Extensions of the Ramanujan-Mordell Formula

by

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Abstract

Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{Z}$ the set of all integers. Explicit formulae are derived for the number of representations of $n \in \mathbb{N}$ by the quadratic form

$$x_1^2 + \cdots + x_r^2 + 2x_{r+1}^2 + \cdots + 2x_{r+s}^2 + 4x_{r+s+1}^2 + \cdots + 4x_{r+s+t}^2$$

under certain conditions on r, s and t , that is, we give formulae for the quantity

$$\text{card}\{(x_1, \dots, x_{r+s+t}) \in \mathbb{Z}^{r+s+t} \mid n = x_1^2 + \cdots + x_r^2 + 2x_{r+1}^2 + \cdots + 2x_{r+s}^2 + 4x_{r+s+1}^2 + \cdots + 4x_{r+s+t}^2\}.$$

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Chapter 1

Introduction

A central problem in the theory of numbers is the determination of an explicit formula for the quantity

$$(1.0.1) \quad r_h(n) := \text{card}\{(x_1, \dots, x_h) \in \mathbb{Z}^h \mid n = x_1^2 + \dots + x_h^2\},$$

which counts the number of representations of a positive integer n as a sum of h squares. This problem has a long and interesting history, which is described in the two books [46], [66]. Relevant references include [9], [12], [13], [15], [16], [22]-[26], [31], [32], [37]-[45], [47]-[52], [60], [61], [63]-[65], [67], [74]-[77], [79], [80], [82], [85].

When $h \equiv 0 \pmod{2}$, say $h = 2m$ ($m \in \mathbb{N}$), Ramanujan [71], [72, pp. 136-162] gave a formula for $r_{2m}(n)$, which was later proved by Mordell [63]. The Ramanujan-Mordell formula expresses $r_{2m}(n)$ as a linear combination of elementary arithmetic functions and the coefficients in the power series expansions of certain modular forms. A proof of this formula has recently been given by Cooper [31] using Ramanujan's ${}_1\psi_1$ summation formula and elliptic functions.

When $m \equiv 0 \pmod{2}$, say $m = 2k$ ($k \in \mathbb{N}$), so $h = 4k$, the elementary arithmetic functions in the Ramanujan-Mordell formula are $\sigma_{2k-1}(n)$, $\sigma_{2k-1}(n/2)$

and $\sigma_{2k-1}(n/4)$, where

$$(1.0.2) \quad \sigma_{2k-1}(l) := \begin{cases} \sum_{\substack{d \in \mathbb{N} \\ d|l}} d^{2k-1}, & \text{if } l \in \mathbb{N}, \\ 0, & \text{if } l \notin \mathbb{N}, \end{cases}$$

and $\sigma(l) := \sigma_1(l)$. Different functions are required when $m \equiv 1 \pmod{2}$. Recently Alaca, Alaca and Williams [9] have given a completely elementary proof of the Ramanujan-Mordell formula for $r_{4k}(n)$. We state the Ramanujan-Mordell formula in the form given in their paper.

Theorem 1.0.1. (*Ramanujan-Mordell formula*) *Let $k \in \mathbb{N}$. Then for $n \in \mathbb{N}$ there exist rational numbers $c(k, m)$ ($m = 1, 2, \dots, [(k-1)/2]$) such that*

$$(1.0.3) \quad r_{4k}(n) = \frac{(-1)^{k+1}4k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}(n) + \frac{(1+(-1)^k)4k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}(n/2) \\ - \frac{2^{2k+2}k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}(n/4) + \sum_{m=1}^{[(k-1)/2]} c(k, m) a(k, m; n),$$

where the B_l ($l \in \mathbb{N}_0$) are the Bernoulli numbers given by

$$(1.0.4) \quad \frac{t}{e^t - 1} = \sum_{l=0}^{\infty} \frac{B_l}{l!} t^l, \quad |t| < 2\pi,$$

and

$$(1.0.5) \quad q^m \prod_{n=1}^{\infty} (1 - q^n)^{24m-8k} (1 - q^{2n})^{20k-48m} (1 - q^{4n})^{24m-8k} = \sum_{n=m}^{\infty} a(k, m; n) q^n.$$

Alaca, Alaca and Williams [9] have given the rational numbers $c(k, m)$ occurring in (1.0.3) as the coefficients in the expansion of a certain polynomial in x [9, eq. (4.11), p. 42] in powers of $x(1-x)$. Alternatively, as noted by Cooper [31], the $c(k, m)$ can be determined for a specific value of k by taking $n = 1, 2, 3, \dots$ in (1.0.3) and solving the resulting equations.

Taking $k = 1, 2, 3, 4, 5, 6$ in the formula (1.0.3), we obtain for $n \in \mathbb{N}$

$$r_4(n) = 8\sigma(n) - 32\sigma(n/4),$$

$$r_8(n) = 16\sigma_3(n) - 32\sigma_3(n/2) + 256\sigma_3(n/4),$$

$$r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 16a(3, 1; n),$$

where

$$q \prod_{n=1}^{\infty} (1 - q^{2n})^{12} = \sum_{n=1}^{\infty} a(3, 1; n)q^n,$$

$$r_{16}(n) = \frac{32}{17} \sigma_7(n) - \frac{64}{17} \sigma_7(n/2) + \frac{8192}{17} \sigma_7(n/4) + \frac{512}{17} a(4, 1; n),$$

where

$$q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{32}}{(1 - q^n)^8(1 - q^{4n})^8} = \sum_{n=1}^{\infty} a(4, 1; n)q^n,$$

$$r_{20}(n) = \frac{8}{31} \sigma_9(n) - \frac{8192}{31} \sigma_9(n/4) + \frac{1232}{31} a(5, 1; n) - \frac{256}{31} a(5, 2; n),$$

where

$$q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{52}}{(1 - q^n)^{16}(1 - q^{4n})^{16}} = \sum_{n=1}^{\infty} a(5, 1; n)q^n,$$

and

$$q \prod_{n=1}^{\infty} (1 - q^n)^8(1 - q^{2n})^4(1 - q^{4n})^8 = \sum_{n=1}^{\infty} a(5, 2; n)q^n,$$

$$r_{24}(n) = \frac{16}{691} \sigma_{11}(n) - \frac{32}{691} \sigma_{11}(n/2) + \frac{65536}{691} \sigma_{11}(n/4)$$

$$+\frac{33152}{691} a(6, 1; n) - \frac{65536}{691} a(6, 2; n),$$

where

$$q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{72}}{(1 - q^n)^{24}(1 - q^{4n})^{24}} = \sum_{n=1}^{\infty} a(6, 1; n) q^n,$$

and

$$q^2 \prod_{n=1}^{\infty} (1 - q^{2n})^{24} = \sum_{n=1}^{\infty} a(6, 2; n) q^n.$$

The formulae for $r_4(n)$ and $r_8(n)$ first occurred implicitly in the work of Jacobi [53]. Many proofs of these formulae are now known, see for example Carlitz [23], Ewell [40], Hirshhorn [50], Spearman and Williams [79], van der Pol [80] and Williams [82]. The formulae for $r_{12}(n)$ and $r_{16}(n)$ are due to Glaisher [43]. The formula for $r_{20}(n)$ has been treated by Rankin [76].

It is clear from the definition of $a(3, 1; n)$ that

$$a(3, 1; n) = 0, \text{ if } n \equiv 0 \pmod{2}.$$

Thus

$$(1.0.6) \quad r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4), \text{ if } n \equiv 0 \pmod{2}.$$

This elegant arithmetic formula was first given by Liouville [57]. A simple proof has recently been given by Williams [85].

It is known that

$$a(6, 1; n) = (-1)^{n-1} \tau(n)$$

and

$$a(6, 2; n) = \tau(n/2),$$

where $\tau(n)$ is Ramanujan's tau function given by

$$(1.0.7) \quad q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Thus we have Ramanujan's beautiful formula [71], [72, p. 162]

$$(1.0.8) \quad \begin{aligned} r_{24}(n) = & \frac{16}{691} \sigma_{11}(n) - \frac{32}{691} \sigma_{11}(n/2) + \frac{65536}{691} \sigma_{11}(n/4) \\ & + \frac{33152}{691} (-1)^{n-1} \tau(n) - \frac{65536}{691} \tau(n/2). \end{aligned}$$

Rankin [75] has proved for $k \geq 3$ that the Ramanujan-Mordell formula for $r_{4k}(n)$ must involve nonzero coefficients of modular forms. Liouville's formula (1.0.6) shows that this may not be the case if n is restricted to an arithmetic progression.

Our purpose in this thesis is to extend the Ramanujan-Mordell theorem to the representation of a positive interger n by the quadratic form

$$(1.0.9) \quad x_1^2 + \cdots + x_r^2 + 2x_{r+1}^2 + \cdots + 2x_{r+s}^2 + 4x_{r+s+1}^2 + \cdots + 4x_{r+s+t}^2,$$

where r, s, t are integers satisfying

$$(1.0.10) \quad r \geq 1, \ s \geq 0, \ t \geq 0, \ r + s + t = 4k, \ k \in \mathbb{N}.$$

The Ramanujan-Mordell theorem is the special case $r = 4k, s = 0, t = 0$. We write $[r, s, t]$ for the form (1.0.9) so that for example we have

$$[4, 2, 2] = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 4x_7^2 + 4x_8^2.$$

In addition we write $N([r, s, t]; n)$ for the number of representations of $n \in \mathbb{N}_0$ by the form $[r, s, t]$, that is

$$(1.0.11) \quad N([r, s, t]; n) = \text{card}\{(x_1, \dots, x_{4k}) \in \mathbb{Z}^{4k} \mid n = x_1^2 + \cdots + x_r^2$$

$$+ 2x_{r+1}^2 + \cdots + 2x_{r+s}^2 + 4x_{r+s+1}^2 + \cdots + 4x_{r+s+t}^2\},$$

so that $N([r, s, t]; 0) = 1$ and

$$N([4k, 0, 0]; n) = r_{4k}(n).$$

When $k = 1$ there are 10 forms of the the type (1.0.9). These are given in Table 1.1.

Table 1.1

quaternary quadratic forms	references
$x_1^2 + x_2^2 + x_3^2 + x_4^2$	$N([4, 0, 0]; n)$ [2, Theorem 1.6, p. 284]
$x_1^2 + x_2^2 + x_3^2 + 2x_4^2$	$N([3, 1, 0]; n)$ [84, p. 225]
$x_1^2 + x_2^2 + x_3^2 + 4x_4^2$	$N([3, 0, 1]; n)$ [2, Theorem 1.7, p. 284]
$x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2$	$N([2, 2, 0]; n)$ [2, Theorem 1.8, p. 284]
$x_1^2 + x_2^2 + 2x_3^2 + 4x_4^2$	$N([2, 1, 1]; n)$ [1, Proposition 4.3, p. 160]
$x_1^2 + x_2^2 + 4x_3^2 + 4x_4^2$	$N([2, 0, 2]; n)$ [2, Theorem 1.11, p. 285]
$x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2$	$N([1, 3, 0]; n)$ [84, p. 225]
$x_1^2 + 2x_2^2 + 2x_3^2 + 4x_4^2$	$N([1, 2, 1]; n)$ [2, Theorem 1.14, p. 285]
$x_1^2 + 2x_2^2 + 4x_3^2 + 4x_4^2$	$N([1, 1, 2]; n)$ [1, Proposition 4.3, p. 161]
$x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2$	$N([1, 0, 3]; n)$ [2, Theorem 1.18, p. 286]

The second column in Table 1.1 gives a reference to where the determination of $N([r, s, t]; n)$ ($r + s + t = 4$) is given. As these determinations are all known we may suppose that $k \geq 2$ in this thesis.

Examination of the evaluations for the 10 forms in Table 1.1 shows that for the 6 forms $[4, 0, 0]$, $[3, 0, 1]$, $[2, 2, 0]$, $[2, 0, 2]$, $[1, 2, 1]$, $[1, 0, 3]$ the evaluations only

involve $\sigma(n), \sigma(n/2), \sigma(n/4), \sigma(n/8)$ and $\sigma(n/16)$, whereas for the 4 forms $[3,1,0]$, $[2,1,1]$, $[1,3,0]$, $[1,1,2]$ the evaluations require the function $\sum_{d|n} \frac{n}{d} \left(\frac{2}{d}\right)$. These two sets of forms are distinguished by the fact that s is even for the first set and that s is odd for the second set. Thus it is unlikely that for $k \geq 2$ the evaluation of $N([r, s, t]; n)$ will only involve the divisor function unless we restrict s to be even.

Under the restrictions

$$r \geq 1, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = 4k, \quad k \geq 2, \quad s \text{ even},$$

we evaluate $N([r, s, t]; n)$ for all $n \in \mathbb{N}$ in Theorems 6.1.3, 6.2.3, 6.3.3. The evaluations are given in terms of $\sigma_{2k-1}(n), \sigma_{2k-1}(n/2), \sigma_{2k-1}(n/4), \sigma_{2k-1}(n/8)$ and $\sigma_{2k-1}(n/16)$ and the coefficients in the power series expansions of certain infinite products. In Chapter 7 the formulae are given explicitly for forms of the type (1.0.9) in 8 ($k = 2$), 12 ($k = 3$) and 16 ($k = 4$) variables. In Chapter 8 relationships between the coefficients δ in these formulae are determined and these are used in Chapter 9 to give simpler evaluations of the $N([r, s, t]; n)$. Several instances were found where $N([r, s, t]; n)$ can be evaluated just in terms of $\sigma_{2k-1}(n), \sigma_{2k-1}(n/2), \sigma_{2k-1}(n/4), \sigma_{2k-1}(n/8)$ and $\sigma_{2k-1}(n/16)$ for certain residue classes of n , in analogy with Liouville's result (1.0.6). For example we find that

$$\begin{aligned} & \text{card} \left\{ (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \in \mathbb{Z}^{12} \mid \right. \\ & \quad \left. n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + 2x_7^2 + 2x_8^2 + 2x_9^2 + 2x_{10}^2 + 2x_{11}^2 + 2x_{12}^2 \right\} \\ &= N([6, 6, 0]; n) = 1032 \sigma_5(n/4) - 1536 \sigma_5(n/8), \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \equiv 0 \pmod{4}$, see Theorem 9.2.2.

In the case of 16 variables we did not find any evaluations of $N([r, s, t]; n)$

$(r + s + t = 16)$ that only depend upon $\sigma_7(n), \sigma_7(n/2), \sigma_7(n/4), \sigma_7(n/8)$ and $\sigma_7(n/16)$. However we found many that depend only upon the coefficients in the power series expansion of a single infinite product. For example

$$\begin{aligned} \text{card} \left\{ (x_1, \dots, x_{16}) \in \mathbb{Z}^{16} \mid n = x_1^2 + \dots + x_{14}^2 + 2x_{15}^2 + 2x_{16}^2 \right\} &= N([14, 2, 0]; n) \\ &= \frac{26112}{17} \sigma_7(n/4) - \frac{253952}{17} \sigma_7(n/8) - \frac{554}{17} a(n), \end{aligned}$$

where

$$q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{32}}{(1 - q^n)^8 (1 - q^{4n})^8} = \sum_{n=1}^{\infty} a(n) q^n,$$

for all $n \in \mathbb{N}$ with $n \equiv 0 \pmod{4}$, see Theorem 9.3.1. When $k = 2$ a few of the evaluations of $N([r, s, t]; n)$ had been carried out by other authors, see for example [5] and [11]. In every case the results of this thesis agreed with those evaluations.

The method used in this thesis follows the ideas of Alaca, Alaca and Williams in their evaluation of $r_{4k}(n)$ [9]. However considerable modification of their argument must be made in order to extend their method to the evaluations of (1.0.11). We conclude this introduction by describing briefly the ideas involved.

We begin by introducing the theta function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

in Chapter 2 and giving some of its properties. Then we show that

$$\sum_{n=-\infty}^{\infty} N([r, s, t]; n) q^n = \varphi(q)^r \varphi(q^2)^s \varphi(q^4)^t,$$

see Theorem 2.8.1, so that in order to determine $N([r, s, t]; n)$ we require the coefficient of q^n in $\varphi(q)^r \varphi(q^2)^s \varphi(q^4)^t$. To do this we introduce the parameters

$$(1.0.12) \quad x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z = \varphi^2(q),$$

see Definition 2.3.2, and show that

$$\begin{aligned} \varphi(q) &= z^{1/2} \\ \varphi(q^2) &= \left(\frac{1 + (1-x)^{1/2}}{2} \right)^{1/2} z^{1/2}, \\ \varphi(q^4) &= \left(\frac{1 + (1-x)^{1/4}}{2} \right) z^{1/2}, \end{aligned}$$

see Theorem 2.5.2. Thus, for $r + s + t = 4k$, we have

$$(1.0.13) \quad \varphi(q)^r \varphi(q^2)^s \varphi(q^4)^t = \left(\frac{1 + (1-x)^{1/2}}{2} \right)^{s/2} \left(\frac{1 + (1-x)^{1/4}}{2} \right)^t z^{2k}.$$

This leads us to study the function $\left(\frac{1 + (1-x)^{1/2}}{2} \right)^{s/2} \left(\frac{1 + (1-x)^{1/4}}{2} \right)^t$, which is carried out in Chapter 3. In the sum of squares case considered by Alaca, Alaca and Williams [9], we have $s = t = 0$, and the analysis is much simpler. In order to find the coefficient of q^n ($n \in \mathbb{N}$) in the right hand side of (1.0.13) in terms of $\sigma_{2k-1}(n), \sigma_{2k-1}(n/2), \sigma_{2k-1}(n/4), \sigma_{2k-1}(n/8)$ and $\sigma_{2k-1}(n/16)$, we require the Eisenstein series

$$(1.0.14) \quad \left\{ \begin{aligned} E_k(q) &= 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \\ E_k(q^2) &= 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n/2) q^n, \\ E_k(q^4) &= 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n/4) q^n, \\ E_k(q^8) &= 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n/8) q^n, \\ E_k(q^{16}) &= 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n/16) q^n. \end{aligned} \right.$$

In Theorem 4.4.1 we show that

$$(1.0.15) \quad E_k(q^r) = e_{k,r}(x)z^{2k}, \quad k \geq 2, \quad r \in \{1, 2, 4, 8, 16\},$$

for some function $e_{k,r}(x)$, where x and z are given in (1.0.12).

In Section 2.10 Ramanujan's discriminant function

$$\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

is recalled and a generalized discriminant function is defined by

$$(1.0.16) \quad \begin{aligned} \Delta(a, b, c; q) &:= \Delta(q)^a \Delta(q^2)^b \Delta(q^4)^c \\ &= q^{a+2b+4c} \prod_{n=1}^{\infty} (1 - q^n)^{24a} (1 - q^{2n})^{24b} (1 - q^{4n})^{24c} \end{aligned}$$

for rational numbers a, b, c satisfying certain conditions, see Theorem 2.10.2. For the rational numbers a, b, c under consideration $\Delta(a, b, c; q)$ has a power series expansion

$$(1.0.17) \quad \Delta(a, b, c; q) = \sum_{n=0}^{\infty} \delta(a, b, c; n) q^n.$$

Moreover we show that $\Delta(a, b, c; q)$ is given in terms of the parameters x and z by

$$(1.0.18) \quad \Delta(a, b, c; q) = \frac{x^{a+2b+4c} (1-x)^{4a+2b+c} z^{12(a+b+c)}}{2^{4(a+2b+4c)}}.$$

In (1.0.13) and (1.0.15) the exponent of z is $2k$, so we will be interested in those $\Delta(a, b, c; q)$ for which $12(a + b + c) = 2k$, that is $a + b + c = k/6$. The main body of this thesis consists of showing that for a complex variable x there exist rational numbers $\alpha_1(r, s, t)$, $\alpha_2(r, s, t)$, $\alpha_4(r, s, t)$, $\alpha_8(r, s, t)$ and $\alpha_{16}(r, s, t)$ such that

$$\begin{aligned} &\left(\frac{1 + (1-x)^{1/2}}{2} \right)^{s/2} \left(\frac{1 + (1-x)^{1/4}}{2} \right)^t (r \geq 1, s \geq 0, t \geq 0, r + s + t = 4k, k \geq 2, s \text{ even}) \\ &= \alpha_1(r, s, t) e_{k,1}(x) + \alpha_2(r, s, t) e_{k,2}(x) + \alpha_4(r, s, t) e_{k,4}(x) \end{aligned}$$

$$\begin{aligned}
& + \alpha_8(r, s, t)e_{k,8}(x) + \alpha_{16}(r, s, t)e_{k,16}(x) \\
& + \text{linear combination of terms of the form } x^{a+2b+4c}(1-x)^{4a+2b+c} \text{ with } a+b+c = k/6.
\end{aligned}$$

Then, taking x and z as in (1.0.12), and multiplying by z^{2k} , we obtain by (1.0.13) and (1.0.15)

$$\begin{aligned}
& \varphi(q)^r \varphi(q^2)^s \varphi(q^4)^t \\
& = \alpha_1(r, s, t)E_k(q) + \alpha_2(r, s, t)E_k(q^2) + \alpha_4(r, s, t)E_k(q^4) \\
& \quad + \alpha_8(r, s, t)E_k(q^8) + \alpha_{16}(r, s, t)E_k(q^{16}) \\
& \quad + \text{linear combination of terms } \Delta(a, b, c; q) \text{ with } a+b+c = k/6.
\end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}$) in this identity, we obtain our extension of the Ramanujan-Mordell theorem, namely

$$\begin{aligned}
(1.0.19) \quad N([r, s, t]; n) & = \text{linear combination of } \sigma_{2k-1}(n), \sigma_{2k-1}(n/2), \sigma_{2k-1}(n/4), \\
& \quad \sigma_{2k-1}(n/8), \sigma_{2k-1}(n/16) \\
& \quad + \text{linear combination of } \delta(a, b, c; n) \text{ with } a+b+c = k/6.
\end{aligned}$$

The precise formulae are given in Theorems 6.1.3, 6.2.3 and 6.3.3. In Chapter 7 all such formulae are given when $k = 2, 3$ and 4 . In Chapter 8 relationships between the $\delta(a, b, c; n)$ ($a+b+c = k/6$) occurring in (1.0.19) are determined. These are used in Chapter 9 to simplify the formulae of Chapter 7.

Chapter 2

The theta function $\varphi(q)$ and the Eisenstein series $E_k(q)$

This chapter consists primarily of definitions and results from the literature. In particular we recall the definition of the theta function $\varphi(q)$, the Eisenstein series $E_k(q)$ and Ramanujan's discriminant function $\Delta(q)$, and we give those properties of these functions that we need in later chapters. Most of our work will be based on the results stated here.

2.1 The arithmetic function $\sigma_k(n)$

We begin by recalling the definition of the arithmetic function $\sigma_k(n)$.

Definition 2.1.1. For $k, n \in \mathbb{N}$ we set $\sigma_k(n) = \sum_{d|n} d^k$, where d runs through the positive integers dividing n . If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We also set $\sigma_1(n) = \sigma(n)$.

It is well known that $\sigma_k(n)$ is a multiplicative function of n for all k . We emphasize that if $n \in \mathbb{N}$, $a \in \mathbb{N}$ and a does not divide n then $n/a \notin \mathbb{N}$ and $\sigma_k(n/a) = 0$.

2.2 The Bernoulli numbers B_n

We recall the definition of the Bernoulli numbers. These numbers are used in the definition of the Eisenstein series $E_k(q)$.

Definition 2.2.1. *The Bernoulli numbers $\{B_n\}_{n \in \mathbb{N}_0}$ are defined by*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad x \in \mathbb{R}, |x| < 2\pi.$$

It is known that $B_{2k+1} = 0$ for $k \geq 1$. The first few values of the B_n are given in Table 2.2.1.

Table 2.2.1

n	B_n	n	B_n	n	B_n
0	1	8	-1/30	18	43867/798
1	-1/2	10	5/66	20	-174611/330
2	1/6	12	-691/2730	22	854513/138
4	-1/30	14	7/6	24	-236364091/2730
6	1/42	16	-3617/510	26	8553103/3

2.3 The theta function $\varphi(q)$

In this section we define the theta function $\varphi(q)$ and prove some of its main properties. We then use $\varphi(q)$ to define the parameters $x(q)$ and $z(q)$, which will play a major role later in the thesis.

We begin by finding the radius of convergence of a certain power series.

Theorem 2.3.1. *The radius of convergence of the complex power series*

$$1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{n^2} \text{ is equal to } 1.$$

Proof. Define the sequence $\{a_n\}_{n \in \mathbb{N}}$ by

$$a_n = \begin{cases} 1, & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

so that

$$\sum_{n=1}^{\infty} q^{n^2} = \sum_{n=1}^{\infty} a_n q^n.$$

The radius of convergence of the power series $\sum_{n=1}^{\infty} a_n q^n$ is given by (see for example [29, Theorem 1.3, p. 31])

$$\frac{1}{\limsup_{n \rightarrow +\infty} |a_n|^{-\frac{1}{n}}} = \frac{1}{1} = 1.$$

Thus $\varphi(q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{n^2}$ has radius of convergence 1. $\square$

We can now define the theta function $\varphi(q)$.

Definition 2.3.1. For $q \in \mathbb{C}$ such that $|q| < 1$ we define the theta function φ by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

For the rest of this thesis, we let

$$(2.3.1) \quad D := \{q \in \mathbb{C} \mid |q| < 1\}.$$

The next lemma gives three basic properties of the theta function φ .

Lemma 2.3.1. For $q \in D$, we have

- (i) $\varphi(q) + \varphi(-q) = 2\varphi(q^4)$,
- (ii) $\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2)$,
- (iii) $\varphi(q)\varphi(-q) = \varphi^2(-q^2)$.

Proof. All these identities are easy to prove.

(i) This is [19, eq. (3.6.1), p. 71].

(ii) This is [19, eq. (3.6.7), p. 72].

(iii) This is [19, eq. (1.3.32), p. 15]. □

Consider the function $f(q) = iq$. We note that $f(D) = D$. This implies that the composite function $\varphi(iq)$ is well-defined on D , and similarly for $\varphi(-iq)$. The next lemma relates the functions $\varphi(iq)$ and $\varphi(-iq)$ to the functions $\varphi(q)$ and $\varphi(-q)$.

Lemma 2.3.2. *For $q \in D$, we have*

$$(i) \quad \varphi(iq) = \left(\frac{\varphi(q) + \varphi(-q)}{2} \right) + i \left(\frac{\varphi(q) - \varphi(-q)}{2} \right),$$

$$(ii) \quad \varphi(-iq) = \left(\frac{\varphi(q) + \varphi(-q)}{2} \right) - i \left(\frac{\varphi(q) - \varphi(-q)}{2} \right).$$

Proof. We only prove (i) as the proof of (ii) is similar. Appealing to Definition 2.3.1 and Lemma 2.3.1(i), we obtain

$$\begin{aligned} \varphi(iq) &= \sum_{n=-\infty}^{\infty} (iq)^{n^2} = \sum_{n=-\infty}^{\infty} i^{n^2} q^{n^2} = \sum_{\substack{n=-\infty \\ n \equiv 0 \pmod{2}}}^{\infty} i^{n^2} q^{n^2} + \sum_{\substack{n=-\infty \\ n \equiv 1 \pmod{2}}}^{\infty} i^{n^2} q^{n^2} \\ &= \sum_{n=-\infty}^{\infty} q^{4n^2} + i \sum_{\substack{n=-\infty \\ n \equiv 1 \pmod{2}}}^{\infty} q^{n^2} = \varphi(q^4) + i \left(\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{\substack{n=-\infty \\ n \equiv 0 \pmod{2}}}^{\infty} q^{n^2} \right) \\ &= \varphi(q^4) + i(\varphi(q) - \varphi(q^4)) = \left(\frac{\varphi(q) + \varphi(-q)}{2} \right) + i \left(\frac{\varphi(q) - \varphi(-q)}{2} \right), \end{aligned}$$

as asserted. □

We conclude this section with the definitions of the very important quantities $x(q)$ and $z(q)$.

Definition 2.3.2. For $q \in D$, we define

$$x = x(q) := 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}$$

and

$$z = z(q) := \varphi^2(q).$$

We remark that in Theorem 2.4.2 we show that $\varphi(q) \neq 0$ for $q \in D$.

2.4 Analytic properties of $\varphi(q)$

A function $f(q)$, which is defined on D and differentiable at every point of D save possibly for a finite number of exceptional points, is said to be analytic on D . The exceptional points are called the singularities of f . If, however, no point of D is a singularity of the analytic function f , we say that f is regular on D .

Analytic properties of the functions $\varphi(q)$, $\varphi(-q)$, $\varphi(iq)$ and $\varphi(-iq)$ are stated and proved in this section. We first determine the domain where these functions are regular.

Theorem 2.4.1. (i) $\varphi(q)$ is a regular function on D .

(ii) $\varphi(-q)$ is a regular function on D .

(iii) $\varphi(iq)$ is a regular function on D .

(iv) $\varphi(-iq)$ is a regular function on D .

Proof. (i) This follows from the fact that a power series is regular within its circle of convergence, see for example [29, Corollary (2.9), p. 37] or [30, p. 38].

- (ii) As the function $f(q) = -q$ is regular on $\mathbb{C}$ (so on D) and $f(D) = D$, we have that the composite function $\varphi(f(q)) = \varphi(-q)$ is regular on D .
- (iii) and (iv) are proved in the same way as (ii) with the functions $g(q) = iq$ and $h(q) = -iq$ respectively. $\square$

Next we show that all four functions are non-vanishing on D .

Theorem 2.4.2. *For $q \in D$, we have*

- (i) $\varphi(q) \neq 0$,
- (ii) $\varphi(-q) \neq 0$,
- (iii) $\varphi(iq) \neq 0$,
- (iv) $\varphi(-iq) \neq 0$.

Proof. (i) Suppose that there exists $q_0 \in D$ with $\varphi(q_0) = 0$. As $\varphi(0) = 1 \neq 0$ we have $q_0 \neq 0$. Set $q_1 = -q_0^2$. Clearly $|q_1| = |-q_0^2| = |q_0|^2 < 1$ so $q_1 \in D$. Also by Lemma 2.3.1 (iii) $\varphi^2(q_1) = \varphi^2(-q_0^2) = \varphi(q_0)\varphi(-q_0) = 0$ so $\varphi(q_1) = 0$. Clearly $q_1 \neq q_0$.

Next we define $q_2 = -q_1^2$. Clearly $|q_2| = |-q_1^2| = |q_1|^2 < 1$ so $q_2 \in D$. Also $\varphi^2(q_2) = \varphi^2(-q_1^2) = \varphi(q_1)\varphi(-q_1) = 0$ so $\varphi(q_2) = 0$. Clearly $q_2 \neq q_0, q_1$.

Continuing in this way we obtain an infinite sequence $q_0, q_1, q_2, \dots$ of zeros of $\varphi(q)$ in D having limit point 0 ($\in D$). Thus, as φ is a regular function on D , $\varphi(q)$ vanishes identically on D . This is a contradiction as $\varphi(0) = 1 \neq 0$. Hence there is no point of D at which φ vanishes.

- (ii)(iii)(iv) These follow from (i) and the fact that $q \in D \Leftrightarrow -q \in D \Leftrightarrow iq \in D \Leftrightarrow -iq \in D$. $\square$

We next deduce the following important results.

Theorem 2.4.3. *Let $x = x(q)$ be the function defined in Definition 2.3.2 for $q \in D$. Then the functions $1 - x(q)$, $1 - x(iq)$, $\frac{\varphi(q)}{\varphi(-q)}$, $\frac{\varphi(iq)}{\varphi(-iq)}$ are regular on D .*

Proof. By Theorem 2.4.1, the functions $\varphi(q)$ and $\varphi(-q)$ are regular on D . Hence $\varphi^4(q)$ and $\varphi^4(-q)$ are regular on D . Thus $\frac{\varphi^4(-q)}{\varphi^4(q)}$ is analytic on D . But, by Theorem 2.4.2, $\varphi(q) \neq 0$ on D . Hence $1 - x(q) = \frac{\varphi^4(-q)}{\varphi^4(q)}$ is regular on D . The rest can be proved similarly. $\square$

The next result gives the region in $\mathbb{C}$ where the complex function $f(q) = q^{1/4}$ is regular.

Theorem 2.4.4. *The function $f(q) = q^{1/4}$ is regular on $\mathbb{C} \setminus \{q \in \mathbb{C} \mid \operatorname{Re} q \leq 0 \text{ and } \operatorname{Im} q = 0\}$.*

Proof. The result can be found in [30, p. 49]. $\square$

The following lemma is useful in the proof of Lemma 2.4.2.

Lemma 2.4.1. *For $q \in D$, let $S = \operatorname{Re} \left(\frac{\varphi(-q)}{\varphi(q)} \right)$ and $T = \operatorname{Im} \left(\frac{\varphi(-q)}{\varphi(q)} \right)$. Then $\operatorname{Re}(1 - x(q)) > 0$ if and only if $S^4 - 6S^2T^2 + T^4 > 0$.*

Proof. As $\frac{\varphi(-q)}{\varphi(q)} = S + iT$, we have

$$1 - x(q) = \frac{\varphi^4(-q)}{\varphi^4(q)} = (S + iT)^4 = (S^4 - 6S^2T^2 + T^4) + i(4S^3T - 4ST^3).$$

Hence $\operatorname{Re}(1 - x(q)) = S^4 - 6S^2T^2 + T^4$ and the result follows. $\square$

Our next goal is to find a subset of D on which the functions $(1 - x(q))^{1/4}$ and $(1 - x(iq))^{1/4}$ are regular. The following results are very important in order to reach that goal.

Lemma 2.4.2. *There exists $\delta \in \mathbb{R}$ with $0 < \delta \leq 1$ such that*

(i) $\operatorname{Re} (1 - x(q)) > 0$, for all $q \in D_\delta$,

(ii) $\operatorname{Re} (1 - x(iq)) > 0$, for all $q \in D_\delta$,

where $D_\delta := \{q \in \mathbb{C} \mid |q| < \delta\}$.

Proof. (i) By Theorem 2.4.3 the function $\frac{\varphi(-q)}{\varphi(q)}$ is regular on D . Thus $\frac{\varphi(-q)}{\varphi(q)}$ is continuous at $q = 0$. Moreover $\left. \frac{\varphi(-q)}{\varphi(q)} \right|_{q=0} = \frac{\varphi(0)}{\varphi(0)} = 1$. Thus there exists $\delta \in \mathbb{R}$ with $0 < \delta \leq 1$ such that

$$\left| \frac{\varphi(-q)}{\varphi(q)} - 1 \right| < \frac{1}{8}$$

for $|q| < \delta$. Then, with $S = \operatorname{Re} \left(\frac{\varphi(-q)}{\varphi(q)} \right)$ and $T = \operatorname{Im} \left(\frac{\varphi(-q)}{\varphi(q)} \right)$, we have

$$\left| (S + iT) - 1 \right| < \frac{1}{8} \quad \text{for } |q| < \delta.$$

Therefore $|S - 1| < \frac{1}{8}$, $|T| < \frac{1}{8}$, for $|q| < \delta$. Thus $\frac{7}{8} < S < \frac{9}{8}$, $-\frac{1}{8} < T < \frac{1}{8}$, for $|q| < \delta$. Then

$$S^4 - 6S^2T^2 + T^4 > \left(\frac{7}{8}\right)^4 - 6 \cdot \frac{9^2}{8^2} \cdot \frac{1}{8^2} = \frac{7^4 - 6 \cdot 81}{8^4} = \frac{2401 - 486}{8^4} > 0.$$

Hence, by Lemma 2.4.1 this implies that $\operatorname{Re} (1 - x(q)) > 0$.

(ii) If $f(q) = iq$, we have $f(D_\delta) = D_\delta$. Thus, from (i) we have that the composite function $1 - x(iq)$ is such that $\operatorname{Re} (1 - x(iq)) > 0$ for $|q| < \delta$. $\square$

For the remainder of this thesis we set

$$D_\delta := \{q \in \mathbb{C} \mid |q| < \delta\} \subseteq D,$$

where δ is the constant determined by Lemma 2.4.2.

We now answer our questions concerning the analyticity of the functions $(1 - x(q))^{1/4}$ and $(1 - x(iq))^{1/4}$.

Theorem 2.4.5. *The functions $(1 - x(q))^{1/4}$ and $(1 - x(iq))^{1/4}$ are regular on D_δ .*

Proof. By Theorem 2.4.3(i) the function $1 - x(q)$ is regular on D_δ , and by Lemma 2.4.2(i)

$$\{1 - x(q) \mid q \in D_\delta\} \subseteq \mathbb{C} \setminus \{q \in \mathbb{C} \mid \operatorname{Re} q < 0, \operatorname{Im} q = 0\}.$$

This implies that the composite function $(1 - x(q))^{1/4}$ is a regular analytic function on D_δ , see [29, § 2.4 p. 34]. Similarly for $(1 - x(iq))^{1/4}$. $\square$

We are now ready to state and prove the main results of this section.

Theorem 2.4.6. *For $q \in D_\delta$, we have*

$$\begin{aligned} \text{(i)} \quad (1 - x(q))^{1/4} &= (1 - x)^{1/4} = \frac{\varphi(-q)}{\varphi(q)}, \\ \text{(ii)} \quad (1 - x(iq))^{1/4} &= \frac{\varphi(-iq)}{\varphi(iq)} = -i \left(\frac{1 + i(1 - x)^{1/4}}{1 - i(1 - x)^{1/4}} \right). \end{aligned}$$

Proof. (i) By Definition 2.3.2 we have

$$(2.4.1) \quad (1 - x(q))^{1/4} = \left(\frac{\varphi^4(-q)}{\varphi^4(q)} \right)^{1/4} = \alpha(q) \frac{\varphi(-q)}{\varphi(q)},$$

where $\alpha(q) = 1, i, -1$ or $-i$. Now $\varphi(0) = 1$ so $x(0) = 0$. Taking $q = 0$ in (2.4.1) we obtain $\alpha(0) = 1^{1/4} = 1$. In D_δ we have by Theorems 2.4.3 and 2.4.5 that

$$\alpha(q) = \frac{\varphi(q)}{\varphi(-q)} (1 - x(q))^{1/4}$$

is regular in D_δ . Thus $\alpha(q)$ is continuous on D_δ . But $0 \in D_\delta$, $\alpha(0) = 1$ and $\alpha(q) = 1, i, -1$ or $-i$, so $\alpha(q) = 1$ for all $q \in D_\delta$. Thus

$$(1 - x(q))^{1/4} = \frac{\varphi(-q)}{\varphi(q)} \quad \text{for all } q \in D_\delta.$$

(ii) Using the same argument as (i) and appealing to Theorems 2.4.3 and 2.4.5, we find that $(1 - x(iq))^{1/4} = \frac{\varphi(-iq)}{\varphi(iq)}$.

By Lemmas 2.3.2 (i)(ii) and 2.3.1(i), we have

$$\begin{aligned}
 \frac{\varphi(-iq)}{\varphi(iq)} &= \frac{\varphi(q^4) - i(\varphi(q) - \varphi(q^4))}{\varphi(q^4) + i(\varphi(q) - \varphi(q^4))} = \frac{\frac{\varphi(q) + \varphi(-q)}{2} - i\left(\frac{\varphi(q) - \varphi(-q)}{2}\right)}{\frac{\varphi(q) + \varphi(-q)}{2} + i\left(\frac{\varphi(q) - \varphi(-q)}{2}\right)} \\
 &= \frac{\varphi(q) + \varphi(-q) - i(\varphi(q) - \varphi(-q))}{\varphi(q) + \varphi(-q) + i(\varphi(q) - \varphi(-q))} = \frac{1 + \frac{\varphi(-q)}{\varphi(q)} - i\left(1 - \frac{\varphi(-q)}{\varphi(q)}\right)}{1 + \frac{\varphi(-q)}{\varphi(q)} + i\left(1 - \frac{\varphi(-q)}{\varphi(q)}\right)} \\
 &= \frac{(1-i) + (1+i)\frac{\varphi(-q)}{\varphi(q)}}{(1+i) + (1-i)\frac{\varphi(-q)}{\varphi(q)}} = \frac{1-i}{1+i} \left(\frac{1 + i\frac{\varphi(-q)}{\varphi(q)}}{1 - i\frac{\varphi(-q)}{\varphi(q)}} \right) = -i \left(\frac{1 + i(1-x)^{1/4}}{1 - i(1-x)^{1/4}} \right),
 \end{aligned}$$

where the last equality was obtained by using (i). This is the desired result. $\square$

2.5 The duplication principle

If $q \in D_\delta$ then $|q| < \delta$. Thus, as $\delta \leq 1$, we have $|q^2| = |q|^2 < |q| < \delta$ so that $q^2 \in D_\delta$. Now suppose we have $q \in D_\delta$ and we transform q into q^2 . How are $x(q^2)$ and $z(q^2)$ related to $x(q)$ and $z(q)$? The formulae giving this relationship comprise the “duplication principle”, which was first discovered by Jacobi and is of crucial importance for the remainder of our work.

Theorem 2.5.1. (*Duplication principle*) For $q \in D_\delta$ we have

$$x(q^2) = \left(\frac{1 - (1 - x(q))^{1/2}}{1 + (1 + x(q))^{1/2}} \right)^2$$

and

$$z(q^2) = \left(\frac{1 + (1 - x(q))^{1/2}}{2} \right) z(q).$$

That is, if $q \rightarrow q^2$ then $x \rightarrow \left(\frac{1 - (1 - x)^{1/2}}{1 + (1 - x)^{1/2}} \right)^2$ and $z \rightarrow \left(\frac{1 + (1 - x)^{1/2}}{2} \right) z$.

Proof. By Definition 2.3.2, Lemma 2.3.1 (ii)(iii) and Theorem 2.4.6(i), we have

$$x(q^2) = 1 - \frac{\varphi^4(-q^2)}{\varphi^4(q^2)} = 1 - \frac{\varphi^2(q)\varphi^2(-q)}{\left(\frac{\varphi^2(q) + \varphi^2(-q)}{2}\right)^2}$$

$$\begin{aligned}
 &= \frac{(\varphi^2(q) + \varphi^2(-q))^2 - 4\varphi^2(q)\varphi^2(-q)}{(\varphi^2(q) + \varphi^2(-q))^2} = \frac{(\varphi^2(q) - \varphi^2(-q))^2}{(\varphi^2(q) + \varphi^2(-q))^2} \\
 &= \left(\frac{1 - \frac{\varphi^2(-q)}{\varphi^2(q)}}{1 + \frac{\varphi^2(-q)}{\varphi^2(q)}} \right)^2 = \left(\frac{1 - (1 - x(q))^{1/2}}{1 + (1 - x(q))^{1/2}} \right)^2.
 \end{aligned}$$

By Definition 2.3.2, Lemma 2.3.1(ii) and Theorem 2.4.6(i) we have

$$z(q^2) = \varphi^2(q^2) = \frac{\varphi^2(q) + \varphi^2(-q)}{2} = \frac{\left(1 + \frac{\varphi^2(-q)}{\varphi^2(q)}\right) \varphi^2(q)}{2} = \left(\frac{1 + (1 - x(q))^{1/2}}{2} \right) z(q).$$

This completes the proof of the duplication principle. $\square$

Next, we use the duplication principle to find expressions for $\varphi(q^2)$ and $\varphi(q^4)$ in terms of x and z as defined in Definition 2.3.2.

Theorem 2.5.2. *For $q \in D_\delta$ and x and z as defined in Definition 2.3.2, we have*

- (i) $\varphi(q) = z^{1/2}$,
- (ii) $\varphi(q^2) = \left(\frac{1 + (1 - x)^{1/2}}{2} \right)^{1/2} z^{1/2}$,
- (iii) $\varphi(q^4) = \left(\frac{1 + (1 - x)^{1/4}}{2} \right) z^{1/2}$.

Proof. From Definition 2.3.2 we have $\varphi(q) = z^{1/2}$. Applying Theorem 2.5.1 to (i), we obtain (ii). Applying Theorem 2.5.1 to (ii), we obtain (iii). $\square$

2.6 The change of sign principle

Similarly, if $q \in D_\delta$ then $-q \in D_\delta$, and suppose we transform q into $-q$. How are $x(-q)$ and $z(-q)$ related to $x(q)$ and $z(q)$? This result, also discovered by Jacobi, is called the “change of sign principle”.

Theorem 2.6.1. (*Change of sign principle*) For $q \in D_\delta$ we have

$$x(-q) = \frac{x(q)}{x(q) - 1}$$

and

$$z(-q) = (1 - x(q))^{1/2} z(q).$$

That is, if $q \rightarrow -q$ then $x \rightarrow \frac{x}{x-1}$ and $z \rightarrow (1-x)^{1/2} z$.

Proof. By Definition 2.3.2 we have

$$\begin{aligned} x(-q) &= 1 - \frac{\varphi^4(q)}{\varphi^4(-q)} = \frac{\varphi^4(-q) - \varphi^4(q)}{\varphi^4(-q)} = \frac{\frac{\varphi^4(-q) - \varphi^4(q)}{\varphi^4(q)}}{\frac{\varphi^4(-q)}{\varphi^4(q)}} \\ &= \frac{(1 - x(q)) - 1}{1 - x(q)} = \frac{-x(q)}{1 - x(q)} = \frac{x(q)}{x(q) - 1}. \end{aligned}$$

By Definition 2.3.2 and Theorem 2.4.6(i) we have

$$z(-q) = \varphi^2(-q) = \frac{\varphi^2(-q)}{\varphi^2(q)} \varphi^2(q) = (1 - x(q))^{1/2} z(q).$$

This completes the proof of the change of sign principle. $\square$

2.7 The rotation principle

If $q \in D_\delta$ then $iq \in D_\delta$. Our final result of this section determines what happens to $x(q)$ and $z(q)$ when q is transformed into iq for $q \in D_\delta$. This result is called the “rotation principle” and is due to Cheng [27].

Theorem 2.7.1. (*Rotation principle*) For $q \in D_\delta$ we have

$$x(iq) = \frac{-8i(1 - x(q))^{1/4} (1 - (1 - x(q))^{1/2})}{(1 - i(1 - x(q))^{1/4})^4}$$

and

$$z(iq) = i \frac{z(q)}{2} (1 - i(1 - x(q))^{1/4})^2.$$

That is, if $q \rightarrow iq$ then $x \rightarrow \frac{-8i(1-x)^{1/4}(1-(1-x)^{1/2})}{(1-i(1-x)^{1/4})^4}$ and $z \rightarrow i \frac{z}{2} (1-i(1-x)^{1/4})^2$.

Proof. Cheng's proof is given in [27]. Here we give a new proof based on some of the results that we proved earlier.

By Definition 2.3.2 and Theorem 2.4.6(ii), we have

$$\begin{aligned} x(iq) &= 1 - \frac{\varphi^4(-iq)}{\varphi^4(iq)} = 1 - \left(-i \left(\frac{1+i(1-x)^{1/4}}{1-i(1-x)^{1/4}} \right) \right)^4 \\ &= 1 - \frac{(1+i(1-x)^{1/4})^4}{(1-i(1-x)^{1/4})^4} = \frac{(1-i(1-x)^{1/4})^4 - (1+i(1-x)^{1/4})^4}{(1-i(1-x)^{1/4})^4} \\ &= \frac{-8i(1-x)^{1/4} + 8i(1-x)^{3/4}}{(1-i(1-x)^{1/4})^4} = \frac{-8i(1-x)^{1/4}(1-(1-x)^{1/2})}{(1-i(1-x)^{1/4})^4}. \end{aligned}$$

By Definition 2.3.2, Lemma 2.3.2(i) and Theorem 2.4.6(i), we have

$$\begin{aligned} z(iq) &= \varphi^2(iq) = \left(\left(\frac{\varphi(q) + \varphi(-q)}{2} \right) + i \left(\frac{\varphi(q) - \varphi(-q)}{2} \right) \right)^2 \\ &= \frac{1}{4} ((\varphi(q) + \varphi(-q)) + i(\varphi(q) - \varphi(-q)))^2 \\ &= \frac{1}{4} \left((\varphi(q) + \varphi(-q))^2 - (\varphi(q) - \varphi(-q))^2 + 2i(\varphi(q) + \varphi(-q))(\varphi(q) - \varphi(-q)) \right) \\ &= \frac{1}{4} \left(4\varphi(q)\varphi(-q) + 2i(\varphi^2(q) - \varphi^2(-q)) \right) = \varphi(q)\varphi(-q) + \frac{i}{2}(\varphi^2(q) - \varphi^2(-q)) \\ &= \left(\frac{\varphi(-q)}{\varphi(q)} + \frac{i}{2} \left(1 - \frac{\varphi^2(-q)}{\varphi^2(q)} \right) \right) \varphi^2(q) = \frac{i}{2} z (-2i(1-x)^{1/4} + (1-(1-x)^{1/2})) \\ &= \frac{i}{2} z (1-i(1-x)^{1/4})^2. \end{aligned}$$

This completes the proof of the rotation principle. □

2.8 The arithmetic function $N([a, b, c]; n)$

The central problem that we address in this thesis is that of determining the number of representations of a positive integer n by a quadratic form of the type

$$x_1^2 + \cdots + x_a^2 + 2x_{a+1}^2 + \cdots + 2x_{a+b}^2 + 4x_{a+b+1}^2 + \cdots + 4x_t^2,$$

where $a \in \mathbb{N}$, $b, c \in \mathbb{N}_0$, $b \equiv 0 \pmod{2}$, b or $c \in \mathbb{N}$ and $t = a + b + c \equiv 0 \pmod{4}$. To this end we make the following definition.

Definition 2.8.1. For $a, b, c, n \in \mathbb{N}_0$ and $t = a + b + c \geq 1$, we define

$$N([a, b, c]; n) :=$$

$$\text{card} \{ (x_1, \dots, x_t) \in \mathbb{Z}^t \mid n = x_1^2 + \cdots + x_a^2 + 2x_{a+1}^2 + \cdots + 2x_{a+b}^2 + 4x_{a+b+1}^2 + \cdots + 4x_t^2 \}.$$

We note that as $(0, \dots, 0) \in \mathbb{Z}^t$ is the only solution to

$$0 = x_1^2 + \cdots + x_a^2 + 2x_{a+1}^2 + \cdots + 2x_{a+b}^2 + 4x_{a+b+1}^2 + \cdots + 4x_t^2,$$

we have $N([a, b, c]; 0) = 1$.

The next theorem expresses the series $1 + \sum_{n=1}^{\infty} N([a, b, c]; n)q^n$ in terms of the theta function φ .

Theorem 2.8.1. For $q \in \mathbb{C}$ such that $|q| < 1$ and $a, b, c \in \mathbb{N}_0$ with $a + b + c \geq 1$, we have

$$\varphi(q)^a \varphi(q^2)^b \varphi(q^4)^c = 1 + \sum_{n=1}^{\infty} N([a, b, c]; n)q^n.$$

Proof. We have by Definition 2.3.1

$$\varphi(q)^a \varphi(q^2)^b \varphi(q^4)^c = \left(\sum_{k=-\infty}^{\infty} q^{k^2} \right)^a \left(\sum_{l=-\infty}^{\infty} q^{2l^2} \right)^b \left(\sum_{m=-\infty}^{\infty} q^{4m^2} \right)^c$$

$$\begin{aligned}
 &= \left(\sum_{k_1, \dots, k_a = -\infty}^{\infty} q^{k_1^2 + \dots + k_a^2} \right) \left(\sum_{l_1, \dots, l_b = -\infty}^{\infty} q^{2l_1^2 + \dots + 2l_b^2} \right) \left(\sum_{m_1, \dots, m_c = -\infty}^{\infty} q^{4m_1^2 + \dots + 4m_c^2} \right) \\
 &= \sum_{k_1, \dots, k_a, l_1, \dots, l_b, m_1, \dots, m_c = -\infty}^{\infty} q^{k_1^2 + \dots + k_a^2 + 2l_1^2 + \dots + 2l_b^2 + 4m_1^2 + \dots + 4m_c^2} \\
 &= \sum_{n=0}^{\infty} \sum_{\substack{k_1, \dots, k_a, l_1, \dots, l_b, m_1, \dots, m_c = -\infty \\ k_1^2 + \dots + k_a^2 + 2l_1^2 + \dots + 2l_b^2 + 4m_1^2 + \dots + 4m_c^2 = n}} q^n \\
 &= \sum_{n=0}^{\infty} N([a, b, c]; n) q^n = 1 + \sum_{n=1}^{\infty} N([a, b, c]; n) q^n,
 \end{aligned}$$

as asserted. □

2.9 The Eisenstein series $E_k(q)$

We begin this section by defining the (normalized) Eisenstein series $E_k(q)$ ($k \in \mathbb{N}$, $q \in \mathbb{C}$, $|q| < 1$) of weight $2k$.

Definition 2.9.1. For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we define the Eisenstein series $E_k(q)$ by

$$E_k(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

We note that $E_k(0) = 1$. For $k \geq 2$ E_k is a modular form of weight $2k$ on the full modular group. On the other hand E_1 is not a modular form.

A very important result for our work is the following result of Ramanujan, which was probably known to Weierstrass.

Theorem 2.9.1. *For $k \in \mathbb{N}$ with $k \geq 2$ and $q \in \mathbb{C}$ with $|q| < 1$ there exist rational numbers $a_k(t, u)$ which depend upon t and u but not on q such that*

$$E_k(q) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) E_2(q)^t E_3(q)^u.$$

Proof. See Berndt [19, eq. (4.2.40), p. 96] or Kilford [54, Prop. 3.6, p. 48]. $\square$

It is known that the rational numbers $a_k(t, u)$ are all positive, see for example Apostol [14, Theorem 1.13 and Ex. 1.11].

Taking $q = 0$ in Theorem 2.9.1 we find that

$$(2.9.1) \quad \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) = 1, \quad k \in \mathbb{N}, \quad k \geq 2.$$

Using Definition 2.9.1 in Theorem 2.9.1, and equating small powers of q , we obtain the values of the $a_k(t, u)$ ($t, u \in \mathbb{N}_0$, $2t + 3u = k$) for $2 \leq k \leq 10$.

Table 2.4.1

k	$a_k(t, u)$
2	$a_2(1, 0) = 1$
3	$a_3(0, 1) = 1$
4	$a_4(2, 0) = 1$
5	$a_5(1, 1) = 1$
6	$a_6(0, 2) = \frac{250}{691}, \quad a_6(3, 0) = \frac{441}{691}$
7	$a_7(2, 1) = 1$

8	$a_8(1, 2) = \frac{2000}{3617}, \quad a_8(4, 0) = \frac{1617}{3617}$
9	$a_9(0, 3) = \frac{5500}{43867}, \quad a_9(3, 1) = \frac{38367}{43867}$
10	$a_{10}(2, 2) = \frac{121250}{174611}, \quad a_{10}(5, 0) = \frac{53361}{174611}$

We now state three theorems that express $E_k(q^r)$, where $k \in \{1, 2, 3\}$ and $r \in \{1, 2, 4\}$, in terms of x and z .

Theorem 2.9.2. *For $q \in \mathbb{C}$ with $|q| < 1$ define x and z as in Definition 2.3.2. Then*

$$(2.9.2) \quad E_1(q) = (1 - 5x)z^2 + 12x(1 - x)z \frac{dz}{dx},$$

$$(2.9.3) \quad E_1(q^2) = (1 - 2x)z^2 + 6x(1 - x)z \frac{dz}{dx},$$

$$(2.9.4) \quad E_1(q^4) = \left(1 - \frac{5}{4}x\right)z^2 + 3x(1 - x)z \frac{dz}{dx}.$$

Proof. Formulae (2.9.2) and (2.9.3) can be found in [19, Theorem 5.4.9, p. 125]. Formula (2.9.4) can be found in [19, Exercise 5.4.13, p. 128]. $\square$

Theorem 2.9.3. *For $q \in \mathbb{C}$ with $|q| < 1$ define x and z as in Definition 2.3.2. Then*

$$(2.9.5) \quad E_2(q) = (1 + 14x + x^2)z^4,$$

$$(2.9.6) \quad E_2(q^2) = (1 - x + x^2)z^4,$$

$$(2.9.7) \quad E_2(q^4) = \left(1 - x + \frac{1}{16}x^2\right)z^4.$$

Proof. Formulae (2.9.5) and (2.9.6) can be found in [19, Theorem 5.4.11, p. 126]. Formula (2.9.7) can be found in [19, Exercise 5.4.13, p. 128]. $\square$

Theorem 2.9.4. *For $q \in \mathbb{C}$ with $|q| < 1$ define x and z as in Definition 2.3.2. Then*

$$(2.9.8) \quad E_3(q) = (1 - 33x - 33x^2 + x^3)z^6,$$

$$(2.9.9) \quad E_3(q^2) = \left(1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3\right)z^6,$$

$$(2.9.10) \quad E_3(q^4) = \left(1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3\right)z^6.$$

Proof. Formulae (2.9.8) and (2.9.9) can be found in [19, Theorem 5.4.12, p. 127]. Formula (2.9.10) can be found in [19, Exercise 5.4.13, p. 128]. $\square$

Next we are interested in expressing the Eisenstein series $E_1(q^8)$ and $E_1(q^{16})$ in terms of x , z and $\frac{dz}{dx}$. In order to do this, we first prove the following simple lemma.

Lemma 2.9.1. *For $q \in \mathbb{C}$ with $|q| < 1$, define x and z as in Definition 2.3.2. Then*

$$2E_1(q^4) - E_1(q^2) = \left(1 - \frac{1}{2}x\right)z^2.$$

Proof. This result follows immediately from (2.9.3) and (2.9.4). $\square$

We now use Lemma 2.9.1 to prove the following theorem.

Theorem 2.9.5. *For $q \in \mathbb{C}$ with $|q| < 1$ define x and z as in Definition 2.3.2. Then for $q \in D_\delta$ we have*

$$E_1(q^8) = \left(\frac{5}{8} - \frac{11}{16}x + \frac{3}{8}(1-x)^{1/2}\right)z^2 + \frac{3}{2}x(1-x)z\frac{dz}{dx},$$

$$E_1(q^{16}) = \left(\frac{11}{32} - \frac{23}{64}x + \frac{3}{16}(1-x)^{1/4} + \frac{9}{32}(1-x)^{1/2} + \frac{3}{16}(1-x)^{3/4} \right) z^2 + \frac{3}{4}x(1-x)z \frac{dz}{dx}.$$

Proof. Using the duplication principle (Theorem 2.5.1) in Lemma 2.9.1, we obtain for $q \in D_\delta$

$$(2.9.11) \quad 2E_1(q^8) - E_1(q^4) = \left(\frac{1}{4} + \frac{3}{4}(1-x)^{1/2} - \frac{x}{8} \right) z^2.$$

Replacing $E_1(q^4)$ by its value given in (2.9.4), and simplifying, we obtain the desired result for $E_1(q^8)$.

Using the duplication principle in (2.9.11), we obtain for $q \in D_\delta$

$$2E_1(q^{16}) - E_1(q^8) = \left(\frac{1}{16} - \frac{1}{32}x + \frac{3}{8}(1-x)^{1/4} + \frac{3}{16}(1-x)^{1/2} + \frac{3}{8}(1-x)^{3/4} \right) z^2.$$

Replacing $E_1(q^8)$ by its value given in the first part of Theorem 2.9.5, and simplifying, we obtain the desired result for $E_1(q^{16})$. $\square$

In the next theorem we obtain new identities involving the Eisenstein series $E_1(q)$ in terms of x , z and $\frac{dz}{dx}$.

Theorem 2.9.6. *For $q \in D_\delta$, x and z as defined in Definition 2.3.2 and the Eisenstein series $E_1(q)$, we have*

- (i) $2E_1(q^2) - E_1(q) = (1+x)z^2$
- (ii) $E_1(-q) = (1-2x)z^2 + 12x(1-x)z \frac{dz}{dx}$
- (iii) $E_1(q) - E_1(-q) = -3xz^2$

Proof. (i) Follows from Theorem 2.9.2.

(ii) Using the change of sign principle (Theorem 2.6.1) in (i), we obtain for $q \in D_\delta$

$$2E_1(q^2) - E_1(-q) = \left(1 + \frac{x}{x-1} \right) (1-x)z^2 = (1-2x)z^2$$

Replacing $E_1(q^2)$ by its value given in (2.9.3), and simplifying, we obtain the desired result for $E_1(-q)$.

(iii) This follows from (2.9.2) and (ii). $\square$

We conclude this section with two theorems that express the Eisenstein series $E_2(q^8)$, $E_2(q^{16})$, $E_3(q^8)$ and $E_3(q^{16})$ in terms of x and z .

Theorem 2.9.7. *For $q \in \mathbb{C}$ with $|q| < 1$ define x and z as in Definition 2.3.2. Then for $q \in D_\delta$ we have*

$$(2.9.12) \quad E_2(q^8) = \frac{1}{256}(136 - 136x + x^2 + (120 - 60x)(1 - x)^{1/2}) z^4,$$

$$(2.9.13) \quad E_2(q^{16}) = \frac{1}{4096} \left(p_2(x) + q_2(x)(1 - x)^{1/4} + r_2(x)(1 - x)^{1/2} + s_2(x)(1 - x)^{3/4} \right) z^4,$$

where

$$p_2(x) = 1096 - 1096x + x^2,$$

$$q_2(x) = 960 - 840x,$$

$$r_2(x) = 1080 - 540x,$$

$$s_2(x) = 960 - 120x.$$

Proof. The formulae follow by applying Theorem 2.5.1 to $E_2(q^4)$ and $E_2(q^8)$. $\square$

Theorem 2.9.8. *For $q \in \mathbb{C}$ with $|q| < 1$ define x and z as in Definition 2.3.2. Then for $q \in D_\delta$ we have*

$$(2.9.14) \quad E_3(q^8) = \frac{1}{4096}(2080 - 3120x + 1038x^2 + x^3 + (2016 - 2016x + 126x^2)(1 - x)^{1/2}) z^6,$$

$$(2.9.15) \quad E_3(q^{16}) = \frac{1}{262144} (p_3(x) + q_3(x)(1-x)^{1/4} + r_3(x)(1-x)^{1/2} + s_3(x)(1-x)^{3/4}) z^6,$$

where

$$p_3(x) = 66592 - 99888x + 33294x^2 + x^3,$$

$$q_3(x) = 64512 - 79632x + 15372x^2,$$

$$r_3(x) = 66528 - 66528x + 4158x^2,$$

$$s_3(x) = 64512 - 49332x + 252x^2.$$

Proof. The formulae follow by applying Theorem 2.5.1 to $E_3(q^4)$ and $E_3(q^8)$. $\square$

2.10 Ramanujan's discriminant function

We begin this section with the definitions of Ramanujan's discriminant function and Ramanujan's tau function.

