^{th} Legendre polynomial with $\cos(\zeta)$ as the argument, and r and r_3 are the distances from the Earth to the satellite and third body, respectively.

As was the case in earlier sections, it is more practical to express the perturbations in terms of the Keplerian elements of the satellite and third body. This coordinate conversion follows the same development as Kaula's model for the non-spherical geopotential effects [17]. The result introduced more summation terms into the disturbing potential and also eliminated the Legendre polynomials in favour of inclination and eccentricity

functions, $F(i)$ and $G(e)$ ¹⁷, respectively, as well as Hansen coefficients ($H(e)$), which are given as infinite power series in e^2 [8]. Kaula also realized that practically, the short-period effects are insignificant and only terms in the summation that result in the mean anomaly being eliminated were used. This gives the new third-body disturbing potential as [36]:

$$\begin{aligned}
 R_{3-body} = & \mu_3 \sum_{l=2}^{\infty} \left(\frac{a^l}{a_3^{l+1}} \right) \sum_{m=1}^l k_m \frac{(l-m)!}{(l+m)!} \sum_{p=0}^l F_{lmp}(i) \sum_{h=0}^l F_{lmh}(i_3) \times \\
 & \times G_{lp(2p-l)}(e) \sum_{q'=-\infty}^{\infty} H_{l+1,hq'}(e_3) \cos(\Theta_{3-body})
 \end{aligned} \tag{2.45}$$

where k_m is the integer part of $(l-m)/2$ and

$$\Theta_{3-body} = (l-2p)\omega - (l-2h)\omega_3 - (l-2h+q')M_3 + m(\Omega - \Omega_3) \tag{2.46}$$

According to Reference [8], the disturbing potential shown in Equation 2.20 represents the second term¹⁸ of the Legendre polynomials assuming a circular orbit for the third body.

The Kaula method for modelling the lunisolar perturbations benefits from higher accuracy by avoiding assumptions about the orbit size in exchange for introducing multiple series summations. However, the resulting complicated disturbing potential, which must still be converted into RSW components, using Equations 2.10 through 2.12, before being inserted into the Gauss ODEs results in an exceedingly complex set of equations compared to Cook's averaged method.

---

¹⁷The inclination and eccentricity functions can be found on page 640 of Reference [8].

¹⁸The first Legendre polynomial term for the third-body potential is zero.

Prado's Method

A third method to model the third-body effects was developed and published in a series of papers between 1998 and 2008 by A.F.B.A. Prado, I.V.D. Costa, R.C. Domingos, and R.V. deMoraes [39, 40, 41]. The authors began their development using the same starting disturbing function as in Kaula's paper in 1962, except that they modelled the mean motion of the third body by combining the mass of the Earth with the Moon or Sun. Rather than expanding the Legendre polynomials using spherical harmonics as was done by Kaula, the authors simply used the direct definition for the Legendre polynomials. From there, the disturbing function was averaged twice: first with respect to the satellite's orbital period and second with respect to the orbital period of the third body. After this extensive development, the following set of ODEs that model the perturbations of the Sun or Moon using the Lagrange Planetary equations was obtained [41]:

$$\frac{da}{dt} = 0 \quad (2.47)$$

$$\frac{de}{dt} = \frac{15\mu_3 n_3^2 e \sqrt{1-e^2}}{8n} \left[1 + \frac{3}{2}e_3^2 + \frac{15}{8}e_3^4 \right] \sin^2 i \sin 2\omega \quad (2.48)$$

$$\frac{di}{dt} = -\frac{15\mu_3 n_3^2 e^2}{16n\sqrt{1-e^2}} \left[1 + \frac{3}{2}e_3^2 + \frac{15}{8}e_3^4 \right] \sin 2i \sin \omega \quad (2.49)$$

$$\frac{d\Omega}{dt} = \frac{3\mu_3 n_3^2 \cos i}{8n\sqrt{1-e^2}} \left[1 + \frac{3}{2}e_3^2 + \frac{15}{8}e_3^4 \right] [5e^2 \cos 2\omega - 3e^2 - 2] \quad (2.50)$$

$$\begin{aligned} \frac{d\omega}{dt} = \frac{3\mu_3 n_3^2}{8n\sqrt{1-e^2}} \left[1 + \frac{3}{2}e_3^2 + \frac{15}{8}e_3^4 \right] & [(5 \cos^2 i - 1 + e^2) \\ & + 5(1 - e^2 - \cos^2(i)) \cos(2\omega)] \end{aligned} \quad (2.51)$$

$$\begin{aligned} \frac{dM}{dt} = & -\frac{\mu_3 n_3^2}{8n} \left[1 + \frac{3}{2}e_3^2 + \frac{15}{8}e_3^4 \right] [(3e^2 + 7)(3\cos^2 i - 1) \\ & + 15(1 + e^2)\sin^2 i \cos^2 \omega] \end{aligned} \quad (2.52)$$

The required development to double-average the equations of motion also removed the dependency of the lunisolar perturbations on the RAAN of the satellite. As previously mentioned, the orientation of the satellite orbit with respect to the third body is a key function of the sign of the third-body perturbations. The resulting equations are essentially the secular components of the lunisolar effects. This is useful for extremely-long periods of analysis (on the order of decades or hundreds of years), but not very useful for the time scales that are the concern of this research.

Kolyuka's Method

In 2009, Y.F. Kolyuka et al. published a paper on the orbit evolution of uncontrolled Molniya-type satellites [42]. In their paper, they presented a unique method to model the third-body perturbations based on the work done in 1962 by M.L. Lidov [43]. The authors divided the third-body effects into three types: short-period, long-period, and secular. As with many other methods, the short-period effects were not considered to be important for long-term mission planning, and were ignored. A significant feature of the method presented by Kolyuka et al. was that it provided a set of ODEs in terms of the Keplerian elements for secular effects that was separate from the long-period effects. Another unique aspect of this method is the coordinate system that it initially uses to model the perturbations. Rather than using the traditional equator-referenced coordinate system, this method uses a system that is referenced to the orbital plane of the third-body equation¹⁹. The following equations model the secular variations of the orbit elements in

---

¹⁹The variable, ' $\tilde{x}$ ' is defined in the ecliptic system, where x is any orbital element.

the ecliptic plane:

$$\delta\tilde{a} \simeq 0 \quad (2.53)$$

$$\delta\tilde{e} = \frac{1}{2}Z\sqrt{1-e^2}\sin^2\tilde{i}\sin 2\tilde{\omega} \quad (2.54)$$

$$\delta\tilde{i} = -\frac{1}{4}Z\frac{e^2}{\sqrt{1-e^2}}\sin^2\tilde{i}\sin 2\tilde{\omega} \quad (2.55)$$

$$\delta\tilde{\Omega} = -Z\frac{1}{\sqrt{1-e^2}}\cos\tilde{i}\left[e^2\sin^2\tilde{\omega} + \frac{1}{5}(1-e^2)\right] \quad (2.56)$$

$$\delta\tilde{\omega} = Z\frac{1}{\sqrt{1-e^2}}\left[(e^2 - \sin^2\tilde{i})\sin^2\tilde{\omega} + \frac{2}{5}(1-e^2)\right] \quad (2.57)$$

where Z is a function of the size and shape of the third-body orbit relative to the size of the satellite orbit:

$$Z = \frac{15}{2}\pi\frac{\mu_3}{\mu}\left(\frac{a}{a_3}\right)^3\sqrt{1-e_3^2} \quad (2.58)$$

Using spherical trigonometry, a coordinate system transformation was used to convert the model into a form relative to the Earth's equator:

$$\delta\tilde{e} = \delta e \quad (2.59)$$

$$\delta\tilde{i} = \cos d \cos \delta\tilde{i} - \sin(\Omega - \Omega_3) \sin i_3 \delta\tilde{\Omega} \quad (2.60)$$

$$\delta\tilde{\Omega} = \frac{1}{\sin i} (\sin d \delta\tilde{i} + \cos d \sin \tilde{i} \delta\tilde{\Omega}) \quad (2.61)$$

$$\delta\tilde{\omega} = \delta\tilde{\omega} + \cos \tilde{i} \delta\tilde{\Omega} - \cos i \delta\Omega \quad (2.62)$$

where d is the difference between the AOP in the equatorial plane and the AOP in the ecliptic plane. The only medium- and long-period analysis that is presented in the paper is for the amplitude of the eccentricity (E) which has a period of $P_3/2$:

$$E = \frac{15}{16} \frac{\mu_3}{\mu_3 + \mu} \frac{P}{P_3} \sqrt{4 \cos^2 \tilde{i} \cos 2\tilde{\omega} + (1 + \cos^2 \tilde{i})^2 \sin^2 2\tilde{\omega}} \quad (2.63)$$

As such, while the Kolyuka method offers insight into the secular effects of the lunisolar perturbations acting on satellite orbits, which is not possible with the Cook method and is somewhat less complicated with the Kaula method, the separation of the variations does not help with the identification of conditions in which the rates of change of the lunisolar perturbations are equal to zero.

Validation of Lunisolar Perturbation Model

Of all the methods to model the lunisolar perturbations, including those not presented in this thesis, Cook's single-averaged method appears to be the most attractive. As explained in the previous subsections, Cook's method is averaged in order to focus on the medium- and long-period oscillations, which the n-body approach does not; but does not average the $2S$ oscillations that are necessary to model in the following chapters, which Prado's and Kolyuka's method do. Finally, in terms of complexity, Kaula's method to model lunisolar perturbation forces, while more robust, is not in a suitable form to derive the conditions such that the rates of change of the orbital elements are zero. Therefore, Cook's method is chosen as the method to model the lunisolar perturbations. This section validates the choice of using the Cook method by comparing simulation results generated in MATLAB with those of STK. As an industry-standard commercial software package, STK is able to numerically simulate the trajectories of a wide range of satellites with very high accuracy [44]. This 'black-box' type toolkit has been used to validate both the dynamic and control developments throughout this thesis.

A custom MATLAB orbit propagator was developed in this research that numerically simulates the orbit evolution of a satellite using a 4th order Runge-Kutta-Fehlberg numerical integrator developed from R.L. Burden [45] with an error tolerance equal to 10^{-8} . The ODEs used in the MATLAB simulator combine the secular effects of J_2 using Equations 2.16 through 2.18 and the lunisolar perturbations using Equations 2.28 through 2.32. Since the equations were averaged over the orbit period of the satellite to predict the long-term evolution of the orbital elements, the integration step size was able to be set to the period of the HEO. Appendix A contains the constants used in the MATLAB simulator [8].

To calculate the rate of change of the effects of the lunisolar perturbation forces,

the orbital elements of the Moon and Sun are required. The National Aeronautics and Space Administration (NASA) Navigation and Ancillary Information Facility has developed an information system called ‘SPICE’ to assist in mission planning and science activities for satellites by creating a database of planetary information including sets of ephemerides²⁰ [46]. STK’s ephemerides were configured to use this system to predict the orbits of the planets and moons of the solar system. Appendix B discusses the variations of the orbital elements of the Moon and Sun as predicted using the SPICE toolkit.

A set of simulations with a mission time of 15 years was run in both MATLAB and STK for each HEO type to validate the use of Cook’s ODEs for modelling lunisolar perturbations. The orbit data of the two methods were compared by calculating the root-mean-squared (RMS) errors and maximum absolute difference for each element. The error values for each orbit type and orbital element are shown in Table 2.1²¹. As expected, all three elements that are perturbed primarily by the lunisolar perturbation forces rather than J_2 , the eccentricity, the inclination, and the AOP, show that the error grows as the size of the HEO increases. This is because the approximations involved in developing Cook’s equations have error values proportional to the squared ratio of the orbital radius of the satellite to the orbital radius of the Moon. As such, Cook’s equations would not be suitable for modelling cis-lunar trajectories²² where the ratio approaches unity. As for the error of the RAAN, since this is primarily an effect due to the Earth’s oblateness, when the orbit type is smaller, the approximations due to including the secular components of J_2 as used in Equations 2.16 through 2.18 become more apparent.

---

²⁰Ephemerides are tables of astronomical objects over a given set of times.

²¹The semi-major axis is not shown in Table 2.1 since lunisolar perturbations have no long-term effect on its variations.

²²Cis-lunar trajectories are the orbits which are close enough to the Moon that the Earth’s gravitational field is no longer dominant.

Table 2.1: Error between variations of the orbital elements between MATLAB and STK.

Orbit Type:	Tundra		TAP		Molniya	
Error type:	RMS	Max	RMS	Max	RMS	Max
Eccentricity:	0.00123	0.00190	0.000385	0.000788	0.000702	0.00144
Inclination (deg):	0.0434	0.0727	0.0329	0.0616	0.0103	0.0192
RAAN (deg):	0.0409	0.105	0.299	0.638	0.637	1.684
AOP (deg):	0.202	0.334	0.135	0.194	0.0974	0.185

To put the scale of the error results from Table 2.1 into context, Figure 2.5 shows the variations of the orbital elements for both the MATLAB and STK simulations for a sample Tundra orbit. Even after a 15-year mission simulation, the MATLAB-based data is shown to closely match the orbit data from STK, with only very small differences visible at the end of the mission period. As such, Cook's equations provide a practical method for modelling the lunisolar perturbations and are well-suited for this application.

An additional aspect of the validation of Cook's equations for modelling lunisolar perturbations is the confirmation of the multiple modes of oscillation seen in the variations of the orbital elements. As previously mentioned, third-body effects cause both medium-period and long-period oscillations. The medium-period oscillations are a result of the revolution of the Moon around Earth, and the Earth around the Sun. These oscillations have a frequency of approximately twice that of the source body and are referred to as the $2M$ and $2S$ oscillations, respectively. Figure 2.6 shows the superposition of the $2M$ and $2S$ oscillations for the eccentricity of a Tundra orbit. The $2M$ oscillations have a period of approximately 14 days and the $2S$ oscillations have a period of approximately 182 days. As stated by Smith, despite the larger magnitude of the lunar effects compared to the solar effects, the shorter period of the $2M$ oscillations compared to the $2S$ oscillations results in the $2S$ oscillations having the more dominant amplitude in the variations of the orbital elements [25].

Figure 2.5: Comparison of the methods to model lunisolar perturbations using the STK and MATLAB propagators for a sample Tundra orbit.

While the time scale in Figure 2.6 is too short to compare all the medium- and long-period oscillations, the long-period oscillations are still partially viewable in Figure 2.5. As stated earlier in this section, the sign of the change of the long-period oscillations is a

Figure 2.6: Multiple oscillation modes of the eccentricity due to lunisolar perturbations.

function of the RAAN of the orbit. The study of the terms in Equations 2.25 through 2.27 show that the trigonometric terms are functions of the difference between the RAANs of the satellite and the third body. It can then be said that long-period oscillations are functions of the RAAN of the satellite ‘relative’ to that of the third body. As such, the term $\Omega - \Omega_3$ is subsequently replaced by the variable $\Delta\Omega$ and is called the ‘relative RAAN’. Therefore, in this thesis the long-period oscillations of an HEO are also called the relative RAAN oscillations. Since each HEO type experiences a different rate of change on its RAAN due to its proximity to the Earth, each orbit has a different period of relative RAAN oscillations. Using the secular rates of change of the RAAN due to J_2 shown in Section 2.2.1, the Molniya orbit has a relative RAAN period of approximately 7.5 years, whereas the TAP and Tundra orbits have periods on the order of 31 years and 125 years, respectively.

2.2.3 Force Model Validation

It cannot be argued that the J_2 effect is the largest component of the geopotential perturbation force. There exist other harmonic terms which can also affect the HEOs with varying degrees. This subsection analyzes the error involved in using various levels of the geopotential harmonics compared to a ‘true’ solution for each HEO using STK. Here, the ‘true’ solution is the result of a simulation in STK using the default High Precision Orbit Propagator (HPOP). The force model for HPOP includes a 21x21 harmonic model for the geopotential perturbations, lunar and solar gravitational attraction, atmospheric drag, and solar radiation pressure. For the long-term mission analysis performed in this thesis, such a complicated model can be unnecessary. Furthermore, it would be impractical to attempt to develop analytic equations to model the geopotential effects using a 21x21 model. As such, the purpose here is to determine the harmonic level necessary to be able to analytically model the geopotential between switch point occurrences with sufficient accuracy. The development of the analytical model is presented in the following section.

To quantify the error for various geopotential harmonic levels, a script was written in MATLAB to run a number of STK simulations, each with a different force model. The geopotential harmonic levels that were used to simulate the orbit in STK were 2x0, 2x2, 4x0, 4x4, and HPOP²³. Each orbit simulation is propagated for a period of 100 days since the time between switch points is approximately 1/4 of a year. After a simulation is completed for each of the force models, the difference between a given model and the HPOP model is calculated over time for all the orbital elements, except for the Mean Anomaly. Three error quantities are then calculated: the RMS error, the mean error, and the maximum error. The RMS error is the root-mean-squared error of the difference in the orbit

---

²³The 2x0 (i.e., J_2) and 4x0 (i.e., J_2 - J_4) harmonic levels only model the zonal harmonics of the Earth and are longitude-independent. The 2x2 and 4x4 harmonic levels model both zonal and tesseral harmonic terms.

data over the entire 100 day time period. The mean error is the average absolute difference in the orbit data, and the maximum error is the largest absolute error value that occurs over the simulation time. Because both the lunisolar and geopotential perturbations are sensitive to the initial conditions of each HEO, specifically the RAAN and the mission's starting epoch²⁴, both of these terms were varied – the RAAN over a full 360° cycle and the epoch over a period of one year. This allows simulations to encapsulate the varied effects of the perturbations and be valid for a wide range of initial orbit conditions. The MATLAB script uses a series of nested FOR loops to calculate the three error values for the variety of initial conditions and the mean values were then calculated for the RMS error and mean error while the maximum value of the maximum error is calculated. This type of simulation was performed for all three HEOs and required 288 individual STK simulations per HEO.

Table 2.2 shows the force model error results for a Molniya orbit. Starting with the semi-major axis, it can be seen that not including the sectorial/tesseral harmonics in the force model increases all types of error by a factor of about 10. It also shows that there is not a large decrease in error when augmenting the order of the force model to include harmonic terms larger than the 2x2 model. For the eccentricity, since it is primarily perturbed by third-body forces as discussed earlier in Section 2, increasing the force model does not affect the error in a significant manner. The inclination behaves similarly to the semi-major axis in that it is strongly dependent on longitude-dependent harmonic terms of the gravitational model. Unlike the semi-major axis, however, there is a noticeable effect in the inclination when using the 4x4 geopotential model compared to the 2x2 model. While the error results of the RAAN show that increasing the order of the force model to at least a 4x0 model results in a very accurate model of the RAAN variations, be-

---

²⁴An epoch is a reference time, and is set to January 1, 2010 for this mission.

cause of the magnitude of the rate of change of the RAAN, the difference in error between the 2x0 model and the 4x0 model can be considered insignificant. Finally for the AOP, both the 2x0 and 4x0 models produce a larger error compared to the models that included tesseral/sectorial harmonics. One unique observation, however, is that the results show that the 2x2 model has less of an error than the 4x4 model. While the exact cause is unknown, it is hypothesized that it is because there are other even higher order harmonic effects that are part of the ‘true’ solution which are offset by the perturbations caused by the 4x4 model.

Table 2.2: Results of force model validation for a Molniya orbit.

		Semi-major axis (km)	Eccentricity $\times 10^{-4}$	Inclination (deg)	RAAN (deg)	Argument of Perigee (deg)
2x0	RMS Error	9.566	0.746	0.0295	0.0126	0.0537
	Mean Error	16.0248	1.24	0.0495	0.0210	0.0949
	Max Error	26.809	2.40	0.0809	0.0507	0.169
2x2	RMS Error	0.687	0.724	0.00238	0.0172	0.0128
	Mean Error	1.139	1.19	0.00395	0.0289	0.0228
	Max Error	2.893	2.84	0.0123	0.0657	0.0568
4x0	RMS Error	9.566	0.746	0.0295	0.00994	0.0536
	Mean Error	16.024	1.24	0.0495	0.0170	0.0945
	Max Error	26.809	2.39	0.0809	0.0384	0.164
4x4	RMS Error	0.767	0.683	0.00215	0.00807	0.0161
	Mean Error	1.235	1.14	0.00346	0.0143	0.0274
	Max Error	2.538	2.04	0.00668	0.0352	0.0712

For the TAP orbit, since the mean radius of the orbit is farther away from the Earth compared to the Molniya orbit, it was suspected that increasing the order of the force model would not have as large an impact in reducing the errors. In Table 2.3, the semi-major axis error shows that the first three force models are all relatively consistent, but that the 4x4 model produces a significant decrease in all three error types. Just as with the Molniya orbit, the eccentricity does not benefit from an increased geopotential model.

Again similar to the semi-major axis, the 4x4 force model for the inclination is a significant improvement over the first three models. Just as for modelling the RAAN of the Molniya orbit, the 2x0 model is more than sufficient for the TAP orbit to account for the RAAN variations because of its high rate of change. Amongst all four force models, there is very little difference in the errors for the AOP and therefore a 2x0 model is sufficient.

Table 2.3: Results of force model validation for a TAP orbit.

		Semi-major axis (km)	Eccentricity $\times 10^{-4}$	Inclination (deg)	RAAN (deg)	Argument of Perigee (deg)
2x0	RMS Error	1.213	0.916	0.00173	0.00443	0.00689
	Mean Error	2.131	1.535	0.00302	0.00756	0.0117
	Max Error	3.586	2.535	0.00494	0.0141	0.0298
2x2	RMS Error	1.212	0.916	0.00173	0.00443	0.00689
	Mean Error	2.126	1.535	0.00300	0.00756	0.0117
	Max Error	3.587	2.534	0.00484	0.0141	0.0298
4x0	RMS Error	1.213	0.916	0.00173	0.00437	0.00689
	Mean Error	2.131	1.535	0.00302	0.00747	0.0117
	Max Error	3.586	2.534	0.00494	0.0126	0.0296
4x4	RMS Error	0.0687	0.929	6.247×10^{-5}	0.00437	0.00678
	Mean Error	0.113	1.551	1.155×10^{-4}	0.00743	0.0113
	Max Error	0.293	2.437	2.809×10^{-4}	0.0128	0.0243

The force model validation data for the final HEO, the Tundra orbit, is shown in Table 2.4. The large distance between the satellite and the Earth lessens the effect of the higher-order harmonic terms, as was seen with the TAP orbit. The semi-major axis still requires a 4x4 force model to accurately model the geopotential effects. This is not surprising considering that many force models used in the control strategy of geostationary satellites use at minimum a 2x2 force model [19]. Just as with the other orbits, the eccentricity error remains relatively constant across all force models. At the higher orbits, the accurate modelling of the inclination still contains some small benefits from the inclusion of the longitude-dependent harmonic terms of the geopotential; however, the error

using the 2x0 model is still sufficiently small that only the oblateness of the Earth effects need to be accounted for to accurately capture the behaviour of the inclination. While there is still a noticeable difference between the zonal force models (2x0 and 2x2) and the sectorial/tesseral force model (4x0 and 4x4) for the RAAN, the percent error between the methods when considering the total variation of the RAAN over a 100 day period is insignificant. Finally, the AOP error amongst the four force models remains relatively constant and therefore the 2x0 model is concluded to be acceptable.

Table 2.4: Results of force model validation for a Tundra orbit.

		Semi-major axis (km)	Eccentricity $\times 10^{-4}$	Inclination $\times 10^{-4}$ (deg)	RAAN (deg)	Argument of Perigee (deg)
2x0	RMS Error	1.164	1.215	2.995	0.00286	0.0133
	Mean Error	2.030	2.203	5.267	0.00487	0.0222
	Max Error	4.125	3.599	13.35	0.0103	0.0620
2x2	RMS Error	0.219	1.198	1.496	0.00280	0.0121
	Mean Error	0.374	2.000	2.645	0.00472	0.0200
	Max Error	0.999	3.135	6.819	0.00874	0.0444
4x0	RMS Error	1.164	1.215	2.994	0.00285	0.0133
	Mean Error	2.030	2.029	5.265	0.00485	0.0222
	Max Error	4.125	3.598	13.35	0.0105	0.0620
4x4	RMS Error	0.0334	1.198	0.2957	0.0285	0.0114
	Mean Error	0.0604	2.008	0.6218	0.00479	0.0190
	Max Error	0.134	3.153	1.363	0.00844	0.0420

2.2.4 Variations of Elements over Time

The dominant perturbations on HEOs, Earth oblateness and lunisolar perturbations, are shown to affect each element in a specific manner depending on the HEO type. For a Tundra orbit, the variations of the orbital elements were previously discussed in Section 2.2.2 using Figure 2.5. The results showed a superposition of oscillation modes in the eccentricity, inclination, and AOP (which are due to lunisolar perturbations) and a

negative secular drift of the RAAN due to J_2 . As cited earlier in this section, the relative RAAN oscillations are highly sensitive to the initial orbit orientations, i.e., the initial relative RAAN. Had a different initial relative RAAN been selected, the oscillations in the eccentricity, inclination, and AOP would behave differently.

A similar set of MATLAB simulations was performed for a sample TAP and Molniya orbit. These simulations model the perturbations due to J_2 , the Moon, and the Sun over a 15-year time interval. For the TAP orbit variations shown in Figure 2.7, the most important distinction compared to the Tundra orbit variations is the difference in the RAAN drift. In Chapter 3, it is explained that the larger drift of the RAAN of the TAP orbit reduces the ratio of the $2S$ oscillation period to relative RAAN period which affects the summation of the various oscillation amplitudes. This difference between the TAP and Tundra orbits becomes a more significant issue when the switch points are exploited to implement the control strategy in Chapter 4.

The results of the Molniya orbit simulations are shown in Figure 2.8. Again the higher rate of change of the RAAN drastically affects the long-term behaviour of the remaining orbital elements compared to the TAP and Tundra orbits. As mentioned in Section 2.2.2, the relative RAAN period of the Molniya orbit is approximately 7.5 years. Therefore, over a 15-year mission, almost two complete cycles of the long-period relative RAAN oscillations are observed. This 15:1 ratio of the $2S$ and relative RAAN oscillation periods blurs the line between the definition of long-period oscillations and medium-period oscillations as it is nearly equal to the ratio between the two medium-period oscillation modes that is approximately 13:1²⁵.

---

²⁵The 13:1 ratio is the ratio of lunar periods per solar period.

Figure 2.7: Variations of the orbital elements using the MATLAB propagator for a sample TAP orbit.

The Molniya orbit's closer proximity to the Earth increases the effects of the neglected higher-order harmonics of the Earth's gravity model. While the use of the ' $J_2 + \text{Moon} + \text{Sun}$ ' perturbation model for a Molniya orbit may be reasonable for small time intervals, of the order of one quarter of a year, for long-duration simulations, there are tesseral har-

monics that can become significant, in particular for the inclination and the AOP. The semi-major axis was not simulated since the perturbations of J_2 (secular) and the Moon and Sun have no significant long-term effect on the variations. It must include higher-order harmonic terms for accurate modelling. The results of including additional harmonic effects are discussed further in the validation of the control strategy using STK in Chapter 4.

The long-term behaviour of the eccentricity, inclination, and AOP due to lunisolar perturbations as shown in Figures 2.5, 2.7, and 2.8 has only briefly been discussed so far. More in-depth analysis into the cause for the lunisolar-induced variations is the subject of Chapter 3. A by-product of the study of the conditions of the switch points enables a greater understanding of the mechanics of the lunisolar perturbations on HEOs.

2.3 Analytical Model of Lunisolar Perturbations

The next step in the dynamic formulation is the development of a method to analytically predict the variations of the eccentricity, inclination, and AOP. The analytical model is necessary for the control strategy as it provides a method to quickly calculate the value of the controlled orbital elements at the switch point time, the result of which is used to compute the ΔV for each possible subsequent required control manoeuvre.

The control strategy, as discussed in Chapter 4, focuses on exploiting the switch points due to the relative apparent orbit of the Sun around the Earth. This is because the medium-period $2M$ oscillations have too small an amplitude to be efficiently exploited. Therefore, the analytical model derived in this section does not take into account the medium-period lunar effects. As a result, the dependency of the perturbations on the

position of the Moon is averaged out of the ODEs for the eccentricity, inclination, and AOP.

Since the switch point analysis predicts the time when the rate of change of the eccen-

Figure 2.8: Variations of the orbital elements using the MATLAB propagator for a sample Molniya orbit.

tricity, inclination, or AOP is zero, it is desired for the analytical model to be primarily a function of time. There are two terms in Equations 2.28 through 2.32 that can be modelled as linear functions of time: the RAAN of the HEO and the AOL of the Sun. For the RAAN of the HEO, it can be approximated as $\Omega(t) = \dot{\Omega}t + \Omega_0$, where $\dot{\Omega} = \dot{\Omega}_{J_2} + \dot{\Omega}_{(S)} + \dot{\Omega}_{(M)}$. The secular perturbations on the RAAN due to the Moon and Sun must also be taken into account for an accurate model of the perturbations. The following equations:

$$\dot{\Omega}_{(M)} = \frac{-3\mu'_M(1 + 4e^2) \cos i}{4(1 - \mu'_M)\sqrt{1 - e^2}} \left(\frac{a}{r_M}\right)^3 \quad (2.64)$$

$$\dot{\Omega}_{(S)} = \frac{-3\mu'_S(1 + 4e^2) \cos i}{4(1 - \mu'_S)\sqrt{1 - e^2}} \left(\frac{a}{r_S}\right)^3 \quad (2.65)$$

model the triple-averaged effects of the Moon and Sun on the RAAN and were developed in Reference [47]. These equations have the position of the satellite and third body averaged out with respect to their orbital periods. Their dependency on the relative RAAN is also removed²⁶. The linear variations of the RAAN are demonstrated in Figures 2.5, 2.7, and 2.8.

The other explicitly time-dependent term in the ODEs due to lunisolar perturbations is the position of the Sun, u_S . Since the $2M$ oscillations are relatively small compared to both the $2S$ oscillations and the relative RAAN oscillations [48], the dependency of the position of the Moon for the ODEs is removed. This is accomplished by using the averaged product of the lunar direction cosines seen in Equations 2.34 through 2.38. For the solar position-dependent terms, since the relative solar orbit is assumed circular, the motion of the Sun is modelled as $u_S(t) = u_{S,0} + n_S t$, where n_S is the mean motion of the Sun [7].

---

²⁶The gravitational parameter terms $\mu'_M = \frac{\mu_M}{\mu_M + \mu_E}$ and $\mu'_S = \frac{\mu_E}{\mu_E + \mu_S}$.

The analytical equation of the eccentricity is derived to be a function of the relative RAAN due to both lunar and solar effects, as well as the positional effect of the Sun. The development of the analytical model of the eccentricity was based on Equation 2.39 which assumes that the AOP is close to 270° . The actual averaging and integration derivations are too complex to be performed by hand so a symbolic integration tool based in MATLAB called MuPAD was used to derive the eccentricity as a function of time due to lunisolar perturbations. The results of the symbolic toolkit are too lengthy to be shown here, but the input equation had the form:

$$e(t) = -\frac{15}{2n}e\sqrt{1-e^2}\left[\frac{\mu_M}{r_M^3}\int(\overline{A_MB_M})dt - \frac{\mu_S}{r_S^3}\int(A_SB_S)dt\right] \quad (2.66)$$

where $\overline{A_MB_M}$ is the lunar-averaged product of the direction cosines A and B of the Moon and A_SB_S is the same direction cosine product but for the Sun with no positional averaging since it is these oscillations that the orbital control strategy is exploiting.

