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Contributions to the Theory of Almost Periodic Differential Equations

by

Zuosheng Hu

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics

Carleton University
Ottawa, Ontario, Canada

May, 2001

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**Contributions to the Theory of Almost Periodic Differential
Equations**

Submitted by

Zuosheng Hu, B.Sc., M.Sc.

in partial fulfillment of the requirements
for the degree of Doctor of Philosophy.

Director, School of Mathematics and Statistics

Dr. A. B. Mingarelli, Thesis Supervisor

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May, 2001

Abstract

In this dissertation we make some contributions to the theory of almost periodic differential equations. Our work deals with the following three topics:

- (i) The non-almost periodicity of all bounded solutions;
- (ii) The existence of at least one almost periodic solution;
- (iii) The almost periodicity of all bounded solutions.

In Chapter 2 we solve the following open problem posed by Professor Mingarelli several years ago:

If $p_i(t)$ are all (Bohr) almost periodic and all the solutions of

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0 \quad (1)$$

are bounded, does it necessarily follow that all solutions are (Bohr) almost periodic?

We do this by constructing a class of Bohr almost periodic differential equations all of whose solutions are bounded, but for which any non-trivial solution is not Bohr almost periodic. We also construct a (Bohr) almost periodic differential equation all of whose solutions are bounded but any non-trivial solution is not S-almost periodic, (i. e. Stepanov almost periodic).

In Chapter 3 we establish some interesting results which extend or refine the famous Favard Theorem for linear differential systems. Not only do we extend Favard's theorem to *general processes*, we also replace the condition

$$\inf_{t \in \mathbb{R}} \|x(t)\| > 0 \quad (2)$$

by

$$\inf_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|x(s)\| ds \right) > 0. \quad (3)$$

On the other hand, if we assume some uniformity on these almost periodic *processes*, the weaker condition

$$\inf_{t \in \mathbb{R}} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|x(s)\| ds \right) > 0 \quad (4)$$

also guarantees the existence of almost periodic solutions of the corresponding processes.

In Chapter 4, we turn to the problem:

Under what condition is any bounded solution of an almost periodic system almost periodic?

We discuss some nonlinear contractive processes on a separable Hilbert space. Under some assumptions, we obtain some results which show that all bounded solutions of such processes are Bohr almost periodic if they have relatively compact range in H . These results generalize several results due to Haraux [64], Levitan and Zhikov [87] and Phong [109].

Finally, in the last chapter, we discuss differential equations with Stepanov almost periodic coefficients. We establish some results on the existence of S-almost periodic solutions for such systems. At the same time, we show that Favard's Theorem holds for such systems.

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Chapter 1

Introduction

1.1 Almost periodic functions

The theory of almost periodic functions was created and developed in its main features by H. Bohr during the 1920's (see [12 ,20] and the literature cited therein). Like many other important mathematical discoveries it is connected with several branches of the modern theory of functions and differential equations. On the one hand, the notion of almost periodicity as a structural property of a function is a generalization of pure periodicity; on the other hand, the theory of almost periodic functions opens a way to studying a wide class of trigonometric series of the general type and even exponential series.

In the theory of pure periodic functions, it is well known that any pure periodic function $f(t)$ defined on $(-\infty, \infty)$ can be developed into a series like

$$\sum_{-\infty}^{\infty} a_n e^{inat} \tag{1.1}$$

i.e., the Fourier series of $f(t)$. Before Bohr, Bohl once established the theory of

functions so called *functions periodic in a more general sense* which corresponds to the case where only vibrations of the form $ae^{i(n_1\alpha_1+\dots+n_m\alpha_m)t}$ are considered, where $\alpha_1, \dots, \alpha_m$ are given linearly independent constants while $n_1, \dots, n_m$ run independently through all integers. Every such function with the *period* $p_1 = \frac{2\pi}{\alpha_1}, \dots, p_m = \frac{2\pi}{\alpha_m}$ can be developed into a series of the form

$$\sum a_{n_1} \dots a_{n_m} e^{i(n_1\alpha_1+\dots+n_m\alpha_m)t} \quad (1.2)$$

Both the theories of pure periodic functions and that of functions in Bohl's sense have as a basic fact that the basis system of frequencies is denumerable (countable). What is Bohr's idea? The set of frequencies may be non-denumerable! The main problem of the theory of almost periodic functions consists in finding those functions $f(t)$ of a real variable which can be resolved into *pure vibrations*, that is, finding those functions which are representable by a trigonometric series of the form $\sum A_n e^{i\lambda_n t}$ (notice that the set of frequencies is not assumed to be denumerable).

Because Bohr's original methods for establishing the fundamental results of the theory were always based on reducing the problems to a problem of pure periodic functions, the proofs of his main results were very difficult and complicated. N. Wiener [132] and H. Weyl [131] used some new methods to extend this theory. At the same time, Bohr's theory of almost periodic functions was restricted to the class of uniformly continuous functions on R . Therefore, there were several authors who made great efforts to generalize this theory.

The first generalization is based on the fact that the class of Bohr almost periodic functions is the class of continuous functions possessing certain structural properties, which amount to a generalization of pure periodicity. W. Stepanov [129] removed

the continuity restrictions and characterized generalized almost periodicity not by values of the functions at each point but by mean values over intervals of fixed length. H. Weyl [131] gave a new structural generalization of almost periodicity, which was wider than Stepanov's type (see [12, 20] and the references cited therein).

The second generalization is based on the fact that the class of Bohr almost periodic functions is the class of limit functions of uniformly convergent sequences of trigonometric polynomials. Besicovitch [12] enlarged Bohr's class of almost periodic functions by considering the convergence of sequences of functions in a more general sense than uniform convergence, and by defining almost periodic functions as limits of such sequences of trigonometric polynomials (see [12]).

In order to apply the theory of almost periodic functions to ergodic theory, Fréchet introduced the concept of asymptotically almost periodic functions in the 1940's (see [54]). Later, there were several authors who used this notion of asymptotically almost periodicity of solutions to discuss the existence of almost periodic solutions of differential equations (see [38, 49, 135, etc.]).

Bochner [15] and Bochner and Neumann [16] extended Bohr's theory to very general abstract spaces. The extension to Banach space is, in particular, of great interest, in view of the fundamental importance of these spaces in theory and application. Such an extension has been the object of research for many decades. At the same time, almost periodicity has been considered both in the strong and in the weak sense. For details, see [3, 4, 26, 41, 55, 87, 111].

For our purposes, we will give the exact definitions of Bohr almost periodic functions and some of their generalizations since we will use them in later chapters. At the same time, we give the more general form of these definitions (almost pe-

riodic functions in Banach spaces). We use the following notation throughout this dissertation.

$R :=$ the real axis $(-\infty, \infty)$;

$R^+ :=$ the half real axis $[0, \infty)$;

$Z :=$ the set of all integers;

$Z^+ :=$ the set of all positive integers;

$R^n :=$ n -dimensional Euclidean space;

$X :=$ real Banach space;

$\|\cdot\| :=$ the norm of X ;

$C(R, X) :=$ the space of all continuous functions defined on R , valued in X ;

$L_1^{loc}(R, X) :=$ the space of all functions defined on R , valued in X , and locally Lebesgue integrable on R ;

If $\{\alpha_n\} \subset R$ is a real sequence, we write it $\alpha = \{\alpha_n\}$. If $\beta = \{\beta_n\}$, then $\beta \subset \alpha$ means that β is a subsequence of α . We also write $\alpha + \beta = \{\alpha_n + \beta_n\}$, $-\alpha = \{-\alpha_n\}$. α' and β' are common subsequences of α and β respectively means that $\alpha'_n = \alpha_{n(k)}$ and $\beta'_n = \beta_{n(k)}$ for some given function $n(k)$.

Definition 1.1 [Bohr] *Let $f \in C(R, X)$. We call f Bohr almost periodic on R if for any given $\varepsilon > 0$, there exists a real number $L = L(\varepsilon) > 0$ such that for any $a \in R$, there exists a number $\tau \in [a, a + L]$ such that*

$$\|f(t + \tau) - f(t)\| < \varepsilon, \text{ for any } t \in R. \quad (1.3)$$

When $X = R$, this definition is Bohr's original definition. We use $AP(R, X)$ to denote the set of all Bohr almost periodic functions defined on R , valued in X .

We won't discuss the basic properties of Bohr almost periodic functions, but only the important ones, those which we will use in the next chapters. We will use the following results.

Theorem 1.1 [Bochner] *Let $f \in C(R, X)$. Then $f \in AP(R, X)$ if and only if for any sequence $\alpha = \{\alpha_n\} \subset R$, one can extract a subsequence $\alpha' \subset \alpha$ such that the sequence $\{f(t + \alpha'_n)\}$ is convergent to some function $g \in C(R, X)$ uniformly on R .*

In fact, Bochner called this property of functions *normality* and he proved that it is equivalent to Bohr's notion of almost periodicity. So, in later literature, some authors defined almost periodic functions in this form (e.g. see [49]).

For convenience, we introduce more notation. If $f, g \in C(R, X)$ and there exists a sequence $\alpha \subset R$ such that

$$\lim_{n \rightarrow \infty} f(t + \alpha_n) = g(t), \text{ pointwise for } t \in R \quad (1.4)$$

then we write this as $T_\alpha f = g$. If the limit (1.4) holds uniformly for all $t \in R$, we write $UT_\alpha f = g$. We use $H(f)$ to denote the **uniform hull** of f , i. e.,

$$H(f) = \{g \in C(R, X) | \text{there exists a sequence } \alpha \text{ such that } UT_\alpha f = g\}.$$

There are several properties of the hull $H(f)$, but we do not present them here (see [49, 99]).

Theorem 1.2 [Bochner] *Let $f \in C(R, X)$. Then $f \in AP(R, X)$ if and only if for any pair of sequences $\alpha, \beta \subset R$, there exist two common subsequences $\alpha' \subset \alpha, \beta' \subset \beta$ such that $T_{\alpha'+\beta'} f, T_{\beta'} f$, and $T_{\alpha'}(T_{\beta'} f)$ all exist and*

$$T_{\alpha'+\beta'} f = T_{\alpha'}(T_{\beta'} f). \quad (1.5)$$

This result plays an important role in discussing the existence of almost periodic solutions to differential equations (see [33, 47, 49, 61, 74, etc.])

Theorem 1.3 *$AP(R, X)$ is an algebra over R . It is closed under uniform limits. Furthermore, if $f \in AP(R, X)$ and F is uniformly continuous on the range of f , then $F(f(t)) \in AP(R, X)$. If $f \in AP(R, X)$ and $\inf_{t \in R} \|f(t)\| > 0$, then $\frac{1}{f(t)} \in AP(R, X)$.*

The proof of this theorem can be found in several sources (e.g. see [4, 12, 20, 49, 135, etc.]).

We also use the following theorems due to Bohr.

Theorem 1.4 *(H. Bohr [20], or M. Fink [49]) Let $F(t)$ be a real continuous function. If $f(t) = \exp iF(t)$ is Bohr almost periodic, then there exist a real number c and a real almost periodic function $G(t)$ such that $F(t) = ct + G(t)$.*

Theorem 1.5 *(Bohr-Bohl) Let $f \in AP(R, R^n)$ and let $F(t) = \int_0^t f(s)ds$, then $F(t) \in AP(R, R^n)$ if and only if $F(t)$ is bounded on R .*

Remark If $f \in AP(R, X)$, where X is a infinite dimensional Banach space, this theorem requires the additional condition: The set $\{F(t)\}$ is relatively compact in X .

For $f \in L_1^{loc}(R, X)$, $\tau \in R$, we define $f_\tau(t) = f(t + \tau)$ for all $t \in R$ and call it the **translate of f at τ** . Obviously, if $f \in L_1^{loc}(R, X)$ (respectively, $C(R, X)$, $AP(R, X)$), then $f_\tau \in L_1^{loc}(R, X)$ (respectively, $C(R, X)$, $AP(R, X)$).

Definition 1.2 *Let $f \in L_1^{loc}(R, X)$, $l > 0$ is a real number. We call f Stepanov almost periodic with respect to l if for any $\varepsilon > 0$, there exists $L = L(\varepsilon) > 0$ such*

that for any $a \in R$, there exists a $\tau \in [a, a + L]$ such that

$$\frac{1}{l} \int_t^{t+l} \|f(s + \tau) - f(s)\| ds < \varepsilon, \text{ for all } t \in R. \quad (1.6)$$

For convenience, we define

$$S_l(t, f) = \frac{1}{l} \int_t^{t+l} \|f(s)\| ds, \text{ for all } t \in R. \quad (1.7)$$

and

$$S_l(f) = \sup_{t \in R} S_l(t, f), \quad (1.8)$$

called the **Stepanov norm** of f . So, (1.6) can be written as

$$S_l(f_\tau - f) < \varepsilon. \quad (1.9)$$

It has been proved that if f is Stepanov almost periodic with respect to $l > 0$, then for any $l' > 0$, f is also Stepanov almost periodic with respect to l' (see [12]). So, we simply say that f is Stepanov almost periodic without the phase, *with respect to l* , and write this as f is Stepanov almost periodic (or simply S-almost periodic). We denote the set of all S-almost periodic functions by $SAP(R, X)$. But whenever we take $f \in SAP(R, X)$, the l in the definition can be taken as any real positive number.

Just like we said above, we won't present definitions of other more general notions of almost periodicity. The definitions of **Weyl almost periodicity** and **Besicovitch almost periodicity** can be found in [12] and the references cited therein, while the definition of **weak almost periodicity** can be found in [26, 55, 87, 111, 113, etc.]. In these papers, weakly almost periodicity of functions also was discussed.

Although the concept of S-almost periodic functions was introduced at almost the same time as that of Bohr's, it has not been discussed as widely as Bohr almost periodic functions. Although as has been observed by Bochner [15], S-almost periodicity can be reduced to almost periodicity in the sense of Bohr in a more general space, there are still some problems that are open. For instance, to this author's knowledge, the answers to the following problems are unknown: *Are some of Bochner's theorems true for S-almost periodic functions? Does Bohr's theorem on the integral of an almost periodic function hold for S-almost periodic functions?* There have been several results on the integral of S-almost periodic functions (see [87]), but all of these results deal with the Bohr almost periodicity of the integral of S-almost periodic functions. So, stronger conditions are required. In Chapter 5, we will discuss these problems and establish some corresponding results. Like the case of Bohr almost periodic functions, we introduce the uniform Stepanov hull of a function and discuss some of its properties. Then, we show that Bochner's theorem (Theorem 1.2) holds for Stepanov almost periodic functions. At the same time, we also extend Bohr's theorem (Theorem 1.5) on the integration of almost periodic functions to the case of S-almost periodic functions without any additional condition, i.e., *if $f(t)$ is Stepanov almost periodic and the range of $F(t) = \int_0^t f(s)ds$ is relatively compact in X , then $F(t)$ is Stepanov almost periodic on R .* These results allow us to establish some results on the solutions of S-almost periodic differential equations.

1.2 Almost Periodic Differential Equations

Among the many reasons that Bohr introduced the concept of an almost periodic function, we find the description of properties of the solutions to some differential equations. It is well known that all solutions of the oscillatory system

$$x'' + \lambda^2 x = 0 \quad (1.10)$$

with $\lambda \in \mathbb{R}$ are pure periodic on $\mathbb{R}$. As well, all the solutions of the system

$$x'' - \lambda^2 x = 0 \quad (1.11)$$

are unbounded on $\mathbb{R}$. But if we consider the system

$$x^{(4)} + 3x'' + 2x = 0, \quad (1.12)$$

what can we say about its solutions? We know that the general solutions of (1.12) can be expressed in the form

$$x(t) = c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t + c_3 \cos t + c_4 \sin t. \quad (1.13)$$

Obviously, all these solutions are bounded on $\mathbb{R}$, but not all are pure periodic. However, we know that they are all Bohr almost periodic. Furthermore, at almost the same time as Bohr introduced the concept of almost periodic functions, he and Neugebauer [21] discussed the linear differential equation

$$x' = Ax + f(t) \quad (1.14)$$

where A is an n th-order constant matrix, $f \in AP(\mathbb{R}, \mathbb{R}^n)$ and they proved that a solution of (1.14) is almost periodic on $\mathbb{R}$ if and only if it is bounded on $\mathbb{R}$ (see [21, 49]).

On account of Floquet theory and Bohr and Neugebauer's result for the equation (1.14), it is easy to show that if $A(t)$ is a pure periodic matrix function and $f(t)$ is a Bohr almost periodic vector function defined on R , then any solution of the system

$$x' = A(t)x + f(t) \quad (1.15)$$

is Bohr almost periodic if and only if it is bounded on R . It is natural that one would expect such a result to hold in the event that $A(t)$ is merely Bohr almost periodic on R . Indeed, some authors (e.g. [1]) have used this claim in proofs. However, it is shown in [36, 98] that this weaker result is false. In Chapter 2, we will give out the details of this problem. We construct a class of Bohr almost periodic differential equations all of whose solutions are bounded, but any nontrivial solution is not Bohr almost periodic. We also construct a Bohr almost periodic differential equation all of whose solutions are bounded, but any nontrivial solution is not S-almost periodic.

It is natural to ask under what condition any bounded solution of (1.15) is Bohr almost periodic. There are few results on this question in the general case. Bochner once established a result on the general case, but the conditions are stronger (see [49, p.108]). There are some results on this question for the special cases such as the first order equation (see [22]) and the second order equation (see [25]).

Another interesting question is under what conditions does (1.15) have at least one Bohr almost periodic solution. Related to this problem there is the famous Favard theorem (see [46]). And this result has been extended several general cases (see [47, 74, 91, 93, 106, 107, etc.]). In Chapter 3, we extend Favard's Theorem to the case of Bohr almost periodic *processes* and almost periodic functional differential

equations. At the same time, we relax the Favard condition

$$\inf_{t \in R} |x(t)| > 0 \quad (1.16)$$

to

$$\inf_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} |x(s)| ds \right) > 0 \quad (1.17)$$

where $l > 0$ is some real constant. In Chapter 5, we extend Favard's Theorem to S-almost periodic systems.

For the homogeneous linear system

$$x' = A(t)x \quad (1.18)$$

where $A(t)$ is a Bohr almost periodic matrix function, there is the following result which displays the character of Bohr almost periodic solutions of (1.18) (see [49, p.85]).

Theorem 1.6 *Let $A(t)$ be a Bohr almost periodic matrix function. If $x(t)$ is a Bohr almost periodic solution of (1.18), then either $x(t) \equiv 0$ on R or*

$$\inf_{t \in R} |x(t)| > 0. \quad (1.19)$$

For the nonlinear system

$$x' = f(t, x) \quad (1.20)$$

where $f(t, x)$ is Bohr almost periodic in t uniformly for $x \in R^n$, it is more difficult to guarantee the Bohr almost periodicity of any bounded solution. It is well-known that if $f(t, x)$ is pure periodic in t , and all solutions of (1.20) are bounded, then (1.20) has at least one pure periodic solution (see [135, p.164, Theorem 15.4]). But is this

result true for the case where $f(t, x)$ is only Bohr almost periodic in t uniformly for $x \in R^n$? Opial [104] constructed such an equation with all its solutions bounded but non-Bohr almost periodic. Later Fink and Frederickson [51] constructed another such equation which has no Bohr almost periodic solution but all solutions of which are uniformly ultimately bounded. Therefore, in order to guarantee the existence of a Bohr almost periodic solution, some additional conditions are required. Amerio [2] introduced the *separation condition* of solutions. Fink [48, 49, 50] extended this condition to a semi-separation condition. F. Nakajima [102] and G Seifert [118] also discussed the almost periodicity of solutions using separation conditions. Others assumed some kind of stability properties for bounded solutions. For example, Miller [96] assumed totally stability, Seifert [118–120] assumed Σ -stability, and Sell [125] assumed stability under a disturbance from the hull. These results all use the theory of dynamical systems and hence the uniqueness of solutions of corresponding initial value problems is assumed at the outset. Many of these results can be obtained by using the property of asymptotic almost periodicity without reference to the uniqueness of solutions (see [38, 133, 135, etc.]). In Chapter 5, we extend some of these results to S-almost periodic differential equations.

Many authors discussed the existence of Bohr almost periodic solutions for the system (1.20) in special cases or even in more general cases (see [17, 33, 38, 42, 44, 52, 67, 79, 80, 88–90, 100–103, 118, 120, 133–136, etc.])

Some of these previous results have been extended to evolution equations in Banach spaces, that is, equations of the form

$$u' = A(t)u + f(t) \tag{1.21}$$

where $A(t)$ is some operator with a parameter $t \in R$ from X to X , $f(t)$ is a function defined on R , valued in X , and $A(t)$ and $f(t)$ are all Bohr almost periodic on R . Most of results deal with the case that $A(t)$ is a linear closed operator. The details can be found in [7, 18, 19, 58, 62, 68, 73, 87, 92, 109, 112, 139–141, etc.]). For the case where X is infinite dimensional, the situation is worse because even though $A(t)$ is independent of t , the bounded solutions of (1.21) may not be Bohr almost periodic (see [87, 109]). But there is the following result.

Theorem 1.7 ([87]) *Let $A : X \rightarrow X$ be a linear bounded operator such that $\sigma(A) \cap iR$ is countable and let $f : R \rightarrow X$ be a Bohr almost periodic function. If $u(t)$ is a bounded and uniformly continuous solution of (1.21), then $u(t)$ is Bohr almost periodic if and only if it has a relatively compact range in X .*

There are very few results in the literature which deal directly with Stepanov almost periodic differential equations, i. e., the right side of the equation is Stepanov almost periodic in t . The author only found several such results in the literature which discussed this field (see [83, 84, 108, 115]).

1.3 Processes in Banach Space

The original work on processes is the fundamental paper of Levinson [85] dealing with periodic differential equations in the plane. Over the years, a tremendous literature on this subject has accumulated. Continuing in the spirit of Levinson for finite dimensions, many authors discussed processes on a general Banach space in order to solve more complicated problems. A process can be considered as being generated by a differential equation. So, the properties of a process can describe some

properties of solutions of corresponding differential equations. At the same time, an abstract process can characterize many differential equations, such as ordinary differential equations, functional differential equations, partial differential equations, and etc.

Definition 1.3 *A process on a Banach Space X is a two-parameter mapping $U(t, \tau) : X \rightarrow X, t \in R, \tau \in R^+$ satisfying the following properties*

- (i) $U(t, 0) = I$, the identity;
- (ii) $U(t, \sigma + \tau) = U(t + \tau, \sigma)U(t, \tau), t \in R, \tau, \sigma \in R^+$;
- (iii) $U(t, \tau, x) = U(t, \tau)x$ is continuous.

Just like differential equations, we can describe some structural properties of a process such as linearity, periodicity or almost periodicity.

A process U is said to be *linear* if $U(t, \tau)$ are bounded linear operators on X for any $t \in R, \tau \in R^+$.

A process U is called *periodic* if there exists a real number $\omega > 0$ such that $U(t + \omega, \tau) = U(t, \tau)$ for any $t \in R, \tau \in R^+$.

A process U is said to be *almost periodic* if for any sequence $\{\sigma_n\} \subset R$, there exists a subsequence $\{\sigma'_n\} \subset \{\sigma_n\}$ such that $\{U(t + \sigma'_n, \tau)x\}$ converges to some $V(t, \tau)x$ in X uniformly in $t \in R$ and pointwise in $(\tau, x) \in R^+ \times X$.

A process U is *autonomous* if $U(t, \tau)$ is independent of t . In this case, $T(\tau) \equiv U(t, \tau)$ is a continuous one-parameter *semigroup* on X : $T(0) = I$ and $T(\tau + \sigma) = T(\tau)T(\sigma), \tau, \sigma \in R^+$ (These kinds of processes have been discussed widely (see [26, 41, 56, etc.])).

Let U be a process on a Banach space X .

A *positive trajectory* of U through $(t, x) \in R \times X$ is defined by the map $U(t, \cdot)x : R^+ \rightarrow X$, i. e. $u(s) = U(t, s)x, s \in R^+$.

A *complete trajectory* of U through $(t, x) \in R \times X$ is a function $u : R \rightarrow X$ satisfying

$$\begin{aligned} u(t) &= x; \\ u(s + \tau) &= U(s, \tau)u(s), \text{ for all } (s, \tau) \in R \times R^+. \end{aligned}$$

We are interested in discussing the properties of trajectories of almost periodic processes. On this subject, Dafermos [40] established some basic properties of trajectories for almost periodic processes. We will refer to some of his results in Chapter 3 and Chapter 4. Ishii [74], Haraux [63] and Vu Quoc Phong [109] discussed some special kind of processes and obtained results on the existence of almost periodic trajectories and almost periodicity of trajectories which have relatively compact range in X . We also refer to some of these results in Chapter 4.

One of our results is a generalization of Favard's theorem. Not only do we extend Favard's theorem to general processes, we also replace the condition

$$\inf_{t \in R} \|u(t)\| > 0 \tag{1.22}$$

by

$$\inf_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\| ds \right) > 0. \tag{1.23}$$

At the same time, if the processes satisfy some uniformity conditions, condition (1.23) can be replaced by

$$\inf_{t \in R} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\| ds \right) > 0. \tag{1.24}$$

These results are given in Chapter 3.

In Chapter 4, we introduce the concept of a compact synchronous Hilbert space and discuss some nonlinear contractive processes on a separable compact synchronous Hilbert space H . Under some assumptions, we obtain results which show that all bounded trajectories of such processes are Bohr almost periodic if they have relatively compact range in H . These results generalize several results due to Haraux [63], Zhikov [87] and Phong [109].

Chapter 2

Non-Almost Periodicity of All Bounded Solutions

2.1 Introduction

In this chapter, we mainly discuss the non-almost periodicity of bounded solutions of almost periodic differential equations by means of some examples. One of the greatest impediments to a thorough understanding of the nature of solutions of linear differential equations with (Bohr) almost periodic coefficients is the lack of a Floquet theory.

It is well-known [49, p.101] that if the $p_i(t)$, $i = 1, 2, \dots, n$ are piecewise continuous periodic functions on R , then every bounded solution of

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0 \quad (2.1)$$

is almost periodic on account of Floquet theory. The analogous result for almost periodic coefficients $p_i(t)$ is known to be false. For example, Conley and Miller [36]

gave an example of an equation (2.1) with $n = 1$ where a bounded solution is not almost periodic, however this result could not extend to $n > 1$. In [98], Mingarelli, Pu and Zheng constructed an example, for each $n > 1$, of an equation (2.1) with almost periodic coefficients in which there exists a bounded solution which is not almost periodic. But for each such case $n > 1$, there is always another unbounded solution. As a result of this peculiarity, the authors of [98] raised the following question:

If $p_i(t)$ are all almost periodic and all the solutions of (2.1) are bounded, does it necessarily follow that all solutions are Bohr almost periodic?

This is a question whose answer has eluded researchers for some time. In this chapter we answer this simple question in the negative thus annihilating any hope of a Floquet-type theory for general linear almost periodic differential equations. We will show that, for each $n > 2$, there exists an equation of form (2.1) for which *every* solution is bounded on $\mathbb{R}$ but yet no solution (except the trivial solution) is almost periodic. Indeed, the situation is worse than we thought originally, as we had hoped for a dichotomy, at the very least.