Definition 2.10.1. *Ramanujan's discriminant function $\Delta(q)$ is defined for $q \in \mathbb{C}$ with $|q| < 1$ by*

$$\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Ramanujan's tau function $\tau(n)$ is defined for all $n \in \mathbb{N}$ by

$$\sum_{n=1}^{\infty} \tau(n) q^n := \Delta(q).$$

With $q = e^{2\pi iz}$, $z \in \mathbb{C}$, $\text{Im } z > 0$ and setting $\tilde{\Delta}(z) := \Delta(e^{2\pi iz}) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}$, it is known that the action on $\tilde{\Delta}$ by the modular group

$$\Gamma = SL(2, \mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is given by

$$\tilde{\Delta}\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\tilde{\Delta}(z),$$

see [54, p. 47]. $\tilde{\Delta}$ is a cusp form of weight 12 on Γ .

The first five values of $\tau(n)$ are

$$(2.10.1) \quad \tau(1) = 1, \tau(2) = -24, \tau(3) = 252, \tau(4) = -1472, \tau(5) = 4830.$$

Mordell [62] has shown that τ is multiplicative, that is

$$(2.10.2) \quad \tau(n_1 n_2) = \tau(n_1)\tau(n_2), \quad n_1, n_2 \in \mathbb{N}, \quad (n_1, n_2) = 1,$$

and that

$$(2.10.3) \quad \tau(p^{n+1}) = \tau(p^n)\tau(p) - p^{11}\tau(p^{n-1}), \quad n \in \mathbb{N}, \quad p \text{ prime}.$$

Both (2.10.2) and (2.10.3) had been conjectured by Ramanujan [71]. Mordell's proofs were later generalized by Hecke and led to the definition and theory of what we now call Hecke operators. Ramanujan also conjectured that $|\tau(p)| \leq 2p^{1/2}$ for all primes p . This estimate was proved by Deligne [33], [34] in the 1970's.

The relationship between $\Delta(q)$ and Eisenstein series is due to Ramanujan [71], [72], namely

$$\Delta(q) = \frac{1}{1728} (E_2(q)^3 - E_3(q)^2).$$

As $E_2(q)$ is a modular form of weight 4 on the full modular group Γ , $E_2(q)^3$ is a modular form of weight 12 on Γ . Similarly, as $E_3(q)$ is a modular form of weight 6 on Γ , $E_3(q)^2$ is also a modular form of weight 12 on Γ . It is easy to check that $E_2(q)^3$ and $E_3(q)^2$ are linearly independent over $\mathbb{Q}$. Thus, as the space of modular forms of

weight 12 has dimension 2, $\{E_2(q)^3, E_3(q)^2\}$ is a basis for this space. But $\Delta(q)$ is a modular form of weight 12 so

$$\Delta(q) = aE_2(q)^3 + bE_3(q)^2$$

for some rational numbers a and b . Equating the coefficients of 1 and q we obtain

$$\begin{cases} 0 &= a + b, \\ 1 &= 720a - 1008b, \end{cases}$$

so $a = -b = \frac{1}{1728}$, which gives Ramanujan's formula.

The next theorem gives the parametrization of $\Delta(q)$, $\Delta(q^2)$ and $\Delta(q^4)$ in terms of x and z as defined in Definition 2.3.2.

Theorem 2.10.1. *For $q \in \mathbb{C}$ such that $|q| < 1$, and x and z as defined in Definition 2.3.2, we have*

$$(2.10.4) \quad \Delta(q) = \frac{x(1-x)^4 z^{12}}{2^4},$$

$$(2.10.5) \quad \Delta(q^2) = \frac{x^2(1-x)^2 z^{12}}{2^8},$$

$$(2.10.6) \quad \Delta(q^4) = \frac{x^4(1-x) z^{12}}{2^{16}}.$$

Proof. These results are proved in [28]. □

We next define a generalized Ramanujan discriminant function.

Definition 2.10.2. *For $a, b, c \in \mathbb{Q}$ with $24a, 24b, 24c \in \mathbb{Z}$, $q \in \mathbb{C}$ and $|q| < 1$, we define a generalized Ramanujan discriminant function $\Delta(a, b, c; q)$ by*

$$(2.10.7) \quad \Delta(a, b, c; q) := \Delta(q)^a \Delta(q^2)^b \Delta(q^4)^c.$$

Clearly $\Delta(1, 0, 0; q) = \Delta(q)$, $\Delta(0, 1, 0; q) = \Delta(q^2)$, $\Delta(0, 0, 1; q) = \Delta(q^4)$.

In 1877 Dedekind introduced the eta function $\eta(z)$ defined by

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}), \quad z \in \mathbb{C}, \operatorname{Im}(z) > 0.$$

Setting $q = e^{2\pi iz}$ we have $|q| < 1$ so that

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

The infinite product $\prod_{n=1}^{\infty} (1 - q^n)$ converges absolutely and is nonzero. The convergence is uniform on compact subsets of $\{z \mid \operatorname{Im}(z) > 0\}$ so $\eta(z)$ is analytic on $\{z \mid \operatorname{Im}(z) > 0\}$. Clearly

$$\eta^{24}(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \Delta(q).$$

Thus

$$\Delta(a, b, c; q) = \eta(z)^{24a} \eta(2z)^{24b} \eta(4z)^{24c}.$$

A product of the form

$$\eta(t_1 z)^{r_1} \eta(t_2 z)^{r_2} \cdots \eta(t_s z)^{r_s},$$

where $s, t_1, \dots, t_s \in \mathbb{N}$ and $r_1, \dots, r_s \in \mathbb{Z}$ is called a η -quotient, see for example [59, p. 4825]. Thus, as $24a, 24b, 24c \in \mathbb{Z}$, the generalized discriminant function $\Delta(a, b, c; q)$ is an η -quotient.

The next theorem gives a generating function for $\Delta(a, b, c; q)$ and a parametrization in terms of x and z .

Theorem 2.10.2. For $a, b, c \in \mathbb{Q}$ with $24a, 24b, 24c \in \mathbb{Z}$, $q \in \mathbb{C}$ such that $|q| < 1$, and x and z as defined in Definition 2.3.2, we have

$$\begin{aligned} \text{(i)} \quad \Delta(a, b, c; q) &= q^{a+2b+4c} \prod_{n=1}^{\infty} (1 - q^n)^{24a} (1 - q^{2n})^{24b} (1 - q^{4n})^{24c}, \\ \text{(ii)} \quad \Delta(a, b, c; q) &= \frac{x^{a+2b+4c} (1-x)^{4a+2b+c} z^{12(a+b+c)}}{2^{4(a+2b+4c)}}. \end{aligned}$$

Proof. These results follow from Definitions 2.10.1 and 2.10.2, and Theorem 2.10.1. $\square$

Given $r, s, t \in \mathbb{Q}$, with $4r, 4s, 4t \in \mathbb{Z}$ it is possible to find $a, b, c \in \mathbb{Q}$ with $24a, 24b, 24c \in \mathbb{Z}$ such that $\Delta(a, b, c; q) = \frac{x^r (1-x)^s z^{2t}}{2^{4r}}$. The following theorem shows us how.

Theorem 2.10.3. For $r, s, t \in \mathbb{Q}$ with $4r, 4s, 4t \in \mathbb{Z}$, $q \in \mathbb{C}$ such that $|q| < 1$, and x, z as defined in Definition 2.3.2, we have

$$(2.10.8) \quad \Delta\left(\frac{1}{3}(r+2s-t), \frac{1}{6}(-6r-6s+5t), \frac{1}{3}(2r+s-t); q\right) = \frac{x^r (1-x)^s z^{2t}}{2^{4r}}.$$

Proof. This follows from Theorem 2.10.2. $\square$

From Theorem 2.10.2 (i) we see that $\Delta(a, b, c; q)$ has a power series expansion in nonnegative powers of q if and only if $a + 2b + 4c \in \mathbb{N}_0$. When this condition holds we define the arithmetic function $\delta(a, b, c; n)$ for all $n \in \mathbb{N}_0$ as follows.

Definition 2.10.3. For $a, b, c \in \mathbb{Q}$ with $24a, 24b, 24c \in \mathbb{Z}$, and $a + 2b + 4c \in \mathbb{N}_0$, we define $\delta(a, b, c; n)$ ($n \in \mathbb{N}_0$) by

$$(2.10.9) \quad \Delta(a, b, c; q) = \sum_{n=0}^{\infty} \delta(a, b, c; n) q^n, \quad q \in \mathbb{C}, \quad |q| < 1.$$

Clearly, we always have

$$(2.10.10) \quad \delta(a, b, c; n) = \begin{cases} 0, & \text{if } n < a + 2b + 4c, \\ 1, & \text{if } n = a + 2b + 4c. \end{cases}$$

2.11 The function $\tau_k(n)$ and six special values of

$$\delta(a, b, c; n)$$

We begin with the definition of the function $\tau_k(n)$.

Definition 2.11.1. For $k \in \mathbb{N}$ with $k|24$, we define $\tau_k(n)$ for all $n \in \mathbb{N}$ by

$$q \prod_{n=1}^{\infty} (1 - q^{kn})^{24/k} = \sum_{n=1}^{\infty} \tau_k(n) q^n.$$

From Definitions 2.10.1 and 2.11.1 we see that $\tau_1 = \tau$. Also it is clear that

$$\tau_k(n) = 0, \quad \text{if } n \not\equiv 1 \pmod{k}.$$

Thus in particular we have $\tau_2(n) = 0$, if $n \equiv 0 \pmod{2}$, and $\tau_4(n) = 0$, if $n \equiv 0, 2, 3 \pmod{4}$.

It is the purpose of this section to determine the values of $\delta(1, 0, 0; n)$, $\delta(-1, 3, -1; n)$, $\delta\left(0, \frac{1}{2}, 0; n\right)$, $\delta\left(0, 0, \frac{1}{4}; n\right)$, $\delta\left(0, 0, \frac{1}{2}; n\right)$ and $\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)$.

From (2.10.10) we see that

$$\begin{aligned} \delta(1, 0, 0; 0) &= \delta(-1, 3, -1; 0) = \delta\left(0, \frac{1}{2}, 0; 0\right) \\ &= \delta\left(0, 0, \frac{1}{4}; 0\right) = \delta\left(0, 0, \frac{1}{2}; 0\right) = \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; 0\right) = 0 \end{aligned}$$

so that we may exclude $n = 0$ and restrict n to be a positive integer.

Theorem 2.11.1. For $n \in \mathbb{N}$ we have $\delta(1, 0, 0; n) = \tau(n)$.

Proof. Taking $a = 1$, $b = 0$ and $c = 0$ in (2.10.9), we see that

$$\sum_{n=0}^{\infty} \delta(1, 0, 0; n) q^n = \Delta(1, 0, 0; q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the asserted result. $\square$

Theorem 2.11.2. For $n \in \mathbb{N}$ we have $\delta(-1, 3, -1; n) = (-1)^{n-1} \tau(n)$.

Proof. We have

$$\prod_{n=1}^{\infty} (1 - (-q)^n) = \prod_{n=1}^{\infty} (1 + q^{2n-1}) (1 - q^{2n}).$$

Now

$$\begin{aligned} \prod_{n=1}^{\infty} (1 + q^{2n-1}) \prod_{n=1}^{\infty} \frac{1 - q^{4n}}{1 - q^{2n}} &= \prod_{n=1}^{\infty} (1 + q^{2n-1}) \prod_{n=1}^{\infty} (1 + q^{2n}) = \prod_{n=1}^{\infty} (1 + q^{2n-1}) (1 + q^{2n}) \\ &= \prod_{n=1}^{\infty} (1 + q^n) = \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^n}, \end{aligned}$$

so

$$\prod_{n=1}^{\infty} (1 + q^{2n-1}) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^n)(1 - q^{4n})}$$

and thus

$$\prod_{n=1}^{\infty} (1 - (-q)^n) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^3}{(1 - q^n)(1 - q^{4n})}.$$

Replacing q by $-q$ in Definition 2.10.1, we obtain

$$-q \prod_{n=1}^{\infty} (1 - (-q)^n)^{24} = \sum_{n=1}^{\infty} \tau(n) (-q)^n$$

so

$$\sum_{n=1}^{\infty} (-1)^{n-1} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - (-q)^n)^{24}$$

$$\begin{aligned}
 &= q \prod_{n=1}^{\infty} (1 - q^n)^{-24} (1 - q^{2n})^{72} (1 - q^{4n})^{-24} \\
 &= \sum_{n=0}^{\infty} \delta(-1, 3, -1; n) q^n.
 \end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the assertion of the theorem. $\square$

Theorem 2.11.3. *For $n \in \mathbb{N}$ we have*

$$\delta\left(0, \frac{1}{2}, 0; n\right) = \tau_2(n) = \frac{1}{2} \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = n}} (x_1^4 - 3x_1^2 x_2^2).$$

Moreover if n is even we have

$$\delta\left(0, \frac{1}{2}, 0; n\right) = \tau_2(n) = 0.$$

Proof. Taking $a = 0$, $b = \frac{1}{2}$ and $c = 0$ in (2.10.9), we obtain

$$\begin{aligned}
 \sum_{n=0}^{\infty} \delta\left(0, \frac{1}{2}, 0; n\right) q^n &= \Delta\left(0, \frac{1}{2}, 0; q\right) = q \prod_{n=1}^{\infty} (1 - q^{2n})^2 \\
 &= \sum_{n=1}^{\infty} \tau_2(n) q^n
 \end{aligned}$$

so that

$$\delta\left(0, \frac{1}{2}, 0; n\right) = \tau_2(n), \quad n \in \mathbb{N}.$$

Bulygin (Boulyguine) [21], see for example Carlitz [24] and Lomadze [58], has shown that for $n \in \mathbb{N}$

$$r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 8 \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = n}} (x_1^4 - 3x_1^2 x_2^2).$$

From the case $k = 3$ of the Ramanujan-Mordell formula, we have for $n \in \mathbb{N}$

$$r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 16 \delta\left(0, \frac{1}{2}, 0; n\right).$$

Thus

$$\delta\left(0, \frac{1}{2}, 0; n\right) = \frac{1}{2} \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = n}} (x_1^4 - 3x_1^2 x_2^2).$$

This completes the proof of the first assertion of the theorem.

Now suppose that n is even. Then $\tau_2(n) = 0$ by the remark following Definition 2.11.1. This proves the second assertion. $\square$

Theorem 2.11.4. For $n \in \mathbb{N}$

$$\delta\left(0, 0, \frac{1}{4}; n\right) = \tau_4(n) = \begin{cases} \frac{1}{2} \sum_{\substack{(x, y) \in \mathbb{Z}^2 \\ x \equiv 1 \pmod{2} \\ y \equiv 0 \pmod{2} \\ n = x^2 + y^2}} (x^2 - y^2), & \text{if } n \equiv 1 \pmod{4}, \\ 0, & \text{if } n \not\equiv 1 \pmod{4}. \end{cases}$$

Proof. Taking $a = 0$, $b = 0$ and $c = 1/4$ in (2.10.9), we obtain

$$\sum_{n=0}^{\infty} \delta\left(0, 0, \frac{1}{4}; n\right) q^n = \Delta\left(0, 0, \frac{1}{4}; q\right) = q \prod_{n=1}^{\infty} (1 - q^{4n})^6 = \sum_{n=1}^{\infty} \tau_4(n) q^n.$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we deduce

$$\delta\left(0, 0, \frac{1}{4}; n\right) = \tau_4(n), \quad n \in \mathbb{N}.$$

Klein and Fricke [55, Vol. II, p. 377] have shown that

$$\tau_4(n) = \begin{cases} \frac{1}{2} \sum_{\substack{(x, y) \in \mathbb{Z}^2 \\ x \equiv 1 \pmod{2} \\ y \equiv 0 \pmod{2} \\ n = x^2 + y^2}} (x^2 - y^2), & \text{if } n \equiv 1 \pmod{4}, \\ 0, & \text{if } n \not\equiv 1 \pmod{4}. \end{cases}$$

This completes the proof of the theorem. $\square$

Theorem 2.11.5. *For $n \in \mathbb{N}$ we have*

$$\delta\left(0, 0, \frac{1}{2}; n\right) = 0, \quad \text{if } n \equiv 1 \pmod{2},$$

and

$$\delta\left(0, 0, \frac{1}{2}; n\right) = \tau_2\left(\frac{n}{2}\right) = \frac{1}{2} \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{n}{2}}} (x_1^4 - 3x_1^2 x_2^2), \quad \text{if } n \equiv 0 \pmod{2}.$$

Proof. By Definition 2.10.3 we have

$$\sum_{n=2}^{\infty} \delta\left(0, 0, \frac{1}{2}; n\right) q^n = q^2 \prod_{n=1}^{\infty} (1 - q^{4n})^{12}$$

and

$$\sum_{n=1}^{\infty} \delta\left(0, \frac{1}{2}, 0; n\right) q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^{12}$$

Thus

$$\sum_{n=1}^{\infty} \delta\left(0, \frac{1}{2}, 0; n\right) q^{2n} = q^2 \prod_{n=1}^{\infty} (1 - q^{4n})^{12} = \sum_{n=2}^{\infty} \delta\left(0, 0, \frac{1}{2}; n\right) q^n$$

Hence

$$\delta\left(0, 0, \frac{1}{2}; n\right) = 0, \quad \text{if } n \equiv 1 \pmod{2},$$

and

$$\delta\left(0, 0, \frac{1}{2}; n\right) = \delta\left(0, \frac{1}{2}, 0; \frac{n}{2}\right), \quad \text{if } n \equiv 0 \pmod{2}.$$

Finally, by Theorem 2.11.3, we have

$$\delta\left(0, 0, \frac{1}{2}; n\right) = \tau_2\left(\frac{n}{2}\right) = \frac{1}{2} \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{n}{2}}} (x_1^4 - 3x_1^2 x_2^2), \quad \text{if } n \equiv 0 \pmod{2}.$$

□

Theorem 2.11.6. *For $n \in \mathbb{N}$ we have*

$$\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) = \frac{1}{2} \sum_{\substack{(x_1, \dots, x_8) \in \mathbb{Z}^8 \\ x_1^2 + \dots + x_8^2 = n}} (x_1^4 - 3x_1^2 x_2^2).$$

Proof. Bulygin (Boulyguine) [21], see for example Carlitz [24] and Lomadze [58], has shown that for $n \in \mathbb{N}$

$$r_{16}(n) = \frac{32}{17} \sigma_7(n) - \frac{64}{17} \sigma_7(n/2) + \frac{8192}{17} \sigma_7(n/4) + \frac{256}{17} \sum_{\substack{(x_1, \dots, x_8) \in \mathbb{Z}^8 \\ x_1^2 + \dots + x_8^2 = n}} (x_1^4 - 3x_1^2 x_2^2)$$

and the Ramanujan-Mordell formula for $k = 4$ gives

$$r_{16}(n) = \frac{32}{17} \sigma_7(n) - \frac{64}{17} \sigma_7(n/2) + \frac{8192}{17} \sigma_7(n/4) + \frac{512}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right).$$

Equating these expressions, we obtain the asserted formula. $\square$

Chapter 3

The function $\left(\frac{1+(1-x)^{1/2}}{2}\right)^r \left(\frac{1+(1-x)^{1/4}}{2}\right)^s$

Throughout this chapter x denotes a variable. In this chapter we determine some of the properties of the function $\left(\frac{1+(1-x)^{1/2}}{2}\right)^r \left(\frac{1+(1-x)^{1/4}}{2}\right)^s$, where $r, s \in \mathbb{N}_0$, which arises naturally in Chapter 6 when we study the number of representations of a positive integer by the positive-definite, diagonal, integral quadratic form in $4k$ variables

$$x_1^2 + \cdots + x_{4k-2r-s}^2 + 2x_{4k-2r-s+1}^2 + \cdots + 2x_{4k-s}^2 + 4x_{4k-s+1}^2 + \cdots + 4x_{4k}^2.$$

This form has 2 as coefficient of $2r$ variables and 4 as the coefficient of s variables. The remaining $4k - 2r - s$ variables have coefficient 1. We always assume that $2r + s < 4k$.

3.1 The polynomials $A_{r,s}(x), B_{r,s}(x), C_{r,s}(x), D_{r,s}(x)$

We begin by defining the four polynomials studied throughout this chapter. The relationship of these polynomials to the function $\left(\frac{1+(1-x)^{1/2}}{2}\right)^r \left(\frac{1+(1-x)^{1/4}}{2}\right)^s$ is given in Lemma 3.1.3.

Definition 3.1.1. For $r, s \in \mathbb{N}_0$ we define

$$(i) \ A_{r,s}(x) := \frac{1}{2^{r+s}} \sum_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s \\ 2i+j \equiv 0 \pmod{4}}} \binom{r}{i} \binom{s}{j} (1-x)^{(2i+j)/4},$$

$$(ii) \quad B_{r,s}(x) := \frac{1}{2^{r+s}} \sum_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s \\ 2i+j \equiv 1 \pmod{4}}} \binom{r}{i} \binom{s}{j} (1-x)^{(2i+j-1)/4},$$

$$(iii) \quad C_{r,s}(x) := \frac{1}{2^{r+s}} \sum_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s \\ 2i+j \equiv 2 \pmod{4}}} \binom{r}{i} \binom{s}{j} (1-x)^{(2i+j-2)/4},$$

$$(iv) \quad D_{r,s}(x) := \frac{1}{2^{r+s}} \sum_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s \\ 2i+j \equiv 3 \pmod{4}}} \binom{r}{i} \binom{s}{j} (1-x)^{(2i+j-3)/4}.$$

From Definition 3.1.1 we see that for $r \in \mathbb{N}_0$ we have

$$B_{r,0}(x) = \frac{1}{2^r} \sum_{\substack{i=0 \\ 2i \equiv 1 \pmod{4}}}^r \binom{r}{i} (1-x)^{(2i-1)/4} = 0,$$

$$D_{r,0}(x) = \frac{1}{2^r} \sum_{\substack{i=0 \\ 2i \equiv 3 \pmod{4}}}^r \binom{r}{i} (1-x)^{(2i-3)/4} = 0.$$

The next lemma gives an upper bound in terms of r and s for the degree of each of the polynomials $A_{r,s}(x)$, $B_{r,s}(x)$, $C_{r,s}(x)$ and $D_{r,s}(x)$.

Lemma 3.1.1. *For $r, s \in \mathbb{N}_0$ we have*

$$(i) \quad \deg(A_{r,s}(x)) \leq \left\lfloor \frac{2r+s}{4} \right\rfloor,$$

$$(ii) \quad \deg(B_{r,s}(x)) \leq \left\lfloor \frac{2r+s-1}{4} \right\rfloor,$$

$$(iii) \quad \deg(C_{r,s}(x)) \leq \left\lfloor \frac{2r+s-2}{4} \right\rfloor,$$

$$(iv) \quad \deg(D_{r,s}(x)) \leq \left\lfloor \frac{2r+s-3}{4} \right\rfloor.$$

Proof. As

$$\max_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s \\ 2i+j \equiv 0 \pmod{4}}} \frac{2i+j}{4} \leq \frac{2r+s}{4},$$

we have, from Definition 3.1.1(i), that

$$\deg(A_{r,s}(x)) \leq \left\lfloor \frac{2r+s}{4} \right\rfloor.$$

The proofs of (ii), (iii), (iv) follow in a similar way. □

The next lemma gives an upper bound for the degree of the polynomials $A_{r,s}(x)$, $B_{r,s}(x)$, $C_{r,s}(x)$ and $D_{r,s}(x)$ in terms of an integer k when $k \geq 2$ and $2r+s < 4k$.

Lemma 3.1.2. *For $k \in \mathbb{N}$, $r, s \in \mathbb{N}_0$ with $k \geq 2$ and $2r+s < 4k$, we have*

- (i) $\deg(A_{r,s}(x)) \leq k-1$,
- (ii) $\deg(B_{r,s}(x)) \leq k-1$,
- (iii) $\deg(C_{r,s}(x)) \leq k-1$,
- (iv) $\deg(D_{r,s}(x)) \leq k-1$.

Proof. (i) As $2r+s < 4k$ implies $\frac{2r+s}{4} < \frac{4k}{4} = k$, we have, from Lemma 3.1.1(i), that

$$\deg(A_{r,s}(x)) \leq \left\lfloor \frac{2r+s}{4} \right\rfloor \leq k-1.$$

(ii) As $2r+s < 4k$ implies $\frac{2r+s-1}{4} < \frac{4k-1}{4} = k - \frac{1}{4}$, we have, from Lemma 3.1.1

(ii), that

$$\deg(B_{r,s}(x)) \leq \left\lfloor \frac{2r+s-1}{4} \right\rfloor \leq k-1.$$

(iii) As $2r + s < 4k$ implies $\frac{2r + s - 2}{4} < \frac{4k - 2}{4} = k - \frac{1}{2}$, we have, from Lemma 3.1.1 (iii), that

$$\deg(C_{r,s}(x)) \leq \left\lfloor \frac{2r + s - 2}{4} \right\rfloor \leq k - 1.$$

(iv) As $2r + s < 4k$ implies $\frac{2r + s - 3}{4} < \frac{4k - 3}{4} = k - \frac{3}{4}$, we have, from Lemma 3.1.1 (iv), that

$$\deg(D_{r,s}(x)) \leq \left\lfloor \frac{2r + s - 3}{4} \right\rfloor \leq k - 1.$$

This completes the proof of Lemma 3.1.2. $\square$

The next lemma relates the polynomials $A_{r,s}(x)$, $B_{r,s}(x)$, $C_{r,s}(x)$ and $D_{r,s}(x)$ to the function $\left(\frac{1 + (1-x)^{1/2}}{2}\right)^r \left(\frac{1 + (1-x)^{1/4}}{2}\right)^s$.

Lemma 3.1.3. *For $r, s \in \mathbb{N}_0$, the polynomials $A_{r,s}(x)$, $B_{r,s}(x)$, $C_{r,s}(x)$ and $D_{r,s}(x)$ are such that*

$$(i) \left(\frac{1 + (1-x)^{1/2}}{2}\right)^r \left(\frac{1 + (1-x)^{1/4}}{2}\right)^s$$

$$= A_{r,s}(x) + B_{r,s}(x)(1-x)^{1/4} + C_{r,s}(x)(1-x)^{1/2} + D_{r,s}(x)(1-x)^{3/4},$$

$$(ii) \left(\frac{1 + (1-x)^{1/2}}{2}\right)^r \left(\frac{1 - (1-x)^{1/4}}{2}\right)^s$$

$$= A_{r,s}(x) - B_{r,s}(x)(1-x)^{1/4} + C_{r,s}(x)(1-x)^{1/2} - D_{r,s}(x)(1-x)^{3/4},$$

$$(iii) \left(\frac{1 - (1-x)^{1/2}}{2}\right)^r \left(\frac{1 + i(1-x)^{1/4}}{2}\right)^s$$

$$= A_{r,s}(x) + iB_{r,s}(x)(1-x)^{1/4} - C_{r,s}(x)(1-x)^{1/2} - iD_{r,s}(x)(1-x)^{3/4},$$

$$\begin{aligned}
 & \text{(iv)} \quad \left(\frac{1-(1-x)^{1/2}}{2}\right)^r \left(\frac{1-i(1-x)^{1/4}}{2}\right)^s \\
 & = A_{r,s}(x) - iB_{r,s}(x)(1-x)^{1/4} - C_{r,s}(x)(1-x)^{1/2} + iD_{r,s}(x)(1-x)^{3/4}.
 \end{aligned}$$

Proof. We begin by noting that for $h, k, l \in \mathbb{N}_0$ we have

$$(-1)^{hk} i^{hl} = i^{2hk+hl} = (i^{2k+l})^h = \begin{cases} 1, & \text{if } 2k+l \equiv 0 \pmod{4}, \\ i^h, & \text{if } 2k+l \equiv 1 \pmod{4}, \\ (-1)^h, & \text{if } 2k+l \equiv 2 \pmod{4}, \\ (-i)^h, & \text{if } 2k+l \equiv 3 \pmod{4}. \end{cases}$$

By the binomial theorem, we have for $h = 0, 1, 2, 3$

$$\begin{aligned}
 & \left(\frac{1+(-1)^h(1-x)^{1/2}}{2}\right)^r \left(\frac{1+i^h(1-x)^{1/4}}{2}\right)^s \\
 & = \left(\frac{1}{2^r} \sum_{k=0}^r \binom{r}{k} (-1)^{hk} (1-x)^{k/2}\right) \left(\frac{1}{2^s} \sum_{l=0}^s \binom{s}{l} i^{hl} (1-x)^{l/4}\right) \\
 & = \frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s}} \binom{r}{k} \binom{s}{l} (-1)^{hk} i^{hl} (1-x)^{(2k+l)/4} \\
 & = \frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s \\ 2k+l \equiv 0 \pmod{4}}} \binom{r}{k} \binom{s}{l} (-1)^{hk} i^{hl} (1-x)^{(2k+l)/4} \\
 & \quad + \left(\frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s \\ 2k+l \equiv 1 \pmod{4}}} \binom{r}{k} \binom{s}{l} (-1)^{hk} i^{hl} (1-x)^{(2k+l-1)/4}\right) (1-x)^{1/4} \\
 & \quad + \left(\frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s \\ 2k+l \equiv 2 \pmod{4}}} \binom{r}{k} \binom{s}{l} (-1)^{hk} i^{hl} (1-x)^{(2k+l-2)/4}\right) (1-x)^{1/2}
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s \\ 2k+l \equiv 3 \pmod{4}}} \binom{r}{k} \binom{s}{l} (-1)^{hk} i^{hl} (1-x)^{(2k+l-3)/4} \right) (1-x)^{3/4}. \\
 & = \frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s \\ 2k+l \equiv 0 \pmod{4}}} \binom{r}{k} \binom{s}{l} (1-x)^{(2k+l)/4} \\
 & + \left(\frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s \\ 2k+l \equiv 1 \pmod{4}}} \binom{r}{k} \binom{s}{l} i^h (1-x)^{(2k+l-1)/4} \right) (1-x)^{1/4} \\
 & + \left(\frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s \\ 2k+l \equiv 2 \pmod{4}}} \binom{r}{k} \binom{s}{l} (-1)^h (1-x)^{(2k+l-2)/4} \right) (1-x)^{1/2} \\
 & + \left(\frac{1}{2^{r+s}} \sum_{\substack{0 \leq k \leq r \\ 0 \leq l \leq s \\ 2k+l \equiv 3 \pmod{4}}} \binom{r}{k} \binom{s}{l} (-i)^h (1-x)^{(2k+l-3)/4} \right) (1-x)^{3/4}.
 \end{aligned}$$

$$= A_{r,s}(x) + i^h B_{r,s}(x) + (-1)^h C_{r,s}(x) + (-i)^h D_{r,s}(x),$$

which gives the asserted results. $\square$

Using Lemma 3.1.3 we obtain explicit expressions for each of $A_{r,s}(x)$, $B_{r,s}(x)$, $C_{r,s}(x)$ and $D_{r,s}(x)$.

Lemma 3.1.4. *For $r, s \in \mathbb{N}_0$ we have*

$$A_{r,s}(x) =$$

$$\begin{aligned}
 & \frac{1}{4} \left(\left(\frac{1+(1-x)^{1/2}}{2} \right)^r \left(\frac{1+(1-x)^{1/4}}{2} \right)^s + \left(\frac{1-(1-x)^{1/2}}{2} \right)^r \left(\frac{1+i(1-x)^{1/4}}{2} \right)^s \right. \\
 & \left. + \left(\frac{1+(1-x)^{1/2}}{2} \right)^r \left(\frac{1-(1-x)^{1/4}}{2} \right)^s + \left(\frac{1-(1-x)^{1/2}}{2} \right)^r \left(\frac{1-i(1-x)^{1/4}}{2} \right)^s \right),
 \end{aligned}$$

$$B_{r,s}(x) =$$

$$\begin{aligned} & \frac{1}{4} \left(\left(\frac{1+(1-x)^{1/2}}{2} \right)^r \left(\frac{1+(1-x)^{1/4}}{2} \right)^s - i \left(\frac{1-(1-x)^{1/2}}{2} \right)^r \left(\frac{1+i(1-x)^{1/4}}{2} \right)^s \right. \\ & \quad \left. - \left(\frac{1+(1-x)^{1/2}}{2} \right)^r \left(\frac{1-(1-x)^{1/4}}{2} \right)^s + i \left(\frac{1-(1-x)^{1/2}}{2} \right)^r \left(\frac{1-i(1-x)^{1/4}}{2} \right)^s \right), \end{aligned}$$

$$C_{r,s}(x) =$$

$$\begin{aligned} & \frac{1}{4} \left(\left(\frac{1+(1-x)^{1/2}}{2} \right)^r \left(\frac{1+(1-x)^{1/4}}{2} \right)^s - \left(\frac{1-(1-x)^{1/2}}{2} \right)^r \left(\frac{1+i(1-x)^{1/4}}{2} \right)^s \right. \\ & \quad \left. + \left(\frac{1+(1-x)^{1/2}}{2} \right)^r \left(\frac{1-(1-x)^{1/4}}{2} \right)^s - \left(\frac{1-(1-x)^{1/2}}{2} \right)^r \left(\frac{1-i(1-x)^{1/4}}{2} \right)^s \right), \end{aligned}$$

$$D_{r,s}(x) =$$

$$\begin{aligned} & \frac{1}{4} \left(\left(\frac{1+(1-x)^{1/2}}{2} \right)^r \left(\frac{1+(1-x)^{1/4}}{2} \right)^s + i \left(\frac{1-(1-x)^{1/2}}{2} \right)^r \left(\frac{1+i(1-x)^{1/4}}{2} \right)^s \right. \\ & \quad \left. - \left(\frac{1+(1-x)^{1/2}}{2} \right)^r \left(\frac{1-(1-x)^{1/4}}{2} \right)^s - i \left(\frac{1-(1-x)^{1/2}}{2} \right)^r \left(\frac{1-i(1-x)^{1/4}}{2} \right)^s \right). \end{aligned}$$

Proof. This follows by solving the four equations in Lemma 3.1.3 for $A_{r,s}(x)$, $B_{r,s}(x)$, $C_{r,s}(x)$ and $D_{r,s}(x)$. □

The main objective of this chapter is to find the values of the polynomials

$A_{r,s}(x)$, $B_{r,s}(x)$, $C_{r,s}(x)$ and $D_{r,s}(x)$ at $x = 0$. From Lemma 3.1.4 we have

Lemma 3.1.5. *For $r, s \in \mathbb{N}_0$*

$$A_{r,s}(0) = \frac{1}{4} \left(1 + 0^r \left(\frac{1+i}{2} \right)^s + 0^s + 0^r \left(\frac{1-i}{2} \right)^s \right),$$

$$B_{r,s}(0) = \frac{1}{4} \left(1 - i \, 0^r \left(\frac{1+i}{2} \right)^s - 0^s + i \, 0^r \left(\frac{1-i}{2} \right)^s \right),$$

$$C_{r,s}(0) = \frac{1}{4} \left(1 - 0^r \left(\frac{1+i}{2} \right)^s + 0^s - 0^r \left(\frac{1-i}{2} \right)^s \right),$$

$$D_{r,s}(0) = \frac{1}{4} \left(1 + i \, 0^r \left(\frac{1+i}{2} \right)^s - 0^s - i \, 0^r \left(\frac{1-i}{2} \right)^s \right),$$

where

$$0^r = \begin{cases} 0, & \text{if } r \geq 1, \\ 1, & \text{if } r = 0. \end{cases}$$

Proof. This follows by replacing x by 0 in Lemma 3.1.4. □

It is clear from Lemma 3.1.5 that we require the values of $(1+i)^s \pm (1-i)^s$. This is the subject of the next section.

3.2 The values of $(1+i)^s \pm (1-i)^s$

The first lemma gives a formula for a sum involving binomial coefficients over an arithmetic progression.

Lemma 3.2.1. *For $a, r, n \in \mathbb{N}_0$ such that $0 \leq a < r$ and $x \in \mathbb{C}$, we have*

$$\sum_{\substack{t=0 \\ t \equiv a \pmod{r}}}^n \binom{n}{t} x^t = \frac{1}{r} \sum_{j=1}^r e^{\frac{-2\pi i a j}{r}} \left(1 + x e^{\frac{2\pi i j}{r}} \right)^n.$$

Proof. For $r \in \mathbb{N}$ and $u \in \mathbb{Z}$ we have

$$\sum_{j=1}^r e^{\frac{2\pi i u j}{r}} = \begin{cases} r, & \text{if } u \equiv 0 \pmod{r}, \\ 0, & \text{if } u \not\equiv 0 \pmod{r}. \end{cases}$$

Thus

$$\begin{aligned} r \sum_{\substack{t=0 \\ t \equiv a \pmod{r}}}^n \binom{n}{t} x^t &= \sum_{t=0}^n \binom{n}{t} x^t \sum_{j=1}^r e^{\frac{2\pi i j(t-a)}{r}} \\ &= \sum_{j=1}^r e^{\frac{-2\pi i a j}{r}} \sum_{t=0}^n \binom{n}{t} x^t e^{\frac{2\pi i j t}{r}} \\ &= \sum_{j=1}^r e^{\frac{-2\pi i a j}{r}} \left(1 + x e^{\frac{2\pi i j}{r}}\right)^n, \end{aligned}$$

and the asserted result follows. □

The next lemma summarizes four simple but useful results involving the greatest integer function.

Lemma 3.2.2. *For $k \in \mathbb{N}_0$, we have*

$$(i) \quad k + 2 = \left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor + 1,$$

$$(ii) \quad \left\lfloor \frac{k+2}{2} \right\rfloor = \begin{cases} \frac{k+2}{2}, & \text{if } k \equiv 0 \pmod{2}, \\ \frac{k+1}{2}, & \text{if } k \equiv 1 \pmod{2}, \end{cases}$$

$$(iii) \left[\frac{k}{4}\right] = \begin{cases} \frac{k}{4}, & \text{if } k \equiv 0 \pmod{4}, \\ \frac{k-1}{4}, & \text{if } k \equiv 1 \pmod{4}, \\ \frac{k-2}{4}, & \text{if } k \equiv 2 \pmod{4}, \\ \frac{k-3}{4}, & \text{if } k \equiv 3 \pmod{4}, \end{cases}$$

$$(iv) \left[\frac{k+1}{4}\right] = \begin{cases} \frac{k}{4}, & \text{if } k \equiv 0 \pmod{4}, \\ \frac{k-1}{4}, & \text{if } k \equiv 1 \pmod{4}, \\ \frac{k-2}{4}, & \text{if } k \equiv 2 \pmod{4}, \\ \frac{k+1}{4}, & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

Proof. (i) This identity is well-known and is easily verified by considering k modulo 2.

(ii) This is easily checked by considering k modulo 2.

(iii) This is easily checked by considering k modulo 4.

(iv) This follows from (iii) by replacing k by $k + 1$. □

We next define an arithmetic function that will be useful throughout this thesis.

Definition 3.2.1. Let $n \in \mathbb{N}_0$. For $r \in \mathbb{N}_0$ and $s \in \mathbb{N}$ with $0 \leq r < s$, we define the arithmetic function $\varepsilon_{r,s}(n)$ by

$$\varepsilon_{r,s}(n) = \begin{cases} 0, & \text{if } n \equiv r \pmod{s}, \\ 1, & \text{otherwise.} \end{cases}$$

To conclude this section, we state and prove a lemma that we will need in the proof of Lemma 3.3.3.

Lemma 3.2.3. *For $s \in \mathbb{N}_0$, we have*

$$(i) \quad (1+i)^s + (1-i)^s = (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s+2}{2} \rfloor} \varepsilon_{2,4}(s),$$

$$(ii) \quad (1+i)^s - (1-i)^s = (-1)^{\lfloor \frac{s}{4} \rfloor} 2^{\lfloor \frac{s+2}{2} \rfloor} i \varepsilon_{0,4}(s).$$

Proof. As $(1+i)^2 = 2i$, $(1+i)^3 = -2(1-i)$ and $(1+i)^4 = -4$, we have

$$(1+i)^s = \begin{cases} (-1)^{\frac{s}{4}} 2^{\frac{s}{2}}, & \text{if } s \equiv 0 \pmod{4}, \\ (-1)^{\frac{s-1}{4}} 2^{\frac{s-1}{2}} (1+i), & \text{if } s \equiv 1 \pmod{4}, \\ (-1)^{\frac{s-2}{4}} 2^{\frac{s}{2}} i, & \text{if } s \equiv 2 \pmod{4}, \\ -(-1)^{\frac{s-3}{4}} 2^{\frac{s-1}{2}} (1-i), & \text{if } s \equiv 3 \pmod{4}. \end{cases}$$

Similarly, as $(1-i)^2 = -2i$, $(1-i)^3 = -2(1+i)$ and $(1-i)^4 = -4$, we have

$$(1-i)^s = \begin{cases} (-1)^{\frac{s}{4}} 2^{\frac{s}{2}}, & \text{if } s \equiv 0 \pmod{4}, \\ (-1)^{\frac{s-1}{4}} 2^{\frac{s-1}{2}} (1-i), & \text{if } s \equiv 1 \pmod{4}, \\ -(-1)^{\frac{s-2}{4}} 2^{\frac{s}{2}} i, & \text{if } s \equiv 2 \pmod{4}, \\ -(-1)^{\frac{s-3}{4}} 2^{\frac{s-1}{2}} (1+i), & \text{if } s \equiv 3 \pmod{4}. \end{cases}$$

This implies that

$$(1+i)^s + (1-i)^s = \begin{cases} (-1)^{\frac{s}{4}} 2^{\frac{s+2}{2}}, & \text{if } s \equiv 0 \pmod{4}, \\ (-1)^{\frac{s-1}{4}} 2^{\frac{s+1}{2}}, & \text{if } s \equiv 1 \pmod{4}, \\ 0, & \text{if } s \equiv 2 \pmod{4}, \\ (-1)^{\frac{s+1}{4}} 2^{\frac{s+1}{2}}, & \text{if } s \equiv 3 \pmod{4}, \end{cases}$$

$$= (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s+2}{2} \rfloor} \varepsilon_{2,4}(s),$$

by Definition 3.2.1 and Lemma 3.2.2(ii)(iv). Similarly, appealing to Definition 3.2.1 and Lemma 3.2.2(ii)(iii), we find that

$$(1+i)^s - (1-i)^s = \begin{cases} 0, & \text{if } s \equiv 0 \pmod{4}, \\ (-1)^{\frac{s-1}{4}} 2^{\frac{s+1}{2}} i, & \text{if } s \equiv 1 \pmod{4}, \\ (-1)^{\frac{s-2}{4}} 2^{\frac{s+2}{2}} i, & \text{if } s \equiv 2 \pmod{4}, \\ (-1)^{\frac{s-3}{4}} 2^{\frac{s+1}{2}} i, & \text{if } s \equiv 3 \pmod{4}, \end{cases}$$

$$= (-1)^{\lfloor \frac{s}{4} \rfloor} 2^{\lfloor \frac{s+2}{2} \rfloor} \varepsilon_{0,4}(s) i,$$

as required. □

3.3 The values of $A_{r,s}(0)$, $B_{r,s}(0)$, $C_{r,s}(0)$, $D_{r,s}(0)$

We are now ready to evaluate the four polynomials of Definition 3.1.1 when $x = 0$.

When $r = s = 0$ we have

$$A_{0,0}(x) = 1, B_{0,0}(x) = C_{0,0}(x) = D_{0,0}(x) = 0,$$

so

$$A_{0,0}(0) = 1, B_{0,0}(0) = C_{0,0}(0) = D_{0,0}(0) = 0.$$

Thus we need only consider those r and s for which either $r \geq 1$ or $s \geq 1$. We consider three cases:

- (i) $r \geq 1, s = 0$ (Lemma 3.3.1),
- (ii) $r \geq 1, s \geq 1$ (Lemma 3.3.2).
- (iii) $r = 0, s \geq 1$ (Lemma 3.3.3),

We begin with case (i), namely $r \geq 1$ and $s = 0$.

Lemma 3.3.1. *For $r \in \mathbb{N}$, we have*

$$(i) \quad A_{r,0}(0) = \frac{1}{2},$$

$$(ii) \quad B_{r,0}(0) = 0,$$

$$(iii) \quad C_{r,0}(0) = \frac{1}{2},$$

$$(iv) \quad D_{r,0}(0) = 0.$$

Proof. Setting $r \geq 1$ and $s = 0$ in Lemma 3.1.5 we have

$$A_{r,0}(0) = \frac{1}{4} (1 + 1) = \frac{1}{2},$$

$$B_{r,0}(0) = \frac{1}{4} (1 - 1) = 0,$$

$$C_{r,0}(0) = \frac{1}{4} (1 + 1) = \frac{1}{2},$$

$$D_{r,0}(0) = \frac{1}{4} (1 - 1) = 0,$$

completing the proof of the lemma. □

Next we turn to case (ii), namely $r \geq 1, s \geq 1$.

Lemma 3.3.2. *For $r, s \in \mathbb{N}$, we have*

$$A_{r,s}(0) = B_{r,s}(0) = C_{r,s}(0) = D_{r,s}(0) = \frac{1}{4}.$$

Proof. Setting $r \geq 1$ and $s \geq 1$ in Lemma 3.1.5 we have

$$A_{r,s}(0) = \frac{1}{4} (1 + 0 + 0 + 0) = \frac{1}{4},$$

$$B_{r,s}(0) = \frac{1}{4} (1 - i \cdot 0 - 0 + i \cdot 0) = \frac{1}{4},$$

$$C_{r,s}(0) = \frac{1}{4} (1 - 0 + 0 - 0) = \frac{1}{4},$$

$$D_{r,s}(0) = \frac{1}{4} (1 + i \cdot 0 - 0 - i \cdot 0) = \frac{1}{4},$$

as asserted. □

Finally, we turn to case (iii), namely $r = 0, s \geq 1$.

Lemma 3.3.3. *For $s \in \mathbb{N}$, we have*

$$(i) \ A_{0,s}(0) = \frac{1}{4} + \frac{(-1)^{\lfloor \frac{s+1}{4} \rfloor} \varepsilon_{2,4}(s)}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}},$$

$$(ii) \ B_{0,s}(0) = \frac{1}{4} + \frac{(-1)^{\lfloor \frac{s}{4} \rfloor} \varepsilon_{0,4}(s)}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}},$$

$$(iii) \ C_{0,s}(0) = \frac{1}{4} - \frac{(-1)^{\lfloor \frac{s+1}{4} \rfloor} \varepsilon_{2,4}(s)}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}},$$

$$(iv) D_{0,s}(0) = \frac{1}{4} - \frac{(-1)^{\lfloor \frac{s}{4} \rfloor} \varepsilon_{0,4}(s)}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}}.$$

Proof. By Lemmas 3.1.5, 3.2.2 and 3.2.3 we have

$$\begin{aligned} A_{0,s}(0) &= \frac{1}{4} \left(1 + \left(\frac{1+i}{2} \right)^s + \left(\frac{1-i}{2} \right)^s \right) = \frac{1}{4} \left(1 + \frac{1}{2^s} ((1+i)^s + (1-i)^s) \right) \\ &= \frac{1}{4} + \frac{(-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s+2}{2} \rfloor} \varepsilon_{2,4}(s)}{2^{s+2}} = \frac{1}{4} + \frac{(-1)^{\lfloor \frac{s+1}{4} \rfloor} \varepsilon_{2,4}(s)}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}}, \end{aligned}$$

$$\begin{aligned} B_{0,s}(0) &= \frac{1}{4} \left(1 - i \left(\frac{1+i}{2} \right)^s + i \left(\frac{1-i}{2} \right)^s \right) = \frac{1}{4} \left(1 - \frac{i}{2^s} ((1+i)^s - (1-i)^s) \right) \\ &= \frac{1}{4} + \frac{(-1)^{\lfloor \frac{s}{4} \rfloor} 2^{\lfloor \frac{s+2}{2} \rfloor} \varepsilon_{0,4}(s)}{2^{s+2}} = \frac{1}{4} + \frac{(-1)^{\lfloor \frac{s}{4} \rfloor} \varepsilon_{0,4}(s)}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}}, \end{aligned}$$

$$\begin{aligned} C_{0,s}(0) &= \frac{1}{4} \left(1 - \left(\frac{1+i}{2} \right)^s - \left(\frac{1-i}{2} \right)^s \right) \\ &= \frac{1}{4} - \frac{(-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s+2}{2} \rfloor} \varepsilon_{2,4}(s)}{2^{s+2}} = \frac{1}{4} - \frac{(-1)^{\lfloor \frac{s+1}{4} \rfloor} \varepsilon_{2,4}(s)}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}}, \end{aligned}$$

$$\begin{aligned} D_{0,s}(0) &= \frac{1}{4} \left(1 + i \left(\frac{1+i}{2} \right)^s - i \left(\frac{1-i}{2} \right)^s \right) = \frac{1}{4} \left(1 + \frac{i}{2^s} ((1+i)^s - (1-i)^s) \right) \\ &= \frac{1}{4} - \frac{(-1)^{\lfloor \frac{s}{4} \rfloor} 2^{\lfloor \frac{s+2}{2} \rfloor} \varepsilon_{0,4}(s)}{2^{s+2}} = \frac{1}{4} - \frac{(-1)^{\lfloor \frac{s}{4} \rfloor} \varepsilon_{0,4}(s)}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}}. \end{aligned}$$

This completes the proof of the lemma. $\square$

Chapter 4

The functions $e_{k,r}(x)$

Throughout this chapter x denotes a complex variable, except in Section 4.4 where x is taken to be $1 - \frac{\varphi^4(-q)}{\varphi^4(q)}$.

In this chapter we define the polynomials $e_{k,1}(x), e_{k,2}(x), e_{k,4}(x)$ (Section 4.1) and functions $e_{k,8}(x), e_{k,16}(x)$ (Sections 4.2 and 4.3) for $k \in \mathbb{N}$ with $k \geq 2$ and show in Section 4.4 that

$$(4.0.1) \quad E_k(q^r) = e_{k,r}(x) z^{2k}, \quad k \geq 2, \quad r \in \{1, 2, 4, 8, 16\},$$

with

$$x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z = \varphi^2(q).$$

We note that the exponent of z in (4.0.1) is the weight of the Eisenstein series E_k .

4.1 The polynomials $e_{k,1}(x)$, $e_{k,2}(x)$ and $e_{k,4}(x)$

We begin by defining the polynomials $e_{k,1}(x), e_{k,2}(x)$ and $e_{k,4}(x)$.

Definition 4.1.1. *For $k \in \mathbb{N}$ with $k \geq 2$ we define*

$$(4.1.1) \quad e_{k,1}(x) := \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (1+14x+x^2)^t (1-33x-33x^2+x^3)^u,$$

$$(4.1.2) \quad e_{k,2}(x) := \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (1-x+x^2)^t \left(1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3\right)^u,$$

$$(4.1.3) \quad e_{k,4}(x) := \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(1-x+\frac{1}{16}x^2\right)^t \left(1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3\right)^u,$$

where the coefficients $a_k(t,u)$ are given in Theorem 2.9.1.

The objective of this section is to determine the values of $e_{k,r}(0)$ and $e_{k,r}(1)$ for $k \geq 2$ and $r \in \{1, 2, 4\}$. We begin with the polynomial $e_{k,1}(x)$.

Lemma 4.1.1. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

- (i) $e_{k,1}(0) = 1$,
- (ii) $e_{k,1}(1) = (-1)^k 2^{2k}$,
- (iii) $\deg(e_{k,1}(x)) = k$,
- (iv) the coefficient of x^k in $e_{k,1}(x)$ is 1.

Proof. (i) By (4.1.1) and (2.9.1) we have

$$e_{k,1}(0) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (1)^t (1)^u = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) = 1.$$

(ii) By (4.1.1) and (2.9.1) we have

$$e_{k,1}(1) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) 2^{4t} (-1)^u 2^{6u} = (-1)^k 2^{2k}.$$

as $2t + 3u = k$ implies $u \equiv k \pmod{2}$ so $(-1)^u = (-1)^k$.

(iii) (iv) For each $(t,u) \in \mathbb{N}_0^2$ with $2t + 3u = k$, we have

$$\deg((1 + 14x + x^2)^t (1 - 33x - 33x^2 + x^3)^u) = k,$$

and the coefficient of x^k in $(1 + 14x + x^2)^t(1 - 33x - 33x^2 + x^3)^u$ is 1. Thus by (4.1.1) the coefficient of x^k in $e_{k,1}(x)$ is

$$\sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) = 1$$

and

$$\deg(e_{k,1}(x)) = k.$$

This completes the proof of Lemma 4.1.1. □

The next lemma gives the corresponding results for $e_{k,2}(x)$.

Lemma 4.1.2. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

- (i) $e_{k,2}(0) = 1$,
- (ii) $e_{k,2}(1) = (-1)^k$,
- (iii) $\deg(e_{k,2}(x)) = k$,
- (iv) the coefficient of x^k in $e_{k,2}(x)$ is 1.

Proof. By (4.1.2) and (2.9.1) we have

$$e_{k,2}(0) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u)(1)^t(1)^u = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) = 1.$$

(ii) By (4.1.2) and (2.9.1) we have

$$e_{k,2}(1) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u)(1)^t(-1)^u = (-1)^k \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) = (-1)^k.$$

(iii) (iv) For each $(t, u) \in \mathbb{N}_0$ with $2t + 3u = k$, we have

$$\deg \left((1 - x + x^2)^t \left(1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3 \right)^u \right) = k,$$

and the coefficient of x^k in $(1-x+x^2)^t \left(1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3\right)^u$ is 1. Thus, by (4.1.2), the coefficient of x^k in $e_{k,2}(x)$ is

$$\sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) = 1$$

and

$$\deg(e_{k,2}(x)) = k.$$

This completes the proof of Lemma 4.1.2 □

Our final lemma of this section gives the corresponding results for $e_{k,4}(x)$.

Lemma 4.1.3. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

- (i) $e_{k,4}(0) = 1$,
- (ii) $e_{k,4}(1) = \frac{(-1)^k}{2^{2k}}$,
- (iii) $\deg(e_{k,4}(x)) = k$,
- (iv) the coefficient of x^k in $e_{k,4}(x)$ is $\frac{1}{2^{2k}}$.

Proof. (i) By (4.1.3) and (2.9.1) we have

$$e_{k,4}(0) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) (1)^t (1)^u = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) = 1.$$

(ii) By (4.1.3) and (2.9.1) we have

$$e_{k,4}(1) = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) (2)^{-4t} (-1)^u (2)^{-6u} = \frac{(-1)^k}{2^{2k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) = \frac{(-1)^k}{2^{2k}}.$$

(iii) (iv) For each $(t, u) \in \mathbb{N}_0^2$ with $2t + 3u = k$, we have

$$\deg \left((1-x + \frac{1}{16}x^2)^t \left(1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3 \right)^u \right) = k,$$

and the coefficient of x^k in $\left(1 - x + \frac{1}{16}x^2\right)^t \left(1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3\right)^u$ is $\frac{1}{16^t 64^u} = \frac{1}{2^{4t+6u}} = \frac{1}{2^k}$. Thus by (4.1.3) the coefficient of x^k in $e_{k,4}(x)$ is

$$\sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} \frac{a_k(t,u)}{2^{2k}} = \frac{1}{2^{2k}}$$

and

$$\deg(e_{k,2}(x)) = k.$$

This completes the proof of Lemma 4.1.3. $\square$

4.2 The function $e_{k,8}(x)$

In this section we define the function $e_{k,8}(x)$ for $k \in \mathbb{N}$ with $k \geq 2$.

Definition 4.2.1. For $k \in \mathbb{N}$ with $k \geq 2$, we define

$$(4.2.1) \quad e_{k,8}(x) := \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (136 - 136x + x^2 + (120 - 60x)(1-x)^{1/2})^t \\ \times \left(2080 - 3120x + 1038x^2 + x^3 + (2016 - 2016x + 126x^2)(1-x)^{1/2} \right)^u$$

$$(4.2.2) \quad = e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2},$$

where $e_{k,8}^{(0)}(x), e_{k,8}^{(1)}(x) \in \mathbb{Q}[x]$ and the coefficients $a_k(t,u)$ are given in Theorem 2.9.1.

Taking $k = 2$ in (4.2.1) we see that (as $t = 1, u = 0, a_2(1,0) = 1$)

$$e_{2,8}^{(0)}(x) = \frac{136 - 136x + x^2}{256}$$

and

$$e_{2,8}^{(1)}(x) = \frac{120 - 60x}{256} = \frac{30 - 15x}{64}.$$

Taking $k = 3$ in (4.2.1) we see that (as $t = 0$, $u = 1$, $a_3(0, 1) = 1$)

$$e_{3,8}^{(0)}(x) = \frac{2080 - 3120x + 1038x^2 + x^3}{4096}$$

and

$$e_{3,8}^{(1)}(x) = \frac{2016 - 2016x + 126x^2}{4096} = \frac{1008 - 1008x + 63x^2}{2048}.$$

The next lemma gives the values of the polynomials $e_{k,8}^{(0)}(x)$ and $e_{k,8}^{(1)}(x)$ at $x = 0$.

Lemma 4.2.1. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

- (i) $e_{k,8}(0) = 1$,
- (ii) $e_{k,8}^{(0)}(0) = \frac{2^{2k} + 1}{2^{2k+1}}$,
- (iii) $e_{k,8}^{(1)}(0) = \frac{2^{2k} - 1}{2^{2k+1}}$.

Proof. We have by (4.2.1), (4.2.2) and (2.9.1)

$$\begin{aligned} e_{k,8}(0) &= e_{k,8}^{(0)}(0) + e_{k,8}^{(1)}(0) = \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) (256)^t (4096)^u \\ &= \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) 2^{8t} 2^{12u} = \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) 2^{4k} = \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) = 1, \end{aligned}$$

which proves (i).

By Definition 4.2.1, we have

$$2 e_{k,8}^{(0)}(x) = \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t, u) \left((136 - 136x + x^2 + (120 - 60x)(1 - x)^{1/2})^t \right)$$

$$\begin{aligned}
 & \times (2080 - 3120x + 1038x^2 + x^3 + (2016 - 2016x + 126x^2)(1-x)^{1/2})^u \\
 & + (136 - 136x + x^2 - (120 - 60x)(1-x)^{1/2})^t \\
 & \times (2080 - 3120x + 1038x^2 + x^3 - (2016 - 2016x + 126x^2)(1-x)^{1/2})^u.
 \end{aligned}$$

$$\text{Thus } 2e_{k,8}^{(0)}(0) = \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (2^{8t+12u} + 2^{4t+6u}) = \frac{1}{2^{4k}} (2^{4k} + 2^{2k}) = \frac{2^{2k} + 1}{2^{2k}},$$

so that $e_{k,8}^{(0)}(0) = \frac{2^{2k} + 1}{2^{2k+1}}$, which is (ii).

Finally we deduce

$$e_{k,8}^{(1)}(0) = e_{k,8}(0) - e_{k,8}^{(0)}(0) = 1 - \left(\frac{1}{2} + \frac{1}{2^{2k+1}} \right) = \frac{2^{2k} - 1}{2^{2k+1}},$$

which is (iii). □

The next lemma gives the value of the polynomial $e_{k,0}^{(0)}(x)$ at $x = 1$.

Lemma 4.2.2. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

$$(4.2.3) \quad e_{k,8}(1) = e_{k,8}^{(0)}(1) = \frac{(-1)^k}{2^{4k}}.$$

Proof. We first note that by (4.2.2), we have $e_{k,8}(1) = e_{k,8}^{(0)}(1) + e_{k,8}^{(1)}(1) \sqrt{1-1} = e_{k,8}^{(0)}(1)$.

Moreover, by (2.9.1) and (4.2.1), we have

$$e_{k,8}(1) = \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (1)^t (-1)^u = \frac{(-1)^k}{2^{4k}},$$

completing the proof of Lemma 4.2.2. □

We conclude this section with a lemma that gives information about the degrees of the polynomials $e_{k,8}^{(0)}(x)$ and $e_{k,8}^{(1)}(x)$.

Lemma 4.2.3. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

- (i) $\deg(e_{k,8}^{(0)}(x)) = k$,
- (ii) $\deg(e_{k,8}^{(1)}(x)) \leq k - 1$,
- (iii) the coefficient of x^k in $e_{k,8}^{(0)}(x)$ is $\frac{1}{2^{4k}}$.