For the orbital control strategy to exploit the $2S$ oscillations of the inclination, just as with the eccentricity, the time-dependent analytical equation for the inclination must account for the averaged direction cosine variations of the Moon, $\overline{C_MA_M}$, as well as the non-averaged direction cosines of the Sun, C_SA_S . As with all the analytical equations developed in this section, the integration of the rate of change of the inclination was derived in MATLAB and the results are too lengthy to be shown here. Additionally, each subsequent analytical equation developed uses the simplified rate of change equations where the AOP is assumed to be a constant 270° . The form of the equation that was integrated is:

$$i(t) = \frac{3\sqrt{1-e^2}}{2n} \left[\frac{\mu_M}{r_M^3} \int (\overline{C_M A_M}) dt + \frac{\mu_S}{r_S^3} \int (C_S A_S) dt \right] \quad (2.67)$$

The final orbital element to be analytically modelled, the AOP, has by far the most complex behaviour. In addition to the lunisolar effects not being modelled by a simple product of the direction cosines, one must also account for the effect of the geopotential as well. As already discussed, the Earth model used for HEOs is limited to J_2 . To accurately model the J_2 effect, the variations of the inclination as perturbed by the Sun, must be taken into account using Equation 2.17. Since the equation is a nonlinear trigonometric function of the inclination, $4 - 5 \sin^2 i$, it will be simplified as a linear function in the form of $m \times i(t) + b$, where the slope, m , equals $-5 \sin 2i_0$ and y-intercept, b , equals $(3 + 5 \cos 2i_0)/2$. The combination of these perturbations on the AOP for the analytical equation is given by:

$$\begin{aligned} \omega(t) = & \frac{3\sqrt{1-e^2}}{2n} \left[\frac{\mu_M}{r_M^3} \int \left((4(\overline{B_M^2}) - (\overline{A_M^2}) - 1) - \chi(\overline{B_M C_M}) \right) dt \right. \\ & + \frac{\mu_S}{r_S^3} \int \left((4(B_S^2)_S - (A_S^2) - 1) - \chi(B_S C_S) \right) dt \Big] \\ & + \frac{3nR_E^2 J_2}{4p^2} \int \left(-5 \sin(2i_0) i(t) + \frac{3 + 5 \cos(2i_0)}{2} \right) dt \end{aligned} \quad (2.68)$$

2.4 Truncation Error for Lunisolar Perturbations

The study of the behaviour of the lunisolar perturbations acting on an Earth-orbiting satellite is essentially a four-body problem, a type of the n-body problem. Numerical simulations of such dynamic systems are sometimes fraught with issues related to numerical precision as the number of significant digits can drastically affect the results [49]. While

these errors are most common with trajectories that utilize equilibrium points in the four-body problem, it was prudent to confirm that such an issue does not plague the numerical simulations performed for this thesis.

A baseline 15-year orbit simulation was performed in MATLAB for a given HEO. Then the results of the second step in the orbit data were truncated from double to single precision and an additional simulation was performed with the single precision orbital data as the initial conditions. The orbital elements of the two sets of simulations were compared to each other by computing the RMS error for the eccentricity, inclination, RAAN, and AOP. One hundred of these pairs of simulations were performed for each HEO with randomized initial RAANs and the RMS values were averaged to demonstrate the effects of the truncation of the initial conditions (ICs). The results of these simulations are shown in Table 2.5. The results show that unlike the sensitivity of the ICs for certain non-Kelperian orbits²⁷ in the four-body problem, the Earth-orbiting HEOs discussed in this thesis are not at risk of their numerical simulations diverging due to loss of numerical precision.

Table 2.5: RMS error of simulations after truncation of ICs.

	Tundra	TAP	Molniya
Eccentricity ($\times 10^{-6}$)	3.67	4.763	8.41
Inclination ($\times 10^{-6}$) (deg)	4.95e	3.931e	3.620
RAAN ($\times 10^{-5}$) (deg)	0.1606	2.629	17.80
AOP ($\times 10^{-6}$) (deg)	2.00	2.698	4.438

---

²⁷A non-Keplerian orbit is an orbit whose shape cannot be approximated by a conic section.

Chapter 3

Dynamic Analysis

As mentioned in the previous chapter, there are multiple modes of oscillations that are imparted by the gravitational attraction of the Moon and Sun to the orbital elements of an HEO. For these oscillations to occur, the rates of change of the orbital elements due to lunisolar perturbations must periodically change sign. The methods to determine the conditions when the rate of change of the eccentricity, inclination, RAAN, and AOP are zero, called the switch points, are developed as well as the methods to predict the future occurrences of the switch points. This chapter focuses on the analysis of the dynamics of lunisolar perturbations to study the switch points of the HEOs.

As discussed in Chapter 2, it is the rate of change of the relative RAAN, $\Delta\Omega$, as well as the position of the Sun (or Moon), i.e., the AOL (u_3), that are responsible for the long- and medium-period oscillations shown in Figures 2.5, 2.7, and 2.8. Since it is the variation of these two terms that is responsible for the behaviour of the lunisolar perturbations, the sensitivity of the rates of change due to solar effects is plotted as a function of both the relative RAAN and the AOL of the Sun as shown in Figure 3.1. The black contour lines on the plots are the switch point curves, which represent conditions when the rate

of change of that particular orbital element is zero. While it is the solar perturbations that are plotted in Figure 3.1, lunar perturbations have a similar relationship except the amplitudes of the oscillations are different.

Figure 3.1: The rates of change of the orbital elements of a Molniya orbit due to solar perturbations as a function of the relative RAAN and the solar argument of latitude.

Starting with the eccentricity in Figure 3.1a, there are a number of observations that can be made which can be correlated to the behaviour of the eccentricity shown in Figures 2.5, 2.7, and 2.8. The behaviour of the long-period oscillations is the easiest to understand from Figure 3.1a. When the relative RAAN is 0° or 180° , as the Sun completes its orbit, the satellite experiences equal times of positive and negative rate of change. When the relative RAAN is greater than 180° , the rate of change of the eccentricity is predominantly positive; when the relative RAAN is less than 180° , it is predominantly negative.

The oscillations due to the position of the Sun, called the $2S$ oscillations, can also be identified using Figure 3.1a. For a given $\Delta\Omega$, over each revolution of the Sun, the satellite crosses the switch point curves four times creating four switch points. Therefore, in an annual period the eccentricity experiences two complete medium-period oscillations, hence the name ‘ $2S$ ’. One aspect to consider when analyzing the scatter plots of Figure 3.1a is that both the argument of latitude and the relative RAAN vary independently as a function of time. While the motion of the Sun is obviously independent of the type of HEO, the drift of the relative RAAN is not. For a Molniya orbit, the RAAN varies approximately -48° per year which means that if the relative RAAN is 90° when u_S is 0° , a straight line can be drawn on the scatter plots starting at $(0,90)$ and ending at $(360,42)$. This line reveals the switch points that occur and is used to understand the future behaviour of orbital elements. Due to the larger size of the TAP and Tundra orbits, the rate of change of the RAAN is significantly smaller. The RAAN drifts for the TAP and Tundra orbits are -12° and -3° per year, respectively. As such, the line drawn on Figure 3.1a to chart the future switch points would be much more horizontal for the two larger HEOs compared to the Molniya orbit.

Unlike the rate of change of the eccentricity, the scatter plot for the rate of change of the inclination, Figure 3.1b, is not affected by the variation of the relative RAAN as noticeably. The switch point curves vary slightly as $\Delta\Omega$ drifts negatively. When the relative RAAN is less than 180° , the rate of change of the inclination is slightly more often positive than negative, creating a net positive effect in the long-period variations. When the relative RAAN has drifted such that it is greater than 180° , the switch point curves behave in such a way that the net effect on the inclination is a net decrease. As can be observed from the scatter plot, when the relative RAAN is 0° or 180° the net effect is zero. The $2S$ oscillations are also easily observable. Regardless of $\Delta\Omega$, every solar period, the inclination experiences four switch points, of approximately equal time spacing between occurrences.

Figure 3.1c shows the rate of change of the RAAN due to the gravitational attraction of the Sun. Since the secular J_2 effect is the dominant perturbation of the RAAN, it is not possible to use this plot to visualize the variation of the RAAN as was done with the eccentricity or inclination; however, the scatter plot can still be gleaned to develop a general understanding of the effects of the third body on the RAAN. The most important aspect to consider when observing the plot is that at any given relative RAAN over a complete solar cycle, the rate of change of the RAAN is primarily negative. Unlike the eccentricity and inclination plots whose rates of change shift from generally positive to generally negative, the RAAN rate is not symmetric about zero. This means that there is an underlying negative secular component of the solar perturbations on the RAAN [8]. In addition to the secular component, however, there is still an oscillatory effect on the RAAN which is the result of the relative motion of the Sun around the Earth. As a result, in a given solar period four switch points always occur.

As previously mentioned, the variations of the AOP due to third-body perturbations are perhaps the most complex to model and understand compared to the eccentricity, inclination, and RAAN. When comparing Equation 2.32 to Equations 2.29 through 2.31, the rate of change of the AOP is clearly more complex. Not only does it depend on many more terms, but it also depends on the rate of change of the RAAN due to third-body perturbations. When comparing the scatter plots of the rate of change of the AOP in Figure 3.1d to Figures 3.1a-c, it clearly has a distinct behaviour compared to the eccentricity, inclination, and RAAN. The most noticeable difference between the plots is the disappearance of the switch points when the relative RAAN is between approximately 225° and approximately 135° ¹. When the relative RAAN is outside this range, the AOP experiences four switch points as the Sun completes its relative orbit around the Earth. These effects can be observed in the variation of the AOP for a Molniya orbit in Figure 2.8d. Between days 2300 and 4000, there are clearly observable $2S$ oscillations in the AOP, which coincide with the RAAN of the satellite orbit being greater than 240° or less than 120° as demonstrated in Figure 2.8c. However, when the time passes the 4000 day mark, the $2S$ oscillations disappear and there begins a period of constant negative drift of the AOP. This occurs because the relative RAAN is within the range where the switch point curves of Figure 3.1d do not exist. Additionally, just as with the perturbations on the RAAN, the AOP is also affected by the J_2 perturbations which can mask some of the third-body effects and make the variations of the AOP even more complex.

On each of the four scatter plots of Figure 3.1, there exist switch point curves which signify that the rate of change of the orbital element is zero. These curves are used to demonstrate the relationship between the relative RAAN and the position of the third body and are quite useful for understanding the behaviour of the lunisolar perturbations.

---

¹The actual values depend on the eccentricity and inclination of the HEO and are discussed in Section 3.1.4.

These curves were plotted numerically and it is not possible to model the relative RAAN as a function of the AOL such that the rate of change of the orbital element is zero. Therefore, alternate methods are needed to develop an analytic method to model the switch points. This is discussed in Sections 3.1.1 through 3.1.4.

In 1962, D.E. Smith published a paper which continued the analysis performed by Cook by performing numerical simulations of the lunar and solar perturbations and developed an analytical equation for the change in the orbital elements averaged over a lunar period [25]. In his paper, the author studied the conditions at which the rate of change of the eccentricity due to lunar perturbations would change sign, which he referred to as ‘turning points’². The analysis investigated a variety of orbit types such as geostationary orbits, low-Earth orbits, and some highly-eccentric orbits. However, his analysis was limited to orbits that had an inclination of zero and also assumed the inclination of the Moon was zero. The analytical development resulted in solving for the location of the Moon when the eccentricity reaches its maximum positive and maximum negative values. Regardless of the assumptions, Smith’s paper established that there is a unique relationship between the position of the third body (i.e., Moon or Sun) and the sign of the rate of change of the eccentricity, and that this relationship can be used to understand the behaviour of lunisolar perturbations.

In Section 3.1 of this chapter, the development of the switch point analysis is presented for the eccentricity, inclination, RAAN, and AOP. While the development shall mostly refer to solar switch points, the solution to the lunar switch point problem is identical. This switch point analysis not only provides a method to determine when the rates of change are equal to zero due to lunar or solar perturbations, but also provides a deeper

---

²‘Turning point’ and ‘switch point’ are equivalent terms.

understanding into the behaviour of the perturbations acting on HEOs. The methods to predict future occurrences of the switch points are developed in Section 3.2. Following that, Section 3.3 validates the switch point analysis and prediction methods by comparing the predicted switch point times to those computed in STK.

3.1 Switch Point Development

3.1.1 Eccentricity

The first step in the development of an analytic method to solve for the occurrence of switch points lies with the simplification of the rate of change of the eccentricity. For the HEOs studied in this thesis, the AOP of the orbit is set to 270° and it is assumed that it remains reasonably close to that value, as explained in Cook's double-averaged method in Section 2.2.2. Using this assumption, the second term inside the square brackets in Equation 2.29 becomes zero, and the equation simplifies to:

$$\frac{de}{dt} = \frac{15}{2n} e \sqrt{1-e^2} \frac{\mu_3}{r_3^3} [AB] \quad (3.1)$$

Since the signs of the variables outside the square brackets remain constant, it is proposed that a switch point occurs when the direction cosines A or B become zero. However, as shown in Equations 2.25 and 2.26, A and B are multi-variable, nonlinear functions of multiple terms such as u_3 and $\Delta\Omega$, and therefore the conditions for A or B to be zero cannot be solved for analytically in terms of u_3 or $\Delta\Omega$. A new set of angles is therefore developed, called the switch point angles. These angles are functions of the orbit and position of the third body, but can be used to solve for the switch points. The projection

of the orbits of the satellite and the Sun onto an Earth-centred unit sphere is shown in Figure 3.2. Using spherical trigonometry, three new angles were identified to solve for the switch points. The first angle, α , is the geocentric arc length between the node of the satellite's orbit and the position of the Sun. The angle γ is the angle between the equatorial plane $\hat{I} - \hat{J}$ and the arc segment defined by α . The third switch point angle, β , is similar to γ , but is referenced to the satellite orbit plane. The equations for the switch point angles are obtained to be³:

$$\alpha = \cos^{-1}(\cos \Delta\Omega \cos u_3 + \cos i_3 \sin u_3 \sin \Delta\Omega) \quad (3.2)$$

$$\beta = i - (180^\circ - \gamma) \quad (3.3)$$

$$\sin(\gamma) = \frac{\sin u_3 \sin i_3}{\sin \alpha} \quad (3.4)$$

$$\cos(\gamma) = \frac{1}{\sin \alpha}(\sin \Delta\Omega \cos u_3 - \cos i_3 \sin u_3 \cos \Delta\Omega) \quad (3.5)$$

Using the unit sphere, and the definition of the angle ζ from Cook defined as the geocentric angle between the satellite and the third body, $\cos \zeta = A \cos u + B \sin u$, the direction cosines are rewritten in terms of the switch point angles α and β as follows:

---

³The directions of the switch point angles are defined in the Nomenclature section.

Figure 3.2: Unit sphere for visualizing the relative geometry of the Sun and satellite orbit.

$$A = \cos \alpha \quad (3.6)$$

$$B = \sin \alpha \cos \beta \quad (3.7)$$

$$C = -\sin \alpha \sin \beta \quad (3.8)$$

With these equations, solving for the occurrences of the switch points for the eccentricity becomes trivial. The rate of change of the eccentricity due to lunar or solar perturbations is zero when α is equal to 90° or when β is equal to 90° or 270° ⁴. From there, two planes are created on the unit sphere, shown in Figure 3.3, which are used to visualize when a switch point occurs based on the position of the Sun. The α plane is defined as a vertical plane along the polar $\hat{K}$ vector while aligned with the satellite's angular momentum vector ($\vec{h}$). When the Sun crosses the α plane, α is equal to 90° and a switch point occurs. The β plane is aligned with both the angular momentum vector of the satellite and node vector of the satellite ($\vec{N}$). When the Sun crosses the β plane, then the switch point angle β is either 90° or 270° and the rate of change of the eccentricity is also zero.

Upon observing both Equation 3.2 and the switch point geometry of Figure 3.3, a few observations can be made regarding the angle α . First, when u_3 is zero, which corresponds to the Sun crossing its ascending node, α is equal to $\Delta\Omega$. Keeping in mind that because of its definition, α must be less than 180° ; therefore, if $\Delta\Omega$ is greater than 180° , then $\alpha = 360^\circ - \Delta\Omega$. Conversely, when the Sun crosses its descending node ($360^\circ > u_3 > 180^\circ$), then either $\alpha = 180^\circ - \Delta\Omega$ or $\alpha = \Delta\Omega - 180^\circ$, depending on the value of $\Delta\Omega$ such that α is bounded between 0° and 180° . There also exists a special case when $\Delta\Omega$ is equal to 0° or 180° . When $\Delta\Omega = 0^\circ$, $\alpha = u_3$ (or $360^\circ - u_3$ if u_3 is greater than 180°) and when $\Delta\Omega = 180^\circ$, $\alpha = 180^\circ - u_3$ (or $u_3 - 180^\circ$). It can also be said that over a given solar period, when the Sun crosses a plane defined by its angular momentum vector and the node vector of the satellite, α has reached its maximum/minimum. Specifically, α is a minimum if it crosses the plane to the side pointing towards the satellite node vector and is a maximum if it crosses the plane to the side pointing away from the satellite node vector.

---

⁴The angle α is limited to within 0° and 180° , inclusive.

Figure 3.3: Unit sphere for the switch point planes.

While the switch point angle α is also a function of the relative RAAN and the inclination of the third body, it is its dependency on the position of the third body (u_3) that provides its largest variations. This is a result of the inclination of the third body not varying considerably⁵ and the period of the relative RAAN oscillations ranging from 7.5 years to 125 years depending on the HEO. Meanwhile, the period of the third body is substantially smaller and its position is thus the cause of the primary variation of α . Fig-

⁵See Appendix B.

Figure 3.4 shows the variations of the α as a function of the argument of latitude of the Sun for a set of relative RAANs. For each curve on the plot, the relative RAAN, as well as the inclination of the Sun, is assumed to be constant. This is to illustrate the basic behaviour of α for various relative geometric conditions between the Sun and satellite orbits. Since the ratio of the relative RAAN period for a Molniya orbit to the solar orbital period is approximately 7.5:1, this means the assumption of a constant relative RAAN over one year is not valid, and the curves in Figure 3.4 are more representative for TAP and Tundra orbits which have significantly higher relative ratios of the periods between $\Delta\Omega$ and u_S .

Figure 3.4: Variation of α over a solar period for a complete cycle of relative RAANs for an HEO.

Using the assumption that the Earth revolves with a near-constant rate around the Sun [7], the argument of latitude of the Sun with respect to the Earth must also have a constant rate of change, the value of which is equal to the mean motion of the Sun. Figure 3.4 shows that α varies almost linearly with respect to u_S , particularly when α is

Figure 3.5: Variation of α over a solar period for a smaller subset of relative RAANs for an HEO.

near 90° . As a result, one can then state that α must also vary linearly with time when it is near 90° .

Figure 3.4 also demonstrates the behaviour of the switch point angle α as the relative RAAN varies by showing the α variation curves at particular values of the relative RAAN. The curves establish that the maximum or minimum values of α depend on the relative RAAN and that the largest maximum or minimum value for α occurs when the orbit of the satellite and third body are aligned ($\Delta\Omega = 0^\circ$ or 180°). Furthermore, observation of Figure 3.4 shows that the values of u_S that result in a switch point (when $\alpha = 90^\circ$) decreases negatively as the relative RAAN drifts negatively. The relationship is further enforced in Figure 3.5 where the range of values of the assumed-constant relative RAAN are smaller. This figure also provides a better demonstration of how the α curve evolves from being completely linear when $\Delta\Omega = 360^\circ$ to showing more gradual oscillations as the

Figure 3.6: Maximum and minimum values of α over a solar period as a function of the relative RAAN.

relative RAAN varies.

To further emphasize the effects of the relative RAAN over the variation of the angle α , two curves were plotted over the entire range of $\Delta\Omega$ which illustrate the maximum potential values for α , as shown in Figure 3.6. These curves establish what can be inferred from Figures 3.4 and 3.5 – that the relative geometry of the Sun and satellite orbits define the magnitude of the oscillations of the switch point angle which affects the determination of the switch point times.

While β is the angle that is used directly to monitor for switch points, it is primarily a function of γ , as shown in Equation 3.3, since the inclination of the satellite remains mostly constant, particularly when compared to the variations of γ [25]. As such, one must understand the behaviour of γ to understand β ; in particular, its relationship with $\Delta\Omega$, u_3 , i_3 , and α . Just as with the analysis for α , one can examine the unit sphere in Figure 3.3 and Equations 3.3, 3.4, and 3.5 to observe a simple set of relationships between the switch

point angles. Firstly, when the Sun crosses the equatorial plane, where $u_3 = 0^\circ$ or 180° , γ is also equal to 0° or 180° , respectively. This signifies that the period of oscillation for γ is exactly equal to the period of u_3 (i.e., the period of the third body). Since it has already been established that $\alpha = u_3$ when $\Delta\Omega = 0^\circ$, using the quadrant-specific arctangent function for Equations 3.4 and 3.5, when $\Delta\Omega = 0^\circ$, $\gamma = 180^\circ - i_3$ (or $360^\circ - i_3$). Similarly, when $\Delta\Omega = 180^\circ$, γ equals i_3 or $180^\circ + i_3$.

A schematic of the relationship between angles γ and β , when the relative RAAN equals 0° and 180° , is shown in Figure 3.7. The unit sphere in part (a) shows the relative geometry when the node vectors of the Sun and satellite are aligned in the same direction ($\Delta\Omega = 0^\circ$). From this view, the Sun traverses from left to right, proceeding from Point I to Point II. Therefore, the spherical triangle defined in Figure 3.3 (Sun- $\vec{N}$ - $\vec{N}_3$) is now a line segment. When this occurs, γ is equal to $360^\circ - i_3$ and β is thus equal to $\Delta i + 180^\circ$, where Δi is equal to $i - i_3$. When the Sun crosses the equatorial plane ($\hat{I} - \hat{J}$), to Point II, both γ and β have an instantaneous increase of 180° since the line segment has flipped from the southern to northern hemisphere. At Point II, γ is now equal to $180^\circ - i_3$, and $\beta = \Delta i$. When the relative RAAN is 180° and the node vectors are aligned, but in the opposite direction as shown in Figure 3.7b, the same type of observations are made. At Point I, when the Sun is in the northern hemisphere of the unit sphere, γ is equal to i_3 and β is equal to $\Sigma i - 180^\circ$, where Σi is equal to $i + i_3$. As the Sun crosses into the southern hemisphere, γ and β instantaneously change by 180° , and thus are equal to $i_3 + 180^\circ$ and Σi , respectively. Of particular note is that when the relative RAAN is equal to 0° or 180° , β is a function solely of the inclinations of both the satellite and the Sun. This plays a large role in the amplitude of the medium-period oscillations of the eccentricity that are a function of the relative RAAN which is discussed later in this section.

Figure 3.7: Schematic of switch point angles γ and β when the relative RAAN is:
 (a) 0° , and (b) 180° .

The behaviour of β over a solar period for various values of the relative RAAN is shown in Figure 3.8. Compared to a similar plot for α in Figure 3.4, the switch point angle β is shown to have much more complex behaviour. Just as with Figure 3.4, the curves in Figure 3.8 assume that the relative RAAN and both inclinations remain constant over the solar period⁶. When the relative RAAN is equal to 0° , β is equal to Δi or $\Delta i + 180^\circ$, depending on the location of the Sun. This is demonstrated by the two disconnected blue lines in Figure 3.8. Since β does not change with the motion of the Sun (with the exception of the 180° step), the slope of β is zero. As the relative RAAN drifts negatively, when $\Delta\Omega$ is equal to 270° , the slope of β is shown to be negative with respect to the argument of latitude of the Sun, and thus with time as well. As the configuration between the two orbits (Sun and satellite) changes, the β curve drifts to the left as the slope changes. When the orbit planes are oppositely aligned and $\Delta\Omega = 180^\circ$, the slope of β has become zero again and the value of β has increased by twice the inclination of the Sun such that it is equal to Σi or $\Sigma i + 180^\circ$. From there, as the relative RAAN continues to drift, the slope of β has now become positive and continues as such until the orbit planes re-align. Thus, it can be said that the slope of β is positive when $0^\circ < \Delta\Omega < 180^\circ$ and is negative when $180^\circ < \Delta\Omega < 360^\circ$.

The change in slope of the β curve is caused by the projection used to define γ . When viewing the unit sphere against the $\vec{N}$ vector as in Figure 3.7a, if the relative RAAN is greater than 180° rather than 0° , the 2-dimensional projection of the Sun's orbit results in viewing the 'northern face' of the Sun's orbit plane. This occurs when the projection of the angular momentum vector of the Sun's orbit along the satellite orbit plane is pointing in the same direction on the unit sphere as the satellite's node vector⁷. Since the Sun always follows its orbit path in a prograde or counter-clockwise manner when viewed from

---

⁶Recall that the inclination of the HEO is 63.4° and the inclination of the Sun is 23.4° .

⁷The angular momentum vector and the node vector of an orbit are always 90° apart.

Figure 3.8: Variations of β over the solar period for a complete cycle of relative RAANs for an HEO.

the celestial north ($\hat{K}$), and β is defined as a function of the position of the Sun, β follows the same counter-clockwise motion, and hence it was defined as negative in its original definition in Figure 3.2. When the relative RAAN is less than 180° , it is shown in Figure 3.8 that β increases with time. This is caused by viewing the orbit plane of the Sun on its ‘southern face’ which makes the Sun appear to complete its orbit in a clockwise direction and that causes β to also move in a clockwise or positive manner.

Figure 3.9 also shows the variation of β over the solar period for various relative RAAN values, but over a smaller range to demonstrate how the drift in the $\Delta\Omega$ affects the β curve. As the relative RAAN decreases from 360° , the 180° jump becomes more gradual. This figure also demonstrates the change in behaviour of the Sun’s position that corresponds to a switch point (when $\beta = 90^\circ$ or 270°). When the relative RAAN is equal to 360° , a switch point occurs when the Sun crosses the Earth’s equatorial plane and u_S is equal

to 0° or 180° . As the relative geometry changes and the β plane drifts with the orbit of the satellite, the argument of latitude of the Sun, where the rate of change of the eccentricity is equal to zero, decreases.

Figure 3.9: Variations of β over the solar period for a smaller subset of relative RAANs for an HEO.

As mentioned earlier in this section, it is the signs of the direction cosines A and B that dictate the sign of the rate of change of the eccentricity. Since the values of A and B are functions solely of α and β , A and B are, therefore, also influenced by the value of $\Delta\Omega$ as shown in Figure 3.10. In this figure, the maximum and minimum values of the direction cosines A , B , and C , are shown as a function of the relative RAAN, thus the curves can be thought of as the amplitude curves of the direction cosines. While the direction cosine C does not affect the rate of change of the eccentricity, it is useful for the inclination and RAAN switch points and is discussed in subsequent sections. Since A is simply a function of $\cos(\alpha)$, the amplitude of A is just a simple oscillation between -1 and +1 caused by the

variations in α as shown in Figure 3.6. Since α only reaches 0° or 180° when the relative RAAN is either 0° or 180° , the direction cosine A only reaches its absolute maximum amplitude when α does.

Figure 3.10: Variations of the amplitudes of direction cosines as a function of the relative RAAN.

The direction cosine B , however, is more complex since the shape of the curve is primarily a function of β . As α is limited to a range between 0° and 180° , and since $B = \sin(\alpha) \cos(\beta)$, the sign of B is only a function of β . It is observed in Figure 3.10 that the B curve resembles that of a beat wave, where the amplitude of the curve varies sinusoidally with the drift of the relative RAAN. Specifically, the amplitude reaches its absolute maximum when $\Delta\Omega = 0^\circ$ and its absolute minimum when $\Delta\Omega = 180^\circ$. Since the magnitude of the rate of change of the eccentricity is a function of the product of A and B , this begins to explain why in Figures 2.8a and 2.7a, the 2S oscillations of the eccentricity mostly vanish when the RAAN is near 180° . Since the amplitude of B is primarily a function of the

term $\cos(\beta)$, the direction cosine A reaches its maximum amplitude of approximately 0.8 when the relative RAAN is 0° because β is equal to Δi or $\Delta i + 180^\circ$. Since the relative inclination of the orbit planes of the satellite and the Sun is approximately 40° , B reaches values up to ± 0.77 when α is equal to 90° . As for the minimum amplitude case when the relative RAAN is 180° , β is equal to Σi or $\Sigma i + 180^\circ$. Since the sum of the inclination angles is around 87° , and $B = f(\cos(\beta))$, the amplitude of B is near zero regardless of the value of α . This is the mechanism that prompts the $2M$ and $2S$ oscillations to disappear when the relative RAAN is near 180° .

Since the angles α and β are shown to be both functions of the position of the Sun and relative RAAN, their variations as a function of time are inter-related. A MATLAB simulation of a Molniya orbit using Cook's single-averaged equations and J_2 over a 15-year period was performed. At each time step, the size of which is equal to the period of the HEO, Equations 3.2 and 3.3 were evaluated to calculate α and β . The resulting scatter plot, shown in Figure 3.11, was created to demonstrate the relationship between α and β , as well as their relationship with the relative RAAN. As shown in Figure 3.11, when $\Delta\Omega$ is less than 180° , the angle β varies from 0° to 360° (according to Figure 3.8); and α starts at less than 90° , reaches its maximum value before $\beta = 180^\circ$, and then proceeds back towards 0° . When the relative RAAN is greater than 180° , the angle β drifts from 360° to 0° , and α begins and ends on the scatter plot above 90° , reaching its minimum value after $\beta = 180^\circ$. Both α and β have a period equal to the apparent orbital period of the Sun.

When the rate of change of the eccentricity is plotted as a function of α and β , its unique behaviour is demonstrated, as shown in Figure 3.12. This figure is used to confirm the ability of α and β to define when a switch point has occurred. The black lines on Figure 3.12, when $\alpha = 90^\circ$ or when $\beta = 90^\circ$ or 270° , are shown to correspond to the

Figure 3.11: Behaviour of α , β , and the relative RAAN for a Molniya orbit over time.

sign change in the rate of change of the eccentricity due to the Sun. The sign of the rate of change of the eccentricity is also shown to be always positive when β is between 90° and 270° and when α is greater than 90° or when β is less than 90° or greater than 270° and α is less than 90° .

There is an additional set of angles that were defined for the purpose of studying and predicting the switch points, called the δ angles. The δ angles are used to determine the intersection points between the orbit of the Sun and the switch point planes: the α plane, the β plane, and the satellite orbit plane (discussed in subsequent sections). Using the node of the Sun's orbit as the reference point, the δ angles consist of: δ_α , δ_β , and δ_{sat} . The angle δ_α is the angle between the node of the third body and the intersection between the third body's orbit plane and the α plane in the direction of motion of the third body. Similarly, the angles δ_β and δ_{sat} determine the intersection points of the third body orbit plane with the β plane and satellite plane, respectively, measured from the node of the

Figure 3.12: Scatter plot of the rate of change of the eccentricity of a Molniya orbit due to the Sun as a function of the switch point angles α and β . The black lines signify when the rate of change is zero.

third body, also in the direction of motion of the third body. The definitions of these angles are shown in Figure 3.13. Since the switch point planes intersect the third-body orbit plane twice, there are in fact two values for each δ angle (with a difference of 180°): one angle greater than 0° , but less than 180° , the other angle less than 0° , but greater than -180° ⁸. Using Figure 3.13, the equations to calculate the δ angles were derived using spherical trigonometry and are shown to be:

⁸The δ angles are defined between $\pm 180^\circ$.

$$\delta_\alpha = \tan^{-1} \left(\frac{-\cot \Delta\Omega}{\cos i_3} \right) \quad (3.9)$$

$$\delta_\beta = \cot^{-1} \left(\frac{\sin i_3 \tan i + \cos i_3 \cos \Delta\Omega}{\sin \Delta\Omega} \right) \quad (3.10)$$

$$\delta_{sat} = \cot^{-1} \left(\frac{-\sin i_3 \cot i + \cos i_3 \cos \Delta\Omega}{\sin \Delta\Omega} \right) \quad (3.11)$$

Figure 3.13: Unit sphere for δ angles.

The relationship between these δ angles and the relative RAAN is shown in Figure 3.14. Angle δ_α is shown to be almost completely linear with respect to the relative RAAN, and therefore time. The slight exception to its linearity occurs when the relative RAAN is near 180° , but this is a small variation on the overall trend. Conversely, δ_{sat} is shown to be primarily linear around $\Delta\Omega = 180^\circ$ but has a noticeable nonlinearity near $\Delta\Omega = 0^\circ$. The angle δ_β has a strong linear relationship with $\Delta\Omega$ when the planes of the satellite and Sun are aligned but as the relative RAAN nears 180° , δ_β temporarily becomes quite nonlinear. The quasi-linear relationship of these angles with respect to time is utilized in the prediction of upcoming switch points. Assuming the Sun's apparent orbit is circular, its mean motion can then be used to predict the time when the AOL of the Sun is equal to δ_α or δ_β , which signifies that the solar rate of change of the eccentricity is equal to zero. This is discussed in more detail in Section 3.2.1.