After answering this question, we pose the following natural question:

Can the boundedness of all solutions imply their Stepanov almost periodicity for the general system

$$x' = A(t)x \tag{2.2}$$

where $A(t)$ is a Bohr almost periodic matrix function?

In this chapter, we shall establish a necessary condition for the Stepanov almost periodicity of solutions for general linear almost periodic systems, and then construct an example of a linear system (2.2) in which all solutions are bounded, but any

nontrivial solution is not Stepanov almost periodic. It is indeed really too bad! The natural conclusion eludes us here.

2.2 Lemmas

In this section, we establish some lemmas in order to prove out main result.

Lemma 2.1 *Suppose that $a_i(t)$, $(i = 1, 2, \dots, n)$ and $g(t)$ have continuous $(n-1)$ th-order derivatives. Then the following equation holds*

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a'_1 + a_1g & a'_2 + a_2g & \cdots & a'_n + a_ng \\ (a'_1 + a_1g)' & (a'_2 + a_2g)' & \cdots & (a'_n + a_ng)' \\ \cdots & \cdots & \cdots & \cdots \\ (a'_1 + a_1g)^{(n-2)} & (a'_2 + a_2g)^{(n-2)} & \cdots & (a'_n + a_ng)^{(n-2)} \end{vmatrix} = W(a_1, \dots, a_n)(t) \quad (2.3)$$

where $W(a_1, \dots, a_n)(t)$ is the Wronskian determinant of $a_1, \dots, a_n$.

Proof We use an induction argument. For $n = 2$, we have

$$\begin{vmatrix} a_1 & a_2 \\ a'_1 + a_1g & a'_2 + a_2g \end{vmatrix} = a_1a'_2 + a_1a_2g - a_2a'_1 - a_1a_2g = \begin{vmatrix} a_1 & a_2 \\ a'_1 & a'_2 \end{vmatrix}.$$

We assume that (2.3) is true for some n . Now, we verify that (2.3) is true for $n + 1$. In fact,

$$\begin{vmatrix}
 a_1 & a_2 & \cdots & a_n \\
 a'_1 + a_1 g & a'_2 + a_2 g & \cdots & a'_n + a_n g \\
 \cdots & \cdots & \cdots & \cdots \\
 (a'_1 + a_1 g)^{(n-2)} & (a'_2 + a_2 g)^{(n-2)} & \cdots & (a'_n + a_n g)^{(n-2)} \\
 (a'_1 + a_1 g)^{(n-1)} & (a'_2 + a_2 g)^{(n-1)} & \cdots & (a'_n + a_n g)^{(n-1)}
 \end{vmatrix}$$

$$= \sum_{i=1}^{n+1} (-1)^{n+1+i} (a'_i + a_i g)^{(n-1)} \times$$

$$\begin{vmatrix}
 a_1 & \cdots & a_{i-1} & a_{i+1} & \cdots & a_{n+1} \\
 a'_1 + a_1 g & \cdots & a'_{i-1} + a_{i-1} g & a'_{i+1} + a_{i+1} g & \cdots & a'_{n+1} + a_{n+1} g \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 (a'_1 + a_1 g)^{(n-1)} & \cdots & (a'_{i-1} + a_{i-1} g)^{(n-1)} & (a'_{i+1} + a_{i+1} g)^{(n-1)} & \cdots & (a'_{n+1} + a_{n+1} g)^{(n-1)}
 \end{vmatrix}$$

$$= \sum_{i=1}^{n+1} (-1)^{n+1+i} (a'_i + a_i g)^{(n-1)} \times W(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n+1})(t)$$

$$= \begin{vmatrix}
 a_1 & a_2 & \cdots & a_{n+1} \\
 a'_1 & a'_2 & \cdots & a'_{n+1} \\
 \cdots & \cdots & \cdots & \cdots \\
 a_1^{(n-1)} & a_2^{(n-1)} & \cdots & a_{n+1}^{(n-1)} \\
 (a'_1 + a_1 g)^{(n-1)} & (a'_2 + a_2 g)^{(n-1)} & \cdots & (a'_{n+1} + a_{n+1} g)^{(n-1)}
 \end{vmatrix}$$

$$\begin{aligned}
 &= W(a_1, \dots, a_{n+1})(t) + \begin{vmatrix} a_1 & \cdots & a_{n+1} \\ a'_1 & \cdots & a'_{n+1} \\ \cdots & \cdots & \cdots \\ a_1^{(n-1)} & \cdots & a_{n+1}^{(n-1)} \\ (a_1 g)^{(n-1)} & \cdots & (a_{n+1} g)^{(n-1)} \end{vmatrix} \\
 &= W(a_1, \dots, a_{n+1})(t).
 \end{aligned}$$

So, by induction, (2.3) holds for any n .

The following lemma also follows by induction on account of Lemma 2.1 .

Lemma 2.2 *Suppose that $a_i(t)$, ($i = 1, 2, \dots, n$) and $g(t)$ are the same as in Lemma 2.1. If*

$$\phi_i(t) = a_i(t) \exp\left(\int_0^t g(s) ds\right), \quad i = 1, 2, \dots, n,$$

then

$$W(\phi_1, \dots, \phi_n)(t) = W(a_1, \dots, a_n)(t) \exp\left(n \int_0^t g(s) ds\right). \quad (2.4)$$

Proof We still use the induction for n . As $n = 2$, by Lemma 2.1, we have that

$$\begin{vmatrix} \phi_1 & \phi_2 \\ \phi'_1 & \phi'_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ a'_1 + a_1 g & a'_2 + a_2 g \end{vmatrix} \exp\left(2 \int_0^t g(s) ds\right) = \begin{vmatrix} a_1 & a_2 \\ a'_1 & a'_2 \end{vmatrix} \exp\left(2 \int_0^t g(s) ds\right).$$

So, as $n = 2$, (2.4) is true. Suppose that (2.4) holds for some $n \geq 2$. We shall show that (2.4) holds for $n + 1$.

We set $b_i(t) = a'_i(t) + a_i(t)g(t)$ and $\psi_i(t) = b_i(t) \exp\left(\int_0^t g(s) ds\right)$, $i = 1, 2, \dots, n$.

Then we have that

$$W(\phi_1, \dots, \phi_{n+1}) = \sum_{i=1}^{n+1} (-1)^i a_i(t) W(\psi_1, \dots, \psi_{i-1}, \psi_{i+1}, \dots, \psi_{n+1})(t) \exp\left(\int_0^t g(s) ds\right)$$

$$\begin{aligned}
 &= \sum_{i=1}^{n+1} (-1)^i a_i(t) W(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{n+1})(t) \exp\left(\int_0^t g(s) ds\right) \\
 &= \begin{vmatrix} a_1 & a_2 & \dots & a_{n+1} \\ b_1 & b_2 & \dots & b_{n+1} \\ \dots & \dots & \dots & \dots \\ b_1^{(n-1)} & b_2^{(n-1)} & \dots & b_{n+1}^{(n-1)} \end{vmatrix} \exp\left((n+1) \int_0^t g(s) ds\right)
 \end{aligned}$$

Using Lemma 2.1, we obtain that

$$W(\phi_1, \dots, \phi_{n+1})(t) = W(a_1, \dots, a_{n+1})(t) \exp\left((n+1) \int_0^t g(s) ds\right). \quad (2.5)$$

By the induction, (2.4) holds for any n and the lemma is proved.

2.3 The case of Bohr almost periodicity

In this section, we establish a general result which shows that there are some equations all of whose solutions are bounded but no nontrivial solution is Bohr almost periodic.

Theorem 2.1 *If there exist functions $a_i(t)$, ($i = 1, 2, \dots, n$) and $g(t)$ such that*

- (i) $a_i(t), a_i^{(k)}(t); (i = 1, 2, \dots, n; k = 1, 2, \dots, n-1)$ are all Bohr almost periodic on R ;
- (ii) $\inf_{t \in R} |W(a_1, \dots, a_n)(t)| \neq 0$;
- (iii) $g(t), g^{(k)}(t), (k = 1, 2, \dots, n-1)$ are all Bohr almost periodic on R ;
- (iv) $\int_0^t g(s) ds \leq 0$ for all $t \in R$;
- (v) $\inf_{t \in R} \int_0^t g(s) ds = -\infty$, then there exist Bohr almost periodic functions $p_i(t), (i = 1, 2, \dots, n)$ such that all the solutions of the equation

$$x^{(n)} + p_n(t)x^{(n-1)} + \dots + p_1(t)x = 0 \quad (2.6)$$

are bounded on R , but any nontrivial solution of (2.6) is not almost periodic.

Proof Let

$$\phi_i(t) = a_i(t) \exp\left(\int_0^t g(s)ds\right), \quad i = 1, 2, \dots, n.$$

By Lemma 2.2 and condition (ii), we have

$$W(\phi_1, \dots, \phi_n)(t) = W(a_1, \dots, a_n)(t) \exp\left(n \int_0^t g(s)ds\right) \neq 0$$

for all $t \in R$. Therefore, the $p_i(t)$, ($i = 1, 2, \dots, n$), can be determined by the following linear system

$$\left\{ \begin{array}{l} \phi_1 p_1(t) + \phi_1' p_2(t) + \dots + \phi_1^{(n-1)} p_n(t) = -\phi_1^{(n)} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \phi_n p_1(t) + \phi_n' p_2(t) + \dots + \phi_n^{(n-1)} p_n(t) = -\phi_n^{(n)} \end{array} \right\} \quad (2.7)$$

and

$$p_i(t) = \frac{W_i(t)}{W(a_1, \dots, a_n)(t) \exp\left(n \int_0^t g(s)ds\right)}$$

where

$$W_i(t) = \begin{vmatrix} \phi_1 & \dots & \phi_1^{(i-1)} & -\phi_1^{(n)} & \phi_1^{(i+1)} & \dots & \phi_1^{(n-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \phi_n & \dots & \phi_n^{(i-1)} & -\phi_n^{(n)} & \phi_n^{(i+1)} & \dots & \phi_n^{(n-1)} \end{vmatrix}, \quad i = 1, 2, \dots, n.$$

We also have

$$\phi_i^{(k)}(t) = b_{i,k}(t) \exp\left(\int_0^t g(s)ds\right), \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots, n-1$$

where $b_{i,k}(t)$ are some algebraic combinations of $a_i^{(k)}(t)$ and $g^{(k)}(t)$. In fact,

$$b_{i,k}(t) = a_i^{(k)}(t) + c_{1,k}(t)a_i^{(k-1)}(t) + \cdots + c_{k,k}(t)a_i(t), \quad i = 1, 2, \dots, n$$

where the $c_{j,k}(t)$ are given by

$$c_{j,k}(t) = d_{1,j,k}g^{(j-1)} + d_{2,j,k}g^{(j-2)} + \cdots + d_{m_j,j,k}g^j$$

and $d_{l,j,k}$ are constants ($l = 1, 2, \dots, m_j$, $j = 1, 2, \dots, k$, $k = 1, 2, \dots, n-1$). So

$$W_i(t) = B_i(t) \exp\left(n \int_0^t g(s) ds\right), \quad i = 1, 2, \dots, n$$

where $B_i(t)$, ($i = 1, 2, \dots, n$) are still some algebraic combinations of $a_i^{(k)}(t)$ and $g^{(k)}(t)$ ($i = 1, 2, \dots, n$; $k = 1, 2, \dots, n-1$). It follows that

$$p_i(t) = \frac{B_i(t)}{W(a_1, \dots, a_n)(t)}, \quad i = 1, 2, \dots, n$$

are almost periodic by Theorem 1.3. For these $p_i(t)$, (2.6) is an almost periodic differential equation and $\{\phi_1(t), \phi_2(t), \dots, \phi_n(t)\}$ is a fundamental system of solutions of (2.6). From the assumptions on $a_i(t)$ and $g(t)$, It is obvious that $\phi_i(t)$, ($i = 1, 2, \dots, n$) are all bounded on R , and thus all solutions of (2.6) are bounded.

Next, we show that any nontrivial solution of (2.6) is not almost periodic. Let $x(t)$ be a nontrivial solution, then there exist constants $C_1, \dots, C_n$ such that

$$x(t) = C_1\phi_1(t) + \cdots + C_n\phi_n(t) = (C_1a_1(t) + \cdots + C_na_n(t)) \exp\left(\int_0^t g(s) ds\right).$$

Let $b(t) = C_1a_1(t) + \cdots + C_na_n(t)$, then $b(t) \neq 0$, ($t \in R$) and inherits the same properties as $a_i(t)$. We write $x^{(k)}(t)$ in the form

$$x^{(k)}(t) = b_k(t) \exp \left(\int_0^t g(s) ds \right), \quad k = 1, 2, \dots, n-1$$

where $b_k(t)$, $k = 1, 2, \dots, n-1$, are some algebraic combinations of $b(t)$, $b^{(k)}(t)$, $g(t)$ and $g^{(k)}(t)$. We see that the $b_k(t)$ are almost periodic and thus bounded ($k = 1, 2, \dots, n-1$).

Let $X(t) = \text{Col}(x(t), x'(t), \dots, x^{(n-1)}(t))$, then $X(t)$ is a solution of the system

$$X'(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -p_1(t) & -p_2(t) & -p_3(t) & \dots & -p_n(t) \end{pmatrix} X(t). \quad (2.8)$$

Now, (2.8) is an almost periodic system and

$$\inf_{t \in R} |X(t)|^2 = \inf_{t \in R} \sum_{k=1}^n |b_k(t)|^2 \exp \left(2 \int_0^t g(s) ds \right) = 0,$$

$$X(t) \neq 0, \quad \text{for all } t \in R.$$

Since $b_k(t)$, $k = 1, 2, \dots, n-1$, are bounded and

$$\inf_{t \in R} \int_0^t g(s) ds = -\infty,$$

$x(t)$ and $x^{(k)}(t)$, $k = 1, 2, \dots, n-1$, are all not almost periodic because if $x(t)$ is almost periodic, so is $x^{(k)}(t)$ for each k (because of the form of $x(t)$) and thus $X(t)$ is almost periodic, contradicting Theorem 1.6, since $X(t)$ is a non-trivial solution. This ends the proof of the theorem.

2.4 The case of Stepanov almost periodicity

Consider general systems of linear differential equations

$$x' = A(t)x + f(t), \quad (2.9)$$

where $x \in R^n$, $A(t)$ is an $n \times n$ matrix function, and $f(t)$ is an n -dimensional vector function, defined on R . Throughout this section, we always suppose that $A(t)$ and $f(t)$ are all Bohr almost periodic on R .

In order to discuss the Stepanov almost periodicity of solutions to the equation (2.9), we use the following notation. Let $l \in R$ be fixed. If $f, g \in C(R, X)$ and there exists a sequence $\alpha \subset R$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_t^{t+l} |f(s + \alpha_n) - g(s)| ds = 0 \quad \text{pointwise for } t \in R, \quad (2.10)$$

then we write this as $ST_\alpha f = g$. If the limit (2.10) holds uniformly for all $t \in R$, we write $UST_\alpha f = g$. We use $\mathcal{SH}(f)$ to denote the **S-uniform hull** of f , i. e.

$$\mathcal{SH}(f) = \{g \in C(R, X) \mid \text{there exists a sequence } \alpha \text{ such that } UST_\alpha f = g\}. \quad (2.11)$$

Lemma 2.3 *Suppose that $\phi(t)$ is a solution of (2.9). If there exist a sequence $\alpha = \{\alpha_n\}$, $g \in \mathcal{SH}(f)$, $B \in \mathcal{SH}(A)$, and $\varphi \in \mathcal{SH}(\phi)$ such that $UST_\alpha A = B$, $UST_\alpha f = g$ and $UST_\alpha \phi = \varphi$, then there exists a solution $\tilde{\varphi}(t)$ of*

$$x' = B(t)x + g(t), \quad (2.12)$$

such that $UST_\alpha \phi = \tilde{\varphi}$ on R . If ϕ is bounded, so is $\tilde{\varphi}$ with the same bound.

Proof By the assumptions, $A(t)$ and $f(t)$ are Bohr almost periodic, so they are Stepanov almost periodic. From (2.11) and (2.10), there exists a real number $l > 0$

such that

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_t^{t+l} |A(s + \alpha_n) - B(s)| ds = 0, \quad (2.13)$$

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_t^{t+l} |f(s + \alpha_n) - g(s)| ds = 0, \quad (2.14)$$

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_t^{t+l} |\phi(s + \alpha_n) - \varphi(s)| ds = 0, \quad (2.15)$$

uniformly on R . So, for any $\epsilon > 0$, there exists a subset I_0 of R , with $m(I_0) < \epsilon$ such that

$$\lim_{n \rightarrow \infty} |A(s + \alpha_n) - B(s)| = 0, \quad (2.16)$$

$$\lim_{n \rightarrow \infty} |f(s + \alpha_n) - g(s)| = 0, \quad (2.17)$$

$$\lim_{n \rightarrow \infty} |\phi(s + \alpha_n) - \varphi(s)| = 0, \quad (2.18)$$

uniformly on $R \setminus I_0$, where $m(I_0)$ is the Lebesgue measure of I_0 .

Now, we pick up $t_0 \in R \setminus I_0$, and define

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t_0) + \int_{t_0}^t (B(s)\varphi(s) + g(s)) ds & t \in I_0 \\ \varphi(t) & t \in R \setminus I_0 \end{cases}$$

It is easy to show that $\tilde{\varphi}(t)$ is a solution of (2.12) and $UST_\alpha \phi = \tilde{\varphi}$ on R . The last conclusion is obvious.

Remark According to this lemma, if ϕ is a solution of (2.9) and assumptions are satisfied, we can simply say that $UST_\alpha \phi$ is a solution of (2.12).

Theorem 2.2 *Let $A(t)$ be Bohr almost periodic on R . If $x(t)$ is a nontrivial Stepanov almost periodic solution of the equation*

$$x' = A(t)x, \quad (2.19)$$

then for any $l > 0$, we have

$$\inf_{t \in R} \frac{1}{l} \int_t^{t+l} |x(s)| ds > 0. \quad (2.20)$$

Proof Assuming the contrary, we suppose that there exists a real number $l > 0$ such that (2.20) does not hold, i. e.,

$$\inf_{t \in R} \frac{1}{l} \int_t^{t+l} |x(s)| ds = 0. \quad (2.21)$$

We can choose a sequence $\alpha = \{\alpha_n\}$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_{\alpha_n}^{\alpha_n+l} |x(s)| ds = 0 \quad (2.22)$$

or

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_0^l |x(s + \alpha_n)| ds = 0. \quad (2.23)$$

Since $A(t)$ is almost periodic on R and $x(t)$ is Stepanov almost periodic on R , we can extract a subsequence $\alpha' \subset \alpha$, $B(t) \in \mathcal{H}(\mathcal{A})$ and $y(t) \in \mathcal{SH}(x)$ such that $y = UST_{\alpha'} x$ and $B = UT_{\alpha'} A$. By Lemma 2.3, there exists a solution $\tilde{y}$ of the equation

$$y' = B(t)y \quad (2.24)$$

such that

$$\tilde{y} = UST_{\alpha'} x \quad (2.25)$$

on R . Since $x(t)$ is a nontrivial solution of (2.19), $\tilde{y}(t)$ is a nontrivial solution of (2.24).

On the other hand, we have

$$\frac{1}{l} \int_0^l |\tilde{y}(s)| ds \leq \frac{1}{l} \int_0^l |\tilde{y}(s) - x(s + \alpha'_n)| ds + \frac{1}{l} \int_0^l |x(s + \alpha'_n)| ds \quad (2.26)$$

Letting $n \rightarrow \infty$, we obtain that

$$\frac{1}{l} \int_0^l |\tilde{y}(s)| ds = 0 \quad (2.27)$$

from (2.22) and (2.25). And thus, $\tilde{y}(t) = 0$ a. e. on $[0, l]$. Since $\tilde{y}(t)$ is a solution of (2.24), we have $\tilde{y}(t) = 0$ for all $t \in R$. This contradicts the fact that $\tilde{y}(t)$ is a nontrivial solution of (2.24). This completes the proof of this theorem.

Corollary 2.1 *Let $a(t)$ be a scalar almost periodic function defined on R . If each bounded solution of the equation*

$$x' = a(t)x \quad (2.28)$$

is Stepanov almost periodic on R , then

$$\sup_{t \in R} \int_0^t a(s) ds < \infty \quad (2.29)$$

implies that for any real number $l > 0$,

$$\inf_{t \in R} \sup_{s \in [t, t+l]} \int_0^s a(\tau) d\tau > -\infty. \quad (2.30)$$

Proof Suppose that (2.29) holds. Then

$$x(t) = e^{\int_0^t a(s) ds} \quad (2.31)$$

is a nontrivial bounded solution of (2.28), so it is Stepanov almost periodic. By Theorem 2.2, for any $l > 0$,

$$\inf_{t \in R} \frac{1}{l} \int_0^{t+l} e^{\int_0^s a(\tau) d\tau} ds > 0 \quad (2.32)$$

Now we show that for any $l > 0$,

$$\inf_{t \in R} \sup_{s \in [t, t+l]} \int_0^s a(\tau) d\tau > -\infty. \quad (2.33)$$

Otherwise, we can choose a sequence $\{t_n\} \subset R$ such that

$$\sup_{s \in [t_n, t_n+l]} \int_0^s a(\tau) d\tau \leq -n, n = 1, 2, \dots \quad (2.34)$$

So,

$$\int_0^s a(\tau) d\tau \leq -n \quad (2.35)$$

for all $s \in [t_n, t_n + l], n = 1, 2, \dots$. Hence,

$$\frac{1}{l} \int_{t_n}^{t_n+l} e^{\int_0^s a(\tau) d\tau} ds \leq \frac{1}{l} \int_{t_n}^{t_n+l} e^{-n} ds = e^{-n}, n = 1, 2, \dots \quad (2.36)$$

This implies that

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_{t_n}^{t_n+l} e^{\int_0^s a(\tau) d\tau} ds = 0. \quad (2.37)$$

This contradicts (2.32) and the proof of this corollary is completed.

Corollary 2.2 *Let $a(t)$ be a Bohr almost periodic function defined on R . If there exists a positive constant M such that*

$$\int_0^t a(s) ds \leq M \quad (2.38)$$

and

$$\inf_{t \in R} \frac{1}{l} \int_t^{t+l} e^{\int_0^s a(\tau) d\tau} ds = 0, \quad (2.39)$$

then the function $e^{\int_0^t a(s) ds}$ is not Stepanov almost periodic on R .

2.5 Examples

In conformity with the results of the previous sections, we construct some examples of equations whose solutions are bounded, but not Bohr or Stepanov almost periodic on R . For any n , we define the function $g(t)$ as

$$g(t) = \sum_{k=1}^{\infty} f_k(t) \quad (2.40)$$

where

$$f_k(t) = -\frac{1}{k^n} \sin \frac{t}{k^{n+1}}, \quad t \in R.$$

Then $g(t)$ satisfies the conditions (iii) – (v) (see [98]) of Theorem 2.1 .

In order to construct the functions $a_i(t)$, $i = 1, 2, \dots, n$, we give the following lemmas, easily proved by induction.

Lemma 2.4 *Let $\lambda_i, i = 1, 2, \dots, m$, be any real numbers. Then the Wronskian determinant of the functions*

$$\{\cos \lambda_1 t, \sin \lambda_1 t, \dots, \cos \lambda_m t, \sin \lambda_m t\}$$

satisfies

$$W(\cos \lambda_1 t, \sin \lambda_1 t, \dots, \cos \lambda_m t, \sin \lambda_m t) = \prod_{i=1}^m \lambda_i \prod_{1 \leq i < j \leq m} (\lambda_i^2 - \lambda_j^2)^2. \quad (2.41)$$

Corollary 2.3 *Let $\lambda_i, [i = 1, 2, \dots, m]$, be any real numbers, Then*

$$W(\cos \lambda_1 t, \sin \lambda_1 t, \dots, \cos \lambda_m t, \sin \lambda_m t) = 0$$

if and only if there are i and j such that $i \neq j$ but $\lambda_i = \pm \lambda_j$ or there is an i such that $\lambda_i = 0$.

Now, we can construct the functions $a_i(t)$, $i = 1, 2, \dots, n$, satisfying the conditions (i) and (ii) of Theorem 2.1 for any n .

If n is even, we let

$$a_{2k-1}(t) = \cos \lambda_k t, \quad a_{2k}(t) = \sin \lambda_k t, \quad k = 1, 2, \dots, \frac{n}{2}.$$

where $\lambda_k \neq 0$, $(k = 1, 2, \dots, \frac{n}{2})$, and $\lambda_i \neq \pm \lambda_j$ as $i \neq j$. By the Corollary, we have that the $a_k(t)$, $k = 1, 2, \dots, n$, satisfy the conditions (i) and (ii) of Theorem 2.1 .

If n is odd, we still take λ_i such that $\lambda_k \neq 0$, $(k = 1, 2, \dots, \frac{n-1}{2})$, and $\lambda_i \neq \pm \lambda_j$ as $i \neq j$, and we let

$$a_1(t) = 1, \quad a_{2k}(t) = \cos \lambda_k t, \quad a_{2k+1}(t) = \sin \lambda_k t, \quad k = 1, 2, \dots, \frac{n-1}{2}.$$

Since $W(a_1, a_2, \dots, a_n)(t) = W(a_2, \dots, a_n)(t)$ and

$$\inf_{t \in R} |W(a_2, \dots, a_n)(t)| \neq 0,$$

the $a_k(t)$ still satisfy the conditions (i) and (ii) of Theorem 2.1 . In particular, when $n = 2$, we take $\lambda_1 = 1$, i.e., $a_1(t) = \cos t$, $a_2(t) = \sin t$. In this case, we let

$$\phi_1(t) = (\cos t) \exp \left(\int_0^t g(s) ds \right), \quad \phi_2(t) = (\sin t) \exp \left(\int_0^t g(s) ds \right).$$

Then $W(\phi_1, \phi_2)(t) \equiv 1$ for all $t \in R$. We let $p_1(t) = 1 + g^2 - g'$, $p_2(t) = -2g$ and consider the second-order linear equation

$$x'' + p_1(t)x' + p_2(t)x = 0 \quad (2.42)$$

Obviously, (2.42) is an almost periodic equation and has two linear independent solution $\phi_1(t)$ and $\phi_2(t)$. So, all solutions are bounded, but no nontrivial solution is almost periodic.