Proof. Using the binomial theorem, the right hand side of (4.2.1) becomes

$$\begin{aligned}
& \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \sum_{\substack{(d,e) \in \mathbb{N}_0^2 \\ d+e=t}} \binom{t}{d} (136 - 136x + x^2)^d (120 - 60x)^e (1-x)^{e/2} \\
& \times \sum_{\substack{(f,g) \in \mathbb{N}_0^2 \\ f+g=u}} \binom{u}{f} (2080 - 3120x + 1038x^2 + x^3)^f (2016 - 2016x + 126x^2)^g (1-x)^{g/2} \\
& = \frac{1}{2^{4k}} \sum_{\substack{(d,e,f,g) \in \mathbb{N}_0^4 \\ 2d+2e+3f+3g=k}} a_k(d+e, f+g) \binom{d+e}{d} \binom{f+g}{f} (136 - 136x + x^2)^d (120 - 60x)^e \\
& \times (2080 - 3120x + 1038x^2 + x^3)^f (2016 - 2016x + 126x^2)^g (1-x)^{(e+g)/2}.
\end{aligned}$$

Thus

$$e_{k,8}^{(1)}(x) = \frac{1}{2^{4k}} \sum_{\substack{(d,e,f,g) \in \mathbb{N}_0^4 \\ 2d+2e+3f+3g=k \\ e+g \equiv 1 \pmod{2}}} a_k(d+e, f+g) \binom{d+e}{d} \binom{f+g}{f} (136 - 136x + x^2)^d (120 - 60x)^e$$

$$\times (2080 - 3120x + 1038x^2 + x^3)^f (2016 - 2016x + 126x^2)^g (1 - x)^{(e+g-1)/2}.$$

The highest power of x in $e_{k,8}^{(1)}(x)$ is

$$\max_{\substack{(d,e,f,g) \in \mathbb{N}_0^4 \\ 2d+2e+3f+3g=k \\ e+g \equiv 1 \pmod{2}}} \left(2d + e + 3f + 2g + \frac{e+g-1}{2} \right) = \frac{1}{2} \max_{\substack{d,e,f,g \geq 0 \\ 2d+2e+3f+3g=k \\ e+g \equiv 1 \pmod{2}}} (4d + 2e + 6f + 4g + e + g - 1)$$

$$= \frac{1}{2} \max_{\substack{d,e,f,g \geq 0 \\ 2d+2e+3f+3g=k \\ e+g \equiv 1 \pmod{2}}} (4d + 3e + 6f + 5g - 1) = \frac{1}{2} \max_{\substack{d,e,f,g \geq 0 \\ 2d+2e+3f+3g=k \\ e+g \equiv 1 \pmod{2}}} (2k - e - g - 1)$$

$$= \frac{1}{2}(2k - 2) = k - 1,$$

proving (ii).

Also

$$(4.2.4) \quad e_{k,8}^{(0)}(x) = \frac{1}{2^{4k}} \sum_{\substack{(d,e,f,g) \in \mathbb{N}_0^4 \\ 2d+2e+3f+3g=k \\ e+g \equiv 0 \pmod{2}}} a_k(d+e, f+g) \binom{d+e}{d} \binom{f+g}{f}$$

$$\times (136 - 136x + x^2)^d (120 - 60x)^e$$

$$\times (2080 - 3120x + 1038x^2 + x^3)^f$$

$$\times (2016 - 2016x + 126x^2)^g (1 - x)^{(e+g)/2}.$$

The highest power of x in $e_{k,8}^{(0)}(x)$ is

$$\begin{aligned} \max_{\substack{(d,e,f,g) \in \mathbb{N}_0^4 \\ 2d+2e+3f+3g=k \\ e+g \equiv 0 \pmod{2}}} \left(2d + e + 3f + 2g + \frac{e+g}{2} \right) &= \frac{1}{2} \max_{\substack{d,e,f,g \geq 0 \\ 2d+2e+3f+3g=k \\ e+g \equiv 0 \pmod{2}}} (4d + 3e + 6f + 5g) \\ &= \frac{1}{2} \max_{\substack{d,e,f,g \geq 0 \\ 2d+2e+3f+3g=k \\ e+g \equiv 0 \pmod{2}}} (2k - e - g) = \frac{1}{2} 2k = k. \end{aligned}$$

From the last equation we note that the highest power of $e_{k,8}^{(0)}(x)$ occurs when $e+g=0$, that is when $e=g=0$. Thus by (4.2.4) and (2.9.1) the coefficient of x^k in $e_{k,8}^{(0)}(x)$ is

$$\frac{1}{2^{4k}} \sum_{\substack{(d,f) \in \mathbb{N}_0^2 \\ 2d+3f=k}} a_k(d, f) = \frac{1}{2^{4k}},$$

proving (iii). □

4.3 The function $e_{k,16}(x)$

Definition 4.3.1. For $k \in \mathbb{N}$ with $k \geq 2$, we define

$$\begin{aligned} (4.3.1) \quad e_{k,16}(x) &:= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t, u) \left(p_2(x) + q_2(x)(1-x)^{1/4} \right. \\ &\quad \left. + r_2(x)(1-x)^{1/2} + s_2(x)(1-x)^{3/4} \right)^t \\ &\quad \times \left(p_3(x) + q_3(x)(1-x)^{1/4} + r_3(x)(1-x)^{1/2} + s_3(x)(1-x)^{3/4} \right)^u \end{aligned}$$

$$(4.3.2) \quad = e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(x)(1-x)^{1/2} + e_{k,16}^{(3)}(x)(1-x)^{3/4},$$

where

$$p_2(x) = 1096 - 1096x + x^2,$$

$$q_2(x) = 960 - 840x,$$

$$r_2(x) = 1080 - 540x,$$

$$s_2(x) = 960 - 120x,$$

$$p_3(x) = 66592 - 99888x + 33294x^2 + x^3,$$

$$q_3(x) = 64512 - 79632x + 15372x^2,$$

$$r_3(x) = 66528 - 66528x + 4158x^2,$$

$$s_3(x) = 64512 - 49332x + 252x^2,$$

where $e_{k,16}^{(j)}(x) \in \mathbb{Q}[x]$ for $j \in \{0, 1, 2, 3\}$ and the coefficients $a_k(t, u)$ are given in Theorem 2.9.1.

We note that for $k = 2$, we have

$$e_{2,16}^{(0)}(x) = p_2(x) = 1096 - 1096x + x^2, \quad e_{2,16}^{(0)}(0) = p_2(0) = 1096,$$

$$e_{2,16}^{(1)}(x) = q_2(x) = 960 - 840x, \quad e_{2,16}^{(1)}(0) = q_2(0) = 960,$$

$$e_{2,16}^{(2)}(x) = r_2(x) = 1080 - 540x, \quad e_{2,16}^{(2)}(0) = r_2(0) = 1080,$$

$$e_{2,16}^{(3)}(x) = s_2(x) = 960 - 120x, \quad e_{2,16}^{(3)}(0) = s_2(0) = 960,$$

$$e_{3,16}^{(0)}(x) = p_3(x) = 66592 - 99888x + 33294x^2 + x^3, \quad e_{3,16}^{(0)}(0) = 66592,$$

$$e_{3,16}^{(1)}(x) = q_3(x) = 64512 - 79632x + 15372x^2, \quad e_{3,16}^{(1)}(0) = q_3(0) = 64512,$$

$$e_{3,16}^{(2)}(x) = r_3(x) = 66528 - 66528x + 4158x^2, \quad e_{3,16}^{(2)}(0) = r_3(0) = 66528,$$

$$e_{3,16}^{(3)}(x) = s_3(x) = 64512 - 49332x + 252x^2, \quad e_{3,16}^{(3)}(0) = s_3(0) = 64512.$$

Lemma 4.3.1. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

$$(i) \quad \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u)(p_2(0) + q_2(0) + r_2(0) + s_2(0))^t \\ \times (p_3(0) + q_3(0) + r_3(0) + s_3(0))^u = 1,$$

$$(ii) \quad \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u)(p_2(0) + q_2(0) - r_2(0) - s_2(0))^t \\ \times (p_3(0) + q_3(0) - r_3(0) - s_3(0))^u = \frac{1}{2^{4k}},$$

$$(iii) \quad \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u)(p_2(0) - q_2(0) - r_2(0) + s_2(0))^t \\ \times (p_3(0) - q_3(0) - r_3(0) + s_3(0))^u = \frac{1}{2^{4k}},$$

$$(iv) \quad \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u)(p_2(0) - q_2(0) + r_2(0) - s_2(0))^t \\ \times (p_3(0) - q_3(0) + r_3(0) - s_3(0))^u = \frac{1}{2^{2k}}.$$

Proof. We have

$$\begin{aligned} & \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u)(p_2(0) + q_2(0) + r_2(0) + s_2(0))^t (p_3(0) + q_3(0) + r_3(0) + s_3(0))^u \\ &= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u)(1096 + 960 + 1080 + 960)^t (66592 + 64512 + 66528 + 64512)^u \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (4096)^t (262144)^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (2)^{12t} (2)^{18u} = \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) 2^{6k} = \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) = 1,
\end{aligned}$$

which proves (i). Next

$$\begin{aligned}
&\frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (p_2(0) + q_2(0) - r_2(0) - s_2(0))^t (p_3(0) + q_3(0) - r_3(0) - s_3(0))^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (1096 + 960 - 1080 - 960)^t (66592 + 64512 - 66528 - 64512)^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (16)^t (64)^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (2)^{4t} (2)^{6u} = \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) 2^{2k} = \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) = \frac{1}{2^{4k}},
\end{aligned}$$

which proves (ii). Further

$$\begin{aligned}
&\frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (p_2(0) - q_2(0) - r_2(0) + s_2(0))^t (p_3(0) - q_3(0) - r_3(0) + s_3(0))^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (1096 - 960 - 1080 + 960)^t (66592 - 64512 - 66528 + 64512)^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (16)^t (64)^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (2)^{4t} (2)^{6u} = \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) 2^{2k} = \frac{1}{2^{4k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) = \frac{1}{2^{4k}},
\end{aligned}$$

which proves (iii). Finally

$$\begin{aligned}
& \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (p_2(0) - q_2(0) + r_2(0) - s_2(0))^t (p_3(0) - q_3(0) + r_3(0) - s_3(0))^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (1096 - 960 + 1080 - 960)^t (66592 - 64512 + 66528 - 64512)^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (256)^t (4096)^u \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (2)^{8t} (2)^{12u} = \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) 2^{4k} = \frac{1}{2^{2k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) = \frac{1}{2^{2k}},
\end{aligned}$$

which proves (iv). $\square$

The next lemma gives the values of the polynomials $e_{k,16}^{(0)}(x)$, $e_{k,16}^{(1)}(x)$, $e_{k,16}^{(2)}(x)$ and $e_{k,16}^{(3)}(x)$ at $x = 0$.

Lemma 4.3.2. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

$$\begin{aligned}
\text{(i)} \quad & e_{k,16}(0) = 1, \\
\text{(ii)} \quad & e_{k,16}^{(0)}(0) = \frac{2^{2k-1}(2^{2k} + 1) + 1}{2^{4k+1}}, \\
\text{(iii)} \quad & e_{k,16}^{(1)}(0) = \frac{2^{2k} - 1}{2^{2k+2}}, \\
\text{(iv)} \quad & e_{k,16}^{(2)}(0) = \frac{2^{2k-1}(2^{2k} + 1) - 1}{2^{4k+1}}, \\
\text{(v)} \quad & e_{k,16}^{(3)}(0) = \frac{2^{2k} - 1}{2^{2k+2}}.
\end{aligned}$$

Proof. (i) By Definition 4.3.1 and Lemma 4.3.1(i), we have

$$\begin{aligned}
e_{k,16}(0) &= e_{k,16}^{(0)}(0) + e_{k,16}^{(1)}(0) + e_{k,16}^{(2)}(0) + e_{k,16}^{(3)}(0) \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (p_2(0) + q_2(0) + r_2(0) + s_2(0))^t (p_3(0) + q_3(0) + r_3(0) + s_3(0))^u
\end{aligned}$$

= 1.

(ii) By Definition 4.3.1 we have

$$\begin{aligned}
4e_{k,16}^{(0)}(x) &= e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(x)(1-x)^{1/2} + e_{k,16}^{(3)}(x)(1-x)^{3/4} \\
&\quad + e_{k,16}^{(0)}(x) - e_{k,16}^{(1)}(x)(1-x)^{1/4} - e_{k,16}^{(2)}(x)(1-x)^{1/2} + e_{k,16}^{(3)}(x)(1-x)^{3/4} \\
&\quad + e_{k,16}^{(0)}(x) - e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(x)(1-x)^{1/2} - e_{k,16}^{(3)}(x)(1-x)^{3/4} \\
&\quad + e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} - e_{k,16}^{(2)}(x)(1-x)^{1/2} - e_{k,16}^{(3)}(x)(1-x)^{3/4} \\
&= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) + q_2(x)(1-x)^{1/4} + r_2(x)(1-x)^{1/2} + s_2(x)(1-x)^{3/4} \right)^t \\
&\quad \times \left(p_3(x) + q_3(x)(1-x)^{1/4} + r_3(x)(1-x)^{1/2} + s_3(x)(1-x)^{3/4} \right)^u \\
&\quad + \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) - q_2(x)(1-x)^{1/4} - r_2(x)(1-x)^{1/2} + s_2(x)(1-x)^{3/4} \right)^t \\
&\quad \times \left(p_3(x) - q_3(x)(1-x)^{1/4} - r_3(x)(1-x)^{1/2} + s_3(x)(1-x)^{3/4} \right)^u \\
&\quad + \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) - q_2(x)(1-x)^{1/4} + r_2(x)(1-x)^{1/2} - s_2(x)(1-x)^{3/4} \right)^t \\
&\quad \times \left(p_3(x) - q_3(x)(1-x)^{1/4} + r_3(x)(1-x)^{1/2} - s_3(x)(1-x)^{3/4} \right)^u \\
&\quad + \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) + q_2(x)(1-x)^{1/4} - r_2(x)(1-x)^{1/2} - s_2(x)(1-x)^{3/4} \right)^t \\
&\quad \times \left(p_3(x) + q_3(x)(1-x)^{1/4} - r_3(x)(1-x)^{1/2} - s_3(x)(1-x)^{3/4} \right)^u.
\end{aligned}$$

Thus, by Lemma 4.3.1, we obtain

$$\begin{aligned}
4e_{k,16}^{(0)}(0) &= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (p_2(0) + q_2(0) + r_2(0) + s_2(0))^t (p_3(0) + q_3(0) + r_3(0) + s_3(0))^u
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (p_2(0) - q_2(0) - r_2(0) + s_2(0))^t (p_3(0) - q_3(0) - r_3(0) + s_3(0))^u \\
& + \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (p_2(0) - q_2(0) + r_2(0) - s_2(0))^t (p_3(0) - q_3(0) + r_3(0) - s_3(0))^u \\
& + \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (p_2(0) + q_2(0) - r_2(0) - s_2(0))^t (p_3(0) + q_3(0) - r_3(0) - s_3(0))^u \\
& = 1 + \frac{1}{2^{4k}} + \frac{1}{2^{2k}} + \frac{1}{2^{4k}} = \frac{2^{2k-1}(2^{2k} + 1) + 1}{2^{4k-1}}.
\end{aligned}$$

Hence we have

$$e_{k,16}^{(0)}(0) = \frac{2^{2k-1}(2^{2k} + 1) + 1}{2^{4k+1}}.$$

(iii) By Definition 4.3.1 we have

$$\begin{aligned}
& 2e_{k,16}^{(0)}(x) + 2e_{k,16}^{(1)}(x) \\
& = \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) + q_2(x)(1-x)^{1/4} + r_2(x)(1-x)^{1/2} + s_2(x)(1-x)^{3/4} \right)^t \\
& \quad \times \left(p_3(x) + q_3(x)(1-x)^{1/4} + r_3(x)(1-x)^{1/2} + s_3(x)(1-x)^{3/4} \right)^u \\
& + \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) + q_2(x)(1-x)^{1/4} - r_2(x)(1-x)^{1/2} - s_2(x)(1-x)^{3/4} \right)^t \\
& \quad \times \left(p_3(x) + q_3(x)(1-x)^{1/4} - r_3(x)(1-x)^{1/2} - s_3(x)(1-x)^{3/4} \right)^u.
\end{aligned}$$

Thus, by Lemma 4.3.1(i)(ii), we deduce

$$\begin{aligned}
& 2e_{k,16}^{(0)}(0) + 2e_{k,16}^{(1)}(0) \\
& = \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (p_2(0) + q_2(0) + r_2(0) + s_2(0))^t (p_3(0) + q_3(0) + r_3(0) + s_3(0))^u \\
& + \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (p_2(0) + q_2(0) - r_2(0) - s_2(0))^t (p_3(0) + q_3(0) - r_3(0) - s_3(0))^u
\end{aligned}$$

$$= 1 + \frac{1}{2^{4k}}.$$

This implies that

$$2 e_{k,16}^{(1)}(0) = \left(1 + \frac{1}{2^{4k}}\right) - 2e_{k,16}^{(0)}(0) = \frac{2^{2k} - 1}{2^{2k+1}}.$$

Hence we have

$$e_{k,16}^{(1)}(0) = \frac{2^{2k} - 1}{2^{2k+2}}.$$

(iv) By Definition 4.3.1 we have

$$\begin{aligned} & 2e_{k,16}^{(0)}(x) + 2e_{k,16}^{(2)}(x) \\ &= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) + q_2(x)(1-x)^{1/4} + r_2(x)(1-x)^{1/2} + s_2(x)(1-x)^{3/4} \right)^t \\ & \quad \times \left(p_3(x) + q_3(x)(1-x)^{1/4} + r_3(x)(1-x)^{1/2} + s_3(x)(1-x)^{3/4} \right)^u \\ &+ \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) - q_2(x)(1-x)^{1/4} + r_2(x)(1-x)^{1/2} - s_2(x)(1-x)^{3/4} \right)^t \\ & \quad \times \left(p_3(x) - q_3(x)(1-x)^{1/4} + r_3(x)(1-x)^{1/2} - s_3(x)(1-x)^{3/4} \right)^u. \end{aligned}$$

Thus, by Lemma 4.3.1(i)(iv), we have

$$\begin{aligned} & 2e_{k,16}^{(0)}(0) + 2e_{k,16}^{(2)}(0) \\ &= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (p_2(0) + q_2(0) + r_2(0) + s_2(0))^t (p_3(0) + q_3(0) + r_3(0) + s_3(0))^u \\ &+ \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0 \\ 2t+3u=k}} a_k(t,u) (p_2(0) - q_2(0) + r_2(0) - s_2(0))^t (p_3(0) - q_3(0) + r_3(0) - s_3(0))^u \\ &= 1 + \frac{1}{2^{2k}}. \end{aligned}$$

This implies that

$$2 e_{k,16}^{(2)}(0) = \left(1 + \frac{1}{2^{2k}}\right) - 2e_{k,16}^{(0)}(0) = \frac{2^{2k-1}(2^{2k} + 1) - 1}{2^{4k}}.$$

Hence we have

$$e_{k,16}^{(2)}(0) = \frac{2^{2k-1}(2^{2k} + 1) - 1}{2^{4k+1}}.$$

(v) Finally as $1 = e_{k,16}(0) = e_{k,16}^{(0)}(0) + e_{k,16}^{(1)}(0) + e_{k,16}^{(2)}(0) + e_{k,16}^{(3)}(0)$, we have

$$e_{k,16}^{(3)}(0) = 1 - e_{k,16}^{(0)}(0) - e_{k,16}^{(1)}(0) - e_{k,16}^{(2)}(0) = \frac{2^{2k} - 1}{2^{2k+2}}.$$

This completes the proof of Lemma 4.3.2. $\square$

The next lemma gives the value of the polynomial $e_{k,16}^{(0)}(x)$ at $x = 1$.

Lemma 4.3.3. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

$$e_{k,16}(1) = e_{k,16}^{(0)}(1) = \frac{(-1)^k}{2^{6k}}.$$

Proof. We first note from Definition 4.3.1 that

$$e_{k,16}(1) = e_{k,16}^{(0)}(1) + e_{k,16}^{(1)}(1)(1-1)^{1/4} + e_{k,16}^{(2)}(1)(1-1)^{1/2} + e_{k,16}^{(3)}(1)(1-1)^{3/4} = e_{k,16}^{(0)}(1).$$

Moreover, from Definition 4.3.1 and (2.9.1), we have

$$\begin{aligned} e_{k,16}(1) &= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(1) + q_2(1)(1-1)^{1/4} + r_2(1)(1-1)^{1/2} + s_2(1)(1-1)^{3/4} \right)^t \\ &\quad \times \left(p_3(1) + q_3(1)(1-1)^{1/4} + r_3(1)(1-1)^{1/2} + s_3(1)(1-1)^{3/4} \right)^u \\ &= \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (p_2(1))^t (p_3(1))^u = \frac{1}{2^{6k}} \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) (1)^t (-1)^u \\ &= \frac{(-1)^k}{2^{6k}}, \end{aligned}$$

as asserted. $\square$

The final lemma of this section gives information on the degree of the polynomial $e_{k,16}^{(i)}(x)$ for $i \in \{0, 1, 2, 3\}$.

Lemma 4.3.4. *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

- i) $\deg(e_{k,16}^{(i)}(x)) \leq k - 1$ for $i \in \{1, 2, 3\}$,
- ii) $\deg(e_{k,16}^{(0)}(x)) = k$,
- iii) the coefficient of x^k in $e_{k,16}^{(0)}(x)$ is $\frac{1}{2^{6k}}$.

Proof. By the multinomial theorem we have

$$\begin{aligned}
& \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(p_2(x) + q_2(x)(1-x)^{1/4} + r_2(x)(1-x)^{1/2} + s_2(x)(1-x)^{3/4} \right)^t \\
& \quad \times \left(p_3(x) + q_3(x)(1-x)^{1/4} + r_3(x)(1-x)^{1/2} + s_3(x)(1-x)^{3/4} \right)^u \\
&= \sum_{\substack{(t,u) \in \mathbb{N}_0^2 \\ 2t+3u=k}} a_k(t,u) \left(\sum_{\substack{(t_0,t_1,t_2,t_3) \in \mathbb{N}_0^4 \\ t_0+t_1+t_2+t_3=t}} \binom{t}{t_0,t_1,t_2,t_3} p_2(x)^{t_0} q_2(x)^{t_1} r_2(x)^{t_2} s_2(x)^{t_3} (1-x)^{(t_1+2t_2+3t_3)/4} \right) \\
& \quad \times \left(\sum_{\substack{(u_0,u_1,u_2,u_3) \in \mathbb{N}_0^4 \\ u_0+u_1+u_2+u_3=u}} \binom{u}{u_0,u_1,u_2,u_3} p_3(x)^{u_0} q_3(x)^{u_1} r_3(x)^{u_2} s_3(x)^{u_3} (1-x)^{(u_1+2u_2+3u_3)/4} \right) \\
&= \sum_{\substack{(t_0,t_1,t_2,t_3,u_0,u_1,u_2,u_3) \in \mathbb{N}_0^8 \\ 2(t_0+t_1+t_2+t_3)+3(u_0+u_1+u_2+u_3)=k}} a_k(t_0+\dots+t_3, u_0+\dots+u_3) \binom{t_0+\dots+t_3}{t_0,t_1,t_2,t_3} \binom{u_0+\dots+u_3}{u_0,u_1,u_2,u_3} \\
& \quad \times p_2(x)^{t_0} q_2(x)^{t_1} r_2(x)^{t_2} s_2(x)^{t_3} p_3(x)^{u_0} q_3(x)^{u_1} r_3(x)^{u_2} s_3(x)^{u_3} (1-x)^{(t_1+2t_2+3t_3+u_1+2u_2+3u_3)/4}.
\end{aligned}$$

Thus, by Definition 4.3.1, for $i \in \{0, 1, 2, 3\}$ we have

$$e_{k,16}^{(i)}(x)$$

$$= \frac{1}{2^{6k}} \sum_{\substack{(t_0, t_1, t_2, t_3, u_0, u_1, u_2, u_3) \in \mathbb{N}_0^8 \\ 2(t_0+t_1+t_2+t_3)+3(u_0+u_1+u_2+u_3)=k \\ t_1+2t_2+3t_3+u_1+2u_2+3u_3 \equiv i \pmod{4}}} a_k(t_0 + \cdots + t_3, u_0 + \cdots + u_3) \binom{t_0 + \cdots + t_3}{t_0, t_1, t_2, t_3} \binom{u_0 + \cdots + u_3}{u_0, u_1, u_2, u_3}$$

$$\times p_2(x)^{t_0} q_2(x)^{t_1} r_2(x)^{t_2} s_2(x)^{t_3} p_3(x)^{u_0} q_3(x)^{u_1} r_3(x)^{u_2} s_3(x)^{u_3} (1-x)^{(t_1+2t_2+3t_3+u_1+2u_2+3u_3-i)/4}.$$

The highest power of x in $e_{k,16}^{(i)}(x)$ is at most

$$\begin{aligned} & \max_{\substack{(t_0, t_1, t_2, t_3, u_0, u_1, u_2, u_3) \in \mathbb{N}_0^8 \\ 2(t_0+t_1+t_2+t_3)+3(u_0+u_1+u_2+u_3)=k \\ t_1+2t_2+3t_3+u_1+2u_2+3u_3 \equiv i \pmod{4}}} \left(2t_0 + t_1 + t_2 + t_3 + 3u_0 + 2u_1 + 2u_2 + 2u_3 \right. \\ & \quad \left. + \frac{t_1 + 2t_2 + 3t_3 + u_1 + 2u_2 + 3u_3 - i}{4} \right) \\ &= \frac{1}{4} \max_{\substack{(t_0, t_1, t_2, t_3, u_0, u_1, u_2, u_3) \in \mathbb{N}_0^8 \\ 2(t_0+t_1+t_2+t_3)+3(u_0+u_1+u_2+u_3)=k \\ t_1+2t_2+3t_3+u_1+2u_2+3u_3 \equiv i \pmod{4}}} \left(8t_0 + 5t_1 + 6t_2 + 7t_3 + 12u_0 + 9u_1 + 10u_2 + 11u_3 - i \right) \\ &= \frac{1}{4} \max_{\substack{(t_0, t_1, t_2, t_3, u_0, u_1, u_2, u_3) \in \mathbb{N}_0^8 \\ 2(t_0+t_1+t_2+t_3)+3(u_0+u_1+u_2+u_3)=k \\ t_1+2t_2+3t_3+u_1+2u_2+3u_3 \equiv i \pmod{4}}} \left(4k - 3t_1 - 2t_2 - t_3 - 3u_1 - 2u_2 - u_3 - i \right) \\ &= \begin{cases} \frac{1}{4}(4k-4) = k-1, & \text{if } i \in \{1, 2, 3\}, \\ \frac{1}{4}(4k) = k, & \text{if } i = 0. \end{cases} \end{aligned}$$

From the last equation we note that the highest power of x in $e_{k,16}^{(0)}(x)$ occurs when $t_1 + 2t_2 + 3t_3 + u_1 + 2u_2 + 3u_3 = 0$, that is when $t_1 = t_2 = t_3 = u_1 = u_2 = u_3 = 0$. This shows that the coefficient of x^k is $\frac{1}{2^{6k}}$. $\square$

4.4 The identity $E_k(q^r) = e_{k,r}(x) z^{2k}$

In this section we show that for $k \in \mathbb{N}$ with $k \geq 2$ and $r \in \{1, 2, 4, 8, 16\}$, the Eisenstein series $E_k(q^r) = e_{k,r}(x) z^{2k}$ with $x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}$ and $z = \varphi^2(q)$.

Theorem 4.4.1. *For $q \in \mathbb{C}$ with $|q| < 1$ define $x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}$ and $z = \varphi^2(q)$. Then, for $k \in \mathbb{N}$ with $k \geq 2$ and $q \in D_\delta$, we have*

- (i) $E_k(q) = e_{k,1}(x) z^{2k}$,
- (ii) $E_k(q^2) = e_{k,2}(x) z^{2k}$,
- (iii) $E_k(q^4) = e_{k,4}(x) z^{2k}$,
- (iv) $E_k(q^8) = e_{k,8}(x) z^{2k}$,
- (v) $E_k(q^{16}) = e_{k,16}(x) z^{2k}$.

Proof. (i) This follows from Theorem 2.9.1, (2.9.5), (2.9.8), (4.1.1) and the fact that $z^{4t+6u} = z^{2k}$ when $2t + 3u = k$.

(ii) This follows from Theorem 2.9.1, (2.9.6), (2.9.9), (4.1.2) and the fact that $z^{4t+6u} = z^{2k}$ when $2t + 3u = k$.

(iii) This follows from Theorem 2.9.1, (2.9.7), (2.9.10), (4.1.3) and the fact that $z^{4t+6u} = z^{2k}$ when $2t + 3u = k$.

(iv) This follows from Theorem 2.9.1, (2.9.12), (2.9.14), (4.2.1) and the fact that $z^{4t+6u} = z^{2k}$ when $2t + 3u = k$. Note that $q \in D_\delta$ is needed for (2.9.12) and (2.9.14).

(v) This follows from Theorem 2.9.1, (2.9.13), (2.9.15), (4.3.1) and the fact

that $z^{4t+6u} = z^{2k}$ when $2t + 3u = k$. Note that $q \in D_\delta$ is needed for (2.9.13) and (2.9.15). $\square$

Chapter 5

The polynomials $P_{k,r,s}(x)$, $Q_{k,r,s}(x)$, $R_{k,r,s}(x)$ and $S_{k,r,s}(x)$

Let $r, s \in \mathbb{N}_0$ with $(r, s) \neq (0, 0)$. Let $k \in \mathbb{N}$ be such that $k \geq 2$ and $2r + s < 4k$. In this chapter we define four polynomials $P_{k,r,s}(x)$, $Q_{k,r,s}(x)$, $R_{k,r,s}(x)$ and $S_{k,r,s}(x)$. These polynomials play an important role in the proofs of our main theorems in Chapter 6. As in Section 3.3 we consider three cases.

- (i) $r \geq 1, s = 0$ and $r < 2k$ (Section 5.1),
- (ii) $r \geq 1, s \geq 1$ and $2r + s < 4k$ (Section 5.2),
- (iii) $r = 0, s \geq 1$ and $s < 4k$ (Section 5.3).

Each section of this chapter considers one of these three particular cases. For each case some properties of the four polynomials are given. In particular, we are interested in the values of $P_{k,r,s}(0)$, $P_{k,r,s}(1)$, $Q_{k,r,s}(0)$, $R_{k,r,s}(0)$, $S_{k,r,s}(0)$ as well as an upper bound for the degree of each of these polynomials.

Let $h \in \mathbb{N}_0$. Throughout this chapter, if $f(x)$ is a polynomial in x over $\mathbb{Q}$

of degree at most h , then

$$f(x) = a_0 + a_1x + \cdots + a_hx^h, \quad a_0, \dots, a_h \in \mathbb{Q}.$$

We abbreviate this by writing

$$f(x) = a_hx^h + \cdots.$$

We emphasize with this notation that a_h may be 0.

5.1 The polynomials $P_{k,r,0}(x)$, $Q_{k,r,0}(x)$, $R_{k,r,0}(x)$ and

$$S_{k,r,0}(x)$$

We begin this section with the definition of the polynomials $P_{k,r,0}(x)$, $Q_{k,r,0}(x)$, $R_{k,r,0}(x)$ and $S_{k,r,0}(x)$. Recall that the polynomials $A_{r,s}(x)$, $B_{r,s}(x)$, $C_{r,s}(x)$ and $D_{r,s}(x)$ are defined in Definition 3.1.1.

Definition 5.1.1. For $k, r \in \mathbb{N}$ with $k \geq 2$ and $r < 2k$, we define

$$(i) \quad P_{k,r,0}(x) = A_{r,0}(x) - \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,1}(x) + \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,2}(x)$$

$$+ \frac{1}{2^{2k}-1} e_{k,4}(x) - \frac{2^{2k}}{2^{2k}-1} e_{k,8}^{(0)}(x),$$

$$(ii) \quad Q_{k,r,0}(x) = 0,$$

$$(iii) \quad R_{k,r,0}(x) = C_{r,0}(x) - \frac{2^{2k}}{2^{2k}-1} e_{k,8}^{(1)}(x),$$

$$(iv) \quad S_{k,r,0}(x) = 0.$$

The following lemma justifies our definition of these four polynomials and plays a key part in the proof of our main theorems.

Lemma 5.1.1. *For $k, r \in \mathbb{N}$ with $k \geq 2$ and $r < 2k$, we have*

$$\begin{aligned} & P_{k,r,0}(x) + Q_{k,r,0}(x)(1-x)^{1/4} + R_{k,r,0}(x)(1-x)^{1/2} + S_{k,r,0}(x)(1-x)^{3/4} \\ &= \left(\frac{1 + (1-x)^{1/2}}{2} \right)^r - \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,1}(x) + \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,2}(x) + \frac{1}{2^{2k}-1} e_{k,4}(x) \\ &\quad - \frac{2^{2k}}{2^{2k}-1} \left(e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2} \right). \end{aligned}$$

Proof. From Definition 5.1.1 and Lemma 3.1.3, we have

$$\begin{aligned} & \left(\frac{1 + (1-x)^{1/2}}{2} \right)^r - \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,1}(x) + \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,2}(x) \\ &+ \frac{1}{2^{2k}-1} e_{k,4}(x) - \frac{2^{2k}}{2^{2k}-1} \left(e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2} \right) \\ &= A_{r,0}(x) + C_{r,0}(x)(1-x)^{1/2} - \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,1}(x) + \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,2}(x) \\ &+ \frac{1}{2^{2k}-1} e_{k,4}(x) - \frac{2^{2k}}{2^{2k}-1} \left(e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2} \right) \\ &= P_{k,r,0}(x) + Q_{k,r,0}(x)(1-x)^{1/4} + R_{k,r,0}(x)(1-x)^{1/2} + S_{k,r,0}(x)(1-x)^{3/4}, \end{aligned}$$

which completes the proof. $\square$

In the next lemma we evaluate the polynomial $P_{k,r,0}(x)$ at $x = 0$ and $x = 1$, and give an upper bound for the degree of $P_{k,r,0}(x)$ when $r < 2k$.

Lemma 5.1.2. *For $k, r \in \mathbb{N}$, $k \geq 2$ and $r < 2k$, we have*

- (i) $P_{k,r,0}(0) = 0$,
- (ii) $P_{k,r,0}(1) = 0$,
- (iii) $\deg(P_{k,r,0}(x)) \leq k - 1$.

Proof. (i) By Definition 5.1.1(i), Lemmas 3.3.1, 4.1.1(i), 4.1.2(i), 4.1.3(i) and 4.2.1 (ii), we have

$$\begin{aligned}
 P_{k,r,0}(0) &= A_{r,0}(0) - \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,1}(0) + \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,2}(0) \\
 &\quad + \frac{1}{2^{2k}-1} e_{k,4}(0) - \frac{2^{2k}}{2^{2k}-1} e_{k,8}^{(0)}(0) \\
 &= \frac{1}{2} - \frac{(-1)^k}{2^r(2^{2k}-1)} + \frac{(-1)^k}{2^r(2^{2k}-1)} + \frac{1}{2^{2k}-1} - \frac{2^{2k}}{2^{2k}-1} \frac{2^{2k}+1}{2^{2k+1}} \\
 &= \frac{1}{2} + \frac{1}{2^{2k}-1} - \frac{2^{2k}+1}{2(2^{2k}-1)} = 0,
 \end{aligned}$$

which proves (i).

(ii) By Lemmas 5.1.1, 4.1.1(ii), 4.1.2(ii), 4.1.3(ii) and 4.2.2, we have

$$\begin{aligned}
 P_{k,r,0}(1) &= \frac{1}{2^r} - \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,1}(1) + \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,2}(1) \\
 &\quad + \frac{1}{2^{2k}-1} e_{k,4}(1) - \frac{2^{2k}}{2^{2k}-1} e_{k,8}^{(0)}(1) \\
 &= \frac{1}{2^r} - \frac{2^{2k}}{2^r(2^{2k}-1)} + \frac{1}{2^r(2^{2k}-1)} + \frac{(-1)^k}{2^{2k}(2^{2k}-1)} - \frac{2^{2k}}{2^{2k}-1} \frac{(-1)^k}{2^{4k}}
 \end{aligned}$$

$$= 0,$$

which proves (ii).

(iii) By Lemmas 3.1.2 (i), 4.1.1 (iii)(iv), 4.1.2 (iii)(iv), 4.1.3 (iii)(iv) and 4.2.3 (i)(iii), we have

$$A_{r,0}(x) = a_{k-1}x^{k-1} + \cdots,$$

$$e_{k,1}(x) = x^k + \cdots,$$

$$e_{k,2}(x) = x^k + \cdots,$$

$$e_{k,4}(x) = \frac{1}{2^{2k}} x^k + \cdots,$$

$$e_{k,8}^{(0)}(x) = \frac{1}{2^{4k}} x^k + \cdots.$$

Then, by Definition 5.1.1(i), we obtain

$$P_{k,r,0}(x) = \left(\frac{-(-1)^k}{2^r(2^{2k}-1)} + \frac{(-1)^k}{2^r(2^{2k}-1)} + \frac{1}{2^{2k}-1} \frac{1}{2^{2k}} - \frac{2^{2k}}{2^{2k}-1} \frac{1}{2^{4k}} \right) x^k + \cdots$$

$$= 0 x^k + \cdots,$$

which shows that $\deg(P_{k,r,0}(x)) \leq k-1$. □

We conclude this section by evaluating the polynomial $R_{k,r,0}(x)$ at $x = 0$ and giving an upper bound for the degree of $R_{k,r,0}(x)$.

Lemma 5.1.3. *For $k, r \in \mathbb{N}$, $k \geq 2$ and $r < 2k$, we have*

$$(i) \quad R_{k,r,0}(0) = 0,$$

$$(ii) \quad \deg(R_{k,r,0}(x)) \leq k-1.$$

Proof. (i) By Definition 5.1.1(iii), Lemmas 3.3.1 and 4.2.1(iii), we have

$$R_{k,r,0}(0) = C_{r,0}(0) - \frac{2^{2k}}{2^{2k}-1} e_{k,8}^{(1)}(0) = \frac{1}{2} - \frac{2^{2k}}{2^{2k}-1} \frac{2^{2k}-1}{2^{2k+1}} = \frac{1}{2} - \frac{1}{2} = 0,$$

which proves (i).

(ii) This follows from Definition 5.1.1(iii), Lemmas 3.1.2(iii) and 4.2.3(ii). $\square$

5.2 The polynomials $P_{k,r,s}(x)$, $Q_{k,r,s}(x)$, $R_{k,r,s}(x)$ and

$$S_{k,r,s}(x)$$

We begin this section with the definition of the polynomials $P_{k,r,s}(x)$, $Q_{k,r,s}(x)$, $R_{k,r,s}(x)$ and $S_{k,r,s}(x)$.

Definition 5.2.1. For $k, r, s \in \mathbb{N}$ with $k \geq 2$ and $2r + s < 4k$, we define

$$(i) \ P_{k,r,s}(x) = A_{r,s}(x) - \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,1}(x) + \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,2}(x)$$

$$+ \frac{1}{2^{2k}-1} e_{k,8}^{(0)}(x) - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(0)}(x),$$

$$(ii) \ Q_{k,r,s}(x) = B_{r,s}(x) - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(1)}(x),$$

$$(iii) \ R_{k,r,s}(x) = C_{r,s}(x) + \frac{1}{2^{2k}-1} e_{k,8}^{(1)}(x) - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(2)}(x),$$

$$(iv) \ S_{k,r,s}(x) = D_{r,s}(x) - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(3)}(x).$$

The following lemma justifies our definition of the four polynomials and plays a key part in the proof of our main theorems.

Lemma 5.2.1. *For $k, r, s \in \mathbb{N}$ with $k \geq 2$ and $2r + s < 4k$, we have*

$$\begin{aligned}
& P_{k,r,s}(x) + Q_{k,r,s}(x)(1-x)^{1/4} + R_{k,r,s}(x)(1-x)^{1/2} + S_{k,r,s}(x)(1-x)^{3/4} \\
&= \left(\frac{1 + (1-x)^{1/2}}{2} \right)^r \left(\frac{1 + (1-x)^{1/4}}{2} \right)^s - \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,1}(x) \\
&+ \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,2}(x) + \frac{1}{2^{2k}-1} (e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2}) \\
&- \frac{2^{2k}}{2^{2k}-1} \left(e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(1-x)^{1/2} + e_{k,16}^{(3)}(1-x)^{3/4} \right).
\end{aligned}$$

Proof. From Lemma 3.1.3(i) and Definition 5.2.1, we have

$$\begin{aligned}
& \left(\frac{1 + (1-x)^{1/2}}{2} \right)^r \left(\frac{1 + (1-x)^{1/4}}{2} \right)^s - \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,1}(x) \\
&+ \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,2}(x) + \frac{1}{2^{2k}-1} (e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2}) \\
&- \frac{2^{2k}}{2^{2k}-1} \left(e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(1-x)^{1/2} + e_{k,16}^{(3)}(1-x)^{3/4} \right) \\
&= A_{r,s}(x) + B_{r,s}(x)(1-x)^{1/4} + C_{r,s}(x)(1-x)^{1/2} + D_{r,s}(x)(1-x)^{3/4} \\
&- \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,1}(x) + \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,2}(x) \\
&+ \frac{1}{2^{2k}-1} (e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2})
\end{aligned}$$

$$\begin{aligned}
& - \frac{2^{2k}}{2^{2k} - 1} \left(e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(1-x)^{1/2} + e_{k,16}^{(3)}(1-x)^{3/4} \right) \\
& = P_{k,r,s}(x) + Q_{k,r,s}(x)(1-x)^{1/4} + R_{k,r,s}(x)(1-x)^{1/2} + S_{k,r,s}(x)(1-x)^{3/4},
\end{aligned}$$

as asserted. $\square$

In the next lemma we evaluate the polynomial $P_{k,r,s}(x)$ at $x = 0$ and $x = 1$, and give an upper bound for the degree of $P_{k,r,s}(x)$ when $2r + s < 4k$.

Lemma 5.2.2. *For $k, r, s \in \mathbb{N}$ with $k \geq 2$ and $2r + s < 4k$, we have*

- (i) $P_{k,r,s}(0) = 0$,
- (ii) $P_{k,r,s}(1) = 0$,
- (iii) $\deg(P_{k,r,s}(x)) \leq k - 1$.

Proof. (i) By Definition 5.2.1(i), Lemmas 3.3.2, 4.1.1(i), 4.1.2(i), 4.2.1(ii) and 4.3.2 (ii), we have

$$\begin{aligned}
P_{k,r,s}(0) &= A_{r,s}(0) - \frac{(-1)^k}{2^{r+s}(2^{2k} - 1)} e_{k,1}(0) + \frac{(-1)^k}{2^{r+s}(2^{2k} - 1)} e_{k,2}(0) \\
&\quad + \frac{1}{2^{2k} - 1} e_{k,8}^{(0)}(0) - \frac{2^{2k}}{2^{2k} - 1} e_{k,16}^{(0)}(0) \\
&= \frac{1}{4} - \frac{(-1)^k}{2^{r+s}(2^{2k} - 1)} + \frac{(-1)^k}{2^{r+s}(2^{2k} - 1)} \\
&\quad + \frac{1}{2^{2k} - 1} \left(\frac{2^{2k} + 1}{2^{2k+1}} \right) - \frac{2^{2k}}{2^{2k} - 1} \left(\frac{2^{2k-1}(2^{2k} + 1) + 1}{2^{4k+1}} \right) \\
&= \frac{1}{4} + \frac{(2^{2k} + 1) - (2^{2k-1}(2^{2k} + 1) + 1)}{(2^{2k} - 1)2^{2k+1}}
\end{aligned}$$

$$= \frac{1}{4} + \frac{2 - (2^{2k} + 1)}{4(2^{2k} - 1)} = \frac{1}{4} + \frac{1 - 2^{2k}}{4(2^{2k} - 1)} = \frac{1}{4} - \frac{1}{4} = 0,$$

which proves (i).

(ii) By Lemmas 5.2.1, 4.1.1(ii), 4.1.2(ii), 4.2.2 and 4.3.3, we have

$$\begin{aligned} P_{k,r,s}(1) &= \frac{1}{2^{r+s}} - \frac{(-1)^k}{2^{r+s}(2^{2k} - 1)} e_{k,1}(1) + \frac{(-1)^k}{2^{r+s}(2^{2k} - 1)} e_{k,2}(1) \\ &\quad + \frac{1}{2^{2k} - 1} e_{k,8}^{(0)}(1) - \frac{2^{2k}}{2^{2k} - 1} e_{k,16}^{(0)}(1) \\ &= \frac{1}{2^{r+s}} - \frac{2^{2k}}{2^{r+s}(2^{2k} - 1)} + \frac{1}{2^{r+s}(2^{2k} - 1)} \\ &\quad + \frac{1}{2^{2k} - 1} \frac{(-1)^k}{2^{4k}} - \frac{2^{2k}}{2^{2k} - 1} \frac{(-1)^k}{2^{6k}} \\ &= 0, \end{aligned}$$

which proves (ii).

(iii) By Lemmas 3.1.2 (i), 4.1.1 (iii)(iv), 4.1.2 (iii)(iv), 4.2.3 (i)(iii) and 4.3.4 (ii)(iii), we have

$$\begin{aligned} A_{r,s}(x) &= a_{k-1}x^{k-1} + \cdots, \\ e_{k,1}(x) &= x^k + \cdots, \\ e_{k,2}(x) &= x^k + \cdots, \\ e_{k,8}^{(0)}(x) &= \frac{1}{2^{4k}} x^k + \cdots, \end{aligned}$$

$$e_{k,16}^{(0)}(x) = \frac{1}{2^{6k}} x^k + \dots$$

Then, by Definition 5.2.1(i), we obtain

$$\begin{aligned} P_{k,r,s}(x) &= \left(-\frac{(-1)^k}{2^{r+s}(2^{2k}-1)} + \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} + \frac{1}{2^{2k}-1} \frac{1}{2^{4k}} - \frac{2^{2k}}{2^{2k}-1} \frac{1}{2^{6k}} \right) x^k + \dots \\ &= 0x^k + \dots, \end{aligned}$$

which shows that $\deg(P_{k,r,s}(x)) \leq k-1$. $\square$

The next three lemmas evaluate the polynomials $Q_{k,r,s}(x)$, $R_{k,r,s}(x)$ and $S_{k,r,s}(x)$ at $x=0$ and give upper bounds for their degrees.

Lemma 5.2.3. *For $k, r, s \in \mathbb{N}$ with $k \geq 2$ and $2r+s < 4k$, we have*

- (i) $Q_{k,r,s}(0) = 0$,
- (ii) $\deg(Q_{k,r,s}(x)) \leq k-1$.

Proof. (i) By Definition 5.2.1(ii), Lemmas 3.3.2 and 4.3.2(iii), we have

$$Q_{k,r,s}(0) = B_{r,s}(0) - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(1)}(0) = \frac{1}{4} - \frac{2^{2k}}{2^{2k}-1} \frac{2^{2k}-1}{2^{2k+2}} = 0.$$

(ii) This follows from Definition 5.2.1(ii), Lemmas 3.1.2(ii) and 4.3.4(i). $\square$

Lemma 5.2.4. *For $k, r, s \in \mathbb{N}$ with $k \geq 2$ and $2r+s < 4k$, we have*

- (i) $R_{k,r,s}(0) = 0$,
- (ii) $\deg(R_{k,r,s}(x)) \leq k-1$.

Proof. (i) By Definition 5.2.1(iii), Lemmas 3.3.2, 4.2.1(iii) and 4.3.2(iv), we have

$$R_{k,r,s}(0) = C_{r,s}(0) + \frac{1}{2^{2k}-1} e_{k,8}^{(1)}(0) - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(2)}(0)$$

$$\begin{aligned}
&= \frac{1}{4} + \frac{1}{2^{2k}-1} \frac{2^{2k}-1}{2^{2k+1}} - \frac{2^{2k}}{2^{2k}-1} \frac{(2^{2k-1}(2^{2k}+1)-1)}{2^{4k+1}} \\
&= \frac{1}{4} + \frac{1}{2^{2k+1}} - \frac{(2^{2k}-1)(2^{2k-1}+1)}{(2^{2k}-1)2^{2k+1}} \\
&= \frac{1}{4} + \frac{1}{2^{2k+1}} - \frac{(2^{2k-1}+1)}{2^{2k+1}} = \frac{1}{4} - \frac{1}{4} = 0,
\end{aligned}$$

which proves (i).

(ii) This follows from Definition 5.2.1 (iii), Lemmas 3.1.2 (iii), 4.2.3 (ii) and 4.3.4(i). $\square$

Lemma 5.2.5. *For $k, r, s \in \mathbb{N}$ with $k \geq 2$ and $2r + s < 4k$, we have*

- (i) $S_{k,r,s}(0) = 0$,
- (ii) $\deg(S_{k,r,s}(x)) \leq k - 1$.

Proof. (i) By Definition 5.2.1(iv), Lemmas 3.3.2 and 4.3.2(v), we have

$$S_{k,r,s}(0) = D_{r,s}(0) - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(3)}(0) = \frac{1}{4} - \frac{2^{2k}}{2^{2k}-1} \frac{2^{2k}-1}{2^{2k+2}} = 0.$$

(ii) This follows from Definition 5.2.1(iv), Lemmas 3.1.2(iv) and 4.3.4(i). $\square$

5.3 The polynomials $P_{k,0,s}(x)$, $Q_{k,0,s}(x)$, $R_{k,0,s}(x)$ and

$$S_{k,0,s}(x)$$

We begin this section with the definition of the polynomials $P_{k,0,s}(x)$, $Q_{k,0,s}(x)$, $R_{k,0,s}(x)$ and $S_{k,0,s}(x)$.

Definition 5.3.1. For $k, s \in \mathbb{N}$ with $k \geq 2$ and $s < 4k$, we define

$$\begin{aligned} \text{(i) } P_{k,0,s}(x) &= A_{0,s}(x) - \frac{(-1)^k}{2^s(2^{2k} - 1)} e_{k,1}(x) \\ &\quad + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k} - 1)} e_{k,2}(x) \\ &\quad - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k} + 1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k} - 1)} e_{k,4}(x) \\ &\quad + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k - \lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k} - 1} e_{k,8}^{(0)}(x) - \frac{2^{2k}}{2^{2k} - 1} e_{k,16}^{(0)}(x), \end{aligned}$$

$$\text{(ii) } Q_{k,0,s}(x) = B_{0,s}(x) - \frac{2^{2k}}{2^{2k} - 1} e_{k,16}^{(1)}(x),$$

$$\text{(iii) } R_{k,0,s}(x) = C_{0,s}(x) + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k - \lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k} - 1} e_{k,8}^{(1)}(x) - \frac{2^{2k}}{2^{2k} - 1} e_{k,16}^{(2)}(x),$$

$$\text{(iv) } S_{k,0,s}(x) = D_{0,s}(x) - \frac{2^{2k}}{2^{2k} - 1} e_{k,16}^{(3)}(x).$$

The following lemma justifies our definition of the four polynomials and plays a key part in the proof of our main theorems.

Lemma 5.3.1. *For $k, s \in \mathbb{N}$ with $k \geq 2$ and $s < 4k$, we have*

$$\begin{aligned}
& P_{k,0,s}(x) + Q_{k,0,s}(x)(1-x)^{1/4} + R_{k,0,s}(x)(1-x)^{1/2} + S_{k,0,s}(x)(1-x)^{3/4} \\
&= \left(\frac{1 + (1-x)^{1/4}}{2} \right)^s - \frac{(-1)^k}{2^s(2^{2k}-1)} e_{k,1}(x) \\
&+ \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k}-1)} e_{k,2}(x) \\
&- \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} e_{k,4}(x) \\
&+ \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k-\lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k}-1} (e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2}) \\
&- \frac{2^{2k}}{2^{2k}-1} \left(e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(1-x)^{1/2} + e_{k,16}^{(3)}(1-x)^{3/4} \right).
\end{aligned}$$

Proof. From Definition 5.3.1 and Lemma 3.1.3, we have

$$\begin{aligned}
& \left(\frac{1 + (1-x)^{1/4}}{2} \right)^s - \frac{(-1)^k}{2^s(2^{2k}-1)} e_{k,1}(x) \\
&+ \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k}-1)} e_{k,2}(x) \\
&- \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} e_{k,4}(x) \\
&+ \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k-\lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k}-1} (e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2})
\end{aligned}$$

$$\begin{aligned}
& - \frac{2^{2k}}{2^{2k} - 1} \left(e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(1-x)^{1/2} + e_{k,16}^{(3)}(1-x)^{3/4} \right) \\
& = A_{0,s}(x) + B_{0,s}(x)(1-x)^{1/4} + C_{0,s}(x)(1-x)^{1/2} + D_{0,s}(x)(1-x)^{3/4} \\
& - \frac{(-1)^k}{2^s(2^{2k} - 1)} e_{k,1}(x) + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k} - 1)} e_{k,2}(x) \\
& - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k} + 1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k} - 1)} e_{k,4}(x) \\
& + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k - \lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k} - 1} (e_{k,8}^{(0)}(x) + e_{k,8}^{(1)}(x)(1-x)^{1/2}) \\
& - \frac{2^{2k}}{2^{2k} - 1} \left(e_{k,16}^{(0)}(x) + e_{k,16}^{(1)}(x)(1-x)^{1/4} + e_{k,16}^{(2)}(1-x)^{1/2} + e_{k,16}^{(3)}(1-x)^{3/4} \right) \\
& = P_{k,0,s}(x) + Q_{k,0,s}(x)(1-x)^{1/4} + R_{k,0,s}(x)(1-x)^{1/2} + S_{k,0,s}(x)(1-x)^{3/4},
\end{aligned}$$

as required. $\square$

In the next lemma we evaluate the polynomial $P_{k,0,s}(x)$ at $x = 0$ and $x = 1$. We also give an upper bound for the degree of the polynomial $P_{k,0,s}(x)$ when $s < 4k$.

Lemma 5.3.2. *For $k, s \in \mathbb{N}$ with $k \geq 2$ and $s < 4k$, we have*

- (i) $P_{k,0,s}(0) = 0$,
- (ii) $P_{k,0,s}(1) = 0$,
- (iii) $\deg(P_{k,0,s}(x)) \leq k - 1$.

Proof. (i) From Definition 5.3.1(i), Lemmas 3.3.3(i), 4.1.1(i), 4.1.2(i), 4.1.3(i), 4.2.1(ii) and 4.3.2(ii), we have

$$\begin{aligned}
P_{k,0,s}(0) &= A_{0,s}(0) - \frac{(-1)^k}{2^s(2^{2k}-1)} e_{k,1}(0) + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k}-1)} e_{k,2}(0) \\
&\quad - \frac{\varepsilon_{2,4}(n) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} e_{k,4}(0) \\
&\quad + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k-\lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k}-1} e_{k,8}^{(0)}(0) - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(0)}(0) \\
&= \frac{1}{4} + \frac{\varepsilon_{2,4}(x) (-1)^{\lfloor \frac{s+1}{4} \rfloor}}{2^{1+\lfloor \frac{s+1}{2} \rfloor}} - \frac{(-1)^k}{2^s(2^{2k}-1)} + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k}-1)} \\
&\quad - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k-\lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k}-1} \frac{2^{2k}+1}{2^{2k+1}} \\
&\quad - \frac{2^{2k}}{2^{2k}-1} \frac{2^{2k-1}(2^{2k}+1)+1}{2^{4k+1}} \\
&= \frac{1}{4} - \frac{(-1)^k}{2^s(2^{2k}-1)} + \frac{(-1)^k}{2^s(2^{2k}-1)} + \frac{2^{2k}+1}{(2^{2k}-1)2^{2k+1}} - \frac{2^{2k-1}(2^{2k}+1)+1}{(2^{2k}-1)2^{2k+1}} \\
&\quad + \varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} \left(\frac{1}{2^{1+\lfloor \frac{s+1}{2} \rfloor}} + \frac{2^{\lfloor \frac{s}{2} \rfloor}}{2^s(2^{2k}-1)} - \frac{2^{2k}+1}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} \right. \\
&\quad \left. + \frac{2^{2k-\lfloor \frac{s+1}{2} \rfloor} (2^{2k}+1)}{(2^{2k}-1)2^{2k+1}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} + \frac{2^{2k} - 2^{4k-1} - 2^{2k-1}}{(2^{2k} - 1)(2^{2k+1})} \\
&\quad + \varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor} \left(\frac{1}{2^{1+\lfloor \frac{s+1}{2} \rfloor}} + \frac{1}{2^{\lfloor \frac{s+1}{2} \rfloor}(2^{2k} - 1)} - \frac{2^{2k} + 1}{2^{\lfloor \frac{s+1}{2} \rfloor}(2^{2k} - 1)} \right. \\
&\quad \left. + \frac{2^{2k} + 1}{(2^{2k} - 1)2^{\lfloor \frac{s+1}{2} \rfloor + 1}} \right) \\
&= \frac{1}{4} + \frac{2^{2k-1} - 2^{4k-1}}{(2^{2k} - 1)2^{2k+1}} \\
&\quad + \varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor} \left(\frac{(2^{2k} - 1) + 2 - 2^{2k+1} - 2 + 2^{2k} + 1}{2(2^{2k} - 1)2^{\lfloor \frac{s+1}{2} \rfloor}} \right) \\
&= \frac{1}{4} + \frac{2^{2k-1}(1 - 2^{2k})}{(2^{2k} - 1)2^{2k+1}} + \varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor}(0) = \frac{1}{4} - \frac{1}{4} = 0,
\end{aligned}$$

which proves (i).

(ii) By Lemmas 5.3.1, 4.1.1(ii), 4.1.2(ii), 4.1.3(ii), 4.2.2 and 4.3.3, we have

$$\begin{aligned}
P_{k,0,s}(1) &= \frac{1}{2^s} - \frac{(-1)^k}{2^s(2^{2k} - 1)} e_{k,1}(1) + \frac{\varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k} - 1)} e_{k,2}(1) \\
&\quad - \frac{\varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k} + 1)}{2^{\lfloor \frac{s+1}{2} \rfloor}(2^{2k} - 1)} e_{k,4}(1) \\
&\quad + \frac{\varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k - \lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k} - 1} e_{k,8}^{(0)}(1) - \frac{2^{2k}}{2^{2k} - 1} e_{k,16}^{(0)}(1),
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^s} - \frac{(-1)^k}{2^s(2^{2k}-1)} e_{k,1}(1) + \frac{(-1)^k}{2^s(2^{2k}-1)} e_{k,2}(1) + \frac{e_{k,8}^{(0)}(x)}{2^{2k}-1} \\
&\quad - \frac{2^{2k}}{2^{2k}-1} e_{k,16}^{(0)}(1) + (-1)^{\lceil \frac{s+1}{4} \rceil} \varepsilon_{2,4}(s) \left(\frac{1}{2^{\lceil \frac{s+1}{2} \rceil} (2^{2k}-1)} e_{k,2}(1) \right. \\
&\quad \left. - \frac{(2^{2k}+1)}{2^{\lceil \frac{s+1}{2} \rceil} (2^{2k}-1)} e_{k,4}(1) + \frac{2^{2k-\lceil \frac{s+1}{2} \rceil}}{2^{2k}-1} e_{k,8}^{(0)}(1) \right) \\
&= \frac{1}{2^s} - \frac{2^{2k}}{2^s(2^{2k}-1)} + \frac{1}{2^s(2^{2k}-1)} + \frac{(-1)^k}{2^{4k}(2^{2k}-1)} - \frac{2^{2k}}{2^{2k}-1} \frac{(-1)^k}{2^{6k}} \\
&\quad + (-1)^{\lceil \frac{s+1}{4} \rceil} \varepsilon_{2,4}(s) \left(\frac{(-1)^k}{2^{\lceil \frac{s+1}{2} \rceil} (2^{2k}-1)} - \frac{(2^{2k}+1)(-1)^k}{2^{\lceil \frac{s+1}{2} \rceil} (2^{2k}-1) 2^{2k}} + \frac{2^{2k-\lceil \frac{s+1}{2} \rceil}}{2^{2k}-1} \frac{(-1)^k}{2^{4k}} \right) \\
&= \frac{(2^{2k}-1)2^{4k} - 2^{6k} + 2^{4k} + (-1)^k 2^s - (-1)^k 2^s}{2^{4k+s}(2^{2k}-1)} \\
&\quad + (-1)^{\lceil \frac{s+1}{4} \rceil} \varepsilon_{2,4}(s) \left(\frac{(-1)^k 2^{2k} - (2^{2k}+1)(-1)^k + (-1)^k}{2^{2k+\lceil \frac{s+1}{2} \rceil} (2^{2k}-1)} \right) \\
&= 0 + (-1)^{\lceil \frac{s+1}{4} \rceil} \varepsilon_{2,4}(s)(0) = 0,
\end{aligned}$$

which proves (ii).