Figure 3.14: Variation of the angles δ_α , δ_β , and δ_{sat} as a function of the relative RAAN.

The δ angles are also particularly useful for determining the amount of time that the rate of change of the eccentricity is either positive or negative. Using the knowledge from

Figure 3.13 that if the Sun is en route to intersect the β plane, i.e., β approaching 90° or 270° , the rate of change is negative; and if the Sun is en route to intersect the α plane, i.e., α approaching 90° , a new angle, $\delta_e = \delta_\beta - \delta_\alpha$, is defined. Angle δ_e is then used to determine the fraction of time of the Sun's orbit where the long-period rate of change is positive or negative. The dependency of δ_e on the relative RAAN is shown in Figure 3.15. Two additional angles, δ_i and δ_Ω , are also plotted in Figure 3.15. Their discussion is presented in Sections 3.1.2 and 3.1.3, respectively.

Figure 3.15: Variation of the angles δ_e , δ_i , and δ_Ω as a function of the relative RAAN.

The angle δ_e can be thought of as a measure of the relative configuration of the satellite orbit and the orbit of the Sun, similar to the relative RAAN. If δ_e is less than 90° , this signifies that the Sun spends more time heading towards the α plane than the β plane and thus the long-period effect on the eccentricity is positive⁹. When δ_e is greater than 90° , the Sun spends more time travelling towards the β plane and the overall rate of change is negative. This relationship is demonstrated in Figure 3.16.

⁹The angle δ_e , bounded by its definition, is defined between 0° and 180° .

Figure 3.16: Scatter plot showing the relationship between the rate of change of the eccentricity and δ_e . The upper curve is the maximum positive rate of change over a given solar period and the lower curve is the maximum negative.

Figure 3.16 plots the maximum positive and negative values of the rate of change of eccentricity over a solar period as a function of the angle δ_e . These curves not only confirm the interconnection between $\frac{de}{dt}$ and δ_e , but also corroborate the previous discussion related to the variations of the amplitude of the $2S$ oscillations as a function of the relative RAAN. When the relative RAAN is 0° , the difference between the maximum positive and maximum negative branches is the largest. As a result, the amplitude of the eccentricity variations due to the position of the Sun is at its maximum. When the relative RAAN is near 180° , however, the distance between the curves has become quite small meaning the solar gravitational effect on the eccentricity has decreased.

This subsection analyzed the behaviour of the rate of change of the eccentricity due to

the Sun by defining the switch point angles which assumed the AOP of the HEO was equal to 270° . The subsequent two sections will perform a similar analysis for the inclination and RAAN using the same set of switch point angles, α and β , as well as δ_α , δ_β , and δ_{sat} .

3.1.2 Inclination

The switch point analysis of the inclination develops a method to solve for the conditions of an inclination switch point, i.e., when the rate of change of the inclination due to the Sun or Moon, is zero. Within the rest of this subsection, any reference to switch point signifies an inclination switch point (as opposed to the eccentricity switch points discussed in the previous section). Starting with Equation 2.6 in Section 2.2.2, the assumption that the AOP is equal to 270° reduces $\frac{di}{dt}$ to:

$$\frac{di}{dt} = \frac{3\sqrt{1-e^2}}{2n} \frac{\mu_3}{r_3^3} [AC] \quad (3.12)$$

The rate of change of the inclination is now in a simplified form, such that it can be stated that a switch point occurs when either of the direction cosines, A or C , is equal to zero. Using the definitions of A and C from Equations 3.6 and 3.8 in terms of the switch point angles, α and β , it can be stated that a switch point will occur when α is equal to 90° , or when β is equal to 0° or 180° .

Half of the required conditions for inclination switch points are shared with those of the eccentricity. This is because both rates of change are functions of A which means that the inclination switch points also share the use of the α plane to determine when a switch occurs. Using the definition of β and γ shown in Equations 3.3 through 3.5, the

other switch point plane for the inclination is the satellite orbit plane since when the Sun intersects the satellite orbit plane, γ is equal to $180^\circ - i$ or $360^\circ - i$. The dependency of the sign of the rate of change of the inclination on whether the Sun is ‘above’ or ‘below’ the orbit of the satellite is logical because the perturbation on the inclination is due to an acceleration in a direction normal to the satellite orbit plane.

The magnitude of the rate of change of the inclination due to the Sun is a function of the product of A and C as shown in Equation 3.12. Returning to Figure 3.10 which plots the amplitude of the direction cosines as a function of the relative RAAN, it is evident that $\frac{di}{dt}$ reaches its maximum absolute value when the $\Delta\Omega$ is equal to 180° , and its minimum absolute value when $\Delta\Omega$ is equal to 0° . Unlike the eccentricity, however, the $2S$ oscillations of the inclination do not disappear since it is only the direction cosine B that experiences a significant reduction in amplitude as the result of the relative geometry between the orbit of the Sun and the HEO. With that said, observation of the inclination variations in Figure 2.8 shows that the amplitudes of the $2S$ oscillations do vary, and are in fact largest when the RAAN of the HEO is near 180° .

Just as with the eccentricity, the rate of change of the inclination is plotted as a function of the α and β switch point angles as shown in Figure 3.17. As expected, the plot demonstrates that whenever α is equal to 90° , or β is equal to 0° or 180° , the rate of change is zero.

The inclination switch points also benefit from using the δ angles as shown in Figure 3.13 to analyze the behaviour of the inclination. Since the rate of change of the inclination is dependent on the Sun’s position relative to the α plane and the satellite orbit plane, the angle $\delta_i = \delta_\alpha - \delta_{sat}$ is used in the same manner as δ_e . The angle δ_i is the angular distance along the orbit of the Sun between the α plane and the satellite orbit

Figure 3.17: Scatter plot of the rate of change of the inclination for a Molniya orbit due to the Sun as a function of the switch point angles α and β . The black lines signify when the rate of change is zero.

plane. The angle's relationship with the relative RAAN is demonstrated in Figure 3.15. Unlike the angle δ_e which was discussed in the previous section, δ_i does not vary far from the 'neutral' value of 90° ¹⁰. Regardless, the value of δ_i indicates whether the net effect of the perturbation on the inclination due to the Sun is positive or negative. If $\delta_i < 90^\circ$, the solar rate of change of the inclination is primarily positive over a solar period, and negative when $\delta_i > 90^\circ$. This is established in Figure 3.18 which plots the maximum positive and maximum negative rates of change of the inclination over a solar period as a function of δ_i and the relative RAAN.

¹⁰The term 'neutral' refers to when the Sun spends equal amounts of time on either side of the switch point planes and thus the net effect is zero.

Figure 3.18: Scatter plot showing the relationship between the rate of change of the inclination and δ_i . The upper curve is the maximum positive rate of change over a given solar period and the lower curve is the maximum negative.

Unlike the equivalent plot for the eccentricity shown in Figure 3.16, the behaviour of $\frac{di}{dt}$ does not drastically change based on the configuration between the orbits of the Sun and satellite. The small changes that $\frac{di}{dt}$ exhibit as δ_i or $\Delta\Omega$ vary are, however, sufficient to still create $2S$ oscillations in the variations of the inclination as shown in Figure 2.5b, 2.7b, and 2.8b. The variation of δ_i is also sufficiently effective to slightly change the amplitude of the $2S$ oscillations, which is particularly evident in the variations of the inclination of a Molniya orbit as shown in Figure 2.8. However, it is not nearly as large as the changes that are observed in the amplitude of the eccentricity. Rather than the maximum amplitude occurring when the relative RAAN is 0° as for the eccentricity, Figure 3.18 confirms that this occurs when the planes of the satellite and Sun are oppositely aligned and $\Delta\Omega = 180^\circ$.

3.1.3 RAAN

When comparing the rates of change of the inclination and the RAAN using Equations 2.30 and 2.31, their similarities are easily observable. Therefore, when the rate of change of the RAAN is simplified for $\omega = 270^\circ$, it is not surprising that it has a similar form to the inclination rate. The rate of change of the RAAN due to solar perturbations is shown to be:

$$\frac{d\Omega}{dt} = \frac{3(1 + 4e^2)}{2n\sqrt{1 - e^2} \sin i} \frac{\mu_3}{r_3^3} [BC] \quad (3.13)$$

The rate of change of the RAAN is zero when B or C is equal to zero. Therefore, in terms of the switch point angles, a RAAN switch point occurs when β is equal to any multiple of 90° , including 0° , and is not dependent on the angle α . This means that the β plane and the satellite orbit plane can be used to determine whether a switch point has occurred for the RAAN. Just as with the inclination, it should not be surprising that the RAAN switch point is a function of the Sun's position relative to the orbit of the satellite since it is a normal force that perturbs the RAAN. Plotting the variation of the rate of change of the RAAN in terms of α and β , shown in Figure 3.19, confirms the independence of the angle α on the switch points and the four values of β that result in a sign change in the RAAN rate. Figure 3.19 also shows that when the angle β is between 0° and 90° , or 180° and 270° , the rate of change of the RAAN is mostly negative. Since each point on the scatter plot occurs at equidistant time intervals equal to the period of the Molniya orbit, the plot confirms that the RAAN rate is predominantly negative.

The RAAN switch points also have their own δ angle, called δ_Ω , which is equal to $\delta_{sat} - \delta_\beta$. The angle δ_Ω determines the long-period behaviour of the rate of the change

Figure 3.19: Scatter plot of the rate of change of the RAAN for a Molniya orbit due to the Sun as a function of the switch point angles α and β . The black lines signify when the rate of change is zero.

due to solar perturbations by expressing the fraction of time that the Sun is in a region on the unit sphere where the rate of change is negative. The behaviour of δ_Ω as a function of the relative RAAN is demonstrated in Figure 3.15. Unlike the angles δ_e and δ_i , δ_Ω is not centred about 90° but instead about 0° . However, its variations as a function of $\Delta\Omega$ are nearly mirrored to δ_e about the angular value of 45° . This is a result of the RAAN switch point being dependent on the satellite orbit plane rather than the α plane and the fact that δ_{sat} and δ_α are almost always 90° apart as shown in Figure 3.14. Using Figure 3.15, it is observed that the sum of δ_e , δ_i , and δ_Ω is equal to 180° regardless of the value of $\Delta\Omega$.

The angle δ_Ω is also used to explain the underlying negative secular component of the

rate of change of the RAAN. Because the RAAN rate is dependent on the β plane and the satellite plane, both of which are anchored to the node of the satellite, there exists no geometrical configuration between the solar orbit and the satellite orbit where δ_Ω is greater than 90° . Therefore, the long-period effects of the Sun on the RAAN are always more negative than positive, thus contributing to an overall negative effect. Figure 3.20 shows how, regardless of the value of $\Delta\Omega$ or δ_Ω , the maximum negative rate of change of the RAAN is always greater than the maximum positive rate of change over a solar period. The overall negative effect notwithstanding, the variation of the angle δ_Ω still has a noticeable effect on the spacing between the maximum and minimum values of the RAAN's rate of change. When the relative RAAN is 0° or 180° , the maximum curve becomes zero whereas the minimum curve becomes smallest when $\Delta\Omega = 180^\circ$.

Figure 3.20: Scatter plot showing the relationship between the rate of change of the RAAN and δ_Ω . The upper curve is the maximum positive rate of change over a given solar period and the lower curve is the maximum negative.

3.1.4 AOP

With the RAAN switch points now analyzed in terms of α and β , the switch point analysis for the first three orbital elements is complete. For each element, it was shown that a switch point occurs whenever α or β are equal to a given constant. Unfortunately, the effect of the lunisolar perturbation forces on the AOP are not as simple as the other orbital elements and a more complicated approach to solve for the switch point curves is required. The rate of change of the AOP with the assumption that the AOP is always approximately 270° is:

$$\frac{d\omega}{dt} = \frac{3\sqrt{1-e^2}}{2n} \frac{\mu_3}{r_3^3} [(4B^2 - A^2 - 1) + \chi BC] \quad (3.14)$$

where the parameter χ is defined as:

$$\chi = -\frac{\cot i(1 + 4e^2)}{1 - e^2} \quad (3.15)$$

Since the rate of change of the AOP is also a function of the rate of change of the RAAN, even with the simplification, Equation 3.14 is still a nonlinear function of all three direction cosines. Regardless, the same method was applied to $\frac{d\omega}{dt}$ as with the other orbital elements. Since the variables outside the square brackets never change sign, the conditions for the switch points are solved for by setting $(4B^2 - A^2 - 1) + \chi BC$ equal to zero. Next, the equations for the direction cosines in terms of the switch point angles α and β are substituted in, reducing the problem to one equation and two unknowns. Therefore, the switch point curves for the AOP are not constant values of α and β , but are defined by an equation $\alpha = f(\beta)$. The following equations¹¹ show values of α as a function of β such that the rate of change of the AOP due to solar perturbations is equal to zero.

---

¹¹There are two equations for $\alpha = f(\beta)$ since it is a closed curve that defines $\alpha_{\dot{\omega}_{(S)}=0}$.

$$\alpha_{\dot{\omega}_{(S)}=0} = \frac{1}{2} \cos^{-1} \left(\frac{-\frac{1}{2} + \cos 2\beta - \frac{\chi \sin 2\beta}{4}}{\frac{3}{2} + \cos 2\beta - \frac{\chi \sin 2\beta}{4}} \right) \quad (3.16)$$

$$\alpha_{\dot{\omega}_{(S)}=0} = \frac{1}{2} \cos^{-1} \left(\frac{\frac{1}{2} - \cos 2\beta + \frac{\chi \sin 2\beta}{4}}{\frac{3}{2} + \cos 2\beta - \frac{\chi \sin 2\beta}{4}} \right) + 90^\circ \quad (3.17)$$

These two equations create a set of closed curves on a plot of α as a function of β , called the ω curves, that define when the solar rate of change of the AOP is zero. These curves are plotted in Figure 3.21 along with a sample plot of α as a function of β as explained below. The ω curves are centred symmetrically with respect to $\alpha = 90^\circ$. The location of the centres of the curves along the β axis, β_c , were solved for by taking the derivative of Equation 3.14 with respect to β after setting α equal to 90° . The corresponding angles are given by:

$$\beta_c = \frac{\tan^{-1}(\frac{-\chi}{4})}{2}, \frac{\tan^{-1}(\frac{-\chi}{4})}{2} + 180^\circ \quad (3.18)$$

The points of which the ω curves intersect the $\alpha = 90^\circ$ line are denoted as β_m reflecting the maximum/minimum values of β at which switch points can occur. These points can be solved for analytically. By setting α equal to 90° in Equations 3.16 and 3.17, an online symbolic mathematical solver, Symbolab, was used to solve for the values of β as a function of χ [50]:

$$\beta_{m1} = \tan^{-1} \left(\frac{-\chi + \sqrt{\chi^2 + 12}}{2} \right) \quad (3.19)$$

$$\beta_{m2} = \tan^{-1} \left(\frac{-\chi - \sqrt{\chi^2 + 12}}{2} \right) \quad (3.20)$$

$$\beta_{m3} = \tan^{-1} \left(\frac{-\chi + \sqrt{\chi^2 + 12}}{2} \right) + 180^\circ \quad (3.21)$$

$$\beta_{m4} = \tan^{-1} \left(\frac{-\chi - \sqrt{\chi^2 + 12}}{2} \right) + 180^\circ \quad (3.22)$$

To solve for the values of β that result in a switch point, the behaviour of α as a function of β must be modelled. Using the unit sphere in Figure 3.3, the following equation was developed:

$$\alpha(\beta) = \cot^{-1} \left(\frac{-\cot i_S \sin(\beta - i) - \cos \Delta\Omega \cos(\beta - i)}{\sin \Delta\Omega} \right) \quad (3.23)$$

The behaviour of $\alpha(\beta)$ is shown to also be a function of the inclination of both the satellite and the Sun, as well as the relative RAAN. A sample curve of Equation 3.23 was plotted on top of the ω curves as β varied from 0° to 360° as shown in Figure 3.21. Note that the curve $\alpha(\beta)$ assumes that the relative RAAN remains constant over the period of β .

To analytically solve for the values of β where $\alpha(\beta)$ crosses the ω curves, Equation 3.23 is set equal to Equation 3.16 or 3.17. Four distinct values of β were derived:

Figure 3.21: Variation of α as a function of β for a constant value of $\Delta\Omega$ with respect to the ω curves.

$$\beta_{swp1} = \frac{1}{2} \cos^{-1} \left(\frac{l_1 l_3 + \sqrt{l_2^2 (l_1^2 + l_2^2 - l_3^2)}}{l_1^2 + l_2^2} \right) \quad (3.24)$$

$$\beta_{swp2} = \frac{1}{2} \cos^{-1} \left(\frac{l_1 l_3 - \sqrt{l_2^2 (l_1^2 + l_2^2 - l_3^2)}}{l_1^2 + l_2^2} \right) \quad (3.25)$$

$$\beta_{swp3} = \beta_{swp1} + 180^\circ \quad (3.26)$$

$$\beta_{swp4} = \beta_{swp2} + 180^\circ \quad (3.27)$$

where β_{swp} refer to values of β at which a switch point occurs for a given $\alpha = f(\beta)$. The variables l_1 , l_2 , and l_3 are defined as:

$$l_1 = 8 \sin^2 \Delta\Omega + 4(\cot^2 i_S - \cos^2 \Delta\Omega) \cos 2i + 8 \cot i_S \cos \Delta\Omega \sin 2i \quad (3.28)$$

$$l_2 = 2\chi \sin^2 \Delta\Omega - 4(\cot^2 i_S - \cos^2 \Delta\Omega) \sin 2i + 8 \cot i_S \cos \Delta\Omega \cos 2i \quad (3.29)$$

$$l_3 = 4(\cot^2 i_S + \cos^2 \Delta\Omega - \sin^2 \Delta\Omega) \quad (3.30)$$

The conditions that result in the rate of change of the AOP due to the Sun equal to zero can then be solved analytically, using Equations 3.24 through 3.30. When the values of β_{swp} are used in Equations 3.16 or 3.17, the result is α_{swp} .

Using the same process as in Figures 3.12, 3.17, and 3.19, Figure 3.22 shows plots of the solar rate of change of the AOP as a function of α and β for a Molniya orbit. The results of Figure 3.22 confirm that the switch point curves derived for the AOP are in fact able to indicate when the rate of change of the AOP is zero. Since the switch point curves and the various β parameters are all functions of χ , which is itself a function of the eccentricity and inclination of the satellite orbit, each HEO has a specific set of switch point curves with a specific set of values, and a slightly different location along the β axis. The scatter plot is also useful for explaining why, depending on the relative RAAN, as shown in Figure 3.1, an AOP switch point may not occur. In Figure 3.22, it can be seen that because of the relationship between α and β , the $\alpha = f(\beta)$ curve may not intersect with the switch point curves and therefore the rate of change of the AOP does not change sign.

While the AOP switch points do not have planes which indicate when a switch point

occurs similar to the other orbital elements, the AOP curves can still be superimposed on the unit sphere as shown in Figure 3.23. The AOP switch point curves, called the ω curves in the figure, are also functions solely of the orbit of the satellite and their location on the unit sphere is a function of the RAAN of the HEO.

Figure 3.22: Scatter plot of the rate of change of the AOP of a Molniya orbit due to the Sun as a function of the switch point angles α and β . The black lines signify when the rate of change is zero.

Following the method to solve for the occurrences of the switch points for the eccentricity, inclination, and RAAN, an angle δ_ω was also defined to calculate the location of the intersection of the solar orbit with the ω curves. Using the spherical triangle on the unit sphere in Figure 3.24, the value δ_ω can be determined using the following equations:

Figure 3.23: Unit sphere showing the various switch point planes: (a) *top view*; (b) *front view*; (c) *side view*.

$$\sin \delta_{\omega} = -\frac{\sin \alpha_{swp} \sin(\beta_{swp} - i)}{\sin \Delta \Omega} \quad (3.31)$$

$$\cos \delta_{\omega} = \cos \Delta \Omega \cos \alpha_{swp} - \sin \Delta \Omega \sin \alpha_{swp} \cos(\beta_{swp} - i) \quad (3.32)$$

Figure 3.24: The unit sphere to define the angle δ_ω .

These equations are functions of the switch point values of α and β which are calculated using Equations 3.24 through 3.27 and Equations 3.16 and 3.17. Since over a given solar period, there are four distinct values of β_{swp} , there are also four values of δ_ω to be calculated. Since the values of β_{swp} are functions of the behaviour of $\alpha(\beta)$ which is in turn a function of the relative RAAN, the values for δ_ω vary with the relative RAAN as well. Figure 3.25 shows the variations of the δ_ω values as the relative RAAN varies from -180° to $+180^\circ$ ¹².

Figure 3.25 shows that as the relative RAAN drifts in the negative direction, all four values of δ_ω also decrease. Since β_{swp1} and β_{swp3} , as well as β_{swp2} and β_{swp4} , are offset

¹²This plot shows the range of $\Delta\Omega$ as being $\pm 180^\circ$ as opposed to 0° to 360° since the AOP switch points disappear when the relative RAAN is near 180° and the values of δ_ω have imaginary components.

Figure 3.25: Variation of the δ_{ω} angles as a function of the relative RAAN.

by 180° , their corresponding values of δ_{ω} are also offset by 180° . The sudden change in the behaviour of the δ_{ω} values near $\Delta\Omega = \pm 135^\circ$ is the result of the absence of AOP switch points due to the relative geometry of the Sun's orbit and the HEO. The discontinuity of the δ_{ω} when $\Delta\Omega = 0^\circ$ is the result of the spherical triangle used to define δ_{ω} becoming a line segment. As opposed to the angles δ_e , δ_i , and δ_{Ω} , the angle δ_{ω} is not useful for developing additional insight into the behaviour of the lunisolar perturbations on the AOP. However, knowing the location of the intersection between the solar orbit and the ω curves is still valuable for the prediction of the AOP switch points. This is discussed further in Section 3.2.3.

The unit sphere now has switch point planes or curves that enables it to be used to visualize when a switch point occurs for any of the four orbital elements described in this section. Now as the Sun completes its relative orbit around the Earth, a quick inspection of the unit sphere enables the sign of the solar rates of change of the eccentricity, inclination, RAAN, and AOP, to be known. Table 3.1 summarizes the switch point analysis

outlined in Section 3.1 by showing the values of the switch point angles that result in the various types of switch points. Since the switch point angle α crosses 90° twice per solar period and there are two ω curves, the switch point summary table shows that each orbital element experiences four switch points per solar period and thus completes two full oscillations in that time frame. Therefore the switch point analysis helps provide a better understanding of the lunisolar perturbations by explaining the mechanism of the $2S$ oscillations that were introduced in Chapter 2. The next section provides a method to predict the future occurrences of the sign changes using the theory developed here.

Table 3.1: Summary of values for angles α and β such that a switch point occurs.

	α (deg)	β (deg)
Eccentricity	90	90 or 270
Inclination	90	0 or 180
RAAN	n/a	0, 90, 180, or 270
AOP	$\frac{1}{2} \cos^{-1} \left(\frac{\mp \frac{1}{2} \pm \cos 2\beta \mp \frac{\chi \sin 2\beta}{4}}{\frac{3}{2} + \cos 2\beta - \frac{\chi \sin 2\beta}{4}} \right)$	$\frac{1}{2} \cos^{-1} \left(\frac{l_1 l_3 \pm \sqrt{l_2^2 (l_1^2 + l_2^2 - l_3^2)}}{l_1^2 + l_2^2} \right) (+180)$

3.2 Switch Point Prediction

The previous section developed the concept of switch points and solved for a method to determine the conditions at which switch points occur. This section deals with the development of a method to predict the times of the future conditions that indicate a switch point. Using analytical methods, a procedure was developed to solve for times when the rate of change of the eccentricity, inclination, or AOP due to lunisolar perturbations is zero¹³. The $2S$ oscillations are of particular interest here because their effects on the orbital elements are large enough to be controlled, compared to the the $2M$ oscillations.

¹³The prediction of the RAAN switch points is not shown here since the J_2 effect is significantly larger than the oscillations induced by the lunisolar perturbation forces. As such, the RAAN does not experience any switch points and its total rate of change remains negative.

The approximate 182-day period of the $2S$ oscillations due to the solar perturbation force results in oscillations of the orbital elements with significant amplitudes compared to the lunar $2M$ oscillations, despite lunar perturbation forces having a larger magnitude [8, 25]. The method to predict the future switch points is used as a prerequisite for an orbit control strategy which exploits the lunisolar perturbations acting on the eccentricity, inclination, and AOP. The development of the control strategy is discussed in the following chapter.

3.2.1 Eccentricity

Since the conditions for an eccentricity switch point are straightforward as shown in Table 3.1, predicting the future times of the switch points is also relatively straightforward. It was stated in Section 3.1.1 that the rate of change of the eccentricity due to solar perturbations is zero when α is equal to 90° or when β is equal to 90° or 270° . Using that knowledge, two switch point planes, the α plane and the β plane, were defined on the unit sphere to help visualize the occurrences of switch points. Since these planes are centred around the Earth along with the orbit of the Sun, both switch point planes intersect with the solar orbit plane twice.

Using the δ angles, δ_α and δ_β , the angular distance between the node of the Sun's orbit and either switch point plane can be calculated at a given time. Since the position of the Sun is also measured from the solar orbit node, the angular distance between the Sun's current position, u_{S_0} , and the switch point planes can also be calculated. This is shown in Figure 3.26. If the eccentricity of the orbit of the Sun is assumed to be sufficiently small, then the rate of change of the AOL of the Sun is equal to its mean motion [8]. While δ_α and δ_β are given by Equations 3.9 and 3.10, respectively, these equations cannot be used to determine the correct quadrant of the δ angles. As such, an alternate method for solving

The spherical triangle defined by $\vec{N}_S$, δ_α , and the intersection point between the equator and the α plane from Figure 3.26 is used to solve for $\sin \delta_\alpha$ and $\cos \delta_\beta$, such that a quadrant-specific arctangent function can be used to produce a result for δ_α in the correct quadrant. The projection of the spherical triangle for δ_α onto a plane is shown in Figure 3.27a. Recalling that δ_α is the angle between the node of the Sun and the intersection between the α plane and the solar orbit plane, $\sin \delta_\alpha$ and $\cos \delta_\alpha$ are modelled as:

$$\lambda = \cos^{-1}(-\sin \Delta\Omega \sin i_S) \quad (3.33)$$

$$\sin \delta_\alpha = \frac{\cos \Delta\Omega}{\sin \lambda} \quad (3.34)$$

$$\cos \delta_\alpha = \cot \lambda \cot i_S \quad (3.35)$$

where λ is a placeholder variable for the angle between the line segments along the α plane and the solar orbit plane.

Figure 3.27: Projection of the spherical triangles to model δ_α (a) and δ_β (b) onto a plane.

Just as with δ_α , to compute δ_β correctly the equations for $\sin \delta_\beta$ and $\cos \delta_\beta$ are also needed. Using the $\delta_\beta \vec{N} \vec{N}_3$ triangle from Figure 3.26 and its planar projection in Figure 3.27b, the following equations were derived to calculate δ_β :

$$\eta_1 = \cos^{-1}(-\cos i_S \sin i + \sin i_S \sin i \cos \Delta\Omega) \quad (3.36)$$

$$\eta_2 = \sin^{-1}\left(\frac{\sin \Delta\Omega \sin i_S}{\sin \eta_1}\right) \quad (3.37)$$

$$\sin \delta_\beta = \frac{\sin \Delta\Omega \cos i}{\sin \eta_1} \quad (3.38)$$

$$\cos \delta_\beta = \cos \Delta\Omega \cos \eta_2 + \sin \Delta\Omega \sin \eta_2 \sin i \quad (3.39)$$

Here, both η_1 and η_2 are intermediary angles used to calculate δ_β . The angle η_1 is defined as the angle between the solar orbit plane and the β plane. Angle η_2 is the arc between the satellite node and the intersection point between the solar orbit plane and the β plane.

With the values of both δ_α and δ_β calculated in their correct quadrants, the angular distance between the current position of the Sun, u_{S_0} , and either of the two switch point planes is now known. However, because the switch point planes are anchored to the node of the HEO, $\vec{N}$, the rate of change of the relative RAAN due to J_2 and lunisolar effects cause the values of the δ angles to also vary. Therefore, in addition to modelling the motion of the Sun to calculate the switch point time, the rates of change of δ_α and δ_β are also required. Since the quadrant issue previously discussed has no bearing on their rates of change, Equations 3.9 and 3.10 are used instead. Assuming the inclination of the Sun and HEO do not vary significantly with time, the derivatives of δ_α and δ_β are shown to be:

$$\dot{\delta}_\alpha = \Delta\dot{\Omega} \frac{\csc^2 \Delta\Omega \cos i_S}{\cos^2 i_S + \cot^2 \Delta\Omega} \quad (3.40)$$

$$\dot{\delta}_\beta = \Delta\dot{\Omega} \frac{\cos i_S + \sin i_S \tan i \cos \Delta\Omega}{\sin^2 \Delta\Omega + (\sin i_S \tan i + \cos i_S \cos \Delta\Omega)^2} \quad (3.41)$$

The variations of δ_α and δ_β as a function of the relative RAAN are shown in Figure 3.14. As discussed in Section 3.1.1, their variations are mostly linear. There is an exception, however, for δ_β when $\Delta\Omega$ is near 180° when its slope drastically increases and δ_β is no longer linear as a function of $\Delta\Omega$. The corresponding plots of the slopes of $\dot{\delta}_\alpha$ and $\dot{\delta}_\beta$ per unit rate of change of the RAAN using Equations 3.40 and 3.41 as a function of $\Delta\Omega$ are shown in Figure 3.28. As expected, $\dot{\delta}_\alpha$ varies very little across the range of the relative RAAN. The rate $\dot{\delta}_\beta$, however, only remains constant when $\Delta\Omega$ is greater than 270° or less than 90° . Outside that interval there is a large degree of variation as a function the relative RAAN. This nonlinearity should not significantly affect the ability to predict the eccentricity switch points. As previously mentioned in Section 3.1.1, the $2S$ oscillations disappear when the relative RAAN is near 180° and as such there are no switch points to predict. Since the rate of change of the relative RAAN is nearly constant over short periods of time, and the relative RAAN does not vary significantly over periods of approximately one half of a year, the rates of change of δ_α and δ_β can be assumed to be constant over short periods of time to enable the prediction of the switch points.

Therefore, the time of the intersection of the α plane and of the β plane, also known as the solar switch point times, can be computed using the following equations:

Figure 3.28: Variations of the rates of change of the δ angles per unit change of the relative RAAN as a function of the relative RAAN.

$$t_{\alpha=90^\circ} = \frac{\delta_\alpha - u_{S_0}}{n_S - \dot{\delta}_\alpha} + t_0 \quad (3.42)$$

$$t_{\beta=90^\circ, 270^\circ} = \frac{\delta_\beta - u_{S_0}}{n_S - \dot{\delta}_\beta} + t_0 \quad (3.43)$$

where u_{S_0} is the location of the Sun with respect to its ascending node at the time of the prediction in the mission, t_0 .

The solar switch point thus is able to be determined with reasonable accuracy and with very little computational effort. Up until now, however, the discussion of the eccentricity switch points have been limited to the solar perturbations because the oscillations due to the relative revolution of the Sun are significant compared to those of the Moon. With that, each third body also perturbs the eccentricity with long-period oscillations due to the relative RAAN between the satellite and the Moon or the Sun [5]. These perturbations

must be accounted for since they produce the largest oscillations. The solar switch point analysis already accounts for the long-period oscillations by including the rate of change of the solar relative RAAN, but the lunar long-period effects must also be accounted for, even if the medium-period effects are not. Since it is the desire of this section to develop a method to predict when the total rate of change of the eccentricity is zero, the predicted solar switch point time must be adjusted since the rates of change of the eccentricity of the Sun and Moon are not zero at the same time.