To construct an example for the case of Stepanov almost periodicity, let $n \geq 3$ and define

$$g_n(t) = \begin{cases} 0, & t \in [0, 1] \cup [2^{n-1} - 1, 2^{n-1}] \\ -\frac{n}{2^{n-1}-1}, & t \in [2, 2^{n-1} - 2] \\ \text{linear}, & t \in [1, 2] \cup [2^{n-1} - 2, 2^{n-1} - 1] \end{cases}$$

Now, extend $g_n(t)$ to be odd and periodic with period 2^n . Then, $g_n(t)$ satisfies the following

$$\begin{aligned} \int_0^t g_n(s)ds &\leq 0, \text{ for all } t \in R; \\ \sup_{t \in R} |g_n(t)| &= \frac{n}{2^{n-1}-1}; \\ \int_0^t g_n(s)ds &= -n \frac{2^{n-1}-3}{2^{n-1}-1}, \text{ for all } t \in [2^{n-1}-1, 2^{n-1}] \end{aligned}$$

for each $n \in \mathbb{Z}^+$. Since

$$\sum_{n=3}^{\infty} \sup_{t \in R} |g_n(t)| = \sum_{n=3}^{\infty} \frac{n}{2^{n-1}-1} < \infty \quad (2.43)$$

the function

$$g(t) = \sum_{n=3}^{\infty} g_n(t) \quad (2.44)$$

is almost periodic on R and

$$\int_0^t g(s)ds = \sum_{n=3}^{\infty} \int_0^t g_n(s)ds \leq 0, t \in R \quad (2.45)$$

Now, let $l = 1$, $t_n = 2^{n-1} - 1$, then

$$\begin{aligned}
 \frac{1}{l} \int_{t_n}^{t_n+l} e^{\int_0^s g(\tau) d\tau} ds &= \int_{2^{n-1}-1}^{2^{n-1}} e^{\int_0^s g(\tau) d\tau} ds \\
 &\leq \int_{2^{n-1}-1}^{2^{n-1}} e^{\int_0^s g_n(\tau) d\tau} ds \\
 &= \int_{2^{n-1}-1}^{2^{n-1}} e^{-n \frac{2^{n-1}-3}{2^{n-1}-1}} ds \\
 &= e^{-n \frac{2^{n-1}-3}{2^{n-1}-1}}.
 \end{aligned}$$

for each $n \in \mathbb{Z}^+$. So,

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_{t_n}^{t_n+l} e^{\int_0^s g(\tau) d\tau} ds = 0. \quad (2.46)$$

This implies that

$$\inf_{t \in \mathbb{R}} \frac{1}{l} \int_t^{t+l} e^{\int_0^s g(\tau) d\tau} ds = 0. \quad (2.47)$$

By Corollary 2.2 , the function

$$e^{\int_0^t g(s) ds} \quad (2.48)$$

is not Stepanov almost periodic on $\mathbb{R}$. Therefore, all nontrivial solutions of equation

$$x' = g(t)x \quad (2.49)$$

are not Stepanov almost periodic on $\mathbb{R}$, but they are all bounded on $\mathbb{R}$ because (2.45) holds.

Chapter 3

The Extensions of Favard's Theorem

3.1 Introduction

In this Chapter, we will mainly discuss extensions of Favard's theorem. This topic is motivated by a former paper [98] in which the authors show that there exists a second order real linear differential equation on the line with almost periodic coefficients for which every solution is bounded but no non-trivial solution is almost periodic (in the classical sense of H. Bohr). This surprising phenomenon shows that boundedness, by itself, is not sufficient to guarantee the existence of almost periodic solutions of even the simplest linear equations. This is in sharp contrast with the periodic case in which Floquet's theory applies and gives the existence of almost periodic solutions in the case of bounded solutions. The authors' task of finding sufficient conditions for the existence of almost periodic solutions of linear

systems of ordinary differential equations was undertaken by the legendary Jean Favard [46], a central figure in the theory of almost periodic differential equations whose work also inspired S. Bochner [15] in his extensions of the theory of almost periodic functions to Banach space (See [46, 49, 126] for further details).

In order to guarantee the existence of at least one (always assumed non-trivial in the sequel) almost periodic solution for the system

$$x' = A(t)x + f(t), \quad (3.1)$$

Favard [46] established the following condition. If any non-trivial solution $x(t)$ of the system

$$x' = B(t)x \quad (3.2)$$

satisfies

$$\inf_{t \in \mathbb{R}} |x(t)| > 0, \quad (3.3)$$

for every $B(t)$ belonging to the *uniform hull* of the almost periodic matrix $A(t)$, where f is also almost periodic, then there exists at least one solution of (3.1) which is almost periodic. There are results in various directions in the literature which imply Favard's Theorem (see [2, 49, 74, 126, 135] and the references cited therein) but none of these results relax condition (3.3). It is indeed a difficult condition to relax as we will see below.

A consequence of the results in this thesis consists in the relaxation of condition (3.3) thereby producing an extension of Favard's Theorem from $\mathbb{R}^n$ to Banach space under a slight weaker assumption than (3.3).

R. A. Johnson [76] constructed an example that shows that if condition (3.3) does not hold, there may not exist any almost periodic solution of the system (3.1), but

there may exist an *almost automorphic solution*. W. Shen and Y. Yi [126] established a general result on the existence of almost automorphic solutions for general linear systems. H. Ishii [74], A. Haraux [61] considered non-linear contractive almost periodic processes on a Banach space and, under some assumptions, they both established the existence of almost periodic solutions in different ways. In particular, Ishii [74] mentioned that when $A(t)$ is a skew symmetric $n \times n$ matrix for all $t \in \mathbb{R}$, the corresponding result can imply Favard's Theorem.

We will also start within the framework of almost periodic processes on a Banach space, but instead of the assumption of contractivity, we assume the convexity and homogeneity of *almost periodic processes*. At the same time, our main point is that we can weaken condition (3.3) by requiring instead that

$$\inf_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} |x(s)| ds \right) > 0 \quad (3.4)$$

where $l > 0$ is some real number. On the other hand, if we assume some uniformity of the almost periodic processes, the weaker condition

$$\inf_{t \in \mathbb{R}} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} |x(s)| ds \right) > 0 \quad (3.5)$$

also guarantees the existence of almost periodic trajectories of the corresponding processes.

3.2 Preliminaries and Lemmas

In this section we first introduce some preliminary results due to C. M. Dafermos (see [40]). Then we introduce the concepts of convexity and homogeneity of processes and establish some simple properties of such processes.

Following [40], we define the σ -**translate** of $U(t, s)$ by $U_\sigma(t, s) = U(t + \sigma, s)$. A process U on X is called **almost periodic** if for any sequence $\{\sigma_n\}$ of R , there exists a subsequence $\{\sigma'_n\} \subset \{\sigma_n\}$ such that the sequence $\{U_{\sigma'_n}(t, s)x\}$ converges to some $V(t, s)$ in X uniformly in t and pointwise in $(s, x) \in R^+ \times X$, where $V(t, s)$ is some process on X . We denote by $\mathcal{H}(U)$ the **uniform hull** of U , i.e.,

$$\begin{aligned} \mathcal{H}(U) = \{ & V(t, s) : X \rightarrow X \mid \text{there exists a sequence } \{\sigma_n\} \text{ of } R \text{ such} \\ & \text{that } U_{\sigma_n}(t, s)x \rightarrow V(t, s)x \text{ uniformly in } t \in R \text{ and pointwise in } (s, x) \in \\ & R^+ \times X \}. \end{aligned}$$

A process U is said **homogenizable** on X if there exists a process $U^0(t, s)$ such that for any $x, y \in X$,

$$U(t, s)x - U(t, s)y = U^0(t, s)(x - y) \quad (3.6)$$

for all $(t, s) \in R \times R^+$. We call $U^0(t, s)$ the **homogeneous process** of $U(t, s)$.

A process U is said **convex** on X if for any $\alpha_i \in [0, 1]$, $x_i \in X$, $i = 1, \dots, n$, $\sum_{i=1}^n \alpha_i = 1$,

$$U(t, s) \sum_{i=1}^n \alpha_i x_i = \sum_{i=1}^n \alpha_i U(t, s)x_i \quad (3.7)$$

for all $(t, s) \in R \times R^+$.

Remark It is easy to show that if $U(t, s)$ is almost periodic and homogenizable (or, convex), then for any $V(t, s) \in \mathcal{H}(U)$, $V(t, s)$ is also almost periodic (see C. M. Dafermos [3]) and homogenizable (or, convex).

Lemma 3.1 (C. M. Dafermos [40]) *Let U be an almost periodic process on X and $V \in \mathcal{H}(U)$. Then, there are two sequences $\{\sigma_n\}$, $\{\tau_n\}$ of R such that $\sigma_n \rightarrow \infty$, $\tau_n \rightarrow -\infty$, and $\{U_{\sigma_n}\}$ and $\{U_{\tau_n}\}$ converge to V pointwise in $R \times R^+ \times X$.*

Lemma 3.2 (C. M. Dafermos [40]) *Let U be an almost periodic process on X and assume that some positive trajectory $U(t, \cdot)x$ has relatively compact range in X . Suppose that $\{\sigma_n\}$ is a sequence in R^+ , $\sigma_n \rightarrow \infty$, such that U_{σ_n} converges to $V \in \mathcal{H}(U)$. Then there exists a subsequence of $\{\sigma_n\}$, denoted again by $\{\sigma_n\}$, such that $\{U(t, s + \sigma_n)x\}$ converges for all $s \in R$ to a complete trajectory $v(s)$ of V with relatively compact range in X .*

Lemma 3.3 (C. M. Dafermos [40]) *Let $u(t)$ be a complete trajectory of an almost periodic process U with relatively compact range. Assume that for some sequence $\{\sigma_n\}$ in R , $\{U_{\sigma_n}\}$ converges to $V \in \mathcal{H}(U)$. Then there is a subsequence $\{\sigma'_n\}$ such that $u(s + \sigma'_n)$ converges for all $s \in R$ to a complete trajectory $v(s)$ of V with relatively compact range in X .*

Lemma 3.4 *Let U be a convex process on X . Then, for any $V \in \mathcal{H}(U)$, if $v_1(t), \dots, v_n(t)$ are complete (respectively, positive) trajectories of V through $(t_0, x_i) \in R \times X$, $\alpha_i \in [0, 1]$, $i = 1, \dots, n$, $\sum_{i=1}^n \alpha_i = 1$, then $v(t) = \sum_{i=1}^n \alpha_i v_i(t)$ is a complete (respectively, positive) trajectory of V through $(t_0, \sum_{i=1}^n \alpha_i x_i)$.*

Proof of Lemma 3.4 Let $v_i(t)$ be the complete trajectory of V through (t_0, x_i) . Then, $v_i(t_0) = x_i$, and $v_i(t + s) = V(t, s)v_i(t)$, $i = 1, \dots, n$, for all $(t, s) \in R \times R^+$. So, $v(t_0) = \sum_{i=1}^n \alpha_i x_i$, and

$$\begin{aligned}
 v(t + s) &= \sum_{i=1}^n \alpha_i v_i(t + s) \\
 &= \sum_{i=1}^n \alpha_i V(t, s) v_i(t) \\
 &= V(t, s) \sum_{i=1}^n \alpha_i v_i(t) \\
 &= V(t, s) v(t).
 \end{aligned}$$

Thus, $v(t)$ is a complete trajectory of V . Similarly, we can prove the case that $v_i(t)$ are positive trajectories of V . This completes the proof of this lemma.

Lemma 3.5 *Let U be a homogenizable process on X . Then, for any $V \in \mathcal{H}(U)$, if $v_1(t), v_2(t)$ are two complete (respectively, positive) trajectories of V , then $v(t) = v_1(t) - v_2(t)$ is a complete (respectively, positive) trajectory of $V^0(t, s)$.*

This lemma can be proved by direct verification.

Now for any $u \in C(R, X)$ and $l > 0$, we define

$$L_l(u) = \sup_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\|^2 ds \right)^{\frac{1}{2}} \quad (3.8)$$

and

$$L(u) = \sup_{t \in R} \left(\overline{\lim}_{l \rightarrow \infty} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\|^2 ds \right)^{\frac{1}{2}} \right). \quad (3.9)$$

Lemma 3.6 *For any $u, v \in C(R, X)$, if there exists a sequence $\{\alpha_n\}$ such that $u(t + \alpha_n)$ converges to $v(t)$ for any $t \in R$, then for any $l > 0$, $L_l(v) \leq L_l(u)$.*

Proof For any $l > 0, t \in R$, $u(t + \alpha_n)$ converges to $v(t)$ uniformly on $[t - l, t + l]$.

So,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n) - v(s)\|^2 ds \right)^{\frac{1}{2}} = 0 \quad (3.10)$$

Using Minkovskii's inequality, we have

$$\begin{aligned} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|v(s)\|^2 ds \right)^{\frac{1}{2}} &\leq \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n)\|^2 ds \right)^{\frac{1}{2}} + \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n) - v(s)\|^2 ds \right)^{\frac{1}{2}} \\ &\leq \sup_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\|^2 ds \right)^{\frac{1}{2}} + \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n) - v(s)\|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$\left(\frac{1}{2l} \int_{t-l}^{t+l} \|v(s)\|^2 ds \right)^{\frac{1}{2}} \leq \sup_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\|^2 ds \right)^{\frac{1}{2}} = L_l(u). \quad (3.11)$$

and thus $L_l(v) \leq L_l(u)$. This completes the proof of this lemma.

In order to obtain a stronger result, we need more assumptions on the processes. Let U be an almost periodic process on X , K a compact subset of X . For any $V, W \in \mathcal{H}(U)$, we assume the following conditions.

- (I) If there exists an sequence $\{\alpha_n\} \subset R$ such that $V_{\alpha_n}(t, s)x \rightarrow W(t, s)x$ uniformly in $t \in R$ and pointwise in $s \in R^+, x \in X$, then, for any compact subset $K \subset X$, $V_{\alpha_n}(0, s)x \rightarrow W(0, s)x$ uniformly in $s \in R^+$ and in $x \in K$.
- (II) The family of mappings $V^0(0, s)x$, parameter $s \in R^+$, is equi-continuous on $x \in X$.
- (III) if $u(t), v(t)$ are complete trajectories of V, W , respectively, with $u(t), v(t) \in K$ for all $t \in R$ and there exists a sequence $\{\alpha_n\} \subset R$ such that $V_{\alpha_n}(t, s)x \rightarrow W(t, s)x$ uniformly in $t \in R$, pointwise in $s \in R^+, x \in X$ and $u(t + \alpha_n) \rightarrow v(t)$ pointwise in $t \in R$, then

$$\lim_{n \rightarrow +\infty} \left(\overline{\lim}_{l \rightarrow +\infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u_{\alpha_n}(s) - v(s)\|^2 ds \right)^{\frac{1}{2}} = 0 \quad (3.12)$$

for each $t \in R$.

Using the ideas of the method in Lemma 3.6 and condition (III), we can prove the following lemma.

Lemma 3.7 *Let U be an almost periodic process on X satisfying condition (III), $V \in \mathcal{H}(U)$, K a compact subset of X . If $u(t)$ is a complete trajectory of U such*

that $u(t) \in K$ for all $t \in R$, then there exists a complete trajectory $v(t)$ of V such that $v(t) \in K$ for all $t \in R$ and $L(v) \leq L(u)$.

Proof By the conditions of this lemma and Lemma 3.2, there exists a trajectory $v(t)$ of V and a sequence $\{\alpha_n\} \subset R$ such that $v(t) \in K$ for any $t \in R$, $U_{\alpha_n}(t, s)x \rightarrow V(t, s)x$ uniformly in t and pointwise in (s, x) and $u(t + \alpha_n) \rightarrow v(t)$ pointwise in $t \in R$. From condition (III), we have that for any t ,

$$\lim_{n \rightarrow \infty} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n) - v(s)\|^2 ds \right)^{\frac{1}{2}} = 0. \quad (3.13)$$

Using Minkovskii's inequality, we have

$$\left(\frac{1}{2l} \int_{t-l}^{t+l} \|v(s)\|^2 ds \right)^{\frac{1}{2}} \leq \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n)\|^2 ds \right)^{\frac{1}{2}} + \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n) - v(s)\|^2 ds \right)^{\frac{1}{2}} \quad (3.14)$$

and so

$$\begin{aligned} & \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|v(s)\|^2 ds \right)^{\frac{1}{2}} \\ & \leq \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n)\|^2 ds \right)^{\frac{1}{2}} + \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n) - v(s)\|^2 ds \right)^{\frac{1}{2}} \\ & \leq \sup_{t \in R} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\|^2 ds \right)^{\frac{1}{2}} + \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n) - v(s)\|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$\left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|v(s)\|^2 ds \right)^{\frac{1}{2}} \leq \sup_{t \in R} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\|^2 ds \right)^{\frac{1}{2}} = L(u), \quad (3.15)$$

and thus $L(v) \leq L(u)$. This completes the proof of this lemma.

3.3 The Existence of an Almost Periodic Solution

In this section, we assume that for any $V \in \mathcal{H}(U)$ and $(t, x) \in R \times X$, the positive trajectory of V through (t, x) is continuous on R^+ , and that there exists a $(t_0, x_0) \in R \times X$ such that the set $\{U(t_0, s)x_0 | s \in R^+\}$ is relatively compact in X . From these assumptions and Lemma 3.1, Lemma 3.2, for any $V \in \mathcal{H}(U)$, there exists a sequence $\{\sigma_n\}$, $\sigma_n \rightarrow \infty$, and a complete trajectory u_V of V such that $U_{\sigma_n} \rightarrow V$ pointwise in $R \times R^+ \times X$ and $u_V(t) = \lim_{n \rightarrow \infty} U(t_0, t + \sigma_n)x_0$ pointwise in $t \in R$. Now, we define

$$K = \text{clco}\{U(t_0, s)x_0 | s \in R^+\} \quad (3.16)$$

and

$$\mathcal{F}_V = \{u_V(t) | u_V(t) \text{ is a complete trajectory of } V, u_V(t) \in K, \text{ for all } t \in R\}$$

where $\text{clco}\{\dots\}$ denotes the compact closure of the relatively compact set $\{\dots\}$.

By Lemma 3.3, for any $V \in \mathcal{H}(U)$, $\mathcal{F}_V$ is not empty.

Lemma 3.8 *For any $V \in \mathcal{H}(U)$ and any $l > 0$, there exists $u_V \in \mathcal{F}_V$ such that*

$$L_l(u_V) = \inf_{u \in \mathcal{F}_V} L_l(u). \quad (3.17)$$

Proof From Lemma 3.3, $\mathcal{F}_V$ is sequentially compact in the topology of pointwise convergence. So, $\mathcal{F}_V$ is closed under the topology of uniform convergence on compact subsets. We define now

$$\lambda = \inf_{u \in \mathcal{F}_V} L_l(u), \quad (3.18)$$

and

$$\lambda_n = \inf_{u \in \mathcal{F}_V} \left(\sup_{|t| \leq n} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\|^2 ds \right)^{\frac{1}{2}} \right). \quad (3.19)$$

Then, $\lambda_n \leq \lambda$, and $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. Take $u_n \in \mathcal{F}_V$ such that

$$\sup_{|t| \leq n} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_n(s)\|^2 ds \right)^{\frac{1}{2}} \leq \lambda_n + \frac{1}{n}. \quad (3.20)$$

So, there exists a $u_V \in \mathcal{F}_V$ such that $u_n(t)$ converges to $u_V(t)$ uniformly on any compact subset of R , i.e. $\|u_n(t) - u_V(t)\| \rightarrow 0$ uniformly on any compact subset of R . Now, for fixed $t \in R$, using Minkovskii's inequality, we have

$$\begin{aligned} & \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s)\|^2 ds \right)^{\frac{1}{2}} \\ & \leq \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s) - u_n(s)\|^2 ds \right)^{\frac{1}{2}} + \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_n(s)\|^2 ds \right)^{\frac{1}{2}} \\ & \leq \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s) - u_n(s)\|^2 ds \right)^{\frac{1}{2}} + \sup_{|t| \leq n} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_n(s)\|^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

where $n > |t|$. Therefore,

$$\left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s)\|^2 ds \right)^{\frac{1}{2}} \leq \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s) - u_n(s)\|^2 ds \right)^{\frac{1}{2}} + \lambda_n + \frac{1}{n}. \quad (3.21)$$

Letting $n \rightarrow +\infty$, we have

$$\left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s)\|^2 ds \right)^{\frac{1}{2}} \leq \lambda, \quad (3.22)$$

so that

$$\sup_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s)\|^2 ds \right)^{\frac{1}{2}} \leq \lambda. \quad (3.23)$$

From the definition of λ and since $u_V \in \mathcal{F}_V$, we have that $L_l(u_V) = \lambda$. This completes the proof of this lemma.

Lemma 3.9 *For any $V, W \in \mathcal{H}(U)$ and any $l > 0$, we have*

$$\inf_{u \in \mathcal{F}_V} L_l(u) = \inf_{u \in \mathcal{F}_W} L_l(u). \quad (3.24)$$

Proof From Lemma 3.8, we can take $u_V \in \mathcal{F}_V$, and $v_W \in \mathcal{F}_W$ such that $L(u_V) = \inf_{u \in \mathcal{F}_V} L_l(u)$, and $L(u_W) = \inf_{u \in \mathcal{F}_W} L_l(u)$. We show that $L(u_V) = L(u_W)$. In fact, since $V, W \in \mathcal{H}(U)$, we have $W \in \mathcal{H}(V)$. From Lemma 3.3 there is a sequence $\{\sigma_n\}$ and an trajectory $\tilde{u}_W$ of W such that $\tilde{u}_W \in \mathcal{F}_W$ and $u(t + \sigma_n)$ converges to $\tilde{u}_W$ pointwise in R . By Lemma 3.6, for any $l > 0$, $L_l(\tilde{u}_W) \leq L_l(u_V)$. Similarly, we have $L_l(u_V) \leq L_l(u_W)$. This completes the proof of this lemma.

Lemma 3.10 *Let U be homogenizable, convex and $l > 0$ is some real number. If $V \in \mathcal{H}(U)$ has the property that for any non-trivial complete trajectory $v(t)$ of V^0 ,*

$$\inf_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|v(s)\| ds \right) > 0, \quad (3.25)$$

then there exists a unique complete trajectory u_V of V such that

$$L_l(u_V) = \inf_{u \in \mathcal{F}_V} L_l(u). \quad (3.26)$$

Proof Lemma 3.8 implies the existence of u_V . Now we show the uniqueness of u_V . Assuming the contrary, suppose that there exists two distinct trajectories u_V, v_V of V satisfying (3.26). Let $w = \frac{1}{2}(u_V + v_V)$ and $y(t) = \frac{1}{2}(u_V - v_V)$. From Lemma 3.4 and Lemma 3.5, we have that w is a trajectory of V , $w \in \mathcal{F}_V$, and that y is a trajectory of V^0 and $y(t) \neq 0$. We define

$$\delta = \inf_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|y(s)\| ds \right), \quad (3.27)$$

and

$$\lambda = \inf_{u \in \mathcal{F}_V} L_l(u). \quad (3.28)$$

From the assumption of this lemma, we have $\delta > 0$. Using the Cauchy inequality we have

$$\inf_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|y(s)\|^2 ds \right)^{\frac{1}{2}} \geq \inf_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|y(s)\| ds \right). \quad (3.29)$$

And so

$$\inf_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|y(s)\|^2 ds \right)^{\frac{1}{2}} \geq \delta > 0. \quad (3.30)$$

Since

$$\begin{aligned} & \left(\frac{1}{2l} \int_{t-l}^{t+l} \|w(s)\|^2 ds \right) + \left(\frac{1}{2l} \int_{t-l}^{t+l} \|y(s)\|^2 ds \right) \\ &= \left(\frac{1}{2l} \int_{t-l}^{t+l} \frac{1}{2} \|u_V(s)\|^2 ds \right) + \left(\frac{1}{2l} \int_{t-l}^{t+l} \frac{1}{2} \|v_V(s)\|^2 ds \right) \\ &\leq \sup_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} \frac{1}{2} \|u_V(s)\|^2 ds \right) + \sup_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} \frac{1}{2} \|v_V(s)\|^2 ds \right), \end{aligned}$$

we have that

$$\frac{1}{2l} \int_{t-l}^{t+l} \|w(s)\|^2 ds \leq \frac{1}{2} L_l^2(u_V) + \frac{1}{2} L_l^2(v_V) - \inf_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|y(s)\|^2 ds \right). \quad (3.31)$$

So, by $L_l(u_V) = L_l(v_V) = \lambda$, we have the following inequalities

$$\begin{aligned} L_l^2(w) &\leq \frac{1}{2} L_l^2(u_V) + \frac{1}{2} L_l^2(v_V) - \inf_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|y(s)\|^2 ds \right) \\ &\leq \frac{1}{2} \lambda^2 + \frac{1}{2} \lambda^2 - \delta^2 \\ &< \lambda^2. \end{aligned}$$

This contradicts the definition of λ and the proof of this lemma is complete.

Theorem 3.1 *Let U be homogenizable, convex and almost periodic on R . Suppose that there is some $l > 0$ such that for every $V \in \mathcal{H}(U)$, then any non-trivial complete trajectory $v(t)$ of V^0 satisfies (3.25). If there exists a $(t_0, x_0) \in R \times X$ such that $K = \{U(t_0, s)x_0 | s \in R^+\}$ is relatively compact in X , then for any $V \in \mathcal{H}(U)$, there exists a complete trajectory of V which is almost periodic on R .*

Proof From Lemma 3.8 and 3.9, for any $V \in \mathcal{H}(U)$, there exists a unique complete trajectory $u_V(t)$ of V such that $u_V(t) \in K$ for all $t \in R$ and

$$L_l(u_V) = \inf_{u \in \mathcal{F}_V} L_l(u). \quad (3.32)$$

Now we show that $u_V(t)$ is almost periodic. From the Bochner theorem (Theorem 1.2) we have that for any pair of sequences $\{\alpha_n\}, \{\beta_n\} \subset R$, there exists common subsequences $\{\alpha'_n\}$ of $\{\alpha_n\}$ and $\{\beta'_n\}$ of $\{\beta_n\}$ such that $\{V_{\alpha'_n}\}$ converges to some $W \in \mathcal{H}(U)$ pointwise, and $\{V_{\alpha'_n + \beta'_n}\}$ and $\{W_{\beta'_n}\}$ converges to some $Y \in \mathcal{H}(U)$ pointwise. By Lemma 3.3, there exists complete trajectories u_W of W , u_Y of Y and $\tilde{u}_Y$ of Y such that $u_W \in \mathcal{F}_W$, $u_Y, \tilde{u}_Y \in \mathcal{F}_Y$ and such that $u_V(t + \alpha'_n)$ converges to $u_W(t)$, $u_W(t + \beta'_n)$ converges to $u_Y(t)$, and $u_V(t + \alpha'_n + \beta'_n)$ converges to $\tilde{u}_Y(t)$ pointwise, respectively. From Lemma 3.6, we have

$$L_l(u_W) \leq L_l(u_V), \quad L_l(\tilde{u}_Y) \leq L_l(u_V), \quad \text{and} \quad L_l(u_Y) \leq L_l(u_W). \quad (3.33)$$

Lemma 3.9 and 3.10 imply

$$u_Y(t) = \tilde{u}_Y(t), \quad (3.34)$$

for all $t \in R$. By Theorem 1.2, we have that $u_V(t)$ is almost periodic on R . This completes the proof of Theorem 3.1.

Using the conditions (I)–(III), we can obtain the following results.

Lemma 3.11 *Let U be an almost periodic process satisfying conditions (I)–(III).*

Then for any $V \in \mathcal{H}(U)$, and $v_n(t) \in \mathcal{F}_V$, there exist a subsequence, denoted by $v_n(t)$ again, and a function $v(t) \in \mathcal{F}_V$, such that

$$\lim_{n \rightarrow +\infty} \left(\overline{\lim}_{l \rightarrow +\infty} \frac{1}{2l} \int_{t-l}^{t+l} \|v_n(s) - v(s)\|^2 ds \right)^{\frac{1}{2}} = 0 \quad (3.35)$$

for all $t \in R$.