(iii) By Lemmas 3.1.2 (i), 4.1.1 (iii)(iv), 4.1.2 (iii)(iv), 4.1.3 (iii)(iv), 4.2.3 (i)(iii) and 4.3.4(ii)(iii), we have

$$A_{0,s}(x) = a_{k-1}x^{k-1} + \dots,$$

$$e_{k,1}(x) = x^k + \dots,$$

$$\begin{aligned}
e_{k,2}(x) &= x^k + \dots, \\
e_{k,4}(x) &= \frac{1}{2^{2k}} x^k + \dots, \\
e_{k,8}^{(0)}(x) &= \frac{1}{2^{4k}} x^k + \dots, \\
e_{k,16}^{(0)}(x) &= \frac{1}{2^{6k}} x^k + \dots.
\end{aligned}$$

Then, by Definition 5.3.1(i) we have

$$\begin{aligned}
P_{k,0,s}(x) &= \left(-\frac{(-1)^k}{2^s(2^{2k}-1)} + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k}-1)} \right. \\
&\quad - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} \frac{1}{2^{2k}} + \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k-\lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k}-1} \frac{1}{2^{4k}} \\
&\quad \left. - \frac{2^{2k}}{2^{2k}-1} \frac{1}{2^{6k}} \right) x^k + \dots \\
&= \left(-\frac{(-1)^k}{2^s(2^{2k}-1)} + \frac{(-1)^k}{2^s(2^{2k}-1)} + \frac{1}{2^{2k}-1} \frac{1}{2^{4k}} - \frac{2^{2k}}{2^{2k}-1} \frac{1}{2^{6k}} \right. \\
&\quad + (-1)^{\lfloor \frac{s+1}{4} \rfloor} \varepsilon_{2,4}(s) \left(\frac{2^{\lfloor \frac{s}{2} \rfloor}}{2^s(2^{2k}-1)} - \frac{2^{2k}+1}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} \frac{1}{2^{2k}} \right. \\
&\quad \left. \left. + \frac{2^{2k-\lfloor \frac{s+1}{2} \rfloor}}{2^{2k}-1} \frac{1}{2^{4k}} \right) \right) x^k + \dots, \\
&= \left(0 + (-1)^{\lfloor \frac{s+1}{4} \rfloor} \varepsilon_{2,4}(s) \left(\frac{1}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} - \frac{2^{2k}+1}{2^{\lfloor \frac{s+1}{2} \rfloor + 2k} (2^{2k}-1)} \right. \right. \\
&\quad \left. \left. + \frac{1}{2^{\lfloor \frac{s+1}{2} \rfloor + 2k} (2^{2k}-1)} \right) \right) x^k + \dots
\end{aligned}$$

$$= 0 \, x^k + \cdots,$$

which shows that $\deg(P_{k,0,s}(x)) \leq k - 1$. $\square$

The next lemma evaluates the polynomial $R_{k,0,s}(x)$ at $x = 0$ and gives an upper bound for the degrees of $Q_{k,0,s}(x)$, $R_{k,0,s}(x)$ and $S_{k,0,s}(x)$.

Lemma 5.3.3. *For $k, s \in \mathbb{N}$ with $k \geq 2$ and $s < 4k$, we have*

- (i) $R_{k,0,s}(0) = 0$,
- (ii) $\deg(R_{k,0,s}(x)) \leq k - 1$,
- (iii) $\deg(Q_{k,0,s}(x)) \leq k - 1$,
- (iv) $\deg(S_{k,0,s}(x)) \leq k - 1$.

Proof. (i) From Definition 5.3.1(iii), Lemmas 3.3.3(iii), 4.2.1(iii) and 4.3.2(iv), we have

$$\begin{aligned}
 R_{k,0,s}(0) &= C_{0,s}(0) + \frac{\varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k - \lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k} - 1} e_{k,8}^{(1)}(0) - \frac{2^{2k}}{2^{2k} - 1} e_{k,16}^{(2)}(0), \\
 &= \frac{1}{4} - \frac{\varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor}}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}} + \left(\frac{\varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k - \lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k} - 1} \right) \left(\frac{2^{2k} - 1}{2^{2k+1}} \right) \\
 &\quad - \frac{2^{2k}}{2^{2k} - 1} \left(\frac{2^{2k-1}(2^{2k} + 1) - 1}{2^{4k+1}} \right) \\
 &= \frac{1}{4} + \frac{1}{2^{2k+1}} - \frac{(2^{2k-1}(2^{2k} + 1) - 1)}{(2^{2k} - 1)2^{2k+1}} \\
 &\quad + \varepsilon_{2,4}(s)(-1)^{\lfloor \frac{s+1}{4} \rfloor} \left(\frac{-1}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}} + \frac{1}{2^{\lfloor \frac{s+1}{2} \rfloor + 1}} \right)
 \end{aligned}$$

$$= \frac{2^{2k-1}(2^{2k} - 1) + 2^{2k} - 1 - 2^{2k-1}(2^{2k} + 1) + 1}{(2^{2k} - 1)2^{2k+1}} = 0.$$

(ii) This follows from Definition 5.3.1(ii), Lemmas 3.1.2(ii) and 4.3.4(i).

(iii) This follows from Definition 5.3.1 (iii), Lemmas 3.1.2 (iii), 4.2.3 (ii) and 4.3.4(i).

(iv) This follows from Definition 5.3.1(iv), Lemmas 3.1.2 (iv) and 4.3.4 (i).

□

Chapter 6

Extensions of the Ramanujan-Mordell formula

We recall that the main problem that we address in this thesis is that of determining the number of representations of a positive integer n by a quadratic form of the type

$$x_1^2 + \cdots + x_a^2 + 2x_{a+1}^2 + \cdots + 2x_{a+b}^2 + 4x_{a+b+1}^2 + \cdots + 4x_t^2,$$

where $a \in \mathbb{N}$, $b, c \in \mathbb{N}_0$ and $t = a + b + c$. In this chapter, we give formulae that determine the number of representations for any positive integer n and any $a, b, c \in \mathbb{N}_0$, where b is an even number and $t = 4k$. These formulae give our extensions of the Ramanujan-Mordell formula (Theorem 1.0.1). As in previous chapters, we have three cases. Each section covers one such case.

In Section 6.1 we obtain explicit formulae for $N([4k - 2r, 2r, 0]; n)$ when $k, r \in \mathbb{N}$ and $r < 2k$.

In Section 6.2 we obtain explicit formulae for $N([4k - 2r - s, 2r, s]; n)$ when $k, r, s \in \mathbb{N}$ and $2r + s < 4k$.

In Section 6.3 we obtain explicit formulae for $N([4k - s, 0, s]; n)$ when $k, s \in \mathbb{N}$ and $s < 4k$.

In order to include the case $k = 1$ in the theorems of this chapter, we define the polynomials

$$(6.0.1) \quad e_{1,1}(x) := 1 - 5x,$$

$$(6.0.2) \quad e_{1,2}(x) := 1 - 2x,$$

$$(6.0.3) \quad e_{1,4}(x) := 1 - \frac{5}{4}x,$$

$$(6.0.4) \quad e_{1,8}(x) := \frac{5}{8} - \frac{11}{16}x + \frac{3}{8}(1-x)^{1/2},$$

$$(6.0.5) \quad e_{1,16}(x) := \frac{11}{32} - \frac{23}{64}x + \frac{3}{16}(1-x)^{1/4} + \frac{9}{32}(1-x)^{1/2} + \frac{3}{16}(1-x)^{3/4}.$$

We emphasize that for x and z as defined in Definition 2.3.2 we have

$$E_1(q^r) \neq e_{1,r}(x)z^2 \quad \text{for every } r \in \{1, 2, 4, 8, 16\}.$$

6.1 Number of representations by the quadratic

form $x_1^2 + \cdots + x_{4k-2r}^2 + 2x_{4k-2r+1}^2 + \cdots + 2x_{4k}^2$.

To begin this section we use all the results that we found in Section 5.1 to obtain the following theorem that is valid for any variables x and z .

Theorem 6.1.1. *For $k, r \in \mathbb{N}$ with $r < 2k$ and variables x and z , we have*

$$\begin{aligned} & \left(\frac{1 + (1-x)^{1/2}}{2} \right)^r z^{2k} \\ &= \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,1}(x) z^{2k} - \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,2}(x) z^{2k} - \frac{1}{2^{2k}-1} e_{k,4}(x) z^{2k} \\ &+ \frac{2^{2k}}{2^{2k}-1} e_{k,8}(x) z^{2k} \end{aligned}$$

$$+ (1-x) \sum_{i=1}^{k-2} a'_{k,r,0}(i) x^i z^{2k} + (1-x)^{1/2} \sum_{i=1}^{k-1} c'_{k,r,0}(i) x^i z^{2k},$$

where $a'_{k,r,0}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-2\}$ and $c'_{k,r,0}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-1\}$.

Proof. When $k = 1$ we have $r = 1$. From (6.0.1)-(6.0.4) we have

$$\frac{1 + (1-x)^{1/2}}{2} z^2 = -\frac{1}{6} e_{1,1}(x) z^2 + \frac{1}{6} e_{1,2}(x) z^2 - \frac{1}{3} e_{1,4}(x) z^2 + \frac{4}{3} e_{1,8}(x) z^2$$

For $k \geq 2$, by Definition 5.1.1, Lemmas 5.1.2 and 5.1.3 we have

$$P_{k,r,0}(x) = (1-x) \left(a'_{k,r,0}(1)x + \dots + a'_{k,r,0}(k-2)x^{k-2} \right),$$

$$R_{k,r,0}(x) = c'_{k,r,0}(1)x + \dots + c'_{k,r,0}(k-1)x^{k-1},$$

where $a'_{k,r,0}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-2\}$ and $c'_{k,r,0}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-1\}$.

This implies, by Lemma 5.1.1 and (4.2.2), that

$$\begin{aligned} (6.1.1) \quad & \left(\frac{1 + (1-x)^{1/2}}{2} \right)^r \\ &= \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,1}(x) - \frac{(-1)^k}{2^r(2^{2k}-1)} e_{k,2}(x) - \frac{1}{2^{2k}-1} e_{k,4}(x) \\ & \quad + \frac{2^{2k}}{2^{2k}-1} e_{k,8}(x) \\ & \quad + (1-x) \sum_{i=1}^{k-2} a'_{k,r,0}(i) x^i + (1-x)^{1/2} \sum_{i=1}^{k-1} c'_{k,r,0}(i) x^i. \end{aligned}$$

Finally, we obtain the asserted result by multiplying both sides of (6.1.1) by z^{2k} . $\square$

Replacing x and z in Theorem 6.1.1 by their values given in Definition 2.3.2, we obtain the following identities involving the theta function φ , Eisenstein series $E_k(q^r)$ with $r \in \{1, 2, 4, 8\}$ and the generalized Ramanujan discriminant function $\Delta(a, b, c; q)$.

Theorem 6.1.2. *For $k, r \in \mathbb{N}$ with $r < 2k$ and $q \in D_\delta$, we have*

$$\begin{aligned} & \varphi(q)^{4k-2r} \varphi(q^2)^{2r} \\ &= \frac{(-1)^k}{2^r(2^{2k}-1)} E_k(q) - \frac{(-1)^k}{2^r(2^{2k}-1)} E_k(q^2) - \frac{1}{2^{2k}-1} E_k(q^4) + \frac{2^{2k}}{2^{2k}-1} E_k(q^8) \\ &+ \sum_{i=1}^{k-2} a_{k,r,0}(i) \Delta\left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q\right) \\ &+ \sum_{i=1}^{k-1} c_{k,r,0}(i) \Delta\left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; q\right), \end{aligned}$$

where $a_{k,r,0}(i) = 2^{4i} a'_{k,r,0}(i)$ for $i \in \{1, \dots, k-2\}$, $c_{k,r,0}(i) = 2^{4i} c'_{k,r,0}(i)$ for $i \in \{1, \dots, k-1\}$ and the coefficients $a'_{k,r,0}(i)$, $c'_{k,r,0}(i)$ are given in Theorem 6.1.1.

Proof. Let $q \in D_\delta$. Set $x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}$, $z = \varphi^2(q)$. For $k = 1$, we have by Theorems 2.5.2 (i) (ii), 2.9.2 and 2.9.5

$$\varphi(q)^2 \varphi(q^2)^2 + \frac{1}{6} E_1(q) - \frac{1}{6} E_1(q^2) + \frac{1}{3} E_1(q^4) - \frac{4}{3} E_1(q^8) = 0,$$

which gives the asserted result on isolating the term $\varphi(q)^2 \varphi(q^2)^2$.

For $k \geq 2$, we note that by Theorem 2.10.3 we have for $i \in \{1, \dots, k-2\}$

$$a'_{k,r,0}(i) x^i (1-x) z^{2k} = 2^{4i} a'_{k,r,0}(i) \frac{x^i (1-x)}{2^{4i}} z^{2k}$$

$$= 2^{4i} a'_{k,r,0}(i) \Delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q \right)$$

$$= a_{k,r,0}(i) \Delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q \right),$$

where $a_{k,r,0}(i) = 2^{4i} a'_{k,r,0}(i)$. Similarly, we obtain for $i \in \{1, \dots, k-1\}$

$$c'_{k,r,0}(i) x^i (1-x)^{1/2} z^{2k} = c_{k,r,0}(i) \Delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; q \right),$$

where $c_{k,r,0}(i) = 2^{4i} c'_{k,r,0}(i)$. Finally, from Theorems 2.5.2, 4.4.1 and 6.1.1, we obtain the desired results. $\square$

Our next result gives the number of representations of $n \in \mathbb{N}$ by the quadratic form

$$x_1^2 + \dots + x_{4k-2r}^2 + 2x_{4k-2r+1}^2 + \dots + 2x_{4k}^2,$$

where $r < 2k$.

Theorem 6.1.3. *For $k, r, n \in \mathbb{N}$ with $r < 2k$, we have*

$$N([4k-2r, 2r, 0]; n)$$

$$\begin{aligned} &= -\frac{(-1)^k 4k}{2^r (2^{2k}-1) B_{2k}} \sigma_{2k-1}(n) + \frac{(-1)^k 4k}{2^r (2^{2k}-1) B_{2k}} \sigma_{2k-1}\left(\frac{n}{2}\right) + \frac{4k}{(2^{2k}-1) B_{2k}} \sigma_{2k-1}\left(\frac{n}{4}\right) \\ &\quad - \frac{2^{2k+2} k}{(2^{2k}-1) B_{2k}} \sigma_{2k-1}\left(\frac{n}{8}\right) \\ &\quad + \sum_{i=1}^{k-2} a_{k,r,0}(i) \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) \\ &\quad + \sum_{i=1}^{k-1} c_{k,r,0}(i) \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right), \end{aligned}$$

where $a_{k,r,0}(i) = 2^{4i} a'_{k,r,0}(i)$ for $i \in \{1, \dots, k-2\}$, $c_{k,r,0}(i) = 2^{4i} c'_{k,r,0}(i)$ for $i \in \{1, \dots, k-1\}$ and the coefficients $a'_{k,r,0}(i)$, $c'_{k,r,0}(i)$ are given in Theorem 6.1.1.

Proof. From Theorem 2.8.1, Definition 2.9.1, Definition 2.10.3 and Theorem 6.1.2 we have for $q \in D_\delta$

$$\begin{aligned}
 1 + \sum_{n=1}^{\infty} N([4k-2r, 2r, 0]; n) q^n &= \varphi(q)^{4k-2r} \varphi(q^2)^{2r} \\
 &= \frac{(-1)^k}{2^r(2^{2k}-1)} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \right) - \frac{(-1)^k}{2^r(2^{2k}-1)} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n} \right) \\
 &\quad - \frac{1}{2^{2k}-1} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{4n} \right) + \frac{2^{2k}}{2^{2k}-1} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{8n} \right) \\
 &\quad + \sum_{i=1}^{k-2} a_{k,r,0}(i) \sum_{n=1}^{\infty} \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) q^n \\
 &\quad + \sum_{i=1}^{k-1} c_{k,r,0}(i) \sum_{n=1}^{\infty} \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; q \right) q^n \\
 &= 1 - \frac{(-1)^k}{2^r(2^{2k}-1)} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \right) + \frac{(-1)^k}{2^r(2^{2k}-1)} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n} \right) \\
 &\quad + \frac{1}{2^{2k}-1} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{4n} \right) - \frac{2^{2k}}{2^{2k}-1} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{8n} \right) \\
 &\quad + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k-2} a_{k,r,0}(i) \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) \right) q^n
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k-1} c_{k,r,0}(i) \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right) \right) q^n \\
& = 1 + \sum_{n=1}^{\infty} \left(- \frac{(-1)^k 4k}{2^r (2^{2k} - 1) B_{2k}} \sigma_{2k-1}(n) + \frac{(-1)^k 4k}{2^r (2^{2k} - 1) B_{2k}} \sigma_{2k-1}\left(\frac{n}{2}\right) \right. \\
& \quad + \frac{4k}{(2^{2k} - 1) B_{2k}} \sigma_{2k-1}\left(\frac{n}{4}\right) - \frac{2^{2k+2} k}{(2^{2k} - 1) B_{2k}} \sigma_{2k-1}\left(\frac{n}{8}\right) \\
& \quad + \sum_{i=1}^{k-2} a_{k,r,0}(i) \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) \\
& \quad \left. + \sum_{i=1}^{k-1} c_{k,r,0}(i) \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right) \right) q^n.
\end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the desired result. $\square$

When $k = 1$ we have $r = 1$ (as $1 \leq r < 2k$). In this case Theorem 6.1.3 gives

$$(6.1.2) \quad N([2, 2, 0]; n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8), \quad n \in \mathbb{N},$$

in agreement with [2, Theorem 1.8, p. 297].

6.2 Number of representations by the quadratic

form $x_1^2 + \cdots + x_{4k-2r-s}^2 + 2x_{4k-2r-s+1}^2 + \cdots + 2x_{4k-s}^2 + 4x_{4k-s+1}^2 + \cdots + 4x_{4k}^2$

To begin this section we use all the results that we found in Section 5.2 to obtain the following theorem that is valid for any variables x and z .

Theorem 6.2.1. *For $k, r, s \in \mathbb{N}$ with $2r + s < 4k$, we have*

$$\begin{aligned} & \left(\frac{1 + (1-x)^{1/2}}{2} \right)^r \left(\frac{1 + (1-x)^{1/4}}{2} \right)^s z^{2k} \\ &= \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,1}(x) z^{2k} - \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,2}(x) z^{2k} - \frac{1}{2^{2k}-1} e_{k,8}(x) z^{2k} \\ &+ \frac{2^{2k}}{2^{2k}-1} e_{k,16}(x) z^{2k} + (1-x) \sum_{i=1}^{k-2} a'_{k,r,s}(i) x^i z^{2k} + (1-x)^{1/4} \sum_{i=1}^{k-1} b'_{k,r,s}(i) x^i z^{2k} \\ &+ (1-x)^{1/2} \sum_{i=1}^{k-1} c'_{k,r,s}(i) x^i z^{2k} + (1-x)^{3/4} \sum_{i=1}^{k-1} d'_{k,r,s}(i) x^i z^{2k}, \end{aligned}$$

where $a'_{k,r,s}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-2\}$ and $b'_{k,r,s}(i), c'_{k,r,s}(i), d'_{k,r,s}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-1\}$.

Proof. When $k = 1$ we have $r = 1$ and $s = 1$. From (6.0.1)-(6.0.5) we have

$$\begin{aligned} \left(\frac{1 + (1-x)^{1/2}}{2} \right) \left(\frac{1 + (1-x)^{1/4}}{2} \right) z^2 &= -\frac{1}{12} e_{1,1}(x) z^2 + \frac{1}{12} e_{1,2}(x) z^2 - \frac{1}{3} e_{1,8}(x) z^2 \\ &+ \frac{4}{3} e_{1,16}(x) z^2. \end{aligned}$$

For $k \geq 2$, by Definition 5.2.1 and Lemmas 5.2.2 - 5.2.5, we have

$$P_{k,r,s}(x) = (1-x) (a_{k,r,s}(1)x + \cdots + a_{k,r,s}(k-2)x^{k-2}),$$

$$Q_{k,r,s}(x) = b'_{k,r,s}(1)x + \cdots + b'_{k,r,s}(k-1)x^{k-1},$$

$$R_{k,r,s}(x) = c'_{k,r,s}(1)x + \cdots + c'_{k,r,s}(k-1)x^{k-1},$$

$$S_{k,r,s}(x) = d'_{k,r,s}(1)x + \cdots + d'_{k,r,s}(k-1)x^{k-1},$$

where $a'_{k,r,s}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-2\}$ and $b'_{k,r,s}(i), c'_{k,r,s}(i), d'_{k,r,s}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-1\}$. This implies, by Lemma 5.2.1, (4.2.2) and (4.3.2) that

$$\begin{aligned}
 (6.2.1) \quad & \left(\frac{1 + (1-x)^{1/2}}{2} \right)^r \left(\frac{1 + (1-x)^{1/4}}{2} \right)^s \\
 &= \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,1}(x) - \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} e_{k,2}(x) - \frac{1}{2^{2k}-1} e_{k,8}(x) \\
 &+ \frac{2^{2k}}{2^{2k}-1} e_{k,16}(x) \\
 &+ (1-x) \sum_{i=1}^{k-2} a'_{k,r,s}(i) x^i + (1-x)^{1/4} \sum_{i=1}^{k-1} b'_{k,r,s}(i) x^i \\
 &+ (1-x)^{1/2} \sum_{i=1}^{k-1} c'_{k,r,s}(i) x^i + (1-x)^{3/4} \sum_{i=1}^{k-1} d'_{k,r,s}(i) x^i.
 \end{aligned}$$

Finally, we obtain the asserted result by multiplying both sides of (6.2.1) by z^{2k} . $\square$

Replacing x and z in Theorem 6.2.1 by their values given in Definition 2.3.2 we obtain the following identities involving the theta function φ , Eisenstein series $E_k(q^r)$ with $r \in \{1, 2, 4, 8, 16\}$ and the generalized Ramanujan discriminant function

$\Delta(a, b, c; q)$.

Theorem 6.2.2. *For $k, r, s \in \mathbb{N}$ with $2r + s < 4k$ and $q \in D_\delta$, we have*

$$\begin{aligned}
 & \varphi^{4k-2r-s}(q) \varphi^{2r}(q^2) \varphi^s(q^4) \\
 &= \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} E_k(q) - \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} E_k(q^2) - \frac{1}{2^{2k}-1} E_k(q^8) + \frac{2^{2k}}{2^{2k}-1} E_k(q^{16}) \\
 &+ \sum_{i=1}^{k-2} a_{k,r,s}(i) \Delta\left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q\right) \\
 &+ \sum_{i=1}^{k-1} b_{k,r,s}(i) \Delta\left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; q\right) \\
 &+ \sum_{i=1}^{k-1} c_{k,r,s}(i) \Delta\left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; q\right) \\
 &+ \sum_{i=1}^{k-1} d_{k,r,s}(i) \Delta\left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; q\right),
 \end{aligned}$$

where $a_{k,r,s}(i) = 2^{4i} a'_{k,r,s}(i)$ for $i \in \{1, \dots, k-2\}$, $b_{k,r,s}(i) = 2^{4i} b'_{k,r,s}(i)$ for $i \in \{1, \dots, k-1\}$, $c_{k,r,s}(i) = 2^{4i} c'_{k,r,s}(i)$ for $i \in \{1, \dots, k-1\}$, $d_{k,r,s}(i) = 2^{4i} d'_{k,r,s}(i)$ for $i \in \{1, \dots, k-1\}$ and the coefficients $a'_{k,r,s}(i)$, $b'_{k,r,s}(i)$, $c'_{k,r,s}(i)$, $d'_{k,r,s}(i)$ are given in Theorem 6.2.1.

Proof. Let $q \in D_\delta$. Set $x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}$, $z = \varphi^2(q)$. For $k = 1$, we have by Theorems 2.5.2 (i)(ii)(iii), 2.9.2 and 2.9.5

$$\varphi(q) \varphi(q^2)^2 \varphi(q^4) + \frac{1}{12} E_1(q) - \frac{1}{12} E_1(q^2) + \frac{1}{3} E_1(q^8) - \frac{4}{3} E_1(q^{16}) = 0,$$

which is the asserted result if we isolate the term $\varphi(q)\varphi(q^2)^2\varphi(q^4)$.

For $k \geq 2$, we note that by Theorem 2.10.3 we have for $i \in \{1, \dots, k-2\}$

$$\begin{aligned} a'_{k,r,s}(i) x^i(1-x)z^{2k} &= 2^{4i} a'_{k,r,s}(i) \frac{x^i(1-x)}{2^{4i}} z^{2k} \\ &= 2^{4i} a'_{k,r,s}(i) \Delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q \right) \\ &= a_{k,r,s}(i) \Delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q \right), \end{aligned}$$

where $a_{k,r,s}(i) = 2^{4i} a'_{k,r,s}(i)$. Similarly, we obtain for $i \in \{1, \dots, k-1\}$

$$b'_{k,r,s}(i) x^i(1-x)^{1/4} z^{2k} = b_{k,r,s}(i) \Delta \left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; q \right),$$

$$c'_{k,r,s}(i) x^i(1-x)^{1/2} z^{2k} = c_{k,r,s}(i) \Delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; q \right),$$

$$d'_{k,r,s}(i) x^i(1-x)^{3/4} z^{2k} = d_{k,r,s}(i) \Delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; q \right),$$

where $b_{k,r,s}(i) = 2^{4i} b'_{k,r,s}(i)$, $c_{k,r,s}(i) = 2^{4i} c'_{k,r,s}(i)$ and $d_{k,r,s}(i) = 2^{4i} d'_{k,r,s}(i)$.

Finally, from Theorems 2.5.2, 4.4.1 and 6.2.1 we obtain the desired results. $\square$

The final result of this section gives the number of representations of $n \in \mathbb{N}$ by the quadratic form

$$x_1^2 + \dots + x_{4k-2r-s}^2 + 2x_{4k-2r-s+1}^2 + 2x_{4k-s}^2 + 4x_{4k-s+1}^2 + \dots + 4x_{4k}^2,$$

where $2r + s < 4k$.

Theorem 6.2.3. *For $k, r, s, n \in \mathbb{N}$ with $2r + s < 4k$, we have*

$$\begin{aligned}
& N([4k - 2r - s, 2r, s]; n) \\
&= -\frac{(-1)^k 4k}{2^{r+s}(2^{2k} - 1)B_{2k}} \sigma_{2k-1}(n) + \frac{(-1)^k 4k}{2^{r+s}(2^{2k} - 1)B_{2k}} \sigma_{2k-1}\left(\frac{n}{2}\right) \\
&+ \frac{4k}{(2^{2k} - 1)B_{2k}} \sigma_{2k-1}\left(\frac{n}{8}\right) - \frac{2^{2k+2}k}{(2^{2k} - 1)B_{2k}} \sigma_{2k-1}\left(\frac{n}{16}\right) \\
&+ \sum_{i=1}^{k-2} a_{k,r,s}(i) \delta\left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n\right) \\
&+ \sum_{i=1}^{k-1} b_{k,r,s}(i) \delta\left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; n\right) \\
&+ \sum_{i=1}^{k-1} c_{k,r,s}(i) \delta\left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n\right) \\
&+ \sum_{i=1}^{k-1} d_{k,r,s}(i) \delta\left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; n\right),
\end{aligned}$$

where $a_{k,r,s}(i) = 2^{4i} a'_{k,r,s}(i)$ for $i \in \{1, \dots, k-2\}$, $b_{k,r,s}(i) = 2^{4i} b'_{k,r,s}(i)$ for $i \in \{1, \dots, k-1\}$, $c_{k,r,s}(i) = 2^{4i} c'_{k,r,s}(i)$ for $i \in \{1, \dots, k-1\}$, $d_{k,r,s}(i) = 2^{4i} d'_{k,r,s}(i)$ for $i \in \{1, \dots, k-1\}$ and the coefficients $a'_{k,r,s}(i)$, $b'_{k,r,s}(i)$, $c'_{k,r,s}(i)$, $d'_{k,r,s}(i)$ are given in Theorem 6.2.1.

Proof. From Theorem 2.8.1, Definition 2.9.1, Definition 2.10.3 and Theorem 6.2.2 we have for $q \in D_\delta$

$$1 + \sum_{n=1}^{\infty} N([4k - 2r - s, 2r, s]; n) q^n = \varphi(q)^{4k-2r-s} \varphi(q^2)^{2r} \varphi(q^4)^s$$

$$\begin{aligned}
&= \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \right) \\
&\quad - \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n} \right) \\
&\quad - \frac{1}{2^{2k}-1} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{8n} \right) \\
&\quad + \frac{2^{2k}}{2^{2k}-1} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{16n} \right) \\
&\quad + \sum_{i=1}^{k-2} a_{k,r,s}(i) \sum_{n=1}^{\infty} \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) q^n \\
&\quad + \sum_{i=1}^{k-1} b_{k,r,s}(i) \sum_{n=1}^{\infty} \delta \left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; n \right) q^n \\
&\quad + \sum_{i=1}^{k-1} c_{k,r,s}(i) \sum_{n=1}^{\infty} \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right) q^n \\
&\quad + \sum_{i=1}^{k-1} d_{k,r,s}(i) \sum_{n=1}^{\infty} \delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; n \right) q^n \\
&= 1 - \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^k}{2^{r+s}(2^{2k}-1)} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n} \right) \\
& + \frac{1}{2^{2k}-1} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{8n} \right) \\
& - \frac{2^{2k}}{2^{2k}-1} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{16n} \right) \\
& + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k-2} a_{k,r,s}(i) \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) \right) q^n \\
& + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k-1} b_{k,r,s}(i) \delta \left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; n \right) \right) q^n \\
& + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k-1} c_{k,r,s}(i) \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right) \right) q^n \\
& + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k-1} d_{k,r,s}(i) \delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; n \right) \right) q^n \\
& = 1 + \sum_{n=1}^{\infty} \left(- \frac{(-1)^k 4k}{2^{r+s}(2^{2k}-1)B_{2k}} \sigma_{2k-1}(n) + \frac{(-1)^k 4k}{2^{r+s}(2^{2k}-1)B_{2k}} \sigma_{2k-1}\left(\frac{n}{2}\right) \right. \\
& \quad \left. + \frac{4k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}\left(\frac{n}{8}\right) - \frac{2^{2k+2}k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}\left(\frac{n}{16}\right) \right)
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^{k-2} a_{k,r,s}(i) \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) \\
 & + \sum_{i=1}^{k-1} b_{k,r,s}(i) \delta \left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; n \right) \\
 & + \sum_{i=1}^{k-1} c_{k,r,s}(i) \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right) \\
 & + \sum_{i=1}^{k-1} d_{k,r,s}(i) \delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; n \right) \Big) q^n.
 \end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the desired results. $\square$

When $k = 1$ we have $r = s = 1$ (as $r, s \in \mathbb{N}$ and $r + 2s < 4k$). Then Theorem 6.2.3 gives

$$(6.2.2) \quad N([1, 2, 1]; n) = 2\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16), \quad n \in \mathbb{N},$$

in agreement with [2, Theorem 1.14, p. 300].

6.3 Number of representations by the quadratic

$$\text{form } x_1^2 + \cdots + x_{4k-s}^2 + 4x_{4k-s+1}^2 + \cdots + 4x_{4k}^2$$

We begin this section by using all the results that we found in Section 5.3 to obtain the following theorem that is valid for any variables x and z .

Theorem 6.3.1. *For $k, s \in \mathbb{N}$ with $s < 4k$, we have*

$$\left(\frac{1 + (1-x)^{1/4}}{2} \right)^s z^{2k}$$

$$\begin{aligned}
 &= \frac{(-1)^k}{2^s(2^{2k}-1)} e_{k,1}(x)z^{2k} - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k}-1)} e_{k,2}(x)z^{2k} \\
 &+ \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor}(2^{2k}-1)} e_{k,4}(x)z^{2k} - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k-\lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k}-1} e_{k,8}(x)z^{2k} \\
 &+ \frac{2^{2k}}{2^{2k}-1} e_{k,16}(x)z^{2k} \\
 &+ (1-x) \sum_{i=1}^{k-2} a'_{k,0,s}(i) x^i z^{2k} + (1-x)^{1/4} \sum_{i=0}^{k-1} b'_{k,0,s}(i) x^i z^{2k} \\
 &+ (1-x)^{1/2} \sum_{i=1}^{k-1} c'_{k,0,s}(i) x^i z^{2k} + (1-x)^{3/4} \sum_{i=0}^{k-1} d'_{k,0,s}(i) x^i z^{2k},
 \end{aligned}$$

where $a'_{k,0,s}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-2\}$, $b'_{k,0,s}(i), d'_{k,0,s}(i) \in \mathbb{Q}$ for $i \in \{0, \dots, k-1\}$ and $c'_{k,0,s}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-1\}$.

Proof. When $k = 1$ we have $s = 1, 2$ or 3 as $s < 4k$. From (6.0.1)-(6.0.5) we have

$$\begin{aligned}
 \left(\frac{1 + (1-x)^{1/4}}{2} \right) &= -\frac{1}{6} e_{1,1}(x)z^2 + \frac{5}{6} e_{1,4}(x)z^2 - e_{1,8}(x)z^2 + \frac{4}{3} e_{1,16}(x)z^2 \\
 &\quad + \frac{1}{4}(1-x)^{1/4} z^2 - \frac{1}{4}(1-x)^{3/4} z^2, \\
 \left(\frac{1 + (1-x)^{1/4}}{2} \right)^2 &= -\frac{1}{12} e_{1,1}(x)z^2 - \frac{1}{12} e_{1,2}(x)z^2 - \frac{1}{3} e_{1,8}(x)z^2 + \frac{4}{3} e_{1,16}(x)z^2 \\
 &\quad + \frac{1}{4}(1-x)^{1/4} z^2 - \frac{1}{4}(1-x)^{3/4} z^2, \\
 \left(\frac{1 + (1-x)^{1/4}}{2} \right)^3 &= -\frac{1}{24} e_{1,1}(x)z^2 + \frac{1}{8} e_{1,2}(x)z^2 - \frac{5}{12} e_{1,8}(x)z^2 + \frac{4}{3} e_{1,16}(x)z^2
 \end{aligned}$$

$$+ \frac{1}{8}(1-x)^{1/4} z^2 - \frac{1}{8}(1-x)^{3/4} z^2.$$

For $k \geq 2$, by Definition 5.3.1, Lemmas 5.3.2 and 5.3.3 we have

$$P_{k,0,s}(x) = (1-x) (a_{k,0,s}(1)x + \cdots + a_{k,0,s}(k-2)x^{k-2}),$$

$$Q_{k,0,s}(x) = b'_{k,0,s}(0) + b'_{k,0,s}(1)x + \cdots + b'_{k,0,s}(k-1)x^{k-1},$$

$$R_{k,0,s}(x) = c'_{k,0,s}(1)x + \cdots + c'_{k,0,s}(k-1)x^{k-1},$$

$$S_{k,0,s}(x) = d'_{k,0,s}(0) + d'_{k,0,s}(1)x + \cdots + d'_{k,0,s}(k-1)x^{k-1},$$

where $a'_{k,0,s}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-2\}$, $b'_{k,0,s}(i), d'_{k,0,s}(i) \in \mathbb{Q}$ for $i \in \{0, \dots, k-1\}$ and $c'_{k,0,s}(i) \in \mathbb{Q}$ for $i \in \{1, \dots, k-1\}$. This implies, by Lemma 5.3.1, that

$$\begin{aligned}
 (6.3.1) \quad & \left(\frac{1 + (1-x)^{1/4}}{2} \right)^s \\
 &= \frac{(-1)^k}{2^s(2^{2k}-1)} e_{k,1}(x) - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k}-1)} e_{k,2}(x) \\
 &+ \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} e_{k,4}(x) \\
 &- \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k-\lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k}-1} e_{k,8}(x) + \frac{2^{2k}}{2^{2k}-1} e_{k,16}(x) \\
 &+ (1-x) \sum_{i=1}^{k-2} a'_{k,0,s}(i) x^i + (1-x)^{1/4} \sum_{i=0}^{k-1} b'_{k,0,s}(i) x^i
 \end{aligned}$$

$$+ (1-x)^{1/2} \sum_{i=1}^{k-1} c'_{k,0,s}(i) x^i + (1-x)^{3/4} \sum_{i=0}^{k-1} d'_{k,0,s}(i) x^i.$$

Finally, we obtain the asserted result by multiplying both sides of (6.3.1) by z^{2k} . $\square$

Replacing x and z in Theorem 6.3.1 by their values given in Definition 2.3.2 we obtain the following identities involving the theta function φ , Eisenstein series $E_k(q^r)$ with $r \in \{1, 2, 4, 8, 16\}$ and the generalized Ramanujan discriminant function $\Delta(a, b, c; q)$.

Theorem 6.3.2. *For $k, s \in \mathbb{N}$ with $s < 4k$ and $q \in D_\delta$, we have*

$$\begin{aligned} & \varphi(q)^{4k-s} \varphi(q^4)^s \\ &= \frac{(-1)^k}{2^s(2^{2k}-1)} E_k(q) - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k}{2^s(2^{2k}-1)} E_k(q^2) \\ &+ \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} (2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1)} E_k(q^4) - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k-\lfloor \frac{s+1}{2} \rfloor} + 1}{2^{2k}-1} E_k(q^8) \\ &+ \frac{2^{2k}}{2^{2k}-1} E_k(q^{16}) \\ &+ \sum_{i=1}^{k-2} a_{k,0,s}(i) \Delta\left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q\right) \\ &+ \sum_{i=0}^{k-1} b_{k,0,s}(i) \Delta\left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; q\right) \\ &+ \sum_{i=1}^{k-1} c_{k,0,s}(i) \Delta\left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; q\right) \end{aligned}$$

$$+ \sum_{i=0}^{k-1} d_{k,0,s}(i) \Delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; q \right),$$

where $a_{k,0,s}(i) = 2^{4i} a'_{k,0,s}(i)$ for $i \in \{1, \dots, k-2\}$, $b_{k,0,s}(i) = 2^{4i} b'_{k,0,s}(i)$ for $i \in \{0, \dots, k-1\}$, $c_{k,0,s}(i) = 2^{4i} c'_{k,0,s}(i)$ for $i \in \{1, \dots, k-1\}$, $d_{k,0,s}(i) = 2^{4i} d'_{k,0,s}(i)$ for $i \in \{0, \dots, k-1\}$ and the coefficients $a'_{k,0,s}(i)$, $b'_{k,0,s}(i)$, $c'_{k,0,s}(i)$, $d'_{k,0,s}(i)$ are given in Theorem 6.3.1.

Proof. Let $q \in D_\delta$. Set $x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}$, $z = \varphi^2(q)$. For $k = 1$, we have by Theorems 2.5.2(ii), 2.9.2 and 2.9.5

$$\begin{aligned} \varphi(q)^3 \varphi(q^4) + \frac{1}{6} E_1(q) - \frac{5}{6} E_1(q^4) + E_1(q^8) - \frac{4}{3} E_1(q^{16}) = \\ \frac{1}{4} (1-x)^{1/4} z^2 - \frac{1}{4} (1-x)^{3/4} z^2 \end{aligned}$$

$$\begin{aligned} \varphi(q)^2 \varphi(q^4)^2 + \frac{1}{12} E_1(q) - \frac{1}{12} E_1(q^2) + \frac{1}{3} E_1(q^8) - \frac{4}{3} E_1(q^{16}) = \\ \frac{1}{4} (1-x)^{1/4} z^2 - \frac{1}{4} (1-x)^{3/4} z^2 \end{aligned}$$

$$\begin{aligned} \varphi(q) \varphi(q^4)^3 + \frac{1}{24} E_1(q) - \frac{1}{8} E_1(q^2) + \frac{5}{12} E_1(q^4) - \frac{4}{3} E_1(q^{16}) = \\ \frac{1}{8} (1-x)^{1/4} z^2 - \frac{1}{8} (1-x)^{3/4} z^2 \end{aligned}$$

which give the asserted results on isolating $\varphi(q)^3 \varphi(q^4)$, $\varphi(q)^2 \varphi(q^4)^2$ and $\varphi(q) \varphi(q^4)^3$.

For $k \geq 2$, we note that from Theorem 2.10.3 we have for $i \in \{1, \dots, k-2\}$

$$\begin{aligned} a'_{k,0,s}(i) x^i (1-x) z^{2k} &= 2^{4i} a'_{k,0,s}(i) \frac{x^i (1-x)}{2^{4i}} z^{2k} \\ &= 2^{4i} a'_{k,0,s}(i) \Delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q \right) \end{aligned}$$

$$= a_{k,0,s}(i) \Delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; q \right),$$

where $a_{k,0,s}(i) = 2^{4i} a'_{k,0,s}(i)$. Similarly, we obtain for $i \in \{0, \dots, k-1\}$

$$b'_{k,0,s}(i) x^i (1-x)^{1/4} z^{2k} = b_{k,0,s}(i) \Delta \left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; q \right),$$

$$d'_{k,0,s}(i) x^i (1-x)^{3/4} z^{2k} = d_{k,0,s}(i) \Delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; q \right),$$

where $b_{k,0,s}(i) = 2^{4i} b'_{k,0,s}(i)$ and $d_{k,0,s}(i) = 2^{4i} d'_{k,0,s}(i)$. For $i \in \{1, \dots, k-1\}$ we have

$$c'_{k,0,s}(i) x^i (1-x)^{1/2} z^{2k} = c_{k,0,s}(i) \Delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; q \right).$$

Finally, from Theorems 4.4.1 and 6.3.1 we obtain the desired results. $\square$

Our final result gives the number of representations of $n \in \mathbb{N}$ by the quadratic form

$$x_1^2 + \dots + x_{4k-s}^2 + 4x_{4k-s+1}^2 + \dots + 4x_{4k}^2,$$

where $s < 4k$.

Theorem 6.3.3. *For $k, s, n \in \mathbb{N}$ with $s < 4k$, we have*

$$N([4k-s, 0, s]; n)$$

$$\begin{aligned} &= -\frac{(-1)^k 4k}{2^s (2^{2k}-1) B_{2k}} \sigma_{2k-1}(n) + \frac{(\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{\lfloor \frac{s}{2} \rfloor} + (-1)^k) 4k}{2^s (2^{2k}-1) B_{2k}} \sigma_{2k-1}\left(\frac{n}{2}\right) \\ &\quad - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 4k(2^{2k}+1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k}-1) B_{2k}} \sigma_{2k-1}\left(\frac{n}{4}\right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{\left(\varepsilon_{2,4}(s) (-1)^{\left[\frac{s+1}{4}\right]} 2^{2k-\left[\frac{s+1}{2}\right]} + 1 \right) 4k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}\left(\frac{n}{8}\right) - \frac{2^{2k+2}k}{(2^{2k}-1)B_{2k}} \sigma_{2k-1}\left(\frac{n}{16}\right) \\
 & + \sum_{i=1}^{k-2} a_{k,0,s}(i) \delta\left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n\right) \\
 & + \sum_{i=0}^{k-1} b_{k,0,s}(i) \delta\left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; n\right) \\
 & + \sum_{i=1}^{k-1} c_{k,0,s}(i) \delta\left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n\right) \\
 & + \sum_{i=0}^{k-1} d_{k,0,s}(i) \delta\left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; n\right),
 \end{aligned}$$

where $a_{k,0,s}(i) = 2^{4i}a'_{k,0,s}(i)$ for $i \in \{1, \dots, k-2\}$, $b_{k,0,s}(i) = 2^{4i}b'_{k,0,s}(i)$ for $i \in \{0, \dots, k-1\}$, $c_{k,0,s}(i) = 2^{4i}c'_{k,0,s}(i)$ for $i \in \{1, \dots, k-1\}$, $d_{k,0,s}(i) = 2^{4i}d'_{k,0,s}(i)$ for $i \in \{0, \dots, k-1\}$ and the coefficients $a'_{k,0,s}(i)$, $b'_{k,0,s}(i)$, $c'_{k,0,s}(i)$, $d'_{k,0,s}(i)$ are given in Theorem 6.3.1.

Proof. From Theorem 2.8.1, Theorem 6.3.2 and Definition 2.10.3, we have for $q \in D_\delta$

$$\begin{aligned}
 & 1 + \sum_{n=1}^{\infty} N([4k-s, 0, s]; n) q^n = \varphi(q)^{4k-s} \varphi(q^4)^s \\
 & = \frac{(-1)^k}{2^s(2^{2k}-1)} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \right) \\
 & \quad - \frac{\varepsilon_{2,4}(s) (-1)^{\left[\frac{s+1}{4}\right]} 2^{\left[\frac{s}{2}\right]} + (-1)^k}{2^s(2^{2k}-1)} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\varepsilon_{2,4}(s) (-1)^{\lceil \frac{s+1}{4} \rceil} (2^{2k} + 1)}{2^{\lceil \frac{s+1}{2} \rceil} (2^{2k} - 1)} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{4n} \right) \\
& - \frac{\varepsilon_{2,4}(n) (-1)^{\lceil \frac{s+1}{4} \rceil} 2^{2k - \lceil \frac{s+1}{2} \rceil} + 1}{2^{2k} - 1} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{8n} \right) \\
& + \frac{2^{2k}}{2^{2k} - 1} \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{16n} \right) \\
& + \sum_{i=1}^{k-2} a_{k,0,s}(i) \sum_{n=1}^{\infty} \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) q^n \\
& + \sum_{i=0}^{k-1} b_{k,0,s}(i) \sum_{n=1}^{\infty} \delta \left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; n \right) q^n \\
& + \sum_{i=1}^{k-1} c_{k,0,s}(i) \sum_{n=1}^{\infty} \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right) q^n \\
& + \sum_{i=0}^{k-1} d_{k,0,s}(i) \sum_{n=1}^{\infty} \delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; n \right) q^n \\
& = 1 - \frac{(-1)^k}{2^s (2^{2k} - 1)} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \right) \\
& + \frac{\varepsilon_{2,4}(s) (-1)^{\lceil \frac{s+1}{4} \rceil} 2^{\lceil \frac{s}{2} \rceil} + (-1)^k}{2^s (2^{2k} - 1)} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n} \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{\varepsilon_{2,4}(n) (-1)^{\lceil \frac{s+1}{4} \rceil} (2^{2k} + 1)}{2^{\lceil \frac{s+1}{2} \rceil} (2^{2k} - 1)} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{4n} \right) \\
& + \frac{\varepsilon_{2,4}(s) (-1)^{\lceil \frac{s+1}{4} \rceil} 2^{2k - \lceil \frac{s+1}{2} \rceil} + 1}{2^{2k} - 1} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{8n} \right) \\
& - \frac{2^{2k}}{2^{2k} - 1} \left(\frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{16n} \right) \\
& + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k-2} a_{k,0,s}(i) \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) \right) q^n \\
& + \sum_{n=1}^{\infty} \left(\sum_{i=0}^{k-1} b_{k,0,s}(i) \delta \left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; n \right) \right) q^n \\
& + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k-1} c_{k,0,s}(i) \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right) \right) q^n \\
& + \sum_{n=1}^{\infty} \left(\sum_{i=0}^{k-1} d_{k,0,s}(i) \delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; n \right) \right) q^n \\
& = 1 + \sum_{n=1}^{\infty} \left(- \frac{(-1)^k 4k}{2^s (2^{2k} - 1) B_{2k}} \sigma_{2k-1}(n) \right. \\
& \quad \left. + \frac{(\varepsilon_{2,4}(s) (-1)^{\lceil \frac{s+1}{4} \rceil} 2^{\lceil \frac{s}{2} \rceil} + (-1)^k) 4k}{2^s (2^{2k} - 1) B_{2k}} \sigma_{2k-1} \left(\frac{n}{2} \right) \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 4k(2^{2k} + 1)}{2^{\lfloor \frac{s+1}{2} \rfloor} (2^{2k} - 1) B_{2k}} \sigma_{2k-1} \left(\frac{n}{4} \right) \\
& + \frac{\left(\varepsilon_{2,4}(s) (-1)^{\lfloor \frac{s+1}{4} \rfloor} 2^{2k - \lfloor \frac{s+1}{2} \rfloor} + 1 \right) 4k}{(2^{2k} - 1) B_{2k}} \sigma_{2k-1} \left(\frac{n}{8} \right) - \frac{2^{2k+2} k}{(2^{2k} - 1) B_{2k}} \sigma_{2k-1} \left(\frac{n}{16} \right) \\
& + \sum_{i=1}^{k-2} a_{k,0,s}(i) \delta \left(\frac{i-k+2}{3}, \frac{-6i+5k-6}{6}, \frac{2i-k+1}{3}; n \right) \\
& + \sum_{i=0}^{k-1} b_{k,0,s}(i) \delta \left(\frac{2i-2k+1}{6}, \frac{-12i+10k-3}{12}, \frac{8i-4k+1}{12}; n \right) \\
& + \sum_{i=1}^{k-1} c_{k,0,s}(i) \delta \left(\frac{i-k+1}{3}, \frac{-6i+5k-3}{6}, \frac{4i-2k+1}{6}; n \right) \\
& + \sum_{i=0}^{k-1} d_{k,0,s}(i) \delta \left(\frac{2i-2k+3}{6}, \frac{-12i+10k-9}{12}, \frac{8i-4k+3}{12}; n \right) \Big) q^n.
\end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}_0$), we obtain the desired results. $\square$

When $k = 1$ we have $s = 1, 2$ or 3 as $s < 4k$. With $k = s = 1$ Theorem 6.3.3 gives

$$\begin{aligned}
(6.3.2) \quad N([3, 0, 1]; n) &= 4\sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) \\
&+ \frac{1}{4} \delta \left(-\frac{1}{6}, \frac{7}{12}, -\frac{1}{4}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}; n \right).
\end{aligned}$$

With $k = 1, s = 2$ Theorem 6.3.3 gives

$$\begin{aligned}
(6.3.3) \quad N([2, 0, 2]; n) &= 2\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16) \\
&+ \frac{1}{4} \delta \left(-\frac{1}{6}, \frac{7}{12}, -\frac{1}{4}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}; n \right).
\end{aligned}$$

With $k = 1$, $s = 3$ Theorem 6.3.3 gives

$$(6.3.4) \quad N([1, 0, 3]; n) = \sigma(n) - 3\sigma(n/2) + 10\sigma(n/4) - 32\sigma(n/16) \\ + \frac{1}{8} \delta\left(-\frac{1}{6}, \frac{7}{12}, -\frac{1}{4}; n\right) - \frac{1}{8} \delta\left(\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}; n\right).$$

We show in Chapter 8 that

$$\delta\left(-\frac{1}{6}, \frac{7}{12}, -\frac{1}{4}; n\right) - \delta\left(-\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}; n\right) = 8\left(\frac{-4}{n}\right) \sigma(n), \quad n \in \mathbb{N},$$

so that

$$N([3, 0, 1]; n) = \left(4 + 2\left(\frac{-4}{n}\right)\right) \sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16), \\ N([2, 0, 2]; n) = \left(2 + 2\left(\frac{-4}{n}\right)\right) \sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16), \\ N([1, 0, 3]; n) = \left(1 + \left(\frac{-4}{n}\right)\right) \sigma(n) - 3\sigma(n/2) + 10\sigma(n/4) - 32\sigma(n/16),$$

in agreement with [2, Theorems 1.7, 1.11 and 1.18].

Chapter 7

Sums with 8, 12 and 16 squares - I

In this chapter we apply Theorems 6.1.3, 6.2.3 and 6.3.3 with $k = 2, 3, 4$ to obtain the number of representations of an integer $n \in \mathbb{N}$ by the quadratic forms

$$x_1^2 + \cdots + x_{4k-2r}^2 + 2x_{4k-2r+1}^2 + \cdots + 2x_{4k}^2, \text{ where } r \geq 1 \text{ and } r < 2k,$$

$$x_1^2 + \cdots + x_{4k-2r-s}^2 + 2x_{4k-2r-s+1}^2 + 2x_{4k-s}^2 + 4x_{4k-s+1}^2 + \cdots + 4x_{4k}^2, \text{ where } r \geq 1, s \geq 1 \\ \text{and } 2r + s < 4k,$$

$$x_1^2 + \cdots + x_{4k-s}^2 + 4x_{4k-s+1}^2 + \cdots + 4x_{4k}^2, \text{ where } s \geq 1 \text{ and } s < 4k.$$

In Section 2, we apply the three theorems for sums with 8 squares. In Section 3, we apply the three theorems for sums with 12 squares and in Section 4, we apply the three theorems for sums with 16 squares.

Explicit formulae in terms of the sigma function $\sigma_k(n)$ and functions of the form $\delta(a, b, c; n)$ are thus obtained for all three cases when $k = 2$, $k = 3$ and $k = 4$.

7.1 Sums with 8 squares

We begin by applying Theorems 6.1.3, 6.2.3 and 6.3.3 with $k = 2$. We first note that from Definition 4.1.1 we have

$$(7.1.1) \quad e_{2,1}(x) = 1 + 14x + x^2,$$

$$(7.1.2) \quad e_{2,2}(x) = 1 - x + x^2,$$

$$(7.1.3) \quad e_{2,4}(x) = 1 - x + \frac{1}{16}x^2.$$

From Definition 4.2.1 we have

$$e_{2,8}(x) = e_{2,8}^{(0)}(x) + e_{2,8}^{(1)}(x)(1 - x)^{1/2},$$

where

$$(7.1.4) \quad e_{2,8}^{(0)}(x) = \frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2,$$

$$(7.1.5) \quad e_{2,8}^{(1)}(x) = \frac{15}{32} - \frac{15}{64}x.$$

From Definition 4.3.1 we deduce

$$e_{2,16}(x) = \left(e_{2,16}^{(0)}(x) + e_{2,16}^{(1)}(x)(1 - x)^{1/4} + e_{2,16}^{(2)}(x)(1 - x)^{1/2} + e_{2,16}^{(3)}(x)(1 - x)^{3/4} \right),$$

where

$$(7.1.6) \quad e_{2,16}^{(0)}(x) = \frac{137}{512} - \frac{137}{512}x + \frac{1}{4096}x^2,$$

$$(7.1.7) \quad e_{2,16}^{(1)}(x) = \frac{15}{64} - \frac{105}{512}x,$$

$$(7.1.8) \quad e_{2,16}^{(2)}(x) = \frac{135}{512} - \frac{135}{1024}x,$$

$$(7.1.9) \quad e_{2,16}^{(3)}(x) = \frac{15}{64} - \frac{15}{512}x.$$

For our first result, we apply Theorem 6.1.3 to the case when $k = 2$ to obtain the number of representations of a positive integer n by the sum of $8 - 2r$ squares and $2r$ squares each multiplied by 2, where $r \in \mathbb{N}$ and $0 < r < 4$. These results were previously obtained in [2].

Theorem 7.1.1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} N([6, 2, 0]; n) &= 8 \sigma_3(n) - 8 \sigma_3(n/2) - 16 \sigma_3(n/4) + 256 \sigma_3(n/8) \\ &\quad + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right), \end{aligned}$$

$$\begin{aligned} N([4, 4, 0]; n) &= 4 \sigma_3(n) - 4 \sigma_3(n/2) - 16 \sigma_3(n/4) + 256 \sigma_3(n/8) \\ &\quad + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right), \end{aligned}$$

$$\begin{aligned} N([2, 6, 0]; n) &= 2 \sigma_3(n) - 2 \sigma_3(n/2) - 16 \sigma_3(n/4) + 256 \sigma_3(n/8) \\ &\quad + 2 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right). \end{aligned}$$

Proof. We only prove the formula for $N([2, 6, 0]; n)$. This is the case $k = 2, r = 3$.

The proofs for the other cases follow in a similar way.

From (7.1.1)-(7.1.5)

$$\begin{aligned} \left(\frac{1 + (1-x)^{1/2}}{2}\right)^r &= \left(\frac{1 + (1-x)^{1/2}}{2}\right)^3 \\ &= \left(\frac{1}{2} - \frac{3}{8}x\right) + \left(\frac{1}{2} - \frac{1}{8}x\right)(1-x)^{1/2} \\ &= \frac{1}{120}(1 + 14x + x^2) - \frac{1}{120}(1 - x + x^2) - \frac{1}{15}\left(1 - x + \frac{1}{16}x^2\right) \\ &\quad + \frac{16}{15}\left(\frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \left(\frac{15}{32} - \frac{15}{64}x\right)(1-x)^{1/2}\right) \\ &\quad + \frac{1}{8}x(1-x)^{1/2} \\ &= \frac{1}{120} e_{2,1}(x) - \frac{1}{120} e_{2,2}(x) - \frac{1}{15} e_{2,4}(x) \\ &\quad + \frac{16}{15} \left(e_{2,8}^{(0)}(x) + e_{2,8}^{(1)}(x)(1-x)^{1/2}\right) + \frac{1}{8}x(1-x)^{1/2}. \end{aligned}$$

Then, by Theorem 6.1.1, we deduce

$$c'_{2,3,0}(1) = \frac{1}{8}.$$

Hence

$$c_{2,3,0}(1) = 16c'_{2,3,0}(1) = 2.$$

Then, by Theorem 6.1.3, we have

$$\begin{aligned} N([2, 6, 0]; n) &= 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/4) + 256\sigma_3(n/8) \\ &\quad + 2 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right), \end{aligned}$$

as asserted. □

The second formula of Theorem 7.1.1 was first proved by Williams [83, Theorem 2, p. 388] in 2006 as a consequence of his determination of the convolution sum $\sum_{m < n/8} \sigma(m)\sigma(n - 8m)$ for all $n \in \mathbb{N}$. All three formulae of Theorem 7.1.1 were proved in 2010 by Alaca, Alaca and Williams [11, Theorem 1.1 (ii)(iv)(ix), pp. 38-39].

For our second result, we apply Theorem 6.2.3 to the case when $k = 2$ to obtain the number of representations of each positive integer n by the sum of $8 - 2r - s$ squares, $2r$ squares each multiplied by 2 and s squares each multiplied by 4, where $r, s \in \mathbb{N}$ and $0 < 2r + s < 8$.

Theorem 7.1.2. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} N([1, 2, 5]; n) &= \frac{1}{4}\sigma_3(n) - \frac{1}{4}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ &\quad + \frac{3}{4} \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{3}{4} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\ &\quad + \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right), \end{aligned}$$

$$\begin{aligned} N([2, 2, 4]; n) &= \frac{1}{2}\sigma_3(n) - \frac{1}{2}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ &\quad + \frac{3}{2} \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{3}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \end{aligned}$$

$$+ \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right),$$

$$\begin{aligned} N([3, 2, 3]; n) &= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ &\quad + 2 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{5}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\ &\quad + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right), \end{aligned}$$

$$\begin{aligned} N([4, 2, 2]; n) &= 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ &\quad + 2 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{7}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\ &\quad + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right), \end{aligned}$$

$$\begin{aligned} N([5, 2, 1]; n) &= 4\sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ &\quad + 2 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{7}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\ &\quad + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right), \end{aligned}$$

$$\begin{aligned} N([1, 4, 3]; n) &= \frac{1}{2}\sigma_3(n) - \frac{1}{2}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ &\quad + \frac{1}{2} \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right), \end{aligned}$$

$$\begin{aligned} N([2, 4, 2]; n) &= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ &\quad + \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{3}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\ &\quad + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right), \end{aligned}$$

$$\begin{aligned}
N([3, 4, 1]; n) &= 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\
&\quad + 2\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + \frac{3}{2}\delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) \\
&\quad + \frac{1}{2}\delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([1, 6, 1]; n) &= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\
&\quad + \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + \frac{1}{2}\delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) \\
&\quad - \frac{1}{2}\delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right).
\end{aligned}$$

Proof. We only prove the formula for $N([2, 4, 2]; n)$. Here $r = s = k = 2$. The proofs for the other cases follow in a similar way.

By (7.1.1), (7.1.2) and (7.1.4) - (7.1.9) we have

$$\begin{aligned}
&\left(\frac{1 + (1-x)^{1/2}}{2}\right)^2 \left(\frac{1 + (1-x)^{1/4}}{2}\right)^2 - \frac{1}{240} e_{2,1}(x) + \frac{1}{240} e_{2,2}(x) \\
&\quad + \frac{1}{15} (e_{2,8}^{(0)}(x) + e_{2,8}^{(1)}(x)(1-x)^{1/2}) \\
&\quad - \frac{16}{15} \left(e_{2,16}^{(0)}(x) + e_{2,16}^{(1)}(x)(1-x)^{1/4} + e_{2,16}^{(2)}(1-x)^{1/2} + e_{2,16}^{(3)}(1-x)^{3/4}\right) \\
&= (1-x)^{1/4} \left(\frac{3}{32}x\right) + (1-x)^{1/2} \left(\frac{1}{16}x\right) \\
&\quad + (1-x)^{3/4} \left(\frac{1}{32}x\right).
\end{aligned}$$

Thus, by Theorem 6.2.1, we have

$$b'_{2,2,2}(1) = \frac{3}{32}, \quad c'_{2,2,2}(1) = \frac{1}{16}, \quad d'_{2,2,2}(1) = \frac{1}{32}.$$

Hence

$$b_{2,2,2}(1) = \frac{3}{2}, \quad c_{2,2,2}(1) = 1, \quad d_{2,2,2}(1) = \frac{1}{2}.$$

Using Theorem 6.2.3 with $k = 2$, $r = 2$ and $s = 2$, we obtain the desired result. $\square$

These formulae are simplified in Theorem 9.1.3. The simplified formulae agree with those obtained by Alaca, Alaca and Williams [11, Theorem 1.1 (iii)(v)(vii)(viii)(x)(xi)(xii)(xiii)(xiv)] in 2010.

For the final result of this section, we apply Theorem 6.3.3 to the case when $k = 2$ to obtain the number of representations of a positive integer n by the sum of $8 - s$ squares and s squares multiplied by 4, where $s \in \mathbb{N}$ and $0 < s < 8$.