Figure 3.29 shows the variations of the solar rate of change of the eccentricity as well as the lunar rate of change and the combined or total rate of change. The lunar rate of change has been averaged to eliminate the medium-period oscillations due to the position of the Moon since, as previously mentioned, the oscillations are small compared to the solar medium-period effects. The red dot on the figure shows the predicted solar switch using the method described earlier in this section, whereas the blue dot represents the time when the total rate of the eccentricity is zero. This point is referred to as the ‘combined’ switch point. The green dot is the magnitude of the long-period lunar rate of change of the eccentricity when $\dot{e}_{(S)} = 0$ which is causing the error between the solar and combined switch points. Together, these three dots makeup a right triangle which is used to predict the combined switch point time. Using the predicted solar switch point time, the combined switch point time can be predicted by assuming that the lunar rate of change remains nearly constant over short periods of time and that the slope of the rate of change due to the Sun can be used as part of a linear approximation of the total rate of change of the eccentricity curve. The approximation of the combined switch point time is shown as:

$$t_{swp} = \frac{-\dot{e}_{(M+S)}(t = t_{swp,S})}{\ddot{e}_{(S)}(t = t_{swp,S})} + t_{swp,S} \quad (3.44)$$

Figure 3.29: Comparison of the solar switch point with the combined switch point for an HEO.

where $t_{swp,S}$ is the solar switch point time computed using Equations 3.42 or 3.43, and t_{swp} is the combined switch point time. The combined rate of change of the eccentricity, $\dot{e}_{(M+S)}$ ($t = t_{swp,S}$), is the sum of the rates of change of the eccentricity due to the Moon and Sun obtained from Equation 2.39 evaluated at the solar switch point time. The derivative of the rate of change of the eccentricity due to the Sun,

$$\ddot{e}_{(S)} = \frac{15}{2n} e \sqrt{1 - e^2} \frac{\mu_S}{r_S^3} \left[A \frac{dB}{dt} + \frac{dA}{dt} B \right] \quad (3.45)$$

only requires the derivatives of the direction cosines A and B since the terms outside the square brackets in Equation 3.45 can be modelled as constant over small periods of time. Based on Equations 2.25 and 2.26, the derivatives of A and B are:

$$\begin{aligned} \frac{dA}{dt} = & -n_S \cos \Delta\Omega \sin u_S + n_S \cos i_S \cos u_S \sin \Delta\Omega - \\ & \Delta\dot{\Omega} \sin \Delta\Omega \cos u_S + \Delta\dot{\Omega} \cos i_S \sin u_S \cos \Delta\Omega \end{aligned} \quad (3.46)$$

$$\begin{aligned} \frac{dB}{dt} = & \cos i \left[n_S \sin \Delta\Omega \sin u_S + n_S \cos i_S \cos u_S \cos \Delta\Omega - \Delta\dot{\Omega} \cos \Delta\Omega \cos u_S - \right. \\ & \left. \Delta\dot{\Omega} \cos i_S \sin u_S \sin \Delta\Omega \right] + n_S \sin i \sin i_S \cos u_S \end{aligned} \quad (3.47)$$

and are now functions of the mean motion of the Sun ($\frac{du_S}{dt} = n_S$) as well as the secular rate of change of the RAAN.

While the solar switch point is predicted to be in the future, a scenario could arise where, after including the effects of the Moon, the combined switch point based on that particular crossing of the switch point plane, is determined to have occurred some time in the past. The opposite could also occur, where the Sun had recently crossed a switch point plane, but the effects of the Moon pushed the combined switch point into the future. Therefore, to ensure that the closest combined switch point in the future is determined, two past solar switch point times and two future solar switch point times are predicted (one for α and one for β in each direction). From there, the combined switch point time for all four predicted solar switch points is computed and the switch point that is the closest future time is selected as the true upcoming switch point time. These scenarios become more common as the magnitude of the long-period lunar effects becomes larger.

3.2.2 Inclination

Since the inclination switch points described in Section 3.1.2 are similar to the eccentricity switch points developed in Section 3.1.1, the method to predict the future occurrences of solar and combined inclination switch points is similar to the method described in the previous section. Recalling that half of the solar switch points for the inclination occur when the Sun crosses the α plane, Equations 3.33 through 3.35, 3.40, and 3.42 from Section 3.2.1 are also used for the inclination. The other half of the solar switch points for the inclination occur when the Sun crosses the satellite orbit plane. The development of the method to predict this time follows the same steps as used for predicting when the Sun crosses the α or β planes.

The first step is to compute the value for δ_{sat} in the correct quadrant. Since Equation 3.11 is unsuitable for this task, a new set of equations is required. Using the spherical triangle $\delta_{sat}\vec{N}\vec{N}_S$ from the unit sphere in Figure 3.26, a projection is shown in Figure 3.30 which is used to solve for δ_{sat} . Using the cosine and sine laws from spherical trigonometry, δ_{sat} is calculated as follows:

$$\psi_1 = \cos^{-1}(-\cos i_S \cos i - \sin i_S \sin i \cos \Delta\Omega) \quad (3.48)$$

$$\psi_2 = \sin^{-1}\left(\frac{\sin \Delta\Omega \sin i_S}{\sin \psi_1}\right) \quad (3.49)$$

$$\sin \delta_{sat} = \frac{-\sin \Delta\Omega \sin i}{\sin \psi_1} \quad (3.50)$$

$$\cos \delta_{sat} = \cos \Delta\Omega \cos \psi_2 + \sin \Delta\Omega \sin \psi_2 \cos i \quad (3.51)$$

Figure 3.30: Projection of the spherical triangle to model δ_{sat} .

Next, the rate of change of δ_{sat} is required to account for the drift of the satellite orbit due to the Earth's oblateness and the lunisolar perturbations on the RAAN. The derivative of Equation 3.11 is:

$$\dot{\delta}_{sat} = \Delta\dot{\Omega} \frac{\cos i_S - \sin i_S \cot i \cos \Delta\Omega}{\sin^2 \Delta\Omega + (-\sin i_S \cot i + \cos i_S \cos \Delta\Omega)^2} \quad (3.52)$$

Just as with Equations 3.40 and 3.41, it shall be assumed with Equation 3.52 that the orbital elements do not vary significantly enough such that $\dot{\delta}_{sat}$ is constant over periods of time of one half of a year. Examining Figure 3.28 for the rate of change of δ_{sat} per unit change of the RAAN, it is shown that the slope of δ_{sat} is nearly constant when $\Delta\Omega$ is less than 270° and greater than 90° , in contrast to that of δ_β . Unlike δ_β , however, the nonlinear portions of δ_{sat} are not nearly as significant. For a given calculated value of $\dot{\delta}_{sat}$, it only needs to be assumed to be constant over small variations of the relative RAAN. The Molniya orbit, which has the largest $\Delta\Omega$ variations compared to the other HEOs, only experiences a change in the relative RAAN of approximately 12° between switch points. For the other HEOs, this variation in the relative RAAN is even smaller. With that, the solar switch point time when the Sun crosses the satellite orbit plane, i.e., when β is equal

to 0° or 180° , is calculated as:

$$t_{\beta=0^\circ, 180^\circ} = \frac{\delta_{sat} - u_{S_0}}{n_S - \dot{\delta}_{sat}} + t_0 \quad (3.53)$$

The solar switch point times for the inclination, calculated using Equations 3.42 and 3.53, must then be converted to the combined switch point times by including the effects of the Moon. To accomplish this, the solar switch point time is converted to the same process as performed in Section 3.2.1. The combined switch point time for the inclination can then be calculated as:

$$t_{swp} = \frac{-\frac{di_{(M+S)}}{dt}(t = t_{swp,S})}{\frac{d^2i_{(S)}}{dt^2}(t = t_{swp,S})} + t_{swp,S} \quad (3.54)$$

where $\frac{di_{(M+S)}}{dt}$ is the combined rate of change of the inclination, which includes the lunar and solar gravitational attraction. The second rate of change of the inclination due to the Sun, $\frac{d^2i_{(S)}}{dt^2}$, evaluated at the solar switch point time $t_{swp,S}$, is:

$$\frac{d^2i_{(S)}}{dt^2} = \frac{3(2 - 2e^2)}{4n\sqrt{1 - e^2}} \frac{\mu_S}{r_S^3} \left[A \frac{dC}{dt} + \frac{dA}{dt} C \right] \quad (3.55)$$

where $\frac{dA}{dt}$ is calculated using Equation 3.46 and the derivative of the direction cosine C is:

$$\begin{aligned} \frac{dC}{dt} = & \sin i \left[-n_S \sin \Delta\Omega \sin u_S - n_S \cos i_S \cos u_S \cos \Delta\Omega + \Delta\dot{\Omega} \cos \Delta\Omega \cos u_S + \right. \\ & \left. \Delta\dot{\Omega} \cos i_S \sin u_S \sin \Delta\Omega \right] + n_S \cos i \sin i_S \cos u_S \end{aligned} \quad (3.56)$$

As in Section 3.2.1, to ensure that the closest future combined switch point time is found, four solar switch points must be calculated (two past, two future) and then adjusted to account for the long-period lunar effects.

Since the prediction of the switch points for the RAAN is not necessary for the orbital control strategy described in Chapter 4, the development of the necessary equations is not presented here. However, since the solar switch point of the RAAN is a function of the times when the Sun crosses the β and satellite orbit planes, the required equations have already been discussed. Equations 3.36 through 3.39, 3.41, and 3.43 are used to calculate when β is 90° or 270° and Equations 3.48 through 3.53 are used to calculate when β is equal to 0° or 180° .

3.2.3 AOP

As discussed in Section 3.1.4, solving for the conditions that indicate that an AOP switch point has occurred is slightly more complicated than for the other three orbital elements. As a result, analytically predicting the future AOP switch points also requires a slightly modified approach compared to the eccentricity and the inclination switch points. The main deviation for predicting the AOP switch points is that there are two ω curves along which the rate of change of the AOP is equal to zero. Therefore, to predict the switch point times one must decide which curve to use based on the initial values of α and β .

The process for predicting the switch point time begins with calculating the four values of β_{swp} using Equations 3.24 through 3.27. From there, using the current value of α and β it must be determined whether the current position is inside the ω curves, or above or be-

low. The result of this, along with the current value of $\Delta\Omega$, dictates which ω curve causes the next switch point. From there, one of the four values of β_{swp} is chosen along with its corresponding α_{swp} . Using these values, the value of δ_ω is then determined. Since δ_ω is an angle along the solar orbit plane, the difference between it and the current position of the Sun is used to determine the time it takes for the Sun to reach the ω curve. However, since ω curves are anchored to the orbit plane of the satellite, just as with the α and β planes, the secular drift of the RAAN causes the location of intersection between the ω curve and the solar orbit plane to vary with time. This is due to the assumption made when solving for β_{swp} using Equation 3.23 which does not account for the drift in the RAAN. Figure 3.31 depicts the discrepancy between a curve that assumes the RAAN is constant ($\alpha = f(\beta)$, $\Delta\Omega = \text{const.}$) and a curve that accounts for the variation of both β and $\Delta\Omega$ ($\alpha = f(\beta, \Delta\Omega)$).

Figure 3.31: Discrepancy in the prediction of *alpha* as a function of β without accounting for the drift in the relative RAAN.

The use of Equations 3.24 through 3.27, 3.16, and 3.17 produce a prediction of the switch point angles when the rate of change of the AOP is zero which are then used to

calculate the angle δ_ω using Equations 3.31 and 3.32; however, to accurately predict the switch point time, the rate of change of δ_ω must be accounted for. For reasons similar to those described in the previous two sections, to determine the rate of change of δ_ω , neither Equation 3.31 nor 3.32 is used. Instead, using the spherical triangle shown in Figure 3.32, a new equation for δ_ω was derived as:

$$\delta_\omega = \cot^{-1} \left(\frac{\sin i_S \cot(\beta_{swp} - i) + \cos i_S \cos \Delta\Omega}{\sin \Delta\Omega} \right) \quad (3.57)$$

Figure 3.32: Projection of the spherical triangle to model δ_ω .

This equation is better suited to derive the rate of change of δ_ω since it is not a function of α_{swp} as was Equations 3.31 and 3.32. The reason to avoid α_{swp} is that the derivative of Equation 3.57 with respect to time would become exceedingly complicated, since both β_{swp} and $\Delta\Omega$ vary with time. Using Equation 3.57, and assuming i and i_S remain constant as before, the rate of change of δ_ω is derived as:

$$\begin{aligned} \dot{\delta}_\omega = & \frac{1}{1 + \cot^2 \delta_\omega} \left(\frac{1}{\sin^2 \Delta\Omega} \right) \left(\dot{\Omega} \cos i_S + \dot{\Omega} \sin i_S \cot(\beta_{swp} - i) \cos \Delta\Omega \right. \\ & \left. + \dot{\beta}_{swp} \sin i_S \csc^2(\beta_{swp} - i) \sin \Delta\Omega \right) \end{aligned} \quad (3.58)$$

A linear rate of change of the relative RAAN is assumed in this equation as well as in calculating $\dot{\beta}_{swp}$ from Equation 3.24 or 3.25. Because of the complex and very nonlinear nature of β_{swp} , its derivative was derived using a commercial symbolic math toolkit. The rate of change of β_{swp} is a very lengthy equation comprised of a very large number of trigonometric functions and is a function of $\Delta\Omega$, i , i_S , χ , and $\delta\dot{\Omega}$. As a result, it is not shown in this thesis. Both β_{swp} and δ_{swp} are functions of the relative RAAN. However, $\Delta\Omega$ is the only variable in $\dot{\beta}_{swp}$ that varies sufficiently with time to cause a change in its behaviour. The behaviour is demonstrated in Figure 3.33. The angular rates $\dot{\delta}_{\omega 3}$ and $\dot{\delta}_{\omega 4}$ are not shown since they are identical to $\dot{\delta}_{\omega 1}$ and $\dot{\delta}_{\omega 2}$, respectively. To account for the variations of $\Delta\Omega$ when predicting the switch point time, the angle $\dot{\delta}_\omega$ is assumed to be constant over short periods of time (less than one half of a year). The RAAN of the Molniya orbit, which has the highest rate of change of the RAAN amongst all three HEOs, secularly drifts just over 20° during that time period. All other HEOs have their RAANs vary significantly less in the same time frame. As previously mentioned, the bulk of switch point analysis occurs when $\Delta\Omega$ is far away from $\pm 180^\circ$.

Using both the calculated value of δ_ω and the current value of the position of the Sun with respect to its node, and both values' respective rates of change, the solar switch point time is predicted using the following equation:

Figure 3.33: Variation of the rate of change of δ_ω as a function of the relative RAAN.

$$t_{sup,S} = t_0 + \frac{\delta_\omega - u_{S_0}}{n_S - \dot{\delta}_\omega} \quad (3.59)$$

This equation has the same form as the prediction of the eccentricity and inclination and involves the same assumptions. Just as with the method to solve for the eccentricity switch points using the switch point angles, the outlined method for the AOP only predicts the time at which the solar rate of change of the AOP is zero. To be used correctly as part of an orbital control strategy, the time at which the total rate of change of the AOP is zero is desired. Not only must the long-period effects of the Moon be accounted for, but unlike the eccentricity and inclination, there are geopotential perturbations on the AOP that must be modelled as well, as described in Chapter 2. Fortunately, for the purposes of predicting the combined AOP switch point time, only the J_2 effect is taken into account to model the geopotential perturbations for reasons described in Section 2.2.3. A method similar to that of the combined eccentricity switch point time was implemented, which assumes that the non-solar perturbations on the AOP are approximately constant over

the small periods of time between the solar switch point time and the combined switch point time. The following equation shows the approximation of the combined switch point time prediction:

$$t_{swp} = -\frac{\dot{\omega}_{(J_2)+(S)+(M)}}{\ddot{\omega}_{(S)}(t = t_{swp,S})} + t_{swp,S} \quad (3.60)$$

where $\dot{\omega}_{(J_2)+(S)+(M)}$ is combined rate of change of the AOP due to J_2 , the Sun, and the Moon. This linear approximation requires the knowledge of the slope of the solar rate of change of the AOP. Just as with the rate of change of the eccentricity, the only time-dependent variables in $\dot{\omega}_{(S)}$ are the direction cosines:

$$\ddot{\omega} = \frac{3\sqrt{1-e^2}}{2n} \frac{\mu_S}{r_S^3} \left[8B \frac{dB}{dt} - 2A \frac{dA}{dt} + \chi \left(B \frac{dC}{dt} + C \frac{dB}{dt} \right) \right] \quad (3.61)$$

Using the rates of change of the direction cosines in Equations 3.46, 3.47, and 3.56, the method to predict the combined AOP switch point time is complete.

3.3 Validation of Switch Point Analysis

Both the method to solve for the occurrence of solar switch points in Section 3.1 and the method to predict the future combined switch points in Section 3.2 rely on several assumptions needed to simplify the process such that an analytical analysis can be employed. To start, the equations of motion to model solar perturbations assumed that the AOP is close enough to 270° such that the terms with $\sin 2\omega$ are neglected. Furthermore, the future variations of the RAAN are predicted to be entirely linear and the apparent

motion of the Sun is modelled as a circle. These types of assumptions may cause errors to accumulate in the prediction of the switch point times. This section quantifies the error of predicted combined switch points for the eccentricity, inclination, and AOP for all three HEOs, and as such validates the methods used to solve for the switch points. It also quantifies the solar switch point time error that is a result of the assumption regarding the AOP being exactly equal to 270° .

To compute the error, a series of MATLAB simulations are performed with randomized initial orbital conditions and starting epoch that are close to the values listed in Table 1.1. For each simulation, the first step is to numerically integrate the equations of motion for an HEO for one half of a year. The force model of the simulation uses the Cook single-averaged equations to model the Sun, the Cook's double-averaged equations for the lunar effects, and models the J_2 effect on the RAAN, AOP, and mean anomaly. After the integration, the combined switch point time is predicted for the eccentricity, inclination, and AOP using the methods described in Section 3.2. Next, the numerically-computed orbit data is put through a FOR loop and the direction cosines are computed at each time step, which is set to 0.5 days. During the one half of a year of orbit data, if the signs of any of the direction cosines change (for the eccentricity or inclination), or if the term $[4B^2 - A^2 - 1 - \chi BC]$ changes sign, then a solar switch point is designated at that time. The solar switch points calculated in this manner are based on the aforementioned approximations and assumptions. To calculate the true combined and solar switch points, the rates of change of the orbital elements are computed at each time step. The true combined switch point is identified whenever $\dot{x}_{(S)} + \dot{\tilde{x}}_{(M)} + \dot{x}_{J_2} = 0$, where x is the eccentricity, inclination, or AOP. The time when $\dot{x}_S = 0$ ignores the effects of the Moon and J_2 in determining the solar switch point. The predicted combined and solar switch points are compared with their true counterparts and the switch point time error is then calculated.

For each HEO, 10,000 of these simulation runs are performed to confirm that the methods developed in this chapter are valid and that they are applicable for the orbital control strategy in Chapter 4. It should be noted, however, that as shown in Figures 2.7 and 2.8, and explained in Section 3.1.1, there exist occurrences when the $2S$ oscillations disappear as a result of the relative geometry between the Sun and HEO. At times such as these, no solar switch points exist. Therefore, while there are 10,000 simulations for each HEO type, there may be less than that number of switch points calculated as a result of the randomized initial conditions that are implemented.

The results of the mean solar switch point time error for all three HEOs are shown in Table 3.2. This quantification of the solar switch point error is used to validate the methods developed in Section 3.1 which determine the conditions at which the rates of change of the orbital elements are equal to zero. Table 3.2 shows that using switch point conditions stated in Table 3.1 results in a mean error of the switch point time of less than one half of a day. This time is considered to be an acceptable level of error since the time between switch points is on average every 91 days; an error of one half of a day over that range is thus quite small. Therefore, the methods developed to use α and β to determine when the rates of change of the eccentricity, inclination, and AOP are equal to zero are valid.

Table 3.2: Mean solar switch time error (hours)

	Eccentricity	Inclination	AOP
Molniya	2.8	5.9	3.1
TAP	4.0	2.8	4.6
Tundra	6.5	1.4	8.1

The second part of the switch point validation is to quantify the error in predicting the future combined switch point times using the methods described in Section 3.2. Those

methods are used as part of the orbital control strategy in Chapter 4 to exploit the sign changes in the rates of change of the eccentricity, inclination, and AOP. Figures 3.34 through 3.36 show the histograms of the predicted combined switch point time error of all three HEOs for the eccentricity, inclination, and AOP, respectively. For the eccentricity switch point time error in Figure 3.34, the Tundra orbit has a mean error of 0.38 days with 94% of the error values as less than one day. Similarly the TAP orbit has a mean error of 0.35 days and 94% of the error values are also less than one day. The mean switch point error of the Molniya orbit for 10,000 simulations is approximately 0.43 days and 91% of those error values are less than one day¹⁴. The error for the eccentricity demonstrates that the prediction method described using Equation 3.44 is indeed quite accurate and shall be implemented into the control strategy in Chapter 4.

Figure 3.34: Predicted combined switch point error for the eccentricity of an HEO.

In Figure 3.35, the results of the error of the predicted combined switch point times

¹⁴While for each HEO, 10,000 simulations were performed, for a given simulation, its initial conditions may result in no computable switch point. Therefore, there are less than the full 10,000 simulation results available to quantify the error.

for the inclination are shown for all three HEOs. The Tundra, TAP, and Molniya orbits have a mean error of 0.43, 0.42, and 0.56 days, respectively. Their corresponding percent of predicted switch point errors that are less than one day are 91%, 93%, and 83%. While the error is slightly higher than the eccentricity errors, the results still signify that the corresponding switch point prediction method is highly accurate for the inclination.

Figure 3.35: Predicted combined switch point error for the inclination of an HEO.

Lastly, the error results for the AOP are shown in Figure 3.36. Since the combined switch points also include the effects from J_2 , it is expected that the mean predicted switch point error will be larger, particularly for the lower orbits as the magnitude of the perturbations due to the Earth's oblateness is larger. Indeed the mean error for the Molniya, TAP, and Tundra orbits are 3.75 days, 1.00 days, and 0.38 days, respectively, and the percentages of error values less than one day are 21%, 59%, and 96%. While the error for the Molniya orbit is large compared to the other HEOs, it still falls within acceptable bounds. Since the predicted switch point times are used with the analytical equations developed in Section 2.3 which have the $2M$ oscillations of the Moon averaged out, the predicted

error is still significantly less than the period of the neglected oscillations. Therefore, the methods described in Section 3.2 are sufficiently accurate to be able to predict, within reasonable accuracy, any future combined switch points for the eccentricity, inclination, or AOP.

Figure 3.36: Predicted combined switch point error for the AOP of an HEO.

Chapter 4

Orbit Control Strategy

The previous two chapters of this thesis focused mainly on the study of the dynamics of lunisolar perturbations acting on HEOs with Chapter 2 dealing with the methods to model the perturbations and Chapter 3 analyzing the perturbations using the development of the switch points. This chapter consists of the development of an orbit control strategy that will maintain the eccentricity, inclination, and AOP of an HEO near some nominal value with a specified tolerance¹. To control the orbital elements, a propulsion system is used which generates a thrust in order to adjust the velocity vector of the satellite. The change to the velocity imparted by the propulsion system is the ΔV . The control approach exploits the oscillations due to the lunisolar perturbations using the dynamic analysis from the previous chapters in order to reduce the total mission ΔV . The reduction of the ΔV is desirable since it directly translates to a reduction in mass of the satellite which affects launch costs. Therefore, the control strategy, the function of which is to efficiently maintain the orbital elements within their tolerance band and estimate the resulting ΔV costs, can be described most accurately as a mission planning tool designed to perform an estimation of the control requirements for a satellite mission in an HEO.

---

¹The ground-track is also controlled by the control strategy, albeit indirectly.

When discussing orbital control strategies for HEOs, the controlled orbital parameters must be identified. Historically, many satellites placed in HEOs had no orbit control implemented since the orbits themselves were reasonably stable for shorter mission durations and with proper selection of the initial conditions [2]. Amongst the select few missions for which payload requirements necessitated performing periodic corrective manoeuvres, the controlled orbital parameters were not universal. For certain missions, only the effects of the non-spherical Earth gravitational model on the semi-major axis were controlled in order to maintain the satellite's ground-track [9]. For others, multiple orbital elements, including the eccentricity, inclination, and AOP, were controlled [51].

G. Rondinelli et. al. published a paper in 1989 that detailed an orbit control strategy for a constellation of three Tundra orbits with their ground-tracks 120° apart [52]. To ensure that the constellation geometry remains intact, resulting in full Earth coverage being maintained, a series of manoeuvres is required to maintain the geographic location of the node of the satellite orbits. The paper analyzed two separate control strategies. The first, called the 'orbit-tended' approach, applied small impulsive manoeuvres for any of the Tundra orbits to adjust the semi-major axis to correct the node as soon as it exceeded its allowable range. The second, called the 'mission-tended' approach, focused primarily on the constellation as a whole and performed the manoeuvres only when the relative phase between two of the orbits became unacceptable. It should be noted that neither method performed a detailed analysis of the perturbation model affecting the semi-major axis or the ground-track; only a simple bang-bang control approach was performed. As such, the perturbations have not been utilized to reduce the required mission ΔV . The results of the paper state that for the 'orbit-tended' approach, maintaining the longitude to within $\pm 5^\circ$ requires approximately 3 m/s/year of ΔV .

A paper in 1989 written by G. Lecohier et al. was published in the Proceedings of the International Symposium: Space Dynamics which detailed a method to control a constellation of Molniya or Tundra orbits in order to improve land mobile communications for Europe [32]. There were three types of constellations considered in this research. The first is a Molniya orbit with three satellites that are 120° apart in both RAAN and true anomaly. The second is a two-satellite Tundra orbit constellation where the RAAN and true anomaly of the satellites are 180° apart, called the Tundra-1 system. The final constellation type is also a Tundra constellation but with a more circular orbit, called the Tundra-2 system. This constellation uses three satellites with a phase angle of 120° for the RAAN and true anomaly. The paper mentions 5 orbit parameters that must be maintained in some capacity to achieve the mission objectives: 1) geographical drift of the ground-track (i.e., the node), 2) global elevation angle of the satellite as seen by a user in Europe, 3) delay at the hand-overs passage of two successive satellites, 4) line of apsides motion, and 5) inclination stability. Objectives 1 and 3 are both achievable indirectly through the control of the semi-major axis; and Objectives 2 and 4 are functions of the control of the AOP. The inclination stability is maintained due to the geopotential and lunisolar perturbations, since the mission duration is capped at 7 years. No inclination manoeuvres are used. Since the Molniya orbit constellation has three satellites in three orbit planes 120° apart, at least one satellite experiences negative effects on the perigee altitude which cause it to decrease and could potentially cause premature orbit decay [2]. Therefore, in addition to the five primary control objectives, the Molniya orbit constellations requires periodic control of the perigee altitude to maintain the eccentricity of the orbit. For the Tundra-1 system, the maintenance of the node to within 5° of the reference value is required to achieve Objectives 1 and 3. This requires at most 10 m/s/year of ΔV . The AOP control, due to the perturbations from the geopotential and lunisolar gravity,

requires a maximum total maintenance cost of 110 m/s/year, but through selection of the initial conditions it can be lowered to approximately 62 m/s/year. For the Tundra-2 system, since the orbit is more circular, the variations of the AOP are smaller and the AOP maintenance costs are typically 60 m/s/year and are capped at 90 m/s/year assuming a maximum allowable range for the AOP of 2° . The node control for the Tundra-2 system is predicted in the range of 6-7 m/s/year for a tolerance of 2.5° . The final constellation, the Molniya system, has similar nodal control requirements as the Tundra-2 system, but also requires the additional perigee altitude maintenance manoeuvres which typically total less than 15 m/s/year of ΔV . Since the period of the RAAN for a Molniya orbit is approximately 7.5 years, for this type of mission, a near-full revolution of the orientation is expected. As such, a large variation in the station-keeping requirements for the AOP is expected since the lunisolar perturbations are strongly a function of the relative RAAN. A mean maintenance cost is predicted for such perturbations to be around 56 m/s/year. The authors conclude the paper by stating that the ΔV requirements for HEOs can be higher than the typical geostationary orbit station-keeping requirements (typically 50 m/s/year), depending on the initial configuration of the orbits with respect to the Moon and Sun. It should be noted, however, that the control strategy developed here does not exploit the perturbations that cause the variations in the orbital elements.

Using the same mission scenario as used in Reference [32], M. K. Khan published a pair of papers in 1993 and 1994 that discussed the station-keeping requirements for a Molniya or TAP orbit constellation [53, 54]. The Molniya orbit station-keeping method proposed by the author focuses on the maintenance of the ground-track through control of the semi-major axis. It also controls the eccentricity to maintain complete coverage by the constellation as well as avoiding premature orbit decay of the perigee altitude [53]. While the tolerances for the station-keeping system are not stated, the author states that

it requires approximately 430 m/s of ΔV for a 15-year mission to maintain the desired constellation configuration. For the 16-hour TAP orbit, a constellation of six satellites in three orbit planes is used. Because of the constellation configuration between the satellites, the control of the eccentricity is stated as being “indispensable” in order to maintain the proper handoff between satellites as they lose sight of the target regions. Khan also describes how the AOP must also be maintained, but that active control is very costly in terms of ΔV . Instead, the author proposes that the three orbit planes have their inclinations specifically set, not at the critical value, such that the drift induced by J_2 counteracts the secular effects of the lunisolar gravity. With this method, the AOP variations are able to be maintained to within a few degrees of the nominal value of 270° . Since there are three distinct planes with different initial RAANs, the eccentricity control requirements of the satellites differ. Reference [54] describes that the ΔV requirements vary from 144 m/s to 383 m/s for a 10-year mission.

In May of 1994, V. Kudielka and W. Drahanowsky performed a feasibility study for a 16-hour HEO (similar to the TAP orbit) [55]. The study covered a range of topics such as transfer methods to the highly-inclined HEO from a geostationary transfer orbit, general effects of the dynamics of HEOs, and de-orbit methods for the spacecraft to reduce space debris. In the study, the authors use a force model that includes J_2 and lunisolar gravity, and assume that the control requirements only necessitate one manoeuvre per year for each controlled element: the eccentricity and the inclination. Kudielka wrote a follow-up paper in 1997 which detailed highly-elliptical orbit planning and covered much of the same topics [56]. The papers describe that station-keeping corrections for an HEO can typically cost 15 to 35 m/s/year for the eccentricity-only control to upwards of 90 m/s/year if the AOP is also controlled and is set to 225° . Since the added control for the AOP is considered extremely high, the inclination is instead set and maintained such that the

stability of the AOP is achieved. The resulting ΔV requirements for the inclination range between 13 and 31 m/s/year depending on the set nominal value of the AOP, which was set between 180° and 225° .

In 1996, T. Ely completed his Ph.D. thesis focusing on the dynamics and control of multiple tesseral resonances including lunisolar perturbations for satellite in non-circular orbits [9]. This work first modelled the primary perturbations affecting the semi-major axis and node location, and then developed a novel control strategy to indirectly maintain the ground-track (node) by adjusting the semi-major axis. The perturbation forces, geopotential harmonics and lunisolar gravity, were modelled using Hamiltonian analysis. A grazing method strategy was employed to exploit these perturbations on the node by designing the manoeuvres such that “targeting burns at one node boundary force the trajectory to drift and graze the other boundary” [9]. Because of the complex dynamics acting on the non-circular satellite orbits, a numerical two-point boundary value problem was employed to determine the required change in the semi-major axis such that the node grazes the edge of its desired deadband range. This control strategy, which uses numerical targeting, was applied to a variety of orbit types with varying values of the eccentricity, inclination, and the AOP. There were no generalizations made as to the required mission ΔV for a given orbit; however, the author did state that the developed algorithm predicted an average ΔV of 0.505 m/s per manoeuvre with a manoeuvre required approximately every 174 days for a Molniya orbit to maintain its longitude to within $\pm 1^\circ$ of its nominal value.