Proof Let $t_k \rightarrow -\infty$ as $k \rightarrow +\infty$. Then, there is a process $W \in \mathcal{H}(V)$ such that condition (I) holds. That is

$$V_{t_k}(0, s)x \rightarrow W(0, s)x \quad \text{uniformly in } s \in R^+, x \in K. \quad (3.36)$$

By Lemma 3.3, there exists $w_n(t) \in \mathcal{F}_W$ such that $v_n(t + t_k) \rightarrow w_n(t)$ for each n and $t \in R$, as $k \rightarrow \infty$. In particular, $v_n(t_k) \rightarrow w_n(0)$ as $k \rightarrow \infty$. Since $\mathcal{F}_V$ is sequentially compact, there exists a complete trajectory of W with range in K such that $w_n(t) \rightarrow w(t)$ pointwise in t as $n \rightarrow \infty$. In particular,

$$w_n(0) \rightarrow w(0), \quad n \rightarrow \infty. \quad (3.37)$$

As $W_{t_k}(t, s)x \rightarrow V(t, s)x$, using Lemma 3.3 again, there exists a complete trajectory $v(t)$ of V , $v(t) \in K$ such that $w(t - t_k) \rightarrow v(t)$, pointwise in t as $k \rightarrow \infty$. By condition (III), we have that

$$\lim_{k \rightarrow \infty} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|w(s - t_k) - v(s)\|^2 ds \right)^{\frac{1}{2}} = 0. \quad (3.38)$$

Now for any t , let k be sufficiently large so that $t - t_k > 0$. We have that

$$\begin{aligned} v_n(t) - v(t) &= \\ V_{t_k}(0, t - t_k)v_n(t_k) - W(0, t - t_k)v_n(t_k) &+ W^0(0, t - t_k)(v_n(t_k) - w_n(0)) \\ + W^0(0, t - t_k)(w_n(0) - w(0)) &+ w(t - t_k) - v(t). \end{aligned} \quad (3.39)$$

Using Minkovskii's inequality and (3.36), (3.37) and (3.38), we can easily derive (3.35).

Lemma 3.12 *Let U be an almost periodic process satisfying conditions (I)–(III). Then for any $V \in \mathcal{H}(U)$, there exists a $u_V \in \mathcal{F}_V$ such that*

$$L(u_V) = \inf_{u \in \mathcal{F}_V} L(u). \quad (3.40)$$

Proof From Lemma 3.3, $\mathcal{F}_V$ is sequentially compact in the topology of pointwise convergence. So, $\mathcal{F}_V$ is closed under the topology of uniform convergence on compact subsets. We define now

$$\lambda = \inf_{u \in \mathcal{F}_V} L(u), \quad (3.41)$$

and

$$\lambda_n = \inf_{u \in \mathcal{F}_V} \left(\sup_{|t| \leq n} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s)\|^2 ds \right)^{\frac{1}{2}} \right). \quad (3.42)$$

Then, $\lambda_n \leq \lambda$, and $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. Take $u_n \in \mathcal{F}_V$ such that

$$\sup_{|t| \leq n} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u_n(s)\|^2 ds \right)^{\frac{1}{2}} \leq \lambda_n + \frac{1}{n}. \quad (3.43)$$

By Lemma 3.11, there exists a $u_V \in \mathcal{F}_V$ and a subsequence, denoted by $u_n(t)$ again, such that

$$\lim_{n \rightarrow +\infty} \left(\overline{\lim}_{l \rightarrow +\infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u_n(s) - u_V(s)\|^2 ds \right)^{\frac{1}{2}} = 0 \quad (3.44)$$

for all $t \in R$.

Now, for fixed $t \in R$, using Minkovskii's inequality, we have

$$\begin{aligned} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s)\|^2 ds \right)^{\frac{1}{2}} &\leq \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s) - u_n(s)\|^2 ds \right)^{\frac{1}{2}} + \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_n(s)\|^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s) - u_n(s)\|^2 ds \right)^{\frac{1}{2}} + \sup_{|s| \leq n} \left(\frac{1}{2l} \int_{s-l}^{s+l} \|u_n(s)\|^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

where $n > |t|$. Therefore,

$$\left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s)\|^2 ds \right)^{\frac{1}{2}} \leq \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s) - u_n(s)\|^2 ds \right)^{\frac{1}{2}} + \lambda_n + \frac{1}{n}. \quad (3.45)$$

Letting $n \rightarrow +\infty$, we have

$$\left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s)\|^2 ds \right)^{\frac{1}{2}} \leq \lambda. \quad (3.46)$$

So,

$$\sup_{t \in \mathbb{R}} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u_V(s)\|^2 ds \right)^{\frac{1}{2}} \leq \lambda. \quad (3.47)$$

From the definition of λ and since $u_V \in \mathcal{F}_V$, we have that $L(u_V) = \lambda$. This completes the proof of this lemma.

Lemma 3.13 *Let U be an almost periodic process satisfying conditions (I)–(III).*

For any $V, W \in \mathcal{H}(U)$, we have that

$$\inf_{u \in \mathcal{F}_V} L(u) = \inf_{u \in \mathcal{F}_W} L(u). \quad (3.48)$$

The proof of this lemma is the same as that of Lemma 3.8 except that we replace $L_l(u)$ by $L(u)$.

Lemma 3.14 *Let U be homogenizable, convex and satisfy conditions (I)–(III). If*

$V \in \mathcal{H}(U)$ is such that for any non-trivial complete trajectory $v(t)$ of V^0 ,

$$\inf_{t \in \mathbb{R}} \left[\overline{\lim}_{l \rightarrow \infty} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|v(s)\|^2 ds \right) \right] > 0, \quad (3.49)$$

then there exists a unique complete trajectory u_V of V such that

$$L(u_V) = \inf_{u \in \mathcal{F}_V} L(u). \quad (3.50)$$

The proof of this lemma is the same as that of Lemma 3.9 except that we replace $L_l(u)$ by $L(u)$.

Using Lemma 3.12, 3.13, 3.14, we can formulate the following theorem.

Theorem 3.2 *Let U be homogenizable, convex, almost periodic on R , and satisfying conditions (I)–(III). Suppose that for every $V \in \mathcal{H}(U)$, any non-trivial complete trajectory $v(t)$ of V^0 satisfies (3.49). If there exists a $(t_0, x_0) \in R \times X$ such that $K = \{U(t_0, s)x_0 | s \in R^+\}$ is relatively compact in X , then for any $V \in \mathcal{H}(U)$, there exists a complete trajectory of V which is almost periodic on R .*

3.4 Special Cases

In this section we obtain results for some special cases using our theorems in the last section. In some cases, we can obtain stronger results under some special assumptions. Firstly, using Theorem 3.1, we can obtain an extension of Favard's theorem for n -dimensional linear systems and for linear functional differential equations. We consider the linear system

$$x' = A(t)x + f(t) \quad (3.51)$$

where $A(t)$ is an $n \times n$ almost periodic matrix function and $f : R \rightarrow R^n$ is an almost periodic function. Obviously, the system (3.51) generates an almost periodic process $U(t, s)$ and it is convex and homogenizable. According to Theorem 3.1, we have the following new extension of Favard's theorem.

Theorem 3.3 *Suppose that for every $B \in \mathcal{H}(A)$, any non-trivial bounded solution $x(t)$ of*

$$x' = B(t)x \quad (3.52)$$

satisfies

$$\inf_{t \in \mathbb{R}} \left(\frac{1}{2l} \int_{t-l}^{t+l} |x(s)| ds \right) > 0. \quad (3.53)$$

If (3.51) has a bounded solution on $\mathbb{R}^+$, then for every $B \in \mathcal{H}(A)$ and every $g \in \mathcal{H}(f)$, the equation

$$x' = B(t)x + g(t) \quad (3.54)$$

has an almost periodic solution on $\mathbb{R}$.

In order to obtain the stronger result, we discuss equation (3.51) in more detail.

Let $U(t, s)$ be the almost periodic process generated by (3.51). Then $U(t, s)$ has the following properties:

1. $U(t, 0)$ is a mapping from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$;
2. $U(t, s + \sigma) = U(t + s, \sigma)U(t, s)$ for all $t, s, \sigma \in \mathbb{R}$;
3. For any $V \in \mathcal{H}(U)$, i.e. V is the process generated by $x' = B(t)x + g(t)$, where $B \in \mathcal{H}(A)$, $g \in \mathcal{H}(f)$, the homogeneous process $V^0(t, s)$ of V is the process generated by the homogeneous system $x' = B(t)x$.

We establish the following assumption on the process $U(t, s)$.

Boundedness Condition: For any $V \in \mathcal{H}(U)$, there exists real number $l > 0$ and constants $M : 0 < M < 1, M^* > 0$ such that the homogeneous process V^0 of V satisfies the following properties:

1. $\|V^0(t, s)\| \leq M^*$ for any $t, s \in R$ and $|t - s| \leq l$;
2. $\|V^0(kl, (k+1)l)x\| \leq M\|x\|$ for any $x \in R^n, k = 0, 1, 2, \dots$;
3. $\|V^0(-kl, -(k+1)l)x\| \leq M\|x\|$ for any $x \in R^n, k = 0, 1, 2, \dots$.

Lemma 3.15 *Let U be the process generated by (3.51) satisfying the Boundedness Condition. Let $V \in \mathcal{H}(U), K \subset R^n$ be compact and $u(t), v(t)$ be complete trajectories of U and V respectively, such that $u(t), v(t) \in K$ for all $t \in R$. If there exists a sequence $\{\alpha_n\} \subset R$ such that*

$$U_{\alpha_n}(t, s)x \rightarrow V(t, s)x \quad \text{uniformly in } t \in R \text{ and pointwise in } s \in R, x \in R^n \quad (3.55)$$

and

$$u(t + \alpha_n) \rightarrow v(t) \quad \text{pointwise in } t \in R, \quad (3.56)$$

then

$$u(t + \alpha_n) \rightarrow v(t) \quad \text{uniformly in } t \in R \quad (3.57)$$

and thus

$$\lim_{n \rightarrow \infty} \sup_{t \in R} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|u(s + \alpha_n) - v(s)\|^2 ds \right) = 0. \quad (3.58)$$

Proof Let $M > 0, l > 0$ be determined by the Boundedness Condition. From (3.55), we have that

$$U_{\alpha_n}(t, s)x \rightarrow V(t, s)x \quad \text{uniformly on } R \times [-l, l] \times K \text{ as } n \rightarrow \infty \quad (3.59)$$

and from (3.56), we know that

$$u(\alpha_n) \rightarrow v(0) \quad \text{as } n \rightarrow \infty. \quad (3.60)$$

In order to prove (3.57), we need to show that for any $\epsilon > 0$, there exists an $N(\epsilon) > 0$ such that for any $n > N(\epsilon)$,

$$||u(t + \alpha_n) - v(t)|| < \epsilon. \quad (3.61)$$

Firstly, from (3.59) and (3.60), we can choose an $N(\epsilon) > 0$ such that if $n \geq N(\epsilon)$,

$$||U_{\alpha_n}(t, s)x - V(t, s)x|| < \epsilon \quad \text{for any } t \in R, s \in [-l, l], x \in K \quad (3.62)$$

and

$$||u(\alpha_n) - v(0)|| < \epsilon. \quad (3.63)$$

For any $t \in R$, we can take an integer k such that $kl < t \leq (k+1)l$, for $t > 0, k \geq 0$, and $-(k+1)l \leq t < -kl$, for $t < 0, k \geq 0$. So, as $t \geq 0$, we have that

$$\begin{aligned} u(t + \alpha_n) - v(t) &= U_{\alpha_n}(0, t)u(\alpha_n) - V(0, t)v(0) \\ &= U_{\alpha_n}(kl, t - kl)u(\alpha_n + kl) - V(kl, t - kl)u(\alpha_n + kl) \\ &\quad + V(kl, t - kl)u(\alpha_n + kl) - V(kl, t - kl)v(kl). \end{aligned}$$

Since $u(\alpha_n + kl) \in K, t - kl \in [-l, l]$, then for $n \geq N$,

$$||U_{\alpha_n}(kl, t - kl)u(\alpha_n + kl) - V(kl, t - kl)u(\alpha_n + kl)|| < \epsilon. \quad (3.64)$$

At the same time,

$$\begin{aligned} &||V(kl, t - kl)u(\alpha_n + kl) - V(kl, t - kl)v(kl)|| \\ &= ||V^0(kl, t - kl)(u(\alpha_n + kl) - v(kl))|| \\ &\leq M^* ||u(\alpha_n + kl) - v(kl)||. \end{aligned}$$

Using induction on k , and (3.62), (3.63), it is easy to prove that for any $k > 0$, the following (recall that $0 < M < 1$)

$$||U(\alpha_n + kl) - v(kl)|| \leq \epsilon(1 + M + M^2 + \cdots + M^k) < \frac{\epsilon}{1 - M} \quad \text{for } n \geq N. \quad (3.65)$$

holds. Therefore,

$$||u(t + \alpha_n) - v(t)|| \leq \epsilon + M^* \cdot \frac{\epsilon}{1 - M} = (1 + \frac{M^*}{1 - M})\epsilon, \quad \text{as } n \geq N. \quad (3.66)$$

Since N is dependent on only ϵ , we have proven that (3.61) holds. This completes the proof of this lemma.

Lemma 3.16 *Let U and K be as in Lemma 3.15. Then for any $V \in \mathcal{H}(U)$, if $\{v_n(t)\}$ is a sequence of complete trajectories of V such that $v_n(t) \in \mathcal{F}_V$ for any $n, t \in R$, then there exists a subsequence, denoted by $v_n(t)$ again, and a complete trajectory $v(t)$ of V such that $v(t) \in \mathcal{F}_V$ and $v_n(t) \rightarrow v(t)$ uniformly in $t \in R$ and thus*

$$\lim_{n \rightarrow \infty} \sup_{t \in R} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} ||v_n(s) - v(s)||^2 ds \right) = 0 \quad (3.67)$$

Proof For any $V \in \mathcal{H}(U)$, V is almost periodic. So we can take a bounded and convergent sequence $\{\beta_n\} \subset R$ and process $W \in \mathcal{H}(U)$ such that

$$V_{\beta_k} x \rightarrow W(t, s)x \quad \text{uniformly in } t \in R \text{ and pointwise in } s \in R, x \in R^n \quad (3.68)$$

and

$$W_{-\beta_k}(t, s)x \rightarrow V(t, s)x \quad \text{uniformly in } t \in R \text{ and pointwise in } s \in R, x \in R^n. \quad (3.69)$$

Since for any n , $v_n(t) \in K$, for all $t \in R$, by Lemma 3.3, there exists a complete trajectory $w_n(t)$ of W such that $w_n(t) \in \mathcal{F}_W$ and

$$v_n(t + \beta_n) \rightarrow w_n(t) \quad \text{pointwise in } t \in R, \text{ as } n \rightarrow \infty. \quad (3.70)$$

Since $\mathcal{F}_W$ is sequentially compact, there is a complete trajectory $w(t)$ of W and a subsequence of $w_n(t)$, denoted by $w_n(t)$ again, such that $w(t) \in \mathcal{F}_W$ and

$$w_n(t) \rightarrow w(t) \quad \text{pointwise in } t \in R \text{ as } n \rightarrow \infty. \quad (3.71)$$

From the above and Lemma 3.3, there exists a complete trajectory $v(t)$ of V such that $v(t) \in \mathcal{F}_V$ and

$$w(t - \beta_k) \rightarrow v(t) \quad \text{pointwise in } t \in R \text{ as } k \rightarrow \infty. \quad (3.72)$$

Using Lemma 3.15, we have

$$w(t - \beta_k) \rightarrow v(t) \quad \text{uniformly in } t \in R \text{ as } k \rightarrow \infty. \quad (3.73)$$

Now, we show that

$$v_n(t) \rightarrow v(t) \quad \text{uniformly in } t \in R, \text{ as } n \rightarrow \infty. \quad (3.74)$$

Since

$$\begin{aligned} v_n(t) - v(t) &= V_{\beta_k}(0, t - \beta_k) - W(0, t - \beta_k)v_n(\beta_k) - W(0, t - \beta_k)v_n(\beta_k) \\ &+ W^0(0, t - \beta_k)(v_n(\beta_k) - w_n(0)) \\ &+ W^0(0, t - \beta_k)(w_n(0) - w(0)) \\ &+ w(t - \beta_k) - v(t) \end{aligned} \quad (3.75)$$

for any fixed n , using arguments similar to those in Lemma 3.15, we can prove that

$$V_{\beta_k}(0, t - \beta_k)v_n(\beta_k) - W(0, t - \beta_k)v_n(\beta_k) \rightarrow 0 \quad \text{uniformly in } t \in R \text{ as } k \rightarrow \infty. \quad (3.76)$$

At the same time,

$$\|W^0(0, t - \beta_k)(v_n(\beta_k) - w_n(0))\| \leq M\|v_n(\beta_k) - w_n(0)\| \rightarrow 0, \quad \text{as } k \rightarrow \infty, \quad (3.77)$$

$$w(t - \beta_k) - v(t) \rightarrow 0 \quad \text{uniformly in } t \in R \text{ as } k \rightarrow \infty, \quad (3.78)$$

$$\|W^0(0, t - \beta_k)(w_n(0) - w(0))\| \leq M\|w_n(0) - w(0)\|. \quad (3.79)$$

Therefore, letting $k \rightarrow \infty$ in (3.75), we have that for any $t \in R$

$$\|v_n(t) - v(t)\| \leq M\|w_n(0) - w(0)\|. \quad (3.80)$$

Since $w_n(0) \rightarrow w(0)$ as $n \rightarrow \infty$, we know that for any $\epsilon > 0$, there exists an N such that for $n \geq N$, $\|w_n(0) - w(0)\| < \frac{\epsilon}{M}$, i.e., $\|v_n(t) - v(t)\| < \epsilon$, for $n \geq N$, where N is only dependent on ϵ . This completes the proof of this lemma.

Using these two lemmas, we can replace the assumptions (I)–(III) in Lemma 3.14 by the Boundedness Condition for this special case. Therefore, we get the following theorem.

Theorem 3.4 *Suppose that for any $B(t) \in \mathcal{H}(A)$, the fundamental matrix $\Phi(t, s)$ of the homogeneous system*

$$x' = B(t)x \quad (3.81)$$

satisfies the Boundedness Condition and any nontrivial solution $v(t)$ of (3.81)

satisfies

$$\inf_{t \in R} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} \|v(s)\| ds \right) > 0. \quad (3.82)$$

If there exists a bounded solution of (3.51), then for any $B \in \mathcal{H}(A)$, $g \in \mathcal{H}(f)$, there exists a solution of

$$x' = B(t)x + g(t) \quad (3.83)$$

which is almost periodic on R .

Now we consider the linear functional differential equation

$$x' = A(t)x_t + f(t) \quad (3.84)$$

where $A(t) : C([0, 1], R^n) \rightarrow R^n$ is an almost periodic linear operator, $x_t \in C([0, 1], R^n)$, $x_t(\theta) = x(t + \theta)$, $\theta \in [0, 1]$, and $f : R \rightarrow R^n$ is almost periodic on R . Let $X = C([0, 1], R^n)$ be our Banach space. It has been known that the system (3.84) generates an almost periodic process $U(t, s)$ on X (see [60]). And, obviously, $U(t, s)$ is convex and homogenizable on X . So, we have the following result.

Theorem 3.5 *Let $A(t) : C([0, 1], R^n) \rightarrow R^n$ be an almost periodic operator on X , $f : R \rightarrow R^n$ an almost periodic function. Suppose that for every $B \in \mathcal{H}(A)$, and any non-trivial solution $x(t)$ of*

$$x' = B(t)x_t \quad (3.85)$$

we have that

$$\inf_{t \in R} \left(\frac{1}{2l} \int_{t-l}^{t+l} \|x_s\| ds \right) > 0. \quad (3.86)$$

If the system (3.84) has a bounded solution $x(t)$ on R^+ such that the set

$$\{x_t | t \in R^+\} \quad (3.87)$$

is relatively compact in X , then for every $B \in \mathcal{H}(A)$ and $g \in \mathcal{H}(f)$, the system

$$x' = B(t)x_t + g(t) \quad (3.88)$$

has an almost periodic solution on R .

Notice that $\|\cdot\|$ here denotes the norm of $C([0, 1], R^n)$, i.e.

$$\|x_t\| = \sup_{\theta \in [0, 1]} |x(t + \theta)|. \quad (3.89)$$

3.5 Example

In this section, we construct a simple example in which Favard's condition (3.3) does not hold, but the key conditions of Theorem 3.4 hold and the system has an almost periodic solution.

Consider the first order linear differential equation

$$x' = a(t)x + f(t) \quad (3.90)$$

where $a(t), f(t) : R \rightarrow R$ are almost periodic functions. Let $x(t + \tau)$ be the general solution through (t, x) of (3.90) for any $\tau \in R$. Then $U(t, \tau) : X \rightarrow X$ defined by $U(t, \tau)x = x(t + \tau)$ is an almost periodic process. For any $b(t) \in \mathcal{H}(a)$ and $g(t) \in \mathcal{H}(f)$, there exists $V \in \mathcal{H}(U)$ such that $V(t, \tau)y = y(t + \tau)$, where $y(t + \tau)$ is the general solution through (t, y) of the equation

$$y' = b(t)y + g(t). \quad (3.91)$$

We assume that the function $a(t)$ satisfies the following conditions:

- (i) $a(t)$ is almost periodic on R ;
- (ii) $\int_0^t a(s)ds < 0$ for all $t \in R$;

(iii) There is a sequence $\{t_n\}$ such that $\lim_{n \rightarrow \infty} \int_0^{t_n} a(s)ds = -\infty$.

It is easy to show that for any $b(t) \in \mathcal{H}(a)$, it satisfies the conditions (i) and (ii).

Now we consider the special differential equation

$$x' = a(t)(x - 1). \quad (3.92)$$

Obviously, this equation has an almost periodic solution $x(t) \equiv 1$, but this equation does not satisfy Favard's condition (3.3). We will show that this system satisfies the key conditions of Theorem 3.4 for the special function $a(t)$. We define

$$U(t, s)x = x(t + s), t \in R, s \in R^+. \quad (3.93)$$

where

$$x(t + s) = (x - 1)e^{\int_t^{t+s} a(\tau)d\tau} + 1 \quad (3.94)$$

is the solution of (3.92) through (t, x) . For any $b(t) \in \mathcal{H}(a)$, let

$$y(t + s) = (y - 1)e^{\int_t^{t+s} b(\tau)d\tau} + 1 \quad (3.95)$$

be the solution of

$$y' = b(t)(y - 1) \quad (3.96)$$

through (t, y) . Then, the process

$$V(t, s)y = y(t + s), t \in R, s \in R^+ \quad (3.97)$$

belongs to $\mathcal{H}(U)$ and for any $V \in \mathcal{H}(U)$, there is a function $b(t) \in \mathcal{H}(a)$ such that V defined by (3.95) and (3.97) belongs to $\mathcal{H}(U)$. According to the assumptions on $a(t)$, the conditions (i)–(iii), we know that for any $V \in \mathcal{H}(U)$, $V^0(0, s)x$ is

equi-continuous . In order to let the key condition of Theorem 3.4 hold, we construct $a(t)$ as follows: Let

$$a(t) = \sum_{n=1}^{\infty} a_n(t) \quad (3.98)$$

and $a_n(t) = \frac{1}{n}g_n(t)$, where

$$g_1(t) = \begin{cases} -\frac{1}{2}x, & t \in [0, 2] \\ x - 3, & t \in [2, 3] \end{cases}$$

$$g_2(t) = \begin{cases} -\frac{1}{3^2-2^1} \frac{2}{2}x, & t \in [0, 3^2 - 2^1] \\ -\frac{1}{2^2-2}, & t = 8 \\ \text{linear}, & t \in [3^2 - 2^1, 3^2 - 2^1 + 1] \cup [3^2 - 1, 3^2] \end{cases}$$

$$g_n(t) = \begin{cases} -\frac{1}{3^n-2^{n-1}} \times \frac{2}{n}x, & t \in [0, 3^n - 2^{n-1}] \\ -\frac{1}{2^{n-2}}, & t \in [3^n - 2^{n-1} + 1, 3^n - 1] \\ \text{linear}, & t \in [3^n - 2^{n-1}, 3^n - 2^{n-1} + 1] \cup [3^n - 1, 3^n]. \end{cases}$$

Now, extend $g_n(t)$ to be odd and periodic with period 2×3^n . Then, $g_n(t)$ satisfies the following

$$\begin{aligned} \int_0^t g_n(s)ds &\leq 0, & \text{for all } t \in R; \\ \sup_{t \in R} |g_n(t)| &= \frac{n}{2^{n-2}}; \\ \int_0^t g_n(s)ds &\geq -\frac{1}{n}, & \text{for all } t \notin [k \cdot 2 \cdot 3^n - 2^{n-1}, k \cdot 2 \cdot 3^n + 2^{n-1}], k = 0, \pm 1, \pm 2, \dots \\ \int_0^{3^n} g_n(s)ds &\leq -1 \end{aligned} \quad (3.99)$$

for each $n \in \mathbb{Z}^+$.

From the definition of $g_n(t)$, we have that $a_n(t)$ is a periodic function for each n and

$$|a_n(t)| \leq \frac{1}{n2^{n-2}}. \quad (3.100)$$

Therefore, $a(t)$ is almost periodic on R .

According to the definition of $a_n(t)$ and $a(t)$, we have that

$$A(t) = \int_0^t a(s)ds = \sum_{n=1}^{\infty} \int_0^t a_n(s)ds \leq 0. \quad (3.101)$$

We let

$$J = R \setminus \cup_{n=2}^{\infty} \cup_{k=-\infty}^{\infty} [3^n + k \cdot 2 \cdot 3^n - 2^{n-1}, 3^n + k \cdot 2 \cdot 3^n + 2^{n-1}]. \quad (3.102)$$

Then, it is easy to show that for any $l > 0$, the length of $J \cap [-l, l] \geq l$. Therefore, for any $t \in R$, we can take $l > 0$ such that the length of $[t-l, t+l] \cap J \geq l$. Thus,

$$\frac{1}{2l} \int_{t-l}^{t+l} e^{A(s)} ds \geq \frac{1}{2l} \int_{J \cap [t-l, t+l]} e^{A(s)} ds \geq \frac{1}{2} e^{-\delta} \quad (3.103)$$

where

$$\delta = \sum_{n=1}^{\infty} \frac{1}{n^2}. \quad (3.104)$$

Therefore, for any nontrivial solution $x(t) = x_0 e^{A(t)}$ of

$$x' = a(t)x \quad (3.105)$$

We have that

$$\inf_{t \in R} \left(\overline{\lim}_{l \rightarrow \infty} \frac{1}{2l} \int_{t-l}^{t+l} |x(s)| ds \right) \geq |x_0| e^{-\delta} > 0 \quad (3.106)$$

Since we have proved that the hypothesis (II) of Theorem 3.2 holds for our equation, for any $b(t) \in \mathcal{H}(a)$ the solution $y(t)$ of

$$y' = b(t)y \quad (3.107)$$

also satisfies (3.106).

Chapter 4

Almost Periodicity of All Bounded Solutions

4.1 Introduction and Preliminaries

In this Chapter, we consider evolution equations in Banach space

$$u'(t) + A(t)u(t) = f(t) \tag{4.1}$$

where $A(t) : X \rightarrow X$ is an operator for any $t \in R$, and $f : R \rightarrow X$ is a continuous function.