Theorem 7.1.3. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} N([1, 0, 7]; n) = & \frac{1}{8} \sigma_3(n) - \frac{9}{8} \sigma_3(n/2) + 17 \sigma_3(n/4) - 32 \sigma_3(n/8) + 256 \sigma_3(n/16) \\ & + \frac{7}{8} \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) - \frac{1}{32} \delta\left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n\right) \\ & + \frac{1}{32} \delta\left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n\right) + \frac{7}{8} \delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) \\ & + \frac{3}{8} \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right), \end{aligned}$$

$$\begin{aligned} N([2, 0, 6]; n) = & \frac{1}{4} \sigma_3(n) - \frac{1}{4} \sigma_3(n/2) - 16 \sigma_3(n/8) + 256 \sigma_3(n/16) \\ & + \frac{7}{4} \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) - \frac{1}{16} \delta\left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n\right) \\ & + \frac{1}{16} \delta\left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n\right) + 2 \delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) \\ & + \frac{1}{2} \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right), \end{aligned}$$

$$\begin{aligned} N([3, 0, 5]; n) = & \frac{1}{2} \sigma_3(n) + \frac{3}{2} \sigma_3(n/2) - 34 \sigma_3(n/4) + 16 \sigma_3(n/8) + 256 \sigma_3(n/16) \\ & + \frac{5}{2} \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) - \frac{1}{16} \delta\left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n\right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{16} \delta \left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n \right) + 3 \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\
& + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([4, 0, 4]; n) &= \sigma_3(n) + 3 \sigma_3(n/2) - 68 \sigma_3(n/4) + 48 \sigma_3(n/8) + 256 \sigma_3(n/16) \\
& + 3 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{7}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\
& + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([5, 0, 3]; n) &= 2 \sigma_3(n) + 2 \sigma_3(n/2) - 68 \sigma_3(n/4) + 48 \sigma_3(n/8) + 256 \sigma_3(n/16) \\
& + 3 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{1}{8} \delta \left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n \right) \\
& - \frac{1}{8} \delta \left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n \right) + \frac{7}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\
& + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([6, 0, 2]; n) &= 4 \sigma_3(n) - 4 \sigma_3(n/2) - 16 \sigma_3(n/8) + 256 \sigma_3(n/16) \\
& + 2 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) + \frac{1}{4} \delta \left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n \right) \\
& - \frac{1}{4} \delta \left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n \right) + \frac{7}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \\
& + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([7, 0, 1]; n) &= 8 \sigma_3(n) - 16 \sigma_3(n/2) + 136 \sigma_3(n/4) - 144 \sigma_3(n/8) + 256 \sigma_3(n/16) \\
& + \frac{1}{4} \delta \left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n \right) - \frac{1}{4} \delta \left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n \right) \\
& + \frac{7}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) + \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right).
\end{aligned}$$

Proof. We only prove the formula for $N([3, 0, 5]; n)$. In this case $k = 2$ and $s = 5$.

The proofs for the other cases follow in a similar way.

By (7.1.1)-(7.1.9) we have

$$\begin{aligned}
 & \left(\frac{1 + (1-x)^{1/4}}{2} \right)^5 - \frac{1}{480} e_{2,1}(x) - \frac{1}{160} e_{2,2}(x) + \frac{17}{120} e_{2,4}(x) \\
 & - \frac{1}{15} (e_{2,8}^{(0)}(x) + e_{2,8}^{(1)}(x)(1-x)^{1/2}) \\
 & - \frac{16}{15} \left(e_{2,16}^{(0)}(x) + e_{2,16}^{(1)}(x)(1-x)^{1/4} + e_{2,16}^{(2)}(1-x)^{1/2} + e_{2,16}^{(3)}(1-x)^{3/4} \right) \\
 & = (1-x)^{1/4} \left(-\frac{1}{16} + \frac{3}{16}x \right) + (1-x)^{1/2} \left(\frac{5}{32}x \right) \\
 & + (1-x)^{3/4} \left(\frac{1}{16} + \frac{1}{32}x \right)
 \end{aligned}$$

Thus we have

$$b'_{2,0,5}(0) = -\frac{1}{16}, b'_{2,0,5}(1) = \frac{3}{16}, c'_{2,0,5}(1) = \frac{5}{32}, d'_{2,0,5}(0) = \frac{1}{16}, d'_{2,0,5}(1) = \frac{1}{32}.$$

Using Theorem 6.3.3 with $k = 2$ and $s = 5$, we obtain the desired result. $\square$

The seven formulae of Theorem 7.1.3 are simplified in Theorem 9.1.6. The simplified formulae agree with those of Alaca, Alaca and Williams [5, Theorem 1.1, pp. 343-344] obtained in 2008.

7.2 Sums with 12 squares

In this section, we apply Theorem 6.1.3, Theorem 6.2.3 and Theorem 6.3.3 when $k = 3$. We first note that from Definition 4.1.1 we have

$$(7.2.1) \quad e_{3,1}(x) = 1 - 33x - 33x^2 + x^3,$$

$$(7.2.2) \quad e_{3,2}(x) = 1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3,$$

$$(7.2.3) \quad e_{3,4}(x) = 1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3.$$

From Definition 4.2.1 we have

$$e_{3,8}(x) = e_{3,8}^{(0)}(x) + e_{3,8}^{(1)}(x)(1-x)^{1/2},$$

where

$$(7.2.4) \quad e_{3,8}^{(0)}(x) = \frac{65}{128} - \frac{195}{256}x + \frac{519}{2048}x^2 + \frac{1}{4096}x^3,$$

$$(7.2.5) \quad e_{3,8}^{(1)}(x) = \frac{65}{128} - \frac{63}{128}x + \frac{63}{2048}x^2.$$

From Definition 4.3.1 we have

$$e_{3,16}(x) = \left(e_{3,16}^{(0)}(x) + e_{3,16}^{(1)}(x)(1-x)^{1/4} + e_{3,16}^{(2)}(x)(1-x)^{1/2} + e_{3,16}^{(3)}(x)(1-x)^{3/4} \right),$$

where

$$(7.2.6) \quad e_{3,16}^{(0)}(x) = \frac{2081}{8192} - \frac{6243}{16384}x + \frac{16647}{131072}x^2 + \frac{1}{262144}x^3,$$

$$(7.2.7) \quad e_{3,16}^{(1)}(x) = \frac{63}{256} - \frac{4977}{16384}x + \frac{3843}{65536}x^2,$$

$$(7.2.8) \quad e_{3,16}^{(2)}(x) = \frac{2079}{8192} - \frac{2079}{8192}x + \frac{2079}{131072}x^2,$$

$$(7.2.9) \quad e_{3,16}^{(3)}(x) = \frac{63}{256} - \frac{3087}{16384}x + \frac{63}{65536}x^2.$$

For our first result, we apply Theorem 6.1.3 to the case when $k = 3$ to obtain the number of representations of an integer n by the sum of $12 - 2r$ squares and $2r$ squares multiplied by 2, where $r \in \mathbb{N}$ and $0 < r < 6$.

Theorem 7.2.1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} N([10, 2, 0]; n) &= 4\sigma_5(n) - 4\sigma_5(n/2) + 8\sigma_5(n/4) - 512\sigma_5(n/8) \\ &\quad + 8\delta\left(0, \frac{1}{2}, 0; n\right) + 8\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 8\delta\left(0, 0, \frac{1}{2}; n\right), \end{aligned}$$

$$\begin{aligned} N([8, 4, 0]; n) &= 2\sigma_5(n) - 2\sigma_5(n/2) + 8\sigma_5(n/4) - 512\sigma_5(n/8) \\ &\quad + 6\delta\left(0, \frac{1}{2}, 0; n\right) + 8\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 8\delta\left(0, 0, \frac{1}{2}; n\right), \end{aligned}$$

$$\begin{aligned} N([6, 6, 0]; n) &= \sigma_5(n) - \sigma_5(n/2) + 8\sigma_5(n/4) - 512\sigma_5(n/8) \\ &\quad + 5\delta\left(0, \frac{1}{2}, 0; n\right) + 6\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 8\delta\left(0, 0, \frac{1}{2}; n\right), \end{aligned}$$

$$\begin{aligned} N([4, 8, 0]; n) &= \frac{1}{2}\sigma_5(n) - \frac{1}{2}\sigma_5(n/2) + 8\sigma_5(n/4) - 512\sigma_5(n/8) \\ &\quad + \frac{7}{2}\delta\left(0, \frac{1}{2}, 0; n\right) + 4\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 8\delta\left(0, 0, \frac{1}{2}; n\right), \end{aligned}$$

$$\begin{aligned} N([2, 10, 0]; n) &= \frac{1}{4}\sigma_5(n) - \frac{1}{4}\sigma_5(n/2) + 8\sigma_5(n/4) - 512\sigma_5(n/8) \\ &\quad + \frac{7}{4}\delta\left(0, \frac{1}{2}, 0; n\right) + 2\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right). \end{aligned}$$

Proof. We only prove the formula for $N([10, 2, 0]; n)$. The proofs for the other cases follow in a similar way.

From (7.2.1)-(7.2.9) and Lemma 5.1.1 we have

$$\begin{aligned} &\left(\frac{1 + (1-x)^{1/2}}{2}\right) + \frac{1}{126} e_{3,1}(x) - \frac{1}{126} e_{3,2}(x) + \frac{1}{63} e_{3,4}(x) \\ &- \frac{64}{63} \left(e_{3,8}^{(0)}(x) + e_{3,8}^{(1)}(x)(1-x)^{1/2}\right) \end{aligned}$$

$$= (1-x) \left(\frac{1}{2}x \right) + (1-x)^{1/2} \left(\frac{1}{2}x - \frac{1}{32}x^2 \right).$$

Thus we have

$$a'_{3,2,0}(1) = \frac{1}{2}, \quad c'_{3,2,0}(1) = \frac{1}{2}, \quad c'_{3,2,0}(2) = -\frac{1}{32}.$$

Using Theorem 6.1.3 with $k = 3$ and $r = 2$, we obtain the desired result. $\square$

We recall from Theorems 2.11.2 and 2.11.4 that $\delta \left(0, \frac{1}{2}, 0; n \right) = \tau_2(n)$ and $\delta \left(0, 0, \frac{1}{2}; n \right) = \tau_2(n/2)$ for $n \in \mathbb{N}$.

For our second result, we apply Theorem 6.2.3 to the case when $k = 3$ to obtain the number of representations of an integer n by the sum of $12 - 2r - s$ squares, $2r$ squares multiplied by 2 and s squares multiplied by 4, where $r, s \in \mathbb{N}$ and $0 < 2r + s < 12$.

Theorem 7.2.2. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} N([1, 2, 9]; n) &= \frac{1}{128} \sigma_5(n) - \frac{1}{128} \sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) \\ &\quad + \frac{71}{128} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{7}{16} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{7}{4} \delta \left(0, 0, \frac{1}{2}; n \right) \\ &\quad + \frac{1}{2} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - 6 \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\ &\quad + \frac{1}{2} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right), \end{aligned}$$

$$\begin{aligned} N([2, 2, 8]; n) &= \frac{1}{64} \sigma_5(n) - \frac{1}{64} \sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) \\ &\quad + \frac{71}{64} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{7}{8} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{7}{2} \delta \left(0, 0, \frac{1}{2}; n \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{15}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{45}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{17}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([3, 2, 7]; n) &= \frac{1}{32} \sigma_5(n) - \frac{1}{32} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{51}{32} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{11}{8} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{21}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{57}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{27}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([4, 2, 6]; n) &= \frac{1}{16} \sigma_5(n) - \frac{1}{16} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{31}{16} \delta \left(0, \frac{1}{2}, 0; n \right) + 2 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{27}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{37}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([5, 2, 5]; n) &= \frac{1}{8} \sigma_5(n) - \frac{1}{8} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{17}{8} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{11}{4} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{35}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{45}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([6, 2, 4]; n) &= \frac{1}{4} \sigma_5(n) - \frac{1}{4} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{9}{4} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{7}{2} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{47}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([7, 2, 3]; n) &= \frac{1}{2} \sigma_5(n) - \frac{1}{2} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{5}{2} \delta \left(0, \frac{1}{2}, 0; n \right) + 4 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{63}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([8, 2, 2]; n) &= \sigma_5(n) - \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + 3 \delta \left(0, \frac{1}{2}, 0; n \right) + 4 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{79}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([9, 2, 1]; n) &= 2 \sigma_5(n) - 2 \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + 4 \delta \left(0, \frac{1}{2}, 0; n \right) + 4 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{79}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([1, 4, 7]; n) &= \frac{1}{64} \sigma_5(n) - \frac{1}{64} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{31}{64} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{1}{2} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{1}{2} \delta \left(0, 0, \frac{1}{2}; n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{7}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{15}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{9}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) + \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([2, 4, 6]; n) &= \frac{1}{32} \sigma_5(n) - \frac{1}{32} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{31}{32} \delta \left(0, \frac{1}{2}, 0; n \right) + \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 3 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{15}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{37}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{17}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([3, 4, 5]; n) &= \frac{1}{16} \sigma_5(n) - \frac{1}{16} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{23}{16} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{3}{2} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{23}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{53}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{25}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([4, 4, 4]; n) &= \frac{1}{8} \sigma_5(n) - \frac{1}{8} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{15}{8} \delta \left(0, \frac{1}{2}, 0; n \right) + 2 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{31}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{33}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([5, 4, 3]; n) &= \frac{1}{4} \sigma_5(n) - \frac{1}{4} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{9}{4} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{5}{2} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{39}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{41}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([6, 4, 2]; n) &= \frac{1}{2} \sigma_5(n) - \frac{1}{2} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{5}{2} \delta \left(0, \frac{1}{2}, 0; n \right) + 3 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{47}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([7, 4, 1]; n) &= \sigma_5(n) - \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + 3 \delta \left(0, \frac{1}{2}, 0; n \right) + 4 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{47}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([1, 6, 5]; n) &= \frac{1}{32} \sigma_5(n) - \frac{1}{32} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{15}{32} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{1}{2} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) + \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{7}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{9}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{9}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) + \frac{3}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([2, 6, 4]; n) &= \frac{1}{16} \sigma_5(n) - \frac{1}{16} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{15}{16} \delta \left(0, \frac{1}{2}, 0; n \right) + \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 2 \delta \left(0, 0, \frac{1}{2}; n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{15}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{29}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{17}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([3, 6, 3]; n) &= \frac{1}{8} \sigma_5(n) - \frac{1}{8} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{11}{8} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{3}{2} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{23}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{45}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{25}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([4, 6, 2]; n) &= \frac{1}{4} \sigma_5(n) - \frac{1}{4} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{7}{4} \delta \left(0, \frac{1}{2}, 0; n \right) + 2 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{31}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{33}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([5, 6, 1]; n) &= \frac{1}{2} \sigma_5(n) - \frac{1}{2} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{5}{2} \delta \left(0, \frac{1}{2}, 0; n \right) + 3 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{31}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{33}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([1, 8, 3]; n) &= \frac{1}{16} \sigma_5(n) - \frac{1}{16} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{7}{16} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{1}{2} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) + 2 \delta \left(0, 0, \frac{1}{2}; n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{7}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{5}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{9}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) + \frac{7}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([2, 8, 2]; n) &= \frac{1}{8} \sigma_5(n) - \frac{1}{8} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
&+ \frac{7}{8} \delta \left(0, \frac{1}{2}, 0; n \right) + \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) + \frac{15}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) \\
&- \frac{29}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) + \frac{17}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) \\
&- \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([3, 8, 1]; n) &= \frac{1}{4} \sigma_5(n) - \frac{1}{4} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
&+ \frac{7}{4} \delta \left(0, \frac{1}{2}, 0; n \right) + 2 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
&+ \frac{15}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{29}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
&+ \frac{17}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([1, 10, 1]; n) &= \frac{1}{8} \sigma_5(n) - \frac{1}{8} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
&+ \frac{7}{8} \delta \left(0, \frac{1}{2}, 0; n \right) + \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \\
&- \frac{1}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) + \frac{19}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
&+ \frac{1}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) + \frac{15}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right).
\end{aligned}$$

Proof. We only prove the formula for $N([1, 6, 5]; n)$. The proofs for the other cases follow in a similar way.

From (7.2.1)-(7.2.9) and Lemma 5.2.1 we have

$$\begin{aligned}
& \left(\frac{1 + (1-x)^{1/2}}{2} \right)^3 \left(\frac{1 + (1-x)^{1/4}}{2} \right)^5 + \frac{1}{16128} e_{3,1}(x) - \frac{1}{16128} e_{3,2}(x) \\
& + \frac{1}{63} (e_{3,8}^{(0)}(x) + e_{3,8}^{(1)}(x)(1-x)^{1/2}) \\
& - \frac{64}{63} \left(e_{3,16}^{(0)}(x) + e_{3,16}^{(1)}(x)(1-x)^{1/4} + e_{3,16}^{(2)}(1-x)^{1/2} + e_{3,16}^{(3)}(1-x)^{3/4} \right) \\
& = (1-x) \left(\frac{15}{512} x \right) + (1-x)^{1/4} \left(\frac{7}{256} x - \frac{9}{1024} x^2 \right) + (1-x)^{1/2} \left(\frac{1}{32} x + \frac{1}{256} x^2 \right) \\
& + (1-x)^{3/4} \left(\frac{9}{256} x + \frac{3}{1024} x^2 \right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
a'_{3,6,5}(1) &= \frac{15}{512}, \quad b'_{3,6,5}(1) = \frac{7}{256}, \quad b'_{3,6,5}(2) = -\frac{9}{1024}, \quad c'_{3,6,5}(1) = \frac{1}{32}, \quad c'_{3,6,5}(2) = \frac{1}{256}, \\
d'_{3,6,5}(1) &= \frac{9}{256}, \quad d'_{3,6,5}(2) = \frac{3}{1024}.
\end{aligned}$$

Using Theorem 6.2.3 with $k = 3$, $r = 6$ and $s = 5$, we obtain the desired result. $\square$

For the final result of this section, we apply Theorem 6.3.3 to the case when $k = 3$ to obtain the number of representations of an integer n by the sum of $12 - s$ squares and s squares multiplied by 4, where $s \in \mathbb{N}$ and $0 < s < 12$.

Theorem 7.2.3. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
N([1, 0, 11]; n) &= \frac{1}{256} \sigma_5(n) - \frac{33}{256} \sigma_5(n/2) + \frac{65}{8} \sigma_5(n/4) - 512 \sigma_5(n/16) \\
&+ \frac{167}{256} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{11}{32} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{11}{4} \delta \left(0, 0, \frac{1}{2}; n \right) \\
&+ \frac{1}{128} \delta \left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n \right) + \frac{15}{32} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{67}{8} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) - \frac{1}{128} \delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n\right) \\
& + \frac{15}{32} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{8} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([2, 0, 10]; n) &= \frac{1}{128} \sigma_5(n) - \frac{1}{128} \sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) \\
&+ \frac{167}{256} \delta\left(0, \frac{1}{2}, 0; n\right) + \frac{11}{16} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - \frac{15}{4} \delta\left(0, 0, \frac{1}{2}; n\right) \\
&+ \frac{1}{64} \delta\left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n\right) + \frac{11}{16} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) \\
&- \frac{51}{4} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) - \frac{1}{64} \delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n\right) \\
&+ \frac{19}{16} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([3, 0, 9]; n) &= \frac{1}{64} \sigma_5(n) + \frac{15}{64} \sigma_5(n/2) - \frac{65}{4} \sigma_5(n/4) + 24\sigma_5(n/8) - 512\sigma_5(n/16) \\
&+ \frac{119}{64} \delta\left(0, \frac{1}{2}, 0; n\right) + \frac{9}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - \frac{15}{4} \delta\left(0, 0, \frac{1}{2}; n\right) \\
&+ \frac{1}{64} \delta\left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n\right) + \frac{15}{16} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) \\
&- \frac{59}{4} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) - \frac{1}{64} \delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n\right) \\
&+ \frac{31}{16} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([4, 0, 8]; n) &= \frac{1}{32} \sigma_5(n) + \frac{15}{32} \sigma_5(n/2) - \frac{65}{2} \sigma_5(n/4) + 40\sigma_5(n/8) - 512\sigma_5(n/16) \\
&+ \frac{71}{32} \delta\left(0, \frac{1}{2}, 0; n\right) + \frac{7}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - \frac{7}{2} \delta\left(0, 0, \frac{1}{2}; n\right) \\
&+ \frac{23}{16} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) - \frac{61}{4} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) \\
&+ \frac{41}{16} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([5, 0, 7]; n) = & \frac{1}{16} \sigma_5(n) + \frac{7}{64} \sigma_5(n/2) - \frac{65}{2} \sigma_5(n/4) + 40 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{37}{16} \delta\left(0, \frac{1}{2}, 0; n\right) + \frac{21}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - \frac{7}{2} \delta\left(0, 0, \frac{1}{2}; n\right) \\
& - \frac{1}{32} \delta\left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n\right) + \frac{37}{16} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) \\
& - \frac{61}{4} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) + \frac{1}{32} \delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n\right) \\
& + \frac{47}{16} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([6, 0, 6]; n) = & \frac{1}{8} \sigma_5(n) - \frac{1}{8} \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{17}{8} \delta\left(0, \frac{1}{2}, 0; n\right) + \frac{15}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 4 \delta\left(0, 0, \frac{1}{2}; n\right) \\
& - \frac{1}{16} \delta\left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n\right) + \frac{55}{16} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) \\
& - \frac{61}{4} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) + \frac{1}{16} \delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n\right) \\
& + \frac{49}{16} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([7, 0, 5]; n) = & \frac{1}{4} \sigma_5(n) - \frac{5}{4} \sigma_5(n/2) + 65 \sigma_5(n/4) - 56 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{7}{4} \delta\left(0, \frac{1}{2}, 0; n\right) + 5 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 5 \delta\left(0, 0, \frac{1}{2}; n\right) \\
& - \frac{1}{16} \delta\left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n\right) + \frac{71}{16} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) \\
& - \frac{61}{4} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) + \frac{1}{16} \delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n\right) \\
& + \frac{49}{16} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([8, 0, 4]; n) = & \frac{1}{2} \sigma_5(n) - \frac{5}{2} \sigma_5(n/2) + 130 \sigma_5(n/4) - 120 \sigma_5(n/8) - 512 \sigma_5(n/16) \\
& + \frac{3}{2} \delta\left(0, \frac{1}{2}, 0; n\right) + 6 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 6 \delta\left(0, 0, \frac{1}{2}; n\right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{79}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([9, 0, 3]; n) &= \sigma_5(n) - 3\sigma_5(n/2) + 130\sigma_5(n/4) - 120\sigma_5(n/8) - 512\sigma_5(n/16) \\
& + 2 \delta \left(0, \frac{1}{2}, 0; n \right) + 6 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 6 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{1}{8} \delta \left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n \right) + \frac{79}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) \\
& - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) - \frac{1}{8} \delta \left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([10, 0, 2]; n) &= 2\sigma_5(n) - 2\sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) \\
& + 4 \delta \left(0, \frac{1}{2}, 0; n \right) + 4 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{1}{4} \delta \left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n \right) + \frac{79}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) \\
& - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) - \frac{1}{4} \delta \left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([11, 0, 1]; n) &= 4\sigma_5(n) - 260\sigma_5(n/4) + 264\sigma_5(n/8) - 512\sigma_5(n/16) \\
& + 8 \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{1}{4} \delta \left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n \right) + \frac{79}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) \\
& - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) - \frac{1}{4} \delta \left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n \right) \\
& + \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right).
\end{aligned}$$

Proof. We only prove the formula for $N([4, 0, 8]; n)$. The proofs for the other cases follow in a similar way.

From (7.2.1)-(7.2.9) and Lemma 5.3.1 we have

$$\begin{aligned}
& \left(\frac{1 + (1-x)^{1/4}}{2} \right)^8 + \frac{1}{16128} e_{3,1}(x) + \frac{5}{5376} e_{3,2}(x) - \frac{65}{1008} e_{3,4}(x) \\
& + \frac{5}{63} (e_{3,8}^{(0)}(x) + e_{3,8}^{(1)}(x)(1-x)^{1/2}) \\
& - \frac{64}{63} \left(e_{3,16}^{(0)}(x) + e_{3,16}^{(1)}(x)(1-x)^{1/4} + e_{3,16}^{(2)}(1-x)^{1/2} + e_{3,16}^{(3)}(1-x)^{3/4} \right) \\
& = (1-x) \left(\frac{71}{512} x \right) + (1-x)^{1/4} \left(\frac{23}{256} x - \frac{61}{1024} x^2 \right) + (1-x)^{1/2} \left(\frac{7}{64} x - \frac{7}{512} x^2 \right) \\
& + (1-x)^{3/4} \left(\frac{41}{256} x - \frac{1}{1024} x^2 \right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
a'_{3,0,8}(1) &= \frac{71}{512}, \quad b'_{3,0,8}(0) = 0, \quad b'_{3,0,8}(1) = \frac{23}{256}, \quad b'_{3,0,8}(2) = -\frac{61}{1024}, \quad c'_{3,0,8}(1) = \frac{7}{64}, \\
c'_{3,0,8}(2) &= -\frac{7}{512}, \quad d'_{3,0,8}(0) = 0, \quad d'_{3,0,8}(1) = \frac{41}{256}, \quad d'_{3,0,8}(2) = -\frac{1}{1024}.
\end{aligned}$$

Using Theorem 6.3.3 with $k = 3$ and $s = 8$, we obtain the desired result. $\square$

7.3 Sums with 16 squares

In this section, we apply Theorem 6.1.3, Theorem 6.2.3 and Theorem 6.3.3 when $k = 4$. We first note that from Definition 4.1.1 we have

$$(7.3.1) \quad e_{4,1}(x) = (e_{2,1}(x))^2 = 1 + 28x + 198x^2 + 28x^3 + x^4,$$

$$(7.3.2) \quad e_{4,2}(x) = (e_{2,2}(x))^2 = 1 - 2x + 3x^2 - 2x^3 + x^4,$$

$$(7.3.3) \quad e_{4,4}(x) = (e_{2,4}(x))^2 = 1 - 2x + \frac{9}{8}x^2 - \frac{1}{8}x^3 + \frac{1}{256}x^4.$$

From Definition 4.2.1 we have

$$e_{4,8}(x) = (e_{2,8}(x))^2 = \left(e_{2,8}^{(0)}(x) + e_{2,8}^{(1)}(x)(1-x)^{1/2} \right)^2 = \left(e_{4,8}^{(0)}(x) + e_{4,8}^{(1)}(x)(1-x)^{1/2} \right),$$

where

$$(7.3.4) \quad e_{4,8}^{(0)}(x) = \frac{257}{512} - \frac{257}{256}x + \frac{1149}{2048}x^2 - \frac{121}{2048}x^3 + \frac{1}{65536}x^4,$$

$$(7.3.5) \quad e_{4,8}^{(1)}(x) = \frac{255}{512} - \frac{765}{1024}x + \frac{1035}{4096}x^2 - \frac{15}{8192}x^3.$$

From Definition 4.3.1 we have

$$\begin{aligned} e_{4,16}(x) &= (e_{2,16}(x))^2 \\ &= \left(e_{2,16}^{(0)}(x) + e_{2,16}^{(1)}(x)(1-x)^{1/4} + e_{2,16}^{(2)}(x)(1-x)^{1/2} + e_{2,16}^{(3)}(x)(1-x)^{3/4} \right)^2, \end{aligned}$$

where

$$(7.3.6) \quad e_{2,16}^{(0)}(x) = \frac{32897}{131072} - \frac{32897}{65536}x + \frac{147069}{524288}x^2 - \frac{15481}{524288}x^3 + \frac{1}{16777216}x^4,$$

$$(7.3.7) \quad e_{2,16}^{(1)}(x) = \frac{255}{1024} - \frac{28575}{65536}x + \frac{51075}{262144}x^2 - \frac{8205}{1048576}x^3,$$

$$(7.3.8) \quad e_{2,16}^{(2)}(x) = \frac{32895}{131072} - \frac{98685}{262144}x + \frac{133515}{1048576}x^2 - \frac{1935}{2097152}x^3,$$

$$(7.3.9) \quad e_{2,16}^{(3)}(x) = \frac{255}{1024} - \frac{20385}{65536}x + \frac{18315}{262144}x^2 - \frac{8205}{1048576}x^3.$$

For the first result of this section, we apply Theorem 6.1.3 to the case when $k = 4$ to obtain the number of representations of an integer n by the sum of $16 - 2r$ squares and $2r$ squares multiplied by 2, where $r \in \mathbb{N}$ and $0 < r < 8$.

Theorem 7.3.1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} &N([14, 2, 0]; n) \\ &= \frac{16}{17}\sigma_7(n) - \frac{16}{17}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/4) + \frac{8192}{17}\sigma_7(n/8) \\ &\quad + \frac{256}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + 12 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) - \frac{1104}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) \end{aligned}$$

$$+ \frac{128}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right),$$

$$N([12, 4, 0]; n)$$

$$\begin{aligned} &= \frac{8}{17} \sigma_7(n) - \frac{8}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/4) + \frac{8192}{17} \sigma_7(n/8) \\ &\quad + \frac{196}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{128}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 12 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{1104}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{128}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right), \end{aligned}$$

$$N([10, 6, 0]; n)$$

$$\begin{aligned} &= \frac{4}{17} \sigma_7(n) - \frac{4}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/4) + \frac{8192}{17} \sigma_7(n/8) \\ &\quad + \frac{166}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{192}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 10 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{1104}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{128}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right), \end{aligned}$$

$$N([8, 8, 0]; n)$$

$$\begin{aligned} &= \frac{2}{17} \sigma_7(n) - \frac{2}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/4) + \frac{8192}{17} \sigma_7(n/8) \\ &\quad + \frac{134}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{224}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 8 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{1104}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{128}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right), \end{aligned}$$

$$N([6, 10, 0]; n)$$

$$\begin{aligned} &= \frac{1}{17} \sigma_7(n) - \frac{1}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/4) + \frac{8192}{17} \sigma_7(n/8) \\ &\quad + \frac{101}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{240}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 6 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{968}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{128}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right), \end{aligned}$$

$$\begin{aligned}
& N([4, 12, 0]; n) \\
&= \frac{1}{34} \sigma_7(n) - \frac{1}{34} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/4) + \frac{8192}{17} \sigma_7(n/8) \\
&\quad + \frac{135}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{180}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 4 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{696}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{128}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right),
\end{aligned}$$

$$\begin{aligned}
& N([2, 14, 0]; n) \\
&= \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/4) + \frac{8192}{17} \sigma_7(n/8) \\
&\quad + \frac{135}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{14}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 2 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{288}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{-416}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right).
\end{aligned}$$

Proof. We only prove the formula for $N([6, 10, 0]; n)$. The proofs for the other cases follow in a similar way.

From (7.3.1)-(7.3.9) and Lemma 5.1.1 we have

$$\begin{aligned}
& \left(\frac{1 + (1-x)^{1/2}}{2}\right)^5 - \frac{1}{8160} e_{4,1}(x) + \frac{1}{8160} e_{4,2}(x) + \frac{1}{255} e_{4,4}(x) \\
& \quad - \frac{256}{255} \left(e_{4,8}^{(0)}(x) + e_{4,8}^{(1)}(x)(1-x)^{1/2}\right) \\
&= (1-x) \left(\frac{101}{272}x - \frac{15}{272}x^2\right) + (1-x)^{1/2} \left(\frac{3}{8}x - \frac{121}{544}x^2 + \frac{1}{544}x^3\right).
\end{aligned}$$

Thus we have

$$a'_{6,10,0}(1) = \frac{101}{272}, a'_{6,10,0}(2) = -\frac{15}{272}, c'_{6,10,0}(1) = \frac{3}{8}, c'_{6,10,0}(2) = -\frac{121}{544}, c'_{6,10,0}(3) = \frac{1}{544}.$$

Using Theorem 6.1.3 with $k = 4$, $r = 10$ and $s = 0$, we obtain the desired result. $\square$

For the second result of this section, we apply Theorem 6.2.3 to the case when $k = 4$ to obtain the number of representations of an integer n by the sum of $16 - 2r - s$ squares, $2r$ squares multiplied by 2 and s squares multiplied by 4, where $r, s \in \mathbb{N}$ and $0 < 2r + s < 16$.

Theorem 7.3.2. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} N([1, 2, 13]; n) &= \frac{1}{8704} \sigma_7(n) - \frac{1}{8704} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{4215}{8704} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{3107}{1088} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{33}{64} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\ &\quad - \frac{1981}{272} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{35}{68} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{531}{1088} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\ &\quad - \frac{2281}{272} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{845}{68} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{557}{1088} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\ &\quad - \frac{1501}{272} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) - \frac{13}{68} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right), \end{aligned}$$

$$\begin{aligned} N([2, 2, 12]; n) &= \frac{1}{4352} \sigma_7(n) - \frac{1}{4352} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{4215}{4352} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{2955}{544} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{33}{32} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\ &\quad - \frac{2019}{136} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{111}{34} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{273}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\ &\quad - \frac{589}{34} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{445}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{271}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \end{aligned}$$

$$- \frac{182}{17} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),$$

$$N([3, 2, 11]; n)$$

$$\begin{aligned} &= \frac{1}{2176} \sigma_7(n) - \frac{1}{2176} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{3195}{2176} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{1859}{272} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{49}{32} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{365}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{835}{544} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &\quad - \frac{439}{17} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{530}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{797}{544} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\ &\quad - \frac{983}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([4, 2, 10]; n)$$

$$\begin{aligned} &= \frac{1}{1088} \sigma_7(n) - \frac{1}{1088} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{2175}{1088} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{1005}{136} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 2 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{1817}{68} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{281}{136} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &\quad - \frac{575}{17} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{263}{136} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\ &\quad - \frac{284}{17} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([5, 2, 9]; n)$$

$$\begin{aligned} &= \frac{1}{544} \sigma_7(n) - \frac{1}{544} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{1393}{544} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{15}{2} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{39}{16} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{2055}{68} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{349}{136} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{694}{17} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{331}{136} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{301}{17} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([6, 2, 8]; n)$$

$$\begin{aligned}
& = \frac{1}{272} \sigma_7(n) - \frac{1}{272} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{849}{272} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{127}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{23}{8} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& - \frac{1087}{34} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{817}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& - \frac{3133}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{815}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([7, 2, 7]; n)$$

$$\begin{aligned}
& = \frac{1}{136} \sigma_7(n) - \frac{1}{136} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{123}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{126}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{27}{8} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& - \frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{919}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& - \frac{3337}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{985}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([8, 2, 6]; n)$$

$$\begin{aligned}
& = \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{271}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{124}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 4 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1021}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& -\frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1155}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([9, 2, 5]; n)$$

$$\begin{aligned}
& = \frac{1}{34} \sigma_7(n) - \frac{1}{34} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{287}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{120}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{19}{4} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& -\frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1157}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& -\frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1291}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([10, 2, 4]; n)$$

$$\begin{aligned}
& = \frac{1}{17} \sigma_7(n) - \frac{1}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{151}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{112}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{11}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& -\frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1361}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& -\frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1359}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([11, 2, 3]; n)$$

$$= \frac{2}{17} \sigma_7(n) - \frac{2}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16)$$

$$\begin{aligned}
& + \frac{83}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{96}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 6 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{552}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{1633}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{1359}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([12, 2, 2]; n)$$

$$\begin{aligned}
& = \frac{4}{17} \sigma_7(n) - \frac{4}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{98}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{64}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 6 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{552}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{1905}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{1359}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([13, 2, 1]; n)$$

$$\begin{aligned}
& = \frac{8}{17} \sigma_7(n) - \frac{8}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{128}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) + 6 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{552}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{1905}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{1359}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([1, 4, 11]; n)$$

$$= \frac{1}{4352} \sigma_7(n) - \frac{1}{4352} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16)$$

$$\begin{aligned}
& + \frac{2175}{4352} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{915}{544} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{1}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{863}{136} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) - \frac{59}{34} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{137}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{1081}{136} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{125}{34} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{135}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{555}{136} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) - \frac{15}{34} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([2, 4, 10]; n)$$

$$\begin{aligned}
& = \frac{1}{2176} \sigma_7(n) - \frac{1}{2176} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{2175}{2176} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{1247}{272} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{229}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{47}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{273}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{286}{17} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{377}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{271}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{313}{34} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([3, 4, 9]; n)$$

$$\begin{aligned}
& = \frac{1}{1088} \sigma_7(n) - \frac{1}{1088} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{1631}{1088} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{869}{136} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{3}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{1341}{68} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{409}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{422}{17} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{513}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{407}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{449}{34} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([4, 4, 8]; n)$$

$$\begin{aligned}
&= \frac{1}{544}\sigma_7(n) - \frac{1}{544}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{1087}{544}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{29}{4}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 2\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{849}{34}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{545}{272}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2181}{68}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{543}{272}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1085}{68}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([5, 4, 7]; n)$$

$$\begin{aligned}
&= \frac{1}{272}\sigma_7(n) - \sigma_7(n/2) - \frac{1}{272}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{679}{272}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{127}{17}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{5}{2}\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{985}{34}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{681}{272}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2633}{68}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{679}{272}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1187}{68}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([6, 4, 6]; n)$$

$$\begin{aligned}
&= \frac{1}{136}\sigma_7(n) - \frac{1}{136}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{407}{136}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{126}{17}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 3\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{535}{17}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{817}{272}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2997}{68}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{815}{272}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1221}{68}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([7, 4, 5]; n)$$

$$\begin{aligned}
&= \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{237}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{124}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{7}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{953}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{3269}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{951}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([8, 4, 4]; n)$$

$$\begin{aligned}
&= \frac{1}{34} \sigma_7(n) - \frac{1}{34} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{135}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{120}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 4 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1089}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{951}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([9, 4, 3]; n)$$

$$\begin{aligned}
&= \frac{1}{17} \sigma_7(n) - \frac{1}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{151}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{112}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{9}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1089}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1223}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
& N([10, 4, 2]; n) \\
&= \frac{2}{17} \sigma_7(n) - \frac{2}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&+ \frac{83}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{96}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 5 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&- \frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1361}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&- \frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1359}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&- \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
& N([11, 4, 1]; n) \\
&= \frac{4}{17} \sigma_7(n) - \frac{4}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&+ \frac{98}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{64}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 6 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&- \frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1361}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&- \frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1359}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&- \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
& N([1, 6, 9]; n) \\
&= \frac{1}{2176} \sigma_7(n) - \frac{1}{2176} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&+ \frac{1087}{2176} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{159}{272} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{1}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&- \frac{93}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) - \frac{89}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{137}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&- \frac{481}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) - \frac{116}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{135}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right)
\end{aligned}$$

$$- \frac{201}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) - \frac{16}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),$$

$$N([2, 6, 8]; n)$$

$$\begin{aligned} &= \frac{1}{1088} \sigma_7(n) - \frac{1}{1088} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{1087}{1088} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{495}{136} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{212}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{30}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{273}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &\quad - \frac{1093}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{275}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{271}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\ &\quad - \frac{541}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([3, 6, 7]; n)$$

$$\begin{aligned} &= \frac{1}{544} \sigma_7(n) - \frac{1}{544} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{815}{544} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{23}{4} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{3}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{314}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{409}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &\quad - \frac{1637}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{479}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{407}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\ &\quad - \frac{813}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([4, 6, 6]; n)$$

$$\begin{aligned} &= \frac{1}{272} \sigma_7(n) - \frac{1}{272} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{543}{272} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{237}{34} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 2 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{399}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{545}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{2113}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{543}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{1017}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([5, 6, 5]; n)$$

$$\begin{aligned}
& = \frac{1}{136} \sigma_7(n) - \frac{1}{136} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{339}{136} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{126}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{5}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& - \frac{467}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{681}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& - \frac{2521}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{679}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& - \frac{1153}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([6, 6, 4]; n)$$

$$\begin{aligned}
& = \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{203}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{124}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 3 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& - \frac{518}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{817}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& - \frac{2861}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{815}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([7, 6, 3]; n)$$

$$\begin{aligned}
& = \frac{1}{34} \sigma_7(n) - \frac{1}{34} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{59}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{120}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{7}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{953}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& -\frac{3133}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{951}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([8, 6, 2]; n)$$

$$\begin{aligned}
& = \frac{1}{17} \sigma_7(n) - \frac{1}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{67}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{112}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 4 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& -\frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1089}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& -\frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1087}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([9, 6, 1]; n)$$

$$\begin{aligned}
& = \frac{2}{17} \sigma_7(n) - \frac{2}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{83}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{96}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 5 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& -\frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1089}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& -\frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1087}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([1, 8, 7]; n)$$

$$= \frac{1}{1088} \sigma_7(n) - \frac{1}{1088} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16)$$

$$\begin{aligned}
& + \frac{543}{1088} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{49}{136} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{1}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{76}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) - \frac{174}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{137}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{413}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) - \frac{303}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{135}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{133}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) - \frac{33}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([2, 8, 6]; n)$$

$$\begin{aligned}
& = \frac{1}{544} \sigma_7(n) - \frac{1}{544} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{543}{544} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{11}{4} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{195}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) - \frac{4}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{273}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{1025}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{139}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{271}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{473}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([3, 8, 5]; n)$$

$$\begin{aligned}
& = \frac{1}{272} \sigma_7(n) - \frac{1}{272} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{407}{272} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{169}{34} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{3}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{297}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{409}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{1569}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{411}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{407}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{745}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([4, 8, 4]; n)$$

$$\begin{aligned}
&= \frac{1}{136} \sigma_7(n) - \frac{1}{136} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{271}{136} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{109}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 2 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{382}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{545}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2045}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{543}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{949}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([5, 8, 3]; n)$$

$$\begin{aligned}
&= \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{169}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{124}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{5}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{450}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{681}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2453}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{679}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1085}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([6, 8, 2]; n)$$

$$\begin{aligned}
&= \frac{1}{34} \sigma_7(n) - \frac{1}{34} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{101}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{120}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 3 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{484}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{817}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2861}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{815}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([7, 8, 1]; n)$$

$$\begin{aligned}
&= \frac{1}{17}\sigma_7(n) - \frac{1}{17}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{67}{17}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{112}{17}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 4\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{552}{17}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{817}{272}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2861}{68}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{815}{272}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{1221}{68}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([1, 10, 5]; n)$$

$$\begin{aligned}
&= \frac{1}{544}\sigma_7(n) - \frac{1}{544}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{271}{544}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + \frac{5}{4}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{1}{2}\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{59}{17}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) - \frac{276}{17}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{137}{272}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{345}{68}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) - \frac{473}{17}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{135}{272}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{65}{68}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) - \frac{67}{17}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([2, 10, 4]; n)$$

$$\begin{aligned}
&= \frac{1}{272}\sigma_7(n) - \frac{1}{272}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{271}{272}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{67}{34}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{178}{17}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) - \frac{72}{71}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{273}{272}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{957}{68}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{3}{17}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{271}{272}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{405}{68}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([3, 10, 3]; n)$$

$$\begin{aligned}
&= \frac{1}{136} \sigma_7(n) - \frac{1}{136} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{203}{136} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{75}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{3}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{280}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{409}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{1501}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{275}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{407}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{677}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([4, 10, 2]; n)$$

$$\begin{aligned}
&= \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{135}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{90}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 2 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{348}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{545}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2045}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{543}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{949}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([5, 10, 1]; n)$$

$$\begin{aligned}
&= \frac{1}{34} \sigma_7(n) - \frac{1}{34} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{101}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{120}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 3 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{484}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{545}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{2045}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{272} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{543}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right)
\end{aligned}$$

$$- \frac{949}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),$$

$$N([1, 12, 3]; n)$$

$$\begin{aligned} &= \frac{1}{272} \sigma_7(n) - \frac{1}{272} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{135}{272} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) + \frac{69}{34} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{1}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{42}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) - \frac{344}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{137}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &\quad - \frac{277}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) - \frac{677}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{135}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\ &\quad + \frac{3}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) - \frac{135}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([2, 12, 2]; n)$$

$$\begin{aligned} &= \frac{1}{136} \sigma_7(n) - \frac{1}{136} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{135}{136} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{7}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{144}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) - \frac{208}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{273}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &\quad - \frac{957}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{3}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{271}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\ &\quad - \frac{405}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([3, 12, 1]; n)$$

$$\begin{aligned} &= \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{135}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{90}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 2 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &\quad - \frac{348}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{273}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{957}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{3}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{271}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{405}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{7} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([1, 14, 1]; n)$$

$$\begin{aligned}
& = \frac{1}{136} \sigma_7(n) - \frac{1}{136} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{135}{136} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{7}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& - \frac{144}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) - \frac{208}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& + \frac{403}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) - \frac{1357}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) - \frac{1}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& + \frac{411}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) - \frac{271}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right).
\end{aligned}$$

Proof. We only prove the formula for $N([7, 2, 7]; n)$. The proofs for the other cases follow in a similar way.

From (7.3.1)-(7.3.9) and Lemma 5.2.1 we have

$$\begin{aligned}
& \left(\frac{1 + (1-x)^{1/2}}{2}\right) \left(\frac{1 + (1-x)^{1/4}}{2}\right)^7 - \frac{1}{65280} e_{4,1}(x) + \frac{1}{65280} e_{4,2}(x) \\
& + \frac{1}{255} (e_{4,8}^{(0)}(x) + e_{4,8}^{(1)}(x)(1-x)^{1/2}) \\
& - \frac{256}{255} \left(e_{4,16}^{(0)}(x) + e_{4,16}^{(1)}(x)(1-x)^{1/4} + e_{4,16}^{(2)}(1-x)^{1/2} + e_{4,16}^{(3)}(1-x)^{3/4}\right) \\
& = (1-x) \left(\frac{123}{544}x - \frac{63}{2176}x^2\right) + (1-x)^{1/4} \left(\frac{919}{4352}x - \frac{3337}{17408}x^2 + \frac{547}{69632}x^3\right) \\
& + (1-x)^{1/2} \left(\frac{27}{128}x - \frac{69}{544}x^2 + \frac{1}{1088}x^3\right)
\end{aligned}$$

$$+(1-x)^{3/4} \left(\frac{985}{4352}x - \frac{1221}{17408}x^2 + \frac{1}{69632}x^3 \right).$$

Thus we have

$$a'_{4,2,7}(1) = \frac{123}{544}, a'_{4,2,7}(2) = -\frac{63}{2176}, b'_{4,2,7}(1) = \frac{919}{4352}, b'_{4,2,7}(2) = -\frac{3337}{17408},$$

$$b'_{4,2,7}(3) = \frac{547}{69632}, c'_{4,2,7}(1) = \frac{27}{128}, c'_{4,2,7}(2) = -\frac{69}{544}, c'_{4,2,7}(3) = \frac{1}{1088}, d'_{4,2,7}(1) = \frac{985}{4352},$$

$$d'_{4,2,7}(2) = -\frac{1221}{17408}, d'_{4,2,7}(3) = \frac{1}{69632}.$$

Using Theorem 6.2.3 with $k = 4$, $r = 2$ and $s = 7$, we obtain the desired result. $\square$

For the final result of this chapter, we apply Theorem 6.3.3 to the case when $k = 4$ to obtain the number of representations of an integer n by the sum of $16 - s$ squares and s squares multiplied by 4, where $s \in \mathbb{N}$ and $0 < s < 16$.

Theorem 7.3.3. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} & N([1, 0, 15]; n) \\ &= \frac{1}{17408} \sigma_7(n) - \frac{129}{17408} \sigma_7(n/2) + \frac{257}{136} \sigma_7(n/4) - \frac{64}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &+ \frac{7751}{17408} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{8775}{2176} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{71}{128} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\ &- \frac{1143}{136} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{253}{136} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) - \frac{1}{512} \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) \\ &+ \frac{1079}{2176} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{2315}{272} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{2591}{136} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\ &+ \frac{1}{512} \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) + \frac{1131}{2176} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1977}{272} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) \\ &- \frac{9}{136} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right), \end{aligned}$$

$$N([2, 0, 14]; n)$$

$$\begin{aligned}
&= \frac{1}{8704} \sigma_7(n) - \frac{1}{8704} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{7751}{8704} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{6643}{1088} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{71}{64} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{4565}{272} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{239}{68} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) - \frac{1}{256} \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) \\
&\quad + \frac{597}{544} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{4933}{272} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{975}{34} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\
&\quad + \frac{1}{256} \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) + \frac{127}{136} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{3337}{272} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) \\
&\quad + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([3, 0, 13]; n)$$

$$\begin{aligned}
&= \frac{1}{4352} \sigma_7(n) + \frac{63}{4352} \sigma_7(n/2) - \frac{257}{68} \sigma_7(n/4) + \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{5983}{4352} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{3809}{544} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{13}{8} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{6539}{272} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{65}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) - \frac{1}{256} \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) \\
&\quad + \frac{115}{68} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{7517}{272} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{1077}{34} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\
&\quad + \frac{1}{256} \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) + \frac{729}{544} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{4221}{272} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) \\
&\quad + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([4, 0, 12]; n)$$

$$\begin{aligned}
&= \frac{1}{2176} \sigma_7(n) + \frac{63}{2176} \sigma_7(n/2) - \frac{257}{34} \sigma_7(n/4) + \frac{96}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{4215}{2176} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{1965}{272} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{33}{16} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{3993}{136} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{66}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{149}{68} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{1235}{34} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{123}{68} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{585}{34} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([5, 0, 11]; n)$$

$$\begin{aligned}
&= \frac{1}{1088} \sigma_7(n) + \frac{31}{1088} \sigma_7(n/2) - \frac{257}{34} \sigma_7(n/4) + \frac{96}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{2821}{1088} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{495}{68} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{77}{32} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{4367}{136} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{66}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1}{128} \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) \\
&\quad + \frac{1379}{544} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{5875}{136} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\
&\quad - \frac{1}{128} \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) + \frac{1307}{544} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{2425}{136} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) \\
&\quad + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([6, 0, 10]; n)$$

$$\begin{aligned}
&= \frac{1}{544} \sigma_7(n) - \frac{1}{544} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{1801}{544} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{15}{2} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{43}{16} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{2191}{68} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1}{64} \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) \\
&\quad + \frac{749}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{3235}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\
&\quad - \frac{1}{64} \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) + \frac{849}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) \\
&\quad + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([7, 0, 9]; n)$$

$$\begin{aligned}
&= \frac{1}{272} \sigma_7(n) - \frac{1}{16} \sigma_7(n/2) + \frac{257}{17} \sigma_7(n/4) - \frac{288}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{1087}{272} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{135}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 3 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{1035}{34} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{60}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1}{64} \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) \\
&\quad + \frac{817}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{3371}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\
&\quad - \frac{1}{64} \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) + \frac{1053}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right)
\end{aligned}$$

$$+ \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),$$

$$N([8, 0, 8]; n)$$

$$\begin{aligned} &= \frac{1}{136} \sigma_7(n) - \frac{1}{8} \sigma_7(n/2) + \frac{514}{17} \sigma_7(n/4) - 32 \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &+ \frac{611}{136} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{142}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{7}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &- \frac{483}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{56}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{953}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &- \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{1223}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\ &- \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([9, 0, 7]; n)$$

$$\begin{aligned} &= \frac{1}{68} \sigma_7(n) - \frac{9}{68} \sigma_7(n/2) + \frac{514}{17} \sigma_7(n/4) - 32 \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &+ \frac{627}{136} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{140}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{35}{8} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &- \frac{483}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{56}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) - \frac{1}{32} \delta \left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n \right) \\ &+ \frac{1191}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) - \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) \\ &+ \frac{1}{32} \delta \left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n \right) + \frac{1325}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) - \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) \\ &+ \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([10, 0, 6]; n)$$

$$\begin{aligned} &= \frac{1}{34} \sigma_7(n) - \frac{1}{34} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &+ \frac{287}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{120}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{23}{4} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &- \frac{552}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) - \frac{1}{16} \delta \left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n \right) \\ &+ \frac{1497}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) - \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) \\ &+ \frac{1}{16} \delta \left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n \right) + \frac{1359}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) - \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) \end{aligned}$$

$$+ \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),$$

$$N([11, 0, 5]; n)$$

$$\begin{aligned} &= \frac{1}{17} \sigma_7(n) + \frac{3}{17} \sigma_7(n/2) - \frac{1028}{17} \sigma_7(n/4) + \frac{992}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &+ \frac{117}{34} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{80}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{15}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &- \frac{690}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{80}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) - \frac{1}{16} \delta \left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n \right) \\ &+ \frac{1769}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) - \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) \\ &+ \frac{1}{16} \delta \left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n \right) + \frac{1359}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) - \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) \\ &+ \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([12, 0, 4]; n)$$

$$\begin{aligned} &= \frac{2}{17} \sigma_7(n) + \frac{6}{17} \sigma_7(n/2) - \frac{2056}{17} \sigma_7(n/4) + \frac{2016}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &+ \frac{49}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{32}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 9 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\ &- \frac{828}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{96}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{1905}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &- \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{1359}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\ &- \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$N([13, 0, 3]; n)$$

$$\begin{aligned} &= \frac{4}{17} \sigma_7(n) + \frac{4}{17} \sigma_7(n/2) - \frac{2056}{17} \sigma_7(n/4) + \frac{2016}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &+ \frac{64}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) + 9 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) - \frac{828}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) \\ &+ \frac{96}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{1}{8} \delta \left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n \right) + \frac{1905}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\ &- \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) - \frac{1}{8} \delta \left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n \right) \\ &+ \frac{1359}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) - \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right), \end{aligned}$$

$$\begin{aligned}
& N([14, 0, 2]; n) \\
&= \frac{8}{17} \sigma_7(n) - \frac{8}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{128}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + 6 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) - \frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) \\
&\quad + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1}{4} \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) + \frac{1905}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) - \frac{1}{4} \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) \\
&\quad + \frac{1359}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$\begin{aligned}
& N([15, 0, 1]; n) \\
&= \frac{16}{17} \sigma_7(n) - \frac{32}{17} \sigma_7(n/2) + \frac{4112}{17} \sigma_7(n/4) - \frac{4128}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{256}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + \frac{1}{4} \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) + \frac{1905}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) - \frac{1}{4} \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) \\
&\quad + \frac{1359}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right).
\end{aligned}$$

Proof. We only prove the formula for $N([13, 0, 3]; n)$. The proofs for the other cases follow in a similar way.

From (7.3.1)-(7.3.9) and Lemma 5.3.1 we have

$$\begin{aligned}
& \left(\frac{1 + (1-x)^{1/4}}{2}\right)^3 - \frac{1}{2040} e_{4,1}(x) - \frac{1}{2040} e_{4,2}(x) + \frac{257}{1020} e_{4,4}(x) \\
& - \frac{21}{85} (e_{4,8}^{(0)}(x) + e_{4,8}^{(1)}(x)(1-x)^{1/2}) \\
& - \frac{256}{255} \left(e_{4,16}^{(0)}(x) + e_{4,16}^{(1)}(x)(1-x)^{1/4} + e_{4,16}^{(2)}(1-x)^{1/2} + e_{4,16}^{(3)}(1-x)^{3/4}\right)
\end{aligned}$$

$$\begin{aligned}
&= (1-x) \left(\frac{4}{17}x \right) + (1-x)^{1/4} \left(\frac{1}{8} + \frac{1905}{4352}x - \frac{3405}{17408}x^2 + \frac{547}{69632}x^3 \right) \\
&\quad + (1-x)^{1/2} \left(\frac{9}{16}x - \frac{207}{1088}x^2 + \frac{3}{2176}x^3 \right) \\
&\quad + (1-x)^{3/4} \left(-\frac{1}{8} + \frac{1359}{4352}x - \frac{1221}{17408}x^2 + \frac{1}{69632}x^3 \right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
a'_{4,0,3}(1) &= \frac{4}{17}, \quad a'_{4,0,3}(2) = 0, \quad b'_{4,0,3}(0) = \frac{1}{8}, \quad b'_{4,0,3}(1) = \frac{1905}{4352}, \quad b'_{4,0,3}(2) = -\frac{3405}{17408}, \\
b'_{4,0,3}(3) &= \frac{547}{69632}, \quad c'_{4,0,3}(1) = \frac{9}{16}, \quad c'_{4,0,3}(2) = -\frac{207}{1088}, \quad c'_{4,0,3}(3) = \frac{3}{2176}, \quad d'_{4,0,3}(0) = -\frac{1}{8}, \\
d'_{4,0,3}(1) &= \frac{1359}{4352}, \quad d'_{4,0,3}(2) = -\frac{1221}{17408}, \quad d'_{4,0,3}(3) = \frac{1}{69632}.
\end{aligned}$$

Using Theorem 6.3.3 with $k = 4$ and $s = 3$, we obtain the desired result. $\square$

Chapter 8

δ identities

In Chapter 7, we applied our main results (Theorems 6.1.3, 6.2.3 and 6.3.3) to sums with 8, 12, 16 squares. Each result depends on a certain number of the arithmetic functions $\delta(a, b, c; n)$. However these functions are not always linearly independent over $\mathbb{Q}$.

The purpose of this chapter is to prove several identities between the functions $\delta(a, b, c; n)$. In Chapter 9, we use the identities proved in this chapter to simplify the formulae obtain in Chapter 7.

In Section 1, we state and prove all the results that we need in order to prove these identities. In Section 2, we prove an identity needed for the sums of 4 squares. In Section 3, we prove all the needed identities for sums with 8 squares. In Section 4, we prove all the needed identities for sums with 12 squares. In Section 5, we prove all the needed identities for the sums with 16 squares. In Section 6, we state and prove several lemmas in order to prove one special identity.

We recall that the set D is defined by

$$D := \{q \in \mathbb{C} \mid |q| < 1\},$$

and that the set D_δ is defined by

$$D_\delta := \{q \in \mathbb{C} \mid |q| < \delta\},$$

where the number δ was defined in Lemma 2.4.2.

8.1 Change of sign, duplication and rotation principles for $\Delta(a, b, c; q)$

For $q \in D$ define x and z as in Definition 2.3.2. In this section, we apply the change of sign, the duplication and the rotation principles to obtain formulae for $\Delta(a, b, c; -q)$, $\Delta(a, b, c; q^2)$, $\Delta(a, b, c; -q^2)$, $\Delta(a, b, c; iq)$ and $\Delta(a, b, c; -iq)$, in terms of x and z .

We also give formulae for the Eisenstein series $E_k(iq)$ and $E_k(-iq)$ when $k = 2, 3$ and 4 .

Theorem 8.1.1. *For $q \in D_\delta$ and x and z as defined in Definition 2.3.2, we have:*

$$(i) \text{ if } q \rightarrow -q \text{ then } \frac{x^r(1-x)^s}{2^{4r}} z^{2t} \longrightarrow \frac{(-1)^r x^r(1-x)^{t-r-s}}{2^{4r}} z^{2t},$$

(ii) if $q \rightarrow q^2$ then

$$\frac{x^r(1-x)^s}{2^{4r}} z^{2t} \longrightarrow \frac{(1-x)^{s/2} (1-\sqrt{1-x})^{2r} (1+\sqrt{1-x})^{2t-2r-2s}}{2^{4r-2s+2t}} z^{2t},$$

(iii) if $q \rightarrow -q^2$ then

$$\frac{x^r(1-x)^s}{2^{4r}} z^{2t} \longrightarrow \frac{(-1)^r (1-x)^{\frac{t-r-s}{2}} (1-\sqrt{1-x})^{2r} (1+\sqrt{1-x})^{2s}}{2^{6r+2s}} z^{2t}.$$

Proof. (i) By Theorem 2.6.1, we have that if $q \rightarrow -q$ then $x \rightarrow \frac{x}{x-1}$ and $z \rightarrow \sqrt{1-x} z$. Hence, if $q \rightarrow -q$ we have

$$\begin{aligned} \frac{x^r(1-x)^s}{2^{4r}} z^{2t} &\longrightarrow \frac{\left(\frac{x}{x-1}\right)^r \left(1 - \frac{x}{x-1}\right)^s (\sqrt{1-x} z)^{2t}}{2^{4r}} = \frac{\frac{(-1)^r x^r}{(1-x)^r} \left(\frac{-1}{x-1}\right)^s (1-x)^t z^{2t}}{2^{4r}} \\ &= \frac{\frac{(-1)^r x^r}{(1-x)^r} \left(\frac{1}{1-x}\right)^s (1-x)^t z^{2t}}{2^{4r}} = \frac{(-1)^r x^r (1-x)^{t-r-s}}{2^{4r}} z^{2t}. \end{aligned}$$

(ii) By Theorem 2.5.1, we have that if $q \rightarrow q^2$, then $x \rightarrow \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2$ and $z \rightarrow \left(\frac{1+\sqrt{1-x}}{2}\right) z$. Hence, if $q \rightarrow q^2$, we have

$$\begin{aligned} \frac{x^r(1-x)^s}{2^{4r}} z^{2t} &\longrightarrow \frac{\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^{2r} \left(1 - \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2\right)^s \left(\frac{1+\sqrt{1-x}}{2}\right)^{2t} z^{2t}}{2^{4r}} \\ &= \frac{\frac{(1-\sqrt{1-x})^{2r}}{(1+\sqrt{1-x})^{2r}} \left(\frac{(1+\sqrt{1-x})^2 - (1-\sqrt{1-x})^2}{(1+\sqrt{1-x})^2}\right)^s \frac{(1+\sqrt{1-x})^{2t}}{2^{2t}} z^{2t}}{2^{4r}} \\ &= \frac{\frac{(1-\sqrt{1-x})^{2r}}{(1+\sqrt{1-x})^{2r}} \frac{(4(1-x)^{1/2})^s}{(1+\sqrt{1-x})^{2s}} (1+\sqrt{1-x})^{2t} z^{2t}}{2^{4r+2t}} \\ &= \frac{(1-\sqrt{1-x})^{2r} 2^{2s} (1-x)^{x/2} (1+\sqrt{1-x})^{2t-2r-2s}}{2^{4r+2t-2s}} z^{2t} \\ &= \frac{(1-x)^{s/2} (1-\sqrt{1-x})^{2r} (1+\sqrt{1-x})^{2t-2r-2s}}{2^{4r+2t-2s}} z^{2t} \end{aligned}$$

(iii) By (i) we have that if $q \rightarrow -q$, then $\frac{x^r(1-x)^s}{2^{4r}} z^{2t} \longrightarrow \frac{(-1)^r x^r (1-x)^{t-r-s}}{2^{4r}} z^{2t}$.