M. J. Bruno and H. J. Pernicka published a paper detailing the design and station-keeping requirements of a constellation of satellites using a Tundra orbit in 2005, which was based on Bruno’s Master’s thesis from the previous year [51, 57]. Rather than focusing solely on the ΔV requirements to maintain the orbits, the authors first studied the

long-period perturbations affecting the orbital elements of a Tundra orbit. In their study, they numerically solved for initial orbit conditions that would minimize the variations of the orbital elements. The resulting analysis placed one of the three satellites in the constellation in a more eccentric orbit than the other two satellites. The other two satellites also had their inclinations set to well below the critical value to approximately 48° . Each satellite was also placed in a distinct orbit plane, with their nodes 120° apart. The station-keeping requirements for this were much higher compared to other papers which developed orbital control strategies for highly-eccentric orbits. Two manoeuvre types were used to maintain the constellation: transverse manoeuvres to control the semi-major axis, eccentricity, and AOP; and normal manoeuvres to control the inclination and RAAN with respect to the other satellite planes. The control strategy developed here performed a simple bang-bang approach where the orbital element was adjusted back to its nominal value using a once-per-year manoeuvre (if necessary). With the exception of the design of initial orbit parameters, no exploitation of the perturbations was used. For the transverse control, the location of the manoeuvres was designed such that a single pair of manoeuvres would adjust the semi-major axis, eccentricity, and AOP by their desired amounts. The locations were solved such that the transverse ΔV component to change the eccentricity and AOP were set equal to each other for a given desired change in those orbital elements. Using a numerical technique, the two true anomalies that allow for such a condition were determined. To adjust the semi-major axis, an iterative approach was used to adjust the ratio between the amount of ΔV used at each manoeuvre until the desired change in the semi-major axis was achieved. The normal manoeuvres only required a single manoeuvre location to adjust the inclination and RAAN. Similar to the transverse manoeuvre, the equations for the ΔV required to change the inclination and RAAN were set equal to each other. The corresponding value of the true anomaly, such that the conditions are met for a given change in the inclination and the RAAN, was then solved analytically. Since each

satellite was placed in a different type of Tundra orbit, the station-keeping requirements varied from satellite to satellite. The average ΔV requirements per year for a satellite was shown to be approximately 24 m/s.

In C. C. Chao's textbook *Applied Orbit Perturbation and Maintenance*, the author describes the basic station-keeping requirements for the node of a satellite in a Molniya orbit [58]. An empirical equation is developed to calculate the station-keeping ΔV per year as a function of the longitude of the ascending node. The ΔV is modelled as a sine function of the node's location and therefore there exist values for the longitude of the node where the station-keeping requirements are zero or at least quite small. These locations are functions of geopotential harmonics that model the Earth gravity field. Using the empirical equation, four approximate longitudes are identified where the ΔV requirements are approximately zero. These are considered as the stable longitudes: 86°W, 23°W, 67°E, and 157°E. While the ΔV equation represents a very approximate approach to determining the required station-keeping costs, it can be used to gain a sense of the range of expected cost as a function of the longitude of the node. According to the author, the maximum expected ΔV to maintain the longitude (though the tolerance is not given) is less than 3 m/s/year.

In 2005, M. S. Konstantinov et al. published a paper entitled "Spacecraft Station-Keeping in the Molniya Orbit using Electric Propulsion", which explicitly sought to create a control system to maintain a constellation of satellites in a Molniya orbit [59]. The authors state that since Molniya orbits are less stable than geostationary orbits, a complex control system is required to maintain the constellation configuration, by controlling three parameters: rate of change of RAAN, argument of perigee, and longitude of the node. To control the parameters, three 'correctable characteristics' were defined as the

perigee altitude, inclination, and period. These characteristics are the elements of the satellite's orbit that the control system alters. It is discussed in the paper that the control law structure includes the direction of the thrust in the RSW system, and the moment of application. To adjust the correctable characteristics, four control points were identified: apogee, perigee, and both the ascending and descending nodes. With the control system modelled and numerically simulated, the author finally stated that approximately 55 m/s per year of ΔV was required to maintain a given orbit configuration, which is comparable to geostationary satellites.

The emergence of satellite radio over the past 20 years has necessitated new orbit solutions to provide radio service to consumers at various locations throughout the world. Instead of using the more common geostationary orbit, which results in low-angle coverage for population centres at higher latitudes, the Sirius Satellite Radio company opted instead with a more unconventional Tundra orbit constellation [60]. Three satellites were launched in 2000 into the constellation with their orbit planes 120° apart. Because of the intense dynamic environment of these orbits, as discussed in the previous two chapters, a control strategy was required to both maintain the constellation and the Tundra orbits themselves. The controlled variations of the orbital elements, acquired through CelesTrak², over a 15-year period are shown in Figure 4.1 [61]. The semi-major axis is shown to be controlled within a 5 km range over the mission period. This is a result of the indirect control of the ground-track to ensure that the Sirius Satellite Radio constellation is centred over the North American continent. The periodic spikes in the semi-major axis are the result of the manoeuvres to maintain the eccentricity and AOP which temporarily shift the semi-major axis. To maintain the shape of the orbits, the eccentricity is also shown to be controlled in Figure 4.1. Because of the initial RAAN of the satellite, there are very few oscillations in

---

²CelesTrak is an online satellite catalogue which publishes orbit data.

the eccentricity. For the first 1200 days of the mission, the inclination variations are also controlled to within a half of a degree of tolerance; however, the remainder of the mission period shows no sign of inclination control. The AOP of the Tundra orbit is maintained near its nominal 270° value to within a tolerance of approximately 1° . Just as with the eccentricity, the variations due to the position of the Sun are masked by the long-period variations on account of the relative RAAN value. Unfortunately, very little published data exist which detail the control strategy concepts or even the ΔV required to maintain the orbital elements shown in Figure 4.1.

Control of the orbital elements of an HEO depends on the type of mission being performed and the payload mounted on the spacecraft. In addition to defining which parameters must be controlled, the tolerances of the control limits are also a function of the mission objectives. As outlined in Chapter 1, the control strategy developed for this thesis is kept non-mission-specific. While the proposed Polar Communications and Weather mission acted as the original inspiration for this research, the control strategy developed in this chapter is designed to demonstrate the ability to exploit the oscillations induced by the gravities of the Moon and Sun. Thus, the primary controlled orbital elements are those which are most significantly affected by lunisolar perturbations: the eccentricity, the inclination, and the AOP. Furthermore, since there are no mission parameters with which to define control tolerances, the amplitudes of the $2S$ oscillations of the orbital elements that are being utilized in the control strategy are used to set the control limits.

In the next section, the different manoeuvre sequences are discussed and the locations of the manoeuvres are derived analytically in an attempt to both maximize the efficiency of the manoeuvre in terms of unit change of the orbital element per unit ΔV and to minimize the effects of the manoeuvre on the other orbital elements. Following that, the sensitivity

Figure 4.1: Variation of the orbital elements for the Sirius-1 satellite [61].

of the controlled orbital elements is studied. The method to exploit the lunisolar perturbations using the grazing method assumes that small adjustments to the orbital elements do not significantly affect the overall behaviour of variations. This sensitivity analysis mea-

sures the effects of a small initial offset of the eccentricity, inclination, or AOP compared to a reference orbit. From there, the analytical prediction and control (APC) method is presented. To maximize the $2S$ oscillations that occur on an HEO, a method to calculate the initial RAAN is developed to ensure that the relative RAAN reaches 0° at the mission mid-point time. As explained in the previous chapter, it is when $\Delta\Omega$ is near 0° that the solar oscillations are at their largest and the relative RAAN oscillations are at their smallest for the eccentricity and AOP. Next, the methodology for the APC is presented. The algorithm used to exploit the lunisolar perturbations using the grazing method by analytically predicting the required ΔV is developed. To validate the results of the APC method, the following section discusses the development of two additional control strategies: the numerical prediction and control (NPC) method and the Baseline method. The numerical method is used to demonstrate that the various analytical methods developed to exploit the lunisolar perturbations in the APC are equivalent to performing a large number of numerical simulations to account for the sign changes in the perturbations. The Baseline control method, which simply sets the orbital elements back to their nominal values, is used to quantify the ΔV savings gained by exploiting the gravities of the Moon and Sun. The results are then presented and discussed and their sensitivities to various mission parameters such as orbit type, mission duration, and control tolerance size are studied. All the previous orbit simulations involved using the custom orbit propagator in MATLAB with a limited force model. To thoroughly validate the control strategy developed in this thesis, a more robust simulator is needed. The software package STK is used in Section 4.6 to provide a more accurate force model for the HEOs. The three orbit control strategies developed in Sections 4.3 and 4.4 are implemented into STK using an external plug-in MATLAB script which contains the control logic. The results of the simulations of all three HEOs are then presented and discussed.

4.1 Manoeuvre Sequence

Since the effects of an impulsive thrust force imparted on the spacecraft have a specific effect on the change in the orbital elements, each controlled orbital element therefore has a distinct manoeuvre sequence. The development of the manoeuvre sequences involves determining the location(s)³ of the manoeuvre(s) and the direction of the thrust vector. The control strategy is developed for a generic spacecraft design, thus no assumptions on the limitations of the propulsion system are made. Therefore it is assumed that the propulsion system can generate a thrust force in any direction. For the design of each specific sequence, the primary objective is to attempt to maximize the efficiency of the thrust vector to change the orbital element⁴. Furthermore, since multiple orbital elements are affected by the same thrust vector, it is desired also to design the manoeuvre sequence such that there is a minimal effect on the other orbital elements. The trade-off between these objectives is discussed in subsequent subsections. It should be noted that regardless of the type of control strategy employed, the manoeuvre sequences for the orbital elements remain the same. The only difference between the control methods is the desired value of the orbital element.

The thrust vector is defined in the RSW coordinate system as described in Section 2.1 such that the ΔV vector can be used with the Gauss planetary equations. Figure 4.2 shows the behaviour of the RSW system for a satellite in a non-circular orbit in relation to the velocity vector. Recall that ϕ is the flight-path angle of the satellite, which is calculated using Equations 2.1 and 2.2. To model the direction of a manoeuvre, an additional angle, the thrust vector direction $\phi_{\Delta V}$ is used. Similar to the definition of the flight-path

---

³For some of the sequences, two manoeuvres are required to perform the desired adjustment of the orbital element.

⁴In terms of the manoeuvre design, maximizing the efficiency refers to solving for the location of the manoeuvre that results in the largest change in an orbital element per unit ΔV

angle, the angle $\phi_{\Delta V}$ is defined with respect to the transverse vector $\hat{S}$, positive counter-clockwise. For the in-plane manoeuvres used in this thesis, the location, as well as the thrust vector direction, are solved for analytically to produce the most efficient manoeuvre.

Figure 4.2: Diagram of the RSW system in the orbit plane.

4.1.1 Eccentricity Manoeuvres

The first manoeuvre sequence, and the most simple one, is for the control of the eccentricity. Using the Gauss planetary equations for the eccentricity in Equation 2.5, the change in the eccentricity due to an impulsive thrust vector is equal to:

$$\Delta e = \frac{\sqrt{1-e^2}}{na} \left\{ \sin(\nu) \Delta V_R + \left(\cos(\nu) + \frac{e + \cos(\nu)}{1 + e \cos(\nu)} \right) \Delta V_S \right\} \quad (4.1)$$

where it is shown that the change in the eccentricity is a function of a propulsive force in either the radial or transverse directions (or both). Similarly, Equation 2.4, which models the change of the semi-major axis, is manipulated to become:

$$\Delta a = \frac{2}{n\sqrt{1-e^2}} \{e \sin(\nu) \Delta V_R + (1 + e \cos \nu) \Delta V_S\} \quad (4.2)$$

Based on Equations 4.1 and 4.2, it is shown that both the eccentricity and the semi-major axis are functions of radial and transverse manoeuvres and that one cannot adjust one orbital element without affecting the other. In 1925, a set of transfer manoeuvres were designed to transfer from one circular orbit to another coplanar circular orbit through a temporary elliptical transfer orbit [8]. This process is called a Hohmann transfer named after its developer, W. Hohmann. A generalized Hohmann transfer can also be applied to eccentric coplanar orbits where the pair of manoeuvres, one at apogee and one at perigee, can set both the eccentricity and semi-major axis to a desired value. This transfer method yields the most efficient use of ΔV to change the two orbital elements, though it was not actually proved by Hohmann⁵ [8]. A diagram of the Hohmann transfer is shown in Figure 4.3.

A secondary benefit of the Hohmann transfer is that since both manoeuvres are performed at the perigee and apogee, only a transverse manoeuvre is required to adjust the semi-major axis and eccentricity. Upon observing Equation 2.8, the AOP is also a function of radial and transverse ΔV components. However, when the true anomaly is 0° or 180° , the coefficient for F_S becomes zero. Therefore, the Hohmann transfer does not affect any orbital element except the semi-major axis and the eccentricity⁶. The direction and location of the first and second manoeuvres dictate whether the eccentricity increases or decreases. The summary of the effects of varying the location and direction

---

⁵There is another transfer sequence called the bi-elliptic method that can be more efficient than the Hohmann transfer, but only when the ratio of the initial semi-major axis to the final semi-major axis exceeds 12.5.

⁶This is only true in theory. In practice, no manoeuvre is perfectly impulsive, nor will it be applied exactly on the line of apsides.

Figure 4.3: Diagram of the generalized Hohmann transfer.

of the Hohmann transfer manoeuvres are shown in Table 4.1⁷. The results demonstrate that there is more than one sequence to lower or raise the eccentricity of the HEO. The Hohmann transfer is also useful for correcting any perturbations acting on the semi-major axis. While the lunisolar perturbations do not significantly affect the size of the orbit, the geopotential gravity causes the semi-major axis to vary away from its nominal value. Since the Hohmann transfer is already adjusting the semi-major axis to correct the eccentricity, the size of the orbit can be set back to its nominal value after each eccentricity manoeuvre sequence. The temporary value of the semi-major axis when the satellite is in the transfer orbit is a function of the desired changes in the eccentricity and semi-major axis, but also the Hohmann transfer sequence. As shown in Table 4.1, depending on the location of the first manoeuvre and its direction, the temporary shift in the semi-major axis can be greater or less than its initial value. Since the variations of the ground-track of the HEO are a function of the value of the semi-major axis relative to its nominal value, the temporary value of the semi-major can be used to offset the drift of the node. This is

⁷The (+) and (−) symbols denote the direction of the ΔV vector and the $\uparrow$ and $\downarrow$ symbols denote the corresponding direction of change in the eccentricity or semi-major axis.

discussed further in Section 4.1.4.

Table 4.1: Type of Hohmann transfers: the resulting change in the eccentricity and the temporary change in the semi-major axis.

ν_1 (deg)	ΔV_1	ν_2 (deg)	ΔV_2	Δe	Δa_{tmp}
0	(+)	180	(-)	$\uparrow$	$\uparrow$
180	(+)	0	(-)	$\downarrow$	$\downarrow$
0	(-)	180	(+)	$\downarrow$	$\uparrow$
180	(-)	0	(+)	$\uparrow$	$\downarrow$

To confirm the effectiveness of the Hohmann for a change in the eccentricity, Equation 4.1 is analyzed over the complete revolution of the orbit. Furthermore, the direction of the thrust vector was varied between $\pm 90^\circ$ within the orbit plane to establish that it is indeed a tangential manoeuvre at the apogee or perigee that produces the largest change in eccentricity per unit ΔV . The results of this analysis are shown in Figure 4.4. As expected, tangential manoeuvres at the apogee and perigee are shown to be the locations where the maximum change in the eccentricity occurs. Additionally, it was observed that there exists a curve that is a function of both the true anomaly and the direction of the manoeuvre where a thrust force has no net change on the eccentricity. This behaviour is useful for the design of the AOP manoeuvre sequence and is discussed further in Section 4.1.3.

With the location and direction of the eccentricity manoeuvres set, the next step in the development of the eccentricity manoeuvre sequence is to calculate the change in the velocity required to set the orbital element to its desired value, e_{des} . The magnitude of the eccentricity manoeuvre ΔV_e is calculated by taking the difference between the desired velocity, which is based on the desired orbital parameters, and the actual velocity of the satellite. The equations used to calculate the ΔV depend on which Hohmann transfer

Figure 4.4: The change in the eccentricity per unit ΔV as a function of the true anomaly and the direction of the manoeuvre. The black curve is the line such that $\Delta e/\Delta V$ is zero. The magenta line is flight path angle of the satellite in a Molniya orbit.

method is used, perigee manoeuvre first or apogee manoeuvre first. For the perigee-manoeuve-first transfer method, the equations are:

$$\Delta V_{e,1} = \sqrt{2\mu\left(\frac{1}{r_p} - \frac{1}{r_p + r_{a,des}}\right)} - \sqrt{2\mu\left(\frac{1}{r_p} - \frac{1}{r_p + r_a}\right)} \quad (4.3)$$

$$\Delta V_{e,2} = \sqrt{2\mu\left(\frac{1}{r_{a,des}} - \frac{1}{r_{p,des} + r_{a,des}}\right)} - \sqrt{2\mu\left(\frac{1}{r_{a,des}} - \frac{1}{r_p + r_{a,des}}\right)} \quad (4.4)$$

To simplify the equations, the orbital elements e and a were replaced by the perigee and apogee radii, r_p and r_a , where $r_p = a(1 - e)$ and $r_a = a(1 + e)$. Therefore, Equations 4.3 and 4.4 are functions of the current values of the perigee and apogee radii, and the desired values of the perigee and apogee radii, $r_{p,des}$ and $r_{a,des}$. For the apogee-manoeuvre-first transfer sequence, the variables r_p and r_a are swapped and the resulting set of equations is:

$$\Delta V_{e,1} = \sqrt{2\mu\left(\frac{1}{r_a} - \frac{1}{r_a + r_{p,des}}\right)} - \sqrt{2\mu\left(\frac{1}{r_a} - \frac{1}{r_a + r_p}\right)} \quad (4.5)$$

$$\Delta V_{e,2} = \sqrt{2\mu\left(\frac{1}{r_{p,des}} - \frac{1}{r_{a,des} + r_{p,des}}\right)} - \sqrt{2\mu\left(\frac{1}{r_{p,des}} - \frac{1}{r_a + r_{p,des}}\right)} \quad (4.6)$$

Depending on whether it is decided to temporarily increase or decrease the semi-major axis during the transfer phase of the Hohmann transfer, Equations 4.3 and 4.4, or 4.5 and 4.6, are used to calculate the magnitude of the transverse ΔV used to set the eccentricity to the desired value.

4.1.2 Inclination Manoeuvres

To control the inclination of a satellite, an out-of-plane manoeuvre is required as demonstrated in Equation 2.6. Just as with the eccentricity and the semi-major axis, Equation 2.6 can be rewritten in terms of the ΔV vector components as⁸:

$$\Delta i = \frac{\sqrt{1-e^2}}{na} \frac{\cos(\omega + \nu)}{1 + e \cos \nu} \Delta V_W \quad (4.7)$$

It is common to perform a manoeuvre to change the inclination at either one of the nodes of the orbit since the orbit-normal thrust vector does not affect the AOP or RAAN at these locations and it is most efficient location for an inclination manoeuvre in terms of $\Delta i/\Delta V$ for circular orbits [8]. However, it is not the most efficient location for an inclination manoeuvre of an HEO. Unlike the eccentricity manoeuvre sequence in the previous section, only one manoeuvre is necessary to adjust the inclination to its desired value. Equation 4.7 is plotted as a function of the true anomaly in Figure 4.5 to establish that the node of the satellite ($\nu = 90^\circ$, since $\omega = 270^\circ$) does not produce the most efficient inclination manoeuvre. Additionally, the change in the RAAN and AOP (due to orbit normal manoeuvres only) per unit of ΔV are also shown as a function of the position of the satellite in Figure 4.5⁹.

As previously mentioned, the use of inclination manoeuvres at the node is beneficial since it is shown that both the change in the RAAN and AOP are zero when a manoeuvre is applied in the orbit normal direction. However, there exists two locations closer to the apogee where the term $\Delta i/\Delta V = f(\nu)$ has reached its maximum. Unfortunately,

---

⁸Note that in this context, Δi signifies the change in inclination due to a manoeuvre, not the relative inclination between the HEO and the solar orbit as discussed in Chapter 3.

⁹Just as with Δi , the variable $\Delta \Omega/\Delta V$ does not refer to the relative RAAN but instead the change in the RAAN of the HEO per unit ΔV .

Figure 4.5: Variation of the change in orbital angles, i , ω , and Ω , per unit ΔV (in the normal direction) for a Molniya orbit.

a manoeuvre in either of these locations results in a change in the AOP and RAAN as well. The inclination manoeuvre sequence is instead designed to account for those unintended changes to ω and Ω . Since the RAAN of the HEO does not need to be controlled, small changes, on the order of less than 1° , are not significant considering the drift rate of the RAAN due to J_2 . The AOP, on the other hand, is controlled by its own manoeuvre sequence which is discussed in the next section. These AOP-specific manoeuvres are designed such that the sign changes in the rate of change of the AOP are capitalized. Regardless, the goal of the control strategy in general is to ensure that the orbital elements do not vary outside some set of margins. Therefore, the inclination manoeuvre sequence is designed such that if the current rate of change of the AOP is positive, the location and direction of the out-of-plane manoeuvre are set such that the AOP decreases while the inclination is set to its desired value. Conversely, if the rate of change of the AOP is negative, the inclination manoeuvre is designed to increase the AOP. The amount of change on the AOP due to the inclination manoeuvre is considerably smaller than the

expected margins imposed on the AOP.

The location of the desired inclination manoeuvre shown in Figure 4.5 is solved for analytically by taking the derivative of Equation 4.7 with respect to the true anomaly, setting it equal to zero, and then solving for ν . The results are:

$$\nu_1 = \sin^{-1} e - 270^\circ \quad (4.8)$$

$$\nu_2 = 270^\circ - \sin^{-1} e \quad (4.9)$$

These locations are shown to be a function of the shape of the HEO as well as the AOP, but this assumes that $\omega = 270^\circ$. While it can be seen in Figure 4.5 that performing the manoeuvre at the values given by Equation 4.8 or 4.9 increases the efficiency of the change in the inclination by approximately 50% for a Molniya orbit, as the orbit becomes more circular, this improvement diminishes. Furthermore, it can be shown that the effects of the inclination manoeuvre on the AOP also diminishes as the eccentricity becomes smaller. These relationships are demonstrated in Figure 4.6.

The percent increase of the change in inclination is calculated by computing the percent error between a manoeuvre performed at the locations determined by Equation 4.8 and a manoeuvre performed at the node. While the smaller eccentricity of the Tundra orbit significantly reduces the gains achieved by performing the inclination manoeuvre at the locations given by Equations 4.8 or 4.9, the change to the AOP is also greatly reduced. Therefore, there is a limited benefit to changing the manoeuvre location depending on the type of HEO.

Figure 4.6: Variation of the change in the AOP per unit ΔV_i and percent increase of the change in the inclination for manoeuvres located as determined by Equation 4.8 or 4.9 compared to nodal manoeuvres.

Table 4.2 summarizes the inclination manoeuvre sequence and details the location and direction of the thrust vector¹⁰ based on whether it is desired to increase or decrease either the inclination or the AOP.

Table 4.2: Summary of the inclination manoeuvre sequence.

$\nu_{\Delta V}$ (deg)	ΔV_W	Δi	$\Delta\omega$
Eq. 4.8	(+)	↑	↓
Eq. 4.9	(+)	↓	↓
Eq. 4.8	(-)	↓	↑
Eq. 4.9	(-)	↑	↑

Since only a single manoeuvre is needed to adjust the inclination, Equation 4.7 is re-arranged in terms of the ΔV to calculate the size of the manoeuvre. The resulting equation:

¹⁰A positive (+) direction is in the direction of the angular momentum vector and a negative (-) direction is opposite to the angular momentum vector.

$$\Delta V_i = (i_{des} - i) \frac{na}{\sqrt{1 - e^2}} \frac{1 + e \cos \nu}{\cos(\omega + \nu)} \quad (4.10)$$

is a function of the desired value of the inclination, i_{des} , which depends on the orbit control strategy used. For the APC or NPC methods, the desired value is set such that the $2S$ oscillations on the inclination are exploited. For the Baseline method, the desired value of the inclination is its nominal value which is the critical inclination.

4.1.3 AOP Manoeuvres

Unlike the eccentricity and inclination, the AOP is affected by both in-plane and out-of-plane manoeuvres as evidenced by Equation 2.8. However, the effects of the orbit-normal thrust component on the AOP are smaller than the combined radial-transverse components, so only in-plane manoeuvres are considered for active control of the AOP¹¹. Modifying Equation 2.8 to calculate the change in the AOP due a thrust in the radial and/or transverse directions results in:

$$\Delta \omega = \frac{\sqrt{1 - e^2}}{nae} \left\{ -\cos(\nu) \Delta V_R + \sin(\nu) \left(1 + \frac{1}{1 + e \cos \nu} \right) \Delta V_S \right\} \quad (4.11)$$

where ΔV_W is set to 0. Unlike the eccentricity, where there is a standard location and direction of the manoeuvre defined by the Hohmann transfer, there is no equivalent manoeuvre sequence for the control of the AOP for an HEO. To solve for a location and direction of the AOP manoeuvre using Equation 4.11, the first step is to replace ΔV_R

---

¹¹Recall, however, that the out-of-plane manoeuvres to control the inclination also alter the AOP slightly.

and ΔV_S by $\Delta V \sin \phi_{\Delta V}$ and $\Delta V \cos \phi_{\Delta V}$, respectively, where ΔV is the magnitude of the thrust vector. The derivative of the term $\Delta\omega/\Delta V$ is then set to zero to calculate the thrust vector direction that results in the maximum change in the AOP per unit ΔV for a given location in the orbit. To solve for the location of the manoeuvre, Equation 4.1 is modified to solve for $\phi_{\Delta V}$ where the change in the eccentricity per unit ΔV is equal to zero. These two equations are then set to equal each other and the resulting manoeuvre locations are solved to be:

$$\nu_1 = \cos^{-1}\left(\frac{-3 - e^2 + \sqrt{e^4 - 10e^2 + 9}}{4e}\right) \quad (4.12)$$

$$\nu_2 = 360^\circ - \cos^{-1}\left(\frac{-3 - e^2 + \sqrt{e^4 - 10e^2 + 9}}{4e}\right) \quad (4.13)$$

There are therefore two locations within an orbit where no change in eccentricity occurs and a maximum change in the AOP occurs at that location based on its thrust direction. To solve for the thrust direction, the derivative of Equation 4.11 with respect to $\phi_{\Delta V}$ is set to zero and the corresponding equation is rearranged in terms of the $\phi_{\Delta V}$. The resulting equation is:

$$\phi_{\Delta V} = \tan^{-1}\left(\frac{-\cos \nu}{\sin \nu \left(1 + \frac{1}{1+e \cos \nu}\right)}\right) \quad (4.14)$$

which shows that the AOP manoeuvre is a combination of radial and transverse thrust force.

The variation of the change in the AOP per unit ΔV as a function of the true anomaly

and the thrust vector direction using Equation 4.11 is shown in Figure 4.7. Superimposed on this figure are the curves which show the relationship between the true anomaly and the thrust direction such that the change in the eccentricity is zero and the change in the AOP per unit ΔV is a maximum. The intersections of these two curves are the locations set for the AOP manoeuvres. As shown, these locations are not on the satellite flight-path angle curve; therefore, the satellite must perform a manoeuvre that is not aligned exactly with the velocity vector as is performed for the eccentricity manoeuvres. It can also be observed that the locations selected for the AOP do not correspond to the peaks of the contour plots of Figure 4.7, though they are close. The locations of the peaks cannot be derived analytically, but the location where the maximum change in the AOP corresponds to when the change in the eccentricity is zero can be. However, it can be observed that the peaks of the contour plot are remarkably close to the intersection of the two curves. While the locations of the peaks are the ideal locations for the AOP manoeuvres, there is very little difference in $\Delta\omega/\Delta V$ between those locations and the locations of the intersections of the curves. For a Molniya orbit, the percent error in using the non-ideal manoeuvre location is approximately 0.40%. This constitutes an extremely small error and thus the locations calculated using Equations 4.12 and 4.13 are acceptable. For the TAP and Tundra orbits, the percent error is even smaller and the distinction between the ideal manoeuvre locations and the locations from Equations 4.12 and 4.13 is also negligible. The percent errors are 0.04% and $5.1 \times 10^{-4}\%$ for the TAP and Tundra orbits, respectively.

While the location and thrust direction for the AOP manoeuvre are designed to provide a near-maximum change to the AOP and a zero change to the eccentricity using a combined radial and transverse manoeuvre, based on Equation 4.2, such a manoeuvre also results in a change to the semi-major axis. As a result, the effects on the semi-major axis must be accounted for to ensure that there is no net change after the manoeuvre

Figure 4.7: The change in AOP per unit ΔV as a function of the true anomaly and the direction of the manoeuvre. The green curve is the line such that $\Delta e/\Delta V$ is equal to zero. The black curve is the line such that $\Delta \omega/\Delta V$ is a maximum. The magenta line is flight path angle of the satellite in a Molniya orbit.

sequence. Figure 4.8 shows the relationship between the change in the semi-major axis and the manoeuvre direction and location along with the curves for the maximum change in ω and zero change in the eccentricity. The results show that both locations for the AOP manoeuvre sequence generate a positive change to the semi-major axis. Therefore, if only

a single manoeuvre was performed to change the AOP, there would also be a change to the semi-major axis. This is undesirable since the period of the HEO must be maintained to ensure proper control of the ground-track, which is discussed in more detail in the next section. To remedy this situation, the AOP sequence performs a pair of manoeuvres, designed such that there is no net change on the semi-major axis. The desired change to the AOP is split up between the locations given by Equation 4.12 and 4.13, and the thrust vector directions are designed so that the second manoeuvre offsets the change in the semi-major axis due to the first manoeuvre.

The summary of the AOP manoeuvre sequence is shown in Table 4.3. The results demonstrate the effect of the manoeuvre locations and signs of the manoeuvres (i.e., thrust direction) on the change of the AOP and the temporary change to the semi-major axis.

Table 4.3: Summary of the AOP manoeuvre sequence.

ν_1	ΔV_1	ν_2	ΔV_2	$\Delta\omega$	Δa_{tmp}
Eq. 4.12	(+)	Eq. 4.13	(-)	$\uparrow$	$\uparrow$
Eq. 4.13	(+)	Eq. 4.12	(-)	$\downarrow$	$\downarrow$
Eq. 4.12	(-)	Eq. 4.13	(+)	$\downarrow$	$\uparrow$
Eq. 4.13	(-)	Eq. 4.12	(+)	$\uparrow$	$\downarrow$

To calculate the magnitude of the ΔV to adjust the AOP to a desired value, ω_{des} , Equation 4.11 is rearranged in terms of the magnitude of the ΔV vector. Since the AOP manoeuvre is a function of both radial and transverse components, the thrust direction must be substituted into Equation 4.11 first in order to solve for the ΔV magnitude. In order for the net effect of the pair of AOP manoeuvres to set the AOP to its desired value, the change to the AOP is divided by two so that each manoeuvre performs one half of the desired change to the AOP. The resulting equation used to calculate the ΔV for both manoeuvre locations is:

Figure 4.8: The change in semi-major axis per unit ΔV as a function of the true anomaly and the direction of the manoeuvre. The green curve is the line such that $\Delta e/\Delta V$ is equal to zero. The black curve is the line such that $\Delta \omega/\Delta V$ is a maximum. The magenta line is flight path angle of the satellite in a Molniya orbit.

$$\Delta V_{\omega} = \frac{1}{2} \frac{\omega_{des} - \omega}{nae\sqrt{1-e^2}} \left[\sin \nu \left(1 + \frac{1}{1+e \cos \nu} \right) \cos \phi_{\Delta V} - \cos \nu \sin \phi_{\Delta V} \right] \quad (4.15)$$

4.1.4 Ground-track Control Sequence

As cited in the literature discussed earlier in this chapter, control of the ground-track¹² of an HEO is typically included when orbit control is implemented. However, as discussed in Reference [9], the primary perturbation source which causes drift of the ground-track is the non-spherical shape of the Earth. A control method has already been developed in Reference [9] which exploits the geopotential perturbations to maintain the node of an HEO. The focus of this thesis is on the study of the dynamics of lunar and solar perturbation forces on acting on HEOs and the development of a control strategy to exploit said forces. Therefore, the primary objective of the orbit control strategy, developed in this chapter, is not to develop a novel ground-track control sequence. Instead, control of the ground-track is performed as a secondary aspect of the manoeuvre sequences which maintain the eccentricity and AOP. Since both manoeuvre sequences result in temporary changes to the semi-major axis which controls the drift of the node, the location order of the manoeuvres for the eccentricity and AOP sequences are set such that the temporary semi-major axis forces the node back towards its nominal value. Furthermore, if the time between the two eccentricity or AOP manoeuvres is not sufficiently long enough to correct the ground-track, the time between the manoeuvres can be increased by one or more orbit periods, or by performing a longer sequence of manoeuvres. This allows more time for the temporary semi-major axis to correct the node back to its nominal value.