In the previous chapters, we have mentioned that if $X = R^n$, A is linear and does not depend on t or if it depends on t in a periodic manner, then all bounded solutions of (4.1) are Bohr almost periodic, but these results cannot be extended to more general cases, for example, the case that $A(t)$ is almost periodic, or the case that X is infinite dimensional (see [109]). Even though all solutions are bounded,

there still may not exist any nontrivial solution which is Bohr almost periodic, or S-almost periodic (see Chapter 2).

The main problem here is then: *Under what conditions is any bounded solution of (4.1) almost periodic?* This is a very complex question and one which has aroused the interest of many mathematicians for the years. Many authors have tried to make some progress on this question since the question was raised and some results were established (see [7, 40, 49, 61, 62, 63, 74, 110–112] and the references cited there)

In [61], A. Haraux dealt with the case that $X = \mathbb{R}^2$ and A is independent of t , nonlinear and monotone. The main result is that under the above assumptions, all bounded solutions of (4.1) are Bohr almost periodic. Encouraged by the idea of this paper, we deal with that case $X = H$, a separable Hilbert space, and $A(t)$ can depend on t in some manner. We introduce the *difference-variation operator* of $A(t)$ and *difference-variation equations* of (4.1). Using some properties of difference-variation equations, we can infer some properties of the solutions of (4.1). Our results can be used to deduce several previous results (see [49, 109, etc.]). In particular, we generalize the result of [61] to the case that X is a separable Hilbert space with certain restrictions.

Let H be a real separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ its orthonormal basis, $\|\cdot\|$ the norm of H . For any $t \in \mathbb{R}$, let $A(t) : D(A(t)) \rightarrow H$ be a densely defined operator (linear or nonlinear), $f(t) : \mathbb{R} \rightarrow H$ is a function. Throughout this chapter, we always assume that

- (a) $A(t)$ and $f(t)$ are almost periodic on $\mathbb{R}$;
- (b) For any $t \in \mathbb{R}$, $A(t)$ is a monotone operator (defined below);

(c) For all $t \in R$, $A(t)$ is Fréchet differentiable on $D(A(t))$ and the derivative operator of $A(t)$, say $A'(t, u)$, is continuous on $D(A(t))$ and for any $u \in D(A(t))$,

$$\lim_{\|h\| \rightarrow 0} \left\| \frac{A(t)(u+h) - A(t)u}{h} - A'(t, u) \right\| = 0, \quad \text{uniformly on } t \in R; \quad (4.2)$$

(d) For any $(t_0, u_0) \in R \times D(A(t_0))$, the equation (4.1) has a unique solution through (t_0, u_0) .

From the assumption (c) and the properties of almost periodic functions, it is easy to see that $A'(t, u)$ is almost periodic in t on R . And if $u(t)$ is an almost periodic solution of (4.1), then, $A'(t, u(t))$ is also almost periodic. If $A'(t, u)$ is continuous on $D(A(t))$, then for any $u_0 \in D(A(t))$, $h \in H$ and $u_0 + sh \in D(A(t))$ for $s \in [0, 1]$, we have

$$A(t)(u_0 + h) - A(t)u_0 = \int_0^1 A'(t, u_0 + sh) h ds \quad (4.3)$$

(cf [117]).

Let $u_0(t)$ be an almost periodic solution of (4.1). The equation

$$y' + \int_0^1 A'(t, u_0(t) + sy) y ds = 0 \quad (4.4)$$

is called the *difference-variation equation* of (4.1) along the solution $u_0(t)$.

Obviously, if $A(t)$ is linear, then the difference-variation equation of (4.1) is simply the homogeneous equation

$$u'(t) + A(t)u(t) = 0 \quad (4.5)$$

associated with (4.1).

Let $U(t, s)$ be the *process* generated by the equation (4.1). Then $U(t, s)$ is an almost periodic process for any $t \in R, s \in R^+$. We denote the trajectory of $U(t, s)$ (or solution of (4.1)) through (s_0, u_0) by $u(t, s_0, u_0)$. We use $U_{(s_0, u_0)}(t, s)$ to denote

the process generated by the equation (4.4) with $u_0(t) = u(t, s_0, u_0)$. We call this the *difference-variation process* of $U(t, s)$ along $u(t, s_0, u_0)$. Evidently, if $u(t, s_0, u_0)$ is almost periodic, then $U_{(s_0, u_0)}(t, s)$ is also almost periodic. At the same time, we have the following fact: If $u(t, s_0, u_0)$ is an trajectory of $U(t, s)$, then for any other trajectory $u(t, s_1, u_1)$ of $U(t, s)$, $u(t, s_1, u_1) - u(t, s_0, u_0)$ is a trajectory of $U_{(s_0, u_0)}(t, s)$.

We say that $U(t, s)$ is *contractive* (or non-expansive) if for any $x, y \in H$, $t \in R$, $s \in R^+$, we have

$$\|U(t, s)x - U(t, s)y\| \leq \|x - y\|. \quad (4.6)$$

A trajectory $u(t)$ of $U(t, s)$ is said to be *relatively compact* in H (or with relatively compact range in H) if the set $\{u(t) | t \in R\}$ is relatively compact in H .

Let $A : H \rightarrow H$ be a nonlinear operator. We call A *monotone* if for any $u, v \in D(A)$ (the domain of A), we have that

$$(Av - Au, v - u) \geq 0. \quad (4.7)$$

We use symbols $\mathcal{H}(A)$, $\mathcal{H}(f)$ to denote the uniform hull of $A(t)$ and $f(t)$, respectively. We also use the following preliminary results for completeness.

Lemma 4.1 (C. M. Dafermos [40]) *Let U be an almost periodic process on H which is contractive and has two (distinct) trajectories with relatively compact range in H . Then each $V \in \mathcal{H}(U)$ has two trajectories $v_1(t)$ and $v_2(t)$, with relatively compact range in H , such that $\|v_1(t) - v_2(t)\| = \text{constant}$ for all $t \in R$.*

Lemma 4.2 (H. Ishii [74], or A. Haraux [61]) *Let U be an almost periodic process. If $u_1(t)$ and $u_2(t)$ are two trajectories of U with relatively compact ranges in H , then the norm $\|u_1(t) - u_2(t)\|$ is nonincreasing in t .*

Lemma 4.3 (H. Ishii [74]). *Let U be an almost periodic process on H which is contractive. Assume that for any $V \in \mathcal{H}(U)$, there exists a trajectory of V , which is continuous on R and has relatively compact range in H . Then, for any $V \in \mathcal{H}(U)$, there exists a trajectory of V which is almost periodic on R .*

In order to obtain more general results, we introduce the following concepts.

Let H be a separable Hilbert space, S a subset of H . We call S *synchronous* if for any $\epsilon > 0$, there exists an integer $N > 0$ such that for any $x \in S$, $x = \sum_{i=1}^{\infty} a_i(x)e_i$, we have

$$\|x - \sum_{i=1}^N a_i(x)e_i\| < \epsilon, \quad (4.8)$$

where $\{e_i\}_{i=1}^{\infty}$ is the basis of H .

We call a separable Hilbert space H *compact synchronous* if any compact subset of H is synchronous.

Obviously, if H is finite dimensional, then it is compact synchronous.

Lemma 4.4 *Let H be compact synchronous and $f : R \rightarrow H$ a function with $f(t) = \sum_{i=1}^{\infty} f_i(t)e_i$. If $f(t)$ has relatively compact range in H and for each i , $f_i(t)$ is almost periodic, then $f(t)$ is almost periodic.*

Proof Since $f(t)$ has relatively compact range in H , there exists a compact subset K of H such that $f(t) = \sum_{i=1}^{\infty} f_i(t)e_i \in K$ for all $t \in R$. Since H is compact synchronous, for any $\epsilon > 0$, we can choose an $N > 0$ such that

$$\|f(t) - \sum_{i=1}^N f_i(t)e_i\| < \epsilon, \quad \text{for all } t \in R. \quad (4.9)$$

Let $F_N(t) = \sum_{i=1}^N f_i(t)e_i$. Then $F_N(t)$ converges to $f(t)$ uniformly for all $t \in R$. Since for each i , $f_i(t)$ is almost periodic, so does $F_N(t)$ and thus $f(t)$ is almost periodic by Theorem 1.3.

4.2 General Results

In this section, we establish some general results which are extensions of previous results (Haraux [61], Phong [109]). First, we have the following obvious result.

Lemma 4.5 *If $A(t)$ is monotone for any $t \in R$, then $U(t, s)$ is contractive.*

According to the definition of $U(t, s)$, it is easy to show this lemma (see [61]).

From this lemma and Lemma 4.3, we have the following result.

Theorem 4.1 *Assume that $A(t)$ is monotone and that for any $(t_0, u_0) \in R \times H$, the trajectory $u(t, t_0, u_0)$ of $U(t, s)$ is continuous on R and there exists a trajectory of $U(t, s)$ which is relatively compact in H . Then there exists a trajectory of $U(t, s)$ which is almost periodic on R .*

We always assume that the hypothesis of Theorem 4.1 hold and denote the almost periodic trajectory of $U(t, s)$ by $u_0(t)$. We also denote the difference-variation process of $U(t, s)$ along $u_0(t)$ simply by $U_0(t, s)$. Now we establish the following hypotheses (to be used at various places in the sequel).

Condition 1 *For any almost periodic trajectory $u(t, t_0, u_0)$ of $U(t, s)$, $U_{(t_0, u_0)}(t, s)$ is such that if $y(t)$ is a relatively compact trajectory of $U_{(t_0, u_0)}(t, s)$ with $\|y(t)\| = \text{constant}$ for all $t \in R$, then $y(t)$ is almost periodic on R .*

Condition 2 *For any $V(t, s) \in \mathcal{H}(U(t, s))$ and any almost periodic trajectory $v(t, t_0, u_0)$ of $V(t, s)$, $V_{(t_0, u_0)}(t, s)$ is such that if $y_1(t), y_2(t)$ are two relatively compact trajectories of $V_{(t_0, u_0)}(t, s)$ with $\|y_1(t)\| = r_1 > 0$, $\|y_2(t)\| = r_2 > 0$, $\|y_1(t) - y_2(t)\| = r_0 > 0$ for all $t \in R$ where $r_i (i = 0, 1, 2)$ are constants and $0 < r_0 < r_1 + r_2$, then $y_1(t), y_2(t)$ are almost periodic on R .*

Theorem 4.2 *Suppose that Condition 1 holds. If $u(t)$ is a bounded uniformly continuous trajectory of $U(t, s)$ which is relatively compact in H , then it is almost periodic on R .*

Proof Let $u_0(t)$ be the almost periodic trajectory of $U(t, s)$ and $u(t)$ be a bounded uniformly continuous trajectory of $U(t, s)$ which is relatively compact in H . Now, since $U(t, s)$ and $u_0(t)$ are almost periodic, we can choose a sequence $\{t_n\} \subset R$ such that as $n \rightarrow +\infty$, $t_n \rightarrow +\infty$, $U(t, s - t_n) \rightarrow U(t, s)$ and $u_0(t - t_n) \rightarrow u_0(t)$ uniformly on R . At the same time, since $u(t)$ is relatively compact in H , we can choose t_n also satisfying $u(t - t_n) \rightarrow v(t)$ uniformly on any compact subset of R as $n \rightarrow +\infty$, where $v(t) : R \rightarrow H$ is a function. Obviously, $v(t)$ is a trajectory of $U(t, s)$ with relatively compact range in H . Let $y(t) = v(t) - u_0(t)$, then $y(t)$ is a trajectory of $U_0(t, s)$ and $\|y(t)\| = \text{constant}$ for all $t \in R$ from Lemma 4.2. By Condition 1, $y(t)$ is almost periodic on R , and so is $v(t)$.

Choose a subsequence $\{t'_n\} \subset \{t_n\}$ such that $v(t + t'_n) \rightarrow u^*(t)$ uniformly on R . Since $\lim_{n \rightarrow \infty} \|u(-t'_n) - u^*(-t'_n)\| = \|v(0) - v(0)\| = 0$, we have that $u(t) = u^*(t)$ for all $t \in R$ by Lemma 4.2. Obviously, $u^*(t)$ is almost periodic, and so is $u(t)$. This completes the proof of this theorem.

Theorem 4.3 *Suppose that Condition 2 holds. If there exist three uniformly continuous trajectories $u_i(t)$ ($i = 0, 1, 2$) of $U(t, s)$ such that $u_i(t)$ has relatively compact range in H ($i = 0, 1, 2$), $u_0(t)$ is almost periodic, and*

$$\|u_0(t) - u_i(t)\| = r_i > 0, t \in R, i = 1, 2. \quad (4.10)$$

$$\|u_1(t) - u_2(t)\| = r_0 > 0, t \in R \quad (4.11)$$

where

$$r_0 < r_1 + r_2, \quad (4.12)$$

then for any $V(t, s) \in \mathcal{H}(U(t, s))$, any bounded uniformly continuous trajectory of $V(t, s)$ with relatively compact range in H is almost periodic on R .

Proof Since $u_0(t) - u_i(t)$ ($i = 1, 2$) is a uniformly continuous trajectory of $U_0(t, s)$ with relatively compact range in H , $u_0(t) - u_i(t)$ ($i = 1, 2$) is almost periodic on R by Condition 2. And so $u_i(t)$ ($i = 1, 2$) is almost periodic. Now for any $V(t, s) \in \mathcal{H}(U(t, s))$, let $v_0(t) \in \mathcal{H}(u_0(t))$ and $v_0(t)$ is a trajectory of $V(t, s)$. By Lemma 4.1, there exist other two trajectories of $V(t, s)$, say $v_1(t)$, $v_2(t)$, such that

$$\|v_0(t) - v_i(t)\| = r_i > 0 \quad (i = 1, 2) \quad (4.13)$$

for all $t \in R$

$$\|v_1(t) - v_2(t)\| = r_0 > 0 \quad (4.14)$$

for all $t \in R$, and $v_i(t)$ ($i = 1, 2$) are all almost periodic on R .

Now let $v(t)$ be any trajectory of $V(t, s)$ with relatively compact range in H . Since $V(t, s), v_i(t), i = 0, 1, 2$ are almost periodic, we can choose a sequence $\{t_n\} \subset R$ with $t_n \rightarrow +\infty$ as $n \rightarrow \infty$ satisfying $V(t - t_n, s)x \rightarrow V(t, s)x$ uniformly in $t \in R$ and pointwise in $(s, x) \in R^+ \times H$, $v_i(t - t_n) \rightarrow v_i(t)$ uniformly in $t \in R$ for $i \in \{0, 1, 2\}$ and $v(t - t_n) \rightarrow w(t)$ uniformly on any bounded interval of R as $n \rightarrow \infty$, where $w(t)$ is a trajectory of $V(t, s)$ with relatively compact range in H . Clearly, $\|w(t) - v_i(t)\|$ is constant on R for $i \in \{0, 1, 2\}$ and from (4.13) and (4.14), we can choose k and l such that $w(t) - v_k(t), w(t) - v_l(t)$ and $v_k(t) - v_l(t)$ satisfy the Condition 2. And thus $w(t)$ is almost periodic on R .

By Lemma 4.3, we can choose a subsequence $\{t_k\}$ of $\{t_n\}$ and a periodic trajectory $\tilde{v}(t)$ of V such that $w(t + t_k)$ converges uniformly to $\tilde{v}(t)$ as $k \rightarrow \infty$. Then, using Lemma 4.1 we have that $\|v(t) - \tilde{v}(t)\|$ is non-increasing on R and $\lim_{k \rightarrow \infty} \|\tilde{v}(-t_k) - v(-t_k)\| = 0$ because $\tilde{v}(-t_k)$ and $v(-t_k)$ both tend to $w(0)$ as $k \rightarrow \infty$. Hence $v(t) = \tilde{v}(t)$ for all $t \in R$. This completes the proof of the theorem.

Lemma 4.6 *Assume Condition 2. Let $v(t)$ be a uniformly continuous trajectory of $U(t, s)$ with relatively compact range in H , $u_0(t)$ as before. If $w(t) = v(t) - u_0(t)$ is not almost periodic and $\|w(t)\| = \text{constant} > 0$ for all $t \in R$, then $u_0(t) - w(t)$ is a trajectory of $U(t, s)$.*

Proof Since $w(t)$ is not almost periodic, there exists a sequence $\{\alpha_n\} \subset R$ such that $\{w(t + \alpha_n)\}$ has no Cauchy subsequence. We can assume, by a suitable refining of $\{\alpha_n\}$, that $U(t + \alpha_n, s)x \rightarrow V(t, s)x$ uniformly in t and pointwise in $(s, x) \in R^+ \times H$, $u_0(t + \alpha_n) \rightarrow u_1(t)$ uniformly in R and $w(t + \alpha_n) \rightarrow w_1(t)$ uniformly on any bounded interval of R , where $V \in \mathcal{H}(U)$, $u_1 \in \mathcal{H}(u_0)$ and $w_1(t) \in C_B(R, H)$. Since $\{w(t + \alpha_n)\}$ is not a Cauchy sequence, there exists a number $\epsilon > 0$, a sequence $\{t_n\} \subset R$ and two subsequences $\{\tau_n\}$ and $\{\sigma_n\}$ of $\{\alpha_n\}$ such that

$$\|w(\tau_n + t_n) - w(\sigma_n + t_n)\| \geq \epsilon, \quad (4.15)$$

$$w(\tau_n + t_n + t) \rightarrow w_\tau(t) \quad \text{uniformly on bounded intervals} \quad (4.16)$$

$$w(\sigma_n + t_n + t) \rightarrow w_\sigma(t) \quad \text{uniformly on bounded intervals} \quad (4.17)$$

$$V(t_n + t, s)x \rightarrow W(t, s)x \quad \text{uniformly in } t \in R \text{ and pointwise in } (s, x) \in R^+ \times H \quad (4.18)$$

$$u_0(t_n + t) \rightarrow u_2(t). \quad (4.19)$$

It is obvious that $u_2(t)$, $u_2(t) + w_\tau(t)$ and $u_2(t) + w_\sigma(t)$ are three different trajectories of W . We choose a sequence $\beta_n \rightarrow +\infty$ such that

$$W(t - \beta_n, s)x \rightarrow W(t, s)x \quad \text{uniformly in } t \text{ and pointwise in } (s, x) \quad (4.20)$$

$$u_2(t - \beta_n) \rightarrow u_2(t) \quad \text{uniformly in } t \in R \quad (4.21)$$

and

$$w_\tau(t - \beta_n) \rightarrow y_\tau(t) \quad \text{uniformly on bounded intervals of } R \quad (4.22)$$

$$w_\sigma(t - \beta_n) \rightarrow y_\sigma(t) \quad \text{uniformly on bounded intervals of } R. \quad (4.23)$$

Then $u_2(t)$, $u_2(t) + y_\tau(t)$ and $u_2(t) + y_\sigma(t)$ are also three different bounded trajectories of W which remain at a constant distance from each other.

There are only two possibilities:

$$(1) \|y_\tau(t) - y_\sigma(t)\| < \|y_\tau(t)\| + \|y_\sigma(t)\|;$$

$$(2) \|y_\tau(t) - y_\sigma(t)\| = \|y_\tau(t)\| + \|y_\sigma(t)\|;$$

In case (1), $u_2(t)$, $u_2(t) + y_\tau(t)$ and $u_2(t) + y_\sigma(t)$ satisfy the conditions of Theorem 4.2. Thus, all trajectories of U are almost periodic because $u \in \mathcal{H}(W)$. This contradicts the hypothesis that $w(t)$ is not almost periodic.

In case (2), since $y_\tau \neq y_\sigma$ and $\|y_\tau\| = \|y_\sigma\|$, the only possibility is that we have $y_\tau = -y_\sigma$. By using negative translations which carry W and u_2 back to U and u_0 , we obtain a function $\tilde{w}(t)$ such that $u_0 - \tilde{w}$ and $u_0 + \tilde{w}$ are two trajectories of W with $\|w - \tilde{w}\|$ and $\|w + \tilde{w}\|$ constant and $\|\tilde{w}(t)\| \equiv \|w(t)\| \equiv r$. This implies that $\tilde{w} = w$ or $\tilde{w} = -w$ and the proof of this lemma is completed.

Lemma 4.7 *Let H be compact synchronous and suppose that Condition 2 holds.*

Let $u(t)$ be a bounded uniformly continuous trajectory of $U(t, s)$ with relatively compact range in H . If $\|u(t) - u_0(t)\| = r$ (constant) > 0 for all $t \in R$, then $u(t)$ is almost periodic.

Proof On the contrary, we assume that $u(t)$ is not almost periodic. This means that $w(t) = u(t) - u_0(t)$ is not almost periodic. By Lemma 4.6, $u_0(t) - w(t)$ is a trajectory of $U(t, s)$. Now, for any $V(t, s) \in \mathcal{H}(U(t, s))$, $v_0(t) \in \mathcal{H}(u_0(t))$, there exists a function $\gamma(t)$ with $\|\gamma(t)\| = r$ such that $v_0(t) + \gamma(t)$ and $v_0(t) - \gamma(t)$ are two trajectories of $V(t, s)$. We assert that for any $y(t) \in C(R, H)$ with $\|y(t)\| = r$ for all $t \in R$ and $v_0(t) + y(t)$ is a trajectory of $V(t, s)$, then $y(t) = \gamma(t)$, or $y(t) = -\gamma(t)$.

In fact, $v_0(t) + \gamma(t)$ and $v_0(t) - \gamma(t)$ are two trajectories of $V(t, s)$. By Lemma 4.2, $\|y(t) - \gamma(t)\|$ and $\|y(t) + \gamma(t)\|$ are both non-increasing, so they are constants. If $y(t) = \gamma(t)$ and $y(t) = -\gamma(t)$ are not both true, then $y(t), \gamma(t)$ are two trajectories of $V_0(t, s)$ such that $\|y(t)\| = \|\gamma(t)\| = r$ and $\|y(t) - \gamma(t)\| = \text{constant} < 2r$. By Condition 2, $\gamma(t)$ is almost periodic and so is $w(t)$. This gives a contradiction. So, $y(t) = \gamma(t)$ or $y(t) = -\gamma(t)$.

Since $u_0(t)$ is almost periodic, $u_0(t)$ has relatively compact range in H by [87, p.2, Property 1]. Again since $u(t)$ has relatively compact range in H , we can choose a compact subset K of H such that $w(t) \in K$ for all $t \in R$. We write $w(t)$ as $w(t) = \sum_{j=1}^{\infty} w_j(t)e_j$, where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of H .

Now, we show that for any $j \in N$, $w_j(t)$ is almost periodic.

For any $j \in N$, we define

$$c_j(w(t)) = w_j(t) + i \operatorname{sign} w_j(t) ||w^j(t)|| \quad (4.24)$$

where $w^j(t) = \sum_{k \neq j} w_k(t) e_k$. Obviously, if $c_j(w(t))$ is almost periodic, then $w_j(t)$ is also almost periodic. And we have that $c_j(-w(t)) = -c_j(w(t))$ and $|c_j(w(t))|^2 = ||w(t)||^2$ for all $t \in R$.

Now we show that for any $j \in N$, $c_j^2(w(t))$ is almost periodic. For any sequences $\{\alpha_n\}$, $\{\beta_n\}$ of R , since $U(t, s)$, $u_0(t)$ are almost periodic, by Theorem 1.2 there exists common subsequences $\{\alpha'_n\} \subset \{\alpha_n\}$, $\{\beta'_n\} \subset \{\beta_n\}$ such that $U(t, s + \alpha'_n)$ converges to some process $U_1(t, s)$, $u_0(t + \alpha'_n)$ converges to some function $u^*(t)$, and $U(t, s + \alpha'_n + \beta'_n)$ and $U_1(t, s + \beta'_n)$ have the same limit $V(t, s)$, and $u_0(t + \alpha'_n)$ and $u^*(t + \beta'_n)$ have the same limit $v_0(t)$. At the same time, we can take $\{\alpha'_n\}$, $\{\beta'_n\}$ such that $u(t + \alpha'_n + \beta'_n)$ converges to some function $v_1(t)$, $u(t + \alpha'_n)$ converges to some function $v_2(t)$ and $v_2(t + \beta'_n)$ converges to some function $v_3(t)$ uniformly on any compact subset of R . Obviously, $v_1(t)$ and $v_3(t)$ are all trajectories of $V(t, s)$. Now, let $w_1(t) = v_1(t) - v_0(t)$, $w_3(t) = v_3(t) - v_0(t)$. Then $||w_1(t)|| = r$, $||w_3(t)|| = r$ for all $t \in R$. By the first part of this lemma, $w_3(t) = w_1(t)$ or $w_3(t) = -w_1(t)$.

Now, $c_j^2(w(t + \alpha'_n + \beta'_n))$ converges to $c_j^2(w_1(t))$, $c_j^2(w(t + \alpha'_n))$ converges to $c_j^2(v_2(t) - u^*(t))$, and $c_j^2(v_2(t + \beta'_n) - u^*(t + \beta'_n))$ converges to

$$c_j^2(v_3(t) - v_0(t)) = c_j^2(w_3(t)) = c_j^2(w_1(t)) \text{ since } w_3(t) = w_1(t) \text{ or } w_3(t) = -w_1(t).$$

From Theorem 1.2, $c_j^2(w(t))$ is almost periodic, and obviously, $|c_j^2(w(t))| = r^2$ for all $t \in R$.

Using Theorem 1.4 , there exists a real number a and a real almost periodic function $\theta(t) : R \rightarrow R$ such that

$$c_j^2(w(t)) = r^2 e^{i(at+\theta(t))} \quad (4.25)$$

for all $t \in R$. Now, let

$$y_j(t) = c_j(w(t)) e^{-\frac{1}{2}i(at+\theta(t))} \quad (4.26)$$

for all $t \in R$. Then, $y_j(t)$ is continuous on R and $y_j^2(t) = c_j^2(w(t)) e^{-i(at+\theta(t))} = r^2$ for all $t \in R$. This implies that $c_j(w(t)) = y_j(t) e^{\frac{1}{2}i(at+\theta(t))}$ is almost periodic on R . But, we have shown that for any $j \in N$, $c_j(w(t))$ is almost periodic. So, $w_j(t)$ is almost periodic for all $j \in N$. From Lemma 4.4 $w(t)$ is almost periodic. This contradiction completes the proof of Lemma 4.7.

Theorem 4.4 *Let H be compact synchronous and suppose that Condition 2 holds. If $u(t)$ is a bounded uniformly continuous trajectory of $U(t, s)$ with relatively compact range in H , then it is almost periodic on R .*

Proof Let $u_0(t)$ be the almost periodic trajectory of $U(t, s)$ and let $u(t)$ be a bounded uniformly continuous trajectory of $U(t, s)$ which is relatively compact in H . Choose $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ such that $U(t, s - t_n) \rightarrow U(t, s)$, $u_0(t - t_n) \rightarrow u_0(t)$ uniformly on R and $u(t - t_n) \rightarrow v(t)$ uniformly on any compact subset of R , where $v(t) : R \rightarrow H$ is a function. Obviously, $v(t)$ is a trajectory of $U(t, s)$ with relatively compact range in H . Let $y(t) = v(t) - u_0(t)$, then $y(t)$ is a trajectory of $U_0(t, s)$ and $\|y(t)\| = \text{constant}$ for all $t \in R$ from Lemma 4.2. By Lemma 4.7, $y(t)$ is almost periodic on R , so is $v(t)$.