If $q \rightarrow q^2$, then by Theorem 2.5.1 we have

$$\begin{aligned}
\frac{(-1)^r x^r (1-x)^{t-r-s}}{2^{4r}} z^{2t} &\rightarrow \frac{(-1)^r \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} \right)^{2r} \left(1 - \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} \right)^2 \right)^{t-r-s} \left(\frac{1+\sqrt{1-x}}{2} \right)^{2t}}{2^{4r}} z^{2t} \\
&= \frac{(-1)^r \frac{(1-\sqrt{1-x})^{2r}}{(1+\sqrt{1-x})^{2r}} \frac{(4\sqrt{1-x})^{t-r-s}}{(1+\sqrt{1-x})^{2t-2r-2s}} (1+\sqrt{1-x})^{2t}}{2^{4r+2t}} z^{2t} \\
&= \frac{(-1)^r (1-x)^{\frac{t-r-s}{2}} (1-\sqrt{1-x})^{2r} (1+\sqrt{1-x})^{2s}}{2^{6r+2s}} z^{2t}
\end{aligned}$$

which proves (iii). $\square$

Next we prove the following lemma. We require this lemma in order to apply the rotation principle to $\Delta(a, b, c; q)$.

Lemma 8.1.1. *For $q \in D_\delta$, x and z as defined in Definition 2.3.2 and $s \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, we have*

$$\left(\frac{(1+i(1-x)^{1/4})^4}{(1-i(1-x)^{1/4})^4} \right)^s = a \frac{(1+i(1-x)^{1/4})^{4s}}{(1-i(1-x)^{1/4})^{4s}},$$

where

$$a = \begin{cases} -i, & \text{if } s = \frac{1}{4}, \\ -1, & \text{if } s = \frac{1}{2}, \\ i, & \text{if } s = \frac{3}{4}, \\ 1, & \text{if } s = 1. \end{cases}$$

Proof. From Theorem 2.4.6(ii) we have

$$(8.1.1) \quad (1-x(iq))^{1/4} = -i \frac{1+i(1-x)^{1/4}}{1-i(1-x)^{1/4}}$$

Raising each side of (8.1.1) to the power 4 we obtain

$$(8.1.2) \quad 1 - x(iq) = \frac{(1 + i(1 - x)^{1/4})^4}{(1 - i(1 - x)^{1/4})^4}$$

Raising each side of equation (8.1.2) to the power $\frac{1}{4}$ we obtain

$$(8.1.3) \quad (1 - x(iq))^{1/4} = \left(\frac{(1 + i(1 - x)^{1/4})^4}{(1 - i(1 - x)^{1/4})^4} \right)^{1/4}$$

Equating (8.1.1) and (8.1.3) we obtain

$$(8.1.4) \quad \left(\frac{(1 + i(1 - x)^{1/4})^4}{(1 - i(1 - x)^{1/4})^4} \right)^{1/4} = -i \frac{1 + i(1 - x)^{1/4}}{1 - i(1 - x)^{1/4}},$$

which is the desired result for $s = 1/4$. Raising (8.1.4) to the powers 2, 3 and 4, we obtain the other results. $\square$

Theorem 8.1.2. *For $q \in D_\delta$, x and z as defined in Definition 2.3.2 and $s \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, we have*

(i) *If $q \rightarrow iq$, then*

$$\frac{x^r(1 - x)^s}{2^{4t}} z^{2t} \longrightarrow \frac{(-1)^{r+t} i^r a (1 - x)^{r/4} (1 - \sqrt{1 - x})^r (1 + i(1 - x)^{1/4})^{4s} (1 - i(1 - x)^{1/4})^{4t-4r-4s}}{2^{r+2t}} z^{2t}.$$

(ii) *If $q \rightarrow -iq$, then*

$$\frac{x^r(1 - x)^s}{2^{4t}} z^{2t} \longrightarrow$$

$$\frac{1}{2^{r+2t}} \left((-1)^{5t-2r-4s} i^{4t-3r} a (1-x)^{r/4} (1-\sqrt{1-x})^r \right. \\ \left. \times (1+i(1-x)^{1/4})^{4t-4r-4s} (1-i(1-x)^{1/4})^{4s} \right) z^{2t},$$

where

$$a = \begin{cases} -i, & \text{if } s = \frac{1}{4}, \\ -1, & \text{if } s = \frac{1}{2}, \\ i, & \text{if } s = \frac{3}{4}, \\ 1, & \text{if } s = 1. \end{cases}$$

Proof. (i) By Theorem 2.7.1, we have that if $q \rightarrow iq$ then

$$x \rightarrow \frac{-8i(1-x)^{1/4} (1-\sqrt{1-x})}{(1-i(1-x)^{1/4})^4}$$

and

$$z \rightarrow i \frac{z}{2} (1-i(1-x)^{1/4})^2.$$

Hence, if $q \rightarrow iq$, we have by Theorem 2.7.1 and Lemma 8.1.1

$$\begin{aligned} & \frac{x^r (1-x)^s}{2^{4r}} z^{2t} \longrightarrow \\ &= \frac{\left(\frac{-8i(1-x)^{1/4} (1-\sqrt{1-x})}{(1-i(1-x)^{1/4})^4} \right)^r \left(1 + \frac{8i(1-x)^{1/4} (1-\sqrt{1-x})}{(1-i(1-x)^{1/4})^4} \right)^s \left(i \frac{z}{2} (1-i(1-x)^{1/4})^2 \right)^{2t}}{2^{4r}} \\ &= \frac{\left(\frac{(-1)^r i^r 2^{3r} (1-x)^{r/4} (1-\sqrt{1-x})^r}{(1-i(1-x)^{1/4})^{4r}} \right) \left(\frac{(1+i(1-x)^{1/4})^4}{(1-i(1-x)^{1/4})^4} \right)^s \frac{(-1)^t z^{2t} (1-i(1-x)^{1/4})^{4t}}{2^{2t}}}{2^{4r}} \\ &= \frac{(-1)^{r+t} i^r (1-x)^{r/4} (1-\sqrt{1-x})^r}{(1-i(1-x)^{1/4})^{4r}} a \frac{(1+i(1-x)^{1/4})^{4s}}{(1-i(1-x)^{1/4})^{4s}} (1-i(1-x)^{1/4})^{4t}}{2^{r+2t}} z^{2t} \end{aligned}$$

$$= \frac{(-1)^{r+t} i^r a (1-x)^{r/4} (1-\sqrt{1-x})^r (1+i(1-x)^{1/4})^{4s} (1-i(1-x)^{1/4})^{4t-4r-4s}}{2^{r+2t}} z^{2t}$$

(ii) By Theorem 2.6.1, we have that if $q \rightarrow -q$ then $x \rightarrow \frac{x}{x-1}$ and $z \rightarrow \sqrt{1-x} z$.

Hence, if $q \rightarrow -q$ we have by (i) and Theorem 2.6.1 that

$$\frac{(-1)^{r+t} i^r a (1-x)^{r/4} (1-\sqrt{1-x})^r (1+i(1-x)^{1/4})^{4s} (1-i(1-x)^{1/4})^{4t-4r-4s}}{2^{r+2t}} z^{2t}$$

becomes

$$\frac{(-1)^{r+t} i^r a \frac{1}{(1-x)^{r/4}} \frac{(-1)^r}{(1-x)^{r/2}} (1-\sqrt{1-x})^r \left(\frac{(1-x)^{1/4}+i}{(1-x)^{1/4}} \right)^{4s} \left(\frac{(1-x)^{1/4}-i}{(1-x)^{1/4}} \right)^{4t-4r-4s}}{2^{r+2t}} z^{2t}$$

$$= \frac{1}{2^{r+2t}} \left((-1)^{2r+t} i^r a (1-x)^{r/4} (1-\sqrt{1-x})^r (i)^{4s} (1-i(1-x)^{1/4})^{4s} \right. \\ \left. \times (-i)^{4t-4r-4s} (1+i(1-x)^{1/4})^{4t-4r-4s} \right) z^{2t}$$

$$= \frac{1}{2^{r+2t}} \left((-1)^{5t-2r-4s} (i)^{4t-3r} a (1-x)^{r/4} (1-\sqrt{1-x})^r \right. \\ \left. \times (1+i(1-x)^{1/4})^{4t-4r-4s} (1-i(1-x)^{1/4})^{4s} \right) z^{2t}$$

which is the asserted result □

In the next theorem we express the quantity $E_1(iq) - E_1(-iq)$ in terms of x and z .

Theorem 8.1.3. *Let $q \in D_\delta$. Define x and z as in Definition 2.3.2 and the Eisenstein series $E_1(q)$ as in Definition 2.9.1. Then*

$$E_1(iq) - E_1(-iq) = -6i \left((1-x)^{1/4} - (1-x)^{3/4} \right) z^2.$$

Proof. Using the rotation principle (Theorem 2.7.1) in Theorem 2.9.6 (iii), we obtain for $q \in D_\delta$

$$\begin{aligned} E_1(iq) - E_1(-iq) &= -3 \left(\frac{-8i(1-x)^{1/4}(1-(1-x)^{1/2})}{(1-i(1-x)^{1/4})^4} \right) \left(i \frac{z}{2} (1-i(1-x)^{1/4})^2 \right)^2 \\ &= -6i(1-x)^{1/4}(1-(1-x)^{1/2})z^2, \end{aligned}$$

which is the desired result. $\square$

We conclude this section with formulae for the Eisenstein series $E_k(iq)$ and $E_k(-iq)$ for $k \geq 2$.

Theorem 8.1.4. *For $q \in D_\delta$, x and z as defined in Definition 2.3.2 and the Eisenstein series $E_2(q)$ and $E_3(q)$ as in Definition 2.9.1, we have*

$$\begin{aligned} \text{(i)} \quad E_2(iq) &= \frac{1}{16} \left(1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \right. \\ &\quad \left. - 840ig^5 - 540g^6 + 120ig^7 + g^8 \right) z^4, \\ \text{(ii)} \quad E_2(-iq) &= \frac{1}{16} \left(1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 \right. \\ &\quad \left. + 840ig^5 - 540g^6 - 120ig^7 + g^8 \right) z^4, \\ \text{(iii)} \quad E_3(iq) &= \frac{1}{64} (1 - 12ig - 6g^2 + 12ig^3 + g^4) \end{aligned}$$

$$\times \left(-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 - 1848ig^5 - 996g^6 + 264ig^7 - g^8 \right) z^6,$$

$$\begin{aligned} \text{(iv)} \quad E_3(-iq) &= \frac{1}{64} (1 + 12ig - 6g^2 - 12ig^3 + g^4) \\ &\times \left(-1 + 264ig - 996g^2 - 1848ig^3 + 1978g^4 + 1848ig^5 - 996g^6 - 264ig^7 - g^8 \right) z^6, \end{aligned}$$

$$\begin{aligned} \text{(v)} \quad E_4(iq) &= \left(\frac{1}{16} (1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \right. \\ &\quad \left. - 840ig^5 - 540g^6 + 120ig^7 + g^8) \right)^2 z^8, \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad E_4(-iq) &= \left(\frac{1}{16} (1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 \right. \\ &\quad \left. + 840ig^5 - 540g^6 - 120ig^7 + g^8) \right)^2 z^8, \end{aligned}$$

where $g = (1 - x)^{1/4}$.

Proof. Results (i)-(iv) were found in [28]. (v) and (vi) follow by Theorem 2.9.1 and (i) to (iv). $\square$

8.2 Identity for the sums of 4 squares

In this section we state and prove an identity relating the functions $\delta \left(-\frac{1}{6}, \frac{7}{12}, -\frac{1}{4}; n \right)$ and $\delta \left(\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}; n \right)$, which occur in the formulae (6.3.2), (6.3.3) and (6.3.4), to the function $\sigma(n)$.

Theorem 8.2.1. *For $n \in \mathbb{N}$, we have*

$$\left(\frac{-4}{n} \right) \sigma(n) = \frac{1}{8} \delta \left(-\frac{1}{6}, \frac{7}{12}, -\frac{1}{4}; n \right) - \frac{1}{8} \delta \left(-\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}; n \right).$$

Proof. Let $q \in D_\delta$. As

$$\frac{i^n - (-i)^n}{2i} = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{4}, \\ -1, & \text{if } n \equiv 3 \pmod{4}, \\ 0, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

$$= \left(\frac{-4}{n} \right),$$

we have

$$\begin{aligned} \frac{E_1(iq) - E_1(-iq)}{2i} &= \frac{1}{2i} \left(\left(1 - 24 \sum_{n=1}^{+\infty} \sigma(n)(iq)^n \right) - \left(1 - 24 \sum_{n=1}^{+\infty} \sigma(n)(-iq)^n \right) \right) \\ &= -24 \sum_{n=1}^{+\infty} \sigma(n) \left(\frac{i^n - (-i)^n}{2i} \right) q^n \\ &= -24 \sum_{n=1}^{+\infty} \left(\frac{-4}{n} \right) \sigma(n) q^n. \end{aligned}$$

Moreover, by Theorem 8.1.3, we have

$$\begin{aligned} \frac{E_1(iq) - E_1(-iq)}{2i} &= \frac{-6i(1-x)^{1/4}(1-(1-x)^{1/2})z^2}{2i} \\ &= -3(1-x)^{1/4}z^2 + 3(1-x)^{3/4}z^2 = -3\Delta \left(-\frac{1}{6}, \frac{7}{12}, -\frac{1}{4}; q \right) + 3\Delta \left(\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}; q \right) \\ &= \sum_{n=1}^{+\infty} \left(-3 \delta \left(-\frac{1}{6}, \frac{7}{12}, -\frac{1}{4}; n \right) + 3 \delta \left(\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}; n \right) \right) q^n. \end{aligned}$$

Equating coefficients and dividing both sides by -24, we obtain the asserted identity.

□

8.3 Identities for the sums of 8 squares

In this section we state and prove four identities relating the functions $\delta(a, b, c; n)$ obtained in Theorems 7.1.1, 7.1.2 and 7.1.3.

Theorem 8.3.1. *For $n \in \mathbb{N}$, we have*

$$\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) = 0, \text{ if } n \equiv 0 \pmod{2}$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) and Theorem 8.1.1(i) we have

$$\begin{aligned} \sum_{n=1}^{+\infty} \delta\left(0, \frac{1}{6}, \frac{1}{6}; 2n\right) q^{2n} &= \frac{1}{2} \left(\sum_{n=1}^{+\infty} \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) q^n \right. \\ &\quad \left. + \sum_{n=1}^{+\infty} \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) (-q)^n \right) \\ &= \frac{1}{2} \left(\Delta\left(0, \frac{1}{6}, \frac{1}{6}; q\right) + \Delta\left(0, \frac{1}{6}, \frac{1}{6}; -q\right) \right) \\ &= \frac{1}{2} \left(\frac{x\sqrt{1-x}}{16} z^4 - \frac{x\sqrt{1-x}}{16} z^4 \right) = 0. \end{aligned}$$

Equating coefficients we obtain $\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) = 0$ if $n \equiv 0 \pmod{2}$. $\square$

Theorem 8.3.2. *For $n \in \mathbb{N}$, we have*

$$\delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right) = \frac{1}{8} \left(\delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) - \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right) \right)$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) we have $\Delta\left(0, \frac{1}{6}, \frac{1}{6}; q\right) = \frac{x(1-x)^{1/2}}{16} z^4$.

By Theorem 8.1.1(ii)

$$\Delta\left(0, \frac{1}{6}, \frac{1}{6}; q^2\right) = \frac{(1-x)^{1/4} (1-\sqrt{1-x})^2 (1+\sqrt{1-x})}{2^7} z^4$$

$$\begin{aligned}
&= \frac{x(1-x)^{1/4} (1 - \sqrt{1-x})}{2^7} z^4 = \frac{x(1-x)^{1/4}}{2^7} z^4 - \frac{x(1-x)^{3/4}}{2^7} z^4 \\
&= \frac{1}{8} \left(\Delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; q \right) - \Delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; q \right) \right).
\end{aligned}$$

This implies that

$$\sum_{n=1}^{+\infty} \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) q^{2n} = \frac{1}{8} \sum_{n=1}^{+\infty} \left(\delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) - \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right) \right) q^n,$$

for all $q \in D_\delta$. Equating the coefficients of q^n , we obtain

$$\delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2 \right) = \frac{1}{8} \left(\delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) - \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right) \right).$$

□

Theorem 8.3.3. *For $n \in \mathbb{N}$, we have*

$$\left(\frac{-4}{n} \right) \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) = \frac{1}{2} \left(\delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right) + \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \right).$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) we have $\Delta \left(0, \frac{1}{6}, \frac{1}{6}; q \right) = \frac{x(1-x)^{1/2}}{16} z^4$.

By Theorem 8.1.2(i) we have

$$\begin{aligned}
&\Delta \left(0, \frac{1}{6}, \frac{1}{6}; iq \right) \\
&= \frac{(-1)^4 i (1-x)^{1/4} (1 - \sqrt{1-x}) (1 + i(1-x)^{1/4})^2 (1 - i(1-x)^{1/4})^2}{2^5} z^4 \\
&= \frac{i(1-x)^{1/4} (1 - \sqrt{1-x}) (1 + \sqrt{1-x})^2}{32} z^4 = \frac{ix(1-x)^{1/4} (1 + \sqrt{1-x})}{32} z^4
\end{aligned}$$

and by Theorem 8.1.2(ii)

$$\Delta \left(0, \frac{1}{6}, \frac{1}{6}; -iq \right)$$

$$\begin{aligned}
&= \frac{(-1)^5 i (1-x)^{1/4} (1-\sqrt{1-x}) (1+i(1-x)^{1/4})^2 (1-i(1-x)^{1/4})^2}{2^5} z^4 \\
&= \frac{i(1-x)^{1/4} (1-\sqrt{1-x}) (1+\sqrt{1-x})^2}{32} z^4 = \frac{-ix(1-x)^{1/4} (1+\sqrt{1-x})}{32} z^4.
\end{aligned}$$

Recall that

$$\frac{i^n - (-i)^n}{2i} = \left(\frac{-4}{n} \right).$$

We have

$$\begin{aligned}
&\sum_{n=1}^{+\infty} \left(\frac{-4}{n} \right) \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) q^n = \sum_{n=1}^{+\infty} \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) \left(\frac{i^n - (-i)^n}{2i} \right) q^n \\
&= \frac{1}{2i} \left(\sum_{n=1}^{+\infty} \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) (iq)^n - \sum_{n=1}^{+\infty} \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) (-iq)^n \right) \\
&= \frac{1}{2i} \left(\Delta \left(0, \frac{1}{6}, \frac{1}{6}; iq \right) - \Delta \left(0, \frac{1}{6}, \frac{1}{6}; -iq \right) \right) \\
&= \frac{1}{2i} \left(\frac{ix(1-x)^{1/4} (1+\sqrt{1-x})}{32} z^4 + \frac{ix(1-x)^{1/4} (1+\sqrt{1-x})}{32} z^4 \right) \\
&= \frac{1}{2i} \left(\frac{ix(1-x)^{1/4} (1+\sqrt{1-x})}{16} \right) z^4 = \frac{x(1-x)^{1/4} (1+\sqrt{1-x})}{32} z^4 \\
&= \frac{x(1-x)^{1/4}}{32} z^4 + \frac{x(1-x)^{3/4}}{32} z^4 \\
&= \frac{1}{2} \left(\Delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; q \right) + \Delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; q \right) \right)
\end{aligned}$$

$$= \frac{1}{2} \sum_{n=1}^{+\infty} \left(\delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right) + \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) \right) q^n.$$

Equating the coefficients of q^n ($n \in \mathbb{N}$), we obtain the asserted identity. $\square$

Theorem 8.3.4. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} \left(\frac{-4}{n} \right) \sigma_3(n) &= -\frac{1}{4} \delta \left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n \right) + \frac{1}{4} \delta \left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n \right) \\ &\quad + \frac{7}{2} \delta \left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n \right) - \frac{1}{2} \delta \left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n \right). \end{aligned}$$

Proof. Let $q \in D_\delta$. As

$$\frac{i^n - (-i)^n}{2i} = \left(\frac{-4}{n} \right),$$

we have

$$\begin{aligned} \frac{E_2(iq) - E_2(-iq)}{2i} &= \frac{1}{2i} \left(\left(1 + 240 \sum_{n=1}^{+\infty} \sigma_3(n) (iq)^n \right) - \left(1 + 240 \sum_{n=1}^{+\infty} \sigma_3(n) (-iq)^n \right) \right) \\ &= 240 \sum_{n=1}^{+\infty} \sigma_3(n) \left(\frac{i^n - (-i)^n}{2i} \right) q^n \\ &= 240 \sum_{n=1}^{+\infty} \left(\frac{-4}{n} \right) \sigma_3(n) q^n. \end{aligned}$$

Moreover, by Theorem 8.1.4(i) and (ii), we have

$$\begin{aligned} \frac{E_2(iq) - E_2(-iq)}{2i} &= -\frac{15}{4} g z^4 + \frac{105}{2} g^3 z^4 - \frac{105}{2} g^5 z^4 + \frac{15}{2} g^7 z^4 \\ &= -\frac{15}{4} (1-x)^{1/4} z^4 + \frac{105}{2} (1-x)^{3/4} z^4 - \frac{105}{2} (1-x)^{5/4} z^4 + \frac{15}{2} (1-x)^{7/4} z^4 \end{aligned}$$

$$\begin{aligned}
&= -\frac{15}{2}(1-x)^{1/4}z^4 + \frac{105}{2}(1-x)^{3/4}z^4 - \frac{105}{2}(1-x)^{1/4}z^4 + \frac{105}{2}x(1-x)^{1/4}z^4 \\
&\quad + \frac{15}{2}(1-x)^{3/4}z^4 - \frac{15}{2}x(1-x)^{3/4}z^4 \\
&= -60(1-x)^{1/4}z^4 + 60(1-x)^{3/4}z^4 + \frac{105}{2}x(1-x)^{1/4}z^4 - \frac{15}{2}x(1-x)^{3/4}z^4 \\
&= -60\Delta\left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; q\right) + 60\Delta\left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; q\right) \\
&\quad + 840\Delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; q\right) - 120\Delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; q\right) \\
&= \sum_{n=1}^{+\infty} \left(-60 \delta\left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n\right) + 60 \delta\left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n\right) \right. \\
&\quad \left. + 840 \delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) - 120 \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right) \right) q^n.
\end{aligned}$$

Equating coefficients and dividing both sides by 240, we obtain the asserted identity.

□

8.4 Identities for the sums of 12 squares

In this section we state and prove eight identities relating the functions $\delta(a, b, c; n)$ obtained in Theorems 7.2.1, 7.2.2 and 7.2.3.

Theorem 8.4.1. *For $n \in \mathbb{N}$, we have*

$$\delta\left(0, \frac{1}{2}, 0; n\right) = 0, \text{ if } n \equiv 0 \pmod{2}.$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) and Theorem 8.1.1(i), we have

$$\begin{aligned} 2 \sum_{n=1}^{+\infty} \delta \left(0, \frac{1}{2}, 0; 2n \right) q^{2n} &= \sum_{n=1}^{\infty} \delta \left(0, \frac{1}{2}, 0; n \right) q^n + \sum_{n=1}^{\infty} \delta \left(0, \frac{1}{2}, 0; n \right) (-q)^n \\ &= \Delta \left(0, \frac{1}{2}, 0; q \right) + \Delta \left(0, \frac{1}{2}, 0; -q \right) \\ &= \frac{x(1-x)}{16} z^6 - \frac{x(1-x)}{16} z^6 = 0. \end{aligned}$$

Equating coefficients we obtain $\delta \left(0, \frac{1}{2}, 0; n \right) = 0$ if $n \equiv 0 \pmod{2}$. $\square$

Theorem 8.4.2. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) &= 8 \delta \left(0, 0, \frac{1}{2}; n \right) = 8 \delta \left(0, \frac{1}{2}, 0; \frac{n}{2} \right) = 8\tau_2 \left(\frac{n}{2} \right) \\ &= 4 \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{n}{2}}} (x_1^4 - 3x_1^2 x_2^2), \text{ if } n \equiv 0 \pmod{2}. \end{aligned}$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) and Theorem 8.1.1(i) we have

$$\begin{aligned} \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; 2n \right) q^{2n} &= \frac{1}{2} \left(\delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) q^n + \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) (-q)^n \right) \\ &= \frac{1}{2} \left(\Delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; q \right) + \Delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; -q \right) \right) \\ &= \frac{1}{2} \left(\frac{x(1-x)^{1/2}}{16} z^6 - \frac{x(1-x)^{3/2}}{16} z^6 \right) = \frac{x^2(1-x)^{1/2}}{32} z^6 \end{aligned}$$

and

$$8 \sum_{n=1}^{+\infty} \delta \left(0, 0, \frac{1}{2}; 2n \right) q^{2n} = 8 \left(\frac{1}{2} \left(\sum_{n=1}^{+\infty} \delta \left(0, 0, \frac{1}{2}; n \right) q^n + \sum_{n=1}^{+\infty} \delta \left(0, 0, \frac{1}{2}; n \right) (-q)^n \right) \right)$$

$$\begin{aligned}
 &= 4 \left(\Delta \left(0, 0, \frac{1}{2}; q \right) + \Delta \left(0, 0, -\frac{1}{2}; -q \right) \right) \\
 &= 4 \left(\frac{x^2(1-x)^{1/2}}{256} z^6 + \frac{x^2(1-x)^{1/2}}{256} z^6 \right) = \frac{x^2(1-x)^{1/2}}{32} z^6.
 \end{aligned}$$

This implies

$$\sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; 2n \right) q^{2n} = \sum_{n=1}^{+\infty} \delta \left(0, 0, \frac{1}{2}; 2n \right) q^{2n}.$$

Equating coefficients, we obtain the asserted identity. $\square$

Theorem 8.4.3. *For $n \in \mathbb{N}$, we have*

$$\delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) = 0, \text{ if } n \equiv 0 \pmod{4}.$$

Proof. Let $n \in \mathbb{N}$. If $n \equiv 0 \pmod{4}$, by Theorem 8.4.2, we have $\delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) = 8 \tau_2(n/2)$. As $\frac{n}{2} \equiv 0 \pmod{2}$, by Theorem 2.11.3, we have $\tau_2 \left(\frac{n}{2} \right) = 0$. Thus $\delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) = 0$ for $n \equiv 0 \pmod{4}$. $\square$

Theorem 8.4.4. *For $n \in \mathbb{N}$, we have*

$$\delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right) = \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2 \right), \text{ if } n \equiv 0 \pmod{2}.$$

Proof. Let $q \in D_\delta$. By Theorems 2.10.2(ii) and 8.1.1(i), we have

$$\begin{aligned}
 \sum_{n=1}^{+\infty} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; 2n \right) q^{2n} &= \frac{1}{2} \left(\sum_{n=1}^{+\infty} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right) q^n \right. \\
 &\quad \left. + \sum_{n=1}^{+\infty} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right) (-q)^n \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\Delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; q \right) + \Delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; -q \right) \right) \\
&= \frac{1}{2} \left(\frac{x^2(1-x)^{3/4}}{256} z^6 + \frac{x^2(1-x)^{1/4}}{256} z^6 \right) \\
&= \frac{x^2(1-x)^{1/4}}{512} z^6 + \frac{x^2(1-x)^{3/4}}{512} z^6
\end{aligned}$$

and by Theorem 8.1.1(ii)

$$\begin{aligned}
\sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) q^{2n} &= \Delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; q^2 \right) \\
&= \frac{(1-x)^{1/4} (1-\sqrt{1-x})^2 (1+\sqrt{1-x})^3}{512} z^6 \\
&= \frac{x^2(1-x)^{1/4} (1+\sqrt{1-x})}{512} z^6 \\
&= \frac{x^2(1-x)^{1/4}}{512} z^6 + \frac{x^2(1-x)^{3/4}}{512} z^6.
\end{aligned}$$

This implies

$$\sum_{n=1}^{+\infty} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; 2n \right) q^{2n} = \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) q^{2n}.$$

Equating coefficients we obtain $\delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; 2n \right) = \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right)$, which is the desired identity. $\square$

Theorem 8.4.5. *For $n \in \mathbb{N}$, we have*

$$\delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) = \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2 \right), \text{ if } n \equiv 0 \pmod{2}.$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) and Theorem 8.1.1(i), we have

$$\begin{aligned}
 \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; 2n \right) q^{2n} &= \frac{1}{2} \left(\sum_{n=1}^{+\infty} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) q^n \right. \\
 &\quad \left. + \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) (-q)^n \right) \\
 &= \frac{1}{2} \left(\Delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; q \right) + \Delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; -q \right) \right) \\
 &= \frac{1}{2} \left(\frac{x^2(1-x)^{1/4}}{256} z^6 + \frac{x^2(1-x)^{3/4}}{256} z^6 \right) \\
 &= \frac{x^2(1-x)^{1/4}}{512} z^6 + \frac{x^2(1-x)^{3/4}}{512} z^6
 \end{aligned}$$

and by Theorem 8.1.1(ii)

$$\begin{aligned}
 \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) q^{2n} &= \Delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; q^2 \right) \\
 &= \frac{(1-x)^{1/4} (1 - \sqrt{1-x})^2 (1 + \sqrt{1-x})^3}{512} z^6 \\
 &= \frac{x^2(1-x)^{1/4} (1 + \sqrt{1-x})}{512} z^6 \\
 &= \frac{x^2(1-x)^{1/4}}{512} z^6 + \frac{x^2(1-x)^{3/4}}{512} z^6.
 \end{aligned}$$

This implies

$$\sum_{n=1}^{+\infty} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; 2n \right) q^{2n} = \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) q^{2n}.$$

Equating coefficients we obtain $\delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; 2n\right) = \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right)$, which is the desired identity. $\square$

Theorem 8.4.6. *For $n \in \mathbb{N}$, we have*

$$\delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) = \begin{cases} 20 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right), & \text{if } n \equiv 0 \pmod{4}, \\ 12 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right), & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) and Theorem 8.1.1(i), we have

$$\begin{aligned} \sum_{n=1}^{+\infty} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; 2n\right) q^{2n} &= \frac{1}{2} \left(\sum_{n=1}^{+\infty} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) q^n \right. \\ &\quad \left. + \sum_{n=1}^{+\infty} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) (-q)^n \right) \\ &= \frac{1}{2} \left(\Delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; q\right) + \Delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; -q\right) \right) \\ &= \frac{1}{2} \left(\frac{x(1-x)^{1/4}}{16} z^6 - \frac{x(1-x)^{7/4}}{16} z^6 \right) \\ &= \frac{x(1-x)^{1/4}}{32} z^6 - \frac{x(1-x)^{3/4}}{32} z^6 + \frac{x^2(1-x)^{3/4}}{32} z^6 \end{aligned}$$

and by Theorem 2.10.2(ii) and Theorem 8.1.1(ii)(iii)

$$\begin{aligned} &\sum_{n=1}^{+\infty} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) (16 + 4(-1)^n) q^{2n} \\ &= 16 \Delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; q^2\right) + 4 \Delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; -q^2\right) \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{x^2(1-x)^{1/4} (1 + \sqrt{1-x})}{32} z^6 - \frac{x(1-x)^{3/4} (1 - \sqrt{1-x})}{32} z^6 \right) \\
 &= \frac{x^2(1-x)^{1/4}}{32} z^6 + \frac{x^2(1-x)^{3/4}}{32} z^6 - \frac{x(1-x)^{3/4}}{32} z^6 + \frac{x(1-x)^{5/4}}{32} z^6 \\
 &= \frac{x^2(1-x)^{1/4}}{32} z^6 + \frac{x^2(1-x)^{3/4}}{32} z^6 - \frac{x(1-x)^{3/4}}{32} z^6 + \frac{x(1-x)^{1/4}}{32} z^6 \\
 &\quad - \frac{x^2(1-x)^{1/4}}{32} z^6 \\
 &= \frac{x^2(1-x)^{3/4}}{32} z^6 - \frac{x(1-x)^{3/4}}{32} z^6 + \frac{x(1-x)^{1/4}}{32} z^6.
 \end{aligned}$$

This implies

$$\sum_{n=1}^{+\infty} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; 2n \right) q^{2n} = \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) (16 + 4(-1)^n) q^{2n}.$$

Equating coefficients we obtain

$$\delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; 2n \right) = (16 + 4(-1)^n) \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right),$$

which is the desired identity. $\square$

Theorem 8.4.7. *For $n \in \mathbb{N}$, we have*

$$\delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) = \begin{cases} -4 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2 \right), & \text{if } n \equiv 0 \pmod{4}, \\ 4 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2 \right), & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) and Theorem 8.1.1(i), we have

$$\sum_{n=1}^{+\infty} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; 2n \right) q^{2n} = \frac{1}{2} \left(\sum_{n=1}^{+\infty} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) q^n \right.$$

$$\begin{aligned}
& + \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) (-q)^n \\
& = \frac{1}{2} \left(\Delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; q \right) + \Delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; -q \right) \right) \\
& = \frac{1}{2} \left(\frac{x(1-x)^{3/4}}{16} z^6 - \frac{x(1-x)^{5/4}}{16} z^6 \right) \\
& = \left(\frac{x(1-x)^{3/4}}{32} z^6 - \frac{x(1-x)^{5/4}}{32} z^6 \right)
\end{aligned}$$

and by Theorem 8.1.1 (iii)

$$\begin{aligned}
4 \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) (-1)^n q^{2n} & = -4 \Delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; -q^2 \right) \\
& = \frac{(1-x)^{3/4} (1-\sqrt{1-x})^2 (1+\sqrt{1-x})}{32} z^6 \\
& = \frac{x(1-x)^{3/4} (1-\sqrt{1-x})}{32} z^6 \\
& = \frac{x(1-x)^{3/4}}{32} z^6 - \frac{x(1-x)^{5/4}}{32} z^6.
\end{aligned}$$

This implies

$$\sum_{n=1}^{+\infty} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; 2n \right) q^{2n} = -4 \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) (-1)^n q^{2n}.$$

Equating coefficients we obtain $\delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; 2n \right) = -4(-1)^n \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right)$, which is the desired identity. $\square$

Theorem 8.4.8. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \left(\frac{-4}{n}\right) \sigma_5(n) &= 2 \delta\left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n\right) - \frac{79}{2} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) + 122 \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) \\ &\quad - 2 \delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n\right) + \frac{49}{2} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - 2 \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right). \end{aligned}$$

Proof. Let $q \in D_\delta$. As

$$\frac{i^n - (-i)^n}{2i} = \left(\frac{-4}{n}\right),$$

we have

$$\begin{aligned} \frac{E_3(iq) - E_3(-iq)}{2i} &= \frac{1}{2i} \left(\left(1 - 504 \sum_{n=1}^{+\infty} \sigma_5(n) (iq)^n\right) - \left(1 - 504 \sum_{n=1}^{+\infty} \sigma_5(n) (-iq)^n\right) \right) \\ &= -504 \sum_{n=1}^{+\infty} \sigma_5(n) \left(\frac{i^n - (-i)^n}{2i}\right) q^n \\ &= -504 \sum_{n=1}^{+\infty} \left(\frac{-4}{n}\right) \sigma_5(n) q^n. \end{aligned}$$

Moreover, by Theorem 8.1.4(iii)(iv), we have

$$\begin{aligned} \frac{E_3(iq) - E_3(-iq)}{2i} &= -\frac{63}{16}gz^6 + \frac{3843}{16}g^3z^6 - \frac{6111}{8}g^5z^6 + \frac{6111}{8}g^7z^6 - \frac{3843}{16}g^9z^6 + \frac{63}{16}g^{11}z^6 \\ (8.4.1) \quad &= -\frac{63}{16}(1-x)^{1/4}z^6 + \frac{3843}{16}(1-x)^{3/4}z^6 - \frac{6111}{8}(1-x)^{5/4}z^6 \\ &\quad + \frac{6111}{8}(1-x)^{7/4}z^6 - \frac{3843}{16}(1-x)^{9/4}z^6 + \frac{63}{16}(1-x)^{11/4}z^6. \end{aligned}$$

Simplifying the right hand side of equation (8.4.1) using MAPLE, we obtain

$$-1008(1-x)^{1/4}z^6 + \frac{4977}{4}x(1-x)^{1/4}z^6 - \frac{3843}{16}x^2(1-x)^{1/4}z^6$$

$$\begin{aligned}
& + 1008(1-x)^{3/4}z^6 - \frac{3087}{4}x(1-x)^{3/4}z^6 + \frac{63}{16}x^2(1-x)^{3/4}z^6 \\
& = -1008\Delta\left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; q\right) + 19908\Delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; q\right) - 61488\Delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; q\right) \\
& \quad + 1008\Delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; q\right) - 12348\Delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; q\right) + 1008\Delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; q\right) \\
& = \sum_{n=1}^{+\infty} \left(-1008 \delta\left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n\right) + 19908 \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) - 61488 \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) \right. \\
& \quad \left. + 1008 \delta\left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n\right) - 12348 \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) + 1008 \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right) \right) q^n.
\end{aligned}$$

Equating coefficients and dividing both sides by -504, we obtain the desired identity.

□

8.5 Identities for the sums of 16 squares

In this section we state and prove eight identities relating the functions $\delta(a, b, c; n)$ obtained in Theorems 7.3.1, 7.3.2 and 7.3.3.

Theorem 8.5.1. *For $n \in \mathbb{N}$, we have*

$$\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) = \frac{1}{8}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \text{ if } n \equiv 0 \pmod{2}.$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) and Theorem 8.1.1(i), we have

$$\sum_{n=1}^{+\infty} \delta\left(0, \frac{1}{3}, \frac{1}{3}; 2n\right) q^{2n} = \frac{1}{2} \left(\sum_{n=1}^{+\infty} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) q^n \right)$$

$$\begin{aligned}
& + \sum_{n=1}^{+\infty} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) (-q)^n \\
& = \frac{1}{2} \left(\Delta \left(0, \frac{1}{3}, \frac{1}{3}; q \right) + \Delta \left(0, \frac{1}{3}, \frac{1}{3}; -q \right) \right) \\
& = \frac{1}{2} \left(\frac{x^2(1-x)}{256} z^8 + \frac{x^2(1-x)}{256} z^8 \right) \\
& = \frac{x^2(1-x)}{256} z^8
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{8} \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; 2n \right) q^{2n} & = \frac{1}{16} \left(\Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; q \right) + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -q \right) \right) \\
& = \frac{1}{16} \left(\frac{x(1-x)}{16} z^8 - \frac{x(1-x)^2}{16} z^8 \right) \\
& = \frac{1}{16} \left(\frac{x(1-x)}{16} z^8 - \frac{x(1-x)}{16} z^8 + \frac{x^2(1-x)}{16} z^8 \right) = \frac{x^2(1-x)}{256} z^8.
\end{aligned}$$

This implies

$$\sum_{n=1}^{+\infty} \delta \left(0, \frac{1}{3}, \frac{1}{3}; 2n \right) q^{2n} = \frac{1}{8} \sum_{n=1}^{+\infty} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; 2n \right) q^{2n}.$$

Equating coefficients we obtain the desired identity. $\square$

Theorem 8.5.2. *For $n \in \mathbb{N}$, we have*

$$\delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) = 0, \text{ if } n \equiv 0 \pmod{2}.$$

Proof. Let $q \in D_\delta$. By Theorem 2.10.2(ii) and Theorem 8.1.1(i), we have

$$\begin{aligned} 2 \sum_{n=1}^{+\infty} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; 2n \right) q^{2n} &= \sum_{n=1}^{\infty} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) q^n + \sum_{n=1}^{\infty} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) (-q)^n \\ &= \Delta \left(0, -\frac{1}{6}, \frac{5}{6}; q \right) + \Delta \left(0, -\frac{1}{6}, \frac{5}{6}; -q \right) \\ &= \frac{x^3 \sqrt{1-x}}{2^{12}} z^8 - \frac{x^3 \sqrt{1-x}}{2^{12}} z^8 = 0. \end{aligned}$$

Equating coefficients we obtain $\delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) = 0$ if $n \equiv 0 \pmod{2}$. □

Theorem 8.5.3. For $n \in \mathbb{N}$ such that $n \equiv 0 \pmod{4}$, we have

$$\begin{aligned} \text{(i)} \quad & \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) = -6 \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\ \text{(ii)} \quad & \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) = -\frac{3}{8} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right). \end{aligned}$$

Proof. (i) Let $q \in D_\delta$. As

$$\frac{(1)^n + (-1)^n + i^n + (-i)^n}{4} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

we have by Theorems 2.10.2(ii), 8.1.1(i) and 8.1.2

$$\begin{aligned} & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) q^n \\ &= \frac{1}{4} \sum_{n=0}^{+\infty} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) ((1)^n + (-1)^n + (i)^n + (-i)^n) q^n \\ &= \frac{1}{4} \left(\Delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; q \right) + \Delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; -q \right) + \Delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; iq \right) \right. \end{aligned}$$

$$\begin{aligned}
& + \Delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; -iq \right) \Bigg) \\
& = \frac{1}{4} \left(\frac{x(1-x)^{1/2}}{16} z^8 - \frac{x(1-x)^{5/2}}{16} z^8 \right. \\
& \quad + \frac{i}{512} (1-x)^{1/4} (1-\sqrt{1-x}) (1+i(1-x)^{1/4})^2 (1-i(1-x)^{1/4})^{10} z^8 \\
& \quad \left. - \frac{i}{512} (1-x)^{1/4} (1-\sqrt{1-x}) (1+i(1-x)^{1/4})^{10} (1-i(1-x)^{1/4})^2 z^8 \right).
\end{aligned}$$

Simplifying the last quantity using MAPLE we obtain

$$-\frac{3}{64} x^2 z^8 + \frac{3}{64} x^3 z^8 + \frac{3}{64} x^2 \sqrt{1-x} z^8 - \frac{3}{128} x^3 \sqrt{1-x} z^8.$$

Moreover, using the same theorems we have

$$\begin{aligned}
& \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} -6 \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) q^n \\
& = -\frac{6}{4} \sum_{n=0}^{+\infty} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) ((1)^n + (-1)^n + (i)^n + (-i)^n) q^n \\
& = -\frac{6}{4} \left(\Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; q \right) + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -q \right) + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; iq \right) \right. \\
& \quad \left. + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -iq \right) \right) \\
& = -\frac{6}{4} \left(\frac{x(1-x)}{16} z^8 - \frac{x(1-x)^2}{16} z^8 \right. \\
& \quad - \frac{i}{512} (1-x)^{1/4} (1-\sqrt{1-x}) (1+i(1-x)^{1/4})^4 (1-i(1-x)^{1/4})^8 z^8 \\
& \quad \left. + \frac{i}{512} (1-x)^{1/4} (1-\sqrt{1-x}) (1+i(1-x)^{1/4})^8 (1-i(1-x)^{1/4})^4 z^8 \right)
\end{aligned}$$

Simplifying the last expression using MAPLE we obtain

$$-\frac{3}{64} x^2 z^8 + \frac{3}{64} x^3 z^8 + \frac{3}{64} x^2 \sqrt{1-x} z^8 - \frac{3}{128} x^3 \sqrt{1-x} z^8.$$

Thus, we have shown that

$$\sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) q^n = \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} -6 \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) q^n.$$

Equating coefficients, we obtain the desired identity. The proof for (ii) follows in exactly the same way. $\square$

Theorem 8.5.4. *For $n \in \mathbb{N}$ such that $n \equiv 0 \pmod{4}$, we have*

$$\begin{aligned} \text{(i)} \quad & \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) = \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) = \frac{1}{15} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right), \\ \text{(ii)} \quad & \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) = -\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right) = \frac{1}{240} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right), \\ \text{(iii)} \quad & \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) = 0. \end{aligned}$$

Proof. We only prove (i) as the proofs of (ii) and (iii) are similar. Let $q \in D_\delta$. As

$$\frac{1 + (-1)^n + i^n + (-i)^n}{4} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

we have by Theorems 2.10.2(ii), 8.1.1(i) and 8.1.2

$$\begin{aligned} & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) q^n \\ &= \frac{1}{4} \sum_{n=0}^{+\infty} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) (1 + (-1)^n + (i)^n + (-i)^n) q^n \\ &= \frac{1}{4} \left(\Delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; q\right) + \Delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; -q\right) + \Delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; iq\right) \right. \\ & \quad \left. + \Delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; -iq\right) \right) \\ &= \frac{1}{4} \left(\frac{x^2(1-x)^{1/4}}{256} z^8 + \frac{x^2(1-x)^{7/4}}{256} z^8 \right. \\ & \quad + \frac{i}{1024} \sqrt{1-x} (1 - \sqrt{1-x})^2 (1 + i(1-x)^{1/4}) (1 - i(1-x)^{1/4})^7 z^8 \\ & \quad \left. - \frac{i}{1024} \sqrt{1-x} (1 - \sqrt{1-x})^2 (1 + i(1-x)^{1/4})^7 (1 - i(1-x)^{1/4}) z^8 \right). \end{aligned}$$

Simplifying the last expression using MAPLE we obtain

$$\begin{aligned} & -\frac{1}{64}x(1-x)^{1/4}z^8 + \frac{5}{256}x^2(1-x)^{1/4}z^8 - \frac{3}{1024}x^3(1-x)^{1/4}z^8 \\ & + \frac{1}{64}x(1-x)^{3/4}z^8 - \frac{3}{256}x^2(1-x)^{3/4}z^8 - \frac{1}{1024}x^3(1-x)^{3/4}z^8. \end{aligned}$$

Moreover, using the same theorems, we have

$$\begin{aligned} & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \frac{1}{15} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) q^n \\ &= \frac{1}{60} \sum_{n=0}^{+\infty} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) ((1)^n + (-1)^n + (i)^n + (-i)^n) q^n \\ &= \frac{1}{60} \left(\Delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; q \right) + \Delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -q \right) + \Delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; iq \right) \right. \\ & \quad \left. + \Delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -iq \right) \right) \\ &= \frac{1}{60} \left(\frac{x(1-x)^{1/4}}{16} z^8 - \frac{x(1-x)^{11/4}}{256} z^8 \right. \\ & \quad - \frac{1}{512} (1-x)^{1/4} (1-\sqrt{1-x}) (1+i(1-x)^{1/4}) (1-i(1-x)^{1/4})^{11} z^8 \\ & \quad \left. - \frac{1}{512} (1-x)^{1/4} (1-\sqrt{1-x}) (1+i(1-x)^{1/4})^{11} (1-i(1-x)^{1/4}) z^8 \right). \end{aligned}$$

Simplifying the last expression using MAPLE we obtain

$$\begin{aligned} & -\frac{1}{64}x(1-x)^{1/4}z^8 + \frac{5}{256}x^2(1-x)^{1/4}z^8 - \frac{3}{1024}x^3(1-x)^{1/4}z^8 \\ & + \frac{1}{64}x(1-x)^{3/4}z^8 - \frac{3}{256}x^2(1-x)^{3/4}z^8 - \frac{1}{1024}x^3(1-x)^{3/4}z^8. \end{aligned}$$

Thus, we have shown that

$$\sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) q^n = \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \frac{1}{15} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) q^n.$$

Equating coefficients, we obtain the identity

$$\delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) = \frac{1}{15} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right),$$

when $n \equiv 0 \pmod{4}$. In the same way, we can obtain the identity

$$\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) = \frac{1}{15} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right),$$

when $n \equiv 0 \pmod{4}$, which proves (i). $\square$

Theorem 8.5.5. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} \left(\frac{-4}{n}\right) \sigma_7(n) &= -34 \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) + \frac{1905}{2} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\ &\quad - 6810 \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + 4376 \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\ &\quad + 34 \delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) - \frac{1359}{2} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\ &\quad + 2442 \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) - 8 \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right). \end{aligned}$$

Proof. Let $q \in D_\delta$. As

$$\frac{i^n - (-i)^n}{2i} = \left(\frac{-4}{n}\right),$$

we have

$$\begin{aligned} \frac{E_4(iq) - E_4(-iq)}{2i} &= \frac{1}{2i} \left(\left(1 + 480 \sum_{n=1}^{+\infty} \sigma_7(n)(iq)^n\right) - \left(1 + 480 \sum_{n=1}^{+\infty} \sigma_7(n)(-iq)^n\right) \right) \\ &= 480 \sum_{n=1}^{+\infty} \sigma_7(n) \left(\frac{i^n - (-i)^n}{2i}\right) q^n = 480 \sum_{n=1}^{+\infty} \left(\frac{-4}{n}\right) \sigma_7(n) q^n. \end{aligned}$$

Moreover, by Theorem 8.1.4 (v)(vi) and the use of MAPLE, we have

$$\frac{E_4(iq) - E_4(-iq)}{2i} = -\frac{15}{16}gz^8 + \frac{8205}{16}g^3z^8 - \frac{73215}{16}g^5z^8 + \frac{179685}{16}g^7z^8 - \frac{179685}{16}g^9z^8$$

$$\begin{aligned}
& + \frac{73215}{16}g^{11}z^8 - \frac{8205}{16}g^{13}z^8 + \frac{15}{16}g^{15}z^8 \\
& = -\frac{15}{16}(1-x)^{1/4}z^8 + \frac{8205}{16}(1-x)^{3/4}z^8 - \frac{73215}{16}(1-x)^{5/4}z^8 + \frac{179685}{16}(1-x)^{7/4}z^8 \\
& \quad - \frac{179685}{16}(1-x)^{9/4}z^8 + \frac{73215}{16}(1-x)^{11/4}z^8 - \frac{8205}{16}(1-x)^{13/4}z^8 + \frac{15}{16}(1-x)^{15/4}z^8.
\end{aligned}$$

Simplifying the last expression using MAPLE, we obtain

$$\begin{aligned}
& -16320(1-x)^{1/4}z^8 + 28575x(1-x)^{1/4}z^8 - \frac{51075}{4}x^2(1-x)^{1/4}z^8 + \frac{8205}{16}x^3(1-x)^{1/4}z^8 \\
& + 16320(1-x)^{3/4}z^8 - 20385x(1-x)^{3/4}z^8 + \frac{18315}{4}x^2(1-x)^{3/4}z^8 - \frac{15}{16}x^3(1-x)^{3/4}z^8 \\
& = -16320\Delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; q\right) + 457200\Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; q\right) \\
& \quad - 3268800\Delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; q\right) + 2100480\Delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; q\right) \\
& + 16320\Delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; q\right) - 326160\Delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; q\right) \\
& + 1172160\Delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; q\right) - 3840\Delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; q\right) \\
& = \sum_{n=1}^{+\infty} \left(-16320 \delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) + 457200 \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \right.
\end{aligned}$$

$$\begin{aligned}
& - 3268800 \, \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + 2100480 \, \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) \\
& + 16320 \, \delta \left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n \right) - 326160 \, \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& + 1172160 \, \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) - 3840 \, \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right) \Big) q^n.
\end{aligned}$$

Equating coefficients and dividing both sides by 480, we obtain the desired identity. $\square$

8.6 An important identity

The goal of this section is to prove an important delta identity that plays a key role in Chapter 9 in simplifying the results involving 16 squares.

Theorem 8.6.1.

$$\delta \left(-5/6, 25/12, -7/12; n \right) := \begin{cases} 45 \, \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), & \text{if } n \equiv 0 \pmod{8}, \\ -15 \, \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), & \text{if } n \equiv 4 \pmod{8}. \end{cases}$$

Several lemmas are needed to prove this identity. We set

$$(8.6.1) \quad A := \varphi(q),$$

$$(8.6.2) \quad B := \varphi(-q).$$

With this notation, the identities of Lemmas 2.3.1 and 2.3.2, give

$$(8.6.3) \quad \varphi(q^2) = \left(\frac{A^2 + B^2}{2} \right)^{1/2},$$

$$(8.6.4) \quad \varphi(-q^2) = (AB)^{1/2},$$

$$(8.6.5) \quad \varphi(q^4) = \frac{A+B}{2},$$

$$(8.6.6) \quad \varphi(-q^4) = \left(\frac{AB(A^2+B^2)}{2} \right)^{1/4},$$

$$(8.6.7) \quad \varphi(iq) = \frac{(A+B)}{2} + i \frac{(A-B)}{2},$$

$$(8.6.8) \quad \varphi(-iq) = \frac{(A+B)}{2} - i \frac{(A-B)}{2}.$$

The following lemma express $\varphi(\omega q), \varphi(i\omega q), \varphi(-\omega q)$ and $\varphi(-i\omega q)$ (where $\omega = e^{\frac{2\pi i}{8}}$) in terms of A and B .

Lemma 8.6.1. *For $q \in \mathbb{C}$ such that $|q| < 1$ and $\omega = e^{\frac{2\pi i}{8}} = \frac{1+i}{\sqrt{2}}$, we have*

$$(i) \quad \varphi(\omega q) = \left(\frac{AB(A^2+B^2)}{2} \right)^{1/4} + \frac{\omega}{2}(A-B),$$

$$(ii) \quad \varphi(i\omega q) = \left(\frac{AB(A^2+B^2)}{2} \right)^{1/4} + \frac{i\omega}{2}(A-B),$$

$$(iii) \quad \varphi(-\omega q) = \left(\frac{AB(A^2+B^2)}{2} \right)^{1/4} - \frac{\omega}{2}(A-B),$$

$$(iv) \quad \varphi(-i\omega q) = \left(\frac{AB(A^2+B^2)}{2} \right)^{1/4} - \frac{i\omega}{2}(A-B).$$

Proof. We only prove (i) as the proofs of (ii), (iii) and (iv) are similar. By (8.6.5)

and (8.6.6) we have

$$\begin{aligned}
\varphi(\omega q) &= \sum_{n=-\infty}^{+\infty} (\omega q)^{n^2} = \sum_{n=-\infty}^{+\infty} \omega^{n^2} q^{n^2} = \sum_{\substack{n=-\infty \\ n \equiv 0 \pmod{2}}}^{+\infty} \omega^{n^2} q^{n^2} + \sum_{\substack{n=-\infty \\ n \equiv 1 \pmod{2}}}^{+\infty} \omega^{n^2} q^{n^2} \\
&= \sum_{n=-\infty}^{+\infty} (-q^4)^{n^2} + \omega \sum_{\substack{n=-\infty \\ n \equiv 1 \pmod{2}}}^{+\infty} q^{n^2} = \varphi(-q^4) + \omega \left(\sum_{n=-\infty}^{+\infty} q^{n^2} - \sum_{\substack{n=-\infty \\ n \equiv 0 \pmod{2}}}^{+\infty} q^{n^2} \right) \\
&= \varphi(-q^4) + \omega (\varphi(q) - \varphi(q^4)) = \varphi(-q^4) + \omega \left(\frac{\varphi(q) - \varphi(-q)}{2} \right) \\
&= \left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} + \frac{\omega}{2} (A - B),
\end{aligned}$$

as asserted. $\square$

The next lemma expresses $\Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; q\right)$ and $\Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -q\right)$ in terms of A and B .

Lemma 8.6.2. *For $q \in D_\delta$ we have*

$$\begin{aligned}
\text{(i)} \quad \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; q\right) &= \frac{1}{16} A^{15} B - \frac{1}{16} A^{11} B^5, \\
\text{(ii)} \quad \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -q\right) &= -\frac{1}{16} A^5 B^{11} + \frac{1}{16} A B^{15}.
\end{aligned}$$

Proof. We only prove (i) as the proof of (ii) is similar. By Definition 2.3.2, Theorem 2.10.2, Theorem 2.4.6(i), (8.6.1) and (8.6.2), we have

$$\begin{aligned}
\Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; q\right) &= \frac{x(1-x)^{1/4}}{16} z^8 = \frac{1}{16} \left(1 - \frac{\varphi^4(-q)}{\varphi^4(q)} \right) \frac{\varphi(-q)}{\varphi(q)} \varphi^{16}(q) \\
&= \frac{1}{16} (\varphi^4(q) - \varphi^4(-q)) \varphi(-q) \varphi^{11}(q) = \frac{1}{16} (\varphi^{15}(q) \varphi(-q) - \varphi^{11}(q) \varphi^5(-q))
\end{aligned}$$

$$= \frac{1}{16}A^{15}B - \frac{1}{16}A^{11}B^5,$$

as required. $\square$

The next lemma expresses $\Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; iq\right) + \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -iq\right)$ in terms of A and B .

Lemma 8.6.3. *For $q \in D_\delta$ we have*

$$\begin{aligned} & \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; iq\right) + \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -iq\right) \\ &= \frac{1}{256} \left(-A^{15}B + 45A^{13}B^3 - 209A^{11}B^5 + 165A^9B^7 + 165A^7B^9 \right. \\ & \quad \left. - 209A^5B^{11} + 45A^3B^{13} - AB^{15} \right). \end{aligned}$$

Proof. Replacing q by iq and $-q$ by $-iq$ in Lemma 8.6.2 and using (8.6.7) and (8.6.8), we have

$$\begin{aligned} & \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; iq\right) + \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -iq\right) \\ &= \frac{1}{16}\varphi^{15}(iq)\varphi(-iq) - \frac{1}{16}\varphi^{11}(iq)\varphi^5(-iq) + \frac{1}{16}\varphi^{15}(-iq)\varphi(iq) - \frac{1}{16}\varphi^{11}(-iq)\varphi^5(iq) \\ &= \frac{1}{16} \left(\left(\frac{(A+B)}{2} + i\frac{(A-B)}{2} \right)^{15} \left(\frac{(A+B)}{2} - i\frac{(A-B)}{2} \right) \right. \\ & \quad - \frac{1}{16} \left(\left(\frac{(A+B)}{2} + i\frac{(A-B)}{2} \right)^{11} \left(\frac{(A+B)}{2} - i\frac{(A-B)}{2} \right)^5 \right. \\ & \quad + \frac{1}{16} \left(\left(\frac{(A+B)}{2} - i\frac{(A-B)}{2} \right)^{15} \left(\frac{(A+B)}{2} + i\frac{(A-B)}{2} \right) \right. \\ & \quad \left. \left. - \frac{1}{16} \left(\left(\frac{(A+B)}{2} - i\frac{(A-B)}{2} \right)^{11} \left(\frac{(A+B)}{2} + i\frac{(A-B)}{2} \right)^5 \right) \right). \end{aligned}$$

Simplifying the last expression with MAPLE, we obtain

$$\begin{aligned} & \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; iq\right) + \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -iq\right) \\ &= \frac{1}{256} \left(-A^{15}B + 45A^{13}B^3 - 209A^{11}B^5 + 165A^9B^7 + 165A^7B^9 \right. \\ & \qquad \qquad \qquad \left. - 209A^5B^{11} + 45A^3B^{13} - AB^{15} \right), \end{aligned}$$

which is the desired result. □

Lemmas 8.6.2 and 8.6.3 allow us to prove the following result.

Lemma 8.6.4. *For $q \in D_\delta$ we have*

$$\begin{aligned} & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) q^n \\ &= \frac{1}{1024} \left(15A^{15}B + 45A^{13}B^3 - 225A^{11}B^5 + 165A^9B^7 + 165A^7B^9 \right. \\ & \qquad \qquad \qquad \left. - 225A^5B^{11} + 45A^3B^{13} + 15AB^{15} \right). \end{aligned}$$

Proof. From Lemmas 8.6.2 and 8.6.3, we have

$$\begin{aligned} & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) q^n \\ &= \frac{1}{4} \sum_{n=0}^{+\infty} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) (1 + (-1)^n + i^n + (-i)^n) q^n \\ &= \frac{1}{4} \left(\Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; q\right) + \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -q\right) + \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; iq\right) \right. \\ & \qquad \qquad \qquad \left. + \Delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -iq\right) \right) \end{aligned}$$

$$\begin{aligned}
& + \Delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; -iq \right) \\
& = \frac{1}{64} A^{15} B - \frac{1}{64} A^{11} B^5 - \frac{1}{64} A^5 B^{11} + \frac{1}{64} A B^{15} \\
& + \frac{1}{1024} \left(-A^{15} B + 45 A^{13} B^3 - 209 A^{11} B^5 + 165 A^9 B^7 + 165 A^7 B^9 \right. \\
& \qquad \qquad \qquad \left. - 209 A^5 B^{11} + 45 A^3 B^{13} - A B^{15} \right) \\
& = \frac{1}{1024} \left(15 A^{15} B + 45 A^{13} B^3 - 225 A^{11} B^5 + 165 A^9 B^7 + 165 A^7 B^9 \right. \\
& \qquad \qquad \qquad \left. - 225 A^5 B^{11} + 45 A^3 B^{13} + 15 A B^{15} \right),
\end{aligned}$$

which is the desired result. $\square$.

The next lemma expresses $\Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; q \right)$ and $\Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -q \right)$ in terms of A and B .

Lemma 8.6.5. *For $q \in \mathbb{C}$ such that $|q| < 1$ we have*

$$(i) \quad \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; q \right) = \frac{1}{16} A^{12} B^4 - \frac{1}{16} A^8 B^8,$$

$$(ii) \quad \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -q \right) = -\frac{1}{16} A^8 B^8 + \frac{1}{16} A^4 B^{12}.$$

Proof. We only prove (i) as the proof of (ii) is similar. By Definition 2.3.2, Theorem 2.10.2, Theorem 2.4.6(i), (8.6.1) and (8.6.2), we have

$$\begin{aligned}
\Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; q \right) &= \frac{x(1-x)}{16} z^8 = \frac{1}{16} \left(1 - \frac{\varphi^4(-q)}{\varphi^4(q)} \right) \frac{\varphi^4(-q)}{\varphi^4(q)} \varphi^{16}(q) \\
&= \frac{1}{16} (\varphi^{12}(q) \varphi^4(-q) - \varphi^8(q) \varphi^8(-q))
\end{aligned}$$

$$= \frac{1}{16}A^{12}B^4 - \frac{1}{16}A^8B^8,$$

as claimed. $\square$

The next lemma expresses $\Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; iq\right) + \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -iq\right)$ in terms of A and B .