In addition to the indirect control of the node, as discussed in References [58] and [62], there exist stable longitudes for Molniya orbits where the perturbations on the ground-track location are near zero. To support the control of the ground-track for Molniya orbits, its nominal value shall be set to a stable longitude to reduce the perturbations. This type

---

¹²With the eccentricity, inclination, and AOP controlled, the only remaining parameter to maintain the ground-track is its geographic location, i.e., the longitude of the ascending node

of ground-track control was utilized for a Soviet communications satellite (Molniya-193). This satellite only performed two manoeuvres over this mission period as shown in Figure 4.9. As a result, the variations of the longitude of the ascending node are bounded within a reasonably small tolerance and no direct control is required. Figure 4.10 depicts the variations of the node over a 9-year period without ground-track control, where the node is maintained to within $\pm 8^\circ$ of its nominal value. For the larger HEOs, the TAP and Tundra orbits, the perturbations on the semi-major axis due to the gravitational Earth model, while much smaller than the Molniya orbit, still require solving for a stable longitude to assist with the indirect control of the node through the eccentricity and AOP manoeuvres. A set of STK simulations for the Tundra and TAP orbits was performed to determine a suitable choice for the desired longitude of the node. For each HEO described in Table 1.1, the orbital elements were simulated for one year with a given initial LAN. The geographic longitudes when the satellite returned to its ascending node was compared to its value at the mission start time. If the variations of the LAN were small, then the initial LAN was considered stable. Just as with the Molniya orbits, there exist multiple values for the larger HEOs. For the Tundra orbit an initial longitude of 153° W was used, and the TAP orbit's longitude was set to 153° E.

4.2 Sensitivity Analysis

The APC method relies on the analytical prediction of the variations of the orbital elements from Chapter 2, as well as the switch point time prediction from Chapter 3, to exploit the lunisolar perturbations. This process, described in detail in Section 4.3, relies on the assumption that a small change in the orbital element does not change the behaviour of the satellite over periods of time of less than one year. This is necessary in

Figure 4.9: Variation of the semi-major axis for the Molniya-193 satellite [61].

Figure 4.10: Variation of the node for the Molniya-193 satellite [61].

order to design a manoeuvre sequence that results in the orbit element reaching its limit when its slope is zero. In this section, the sensitivity of the eccentricity, inclination, and AOP to small changes in their initial values is studied to ensure that the APC method for calculating the desired values for the controlled orbital elements is valid.

The method to quantify the error associated with a small disturbance to the initial orbital element is as follows. For a given HEO, a random initial epoch, eccentricity (near the value stated in Table 1.1), and RAAN are selected¹³. An orbit simulation is then performed for a duration of 100 days using MATLAB. This uses the same numerical simulation tools described in Section 3.3. The orbit data from this simulation represents the ‘reference’ solution. From there, the initial eccentricity of the reference solution is offset to replicate a change to the eccentricity due to a manoeuvre. The amount of the adjustment is a normally distributed random number with a mean of zero and a 3σ value of 0.003¹⁴. This value is set to represent the largest expected manoeuvre. A second numerical simulation then takes place with the offset eccentricity. These results are the ‘offset’ data. Then, the RMS error between the reference eccentricity data and the offset eccentricity data is computed to capture the differences in behaviour between the two data sets over time. Next, the percent error between the RMS error and the difference between the offset eccentricity data and the reference eccentricity data is calculated. This percent error value is used to confirm that the changes in behaviour of the eccentricity after a manoeuvre are very small and that the data simply shifts positively or negatively. To capture the general behaviour of the eccentricity discrepancies between the reference and offset data, 10,000 simulation values of the percent error were calculated. For each of the 200 reference orbit simulations, 50 offset simulations were performed, altering the initial eccentricity both positively and negatively, to acquire the total expected range of manoeuvres. The result of the analysis is used to confirm that the reference data and the offset data can be approximated as parallel curves.

---

¹³This procedure is for the study of the sensitivity of the eccentricity. For equivalent studies for the inclination and AOP, the orbital element is replaced with the corresponding one.

¹⁴This value is twice the $2S$ oscillation amplitude of the eccentricity for a Tundra orbit.

The results of the 10,000 simulations for the eccentricity sensitivity analysis were compiled into a histogram to quantify the error for the APC method to exploit the lunisolar perturbations. The results for each HEO are shown in Figure 4.11. The results of the TAP and Tundra orbit simulations show that there is less than 1% error between the initial change in the eccentricity and the overall difference between the reference and offset data. The Molniya orbit has a slightly larger mean percent error, but its maximum expected percent error is just over 1%. Therefore, it is demonstrated in Figure 4.11 that the sensitivity of the eccentricity to a small initial offset is small and that the procedure for determining the desired value of the eccentricity using the APC method is valid.

The same style of simulations was performed for the inclination of the HEO, the results of which are shown in Figure 4.12. The data for all three HEOs show that there is a very small discrepancy in the behaviour of the inclination due to a manoeuvre typical of what one sees in the APC method developed in Section 4.3. For the study of the sensitivity of the AOP, Figure 4.13 shows the results of the simulations for all three HEOs. While the percent error for the Tundra orbit is larger than the percent errors of either of Figure 4.11 or 4.12, it is still a very small variation. As such, the analytical control of the AOP can be accomplished using the analytical equations from Section 2.3 and the switch point prediction methods from Section 3.2.

4.3 Analytical Prediction and Control Strategy

The combination of the lunisolar and geopotential perturbation forces acting on the eccentricity, inclination, and AOP of HEOs has been shown to be significant. If there is a mission objective that requires that these elements remain close to their specified val-

Figure 4.11: Histogram of the sensitivity of the eccentricity.

Figure 4.12: Histogram of the sensitivity of the inclination.

ues, such as maintaining a narrow position control of the satellite as it reaches its apogee for the coverage of the Arctic region, the natural dynamics of the HEO cannot be relied upon and an orbit control strategy must be employed. As with any orbit control strategy, reducing the ΔV required for the mission is a primary objective. It has a direct effect

Figure 4.13: Histogram of the sensitivity of the AOP.

on the mass of the satellite and therefore the cost of the mission. To reduce the total mission ΔV , a novel control strategy is developed. The control strategy discussed, called the APC method, uses analytical means to calculate the desired value of the controlled orbital elements such that the lunisolar perturbations are exploited using the grazing method.

4.3.1 Initial RAAN

Before the development of the APC method, some discussion is needed for the setup of the orientation of the HEO with respect to the Moon and Sun at the start of the mission. The initial RAAN of the satellite orbit was not set until the study of the dynamics of the lunisolar perturbations was complete. Throughout Chapters 2 and 3, the sensitivity and direction of the variations of the orbital elements were shown to be functions of the initial RAAN of the HEO. It was observed in Figures 2.5, 2.7, and 2.8 that the $2S$ oscillations are large when the RAAN of the HEO is near zero and the long-period variations of the eccentricity change sign as the RAAN crosses 0° . The source of this behaviour was explained

in Chapter 3 using the switch point analysis. The relative geometry of the Sun's orbit with respect to the switch point planes dictates the fraction of the solar period where the rates of change of orbital elements are positive or negative which cause the changes in the long-period behaviour of the eccentricity, inclination, and AOP as the RAAN crosses 0° .

It was further discussed in Chapter 3 how the amplitude of the $2S$ oscillations behaves over a cycle of the relative RAAN. Because the sum of the inclinations of the HEO and the Sun is near 90° , the $2S$ oscillations of the eccentricity disappear when $\Delta\Omega$ is near 180° . Furthermore, it was shown that for a Molniya orbit, when the relative RAAN is near 180° , no AOP switch points exist. Again this is due to the configuration between the solar orbit and the ω curves, demonstrated in Figure 3.23. Since the APC method exploits the $2S$ oscillations to reduce the amount of ΔV required for maintaining the orbital elements, the initial RAAN must be set such that there are a maximum number of $2S$ oscillations to exploit.

The calculation of the initial RAAN is based on keeping the mean value of the relative RAAN near 0° , without active control, over the course of the mission. However, there are two values of $\Delta\Omega$ that must be accounted for: $\Delta\Omega_S$ and $\Delta\Omega_M$. Both the Sun and Moon perturb the orbital elements with long-period oscillations which depend on their respective relative RAANs. It is then the combined effect of both third bodies that must be modelled to determine the initial RAAN. As detailed in Appendix B, the RAAN of the Sun is effectively constant at 0° in an Earth-centred TOD frame; however, the RAAN of the Moon oscillates between $\pm 13^\circ$ over an 18.6° period. Therefore, as the RAAN negatively drifts over time, $\Delta\Omega_S$ and $\Delta\Omega_M$ will have different values. As such, the times at which the two relative RAAN values are zero are different. To maximize the number of $2S$ oscillations as well as ensuring that the long-period variations are symmetric over the total mission

duration, it is desired that the relative RAAN is zero at the mission midpoint time. Accounting for both relative RAANs, however, results in neither value being set exactly to zero at the mission midpoint. Instead the initial RAAN is a function of both solar and lunar relative RAANs. Because the effects of the Moon acting on the RAAN and AOP of the HEO are approximately twice as large as those of the Sun [8], a weighted-average calculation is required to determine the initial RAAN that accounts for the long-period effects of both the Sun and Moon.

The derivation of the equation to calculate the initial RAAN begins with developing a function for the average relative RAAN which accounts for the difference in magnitude of the solar and lunar effects. According to Vallado, the ratio of the perturbations is approximated as $(\mu_M/\mu_S)(r_S/r_M)^3$ [8]. The average relative RAAN is then set to zero at the mission midpoint time, t_{mid} , and the variation of the RAAN is approximated using the linear sum of Equations 2.16, 2.64, and 2.65. The resulting equation for the initial RAAN, Ω_0 , is:

$$\Omega_0 = \Omega_M(t = t_{mid}) \frac{f}{1+f} - \dot{\Omega} t_{mid} \quad (4.16)$$

where $f = (\mu_M/\mu_S)(r_S/r_M)^3$, which is approximately equal to 2.2, and the variable $\Omega_M(t = t_{mid})$ is the RAAN of the Moon at the mission midpoint time. Knowing the desired duration of the mission and the initial date, the RAAN of the Moon is calculated using the SPICE Toolkit [46].

This method to calculate the initial RAAN to enable the maximum opportunity to exploit the $2S$ oscillations over the mission is based on the approximation of the combined

effect of the Moon and Sun. When the length of the mission is 15 years, the error due to this approximation is not as noticeable as for shorter missions (~ 5 years). However, the shorter missions do not necessarily suffer from this error since its shorter duration allows for more flexibility in the initial RAAN. This is because more of the mission period experiences conditions where the $2S$ oscillations are dominant over the long-period effects (since the slope of the long-period effects is approximately zero when the relative RAAN is near zero). The variation of the RAAN for a Molniya orbit primarily due to J_2 is such that the RAAN can complete one or more revolutions during a mission, depending on its duration. For situations such as this, the mean average RAAN is set to zero; however, the value of the average RAAN at the mission midpoint is not necessarily 0° .

To specify both the initial longitude of the node and the initial RAAN, the only remaining free variable is the mission start time. Using an approximate start time of January 1, 2010, a small shift in time is calculated using both the initial RAAN and the initial longitude of the node. The equation to calculate the small time change was derived as:

$$\Delta t = -\frac{\lambda_0 + \theta_{GMST_0} - \Omega_0}{w_E} \quad (4.17)$$

where λ_0 is the initial geographic longitude of the ascending node of the HEO and θ_{GMST_0} is the Greenwich Hour Angle at the approximate start time.

4.3.2 Methodology

The perturbations acting on the HEO are exploited using a method commonly called the grazing method. The idea is that for an orbital element that is desired to be con-

trolled, if there are periodic oscillations in its variations that are approximately the size of the control margin, one can exploit the change in sign of the rate of change of the orbital element and thus reduce the amount of ΔV required, or at the very least, extend the time between manoeuvres [8]. A schematic of the grazing method is shown in Figure 4.14 where the oscillations of the eccentricity¹⁵ are exploited for a Tundra orbit, where it is desired to maintain the eccentricity near e_{nom} with some tolerance, e_{tol} ¹⁶. When the eccentricity reaches the edge of its control tolerance, e_{max} , the combined switch point time of the eccentricity is calculated using Equation 3.44. The analytical approximation of the future variations of the eccentricity, assuming no manoeuvre is performed using Equation 2.66, and denoted by a dashed black line in Figure 4.14. Specifically, Equation 2.66 is evaluated at the switch point time, t_{sup} , corresponding to e_{sup} . Using the difference between the switch point value of the eccentricity and the maximum allowable value of the eccentricity, the desired eccentricity after the manoeuvre is computed, $e_{des} = 2e_{max} - e_{sup}$. After the eccentricity has been set to e_{des} , it is allowed to drift without control until it reaches its tolerance. It is shown that the time at which the rate of change of the eccentricity is zero coincides with the time when the eccentricity reaches e_{max} . Thus, the eccentricity is said to have ‘grazed’ its tolerance limit before the sign change in the perturbation forces the eccentricity back towards its nominal value. The necessity of the sensitivity analysis from Section 4.2 is demonstrated here since the grazing method requires that the behaviour of both the analytical prediction curve and the actual eccentricity variations are the same in order to ensure that the eccentricity only grazes e_{max} . When the eccentricity reaches e_{min} the process for exploiting the sign changes in the oscillations repeats itself. This is the strategy to analytically predict and control the orbital elements of an HEO.

---

¹⁵The same process is used for the control of the inclination or the AOP.

¹⁶In actual satellite missions, the tolerance value will also have a factor of safety added to it to ensure the orbital elements never exceed their allowable margins.

Figure 4.14: Schematic of the grazing method.

To test the APC method, the strategy was implemented into the previously-discussed MATLAB orbit propagator. As discussed in Section 2.2.3, a force model to account for the major perturbations acting on an HEO includes the J_2 effect due to the Earth's oblateness, and the lunar and solar gravitational attractions. To model the lunisolar perturbations, Cook's averaged method from Section 2.2.2 is used. Since in the absence of any manoeuvres the force model is averaged over the period of the satellite orbit, the time step size between the manoeuvre sequences is set to the orbital period of the HEO. As discussed in Section 4.1, during a manoeuvre sequence, the location of the manoeuvres depends on the orbital element being controlled as well as desired change to the LAN. As such, to simulate the motion of the satellite during a manoeuvre sequence, the time step is reduced. Furthermore, the Gauss planetary equations (Equations 2.4 through 2.8) are also added to the force model, where the disturbance force $\vec{F} = F_R\hat{R} + F_S\hat{S} + F_W\hat{W}$ models the impulsive thrust of the propulsion system.

The overall orbit control strategy architecture is shown in Figure 4.15¹⁷. The initial conditions are set using Table 1.1 as well as Equation 4.16 for the initial RAAN and Equation 4.17 for the mission start date. Additionally, the satellite's initial position is set at its ascending node. This is to ensure that the satellite's geographic location at the node is used to track the variations of the ground-track, which is a function of its nodal period as discussed in Chapter 2. The satellite is then propagated for one nodal period, until it reaches the next ascending node. After the propagation period, a tolerance check is performed for the three orbital elements. If all three elements are within their respective tolerance limits, then another propagation period is run, and the tolerance check is performed again. If any of the orbital elements have exceeded their tolerance during the propagation period, their respective manoeuvre sequence is initiated. The first step in the sequence is to propagate the satellite to its first manoeuvre location which depends on the orbital element being controlled, as discussed in Section 4.1. Next, the desired eccentricity, inclination, or AOP is computed. Following that, the amount of ΔV_I for Part I of the manoeuvre sequence is calculated in terms of the desired change in the orbital element. With ΔV_I calculated, the satellite orbit is propagated with the impulsive manoeuvre in place. From there, the second part of the manoeuvre sequence (if necessary) begins, starting with the propagation of the satellite to the second manoeuvre location. The remainder of the manoeuvre sequence follows Part I, and then the satellite is propagated forward to the node, before exiting the control loop. The satellite is brought forward to the node to ensure that the time steps outside the control sequences always occur at the same location within the orbit.

For the APC method, the computation of the desired eccentricity, inclination, or AOP involves a multi-step process with conditions that depend on the magnitude of the lunar

---

¹⁷Recall that the inclination manoeuvre only requires a single manoeuvre.

Figure 4.15: Block diagram of orbit control strategy.

and solar perturbations acting on that particular HEO, as well as the location of the Sun with respect to the satellite orbit. The process is summarized in Figure 4.16¹⁸. The first

¹⁸This figure depicts the process for solving for the desired eccentricity, but it is equally valid for the desired inclination or AOP.

step is to calculate the required switch point time, t_{swp} . As explained in Chapter 3, it is possible that no switch points exist based on the relative geometry of the Sun and satellite orbits. If this is the case, then there are no oscillations to exploit in the near future and the grazing method cannot be used for that particular manoeuvre sequence. Instead, the desired orbital element is set to the opposite end of the control tolerance. If there are no switch points over a given period of time, then there are no changes in the rate of change. Therefore, the perturbations cause the element to drift back towards the nominal value after the manoeuvre sequence has been performed. If a switch point time was calculated, the value of the orbit element at that time, e_{swp} , i_{swp} , or ω_{swp} , is predicted using the analytical equations developed in Section 2.3. There is still the possibility that the switch point cannot be exploited, however. If the predicted value of the controlled orbital element is found to be greater/less than the nominal orbital element value plus/minus three times the tolerance size, then the size of the adjustment of the orbital element to exploit the switch point is too large for the tolerance range. Therefore, the switch point cannot be exploited at that manoeuvre sequence, and the desired orbital element must be instead set to the other end of the control tolerance, just as when no switch point time existed. However, since a switch point was still predicted, there is a forthcoming manoeuvre sequence that is able to exploit the switch point¹⁹. If the predicted orbital element is inside the aforementioned range, the switch point can be exploited and the desired value for the eccentricity, inclination, or AOP is calculated.

---

¹⁹Depending on how far in the future the switch point occurs, the sign change may not be able to be exploited in the following manoeuvre sequence, but instead in the sequence following that.

Figure 4.16: Block diagram of the calculation of the desired eccentricity (e_{des}), inclination (i_{des}), or AOP (ω_{des}) using the APC method.

4.4 Numerical Prediction and Control and Baseline Control Strategies

To confirm that the analytical method to determine the desired values of the controlled orbital elements is a valid approach to exploiting the lunisolar perturbations, an equivalent numerical approach was developed called the Numerical Prediction and Control (NPC) method. The NPC strategy also uses the grazing method to exploit the sign changes in the rates of change of the orbital elements; however, it does not rely on the approximations and assumptions that are a part of the APC strategy. These include neglecting terms in the lunisolar perturbation model that are a function of $\sin 2\omega$, modelling the orbits of the

Sun and Moon as circular, assuming the rate of change of the RAAN is constant, as well as other assumptions that were mentioned. To calculate the desired eccentricity, inclination, or AOP, the NPC method performs a numerical integration of the force model until the $2S$ oscillations change sign. Just as with the APC method, the value of orbital elements at the switch point time is subtracted with the nominal value plus/minus its tolerance size to calculate the desired controlled orbital element that results in making use of the natural dynamics acting on the HEO. The control architecture of the NPC strategy follows that of the APC strategy shown in Figures 4.15 and 4.16, except that in Figure 4.16, the switch point value and time are calculated by performing an independent numerical simulation of the orbital elements assuming no manoeuvre occurs. This results in the actual value of the orbital element when its combined rates of changes are equal to zero. Using the numerically-calculated switch point value of the eccentricity, inclination, or AOP, their respective desired values are calculated just as with the APC method. Since the NPC approach for controlling the orbital elements is used to validate the APC approach, the same set of initial conditions are used, including the calculation of the initial RAAN discussed in the previous section.

In addition to validating the analytical method to control the orbital elements, it is also desired to confirm that utilizing the sign changes in the rates of change of the eccentricity, inclination, and AOP, results in a decrease in the required mission ΔV compared to other approaches. As such, a third control strategy, called the Baseline method, was also developed. This method uses no future knowledge of the orbital elements or the behaviour of the perturbations to control the orbital elements. Instead, whenever a controlled element reaches the limit of its tolerance, a manoeuvre sequence is performed to set the element back to its nominal value. This was seen as a common approach in the literature review earlier in this chapter, which sometimes required large amounts of ΔV for a mission. This

control strategy follows the outline described in Figure 4.15 with the exception of the aforementioned calculation of the desired value of the eccentricity, inclination, or AOP. Just as with the NPC strategy, the Baseline strategy also uses the analysis performed in Section 4.3.1 to calculate the initial RAAN of the HEO. The results of the Baseline method are to be compared to the APC and NPC methods to quantify the percent decrease of the total mission ΔV that occurs when the lunisolar perturbations are exploited. Furthermore, in the next section, it will also be shown how the enhanced knowledge of the perturbations using the switch point analysis also typically reduces the number of manoeuvres required for a mission.

4.5 Results

In this section, the results of the MATLAB orbit simulations using the three types of control strategies are presented and discussed. However, a discussion on the choice of the tolerance sizes must first be made. As explained at the beginning of this chapter, the development of the control strategy was not based on any specific mission: past, present, or proposed. The concept of the control strategy stemmed from determining if the oscillations of the satellite's orbital elements due to the gravity of the Moon and Sun could be utilized to reduce the required amount of control input. The effectiveness of such a strategy depends significantly on the size of the tolerance limits on the orbital elements. If the tolerance size is too large, the medium-period oscillations may not cause the orbital elements to cross their control limits and the ΔV savings between the APC or NPC methods and the Baseline method may be quite small. Conversely, if the tolerance limits are too small, the majority of the manoeuvres that occur do not have a switch point to exploit since only approximately four switch points occur per year per element. As such,

the tolerance limits are varied as a function of the amplitude of the $2S$ oscillations to determine the tolerance size which results in the greatest amount of ΔV savings. Since the magnitude of the $2S$ oscillations depend on the orbit type, each HEO will have a different size of tolerances studied.

While it is the $2S$ oscillations that are exploited, the occurrence of the $2M$ oscillations must be accounted for as well. Since oscillations due to the position of the Moon are much smaller in magnitude than those of the Sun, an additional margin must be added to the control tolerance in order to ensure that the orbital elements are maintained within their margins. Figure 4.17 shows the variation of an orbital element within its tolerances, where x is used to represent the eccentricity, inclination, or AOP. Centred about the nominal value, x_{nom} , the orbital element is allowed to freely vary between $x_{nom} \pm (x_{tol} + x_{2M})$. The tolerance size, x_{tol} , is a function of the amplitude of the $2S$ oscillations for the particular orbital element and HEO. The variable x_{2M} is the approximate amplitude of the $2M$ oscillations which adds a margin to the control tolerance. To ensure that switch points are still exploitable with the added margin, the manoeuvre size is set such that x reaches $x_{nom} \pm x_{tol}$ at the future switch point time shown on Figure 4.17 at around day 75. Additionally, if there are no switch points to exploit, when x is sent to the other end of the control tolerance as described in Section 4.3, its desired value is set to $x_{nom} \pm x_{tol}$.

Before the results of the APC and NPC methods are shown, some general results of the Baseline control strategy are first discussed. Disregarding, for the moment, the development of the initial RAAN calculations to ensure the satellite experiences a maximum number of $2S$ oscillations, a series of simulations were performed in MATLAB for each HEO while varying the initial RAAN from 0° to 360° . These simulations were performed for both a 15-year and a 5-year mission. The results of the required ΔV depending on the

Figure 4.17: Control tolerances for the orbit control strategies.

initial RAAN for a 15-year mission are shown in Figure 4.18.

For the Tundra orbit shown in Figure 4.18(a), there is a strong correlation shown between the choice of the initial RAAN and the amount of ΔV required, particularly for the eccentricity and AOP control. Indeed, this is one of the primary reasons for the development in Section 4.3.1 where the ideal initial RAAN is calculated. The eccentricity control requirements show that there are two instances where the ΔV is at a minimum: near RAAN values of 45° and 225° . This is a result of the relative RAAN reaching 0° or 180° during the mission midpoint where the long-period oscillations of the eccentricity become zero. Since the drift of the RAAN is very small for a Tundra orbit, if the initial RAAN is set to be marginally larger than 0° or 180° , then throughout the 15-year mission, the long-period eccentricity variations are near-zero and it is primarily the $2S$ oscillations that cause the eccentricity to exceed its control limits. For control of the AOP, the maximum variations of the AOP occur then the initial RAAN is set to just above 180° . This is caused by the reduction of AOP solar switch points due to the relative geometry of the ω

curves with the Sun's orbit. In this geometric configuration, the solar rate of change is primarily negative over the semi-annual cycle and thus creates a larger net negative effect on the AOP which requires more ΔV to compensate for said effects.

For the TAP orbit in Figure 4.18(b), there is a slightly less significant relationship between Ω_0 and the mission ΔV compared to the Tundra orbit, but it is still important to note. Since the TAP orbit has a greater rate of change of the RAAN due to J_2 , the initial RAAN that requires a minimum amount of orbit control shall be further away from 0° . Just as with the Tundra orbit, the ΔV requirements for the inclination are shown not to be significantly affected by the choice of Ω_0 . This is a result of the behaviour of $\delta_i = \delta_\alpha - \delta_{sat}$ using Equations 3.9 and 3.11 as the $\Delta\Omega$ varies over time as shown in Figure 3.15. Because of the behaviour between the relative geometry between the α and satellite planes (which identifies the inclination switch points) and the solar orbit plane, the long-period effects of the inclination are small compared to the eccentricity and the AOP. As a result, the long-period oscillations do not significantly depend on the initial RAAN of the mission.

In Figure 4.18(c), the orbit control requirements for a Molniya orbit are shown. Unlike the previous two HEOs, there is almost no change in the ΔV requirements for any of the orbital elements as the initial RAAN of the mission varies. The cause of this behaviour is the rate of change of the RAAN compared to the mission duration. Over a 15-year period, the relative RAAN of a Molniya orbit drifts approximately 720° , completing two full RAAN cycles. As a result, regardless of the initial RAAN, the Molniya orbit experiences the same set of lunisolar effects which perturb the orbital elements that are controlled. Therefore, in general, one should not expect to observe any differences in the ΔV requirements based on the choice of the initial RAAN if the mission duration is set to an integer number of relative RAAN cycles. In summary, the baseline ΔV results for all three HEOs

show that a large amount of the ΔV is required to maintain the orbital elements with a tight control using a classical control approach that does not account for the sign changes in the variations of the orbital elements.

Figure 4.19 shows the Baseline control results of each HEO with a control tolerance of $1/2 \times 2S$ over a 5-year mission. Since the mission time is short, it is shown that for the Tundra and TAP orbits, the initial RAAN that results in the minimum ΔV requirements is closer to 0° than for the 15-year mission. This is a result of the RAAN drifting less over the mission period. For the Tundra orbit, Figure 4.19(a) shows that the same trends for the eccentricity and AOP exist as for the 15-year mission. However, it is observed that there is a noticeable change in the inclination ΔV requirements depending on the initial RAAN. As shown in Figure 3.18, when the relative RAAN is nearly zero, the magnitudes of the perturbation forces acting on the inclination due to the Sun or Moon are slightly lower than the magnitudes of the perturbations when the relative RAAN is near 180° . Therefore, one expects that the control requirements for the inclination to also decrease slightly if the initial RAAN is set such that the relative RAAN averages to approximately 0° . This is only valid for shorter duration missions where the drift of the RAAN is limited. The most significant change in the behaviour between the 5- and 15-year missions is that the TAP orbit in Figure 4.19(b) is shown to be significantly more sensitive to the initial RAAN. Furthermore, the Molniya orbit also begins to show sensitivity to the initial RAAN with the reduced mission period. Since the drift of the RAAN for a Molniya orbit is approximately $-0.13^\circ/\text{day}$, based on the calculations from Section 4.3.1, one would expect that the initial RAAN should be approximately 120° . Indeed, this is observed in Figure 4.19(c). It should be noticed that when comparing the minimum ΔV 's in Figure 4.18 with those in Figure 4.19 the results of the 5-year mission are not equal to one third of the 15-year mission results, but less than that amount. This is a result of the 5-year missions having

Figure 4.18: The ΔV requirements using the Baseline control strategy for all three HEOs with various initial mission RAANs with the control tolerance set at $1/2 \times 2S$ (15-year mission).

less time away from the relative RAAN being zero than the 15-year missions. As a result, the 15-year missions must account for the longer duration of long-period variations which drives up the ΔV costs.

The remainder of this section focuses on demonstrating that the APC and NPC methods are able to reduce that amount of ΔV required for an HEO in a 5- or 15-year mission with various control tolerance sizes. The APC and NPC methods as described in Sections 4.3.2 and 4.4 were implemented into the custom MATLAB orbit simulation script used to calculate the mission ΔV . Since this thesis is focused on the control of the eccentricity, inclination, and AOP due to lunisolar effects, the discussion of the ground-track control is limited to the STK simulations which are used to validate the results shown here. The STK simulations are discussed in detail in Section 4.6.

The controlled variations of the semi-major axis, eccentricity, inclination, and AOP of a Tundra orbit over a 5-year period using the APC method with a tolerance size of $1/2 \times 2S$ is shown in Figure 4.20. Since the force model only includes J_2 and the lunisolar effects modelled using Cook's single-averaged equations, there is no long-period effect on the semi-major axis and therefore it remains constant except when temporarily altered during an eccentricity or AOP manoeuvre sequence [8, 28]. The lower three plots of Figure 4.20 show the variations of the eccentricity, inclination, and AOP over time. For the majority of the manoeuvre sequences shown for the orbital elements, the switch point time is able to be exploited. A few exceptions to this include the first manoeuvre for the inclination, the second-to-last manoeuvre for the eccentricity, and the second-to-last manoeuvre for the AOP. In these cases, a switch point time was predicted; however, the analytically-calculated value of the orbital element was either too large or too small for the size of the control tolerances. As a result, the desired orbital element was set to the opposite end of

Figure 4.19: The ΔV requirements using the Baseline control strategy for all three HEOs with various initial mission RAANs with the control tolerance set at $1/2 \times 2S$ (5-year mission).

the controlled region (marked by the ‘red’ lines). However, since the switch points were able to be predicted, the subsequent manoeuvres were all able to properly utilize the sign changes in the rate of change of the orbital elements due to the Sun and Moon. These figures also demonstrate that for a 5-year mission in a Tundra orbit, the long-period oscillations are quite small and the $2S$ oscillations are the dominant variations of the orbital elements.

Equivalent plots to Figure 4.20 are shown for the TAP and Molniya orbits in Figures C.1 and C.2 located in Appendix C. While the 5-year TAP orbit mission demonstrates that the switch points are almost always able to be exploited for all three orbital elements, the Molniya orbit data shows many instances where no switch points exist. While the sign changes in the rates of change of the orbital elements cannot be taken advantage of to reduce the required ΔV , the knowledge that there are no upcoming switch points is still useful. For the case of the eccentricity of the Molniya orbit, the first switch point does not occur until approximately day 600. Before it occurs, the eccentricity is consistently adjusted to the minimum allowable value of the control tolerance since it is known that the rate of change of the eccentricity will continue to be positive. As such, instead of following the Baseline control approach by sending the eccentricity to its nominal value, the APC and NPC methods are able to decrease the number of manoeuvres required for a mission.

As discussed earlier, the tolerance size bears significant weight with regards to both the amount of ΔV required to maintain the orbital elements as well as the percent difference between the APC method and the Baseline control method. Figures 4.21 and 4.22 show the orbit control results for a Tundra orbit with 5- and 15-year missions, respectively. Each figure contains four graphs (total, eccentricity, inclination, and AOP) which plot the ΔV as a function of the control tolerance size. The numbers in brackets at the top of the bars

Figure 4.20: Controlled variations of the orbital elements of a Tundra orbit over a 5-year period using the APC method with a tolerance size of $1/2 \times 2S$.

represent the number of manoeuvres. Table 4.4 shows the sizes of the control tolerances for the eccentricity, inclination, and AOP, as well as the maximum amount of ΔV per manoeuvre sequence for each controlled orbital element based on the tolerance size.

Figure 4.21: ΔV results for a Tundra orbit comparing different orbit control strategies at various control tolerances for a 5-year mission. The numbers in brackets are the number of manoeuvres required.

Table 4.4: Control tolerances and parameters for a Tundra orbit mission.