Choose a subsequence $\{t'_n\} \subset \{t_n\}$ such that $v(t + t'_n) \rightarrow u^*(t)$ uniformly on R . Since $\lim_{n \rightarrow \infty} \|u(-t'_n) - u^*(-t'_n)\| = \|v(0) - v(0)\| = 0$, we have that $u(t) = u^*(t)$ for all $t \in R$ by Lemma 4.2. Obviously, $u^*(t)$ is almost periodic, and so is $u(t)$. This completes the proof of this theorem.

4.3 Applications to Special Cases

In this section, we apply our main results (Theorem 4.2 , 4.4) to some special cases. First, we have the following result.

Theorem 4.5 *Let $A(t)$ be linear and monotone for any $t \in R$. Suppose that any bounded uniformly continuous solution $y(t)$ of the homogeneous equation of (4.1)*

$$y' + A(t)y = 0 \tag{4.27}$$

has the property that if $y(t)$ has relatively compact range in H and $\|y(t)\| = \text{constant}$ for any $t \in R$, then $y(t)$ is almost periodic. Then any bounded uniformly continuous solution of (4.1) with relatively compact range in H is almost periodic.

Corollary 4.1 *Let $H = R^n$, $A(t)$ be linear for any $t \in R$. If A is not dependent on t or depends on t in a periodic manner, then any bounded solution of (4.1) is almost periodic (cf. [49 , Theorem 5.8, 6.4]).*

Proof Under the assumptions of this corollary, any bounded solution of (4.27) is almost periodic. So, if $\|y(t)\| = \text{constant}$ for all $t \in R$, then $y(t)$ is bounded, and thus is almost periodic from Theorem 4.5.

Corollary 4.2 *Let A be linear and independent of t . Suppose that $\sigma(A) \cap iR$ is countable. Then any bounded uniformly continuous solution of (4.1) with relatively compact range is almost periodic (cf. [87 , p.94, Theorem 5]).*

Corollary 4.3 *Let $A(t)$ be linear and $A(t+1) = A(t)$ for all $t \in R$. Let V be the monodromy operator of the semi-group generated by $A(t)$. Assume that $\sigma(V) \cap \{\lambda \in C : |\lambda| = 1\}$ is countable. Then any bounded uniformly continuous solution of (4.1) with relatively compact range is almost periodic (cf. [109 , Theorem 4.2]).*

Remark Compared with the results we cited, these Corollaries require a stronger condition, namely, the monotonicity of A . This is because they are implied by a more general result, Theorem 4.5, which can be applied to more general almost periodic cases (see the example in the next section). But the previous results can only be applied to special cases.

Lemma 4.8 *Let A be independent of t . Suppose that $u_1, u_2, u_3 \in D(A)$ and u_1, u_2, u_3 are affinely independent such that*

$$(Au_i - Au_j, u_i - u_j) = 0, \quad i, j = 1, 2, 3. \quad (4.28)$$

where $(\cdot, \cdot)$ is the inner product of H . Then there exist a vector $a \in H$ and a skew-symmetric linear operator $L \in \mathcal{L}(H)$ such that for any $u \in \text{Int}\{u | u = \sum_{i=1}^3 \alpha_i u_i, 0 \leq \alpha_i \leq 1, \sum_{i=1}^3 \alpha_i = 1\}$,

$$Au = Lu + a. \quad (4.29)$$

Furthermore, if A is compact, so is L .

In [61], A. Haraux proved this lemma for the case $H = R^2$, and he also mentioned that it is valid in a more general framework. See Lemma 1.3 in [61] for details.

Theorem 4.6 *Let H be compact synchronous and A be independent of t and compact. Then any bounded uniformly continuous solution of (1) with relatively compact range in H is almost periodic.*

Proof By Theorem 4.3, it is sufficient to show that Condition 2 holds under such an assumption. Let $u_0(t)$ be the almost periodic solution of (4.1), and $U_0(t, s)$ be the difference-variation process of $U(t, s)$ along $u_0(t)$. Since A is independent of t , we only need to show that Condition 2 holds for $U_0(t, s)$. Let $\xi_i(t)$ ($i = 1, 2$) be any two distinct trajectories of $U_0(t, s)$ such that $\xi_i(t)$ is relatively compact in H , $\|\xi_i(t)\| \equiv r_i > 0$, $\|\xi_1(t) - \xi_2(t)\| \equiv r_0 > 0$ for all $t \in R$, and $0 < r_0 < r_1 + r_2$. We will show that $\xi_i(t)$ are almost periodic ($i = 1, 2$). Let $\eta_i = u_0(t) - \xi_i(t)$ ($i = 1, 2$). According to the definition of $U_0(t, s)$, $\eta_i(t)$ is a trajectory of $U(t, s)$ for $i = 1, 2$, and we have $\|u_0(t) - \eta_i(t)\| \equiv r_i > 0$, $\|\eta_1(t) - \eta_2(t)\| \equiv r_0 > 0$, for all $t \in R$. Since $0 < r_0 < r_1 + r_2$, $u_0(t), \eta_1(t), \eta_2(t)$ are affinely independent and

$$(A\eta_i(t) - A\eta_j(t), \eta_i(t) - \eta_j(t)) \equiv 0, \quad i, j = 0, 1, 2 \quad (4.30)$$

for all $t \in R$, where $\eta_0(t) = u_0(t)$. Let

$\Omega(t) = \{u | u = \sum_{j=0}^2 \alpha_j \eta_j(t), 0 \leq \alpha_j \leq 1, \sum_{j=0}^2 \alpha_j = 1\}$. From Lemma 4.8, there exists a compact linear operator $L(t)$ and an element $a(t) \in H$ such that for any $u(t) \in \text{Int}\Omega(t)$, we have

$$Au = L(t)u + a(t). \quad (4.31)$$

Since $\Omega(t) \cap \Omega(\tau) \neq \emptyset$ for τ close to t , a and L , in fact, are independent of t . Now let $u(t) = \sum_{j=0}^2 \alpha_j \eta_j$, $0 < \alpha_j < 1$, $\sum_{j=0}^2 \alpha_j = 1$. Then, $u(t)$ is a bounded uniformly continuous solution of the equation

$$u' + Lu = f(t) - a \quad (4.32)$$

and $u(t)$ is relatively compact in H . Since L is compact, $\sigma(L) \cap iR$ is countable. [87, P_{94} , Theorem 5] implies that $u(t)$ is almost periodic. Letting $\alpha_2 \rightarrow 0$, we get that $\eta_1(t)$ is almost periodic and we have the same result for $\eta_2(t)$. Finally, since $u_0(t)$ is almost periodic, so are $\xi_1(t)$ and $\xi_2(t)$. This completes the proof of the theorem.

Remark When $H = R^2$, this theorem is Theorem 2.1 in [61].

4.4 Examples

Example 1 Consider the linear system

$$x' + A(t)x = f(t) \quad (4.33)$$

where $x \in R^3$, $f : R \rightarrow R^3$ is almost periodic, and

$$A(t) = \begin{pmatrix} 0 & -a(t) & 0 \\ a(t) & 0 & 0 \\ 0 & 0 & b(t) \end{pmatrix}$$

such that $a(t)$ and $b(t)$ are real almost periodic functions.

Suppose that $a(t) = a_0 + a^*(t)$ where a_0 is a constant, $a^*(t)$ is almost periodic, $\int_0^t a^*(s)ds$ is bounded on R , and $b(t) \geq 0$ for all $t \in R$. Then, any bounded solution

of (4.33) is almost periodic. In fact, for any $t \in R$, $A(t)$ is monotone. Since the homogeneous equation of (4.33)

$$y' + A(t)y = 0 \quad (4.34)$$

has the three linear independent solutions

$$y_1(t) = \begin{pmatrix} \cos \int_0^t a(s)ds \\ -\sin \int_0^t a(s)ds \\ 0 \end{pmatrix}, y_2(t) = \begin{pmatrix} \sin \int_0^t a(s)ds \\ \cos \int_0^t a(s)ds \\ 0 \end{pmatrix}, y_3(t) = \begin{pmatrix} 0 \\ 0 \\ e^{-\int_0^t b(s)ds} \end{pmatrix},$$

it follows that for any solution $y(t)$ of (4.34), with $\|y(t)\| \equiv \text{constant}$ for all $t \in R$, then $y(t) = c_1 y_1(t) + c_2 y_2(t)$, unless $b(t) \equiv 0$. From the assumption on $a(t)$ and Lemma 4.1, $y(t)$ is almost periodic. So, by Theorem 4.3, any bounded solution of (4.33) is almost periodic.

Example 5.2 Let $l > 0$ and $F : R \rightarrow R$ be a monotone function and $f : R \rightarrow R$ be almost periodic. Consider the third order differential equation

$$u''' + lu'' + u' - u + F(u'' + u) = f(t). \quad (4.35)$$

Equation (4.35) is equivalent to the third dimensional system

$$\begin{cases} x' - y = 0 \\ y' + x - lz = 0 \\ z' + ly + \frac{1}{l}F(lz) = \frac{1}{l}f(t) \end{cases}$$

or

$$X'(t) + AX(t) = G(t) \quad (4.36)$$

where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3, AX = \begin{pmatrix} -y \\ x - ly \\ ly + \frac{1}{l}F(lz) \end{pmatrix}, \text{ and } G(t) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{l}f(t) \end{pmatrix}$$

Since F is monotone, it is easy to verify that A is monotone. By Theorem 4.6 , any bounded solution of (4.36) is almost periodic if it exists. From this, we obtain that if a solution $x(t)$ of (4.35) satisfies that $x(t)$, $x'(t)$, $x''(t)$ are all bounded on R , then $x(t)$ is almost periodic.

Chapter 5

Further Research: Stepanov's Almost Periodic Differential Equations

5.1 Introduction

In this chapter, we describe further research problems and list some established results. The main topic is on Stepanov's almost periodic functions and on Stepanov's almost periodic differential equations.

Although the concept of Stepanov almost periodic functions has been introduced for more than sixty years, some properties, which play an important role in discussing the properties of solutions of differential equations, have not been established until now. Similar to the case of Bohr almost periodic functions, we establish some properties of Stepanov almost periodic functions. It is well-known

that for (Bohr) almost periodic functions, there is the famous Integral Theorem (Theorem 1.5) and Bochner's Theorem (Theorem 1.2) which play an important role in the theory of almost periodic differential equations. It is natural to pose the following question:

Does the Integral Theorem and Bochner's Theorem hold for S -almost periodic (Stepanov's almost periodic) functions?

As far as we know, this is still an open problem. In this chapter, we will discuss this problem and give a positive answer.

In Chapter 2, we established several results which show that even though we have Bohr almost periodic systems all solutions of which are bounded, there still may not exist any Bohr or Stepanov almost periodic solution. Therefore, in discussing the existence of almost periodic solutions, some extra properties are required. One such extra condition involves some stability property of the bounded solution such as Miller [96], or total stability, Seifert [118–120], Σ -stability, Sell [125], or stability under a disturbance from the hull, and so on. Finally, there is the separation property posed by Amerio [2]. In fact, the separation property of a solution is a kind of generalization of Favard's condition. Later, Fink [48] considered the semi-separation property of a solution. These results have been extended to evolution equations in Banach spaces (see [87]). There are few results on S -almost periodic differential equations in the literature. In this chapter, we discuss the existence of S -almost periodic solutions for S -almost periodic differential equations. At the same time, we extend Favard's Theorem to S -almost periodic differential systems.

5.2 Properties of S-almost Periodic Functions

For completeness, we first give some basic properties of S-almost periodic functions.

We recall the definition of the Stepanov norm $S_l(f)$ of a function $f \in L_1^{loc}(R, X)$.

The quantity

$$S_l(f) = \sup_{t \in R} \frac{1}{l} \int_t^{t+l} \|f(s)\| ds$$

where $l > 0$ is some constant, is the Stepanov norm (or S_l -norm) of f .

Replacing the supremum norm by the S_l -norm in the definition of continuity (respectively, uniform continuity, boundedness) of f , we can introduce the concept of S_l -continuity (respectively, S_l -uniform continuity, S_l -boundedness) of f . For example, we call $f \in L_1^{loc}(R, X)$ S_l -bounded if there exists a constant $M > 0$ such that $S_l(f) \leq M$. It is easy to show that S_l -boundedness (S_l -continuity, S_l -uniform continuity) is not dependent on the constant l . So, we simply call such functions S -bounded, S -continuous, and S -uniformly continuous whenever these notions apply.

We recall the definition of $S_l(t, f)$ as follows:

$$S_l(t, f) = \frac{1}{l} \int_t^{t+l} \|f(s)\| ds \quad \text{for all } t \in R. \quad (5.1)$$

From (5.1) we have that for any $t, s \in R$,

$$S_l(t, f_s) = S_l(t + s, f).$$

where f_s is the translate of f . We use $S_l C(R, X)$ to denote the set of all S_l -continuous functions. Obviously, $C(R, X) \subset S_l C(R, X)$. As in the case of Bohr almost periodic functions, we introduce another definition of a Stepanov almost periodic function.

Definition 5.1 Let $f \in S_l C(R, X)$. If for any sequence $\{\alpha_n\} \subset R$, there exist a subsequence $\{\alpha'_n\}$ of $\{\alpha_n\}$ and a function $g \in SC(R, X)$ such that

$$\lim_{n \rightarrow \infty} S_l(t, f_{\alpha'_n} - g) = 0, \quad \text{uniformly on } R,$$

then f is called S_l -almost periodic on R .

This definition is equivalent to that of Stepanov's which is stated in Chapter 1. We now state the basic properties of Stepanov almost periodic functions, which were proved in different ways (see [12]). Our proofs are based on the new definition, so they are different from others.

We still use $SAP(R, X)$ to denote the set of all S_l -almost periodic functions.

Proposition 5.1 For any $f \in SAP(R, X)$, f is S -bounded on R .

Proof Suppose that f is not S -bounded on R . Then there exists a sequence $\{t_n\} \subset R$ such that

$$S_l(t_n, f) \geq n, \quad n = 1, 2, \dots \quad (5.2)$$

where $l > 0$ is some real number. We can always assume that $l = 1$, without loss generality, and we can omit l so that we can write (5.2) as

$$S(t_n, f) \geq n, \quad n = 1, 2, \dots \quad (5.3)$$

Since f is S -almost periodic, there exist a subsequence $\{t'_n\}$ of $\{t_n\}$ and a function $g \in SC(R, X)$ such that

$$\lim_{n \rightarrow \infty} S(t, f_{t'_n} - g) = 0 \quad \text{uniformly on } R.$$

In particular,

$$\lim_{n \rightarrow \infty} S(0, f_{t'_n} - g) = 0.$$

So, we choose a sufficiently large integer $N > 0$ such that

$$S(0, f_{t'_n} - g) \leq 1, \quad \text{for all } n \geq N.$$

Then

$$S(0, f_{t'_n}) \leq S(0, f_{t'_n} - g) + S(0, g) \leq 1 + S(0, g), \quad n \geq N.$$

This means that

$$S(t_n, f) \leq 1 + S(0, g), \quad n \geq N.$$

This contradicts (5.2) and the proof is completed.

Proposition 5.2 *For any $f \in SAP(R, X)$, f is S -uniformly continuous on R .*

Proof By the contrary, suppose that f is not S -uniformly continuous on R .

Then there exist an $\varepsilon_0 > 0$ and sequences $\{t_n\}, \{\delta_n\}$ such that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$\frac{1}{l} \int_{t_n}^{t_n+l} \|f(s + \delta_n) - f(s)\| ds \geq \varepsilon_0. \quad (5.4)$$

Since f is S -almost periodic, there exist a subsequence $\{t'_n\} \subset \{t_n\}$ and a function $g \in SC(R, X)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{l} \int_t^{t+l} \|f(s + t'_n) - g(s)\| ds = 0 \quad \text{uniformly on } R.$$

So, for $\varepsilon = \frac{\varepsilon_0}{3} > 0$, we can choose an integer $N_1 > 0$ such that whenever $n \geq N_1$

$$\frac{1}{l} \int_t^{t+l} \|f(s + t'_n) - g(s)\| ds < \varepsilon \quad \text{for any } t \in R.$$

In particular,

$$\frac{1}{l} \int_0^l \|f(s + t'_n) - g(s)\| ds < \varepsilon$$

and

$$\frac{1}{l} \int_{\delta'_n}^{\delta'_n+l} \|f(s+t'_n) - g(s)\| ds < \varepsilon$$

where $\{\delta'_n\}$ is the subsequence of $\{\delta_n\}$ with the same subscripts as $\{t'_n\}$.

On the other hand, since $g \in SH(f)$, g is S-continuous at $t = 0$. So for $\varepsilon > 0$, there exists a $\delta_0 > 0$ such that

$$\frac{1}{l} \int_0^l \|g(s+\delta) - g(s)\| ds < \varepsilon \quad \text{whenever } |\delta| < \delta_0.$$

Since $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, we can choose an integer $N_2(> N_1)$ such that $|\delta_n| < \delta_0$ whenever $n > N_2$. Now we have

$$\begin{aligned} \frac{1}{l} \int_{t'_n}^{t'_n+l} \|f(s+\delta'_n) - f(s)\| ds &= \frac{1}{l} \int_0^l \|f(s+t'_n+\delta'_n) - f(s+t'_n)\| ds \\ &\leq \frac{1}{l} \int_0^l \|f(s+t'_n+\delta'_n) - g(s+\delta'_n)\| ds \\ &\quad + \frac{1}{l} \int_0^l \|g(s+\delta'_n) - g(s)\| ds \\ &\quad + \frac{1}{l} \int_0^l \|f(s+t'_n) - g(s)\| ds \\ &< 3\varepsilon = \varepsilon_0 \end{aligned}$$

This contradicts (5.4) and the proof is complete.

Proposition 5.3 *SAP(R, X) is closed in the sense of the S-norm.*

Proof Let $\{f_n\} \subset SAP(R, X)$, $f \in SC(R, X)$ and

$$S_l(f_n - f) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We show that $f \in SAP(R, X)$.

For any sequence $\{\alpha_k\} \subset R$, since $f_1 \in SAP(R, X)$ we can choose a subsequence $\{\alpha_k^{(1)}\}$ of $\{\alpha_k\}$ and a function $g_1 \in SC(R, X)$ such that

$$\lim_{n \rightarrow \infty} S_l(t, (f_1)_{\alpha_k^{(1)}} - g_1) = 0 \quad \text{uniformly on } R,$$

where $(f_1)_{\alpha_k^{(1)}}$ is the translate of f_1 at $\alpha_k^{(1)}$. So, for any $\varepsilon > 0$, there exists a $K_1 > 0$ such that, whenever $k \geq K_1$, $m \geq K_1$, we have

$$S_l(t, (f_1)_{\alpha_k^{(1)}} - (f_1)_{\alpha_m^{(1)}}) < \varepsilon \quad \text{for any } t \in R.$$

Again, since $f_2 \in SAP(R, X)$, we can choose a subsequence $\{\alpha_k^{(2)}\}$ of $\{\alpha_k^{(1)}\}$ and a function $g_2 \in SC(R, X)$ such that

$$\lim_{n \rightarrow \infty} S_l(t, (f_2)_{\alpha_k^{(2)}} - g_2) = 0 \quad \text{uniformly on } R.$$

And so for any $\varepsilon > 0$, there exists a $K_2 > 0$ such that whenever $k \geq K_2$, $m \geq K_2$, we have

$$S_l(t, (f_2)_{\alpha_k^{(2)}} - (f_2)_{\alpha_m^{(2)}}) < \varepsilon \quad \text{for any } t \in R.$$

Similarly, we can choose a set of sequences $\{\alpha_k^{(n)}\}$ such that

$$\{\alpha_k^{(n)}\} \subset \{\alpha_k^{(n-1)}\} \subset \cdots \subset \{\alpha_k^{(1)}\} \subset \{\alpha_k\} \quad \text{for } n = 1, 2, \dots$$

and for any $\varepsilon > 0$, there exists $K_n > 0$ such that whenever $k \geq K_n$, $m \geq K_n$,

$$S_l(t, (f_n)_{\alpha_k^{(n)}} - (f_n)_{\alpha_m^{(n)}}) < \varepsilon \quad \text{for any } t \in R \quad n = 1, 2, \dots \quad (5.5)$$

Let $\beta_n = \alpha_n^{(n)}$, be the diagonal subsequence. Then $\{\beta_n\} \subset \{\alpha_n^{(n)}\} \subset \{\alpha_n\}$. Now we show that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$S_l(f_{\beta_k} - f_{\beta_m}) < \varepsilon, \quad \text{whenever } k \geq N, m \geq N. \quad (5.6)$$

In fact, for any $\varepsilon > 0$, we can choose an integer $N_1 > 0$ such that for $n \geq N_1$

$$S_l(f_n - f) < \varepsilon.$$

From (5.5), we can choose $N > N_1$ such that, whenever $k \geq N, m \geq N$,

$$S_l((f_{N_1})_{\alpha_k^{(N_1)}} - (f_{N_1})_{\alpha_m^{(N_1)}}) < \varepsilon.$$

So,

$$\begin{aligned} S_l(f_{\beta_k} - f_{\beta_m}) &= S_l(f_{\alpha_k^{(k)}} - f_{\alpha_m^{(m)}}) \\ &\leq S_l(f_{\alpha_k^{(k)}} - (f_{N_1})_{\alpha_k^{(k)}}) + S_l((f_{N_1})_{\alpha_k^{(k)}} - (f_{N_1})_{\alpha_m^{(m)}}) + S_l((f_{N_1})_{\alpha_m^{(m)}} - f_{\alpha_m^{(m)}}) \\ &< 3\varepsilon \end{aligned}$$

whenever $k \geq N, m \geq N$. This shows that (5.6) is true using standard arguments.

From (5.6), it is easy to show that f is Stepanov almost periodic and the proposition is therefore proved.

Proposition 5.4 *Let $f \in SAP(R, X)$. If for a given l , $S_l(t, f) \rightarrow 0$ as $t \rightarrow \infty$, then $S_l(t, f) \equiv 0$ for all $t \in R$.*

Proof Assuming to the contrary, suppose that there exists a $t_0 \in R$ such that

$$S_l(t_0, f) = \varepsilon_0 > 0. \tag{5.7}$$

We shall obtain a contradiction. Take a sequence $\alpha = \{\alpha_n\}$ such that $\alpha_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Now, for any $t \in R$,

$$\lim_{n \rightarrow +\infty} S_l(t + \alpha_n, f) = 0.$$

According to the definition of S-almost periodic functions, we can choose a subsequence $\alpha' \subset \alpha$ and function $g \in SC(R, X)$ such that $UST_{\alpha'} f = g$, i.e.,

$$\lim_{n \rightarrow +\infty} S_l(t, f_{\alpha'_n} - g) = 0, \quad \text{uniformly for } t \in R.$$

Since

$$S_l(t, g) \leq S_l(t, f_{\alpha'_n} - g) + S_l(t, f_{\alpha'_n}) = S_l(t, f_{\alpha'_n} - g) + S_l(t + \alpha'_{n'}, f),$$

we can let $n \rightarrow \infty$, and we have $S_l(t, g) = 0$ for any $t \in R$. Thus,

$$\begin{aligned} S_l(t_0, f) &= S_l(t_0 - \alpha'_{n'}, f_{\alpha'_n} - g) + S_l(t_0 - \alpha'_{n'}, g) \\ &= S_l(t_0 - \alpha'_{n'}, f_{\alpha'_n} - g) \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$. This contradicts (5.7).

Now we give the definition of the *uniform Stepanov hull* of a Stepanov almost periodic function, and then establish some properties of the hull.

Definition 5.2 Let $f \in SC(R, X)$. We call the set

$$\{g | g \in SC(R, X), \text{ there exists a sequence } \{\alpha_n\} \subset R \text{ such that } UST_{\alpha} f = g\} \quad (5.8)$$

the **uniform Stepanov hull**, or simply **uniform S-hull**, denoted by $SH(f)$.

Obviously, for any $f \in SC(R, X)$, $SH(f)$ is not empty since $f \in SH(f)$.

Now we discuss the properties of the uniform S-hull of a function .

Proposition 5.5 If f is Bohr almost periodic on R , then

$$H(f) \subseteq SH(f)$$

where $H(f)$ is the uniform Bohr hull of f .

Proof This is obvious by definition.

Proposition 5.6 *$SH(f)$ is compact in the sense of the S -norm if and only if f is S -almost periodic on R .*

Proof Necessity. Suppose that $SH(f)$ is compact. For any sequence $\{\alpha_n\} \subset R$, let $f_n(t) = f_{\alpha_n}(t)$. Then $f_n \in SH(f)$ for each $n = 1, 2, \dots$. So there exists a subsequence $\{n_k\} \subseteq \{n\}$ and a function $g \in SH(f)$ such that $S_l(t, f_{n_k} - g) \rightarrow 0$ uniformly on R as $k \rightarrow \infty$. So, $S_l(t, f_{\alpha_n} - g) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on R . This implies that f is S -almost periodic on R .

Sufficiency. Let $f \in SAP(R, X)$, $g_n \in SH(R, X)$, $n = 1, 2, \dots$. For any n , choose $\{\alpha_k^{(n)}\} \subset R$ such that

$$S_l(f_{\alpha_k^{(n)}} - g_n) = \sup_{t \in R} S_l(t, f_{\alpha_k^{(n)}} - g_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

So, we can choose a sequence $\{\alpha_n\} \subset R$ such that

$$S_l(f_{\alpha_n} - g_n) < \frac{1}{n}, \quad \text{for each } n = 1, 2, \dots$$

Since $f \in SAP(R, X)$, there exists $\{\alpha_{n_k}\} \subseteq \{\alpha_n\}$ and a function $g \in SC(R, X)$ such that

$$\lim_{n \rightarrow \infty} S_l(t, f_{\alpha_{n_k}} - g) = 0, \quad \text{uniformly on } R.$$

So, $g \in SH(R, X)$ and

$$\begin{aligned} S_l(t, g_{n_k} - g) &\leq S_l(t, g_{n_k} - f_{\alpha_{n_k}}) + S_l(t, f_{\alpha_{n_k}} - g) \\ &\leq S_l(t, g_{n_k} - f_{\alpha_{n_k}}) + \sup_{t \in R} S_l(t, f_{\alpha_{n_k}} - g) \\ &\leq \frac{1}{n_k} + S_l(f_{\alpha_{n_k}} - g). \end{aligned}$$

Therefore,

$$\lim_{k \rightarrow \infty} S_l(t, g_{n_k} - g) = 0.$$

This means that g_{n_k} converges to g in the S-norm sense and so, $SH(f)$ is compact.