Lemma 8.6.6. *For $q \in \mathbb{C}$ such that $|q| < 1$ we have*

$$\begin{aligned} & \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; iq\right) + \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -iq\right) \\ &= \frac{1}{64} \left(-A^{14}B^2 - 2A^{12}B^4 + A^{10}B^6 + 4A^8B^8 + A^6B^{10} - 2A^4B^{12} - A^2B^{14} \right). \end{aligned}$$

Proof. Replacing q by iq and $-q$ by $-iq$ in Lemma 8.6.5 and using (8.6.7) and (8.6.8), we have

$$\begin{aligned} & \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; iq\right) + \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -iq\right) \\ &= \frac{1}{16} \left(\varphi^{12}(iq)\varphi^4(-iq) - 2\varphi^8(iq)\varphi^8(-iq) + \varphi^{12}(-iq)\varphi^4(iq) \right) \\ &= \frac{1}{16} \left(\left(\frac{(A+B)}{2} + i\frac{(A-B)}{2} \right)^{12} \left(\frac{(A+B)}{2} - i\frac{(A-B)}{2} \right)^4 \right. \\ & \quad \left. - 2 \left(\frac{(A+B)}{2} + i\frac{(A-B)}{2} \right)^8 \left(\frac{(A+B)}{2} - i\frac{(A-B)}{2} \right)^8 \right. \\ & \quad \left. + \left(\frac{(A+B)}{2} + i\frac{(A-B)}{2} \right)^4 \left(\frac{(A+B)}{2} - i\frac{(A-B)}{2} \right)^{12} \right). \end{aligned}$$

Simplifying the last expression with MAPLE, we obtain

$$\begin{aligned}
& \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; iq\right) + \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -iq\right) \\
&= \frac{1}{64} \left(-A^{14}B^2 - 2A^{12}B^4 + A^{10}B^6 + 4A^8B^8 + A^6B^{10} - 2A^4B^{12} - A^2B^{14} \right),
\end{aligned}$$

which is the desired result. $\square$

Lemmas 8.6.5 and 8.6.6 allow us to prove the following result.

Lemma 8.6.7. *For $q \in \mathbb{C}$ such that $|q| < 1$ we have*

$$\begin{aligned}
& \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) q^n \\
&= \frac{1}{256} \left(-A^{14}B^2 + 2A^{12}B^4 + A^{10}B^6 - 4A^8B^8 + A^6B^{10} + 2A^4B^{12} - A^2B^{14} \right).
\end{aligned}$$

Proof. From Lemmas 8.6.5 and 8.6.6, we have

$$\begin{aligned}
& \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) q^n \\
&= \frac{1}{4} \sum_{n=0}^{+\infty} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) (1 + (-1)^n + i^n + (-i)^n) q^n \\
&= \frac{1}{4} \left(\Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; q\right) + \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -q\right) + \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; iq\right) \right. \\
&\quad \left. + \Delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -iq\right) \right) \\
&= \frac{1}{64} A^{12}B^4 - \frac{1}{64} A^8B^8 - \frac{1}{64} A^8B^8 + \frac{1}{64} A^4B^{12} \\
&+ \frac{1}{256} \left(-A^{14}B^2 - 2A^{12}B^4 + A^{10}B^6 + 4A^8B^8 + A^6B^{10} - 2A^4B^{12} - A^2B^{14} \right) \\
&= \frac{1}{256} \left(-A^{14}B^2 + 2A^{12}B^4 + A^{10}B^6 - 4A^8B^8 + A^6B^{10} + 2A^4B^{12} - A^2B^{14} \right),
\end{aligned}$$

which is the desired result. $\square$.

Lemma 8.6.8. *For $q \in \mathbb{C}$ such that $|q| < 1$ we have*

$$\begin{aligned} & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} (-1)^{n/4} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) q^n \\ &= \frac{1}{2048} \left(A^{15}B + 4A^{14}B^2 + 3A^{13}B^3 - 8A^{12}B^4 - 15A^{11}B^5 - 4A^{10}B^6 + 11A^9B^7 \right. \\ & \quad \left. + 16A^8B^8 + 11A^7B^9 - 4A^6B^{10} - 15A^5B^{11} - 8A^4B^{12} + 3A^3B^{13} \right. \\ & \quad \left. + 4A^2B^{14} + AB^{15} \right). \end{aligned}$$

Proof. From Lemma 8.6.7, (8.6.1) and (8.6.2), we have

$$\begin{aligned} & \left(\Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; q \right) + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -q \right) + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; iq \right) \right. \\ & \quad \left. + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -iq \right) \right) \\ &= \frac{1}{64} \left(-A^{14}B^2 + 2A^{12}B^4 + A^{10}B^6 - 4A^8B^8 + A^6B^{10} + 2A^4B^{12} - A^2B^{14} \right) \\ &= \frac{1}{64} \left(-\varphi^{14}(q)\varphi^2(-q) + 2\varphi^{12}(q)\varphi(-q)^4 + \varphi(q)^{10}\varphi(-q)^6 - 4\varphi(q)^8\varphi^8(-q) \right. \\ & \quad \left. + \varphi^6(q)\varphi^{10}(-q) + 2\varphi^4(q)\varphi^{12}(-q) - \varphi^2(q)\varphi^{14}(-q) \right). \end{aligned}$$

Replacing q by ωq in the last equations and using Lemma 8.6.1, we obtain

$$\begin{aligned}
& \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} (-1)^{n/4} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) q^n \\
&= \frac{1}{4} \left(\Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; \omega q \right) + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -\omega q \right) \right. \\
&\quad \left. + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; i\omega q \right) + \Delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; -i\omega q \right) \right) \\
&= \frac{1}{256} \left(- \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^{14} \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^2 \right. \\
&\quad + 2 \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^{12} \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^4 \\
&\quad + \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^{10} \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^6 \\
&\quad - 4 \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^8 \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^8 \\
&\quad + \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^6 \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^{10} \\
&\quad + 2 \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^4 \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^{12} \\
&\quad \left. - \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^2 \left(\left(\frac{AB(A^2 + B^2)}{2} \right)^{1/4} \omega (A - B) \right)^{14} \right) \\
&= \frac{1}{2048} \left(A^{15}B + 4A^{14}B^2 + 3A^{13}B^3 - 8A^{12}B^4 - 15A^{11}B^5 - 4A^{10}B^6 + 11A^9B^7 \right. \\
&\quad + 16A^8B^8 + 11A^7B^9 - 4A^6B^{10} - 15A^5B^{11} - 8A^4B^{12} + 3A^3B^{13} + 4A^2B^{14} \\
&\quad \left. + AB^{15} \right),
\end{aligned}$$

where the last equation was found using MAPLE. $\square$

Lemma 8.6.9. For $q \in \mathbb{C}$ such that $|q| < 1$ we have

$$\begin{aligned} & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} (15 + 30(-1)^{n/4}) \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) q^n \\ &= \frac{1}{1024} \left(15A^{15}B + 45A^{13}B^3 - 225A^{11}B^5 + 165A^9B^7 + 165A^7B^9 - 225A^5B^{11} \right. \\ & \qquad \qquad \qquad \left. + 45A^3B^{13} + 15AB^{15} \right). \end{aligned}$$

Proof. By Lemmas 8.6.7 and 8.6.8 we have

$$\begin{aligned} & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} (15 + 30(-1)^{n/4}) \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) q^n \\ &= 15 \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) q^n + 30 \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} (-1)^{n/4} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) q^n \\ &= \frac{15}{256} \left(-A^{14}B^2 + 2A^{12}B^4 + A^{10}B^6 - 4A^8B^8 + A^6B^{10} + 2A^4B^{12} - A^2B^{14} \right) \\ & \quad + \frac{30}{2048} \left(A^{15}B + 4A^{14}B^2 + 3A^{13}B^3 - 8A^{12}B^4 - 15A^{11}B^5 - 4A^{10}B^6 + 11A^9B^7 \right. \\ & \qquad \qquad \qquad + 16A^8B^8 + 11A^7B^9 - 4A^6B^{10} - 15A^5B^{11} - 8A^4B^{12} + 3A^3B^{13} + 4A^2B^{14} \\ & \qquad \qquad \qquad \left. + AB^{15} \right). \end{aligned}$$

Simplifying the last expression with MAPLE, we obtain

$$\begin{aligned} & \frac{1}{1024} \left(15A^{15}B + 45A^{13}B^3 - 225A^{11}B^5 + 165A^9B^7 + 165A^7B^9 - 225A^5B^{11} \right. \\ & \qquad \qquad \qquad \left. + 45A^3B^{13} + 15AB^{15} \right), \end{aligned}$$

which gives the desired result. $\square$

We are now ready to prove Theorem 8.6.1.

Proof of Theorem 8.6.1 By Lemmas 8.6.4 and 8.6.9, we have

$$\begin{aligned}
 & \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) q^n \\
 &= \frac{1}{1024} \left(15A^{15}B + 45A^{13}B^3 - 225A^{11}B^5 + 165A^9B^7 + 165A^7B^9 \right. \\
 & \qquad \qquad \qquad \left. - 225A^5B^{11} + 45A^3B^{13} + 15AB^{15} \right) \\
 &= \sum_{\substack{n=0 \\ n \equiv 0 \pmod{4}}}^{+\infty} (15 + 30(-1)^{n/4}) \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) q^n.
 \end{aligned}$$

Equating coefficients of q^n we obtain

$$\delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) = (15 + 30(-1)^{n/4}) \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \text{ when } n \equiv 0 \pmod{4},$$

which is the desired result. $\square$

Chapter 9

Sums with 8, 12 and 16 squares - II

In Chapter 7, we saw the application of our main results (Theorems 6.1.3, 6.2.3 and 6.3.3) to forms involving 8, 12 or 16 squares. Inevitably these results depend on a certain number of arithmetic functions $\delta(a, b, c; n)$. The purpose of this chapter is to use the identities found in Chapter 8 to reduce the number of functions $\delta(a, b, c; n)$ occurring in the formulae in the theorems of Chapter 7.

In Section 1 we show that when n is even certain of the formulae of Theorems 7.1.1 - 7.1.3 depend only on $\sigma_3(n/2), \dots, \sigma_3(n/16)$, whereas other require in addition the quantity $\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right)$. These results reprove those of Alaca, Alaca and Williams [5], [11] in a different way.

In Section 2 we show that when n is even certain of the formulae of Theorems 7.2.1 - 7.2.3 depend only on $\sigma_5(n/2), \dots, \sigma_5(n/16)$, whereas others require additionally the quantity $\tau_2(n)$. These results are new. For example we establish in Theorem 9.2.1 the elegant formula

$$\begin{aligned} \text{card} \left\{ (x_1, \dots, x_{12}) \in \mathbb{Z}^{12} \mid n = x_1^2 + \dots + x_6^2 + 2x_7^2 + \dots + 2x_{12}^2 \right\} \\ = 32 \sigma_5(n/2) - 24 \sigma_5(n/4) - 512 \sigma_5(n/8) + 40 \tau_2(n/2), \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \equiv 0 \pmod{2}$, which is an analogue of Ramanujan's beautiful formula for $r_{24}(n)$ (see (1.0.8)).

In Section 3 we do the same for Theorems 7.3.1 - 7.3.3. The results obtained are new. The following result from Theorem 9.3.2 illustrates the type of results obtained. For $n \in \mathbb{N}$ with $n \equiv 0 \pmod{8}$ we show in Theorem 9.3.2 that

$$\begin{aligned} & \text{card} \left\{ (x_1, \dots, x_{16}) \in \mathbb{Z}^{16} \mid n = x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2 + 4x_5^2 + \dots + 4x_{16}^2 \right\} \\ &= 480 \sigma_7(n/8) + \frac{3}{8} \sum_{\substack{(x_1, \dots, x_8) \in \mathbb{Z}^8 \\ x_1^2 + \dots + x_8^2 = n}} (x_1^4 - 3x_1^2 x_2^2). \end{aligned}$$

The following short table gives the values occurring in this formula for $n = 8, 16, 24, 32, 40$.

n	$N([2, 2, 12]; n)$	$\sigma_7(n/8)$	$\sum_{\substack{(x_1, \dots, x_8) \in \mathbb{Z}^8 \\ x_1^2 + \dots + x_8^2 = n}} (x_1^4 - 3x_1^2 x_2^2)$
8	864	1	1024
16	58848	129	-8192
24	1054848	2188	12288
32	7950816	16513	65536
40	37419840	78126	-215040

9.1 Sums with 8 squares

In this section we restrict n to particular congruence classes modulo 2 and modulo 4 to reduce the number of delta functions $\delta(a, b, c; n)$ occurring in Theorems 7.1.1, 7.1.2 and 7.1.3.

We make use of the following simple properties of σ_3 . Let $n \in \mathbb{N}$.

If $n \equiv 0 \pmod{2}$ then

$$(9.1.1) \quad \sigma_3(n) = 9\sigma_3(n/2) - 8\sigma_3(n/4),$$

if $n \equiv 2 \pmod{4}$ then

$$(9.1.2) \quad \sigma_3(n) = 9\sigma_3(n/2),$$

and if $n \equiv 0 \pmod{4}$ then

$$(9.1.3) \quad \begin{cases} \sigma_3(n) = 73\sigma_3(n/4) - 72\sigma_3(n/8), \\ \sigma_3(n/2) = 9\sigma_3(n/4) - 8\sigma_3(n/8). \end{cases}$$

We begin with the simplification of Theorem 7.1.1 to the cases $n \equiv 0 \pmod{2}$ and $n \equiv 2 \pmod{4}$.

Theorem 9.1.1. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{2}$, we have*

$$N([6, 2, 0]; n) = 64 \sigma_3(n/2) - 80 \sigma_3(n/4) + 256 \sigma_3(n/8),$$

$$N([4, 4, 0]; n) = 32 \sigma_3(n/2) - 48 \sigma_3(n/4) + 256 \sigma_3(n/8),$$

$$N([2, 6, 0]; n) = 16 \sigma_3(n/2) - 32 \sigma_3(n/4) + 256 \sigma_3(n/8).$$

Proof. The results follow by applying Theorem 8.3.1 to Theorem 7.1.1 and simplifying using (9.1.1). $\square$

This theorem follows from Alaca, Alaca and Williams [11, Theorem 1.1 (ii)(iv)(ix), pp. 38-39] as $c_8(n) = 0$ for $n \equiv 0 \pmod{2}$. The second formula also follows from Williams [83, Theorem 2, p. 388].

Corollary 9.1.1. *Let $n \in \mathbb{N}$. If $n \equiv 0 \pmod{2}$ then*

$$\begin{aligned} & \text{card} \left\{ (x_1, \dots, x_8) \in \mathbb{Z}^8 \mid n = x_1^2 + x_2^2 + 2x_3^2 + \dots + 2x_8^2, x_1 \equiv x_2 \equiv 1 \pmod{2} \right\} \\ & = 16 \sigma_3(n/2) - 96 \sigma_3(n/4) + 336 \sigma_3(n/8) - 256 \sigma_3(n/16). \end{aligned}$$

Proof. As $n \equiv 0 \pmod{2}$ the solutions of

$$n = x_1^2 + x_2^2 + 2x_3^2 + \cdots + 2x_8^2$$

satisfy $x_1 \equiv x_2 \equiv 0 \pmod{2}$ or $x_1 \equiv x_2 \equiv 1 \pmod{2}$. The former satisfy

$$\frac{n}{2} = x_3^2 + \cdots + x_8^2 + 2 \left(\frac{x_1}{2} \right)^2 + 2 \left(\frac{x_2}{2} \right)^2$$

so there are $N([6, 2, 0]; n/2)$ such solutions.

Hence the required number is

$$\begin{aligned} N([2, 0, 6]; n) - N([6, 2, 0]; n/2) \\ = 16 \sigma_3(n/2) - 96 \sigma_3(n/4) + 336 \sigma_3(n/8) - 256 \sigma_3(n/16) \end{aligned}$$

by Theorem 9.1.1. □

Theorem 9.1.2. *Let $n \in \mathbb{N}$. For $n \equiv 2 \pmod{4}$, we have*

$$N([6, 2, 0]; n) = 64 \sigma_3(n/2),$$

$$N([4, 4, 0]; n) = 32 \sigma_3(n/2),$$

$$N([2, 6, 0]; n) = 16 \sigma_3(n/2).$$

Proof. This follows from Theorem 9.1.1, the facts that $\sigma_3(n/4)$ and $\sigma_3(n/8)$ are both equal to 0 when $n \equiv 2 \pmod{4}$, and (9.1.2). □

The next result shows that Theorem 7.1.2 can be written in terms of the function $\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right)$. This theorem is due to Alaca, Alaca and Williams [11, Theorem 1.1 (iii)(v)(vii)(viii)(x)(xi)(xii)(xiii)(xiv), pp. 38-39].

Theorem 9.1.3. *For $n \in \mathbb{N}$ we have*

$$N([1, 2, 5]; n) = \frac{1}{4}\sigma_3(n) - \frac{1}{4}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \left(\frac{3}{4} + \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 2 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([2, 2, 4]; n) = \frac{1}{2}\sigma_3(n) - \frac{1}{2}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \left(\frac{3}{2} + 2 \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([3, 2, 3]; n) = \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \left(2 + 3 \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 8 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([4, 2, 2]; n) = 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \left(2 + 4 \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 12 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([5, 2, 1]; n) = 4\sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \left(2 + 4 \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 12 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([1, 4, 3]; n) = \frac{1}{2}\sigma_3(n) - \frac{1}{2}\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \left(\frac{1}{2} + \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([2, 4, 2]; n) = \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \left(1 + 2 \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([3, 4, 1]; n) = 2\sigma_3(n) - 2\sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \left(2 + 2 \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([1, 6, 1]; n) = \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ + \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right).$$

Proof. We only prove the formula for $N([3, 2, 3]; n)$ as the proofs for the other cases are similar. By Theorems 7.1.2, 8.3.2 and 8.3.3 we have

$$\begin{aligned}
N([3, 2, 3]; n) &= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\
&\quad + 2\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + \frac{5}{2}\delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) \\
&\quad + \frac{1}{2}\delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right) \\
&= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) + 2\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) \\
&\quad + \left(\delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) - \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right)\right) \\
&\quad + \left(\frac{3}{2}\delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) + \frac{3}{2}\delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right)\right) \\
&= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) + 2\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) \\
&\quad + 8\delta(0, 1/6, 1/6; n/2) + 3\left(\frac{-4}{n}\right)\delta(0, 1/6, 1/6; n) \\
&= \sigma_3(n) - \sigma_3(n/2) - 16\sigma_3(n/8) + 256\sigma_3(n/16) \\
&\quad + \left(2 + 3\left(\frac{-4}{n}\right)\right)\delta(0, 1/6, 1/6; n) + 8\delta(0, 1/6, 1/6; n/2),
\end{aligned}$$

as asserted. □

Restricting Theorem 9.1.3 to the special cases $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$, we obtain the following two results.

Theorem 9.1.4. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{4}$, we have*

$$N([1, 2, 5]; n) = 16\sigma_3(n/4) - 32\sigma_3(n/8) + 256\sigma_3(n/16),$$

$$N([2, 2, 4]; n) = 32\sigma_3(n/4) - 48\sigma_3(n/8) + 256\sigma_3(n/16),$$

$$N([3, 2, 3]; n) = 64 \sigma_3(n/4) - 80 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([4, 2, 2]; n) = 128 \sigma_3(n/4) - 144 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([5, 2, 1]; n) = 256 \sigma_3(n/4) - 272 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([1, 4, 3]; n) = 32 \sigma_3(n/4) - 48 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([2, 4, 2]; n) = 64 \sigma_3(n/4) - 80 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([3, 4, 1]; n) = 128 \sigma_3(n/4) - 144 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([1, 6, 1]; n) = 64 \sigma_3(n/4) - 80 \sigma_3(n/8) + 256 \sigma_3(n/16).$$

Proof. This follows from Theorem 8.3.1, Theorem 9.1.3 and (9.1.3). □

Theorem 9.1.5. *Let $n \in \mathbb{N}$. For $n \equiv 2 \pmod{4}$ we have*

$$N([1, 2, 5]; n) = 2 \sigma_3(n/2) + 2 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([2, 2, 4]; n) = 4 \sigma_3(n/2) + 4 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([3, 2, 3]; n) = 8 \sigma_3(n/2) + 8 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([4, 2, 2]; n) = 16 \sigma_3(n/2) + 12 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([5, 2, 1]; n) = 32 \sigma_3(n/2) + 12 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([1, 4, 3]; n) = 4 \sigma_3(n/2) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([2, 4, 2]; n) = 8 \sigma_3(n/2) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([3, 4, 1]; n) = 16 \sigma_3(n/2) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([1, 6, 1]; n) = 8 \sigma_3(n/2) + 4 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right).$$

Proof. The results follow from Theorem 8.3.1, Theorem 9.1.3, the fact that $\sigma_3(n/8)$ and $\sigma_3(n/16)$ are both equal to 0 when $n \equiv 2 \pmod{4}$, and the use of (9.1.2) for simplification. $\square$

The next result shows that Theorem 7.1.3 can be written in terms of the function $\delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right)$ for all $n \in \mathbb{N}$. This theorem is due to Alaca, Alaca and Williams [5, Theorem 1.1, pp. 343-344].

Theorem 9.1.6. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} N([1, 0, 7]; n) &= \left(\frac{1}{8} + \frac{1}{8} \left(\frac{-4}{n}\right)\right) \sigma_3(n) - \frac{9}{8} \sigma_3(n/2) + 17 \sigma_3(n/4) - 32 \sigma_3(n/8) \\ &\quad + 256 \sigma_3(n/16) + \left(\frac{7}{8} + \frac{7}{8} \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right), \end{aligned}$$

$$\begin{aligned} N([2, 0, 6]; n) &= \left(\frac{1}{4} + \frac{1}{4} \left(\frac{-4}{n}\right)\right) \sigma_3(n) - \frac{1}{4} \sigma_3(n/2) - 16 \sigma_3(n/8) \\ &\quad + 256 \sigma_3(n/16) + \left(\frac{7}{4} + \frac{7}{4} \left(\frac{-4}{n}\right)\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) \\ &\quad + 2 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right), \end{aligned}$$

$$\begin{aligned}
N([3, 0, 5]; n) &= \left(\frac{1}{2} + \frac{1}{4} \left(\frac{-4}{n} \right) \right) \sigma_3(n) + \frac{3}{2} \sigma_3(n/2) - 34 \sigma_3(n/4) + 16 \sigma_3(n/8) \\
&\quad + 256 \sigma_3(n/16) + \left(\frac{5}{2} + \frac{11}{4} \left(\frac{-4}{n} \right) \right) \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) \\
&\quad + 6 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2 \right),
\end{aligned}$$

$$\begin{aligned}
N([4, 0, 4]; n) &= \sigma_3(n) + 3 \sigma_3(n/2) - 68 \sigma_3(n/4) + 48 \sigma_3(n/8) \\
&\quad + 256 \sigma_3(n/16) + \left(3 + 4 \left(\frac{-4}{n} \right) \right) \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) \\
&\quad + 12 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2 \right),
\end{aligned}$$

$$\begin{aligned}
N([5, 0, 3]; n) &= \left(2 - \frac{1}{2} \left(\frac{-4}{n} \right) \right) \sigma_3(n) + 2 \sigma_3(n/2) - 68 \sigma_3(n/4) + 48 \sigma_3(n/8) \\
&\quad + 256 \sigma_3(n/16) + \left(3 + \frac{11}{2} \left(\frac{-4}{n} \right) \right) \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) \\
&\quad + 20 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2 \right),
\end{aligned}$$

$$\begin{aligned}
N([6, 0, 2]; n) &= \left(4 - \left(\frac{-4}{n} \right) \right) \sigma_3(n) - 4 \sigma_3(n/2) - 16 \sigma_3(n/8) \\
&\quad + 256 \sigma_3(n/16) + \left(2 + 7 \left(\frac{-4}{n} \right) \right) \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) \\
&\quad + 28 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2 \right),
\end{aligned}$$

$$\begin{aligned}
N([7, 0, 1]; n) &= \left(8 - \left(\frac{-4}{n} \right) \right) \sigma_3(n) - 16 \sigma_3(n/2) + 136 \sigma_3(n/4) - 144 \sigma_3(n/8) \\
&\quad + 256 \sigma_3(n/16) + 7 \left(\frac{-4}{n} \right) \delta \left(0, \frac{1}{6}, \frac{1}{6}; n \right) \\
&\quad + 28 \delta \left(0, \frac{1}{6}, \frac{1}{6}; n/2 \right).
\end{aligned}$$

Proof. We only prove the formula for $N([2, 0, 6]; n)$ as the proofs for the other cases

are similar. By Theorems 7.1.1, 8.3.2, 8.3.3 and 8.3.4, we have

$$\begin{aligned}
& N([2, 0, 6]; n) \\
&= \frac{1}{4} \sigma_3(n) - \frac{1}{4} \sigma_3(n/2) - 16 \sigma_3(n/8) + 256 \sigma_3(n/16) \\
&\quad + \frac{7}{4} \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) - \frac{1}{16} \delta\left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n\right) + \frac{1}{16} \delta\left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n\right) \\
&\quad + 2 \delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) + \frac{1}{2} \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right) \\
&= \frac{1}{4} \sigma_3(n) - \frac{1}{4} \sigma_3(n/2) - 16 \sigma_3(n/8) + 256 \sigma_3(n/16) \\
&\quad + \frac{7}{4} \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + \left(-\frac{1}{16} \delta\left(-\frac{1}{2}, \frac{17}{12}, -\frac{7}{12}; n\right) + \frac{1}{16} \delta\left(-\frac{1}{6}, \frac{11}{12}, -\frac{5}{12}; n\right)\right. \\
&\quad \left.+ \frac{7}{8} \delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) - \frac{1}{8} \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right)\right) \\
&\quad + \left(\frac{7}{8} \delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) + \frac{7}{8} \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right)\right) \\
&\quad + \left(\frac{1}{4} \delta\left(-\frac{1}{6}, \frac{5}{12}, \frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{12}, \frac{1}{4}; n\right)\right) \\
&= \frac{1}{4} \sigma_3(n) - \frac{1}{4} \sigma_3(n/2) - 16 \sigma_3(n/8) + 256 \sigma_3(n/16) \\
&\quad + \frac{7}{4} \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) + \frac{1}{4} \left(\frac{-4}{n}\right) \sigma_3(n) + \frac{7}{4} \left(\frac{-4}{n}\right) \delta\left(0, \frac{1}{6}, \frac{1}{6}; n\right) \\
&\quad + 2 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),
\end{aligned}$$

which is the desired result. $\square$

We note that when $n \equiv 3 \pmod{4}$ the first assertion of Theorem 9.1.6 gives $N([1, 0, 7]; n) = 0$. This is clear as the congruence

$$x_1^2 + 4x_2^2 + \cdots + 4x_8^2 \equiv n \pmod{4}$$

is insolvable for $n \equiv 3 \pmod{4}$.

In the next two theorems we restrict Theorem 9.1.6 to the cases $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$. It is interesting that in the case $n \equiv 0 \pmod{4}$ (Theorem 9.1.7) all the formulae can be expressed in terms of $\sigma_3(n), \sigma_3(n/2), \dots, \sigma_3(n/16)$ only.

Theorem 9.1.7. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{4}$, we have*

$$N([1, 0, 7]; n) = 16 \sigma_3(n/4) - 32 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([2, 0, 6]; n) = 16 \sigma_3(n/4) - 32 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([3, 0, 5]; n) = 16 \sigma_3(n/4) - 32 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([4, 0, 4]; n) = 32 \sigma_3(n/4) - 48 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([5, 0, 3]; n) = 96 \sigma_3(n/4) - 112 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([6, 0, 2]; n) = 256 \sigma_3(n/4) - 272 \sigma_3(n/8) + 256 \sigma_3(n/16),$$

$$N([7, 0, 1]; n) = 576 \sigma_3(n/4) - 592 \sigma_3(n/8) + 256 \sigma_3(n/16).$$

Proof. The results follow from Theorem 9.1.6, Theorem 8.3.1, the fact that $\left(\frac{-4}{n}\right)$ is equal to 0 when $n \equiv 0 \pmod{4}$, and the use of (9.1.3) for simplification. $\square$

We remark that the formula for $N([1, 0, 7]; n)$ ($n \equiv 0 \pmod{4}$) can be derived as follows from that for $r_8(n)$: we have

$$n = x_1^2 + 4x_2^2 + \dots + 4x_8^2 \iff \frac{n}{4} = \left(\frac{x_1}{2}\right)^2 + x_2^2 + \dots + x_8^2$$

so that

$$N([1, 0, 7]; n) = r_8(n/4) = 16 \sigma_3(n/4) - 32 \sigma_3(n/8) + 256 \sigma_3(n/16).$$

Similarly the formula for $N([2, 0, 6]; n)$ ($n \equiv 0 \pmod{4}$) can also be derived from that for $r_8(n)$. We have

$$n = x_1^2 + x_2^2 + 4x_3^2 + \cdots + \cdots + 4x_8^2 \iff \frac{n}{4} = \left(\frac{x_1}{2}\right)^2 + \left(\frac{x_2}{2}\right)^2 x_2^2 + x_3^2 + \cdots + x_8^2$$

so that

$$N([2, 0, 6]; n) = r_8(n/4) = 16 \sigma_3(n/4) - 32 \sigma_3(n/8) + 256 \sigma_3(n/16).$$

The same argument applies to $N([3, 0, 5]; n)$ as $x_1^2 + x_2^2 + x_3^2 \equiv 0 \pmod{4}$ implies $x_1 \equiv x_2 \equiv x_3 \equiv 0 \pmod{2}$. However we cannot apply the same argument to $N([4, 4, 0]; n)$ as $x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv 0 \pmod{4}$ does not imply $x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv 0 \pmod{2}$.

Theorem 9.1.8. *Let $n \in \mathbb{N}$. For $n \equiv 2 \pmod{4}$, we have*

$$N([1, 0, 7]; n) = 0,$$

$$N([2, 0, 6]; n) = 2 \sigma_3(n/2) + 2 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([3, 0, 5]; n) = 6 \sigma_3(n/2) + 6 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([4, 0, 4]; n) = 12 \sigma_3(n/2) + 12 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([5, 0, 3]; n) = 20 \sigma_3(n/2) + 20 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([6, 0, 2]; n) = 32 \sigma_3(n/2) + 28 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right),$$

$$N([7, 0, 1]; n) = 56 \sigma_3(n/2) + 28 \delta\left(0, \frac{1}{6}, \frac{1}{6}; n/2\right).$$

Proof. The results follow from Theorem 9.1.6, Theorem 8.3.1, the fact that $\left(\frac{-4}{n}\right), \sigma_3(n/4), \sigma_3(n/8)$ and $\sigma_3(n/16)$ are all equal to 0 when $n \equiv 2 \pmod{4}$, and the use of (9.1.2) for simplification. $\square$

We remark that the first assertion of Theorem 9.1.8 is clear as the congruence

$$n \equiv x_1^2 + 4x_2^2 + \cdots + 4x_8^2 \pmod{4}$$

is insolvable when $n \equiv 2 \pmod{4}$.

9.2 Sums with 12 squares

In this section we restrict n to particular congruence classes modulo 4 and modulo 8 to reduce the number of delta functions $\delta(a, b, c; n)$ occurring in Theorems 7.2.1, 7.2.2 and 7.2.3. The results obtained are new.

We make use of the following simple properties of σ_5 . Let $n \in \mathbb{N}$.

If $n \equiv 0 \pmod{2}$ then

$$(9.2.1) \quad \sigma_5(n) = 33 \sigma_5(n/2) - 32 \sigma_5(n/4),$$

if $n \equiv 2 \pmod{4}$ then

$$(9.2.2) \quad \sigma_5(n) = 33 \sigma_5(n/2),$$

if $n \equiv 0 \pmod{4}$ then

$$(9.2.3) \quad \begin{cases} \sigma_5(n) = 1057 \sigma_5(n/4) - 1056 \sigma_5(n/8), \\ \sigma_5(n/2) = 33 \sigma_5(n/4) - 32 \sigma_5(n/8), \end{cases}$$

and if $n \equiv 0 \pmod{8}$ then

$$(9.2.4) \quad \begin{cases} \sigma_5(n) = 33825 \sigma_5(n/8) - 33824 \sigma_5(n/16), \\ \sigma_5(n/2) = 1057 \sigma_5(n/8) - 1056 \sigma_5(n/16), \\ \sigma_5(n/4) = 33 \sigma_5(n/8) - 32 \sigma_5(n/16). \end{cases}$$

We begin with the simplification of Theorem 7.2.1 when $n \equiv 0 \pmod{2}$. It is interesting that in this case the formulae depend only on $\tau_2(n/2)$ in addition to $\sigma_3(n), \dots, \sigma_3(n/8)$.

Theorem 9.2.1. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{2}$, we have*

$$N([10, 2, 0]; n) = 128 \sigma_5(n/2) - 120 \sigma_5(n/4) - 512 \sigma_5(n/8) + 56 \tau_2(n/2),$$

$$N([8, 4, 0]; n) = 64 \sigma_5(n/2) - 56 \sigma_5(n/4) - 512 \sigma_5(n/8) + 56 \tau_2(n/2),$$

$$N([6, 6, 0]; n) = 32 \sigma_5(n/2) - 24 \sigma_5(n/4) - 512 \sigma_5(n/8) + 40 \tau_2(n/2),$$

$$N([4, 8, 0]; n) = 16 \sigma_5(n/2) - 8 \sigma_5(n/4) - 512 \sigma_5(n/8) + 24 \tau_2(n/2),$$

$$N([2, 10, 0]; n) = 8 \sigma_5(n/2) - 512 \sigma_5(n/8) + 16 \tau_2(n/2).$$

Proof. If $n \equiv 0 \pmod{2}$, we have by Theorems 7.2.1, 8.4.1 and 8.4.2

$$\begin{aligned} N([4, 8, 0]; n) &= \frac{1}{2} \sigma_5(n) - \frac{1}{2} \sigma_5(n/2) + 8 \sigma_5(n/4) - 512 \sigma_5(n/8) \\ &\quad + \frac{7}{2} \delta\left(0, \frac{1}{2}, 0; n\right) + 4 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 8 \delta\left(0, 0, \frac{1}{2}; n\right) \\ &= \frac{1}{2} \sigma_5(n) - \frac{1}{2} \sigma_5(n/2) + 8 \sigma_5(n/4) - 512 \sigma_5(n/8) \end{aligned}$$

$$\begin{aligned}
& + 3 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \\
& = \frac{1}{2} \sigma_5(n) - \frac{1}{2} \sigma_5(n/2) + 8 \sigma_5(n/4) - 512 \sigma_5(n/8) \\
& \quad + 24 \tau_2(n/2).
\end{aligned}$$

Simplifying using (9.2.1), we obtain the desired result. $\square$

In the next two theorems we restrict Theorem 9.2.1 to the cases $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$.

Theorem 9.2.2. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{4}$, we have*

$$N([10, 2, 0]; n) = 4104 \sigma_5(n/4) - 4608 \sigma_5(n/8),$$

$$N([8, 4, 0]; n) = 2056 \sigma_5(n/4) - 2560 \sigma_5(n/8),$$

$$N([6, 6, 0]; n) = 1032 \sigma_5(n/4) - 1536 \sigma_5(n/8),$$

$$N([4, 8, 0]; n) = 520 \sigma_5(n/4) - 1024 \sigma_5(n/8),$$

$$N([2, 10, 0]; n) = 264 \sigma_5(n/4) - 768 \sigma_5(n/8).$$

Proof. For $n \equiv 0 \pmod{4}$ we have $\frac{n}{2} \equiv 0 \pmod{2}$ so $\tau_2\left(\frac{n}{2}\right) = 0$. The theorem now follows from Theorem 9.2.1 and (9.2.3). $\square$

Theorem 9.2.3. *Let $n \in \mathbb{N}$. For $n \equiv 2 \pmod{4}$, we have*

$$N([10, 2, 0]; n) = 128 \sigma_5(n/2) + 56 \tau_2(n/2),$$

$$N([8, 4, 0]; n) = 64 \sigma_5(n/2) + 56 \tau_2(n/2),$$

$$N([6, 6, 0]; n) = 33 \sigma_5(n/2) + 40 \tau_2(n/2),$$

$$N([4, 8, 0]; n) = 16 \sigma_5(n/2) + 24 \tau_2(n/2),$$

$$N([2, 10, 0]; n) = 8 \sigma_5(n/2) + 16 \tau_2(n/2).$$

Proof. This follows immediately from Theorem 9.2.1 as $\sigma_5(n/4) = \sigma_5(n/8) = 0$ when $n \equiv 2 \pmod{4}$. $\square$

In the next three theorems we simplify Theorem 7.2.2 in the cases $n \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{8}$ and $n \equiv 2 \pmod{4}$.

Theorem 9.2.4. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{4}$, we have*

$$N([1, 2, 9]; n) = 8 \sigma_5(n/4) - 512 \sigma_5(n/16) + 16 \tau_2(n/4),$$

$$N([2, 2, 8]; n) = 16 \sigma_5(n/4) - 8 \sigma_5(n/8) - 512 \sigma_5(n/16) + 24 \tau_2(n/4),$$

$$N([3, 2, 7]; n) = 32 \sigma_5(n/4) - 24 \sigma_5(n/8) - 512 \sigma_5(n/16) + 40 \tau_2(n/4),$$

$$N([4, 2, 6]; n) = 64 \sigma_5(n/4) - 56 \sigma_5(n/8) - 512 \sigma_5(n/16) + 72 \tau_2(n/4),$$

$$N([5, 2, 5]; n) = 128 \sigma_5(n/4) - 120 \sigma_5(n/8) - 512 \sigma_5(n/16) + 136 \tau_2(n/4),$$

$$N([6, 2, 4]; n) = 256 \sigma_5(n/4) - 248 \sigma_5(n/8) - 512 \sigma_5(n/16) + 248 \tau_2(n/4),$$

$$N([7, 2, 3]; n) = 512 \sigma_5(n/4) - 504 \sigma_5(n/8) - 512 \sigma_5(n/16) + 408 \tau_2(n/4),$$

$$N([8, 2, 2]; n) = 1024 \sigma_5(n/4) - 1016 \sigma_5(n/8) - 512 \sigma_5(n/16) + 568 \tau_2(n/4),$$

$$N([9, 2, 1]; n) = 2048 \sigma_5(n/4) - 2040 \sigma_5(n/8) - 512 \sigma_5(n/16) + 568 \tau_2(n/4),$$

$$N([1, 4, 7]; n) = 16 \sigma_5(n/4) - 8 \sigma_5(n/8) - 512 \sigma_5(n/16) + 24 \tau_2(n/4),$$

$$N([2, 4, 6]; n) = 32 \sigma_5(n/4) - 24 \sigma_5(n/8) - 512 \sigma_5(n/16) + 40 \tau_2(n/4),$$

$$N([3, 4, 5]; n) = 64 \sigma_5(n/4) - 56 \sigma_5(n/8) - 512 \sigma_5(n/16) + 72 \tau_2(n/4),$$

$$N([4, 4, 4]; n) = 128 \sigma_5(n/4) - 120 \sigma_5(n/8) - 512 \sigma_5(n/16) + 120 \tau_2(n/4),$$

$$N([5, 4, 3]; n) = 256 \sigma_5(n/4) - 248 \sigma_5(n/8) - 512 \sigma_5(n/16) + 184 \tau_2(n/4),$$

$$N([6, 4, 2]; n) = 512 \sigma_5(n/4) - 504 \sigma_5(n/8) - 512 \sigma_5(n/16) + 248 \tau_2(n/4),$$

$$N([7, 4, 1]; n) = 1024 \sigma_5(n/4) - 1016 \sigma_5(n/8) - 512 \sigma_5(n/16) + 248 \tau_2(n/4),$$

$$N([1, 6, 5]; n) = 32 \sigma_5(n/4) - 24 \sigma_5(n/8) - 512 \sigma_5(n/16) + 40 \tau_2(n/4),$$

$$N([2, 6, 4]; n) = 64 \sigma_5(n/4) - 56 \sigma_5(n/8) - 512 \sigma_5(n/16) + 56 \tau_2(n/4),$$

$$N([3, 6, 3]; n) = 128 \sigma_5(n/4) - 120 \sigma_5(n/8) - 512 \sigma_5(n/16) + 88 \tau_2(n/4),$$

$$N([4, 6, 2]; n) = 256 \sigma_5(n/4) - 248 \sigma_5(n/8) - 512 \sigma_5(n/16) + 120 \tau_2(n/4),$$

$$N([5, 6, 1]; n) = 512 \sigma_5(n/4) - 504 \sigma_5(n/8) - 512 \sigma_5(n/16) + 120 \tau_2(n/4),$$

$$N([1, 8, 3]; n) = 64 \sigma_5(n/4) - 56 \sigma_5(n/8) - 512 \sigma_5(n/16) + 56 \tau_2(n/4),$$

$$N([2, 8, 2]; n) = 128 \sigma_5(n/4) - 120 \sigma_5(n/8) - 512 \sigma_5(n/16) + 56 \tau_2(n/4),$$

$$N([3, 8, 1]; n) = 256 \sigma_5(n/4) - 248 \sigma_5(n/8) - 512 \sigma_5(n/16) + 56 \tau_2(n/4),$$

$$N([1, 10, 1]; n) = 128 \sigma_5(n/4) - 120 \sigma_5(n/8) - 512 \sigma_5(n/16) + 56 \tau_2(n/4).$$

Proof. We only prove the formula for $N([9, 2, 1]; n)$ as the proofs for the other cases are similar. By Theorems 7.2.2 and 8.4.1 - 8.4.7 with $n \equiv 0 \pmod{4}$, we have

$$\begin{aligned} N([9, 2, 1]; n) &= 2\sigma_5(n) - 2\sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) \\ &\quad + 4 \delta\left(0, \frac{1}{2}, 0; n\right) + 4 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 4 \delta\left(0, 0, \frac{1}{2}; n\right) \\ &\quad + \frac{79}{16} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) - \frac{61}{4} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) \\ &\quad + \frac{49}{16} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right) \\ &= 2\sigma_5(n) - 2\sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) \\ &\quad + \frac{79}{16} \left(20 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right)\right) - \frac{61}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &\quad + \frac{49}{16} \left(-4\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right)\right) - \frac{1}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &= 2\sigma_5(n) - 2\sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) \\ &\quad + 71 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &= 2\sigma_5(n) - 2\sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) + 568 \tau_2(n/4). \end{aligned}$$

Simplifying using (9.2.3), we obtain the desired result. $\square$

Theorem 9.2.5. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{8}$, we have*

$$N([1, 2, 9]; n) = 264 \sigma_5(n/8) - 768 \sigma_5(n/16),$$

$$N([2, 2, 8]; n) = 520 \sigma_5(n/8) - 1024 \sigma_5(n/16),$$

$$N([3, 2, 7]; n) = 8 \sigma_5(n/8) - 1536 \sigma_5(n/16),$$

$$N([4, 2, 6]; n) = 2056 \sigma_5(n/8) - 2560 \sigma_5(n/16),$$

$$N([5, 2, 5]; n) = 4101 \sigma_5(n/8) - 4608 \sigma_5(n/16),$$

$$N([6, 2, 4]; n) = 8200 \sigma_5(n/8) - 8704 \sigma_5(n/16),$$

$$N([7, 2, 3]; n) = 16392 \sigma_5(n/8) - 16896 \sigma_5(n/16),$$

$$N([8, 2, 2]; n) = 32776 \sigma_5(n/8) - 33280 \sigma_5(n/16),$$

$$N([9, 2, 1]; n) = 65544 \sigma_5(n/8) - 66048 \sigma_5(n/16),$$

$$N([1, 4, 7]; n) = 520 \sigma_5(n/8) - 1024 \sigma_5(n/16),$$

$$N([2, 4, 6]; n) = 1032 \sigma_5(n/8) - 1536 \sigma_5(n/16),$$

$$N([3, 4, 5]; n) = 2056 \sigma_5(n/8) - 2560 \sigma_5(n/16),$$

$$N([4, 4, 4]; n) = 4104 \sigma_5(n/8) - 4608 \sigma_5(n/16),$$

$$N([5, 4, 3]; n) = 8200 \sigma_5(n/8) - 8704 \sigma_5(n/16),$$

$$N([6, 4, 2]; n) = 16392 \sigma_5(n/8) - 16896 \sigma_5(n/16),$$

$$N([7, 4, 1]; n) = 32776 \sigma_5(n/8) - 33280 \sigma_5(n/16),$$

$$N([1, 6, 5]; n) = 1032 \sigma_5(n/8) - 1536 \sigma_5(n/16),$$

$$N([2, 6, 4]; n) = 2056 \sigma_5(n/8) - 2560 \sigma_5(n/16),$$

$$N([3, 6, 3]; n) = 4104 \sigma_5(n/8) - 4608 \sigma_5(n/16),$$

$$N([4, 6, 2]; n) = 8200 \sigma_5(n/8) - 8704 \sigma_5(n/16),$$

$$N([5, 6, 1]; n) = 16392 \sigma_5(n/8) - 16896 \sigma_5(n/16),$$

$$N([1, 8, 3]; n) = 2056 \sigma_5(n/8) - 2560 \sigma_5(n/16),$$

$$N([2, 8, 2]; n) = 4104 \sigma_5(n/8) - 4608 \sigma_5(n/16),$$

$$N([3, 8, 1]; n) = 8200 \sigma_5(n/8) - 8704 \sigma_5(n/16),$$

$$N([1, 10, 1]; n) = 4104 \sigma_5(n/8) - 4608 \sigma_5(n/16).$$

Proof. As $n \equiv 0 \pmod{8}$ we have $\frac{n}{4} \equiv 0 \pmod{2}$ so $\tau_2(n/4) = 0$. The result now follows from Theorem 9.2.4 and (9.2.4). $\square$

In the next theorem we have chosen to leave $\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right)$ rather than replace it by $\tau_2(n/2)$ as $\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right)$ also occurs. We have been unable to find an expression for $\delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right)$ in terms of $\tau(n)$, $\tau_2(n)$, $\tau_4(n)$ when n is odd.

Theorem 9.2.6. *Let $n \in \mathbb{N}$. For $n \equiv 2 \pmod{4}$, we have*

$$N([1, 2, 9]; n) = \frac{1}{4} \sigma_5(n/2) + \frac{7}{32} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 2 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([2, 2, 8]; n) = \frac{1}{2} \sigma_5(n/2) + \frac{7}{16} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 4 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([3, 2, 7]; n) = \sigma_5(n/2) + \frac{7}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 8 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([4, 2, 6]; n) = 2 \sigma_5(n/2) + \frac{3}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 14 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([5, 2, 5]; n) = 4 \sigma_5(n/2) + \frac{9}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 22 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([6, 2, 4]; n) = 8 \sigma_5(n/2) + 3 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 32 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([7, 2, 3]; n) = 16 \sigma_5(n/2) + \frac{7}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 44 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([8, 2, 2]; n) = 32 \sigma_5(n/2) + \frac{7}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 56 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([9, 2, 1]; n) = 64 \sigma_5(n/2) + \frac{7}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 56 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([1, 4, 7]; n) = \frac{1}{2} \sigma_5(n/2) + \frac{7}{16} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 4 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([2, 4, 6]; n) = \sigma_5(n/2) + \frac{5}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 6 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([3, 4, 5]; n) = 2 \sigma_5(n/2) + \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 10 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([4, 4, 4]; n) = 4 \sigma_5(n/2) + \frac{3}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 16 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([5, 4, 3]; n) = 8 \sigma_5(n/2) + 2 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 24 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([6, 4, 2]; n) = 16 \sigma_5(n/2) + \frac{5}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 32 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([7, 4, 1]; n) = 32 \sigma_5(n/2) + \frac{7}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 32 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([1, 6, 5]; n) = \sigma_5(n/2) + \frac{5}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 6 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([2, 6, 4]; n) = 2 \sigma_5(n/2) + \frac{3}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 8 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([3, 6, 3]; n) = 4 \sigma_5(n/2) + \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 12 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([4, 6, 2]; n) = 8 \sigma_5(n/2) + \frac{3}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 16 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([5, 6, 1]; n) = 16 \sigma_5(n/2) + \frac{5}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 16 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([1, 8, 3]; n) = 2 \sigma_5(n/2) + \frac{3}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 8 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([2, 8, 2]; n) = 4 \sigma_5(n/2) + \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 8 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([3, 8, 1]; n) = 8 \sigma_5(n/2) + \frac{3}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 8 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([1, 10, 1]; n) = 4 \sigma_5(n/2) + \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 8 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right).$$

Proof. We only prove the formula for $N([3, 6, 3]; n)$ as the proofs for the other cases are similar. By Theorems 7.2.2 and 8.4.1 - 8.4.7 with $n \equiv 2 \pmod{4}$, we have

$$\begin{aligned} N([3, 6, 3]; n) &= \frac{1}{8}\sigma_5(n) - \frac{1}{8}\sigma_5(n/2) + 8\sigma_5(n/8) - 512\sigma_5(n/16) \\ &\quad + \frac{11}{8} \delta\left(0, \frac{1}{2}, 0; n\right) + \frac{3}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 4 \delta\left(0, 0, \frac{1}{2}; n\right) \\ &\quad + \frac{23}{16} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) - \frac{45}{4} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) \\ &\quad + \frac{25}{16} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{4} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right) \\ &= \frac{1}{8}\sigma_5(n) - \frac{1}{8}\sigma_5(n/2) \\ &\quad + \frac{3}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 4 \left(\frac{1}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right)\right) \\ &\quad + \frac{23}{16} \left(12 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right)\right) - \frac{45}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &\quad + \frac{25}{16} \left(4 \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right)\right) - \frac{1}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &= \frac{1}{8}\sigma_5(n) - \frac{1}{8}\sigma_5(n/2) \\ &\quad + \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 12 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right). \end{aligned}$$

Simplifying using (9.2.2), we obtain the desired result. $\square$

In the next result, we use an identity of Chapter 8 (Theorem 8.4.8) to reduce the number of delta functions $\delta(a, b, c; n)$ occurring in Theorem 7.2.3 by two. Note that by Theorems 2.11.3 and 2.11.5 we have $\delta\left(0, \frac{1}{2}, 0; n\right) = \tau_2(n)$ and $\delta\left(0, 0, \frac{1}{2}; n\right) = \tau_2(n/2)$.

Theorem 9.2.7. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}
 N([1, 0, 11]; n) &= \left(\frac{1}{256} + \frac{1}{256} \left(\frac{-4}{n} \right) \right) \sigma_5(n) - \frac{33}{256} \sigma_5(n/2) + \frac{65}{8} \sigma_5(n/4) \\
 &\quad - 512 \sigma_5(n/16) \\
 &\quad + \frac{167}{256} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{11}{32} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{11}{4} \delta \left(0, 0, \frac{1}{2}; n \right) \\
 &\quad + \frac{319}{512} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{1133}{128} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
 &\quad + \frac{191}{512} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{15}{128} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
 \end{aligned}$$

$$\begin{aligned}
 N([2, 0, 10]; n) &= \left(\frac{1}{128} + \frac{1}{128} \left(\frac{-4}{n} \right) \right) \sigma_5(n) - \frac{1}{128} \sigma_5(n/2) + 8 \sigma_5(n/8) \\
 &\quad - 512 \sigma_5(n/16) \\
 &\quad + \frac{167}{256} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{11}{16} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{15}{4} \delta \left(0, 0, \frac{1}{2}; n \right) \\
 &\quad + \frac{255}{256} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{877}{64} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
 &\quad + \frac{255}{256} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{15}{64} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
 \end{aligned}$$

$$\begin{aligned}
 N([3, 0, 9]; n) &= \left(\frac{1}{64} + \frac{1}{128} \left(\frac{-4}{n} \right) \right) \sigma_5(n) + \frac{15}{64} \sigma_5(n/2) - \frac{65}{4} \sigma_5(n/4) + 24 \sigma_5(n/8) \\
 &\quad - 512 \sigma_5(n/16) \\
 &\quad + \frac{119}{64} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{9}{8} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{15}{4} \delta \left(0, 0, \frac{1}{2}; n \right) \\
 &\quad + \frac{319}{256} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{1005}{64} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
 &\quad + \frac{447}{256} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{15}{64} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
 \end{aligned}$$

$$N([4, 0, 8]; n) = \frac{1}{32} \sigma_5(n) + \frac{15}{32} \sigma_5(n/2) - \frac{65}{2} \sigma_5(n/4) + 40 \sigma_5(n/8) - 512 \sigma_5(n/16)$$

$$\begin{aligned}
& + \frac{71}{32} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{7}{4} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{7}{2} \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{23}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{41}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([5, 0, 7]; n) &= \left(\frac{1}{16} - \frac{1}{64} \left(\frac{-4}{n} \right) \right) \sigma_5(n) + \frac{7}{64} \sigma_5(n/2) - \frac{65}{2} \sigma_5(n/4) + 40 \sigma_5(n/8) \\
& - 512 \sigma_5(n/16) \\
& + \frac{37}{16} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{21}{8} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{7}{2} \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{217}{128} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{427}{32} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{425}{128} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{9}{32} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([6, 0, 6]; n) &= \left(\frac{1}{8} - \frac{1}{32} \left(\frac{-4}{n} \right) \right) \sigma_5(n) - \frac{1}{8} \sigma_5(n/2) + 8 \sigma_5(n/8) \\
& - 512 \sigma_5(n/16) \\
& + \frac{17}{8} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{15}{4} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{141}{64} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{183}{16} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\
& + \frac{245}{64} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{5}{16} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([7, 0, 5]; n) &= \left(\frac{1}{4} - \frac{1}{32} \left(\frac{-4}{n} \right) \right) \sigma_5(n) - \frac{5}{4} \sigma_5(n/2) + 65 \sigma_5(n/4) - 56 \sigma_5(n/8) \\
& - 512 \sigma_5(n/16) \\
& + \frac{7}{4} \delta \left(0, \frac{1}{2}, 0; n \right) + 5 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 5 \delta \left(0, 0, \frac{1}{2}; n \right) \\
& + \frac{205}{64} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{183}{16} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right)
\end{aligned}$$

$$+ \frac{245}{64} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{5}{16} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right),$$

$$\begin{aligned} N([8, 0, 4]; n) &= \frac{1}{2} \sigma_5(n) - \frac{5}{2} \sigma_5(n/2) + 130 \sigma_5(n/4) - 120 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ &+ \frac{3}{2} \delta \left(0, \frac{1}{2}, 0; n \right) + 6 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 6 \delta \left(0, 0, \frac{1}{2}; n \right) \\ &+ \frac{79}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\ &+ \frac{49}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right), \end{aligned}$$

$$\begin{aligned} N([9, 0, 3]; n) &= \left(1 + \frac{1}{16} \left(\frac{-4}{n} \right) \right) \sigma_5(n) - 3 \sigma_5(n/2) + 130 \sigma_5(n/4) - 120 \sigma_5(n/8) \\ &- 512 \sigma_5(n/16) \\ &+ 2 \delta \left(0, \frac{1}{2}, 0; n \right) + 6 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 6 \delta \left(0, 0, \frac{1}{2}; n \right) \\ &+ \frac{237}{32} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{183}{8} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\ &+ \frac{49}{32} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{8} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right), \end{aligned}$$

$$\begin{aligned} N([10, 0, 2]; n) &= \left(2 + \frac{1}{8} \left(\frac{-4}{n} \right) \right) \sigma_5(n) - 2 \sigma_5(n/2) + 8 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ &+ 4 \delta \left(0, \frac{1}{2}, 0; n \right) + 4 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - 4 \delta \left(0, 0, \frac{1}{2}; n \right) \\ &+ \frac{79}{8} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{2} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right), \end{aligned}$$

$$\begin{aligned} N([11, 0, 1]; n) &= \left(4 + \frac{1}{8} \left(\frac{-4}{n} \right) \right) \sigma_5(n) - 260 \sigma_5(n/4) + 264 \sigma_5(n/8) \\ &- 512 \sigma_5(n/16) \\ &+ 8 \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{79}{8} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) \end{aligned}$$

$$- \frac{61}{2} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right).$$

Proof. We only prove the formula for $N([5, 0, 7]; n)$ as the proofs for the other cases are similar. By Theorems 7.2.3 and 8.4.8, we have

$$\begin{aligned} & N([5, 0, 7]; n) \\ &= \frac{1}{16} \sigma_5(n) + \frac{7}{64} \sigma_5(n/2) - \frac{65}{2} \sigma_5(n/4) + 40 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ &\quad + \frac{37}{16} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{21}{8} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{7}{2} \delta \left(0, 0, \frac{1}{2}; n \right) \\ &\quad - \frac{1}{32} \delta \left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n \right) + \frac{37}{16} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{61}{4} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \\ &\quad + \frac{1}{32} \delta \left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n \right) + \frac{47}{16} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - \frac{1}{4} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right). \\ \\ &= \frac{1}{16} \sigma_5(n) + \frac{7}{64} \sigma_5(n/2) - \frac{65}{2} \sigma_5(n/4) + 40 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ &\quad + \frac{37}{16} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{21}{8} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{7}{2} \delta \left(0, 0, \frac{1}{2}; n \right) \\ &\quad - \frac{1}{64} \left(2 \delta \left(-\frac{5}{6}, \frac{9}{4}, -\frac{11}{12}; n \right) - \frac{79}{2} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) + 122 \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) \right. \\ &\quad \left. - 2 \delta \left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n \right) + \frac{49}{2} \delta \left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n \right) - 2 \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right) \right) \\ &\quad + \frac{217}{128} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{427}{32} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) + \frac{425}{128} \delta \left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n \right) \\ &\quad - \frac{9}{32} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right) \\ \\ &= \left(\frac{1}{16} - \frac{1}{64} \left(\frac{-4}{n} \right) \right) \sigma_5(n) + \frac{7}{64} \sigma_5(n/2) - \frac{65}{2} \sigma_5(n/4) + 40 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ &\quad + \frac{37}{16} \delta \left(0, \frac{1}{2}, 0; n \right) + \frac{21}{8} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) - \frac{7}{2} \delta \left(0, 0, \frac{1}{2}; n \right) \\ &\quad + \frac{217}{128} \delta \left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n \right) - \frac{427}{32} \delta \left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n \right) + \frac{425}{128} \delta \left(-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4}; n \right) \\ &\quad - \frac{9}{32} \delta \left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n \right), \end{aligned}$$

which is the desired result. $\square$

In the last three theorems of this section, we simplify Theorem 9.2.7 in the cases $n \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{8}$ and $n \equiv 2 \pmod{4}$.

Theorem 9.2.8. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{4}$, we have*

$$N([1, 0, 11]; n) = 8 \sigma_5(n/4) - 512 \sigma_5(n/16) + 16 \tau_2(n/4),$$

$$N([2, 0, 10]; n) = 8 \sigma_5(n/4) - 512 \sigma_5(n/16) + 16 \tau_2(n/4),$$

$$N([3, 0, 9]; n) = 8 \sigma_5(n/4) - 512 \sigma_5(n/16) + 16 \tau_2(n/4),$$

$$N([4, 0, 8]; n) = 16 \sigma_5(n/4) - 8 \sigma_5(n/8) - 512 \sigma_5(n/16) + 24 \tau_2(n/4),$$

$$N([5, 0, 7]; n) = 48 \sigma_5(n/4) - 40 \sigma_5(n/8) - 512 \sigma_5(n/16) + 56 \tau_2(n/4),$$

$$\begin{aligned} N([6, 0, 6]; n) &= 128 \sigma_5(n/4) - 120 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ &\quad + 136 \tau_2(n/4), \end{aligned}$$

$$\begin{aligned} N([7, 0, 5]; n) &= 288 \sigma_5(n/4) - 280 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ &\quad + 296 \tau_2(n/4), \end{aligned}$$

$$\begin{aligned} N([8, 0, 4]; n) &= 576 \sigma_5(n/4) - 568 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ &\quad + 568 \tau_2(n/4), \end{aligned}$$

$$N([9, 0, 3]; n) = 1088 \sigma_5(n/4) - 1080 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ + 952 \tau_2(n/4),$$

$$N([10, 0, 2]; n) = 2048 \sigma_5(n/4) - 2040 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ + 1336 \tau_2(n/4),$$

$$N([11, 0, 1]; n) = 3968 \sigma_5(n/4) - 3960 \sigma_5(n/8) - 512 \sigma_5(n/16) \\ + 1336 \tau_2(n/4).$$

Proof. We only prove the formula for $N([9, 0, 3]; n)$ as the proofs for the other cases are similar. By Theorems 9.2.7 and 8.4.1 - 8.4.7, and the fact that $\left(\frac{-4}{n}\right) = 0$ when $n \equiv 0 \pmod{4}$, we have when $n \equiv 0 \pmod{4}$.

$$\begin{aligned} N([9, 0, 3]; n) &= \left(1 + \frac{1}{16} \left(\frac{-4}{n}\right)\right) \sigma_5(n) - 3\sigma_5(n/2) + 130\sigma_5(n/4) - 120\sigma_5(n/8) \\ &\quad - 512\sigma_5(n/16) \\ &\quad + 2 \delta\left(0, \frac{1}{2}, 0; n\right) + 6 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - 6 \delta\left(0, 0, \frac{1}{2}; n\right) \\ &\quad + \frac{237}{32} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) - \frac{183}{8} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) \\ &\quad + \frac{49}{32} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{1}{8} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right) \\ &= \sigma_5(n) - 3\sigma_5(n/2) + 130\sigma_5(n/4) - 120\sigma_5(n/8) - 512\sigma_5(n/16) \\ &\quad + \frac{237}{32} \left(20 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right)\right) - \frac{183}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &\quad + \frac{49}{32} \left(-4 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right)\right) - \frac{1}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &= \sigma_5(n) - 3\sigma_5(n/2) + 130\sigma_5(n/4) - 120\sigma_5(n/8) - 512\sigma_5(n/16) \end{aligned}$$

$$+ 119 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2 \right).$$

Simplifying using (9.2.3), we obtain the desired result. $\square$

We remark that when $n \equiv 0 \pmod{4}$

$$n = x_1^2 + 4x_2^2 + \cdots + 4x_{12}^2 \iff \frac{n}{4} = \left(\frac{x_1}{2}\right)^2 + x_2^2 + \cdots + x_{12}^2$$

so that

$$\begin{aligned} N([1, 0, 11]; n) &= r_{12}(n/4) \\ &= 8\sigma_5(n/4) - 512\sigma_5(n/16) + 16 \delta \left(0, \frac{1}{2}, 0; n/4 \right) \\ &= 8\sigma_5(n/4) - 512\sigma_5(n/16) + 16 \tau_2(n/4) \end{aligned}$$

by Liouville's twelve squares theorem and Theorem 8.4.2. The formulae for $N([2, 0, 10]; n)$ and $N([3, 0, 9]; n)$ can also be proved in this way when $n \equiv 0 \pmod{4}$.