Tolerance size	$(1/2) \times 2S$ osc.	$2S$ osc.	$2 \times 2S$ osc.
Eccentricity tolerance size	0.00075	0.0015	0.0030
Inclination tolerance size (deg)	0.01835	0.0367	0.0734
AOP tolerance size (deg)	0.09	0.18	0.36
Max. ΔV_e per manoeuvre sequence (m/s)	3.3	5.6	10.0
Max. ΔV_i per manoeuvre sequence (m/s)	2.7	4.6	8.5
Max. ΔV_ω per manoeuvre sequence (m/s)	2.5	4.3	5.4

Figure 4.22: ΔV results for a Tundra orbit comparing different orbit control strategies at various control tolerances for a 15-year mission. The numbers in brackets are the number of manoeuvres required.

The results of Figure 4.21 demonstrate that both APC and NPC are essentially equivalent for a 5-year mission. This is shown in terms of both the amount of ΔV and the number of manoeuvres required for all three orbital elements. When comparing the APC/NPC methods with the Baseline control method, significant savings are shown, particularly for the 5-year mission. The largest savings are shown for the inclination and AOP when the tolerance size is smallest; however, for the eccentricity, when the tolerance is set equal to the $2S$ oscillations, the percent savings using the APC method is approximately 60%

compared to the Baseline method. The cause of the difference between the tolerance size that results in the largest ΔV savings is a function of the relationship between the $2S$ oscillations and the long-period oscillations. The savings compared to using the Baseline method are drastically reduced when the tolerance size is increased to $2 \times 2S$. At this tolerance size, multiple cycles of the $2S$ oscillations can be completed without any corrections since, for the 5-year mission, the long-period oscillations are comparatively insignificant for a Tundra orbit, particularly for the eccentricity and the inclination. For the ΔV results of a 15-year mission in a Tundra orbit, shown in Figure 4.22, the savings achieved by using the grazing method, i.e., the APC or NPC strategies, are reduced significantly compared to the 5-year mission when the tolerance size is equal to or less than the $2S$ oscillations, but still shown to exist. The effectiveness of the reduction in the number of manoeuvres using the grazing method-based control strategies is still seen to exist, particularly when the tolerance size is small. The exception for this is the control of the AOP when the tolerance size is equal to $1/2 \times 2S$. For this particular case, a reduction of almost 50 m/s is seen. When the tolerance size is increased to $2 \times 2S$ however, all three control methods essentially become equivalent, both in terms of the total amount of ΔV and the total number of manoeuvres. At this tolerance size, the maximum amount of ΔV per manoeuvre sequence for the orbital elements has grown larger as shown in Table 4.4. Therefore, the smaller differences in the ΔV are less meaningful. If the difference between the Baseline and APC/NPC ΔV results for an orbital element is approximately equal to maximum ΔV per manoeuvre sequence, any savings due to the APC/NPC method are most likely due to the Baseline control method having performed an extra set of manoeuvres near the end of the mission. As such, the savings do not adequately capture the effectiveness of the control approach using the grazing methods.

Similar plots for the TAP and Molniya orbits for the 5- and 15-year missions are shown

in Appendix C, Figures C.4 through C.5. For the 5-year TAP mission, the behaviour of the required ΔV for each orbital element, as a function of tolerance size, is similar to the 5-year Tundra mission. The notable exception is for the eccentricity when the tolerance is set to $2 \times 2S$. The cause of the Baseline control being less than the APC or NPC methods relates to the final manoeuvre sequence at the end of the mission. The situation can occur where the APC/NPC methods require a set of manoeuvres just before the end of the mission, whereas the mission ends right before a manoeuvre is required for the Baseline control method. As previously explained, if the difference between the Baseline method and the APC/NPC method is approximately equal to the maximum ΔV per manoeuvre sequence, then the difference is considered inconsequential. When the mission duration for the TAP orbit increases to 15 years, again it is shown that the savings of the APC/NPC method are reduced, except for the AOP when the tolerance size is equal to $2S$. Despite the reduction of the ΔV savings using the APC method, Figure C.3 shows that the number of manoeuvres required to maintain the eccentricity, inclination, and AOP are still significantly less than the Baseline approach, which makes the APC method an operationally-desirable one. The 5-year Molniya orbit mission, for which the ΔV results are shown in Figure C.6, show that the savings gained using the APC or NPC methods compared to the Baseline approach have been reduced such that, in terms of ΔV required, there is very little difference between all three control strategies. The only exceptions for this general behaviour are inclination control results when the tolerance size is $1/2 \times 2S$ and the AOP control results when the tolerance size is $2 \times 2S$. For all other scenarios, the long-period oscillations acting on the Molniya orbit are too large to effectively exploit the $2S$ oscillations. This is a result of the large drift of the Molniya orbit which completes over $3/4$ of its 7-year period during the mission time. Therefore, for only a small fraction of the mission, when the relative RAAN is near zero, are the long-period oscillations at a minimum and the $2S$ oscillations at a maximum. These are necessary conditions for

the APC/NPC method to exploit the lunisolar perturbations and reduce the required ΔV compared to the Baseline method. For the 15-year mission, the relative RAAN completes approximately two full cycles of its period and is again unable to utilize the lunisolar perturbations to reduce the ΔV for the eccentricity and AOP. The inclination, however, shows small but noticeable savings for all three tolerance sizes. As described in Chapter 3, the long-period oscillations of the inclination are not as significant as those of the eccentricity and AOP. Therefore, the $2S$ oscillations of the inclination are still able to be exploited when the mission time is large for the Molniya orbit. Regardless of the reduction in the savings for the ΔV for the Molniya orbit, Figures C.5 and C.6 show that the number of manoeuvres required over the 5- and 15-year missions are still significantly reduced when using the APC/NPC methods compared to the Baseline method, even when the tolerance size is set to $2 \times 2S$.

When analyzing the general behaviour of the ΔV results for the three HEOs for both the 5- and 15-year missions, several conclusions can be made. First, it is shown that as the orbit size gets smaller (i.e., from Tundra orbit to the Molniya orbit) the effectiveness of the APC/NPC method to take advantage of the switch points of the orbital elements compared to the Baseline approach is significantly reduced. That being said, only the reduction in ΔV is affected by the orbit size change. By knowing the behaviour of the future lunisolar perturbations, the APC/NPC methods are still able to reduce the number of manoeuvres required to control the orbital elements, even if the switch points cannot be exploited. As previously explained, the cause of the reduction of the APC/NPC methods ability to exploit the lunisolar perturbations is due to the larger drift of the RAAN of the orbit. Recall that the smaller HEOs have a larger drift since they are closer to the Earth. This larger drift causes the relative geometry between the HEO and the orbits of the Moon and Sun to spend less time aligned such that the long-period oscillations of the

orbital elements are nearly zero. Consequently, the $2S$ oscillations have less time to be the dominant oscillation over that time period and there are fewer opportunities to exploit the switch points.

The second conclusion, which is related to the first, is that the mission time is also shown to significantly affect the grazing-method-based control strategies compared to the simple Baseline strategy. As explained in Section 4.3.1, the initial RAAN of the HEO is set such that when the mission midpoint time occurs, the weighted combination of the solar and lunar relative RAANs is set to zero. This is done to ensure that there are as many opportunities to exploit the $2S$ oscillations as possible. With a large mission duration, there is an increased amount of time during the mission where the long-period oscillations are more dominant than the $2S$ oscillations. As a result, the APC/NPC methods are less able to exploit the switch points.

Third, in general, the ability of the grazing method to utilize the sign changes in the rate of change of the orbital elements depends on the size of the control tolerance with respect to the amplitude of the oscillations. As such, it can be seen for all three orbits, that as the tolerance size increases, the difference between the APC/NPC and the Baseline ΔV requirements decrease. Such a relationship is a result of the $2S$ oscillations being dwarfed by the large control tolerance size so that when the switch points do occur near the mission midpoint time, they are not exploited because there are less control requirements.

This section focused on the orbit control results of various HEO mission scenarios using the MATLAB propagator, which uses a simplified force model of J_2 and the lunar and solar gravity. In order to validate the methods, the three control methods are to be implemented into STK which has a much more accurate force model. With that, the control

requirements can be estimated more accurately and the ability of the analytical method to predict and control the lunisolar perturbations confirmed.

4.6 Validation of Orbit Control Strategy Using STK

The validation of the orbit control strategy developed in the previous section with STK is necessary to demonstrate that the analytical methods to predict the switch points are detailed enough to accurately capture the ‘real’ behaviour of the variations of the orbital elements compared to the MATLAB-based simulations which use a simplified perturbation model. If the results of the orbit control strategy using MATLAB are considered the theoretical results, the output ΔV using STK are the reference results used to validate the theoretical ones. The accuracy of the orbit simulations in STK is considered very high (when using the HPOP force model) as the software package is used by multiple space agencies and companies to solve a variety of orbital dynamics problems [63]. As such, the results of these simulations are assumed to be as close of an approximation to ‘real-life’ perturbations as is practical.

While STK offers an accurate perturbation model and integration tools, it lacks a level of internal customization to certain specific tasks. Instead, the software package was developed to be able to be integrated with external software, such as MATLAB, to customize any type of simulation. The simulations using STK are then still based in MATLAB with a connection to STK using Mexconnect to act as a numerical integration toolkit for the dynamics of the HEO. All preliminary orbit control calculations and decisions, such as the block diagram in Figures 4.15 and 4.16, are performed using the same code developed in the MATLAB-only simulations.

Figure 4.23 shows the block diagram of the integration of STK and MATLAB used to simulate an HEO under an accurate perturbation model using the control strategy developed in the previous section. The simulator is initialized in MATLAB where the HEO type, control strategy, tolerance limits, and other initial conditions are set. This data is then fed into STK and the orbit is propagated until one of the control tolerances is reached or the mission time has elapsed. Within this piecewise simulation, no orbit control is implemented and the satellite is modelled using HPOP, which was discussed in Section 2.2.3. From there, the orbit data is fed back into MATLAB where it is stored, and the desired orbital element for the eccentricity, inclination, or AOP is calculated. When using the APC method, Figure 4.16 is used. For the NPC method, however, the current orbit data is sent back into STK to the NPC simulator. This simulation involved propagating the orbit data 100 days forward in time assuming no manoeuvre takes place. As a result of the NPC method requiring additional numerical simulations for each manoeuvre sequence, the length of time required to calculate the mission ΔV using the NPC method is significantly longer than either the APC or Baseline methods. The additional simulation time depends on several factors such as simulation method (MATLAB or STK) and the number of manoeuvre sequences, which are a function of the orbit type and the tolerance size. Per manoeuvre sequence, the NPC simulation requires an additional 3 to 8 seconds, depending on the HEO. The NPC orbit data is then sent back to MATLAB where the switch point value of the to-be-controlled orbital element is computed. With the desired orbital element calculated to exploit the switch points for the APC/NPC method or set to the nominal value for the Baseline method, the next step is to calculate the characteristics of the manoeuvre sequence. These include the amount of ΔV , the location of the manoeuvres as described in Sections 4.1.1 through 4.1.3, and the thrust direction. These values are then sent to STK to simulate the satellite motion during the manoeuvre sequence. The nominal

value of the semi-major axis based on the desired nodal period of the HEO is also sent to the STK block. STK has a built-in differential corrector which is utilized to ensure that the post-manoeuvre sequence semi-major axis is set to correct the nodal period to ensure the ground-track remains constant. After the manoeuvre simulation is complete, the orbit data is sent back to MATLAB and from there back to the STK propagation block. This process repeats itself until the stopping condition of the propagation block is the end of the mission.

Figure 4.23: Block diagram of the MATLAB-STK orbit control simulator.

Control of the ground-track of the HEO ensures that the satellite is able to repeat its coverage of the Earth which has shown to be a common requirement for many missions utilizing HEOs as described at the start of this chapter. As previously mentioned, however, the focus of this research is not developing a novel method for controlling the ground-track, nor was it to exploit the perturbations of the geopotential harmonics acting on the semi-major axis to minimize the control effort required. Both the eccentricity and AOP manoeuvre sequences necessitate a temporary adjustment to the semi-major axis; the opportunity to utilize the adjustments to indirectly maintain the node was then implemented.

Figures 4.24 and 4.25 show the variations of the semi-major axis and LAN for a Tundra orbit with tolerance size of $2S$ and $1/2 \times 2S$, respectively. Figure 4.24 shows that when the eccentricity or AOP manoeuvres are required approximately every 100 days (shown by impulsive changes to the semi-major axis), the small corresponding corrections to the node location are shown to maintain the ground-track reasonably well to within a few degrees. It is only when there are long periods of time where no change to the semi-major axis occurs, most likely as a result of exploiting the switch points of the eccentricity and AOP, does the drift of the LAN exceed the ability of the control strategy to adjust the LAN. This is shown around days 200, 900, and 1100. When the tolerance size is decreased to $1/2 \times 2S$, the variations shown in Figure 4.25 demonstrate that a very tight control of the LAN can be achieved. The smaller tolerance size for the directly controlled orbital elements results in more frequent changes to the semi-major axis which in turn means more frequent corrections to the LAN. As a result, there are fewer opportunities for the LAN to vary without being indirectly controlled and thus the limits on its variations become tighter. Figure 4.25 also demonstrates how the direction of the change to the semi-major axis affects the shift in the LAN. Indeed, part of the control strategy for selecting the locations and order of the eccentricity and AOP manoeuvres is based on the value of the LAN compared to its nominal value. This was described in Sections 4.1.1 through 4.1.4. When the LAN is less than its nominal value, the semi-major axis is reduced which decreases the nodal period of the HEO. This small change to the nodal period results in the satellite arriving at its next node crossing with the Earth rotated more than it had been previously. Consequently, the longitude of the satellite at the node crossing is greater than before the manoeuvre.

The controlled behaviour of the ground-track variations are similar for the TAP and Molniya orbits. However, the lower orbits typically required more manoeuvres than the

Figure 4.24: Variation of the semi-major axis and LAN for a Tundra orbit using a 2S tolerance size.

Tundra orbit since the long-period oscillations are more significant as a result of their larger drifts of the relative RAAN. Therefore, even when the tolerance sizes are larger, there are still sufficient manoeuvres that the variations of the LAN are maintained to within a small range similar to those seen in Figure 4.25. While the addition of the indirect control of the ground-track for the STK simulations, along with the increased force model, causes an increase in the ΔV requirements when compared to the MATLAB simulations, this amount is not considered significant. When comparing the results of Figure 4.21 from MATLAB with the equivalent results from STK, the control required for the eccentricity only increases by 5 m/s to 48 m/s. Similarly, the inclination control requirements increase

Figure 4.25: Variation of the semi-major axis and LAN for a Tundra orbit using a $1/2 \times 2S$ tolerance size.

from 39 m/s to 45 m/s. For the AOP, while the amount of ΔV increases by 10 m/s, the increase per manoeuvre averages to approximately an extra 0.2 m/s per manoeuvre. For the other HEOs, similar trends are seen. Again, the AOP is the only orbital element that shows a significant increase in the required ΔV . The cause for the increase is primarily the additional perturbations acting on the AOP in the STK simulations which are not modelled in MATLAB.

One of the relationships observed from the MATLAB simulations from the previous section was that the mission duration affects the amount of ΔV required per year due to

the dominant long-period oscillations. A significant gap in the amount of ΔV per year is shown as a result for each of the HEOs. Since the orbit control strategy is defined without strictly-defined mission parameters, a supplementary 10-year mission was added to the set of simulations in order to better understand how the mission duration affects the orbit control requirements. Furthermore, there is also a large change seen in the ΔV requirements between the three tolerance sizes. As such, additional tolerance sizes based on the amplitude of the $2S$ oscillations are considered. Table 4.5 shows the list of the new tolerance sizes that are used for the STK simulations. The results from the Molniya orbit simulations in MATLAB show that the total ΔV per year is well over the standard 50 m/s per year for geostationary-class satellites. Additionally, the examination of the effectiveness of the APC strategy compared to the NPC strategy for the Molniya orbit showed that the analytical equations to predict the future values of the orbital elements at the switch point times are not as accurate as for the TAP and Tundra orbits, particularly for the AOP. As such, the APC method is unable to exploit the lunisolar perturbations as effectively as the NPC method which predicts the switch point times and values as accurately as the force model itself. This is a result of the additional geopotential harmonics acting on the AOP which are not modelled by the analytical equations from Chapter 2. The Molniya orbit with its perigee altitude at 1000 km, is therefore not an ideal candidate for use with the APC method. Instead, a supplementary Molniya orbit with a perigee altitude raised to 3500 km is added to the list of HEOs being investigated. The higher perigee altitude decreases the effects of the tesseral harmonics which causes the discrepancy between the analytically-modelled variations of the AOP and the numerically-modelled variations. Furthermore, the higher perigee altitude also decreases the drift of the relative RAAN of the higher Molniya orbit. This provides the extra benefit of slightly increasing the amount of time that the Molniya orbit spends with the $2S$ oscillations being the dominant variations over the long-period oscillations. The drift of the RAAN of the 3500 km Molniya orbit

is $-0.086^\circ/\text{day}$ which results in a relative RAAN period of approximately 11.5 years.

Table 4.5: Various tolerances sizes implemented with the orbit control strategies.

Tolerance Level	1	2	3	4	5	6	7	8	9
Tolerance Size ($\times 2S$)	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2

With the new HEO, the additional mission duration, and nine tolerance sizes for the three control strategies, 324 STK orbit simulations were performed to estimate the ΔV to maintain the eccentricity, inclination, and AOP, which also includes indirect control of the ground-track. The results of the Tundra orbit are shown in Figure 4.26 which compares the individual orbit elements' ΔV results amongst the Baseline, APC, and NPC strategies for various missions durations and tolerance sizes. The STK results confirm those calculated in the MATLAB simulations. For the larger HEOs, the APC and NPC methods produce very similar estimations for the required ΔV and the number of manoeuvres, despite the more accurate perturbation model for the prediction of the switch points used in the NPC method. Both control methods are able to achieve significant savings in ΔV and in the number of manoeuvres, particularly for shorter missions. When the tolerance sizes are too small or too large, the distinction between the Baseline method and the APC or NPC methods can become very small. For the 5-year mission, when the tolerance level is set to around 5 or 6, which is approximately equal to the $2S$ oscillation amplitude, there is a considerable difference between the Baseline approach and the grazing method approaches. This is a result of the very small variations of the long-period oscillations of the orbital elements over the mission time. The $2S$ oscillations are able to be exploited with the APC/NPC methods and the tolerance size is just large enough to require few manoeuvres afterwards. The Baseline method, however, is required to perform many manoeuvres resulting in a saw-tooth pattern of controlled variations. Figure 4.26 also shows that each

orbital element requires a similar value for ΔV for control when the tolerance size is larger.

The ΔV results for the TAP, Molniya 3500²⁰, and the Molniya 1000²¹ are shown in Figures C.7, C.8, and C.9. For the TAP orbit, the most noticeable difference between it and the Tundra orbit is that each orbital element requires a very different amount of ΔV . The eccentricity is shown to require the largest amount of ΔV , whereas the inclination requires the least. The largest ΔV savings are shown for the AOP, particularly for the 5-year mission. This is a result of the near-zero long-period oscillations of the AOP over a long period of time and the tolerance size of the AOP being larger than the amplitude of the $2S$ oscillations.

---

²⁰The Molniya orbit with a perigee altitude of 3500 km.

²¹The Molniya orbit with a perigee altitude of 1000 km.

Figure 4.26: The ΔV results of the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Tundra orbit).

Much smaller, though not insignificant, savings are shown for the eccentricity and inclination. The larger AOP-based ΔV savings are due the longer period of time where the $2S$ oscillations are dominant and exploitable compared to the inclination and eccentricity. For the inclination, during the discussion of δ_i in Section 3.1.2, it was stated how the maximum amplitude of the $2S$ oscillations does not occur when the relative RAAN is near zero such as for the eccentricity. The inclination is dependent on the direction cosine C , and C is shown to be at its largest amplitude when the relative RAAN is 180° and at its smallest amplitude when it is 0° as shown in Figure 3.10. Consequently, the amplitude of the $2S$ oscillations for the inclination is not at its peak throughout the mission since the initial RAAN is set such that the mean relative RAAN is 0° at the mission midpoint time. The eccentricity on the other hand, is designed to have the amplitude of its $2S$ oscillation at a maximum during the mission; however, there is only a small window where the magnitude of the $2S$ oscillations exceed that of the long-period variations. The rate of change of the eccentricity is a function of the direction cosine B , and though it reaches its maximum amplitude when the $\Delta\Omega$ is 0° , it quickly diminishes the closer the relative RAAN is to 180° . As a result, the $2S$ oscillations are no longer dominant and the large amplitude of the long-period oscillations negates the APC/NPC method's ability to exploit the switch points. For the 5-year mission, the Baseline control of the AOP is shown to have a unique behaviour compared to other orbital elements and/or mission durations as the tolerance level increases. Despite the larger tolerance size, Figure C.7 shows that it requires more Baseline ΔV for tolerance level 5 and 7 than for 4. The cause of this phenomenon is that the lack of the Baseline control strategy's ability to utilize the future-predicted variations due to the Sun and Moon resulted in control of the AOP in such a manner that the manoeuvres occurred at the switch point times. This incidental control behaviour resulted in the AOP being set back to the nominal value and then the sign change in the rate of change of the AOP continuing to pull the orbital element to-

wards that other end of the control limits. As such, the savings for the AOP may not necessarily represent expected ΔV savings compared to a practical control strategy that does not exploit the lunisolar perturbations.

The results of the Molniya 3500 orbit are shown in Figure C.8. Unlike the TAP orbit, where the eccentricity requirements are largest, for the Molniya 3500 orbit the AOP requires three times as much ΔV as the inclination and one and a half times as much as the eccentricity for a 15-year mission. As the mission duration is lowered, however, the amount of ΔV for the AOP decreases more quickly than for the eccentricity or inclination. As shown, for a 5-year mission the AOP and inclination require almost the same amount of ΔV . This behaviour is a result of the AOP $2S$ oscillations being dominant for the entire duration of the 5-year mission, whereas the $2S$ oscillations are only dominant for one third of the 15-year mission period. The comparison of the controlled variations of the AOP for the two mission durations are shown in Figure C.13. The largest savings occur for the AOP in a 5-year mission, but the inclination also shows reduction in the ΔV , particularly for the 10- and 15-year missions. This is a result of there only being a few switch points to exploit for the 5-year mission, and many switch points to exploit when the mission time is larger. Just as with the TAP orbit, the variations of the inclination for the shorter mission period have fewer switch points to exploit than for the longer mission. Interestingly, the results of the AOP control for a 15-year mission show that the tolerance level does not affect the required ΔV as significantly as one would expect. The result of this behaviour is lack of occurrence of the switch points. As explained in Section 3.1.4, there exist values of the relative RAAN where no switch points occur due to the inclination of the Sun relative to the size of the ω curves on the unit sphere. Such a three-body relative configuration only occurs for the Molniya orbit missions because the drift of their RAANs is high enough that over a 15-year mission, there will be at least one complete revolution of the relative

RAAN. When these conditions occur, the rate of change of the AOP is negative over long periods of time (approximately 2/3 of the total mission time). The APC or NPC methods therefore can only utilize the knowledge that no switch points means the desired value of the AOP should be set to the upper allowable limit of the AOP. As a result, in comparison to the Baseline strategy the grazing-method-based control strategies are able to reduce the number of manoeuvres required for the control of the AOP for a 15-year mission by a little over half. The constant negative rate of change of the AOP during the no-switch-point time periods means that small changes to the tolerance size do not significantly change the required ΔV .

For the Molniya 1000 orbit, the ΔV results from STK are shown in Figure C.9. Just as with the other Molniya orbit, the AOP control requirements are largest amongst the three orbital elements when the mission time is 15 years; however, for the 5-year mission, they are shown to be less than the eccentricity requirements (though still significantly larger than the inclination). Unlike the Molniya 3500 orbit, however, the lower Molniya orbit results show that the 15-year control of the AOP is affected by the tolerance level. As one would expect, the larger tolerance size requires less ΔV . As previously mentioned, this particular Molniya orbit completes two full periods of the relative RAAN cycle. Less time is then spent with the geometric configuration where the AOP switch points do not exist. The non- J_2 perturbations acting on the Molniya 1000 orbit due to its lower perigee altitude are much larger than all other HEOs studied in this investigation. Therefore, the usefulness of the APC method, which does not account for these extra perturbations, has decreased. Only the NPC method, which uses STK's highly accurate perturbation model to predict the switch point times and the corresponding values of the orbital elements can accurately calculate the ΔV to exploit the perturbations.

To emphasize the sensitivity control requirements to the mission duration, the mean ΔV per year²² was calculated for the APC method based on the results from Figures 4.26, and C.7 through C.9. The results for the Tundra orbit are shown in Figure 4.27. Both the eccentricity and inclination show that the mean amount of ΔV per year decreases significantly as the mission duration decreases. The AOP, on the other hand shows that all three mission durations require approximately the same amount of mean ΔV per year, though the 10-year mission requires slightly more ΔV per year. This is a result of the size of the ω curves for the Tundra orbit being large enough that the $2S$ oscillations do not disappear even when outside the region where the $2S$ oscillations are dominant. For the TAP orbit results, shown in Figure C.10, all three controlled orbital elements show that as mission duration increases, so does the mean amount of ΔV per year. The deviation of these results compared to the Tundra orbit is the TAP orbit's smaller ω curve size which results in a higher sensitivity to the duration for the amount of ΔV required for the AOP. Figure C.11 shows that for the eccentricity of the Molniya 3500 orbit, the 10-year mission requires the most amount of ΔV . This is due to the period of the relative RAAN of the being approximately equal to the mission period. Any mission duration that is greater or less than this amount requires less mean ΔV per year since the fraction of the mission time where the $2S$ oscillations are dominant grows. For the inclination, the 5-year mission requires more ΔV per year; however, it is not a significant increase. For the AOP, it is shown that the mission duration is a significant factor for the mean ΔV per year and that an increased mission duration requires more ΔV per year. The results of the Molniya 1000 orbit are shown in Figure C.12. The mean ΔV per year for the eccentricity is shown to be largest when the mission time is 5 years. The 10- and 15-year missions require less mean ΔV per year since there are more instances where the long-period oscillations are near zero (when $\Delta\Omega$ is equal to 0° or 180°). Just as with the Molniya 3500 orbit, Fig-

---

²²The mean ΔV per year is the total required ΔV divided by the mission duration. The actual amount of ΔV per year varies as the relative geometry of the solar orbit with satellite orbit changes.

ure C.12 shows that the inclination mean ΔV requirements per year are slightly more for the 5-year mission, though the amount is fairly small. The cause of this behaviour is due to the long-period oscillations of the inclination being zero when the relative RAAN is 180° instead of 0° as with the eccentricity. As a result, a longer mission duration coupled with the large drift rate of the relative RAAN of the Molniya orbits result in more opportunities to exploit the inclination switch points compared to the 5-year mission. Finally for the AOP of the Molniya 1000 orbit, it is shown that the 5-year mission requires the least amount of mean ΔV per year. The fraction of the mission time when the $2S$ oscillations of the AOP are dominant is such that the 15-year mission experiences three distinct periods of time with these conditions, as oppose to the 10-year mission which only experiences one such opportunity. As a result, the 10-year mission requires the largest amount of mean ΔV per year compared to the other mission durations because of the relative geometry of the satellite orbit with respect to the orbit of the Sun.

The literature review of existing control strategies developed for HEOs at the beginning of this chapter often quoted 50 m/s/year of ΔV as the standard for a ΔV budget. This value originates from geostationary satellite orbit control requirements to control both the longitude and the latitude of the satellite. In comparison, the APC method is designed to control the AOP and the inclination, which are equivalent control parameters for the apogee location of an HEO. Furthermore, the eccentricity is controlled to maintain the altitude of the apogee, which is typically not needed for geostationary satellite control. As result, the 50 m/s/year target shall be used here to determine if the ΔV estimations are practical, i.e., if the required propellant mass is an achievable value. The total mission ΔV s for the HEOs are shown in Figure 4.28 and 4.29. For the Tundra orbit, regardless of the mission duration or tolerance level, the estimated total ΔV using the APC method

is shown to be equal to or less than 50 m/s/year²³. This means that the Tundra orbit, which has the same period as the geostationary satellite, can have its apogee maintained to within ± 11 km of altitude, $\pm 0.01^\circ$ for the inclination, and $\pm 0.03^\circ$ for the AOP all while using approximately the same amount of ΔV as a geostationary satellite. This equates to maintaining the satellite at apogee to within a cuboid measuring 21 km by 44 km by 9 km in the RSW system. It should be noted, however, that none of the research performed for the development of the control strategy involved the navigation of the satellite, i.e., determining its position and velocity. As such, it is not within the scope of this thesis to comment on the practicality of trying to determine the orbital elements to such a precision. It is only stating that there exists a sequence of manoeuvres whereby the orbital elements could be maintained to those tolerances assuming the accuracy of the navigation solution allowed for such tight control. There is development into the navigation problem of HEOs using the GPS constellation; however, the results are theoretical and have only been applied to Molniya orbits [64]. Even though all tolerance levels of the 15-year mission require 50 m/s/year of ΔV or less, the results of the 10- and 5-year missions further demonstrate that the mission time is a significant factor when estimating the total ΔV for an HEO. By doubling or tripling the mission time, one cannot simply double or triple the estimated ΔV budget.

The results of the TAP orbit in Figure 4.28 show that only the less stringent tolerances for the 15-year mission are reasonably close to the 50 m/s/year benchmark. The more stringent tolerances are shown to require too much ΔV . This is a result of the TAP orbit's larger drift of the RAAN whose effects on the perturbations and thus estimated mission ΔV were previously discussed. However, when the mission time is reduced to 10 years, all but the lowest tolerance level are able to meet the cap for the ΔV . The trend

---

²³To achieve this limitation, the total ΔV must be less than 750 m/s, 500 m/s, and 250 m/s for the 15-, 10-, and 5-year missions, respectively.

of significant decrease for the ΔV continues as the mission time is lowered to 5 years. All of the tolerance levels studied are able to be controlled while still requiring less than 50 m/s/year. For a TAP orbit, this translates to maintaining the apogee within a cuboid measuring 12.9 km in the radial direction, 26.2 km in the velocity direction, and 4.2 km in the orbit normal direction.

For the Molniya orbits, their lower semi-major axis results in larger perturbations due to the Earth, which indirectly increases the magnitude of the variations of the orbital elements caused by the lunisolar perturbations. Therefore, both Molniya orbits require significantly more ΔV than either of the previous two HEOs. Figure 4.29 shows the total ΔV requirements predicted using the APC method for both Molniya orbits. The results for these HEOs show that both the 15- and 10-year missions require well over 50 m/s/year—particularly the Molniya 1000 orbit. For the 5-year mission, however, the Molniya 3500 orbit is shown to meet the ΔV limitations for most of the tolerance levels. As mentioned early on in this section, it was suspected that the Molniya 1000 orbit would exceed the maximum per year ΔV , thus the additional Molniya 3500 orbit was added to the set of HEOs. Indeed, the STK simulations show this to be true. The Molniya 1000 orbit has such a large drift that the long-period oscillations are almost always the dominant mode of oscillation in the variations of the orbital elements. This causes the ΔV requirements to increase significantly since the APC method is unable to exploit the switch points effectively to reduce the required ΔV . For the Molniya 3500 orbit, however, the APC strategy is able to maintain the apogee of the satellite down to a cuboid measuring $17 \times 4.8 \times 29$ km for a 5-year mission requiring less than 50 m/s/year.

This chapter presented the development of an orbit control strategy used to estimate the ΔV for a satellite in an HEO by exploiting the lunisolar perturbations. The result is

a mission planning tool used to analyze the feasibility of a mission based on conditions such as the control tolerance size and the mission duration. This research also develops the framework for a guidance system to be used as part of an orbit control system that could be implemented for actual mission. Additionally, the development of the control strategy required research into the various manoeuvre sequences required to maintain the eccentricity, inclination, and AOP. The eccentricity and AOP manoeuvres were also used to indirectly control/maintain the ground-track of the satellite.

Figure 4.27: The mean required ΔV per year using the APC method from the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Tundra orbit).

Figure 4.28: Total required mission ΔV using the APC method for the Tundra and TAP orbits over various control tolerance sizes for 15-, 10-, and 5-year missions.