Proposition 5.7 *If $f \in SAP(R, X)$, $g \in SH(f)$, then $g \in SAP(R, X)$ and $f \in SH(g)$.*

Proof Since $g \in SH(f)$, there exists a sequence $\{\alpha_n\} \subset R$ such that

$$\lim_{n \rightarrow \infty} S_l(t, f_{\alpha_n} - g) = 0 \quad \text{uniformly on } R.$$

For any sequence $\{\beta_n\} \subset R$, let $\gamma_n = \beta_n + \alpha_n, n = 1, 2, \dots$. Since $f \in SAP(R, X)$, there exists a subsequence $\{\gamma'_n\} \subseteq \{\gamma_n\}$ and a function $g_1 \in SH(f)$ such that

$$\lim_{n \rightarrow \infty} S_l(t, f_{\gamma'_n} - g_1) = 0 \quad \text{uniformly on } R,$$

so,

$$\begin{aligned} S_l(t, g_{\beta'_n} - g_1) &\leq S_l(t, g_{\beta'_n} - f_{\gamma'_n}) + S_l(t, f_{\gamma'_n} - g_1) \\ &= S_l(t + \beta'_n, g - f_{\alpha'_n}) + S_l(t, f_{\gamma'_n} - g_1) \end{aligned}$$

and thus

$$\lim_{n \rightarrow \infty} S_l(t, g_{\beta'_n} - g_1) = 0 \quad \text{uniformly on } R.$$

This implies that $g \in SAP(R, X)$. Choosing $\delta_n = -\alpha_n$, we get $f \in SH(g)$.

Proposition 5.8 *If $f \in SAP(R, X)$, then for any $g \in SH(f)$, $SH(g) = SH(f)$.*

Proof We first show that for any $f \in SH(f)$, $SH(g) \subset SH(f)$.

Let $h \in SH(g)$. Then there are two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that

$$\lim_{n \rightarrow \infty} S_l(f_{\alpha_n} - g) = 0 \quad \text{uniformly on } R,$$

and

$$\lim_{n \rightarrow \infty} S_l(g_{\beta_n} - h) = 0 \quad \text{uniformly on } R.$$

So,

$$\lim_{n \rightarrow \infty} S_l(f_{\alpha_n + \beta_n} - h) \leq \lim_{n \rightarrow \infty} S_l(f_{\alpha_n + \beta_n} - g_{\beta_n}) + \lim_{n \rightarrow \infty} S_l(g_{\beta_n} - h) = 0 \quad \text{uniformly on } R.$$

Therefore, $h \in SH(f)$. This implies that $SH(g) \subset SH(f)$. From Proposition 5.7,

$f \in SH(g)$. Using what we have proved above, we have that $SH(f) \subset SH(g)$.

Thus, $SH(g) = SH(f)$ and the proof is complete.

5.3 Generalization of Bochner's Theorem

In this section, we extend Bochner's Theorem on Bohr almost periodic functions to the case of Stepanov almost periodic functions. Just like the case of Bohr almost periodic functions, this theorem plays an important role in discussing the existence of Stepanov almost periodic solutions for Stepanov almost periodic differential equations.

Now we state Bochner's Theorem (Theorem 1.2) for Stepanov almost periodic functions.

Theorem 5.1 *Let $f \in SC(R, X)$. Then f is Stepanov almost periodic on R if and only if for any pair of sequences $\alpha, \beta \subset R$, one can extract common subsequences*

$\alpha' \subset \alpha, \quad \beta' \subset \beta$ such that

$$ST_{\alpha'+\beta'} f = ST_{\alpha'}(ST_{\beta'} f),$$

i.e. there exist two functions $g, h \in SC(R, X)$ such that

$$\lim_{n \rightarrow \infty} S_l(t, f_{\alpha'_n + \beta'_n} - g) = 0 \quad \text{for all } t \in R$$

$$\lim_{n \rightarrow \infty} S_l(t, f_{\beta'_n} - h) = 0 \quad \text{for all } t \in R$$

$$\lim_{n \rightarrow \infty} S_l(t, h_{\alpha'_n} - g) = 0 \quad \text{for all } t \in R.$$

Proof Necessity. Let $f \in SAP(R, X)$. For any sequences $\alpha = \{\alpha_n\} \subset R, \beta = \{\beta_n\} \subset R$, we can extract a subsequence $\beta'' \subset \beta$ and a function $h \in SC(R, X)$ such that $UST_{\beta''} f = h$ and $h \in SAP(R, X)$ by Proposition 5.7. So, we can extract a subsequence $\alpha' = \{\alpha'_n\} \subset \alpha''$, which is a subsequence of α with the same subscripts as β'' , and a function $g \in SC(R, X)$ such that $UST_{\alpha'} h = g$. Now let $\beta' \subset \beta''$ have the same subscripts as α' and let $\gamma' = \alpha' + \beta'$. Then we can choose a subsequence $\gamma \subset \gamma'$ and a function $k \in SC(R, X)$ such that $UST_{\gamma} f = k$. Since

$$S_l(t, k - g) \leq S_l(t, k - f_{\gamma_n}) + S_l(t + \alpha_n, f_{\beta_n} - h) + S_l(t, h_{\alpha_n} - g),$$

we can let $n \rightarrow \infty$. Then we see that

$$S_l(t, k - g) = 0 \quad \text{for any } t \in R.$$

But

$$S_l(t, f_{\gamma} - g) \leq S_l(t, f_{\gamma} - k) + S_l(t, k - g) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This implies that $ST_\gamma f = g$. Since $\gamma \subset \gamma' = \alpha' + \beta'$, we can write $\gamma = \alpha + \beta$ where $\alpha \subset \alpha'$ and $\beta \subset \beta'$ have the same subscripts and $ST_\alpha h = g$, $ST_\beta f = h$. This completes the proof of the necessity.

Sufficiency. For any sequence $\gamma = \{\gamma_n\}$, we will show that there exist a subsequence $\gamma' \subset \gamma$ and a function $g \in SC(R, X)$ such that $UST_{\gamma'} f = g$.

For any sequence $\gamma = \{\gamma_n\}$, let $\alpha = 0, \beta = \gamma$. From the assumptions, we can choose common sequences $\alpha' \subset \alpha, \beta' \subset \beta$ and the functions $g, h \in SC(R, X)$ such that $ST_{\alpha'+\beta'} f = g, ST_{\beta'} f = h, ST_{\alpha'} h = g$. So, $ST_{\beta'} f = g$, and $ST_{\beta'} f = h$. Therefore,

$$S_l(t, g - h) \leq S_l(t, g - f_{\beta'_n}) + S_l(t, f_{\beta'_n} - h) \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ for any } t \in R.$$

So,

$$S_l(t, g - h) \equiv 0 \quad \text{for any } t \in R.$$

Now we will show that $UST_{\beta'} f = g$. i.e.,

$$\lim_{n \rightarrow \infty} S_l(t, f_{\beta'_n} - g) = 0 \quad \text{uniformly on } R.$$

Assuming to the contrary, we suppose that the convergence of the above is not uniform on R . Then there exists a real number $\varepsilon_0 > 0$, a subsequence $\nu = \{\nu_n\}$ of β and a sequence $t = \{t_n\} \subset R$ such that

$$S_l(t_n, f_{\nu_n} - g) \geq \varepsilon_0 \quad \text{for any } n = 1, 2, \dots$$

From the given conditions, we can choose a pair of common subsequences

$\nu' \subset \nu, \quad t' = \{t'_n\} \subset t$ and functions $g_1, h_1 \in SC(R, X)$ such that $ST_{\nu'+\nu'} f = g_1$, $ST_{\nu'} f = h_1$ and $ST_{\nu'} h_1 = g_1$.

Since $\nu' \subset \nu \subset \beta$, we have that

$$S_l(t, h - h_1) \leq S_l(t, f_{\nu'_n} - h) + S_l(t, f_{\nu'_n} - h_1).$$

Letting $n \rightarrow \infty$, we have that $S_l(t, h - h_1) \equiv 0$ for all $t \in R$.

But,

$$S_l(t, f_{t'_n + \nu'_n} - g_{t'_n}) \leq S_l(t, f_{t'_n + \nu'_n} - g_1) + S_l(t, g_1 - h_1 \nu'_n) + S_l(t, h_1 \nu'_n - g_{t'_n}).$$

So, if we let $n \rightarrow \infty$, we have that

$$S_l(t, f_{t'_n + \nu'_n} - g_{t'_n}) \rightarrow 0, \text{ for any } t \in R.$$

We set $t = 0$, and use the fact that $S_l(0, f_{t+s}) = S_l(t, f_s)$. We have that

$$S_l(t'_n, f_{\nu'_n} - g) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $t' \subset t$, we obtain a contradiction because

$$S_l(t_n, f_{\nu_n} - g) \geq \varepsilon_0 \quad \text{for any } n = 1, 2, \dots$$

This completes the proof of this theorem.

5.4 An Integration Theorem

In this section we discuss the Stepanov almost periodicities of the integration of S-almost periodic functions. For the integral of Bohr almost periodic functions, we have stated the famous Bohr Theorem (see Theorem 1.5.) in Chapter 1, i.e.:

Let $f \in AP(R, R^n)$ and let $F(t) = \int_0^t f(s)ds$, then $F(t) \in AP(R, R^n)$ if and only if $F(t)$ is bounded on R .

The question is whether this result can be extended to the case of S-almost periodic functions. Our results answer this question. In order to discuss the

integral of almost periodic functions, for any $f \in L_1^{loc}(R, X)$, we let

$$F(t) = \int_0^t f(s)ds \quad \text{for any } t \in R.$$

It is more difficult to discuss the integration of S-almost periodic functions. For more general almost periodic functions, the boundedness of the integral cannot imply its almost periodicity. Additional conditions are needed. For the case where X is infinite dimensional, additional conditions are: either $F(R)$ is relatively compact in X or X does not contain a subspace C_0 (see [87]). For the case that f is asymptotically almost periodic, an additional condition is that $F(t)$ has uniformly convergent mean on R (see [112, 113]). These results play an important role in discussing the almost periodicity of bounded solutions for linear systems. Most results on this topic were collected in Amerio and Prouse's book (see [4]). In these previous results, more conditions were required to guarantee the almost periodicity of the integral of S-almost periodic functions (see Bochner[15]). The following results show that the boundedness of f implies the S-almost periodicity of the integral of f . In order to prove our result, we need some preliminaries. Firstly, we introduce the following theorem which in some sense is a weak form of the Ascoli Theorem.

Theorem 5.2 *Let $I = [a, b]$ be an closed interval of R , $l > 0$ a real number. Set $I_l = [a, b + l]$. Let $\{f_n(t)\}$ be a sequence of functions defined on I_l with values in X . If $\{f_n(t)\}$ satisfies*

(i) *For any $t \in I$, and any subsequence $\{f_{n_k}(t)\} \subset \{f_n(t)\}$, there exists a subsequence $\{\tilde{f}_k(t)\}$ of $\{f_{n_k}(t)\}$ such that $\{\tilde{f}_k(t)\}$ is convergent in the sense of the S-norm;*

(ii) $\{f_n(t)\}$ is equi-S-continuous on I , i.e., for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that whenever $|\tau| < \delta$, $S_l(t, (f_n)_\tau - f_n) < \varepsilon$ for all $t \in I$, where $(f_n)_\tau$ is the translate of f_n at τ .
Then $\{f_n(t)\}$ contains a subsequence which is convergent uniformly on I in the sense of the S-norm.

Proof Let $I_r = \{r_1, r_2, \dots\}$ be an enumeration of all the rationals in I . From condition (i), there exists a subsequence $\{f_{1,n}(t)\}$ of $\{f_n(t)\}$ such that $\{f_{1,n}(t)\}$ is convergent at r_1 in the sense of the S-norm. Again using condition (i), there exists a subsequence $\{f_{2,n}(t)\}$ of $\{f_{1,n}(t)\}$ such that $\{f_{2,n}(t)\}$ is convergent at r_2 in the sense of the S-norm. Similarly, we can obtain a set of subsequences $\{f_{k,n}(t)\}$ such that $\{f_{k,n}(t)\}$ is convergent at r_k in the sense of the S-norm and $\{f_{k,n}(t)\} \subset \{f_{k-1,n}(t)\}$. Now, we set $\tilde{f}_k(t) = f_{k,k}(t)$. Then, $\{\tilde{f}_k(t)\}_{k=n}^\infty \subset \{f_{n,k}(t)\}$. So, $\{\tilde{f}_k(t)\}$ is convergent at $r_k, k = 1, 2, \dots$.
Now we show that $\{\tilde{f}_k(t)\}$ is convergent uniformly on I . From condition (ii), for any $\varepsilon > 0$, we can choose a $\delta = \delta(\varepsilon) > 0$ such that

$$S_l(t, (f_n)_\tau - f_n) < \varepsilon \quad \text{for all } t \in I, \quad \text{whenever } |\tau| < \delta. \quad (5.9)$$

Let $\{\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_K\} \subset I_r$ such that $I \subset \bigcup_{i=1}^K (\tilde{r}_i - \delta, \tilde{r}_i + \delta)$. So, we can find a large positive integer N (independent of $\tilde{r}_i \in \{\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_K\}$) such that

$$S_l(\tilde{r}_i, \tilde{f}_k - \tilde{f}_m) < \varepsilon \quad \text{whenever } k > N, m > N \quad \text{and} \quad \tilde{r}_i \in \{\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_K\}. \quad (5.10)$$

For any $t \in I$, there is an $\tilde{r}_i \in \{\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_K\}$ such that $t \in (\tilde{r}_i - \delta, \tilde{r}_i + \delta)$. From (5.9) and (5.10), we obtain that whenever $k > N, m > N$,

$$S_l(t, \tilde{f}_k - \tilde{f}_m) \leq S_l(t, \tilde{f}_k - (\tilde{f}_k)_\Delta) + S_l(t, (\tilde{f}_k)_\Delta - (\tilde{f}_m)_\Delta) + S_l(t, (\tilde{f}_m)_\Delta - \tilde{f}_m)$$

$$\begin{aligned}
&= S_l(t, \tilde{f}_k - (\tilde{f}_k)_\Delta) + S_l(\tilde{r}_i, \tilde{f}_k - \tilde{f}_m) + S_l(t, (\tilde{f}_m)_\Delta - \tilde{f}_m) \\
&< 3\varepsilon
\end{aligned}$$

where $\Delta = t - \tilde{r}_i$ and $(\tilde{f}_m)_\Delta$ is the translate of $\tilde{f}_m$ at Δ . This shows that $\{\tilde{f}_k(t)\}$ is convergent uniformly on I and the proof is complete.

Lemma 5.1 *Let $f \in SAP(R, X)$, $F(t) = \int_0^t f(s)ds$. Then for any $\alpha = \{\alpha_n\} \subset R$, $\{F_{\alpha_n}(t)\}$ is equi-S-continuous on R .*

Proof Since $f \in SAP(R, X)$, f is S-bounded on R . So, there exists a positive constant $M > 0$ such that $S_l(t, f) \leq M$, for all $t \in R$. Now for any $\varepsilon > 0$, we set $\delta = \frac{\varepsilon}{3M}$. Since

$$\begin{aligned}
S_l(t, (F_{\alpha_n})_\tau - F_{\alpha_n}) &= \frac{1}{l} \int_t^{t+l} \left| \int_0^{s+\alpha_n+\tau} f(u)du - \int_0^{s+\alpha_n} f(u)du \right| ds \\
&\leq \frac{1}{l} \int_t^{t+l} \left| \int_{s+\alpha_n}^{s+\alpha_n+\tau} f(u)du \right| ds,
\end{aligned}$$

whenever $0 < \tau < \delta (< l)$, we have that

$$\begin{aligned}
S_l(t, (F_{\alpha_n})_\tau - F_{\alpha_n}) &\leq \frac{1}{l} \int_{t+\alpha_n}^{t+\alpha_n+\tau} \int_t^{u-\alpha_n} |f(u)| ds du + \frac{1}{l} \int_{t+\alpha_n+\tau}^{t+\alpha_n+l} \int_{u-\alpha_n-\tau}^{u-\alpha_n} |f(u)| ds du \\
&\quad + \frac{1}{l} \int_{t+\alpha_n+l}^{t+\alpha_n+l+\tau} \int_{u-\alpha_n-\tau}^{t+l-\alpha_n} |f(u)| ds du \\
&= \frac{1}{l} \int_{t+\alpha_n}^{t+\alpha_n+\tau} |f(u)|(u - \alpha_n - t) du + \frac{1}{l} \int_{t+\alpha_n+\tau}^{t+\alpha_n+l} |f(u)|\tau du \\
&\quad + \frac{1}{l} \int_{t+\alpha_n+l}^{t+\alpha_n+l+\tau} |f(u)|(t + l + \tau - u) du \\
&\leq 3 \frac{1}{l} \int_{t+\alpha_n}^{t+\alpha_n+l} |f(u)|\tau du \leq 3M|\tau| < \varepsilon.
\end{aligned}$$

Similarly, whenever $-\delta < \tau < 0$, we also have

$$S_l(t, (F_{\alpha_n})_\tau - F_{\alpha_n}) \leq 3M|\tau| < \varepsilon.$$

This shows that $\{F_{\alpha_n}(t)\}$ is equi-S-continuous on R and this is what we desired to show.

Lemma 5.2 *Let $f \in SAP(R, X)$, $F(t) = \int_0^t f(s)ds$, and let $\alpha = \{\alpha_n\} \subset R$, $F_n(t) = F(t + \alpha_n)$. If $\{F(t)|t \in R\}$ is relatively compact in X , then for any $\alpha' \subset \alpha$, there exists a subsequence $\alpha'' \subset \alpha'$ and a function $G \in SC(R, X)$ such that*

$$\lim_{n \rightarrow +\infty} S_l(t, F_{\alpha_n''} - G) = 0 \quad \text{for any fixed } t \in R. \quad (5.11)$$

Proof Since $\{F(t)\}$ is relatively compact in X , for $\{F(\alpha_n')\}$, there exists a subsequence of α' , denoted by α'' again, and a point $G_0 \in X$ such that

$$\lim_{n \rightarrow \infty} F(\alpha_n'') = G_0. \quad (5.12)$$

Now, for α'' , we can choose common subsequences $\alpha'' \subseteq \alpha'$ and a function $g \in SC(R, X)$ such that $ST_{\alpha''}f = g$. Set

$$G(t) = G_0 + \int_0^t g(s)ds. \quad (5.13)$$

Now, we will prove that (5.11) holds.

Firstly, for any $s \in R$, there exists an integer $m \in \mathbb{Z}$ such that $ml \leq s < (m+1)l$.

So, we have that

$$\begin{aligned} \|F(s + \alpha_n'') - G(s)\| &= \left\| \int_0^{s+\alpha_n''} f(u)du - G_0 - \int_0^s g(u)du \right\| \\ &\leq \left\| \int_0^{\alpha_n''} f(s)ds - G_0 \right\| + \left\| \int_0^s [f(u + \alpha_n'') - g(u)]du \right\| \\ &\leq \left\| \int_0^{\alpha_n''} f(s)ds - G_0 \right\| \\ &\quad + \left\| \sum_{k=0}^{m-1} \int_{kl}^{(k+1)l} [f(u + \alpha_n'') - g(u)]du + \int_{ml}^s [f(u + \alpha_n'') - g(u)]du \right\|. \end{aligned}$$

Now, for any $t \in R$, fixed, let $m \in Z$ be such that $ml \leq t < (m+1)l$. Then we have that

$$\begin{aligned}
S_l(t, F_{\alpha_n''} - G) &\leq \left\| \int_0^{\alpha_n''} f(s)ds - G_0 \right\| + \frac{1}{l} \int_t^{t+l} \left\| \int_0^s [f(u + \alpha_n'') - g(u)]du \right\| ds \\
&\leq \left\| \int_0^{\alpha_n''} f(s)ds - G_0 \right\| \\
&\quad + \left\| \int_0^t [f(u + \alpha_n'') - g(u)]du \right\| + \frac{1}{l} \int_t^{t+l} \left\| \int_t^s [f(u + \alpha_n'') - g(u)]du \right\| ds \\
&\leq \left\| \int_0^{\alpha_n''} f(s)ds - G_0 \right\| \\
&\quad + \left\| \sum_{k=0}^{m-1} \int_{kl}^{(k+1)l} [f(s + \alpha_n'') - g(s)]ds + \int_{ml}^t [f(s + \alpha_n'') - g(s)]ds \right\| \\
&\quad + \left\| \int_t^{t+l} [f(u + \alpha_n'') - g(u)]du \right\| \\
&\leq \left\| \int_0^{\alpha_n''} f(s)ds - G_0 \right\| + l \sum_{k=0}^m \frac{1}{l} \int_{kl}^{(k+1)l} \|f(s + \alpha_n'') - g(s)\| ds \\
&\quad + \int_t^{t+l} \|f(s + \alpha_n'') - g(s)\| ds.
\end{aligned}$$

Letting $n \rightarrow \infty$, we have that

$$\lim_{n \rightarrow \infty} S_l(t, F_{\alpha_n''} - G) = 0, \quad \text{for any fixed } t \in R.$$

This show that (5.11) holds and this completes the proof of this lemma.

From Theorem 5.2 and the above lemmas, we can obtain the following corollary.

Corollary 5.1 *Let $f \in SAP(R, X)$, $F(t) = \int_0^t f(s)ds$. Let $\{\alpha_n\} \subset R$ be a sequence. Set $x_n(t) = F_{\alpha_n}(t)$, $n = 1, 2, \dots$. If the set $\{F(t) | t \in R\}$ is relatively compact in X , then $\{x_n(t)\}$ contains a subsequence which is uniformly convergent in the sense of the S -norm on any compact subset of R .*

Let K be a compact convex subset of X . Assume that $\{F(t) | t \in R\} \subset K$. For any $g \in SH(f)$, there exists a sequence $\alpha = \{\alpha_n\} \subset R$ such that $UST_\alpha f = g$. From

Corollary 5.1 there exists a subsequence $\{\alpha'_n\} \subset \{\alpha_n\}$ and a function

$G(t) \in SC(R, X)$ such that $ST_{\alpha'}F = G$ uniformly on any compact subset of R .

From the proof of Lemma 5.2, we know that $G(t) = G_0 + \int_0^t g(s)ds$ and $G(t) \in K$ for all $t \in R$.

We define

$$\Omega_F = \{x(t) | x(t) = c + F(t), c \in X, x(t) \in K, \text{ for all } t \in R\}.$$

For any $x \in \Omega_F$, we define

$$N(x) = \sup_{t \in R} \left(\frac{1}{l} \int_t^{t+l} |x(s)|^2 ds \right)^{\frac{1}{2}}.$$

Lemma 5.3 *There exists a function $x_0(t) \in \Omega_F$ such that $N(x_0) = \inf_{x \in \Omega_F} N(x)$.*

Proof Let

$$\lambda = \inf_{x \in \Omega_F} N(x),$$

and

$$\lambda_n = \inf_{x \in \Omega_F} \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x(s)|^2 ds \right)^{\frac{1}{2}}.$$

Then $\lambda \leq N(F)$ exists and $\lambda_n \leq \lambda_{n+1}$, $\lim_{n \rightarrow +\infty} \lambda_n = \lambda$. According to the definition of λ_n , we can choose a sequence $\{c_n\} \subset K$ such that

$$\sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x_n(s)|^2 ds \right)^{\frac{1}{2}} \leq \lambda_n + \frac{1}{n}$$

where $x_n(t) = c_n + F(t)$, $n = 1, 2, \dots$ and $x_n(t) \in K$ for all $t \in R$ and $n \in \mathbb{Z}^+$.

Since $F(0) = 0$, $\{c_n\} \subset K$. So, there exists a subsequence $\{c_{n_k}\} \subset \{c_n\}$ and a $c_0 \in K$ such that $\{c_{n_k}\}$ is convergent to c_0 and thus $\{x_{n_k}(t)\}$ is convergent to $x_0(t) = c_0 + F(t)$, i.e.,

$$\lim_{k \rightarrow \infty} |x_{n_k}(t) - x_0(t)| = 0 \text{ uniformly on } t \in R.$$

Since $x_k(t) \in K$, $x_0(t) \in K$ for all $t \in R$. Therefore

$$\lim_{k \rightarrow \infty} \left(\frac{1}{l} \int_t^{t+l} |x_{n_k}(s) - x_0(s)|^2 ds \right)^{\frac{1}{2}} = 0 \quad \text{for all } t \in R.$$

Thus,

$$\begin{aligned} N(x_0) &\leq \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x_0(s) - x_n(s)|^2 ds \right)^{\frac{1}{2}} + \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x_n(s)|^2 ds \right)^{\frac{1}{2}} \\ &\leq \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x_0(s) - x_n(s)|^2 ds \right)^{\frac{1}{2}} + \lambda_{n_k} + \frac{1}{n_k}. \end{aligned}$$

Letting $k \rightarrow +\infty$, we have that $N(x_0) \leq \lambda$. Since $x_0 \in \Omega_F$, we have $N(x_0) = \lambda$ and the proof is completed.

It is easy to show the following lemma and so we omit its proof.

Lemma 5.4 *Let $g \in SH(f)$, $\alpha = \{\alpha_n\} \subset R$ such that $UST_\alpha f = g$. For any $x \in \Omega_F$, if there exists $y \in SC(R, X)$ such that $ST_\alpha x = y$, then $y \in \Omega_G$ and $N(y) \leq N(x)$. Furthermore, if $N(x) = \inf_{x \in \Omega_F} N(x)$, then $N(y) = \inf_{y \in \Omega_G} N(y)$, where $G(t) = \int_0^t g(s) ds$.*

From this lemma, we can easily prove that for any $g \in SH(f)$, we have

$$\inf_{x \in \Omega_F} N(x) = \inf_{y \in \Omega_G} N(y).$$

Lemma 5.5 *For any $g \in SH(f)$, there exists a unique $y_0 = d_0 + G(t)$, for some $d_0 \in X$, such that*

$$N(y_0) = \inf_{y \in \Omega_F} N(y).$$

Proof The existence of y_0 follows from Lemma 5.3. We show the uniqueness now.

Assume the contrary. Suppose that there exist $d_1, d_2 \in X$ such that

$d_1 \neq d_2$, $y_i(t) = d_i + G(t) \in K$, ($i = 1, 2$), for all $t \in R$ and

$$N(y_1) = N(y_2) = \inf_{y \in \Omega_G} N(y).$$

Now, we set $u(t) = \frac{1}{2}(y_1(t) + y_2(t))$, $v(t) = \frac{1}{2}(y_1(t) - y_2(t))$. Since K is convex, $u(t), v(t) \in K$ for all $t \in R$. Thus, we have that

$$\begin{aligned} \frac{1}{l} \int_t^{t+l} (u(s))^2 ds + \frac{1}{l} \int_t^{t+l} (v(s))^2 ds &= \frac{1}{l} \int_t^{t+l} \frac{1}{4} (y_1(s) + y_2(s))^2 ds + \frac{1}{l} \int_t^{t+l} \frac{1}{4} (y_1(s) - y_2(s))^2 ds \\ &= \frac{1}{2l} \int_t^{t+l} (y_1(s))^2 ds + \frac{1}{2l} \int_t^{t+l} (y_2(s))^2 ds \\ &= \left(\inf_{y \in \Omega_G} N(y) \right)^2. \end{aligned}$$

But, $v(t) \equiv d_1 - d_2 \neq 0$, so, $\inf_{y \in \Omega_G} N(y) > 0$, and thus

$$\frac{1}{l} \int_t^{t+l} (u(s))^2 ds < \left(\inf_{y \in \Omega_G} N(y) \right)^2.$$

i.e., $N(u) < \inf_{y \in \Omega_G} N(y)$. This contradicts the fact $u(t) \in K$ for all $t \in R$ and the proof is ended.