Theorem 9.2.9. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{8}$, we have*

$$N([1, 0, 11]; n) = 264 \sigma_5(n/8) - 768 \sigma_5(n/16),$$

$$N([2, 0, 10]; n) = 264 \sigma_5(n/8) - 768 \sigma_5(n/16),$$

$$N([3, 0, 9]; n) = 264 \sigma_5(n/8) - 768 \sigma_5(n/16),$$

$$N([4, 0, 8]; n) = 520 \sigma_5(n/8) - 1024 \sigma_5(n/16),$$

$$N([5, 0, 7]; n) = 1544 \sigma_5(n/8) - 2048 \sigma_5(n/16),$$

$$N([6, 0, 6]; n) = 4104 \sigma_5(n/8) - 4608 \sigma_5(n/16),$$

$$N([7, 0, 5]; n) = 9224 \sigma_5(n/8) - 9728 \sigma_5(n/16),$$

$$N([8, 0, 4]; n) = 18440 \sigma_5(n/8) - 18944 \sigma_5(n/16),$$

$$N([9, 0, 3]; n) = 34824 \sigma_5(n/8) - 35328 \sigma_5(n/16),$$

$$N([10, 0, 2]; n) = 65544 \sigma_5(n/8) - 66048 \sigma_5(n/16),$$

$$N([11, 0, 1]; n) = 126984 \sigma_5(n/8) - 127488 \sigma_5(n/16).$$

Proof. As $n \equiv 0 \pmod{8}$ we have $\frac{n}{4} \equiv 0 \pmod{2}$ so $\tau_2(n/4) = 0$. The results now follow using (9.2.4). $\square$

Theorem 9.2.10. *Let $n \in \mathbb{N}$. For $n \equiv 2 \pmod{4}$, we have*

$$N([1, 0, 11]; n) = 0,$$

$$N([2, 0, 10]; n) = \frac{1}{4} \sigma_5(n/2) + \frac{7}{32} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 2 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([3, 0, 9]; n) = \frac{3}{4} \sigma_5(n/2) + \frac{21}{32} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 6 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([4, 0, 8]; n) = \frac{3}{2} \sigma_5(n/2) + \frac{21}{16} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 12 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([5, 0, 7]; n) = \frac{5}{2} \sigma_5(n/2) + \frac{35}{16} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 20 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([6, 0, 6]; n) = 4 \sigma_5(n/2) + \frac{13}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 30 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([7, 0, 5]; n) = 7 \sigma_5(n/2) + \frac{35}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 42 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([8, 0, 4]; n) = 14 \sigma_5(n/2) + \frac{21}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 56 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([9, 0, 3]; n) = 30 \sigma_5(n/2) + \frac{21}{4} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 72 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([10, 0, 2]; n) = 64 \sigma_5(n/2) + \frac{7}{2} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) + 88 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right),$$

$$N([11, 0, 1]; n) = 132 \sigma_5(n/2) + 88 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right).$$

Proof. We only prove the formula for $N([2, 0, 10]; n)$ as the proofs for the other cases are similar. By Theorem 9.2.7 and Theorems 8.4.1 - 8.4.7, the fact that $\left(\frac{-4}{n}\right) = 0$ when $n \equiv 2 \pmod{4}$, and as $\sigma_5(n/4), \sigma_5(n/8), \sigma_5(n/16)$ are all equal to zero when $n \equiv 2 \pmod{4}$, we have for $n \equiv 2 \pmod{4}$

$$\begin{aligned} N([2, 0, 10]; n) &= \left(\frac{1}{128} + \frac{1}{128} \left(\frac{-4}{n}\right)\right) \sigma_5(n) - \frac{1}{128} \sigma_5(n/2) + 8 \sigma_5(n/8) \\ &\quad - 512 \sigma_5(n/16) \\ &\quad + \frac{167}{256} \delta\left(0, \frac{1}{2}, 0; n\right) + \frac{11}{16} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - \frac{15}{4} \delta\left(0, 0, \frac{1}{2}; n\right) \\ &\quad + \frac{255}{256} \delta\left(-\frac{1}{2}, \frac{5}{4}, -\frac{1}{4}; n\right) - \frac{877}{64} \delta\left(-\frac{1}{6}, \frac{1}{4}, \frac{5}{12}; n\right) \\ &\quad + \frac{255}{256} \delta\left(-\frac{1}{6}, \frac{3}{4}, -\frac{1}{12}; n\right) - \frac{15}{64} \delta\left(\frac{1}{6}, -\frac{1}{4}, \frac{7}{12}; n\right) \\ &= \frac{1}{128} \sigma_5(n) - \frac{1}{128} \sigma_5(n/2) \\ &\quad + \frac{11}{16} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) - \frac{15}{32} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) \\ &\quad + \frac{255}{256} \left(12 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right)\right) - \frac{877}{64} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &\quad + \frac{255}{256} \left(4 \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right)\right) - \frac{15}{64} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2\right) \\ &= \frac{1}{128} \sigma_5(n) - \frac{1}{128} \sigma_5(n/2) \end{aligned}$$

$$+ \frac{7}{32} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) + 2 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n/2 \right).$$

Simplifying using (9.2.2), we obtain the desired result. $\square$

9.3 Sums with 16 squares

In this section we restrict n to particular congruence classes modulo 4 and modulo 8 in order to reduce the number of delta functions $\delta(a, b, c; n)$ occurring in Theorems 7.3.1, 7.3.2 and 7.3.3.

We make use of the following simple properties of σ_7 . Let $n \in \mathbb{N}$.

If $n \equiv 0 \pmod{4}$ then

$$(9.3.1) \quad \begin{cases} \sigma_7(n) = 16513 \sigma_7(n/4) - 16512 \sigma_5(n/8), \\ \sigma_7(n/2) = 129 \sigma_7(n/4) - 128 \sigma_7(n/8), \end{cases}$$

if $n \equiv 4 \pmod{8}$ then

$$(9.3.2) \quad \begin{cases} \sigma_7(n) = 16513 \sigma_7(n/4), \\ \sigma_7(n/2) = 129 \sigma_7(n/4), \end{cases}$$

and if $n \equiv 0 \pmod{8}$ then

$$(9.3.3) \quad \begin{cases} \sigma_7(n) = 2113665 \sigma_7(n/8) - 2113664 \sigma_7(n/16), \\ \sigma_7(n/2) = 16513 \sigma_7(n/8) - 16512 \sigma_7(n/16), \\ \sigma_7(n/4) = 129 \sigma_7(n/8) - 128 \sigma_7(n/16). \end{cases}$$

The quantity $\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)$ ($n \in \mathbb{N}$) will play a major role in this section. Its value is given in Theorem 2.11.6, namely

$$\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) = \frac{1}{2} \sum_{\substack{(x_1, \dots, x_8) \in \mathbb{Z}^8 \\ x_1^2 + \dots + x_8^2 = n}} (x_1^4 - 3x_1^2 x_2^2).$$

We begin with the simplification of Theorem 7.3.1 when $n \equiv 0 \pmod{4}$.

Theorem 9.3.1. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{4}$, we have*

$$N([14, 2, 0]; n) = \frac{262112}{17} \sigma_7(n/4) - \frac{253952}{17} \sigma_7(n/8) - \frac{554}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([12, 4, 0]; n) = \frac{131040}{17} \sigma_7(n/4) - \frac{122880}{17} \sigma_7(n/8) - \frac{630}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([10, 6, 0]; n) = \frac{65504}{17} \sigma_7(n/4) - \frac{57344}{17} \sigma_7(n/8) - \frac{464}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([8, 8, 0]; n) = \frac{32736}{17} \sigma_7(n/4) - \frac{24576}{17} \sigma_7(n/8) - \frac{296}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([6, 10, 0]; n) = \frac{16352}{17} \sigma_7(n/4) - \frac{8192}{17} \sigma_7(n/8) - \frac{178}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([4, 12, 0]; n) = 480 \sigma_7(n/4) - 6 \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([2, 14, 0]; n) = \frac{4064}{17} \sigma_7(n/4) - \frac{4096}{17} \sigma_7(n/8) - \frac{64}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right).$$

Proof. We only prove the formula for $N([8, 8, 0]; n)$ as the proofs for the other cases are similar. By Theorems 7.3.1, 8.5.1, 8.5.2 and 8.5.3 we have

$$N([8, 8, 0]; n)$$

$$\begin{aligned}
&= \frac{2}{17}\sigma_7(n) - \frac{2}{17}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{134}{17}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{224}{17}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 8\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{1104}{17}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{128}{17}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right). \\
\\
&= \frac{2}{17}\sigma_7(n) - \frac{2}{17}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{134}{17}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{224}{17}\left(\frac{1}{8}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\
&\quad + 8\left(-6\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right)\right) - \frac{1104}{17}\left(-\frac{3}{8}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right)\right) \\
\\
&= \frac{2}{17}\sigma_7(n) - \frac{2}{17}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad - \frac{296}{17}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right).
\end{aligned}$$

Simplifying using (9.3.1), we obtain the desired result. $\square$

In the next two theorems we simplify Theorem 7.3.2 to when $n \equiv 0 \pmod{8}$ and $n \equiv 4 \pmod{8}$.

Theorem 9.3.2. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{8}$, we have*

$$N([1, 2, 13]; n) = \frac{4064}{17}\sigma_7(n/8) + \frac{4096}{17}\sigma_7(n/16) + \frac{8}{17}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([2, 2, 12]; n) = 480\sigma_7(n/8) + \frac{3}{4}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([3, 2, 11]; n) = \frac{16352}{17}\sigma_7(n/8) - \frac{8192}{17}\sigma_7(n/16) + \frac{89}{68}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([4, 2, 10]; n) = \frac{32736}{17}\sigma_7(n/8) - \frac{24576}{17}\sigma_7(n/16) + \frac{165}{68}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([5, 2, 9]; n) = \frac{65504}{17} \sigma_7(n/8) - \frac{57344}{17} \sigma_7(n/16) + \frac{317}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([6, 2, 8]; n) = \frac{131040}{17} \sigma_7(n/8) - \frac{122880}{17} \sigma_7(n/16) + \frac{151}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([7, 2, 7]; n) = \frac{262112}{17} \sigma_7(n/8) - \frac{253952}{17} \sigma_7(n/16) + \frac{555}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([8, 2, 6]; n) = \frac{524256}{17} \sigma_7(n/8) - \frac{516096}{17} \sigma_7(n/16) + \frac{488}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([9, 2, 5]; n) = \frac{1048544}{17} \sigma_7(n/8) - \frac{1040384}{17} \sigma_7(n/16) + \frac{824}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([10, 2, 4]; n) = 123360 \sigma_7(n/8) - 122880 \sigma_7(n/16) + \frac{315}{4} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([11, 2, 3]; n) = \frac{4194272}{17} \sigma_7(n/8) - \frac{4186112}{17} \sigma_7(n/16) + \frac{8249}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([12, 2, 2]; n) = \frac{8388576}{17} \sigma_7(n/8) - \frac{8380416}{17} \sigma_7(n/16) + \frac{11385}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([13, 2, 1]; n) = \frac{16777184}{17} \sigma_7(n/8) - \frac{16769024}{17} \sigma_7(n/16) + \frac{11537}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([1, 4, 11]; n) = 480 \sigma_7(n/8) + \frac{3}{4} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([2, 4, 10]; n) = \frac{16352}{17} \sigma_7(n/8) - \frac{8192}{17} \sigma_7(n/16) + \frac{89}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([3, 4, 9]; n) = \frac{32736}{17} \sigma_7(n/8) - \frac{24576}{17} \sigma_7(n/16) + \frac{165}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([4, 4, 8]; n) = \frac{65504}{17} \sigma_7(n/8) - \frac{57344}{17} \sigma_7(n/16) + \frac{75}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([5, 4, 7]; n) = \frac{131040}{17} \sigma_7(n/8) - \frac{122880}{17} \sigma_7(n/16) + \frac{134}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([6, 4, 6]; n) = \frac{262112}{17} \sigma_7(n/8) - \frac{253952}{17} \sigma_7(n/16) + \frac{235}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([7, 4, 5]; n) = \frac{524256}{17} \sigma_7(n/8) - \frac{516096}{17} \sigma_7(n/16) + \frac{403}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([8, 4, 4]; n) = \frac{1048544}{17} \sigma_7(n/8) - \frac{1040384}{17} \sigma_7(n/16) + \frac{2667}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([9, 4, 3]; n) = 123360 \sigma_7(n/8) - 122880 \sigma_7(n/16) + \frac{243}{4} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([10, 4, 2]; n) = \frac{4194272}{17} \sigma_7(n/8) - \frac{4186112}{17} \sigma_7(n/16) + \frac{5597}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([11, 4, 1]; n) = \frac{8388576}{17} \sigma_7(n/8) - \frac{8380416}{17} \sigma_7(n/16) + \frac{5265}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([1, 6, 9]; n) = \frac{16352}{17} \sigma_7(n/8) - \frac{8192}{17} \sigma_7(n/16) + \frac{89}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([2, 6, 8]; n) = \frac{32736}{17} \sigma_7(n/8) - \frac{24576}{17} \sigma_7(n/16) + \frac{37}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([3, 6, 7]; n) = \frac{65504}{17} \sigma_7(n/8) - \frac{57344}{17} \sigma_7(n/16) + \frac{133}{34} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([4, 6, 6]; n) = \frac{131040}{17} \sigma_7(n/8) - \frac{122880}{17} \sigma_7(n/16) + \frac{117}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([5, 6, 5]; n) = \frac{262112}{17} \sigma_7(n/8) - \frac{253952}{17} \sigma_7(n/16) + \frac{201}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([6, 6, 4]; n) = \frac{524256}{17} \sigma_7(n/8) - \frac{516096}{17} \sigma_7(n/16) + \frac{1323}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([7, 6, 3]; n) = \frac{1048544}{17} \sigma_7(n/8) - \frac{1040384}{17} \sigma_7(n/16) + \frac{2021}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([8, 6, 2]; n) = 123360 \sigma_7(n/8) - 122880 \sigma_7(n/16) + \frac{157}{4} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([9, 6, 1]; n) = \frac{4194272}{17} \sigma_7(n/8) - \frac{4186112}{17} \sigma_7(n/16) + \frac{2333}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([1, 8, 7]; n) = \frac{32736}{17} \sigma_7(n/8) - \frac{24576}{17} \sigma_7(n/16) + \frac{37}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([2, 8, 6]; n) = \frac{65504}{17} \sigma_7(n/8) - \frac{57344}{17} \sigma_7(n/16) + \frac{58}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([3, 8, 5]; n) = \frac{131040}{17} \sigma_7(n/8) - \frac{122880}{17} \sigma_7(n/16) + \frac{100}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([4, 8, 4]; n) = \frac{262112}{17} \sigma_7(n/8) - \frac{253952}{17} \sigma_7(n/16) + \frac{651}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([5, 8, 3]; n) = \frac{524256}{17} \sigma_7(n/8) - \frac{516096}{17} \sigma_7(n/16) + \frac{983}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([6, 8, 2]; n) = \frac{1048544}{17} \sigma_7(n/8) - \frac{1040384}{17} \sigma_7(n/16) + \frac{1273}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([7, 8, 1]; n) = 123360 \sigma_7(n/8) - 122880 \sigma_7(n/16) + \frac{61}{4} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([1, 10, 5]; n) = \frac{65504}{17} \sigma_7(n/8) - \frac{57344}{17} \sigma_7(n/16) + \frac{58}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([2, 10, 4]; n) = \frac{131040}{17} \sigma_7(n/8) - \frac{122880}{17} \sigma_7(n/16) + \frac{315}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([3, 10, 3]; n) = \frac{262112}{17} \sigma_7(n/8) - \frac{253952}{17} \sigma_7(n/16) + \frac{481}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([4, 10, 2]; n) = \frac{524256}{17} \sigma_7(n/8) - \frac{516096}{17} \sigma_7(n/16) + \frac{609}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([5, 10, 1]; n) = \frac{1048544}{17} \sigma_7(n/8) - \frac{1040384}{17} \sigma_7(n/16) + \frac{457}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([1, 12, 3]; n) = \frac{131040}{17} \sigma_7(n/8) - \frac{122880}{17} \sigma_7(n/16) + \frac{315}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([2, 12, 2]; n) = \frac{262112}{17} \sigma_7(n/8) - \frac{253952}{17} \sigma_7(n/16) + \frac{277}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([3, 12, 1]; n) = \frac{524256}{17} \sigma_7(n/8) - \frac{516096}{17} \sigma_7(n/16) + \frac{201}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([1, 14, 1]; n) = \frac{262112}{17} \sigma_7(n/8) - \frac{253952}{17} \sigma_7(n/16) + \frac{277}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right).$$

Proof. We only prove the formula for $N([4, 4, 8]; n)$ as the proofs for the other cases are similar. By Theorem 7.3.2, Theorems 8.5.1 - 8.5.4, and Theorem 8.6.1 with $n \equiv 0 \pmod{8}$, we have, for $n \equiv 0 \pmod{8}$.

$$\begin{aligned} N([4, 4, 8]; n) &= \frac{1}{544} \sigma_7(n) - \frac{1}{544} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{1087}{544} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{29}{4} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 2 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\ &\quad - \frac{849}{34} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{545}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\ &\quad - \frac{2181}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{543}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\ &\quad - \frac{1085}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right). \end{aligned}$$

$$\begin{aligned} &= \frac{1}{544} \sigma_7(n) - \frac{1}{544} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &\quad + \frac{1087}{544} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{29}{4} \left(\frac{1}{8} \delta\left(-\frac{1}{3}, 1, -\frac{1}{6}; n\right) \right) \end{aligned}$$

$$\begin{aligned}
& + 2 \left(-6 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \right) - \frac{849}{34} \left(-\frac{3}{8} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \right) \\
& + \frac{545}{272} \left(45 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \right) - \frac{2181}{68} \left(3 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \right) \\
& + \frac{547}{17} \left(\frac{3}{16} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \right) + \frac{543}{272} \left(3 \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \right) \\
& + \frac{1}{17} \left(-\frac{3}{16} \delta \left(-\frac{1}{3}, 1, -\frac{1}{6}; n \right) \right) \\
& = \frac{1}{544} \sigma_7(n) - \frac{1}{544} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{75}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right).
\end{aligned}$$

Simplifying using (9.3.3), we obtain the desired result. $\square$

Theorem 9.3.3. *Let $n \in \mathbb{N}$. For $n \equiv 4 \pmod{8}$, we have*

$$\begin{aligned}
N([1, 2, 13]; n) &= \frac{32}{17} \sigma_7(n/4) - \frac{8}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([2, 2, 12]; n) &= \frac{64}{17} \sigma_7(n/4) - \frac{47}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([3, 2, 11]; n) &= \frac{128}{17} \sigma_7(n/4) - \frac{77}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([4, 2, 10]; n) &= \frac{256}{17} \sigma_7(n/4) - \frac{137}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([5, 2, 9]; n) &= \frac{512}{17} \sigma_7(n/4) - \frac{257}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([6, 2, 8]; n) &= \frac{1024}{17} \sigma_7(n/4) - \frac{120}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([7, 2, 7]; n) &= \frac{2048}{17} \sigma_7(n/4) - \frac{429}{34} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),
\end{aligned}$$

$$N([8, 2, 6]; n) = \frac{4096}{17} \sigma_7(n/4) - \frac{361}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([9, 2, 5]; n) = \frac{8192}{17} \sigma_7(n/4) - \frac{569}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([10, 2, 4]; n) = \frac{16384}{17} \sigma_7(n/4) - \frac{3345}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([11, 2, 3]; n) = \frac{32768}{17} \sigma_7(n/4) - \frac{4531}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([12, 2, 2]; n) = \frac{65536}{17} \sigma_7(n/4) - \frac{5475}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([13, 2, 1]; n) = \frac{131072}{17} \sigma_7(n/4) - \frac{5323}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([1, 4, 11]; n) = \frac{64}{17} \sigma_7(n/4) - \frac{47}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([2, 4, 10]; n) = \frac{128}{17} \sigma_7(n/4) - \frac{77}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([3, 4, 9]; n) = \frac{256}{17} \sigma_7(n/4) - \frac{137}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([4, 4, 8]; n) = \frac{512}{17} \sigma_7(n/4) - \frac{60}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([5, 4, 7]; n) = \frac{1024}{17} \sigma_7(n/4) - \frac{103}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([6, 4, 6]; n) = \frac{2048}{17} \sigma_7(n/4) - \frac{172}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([7, 4, 5]; n) = \frac{4096}{17} \sigma_7(n/4) - \frac{276}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([8, 4, 4]; n) = \frac{8192}{17} \sigma_7(n/4) - \frac{1681}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([9, 4, 3]; n) = \frac{16384}{17} \sigma_7(n/4) - \frac{2393}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([10, 4, 2]; n) = \frac{32768}{17} \sigma_7(n/4) - \frac{3103}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([11, 4, 1]; n) = \frac{65536}{17} \sigma_7(n/4) - \frac{3435}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([1, 6, 9]; n) = \frac{128}{17} \sigma_7(n/4) - \frac{77}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([2, 6, 8]; n) = \frac{256}{17} \sigma_7(n/4) - \frac{30}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([3, 6, 7]; n) = \frac{512}{17} \sigma_7(n/4) - \frac{103}{34} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([4, 6, 6]; n) = \frac{1024}{17} \sigma_7(n/4) - \frac{86}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([5, 6, 5]; n) = \frac{2048}{17} \sigma_7(n/4) - \frac{138}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([6, 6, 4]; n) = \frac{4096}{17} \sigma_7(n/4) - \frac{849}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([7, 6, 3]; n) = \frac{8192}{17} \sigma_7(n/4) - \frac{1239}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([8, 6, 2]; n) = \frac{16384}{17} \sigma_7(n/4) - \frac{1679}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([9, 6, 1]; n) = \frac{32768}{17} \sigma_7(n/4) - \frac{2015}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([1, 8, 7]; n) = \frac{256}{17} \sigma_7(n/4) - \frac{30}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([2, 8, 6]; n) = \frac{512}{17} \sigma_7(n/4) - \frac{43}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([3, 8, 5]; n) = \frac{1024}{272} \sigma_7(n/4) - \frac{69}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([4, 8, 4]; n) = \frac{2048}{17} \sigma_7(n/4) - \frac{433}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([5, 8, 3]; n) = \frac{4096}{17} \sigma_7(n/4) - \frac{645}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([6, 8, 2]; n) = \frac{8192}{17} \sigma_7(n/4) - \frac{899}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([7, 8, 1]; n) = \frac{16384}{17} \sigma_7(n/4) - \frac{1135}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([1, 10, 5]; n) = \frac{512}{17} \sigma_7(n/4) - \frac{43}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([2, 10, 4]; n) = \frac{1024}{17} \sigma_7(n/4) - \frac{225}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([3, 10, 3]; n) = \frac{2048}{17} \sigma_7(n/4) - \frac{331}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([4, 10, 2]; n) = \frac{4096}{17} \sigma_7(n/4) - \frac{475}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([5, 10, 1]; n) = \frac{8192}{17} \sigma_7(n/4) - \frac{627}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([1, 12, 3]; n) = \frac{1024}{17} \sigma_7(n/4) - \frac{225}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),$$

$$N([2, 12, 2]; n) = \frac{2048}{17} \sigma_7(n/4) - \frac{263}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([3, 12, 1]; n) = \frac{4096}{17} \sigma_7(n/4) - \frac{339}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),$$

$$N([1, 14, 1]; n) = \frac{2048}{17} \sigma_7(n/4) - \frac{263}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right).$$

Proof. We only prove the formula for $N([5, 8, 3]; n)$ as the proofs for the other cases are similar. By Theorem 7.3.2, Theorems 8.5.1 - 8.5.4, and Theorem 8.6.1 with $n \equiv 4 \pmod{8}$ we have, for $n \equiv 4 \pmod{8}$.

$$\begin{aligned} & N([5, 8, 3]; n) \\ &= \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\ &+ \frac{169}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{124}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{5}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\ &- \frac{450}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{681}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\ &- \frac{2453}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{679}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\ &- \frac{1085}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right) \\ &= \frac{1}{68} \sigma_7(n) - \frac{1}{68} \sigma_7(n/2) \\ &+ \frac{169}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{124}{17} \left(\frac{1}{8} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\ &+ \frac{5}{2} \left(-6 \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) - \frac{450}{17} \left(-\frac{3}{8} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\ &+ \frac{681}{272} \left(-15 \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) - \frac{2453}{68} \left(-\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\ &+ \frac{547}{17} \left(-\frac{1}{16} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) + \frac{679}{272} \left(-\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right)\right) \\ &+ \frac{1}{17} \left(\frac{1}{16} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \end{aligned}$$

$$= \frac{1}{68}\sigma_7(n) - \frac{1}{68}\sigma_7(n/2) - \frac{645}{68}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right).$$

Simplifying using (9.3.2), we obtain the desired result. $\square$

In the next result, we use an identity of Chapter 8 to reduce the number of delta functions $\delta(a, b, c; n)$ occurring in Theorem 7.3.3 by two.

Theorem 9.3.4. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} N([1, 0, 15]; n) &= \left(\frac{1}{17408} + \frac{1}{17408}\left(\frac{-4}{n}\right)\right)\sigma_7(n) - \frac{129}{17408}\sigma_7(n/2) + \frac{257}{136}\sigma_7(n/4) - \frac{64}{17}\sigma_7(n/8) \\ &\quad + \frac{8192}{17}\sigma_7(n/16) \\ &\quad + \frac{7751}{17408}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{8775}{2176}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{71}{128}\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\ &\quad - \frac{1143}{136}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{253}{136}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{15359}{34816}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\ &\quad - \frac{70675}{8704}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{40909}{2176}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{19455}{34816}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\ &\quad - \frac{64485}{8704}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) - \frac{143}{2176}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right), \end{aligned}$$

$$\begin{aligned} N([2, 0, 14]; n) &= \left(\frac{1}{8704} + \frac{1}{8704}\left(\frac{-4}{n}\right)\right)\sigma_7(n) - \frac{1}{8704}\sigma_7(n/2) - \frac{32}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\ &\quad + \frac{7751}{8704}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{6643}{1088}\delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{71}{64}\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\ &\quad - \frac{4565}{272}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{239}{68}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{17199}{17408}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\ &\quad - \frac{75523}{4342}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{30653}{1088}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{17615}{17408}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\ &\quad - \frac{54613}{4352}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{65}{1088}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right), \end{aligned}$$

$$\begin{aligned} N([3, 0, 13]; n) &= \left(\frac{1}{4352} + \frac{1}{8704}\left(\frac{-4}{n}\right)\right)\sigma_7(n) + \frac{63}{4352}\sigma_7(n/2) - \frac{257}{68}\sigma_7(n/4) + \frac{32}{17}\sigma_7(n/8) \end{aligned}$$

$$\begin{aligned}
& + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{5983}{4352} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{3809}{544} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{13}{8} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& - \frac{6539}{272} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{65}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{27535}{17408} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& - \frac{116867}{4352} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{33917}{1088} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{24687}{17408} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& - \frac{68757}{4352} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{65}{1088} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([4, 0, 12]; n)$$

$$\begin{aligned}
& = \frac{1}{2176} \sigma_7(n) + \frac{63}{2176} \sigma_7(n/2) - \frac{257}{34} \sigma_7(n/4) + \frac{96}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{4215}{2176} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{1965}{272} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{33}{16} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& - \frac{3993}{136} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{66}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{149}{68} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& - \frac{1235}{34} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{123}{68} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& - \frac{585}{34} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([5, 0, 11]; n)$$

$$\begin{aligned}
& = \left(\frac{1}{1088} - \frac{1}{4352} \left(\frac{-4}{n} \right) \right) \sigma_7(n) + \frac{31}{1088} \sigma_7(n/2) - \frac{257}{34} \sigma_7(n/4) + \frac{96}{17} \sigma_7(n/8) \\
& + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{2821}{1088} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{495}{68} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{77}{32} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
& - \frac{4367}{136} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{66}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{23969}{8704} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& - \frac{97405}{2176} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{18051}{544} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{19553}{8704} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& - \frac{37579}{2176} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{31}{544} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([6, 0, 10]; n)$$

$$= \left(\frac{1}{544} - \frac{1}{2176} \left(\frac{-4}{n} \right) \right) \sigma_7(n) - \frac{1}{544} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16)$$

$$\begin{aligned}
& + \frac{1801}{544} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{15}{2} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{43}{16} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{2191}{68} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{817}{256} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{3245}{64} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{16} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{12225}{4352} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{18315}{1088} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{15}{272} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([7, 0, 9]; n)$$

$$\begin{aligned}
& = \left(\frac{1}{272} - \frac{1}{2176} \left(\frac{-4}{n} \right) \right) \sigma_7(n) - \frac{1}{16} \sigma_7(n/2) + \frac{257}{17} \sigma_7(n/4) - \frac{288}{17} \sigma_7(n/8) \\
& + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{1087}{272} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{135}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 3 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{1035}{34} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{60}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{881}{256} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{3373}{64} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{16} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{15489}{4352} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{18315}{1088} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{15}{272} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([8, 0, 8]; n)$$

$$\begin{aligned}
& = \frac{1}{136} \sigma_7(n) - \frac{1}{8} \sigma_7(n/2) + \frac{514}{17} \sigma_7(n/4) - 32 \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{611}{136} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{142}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{7}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{483}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{56}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{953}{272} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{3405}{68} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{547}{17} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{1223}{272} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{1221}{68} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{1}{17} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([9, 0, 7]; n)$$

$$\begin{aligned}
& = \left(\frac{1}{68} + \frac{1}{1088} \left(\frac{-4}{n} \right) \right) \sigma_7(n) - \frac{9}{68} \sigma_7(n/2) + \frac{514}{17} \sigma_7(n/4) - 32 \sigma_7(n/8) \\
& + \frac{8192}{17} \sigma_7(n/16)
\end{aligned}$$

$$\begin{aligned}
& + \frac{627}{136} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{140}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{35}{8} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{483}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{56}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{7623}{2176} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{23835}{544} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{3829}{136} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{11959}{2176} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{10989}{544} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{9}{136} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([10, 0, 6]; n)$$

$$\begin{aligned}
& = \left(\frac{1}{34} + \frac{1}{544} \left(\frac{-4}{n} \right) \right) \sigma_7(n) - \frac{1}{34} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{287}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{120}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{23}{4} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{552}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{64}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{4083}{1088} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{10215}{272} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{1641}{68} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{6795}{1088} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{6105}{272} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{5}{68} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([11, 0, 5]; n)$$

$$\begin{aligned}
& = \left(\frac{1}{17} + \frac{1}{544} \left(\frac{-4}{n} \right) \right) \sigma_7(n) + \frac{3}{17} \sigma_7(n/2) - \frac{1028}{17} \sigma_7(n/4) + \frac{992}{17} \sigma_7(n/8) \\
& + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{117}{34} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{80}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + \frac{15}{2} \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right) \\
& - \frac{690}{17} \delta \left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n \right) + \frac{80}{17} \delta \left(0, -\frac{1}{6}, \frac{5}{6}; n \right) + \frac{5171}{1088} \delta \left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n \right) \\
& - \frac{10215}{272} \delta \left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n \right) + \frac{1641}{68} \delta \left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n \right) + \frac{6795}{1088} \delta \left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n \right) \\
& - \frac{6105}{272} \delta \left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n \right) + \frac{5}{68} \delta \left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n \right),
\end{aligned}$$

$$N([12, 0, 4]; n)$$

$$\begin{aligned}
& = \frac{2}{17} \sigma_7(n) + \frac{6}{17} \sigma_7(n/2) - \frac{2056}{17} \sigma_7(n/4) + \frac{2016}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{49}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) - \frac{32}{17} \delta \left(0, \frac{1}{3}, \frac{1}{3}; n \right) + 9 \delta \left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{828}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{96}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1905}{272} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
& -\frac{3405}{68} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1359}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
& -\frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([13, 0, 3]; n)$$

$$\begin{aligned}
& = \left(\frac{4}{17} - \frac{1}{272} \left(\frac{-4}{n}\right)\right) \sigma_7(n) + \frac{4}{17} \sigma_7(n/2) - \frac{2056}{17} \sigma_7(n/4) + \frac{2016}{17} \sigma_7(n/8) \\
& + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{64}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + 9 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) - \frac{828}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) \\
& + \frac{96}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{5715}{544} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{10215}{136} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) \\
& + \frac{1641}{34} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{1359}{544} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1221}{136} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) \\
& + \frac{1}{34} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

$$N([14, 0, 2]; n)$$

$$\begin{aligned}
& = \left(\frac{8}{17} - \frac{1}{136} \left(\frac{-4}{n}\right)\right) \sigma_7(n) - \frac{8}{17} \sigma_7(n/2) - \frac{32}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{128}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + 6 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) - \frac{552}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) \\
& + \frac{64}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1905}{136} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{3405}{34} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) \\
& + \frac{1094}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right),
\end{aligned}$$

$$N([15, 0, 1]; n)$$

$$\begin{aligned}
& = \left(\frac{16}{17} - \frac{1}{136} \left(\frac{-4}{n}\right)\right) \sigma_7(n) - \frac{32}{17} \sigma_7(n/2) + \frac{4112}{17} \sigma_7(n/4) - \frac{4128}{17} \sigma_7(n/8) \\
& + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{256}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + \frac{1905}{136} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{3405}{34} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) \\
& + \frac{1094}{17} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right).
\end{aligned}$$

Proof. We only prove the formula for $N([13, 0, 3]; n)$ as the proofs for the other cases are similar. By Theorems 7.3.3 and 8.5.5, we have

$$\begin{aligned}
& N([13, 0, 3]; n) \\
&= \frac{4}{17}\sigma_7(n) + \frac{4}{17}\sigma_7(n/2) - \frac{2056}{17}\sigma_7(n/4) + \frac{2016}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{64}{17}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + 9\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) - \frac{828}{17}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) \\
&\quad + \frac{96}{17}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{1}{8}\delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) + \frac{1905}{272}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{3405}{68}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{17}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) - \frac{1}{8}\delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right) \\
&\quad + \frac{1359}{272}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1221}{68}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{17}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right). \\
\\
&= \frac{4}{17}\sigma_7(n) + \frac{4}{17}\sigma_7(n/2) - \frac{2056}{17}\sigma_7(n/4) + \frac{2016}{17}\sigma_7(n/8) + \frac{8192}{17}\sigma_7(n/16) \\
&\quad + \frac{64}{17}\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + 9\delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) - \frac{828}{17}\delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) \\
&\quad + \frac{96}{17}\delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) \\
&\quad - \frac{1}{272}\left(-34\delta\left(-\frac{7}{6}, \frac{37}{12}, -\frac{5}{4}; n\right) + \frac{1905}{2}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right)\right. \\
&\quad \left.- 6810\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + 4376\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + 34\delta\left(-\frac{5}{6}, \frac{31}{12}, -\frac{13}{12}; n\right)\right. \\
&\quad \left.- \frac{1359}{2}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) + 2442\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) - 8\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right)\right) \\
&\quad + \frac{5715}{544}\delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{10215}{136}\delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{1641}{34}\delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\
&\quad + \frac{1359}{272}\delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1221}{68}\delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{34}\delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right) \\
\\
&= \left(\frac{4}{17} - \frac{1}{272}\left(\frac{-4}{n}\right)\right)\sigma_7(n) + \frac{4}{17}\sigma_7(n/2) - \frac{2056}{17}\sigma_7(n/4) + \frac{2016}{17}\sigma_7(n/8)
\end{aligned}$$

$$\begin{aligned}
& + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{64}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) + 9 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) - \frac{828}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) \\
& + \frac{96}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) \\
& + \frac{5715}{544} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) - \frac{10215}{136} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{1641}{34} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) \\
& + \frac{1359}{272} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) - \frac{1221}{68} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{1}{34} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right),
\end{aligned}$$

which is the desired result. $\square$

In the last two theorems of this section, we simplify Theorem 9.3.4 when $n \equiv 0 \pmod{8}$ and $n \equiv 4 \pmod{8}$.

Theorem 9.3.5. *Let $n \in \mathbb{N}$. For $n \equiv 0 \pmod{8}$, we have*

$$\begin{aligned}
N([1, 0, 15]; n) &= \frac{4064}{17} \sigma_7(n/8) + \frac{4096}{17} \sigma_7(n/16) + \frac{8}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([2, 0, 14]; n) &= \frac{4064}{17} \sigma_7(n/8) + \frac{4096}{17} \sigma_7(n/16) + \frac{8}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([3, 0, 13]; n) &= \frac{4064}{17} \sigma_7(n/8) + \frac{4096}{17} \sigma_7(n/16) + \frac{8}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([4, 0, 12]; n) &= 480 \sigma_7(n/8) + \frac{3}{4} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([5, 0, 11]; n) &= \frac{24544}{17} \sigma_7(n/8) - \frac{16384}{17} \sigma_7(n/16) + \frac{127}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([6, 0, 10]; n) &= \frac{65504}{17} \sigma_7(n/8) - \frac{57344}{17} \sigma_7(n/16) + \frac{317}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([7, 0, 9]; n) &= 8672 \sigma_7(n/8) - 8192 \sigma_7(n/16) + \frac{41}{4} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([8, 0, 8]; n) &= \frac{294880}{17} \sigma_7(n/8) - \frac{286720}{17} \sigma_7(n/16) + \frac{341}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([9, 0, 7]; n) &= \frac{557024}{17} \sigma_7(n/8) - \frac{548864}{17} \sigma_7(n/16) + \frac{611}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([10, 0, 6]; n) &= \frac{1048544}{17} \sigma_7(n/8) - \frac{1040384}{17} \sigma_7(n/16) + \frac{1028}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),
\end{aligned}$$

$$\begin{aligned}
N([11, 0, 5]; n) &= \frac{2031584}{17} \sigma_7(n/8) - \frac{2023424}{17} \sigma_7(n/16) + \frac{1658}{17} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([12, 0, 4]; n) &= \frac{4063200}{17} \sigma_7(n/8) - \frac{4055040}{17} \sigma_7(n/16) + \frac{10395}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([13, 0, 3]; n) &= \frac{8257504}{17} \sigma_7(n/8) - \frac{8249344}{17} \sigma_7(n/16) + \frac{15775}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([14, 0, 2]; n) &= \frac{16777184}{17} \sigma_7(n/8) - \frac{16769024}{17} \sigma_7(n/16) + \frac{22145}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([15, 0, 1]; n) &= \frac{33816544}{17} \sigma_7(n/8) - \frac{33808384}{17} \sigma_7(n/16) + \frac{24277}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right).
\end{aligned}$$

Proof. We only prove the formula for $N([11, 0, 5]; n)$ as the proofs for the other cases are similar. By Theorem 9.3.4, Theorems 8.5.1 - 8.5.4, Theorem 8.6.1 and the fact that $\left(\frac{-4}{n}\right) = 0$ when $n \equiv 0 \pmod{8}$, we have, for $n \equiv 0 \pmod{8}$

$$\begin{aligned}
&N([11, 0, 5]; n) \\
&= \left(\frac{1}{17} + \frac{1}{544} \left(\frac{-4}{n}\right)\right) \sigma_7(n) + \frac{3}{17} \sigma_7(n/2) - \frac{1028}{17} \sigma_7(n/4) + \frac{992}{17} \sigma_7(n/8) \\
&\quad + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{117}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{80}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + \frac{15}{2} \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{690}{17} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{80}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{5171}{1088} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{10215}{272} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{1641}{68} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{6795}{1088} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{6105}{272} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{5}{68} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right) \\
&= \frac{1}{17} \sigma_7(n) + \frac{3}{17} \sigma_7(n/2) - \frac{1028}{17} \sigma_7(n/4) + \frac{992}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{117}{34} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{80}{17} \left(\frac{1}{8} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\
&\quad + \frac{15}{2} \left(-6 \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) - \frac{690}{17} \left(-\frac{3}{8} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\
&\quad + \frac{5171}{1088} \left(45 \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) - \frac{10215}{272} \left(3 \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1641}{68} \left(\frac{3}{16} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) \right) + \frac{6795}{1088} \left(3 \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) \right) \\
& + \frac{5}{68} \left(-\frac{3}{16} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right) \right) \\
& = \frac{1}{17} \sigma_7(n) + \frac{3}{17} \sigma_7(n/2) - \frac{1028}{17} \sigma_7(n/4) + \frac{992}{17} \sigma_7(n/8) + \frac{8192}{17} \sigma_7(n/16) \\
& + \frac{117}{34} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right).
\end{aligned}$$

Simplifying using (9.3.3), we obtain the desired result. $\square$

Theorem 9.3.6. *Let $n \in \mathbb{N}$. For $n \equiv 4 \pmod{8}$, we have*

$$\begin{aligned}
N([1, 0, 15]; n) &= \frac{32}{17} \sigma_7(n/4) - \frac{8}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([2, 0, 14]; n) &= \frac{32}{17} \sigma_7(n/4) - \frac{8}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([3, 0, 13]; n) &= \frac{32}{17} \sigma_7(n/4) - \frac{8}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([4, 0, 12]; n) &= \frac{64}{17} \sigma_7(n/4) - \frac{47}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([5, 0, 11]; n) &= \frac{192}{17} \sigma_7(n/4) - \frac{107}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([6, 0, 10]; n) &= \frac{512}{17} \sigma_7(n/4) - \frac{257}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([7, 0, 9]; n) &= \frac{1152}{17} \sigma_7(n/4) - \frac{557}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([8, 0, 8]; n) &= \frac{2304}{17} \sigma_7(n/4) - \frac{270}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([9, 0, 7]; n) &= 256 \sigma_7(n/4) - 28 \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([10, 0, 6]; n) &= \frac{8192}{17} \sigma_7(n/4) - \frac{773}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([11, 0, 5]; n) &= \frac{15872}{17} \sigma_7(n/4) - \frac{1163}{17} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right), \\
N([12, 0, 4]; n) &= \frac{31744}{17} \sigma_7(n/4) - \frac{6465}{68} \delta \left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n \right),
\end{aligned}$$

$$\begin{aligned}
N([13, 0, 3]; n) &= \frac{64512}{17} \sigma_7(n/4) - \frac{8157}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([14, 0, 2]; n) &= \frac{131072}{17} \sigma_7(n/4) - \frac{8859}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right), \\
N([15, 0, 1]; n) &= \frac{264192}{17} \sigma_7(n/4) - \frac{6727}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right),
\end{aligned}$$

Proof. We only prove the formula for $N([7, 0, 9]; n)$ as the proofs for the other cases are similar. By Theorem 9.3.4, Theorems 8.5.1 - 8.5.4, Theorem 8.6.1 and the fact that $\left(\frac{-4}{n}\right) = 0$ when $n \equiv 4 \pmod{8}$, we have, for $n \equiv 4 \pmod{8}$

$$\begin{aligned}
&N([7, 0, 9]; n) \\
&= \left(\frac{1}{272} - \frac{1}{2176} \left(\frac{-4}{n}\right)\right) \sigma_7(n) - \frac{1}{16} \sigma_7(n/2) + \frac{257}{17} \sigma_7(n/4) - \frac{288}{17} \sigma_7(n/8) \\
&\quad + \frac{8192}{17} \sigma_7(n/16) \\
&\quad + \frac{1087}{272} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{135}{17} \delta\left(0, \frac{1}{3}, \frac{1}{3}; n\right) + 3 \delta\left(-\frac{2}{3}, \frac{11}{6}, -\frac{1}{2}; n\right) \\
&\quad - \frac{1035}{34} \delta\left(-\frac{1}{3}, \frac{5}{6}, \frac{1}{6}; n\right) + \frac{60}{17} \delta\left(0, -\frac{1}{6}, \frac{5}{6}; n\right) + \frac{881}{256} \delta\left(-\frac{5}{6}, \frac{25}{12}, -\frac{7}{12}; n\right) \\
&\quad - \frac{3373}{64} \delta\left(-\frac{1}{2}, \frac{13}{12}, \frac{1}{12}; n\right) + \frac{547}{16} \delta\left(-\frac{1}{6}, \frac{1}{12}, \frac{3}{4}; n\right) + \frac{15489}{4352} \delta\left(-\frac{1}{2}, \frac{19}{12}, -\frac{5}{12}; n\right) \\
&\quad - \frac{18315}{1088} \delta\left(-\frac{1}{6}, \frac{7}{12}, \frac{1}{4}; n\right) + \frac{15}{272} \delta\left(\frac{1}{6}, -\frac{5}{12}, \frac{11}{12}; n\right) \\
&= \frac{1}{272} \sigma_7(n) - \frac{1}{16} \sigma_7(n/2) + \frac{257}{17} \sigma_7(n/4) \\
&\quad + \frac{1087}{272} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right) - \frac{135}{17} \left(\frac{1}{8} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\
&\quad + 3 \left(-6 \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) - \frac{1035}{34} \left(-\frac{3}{8} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\
&\quad + \frac{881}{256} \left(-15 \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) - \frac{3373}{64} \left(-\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\
&\quad + \frac{547}{16} \left(-\frac{1}{16} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) + \frac{15489}{4352} \left(-\delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right) \\
&\quad + \frac{15}{272} \left(\frac{1}{16} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right)\right)
\end{aligned}$$

$$= \frac{1}{272} \sigma_7(n) - \frac{1}{16} \sigma_7(n/2) + \frac{257}{17} \sigma_7(n/4) - \frac{557}{68} \delta\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}; n\right).$$

Simplifying using (9.3.2), we obtain the desired result. $\square$

We close by remarking that the first three formulae of Theorems 9.3.5 and 9.3.6 can be derived from the Ramanujan-Mordell formula for $r_{16}(n)$, see the remark following Theorem 9.2.8.

Conclusion

The research of this thesis suggests several possibilities for further development. First, the analogous formulae for $N([r, s, t]; n)$ ($r + s + t = 4k$) should be determined when s is odd. Perhaps this will involve the sums

$$\sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(\frac{8}{d}\right) d^{2k-1}, \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(\frac{8}{d}\right) (n/d)^{2k-1} \quad ?$$

Secondly the formulae for $N([r, s, t]; n)$ ($r + s + t = 4k + 2$) should be found. In this case it is possible that the evaluations will require sums of the form

$$\begin{aligned} \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(\frac{-4}{d}\right) d^{2k}, \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(\frac{-4}{n/d}\right) d^{2k}, \quad s \text{ even,} \\ \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(\frac{-8}{d}\right) d^{2k}, \sum_{\substack{d \in \mathbb{N} \\ d|n}} \left(\frac{-8}{n/d}\right) d^{2k}, \quad s \text{ odd.} \end{aligned}$$

Finally it would be interesting to determine the minimum number of δ -coefficients that are required for the evaluations of $N([r, s, t]; n)$ including the cases when n is restricted to certain residue classes.

List of References

- [1] A. Alaca, Ş. Alaca, M. F. Lemire and K. S. Williams, *Jacobi's identity and representations of integers by certain quaternary quadratic forms*, Int. J. Modern Math. **2** (2007), 143-176.
- [2] A. Alaca, Ş. Alaca, M. F. Lemire and K. S. Williams, *Nineteen quaternary quadratic forms*. Acta Arith. **130** (2007), 277-310.
- [3] A. Alaca, Ş. Alaca, M. F. Lemire and K. S. Williams, *The number of representations of a positive integer by certain quaternary quadratic forms*, Int. J. Number Theory **5** (2009), 13-40.
- [4] A. Alaca, Ş. Alaca and K. S. Williams, *The simplest proof of Jacobi's six squares theorem*, Far East J. Math. Sci. **27** (2007), 187-192.
- [5] A. Alaca, Ş. Alaca and K. S. Williams, *Seven octonary quadratic forms*, Acta Arith. **135** (2008), 339-350.
- [6] A. Alaca, Ş. Alaca and K. S. Williams, *Berndt's curious formula*, Int. J. Number Theory **4** (2008), 677-689.
- [7] A. Alaca, Ş. Alaca and K. S. Williams, *The convolution sum*

$$\sum_{m < n/16} \sigma(m)\sigma(n - 16m)$$
, Canad. Math. Bull. **51** (2008), 3-14.
- [8] A. Alaca, Ş. Alaca and K. S. Williams, *Liouville's sextenary quadratic forms*
 $x^2 + y^2 + z^2 + t^2 + 2u^2 + 2v^2$, $x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2$ and $x^2 + 2y^2 + 2z^2 + 2t^2 + 2u^2 + 4v^2$, Far East J. Math. Soc. **30** (2008), 547-556.
- [9] A. Alaca, Ş. Alaca and K. S. Williams, *Sums of $4k$ squares: a polynomial approach*, J. Combinatorics and Number Theory 1 (2) (2009), 33-52.
- [10] A. Alaca, Ş. Alaca and K. S. Williams, *Some identities involving theta functions*, J. Number Theory 129 (2009), 1404-1431.

- [11] A. Alaca, Ş. Alaca and K. S. Williams, *Fourteen octonary quadratic forms*, Int. J. Number Theory **6** (2010), 37-50.
- [12] K. Ananda Rau, *On the representation of a number as a sum of an even number of squares*, J. Madras Univ. Part B **24** (1954), 61-89.
- [13] N. C. Ankeny, *Sums of 3 squares*, Proc. Amer. Math. Soc. **8** (1957), 316-319.
- [14] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer Verlag, New York 1976.
- [15] P. T. Bateman, *On the representation of a number as a sum of three squares*, Trans. Amer. Math. Soc. **71** (1951), 70-101.
- [16] E. T. Bell, *On the number of representations of $2n$ as a sum of $2r$ squares*, Bull. Amer. Math. Soc. **26** (1919), 19-25.
- [17] E. Benz, *Über die Anzahl Darstellungen einer Zahlen durch gewisse quaternäre quadratische Formen: Beweise, welche auf Identiäten aus dem Gebiete der Thetafunktionen basieren. Dissertation, Zürich 1964*, in "Studien zur Theorie der quadratischen Formen", editors B. L. van der Waerden und H. Gross, Stuttgart, 1968, pp. 165-198.
- [18] B. C. Berndt, *Ramanujan's Notebooks*, Part I (1985), Part II (1989), Part III (1991), Part IV (1994), Part V (1998), Springer-Verlag, New York.
- [19] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Amer Math. Soc., Providence, Rhode Island, USA, 2006.
- [20] J. M. Borwein, P. B. Borwein and F. G. Garvan, *Some cubic modular identities of Ramanujan*, Trans. Amer. Math. Soc. **343** (1994), 35-47.
- [21] V. Bulygin (V. Boulyguine), *Sur une application des fonction elliptiques au problème de représentation des nombres entiers par une somme de carrés*, Bull. Acad. Imp. Sci. St. Petersburg **6** (1914), 389-404.
- [22] L. Carlitz, *On the representations of an integer as the sum of 24 squares*, Nederl. Akad. Wetensch. Indag. Math **17** (1955), 504-506.
- [23] L. Carlitz, *Note on sums of 4 and 6 squares*, Proc. Amer. Math. Soc. **8** (1957), 120-124.
- [24] L. Carlitz, *Bulygin's method for sums of squares*, J. Number Theory **5** (1973), 405-412.

- [25] H. H. Chan, S. Cooper and W.-C. Liaw, *On $\eta^3(a\tau)\eta^3(b\tau)$ with $a + b = 8$* , J. Austral. Math. Soc. **84** (2008), 301–313.
- [26] H. H. Chan and K. S. Chua, *Representations of integers as sums of 32 squares*, Ramanujan J. **7** (2003), 79–89.
- [27] N. Cheng, *Convolution sums involving divisor functions*, M.Sc. thesis, Carleton University, Ottawa, Canada, 2003.
- [28] N. Cheng and K. S. Williams, *Evaluation of some convolution sums involving the sum of divisors functions*, Yokohama Math. J. **52** (2005), 39–57.
- [29] J. B. Conway, *Functions of One Complex Variable*, Second Edition, Springer-Verlag, Providence, New York, USA, 1978.
- [30] E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, 1946.
- [31] S. Cooper, *On sums of an even number of squares, and an even number of triangular numbers: an elementary approach based on Ramanujan's ${}_1\psi_1$ summation formula*, in *q-series with Applications to Combinatorics, Number Theory, and Physics*, editors B. C. Berndt and K. Ono, Contemporary Mathematics **291** (2001) Amer. Math. Soc., Providence, RI, 2001, pp. 115–137.
- [32] S. Cooper and H. Y. Lam, *Sums of two, four, six and eight squares and triangular numbers: an elementary approach*, Indian J. Math. **44** (2002), 21–40.
- [33] P. Deligne, *Formes modulaires et représentations l -adiques*, Lecture Notes in Mathematics, No. 179, Berlin: Springer-Verlag, (1971), 139–172.
- [34] P. Deligne, *La conjecture de Weil I*, Publ. Math. IHES, **43** (1974), 273–307.
- [35] P. Demuth, *Die Zahl der Darstellungen einer natürlichen Zahl durch spezielle quaternäre quadratische Formen aufgrund der Siegelschen Massformel*, in “*Studien zur Theorie der quadratischen Formen*”, editors B. L. van der Waerden und H. Gross, Stuttgart, 1968, pp. 224–254.
- [36] L. E. Dickson, *History of the Theory of Numbers*, Vols. I–III, Chelsea, 1966.
- [37] Th. Estermann, *On the representations of a number as a sum of 2 squares*, Acta. Arith. **2** (1936), 47–79.
- [38] Th. Estermann, *On the representations of a number as a sum of 3 squares*, Proc. London Math. Soc. (3) **9** (1959), 575–594.

- [39] Th. Estermann, *On the representations of a number as a sum of squares*, Acta Arith. **2** (1936), 47-79.
- [40] J. A. Ewell, *A simple derivation of Jacobi's four-square formula*, Proc. Amer. Math. Soc. **85** (1982), 323-326.
- [41] J. A. Ewell, *On sums of sixteen squares*, Rocky Mountain J. Math. **17** (1987), 295-299.
- [42] J. W. L. Glaisher, *On the numbers of representations of a number as a sum of $2r$ squares, where $2r$ does not exceed eighteen*, Proc. London Math. Soc. (2) **5** (1907), 479-490.
- [43] J. W. L. Glaisher, *On the representations of a number as a sum two, four, six, eight, ten and twelve squares*, Quart. J. Pure Appl. Math. Oxford **38** (1907), 1-62.
- [44] J. W. L. Glaisher, *On the representations of a number as a sum of fourteen and sixteen squares*, Quart. J. Pure Appl. Math. Oxford **38** (1907), 178-236.
- [45] J. W. L. Glaisher, *On the representations of a number as a sum of eighteen squares*, Quart. J. Pure Appl. Math. Oxford **38** (1907), 289-351.
- [46] E. Grosswald, *Representation of Integers as Sums of Squares*, Springer-Verlag, New York, 1985.
- [47] K.-B. Gundlach, *On the representation of a number as a sum of squares*, Glasgow Math. J. **19** (1978), 173-197.
- [48] G. H. Hardy, *On the representation of a number as a sum of any number of squares and in particular of five and seven*, Proc. Nat. Acad. Sci., U.S.A. **4** (1918), 189-193.
- [49] G. H. Hardy, *On the representation of a number as a sum of any number of squares and in particular of five*, Trans. Amer. Math. Soc. **21** (1920), 255-284.
- [50] M. D. Hirschhorn, *A simple proof of Jacobi's four-square theorem*, J. Austral. Math. Soc. Ser. A **32** (1982), 61-67.
- [51] J. G. Huard and K. S. Williams, *Sums of twelve squares*, Acta Arith. **109** (2003), 195-204.
- [52] J. G. Huard and K. S. Williams, *Sums of sixteen squares*, Far East J. Math. Sci. **7** (2002), 147-164.

- [53] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, 1829, in *Gesammelte Werke (Erster Band)*, Chelsea Publishing Co., New York, 1969, pp. 49-239.
- [54] L. J. P. Kilford, *Modular Forms*, Imperial College Press, 2008.
- [55] F. Klein and R. Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, Vol. I (1890), Vol. II (1892), Teubner, Leipzig.
- [56] M. I. Knopp, *Modular Functions in Analytic Number Theory*, Chelsea Publishing Co., New York, 1993.
- [57] J. Liouville, *Extrait d'une lettre adressée à M. Besge*, J. Math. Pures Appl. **9** (1864), 296-298.
- [58] G. A. Lomadze, *Representation of numbers by sums of the quadratic forms $x_1^2 + x_1x_2 + x_2^2$* , Acta Arith. **54** (1989), 9-36. (in Russian)
- [59] Y. Martin, *Multiplicative η -quotients*, Trans. Amer. Math. Soc. **348** (1996), 4825-4856.
- [60] E. McAfee and K. S. Williams, *Sums of six squares*, Far East J. Math. Sci. **16** (2005), 17-41.
- [61] S. C. Milne, *Infinite families of exact sums of squares formulas, Jacobi elliptic functions, continued fractions, and Schur functions*, The Ramanujan Journal, **6** (2002), no. 1, 1-151.
- [62] L. J. Mordell, *On Mr. Ramanujan's empirical expansions of modular functions*, Proc. Cambridge Philos. Soc. **19** (1917), 117-124.
- [63] L. J. Mordell, *On the representations of numbers as a sum of $2r$ squares*, Quart. J. Pure Appl. Math. **48** (1917), 93-104.
- [64] L. J. Mordell, *On the representations of a number as a sum of 3 squares*, Rev. Math. Pures Appl. **3** (1958), 25-27.
- [65] L. J. Mordell, *On the representations of a number as a sum of an odd number of squares*, Trans. Cambridge Philos. Soc. **22** (1919), 361-372.
- [66] C. J. Moreno and S. S. Wagstaff, *Sums of Squares of Integers*, Chapman and Hall, Boca Raton, Florida, 2006.

- [67] K. Ono, *Representations of integers as sums of squares*, J. Number Theory **95** (2002), 253–258.
- [68] T. Pepin, *Étude sur quelques formules d'analyse utiles dans la théorie des nombres*, Atti Accad. Pont. Nuovi Lincei **38** (1884-5), 139-196.
- [69] T. Pepin, *Sur quelques quadratiques quaternaires*, J. Math. Pures Appl. **6** (1890), 5-67.
- [70] H. Petersson, *Modulfunktionen und Quadraische Formen*, Springer-Verlag, Berlin, 1982.
- [71] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), 159-184.
- [72] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.
- [73] S. Ramanujan, *Notebooks, 2 vols*, Tata Institute of Fundamental Research, Bombay, 1957.
- [74] R. A. Rankin, *On the representations of a number as a sum of squares and certain related identities*, Proc. Cambridge Philos. Soc. **41** (1945), 1-11.
- [75] R. A. Rankin, *Representations of a number as the sum of a large number of squares*, Proc. Roy. Soc. Edinburgh Sect. A **65** (1960/61), 318-331.
- [76] R. A. Rankin, *On the representation of a number as the sum of any number of squares and in particular of 20*, Acta Arith. **7** (1961/62), 399-407.
- [77] R. A. Rankin, *Sums of squares and cusp forms*, Amer. J. Math. **87** (1965), 857-860.
- [78] R. A. Rankin, *Modular Forms and Functions*, Cambridge University Press, Cambridge, 1977.
- [79] B. K. Spearman and K. S. Williams, *The simplest arithmetic proof of Jacobi's four squares theorem*, Far East J. Math. Sci. **2** (2000), 433-439.
- [80] B. van der Pol, *The representation of numbers as sums of 8, 16 and 24 squares*, Indag. Math. **16** (1954), 349-361.

- [81] H. Wild, *Die Anzahl der Darstellungen einer natürlichen Zahl durch die Form $x^2 + y^2 + z^2 + 2t^2$* , Abh. Math. Sem. Univ. Hamburg **40** (1974), 132-135.
- [82] K. S. Williams, *An arithmetic proof of Jacobi's eight squares theorem*, Far East J. Math. Sci. **3** (2001), 1001-1005.
- [83] K. S. Williams, *The convolution sum $\sum_{m < n/8} \sigma(m)\sigma(n - 8m)$* , Pacific J. Math. **228** (2006), 387-396.
- [84] K. S. Williams, *On the representations of a positive integer by the forms $x^2 + y^2 + z^2 + 2t^2$ and $x^2 + 2y^2 + 2z^2 + 2t^2$* , Int. J. Modern Math. **3** (2008), 225-230.
- [85] K. S. Williams, *On Liouville's twelve squares theorem*, Far East J. Math. Sci. **29** (2008), 239-242.