Figure 4.29: Total required mission ΔV using the APC method for the Molniya 3500 and 1000 orbits over various control tolerance sizes for 15-, 10-, and 5-year missions.

Two types of control approaches using the grazing method to exploit the switch points were developed in this chapter. The primary method (APC) uses analytical predictions of the switch point times and values to calculate the manoeuvre size. The second method (NPC), uses numerically-simulated orbit data to determine the manoeuvre size such that the switch point is exploited. The NPC method is technically more accurate compared to the APC method since it uses a highly accurate force model within STK to compute the future variations of the orbital elements. The APC method, on the other hand uses the equations from Section 3.2 to predict the switch point time and the analytical equations of motion from Section 2.3 to evaluate the eccentricity, inclination and AOP at the switch point. For the larger HEOs, however, such as the Tundra and TAP orbits, the ΔV results from this section show that the APC and NPC simulations are extremely similar. The primary difference then is that the simulations using the NPC method need more time for a given simulation since they require many extra numerical simulations to calculate the manoeuvre sizes to utilize the switch points. The APC method, however, can calculate the manoeuvre size in a short period of time since only a small set of equations is needed. The smaller HEOs, i.e., the two Molniya orbits, have slightly larger discrepancy between the APC and NPC results. This discrepancy is due to the more complicated force model for these orbits. Consequently, the APC method may not be able to exploit the switch points as efficiently as the NPC method. This can result in a slightly larger estimation of the ΔV requirements for the mission or even overestimate the number of manoeuvres required. The similar results of the APC and NPC methods for the larger HEOs, however, demonstrates that the analytical method can be an effective and equivalent approach to the control problem for a certain class of HEOs.

Besides the HEO type and mission duration, discussed later in this section, the control parameter that was shown to affect the effectiveness the APC method was the tolerance

size of the control strategy. The concept of the grazing method is to extend the time between the manoeuvres by designing the change in the orbital element such that the time when its rate of change is zero coincides with the time when it reaches the upper or lower limit of its allowable range. As such the size of the allowable range plays an important role in the effectiveness of the grazing method to reduce the required ΔV . Stated earlier in this chapter, the range of values for the tolerance are not based on any specific mission parameters, but the amplitude of the oscillations that the APC method attempts to exploit. Therefore, the values of the tolerance size were based on factors of the amplitude of the $2S$ -oscillations from $1/6$ to 2 . Too small of a tolerance size results in requiring too many manoeuvres and not being able to reduce the ΔV compared to the Baseline method. Too large of a tolerance size and the oscillations that the grazing method is attempting to utilize are completely encapsulated within the allowable limits and again very little ΔV savings occur. The results of the STK simulations in this section demonstrated that the largest savings are typically seen when the tolerance size is between $1/2 \times 2S$ and $4/3 \times 2S$. The wide range of acceptable tolerance sizes is a result of the overlap of the long-period oscillations with the $2S$ -oscillations. This causes the effective amplitude of the $2S$ -oscillations to differ.

Since each HEO has a different mean rate of change of the RAAN, the lunisolar perturbations, as discussed using the switch point analysis from Chapter 3, have a distinct effect on each HEO. The difference in the perturbations acting on the HEOs results in varying ΔV requirements. The dynamic analysis from Chapter 3 showed that an HEO experiences multiple modes of oscillation in the variations of the orbital elements. The two primary modes are the long-period oscillations due to the drift of the relative RAAN and the $2S$ oscillations due to revolution of the Earth around the Sun. The relative RAAN oscillations are primarily the dominant mode of the two; however, the switch point

analysis showed that the rate of change of this oscillation mode is zero when the relative RAAN is 0° or 180° . For the eccentricity and the AOP, the switch point analysis also showed that the $2S$ oscillations reach their largest magnitude when the relative RAAN is 0° . The $2S$ oscillations of the inclination, however, reaches their maximum magnitude when the relative RAAN is 180° . That said, the amplitude of the $2S$ oscillations of the inclination when the $\Delta\Omega = 0^\circ$ are still significant, unlike the $2S$ oscillations of the eccentricity and AOP where their amplitudes become very small or disappear completely when $\Delta\Omega = 180^\circ$. Therefore, the initial RAAN of the HEO is designed such that the relative RAAN reaches 0° near its mid-mission point time. This ensures the maximum availability of the switch points to exploit by setting the relative geometry of the satellite and third bodies such that the rate of change of the long-period oscillations is at its minimum. This creates a ‘window’ of opportunity where there is a maximum potential to utilize the switch points and reduce the ΔV compared to the Baseline method. Proper selection of the initial RAAN ensures that this window occurs as long as possible throughout the mission. The different RAAN rate of the HEOs means that the window of each orbit is unique. A higher orbit has a lower RAAN rate which translates to a larger switch point window. Therefore, the Tundra orbit has the largest window whereas the Molniya 1000 orbit has the smallest. The duration of the window relative to the mission duration is a primary factor in the grazing-method-based control strategies ability to exploit the switch points to significantly reduce the mission ΔV . The small switch point window of the Molniya 1000 orbit relative to a 15-year mission results in a very small (if any) decrease in the amount of ΔV required to maintain the orbital elements²⁴. Conversely, the Tundra orbit’s large switch point window relative to a 5-year mission duration sets the relative geometry of the mission where the relative RAAN oscillations are nearly zero for the entire mission period. As a result, the switch points are able to be exploited throughout the mission and

---

²⁴It is still, however, able to significantly reduce the number of manoeuvres required, depending on the tolerance level.

the APC method is able to reduce the total ΔV . The suitability of the APC method, based on the results of Figure 4.26, as well as Figures C.7 through C.9, shows that the Molniya 1000 orbit is unable to effectively use the APC method to take advantage of the variations of the orbital elements compared to the Baseline or NPC methods. This is a result of the combination of the more complex perturbation acting on the orbit which creates error in predicting the desired orbital element value for a given manoeuvre sequence, as well as the very small switch point window. The larger Molniya orbit suffers from the same problems as the lower one; however, the effects of the problems are less significant and some savings in the ΔV are observed for the inclination and AOP, depending on the mission duration and tolerance size. For the Tundra and TAP orbit, their respective switch point window durations make them ideal for control of the orbital elements using the grazing method. Their eccentricities, inclinations, and AOPs are modelled with reasonable accuracy using only J_2 , the Moon, and the Sun. The APC method is thus able to effectively predict the times and values of the switch points and reduce the mission ΔV .

Chapter 5

Conclusions

The lunisolar gravity coupled with indirect gravitational attraction of the Earth's oblate shape impart¹ multiple modes of oscillations in the orbital elements of a satellite in an HEO that is critically inclined with an AOP set to 270° . The focus of this thesis is to analyze the dynamics of the HEO's orbital elements to determine the geometric conditions of the Moon or Sun relative to the satellite orbit that result in the rate of change of an orbital element being zero. The conditions for which this occurs are called switch points. The study of the switch points aids in the understanding of the fundamental behaviour of the lunar and solar perturbations acting on an HEO in orbit around the Earth. The identification of the switch point planes also provides a method to visualize how the relative geometry of the satellite, Earth, Moon, and Sun results in varying oscillations for the orbital elements of the satellite. As the relative RAAN between the HEO and the third body drifts (primarily due to J_2), the four-body configuration changes. It is this mechanism that is responsible for the long-period behaviour of the orbital elements. This knowledge can aid a mission planner in understanding the nature of lunisolar perturbations acting on HEOs.

---

¹The oblateness due to the Earth causes the RAAN of the HEO to drift which is responsible for the long-period relative RAAN oscillations discussion in the previous chapters.

Furthermore, using the switch point planes in relation to the orbit of the Sun or Moon, an analytically-driven geometric method to predict the future switch point occurrences was developed for the eccentricity, inclination, RAAN, and AOP. Using the enhanced knowledge of the future perturbations, an orbit control strategy was developed which exploits the sign changes in the lunisolar perturbations acting on the eccentricity, inclination, and AOP. Such a control strategy uses analytical means to predict the switch points and calculate the manoeuvre size such that the time when the rate of change of the orbital elements is zero corresponds to when the orbital element reaches the upper or lower tolerance limit. Development of the control strategy also includes the derivation of near-optimal locations of the inclination and AOP manoeuvres for HEOs. The inclination manoeuvre location is set such that an out-of-plane manoeuvre has a maximum effect on the inclination. This results in a small secondary perturbation of the AOP, but it is taken into account in the control decisions. The AOP manoeuvre location is designed as a combination radial/transverse manoeuvre that has a near-maximum effect on the AOP without any secondary effect on the eccentricity. Furthermore, the eccentricity and AOP manoeuvre sequences are designed such that the ground-track of the HEO is indirectly maintained through temporary adjustments to the semi-major axis. The control method developed for the HEO is a mission analysis tool to be used for estimating the control requirements for a satellite mission during early conceptual design. No specific mission was used during the development of the control strategy to ensure that its applicability for a variety of missions using HEOs remained.

The analytical control method was first tested using a custom orbit propagator in MATLAB with a simplified perturbation model and then using STK's high-precision orbit propagator to validate the findings from MATLAB. The results for both simulation methods were compared to two other control methods: a Baseline control strategy which

has no knowledge of the future perturbations affecting the HEO and a numerical control strategy which is similar to the analytical method except that it uses numerical simulations to compute the switch points. Additionally, the simulations were run with various initial conditions such as mission duration and tolerance size. The results demonstrate that, primarily for the Tundra orbit along with the TAP orbit, the grazing-method-based control strategies are able to significantly reduce both the ΔV and the number of manoeuvres for a mission compared to the Baseline method, particularly for shorter missions. In these orbits, the APC method is also shown to be equivalent to the NPC method. For the Molniya orbits, the more complex perturbation environment as a result of the higher dynamic load due to their low perigee culminates in the APC method's inability to significantly reduce the mission ΔV nor accurately exploit the switch points in comparison to the NPC method.

While the work developed for this thesis is comprehensive enough to stand on its own as a research topic, there exist opportunities to expand on the research developed here. They include the following.

Low-thrust propulsion Many satellite missions are foregoing the use of traditional impulsive manoeuvres in favour of a low-thrust propulsion system. Such systems generate an electric or magnetic field to accelerate propellants such as Xenon thereby imparting a small net force on the satellite. The cumulative effect results in the ability to adjust the orbital elements over long periods of time. Knowledge of the future switch point time in conjunction with a low-thrust propulsion system could prove to be a unique and interesting control problem.

Integrate tesseral harmonics study by Ely et al. The control of the ground-track of the HEO was not the primary function of the control strategy developed in this thesis. Instead, it was included after the control sequences of the eccentricity and

AOP were developed when the temporary change in the semi-major axis was found to be able to assist in maintaining the ground-track. In his Ph.D. thesis, Ely performed an intensive analysis of the higher-order tesseral harmonics, as well as the lunar and solar effects [65]. A novel control strategy was then developed to maintain the satellite's ground-track using the grazing method. The ability to combine both Ely's control approach as well as the APC method would be valuable for a mission requiring robust control of multiple orbital parameters.

Optimal control theory The use of the grazing method to exploit the lunisolar perturbations cannot be stated as the minimization of the required ΔV since no optimization was applied to the control problem. One could then compare the relatively-simple APC control approach to one that uses optimal control theory such as that described in Reference [66] to determine the differences between the results.

Coverage analysis The tolerance levels discussed in Chapter 4 are based on the magnitude of the perturbations acting on the HEO. No analysis was performed to determine which tolerance level was sufficient since there are no mission constraints. It would be worthwhile to apply the control strategy to a satellite mission with specific requirements and then study the relationship between the various tolerance levels and whether the mission achieves its objectives.

Constellation mission The proposed mission that inspired this project, PCW, is currently envisioned as a constellation of two satellites in an HEO. Depending on the constellation configuration (same orbit plane or 90° out-of-phase), the combined control of the constellation, i.e., the control of both satellites simultaneously, would be interesting. The relative spacing of the satellites would most likely become a primary control requirement. Therefore, control of the ground-track of each satellite would need to be more robust than that developed for this thesis.

Post-mission orbit planning Many space agencies and private companies are putting more emphasis on proper end-of-mission planning to reduce the amount of space debris in orbit, or at the very least place the satellites in a graveyard orbit². The HEOs under study here have significant orbital energy and very high perigee altitudes. As such, neither an end-of-mission de-orbit manoeuvre nor using atmospheric drag would be able or practical to force the satellite to de-orbit (with the exception of the Molniya 1000 orbit, and possibly the Molniya 3500 orbit). It is the responsibility of any mission designer to ensure that steps are taken to reduce the risk of post-mission collision with another satellite or debris. Analyzing the likely behaviour of the satellite over very long time periods (approximately 50 years) is necessary in order to determine a safe location for the satellite remains.

As the number of satellites placed in the geostationary belt increases over time, demand for other orbit regimes will grow. HEOs are able to offer relief to that demand particularly for nations located near the Arctic such as Canada. The suitability of these orbits for such missions is a result of their high-apogee altitude relative to their perigee altitude, along with their critical inclination and AOP of 270° . As a result, HEOs around the Earth have a complex and dynamic set of perturbations acting on them causing potentially-significant variations in the orbital elements. The unique perturbation environment of the HEOs, which is due to their large eccentricity, means that traditional methods for control of the orbit of a satellite, for example geostationary orbits, are ineffective and a new approach is needed. The new approach, described in this thesis, has been shown to be enhanced by intensive study of the perturbations in terms of the orbital geometry of the Sun and Moon relative to the HEO.

---

²A graveyard orbit is an orbit where the risk of collision with other satellites is minimal.

References

- [1] A. E. Roy. Luni-solar perturbations of an earth satellite. *Astrophysics and Space Science*, 4:375–386, 1969.
- [2] D. G. King-Hele. The orbital lifetimes of molniya satellites. *Journal of the British Interplanetary Society*, 28:783–796, 1975.
- [3] L. Garand and G. Kroupnik. Status of the polar communications and weather (PCW) mission to serve the arctic. Technical report, Canadian Space Agency, <http://www.gfy.ku.dk/~awh/satellite-imaging/files/Presentations/Kroupnik.ppt>, 2010. Accessed Oct. 2010.
- [4] F. Delhaise. The effects of luni-solar gravitation and solar radiation pressure on the molniya and tundra orbits. MAS Working Paper 295, European Space Agency - ESOC, 1989.
- [5] M. Bourassa, B. Burlton, R. Langlois, and F. Afagh. Switch points for highly eccentric orbits: Modelling the occurrences of sign changes in the rate of change of the eccentricity. In *64th International Astronautical Congress*, 2013.
- [6] M. Bourassa, B. Burlton, F. Afagh, and R. Langlois. Additional switch points for highly eccentric orbits: Modelling the occurrences of sign changes in the rate of change of the inclination, raan, and argument of perigee. Technical Report Applied Dynamics Laboratory Report No. ADL/14/MB/1, Carleton University, 2014.

- [7] M. T. Lane. On analytic modelling of lunar perturbations of artificial satellites of the earth. *Celestial Mechanics and Dynamical Astronomy*, 46:287–305, 1989.
- [8] D. A. Vallado. *Fundamentals of Astrodynamics and Applications*. The Space Technology Library. Microcosm Press and Springer, Hawthorne, CA and New York, NY, 3rd edition, 2007.
- [9] T. A. Ely. *Dynamics and Control of Artificial Satellite Orbits with Multiple Tesseral Resonances*. PhD thesis, Purdue University, 1996.
- [10] A. P. Trishchenko, L. Garand, and L. D. Trishchenko. Three apogee 16-h highly elliptical orbit as optimal choice for continuous meteorological imaging of polar regions. *Journal of Atmospheric and Oceanic Technology*, 2011.
- [11] B. Evans. Sirius rising: Proton-m ready to launch digital radio satellite into orbit. Online. Accessed Nov. 25, 2014.
- [12] R. R. Bate, D. D. Mueller, and J. E. White. *Fundamentals of Astrodynamics*. Dover Publications Inc., New York, 1st edition, 1971.
- [13] D. Mortari, M. P. Wilkins, and C. Bruccoleri. The flower constellations. *The Journal of the Astronautical Sciences*, 52:107–127, 2004.
- [14] Y. Kozai. The motion of a close earth satellite. *Smithsonian Astrophysical Observatory*, 64:367–377, 1959.
- [15] F. Delhaise and J. Henrard. The problem of critical inclination combined with a resonance in mean motion in artificial satellite theory. *Celestial Mechanics and Dynamical Astronomy*, 55:261–280, 1993.
- [16] W. M. Kaula. *Theory of Satellite Geodesy*. Blaisdell Publishing Company, Waltham, MA, 1966.

- [17] W. M. Kaula. Analysis of gravitational and geometric aspects of geodetic utilization of satellites. *Geophysical Journal of the Royal Astronomical Society*, 5:104–133, 1961.
- [18] W. M. Kaula. A geoid and world geodetic system based on a combination of gravimetric, astrogeodetic, and satellite data. *Journal of Geophysical Research*, 66(6):1799–1811, 1961.
- [19] S. Hardacre. *Control of Colocated Geostationary Satellites*. PhD thesis, Cranfield University, 1996.
- [20] T. A. Ely and K. C. Howell. Long term evolution of artificial satellite orbits due to resonant tesseral harmonics. *American Astronomical Society*, 196:1397–1413, 1995.
- [21] L. Blitzer. Lunar-solar perturbations of an earth satellite. *American Journal of Physics*, 27(9):634–645, 1959.
- [22] E. Upton, A. Bailie, and P. Musen. Lunar and solar perturbations on satellite orbits. *Science, New Series*, 130(3390):1710–1711, 1959.
- [23] P. Musen. On the long-period lunar and solar effects on the motion of an artificial satellite - 2. *Journal of Geophysical Research*, 66(9):2797–2805, 1961.
- [24] P. Musen. On the long-period lunisolar effect in the motion of the artificial satellite - 1. *Journal of Geophysical Research*, 66(6):1659–1665, 1961.
- [25] D. E. Smith. The perturbation of satellite orbits by extra-terrestrial gravitation. *Planetary Space Sciences*, 9:659–674, 1962.
- [26] R. R. Allan. Satellite orbit perturbations due to radiation pressure and luni-solar forces. *Quarterly Journal of Mechanics and Applied Mathematics*, 15(3):283–301, 1962.

- [27] B. A. Shute and J. Chiville. The lunar-solar effect on the orbital lifetimes of artificial satellites with highly eccentric orbits. *Planetary Space Sciences*, 14:361–369, 1966.
- [28] G. E. Cook. Luni-solar perturbations of the orbit of an earth satellite. *The Geophysical Journal of the Royal Astronomical Society*, 6(3):271–291, 1962.
- [29] S. Hughes. Earth satellite with resonant lunisolar perturbations 1 - resonances dependent only of inclination. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 372:243–264, 1980. 1749.
- [30] M. E. Hough. Orbits near critical inclination, including lunisolar perturbations. *Celestial Mechanics*, 25:111–136, 1981.
- [31] G. E. O. Giacaglia. Lunar perturbations of artificial satellites of the earth. *Celestial Mechanics*, 9:239–267, 1974.
- [32] G. Lecohier, V. Guermonprez, and F. Delhaise. European molniya and tundra orbit control. *Proceedings of the International Symposium: Space Dynamics*, pages 165–191, 1990. Toulouse, France.
- [33] F. Delhaise and A. Morbidelli. Luni-solar effects of geosynchronous orbits at the critical inclination. *Celestial Mechanics and Dynamical Astronomy*, 57:155–173, 1993.
- [34] W. M. Smart. *Celestial Mechanics*. Longmans Green and Co., 1953.
- [35] F. T. Geyling and H. R. Westerman. *Introduction to Orbital Mechanics*. Addison-Wesley Aerospace Series. Addison-Wesley Publishing Company, Whippany, NJ, 1971.
- [36] W. M. Kaula. Development of the lunar and solar disturbing functions for a close satellite. *The Astronomical Journal*, 67(5):300–303, 1962.
- [37] A. Cayley. On the development of the disturbing function in lunar theory. *Memoirs of the Royal Astronomical Society*, 27:69–95, 1859.

- [38] E. Upton, A. Bailie, and P. Musen. Development of the lunar and solar perturbations in the motion of an artificial satellite. Technical Report D-494, Goddard Space Flight Center, NASA, 1961.
- [39] A. F. B. A. Prado and I. V. D. Costa. Third-body perturbation in spacecraft trajectory. In *International Astronautical Conference*, 1998.
- [40] I. V. D. Costa and A. F. B. A. Prado. Orbital evolution of a satellite perturbed by a third-body. *Advances in Space Dynamics*, pages 176–194, 2000.
- [41] R. C. Domingos, R. V. de Moraes, and A. F. B. A. Prado. Third-body perturbation in the case of elliptic orbits for the disturbing body. *Hindawi Publishing Corporation*, 2008, 2008.
- [42] Y. F. Kolyuka, N. V. Yashina, and Gridchina T.A. Orbit evolution and uncontrolled re-entry of the molniya type satellites. In *5th European Conference on Space Debris*, 2009.
- [43] M. L. Lidov. Evolution of orbits of artificial satellites of planets due to gravitational perturbations of outer bodies. *Artificial Earth Satellites*, 8:168–207, 1961.
- [44] Systems ToolKit (STK). <http://www.agi.com>.
- [45] R. L. Burden and J. D. Faires. *Numerical Analysis*. Brooks/Cole, Pacific Grove, CA, 7th edition, 2001.
- [46] The JPL SPICE toolkit. <http://naif.jpl.nasa.gov/naif/toolkit.html>.
- [47] E. M. Alessi, C. Colombo, J. P. S. Cuartiellas, and M. Landgraf. Out-of-plane extension of resonant encounters for escape and capture. In *63rd International Astronautical Congress*, 2013.

- [48] M. Bourassa, B. Burlton, F. Afagh, and R. Langlois. Modelling luni-solar perturbations acting on highly eccentric earth-orbiting satellites. In *Proceedings of Canadian Aeronautics and Space Institute (CASI) ASTRO 2012 Conference*, Quebec City, Canada, 2012.
- [49] W. S. Koon, M. W. Lo, J. E. Marsden, and S. D. Ross. *Dynamical Systems, the Three-Body Problem and Space Mission Design*. 2011.
- [50] Symbolab. <https://www.symbolab.com>. Last accessed Feb. 23, 2015.
- [51] M. J. Bruno and H. J. Pernicka. Tundra constellation design and stationkeeping. *Journal of Spacecraft and Rockets*, 42(5):902–912, 2005.
- [52] G. Rondinelli, A. Cramarossa, and F. Graziani. Orbit control strategy for a constellation of three satellites in tundra orbits. *Proceedings of the AAS/AIAA Astrodynamics Conference*, Part 1:697–708, 1989.
- [53] M. Khan. Design of an optimal stationkeeping strategy for a system of highly eccentric, highly inclined orbits. In *AAS/AIAA Spaceflight Mechanics Meeting, Paper No 94-109*, pages 249–259, 1994.
- [54] M. Khan. The critically inclined 16-hour orbit: An unconventional option for improved telecommunications on the northern hemisphere. Technical report, mpb Aerospace Dept., 1993.
- [55] V. Kudielka and W. Drahanowsky. Phase 3d - feasibility study of launch sequences and orbits. Technical report, OeVSV/RTU Report, 1994.
- [56] V. Kudielka. Planning for an unconventional, highly-elliptic orbit. In *Proceedings of the 12th International Symposium on ‘Space Flight Dynamics’*, 1997.

- [57] M. J. Bruno. Tundra constellation design and stationkeeping. Master's thesis, San Jose State University, 2004.
- [58] C. C. Chao. *Applied Orbit Perturbation and Maintenance*. The Aerospace Corporation, 2005.
- [59] M. S. Konstantinov, V. A. Obukhov, V. G. Petukhov, G. A. Popov, and G. G. Tchernobelsky. Spacecraft station-keeping in the molniya orbit using electric propulsion. *56th International Astronautical Congress*, 4:2693–2702, 2005.
- [60] L. Barker and J. Stoen. Sirius satellite design: The challenges of the tundra orbit in commercial spacecraft design. In *AAS Guidance and Control Conference*, volume 107, pages 575–596, 2001.
- [61] T. S. Kelso. Celestrak. <http://celestrak.com>. Last accessed Feb. 25, 2015.
- [62] C. A. Wagner. Stable longitudes for 12-hr eccentric orbit satellites. *Journal of Spacecraft and Rockets*, 9(10):757–763, 1972.
- [63] C. C. Chao, J. W. Cook, J. Cox, T. F. Starchville, R. C. Thompson, and L. F. Warner. Independent verification and validation for analytical graphics, inc. of three astrodynamics functions of the satellite tool kit: Version 4.1.0. Technical Report ATR-2000(7605)-1, The Aerospace Corporation, 2000.
- [64] A. Vigneron. Nonlinear filtering for autonomous navigation of spacecraft in highly elliptical orbit. Master's thesis, Carleton University, 2014.
- [65] T. A. Ely and K. C. Howell. Tesseral resonance overlap including the effects of luni-solar perturbations. In *AIAA/AAS Astrodynamics Conference*, pages 119–129, 1996.
- [66] A. H. Schubert. Optimal orbit control for spacecraft on elliptical orbits with atmospheric drag. Technical report, ESTEC - ESA, 2001.

APPENDIX A

Constants

Parameter	Variable	Value
Universal gravity constant	G	$6.67384 \times 10^{-11} \text{ m}^3/(\text{kg} \times \text{s}^2)$
Earth zonal gravity coefficient (order 2)	J_2	$1.08262668355 \times 10^{-3}$
Earth zonal gravity coefficient (order 3)	J_3	$-2.53265648533 \times 10^{-6}$
Earth zonal gravity coefficient (order 4)	J_4	$-1.61962159137 \times 10^{-6}$
Earth radius	R_E	6378.1363 km
Mass of the Earth	m_E	$5.972 \times 10^{24} \text{ kg}$
Mass of the Sun	m_S	$1.989 \times 10^{30} \text{ kg}$
Mass of the Moon	m_M	$7.348 \times 10^{22} \text{ kg}$
Apparent mean orbital radius of the Sun	r_S	$1.496 \times 10^8 \text{ km}$
Mean orbital radius of the Moon	r_M	384400 km
Gravitational parameter of the Earth	μ_E	$3.986004415 \times 10^5 \text{ km}^3/\text{s}^2$
Gravitational parameter of the Sun	μ_S	$1.32712440018 \times 10^{11} \text{ km}^3/\text{s}^2$
Gravitational parameter of the Moon	μ_M	$4902.7779 \text{ km}^3/\text{s}^2$
Rotational rate of the Earth	ω_E	$7.292115 \times 10^{-5} \text{ rad/s}$

APPENDIX B

Lunar and Solar Orbital Elements

The variations of the orbital elements of the Moon and Sun are modelled using the SPICE toolkit developed by the National Ancillary Information Facility (NAIF), a subdivision of the Jet Propulsion Laboratory [46]. The elements are modelled in an Earth-centred True-of-Date coordinate system with the $X - Y$ -plane defined along the equator. The variations of the RAAN over a 25-year period are shown in Figure B.1. The RAAN of the Moon is shown to oscillate between $\pm 13^\circ$ over a 18.6-year period. The RAAN of the Sun is effectively zero. Figure B.2 shows the variations of the inclination. The inclination of the Sun is constant at 23.45° , which is equal to the angle of the ecliptic. Just as with the lunar RAAN, the lunar inclination oscillates over the 18.6-year period, with a maximum value of approximately 28.5° and a minimum value of 18.3° . Figures B.3 and B.4 show the variations of the AOL and radius, respectively, for the Sun and Moon over a one year period. Figure B.3 demonstrates the approximately 13:1 ratio of lunar cycles per solar cycle, which is the cause of the amplitude of the $2S$ oscillations being larger than the $2M$ oscillations. This ratio is also observed in Figure B.4.

Figure B.1: Variations of the RAAN of the Moon and Sun over a 25-year period.

Figure B.2: Variations of the inclination of the Moon and Sun over a 25-year period.

Figure B.3: Variations of the argument of latitude of the Moon and Sun over a 1-year period.

Figure B.4: Variations of the orbital radius of the Moon and Sun over a 1-year period.

APPENDIX C

Supplemental Results

Appendix C contains the supplemental results of the control strategy simulations from Chapter 4. Both the MATLAB and STK simulation results are included.

C.1 MATLAB Results

The results from this section are those computed using the custom MATLAB orbit propagator. This propagator uses a simplified force model that is specific for HEOs. The perturbations include the Earth equatorial oblateness (J_2) and the gravity of the Moon and Sun.

Table C.1: Control tolerances and parameters for a TAP orbit mission.

Tolerance size	$(1/2) \times 2S$ osc.	$2S$ osc.	$2 \times 2S$ osc.
Eccentricity tolerance size	0.0006	0.0012	0.0024
Inclination tolerance size (deg)	0.0112	0.0224	0.0447
AOP tolerance size (deg)	0.07	0.14	0.28
Max ΔV_e per manoeuvre sequence (m/s)	3.3	5.8	10.7
Max ΔV_i per manoeuvre sequence (m/s)	1.9	3.1	5.8
Max ΔV_w per manoeuvre sequence (m/s)	3.5	6.2	11.4

Figure C.1: Controlled variations of the orbital elements of a TAP orbit over a 5-year period using the APC method with a tolerance size of $1/2 \times 2S$.

Figure C.2: Controlled variations of the orbital elements of a Molniya orbit over a 5-year period using the APC method with a tolerance size of $1/2 \times 2S$.

Figure C.3: The ΔV results for a TAP orbit comparing different orbit control strategies at various control tolerances for a 15-year mission. The numbers in brackets are the number of manoeuvres required.

Table C.2: Control tolerances and parameters for a Molniya orbit mission.

Tolerance size	$(1/2) \times 2S$ osc.	$2S$ osc.	$2 \times 2S$ osc.
Eccentricity tolerance size	0.0004	0.0008	0.0016
Inclination tolerance size (deg)	0.0072	0.01435	0.0287
AOP tolerance size (deg)	0.055	0.11	0.22
Max ΔV_e per manoeuvre sequence (m/s)	3.1	5.2	9.6
Max ΔV_i per manoeuvre sequence (m/s)	1.5	2.3	4.2
Max ΔV_ω per manoeuvre sequence (m/s)	4.5	7.9	14.5

Figure C.4: The ΔV results for a TAP orbit comparing different orbit control strategies at various control tolerances for a 5-year mission. The numbers in brackets are the number of manoeuvres required.

Figure C.5: The ΔV results for a Molniya orbit comparing different orbit control strategies at various control tolerances for a 15-year mission. The numbers in brackets are the number of manoeuvres required.

Figure C.6: The ΔV results for a Molniya orbit comparing different orbit control strategies at various control tolerances for a 5-year mission. The numbers in brackets are the number of manoeuvres required.

C.2 STK Results

The results in this section are computed using the STK simulations with the HPOP force model described in Section 4.6 which includes a 21x21 geopotential model, lunar and solar gravity, atmospheric drag, and solar radiation pressure. The results of the simulations represent the most accurate simulations of a satellite orbit that are practical for this research. As such, they are considered to be the ‘true’ solution.

Table C.3: Control tolerances and parameters for a Molniya 3500 orbit mission.

Tolerance size	$(1/2) \times 2S$ osc.	$2S$ osc.	$2 \times 2S$ osc.
Eccentricity tolerance size	0.0005	0.001	0.002
Inclination tolerance size (deg)	0.0077	0.0154	0.0307
AOP tolerance size (deg)	0.0469	0.0939	0.188
Max ΔV_e per manoeuvre sequence (m/s)	4.3	6.6	11.7
Max ΔV_i per manoeuvre sequence (m/s)	1.8	2.8	4.9
Max ΔV_ω per manoeuvre sequence (m/s)	4.5	6.9	11.4

Figure C.7: The ΔV results of the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (TAP orbit).

Figure C.8: The ΔV results of the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Molniya 3500 orbit).

Figure C.9: The ΔV results of the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Molniya 1000 orbit).

Figure C.10: The mean required ΔV per year using the APC method from the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (TAP orbit).

Figure C.11: The mean required ΔV per year using the APC method from the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Molniya 3500 orbit).

Figure C.12: The mean required ΔV per year using the APC method from the STK simulations for the eccentricity, inclination, and AOP as a function of the tolerance size for various mission durations (Molniya 1000 orbit).

Figure C.13: Comparison of the AOP variations for a 15- and 5-year mission using the APC method.