Theorem 5.3 *Let $f \in SAP(R, X)$, $F(t) = \int_0^t f(s)ds$. If the set $\{F(t) | t \in R\}$ is relatively compact in X , then for any $g \in SH(f)$, $G(t) = \int_0^t g(s)ds \in SAP(R, X)$.*

Proof From the assumption, we can take a compact convex subset K of X such that $F(t) \in K$ for all $t \in R$. From Lemma 5.5, for any $g \in SH(f)$, there exists unique $y_0(t) = d_0 + G(t)$, for some $d_0 \in K$, such that $N(y_0) = \inf_{y \in \Omega_G} N(y)$. We show that $y_0(t) \in SAP(R, X)$.

For any sequences $\alpha = \{\alpha_n\}$, $\beta = \{\beta_n\} \subset R$, from Theorem 5.1, there exists common subsequences $\alpha' \subset \alpha$, $\beta' \subset \beta$ and the functions $h_1, h_2 \in SH(f)$ such that $UST_{\alpha' + \beta'} g = h_1$, $UST_{\alpha'} g = h_2$ and $UST_{\beta'} h_2 = h_1$. From Corollary 5.1 and Lemma

5.4, there exist functions $u(t) \in \Omega_{H_2}, v(t), w(t) \in \Omega_{H_1}$ such that $ST_{\alpha'}y_0 = u, ST_{\alpha'+\beta'}y_0 = v, ST_{\beta'}u = w$, take subsequences if necessary, and $N(v) = N(w) = \inf_{y \in \Omega_G} N(y)$. From Lemma 5.5, we have $v(t) \equiv w(t)$, for all $t \in R$, i.e., $ST_{\beta'}(ST_{\alpha'}y_0) = ST_{\alpha'+\beta'}y_0$. By Theorem 5.1, $y_0(t) \in SAP(R, X)$, and so $G(t) \in SAP(R, X)$. This completes the proof of this Theorem.

5.5 Existence of S-almost Periodic Solutions

In this section we consider S-almost periodic differential equations. Firstly, we define some concepts related to S-almost periodic functions with parameters, i.e., $f(t, x) : R \times X \rightarrow X$. We assume that for any $x \in X, f(t, x) \in L_1^{loc}(R, X)$ and for any $t \in R, f(t, x)$ is continuous on X . We write the set of all such functions as $L_t^{loc}C_x(R \times X, X)$. Let K be a compact subset of X . For any $f \in L_t^{loc}C_x(R \times X, X)$, we write

$$S_l(t, f, K) = \sup_{x \in K} S_l(t, f(\cdot, x)) \quad (5.14)$$

and

$$S_l(f, K) = \sup_{t \in R} S_l(t, f, K) \quad (5.15)$$

Let $f, g \in L_t^{loc}C_x(R \times X, X), \alpha = \{\alpha_n\} \subset R$ be a sequence, and K a compact subset of X . If

$$S_\ell(t, f_{\alpha_n} - g) \rightarrow 0 \quad \text{as } n \rightarrow +\infty \quad \text{pointwise for } t \in R \quad (5.16)$$

we write $S_K T_\alpha f = g$. If (5.16) also holds uniformly for $t \in R$, we write it as $US_K T_\alpha f = g$. Similarly, we denote the uniform S-hull of f by $S_K H(f)$, i. e.,

$$S_K H(f) = \{g \mid \text{there is a sequence } \alpha \subset R \text{ such that } US_K T_\alpha f = g\}.$$

Definition 5.3 Let $f \in L_t^{loc}C_x(R \times X, X)$. We call f *S-almost periodic in t uniformly in x* if for any compact subset K of X , and any sequence $\alpha = \{\alpha_n\} \subset R$, one can extract a subsequence $\alpha' \subset \alpha$ and a function $g \in L_t^{loc}C_x(R \times X, X)$ such that $US_K T_{\alpha} f = g$.

Now, we consider S-almost periodic differential equations

$$x' = f(t, x) \quad (5.17)$$

where $f \in L_t^{loc}C_x(R \times R^n, R^n)$ is S-almost periodic in t uniformly in $x \in R^n$. We always assume that for any $(t_0, x_0) \in R \times R^n$, the equation (5.17) has a solution through (t_0, x_0) , defined in $[t_0, +\infty)$ and we denote such solution by $x(t, t_0, x_0, f)$. Our main goal is to discuss the existence of S-almost periodic solutions of (5.17). Let I be a subset of R . We use $m(I)$ to denote the Lebesgue measure of I . If I' is another subset of R , $I \setminus I'$ is the set

$$\{t | t \in I, \text{ but } t \notin I'\} \quad (5.18)$$

Let $I = [a, b]$ be a closed interval of R and $F = \{f\}$ a subset of $SC(R, R^n)$. We say F is *almost everywhere equi-continuous* on I if there exists at most a subset I' of I with $m(I') = 0$ such that F is equi-continuous on $I \setminus I'$. F is said *almost everywhere uniformly bounded* on I if there exists at most a subset I' of I with $m(I') = 0$ such that F is uniformly bounded on $I \setminus I'$. Let $\{f_n\} \subset F$ be a sequence of functions. $\{f_n\}$ is *almost everywhere uniformly convergent* on I if there exists at most a subset I' of I with $m(I') = 0$ such that $\{f_n\}$ is uniformly convergent on $I \setminus I'$.

Lemma 5.6 *Let $I = [a, b]$ be a closed interval of $\mathbb{R}$. For any subset I_0 of I with $m(I_0) = 0$, there exists a subset Q_0 of $I \setminus I_0$, which is countable and dense in $I \setminus I_0$.*

Proof Let $Q_I = \{r_1, r_2, \dots, r_n, \dots\}$ be the all rationals in I . Then Q_I is dense in I . For any subset I_0 of I with $m(I_0) = 0$, we show that for each $r_n \in Q_I$, there exists a sequence $\{t_k^n\} \subset I \setminus I_0$ such that $t_k^n \rightarrow r_n$ as $n \rightarrow \infty$. In fact, if there exists an r_n , and there exists no such sequence, then there must exist a real number $\delta > 0$ such that $(r_n - \delta, r_n + \delta) \cap (I \setminus I_0) = \emptyset$. We can take $\delta > 0$ sufficiently small so that $(r_n - \delta, r_n + \delta) \subset I$. So, $(r_n - \delta, r_n + \delta) \subset I_0$. This contradicts the fact that $m(I_0) = 0$. Now, let $T = \bigcup_{n=1}^{\infty} \{t_k^n\}$. Obviously, $T \subset I \setminus I_0$ and it is countable and dense in $I \setminus I_0$. This completes the proof of this lemma.

Theorem 5.4 *Let $F = \{f\}$ be a set of functions defined on a bounded interval I . If F is almost everywhere uniformly bounded and almost everywhere equi-continuous on I , then F contains a sequence which is almost everywhere uniformly convergent on I .*

Proof Let I_0 is subset of I with $m(I_0) = 0$ such that F is uniformly bounded and equi-continuous on $I \setminus I_0$. From Lemma 5.6, there exists a subset T of $I \setminus I_0$, which is countable and dense in $I \setminus I_0$.

The rest of the proof is the same as that of the Ascoli Theorem (see [35]), by replacing the set of all rationals in I by the set T . We omit it.

Corollary 5.2 *Let $\{f_n\}$ be a sequence of functions defined on $\mathbb{R}$. If $\{f_n\}$ is almost everywhere bounded and almost everywhere equi-continuous on $\mathbb{R}$, then there exists a subsequence of $\{f_n\}$, which is almost everywhere convergent on every compact subset of $\mathbb{R}$.*

Proof Using Theorem 5.4 and a diagonalization argument, we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which is almost everywhere uniformly convergent on every compact subset of R . For details, see [49].

Now, we discuss the differential equation

$$x' = f(t, x) \quad (5.19)$$

where $f(t, x) \in L_t^{loc}C_x(R \times R^n, R^n)$ and the equation

$$x' = g(t, x) \quad (5.20)$$

where $g(t, x) \in S_K H(f)$. We will establish some properties of the bounded solutions of (5.20). We have the following lemma.

Lemma 5.7 *Let K be a compact subset of R^n , ϕ a solution of (5.19) with $\{\phi(t) | t \in R\} \subset K$. If there exist a sequence $\alpha = \{\alpha_n\} \subset R$, a function $g \in SH_K(f)$ and a function $\varphi \in S_K H(\phi)$ such that $S_K T_\alpha f = g$ and $ST_\alpha \phi = \varphi$ holds uniformly in any compact subset of R , then there exists a solution of (5.20), say $\tilde{\varphi}$, such that $ST_\alpha \phi = \tilde{\varphi}$ holds uniformly in any compact subset of R , and $\tilde{\varphi} \in K$ for all $t \in R$.*

Proof From the assumptions, for any compact subset I of R , we have that

$$\lim_{n \rightarrow \infty} S_l(t, f_{\alpha_n} - g, K) = 0, \text{ uniformly on } I \quad (5.21)$$

and

$$\lim_{n \rightarrow \infty} S_l(t, \phi_{\alpha_n} - \varphi) = 0, \text{ uniformly on } I. \quad (5.22)$$

So, there exists a subset I_0 of R with $m(I_0) = 0$ such that

$$\lim_{n \rightarrow \infty} |f(t + \alpha_n, x) - g(t, x)| = 0, \text{ uniformly on } (I \setminus I_0) \times K \quad (5.23)$$

and

$$\lim_{n \rightarrow \infty} |\phi(t + \alpha_n) - \varphi(t)| = 0, \text{ uniformly on } I \setminus I_0. \quad (5.24)$$

Now, we take $t_0 \notin I_0$ and define

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t_0) + \int_{t_0}^t g(s, \varphi(s)) ds & t \in I_0 \\ \varphi(t) & t \in R \setminus I_0 \end{cases}$$

It is easy to show that $\tilde{\varphi}(t)$ is solution of (5.20), $\tilde{\varphi}(t) \in K$ for all $t \in R$ and $ST_\alpha \phi = \tilde{\varphi}$ holds uniformly on any compact subset of R . This ends the proof of this lemma.

Remark According to this lemma, if ϕ is a solution of (5.19), and there exists a sequence $\alpha \subset R$ such that (5.21), (5.22) hold, we can simply write $ST_\alpha \phi = \tilde{\varphi}$ as a solution of (5.20).

Theorem 5.5 Suppose that $f(t, x)$ is S -almost periodic in t uniformly in $x \in R^n$ and bounded almost everywhere in t on $R \times R^n$. Let K be a compact subset of R^n , $\phi(t)$ a solution of (5.19) with $\phi(t) \in K$ for all $t \in R$. Then, for any given sequence $\alpha \subset R$, one can extract a subsequence α' of α and functions $g \in S_K H(f)$, $\varphi, \phi_1 \in SC(R, R^n)$ such that $ST_{\alpha'} \phi = \varphi$, $ST_{-\alpha'} \varphi = \phi_1$ holds uniformly on any compact subset of R , φ and ϕ_1 are solutions of (5.20) and (5.19), respectively, and $\varphi(t) \in K, \phi_1(t) \in K$ for all $t \in R$.

Proof From the assumptions, for any given sequence $\alpha \subset R$, we can choose a subsequence $\beta \subset \alpha$ and a function $g \in S_K H(f)$ such that $g = US_K T_\beta f$ and $f = US_K T_{-\beta} g$. Let $I_N = [-N, N]$. From the assumptions on f , we have that $f(t, x), f(t + \beta_n, x)$ are uniformly bounded almost everywhere on I_N for all $x \in K$.

So, $\phi(t + \beta_n)$ is uniformly bounded and almost everywhere equi-continuous on I_N .

Using Corollary 5.2. , we can extract a subsequence $\gamma \subset \beta$ and a function $\tilde{\varphi}$ such that $\{\phi(t + \gamma_n)\}$ is almost everywhere uniformly convergent to $\tilde{\varphi}$ on I_N . So, we have that $S_K T_\gamma f = g$ and $ST_\gamma \phi = \tilde{\varphi}$ holds uniformly on each $I_N, N = 1, 2, \dots$.

From Lemma 5.7 , there exists a solution φ of (5.20) such that $ST_\gamma \phi = \varphi$ holds in each I_N and $\varphi(t) \in K$ for all $t \in R$.

Using the same arguments, we can pick up a subsequence $\alpha' \subset \gamma$ and $\tilde{\phi}$ such that $US_K T_{-\alpha} g = f$ and $US_K T_{-\alpha} \varphi = \tilde{\phi}$ hold on each I_N . Again using Lemma 5.7 , there exists a solution ϕ_1 of (5.19) such that $UST_{-\alpha} \varphi = \phi_1$ holds on each I_N and $\phi_1(t) \in K$ for all $t \in R$. This completes the proof of the theorem.

Theorem 5.6 *Suppose that $f(t, x)$ satisfies all conditions in Theorem 5.5. Let K be compact subset of R^n , $\phi(t)$ a solution of (5.19) defined on $[t_0, +\infty)$ for some $t_0 \in R$, valued in K . Then, for any $g \in S_K H(f)$, the equation (5.20) has a solution φ defined on R with values in K .*

Proof Let $\alpha_n = n$. Then $\phi(t + n)$ is a solution of

$$x' = f(t + n, x) \tag{5.25}$$

defined on $[t_0 - n, +\infty)$. Using arguments similar to Theorem 5.5, there exist a function $g \in S_K H(f)$, φ and $\alpha' \subset \alpha$ such that $S_K T_{\alpha} f = g$, $ST_{\alpha} \phi = \varphi$ and φ is a solution of (5.20). Obviously, φ is defined on R . At the same time, there exists ϕ_1 with $\phi_1 = ST_{-\alpha} \varphi$ and ϕ_1 is a solution of (5.19) defined on R .

Now, using Theorem 5.5, for any $g \in S_K H(f)$, we can get a solution of (5.20), which is defined on R with the values in K . The proof is complete.

We can prove the following lemma easily using the definitions of Stepanov hull and of S-norm.

Lemma 5.8 *Let $\phi \in SC(R, R^n)$, $l > 0$ a real number. Then for any $\varphi \in SH(\phi)$, we have $S_l(\varphi) \leq S_l(\phi)$.*

We also have

Lemma 5.9 *Let $\{x_n(t)\}$ be a sequence of solutions of (5.19), defined on R . If $\{x_n(t)\}$ is almost everywhere uniformly convergent on any compact subset of R , then there exists a solution $\phi(t)$ of (5.19) such that $\{x_n(t)\}$ almost everywhere uniformly converges to $\phi(t)$ on any compact subset of R .*

Proof From the conditions, we can take a function $x^*(t)$, defined almost everywhere on R and $\{x_n(t)\}$ almost everywhere uniformly converges to $x^*(t)$ on any compact subset of R . Let $I_0 \subset R$, with $m(I_0) = 0$ and let $x^*(t)$ be defined on $R \setminus I_0$. Now choose $t_0 \in R \setminus I_0$ and define

$$\phi(t) = \begin{cases} x^*(t_0) + \int_{t_0}^t f(s, x^*(s)) ds & t \in I_0 \\ x^*(t) & t \in R \setminus I_0. \end{cases}$$

It is easy to show that $\phi(t)$ is a solution of (5.19) and $\{x_n(t)\}$ almost everywhere uniformly converges to $\phi(t)$ on any compact subset of R . This ends the proof.

Theorem 5.7 *Suppose that $f(t, x)$ is almost everywhere bounded in t uniformly in $x \in R$. Let K be a compact subset of R^n , ϕ a solution of (5.19) with $\phi(t) \in K$ for all $t \in R$. Then there is a solution ϕ_0 of (5.19) such that $\phi_0(t) \in K$ for all $t \in R$ and $\phi_0(t)$ minimizes the S-norm in K , i.e., for any solution φ of (5.19) with $\varphi(t) \in K$ for all $t \in R$, $S_l(\phi_0) \leq S_l(\varphi)$.*

Proof Let

$\mathcal{F} = \{x(t) | x(t) \text{ is a solution of (5.19), } x(t) \in K \text{ for all } t \in R\}$

and define

$$\lambda = \inf_{x \in \mathcal{F}} \{S_l(x)\}. \quad (5.26)$$

Then λ exists and $\lambda \leq S_l(\phi)$. Now we define

$$\lambda_n = \inf_{x \in \mathcal{F}} \left\{ \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x(s)| ds \right) \right\}. \quad (5.27)$$

Obviously, $\lambda_n \leq \lambda_{n+1}$ and $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. So, we can find a sequence $\{x_n(t)\}$ such that $x_n(t) \in \mathcal{F}$ and for each $n \in Z^+$

$$\sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x(s)| ds \right) \leq \lambda_n + \frac{1}{n}. \quad (5.28)$$

According to the assumptions, $\{x_n(t)\}$ are uniformly bounded and almost everywhere equi-continuous. So, from Theorem 5.4, there exists a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}$ which is almost everywhere uniformly convergent on any compact subset of R . From Lemma 5.9, there exists a function $\phi_0 \in \mathcal{F}$ such that $\{x_{n_k}(t)\}$ converges to ϕ_0 almost everywhere uniformly. Now, using Minkovskii's inequality and (5.28), we have that

$$\begin{aligned} \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |\phi_0(s)| ds \right) &\leq \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |\phi_0(s) - x_{n_k}(s)| ds \right) + \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |x_{n_k}(s)| ds \right) \\ &\leq \sup_{|t| \leq n} \left(\frac{1}{l} \int_t^{t+l} |\phi_0(s) - x_{n_k}(s)| ds \right) + \lambda_n + \frac{1}{n} \end{aligned}$$

for each $n \in Z^+$. Letting $n \rightarrow \infty$, we have that $S_l(\phi_0) \leq \lambda$. Since $\phi_0 \in \mathcal{F}$, from the definition of λ , we obtain $S_l(\phi_0) = \lambda$. This ends the proof of the theorem.

Lemma 5.10 *Suppose that $f(t, x)$ is S -almost periodic in t uniformly in $x \in R^n$ and almost everywhere bounded in t uniformly in $x \in R$. Let K be a compact subset*

of R^n , ϕ a solution of (5.19) defined on R with minimizing S -norm in K . If there are $g \in S_K H(f)$ and a sequence $\alpha \subset R$ such that $UST_\alpha f = g$ and $ST_\alpha \phi$ exists uniformly on any compact subset of R , then, $ST_\alpha \phi$ is a solution of the equation

$$x' = g(t, x) \quad (5.29)$$

with minimizing S -norm in K .

Proof From the Remark of Lemma 5.7, we know that $ST_\alpha \phi$ is a solution of (5.29) valued in K . Firstly, we show that $S_l(ST_\alpha \phi) = S_l(\phi)$. In fact, by Lemma 5.8, we have $S_l(ST_\alpha \phi) \leq S_l(\phi)$. On the other hand, from Theorem 5.5 we can take a subsequence $\alpha' \subset \alpha$ such that $ST_{-\alpha'}(ST_\alpha \phi)$ is a solution of (5.19) valued in K . Using Lemma 5.8 again, we have that $S_l(ST_{-\alpha'}(ST_\alpha \phi)) \leq S_l(ST_{\alpha'} \phi) \leq S_l(\phi)$. But, according to the assumption, ϕ minimizes the S -norm in K , and so $S_l(\phi) \leq S_l(ST_{-\alpha'}(ST_\alpha \phi))$. Therefore we have proved that $S_l(ST_\alpha \phi) = S_l(\phi)$. Now, we shall show that $ST_\alpha \phi$ minimizes S -norm in K . Assuming the contrary, the equation (5.29) has another solution φ such that $S_l(\varphi) < S_l(ST_\alpha \phi)$. Then, there exists a subsequence $\alpha' \subset \alpha$ such that $US_K T_{-\alpha'} g = f$ and from Theorem 5.5, there exists a solution ϕ_1 of (5.19) such that $ST_{-\alpha'} \varphi = \phi_1$ and $\phi_1(t) \in K$ for all $t \in R$. From Lemma 5.8, we have that $S_l(\phi_1) = S_l(ST_{-\alpha'} \varphi) \leq S_l(\varphi) < S_l(ST_\alpha \phi) = S_l(\phi)$. This contradicts the fact that ϕ minimizes the S -norm in K . This ends the proof of this lemma.

Now we state and prove the main result of this section, i.e., an existence theorem for an S -almost periodic solution of (5.19).

Theorem 5.8 *Suppose that $f(t, x)$ is S -almost periodic in t uniformly in $x \in R$ and almost everywhere bounded in t uniformly in $x \in R^n$. Let K be a compact*

subset of R^n . If (5.19) has a solution $\phi(t)$ defined on $[t_0, \infty)$ for some $t_0 \in R$ with $\phi(t) \in K$ for all $t \in [t_0, \infty)$ and for any $g \in S_K H(f)$, the equation (5.29) has at most one solution minimizing the S-norm in K , then, for every $g \in S_K H(f)$, the equation (5.29) has an S-almost periodic solution on R .

Proof By Theorem 5.6 and Theorem 5.7, there exists a solution ϕ_1 of (5.19) defined on R with minimizing S-norm in K . We show that ϕ_1 is an S-almost periodic solution of (5.19).

To this end, we use the generalization of Bochner's theorem and show that for any pair of sequences $\alpha, \beta \subset R$, there are two common sequences $\alpha' \subset \alpha, \beta' \subset \beta$ such that $ST_{\alpha'}(ST_{\beta'}\phi_1) = ST_{\alpha'+\beta'}\phi_1$.

Now, let $\alpha, \beta \subset R$ be any pair of sequences. Since $f(t, x)$ is S-almost periodic in t uniformly in $x \in R^n$, we can find two common sequences $\alpha' \subset \alpha, \beta' \subset \beta$ and functions $g, g_1 \in S_K H(f)$ such that $g_1 = UST_{\beta'}f$, $g = UST_{\alpha'}g_1 = UST_{\alpha'+\beta'}f$. From Theorem 5.5 and the Remark of Lemma 5.7, we can take the sequences α', β' such that $ST_{\beta'}\phi_1, ST_{\alpha'}(ST_{\beta'}\phi_1), ST_{\alpha'+\beta'}\phi_1$ exist uniformly on any compact subset of R and $ST_{\beta'}\phi_1$ is a solution of the equation $x' = g_1(t, x)$, while $ST_{\alpha'}(ST_{\beta'}\phi_1), ST_{\alpha'+\beta'}\phi_1$ are all solutions of (5.29). Furthermore, from Lemma 5.8, $ST_{\alpha'}(ST_{\beta'}\phi_1), ST_{\alpha'+\beta'}\phi_1$ are all solutions of (5.29) with minimizing S-norm in K . By the assumption of the theorem, we have that $ST_{\alpha'}(ST_{\beta'}\phi_1) = ST_{\alpha'+\beta'}\phi_1$ for all $t \in R$. This is what we desire.

Obviously, for any $g \in S_K H(f)$, there exists a sequence $\alpha \subset R$ such that $g = US_K T_\alpha f$ and therefore $ST_\alpha\phi$ is a S-almost periodic solution of (5.29). This completes the proof of the theorem.

5.6 Favard's Theorem for S-almost Periodic Differential Systems

We consider a linear S-almost periodic system

$$x' = A(t)x + f(t) \quad (5.30)$$

where $A(t)$ is an $n \times n$ matrix function and $f(t)$ is vector function. We always assume that $A(t)$ and $f(t)$ are S-almost periodic on R , but do not indicate this again in this section.

In order to establish our results, we introduce the square norm of a function as follows: For any $x(t) \in SC(R, R^n)$, we define

$$S_l^2(x) = \sup_{t \in R} \left(\frac{1}{l} \int_t^{t+l} |x(s)|^2 ds \right)^{\frac{1}{2}} \quad (5.31)$$

where $l > 0$ is a constant. It is easy to prove the following lemma.

Lemma 5.11 *Let $\phi(t) \in SC(R, R^n)$. Then for any $\varphi \in SH(\phi)$, we have that $S_l^2(\varphi) \leq S_l^2(\phi)$.*

Lemma 5.12 *Suppose that $A(t)$ and $f(t)$ are almost everywhere bounded on any compact subset of R . Let K be a compact subset of R^n , ϕ a solution of (5.30) with $\phi(t) \in K$ for all $t \in R$. Then there is a solution ϕ_0 of (5.30) such that $\phi_0(t) \in K$ for all $t \in R$ and $\phi_0(t)$ minimizes square norm in K , i.e., for any solution φ of (5.30) with $\varphi(t) \in K$ for all $t \in R$, $S_l^2(\phi_0) \leq S_l^2(\varphi)$.*

The proof of this lemma is almost the same as that of Theorem 5.7. Only Minkovskii's inequality is required here. So we omit the details.

Lemma 5.13 *Let $A(t)$ and $f(t)$ be almost everywhere bounded on any compact subset of R . Let $B \in SH(A)$ and $g \in SH(f)$ be such that there exists a sequence $\alpha \subset R$ such that $UST_\alpha A = B$, $UST_\alpha f = g$. If every non-trivial bounded solution $y(t)$ of the equation*

$$x' = B(t)x \quad (5.32)$$

satisfies

$$\inf_{t \in R} S_l(t, y) > 0, \quad (5.33)$$

then the equation

$$x' = B(t)x + f(t) \quad (5.34)$$

has at most one solution in K with minimizing S_l^2 -norm.

Proof Assuming the contrary. Suppose that there are two distinct solutions $x_1(t), x_2(t)$ of (5.34) with minimizing S^2 -norm in K , i.e., $S_l^2(x_1) = S_l^2(x_2)$. Let $y_1(t) = \frac{1}{2}(x_1(t) + x_2(t))$, $y_2(t) = \frac{1}{2}(x_1(t) - x_2(t))$. Then, $y_1(t)$ is a solution of (5.34), but $y_2(t)$ is a solution of (5.32). From the condition (5.33), we have

$$\delta = \inf_{t \in R} S_l(t, y_2) > 0. \quad (5.35)$$

Using Cauchy's inequality, we have

$$\inf_{t \in R} (S_l(t, |y_2|^2))^{\frac{1}{2}} \geq \delta > 0. \quad (5.36)$$

Now we have

$$\begin{aligned} S_l(t, |y_1|^2) + S_l(t, |y_2|^2) &= \frac{1}{2}S_l(t, |x_1|^2) + \frac{1}{2}S_l(t, |x_2|^2) \\ &\leq (S_l^2(x_1))^2. \end{aligned}$$

Thus,

$$\begin{aligned}
 (S_l^2(y_1))^2 &\leq (S_l^2(x_1))^2 - S_l(t, |y_2|^2) \\
 &\leq (S_l^2(x_1))^2 - \delta^2 \\
 &< (S_l^2(x_1))^2.
 \end{aligned}$$

So, we get $S_l^2(y_1) < S_l^2(x_1)$. This contradicts the fact that x_1 minimizes the S_l^2 -norm. This ends the proof.

Now, we can state a generalization of Favard's Theorem for S-almost periodic differential equations.

Theorem 5.9 *Suppose that $A(t)$ and $f(t)$ are almost everywhere bounded on any compact subset of R and that for any $B \in SH(A)$, any nontrivial solution of (5.32) satisfies (5.33). Let K be a compact subset of R^n , $\phi(t)$ a solution of (5.30) defined on $[\tau, +\infty)$, for $\tau \in R$, with values in K . Then for any $B \in SH(A)$, $g \in SH(f)$, the equation (5.30) has an S-almost periodic solution on R .*

This theorem can easily follow from Theorem 5.8 and Lemma 5.13. We leave out the details.

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